THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

CONDUCTED BY

LORD KELVIN, LL.D. F.R.S. &c.

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AND

WILLIAM FRANCIS, Ph.D. F.L.S. F.R.A.S. F.C.S.

> "Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.

VOL. XLIV.—FIFTH SERIES.

JULY—DECEMBER 1897.

LONDON:

TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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ERRATA.

Page 256, last line. *For* $1{\cdot}6.10^{-10}$ *read* $1{\cdot}6.10^{7}$ C.G.S.
,, 461 (Dr. Wildermann's paper) in lines 19 and 21, p. 463, lines 2–3, and throughout the paper, *for* solid solvend *read* solidified solvent.

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[FIFTH SERIES.]

JULY 1897.

I. *The Ionizing Power of Solvents.* *By* W. C. Dampier Whetham, *M.A., Fellow of Trinity College, Cambridge*[*].

THE exceptionally high power which water possesses of making many substances dissolved in it good conductors of electricity was noticed as soon as solutions in other liquids began to be studied. Even in cases where some conductivity is produced, as in those of salts and acids dissolved in alcohol, the electrical resistance is much greater than if water were used as solvent.

Now the dielectric constant of water is about 75, a number greater than that of any other substance which has been examined, and it seems likely that there is a relation between these two properties. Thus J. J. Thomson[†] and W. Nernst[‡] have pointed out that the effect of immersing a system formed of two oppositely electrified bodies in a medium of high dielectric capacity is to diminish the attractive force between the bodies. The result will be that the oppositely charged ions, which we must suppose to make up a salt or acid, can, under favourable circumstances, separate from each other, and the number of molecules thus dissociated at any instant, and therefore the conductivity, will depend on the dielectric constant of the solvent.

Several experimental investigations on the relative conductivities of substances when dissolved in different solvents have

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[*] Communicated by Prof. J. J. Thomson, F.R.S.

[†] Phil. Mag. 1893, xxxvi. p. 320.

[‡] *Zeits. physikal. Chem.* 1894, xiii. p. 531.

already been carried out, and an attempt to connect such results quantitatively with the dielectric constants has been made by the present writer*. Since then further experiments have been made by Völlmer† on solutions in methyl and ethyl alcohols, and he finds that the ratios of the conductivities of various salts in water, methyl alcohol, and ethyl alcohol are 100 : 73 : 34. If we assume that the frictional resistances which the three solvents oppose to the passage of the ions through them are in the same ratios as their viscosities (a more or less plausible assumption for which there is no experimental evidence), we can find relative values for the ionizing powers by taking them to vary as the dielectric constants and as the ionic fluidities. The numbers for the three liquids come out as 100 : 63 : 26, a rough agreement with observation, which goes to show that we are probably working somewhat in the right direction.

The experiments now to be described were undertaken in order to investigate the same idea by another method. It seemed possible, if a liquid could be found of a suitable nature and used as solvent, that water, used as solute, should itself be ionized, and make a solution of high conductivity.

A search for data on dielectric constants did not, however, discover any liquid of higher constant than water. Nevertheless, it seemed likely that an examination of the conductivities of mixtures of water and other liquids having various constants would throw some light on the question at issue.

In the choice of liquids for this purpose three things had to be borne in mind. Firstly, the substances must be liquid at convenient temperatures and pressures; secondly, they must, when pure, have high specific resistances; and thirdly, their dielectric constants must vary as much as possible. The three acids—acetic, trichloracetic, and formic—were finally selected. Many dielectric constants were determined by Thwing‡, whose numbers for water, formic acid, and acetic acid are 75·5, 62·0, and 10·3 respectively. The constant for trichloracetic acid is not given, but from the numbers for aldehyde (18·55) and chloral (5·47) it appears that the replacement of three hydrogen by three chlorine atoms lowers the dielectric constant considerably. The value for trichloracetic acid is, therefore, probably less than 10. Formic acid appears to have a higher dielectric capacity than any liquid other than water which has yet been examined.

The work was done in the summer of 1896, but its publication was delayed because it seemed desirable that the

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* Phil. Mag. 1894, xxxviii. p. 392.

† Wied. *Ann.* 1894, lii. p. 328.

‡ *Zeits. physikal. Chem.* 1894, xiv. p. 286.

remarkable result obtained for mixtures of formic acid and water should be confirmed. This has now incidentally been done in the course of a careful series of experiments by Dr. Vladimír Novák, whose paper follows this one.

As the object of the investigation was merely to test the general behaviour of the mixtures, no great accuracy was sought, and a simple form of apparatus was sufficient. Kohlrausch's telephone method was adopted for the conductivity measurements, a straight-wire bridge being arranged in the usual manner. Instead, however, of using a battery and induction-coil as the source of current, the town supply, alternating about 100 times a second, was sent through the longer coil of a simple transformer, the shorter coil of thick wire being connected with the bridge. This arrangement is in constant use at the Cavendish Laboratory, and gives excellent results, the sensitiveness being both greater and more uniform than with the usual arrangement.

Two cells were used to contain the solutions, one giving a much higher resistance than the other. Their forms are represented in figs. 1 and 2. When in use each cell was immersed in a water-bath, the temperature of which could be easily adjusted.

Fig. 1.

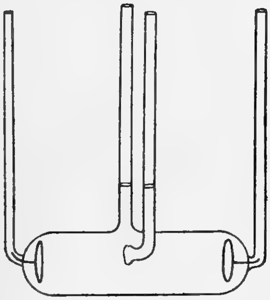


Fig. 2.

The process of the experiments was as follows:—The acid sold as pure was placed in cell No. 2 and weighed. Its resistance at a known temperature was then measured in the water-bath. Successive quantities of water were then added, first from a pipette which had been adjusted to deliver half a gram of water at the temperature of the laboratory, and then from larger pipettes. When the volume of liquid in the cell became inconveniently large, the additions of water were made in a separate vessel, into which the contents of the cell were emptied after each electrical measurement.

The electrical constants of the cells were determined by filling them with solutions of barium chloride of known concentrations, and measuring the resistances. The constants were then calculated from Kohlrausch's determinations of the conductivities of these solutions. Thus in cell No. 1 a solution of 0·01 normal strength gave a resistance of 2142 ohms at 18°, and a solution of 0·1 normal strength a resistance of 251·5 ohms at 18°·2. The constants calculated from these numbers came out $2{\cdot}307 \times 10^{-9}$ and $2{\cdot}305 \times 10^{-9}$ respectively, the mean value being $2{\cdot}306 \times 10^{-9}$. For cell No 2, determinations with 0·01 normal and 0·001 normal solutions gave 131 0 and 1132 ohms, which lead to values for the constant of $1{\cdot}412 \times 10^{-10}$ and $1{\cdot}408 \times 10^{-10}$, giving a mean of $1{\cdot}410 \times 10^{-10}$.

The following results were obtained :—

Acetic Acid and Water.

Per cent. of water by weight.	Resistance at 18° in cell No. 2.	Conductivity in C.G.S. units.	Conductivity, corrected for that of pure solvent.
0·0	100000	$1{\cdot}41 \times 10^{-15}$	0
2·15	45000	3·14 ,,	$1{\cdot}73 \times 10^{-15}$
4·22	24000	5·88 ,,	4·47 ,,
6·20	16000	8·81 ,,	7·40 ,,
8·10	10100	13·96 ,,	12·55 ,,
9·92	7000	20·15 ,,	18·74 ,,
11·67	5990	23·55 ,,	22·14 ,,
14·98	2780	50·73 ,,	49·32 ,,
18·05	1760	80·13 ,,	78·72 ,,
24·83	817	173·6 ,,	172·2 ,,
30·58	497	283·6 ,,	282·2 ,,
39·79	270	522·2 ,,	520·8 ,,
46·84	194	727 ,,	726 ,,
52·41	152	928 ,,	927 ,,
56·93	131	1076 ,,	1075 ,,
66·47	100·2	1408 ,,	1407 ,,
75·51	85·6	1648 ,,	1647 ,,
84·14	82·6	1707 ,,	1706 ,,
88·22	87·7	1608 ,,	1607 ,,
92·25	94·9	1486 ,,	1485 ,,
96·12	124	1137 ,,	1136 ,,
98·06	168	839 ,,	838 ,,
99·40	......	......	460 ,,
99·82	......	......	252 ,,

The numbers below the dotted line are taken from Kohlrausch's results for solutions of acetic acid in water.

Trichloracetic Acid and Water.—Pure trichloracetic acid melts at about 52°. The measurements were made at 60°.

Per cent. of water by weight.	Cell.	Resistance at 60°.	Conductivity at 60° in C.G.S. units.
0	2	>1 million.	0
0·80	„	280000	$0{\cdot}5 \times 10^{-15}$
1·59	„	70000	2·0 „
3·13	„	8700	16·2 „
4·62	„	2040	68·9 „
6·06	„	665	211·5 „
7·47	„	277	507·6 „
10·15	„	76·8	1830 „
13·90	„	22·8	6167 „
19·48	1	109	21150 „
28·74	„	33·5	68800 „
53·0	„	9·7	238000 „
68·8	„	7·5	307000 „
80·5	„	7·9	292000 „
91·8	......	......	185000 „
95·9	......	......	105000 „
97·95	......	......	57100 „
98 98	......	......	29900 „

The numbers below the dotted line are calculated from Ostwald's results for solutions of trichloracetic acid in water.

Formic Acid and Water.—The formic acid sold as pure had a freezing-point of +1°·60. On adding 0·5 c.c. of water to 25·99 grams of acid it froze at −1°·1, a lowering of 2°·7, and a further addition of 0·5 c.c., making 1 c.c. in all, gave a mixture freezing at −3°·8. Thus the lowering produced by the second half-gram of water was also 2°·7, and the depression of the freezing-point is proportional to the quantity of water present. The freezing-point of the anhydrous acid is taken by Raoult to be 8·5, so that the acid used in these experiments contained about 2·47 per cent. of water to begin with. Allowing for this, and correcting the conductivity results for the calculated conductivity of the pure acid, we get:—

Per cent. of water by weight.	Cell.	Resistance at 18°.	Conductivity in C.G.S. units.
2·47	2	228	$3{\cdot}675 \times 10^{-13}$
3·41	„	185	5·111 „
4·33	„	160	6·304 „
6·11	„	118	9·435 „
11·07	„	68	18·54 „
18·28	„	40·4	32·76 „
29·67	1	423	51·99 „
45·01	„	285	78·41 „
61·71	„	226	99·48 „
78·23	„	231	97·31 „
84·79	„	260	86·17 „
90·51	„	315	70·69 „
97·70	......	......	39·50 „
99·43	......	......	19·38 „

The numbers below the dotted line are calculated from Ostwald's results for solutions of formic acid in water.

If we plot these results as curves, showing the relation between the percentage of water and the electrical conductivity, we get the diagrams represented in figs. 3, 4, and 5. It must be noticed that these curves are not drawn on the same vertical scale. The ascending branch of each curve relates to a solution of water in the acid, while the descending branch is concerned with a solution of the acid in water. These latter solutions all give curves of the same general form, asymptotic to the vertical axis, and this may be considered as the typical curve of a conducting solution. The actual conductivity has not much influence on the general form of the curve, though it will, of course, alter its constants. This is shown by the fact that the general form of the curve for acetic acid in water is the same as that for trichloracetic acid, which has a very much higher conductivity. The general form of the curve may thus be considered to be determined by the general electrical nature of the liquid, about which it will therefore give us valuable information.

Let us now examine the ascending portions of the curves, which relate to mixtures containing small percentages of water, that is, to solutions of water in the acids. In the cases of acetic and trichloracetic acids, which are liquids with dielectric constants very small as compared with that of water, the form of the upward curve is entirely different from that of the descending branch. Instead of approaching the vertical axis asymptotically, it approaches the horizontal axis in this manner, and eventually cuts the vertical axis at right angles.

Fig. 3.—Water and Acetic Acid.

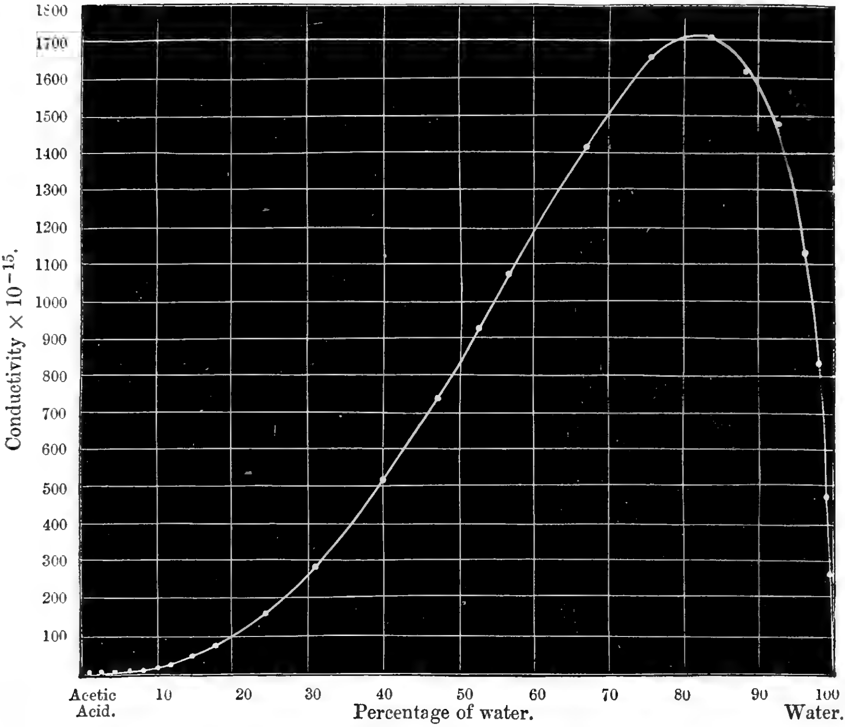


Fig. 4.—Water and Trichloracetic Acid.

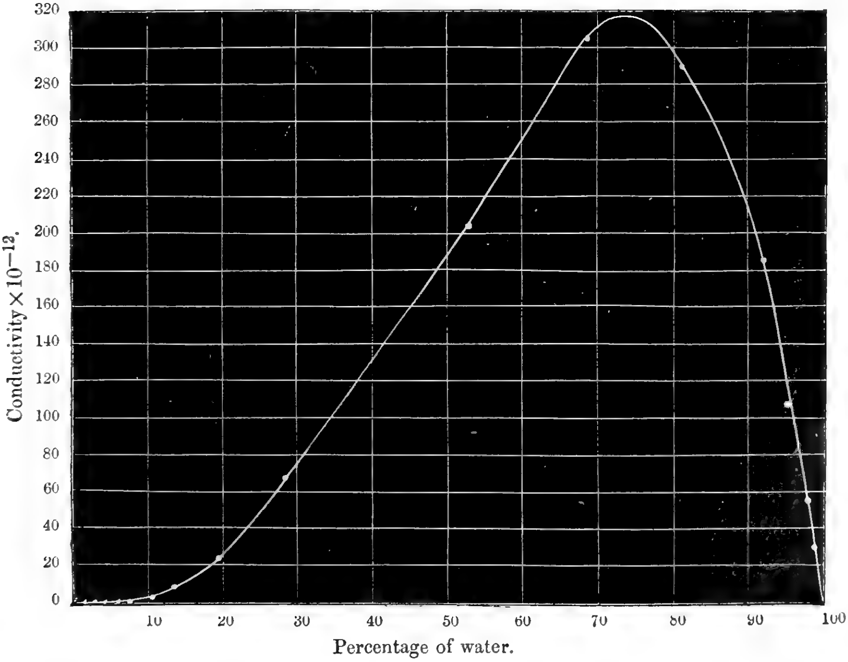
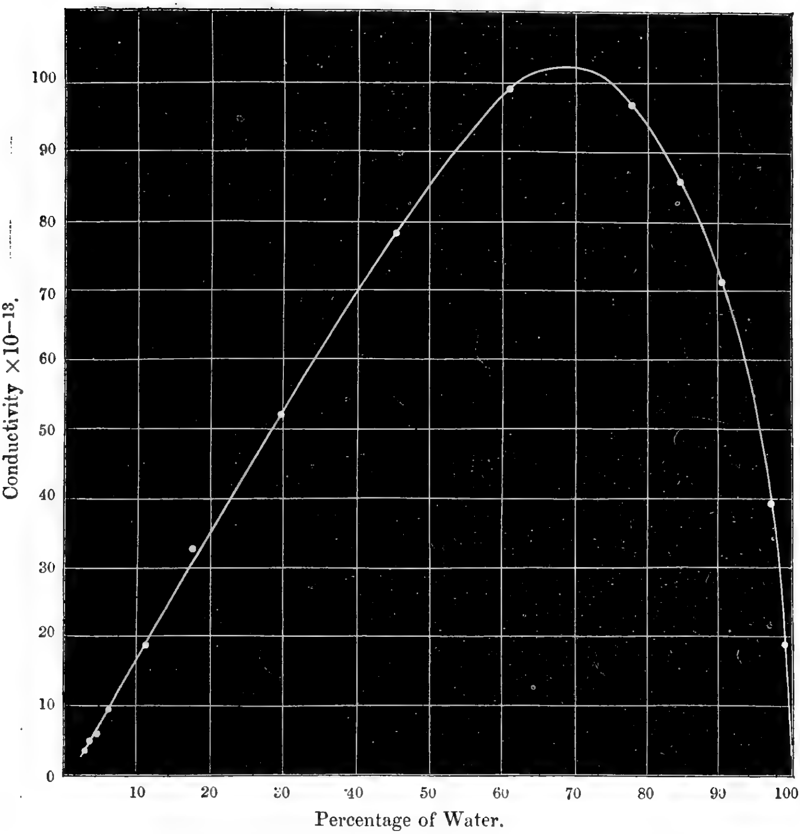


Fig. 5.—Water and Formic Acid.



The curve for water in formic acid, however, does not resemble these. It is, till the proportion of water reaches about 40 per cent., a straight line. This has been confirmed by Dr. Novák's much more elaborate experiments. We have here, then, a form of curve which approaches that of a typical electrolytic solution, though it does not quite coincide with it. Now formic acid, as we have seen, has a dielectric constant of 62, a number which approaches that of water, 75. Could we find some liquid whose constant was greater than 75, it seems probable that it would form a solvent in which water would be an electrolyte, just as common mineral salts and acids are electrolytes when dissolved in water.

The forms of the curves indicate that the solutions of water in these three acids are not of the same nature as those of normal electrolytic liquids. This difference is less marked in the case of formic acid, but, even there, a true difference in kind seems to exist, and it is probable that no true ionization of the water occurs. In order to test this conclusion we may make use of the freezing-point determinations which have already been described.

It was found that 0·5 gram of water depressed the freezing-point of 25·99 grams of strong formic acid by 2°·7. This quantity of water corresponds to 1·07 gram-molecule in 1000 grams of solvent, so that 1 gram-molecule in the same weight of solvent would give a depression equal to 2°·52. On the assumption that there is no ionization, van't Hoff's formula gives 2°·84 for this molecular depression, and Raoult's experimental results for various organic substances dissolved in formic acid vary from 2°·6 to 2°·9. The phenomena of the freezing-points thus confirm our previous conclusion that water dissolved in formic acid is not ionized.

Nevertheless, such a solution possesses considerable conducting power, of which an explanation is required. Possibly it is due to the distribution through the general non-conducting or badly conducting mass of a small quantity of what may be considered a solution of acid in the water which has been added. The curves, however, indicate that such a state is essentially different from that which holds in a normal electrolytic solution.

In conclusion we may say that there seems to be considerable evidence to show that, if a liquid could be found possessing a dielectric constant higher than that of water, it would, when used as a solvent, cause water dissolved in it to be ionized, and give it a conductivity depending on the amount of such ionization and on the ionic fluidity. The conductivity curve would then have the same general form as those which we have seen to be typical of solutions of electrolytes.

II. *Specific Electric Conductivities and Freezing-points of Solutions of Water in Formic Acid. By* Dr. VLADIMÍR NOVÁK, *Privatdocent of the Bohemian University in Prague* *.

I was induced to do this work by Mr. W. C. D. Whetham (Trinity College, Cambridge), who found in some experiments that formic acid, if it contains a small quantity of water, shows extremely simple changes in its specific electric conductivity and freezing-point.

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* Communicated by Prof. J. J. Thomson, F.R.S.

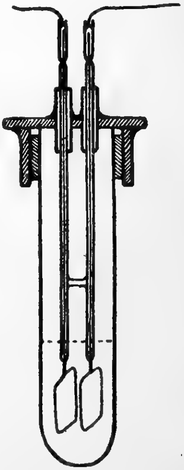
To test this, I measured the electric conductivity of a series of formic acid solutions which contained only a small amount of water, the freezing-point of every solution being observed.

The formic acid was obtained from Harrington (London); its freezing-point was 1°·47. I treated this acid according to the method given by W. Saposchnikoff *, and I prepared, by repeated distillation and recrystallization, an acid which froze at 7°·76.

The electric measurements were made by F. Kohlrausch's method by means of Wheatstone's bridge and telephone. The alternating current was the town supply, which was sent through the longer coil of a transformer, the shorter coil being connected with the ends of the bridge.

The resistance-cell was so arranged that it was possible to use only a small amount of the liquid. A tube ground on the top was closed by another ground-glass ring, which was cemented to a plain thick glass plate through which two insulated platinum electrodes passed. The apparatus is shown in fig. 1.

Fig. 1.



I preferred this form to those of Kohlrausch or Arrhenius because the distance between the electrodes was absolutely constant, and the cleaning of the tube and flushing and drying of the electrodes was extremely easily done.

The minimum of the sound in the telephone was very sharp if the resistance used was some hundred ohms. If smaller resistances were used, a condenser of suitable capacity had to be added on the side of the rheostat.

The resistance-cell was put in a larger vessel filled with water, stirred by a mechanical stirrer, the temperature being measured by two mercury thermometers.

The resistance-cell was calibrated with a $\frac{N}{100}$ solution of potassium chloride, the specific molecular conductivity of which was adopted as $1147 . 10^{-8}$ † in terms of mercury. The distilled water used had a conductivity $0{\cdot}9 . 10^{-10}$, which (according to Kohlrausch's *Leitfaden d. prakt. Physik*, viii. Aufl. 1896) may be considered as very low.

The freezing-points of different solutions were observed by

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* W. Saposchnikoff, *Journ. russ. phys. chem. Ges.* 25. ii. p. 626 (1893).

† Kohlrausch, Wied. *Ann.* xxvi. p. 195.

means of Beckmann's apparatus on a mercury thermometer (J. Hicks, London) divided in $\frac{1}{10}°$, corrections of which were given by the Kew Observatory.

Solutions of water in formic acid were prepared in a glass bottle (of 50 c. c.) with a glass stopper, by weighing the bottle, then the bottle and the acid, and finally weighing the whole, when some drops of water had been very carefully added.

I first tested whether the formic acid, which is very hygroscopic and easily decomposes, remained in a constant state for the time (some hours) during which the observations of the resistances and freezing-points were taken. The best way to test this is, of course, by the measurement of the resistance of the acid during some days.

The following table (Table I.) shows three such observations for three different solutions, freezing at 7·69, 5·46, 5·38 respectively:—

Every measurement started with a solution, which was prepared by freezing; the second solution (5·46 f.-p.) was left in the resistance-cell for the whole time (145 hours); the others were put immediately in the bottle after the measurement.

The conductivity increases during the time, more rapidly the first day than afterwards, showing that the acid takes some water from the air or that it decomposes slightly, especially if in contact with platinum black on the electrodes.

More concentrated acid seems to be more hygroscopic than dilute acid—the observations show that it is as difficult to keep the concentrated acid unchanged as to prepare it. It follows clearly that all the measurements referring to one solution must be made in a very short time, and that it is not possible to get a high accuracy in results.

Every solution was measured at four or five different temperatures, beginning from a temperature near the freezing-point and repeating the reading at each temperature four times at equal intervals so as to be sure that the temperatures of the solution and of the bath were the same.

Every solution was prepared from the fresh, most concentrated, acid, or from acid which had been distilled and frozen just before it was used for the measurements. All this treatment took up a great deal of time, because the greatest care was spent in preparing the acid pure.

Table II. shows the results. In the first column are freezing-points (F. P.), in the second the specific conductivity in terms of mercury at 0° ($c \,.\, 10^9$), in the third the temperature of the solution (t), in the fourth the temperature-coefficient (K), and in the fifth the concentration in per cents. of the water (p). The other part of the table contains data for the specific molecular conductivity (c_m) and the concentration calculated in equivalent gram-molecules per litre (m).

Table I.

Freezing-point, °C.	Time, hours.	Conductivity, 10^{-10} cm. g. sec.	At the temp., °C.	Conductivity at 15° cm. g. sec.	Change in the conductivity for the first day per cent.
7·69	0	12·23	14·73	12·3	1·7 per cent.
	0·5	12·64	16·91		
	1·0	13·01	18·85		
	48·0	11·04	8·10	12·7	
	48·5	11·49	10·10		
	49·0	11·96	12·65		
5·46	0	20·71	13·92	21·0	1·9 per cent.
	24	21·33	14·59	21·4	
	24·5	21·98	16·64		
	25·0	23·07	20·04		
	47·0	17·37	2·73	21·7	
	47·5	21·45	14·23		
	48·0	22·22	16·53		
	72·0	21·81	14·80	21·8	
	72·5	23·31	19·32		
	144·0	21·53	12·76	22·3	
	144·5	22·32	15·05		
	145·0	23·26	17·77		
	192·0	22·35	14·20	22·6	
	192·5	23·08	16·42		
	193·0	23·53	17·48		
5·38	0	24·89	6·11	28·7	1·0 per cent.
	0·5	26·36	9·53		
	24·0	27·36	11·33	29·0	
	24·5	28·89	14·56		
	25·0	29·61	16·27		
	72·0	28·27	12·96	29·2	
	75·0	27·56	11·18		
	75·5	26·31	8·50		
	76·0	29·14	14·65		
	192·0	25·24	5·25	29·7	
	192·5	27·10	9·06		
	193·0	28·71	12·66		
	193·5	29·86	15·33		
	194·0	31·15	18·20		

TABLE II.

F.-P., °C.	$c . 10^9$, 0° Hg.	t.	$K . 10^4$.	p, per cent.	$c_m . 10^9$, 0° Hg.	t, °C.	m, g. m. p. l.
7·69	(10·89)	(7·69)	200	0·58	27·49	7·69	0·396
	12·23	14·73					
	12·64	16·91					
	13·01	18·85					
7·29	(14·00)	(7·29)	189	0·80	25·63	7·29	0·526
	14·57	9·83					
	15·37	13·05					
	15·91	15·41					
	16·03	15·86					
	16·28	16·97					
	16·66	18·82					
7·00	(16·57)	(7·00)	203	0·98	24·73	7·00	0·670
	16·72	7·62					
	17·15	8·98					
	18·06	11·95					
	18·85	14·52					
	19·91	18·48					
6·10	(22·86)	(6·10)	208	1·58	21·00	6·10	1·089
	24·24	9·25					
	24·51	9·63					
	25·68	12·71					
	26·79	15·26					
	27·73	17·65					
5·41	(27·50)	(5·41)	207	2·03	19·57	5·41	1·405
	28·06	6·52					
	29·95	10·16					
	31·27	12·81					
	32·70	15·93					
	34·13	19·18					
4·71	(31·70)	(4·71)	190	2·44	18·67	4·71	1·698
	32·06	5·40					
	34·09	8·93					
	34·88	10·50					
	35·02	10·76					
	35·42	11·42					
	38·64	16·91					
4·35	(34·87)	(4·35)	186	2·69	18·50	4·35	1·876
	35·26	5·00					
	36·93	7·79					
3·87	(37·88)	(3·87)	191	3·00	18·02	3·87	2·102
	37·72	3·68					
	39·71	6·64					
	41·28	8·93					
	42·70	10·98					
	45·13	14·78					
	46·72	17·63					

TABLE II. (*continued*).

F.-P., ° C.	$c . 10^9$, 0° Hg.	t.	K . 10^4.	p, °/°.	$C_m . 10^9$, 0° Hg.	t. ° C.	m, g. m. p. l.
3·60	(39·47)	(3·60)	191	3·16	17·79	3·60	2·218
	39·79	4·04					
	41·30	6·19					
	43·84	9·75					
	44·81	11·18					
	47·01	14·68					
	48·73	17·32					
3·41	(41·22)	(3·41)	174	3·31	17·74	3·41	2·323
	41·37	3·62					
	43·77	6·93					
	45·86	9·94					
	48·63	13·80					
	50·25	16·27					
	50·54	16·84					
2·91	(44·47)	(2·91)	195	3·65	17·32	2·91	2·569
	45·05	3·62					
	47·37	6·22					
	51·42	11·43					
	53·51	14·06					
	55·18	16·64					
2·21	(49·54)	(2·21)	187	4·14	16·90	2·21	2·932
	50·31	3·07					
	54·41	7·70					
	54·58	7·88					
	56·84	10·48					
	57·45	11·04					
	57·57	11·34					
	61·13	15·64					
	62·56	17·58					
1·60	(53·66)	(1·60)	185	4·50	16·76	1·60	3·202
	55·44	3·45					
	56·84	4·90					
	59·16	7·25					
	63·40	11·45					
	65·58	13·96					
	68·90	17·85					
−0·40	(67·39)	(−0·40)	1·24	5·82	16·05	−0·40	4·198
	70·60	2·19					
	72·78	3·95					
	76·32	6·82					
	80·28	9·87					
	87·90	17·11					
−1·81	(75·36)	(−1·81)	......	6·75	15·33	−1·81	4·915
	79·87	1·19					
	83·35	3·51					
	89·58	7·88					
	90·35	8·26					
	94·01	11·06					

Table II. (*continued*).

F.-P., °C.	$c.10^9$, 0° Hg.	t.	$K.10^4$.	p, %	$C_m.10^9$, 0° Hg.	t. °C.	m, g. m. p. l.
......	110·0	2·11	......	9·02	......	......	6·733
	122·7	8·36					
	123·6	8·79					
	124·3	9·56					
	129·6	12·18					
	132·6	13·80					
	134·4	15·15					
	140·9	19·54					
......	127·9	3·53	......	10·34	......	......	7·829
	141·6	9·73					
	150·3	14·06					
	156·6	17·85					
......	200·2	1·89	......	15·78	......	......	12·72
	207·6	4·16					
	223·0	8·62					
	228·7	10·60					
	237·7	13·56					
	248·1	17·13					
......	345·2	9·67	......	24·52	......	......	22·05
	349·6	10·65					
	358·8	12·75					
	372·2	16·24					

Fig. 2 shows the dependence between specific conductivity (c) and the temperature (t). The first number for each solution (put in brackets) is calculated for $t=$ F.-P., supposing the curves are straight lines, that being so at least for the lower temperature. The temperature-coefficient "K" is calculated for the first seven solutions from all the observations, for the others only from the observations at the low temperatures.

The diagram (fig. 2) shows some very interesting results. The points for the specific conductivity (c) of different solutions at the temperature of freezing are on a *straight line*, which could be easily extended to the freezing-point of the anhydrous acid, which is, no doubt, that of Raoult, freezing at $8^\circ{\cdot}52$.

The equation of this straight line, using observations from 1 to 14, is

$$c_t = 64{\cdot}77 - 6{\cdot}937 \text{ F.-P.} \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Fig. 2.

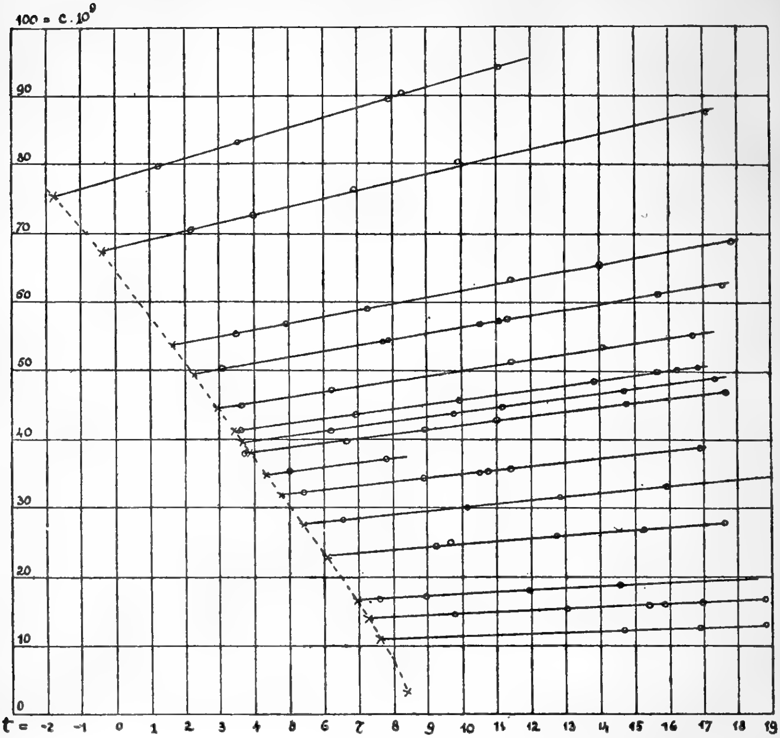


TABLE III.

c_t*.	c_t.	Diff.	F.-P.*	F.-P.	Diff.	c_t*.	c_t.	Diff.
11 40	10·89	0·51	7·63	7·69	−0·06			
14·18	14·00	0·18	7·29	7·29	0 00	14·52	14·00	0 52
16·19	16·57	−0·38	7·02	7·00	0·02	16·40	16·57	−0·17
22·43	22·86	−0·43	6·09	6·10	−0·01	22·74	22·86	−0·12
27·22	27·50	−0·28	5·40	5·41	−0·01	27·47	27·50	−0·03
32·07	31·70	0·37	4·77	4·71	0·06	31·82	31·70	0·12
34·57	34 87	−0·30	4·38	4·35	0·03	34·46	34·87	−0·41
37 90	37·88	+0·02	3·91	3·87	0·04	37·76	37·88	−0·12
39·78	39·47	0·31	3·66	3·60	0·06	39·46	39·47	−0·01
41·10	41·22	−0·12	3·43	3·41	0·02	40 99	41·22	−0·23
44·56	44·47	0·09	2·91	2·91	0·00	44·54	44·47	0·27
49·42	49·54	−0·12	2·16	2·21	−0·05	49·75	49·54	0·21
53·65	53·66	−0·01	1·60	1·60	0·00	53·59	53·66	−0·07
67·53	67·39	0·14	−0·42	−0·40	−0·02	67·50	67·39	0 11
......	......	......	−1·85	−1·81	−0·04			

In Table III. there are put down : $c_t{}^*$ calculated from the equation above, and c_t taken from Table II., as also the differences ($c_t{}^* - c_t =$ diff.). Putting $t = 8{\cdot}52$ in the equation, we get for the anhydrous formic acid, $c_t = 5{\cdot}65 \, . \, 10^{-9}$ at 8°·52 C. in terms of mercury at 0°.

The temperature-coefficient can be determined according to K . 10^4 from Table II. as 0·020, because the numbers for K are nearly equal, which is shown in the diagram (fig. 2) by straight lines passing through the same point.

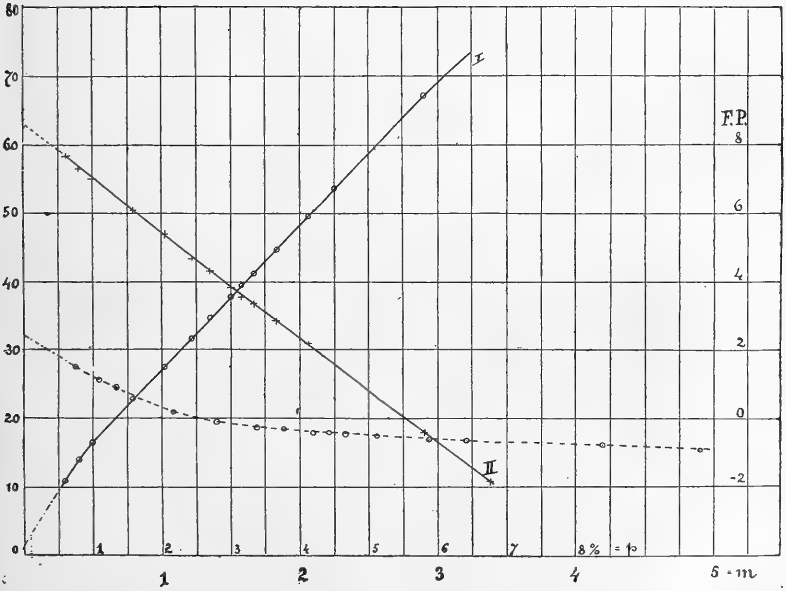
W. Saposchnikoff got for the most concentrated formic acid the following results :—

Freezing-points of the acid.	$c \, . \, 10^9$.	t°.	C at freezing-point.
8·30	4 8	16·5	3·7
8·344	4·7	17·6	3·5
8·356	4·6	18·5	3·4
8·39	4·0	18·8	2·9

The last column is calculated according to the temperature-coefficient 0·020. These numbers are not on the straight line, but if we extrapolate a curve through the first three observations, we get, from the diagram 2, for 8·39 F.-P. the value 3·0, which agrees with that of Saposchnikoff (2·9) very well indeed.

Fig. 3.

$c \, . \, 10^9$ or $c_m \, . \, 10^9$ (the dotted line).



Another diagram (fig. 3) shows the relation between the *specific conductivity* c and *concentration* p (see the curve I.). The second curve—practically a straight line—is constructed from p, the *concentrations*, as abscissæ and *freezing-points* as ordinates (II.). The dotted curve shows the relation between *molecular conductivity* c_m and *concentration* measured in gram-molecules per litre m.

Lines I. and II. are nearly straight lines, which fact is very interesting. It means that if we add a small quantity of water to formic acid, the lowering in freezing-point, as also the change of specific conductivity, is *proportional* to that *quantity*. The connexion between freezing-point and concentration (p), calculated from all the observations, gives the equation

$$\text{F.-P.} = 8{\cdot}52 - 1{\cdot}537\,p. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The second part of Table III. gives the values for F.-P.* calculated.

The dependence between specific conductivity (c_t) and concentration (p) (the conductivity being taken at the temperature of freezing) calculated from observations 2 to 14 gives the relation

$$c = 6{\cdot}10 + 10{\cdot}546\,p, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

which can be used as an interpolation formula for $p=0{\cdot}58$ to 5 82. The first two observations seem to show that the beginning of the line I. is a curve. Extrapolating the line I. as a curve, we get for the concentration $p=0{\cdot}09$ (which is that of the best acid of Saposchnikoff), the value of $c=3{\cdot}0\,.\,10^{-9}$, which agrees with his value $2{\cdot}9\,.\,10^{-9}$ very well. But I will not conceal that the difficulty of keeping the concentrated acid unchanged and determining the very small amount of water in the first solutions, might easily be the cause why the first two points are not on the straight line.

Now let us compare the results with theory.

For "weak solutions" we can write van 't Hoff's formula

$$\delta \mathrm{T} = \frac{0{\cdot}001976\,\mathrm{T}^2\,.\,m}{\lambda\rho},$$

where δT is the depression in freezing-point, T the absolute temperature of freezing, m the number of gram-molecules per litre, λ the latent heat, ρ the specific gravity of the solvent. Formic acid being here solvent we get

$$\mathrm{T}=281{\cdot}52, \quad \lambda=55{\cdot}6, \quad \rho=1{\cdot}223,$$

or

$$\delta\mathrm{T} = 2{\cdot}303\,m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Using this formula, we can calculate the freezing-points for the solutions employed, and we get :—

Freezing-points.		
Calculated.	Observed.	Difference.
7·61	7·69	−0·08
7·26	7·29	−0·03
6·98	7·00	−0·02
6·01	6·10	−0·09
5·28	5·41	−0·13
4·61	4·71	−0·10
4·20	4·35	−0·15
3·68	3·87	−0·19
3·41	3·60	−0·19
3·17	3·41	−0·24
2·60	2·91	−0·31
1·87	2·21	−0·34
1·15	1·60	−0·45
−1·15	−0·40	−0·75
−3·80	−1·81	−1·99

The increasing differences between calculated and observed values for the freezing-points show clearly that proportionality between depression in freezing and concentration *measured in number of gram-molecules per litre* only holds for dilute solutions, which of course could be seen from the diagram (fig. 3) and equation (3), which shows the proportionality between depression in freezing-point and concentration *measured in per cents. of water.* Putting then for p its value in m,

$$p = \frac{1801\,m}{1223 + 18{\cdot}01\,m}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we can calculate the value of $1{\cdot}537 \dfrac{1801\,m}{1223 + 18{\cdot}01\,m}$ for different values of m.

We get :—

m.	$1{\cdot}537 \dfrac{1801\,m}{1223+18{\cdot}01\,m}$.
0	2·263
0·1	2·260
1	2·230
2	2·198
3	2·168
4	2·131
5	2·108

Results.—Formic acid, if it dissolves a small quantity of water, becomes more conductive and its freezing-point falls. The *increase of conductivity* is proportional to *the amount of water* added. This law holds from 1 to 6 °/₀ of water. Similar *proportionality* appears between the *freezing-point* and *conductivity* and between the *freezing-point* and *concentration* measured in *per cents.*

The *specific conductivity* of anhydrous formic acid freezing at 8°·52 is $1·5 . 10^{-9}$ in terms of mercury at 8°·52, which value agrees very well with the observations of Saposchnikoff.

The molecular conductivity of anhydrous formic acid freezing at 8°·52 is $32 . 10^{-9}$ in terms of mercury at 8°·52.

The temperature-coefficient, or the change of the conductivity in per cents. for 1°, is 0·020, and it is practically the same for all solutions of water in formic acid up to 6 °/₀.

The relations can be written :—

Specific conductivity at the temperature of freezing
$$= 6·10 + 10·546 \text{ (per cents. of water)}.$$
$$= 64·77 - 6·937 \text{ (freezing-point)}.$$
$$\text{Freezing-point} = 8·52 - 1·537 \text{ (per cents. of water)},$$
using the limits from above.

My best thanks are due to Prof. J. J. Thomson who supported me with his kind advice, and to Mr. W. C. D. Whetham, who followed this work with the greatest interest.

Cambridge, Cavendish Laboratory,
1896–97.

III. *Experiments on the Motion of the Æther in an Electromagnetic Field.* *By* W. Craig Henderson, *M.A., B.Sc., and* J. Henry, *M.A., B.E.,* 1851 *Exhibition Science Scholars**.

AN electric wave when it passes through the æther produces a displacement current and a magnetic force at right angles to this current ; there will thus be a mechanical force acting on the æther in a direction at right angles both to the displacement current and to the magnetic force, which will therefore at some distance from the radiator be at right angles to the front of the wave. If the oscillations are not damped this force will be periodic and its mean value zero ; it will, however, have a finite value if the oscillations are rapidly damped. The following experiments were undertaken at the suggestion of Professor J. J. Thomson to see whether this force would set the æther in motion.

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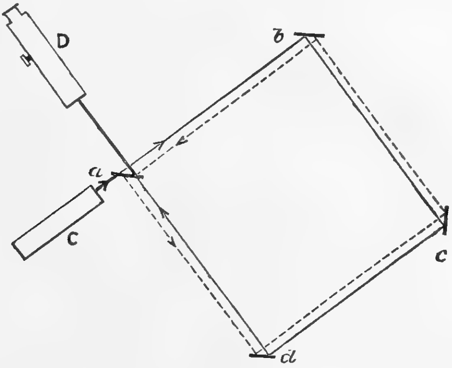
* Communicated by Prof. J. J. Thomson, F.R.S.

The method adopted for the detection and measurement of this motion was similar to that used by Professor Oliver Lodge*. Three small plane mirrors are placed vertically at three corners of a square marked out on a rigid table. These mirrors were mounted in such a way that, by the pressure of screws behind, a slight rotation round a vertical or round a horizontal axis could be given to each. The mirrors were placed with their planes perpendicular to the diagonals of the square. At the fourth corner and with its plane in the diagonal there was placed a fourth mirror, vertical as the others, but "silvered" only to such an extent that a beam of light incident at 45° is divided into two beams of approximately equal intensity, one transmitted, the other reflected.

Fig. 1.



A beam of parallel light from the collimator C (fig. 1) falls on this mirror (*a*) at the required angle of 45°. The transmitted beam passes along the side *ab*, while the reflected beam goes at right angles to this along *ad*. By adjustments of the mirrors *b*, *c*, *d* these two beams are made to pass round the square to *a* again, where they are partly transmitted and partly reflected, so that both now travel in the same direction along *a*D.

By carefully adjusting the mirror *a* these two beams are superposed so as to produce interference-fringes, which are observed through the telescope D. In actual experiment we found that the two beams produced by the semitransparent mirror were of unequal brilliancy, the reflected beam being the more intense; and in order to obtain the best effect we found it necessary to move that mirror in the direction of the transmitted beam so as to decrease the angle of incidence, thereby

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* Phil. Trans. vol. clxxxiv. (1893).

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decreasing the amount of reflected light and increasing that of transmitted light. By this means we obtained in the field of the telescope perfectly defined interference-bands, the slightest motion of which could be at once detected.

If now the æther can be made to move in the direction of the transmitted beam, the increase thus caused in the velocity of light in that direction and the corresponding diminution of the velocity of light in the contrary direction should alter the difference of phase between the beams on arrival at the mirror a, and so produce a displacement of the bands.

To endeavour to produce this motion of the æther our apparatus was arranged as follows:—Two condensers, consisting each of two plates of sheet tin (56 centim. × 39 centim.) separated by a glass plate (3 millim. in thickness), were laid horizontally side by side on paraffin blocks on a table. The bottom plates were connected to the terminals of the secondary of a Ruhmkorff coil. The upper plates were joined through a circuit containing two spark-gaps—one placed at the slit of the collimator so as to be the source of light, while the other was situated at a distance of 4 centim. from the edge of the mirror c and in the plane of the mirror.

The primary plates of the condenser were also connected through a spark-gap. When these plates are raised to a high potential by the coil they suddenly discharge themselves through the spark-gap, thus causing the secondary plates to discharge themselves through their connecting circuit with its spark-gaps.

The equation for the discharge of a condenser of capacity C through a circuit of self-induction L and resistance R is

$$\mathrm{L}\frac{d^2\mathrm{Q}}{dt^2}+\mathrm{R}\frac{d\mathrm{Q}}{dt}+\frac{\mathrm{Q}}{\mathrm{C}}=0,$$

whence

$$\mathrm{Q}=\mathrm{A}\epsilon^{\left(-\frac{\mathrm{R}}{2\mathrm{L}}\pm\frac{\sqrt{\mathrm{R}^2-\frac{4\mathrm{L}}{\mathrm{C}}}}{2\mathrm{L}}\right)t+\alpha},$$

where Q is charge in condenser at time t after beginning of discharge.

If $\mathrm{R}^2<\frac{4\mathrm{L}}{\mathrm{C}}$, the discharge is oscillatory, and

$$\mathrm{Q}=\mathrm{A}\epsilon^{-\frac{\mathrm{R}}{2\mathrm{L}}t}.\cos\left[\sqrt{\frac{1}{\mathrm{LC}}-\frac{\mathrm{R}^2}{4\mathrm{L}^2}}.t+\alpha\right].$$

The amplitude of the oscillation is

$$\mathrm{A}\epsilon^{-\frac{\mathrm{R}}{2\mathrm{L}}.t},$$

and the period is

$$T=\frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}.$$

In order to compare the magnitudes of successive oscillations, we have

(i.) $t=0$, amplitude $= A$.

(ii.) $t=T$, amplitude $= A\epsilon^{-\frac{R}{2L}\cdot\frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}}$.

As R is increased, since $R^2<\frac{4L}{C}$, both factors in the index of ϵ in (ii.) increase, and therefore the amplitude diminishes.

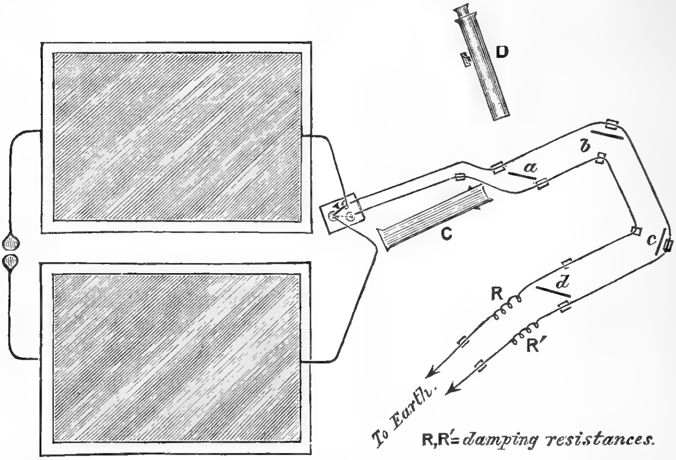
By introducing a resistance, therefore, into the circuit the effects of second and succeeding oscillations may be rendered negligible. In the secondary circuit of our condensers we accordingly introduced a resistance which experiment showed to be sufficient to fulfil this object.

The object of the spark near *c* is to produce electric waves in the direction *cb*, those in the direction *cd* being screened off by a copper plate. If, then, these electric waves are accompanied by an actual motion of the æther in the direction of propagation, it should be at once apparent by the displacement of the interference-bands. In actual experiment this spark-gap was at first short-circuited; then one of us observed the bands produced by the spark at the collimator, and watched if he could detect any displacement when the short circuit was suddenly removed. In repeated experiments, however, no motion whatever was discernible, although a displacement for one fifth of the breadth of a band—a displacement which would be produced by an alteration of the difference of phase by one tenth of a period—would have been visible to the eye.

In our next experiment the spark-gap at *c* was removed, and the wires of the secondary circuit were led to earth after passing round the three sides *ab*, *bc*, *cd* of the square—parallel to the beams and so as to include the beams between them. The same damping was kept in this circuit. The wires were insulated on paraffin blocks and strips of tinfoil hung over them to a depth of 3 centim. throughout their length, thus giving sides to the electric waves (fig. 2).

When the secondary plates discharge, a positive current runs down the one wire and a negative current down the other. We may therefore consider the space between the strips to be filled with tubes of forces stretching horizontally across perpendicular to the beams of light and moving along in the direction of the transmitted beam. If these tubes of force drag the æther with them, a displacement as already described should be observed in the interference-bands, when, after being short-circuited, these parallel wires are suddenly introduced into the circuit.

Fig. 2.



In our experiments, however, we again failed to detect any motion. At this point we may find what minimum velocity of the æther would have been detected by this method.

Let u = velocity of the æther,

v = „ „ light,

d = path of light in moving æther,

λ = wave-length of light used.

Then $v+u$ = velocity of one beam,

$v-u$ = velocity of other beam ;

$\therefore \quad \frac{d}{v+u}$ = time taken by first beam to travel distance d,

and $\frac{d}{v-u}$ = „ „ second „ „ „

$\therefore \quad \frac{2ud}{v^2-u^2}$ = difference of time occupied by the two beams in passing round the square.

Denote this time by τ.

Let $\tau = n\mathrm{T}$, where T is the period of vibration of the light used. There will therefore be a displacement of the fringes through $2n$ bands. Taking one fifth of the breadth of a band as the minimum displacement which the eye could have detected—and such a displacement could certainly have been observed—we have

$$n = \frac{1}{10},$$

$$\therefore \quad \frac{2ud}{v^2 - u^2} = \frac{1}{10} \cdot \mathrm{T}.$$

Neglecting u^2, for a first approximation we have

$$u = \frac{1}{20} \cdot \frac{v^2 \mathrm{T}}{d},$$

$$= \frac{1}{20} \cdot \frac{v\lambda}{d},$$

For

$$v = 3 \times 10^{10} \text{ centim. per second,}$$

$$\lambda = 0{\cdot}00059 \text{ millim.,}$$

$$d = 77 \text{ centim.,}$$

we have

$$u \doteqdot 11{\cdot}5 \text{ metres per second.}$$

So that any velocity of the æther exceeding 11·5 metres per second would have been detected; and the experiment therefore shows that if the æther does move at all, its velocity does not attain to even this small magnitude.

In our first experiment, as $d = 27$ centim., a velocity of the æther of at least $\frac{77}{27} \times 11{\cdot}5$ metres per second would have been detected.

In our third experiment we dispensed with the condensers and circuit round the mirrors, and interposed in the path of the light along the side *bc* of the square (fig. 1) an electrolytic cell (breadth 45 millim., length 61 millim., height 42 millim.) with bottom and sides of plate glass and ends of brass. These brass plates formed the electrodes of the cell, so that when connected to storage-cells there was a uniform current through the electrolyte (acidulated water). The cell rested on one pole of an electromagnet with the other pole vertically over it, and was so placed that the beams of light traversed it at right angles to the current. We therefore had a vertical magnetic field, a horizontal electric current, and the beams of light at right angles to both. The interference-fringes were now obtained by a sodium-flame at the slit of the collimator, and were much brighter than those obtained from the electric spark; but assuming as before one fifth of a band's breadth as the minimum displacement which would be visible to the eye, we have, since $d = 45$ millim.,

$$u \doteq 200 \text{ metres per second}$$

as minimum velocity of the æther in the cell which would be detected. The strength of the magnetic field was 300 C.G.S. units.

In our final experiment, while retaining the electromagnet in position, we removed the electrolytic cell and set up in its place a horizontal electrostatic field by means of a condenser, consisting of two tin plates $4\frac{1}{2}$ centim. apart. These plates measured 17 centim. × 12 centim., but one of them was cut at the middle of its length so as to pass between the pole-pieces of the electromagnet. The position of the apparatus was therefore this:—The plates of the condenser were in vertical planes parallel to the direction of the beams of light. Between these plates were the poles of the electromagnet, the one vertically above the other. The plates were connected with the poles of a Wimshurst machine, and also with the terminals of a Kelvin electrostatic voltmeter. The negative pole of the machine was connected further with the case of the voltmeter and led to earth. The positive plate of the condenser was insulated on glass plates.

The strength of the magnetic field was the same as before, 300 C.G.S. units; and between the plates of the condenser the difference of potential was varied, the maximum being 6600 volts. During each set of observations the potential was kept constant; but in none of the observations was any motion of the interference-bands discerned.

Cavendish Laboratory,
Cambridge, 1897.

IV. *A Nickel Stress Telephone. By* T. A. Garrett, *M.A.*, *and* William Lucas, *M.A.*[*]

It is well known that, if a magnetized nickel wire be subjected to longitudinal stress, variations of this stress produce variations in the magnetization of the wire. If the stress be a pressure, an increase in the pressure produces an increase in the magnetization, and a decrease in the pressure a decrease in the magnetization.

Suppose then that a magnetized piece of nickel wire,

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[*] Communicated by the Physical Society: read April 9, 1897.

round which a coil of insulated wire has been wound, is fixed in a vertical position by a clamp at its lower end, and that its upper end is fastened to the centre of a light horizontal plate, which serves as a diaphragm. On speaking against the top of the diaphragm variations of longitudinal pressure and consequently of magnetization will be produced in the nickel wire, and hence a varying current will be induced in the coil. If the coil be connected with any kind of receiving telephone the speech will be reproduced.

We have made a large number of experiments with the object of determining the form of instrument which would give the best results; but owing to the difficulties of comparing the speaking powers of different instruments we found it impossible to say with certainty which form was the best. The details of one of the forms with which good results were obtained are as follows:—The nickel wire was 10 cm. long and 1 mm. in diameter. The clamp was formed of a massive piece of gunmetal (an ordinary workshop vice will do quite as well) and the diaphragm was a circular disk of pine wood 3·5 mm. thick and 12·5 cm. in diameter. A hole was made through it in the centre into which the nickel wire was stuck with sealing-wax. This diaphragm was entirely supported by the nickel wire. The coil consisted of 3·9 grms. of No. 40 B.W.G. silk-covered wire wound directly on the nickel wire. The length of the coil was about 7·5 cm. An ordinary watch telephone, the resistance of the coils of which was about 136 ohms, was used as a receiver. The resistance of the line was quite small, as all the experiments were made from one room to another in the same building. The nickel wire was sometimes magnetized by stroking it with a magnet, sometimes by sending a current from a few Leclanché cells through the coil. Whether the wire was annealed or not appeared to be immaterial. Wires of various dimensions were tried, but the results obtained were not very different. With a diaphragm made of pine wood, the results were decidedly better than with a metallic diaphragm articulation then being very clear. It was found to be better to use a clamp of considerable mass.

When the nickel wire was replaced by an iron wire the results obtained were very feeble.

The same instrument was also used as a receiver. With an iron wire it worked very badly, with a nickel wire it worked rather better but not at all well. Since beginning our experiments we have found that M. Ader* has used as a receiver

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* *Comp. Rend.* 1879, t. lxxxviii. p. 575.

an instrument somewhat similar to the above, except that the wire was of iron instead of nickel.

In order to show that the current induced in the coil is at any rate chiefly due to variations in the magnetization of the nickel wire produced by the variations of stress, and not to the relative motion of the nickel wire and the coil, the following experiment was made :—A steel wire and a nickel wire of the same dimensions were attached to similar wooden diaphragms. These wires having been magnetized by stroking them with a permanent magnet were in turn inserted into the same solenoid and clamped as before at their lower ends. The same watch telephone was used as a receiver with each. The results obtained with the weakly magnetized nickel wire were enormously better than those obtained with the strongly magnetized steel wire. If the induced currents were chiefly due to the relative motion of the coil and magnetized wire the best results would have been obtained with the strongly magnetized steel wire, since it can hardly be supposed that the relative motions of the coil and magnetized wire differed so much in the two cases as to cause such an enormous difference in the results.

V. *On the Incidence of Aerial and Electric Waves upon Small Obstacles in the form of Ellipsoids or Elliptic Cylinders, and on the Passage of Electric Waves through a circular Aperture in a Conducting Screen. By* Lord RAYLEIGH, *F.R.S.*[*]

THE present paper may be regarded as a development of previous researches by the author upon allied subjects. When the character of the obstacle differs only infinitesimally from that of the surrounding medium, a solution may be obtained independently of the size and the form which it presents. But when this limitation is disregarded, when, for example, in the case of aerial vibrations the obstacle is of arbitrary compressibility and density, or in the case of electric vibrations when the dielectric constant and the permeability are arbitrary, the solutions hitherto given are confined to the case of small spheres, or circular cylinders. In the present investigation extension is made to ellipsoids, including flat circular disks and thin blades.

The results arrived at are limiting values, strictly applicable only when the dimensions of the obstacles are infinitesimal,

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[*] Communicated by the Author.

and at distances outwards which are infinitely great in comparison with the wave-length (λ). The method proceeds by considering in the first instance what occurs in an intermediate region, where the distance (r) is at once great in comparison with the dimensions of the obstacle and small in comparison with λ. Throughout this region and within it the calculation proceeds as if λ were infinite, and depends only upon the properties of the common potential. When this problem is solved, extension is made without much difficulty to the exterior region where r is great in comparison with λ, and where the common potential no longer avails.

At the close of the paper a problem of some importance is considered relative to the escape of electric waves through small circular apertures in metallic screens. The case of narrow elongated slits has already been treated *.

Obstacle in a Uniform Field.

The analytical problem with which we commence is the same whether the flow be thermal, electric, or magnetic, the obstacle differing from the surrounding medium in conductivity, specific inductive capacity, or permeability respectively. If ϕ denote its potential, the uniform field is defined by

$$\phi = ux + vy + wz; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

u, v, w being the fluxes in the direction of fixed arbitrarily chosen rectangular axes. If ψ be the potential in the uniform medium due to the obstacle, so that the complete potential is $\phi+\psi$, ψ may be expanded in the series of spherical harmonics

$$\psi = \frac{S_0}{r} + \frac{S_1}{r^2} + \frac{S_2}{r^3} + \ldots, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the origin of r being within the obstacle. Since there is no source, S_0 vanishes. Further, at a great distance S_2, S_3, ... may be neglected, so that ψ there reduces to

$$\psi = \frac{S_1}{r^2} = \frac{A'x + B'y + C'z}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The disturbance (3) corresponds to (1). If we express separately the parts corresponding to u, v, w, writing $A' = A_1u + A_2v + A_3w$, &c., we have

$$\begin{aligned} r^3\psi = &\ u(A_1x + B_1y + C_1z) \\ &+ v(A_2x + B_2y + C_2z) \\ &+ w(A_3x + B_3y + C_3z); \quad . \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

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* Phil. Mag. vol. xliii. p. 272.

but the nine coefficients are not independent. By the law of reciprocity the coefficient of the x-part due to v must be the same as that of the y-part due to u, and so on*. Thus $B_1 = A_2$, &c., and we may write (4) in the form

$$r^3\psi = u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$F = \tfrac{1}{2}A_1x^2 + \tfrac{1}{2}B_2y^2 + \tfrac{1}{2}C_3z^2 + B_1xy + C_2yz + C_1zx. \quad . \quad (6)$$

In the case of a body, like an ellipsoid, symmetrical with respect to three planes chosen as coordinate planes,

$$B_1 = C_2 = C_1 = 0,$$

and (4) reduces to

$$r^3\psi = A_1ux + B_2vy + C_3wz. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It will now be shown that by a suitable choice of coordinates this reduction may be effected in any case. Let u, v, w originate in a source at distance R, whose coordinates are x', y', z', so that $u = x'/R^3$, &c. Then (5) becomes

$$r^3R^3\psi = x'\frac{dF}{dx} + y'\frac{dF}{dy} + z'\frac{dF}{dz} = A_1xx' + B_2yy' + C_3zz'$$
$$+ B_1(x'y + y'x) + C_2(y'z + z'y) + C_1(z'x + x'z)$$
$$= F(x + x', y + y', z + z') - F(x, y, z) - F(x', y', z').$$

Now by a suitable transformation of coordinates $F(x, y, z)$, and therefore $F(x', y', z')$ and $F(x + x', y + y', z + z')$, may be reduced to the form $A_1x^2 + B_2y^2 + C_3z^2$, &c. If this be done,

$$r^3R^3\psi = A_1xx' + B_2yy' + C_3zz',$$

or reverting to u, v, w, reckoned parallel to the new axes,

$$r^3\psi = A_1ux + B_2vy + C_3wz, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as in (7) for the ellipsoid. It should be observed that this reduction of the potential at a distance from the obstacle to the form (8) is independent of the question whether the material composing the obstacle is uniform.

For the case of the ellipsoid (a, b, c) of uniform quality the solution may be completely carried out. Thus †, if T be

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* 'Theory of Sound,' § 109. u and v may be supposed to be due to point-sources situated at a great distance R along the axes of x and y respectively.

† The magnetic problem is considered in Maxwell's 'Electricity and Magnetism,' 1873, § 437, and in Mascart's *Leçons*, 1896, §§ 52, 53, 276.

the volume, so that

$$T=\tfrac{4}{3}\pi abc, \quad \ldots \ldots \quad (9)$$

we have

$$A_1u=-AT, \quad B_2v=-BT, \quad C_3w=-CT, \quad . \quad (10)$$

$$A=\frac{\kappa u}{1+\kappa L}, \quad B=\frac{\kappa v}{1+\kappa M}, \quad C=\frac{\kappa w}{1+\kappa N}, \quad . \quad (11)$$

where

$$L=2\pi abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}}, \quad . \quad (12)$$

with similar expressions for M and N.

In (11) κ denotes the susceptibility to magnetization. In terms of the permeability μ, analogous to conductivity in the allied problems, we have, if μ' relate to the ellipsoid and μ to the surrounding medium,

$$1+4\pi\kappa=\mu'/\mu, \quad \ldots \ldots \quad (13)$$

so that

$$A=\frac{(\mu'-\mu)u}{4\pi\mu+(\mu'-\mu)L}, \quad \ldots \ldots \quad (14)$$

with similar equations for B and C.

Two extreme cases are worthy of especial notice. If $\mu'/\mu=\infty$, the general equation for ψ becomes

$$-\frac{r^3\psi}{T}=\frac{ux}{L}+\frac{vy}{M}+\frac{wz}{N}. \quad \ldots \quad (15)$$

On the other hand, if $\mu'/\mu=0$,

$$-\frac{r^3\psi}{T}=\frac{ux}{L-4\pi}+\frac{vy}{M-4\pi}+\frac{wz}{N-4\pi}. \quad . \quad . \quad (16)$$

In the case of the sphere (a)

$$L=M=N=\tfrac{4}{3}\pi; \quad \ldots \ldots \quad (17)$$

so that (15) becomes

$$\psi=-\frac{a^3}{r^3}(ux+vy+wz), \quad \ldots \ldots \quad (18)$$

giving, when $r=a$, $\phi+\psi=0$. This is the case of the perfect conductor.

In like manner for the non-conducting sphere (16) gives

$$\psi=\frac{a^3}{2r^3}(ux+vy+wz). \quad \ldots \ldots \quad (19)$$

If the conductivity of the sphere be finite (μ'),

$$\psi = -\frac{a^3}{r^3}\,\frac{\mu'-\mu}{\mu'+2\mu}\,(ux+vy+wz), \quad . \quad . \quad . \quad (20)$$

which includes (18) and (19) as particular cases.

If the ellipsoid has two axes equal, and is of the planetary or flattened form,

$$b = c = \frac{a}{\sqrt{(1-e^2)}}, \qquad \mathrm{T} = \tfrac{4}{3}\pi c^3\sqrt{(1-e^2)}\,; \quad . \quad (21)$$

$$\mathrm{L} = 4\pi\left\{\frac{1}{e^2} - \frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right\}, \quad . \quad . \quad . \quad (22)$$

$$\mathrm{M} = \mathrm{N} = 2\pi\left\{\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}. \quad . \quad (23)$$

In the extreme case of a disk, when $e=1$ nearly,

$$\mathrm{L} = 4\pi - 2\pi^2\sqrt{(1-e^2)}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\mathrm{M} = \mathrm{N} = \pi^2\sqrt{(1-e^2)}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Thus in the limit from (14), (21) $\mathrm{TA}=0$, unless $\mu'=0$; and when $\mu'=0$,

$$\mathrm{TA} = -\frac{2c^3u}{3\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In like manner the limiting values of TB, TC are zero, unless $\mu'=\infty$, and then

$$\mathrm{TB} = \frac{4c^3v}{3\pi}, \qquad \mathrm{TC} = \frac{4c^3w}{3\pi}. \quad . \quad . \quad . \quad . \quad (27)$$

In all cases

$$\psi = -\frac{\mathrm{T}(\mathrm{A}x+\mathrm{B}y+\mathrm{C}z)}{r^3} \quad . \quad . \quad . \quad . \quad (28)$$

gives the disturbance due to the ellipsoid.

If the ellipsoid of revolution be of the ovary or elongated form,

$$a = b = c\sqrt{(1-e^2)}\,; \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\mathrm{L} = \mathrm{M} = 2\pi\left\{\frac{1}{e^2} - \frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad . \quad . \quad (30)$$

$$\mathrm{N} = 4\pi\left\{\frac{1}{e^2}-1\right\}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\}. \quad (31)^*$$

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* There are slight errors in the values of L, M, N recorded for this case in both the works cited.

In the case of a very elongated ovoid L and M approximate to the value 2π, while N approximates to the form

$$N=4\pi\frac{a^2}{c^2}\left(\log\frac{2c}{a}-1\right), \quad . \quad . \quad . \quad . \quad (32)$$

vanishing when $e=1$.

In Two Dimensions.

The case of an elliptical cylinder in two dimensions may be deduced from (12) by making c infinite, when the integration is readily effected. We find

$$L=\frac{4\pi b}{a+b}, \qquad M=\frac{4\pi a}{a+b}. \quad . \quad . \quad . \quad (33)$$

A and B are then given by (14) as before, and finally

$$\psi=-\frac{ab(a+b)}{2r^2}\left\{\frac{(\mu'-\mu)ux}{\mu a+\mu' b}+\frac{(\mu'-\mu)vy}{\mu b+\mu' a}\right\}, \quad . \quad (34)$$

corresponding to

$$\phi=ux+vy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In the case of circular section $L=M=2\pi$, so that

$$\psi=-\frac{a^2}{r^2}\frac{\mu'-\mu}{\mu'+\mu}(ux+vy). \quad . \quad . \quad . \quad . \quad (36)$$

When $b=0$, that is when the obstacle reduces itself to an infinitely thin blade, ψ vanishes unless $\mu'=0$ or $\mu'=\infty$. In the first case

$$(\mu'=0) \qquad \psi=\frac{a^2vy}{2r^2}; \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in the second

$$(\mu'=\infty) \qquad \psi=-\frac{a^2ux}{2r^2}. \quad . \quad . \quad . \quad . \quad (38)$$

Aerial Waves.

We may now proceed to investigate the disturbance of plane aerial waves by obstacles whose largest diameter is small in comparison with the wave-length (λ). The volume occupied by the obstacle will be denoted by T; as to its shape we shall at first impose no restriction beyond the exclusion of very special cases, such as would involve resonance in spite of the small dimensions. The compressibilities and densities of the medium and of the obstacle are denoted by m, m'; σ, σ'; so that if V, V′ be the velocities of propagation

$$V^2=m/\sigma, \quad V'^2=m'/\sigma'. \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

The velocity-potential of the undisturbed plane waves is represented by

$$\phi=e^{ikVt}.e^{ikx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

in which $k=2\pi/\lambda$. The time factor e^{ikVt}, which operates throughout, may be omitted for the sake of brevity.

The velocity-potential (ψ) of the disturbance propagated outwards from T may be expanded in spherical harmonic terms*

$$r\psi=e^{-ikr}\{S_0+S_1f_1(ikr)+S_2f_2(ikr)+..\}, \quad . \qquad (41)$$

where

$$f_n(ikr)=1+\frac{n(n+1)}{2\,.\,ikr}+\frac{(n-1)\ldots(n+2)}{2\,.\,4\,.\,(ikr)^2}$$

$$+\ldots\ldots+\frac{1\,.\,2\,.\,3\ldots2n}{2\,.\,4\,.\,6\ldots2n\,(ikr)^n}. \quad . \quad . \quad . \quad . \qquad (42)$$

At a great distance from the obstacle $f_n(ikr)=1$; and the relative importance of the various harmonic terms decreases in going outwards with the order of the harmonic. For the present purpose we shall need to regard only the terms of order 0 and 1. Of these the term of order 0 depends upon the variation of compressibility, and that of order 1 upon the variation of density.

The relation between the variable part of the pressure δp, the condensation s, and ϕ is

$$V^2s=-\frac{d\phi}{dt}=\frac{\delta p}{\sigma};$$

so that during the passage of the undisturbed primary waves the rate at which fluid enters the volume T (supposed for the moment to be of the same quality as the surrounding medium) is

$$T\frac{ds}{dt}=-\frac{T}{V^2}\frac{d^2\phi}{dt^2}=k^2T. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

If the obstacle present an unyielding surface, its effect is to prevent the entrance of the fluid (43); that is, to superpose upon the plane waves such a disturbance as is caused by the *introduction* of (43) into the medium. Thus, if the potential of this disturbance be

$$\psi=S_0\frac{e^{-ikr}}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

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* 'Theory of Sound,' §§ 323, 324.

S_0 is to be determined by the condition that when $r=0$

$$4\pi r^2 d\psi/dr = k^2\mathrm{T},$$

so that $S_0 = -k^2\mathrm{T}/4\pi$, and

$$\psi = -\frac{k^2\mathrm{T}}{4\pi}\frac{e^{-ikr}}{r} = -\frac{\pi\mathrm{T}}{\lambda^2}\frac{e^{-ikr}}{r}. \quad . \quad . \quad . \quad . \quad (45)$$

This result corresponds with $m'=\infty$ representing absolute incompressibility. The effect of finite compressibility, differing from that of the surrounding medium, is readily inferred by means of the pressure relation ($\delta p = ms$). The effect of the variation of compressibility at the obstacle is to increase the rate of introduction of fluid into T from what it would otherwise be in the ratio $m : m'$; and thus (45) now becomes

$$\psi = -\frac{\pi\mathrm{T}}{\lambda^2}\frac{m'-m}{m'}\frac{e^{-ikr}}{r}; \quad . \quad . \quad . \quad . \quad . \quad (46)$$

or if we restore the factor $e^{ik\mathrm{V}t}$ and throw away the imaginary part of the solution,

$$\psi = -\frac{\pi\mathrm{T}}{\lambda^2 r}\frac{m'-m}{m'}\cos k(\mathrm{V}t-r). \quad . \quad . \quad . \quad (47)$$

This is superposed upon the primary waves

$$\phi = \cos k(\mathrm{V}t + x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

When $m'=0$, *i. e.*, when the material composing the obstacle offers no resistance to compression, (47) fails. In this case the condition to be satisfied at the surface of T is the evanescence of δp, or of the total potential $(\phi+\psi)$. In the neighbourhood of the obstacle $\phi=1$; and thus if M′ denote the electrical "capacity" of a conducting body of form T situated in the open, $\psi = -\mathrm{M}'/r$, r being supposed to be large in comparison with the linear dimension of T but small in comparison with λ. The latter restriction is removed by the insertion of the factor e^{-ikr}, and thus, in place of (46), we now have

$$\psi = -\frac{\mathrm{M}'e^{-ikr}}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The value of M′ may be expressed when T is in the form of an ellipsoid. For a sphere of radius R,

$$\mathrm{M}' = \mathrm{R}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

for a circular plate of radius R,

$$\mathrm{M}' = 2\mathrm{R}/\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

When the density of the obstacle (σ') is the same as that of the surrounding medium, (47) constitutes the complete solution. Otherwise the difference of densities causes an interference with the flow of fluid, giving rise to a disturbance of order 1 in spherical harmonics. This disturbance is independent of that already considered, and the flow in the neighbourhood of the obstacle may be calculated as if the fluid were incompressible. We thus fall back upon the problem considered in the earlier part of this paper, and the results will be applicable as soon as we have established the correspondence between density and conductivity.

In the present problem, if χ denote the whole velocity-potential, the conditions to be satisfied at any part of the surface of the obstacle are the continuity of $d\chi/dn$ and of $\sigma\chi$, the latter of which represents the pressure. Thus, if we regard $\sigma\chi$ as the variable, the conditions are the continuity of $(\sigma\chi)$ and of $\sigma^{-1}d(\sigma\chi)/dn$. In the conductivity problem the conditions to be satisfied by the potential (χ') are the continuity of χ' and of $\mu\, d\chi'/dn$.

In an expression relating only to the external region where σ is constant, it makes no difference whether we are dealing with $\sigma\chi$ or with χ; and accordingly there is correspondence between the two problems provided that we suppose the ratio of μ's in the one problem to be the reciprocal of the ratio of the σ's in the other.

We may now proceed to the calculation of the disturbance due to an obstacle, based upon the assumption that there is a region over which r is large compared with the linear dimension of T, but small in comparison with λ. Within this region ψ is given by (8) if the motion be referred to certain principal axes determined by the nature and form of the obstacle, the quantities u, v, w being the components of flow in the primary waves. By (41), (42), this is to be identified with

$$\psi = \mathrm{S}_1 \frac{e^{-ikr}}{r}\left(1+\frac{1}{ikr}\right), \quad . \quad . \quad . \quad . \quad (52)$$

when r is small in comparison with λ; so that

$$\mathrm{S}_1 = \frac{ik(\mathrm{A}_1\, ux + \mathrm{B}_2\, vy + \mathrm{C}_3\, wz)}{r}. \quad . \quad . \quad . \quad (53)$$

At a great distance from T, (52) reduces to

$$\psi = \frac{ik(\mathrm{A}_1\, ux + \mathrm{B}_2\, vy + \mathrm{C}_3\, wz)e^{-ikr}}{r^2}, \quad . \quad . \quad (54)$$

—a term of order 1, to be added to that of zero order given in (46).

In general, the axis of the harmonic in (54) is inclined to the direction of propagation of the primary waves; but there are certain cases of exception. For example, v and w vanish if the primary propagation be parallel to x (one of the principal axes). Again, as for a sphere or a cube, A_1, B_2, C_3 may be equal.

We will now limit ourselves to the case of the ellipsoid, and for brevity will further suppose that the primary waves move parallel to x, so that $v=w=0$. The terms corresponding to u and v, if existent, are simply superposed. If, as hitherto, $\phi=e^{ikx}$, $u=ik$; so that by (14), σ being substituted for μ' and σ' for μ,

$$A=\frac{ik(\sigma-\sigma')}{4\pi\sigma'+(\sigma-\sigma')L}. \quad . \quad . \quad . \quad . \qquad (55)$$

In the intermediate region by (28) $\psi=-\mathrm{T}Ax/r^3$, and thus at a great distance

$$\psi=-\frac{ikx\,\mathrm{T}Ae^{-ikr}}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

or on substitution of the values of A and k,

$$\psi=-\frac{\pi \mathrm{T}x\,e^{-ikr}}{\lambda^2 r^2}\,\frac{4\pi(\sigma'-\sigma)}{4\pi\sigma'+(\sigma-\sigma')L}. \quad . \quad . \qquad (57)$$

Equations (46), (57) express the complete solution in the case supposed.

For an obstacle which is rigid and fixed, we may deduce the result by supposing in our equations $m'=\infty$, $\sigma'=\infty$. Thus

$$\psi=-\frac{\pi \mathrm{T}e^{-ikr}}{\lambda^2 r}\left\{1+\frac{x}{r}\,\frac{4\pi}{4\pi-L}\right\}. \quad . \quad . \qquad (58)$$

Certain particular cases are worthy of notice. For the sphere $L=\frac{4}{3}\pi$, and

$$\psi=-\frac{\pi \mathrm{T}e^{-ikr}}{\lambda^2 r}\left\{1+\frac{3x}{2r}\right\}. \quad . \quad . \quad . \quad . \qquad (59)\,*$$

If the ellipsoid reduce to an infinitely thin circular disk of radius c, $\mathrm{T}=0$ and the term of zero order vanishes. The term of the first order also vanishes if the plane of the disk be parallel to x. If the plane of the disk be perpendicular to

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[*] 'Theory of Sound,' § 334.

x, $4\pi-\mathrm{L}$ is infinitesimal. By (21), (24) we get in this case

$$\frac{4\pi\mathrm{T}}{4\pi-\mathrm{L}}=\frac{8c^3}{3};$$

so that

$$\psi=-\frac{8\pi c^3}{3\lambda^2}\frac{x}{r}\frac{e^{-ikr}}{r}. \quad . \quad . \quad . \quad . \quad (60)$$

If the axis of the disk be inclined to that of x, ψ retains its symmetry with respect to the former axis, and is reduced in magnitude in the ratio of the cosine of the angle of inclination to unity.

In the case of the sphere the general solution is

$$\psi=-\frac{\pi\mathrm{T}e^{-ikr}}{\lambda^2 r}\left\{\frac{m'-m}{m'}+\frac{3x}{r}\frac{\sigma'-\sigma}{2\sigma'+\sigma}\right\}. \quad . \quad (61)^*$$

Waves in Two Dimensions.

In the case of two dimensions (x, y) the waves diverging from a cylindrical obstacle have the expression, analogous to (41),

$$\psi=\mathrm{S}_0\,\mathrm{D}_0(kr)+\mathrm{S}_1\,\mathrm{D}_1(kr)+\ldots, \quad . \quad . \quad (62)\dagger$$

where S_0, $\mathrm{S}_1\ldots$ are the plane circular functions of the various orders, and

$$\mathrm{D}_0(kr)=-\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}e^{-ikr}\left\{1-\frac{1^2}{1\,.\,8ikr}+\ldots\right\}$$

$$=\left(\gamma+\log\frac{ikr}{2}\right)\left\{1-\frac{k^2r^2}{2^2}+\ldots\right\}+\frac{k^2r^2}{2^2}-\frac{3}{2}\frac{k^4r^4}{2^2\,.\,4^2}+\ldots, \quad . \quad . \quad . \quad (63)$$

$$\mathrm{D}_1(kr)=\frac{d\mathrm{D}_0(kr)}{d(kr)}=\left(\frac{\pi i}{2kr}\right)^{\frac{1}{2}}e^{-ikr}\left\{1-\frac{-1\,.\,3}{1\,.\,8ikr}+\ldots\right\}$$

$$=\frac{1}{kr}\left\{1-\frac{k^2r^2}{2^2}+\ldots\right\}+\left(\gamma+\log\frac{ikr}{2}\right)\left\{\frac{kr}{2}-\frac{k^3r^3}{2^2\,.\,4}+\ldots\right\}$$

$$+\frac{kr}{2}-\frac{3}{2}\frac{k^3r^3}{2^2\,.\,4}+\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

As in the case of three dimensions already considered, the term of zero order in ψ depends upon the variation of compressibility. If we again begin with the case of an unyielding

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* 'Theory of Sound,' § 335.

† See 'Theory of Sound,' § 341; Phil. Mag. April 1897, p. 266.

boundary, the constant S_0 is to be found from the condition that when $r=0$

$$2\pi r\, d\psi/dr = k^2\mathrm{T},$$

T denoting now the area of cross-section. When r is small,

$$\frac{d\mathrm{D}_0(kr)}{dr} = \frac{1}{r};$$

and thus $S_0 = k^2\mathrm{T}/2\pi$,

$$\psi = \frac{k^2\mathrm{T}}{2\pi}\,\mathrm{D}_0(kr) = -\frac{k^2\mathrm{T}}{2\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}, \;\ldots \qquad (65)$$

when r is very great. This corresponds to (45).

In like manner, if the compressibility of the obstacle be finite,

$$\psi = -\frac{k^2\mathrm{T}}{\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} \frac{m'-m}{2m'} e^{-ikr}. \;\ldots \qquad (66)$$

The factor $i^{-\frac{1}{2}} = e^{-\frac{1}{4}i\pi}$; and thus if we restore the time-factor $e^{ik\mathrm{V}t}$, and reject the imaginary part of the solution, we have

$$\psi = -\frac{2\pi\mathrm{T}}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}\,\frac{m'-m}{2m'}\cos\frac{2\pi}{\lambda}(\mathrm{V}t - r - \tfrac{1}{8}\lambda), \qquad (67)$$

corresponding to the plane waves

$$\phi = \cos\frac{2\pi}{\lambda}(\mathrm{V}t + x). \qquad (68)$$

In considering the term of the first order we will limit ourselves to the case of the cylinder of elliptic section, and suppose that one of the principal axes of the ellipse is parallel to the direction (x) of primary wave-propagation. Thus in (34), which gives the value of ψ at a distance from the cylinder which is great in comparison with a and b, but small in comparison with λ, we are to suppose $u=ik$, $v=0$, at the same time substituting σ, σ' for μ', μ respectively. Thus for the region in question

$$\psi = \frac{ab\,.\,ikx}{2r^2}\,\frac{(\sigma'-\sigma)(a+b)}{\sigma' a + \sigma b}; \qquad (69)$$

and this is to be identified with $S_1 D_1(kr)$ when kr is small, *i. e.* with S_1/kr. Accordingly

$$S_1 = \frac{x}{r}\,\frac{ik^2 ab}{2}\,\frac{(\sigma'-\sigma)(a+b)}{\sigma' a + \sigma b};$$

so that, at a distance r great in comparison with λ, ψ becomes

$$\psi = -\frac{k^2\mathrm{T}}{\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\frac{(\sigma'-\sigma)(a+b)}{2(\sigma'a+\sigma b)}\,\frac{x}{r}\,e^{-ikr}, \quad . \; . \; . \qquad (70)$$

T being written for πab. The complete solution for a great distance is given by addition of (66) and (70), and corresponds to $\phi = e^{ikx}$.

In the case of circular section ($b=a$) we have altogether *

$$\psi = -k^2a^2\,e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{m'-m}{2m'} + \frac{\sigma'-\sigma}{\sigma'+\sigma}\,\frac{x}{r}\right\}, \qquad (71)$$

which may be realized as in (67). If the material be unyielding, the corresponding result is obtained by making $m'=\infty$, $\sigma'=\infty$ in (71). The realized value is then †

$$\psi = -\frac{2\pi\,.\,\pi a^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}\left(\frac{1}{2}+\frac{x}{r}\right)\cos\frac{2\pi}{\lambda}\left(\mathrm{V}t - r - \tfrac{1}{8}\lambda\right). \quad . \qquad (72)$$

In general, if the material be unyielding, we get from (66), (70)

$$\psi = -k^2ab\,e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left(\frac{1}{2}+\frac{a+b}{2a}\,\frac{x}{r}\right). \quad . \; . \; . \qquad (73)$$

The most interesting case of a difference between a and b is when one of them vanishes, so that the cylinder reduces to an infinitely thin blade. If $b=0$, ψ vanishes as to both its parts; but if $a=0$, although the term of zero order vanishes, that of the first order remains finite, and we have

$$\psi = -\tfrac{1}{2}k^2b^2\,e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\frac{x}{r}, \quad . \; . \; . \; . \qquad (74)$$

in agreement with the value formerly obtained ‡.

It remains to consider the extreme case which arises when $m'=0$. The term of zero order in circular harmonics, as given in (66), then becomes infinite, and that of the first order (70) is relatively negligible. The condition to be satisfied at the surface of the obstacle is now the evanescence of the total potential $(\phi+\psi)$, in which $\phi=1$.

It will conduce to clearness to take first the case of the circular cylinder (*a*). By (62), (63) the surface condition is

$$\mathrm{S}_0\{\gamma + \log(\tfrac{1}{2}ika)\} + 1 = 0. \quad . \; . \; . \; . \; . \qquad (75)$$

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* 'Theory of Sound,' § 343.

† *Loc. cit.* equation (17).

‡ Phil. Mag. April 1897, p. 271. The primary waves are there supposed to travel in the direction of $+x$, but here in the direction of $-x$.

Thus at a distance r great in comparison with λ we have

$$\psi = \frac{e^{-ikr}}{\gamma+\log(\frac{1}{2}ika)}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \qquad (76)$$

When the section of the obstacle is other than circular, a less direct process must be followed. Let us consider a circle of radius ρ concentric with the obstacle, where ρ is large in comparison with the dimensions of the obstacle but small in comparison with λ. Within this circle the flow may be identified with that of an incompressible fluid. On the circle we have

$$\phi+\psi=1+S_0\{\gamma+\log(\tfrac{1}{2}ik\rho)\}, \quad . \quad . \qquad (77)$$

$$2\pi\, d(\phi+\psi)/dr=2\pi S_0, \quad . \quad . \quad . \quad . \quad . \qquad (78)$$

of which the latter expresses the flow of fluid across the circumference. This flow in the region between the circle and the obstacle corresponds to the potential-difference (77). Thus, if R denote the electrical resistance between the two surfaces (reckoned of course for unit length parallel to z),

$$S_0\{\gamma+\log(\tfrac{1}{2}ik\rho)-2\pi R\}=1, \quad . \quad . \quad . \quad . \qquad (79)$$

and $\psi=S_0D_0(kr)$, as usual.

The value of S_0 in (79) is of course independent of the actual value of ρ, so long as it is large. If the obstacle be circular,

$$2\pi R=\log(\rho/a).$$

The problem of determining R for an elliptic section (a, b) can, as is well known, be solved by the method of conjugate functions. If we take

$$x=c\cosh\xi\cos\eta, \qquad y=c\sinh\xi\sin\eta, \quad . \quad . \quad . \qquad (80)$$

the confocal ellipses

$$\frac{x^2}{\cosh^2\xi}+\frac{y^2}{\sinh^2\xi}=c^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (81)$$

are the equipotential curves. One of these, for which ξ is large, can be identified with the circle of radius ρ, the relation between ρ and ξ being

$$\xi=\log(2\rho/c).$$

An inner one, for which $\xi=\xi_0$, is to be identified with the ellipse (a, b), so that

$$a=c\cosh\xi_0, \qquad b=c\sinh\xi_0,$$

whence

$$c^2 = a^2 - b^2, \qquad \tanh \xi_0 = b/a.$$

Thus

$$2\pi \mathrm{R} = \xi - \xi_0 = \log \frac{2\rho}{a+b}; \quad . \quad . \quad . \quad . \quad (82)$$

and then (79) gives as applicable at a great distance

$$\psi = \frac{e^{-ikr}}{\gamma + \log\{\frac{1}{4}ik(a+b)\}} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad (83)$$

The result for an infinitely thin blade is obtained by merely putting $b=0$ in (83).

For some purposes the imaginary part of the logarithmic term may be omitted. The realized solution is then

$$\psi = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \frac{\cos k(\mathrm{V}t - r - \frac{1}{8}\lambda)}{\gamma + \log\{\frac{1}{4}k(a+b)\}}, \quad . \quad . \quad . \quad (84)$$

corresponding, as usual, to

$$\phi = \cos k(\mathrm{V}t + x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Electrical Applications.

The problems in two dimensions for aerial waves incident upon an obstructing cylinder of small transverse dimensions are analytically identical with certain electric problems which will now be specified. The general equation $(\nabla^2 + k^2) = 0$ is satisfied in all cases. In the ordinary electrical notation $\mathrm{V}^2 = 1/\mathrm{K}\mu$, $\mathrm{V}'^2 = 1/\mathrm{K}'\mu'$; while in the acoustical problem $\mathrm{V}^2 = m/\sigma$, $\mathrm{V}'^2 = m'/\sigma'$. The boundary conditions are also of the same general form. Thus if the primary waves be denoted by $\gamma = e^{ikx}$, γ being the magnetic force parallel to z, the conditions to be satisfied at the surface of the cylinder are the continuity of γ and of $\mathrm{K}^{-1}\, d\gamma/dn$. Comparing with the acoustical conditions we see that K replaces σ, and consequently (by the value of V^2) μ replaces $1/m$. These substitutions with that of γ, or c (the magnetic induction), for ψ and ϕ suffice to make (66), (70) applicable to the electrical problem. For example, in the case of the circular cylinder, we have for the dispersed wave

$$c = -\, k^2 a^2 e^{-ikr} \left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} \left\{ \frac{\mu - \mu'}{2\mu} + \frac{\mathrm{K}' - \mathrm{K}}{\mathrm{K}' + \mathrm{K}} \frac{x}{r} \right\}, \quad . \quad (86)$$

corresponding to the primary waves

$$c = e^{ikx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

An important particular case is obtained by making $K' = \infty$, $\mu' = 0$, in such a way that V' remains finite. This is equivalent to endowing the obstacle with the character of a perfect conductor, and we get

$$c = -k^2a^2e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{1}{2}+\frac{x}{r}\right\}, \quad . \quad . \quad (88)$$

which, when realized, coincides with (72).

The other two-dimensional electrical problem is that in which everything is expressed by means of R, the electromotive intensity parallel to z. The conditions at the surface are now the continuity of R and of $\mu^{-1}dR/dn$. Thus K and μ are simply interchanged, μ replacing σ and K replacing $1/m$ in (66), (70), ϕ and ψ also being replaced by R. In the case of the circular cylinder

$$R = -k^2a^2e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{K-K'}{2K}+\frac{\mu'-\mu}{\mu'+\mu}\frac{x}{r}\right\}, \quad (89)$$

corresponding to the primary waves

$$R = e^{ikx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

If in order to obtain the solution for a perfectly conducting obstacle we make $K' = \infty$, $\mu' = 0$, (89) becomes infinite, and must be replaced by the analogue of (83). Thus for the perfectly conducting circular obstacle

$$R = \frac{e^{-ikr}}{\gamma + \log\left(\frac{1}{2}ika\right)}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (91)$$

which may be realized as in (84).

The problem of a conducting cylinder is treated by Prof. J. J. Thomson in his valuable 'Recent Researches in Electricity and Magnetism,' § 364; but his result differs from (84), not only in respect to the sign of $\frac{1}{8}\lambda$, but also in the value of the denominator *. The values here given are those which follow from the equations (9), (17) of § 343 'Theory of Sound.'

Electric Waves in Three Dimensions.

In the problems which arise under this head the simple acoustical analogue no longer suffices, and we must appeal to the general electrical equations of Maxwell. The components of electric polarization (f, g, h) and of magnetic force (α, β, γ),

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* It should be borne in mind that γ here is the same as Prof. Thomson's $\log \gamma$.

being proportional to e^{ikVt}, all satisfy the fundamental equation

$$(\nabla^2+k^2)=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

and they are connected together by such relations as

$$4\pi\frac{df}{dt}=\frac{d\gamma}{dy}-\frac{d\beta}{dz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

or

$$\frac{d\alpha}{dt}=4\pi V^2\left(\frac{dg}{dz}-\frac{dh}{dy}\right), \quad . \quad . \quad . \quad . \quad . \quad (94)$$

in which any differentiation with respect to t is equivalent to the introduction of the factor ikV. Further

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0, \qquad \frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \qquad (95)$$

The electromotive intensity (P, Q, R) and the magnetization (a, b, c) are connected with the quantities already defined by the relations

$$f, g, h=K(P, Q, R)/4\pi; \qquad a, b, c=\mu(\alpha, \beta, \gamma); \quad (96)$$

in which K denotes the specific inductive capacity and μ the permeability; so that $V^{-2}=K\mu$.

The problem before us is the investigation of the disturbance due to a small obstacle (K', μ') situated at the origin, upon which impinge primary waves denoted by

$$f_0=0, \qquad g_0=0, \qquad h_0=e^{ikx}, \quad . \quad . \quad . \quad . \quad . \quad (97)$$

or, as follows from (94),

$$\alpha_0=0, \qquad \beta_0=4\pi Ve^{ikx}, \qquad \gamma_0=0. \quad . \quad . \quad . \quad (98)$$

The method of solution, analogous to that already several times employed, depends upon the principle that in the neighbourhood of the obstacle and up to a distance from it great in comparison with the dimensions of the obstacle but small in comparison with λ, the condition at any moment may be identified with a steady condition such as is determined by the solution of a problem in conduction. When this is known, the disturbance at a distance from the obstacle may afterwards be derived.

We will commence with the case of the *sphere*, and consider first the magnetic functions as disturbed by the change of permeability from μ to μ'. Since in the neighbourhood of the sphere the problem is one of steady distribution, α, β, γ are derivable from a potential. By (98), in which we may write

$e^{ikx}=1$, the primary potential is $4\pi V y$; so that in (1) we are to take $u=0$, $v=4\pi V$, $w=0$. Hence by (20) α, β, γ for the *disturbance* are given by

$$\alpha=d\psi/dx, \qquad \beta=d\psi/dy, \qquad \gamma=d\psi/dz,$$

where

$$\psi=-4\pi V\frac{\mu'-\mu}{\mu'+2\mu}\frac{a^3y}{r^3}. \quad . \quad . \quad . \quad . \quad (99)$$

In like manner f, g, h are derivable from a potential χ. The primary potential is z simply, so that in (1), $u=0$, $v=0$, $w=1$. Hence by (20)

$$\chi=-\frac{K'-K}{K'+2K}\frac{a^3z}{r^3}, \quad . \quad . \quad . \quad . \quad (100)$$

from which f, g, h for the disturbance are derived by simple differentiations with respect to x, y, z respectively.

Since f, g, h, α, β, γ all satisfy (92), the values at a distance can be derived by means of (41). The terms resulting from (99), (100) are of the second order in spherical harmonics. When r is small,

$$r^{-1}e^{-ikr}f_2(ikr)=-3/k^2r^3,$$

and when r is great

$$r^{-1}e^{-ikr}f_2(ikr)=r^{-1}e^{-ikr};$$

so that, as regards an harmonic of the second order, the value at a distance will be deduced from that in the neighbourhood of the origin by the introduction of the factor $-\frac{1}{3}k^2r^2e^{-ikr}$. Thus, for example, f in the neighbourhood of the origin is

$$f=\frac{d\chi}{dx}=\frac{K'-K}{K'+2K}\frac{3a^3xz}{r^5}; \quad . \quad . \quad . \quad (101)$$

so that at a great distance we get

$$f=-\frac{K'-K}{K'+2K}\frac{k^2a^3xz\,e^{-ikr}}{r^3}. \quad . \quad . \quad . \quad (102)$$

In this way the terms of the second order in spherical harmonics are at once obtained, but they do not constitute the complete solution of the problem. We have also to consider the possible occurrence of terms of other orders in spherical harmonics. Terms of order higher than the second are indeed excluded, because in the passage from r small to r great they suffer more than do the terms of the second order. But for a like reason it may happen that terms of order zero and 1 in spherical harmonics rise in relative importance so as to be comparable at a distance with the term of the second order, although relatively negligible in the neighbourhood of the obstacle. The factor, analogous to $-\frac{1}{3}k^2r^2e^{-ikr}$ for the second order, is for the first order $ikre^{-ikr}$, and for zero order e^{-ikr}. Thus, although (101) gives the value of f with sufficient completeness for the neighbourhood of the obstacle, (102) may need to be supplemented by terms of the first and zero orders in spherical harmonics of the same importance as itself. The supplementary terms may be obtained without much difficulty from those already arrived at by means of the relations (93), (94), (95); but the process is rather cumbrous, and it seems better to avail ourselves of the forms deduced by Hertz * for electric vibrations radiated from a centre.

If we write $\Pi = \mathrm{A}e^{-ikr}/r$, the solution corresponding to an impressed electric force acting at the origin parallel to z is

$$f = -\frac{d^2\Pi}{dx\,dz}, \qquad g = -\frac{d^2\Pi}{dy\,dz}, \qquad h = \frac{d^2\Pi}{dx^2} + \frac{d^2\Pi}{dy^2}; \qquad (103)$$

$$\alpha = -4\pi\frac{d^2\Pi}{dy\,dt}, \qquad \beta = 4\pi\frac{d^2\Pi}{dx\,dt}, \qquad \gamma = 0. \qquad . \qquad (104)$$

These values evidently satisfy (92) since Π does so, and they harmonize with (93), (94), (95).

In the neighbourhood of the origin, where kr is small, e^{-ikr} may be identified with unity, so that $\Pi = \mathrm{A}/r$. In this case (103) may be written

$$f = -\frac{d^2\Pi}{dx\,dz}, \qquad g = -\frac{d^2\Pi}{dx\,dz}, \qquad h = -\frac{d^2\Pi}{dz^2},$$

and all that remains is to identify $-d\Pi/dz$ with χ in (100). Accordingly

$$\mathrm{A} = -a^3\frac{\mathrm{K}' - \mathrm{K}}{\mathrm{K}' + 2\mathrm{K}}. \quad . \quad . \quad . \quad . \quad (105)$$

The values of f, g, h in (103) are now determined. Those of α, β, γ are relatively negligible in the neighbourhood of the origin. At a great distance we have

$$f = -\mathrm{A}\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right) = -\frac{\mathrm{A}}{r}\frac{d^2e^{-ikr}}{dx\,dz} = \frac{k^2\mathrm{A}e^{-ikr}}{r}\frac{xz}{r^2};$$

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* *Ausbreitung der electrischen Kraft*, Leipzig, 1892, p. 150. It may be observed that the solution for the analogous but more difficult problem relating to an elastic solid was given much earlier by Stokes (Camb. Trans. vol. ix. p. 1, 1849). Compare 'Theory of Sound,' 2nd ed. § 378.

so that (103), (104) may be written

$$f,\ g,\ h=\frac{K'-K}{K'+2K}\,\frac{k^2a^3e^{-ikr}}{r}\left(-\frac{xz}{r^2},\ -\frac{yz}{r^2},\ \frac{x^2+y^2}{r^2}\right), \quad (106)$$

$$\frac{\alpha,\ \beta,\ \gamma}{4\pi V}=\frac{K'-K}{K'+2K}\,\frac{k^2a^3e^{-ikr}}{r}\left(\frac{y}{r},\ -\frac{x}{r},\ 0\right). \quad . \quad (107)$$

These equations give the values of the functions for a disturbance radiating from a small spherical obstacle, so far as it depends upon (K′−K). We have to add a similar solution dependent upon the change from μ to μ'. In this (103), (104) are replaced by

$$\frac{\alpha}{V^2}=-\frac{d^2\Pi}{dx\,dy},\qquad \frac{\beta}{V^2}=\frac{d^2\Pi}{dx^2}+\frac{d^2\Pi}{dz^2},\qquad \frac{\gamma}{V^2}=-\frac{d^2\Pi}{dz\,dy}; \quad (108)$$

$$4\pi f=-\frac{d^2\Pi}{dz\,dt},\qquad g=0,\qquad 4\pi h=\frac{d^2\Pi}{dx\,dt}, \quad (109)$$

where $\Pi=Be^{-ikr}/r$, corresponding to an impressed magnetic force parallel to y. In the neighbourhood of the origin (108) becomes

$$\frac{\alpha}{V^2}=-\frac{d^2\Pi}{dx\,dy},\qquad \frac{\beta}{V^2}=-\frac{d^2\Pi}{dy^2},\qquad \frac{\gamma}{V^2}=-\frac{d^2\Pi}{dz\,dy},$$

so that ψ in (99) is to be identified with $-V^2d\Pi/dy$. Thus

$$B=-\frac{4\pi a^3}{V}\,\frac{\mu'-\mu}{\mu'+2\mu}. \quad . \quad . \quad . \quad . \quad (110)$$

At a great distance we have

$$f,\ g,\ h=\frac{\mu'-\mu}{\mu'+2\mu}\,\frac{k^2a^3e^{-ikr}}{r}\left(\frac{z}{r},\ 0,\ -\frac{x}{r}\right); \quad . \quad (111)$$

$$\frac{\alpha,\ \beta,\ \gamma}{4\pi V}=\frac{\mu'-\mu}{\mu'+2\mu}\,\frac{k^2a^3e^{-ikr}}{r}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{zy}{r^2}\right). \quad (112)$$

By addition of (111) to (106) and of (112) to (107) we obtain the complete values of f, g, h, α, β, γ when both the dielectric constant and the permeability undergo variation. The disturbance corresponding to the primary waves $h=e^{ikx}$ is thus determined.

When the changes in the electric constants are small, (106), (111) may be written

$$f=\frac{\pi \mathrm{T}}{\lambda^2 r}e^{-ikr}\left(-\frac{\Delta \mathrm{K}}{\mathrm{K}}\frac{xz}{r^2}+\frac{\Delta\mu}{\mu}\frac{z}{r}\right), \quad . \quad . \quad (113)$$

$$g=\frac{\pi \mathrm{T}}{\lambda^2 r}e^{-ikr}\left(-\frac{\Delta \mathrm{K}}{\mathrm{K}}\frac{yz}{r^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

$$h=\frac{\pi \mathrm{T}}{\lambda^2 r}e^{-ikr}\left(\frac{\Delta \mathrm{K}}{\mathrm{K}}\frac{x^2+y^2}{r^2}-\frac{\Delta\mu}{\mu}\frac{x}{r}\right), \quad . \quad . \quad (115)$$

where $\mathrm{T}=\frac{4}{3}\pi a^3$, $k=2\pi/\lambda$. These are the results given formerly * as applicable in this case to an obstacle of volume T and of arbitrary form. When the obstacle is spherical and $\Delta\mathrm{K}/\mathrm{K}$ is not small, it was further shown that $\Delta\mathrm{K}/\mathrm{K}$ should be replaced by $(\mathrm{K}'-\mathrm{K})/(\mathrm{K}'+2\mathrm{K})$, and similar reasoning would have applied to $\Delta\mu/\mu$.

The solution for the case of a spherical obstacle having the character of a perfect conductor may be derived from the general expressions by supposing that $\mathrm{K}'=\infty$, and (in order that V' may remain finite) $\mu'=0$. We get from (106), (111),

$$f=-\frac{k^2a^3e^{-ikr}}{r}\left(\frac{xz}{r^2}+\frac{z}{2r}\right), \quad . \quad . \quad . \quad . \quad (116)$$

$$g=-\frac{k^2a^3e^{-ikr}}{r}\frac{yz}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

$$h=+\frac{k^2a^3e^{-ikr}}{r}\left(\frac{x^2+y^2}{r^2}+\frac{x}{2r}\right), \quad . \quad . \quad . \quad (118)$$

in agreement with the results of Prof. J. J. Thomson †. As was to be expected, in every case the vectors (f, g, h), (α, β, γ), (x, y, z) are mutually perpendicular.

Obstacle in the Form of an Ellipsoid.

The case of an ellipsoidal obstacle of volume T, whose principal axes are parallel to those of x, y, z, *i. e.* parallel to the directions of propagation and of vibration in the primary waves, is scarcely more complicated. The passage from the values of the disturbance in the neighbourhood of the obstacle to that at a great distance takes place exactly as in the case of the sphere. The primary magnetic potential in the neighbourhood of the obstacle is $4\pi\mathrm{V}y$, and thus, as before, $u=0$, $v=4\pi\mathrm{V}$, $w=0$ in (1). Accordingly, by (14), $\mathrm{A}=0$, $\mathrm{C}=0$; and (28) gives

$$\psi=-4\pi\mathrm{V}\frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)\mathrm{M}}\frac{\mathrm{T}y}{r^3}, \quad . \quad . \quad (119)$$

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* "Electromagnetic Theory of Light," Phil. Mag. vol. xii. p. 90 (1881).

† 'Recent Researches,' § 377.

corresponding to (99) for the sphere. In like manner the electric potential is

$$\chi = -\frac{\mathrm{K}'-\mathrm{K}}{4\pi\mathrm{K}+(\mathrm{K}'-\mathrm{K})\mathrm{N}}\,\frac{\mathrm{T}z}{r^3}. \quad . \quad . \quad . \quad (120)$$

These potentials give by differentiation the values of α, β, γ and f, g, h respectively in the neighbourhood of the ellipsoid. Thus at a great distance we obtain for the part dependent on $(\mathrm{K}'-\mathrm{K})$, as generalizations of (106), (107),

$$f, g, h = \frac{\mathrm{K}'-\mathrm{K}}{4\pi\mathrm{K}+(\mathrm{K}'-\mathrm{K})\mathrm{N}}\,\frac{k^2\mathrm{T}e^{-ikr}}{r}\left(-\frac{xz}{r^2}, \ -\frac{yz}{r^2}, \ \frac{x^2+y^2}{r^2}\right); \quad . \quad . \quad . \quad (121)$$

$$\frac{\alpha, \beta, \gamma}{4\pi\mathrm{V}} = \frac{\mathrm{K}'-\mathrm{K}}{4\pi\mathrm{K}+(\mathrm{K}'-\mathrm{K})\mathrm{N}}\,\frac{k^2\mathrm{T}e^{-ikr}}{r}\left(\frac{y}{r}, \ -\frac{x}{r}, \ 0\right). \quad . \quad (122)$$

To these are to be added corresponding terms dependent upon $(\mu'-\mu)$, viz.:—

$$f, g, h = \frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)\mathrm{M}}\,\frac{k^2\mathrm{T}e^{-ikr}}{r}\left(\frac{z}{r}, \ 0, \ -\frac{x}{r}\right); \quad . \quad . \quad (123)$$

$$\frac{\alpha, \beta, \gamma}{4\pi\mathrm{V}} = \frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)\mathrm{M}}\,\frac{k^2\mathrm{T}e^{-ikr}}{r}\left(-\frac{xy}{r^2}, \ \frac{x^2+z^2}{r^2}, \ -\frac{zy}{r^2}\right). \quad . \quad . \quad . \quad (124)$$

The sum gives the disturbance at a distance due to the impact of the primary waves,

$$h_0 = e^{ikx}, \qquad \beta_0 = 4\pi\mathrm{V}e^{ikx}, \quad . \quad . \quad . \quad . \quad (125)$$

upon the ellipsoid T of dielectric capacity K′ and of permeability μ'.

As in the case of the sphere, the result for an ellipsoid of perfect conductivity is obtained by making $\mathrm{K}'=\infty$, $\mu'=0$. Thus

$$f = -\frac{k^2e^{-ikr}}{r}\left(\frac{\mathrm{T}}{\mathrm{N}}\,\frac{xz}{r^2} + \frac{\mathrm{T}}{4\pi-\mathrm{M}}\,\frac{z}{r}\right), \quad . \quad (126)$$

$$g = -\frac{k^2e^{-ikr}}{r}\,\frac{\mathrm{T}}{\mathrm{N}}\,\frac{yz}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

$$h = +\frac{k^2e^{-ikr}}{r}\left(\frac{\mathrm{T}}{\mathrm{N}}\,\frac{x^2+y^2}{r^2} + \frac{\mathrm{T}}{4\pi-\mathrm{M}}\,\frac{x}{r}\right). \quad . \quad (128)$$

Next to the sphere the case of greatest interest is that of a flat circular disk (radius=R). The volume of the obstacle then vanishes, but the effect remains finite in certain cases notwithstanding. Thus, if the axis of the disk be parallel to x, that is to the direction of primary propagation, we have (21), (25),

$$\frac{\mathrm{T}}{\mathrm{N}}=\frac{4\mathrm{R}^3}{3\pi}, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=0. \quad . \quad . \quad . \quad . \qquad (129)$$

In spite of its thinness, the plate being a perfect conductor disturbs the electric field in its neighbourhood; but the magnetic disturbance vanishes, the zero permeability having no effect upon the magnetic flow parallel to its face. If the axis of the disk be parallel to y {see (24)},

$$\frac{\mathrm{T}}{\mathrm{N}}=\frac{4\mathrm{R}^3}{3\pi}, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=\frac{2\mathrm{R}^3}{3\pi}; \quad . \quad . \quad . \quad . \qquad (130)$$

and if the axis be parallel to z,

$$\frac{\mathrm{T}}{\mathrm{N}}=0, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=0, \quad . \quad . \quad . \quad . \quad . \qquad (131)$$

so that in this case the obstacle produces no effect at all.

Circular Aperture in Conducting Screen.

The problem proposed is the incidence of plane waves ($h_0=e^{ikx}$) upon an infinitely thin screen at $x=0$ endowed with perfect electric conductivity and perforated by a circular aperture. In the absence of a perforation there would of course be no waves upon the negative side, and upon the positive side the effect of the screen would merely be to superpose the reflected waves denoted by $h_0=-e^{-ikx}$. We wish to calculate the influence of a small circular aperture of radius R.

In accordance with the general principle the condition of things is determined by what happens in the neighbourhood of the aperture, and this is substantially the same as if the wave-length were infinite. The problem is then expressible by means of a common potential. The magnetic force at a distance from the aperture on the positive side is altogether $8\pi\mathrm{V}$, and on the negative side zero; while the condition to be satisfied upon the faces of the screen is that the force be entirely tangential. The general character of the flow is indicated in fig. 1.

The problem here proposed is closely connected with those which we have already considered where no infinite screen was present, but a flat finite obstacle, which may be imagined to coincide with the proposed aperture. The primary magnetic field being $\beta=4\pi V$, and the disk of radius R being of infinite permeability, the potential at a distance great

Fig. 1.

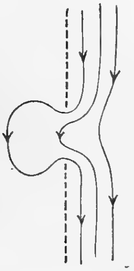


Fig. 2.

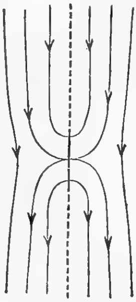
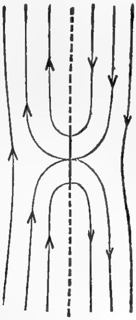


Fig. 3.



compared with R (but small compared with λ) is by (27) (28)

$$\psi=-4\pi V\frac{4R^3}{3\pi}\frac{y}{r^3}. \quad . \quad . \quad . \quad . \quad (132)$$

By the symmetry the part of the plane $x=0$ external to the disk is not crossed by the lines of flow, and thus it will make no difference in the conditions if this area be filled up by a screen of zero permeability. On the other hand, the part of the plane $x=0$ represented by the disk is met normally by the lines of flow. This state of things is indicated in fig. 2.

The introduction of the lamina of zero permeability effects the isolation of the positive and negative sides. We may therefore now reverse the flow upon the negative side, giving the state of things indicated in fig. 3. But the plate of infinite permeability then loses its influence and may be removed, so as to re-establish a communication between the positive and negative sides through an aperture. The passage from the present state of things to that of fig. 1 is effected by superposition upon the whole field of $\beta=4\pi V$, so as to destroy the field at a distance from the aperture upon the negative side and upon the positive side to double it.

As regards the solution of the proposed problem we have then on the positive side

$$\psi=8\pi Vy-4\pi V\frac{4R^3}{3\pi}\frac{y}{r^3}, \quad . \quad . \quad . \quad . \quad (133)$$

and on the negative side

$$\psi=4\pi V\frac{4R^3}{3\pi}\frac{y}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Thus on the negative side at a distance great in comparison with the wave-length we get, as in (99), (111), (112),

$$f, g, h = -\frac{4R^3}{3\pi}\frac{k^2e^{-ikr}}{r}\left(\frac{z}{r}, 0, -\frac{x}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad (135)$$

$$\frac{\alpha, \beta, \gamma}{4\pi V} = -\frac{4R^3}{3\pi}\frac{k^2e^{-ikr}}{r}\left(-\frac{xy}{r^2}, \frac{x^2+z^2}{r^2}, -\frac{zy}{r^2}\right). \quad . \quad (136)$$

On the positive side these values are to be reversed, and addition made of

$$h_0 = e^{ikx} - e^{-ikx}, \qquad \beta_0 = 4\pi V(e^{ikx} + e^{-ikx}), \quad . \quad (137)$$

representing the plane waves incident and reflected.

The solution for h in (135) may be compared with that obtained (27), (28) in a former paper *, where, however, the primary waves were supposed to travel in the positive, instead of, as here, in the negative direction. It had at first been supposed that the solution for ϕ there given might be applied directly to h, which satisfies the condition (imposed upon ϕ) of vanishing upon the faces of the screen. If this were admitted, as also $g=0$ throughout, the value of h would follow by (95). The argument was, however, felt to be insufficient on account of the discontinuities which occur at the *edge* of the aperture, and the value now obtained, though of the same form, is doubly as great.

Terling Place, Witham.

VI. *Thermal Transpiration and Radiometer Motion.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN the Phil. Mag. for Feb. 1897, Prof Osborne Reynolds, in commenting upon my paper on Thermal Transpiration and Radiometer Motion, remarks freely on errors into which I have fallen therein. I fancy that most readers of my paper will recognize that the particular errors mentioned by Prof. Reynolds are rather the result of his own misinterpretation than of my blundering; but still, as he has taken six pages of the Phil. Mag. in which to lay these errors to my charge, I should like to point out briefly how the errors are his own.

First:—On page 143 Prof. Reynolds writes:—" while Mr. Sutherland expressly excludes the action of these walls

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* " On the Passage of Waves through Apertures in Plane Screens, and Allied Problems," Phil. Mag. vol. xliii. p. 264 (1897).

(or the walls themselves) from any part in thermal transpiration, which he rests solely on the action of direct conduction of heat." Now I do no such thing: what I really say is, that the solid walls dominate the temperature of the gas so completely that the temperature of the gas in any section of the tube may be taken as that of the solid walls bounding it; then suppose this distribution of temperature maintained in any way, and the solid walls may be supposed to be abolished as far as the calculation of the number of molecules crossing any section is concerned. Yet although the action of the walls on the gas through viscosity is subsequently treated as of fundamental importance, Prof. Reynolds lays this simple little mathematical artifice of supposing the walls abolished for the purpose of minimizing the verbiage of a certain calculation, to my charge as a total ignoring of the walls. It is quite plain that I do not rest thermal transpiration "solely on the action of direct conduction of heat," if Prof. Reynolds means by this the normal conduction in an indefinite mass of gas; I take account of the action of the walls both when I ascribe their temperature to the gas, and when I consider their frictional resistance to the motion of the gas.

Second:—On pages 144 and 145 Prof. Reynolds gives an account of Clausius's discovery that the colliding molecules in a gas conducting heat must be considered to have a uniform drift from hot to cold, because, while equal numbers of molecules are travelling in opposite directions across any section, those crossing from the hot side are moving more swiftly, and he then seeks to imply that I have made the "somewhat common and certainly historical error in mechanical analysis" of neglecting this drift. On page 147 he states of me:—"he gives reasons for thinking that the walls of the tube produce no effect on the characteristics of the molecules which rebound from them. And thus having cut away the only cause of thermal transpiration he proceeds:" I have already pointed out what a complete misstatement this is of my contention. In my paper I gave a simplified and brief calculation, after Clausius, of the number of molecules that must cross unit area in unit time in a gas at rest on account of a given density and rate of variation of density and a given temperature and given rate of variation of temperature across that section. I did not assert that the gas was to be at rest to begin with, but as it was not stated to be moving, that was assumed, and I only calculated in the usual way, only more briefly, what must be the number crossing unit area in unit time so long as the given conditions are maintained. Of this little calculation Prof. Reynolds says :—" Having thus fallen into the error from which Clausius released Maxwell, the inevitable result is that he arrives at the absurd conclusion that in a gas at rest more molecules cross the fixed plane from the cold side than from the hot side." The absurdity appears to me to be all Prof. Reynolds's, when he objects to my deducing motion as the result of certain conditions in a system supposed to be initially at rest, merely because it was supposed to be initially at rest. I presume that what Prof. Reynolds is really driving at, is the expression of his feeling that he would like the motion investigated not merely at its initial stage, but at a general stage, and with this everybody can sympathize. But when the initial motion for given permanent conditions has been calculated, it is plain how the drift from hot to cold comes in in the theory of conduction in gases, because as there is to be no resultant transfer of matter across any area, the gas as a whole must have a drift equal and opposite to that flow due to variation of density and temperature, which is now not only initial, but permanent, because the initial conditions are kept permanent. And yet on page 148 Prof. Reynolds calmly charges it against me that I fail to notice that the drift discovered by Clausius is opposite to that which I consider fundamental in thermal transpiration ; whereas my references to Clausius, apart from their being a just acknowledgement of indebtedness, were made to obviate the necessity of explaining a matter that had already been cleared up as this had. On page 148 I am charged with a complete misstatement of Clausius's position when I say that under ordinary circumstances the motion occurring in connexion with conduction in gases cannot establish appreciable difference of pressure. There is to my mind no misstatement here, but simply a bare statement of fact ; consider a mass of gas at rest and at uniform temperature in a cylinder whose curved surface is impervious to heat, and suppose one end of the cylinder suddenly heated and the other cooled so that the average temperature will remain the same, then according to Clausius's theory there will at first be a motion of the gas which will finally produce quite a different distribution of temperature and density from the original, and then a permanent state will be established in which the drift neutralizes the motion caused by variation of density and temperature ; but the initial and permanent motions will never be associated with establishment of an appreciable difference of pressure, and I maintain that my assertion of this fact is not a misstatement of Clausius's position, but only a restatement in terms which bring it into its natural relation with thermal transpiration.

One statement in Prof. Reynolds's paper I must protest against; on page 147 he says of me: "showing that he has in no way realised the point of Clausius's paper which he quotes—showing, in fact, that he cannot even have read it." I am sorry that without just cause Prof. Reynolds should disfigure a scientific discussion with an assertion such as the last part of this.

While disclaiming the errors that Prof. Reynolds would saddle me with, I must confess that the theory of my paper is far less complete than I should like to see it, but I have a partial excuse in that I cut the space devoted to the bare essentials of the theory down to a minimum, seeing that its applications made a demand on your space which, it appeared to me, might jeopardize their being published at all.

Yours obediently,

Melbourne, 19th April, 1897. WILLIAM SUTHERLAND.

VII. *Doublets and Triplets in the Spectrum produced by External Magnetic Forces.* *By* Dr. P. ZEEMAN[*].

1. ON a former occasion[†] I have remarked that the elementary treatment of the general idea of the Lorentz theory of the magnetic broadening of the spectral lines indicates that the broadened line must in some cases be broken up into a triplet. I have examined this subject somewhat more in detail. Further consideration shows that with a very strong magnetic field, a magnetically broadened line must be broken up into doublets or triplets according as the light is emitted in a direction parallel or perpendicular respectively to the lines of force. With a smaller intensity of the field, the line will be simply widened. The prediction of Lorentz with regard to the polarization of the edges of the broadened lines having been fulfilled, it seemed worth while to pursue still further the study of the polarization of the lines. Even with magnetic forces insufficient to break up the line into a triplet, it might be possible to learn something more about the constitution of the widened line. It seemed to me to be of interest to investigate this point, and to see whether the explanation formerly given, intended as the very first sketch of the motion of ions according to the theory of Lorentz, was further confirmed by experiment.

Prof. Oliver Lodge[‡] has suggested that, under some sup-

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[*] Communicated by Prof. Oliver Lodge.

[†] Phil. Mag. for March 1897.

[‡] 'The Electrician' for February 28, 1897, p. 569

positions, we might conclude also that the line will be simply widened in a strong field. I take the liberty of stating briefly to the readers of the 'Philosophical Magazine' the preliminary results I have obtained in regard to these questions.

2. In § 18 of the cited paper, the motion of an ion in a magnetic field was treated according to Lorentz's theory. The system of coordinates was chosen in such a manner that the axis of z was parallel to the magnetic force, the plane of (x, y) being perpendicular to it. Referring for the rest to the cited paper, I recapitulate that the considered motions of the ions, existing before the putting on of the magnetic force, were resolved into a rectilinear harmonic motion parallel to the axis of z and two circular (right-handed and left-handed) motions in the plane of (x, y).

The first remains unchanged under the influence of the magnetic force, the periods of the last are changed.

3. Using the rule given by Lorentz, therefore, we see that in the direction of the lines of force, right-handed and left-handed circularly polarized light of changed period must be propagated. The ions vibrating parallel to the lines of force of course do not cause any disturbance in the æther equivalent to light in that direction.

In a direction perpendicular to the lines of force, those motions of the ions which have a component parallel to the plane of (x, y) give plane-polarized light vibrating in a vertical plane and with changed period (I suppose the axis of z in a horizontal plane). The ions, however, moving parallel to the axis of z give electrical vibrations in a horizontal direction with unchanged period.

4. *Magnetic Doublets.*—Hence with every value of the magnetic force, an infinitely narrow spectral line breaks up into two lines, the light being emitted in the direction of the lines of force. However, for lines of finite width the magnetic change of the period must exceed somewhat that corresponding to *half the width* of the original line in order that the doublet may be seen. One of the components then must be over the whole width left-handed, the other right-handed circularly polarized.

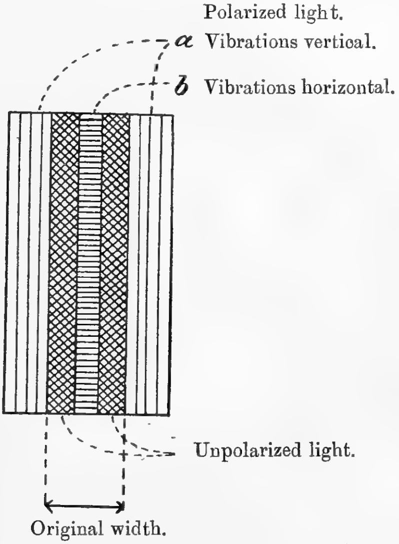
5. *Magnetic Triplets.*—With finite width of the spectral line and observing in a direction perpendicular to the lines of force, the line is broken up into a triplet if the magnetic change of the period corresponds to *the whole width* of the original spectral line. Hence the magnetic force necessary to produce a triplet is equal to twice the value necessary for a doublet.

6. *Intermediate Forms of Magnetic Doublets and of Triplets* —With magnetic forces less than the ones supposed in §§ 4

and 5, forms intermediate between the unchanged spectral lines and the doublets and triplets may be expected. With the doublet only one particular intermediate form exists, viz., a line the edges of which are circularly polarized, the central part emitting unpolarized light. This case I described in my former paper.

Of triplets two intermediate forms may be distinguished.

I will call a line "Triplet *a*," if the magnetic change corresponds to somewhat more than half the width of the unchanged line. In this case the broadened line is composed in the following manner: the central part will emit horizontal vibrations, at both its sides bands of (chiefly, but not entirely) unpolarized light border it, which again are enclosed by bands of vertically vibrating light. The accompanying diagram gives a rough scheme of the constitution of the line.



In "Triplet *b*" the magnetic change amounts to less than half the width of the original line. This case I described in my cited paper.

7. Hence we may expect that if we can just resolve the doublet, the magnetic force also is sufficient for triplet *a*.

As with sodium (*cf.* § 11) I did not succeed completely with the means at my disposal in observing the expected doublets and triplets, so with other substances I looked in the first place for magnetic doublets. We might expect then that further inquiry would show something about the triplets. At last I succeeded in observing with the blue line ($\lambda = 480\ \mu\mu$) of cadmium the doublet and the triplet *a*. It has already been remarked by

Egoroff and Georgiewsky * that cadmium also exhibits the broadening of the spectral lines under the action of magnetism.

8. The method of experiment was principally the same as that formerly used. As in my former experiments, I used a Ruhmkorff electromagnet. I missed, however, the beautiful Rowland grating I used in the laboratory of Prof. Onnes. I now only had at my disposal a smaller one with a radius of 6 ft., but like the Leyden one with 14438 lines to the inch. The second spectrum was very satisfactory. For the cadmium spectrum a spark was used between cadmium electrodes; with a lens, as often used in this manner by Lockyer, an image of spark and electrodes was formed on the slit; it is then easily verified that the part of the spark just between the poles is analysed.

The above-mentioned cadmium line is especially sharp at the violet side.

Now I succeeded, indeed, in observing the expected phenomena (doublet and triplet *a*).

9. For observing the doublet along the lines of force a perforated pole was used. With a current of 30 amp., the distance of the poles being as small as possible without interfering with the cadmium electrodes, a perfectly defined doublet was seen. One of the components appeared to be left-handed, the other right-handed circularly polarized † *over the whole width.* It might still be argued that the intervening dark space between the components is caused by a reversal and that it is not a doubling of the lines. However, then it must be supposed that this absorption-line just intercepts that part of a magnetically broadened line emitting unpolarized light. Now with a smaller value of the magnetic force the dark space becomes narrower. I grant that this is also the case with the supposed absorption-line. However, then a greater part of the broadened emission-line should emit unpolarized light, the smaller absorption-line now intercepting but a part of it. It appeared, however, that even with very narrow intervening dark space, the components of the doublet were entirely circularly polarized, and hence the intervening dark space seems to be different from an absorption-line. There is yet another argument against the

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* *C. R.* 1897, t. cxxiv. pp. 748 & 242.

† The sign of the circular polarization is the same in the case of this cadmium line and of the D-lines. I must, however, correct my statement in § 24 of my former paper. I now see that if the lines of force are running towards the grating, the right-handed circularly polarized rays appear to have the greater period. Hence the radiation is due chiefly to the motion of a *negatively*-charged particle. Probably my mistake arose from a faulty indication of the axis of the λ/4-plate used.

interpretation that the dark space is of the nature of an absorption-band. Indeed, one should expect to see also reversal if looking across the lines of force. However, in this direction no trace of an ordinary absorption-line was visible with the same magnetic intensity. Hence we must conclude, I think, that I have really observed the doublet indicated by Lorentz's theory.

10. Looking across the lines of force I succeeded in observing the phenomenon indicated as triplet *a* (§ 6, diagram). If no nicol was used only a broadened line was visible. A nicol oriented so as to get rid of light whose plane of polarization is vertical makes a dark well-defined line appear in the midst of the broadened line. Turning the nicol over 90° this dark line disappears, the lateral wings now being quenched, because these are polarized in a horizontal plane. All this is to be expected according to § 6. The analysers in §§ 9 and 10 were inserted according to Lodge's * advice before the light reaches the grating.

Till now I have not had the opportunity of measuring the intensity of the field used in §§ 9 and 10. I took care, however, to make them nearly equal in the two cases.

11. Finally it may be mentioned that I observed also traces of the phenomena described in §§ 9 and 10 with sodium. As the D-lines reverse so very easily by non-magnetic agency and as they show sometimes complicated phenomena (Lockyer and Ciamician †), it seems difficult to see the phenomena pure and simple with sodium. It is true that it is not difficult to avoid perturbations caused by the motion of the piece of asbestos, as mentioned by Egoroff and Georgiewsky ‡ in their interesting note. To avoid disturbances of this kind I simply moistened the mouth-piece of the burner with a very concentrated solution of NaCl or NaBr. Small disturbances nevertheless easily make their appearance.

12. Although my inquiry is not yet closed, I think, however, that we can say that it seems to give new evidence in favour of the interpretation of the magnetization of the spectral lines given by Lorentz's theory. When I have determined the intensity of my field, we can decide the question whether we can hope to obtain a pure triplet by augmentation of the magnetic force or by using narrower spectral lines, though, of course, it may yet turn out that the phenomena prove to be less simple. For a future paper also must be reserved the accurate measurement of the amount of the magnetic change

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* 'The Electrician' of 26 February, 1897.

† *cf.* Kayser, *Spectralanalyse*, p. 305.

‡ *C. R.* t. cxxiv. p. 949.

with Cd and other substances [*], and therewith the discussion about the ratio between mass and charge of the ions in Lorentz's theory. It is very probable that these "ions" differ from the electrolytical. It is true that by means of the latter many phenomena can be interpreted, as also is done in a paper by Richarz [†], where, *e. g.*, the molecular magnetism is explained by the motion of the ions, but the high value of e/m which I have found makes it extremely improbable that we have to deal with the same mass in the two cases. And Lodge [‡] has even pointed out in an extremely interesting paper that my value of e/m can be explained without the necessary motion of any "matter" at all.

Appendix.—On communicating the observations of this paper to Prof. Lodge, he most kindly informed me that he had already seen some of them himself. Especially he had seen the doubling of the lines and shown them at the Royal Society soirée on May 20. He, however, did not suppose it to be a real doubling of the line, but thought it to be a broadening and a reversal. I have given my reasons (the chief being the circular polarization over the whole width) for thinking that I observed a real doubling. I am confirmed in this opinion because it corroborates the observation of the perpendicular polarizations of the middle and of the edges of the triplet *a* (§ 10). As Prof. Lodge does not mention that he observed this last phenomenon, and as my point of view is different from his, his object not being in the first place to test Lorentz's theory, I publish the paper in its original form, though it turns out that part of its contents has been observed by both of us independently [§].

Amsterdam, June 4.

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[*] I have experiments in progress (measurements of the photographed magnetized spectrum) by which I hope to be able to obtain fairly accurate values of the magnetic change.

[†] Wied. *Ann.* lii. p. 385 (1894).

[‡] 'The Electrician,' Mar. 12, 1897.

[§] *Disclaimer by* Prof. LODGE.

Having been asked to exhibit Prof. Zeeman's discovery at the Royal Society soirée I arranged apparatus to obtain it more powerfully, and thereby saw the new effects; but without any intention of trespassing on the prerogative of the discoverer.

What I saw concerning polarization was that, when looking across the lines of force, a nicol brought out the doubling (or tripling or quadrupling as the case might be) more sharply, no doubt by quenching the residual light; and that it restored the original appearance of the line when rotated through 90°. I mention this here as partially confirming Prof. Zeeman's most interesting and much more complete investigation, on the theoretical aspect of which at present I offer no further opinion.—Liverpool, June 6.

VIII. *Alternating Currents in Concentric Cables.*

By W. A. PRICE, *M.A.*[*]

Fig. 1.

§ 1.

AOB is part of an infinite electrical conductor possessing resistance and capacity, the inductance being negligible. The characteristic differential equation is

$$\frac{d^2u}{dx^2}=\rho\gamma\frac{du}{dt},$$

ρ being the resistance, and γ the capacity per unit of length. At A, B are inserted equal alternating electromotive forces in the same phase, each represented by $2\text{E}\sin\omega t$, so that there is no current at O the middle point of AB. The distance AB is 2 L. At any point P distant x from O the potential due to the electromotive force acting at A is

$$\text{E}\,e^{-\alpha(\text{L}-x)}\sin\{\omega t-\alpha(\text{L}-x)\}$$

where

$$\alpha=\sqrt{\frac{\omega\rho\gamma}{2}},$$

and the current at P due to this force acting at A is

$$\frac{\text{E}}{\sqrt{2}}\frac{\alpha}{\rho}e^{-\alpha(\text{L}-x)}[\sin\{\omega t-\alpha(\text{L}-x)\}+\cos\{\omega t-\alpha(\text{L}-x)\}].$$

The current at P due to the electromotive force acting at B is

$$\frac{\text{E}}{\sqrt{2}}\cdot\frac{\alpha}{\rho}e^{-\alpha(\text{L}+x)}[\sin\{\omega t-\alpha(\text{L}+x)\}+\cos\{\omega t-\alpha(\text{L}+x)\}]$$

acting in an opposite direction to the first.

The potential at P due to the electromotive force acting at B is

$$\text{E}e^{-\alpha(\text{L}+x)}\sin\{\omega t-\alpha(\text{L}+x)\}.$$

Then the resultant potential at A is

$$\text{E}\,\{\sin\omega t+e^{-2\alpha\text{L}}\sin(\omega t-2\alpha\text{L})\};$$

the potential at O is

$$2\text{E}e^{-\alpha\text{L}}\sin(\omega t-\alpha\text{L});$$

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[*] Communicated by the Physical Society: read April 9, 1897.

the current at A is

$$\frac{\mathrm{E}}{\sqrt{2}}\cdot\frac{\alpha}{\rho}\{1-2e^{-2\alpha\mathrm{L}}\cos 2\alpha\mathrm{L}+e^{-4\alpha\mathrm{L}}\}^{\frac{1}{2}}\cos\omega t.$$

Changing the time from which t is reckoned, so that ωt becomes $\omega t+\phi$, where

$$\tan\phi=\frac{1-e^{-2\alpha\mathrm{L}}(\cos 2\alpha\mathrm{L}+\sin 2\alpha\mathrm{L})}{1-e^{-2\alpha\mathrm{L}}(\cos 2\alpha\mathrm{L}-\sin 2\alpha\mathrm{L})},$$

and writing

$$\frac{\mathrm{E}}{\sqrt{2}}\cdot\frac{\alpha}{\rho}\{1-2e^{-2\alpha\mathrm{L}}\cos 2\alpha\mathrm{L}+e^{-4\alpha\mathrm{L}}\}^{\frac{1}{2}} \text{ as A, and } \mathrm{L}\gamma \text{ as } \Gamma,$$

the current at A becomes $\quad \mathrm{A}\cos\omega t$;

the potential at A becomes

$$\frac{\mathrm{A}}{\omega\Gamma}\alpha\mathrm{L}\frac{(\sinh 2\alpha\mathrm{L}+\sin 2\alpha\mathrm{L})\sin\omega t+(\sinh 2\alpha\mathrm{L}-\sin 2\alpha\mathrm{L})\cos\omega t}{\cosh 2\alpha\mathrm{L}-\cos 2\alpha\mathrm{L}};$$

and the potential at O becomes

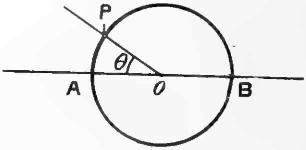
$$\frac{2\mathrm{A}}{\omega\Gamma}\alpha\mathrm{L}\frac{(\cos a\mathrm{L}\sinh a\mathrm{L}+\sin a\mathrm{L}\cosh a\mathrm{L})\sin\omega t+(\cos a\mathrm{L}\sinh a\mathrm{L}-\sin a\mathrm{L}\cosh a\mathrm{L})\cos\omega t}{\cosh 2a\mathrm{L}-\cos 2a\mathrm{L}}.$$

The form and notation of these expressions are those given in 'Alternating Currents of Electricity,' T. H. Blakesley, London, 1889.

Consider the circular electrical conductor AB (fig. 2), having a total resistance $2\pi r$ and a total capacity $2\pi c$: the characteristic equation being

$$\frac{d^2u}{d\theta^2}=rc\cdot\frac{du}{dt}.$$

Fig. 2.



At a time $t=0$, when the conductor is uncharged, a charge $2\pi\mathrm{S}$ is introduced at A. The charge at any point P at a subsequent time t is

$$s=\mathrm{S}\sum_{-\infty}^{\infty}\cos i\theta e^{-\mu i^2 t}, \text{ where } \mu=\frac{1}{rc}:$$

$s\,d\theta$ being the charge at P, where $\mathrm{AOP}=\theta$, of a small arc subtending $d\theta$ at the centre.

If a series of charges, $2\pi\mathrm{S}\cos\omega\tau\,.\,d\tau$, is introduced at A at intervals of time $d\tau$ beginning when $\tau=0$, s, the charge at P at a time t,

$$=\mathrm{S}\sum_{-\infty}^{\infty}\cos i\theta\left\{\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4}-\frac{\mu i^2}{\omega^2+\mu^2 i^4}e^{-\mu i^2 t}\right\}.$$

If this succession of charges equivalent to a permanent alternating current $2\pi S\cos\omega t$ be continued for a long time, the term involving $e^{-\mu i^2}$ becomes very small and we have

$$s = S\sum_{-\infty}^{\infty}\cos i\theta\,\frac{\mu i^2\cos\omega t + \omega\sin\omega t}{\omega^2+\mu^2 i^4}.$$

The potential at A is

$$\frac{S}{\omega c}\sin\omega t\sum_{-\infty}^{\infty}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1} + \frac{S}{\omega c}\cos\omega t\sum_{-\infty}^{\infty}\frac{\mu i^2}{\omega}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1};$$

the potential at B is

$$\frac{S}{\omega c}\sin\omega t\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{\mu^2 i^4}{\omega^2}\right) + \frac{S}{\omega c}\cos\omega t\sum_{-\infty}^{\infty}(-1)^i\frac{\mu i^2}{\omega}\left(1+\frac{\mu^2 i^4}{\omega^2}\right)^{-1}.$$

It may be seen that the electrical actions in the two cases discussed are identical so far as the portion AB of the conductor in the first case is concerned.

Writing $L\gamma=\pi c$, $L\rho=\pi r$, $\pi S=A$, and $\sqrt{\frac{\omega}{2\mu}}$ as β, so that $\alpha L=\beta\pi$,

$$\sum_{-\infty}^{\infty}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \beta\pi\frac{\sinh 2\beta\pi+\sin 2\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_1(\beta)$];

$$\sum_{-\infty}^{\infty}\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \beta\pi\frac{\sinh 2\beta\pi-\sin 2\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_2(\beta)$];

$$\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = 2\beta\pi\frac{\cos\beta\pi\sinh\beta\pi+\sin\beta\pi\cosh\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_3(\beta)$];

$$\sum_{-\infty}^{\infty}(-1)^i\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = 2\beta\pi\frac{\cos\beta\pi\sinh\beta\pi-\sin\beta\pi\cosh\beta\pi}{\cosh 2\beta\pi-\cos 2\beta\pi}$$

[this expression I shall refer to as $f_4(\beta)$].

In the particular case where β is integral

$$\sum_{-\infty}^{\infty}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \sum_{-\infty}^{\infty}\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \beta\pi\coth\beta\pi,$$

$$\sum_{-\infty}^{\infty}(-1)^i\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \sum_{-\infty}^{\infty}(-1)^i\frac{i^2}{2\beta^2}\left(1+\frac{i^4}{4\beta^4}\right)^{-1} = \beta\pi\,\mathrm{cosech}\,\beta\pi.$$

These last expressions may also be obtained by expanding $(\cos\beta\theta \pm \sin\beta\theta)e^{\beta\theta}$ in Fourier series of cosines.

For

$$\beta\pi(\cos\beta\theta - \sin\beta\theta)e^{\beta\theta} = \sum_{-\infty}^{\infty}(-1)^i\{(-1)^\beta e^{\beta\pi} - (-1)^i\}\left\{1+\frac{i^4}{4\beta^4}\right\}^{-1}\cos i\theta,$$

$$\beta\pi(\cos\beta\theta + \sin\beta\theta)e^{\beta\theta} = \sum_{-\infty}^{\infty}(-1)^i\{(-1)^\beta e^{\beta\pi} - (-1)^i\}\frac{i^2}{2\beta^2}\left\{1+\frac{i^4}{4\beta^4}\right\}^{-1}\cos i\theta.$$

Putting θ equal to 0 and π successively in these series, and adding and subtracting the series thus obtained in pairs, the four last expressions are obtained. The method is applicable only when β is integral.

In a paper by Dr. Glaisher ('Proc. London Math. Soc.,' vol. vii.) the value of $\sum_{-\infty}^{\infty}\frac{1}{r^4+x^4}$ is determined by another method which gives the same result as the above.

The physical significance of β being integral is that in that case the value of ω is such that the alternations at the sending and receiving ends of the cable are in the same phase. The charge at the sending end being

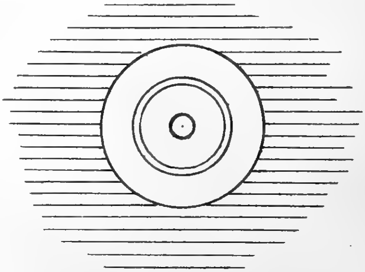
$$s = \frac{S}{\omega}\beta\pi\cosh\beta\pi(\sin\omega t + \cos\omega t)$$

and at the receiving end

$$s = \frac{S}{\omega}\beta\pi\,\mathrm{cosech}\,\beta\pi(\sin\omega t + \cos\omega t),$$

the ratio of the amplitudes being $\cosh\beta\pi$.

Fig. 3.



§ 2. A cable, shown in section in fig. 3, contains two conductors, an inner central conductor and an outer concentric conductor, insulated from one another and the water in which the whole is immersed. The whole cable is supposed to be arranged in the circular form of fig. 2.

$2\pi r$, $2\pi r'$ are the total resistances of the inner and outer conductors ;

$2\pi m$ the total capacity of the inner dielectric measured between the two conductors ;

$2\pi n$ the total capacity of the outer dielectric measured between the outer conductor and the water.

Then $2\pi \frac{mn}{m+n}$ is the capacity of the central conductor, the outer one being insulated; and $2\pi n$ the capacity of the outer conductor, the inner one being insulated.

$u\ u'$, C C$'$ are the potentials of and the currents in the two conductors at point P and time t, and $s\ s'$ the density of the charges on the conductors. All these quantities being referred to a unit of length subtending unit angle at the centre, then

$$r\mathrm{C} = -\frac{du}{d\theta}, \qquad r'\mathrm{C}' = -\frac{du'}{d\theta};$$

$$-\frac{d\mathrm{C}}{d\theta} = m\left(\frac{du}{dt} - \frac{du'}{dt}\right), \quad -\frac{d\mathrm{C}'}{d\theta} = m\left(\frac{du'}{dt} - \frac{du}{dt}\right) + n\frac{du'}{dt};$$

$$s + s' = nu', \qquad \text{and} \qquad s = m(u - u').$$

Also
$$\left(\frac{d^2}{d\theta^2} - mr\frac{d}{dt}\right)s + mr'\frac{d}{dt}s' = 0,$$

and
$$\left(\frac{d^2}{d\theta^2} - (m+n)r'\frac{d}{dt}\right)s' + mr\frac{d}{dt}s = 0.$$

Hence s and s' satisfy the equation

$$\left(\mu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)s = 0,$$

where μ, ν are roots of $mnrr'x^2 - (mr + (m+n)r')x + 1 = 0$.

§3. A uniform conductor Q, along which the propagation of charges due to any electrical disturbance is determined by the differential equation

$$\left(\mu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)s = 0,$$

is arranged in a circle as in fig. 2. Its total resistance is $2\pi r$, and its total capacity $2\pi c$. Any disturbance will be propagated along the conductor in two distinct systems, one determined by the differential equation

$$\left(\mu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)s = 0, \qquad \text{the other by} \left(\nu\frac{d^2}{d\theta^2} - \frac{d}{dt}\right)s = 0;$$

and the disturbance produced at any given point will be the sum of two disturbances conveyed by the two systems.

Since each propagation is determined by a differential equation of the same form as that which applies to a single isolated conductor, such as is discussed in § 1, the action of Q is equivalent to that of a pair of single conductors, and the propagation along Q is the sum of the propagations along the equivalent pair.

Let the total resistances of the two equivalent cables be $2\pi r_1$ and $2\pi r_2$, and their capacities $2\pi c_1$ $2\pi c_2$. Then $\mu r_1 c_1 = 1$ and $\nu r_2 c_2 = 1$.

Since the pair is electrically equivalent to Q, they must convey the same steady current as Q, when subject to a steady E.M.F., and must be charged to the same potential by a given charge. So $r_1^{-1} + r_2^{-1} = r^{-1}$, and $c_1 + c_2 = c$. From these four equations $r_1\ r_2\ c_1\ c_2$ can be determined.

Then if a charge $2\pi S$ be introduced into this conductor at any point, the charge at any point after a time t will be the sum of the charges at the corresponding points of the equivalent pair, if the same charge had been introduced into them in the proportions $2\pi S\frac{c_1}{c}$, $2\pi S\frac{c_2}{c}$.

If an alternating current $2\pi S \cos \omega t$ be introduced into the conductor at A where $\theta = 0$, the charge at P will be

$$s = \frac{S}{\omega}\frac{c_1}{c}\sum_{-\infty}^{\infty} \cos i\theta\, \frac{\mu i^2 \cos \omega t + \omega \sin \omega t}{\omega^2 + \mu^2 i^4}$$

$$+ \frac{S}{\omega}\frac{c_2}{c}\sum_{-\infty}^{\infty} \cos i\theta\, \frac{\nu i^2 \cos \omega t + \omega \sin \omega t}{\omega^2 + \nu^2 i^4} = s_1 + s_2.$$

§ 4. Applying this argument to the conductors of the cable described in § 2, let $r_1\ r_2\ c_1\ c_2$ be the resistances and capacities of the pair equivalent to the central conductor, and $r_1'\ r_2'\ c_1'\ c_2'$ of the pair equivalent to the concentric; so that

$$r_1^{-1} + r_2^{-1} = r^{-1}, \qquad r_1'^{-1} + r_2'^{-1} = r'^{-1}\ ;$$

$$r_1 c_1 \mu = r_2 c_2 \nu = r_1' c_1' \mu = r_2' c_2' \nu = 1\ ;$$

$$c_1 + c_2 = \frac{mn}{m+n} = c, \qquad c_1' + c_2' = c' = n.$$

Then
$$\frac{c_1}{c} = \frac{c_2'}{c'} = \frac{\mu}{\mu - \nu}(1 - m\nu r),$$

and
$$\frac{c_2}{c} = \frac{c_1'}{c'} = \frac{\nu}{\nu - \mu}(1 - m\mu r).$$

Then the charge at any point P due to an alternating current $2\pi S \cos \omega t$ introduced at A, where $\theta = 0$, is $s = s_1 + s_2$,

where

$$s_1 = \mathrm{S}\frac{\mu}{\mu-\nu}(1-m\mu r)\sum_{-\infty}^{\infty}\cos i\theta\,\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4},$$

$$s_2 = \mathrm{S}\frac{\nu}{\nu-\mu}(1-m\nu r)\sum_{-\infty}^{\infty}\cos i\theta\,\frac{\nu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\nu^2 i^4}.$$

§ 5. If an alternating current $2\pi\mathrm{S}\cos\omega t$ be introduced into the concentric conductor at A, the charge s' at any point P $=s_1'+s_2'$, where

$$s_1' = \frac{\mathrm{S}}{\omega}\,\frac{\nu(1-\mu m r)}{\nu-\mu}\sum_{-\infty}^{\infty}\cos i\theta\,\frac{\mu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\mu^2 i^4},$$

$$s_2' = \frac{\mathrm{S}}{\omega}\,\frac{\mu(1-\nu m r)}{\mu-\nu}\sum_{-\infty}^{\infty}\cos i\theta\,\frac{\nu i^2\cos\omega t+\omega\sin\omega t}{\omega^2+\nu^2 i^4}.$$

§ 6. Consider the charge s' induced in the concentric conductor by the charge s_1+s_2 in the central conductor.

From § 2,

$$\left(\frac{d^2}{d\theta^2}-mr\frac{d}{dt}\right)(s_1+s_2)+mr'\frac{d}{dt}s'=0,$$

$$\frac{d^2}{d\theta^2}s_1=\frac{1}{\mu}\frac{d}{dt}s_1,\qquad\qquad \frac{d^2}{d\theta^2}s_2=\frac{1}{\nu}\frac{d}{dt}s_2;$$

hence
$$\frac{1-m\mu r}{\mu}\frac{ds_1}{dt}+\frac{1-m\nu r}{\nu}\frac{ds_2}{dt}+mr'\frac{ds'}{dt}=0.$$

Integrating, $mr's'=-\frac{1}{\mu}(1-m\mu r)s_1-\frac{1}{\nu}(1-m\nu r)s_2+$ some expression Θ independent of t.

Θ is periodic in θ, satisfies $\left(\mu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\left(\nu\frac{d^2}{d\theta^2}-\frac{d}{dt}\right)\Theta=0,$ and has a mean value zero. Hence $\Theta=0$, and

$$mr's'=-\frac{1}{\mu}(1-m\mu r)s_1-\frac{1}{\nu}(1-m\nu r)s_2.$$

In the same way the charge s induced in the central conductor by the distribution $s_1'+s_2'$ in the concentric conductor is given by the equations

$$\left\{\frac{d^2}{d\theta^2}-(m+n)r'\frac{d}{dt}\right\}(s_1'+s_2')+mr\frac{d}{dt}s=0,$$

$$\frac{d^2}{d\theta^2}s_1'=\frac{1}{\mu}\frac{d}{dt}s_1',\qquad\qquad \frac{d^2}{d\theta^2}s_2'=\frac{1}{\nu}\frac{d}{dt}s_2',$$

and
$$mrs=\frac{1}{\nu}(1-mr\nu)s_1'+\frac{1}{\mu}(1-mr\mu)s_2'.$$

§ 7. Suppose an alternating current $2\pi A \cos \omega t$ to be introduced into the central conductor at A where $\theta=0$, and a current $2\pi B \cos(\omega t+\phi)$ into the same conductor at B where $\theta=\pi$.

The charge at B on the central conductor due to the current introduced at A is, writing

$$\sqrt{\frac{\omega}{2\mu}} \text{ as } \beta, \text{ and } \sqrt{\frac{\omega}{2\nu}} \text{ as } \gamma,$$

$$\frac{A}{\omega}\frac{c_1}{c}\{\sin \omega t\, f_3(\beta) + \cos \omega t\, f_4(\beta)\}$$
$$+\frac{A}{\omega}\frac{c_2}{c}\{\sin \omega t\, f_3(\gamma) + \cos \omega t\, f_4(\gamma)\}.$$

The charge at B on the central conductor due to the current introduced at B is

$$\frac{B}{\omega}\frac{c_1}{c}\{\sin(\omega t+\phi) f_1(\beta) + \cos(\omega t+\phi) f_2(\beta)\}$$
$$+\frac{B}{\omega}\frac{c_2}{c}\{\sin(\omega t+\phi) f_1(\gamma) + \cos(\omega t+\phi) f_2(\gamma)\}.$$

The charge at B induced on the concentric conductor by the current introduced at A is

$$-\frac{1}{m\mu r'}(1-m\mu r)\frac{A}{\omega}\frac{c_1}{c}\{\sin \omega t\, f_3(\beta) + \cos \omega t\, f_4(\beta)\}$$
$$-\frac{1}{m\nu r'}(1-m\nu r)\frac{A}{\omega}\frac{c_2}{c}\{\sin \omega t\, f_3(\gamma) + \cos \omega t\, f_4(\gamma)\}.$$

The charge at B induced on the concentric conductor by the current introduced at B is

$$-\frac{1}{m\mu r'}(1-m\mu r)\frac{B}{\omega}\frac{c_1}{c}\{\sin(\omega t+\phi) f_1(\beta) + \cos(\omega t+\phi) f_2(\beta)\}$$
$$-\frac{1}{m\nu r'}(1-m\nu r)\frac{B}{\omega}\frac{c_2}{c}\{\sin(\omega t+\phi) f_1(\gamma) + \cos(\omega t+\phi) f_2(\gamma)\}.$$

The potential of the central conductor at the point B is $\frac{(m+n)s+ms'}{mn}$, where $s\ s'$ are the total charges on the two conductors respectively, and is

$$\mu r \frac{A}{\omega}\frac{c_1}{c}\{\sin \omega t\, f_3(\beta) + \cos \omega t\, f_4(\beta)\}$$

$$+ \nu r \frac{A}{\omega}\frac{c_2}{c}\{\sin \omega t\, f_3(\gamma) + \cos \omega t\, f_4(\gamma)\},$$

$$+ \mu r \frac{B}{\omega}\frac{c_1}{c}\{\sin(\omega t + \phi) f_1(\beta) + \cos(\omega t + \phi) f_2(\beta)\}$$

$$+ \nu r \frac{B}{\omega}\frac{c_2}{c}\{\sin(\omega t + \phi) f_1(\gamma) + \cos(\omega t + \phi) f_2(\gamma)\}.$$

If a known current $2\pi A \cos \omega t$ be introduced at A, and the point B be kept at zero potential, the equation of the last expression to zero for every value of t gives two equations to determine B and ϕ, the amplitude and phase of the current at the receiving end of the cable. These are

$$A\{\mu c_1 f_3(\beta) + \nu c_2 f_3(\gamma)\} + B[\mu c_1\{\cos\phi\, f_1(\beta) - \sin\phi\, f_2(\beta)\}$$
$$+ \nu c_2\{\cos\phi\, f_1(\gamma) - \sin\phi\, f_2(\gamma)\}] = 0;$$

$$A\{\mu c_1 f_4(\beta) + \nu c_2 f_4(\gamma)\} + B[\mu c_1\{\sin\phi\, f_1(\beta) + \cos\phi\, f_2(\beta)\}$$
$$+ \nu c_2\{\sin\phi\, f_1(\gamma) + \cos\phi\, f_2(\gamma)\}] = 0.$$

Similarly, if the currents had been introduced into the concentric conductor, the central one being insulated throughout, we should have for the potential of the concentric conductor at B, $\frac{1}{n}(s + s')$, which is exactly the same expression as for the central conductor substituting r' for r, $\frac{c_1'}{c}$ for $\frac{c_1}{c}$, and $\frac{c_2'}{c'}$ for $\frac{c_2}{c}$. Remembering that $\frac{c_1}{c} = \frac{c_2'}{c'}$ and $\frac{c_2}{c} = \frac{c_1'}{c'}$, it is seen that the equations for determining the phase and amplitude of the current received at B when that end of the concentric conductor is earthed are the same as in the case of the central conductor, only interchanging c_1 and c_2.

§ 8. So if an alternating current, $2\pi A \cos(\omega t + a)$, be introduced into the central conductor at A, and a current $2\pi B \cos(\omega t + b)$ pass out of the concentric conductor at B, the potentials of the conductors at A, B may be determined.

The potential of the central conductor at A is

$$\frac{1}{mn}\{(m + n)s + ms'\}$$

where s s' are the charges on the two conductors.

$$\text{This} = \frac{A}{\omega}\cdot\frac{\mu^2 r}{\mu-\nu}(1-m\nu r)\{\sin(\omega t+a)f_1(\beta)+\cos(\omega t+a)f_2(\beta)\}$$

$$+\frac{A}{\omega}\cdot\frac{\nu^2 r}{\nu-\mu}(1-m\mu r)\{\sin(\omega t+a)f_1(\gamma)+\cos(\omega t+a)f_2(\gamma)\}$$

$$-\frac{B}{\omega}\cdot\frac{\mu}{n(\mu-\nu)}\{\sin(\omega t+b)f_3(\beta)+\cos(\omega t+b)f_4(\beta)\}$$

$$-\frac{B}{\omega}\cdot\frac{\nu}{(\nu-\mu)n}\cdot\{\sin(\omega t+b)f_3(\gamma)+\cos(\omega t+b)f_4(\gamma)\}.$$

The potential of the concentric conductor at B is $\frac{1}{n}(s+s')$.

$$\text{This} = \frac{A}{\omega}\cdot\frac{\mu}{n(\mu-\nu)}\{\sin(\omega t+a)f_3(\beta)+\cos(\omega t+a)f_4(\beta)\}$$

$$+\frac{A}{\omega}\cdot\frac{\nu}{n(\nu-\mu)}\{\sin(\omega t+a)f_3(\gamma)+\cos(\omega t+a)f_4(\gamma)\}$$

$$+\frac{B}{\omega}\cdot\frac{\mu\nu r'}{\mu-\nu}(1-m\mu r)\{\sin(\omega t+b)f_1(\beta)+\cos(\omega t+b)f_2(\beta)\}$$

$$+\frac{B}{\omega}\cdot\frac{\mu\nu r'}{\nu-\mu}(1-m\nu r)\{\sin(\omega t+b)f_1(\gamma)+\cos(\omega t+b)f_2(\gamma)\}.$$

§ 9. Consider a system of circular conductors, of which fifteen are shown in fig. 4.

Of these Nos. 1, 5, 6, 10, 11, 15 have each a resistance $2\pi r$; they represent the central conductors of such concentric cables as we have been considering. Nos. 2, 4, 7, 9, 12, 14 have each a resistance $2\pi r'$. They represent the outer or concentric conductors. Nos. 3, 8, 13, have no resistance, and are maintained always at zero potential throughout. They represent the water or sheathing of the cables.

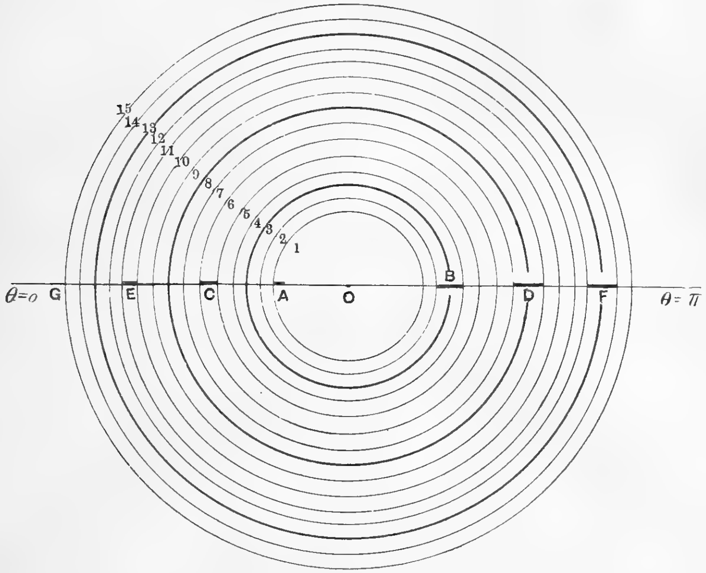
The capacity measured between each of the pairs 1, 2 : 4, 5 : 6, 7 : 9, 10 : 11, 12 : 14, 15 respectively is $2\pi c$, where $c=m$; and between the pairs 2, 3 : 3, 4 : 7, 8 : 8, 9 : 12, 13 : 13, 14 is $2\pi c'$, where $c'=n$. The capacity between the pairs 5, 6 : 10, 11 is zero.

At B conductors 2, 4 are connected by a piece having no resistance or capacity.

Similarly at C, conductors 5, 6 are connected; at D, 7, 9; at E, 10, 11; at F, 12, 14, and so on. The series may be indefinitely extended.

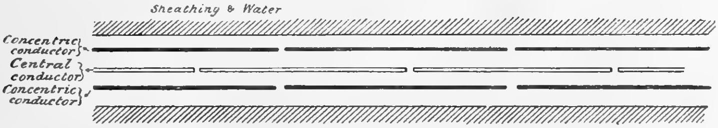
The system is electrically equivalent to a length of concentric cable of which the two conductors are cut alternately

Fig. 4.



at equidistant points in the way shown in fig. 5, representing a longitudinal section of the cable. If an alternating current be introduced into the system of fig. 4 at any link, say E, and the point A be kept at zero potential, alternating currents of the same period, but of different phases and amplitudes, will be found at every point of both conductors.

Fig. 5.



Suppose the currents at the points A, B, C, D, . . G to be

$$A \cos (\omega t + a),\ B \cos (\omega t + b),\ .\ .\ G (\cos \omega t + g),$$

so that the values of Gg being known, and the point A being kept at zero potential, the values of Aa, Bb, Cc, . . Ff are to be determined, and especially the values of Aa.

The potential at the point C on the conductor 5 has been expressed in terms of Cc, Bb; and the potential at the point C on the conductor 6 in terms of Cc, Dd. These two are equal for all values of t, and we have an equation which can be separated into two others between Bb, Cc, Dd. In the same way we obtain two equations at A, B, D, E, and F, and between these all the quantities required can be obtained.

In the particular case where ω has such a value that β and γ are integral, and the currents at all the points are in the same phase, let us write in § 8,

$$\frac{\mu^2 r}{\mu-\nu}(1-m\nu r)f_1(\beta)+\frac{\nu^2 r}{\nu-\mu}(1-m\mu r)f_1(\gamma) \text{ as P,}$$

$$\frac{\mu}{n(\mu-\nu)}f_3(\beta)+\frac{\nu}{n(\nu-\mu)}f_3(\gamma) \text{ as Q,}$$

$$\frac{\mu\nu r'}{\nu-\mu}(1-m\mu r)f_1(\beta)+\frac{\mu\nu r'}{\mu-\nu}(1-m\nu r)f_1(\gamma) \text{ as R,}$$

remembering that when β γ are integral, $f_1(\beta)=f_2(\beta)$, and $f_3(\beta)=f_4(\beta)$, and the same with γ.

The condition at A gives AP=BQ,
" " B " (A+C)Q=2BR,
" " C " (B+D)Q=2CP,
" " D " (C+E)Q=2DR, and so on.

If the known current is introduced at B, and A is kept to earth, the ratio of the amplitude of the received current to the sent current is given by

$$\frac{B}{A}=\frac{P}{Q} \quad . \; . \; . \; . \; . \; . \; . \quad \text{Case I.}$$

If the known current is introduced at C, we have

$$\frac{C}{A}=2\frac{PR}{Q^2}-1 \quad . \; . \; . \; . \; . \quad \text{Case II.}$$

If the known current is introduced at E, we have

$$\frac{E}{A}=8\left(\frac{PR}{Q^2}\right)^2-8\frac{PR}{Q^2}+1 \quad . \; . \quad \text{Case III.}$$

If the known current is introduced at G, we have

$$\frac{G}{A}=32\left(\frac{PR}{Q^2}\right)^3-48\left(\frac{PR}{Q^2}\right)^2+18\frac{PR}{Q^2}-1 \quad . \quad \text{Case IV.}$$

§ 10. Applying these results to a cable having the following constants, represented by half the circle of fig. 4:

Length 4000 kilometres $= 4.10^8$ units.

Resistance of each conductor, 5 ohms per kilometre, *i.e.* 2.10^{13} units for the whole cable.

Capacity measured between the conductors, $\frac{5+\sqrt{7}}{18}10^{-15}$ units per kilometre (about ·425 mfds.), *i.e.* $\frac{5+\sqrt{7}}{18}4.10^{-12}$ units for the whole cable.

Capacity measured between the outer conductor and the sheathing, $\frac{5-\sqrt{7}}{9}10^{-15}$ units per kilometre (about ·26 mfds.), *i.e.* $\frac{5-\sqrt{7}}{9}.4.10^{-12}$ units for the whole cable.

Then

$$m=\frac{2}{\pi}\frac{5+\sqrt{7}}{9}10^{-12}, \qquad n=\frac{4}{\pi}\frac{5-\sqrt{7}}{9}10^{-12},$$

$$r=r'=\frac{2}{\pi}10^{13}, \qquad \mu=\frac{\pi^2}{80}, \qquad \nu=\frac{9\pi^2}{80},$$

$$\frac{c_1}{c}=\frac{c_2'}{c'}=\frac{3+\sqrt{7}}{16}, \qquad \frac{c_2}{c}=\frac{c_1'}{c'}=\frac{13-\sqrt{7}}{16}.$$

Consider the cable in four cases :—

Case I. when both conductors are continuous.

Case II. when the inner conductor is cut in the middle so that the cable is in two sections.

Case III. when the inner conductor is cut in two places, and the outer in one, so that the cable is in four sections.

Case IV. when the inner conductor is cut in three places, and the outer in two, so that the cable is in six sections.

Take a value of $\omega.\frac{9.144.\pi^2}{80}$ which makes $\beta.\gamma$ integral in all cases. This corresponds to about 25 alternations per second.

Then if the cable be divided into n equal sections the new values of $rr'mn\beta\gamma$ are inversely proportional to n, while the values of μ, ν are proportional to the square of n. The value of $\frac{\mathrm{PR}}{\mathrm{Q}^2}$ is unchanged in every case.

In Case I. $\beta=30\ \gamma=12$, and $A=1\cdot26\,e^{-12\pi}B$;
Case II. $\beta=18\ \gamma=\ 6$, and $A=1\cdot07\,e^{-12\pi}C$;
Case III. $\beta=\ 9\ \gamma=\ 3$, and $A=\ \cdot57\,e^{-12\pi}E$;
Case IV. $\beta=\ 6\ \gamma=\ 2$, and $A=\ \cdot31\,e^{-12\pi}G$.

Thus the effect of each successive subdivision is to further reduce the amplitude of the received current.

§11. Returning to the equation of § 2 for determining $\mu\nu$, viz. :—

$$mnrr'x^2-x\{\overline{m+n}\,\nu'+m\nu\}+1=0,$$

write in this

$$\frac{1}{r}=\gamma,\quad \frac{1}{r'}=\gamma',\quad \frac{1}{m}=k,\quad \frac{1}{n}=k',\quad \text{and } \gamma+\gamma'=\Gamma,$$

so that γ, γ' are the conductivities of the two conductors, k, k' the rigidities to electrostatic stress of the two dielectrics, and Γ the total conductivity of the two conductors.

The equation becomes

$$x^2-x\{\gamma k+\Gamma k'\}+\gamma\gamma'kk'=0,$$

and the roots of this equation represent the signalling speeds of the two equivalent cables.

If k be very small, $x=\Gamma k'$ or 0. Hence if the layer of dielectric between the conductors be very thin, though at the same time perfectly insulating, the speed through the central conductor, however small its section may be, is the same as if the whole of the two conductors were solid, and the whole used for the conducting circuit.

IX. *The Formation of Mercury Films by an Electrical Process.*

By Rollo Appleyard[*].

If a sheet of gelatine, damp leather, or similar permeable substance is used as a separating diaphragm between two bodies of mercury, and a current is sent through it, a film of mercury is deposited upon the surface connected to the positive pole of the battery. And the film remains on the diaphragm after removal from the apparatus.

If the diaphragm is replaced in the apparatus and subjected to a current in the reverse direction, the film vanishes from that surface, and a second film appears on the other side. That is to say, the film is always on the side of the diaphragm connected to the positive pole of the battery, and there is no film on the negative surface.

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[*] Communicated by the Physical Society: read May 14, 1897.

In this way I have deposited mercury-films upon such different substances as filter-paper, plaster-of-paris, Woodbury-type-gelatine, porous earthenware, asbestos-paper, and sheep's skin. As a rule, the only preparation of the substances is damping with distilled water. Asbestos mill-board is one of the best materials to operate upon.

The coating of mercury formed by this electrical process is far more perfect than any I have been able to obtain by the methods of "pickling" or contact-pressure. I have no definite views as to the cause of the phenomenon, but I am inclined to regard it as a secondary action, the result of electrolysis, aided to some degree by electric osmosis. The experiments of Mr. C. K. Falkenstein* upon the electric tanning of leather are evidence in favour of the osmosis theory. I understand also that electric osmosis has lately been used for administering cocaine, replacing the ordinary hypodermic injections.

It will be remembered that in some experiments† I recently described to the Physical Society a globule of mercury, electrified in a partially conducting liquid, sent forth a tentacle in a direction towards the positive electrode. It is perhaps significant that in the present experiments the film is always on the positive surface. In the case of liquid "coherers" it is possible to conceive that, under the influence of steady or transitory currents, mobile conductors attempt to creep in this fashion through the dielectric films. And, again, though the effect is probably very slight and entirely masked by electrolytic disturbances, sea-water may thus penetrate the dielectric at a "fault" in a cable. The film-forming process may be of practical utility where large surfaces are required to be coated with mercury, as in gold extraction; and possibly in the preparation of plates for primary or secondary batteries.

The following experiments are given in illustration of the action :—

(1) A sheet of filter-paper, damped with distilled water, is folded upon itself four or five times. It is then laid flat upon a conducting surface. An indiarubber ring is pressed down on the folded paper, and mercury is poured in to fill the ring. A battery of 200 volts is then connected for about three minutes between the mercury and the lower conducting plate, the mercury being positive. After stopping the current the ring is removed from the filter-paper and superfluous mer-

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* Journ. Inst. Elec. Engineers, vol. xxii. p. 225 (1893).

† "Liquid Coherers and Mobile Conductors," Phil. Mag. May 1897.

cury is poured off, leaving a circular film upon the upper surface of the paper. On unfolding the sheet, a blackish substance is found distributed through it, graduated from the positive towards the negative pole.

(2) If a sheet of tinfoil is included within the folds of filter-paper, while the battery is applied, the tinfoil becomes perforated with pin-holes. This, however, must not be taken *per se* as a proof of any direct transition of metallic mercury through the diaphragm; for if, instead of the mercury contained in the ring, a sheet of metal is used as the upper electrode, pinholes still appear in a sheet of tinfoil included in the folds of damp filter-paper. Moreover, if a sheet of tinfoil is used as the upper electrode, this also becomes perforated.

(3) Again, if the sheets of tinfoil are removed, and a gold coin is used for the upper (positive) electrode, upon the top of the damp filter-paper, the current produces a gold discoloration which penetrates the folds. I have not yet had time to examine this result, but it seems to throw some light upon the formation of metallic lodes and veins. And although the discoloration is the effect of a steady current, and not sparks, it may help to explain the "inductoscripts"* of Mr. F. J. Smith.

X. *On the Isothermals of Isopentane.*

By J. Rose-Innes, *M.A., B.Sc.*†

THE recent publication by Prof. Sydney Young of a long series of researches on isopentane (Proc. Phys. Soc. Session 1894–95, pp. 602–657) offers a great opportunity to those interested in the theory of gases. Since isopentane is a saturated hydrocarbon, there seems to be a reasonable hope that we are here dealing with a substance which will not tend to form complex molecules at low volumes, and whose behaviour may therefore be treated as normal; and this fact, together with the wide range of volume over which the experiments have been conducted, renders Prof. Young's results well fitted to test the various formulæ that have been from time to time proposed. Among these formulæ there is none more important than that formerly suggested by Prof. Young himself, in conjunction with Prof. Ramsay, that the pressure of a gas kept at constant volume is a linear function of the temperature; and the bearing of the experimental results

* "Inductoscripts," by F. J. Smith, Proc. Phys. Soc. vol. xi. p. 353 (1892).

† Communicated by the Physical Society: read May 28, 1897.

with isopentane upon this formula is fully considered by Prof. Young in his paper. Accepting the formula, and writing it as

$$p = b\mathrm{T} - a,$$

where b and a are functions of the volume only, the values of b and a for a large number of volumes are given in the paper (*loc. cit.* pp. 650–655), and they are sufficiently numerous to enable us to fully test any algebraic expression that endeavours to represent them. I spent a considerable amount of time examining the values of a and b, testing the formulæ that have been proposed by various physicists, as well as others of my own devising, without arriving at any that gave complete satisfaction; and it occurred to me afterwards that possibly more definite results could be secured by examining some physical quantity which depended upon both a and b than by examining b and a separately by themselves. Thus if we accept Ramsay and Young's linear law, there will be one and only one temperature for each volume at which the gas has its pressure equal to that given by the laws of a perfect gas. In effect, if we put

$$p = b\mathrm{T} - a$$

as giving the actual pressure, we may also write

$$p = \frac{\mathrm{RT}}{v} + \left(b - \frac{\mathrm{R}}{v}\right)\mathrm{T} - a,$$

and this shows that $p = \frac{\mathrm{RT}}{v}$ if we take $\mathrm{T} = \frac{a}{b - \mathrm{R}/v}$. Call the temperature so found τ; the values of τ have been calculated and are given in the following table (p. 78); the results above vol. 400 are not included, as they vary within such wide limits owing to experimental errors.

In calculating this table, the value of R was taken $= \frac{1}{\cdot 001158}$.

An examination of the table shows that the temperature in question is very much the same for all large volumes down to about vol. 8. Of course the actual numbers obtained vary a good deal, but these variations are without method, sometimes in one direction and sometimes in another, and when the numbers are plotted against $v^{-\frac{1}{3}}$ it appears to me impossible to tell from an inspection of the diagram whether the value above vol. 8 is on the whole increasing or decreasing. These variations may therefore be attributed to experimental error; and they may be to a large extent got rid of by employing "smoothed" values of b and a, as was done in Ramsay and

Volume.	τ.	Volume.	τ.	Volume.	τ.
350	851·9	22	835·8	4·3	804·6
300	835·4	20	845·9	4·0	799·1
260	820·8	19	854·7	3·8	792·3
230	825·6	18	848·7	3·6	777·3
200	969·6	17	850·9	3·4	764·0
180	935·6	16	845·3	3·2	737·2
160	846·7	15	854·1	3·0	715·5
140	787·0	14	848·9	2·9	696·5
120	808·9	13	847·8	2·8	675·9
100	804·1	12	844·1	2·7	656·4
90	793·8	11	840·1	2·6	637·7
80	773·4	10	843·5	2·5	615·2
70	854·7	9·5	846·7	2·4	593·7
60	876·2	9	838·3	2·3	565·9
50	871·6	8·5	834·5	2·2	541·1
45	853·1	8	833·7	2·1	511·2
40	816·2	7·5	833·2	2·0	478·7
36	832·8	7	832·2	1·9	446·3
33	820·0	6·5	828·1	1·8	408·6
30	833·7	6	827·2	1·7	367·2
28	835·1	5·5	819·8	1·6	318·4
26	846·7	5	813·7		
24	848·0	4·6	809·5		

Young's paper on ether (Phil. Mag. xxiii. p. 441). Prof. Young has not given any smoothed values of b for isopentane, however, and I have thought it better not to use my own smoothed values, so as to employ numbers that are above the suspicion of bias towards any particular theory.

The mean value of τ for all volumes above 8 is 842·4; and this number could be taken as the true value for all such volumes without introducing any serious error. Below vol. 8, however, the value of τ distinctly diminishes with the volume; and by the time the critical point is reached, τ is not more than about 804. It is not easy to find a formula for τ which will give the experimental results throughout the whole range of volume; but whatever may turn out to be its proper algebraic expression, there appears to be little doubt that its initial decrease from the maximum value varies as a higher power of the density than the first.

The result appears to me to be of importance, since it is by itself sufficient to reveal the inadequacy for isopentane of many of the gas-formulæ that have been proposed. We have seen that

$$\tau = \frac{av}{bv - \mathrm{R}}.$$

Suppose that we can put

$$b=\frac{\mathrm{R}}{v}+\frac{\mathrm{B}_2}{v^2}+\frac{\mathrm{B}_3}{v^3}+\text{ higher powers of }\frac{1}{v},$$

$$a=\frac{\mathrm{A}_2}{v^2}+\frac{\mathrm{A}_3}{v^3}+\text{ higher powers of }\frac{1}{v};$$

Then

$$\tau=\frac{\mathrm{A}_2+\mathrm{A}_3v^{-1}+\ldots}{\mathrm{B}_2+\mathrm{B}_3v^{-1}+\ldots},$$

$$=\frac{\mathrm{A}_2}{\mathrm{B}_2}+\frac{\mathrm{A}_3\mathrm{B}_2-\mathrm{A}_2\mathrm{B}_3}{\mathrm{B}_2{}^2}v^{-1}+\text{higher powers of }\frac{1}{v}.$$

If b and a in any gas-formula can be expanded in powers of v^{-1} as assumed, and if $\mathrm{A}_3\mathrm{B}_2-\mathrm{A}_2\mathrm{B}_3$ is different from zero, which will in general happen, the decrease of τ from the maximum value will be as v^{-1}; so that the proposed formula is inadmissible.

Let us now return to the quantities a and b. The values of $\frac{1}{av^2}$ were plotted against $v^{-\frac{1}{3}}$ by Prof. Young, and the resulting diagram is given in his paper (*loc. cit.* p. 653). The curve determined by the points plotted would seem to be of a somewhat complicated character, and I do not think it possible to obtain any simple formula that will reproduce it entirely. On the other hand, there seems to be considerable evidence of discontinuity in the neighbourhood of vol. 3·4; and even if there is not discontinuity in the true mathematical sense of the word, there appears to be such a rapid alteration of behaviour as to amount in practice to the same thing. The easiest plan is to treat the curve as consisting of two parts, the formula passing abruptly from one expression to another somewhere in the neighbourhood of vol. 3·4. Whether this accurately represents what takes place in nature is uncertain, but there is no doubt that it immensely simplifies the problem. We may therefore confine our attention to volumes above 3·4; and of the formulæ already proposed I found the best to be that suggested by Mr. W. Sutherland, who has given a gas-formula equivalent to putting

$$a=\frac{l}{v(v+k)},$$

where l and k are two constants characteristic of the gas (Phil. Mag. xxxv. p. 215). I have taken the following values of the constants

$$l=5{,}420{,}800,\qquad k=3\cdot636.$$

The next step was to find a formula for b. Now

$$b = \frac{\mathrm{R}}{v} + \frac{a}{\tau},$$

and the conditions already discovered with respect to the algebraic expression for τ considerably restrict the field of research; this is an advantage, as it lessens the amount of arithmetical work to be performed. I found that fairly good results could be obtained by putting

$$b = \frac{\mathrm{R}}{v}\left(1 + \frac{e}{v + k - \frac{g}{v^2}}\right),$$

where R and k have the values already given and

$$e = 7{\cdot}473, \qquad g = 6{\cdot}2318.$$

By combining the formulæ given above for a and for b, we obtain as the formula for the isothermals

$$p = \frac{\mathrm{RT}}{v}\left(1 + \frac{e}{v + k - \frac{g}{v^2}}\right) - \frac{l}{v + k},$$

where R, e, k, g, and l are constants, and have the values already given.

In order to test this formula it is desirable to draw a system of isothermals, but if this be done in the ordinary Andrews' diagram the result is not satisfactory, as the range of p is so large. It was found possible to obtain a good diagram, however, by calculating pv and plotting it against $v^{-\frac{1}{3}}$; the calculated isothermals are shown as continuous lines, while the experimental values are put in as dots. It will be seen that there is a fair agreement between calculation and experiment down to about vol. 3·4. Below this volume there is no longer any agreement; we should naturally expect such a result, since the formula for a admittedly holds up to the neighbourhood of vol. 3·4 only. Therefore, even if the calculated isothermal were to agree with the found isothermal below vol. 3·4 for some one temperature, this would only happen by a compensation of errors, and could not occur for any second temperature.

It will be noticed that in the neighbourhood of vol. 16 there is a sensible divergence between the calculated and found isothermals amounting to slightly over 1 per cent. This divergence is certainly unsatisfactory as far as it goes;

but differences just as great have occurred in the past in inquiries of this kind between the results of independent observers experimenting on the same substance. For the present, then, and pending the confirmation of Prof. Young's results by some other observer, we may take the simple formula given above as representing all that we certainly know concerning the behaviour of isopentane under the conditions of volume specified.

The formula proposed may be employed to calculate the critical constants; this may be done by a method depending solely on Ramsay and Young's linear law. Let us take the equation

$$p=b\mathrm{T}-a,$$

and differentiate it with respect to v, keeping T constant,

$$\frac{dp}{dv}=\mathrm{T}\frac{db}{dv}-\frac{da}{dv}.$$

Differentiate again

$$\frac{d^2p}{dv^2}=\mathrm{T}\frac{d^2b}{dv^2}-\frac{d^2a}{dv^2}.$$

At the critical point $\frac{dp}{dv}$ and $\frac{d^2p}{dv^2}$ vanish together, so we have

$$\mathrm{T}\frac{db}{dv}-\frac{da}{dv}=0,$$

$$\mathrm{T}\frac{d^2b}{dv^2}-\frac{d^2a}{dv^2}=0.$$

Eliminating T this gives

$$\frac{db}{dv}\frac{d^2a}{dv^2}-\frac{d^2b}{dv^2}\frac{da}{dv}=0.$$

When v has been calculated from this equation we may obtain T by putting

$$\mathrm{T}=\frac{da}{dv}\div\frac{db}{dv},$$

and when v and T are known the original isothermal equation will give p.

In this way we may obtain the following results:—

$$\begin{cases}\text{critical volume }=4\cdot5,\\ \text{critical temperature}=191^\circ\cdot7\text{ C.},\\ \text{critical pressure}=26250,\end{cases}$$

which agree fairly well with the numbers found experimentally by Prof. Young.

There is one of the isothermals in the set determined by our formula which is of peculiar interest. The biggest value of τ occurs when v is indefinitely large, its value being then $\frac{l}{\mathrm{R}e}$. Let us call this quantity τ_0 for shortness, and let us see what form the isothermal for this particular temperature takes. We shall easily obtain

$$p=\frac{\mathrm{R}\tau_0}{v}+\frac{lg}{v^3(v+k-\frac{g}{v^2})(v+k)},$$

so that the pressure for a long range of volume is practically given by Boyle's law; and when an increase does occur, it varies as the *fifth* power of the density. In the case of isopentane this is merely a deduction from theory, since the temperature required is 567° C.; much too high to allow of any experiments being carried out for verification. Indeed, for most other substances, the temperature of minimum deviation from Boyle's law, being much higher than the critical temperature, is much too high to allow of any experiments being performed there. An exception may be made, however, in the case of nitrogen, which has a very low critical temperature. The following are the values of pv at 16° C. for nitrogen under different pressures determined by M. Amagat ("Mémoires sur l'élasticité et la dilatabilité des Fluides jusq'aux très-hautes Pressions," *Ann. Chim. Phys.* 6th series, vol. xxix. p. 107).

Pressure in metres of mercury.	*pv.*	Pressure in metres of mercury.	*pv.*
·76	1·0000	45	·9895
20	·9930	50	·9897
25	·9919	55	·9902
30	·9908	60	·9908
35	·9899	65	·9913
40	·9896		

From this table it is evident that at 16° C. for nitrogen we are only a short way off the temperature of minimum deviation from Boyle's law. At temperatures higher than the temperature of minimum deviation the gas would have pv increasing with pressure from the first, and this is what happens with hydrogen at ordinary temperatures.

XI. *On the Conditions which Determine the Ultimate Optical Efficiency of Methods for Observing Small Rotations, and on a Simple Method of Doubling the Accuracy of the Mirror and Scale Method.* *By* F. L. O. Wadsworth*.

In a paper on the "Optical Comparison of Methods for Observing Small Rotations," Rayleigh discussed some years ago the relative limiting accuracy of the two methods then commonly in use for observing small angular deflexions of a suspended system ; *i. e.*, the Gaussian method of scale and mirror, and the method of the pointer and microscope. The conclusion was reached that theoretically the two systems were on a par with each other when the length of the pointer was equal to the diameter of the mirror†.

It would therefore seem that the latter method has a considerable advantage over the former, since for a given weight, or, what is more important, for a given moment of inertia, it is possible to make a pointer of a length of at least ten times the diameter of a mirror, and thereby attain with a given time of swing a tenfold greater sensitiveness. But notwithstanding this apparent theoretical superiority of the microscope and pointer, the mirror is in practice nearly always preferred, not only because of its greater convenience, but also because experiment has shown that it is *really not so inferior in accuracy* as the above comparison would lead one to expect. The reason for this is that the conditions assumed by Lord Rayleigh as a basis for this comparison do not always hold. Let us consider first what these conditions are. In the case of a revolving mirror Rayleigh assumed that the smallest angular motion which would be perceptible through its effect on a reflected image would be (supposing the mirror rectangular and the incidence nearly normal) such that "one edge of the mirror advances $\frac{1}{4}\lambda$ (while the other edge retreats to a like amount)," and thus "introduces a phase discrepancy of a whole period where before the rotation there was complete agreement." This gives us for the limiting angular motion α_r the expression

$$\alpha_r = \frac{\lambda}{2b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where b is the length of the mirror perpendicular to the axis

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* Communicated by the Author.

† "Optical Comparison of Methods for Observing Small Rotations," Lord Rayleigh, Phil. Mag., Oct. 1885. See also art. "Wave Theory," *Enc. Brit.* vol. xxiv. § 13.

of rotation, and λ the mean wave-length of light. It is readily seen that this movement is just that required to separate the superposed images of two lines, one of which is fixed, the other seen by reflexion in the mirror, by the amount necessary for "resolution"*.

The corresponding expression for the microscope is deduced on the assumption that the smallest linear movement that can be observed with this instrument is also equal to the limit of resolving power. This, according to the researches of Helmholtz †, is

$$\epsilon = \tfrac{1}{2}\lambda\,‡. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The angular movement corresponding to this linear movement at the end of a pointer of length l is

$$\alpha_m = \epsilon / l = \frac{\lambda}{2l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which at once follows the conclusion already stated, that the two methods are equally accurate when

$$l = b.$$

It will be noticed in the first place that both of these expressions are based on the assumption that the smallest measurable angle or distance is the same as the corresponding angular or linear resolution. But in general there is a difference between these two quantities. The smallest measurable quantity depends not on the resolution of one object from another, but on the accuracy with which one object can be located with respect to another. According to Michelson this "accuracy" for any optical instrument is directly proportional to the resolving power, and is therefore represented by an expression

$$\mathrm{A} = a\mathrm{R},$$

where A represents the accuracy (the reciprocal of the smallest length or angle that can be detected); R is the resolution (the reciprocal of the angular or linear resolving

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* The term "resolving power" has been erroneously applied by some to the expression (1). In its commonly accepted meaning the term resolving power refers simply to the capacity of an optical system to separate two or more lines or points whose positions with respect to each other are fixed. A mirror, therefore, whether used alone or with a telescope, has exactly the same resolving power as a telescope of the same aperture.

† Pogg. *Ann.* Jubelband, 1874. See also paper by Rayleigh, "On the Theory of Optical Images, with Special Reference to the Microscope," Phil. Mag. Aug. 1896.

‡ In Rayleigh's first paper this limit is given, probably through a typographical error, as $\frac{1}{4}\lambda$ instead of $\frac{1}{2}\lambda$.

power); and a is a constant whose value varies with different instruments, but is, according to Michelson, principally dependent upon the magnification of the object whose position is to be determined in comparison to the reference object[*].

If the magnification is so high that the diffraction-fringes about the measured object are much broader than the reference object (usually a cross-wire in the eyepiece of the observing instrument), the value of a may be as large as 100. Under practical conditions, however, no such degree of magnification can be attained, for reasons which will be presently considered, and the value of a will be correspondingly lower. For the microscope Michelson fixes the limiting value of a at 5 (from experiments of Rogers and Morley), corresponding to an error of about $\frac{1}{10}\lambda$, and at 4 in the case of the telescope, or 8 in the case of the revolving mirror, corresponding to an error in angle of $0''{\cdot}05$ for a mirror of 25 centim. aperture. In a later paper[†] the same author assumes a slightly higher value as the limiting value of a (*i. e.*, 10) in the case of the mirror.

In a former paper describing some results of some work on the testing of straight edges[‡], I have pointed out that with a mirror of 4 centim. aperture an angular movement of $0''{\cdot}15$ could be detected, and one of not more than $0''{\cdot}2$ to $0''{\cdot}3$ measured with certainty. From (1) we find for α_r for a 4 centim. aperture

$$\alpha_r = \frac{\lambda = \text{about } 0{\cdot}00055}{80} \simeq 1''{\cdot}5.$$

The two corresponding values of a in this case are therefore 20 and 10. More recently Boys, in his great memoir on the Newtonian Constant, has also pointed out that with his apparatus (which had a mirror of about 2·3 centim. aperture) the accuracy of measurement was at least seven times the resolution, or nearly four times the theoretical measuring power[§] as given by Rayleigh. Boys, however, ascribes this

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[*] "Measurement of Light Waves," A. Michelson, Amer. Journ. Science, vol. xxxix. p. 115.

[†] "Light Waves and their Application to Metrology," 'Nature,' vol. xlix. p. 56, Nov. 16, 1893.

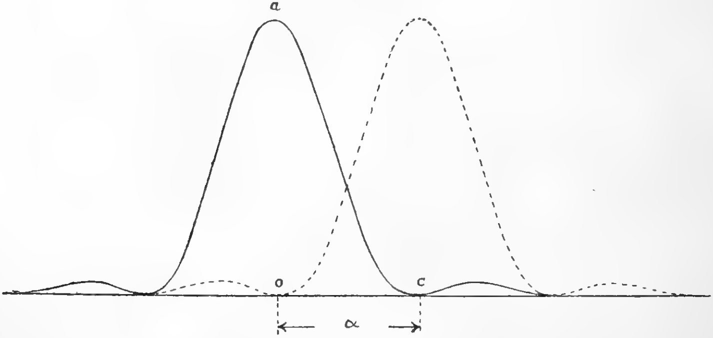
[‡] "The Manufacture of Very Accurate Straight Edges," Journal Franklin Inst., vol. cxxxviii. p. 1, July 1894; 'American Machinist,' Aug. 23rd, 1894.

[§] "On the Newtonian Constant of Gravitation," Phil. Trans. vol. clxxxvi. A, pp. 1–72 (1895). See particularly p. 33, and also review of this memoir by the writer, in the 'Astrophysical Journal' for April 1896, vol. iii. p. 303. Prof. Boys states that this degree of accuracy could easily have been exceeded if the divisions on his scale had been finer.

to the *narrowness* of the diffraction images of the lines of his scale [*]; a condition exactly the reverse of that assumed by Michelson as necessary for accuracy of measurement. To explain this apparent contradiction we must consider a little more closely just how angular measurements of position are usually made. As a rule the angular motion is determined by observing the motion of the image of a point either actually moving with the system (microscope and pointer), or apparently moving with it [†] (mirror and scale), with reference to some fixed point. The image and fixed point are examined either with the eye directly or with the aid of an eyepiece. In the first case the reference-point is a line or series of lines engraved on a screen upon which the moving image is projected; in the second case it is generally a cross-wire placed in the focus of the observing eyepiece. In either case the object examined is, when the highest degree of accuracy is aimed at, a narrow line whose image is represented at the focal plane of the mirror or objective (rectangular aperture) by the diffraction pattern of fig. 1.

The problem is to determine the position of the centre of this pattern with reference to a fixed line (or wire) parallel to the axis of symmetry *ao*. When the relative position is judged by means of the eye it makes no difference whether the image is thrown on a ruled screen or seen through an eyepiece. In either case the image of the diffraction pattern, as formed on the retina, is another diffraction pattern similar

Fig. 1.



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[*] The scale having black lines on a very brightly illuminated field, the effect of diffraction would be to make the images of the lines appear narrower than their geometrical images.

[†] It does not matter which, for it is the motion of the image only with which we are concerned.

in every respect to the first save that it is broadened in the inverse ratio of the aperture of the eye to the aperture of the mirror or objective; the only effect of interposing the latter* being to change the apparent magnitude of an object of finite dimensions or the apparent brightness of point sources (such as stars). The retinal image of the reference-line or cross-wire will also be a diffraction pattern which will be similar to that of the line under examination. These two patterns will be of the same width when the angular width of the line itself as viewed from the objective (or mirror) is the same as the angular diameter of the cross-wire as viewed from the eye-lens, and when the aperture of this latter is just sufficient to admit the full cone of light from the first aperture. Under these circumstances the conditions are exactly the same as they would be if we were looking through any instrument at two parallel lines of the same width, except that when the observing instrument is moved only one of the images at the focal plane changes its position with reference to the axis of the instrument. In considering the limiting degree of accuracy with which the position of the (apparently) moving image can be determined with reference to the other we must distinguish between at least three cases.

I. Suppose both reference-line and object are of the same intensity (both light or both dark) and are initially superposed. It is obvious that no motion of one image with respect to the other will be evident until the separation is at least as great as that required for the "resolution" of a double line. For lines of negligible width viewed through a rectangular aperture this separation (angular) is, as is well known†,

$$\Sigma=\frac{\lambda}{b}=\alpha.$$

In this case therefore the smallest angular movements observable with a mirror and with a microscope and pointer are given by expressions (1) and (3) as assumed by Rayleigh.

For lines of finite width the smallest angular separation at which resolution occurs is given by the expression ‡

$$\Sigma=\sigma+\frac{\alpha^2}{2\sigma+\alpha},$$

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* Provided only the angular aperture of the eyepiece lens (and the pupil of the eye) is the same viewed from the focal plane as that of the objective or image-forming lens; and provided also the magnifying power of the eyepiece be not too high (see paper by Helmholtz already referred to, Pogg. *Ann.* 1874).

† See Rayleigh, "Wave Theory," *Enc. Brit.* vol. xxiv.

‡ See paper "On the Resolving Power of Telescopes and Spectroscopes for Lines of Finite Width," Phil. Mag. vol. xliii. p. 317, May 1897.

where σ is angular width of the lines and α, as just defined, the resolving power of the aperture through which they are viewed. The value of Σ is slightly less for very small values of σ than for $\sigma=0$. Thus the minimum value of Σ occurs when

$$\sigma=\frac{\alpha}{2(\sqrt{2}+1)}\simeq\tfrac{1}{5}\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For this value of σ

$$\Sigma\simeq 0\cdot 9\alpha.$$

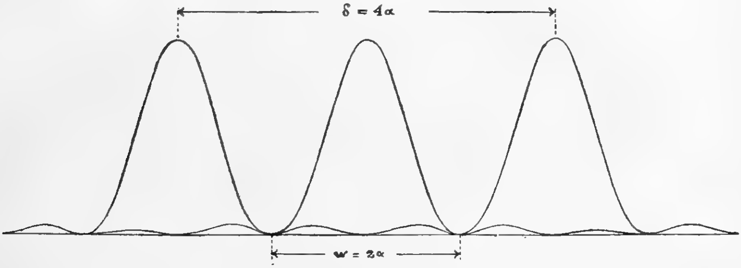
It is therefore an advantage that the cross-wire and reference-lines have a certain small finite width, not only on account of greater distinctness but also because, as long as the width does not exceed the value given in (4), they are more easily resolved, *i. e.*, a smaller angular movement of one with respect to the other can be measured.

II. Suppose as before that both objects are of the same character but are not superposed. The smallest relative movement that can be detected in this case depends upon a number of things, amongst others the initial distance between the centre O of the image and of the wire or reference-line ; the width of the image and wire ; and to a greater extent than in the preceding case on the ability of the observer. In any case of this kind it adds greatly to the accuracy of the determination to have either two reference-wires, one on each side of the image, or two images moving together on each side of the single fixed reference-wire. The first condition corresponds to the case of locating the position of one engraved line on a bar (or narrow pointer) in a microscope by means of the double (parallel) cross-wires ; the second to the case of locating the lines of a scale seen by reflexion from the revolving mirror with reference to a single fixed cross-wire in the eyepiece of the observing telescope. In either case, if the central object is nearly midway between the outer two, and if these latter are sufficiently far apart in comparison with the width of the former, it will be possible to estimate a shift of one system with respect to the other of from $\frac{1}{10}$ to $\frac{1}{50}$ the whole interval δ between the two outer objects. The ultimate accuracy therefore depends directly on the product of two factors, the absolute interval δ, and the percentage error of subdividing this interval, which we will call Δ. As δ increases Δ diminishes, at first slowly, then more rapidly, and then more slowly again. The product of δ and Δ will therefore be larger for either a very small or a very large value of δ than it will be for an intermediate value. That is, it is important not to make the distance either between the double cross-wires or the lines on the scale too small, nor on the other hand too large. When we know the law of variation of Δ with δ we can determine exactly what separation would give the best result in a given case. Let us suppose as before that the conditions are such that the final (retinal) images of object and reference-wire have the same width. Then when δ is zero the two outer images merge into one and we have the case previously considered, in which we found that the smallest observable movement under the most favourable circumstances was that which produced a separation

$$\Sigma = 0{\cdot}9\alpha.$$

Suppose next that the two outer objects are separated by an amount equal to about twice their own width, as in fig. 2.

Fig. 2.



The central image, if midway between the outer two, would be clearly separated from each of them, and it would be easy to determine its position to within, say, $\frac{1}{5}$ the whole interval δ, *i. e.*, to within about $0{\cdot}8\,\alpha$, a somewhat higher degree of accuracy than before. When the separation δ is made greater (say five times the width (2α) of the images, as in fig. 3), experience shows that it is possible to locate the central object to within $\frac{1}{20}$ to $\frac{1}{50}$ the interval, *i. e.*, to within $\frac{1}{2}$ to $\frac{1}{5}$ the resolving power of the instrument*. With still greater separation the accuracy may be slightly, but only slightly, increased. It is obvious that the narrower the

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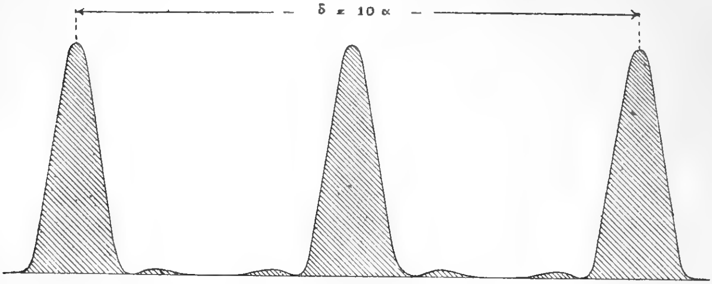
* This is on the assumption that the position can be located by micrometric methods, one object being made to bisect the interval between the other two. If simple eye estimates only can be made, the accuracy will be less, probably at most not more than $\frac{1}{20}$ the separation or $\frac{1}{4}$ the width of the image. In such a case better results would generally be secured by making δ smaller.

diffraction images the greater the attainable accuracy, because a given percentage error of subdivision Δ may be attained

Fig. 3.



with a smaller absolute value of δ. Hence in this case the conditions assumed by Boys, *i.e.* dark images on a bright field, are more favourable to accuracy of reading (because the images *appear* narrower) than the reverse arrangement. Under the most favourable circumstances (see experimental results quoted on p. 85) the smallest observable movement of one image with respect to the other seems to be between $\frac{1}{5}$ and $\frac{1}{8}$ the limit of resolving power. This corresponds to a linear movement of about $\frac{1}{10}\lambda$ in the case of the microscope, and to an angular movement of from $\frac{1}{10}\frac{\lambda}{b}$ to $\frac{1}{5}\frac{\lambda}{b}$ in the case of the mirror.

III. Suppose the cross-wire and the image unlike in character—one light, the other dark. In this case the accuracy in setting will *increase* with the width of the image with respect to the reference-wire (Michelson) provided the former is bright enough to be distinctly seen. If the width is great enough the position of the image with respect to the wire can be determined to within one per cent., *i.e.* the accuracy may be 50 times the resolution of the instrument. But from what has already been said (p. 87) it is evident that the only method of broadening the image of a narrow line, as seen through a telescope or microscope, with reference to the image of the cross-wire as seen through the eyepiece alone, is to make the effective angular aperture of the telescope or microscope objective much smaller than that of the eyepiece. This entails a loss not only in contrast between the two images, but also (if carried too far) a decrease in the resolving power of the instrument; for, as Rayleigh has pointed out, "the limit of the resolving power of telescopes is attained with a very moderate magnification, probably about 20 for each inch of aperture of the object-glass or mirror"*. The attainable accuracy in this case, therefore, depends very largely on the *brightness* of the object whose position is to be determined, and in the great majority of cases this is not sufficiently great to enable the limit indicated above to be reached with either the microscope or mirror and telescope†. Under the most favourable conditions as regards illumination, &c., the practical limit of accuracy seems to be about the same in this case as in the preceding, *i. e.*, about $\frac{1}{5}$ the limit of resolution for the microscope, and $\frac{1}{10}$ to $\frac{1}{15}$ that limit for the mirror.

From a review of the preceding cases, which cover all those usually met with in practice, we see that the mirror has a somewhat greater efficiency with respect to the microscope than is indicated by Rayleigh's comparison. The limit of accuracy in the case of the microscope is at most $\frac{1}{5}$ the limit of theoretical resolution given by (2); in the case of the mirror it is (in the last two cases) from $\frac{1}{7}$ to $\frac{1}{8}$ the limit given by (1). This higher efficiency is largely due to the fact that the most favourable conditions are far more easily attained with the mirror and telescope than with the microscope. Indeed, the limit of resolution indicated by (2) for the latter supposes that the angular aperture of the microscope objective is 180°, a condition *never* fulfilled in practice. If we assume the greatest practicable aperture that can be used for the purpose under consideration to be 90°, the resolution will be less than the limit indicated in the proportion of 7 to 10. Again, it is far more difficult to secure the requisite degree of illumination with the microscope than with the mirror; and this will still further decrease its measuring power in comparison with the latter.

In general, I think it would be safe to assume from both the preceding theoretical considerations and from the results of experiment, that in order to make the microscope and pointer method as *accurate* as the mirror method for observing small angular movements, the length of the pointer would have to be at least two to three times the diameter of the mirror. The greater convenience and efficiency in other directions of the first method makes it, as stated in the first part of the paper, far more generally useful.

To these two methods of measuring angular motions we

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* Phil. Mag. Oct. 1879, p. 263.

† It is, however, readily reached in the case of the interferometer, because with this instrument the width of the fringes may be made as great as we please without diminishing their brightness.

must now add a third, the interferometer method, first suggested[*] by Michelson. I think that the only applications of this method to the measurements of the deflexions of a suspended system that have yet been made, are those recently described by Barus[†] and by myself[‡], but these are sufficient to prove its practicability and determine its accuracy as compared with those just discussed. Professor Michelson has shown that theoretically this method of angular measurement may be made from 20 to 50 times as accurate as either of the preceding. Practically, however, the accuracy in the case of a freely suspended system is considerably less, because the unsteadiness of the fringes, due to irregular motions of the system, prevents the estimation of as small a fraction of a fringe as is possible with the more stable forms of the instrument. From the experiments just referred to the writer has concluded that the smallest movement that can be *measured* with certainty by this means is that corresponding to a shift of $\frac{1}{20}$ of a fringe[§]. This means, in the form of instrument used in these experiments, an angular movement of

$$\theta = \frac{1}{40}\frac{\lambda}{b'},$$

b' being the distance between the centres of the two mirrors B and D (see fig. 4) on the suspended system.

The smallest movement observable with the mirror is, as just stated, about $\frac{1}{15}$ the limit of resolution, or

$$\theta_r = \frac{1}{15}\frac{\lambda}{b} = \frac{2}{15}\alpha_r.$$

Hence, when $b = b'$ the interferometer method is at most about three times as accurate as the mirror method[‖]. It has

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[*] See papers: "Measurement by Light Waves," Am. Jour. Science, vol. xxxix. p. 115; "Application of Interference Methods to Astronomical Measurements," Phil. Mag. vol. xxx. p. 1; "Application of Light Waves to Metrology," 'Nature,' Nov. 16, 1893.

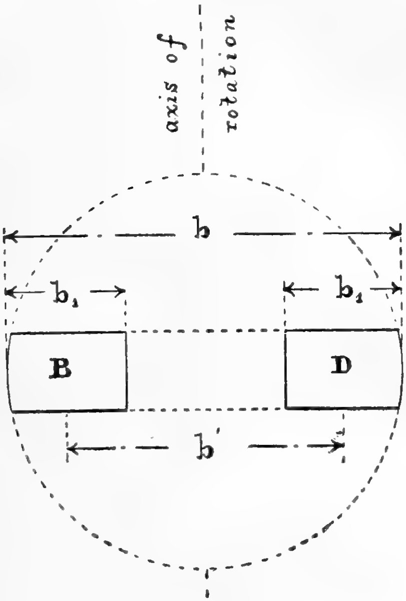
[†] "On a Possible Development of the Ideostatic Electrometer," 'The Physical Review,' vol. iv. p. 400, March–April 1897.

[‡] "On the Application of the Interferometer to the Measurement of Small Angular Deflections of a Suspended System," ibid. vol. iv. p. 480, May–June 1897.

[§] Dr. Barus assumes the limit to be $\frac{1}{10}$ of a fringe.

[‖] If we could get rid entirely of the effects of vibration the advantage of the interferometer over the simple mirror would be much greater. For such purposes as measuring the twist or the bending of bars, or the errors of straight edges, &c., the mountings of the mirrors may be made so stable that irregular motions are almost entirely eliminated, and it been stated that the former has a considerable advantage in requiring only two portions, B and D, at the ends of a diameter, instead of the full mirror. But this advantage disappears if we simply cut away those portions of the mirror outside of the rectangle enclosing B and D*. When this is done the simple mirror becomes considerably more efficient than is indicated by the preceding comparison. This is because the effective aperture b of the mirror is the distance between the extreme edges, while the aperture b' of the other

Fig. 4.



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then becomes possible to estimate and measure angular movements corresponding to less than $\frac{1}{100}$ of a fringe, or less than $\frac{1}{5}$ the amount indicated as measurable in the case of a freely suspended system. For such purposes, also, the interferometer possesses other considerable advantages in the way of lightness of the moving parts and (at least for very large apertures) cheapness.

* The advantages of this form of mirror, as compared with the full circular mirror, have also been pointed out by Boys, who used it in his work on the determination of the Newtonian Constant. (See memoir already referred to, p. 32.)

system is the distance between the centres of the portions B and D*. Hence, for a constant moment of inertia I, the horizontal aperture b (which alone determines resolution) will be greater for the narrow rectangular mirror than for either the circular mirror or the interferometer system. Thus we have for the circular mirror (if thin)

$$I=\frac{1}{64}\pi\mu b^4 \cong \frac{1}{21}\mu b^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

μ being the mass per unit surface of the plate. For the rectangular mirror, height c, we have similarly

$$I_1=\frac{1}{12}\mu c b^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the height c is taken as $\frac{1}{5}$ the breadth (fig. 4) we have

$$I_1=\frac{1}{60}\mu b^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, if the thickness of the mirrors is the same in the two cases, the horizontal apertures will for a constant moment of inertia vary nearly in the proportion of four to three. The resolving powers will therefore (since the resolving power of a circular aperture is about 0·9 that of a rectangular aperture of the same horizontal diameter) be about in the proportion of 10 to 7.

The moment of inertia of the interferometer system will similarly be (not allowing for the support of the mirrors)

$$I_2=2\left(\frac{1}{12}\mu c b_{\prime}^{\,2}+\mu c b_{\prime}\left(\frac{b'}{2}\right)^2\right), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$b_{\prime}$ being the length of each of the interferometer mirrors B and D (fig. 4) and b' the distance between the centres. For comparison with the rectangular mirror, assume that the height c is the same as in the case of the mirror, *i. e.* $\frac{1}{5}b=\frac{1}{5}(b_{\prime}+b')$. The length of each mirror, $b_{\prime\prime}$, should be for practical reasons at least one and one-half times the height

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* If it were only possible to make these two independent surfaces part of the same optical plane (which is not necessary in the case of the interferometer) we might also dispense with the intermediate dotted portion in the ordinary method of mirror reading. In such a case the effectiveness of the mirror method would be quite equal to that of the interferometer method.

c*. When this is the case, $c=\frac{2}{7}b'$, and $b_{/}=\frac{3}{7}b'$. Hence

$$\mathrm{I}=\frac{156}{(7)^4}\mu b'^4.$$

Comparing this with (7) we find that for equal moments of inertia

$$\frac{b}{b'}=\frac{\sqrt[4]{9360}}{7}=\frac{9.8}{7}.$$

In this case, therefore, the mirror has a decided advantage over the interferometer system, *i. e.* for a given aperture $b=b'$ the moment of inertia of the simple mirror is actually less than that of the two mirrors. When the width of the mirrors is $\frac{1}{10}$ the aperture, we have

$$\frac{b}{b'}=\frac{\sqrt[4]{105120}}{17}=\frac{18}{17}+$$

or the simple mirror still has the advantage. And it is not until the horizontal aperture is over twelve times the height that the interferometer system has any advantage over the mirror of the same effective aperture, as regards the moment of inertia and consequent effect on the time of swing of the suspended mass to which it is attached. It is true that a mirror of this length would probably have to be thicker than the two small mirrors B and D; but the increase in the moment of inertia due to this would be more than counterbalanced by the inertia of the mounting of the two mirrors, of which no account is taken in (8)†.

In attaining the highest degree of accuracy in the use of the mirror, the German method of reading (telescope and scale) is greatly superior to the English method, chiefly because of the greater fineness of the reference-line (the cross-wire of the telescope, instead of a ruled line on a scale). The mirror ought also to be as nearly optically flat as possible, and the greatest care should be taken in selecting the glass

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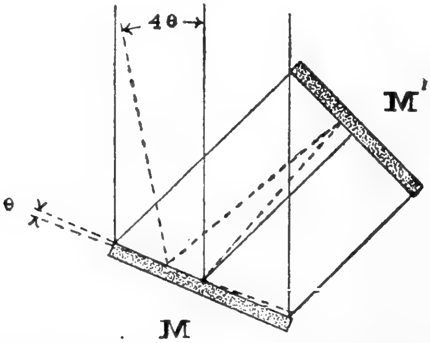
* It would be better to have the length twice the height (see paper in 'Physical Review' already referred to). But this would make the moment of inertia for a given aperture b' considerably greater.

† A practical disadvantage of the very narrow mirror that has been pointed out by Boys is that the vertical definition is not sufficiently good to enable the figures on the scale to be read. This, however, could be overcome by making these figures very large, or by using in place of them characters similar to those used on stadia rods.

for the purpose. It would be better in general to use a short, thick mirror with an optically worked surface, rather than a longer thin one of ordinary quality[*].

In the ordinary method of using a mirror the only possible way of increasing the ultimate measuring power is to increase its aperture. Some years[†] ago a simple method suggested itself to me, by means of which the angular deviation of the image for a given movement of the mirror might be doubled, and the ultimate measuring power thus increased without changing the dimensions of the reflecting surface. I have since learned that Michelson and Morley had previously proposed to employ the essential feature of this method (that of multiple reflexion) as a possible means of solution of a different problem; but as far as I am aware no one has suggested the use of it in ordinary measurements of deflexion by the mirror method. If it has been suggested, it is at least not generally known, and a description of it may therefore not be without interest. The plan is simply to place close to the revolving mirror m a second mirror m' (fig. 5), which receives the rays after the first reflexion and returns them to m, from which they are again reflected in nearly the original direction.

Fig. 5.



Any angular movement θ of the mirror m then causes an angular deviation of the doubly reflected ray of four times that amount. In order to realize the effect of the full aperture of the mirror it is necessary that the two mirrors should be as nearly parallel as possible, and as close together as possible. These two conditions evidently conflict with each other. If we need only read deflexions on one side of the middle position the mirror may be placed as in fig. 5, so that

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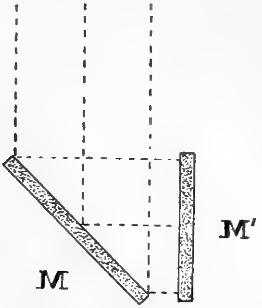
[*] An accurately cylindrical surface is, however, nearly as good as a "flat" if the axis of the surface is parallel to the axis of rotation.

[†] The entry in my note-book is dated November 12th, 1889.

its edge just clears the incident beam. If the distance between the surfaces is equal to the diameter of the moving mirror the angle of incidence will be about 20° and the aperture effective for resolution will therefore be about 92 per cent. of the full aperture. If we need to read deflexions on both sides it will be necessary to place the second mirror m' so that it makes a larger angle with the first, as in fig. 6. In this case the effective aperture will be less, but may still be made as large as 75 per cent. of the full aperture*.

Fig. 6.



It is very easy to add the additional mirror to a magnetometer or electrometer, or to a galvanometer in which the mirror is between the coils instead of inside of one, as in the older forms of the Kelvin instruments. The doubling of the deflexion for a given movement of the suspended system is particularly advantageous in instruments intended for automatic registration (photographic or otherwise), because in such instruments the requisite linear movement on the record sheet can be secured when the distance of the latter from the instrument is only half what is necessary with the customary mirror arrangement.

Yerkes Observatory, University of Chicago,
March 1897.

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* When the angular deflexion is large there is a further slight diminution of aperture due to the lateral displacement of the beam at the second reflexion. When the distance between the mirrors is equal to the diameter of m this lateral displacement is about 2 per cent. of the aperture for each degree of deflexion of the image.

ERRATA in my preceding article "On the Resolving Power of Telescopes and Spectroscopes for Lines of Finite Width," in the May number of this Journal.

Page 328, line 5, *for* maximum *read* minimum.
„ 341, „ 17, *for* brought *read* bright.
„ 341, „ 26, *after* telescope-lens *insert* of rectangular aperture.
„ 342, Table V., 4th column, in the heading, *for* 48 *read* 4·8.

XII. *On the Proof of a Theorem in Wave-motion.*

To the Editors of the Philosophical Magazine.

Gentlemen,

Were it not for the unexpected avowal made by Mr. Preston on p. 460 of the June number of the Philosophical Magazine, that he "feels bound" to "protest against" MacCullagh's method of investigating wave-motion (which is the method I had employed), I should scarcely have felt it incumbent on me to trouble you with any comments upon his letter; for though there are fallacies in its earlier paragraphs, they seem sufficiently transparent not to need formal reply. Nevertheless I will further on briefly refer to them. However, what I desire chiefly to do is to apply myself to the more important task of rescuing from oblivion one of the several additions to the resources of mathematics in its application to physical science which were made by MacCullagh, and which, to my surprise, seem now to be forgotten in his own University.

For about thirty years—that is, in the thirties, the forties, and most of the fifties of the present century—the University of Dublin was a great school of geometrical teaching. It had risen to this position owing to the profound reform in the teaching of the University which had been effected by Provost Bartholomew Lloyd in the twenties. As a school of geometry Dublin had then no rival except Paris. And perhaps there are no greater names among those who have advanced geometry, treated as a branch of Pure Mathematics, than those of the great Frenchman Chasles, of Sir William Hamilton, and of Dr. Salmon the present venerable Provost of Trinity College. The last two received their training in the University of Dublin at the time above referred to. The great feature of the methods then pursued was the close *interweaving* of analytical and geometrical conceptions and methods of treatment, often with a preponderance of the contribution from geometry. Where such a combination was not practicable the geometrical method was advanced side by side with the analytical. But where the intimate union of the two could be effected it furnished tools for discovery more effective than those of either method handled separately, and at the same time kept before the mind of the inquirer a clearness of vision in regard to what was being done, which was more continuous and of an altogether higher order than can be reached by the analytical treatment applied separately or when it is illustrated only by the rough geometrical conceptions ordinarily employed.

Another distinctive feature of the Dublin School of Mathematics of those decades—one which was perhaps peculiar to that school—was the large introduction of these same accurate methods into the study of the various branches of Physics. This was chiefly due to the influence of MacCullagh on the teaching of the University; and his suggestions were being actively developed and extended in the instruction given to undergraduates in the student days of the present writer—from 1844 to 1847—which were the last years of MacCullagh's life.

Some years afterwards a change was made in the curriculum of Trinity College, Dublin, which had the effect of diminishing the stress which the College until then had laid upon the earlier and necessary steps of the training of the mind for such pursuits; and as a consequence the attention paid by schools preparing for the University, to developing the geometrical insight and skill of their pupils, gradually fell off. One result, probably unforeseen and certainly unfortunate, has been that much valuable unrecorded work that was done in the University of Dublin in those earlier times was not followed up; and has been, or is in risk of being, lost. Circumstances have occurred within the last few months, culminating in Mr. Preston's announcement in the June number of the Philosophical Magazine, which have brought forcibly to the knowledge of the present writer that methods of great value which were then employed in physics have become a lost art, unknown to the present teachers and students of the University.

In what I have written within the last nine months about wave-motion, and especially in the paper at p. 273 of the April number of the Philosophical Magazine, I applied what used to be called "MacCullagh's method," or "Proof by the Principle of Reversal," to the study of wave-motion, for which it is in a special degree adapted. In doing this I assumed that my readers understood the fundamental principles upon which all accurate geometrical proofs in this branch of physics ultimately rest; but the doubts expressed and the questions put by Mr. Preston on p. 460 of the June number of the Philosophical Magazine, show that these as they used to be studied in the University of Dublin are now forgotten, and that even so well-informed a reader as Mr. Preston is unacquainted with them. Accordingly explanations must now be given which would have been superfluous forty years ago. I am endeavouring to do this in a paper I am preparing, and which I hope shortly to have ready for publication. It will, I trust, effectually rescue from oblivion at least one method of proof of great value; and I shall endeavour sufficiently to go back to first principles to meet all Mr. Preston's difficulties and remove his scruples. It may be hoped that others with more leisure than the present writer will devote some of it to recovering other parts of the distinctively Dublin work* in Mathematical Physics of the above-mentioned period, which after MacCullagh's death was far too much left to the chances of tradition.

A few words suffice for the rest of Mr. Preston's letter. In his second paragraph, on p. 458, he says "I fear Dr. Stoney has misunderstood my communication. What I intended to convey was &c." What Mr. Preston did convey was not merely that he was making the artificial analysis he proceeds here to set forth, but, to use his own words on p. 238, "This then" (viz. an equation furnished by Mr. Preston's artificial analysis) "is the analytical expression of the general theorem enunciated by Dr. Stoney" (which dealt with the analysis effected by nature); and what I pointed out is that this is an entire delusion.

The contrast between the two is well shown by the case discussed by Mr. Preston at the foot of p. 459 of his letter and top of the following page. To give the problem definiteness let us suppose that what we have to deal with is the radiation by which we see an object which is illuminated by monochromatic light. Here all the luminous waves emitted by the object are of one wave-length, and all have transversals lying in the wave-fronts. The analysis of this light under my theorem is into trains of plane waves which have both these characteristics. The plane-wave components have all of them the one wave-length of the monochromatic light, and the transversals everywhere lie, as they should, in the

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* Another feature which then distinguished the teaching of the University of Dublin in Mathematical Physics was the almost exclusive study of *great* writers—Newton, Lagrange, Laplace, Poisson, Gauss, MacCullagh, Ampère, &c.—instead of re-castings of their work by compilers of text-books; and all were illuminated by incorporating into them the geometrical methods peculiar to the University, wherever this was practicable. The great achievements of the fifty years that have since elapsed—Thermodynamics; the Kinetic Theory of gas; Spectroscopy; the Electricity of Faraday, Clerk-Maxwell, and Lord Kelvin; the Electromagnetic Theory of light—were then unknown; but while the teaching of the University has since gained so immensely, something has been lost, and especially in regard to the methods which were distinctive of a Dublin training and which in a marked degree tended to produce original thinkers.

planes of the waves to which they belong. Each of these trains of plane waves the luminiferous æther can propagate forward unaltered, whether it exist alone in the æther or accompanied by other motions. Finally, this analysis is of great use for investigating numberless optical problems.

Mr. Preston's analysis decomposes each transversal into its ξ, η, and ζ components. Of these he forms three groups. He groups all the ξ components together, all the η components together, and all the ζ components together; and by his theorem he decomposes each of these groups into trains of non-natural waves with lengths ranging from infinity down to cypher, and with transversals which, in all but three directions, stand obliquely to the front of the wave. Trains of waves of this kind, if isolated, cannot be propagated by the æther. The instant one of them is left to itself it breaks up, and what is thenceforward propagated is something quite different. Finally, the analysis is of no known use in physics. From all this it may be judged how much what Mr. Preston says about this case, at the bottom of p. 259 and top of p. 260, needs correction.

The useful theorem—that which supplies us with a resolution into uniform plane waves which the medium is competent to propagate—has been established by the geometrical method of proof which we owe to MacCullagh, and which is based on the Principle of Reversal. No analytical proof has yet been discovered, but the difficulties in the way of such a proof seem such as ought not to be insurmountable. We have seen, on p. 273 of the April number of this Magazine, that the symbolical expression of the results arrived at by the geometrical proof is

$$\mathrm{F}(x,y,z,t)=\iint\Sigma\left[\mathrm{M}\sin\left(2\pi\frac{r-vt}{v\tau}+\alpha\right)\right].\sin\theta\,d\theta\,d\phi,$$

where $r=x\cos\theta+y\sin\theta\cos\phi+z\sin\theta\sin\phi$, and in which the M's, the transversals, are directed quantities. In this expression the vector components of the M's may be carried outside the sign Σ, but no farther. Accordingly, the double integral is of a new kind, in which the summation is a geometrical summation, and not an algebraic summation as it is in ordinary integrals. This presumably is the source of the difficulty. It seems, however, to be a state of things which may be dealt with, if we can succeed in finding a dynamical problem in which the M's shall be forces, but which shall lead to the above symbolical expression in all other respects. and there is no apparent reason for regarding it as hopeless that such a dynamical analogue may yet be found and successfully treated. I am, Gentlemen,

Faithfully yours,
G. JOHNSTONE STONEY.

8 Upper Hornsey Rise, N.,
June 11, 1897.

[*Note added June* 22.—Since the above was written I learn from Dr. Larmor that MacCullagh himself refers to his method of proof in a footnote to one of his papers. I had supposed that our knowledge of it depended solely on tradition.—G. J. S.]

P.S.—The reader is requested to make the following corrections :—

In the footnote on p. 279 of the April number of the Magazine, third line from the bottom of the page—

Change "deducted" into "allowed for." The allowance to be made is *not* a simple deduction.

In the first footnote on p. 372 of the May number of the Magazine, transfer the words "so as to receive all the light which emerges from it" from the end of the first sentence of the footnote, to the end of the second sentence. The words were inadvertently written in at the wrong place.

XIII. *On the Electrification of Air by Uranium and its Compounds. By* J. CARRUTHERS BEATTIE, *D.Sc., F.R.S.E. With a Note by* LORD KELVIN, *G.C.V.O., F.R.S.L. & E., &c., &c.*[*]

§ 1. IT is proposed in the following paper to describe experiments made to test the electric state of the air in the neighbourhood of metallic uranium, or of other metals on which a salt of uranium had been deposited from a solution when these metals were charged to a positive or negative potential.

§ 2. *Method employed.*—To test the electric state of the air, the electric-filter method due to Kelvin, Maclean, and Galt[†] was employed. The special filter used in the experiments to be described was a block-tin tube 10 cm. long and 1 cm. diameter, filled with brass filings. This was insulated on two tunnelled pieces of paraffin, and put in metallic connexion with the insulated pair of quadrants of a quadrant electrometer, whose capacity was $\frac{1}{10,000}$ of a microfarad. From one of the tunnelled pieces of paraffin a metal tube led to an air-pump; from the other a piece of indiarubber tubing led to the place where the air to be tested was. This air was then

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[*] Read before the Royal Society of Edinburgh, June 7th, 1897. Communicated by Kord Kelvin.

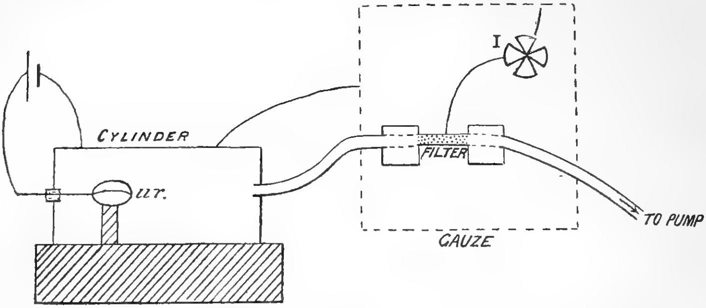
[†] Kelvin, Maclean, Galt, Proc. Roy. Soc. London, March 14th, 1895.

drawn through the electric filter, and the deviation of the electrometer-reading from the metallic zero of the instrument when the two pairs of quadrants were insulated, was noted. To give some idea of the efficiency of the filter the results obtained when air was drawn from the neighbourhood of the electrodes of a Ruhmkorff coil—a 10 inch Apps—will be given. When the electrodes were near enough to admit of sparks passing it was found that the air in the neighbourhood of the positive electrode gave a positive electrification of +1·5 volts after one minute of pumping at the rate of 12 strokes per minute. When the air was drawn from the negative electrode's vicinity a negative electrification of −2·5 volts in half a minute was observed. When the electrodes were drawn so far apart that sparking did not take place, air drawn from the positive electrode raised the filter to a potential of +·7 of a volt after one minute of pumping; air drawn from the neighbourhood of the negative electrode gave a reading of −1·5 volts in the same time.

When the negative electrode was connected metallically to the cathode of a Crookes tube, air drawn from the neighbourhood of the cathode through the filter caused an electrometer deviation equivalent to a potential of −·8 of a volt per minute of pumping. Air drawn from the neighbourhood of the wire connecting the negative electrode to the cathode caused a deviation equivalent to −·7 of a volt. Air drawn from the neighbourhood of the wire joining the anode of the Crookes tube to the positive terminal of the coil gave a positive reading on the electrometer less in amount than the corresponding negative reading.

§ 3. *Electrification of Air by Metallic Uranium.*—To examine the electric state of the air in the neighbourhood of metallic uranium charged to a given potential, the following experimental arrangement was made:—A metallic cylinder 24 cm. in length, 8 cm. in diameter, was constructed. One end of this was closed with a piece of the same metal; in this end was a hole into which a plug of tunnelled paraffin was placed. From the paraffin a piece of indiarubber tubing about 1½ foot long led to the electric filter. The other end of the metallic cylinder was closed with a disk of paraffin, in the centre of which a hole was made to admit air from the laboratory. This hole was filled with cotton wool. The uranium, a circular disk 5·5 cm. diameter, ·5 cm. thickness, was insulated on paraffin inside the cylinder. The cylinder itself was laid on a block of paraffin. The arrangement will easily be understood from the following diagram.

In the first instance a piece of lead of the same size as the uranium was insulated on paraffin in the metallic cylinder, and was connected to the electrometer sheaths. The metal cylinder was also connected to the sheaths. Air drawn from the cylinder through the filter caused no deviation from the metallic zero of the electrometer. When the lead was disconnected from the sheaths and joined to one terminal of a battery, and the other terminal connected to sheaths, no electrification of the air was observed with the lead at potentials up to ±95 volts.



When the uranium was insulated in the metallic cylinder, and this latter was of polished copper, the following results were obtained:—

(*a*) Uranium and copper both connected to sheaths and air drawn from the cylinder through the filter. It was found that an electrometer deviation of +0·05 of a volt per minute of pumping was produced, the pump being worked at 12 strokes per minute.

(*b*) Uranium disconnected from the electrometer sheaths and joined to one terminal of a battery, the other terminal and the copper cylinder being connected to the electrometer sheaths.

Uranium kept at	Electrometer deviation in volts per minute of pumping 12 strokes per minute.
+ 2 volts	+·36
+10 ,,	+·65
+22 ,,	+·53
+45 ,,	+·35
+68 ,,	+·25
+93 ,,	+·235
− 2 ,,	−·32
−10 ,,	−·53
−22 ,,	−·40
−45 ,,	−·25
−68 ,,	−·22
−93 ,,	−·20

(*c*) Uranium and one terminal of battery connected to the electrometer sheaths, copper cylinder connected to the other terminal of the battery.

With this arrangement it was found that there was a negative electrification of the air with the positive terminal to the copper cylinder. The deviation on the electrometer per minute of pumping was exactly equal to the negative electrification obtained with arrangement (*b*) with the uranium kept connected to the negative terminal of the battery and the positive terminal to sheaths. With the negative terminal to the copper cylinder the air was found to be positively electrified and to the same extent as the corresponding positive electrifications in arrangement (*b*).

Similar results were obtained after the inside of the copper cylinder had been oxidized.

With a zinc cylinder and with the same connexions as in (*a*) an electrometer reading equivalent to $-0{\cdot}1$ of a volt per minute of pumping was observed.

With an aluminium cylinder and arrangement (*a*) a slight positive electrification of the air was observed. With a lead cylinder a negative electrification of $-{\cdot}05$ of a volt per minute of pumping was obtained. The various cylinders were all of the same size.

With zinc, aluminium, or lead cylinder and arrangements (*b*) and (*c*), similar results were observed. That is, the nature of the metal of the vessel surrounding the uranium did not influence the extent to which the air was electrified.

In all cases the air in the metallic cylinder was electrified positively when the uranium was electrified positively; negatively when the uranium was electrified negatively. The positive electrification given to the air for a given positive potential of the uranium or an equal negative potential of the metallic cylinder was always greater than the negative electrification of the air when the uranium was kept at the corresponding negative potential or the copper cylinder at an equal positive potential. The electrification of the air attained a maximum when the potential was between ± 10 and ± 22 volts. These results may be compared with the results obtained with Röntgen rays by Mr. Rutherford [*]. The explanation of the kind of the electrification of the air, and of the fact that a maximum value is reached, is given in Lord Kelvin's note appended below.

The uranium was next wrapped in aluminium foil, and again insulated in the lead cylinder. The air drawn away from the lead cylinder was found to be negatively or positively electrified according as the uranium wrapped in aluminium

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[*] Phil. Mag. April 1897.

foil was negatively or positively electrified. The electrifications were less than those obtained with the uranium bare. For example, with the uranium wrapped in aluminium foil kept at +2 volts and the lead cylinder connected to sheaths, or with the first to sheaths and the cylinder at −2 volts, the air when drawn through the filter raised it to a potential of +·11 of a volt per minute of pumping. With +10 volts, the potential per minute of pumping was +0·05 of a volt, with 95 volts, +·015 of a volt per minute. The negative electrifications of the air with the uranium wrapped in aluminium foil kept at −2, −10, or −95 volts, and the metallic cylinder to the sheaths of the electrometer, were smaller than the corresponding positive electrifications given above for +2, +10, and +95 volts.

Finally the uranium was placed inside a paraffin cylinder. Round the outside of this cylinder tin-foil was wrapped. With both uranium and tin-foil connected to case—arrangement (*a*)—there having been no previous electrification of the uranium in the paraffin cylinder, the air drawn from the cylinder showed no sign of electrification.

With arrangements (*b*) and (*c*) similar results were obtained to those found when the cylinder used was of metal, with this difference, however, that the amount of electrification with any given voltage depended on the voltage which had been used in the experiment immediately preceding. For example, after the tinfoil round the cylinder of paraffin had been connected to the positive terminal of the battery, and the uranium to sheaths, it was found that the air drawn through the filter raised it to a potential of −·235 of a volt per minute of pumping. In the succeeding six minutes, while the tinfoil was kept at −10 volts, and the uranium to sheaths, the electrometer deviation was equivalent to a rise in potential of the filter of +·7 of a volt per minute. In the succeeding two minutes the rise of potential was +·6 of a volt per minute, and six minutes later it was +·3 of a volt per minute.

Both uranium and tinfoil were next connected to sheaths —arrangement (*a*)—and in the first two minutes of pumping the rise of potential of the filter was −·6 of a volt per minute. In the second two minutes the rise was only −·13 of a volt per minute.

With the uranium connected to sheaths and the tinfoil at +10 volts, the air was found to be negatively electrified; the amount of this electrification was such as to raise the potential of the filter −·65 of a volt per minute in the first minute and −·2 of a volt per minute in the tenth minute.

On again connecting both uranium and tinfoil to case the air was found to be positively electrified, the deviation on the electrometer in the first minute indicating a rise of potential of +·65 of a volt.

It will be seen from these results that, in addition to the electrification deposited in the air and carried to the filter, electricity also passes through the air to the sides of the paraffin cylinder. This electricity, when the uranium and the tinfoil are connected to case, induces the opposite electricity on the uranium, and this is then again given off to the air by the uranium. It will easily be seen from the note appended below why the electrification of the air in the paraffin cylinder at any given potential is greatest when the uranium had not been previously electrified or had been electrified previously with electricity of the opposite kind.

§ 4. *Electrification of Air by Uranium Salts.*—Two salts of uranium were used, uranium acetate and uranium nitrate. The nitrate was deposited on a strip of platinum which was then insulated on paraffin in the copper cylinder described above. The same series of experiments was gone through and results of a similar nature to those described in § 3 were obtained. The electrifications of air observed were, however, much smaller in amount.

The uranium acetate was deposited on tinfoil and placed in a zinc cylinder which was provided with a mica window. By means of this window ultra-violet light from an arc lamp could be caused to shine on the uranium acetate. The electrifications of the air were similar in kind to those already described, and were of about the same amount as those obtained with the nitrate of uranium. No change in the electrification of the air from that produced in ordinary daylight was observed when the ultra-violet light was shone on the acetate.

Physical Laboratory,
University of Glasgow, May 1897.

Note by Lord KELVIN *on the sign of the Electrification found in Air drawn from Space surrounding Electrified Uranium.*

In some of our experiments with high voltages we found sparks to pass between uranium and other metals * apparently according to the laws of disruptive discharge subject to but little modification by the special quasi-conductivity produced in air by the "uranium rays." On the other hand, all our

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* Kelvin, Beattie, Smolan, Proc. R. S. E., April 4th, 1897.

experiments with voltages less than 500 or 600 volts per cm. of line of force in the air at ordinary atmospheric pressure, seem to be not sensibly influenced by disruptive charges or by brushes ; and the quasi-conductivity of air produced by uranium was the dominant factor. This is undoubtedly the case in the experiments described by Dr. Beattie, and I assume it to be so in what follows except when I give express warning of possible liability to disruptive discharges.

The effective conductivity produced in the air by the uranium influence is, of course, greatest in the immediate neighbourhood of the uranium, but there is something of it throughout the enclosure. Hence it may be expected that electricity of the same kind as that of the uranium will be deposited in the air close around it, and electricity of the opposite kind in the air near the enclosing metal surface. According to our former experiments, the quantity flowing from either the uranium or from the surrounding metal per sq. cm. of its surface increases but little with increased voltage when this exceeds 5 or 10 volts per cm. Now, when the greatest diameter of the uranium is small in comparison with distances to the outer metal surface, the voltage per cm. is much greater along the lines of force near the uranium surface than near their outer ends on the surrounding metal. Hence the rate of discharge of electricity into the air from the uranium will cease to increase sensibly with increasing difference of potential between the uranium and the surrounding metal ; while the rate of discharge of the opposite electricity from the large surrounding metal surface is still notably increasing. Hence if the dimensions and shapes of the uranium and of the surrounding metallic surface are such that for small voltages, such as 10 or 20 volts of difference between the uranium and the surrounding metal, the electricity lodged in the air by discharge from the uranium preponderates over that discharged from the surrounding metal, the excess come to a maximum and diminish, possibly even down to zero, with greater differences of potential. At potential-differences still greater the electricity lodged in the air from the outer metal may preponderate, and the electricity in the air drawn off, and given to the filter, be of opposite sign to that of the uranium which was found with the lower voltages: *provided the configurations are such, and the voltages are so moderate, that disruptive discharge does not intervene to any practically disturbing extent.*

XIV. *Radiation in a Magnetic Field.*

By ALBERT A. MICHELSON[*].

IN the interesting and important paper of Zeeman "On the Influence of Magnetism on the Nature of the Light emitted by a Substance"[†], there is a reference to the work of the late M. Fievez, who found that instead of a broadening of the spectral lines there were reversals and double reversals, which Zeeman has not observed.

In some cases the magnitudes to be observed are of the order of a fortieth of the distance between the sodium lines, and should be clearly seen in a good spectroscope under proper conditions; but others occur in which they are but a third or a fourth as large, and in these cases all detail is lost in diffraction effects and optical imperfections.

For the investigation of just such cases the *interferometer* is particularly adapted, and it was determined to investigate the problem with the aid of this instrument.

The first substance tried was sodium. A bead of sodic carbonate was placed in the flame of a small hand blowpipe, which could be kept under better control than a Bunsen burner. This was placed between the flat pole-pieces of a moderately large electromagnet in the manner described by Zeeman, and the light after passing through a collimating lens entered the interferometer. The difference of path commencing at zero was increased by single turns of the millimeter screw, noting at each turn the clearness or visibility of the interference-fringes[‡], first without and then with the magnetizing current.

The curves A and D, fig. 1[§], show the results of this experiment. They are the envelopes of the visibility curves, the alternations of which are too rapid to show on this scale. The abscissæ are differences in path of the interfering pencils in millimetres. From these the distribution of light in the source is found as described in a previous article[||]. The results are shown at A and D (fig. 2), the first representing the appearance of one of the sodium lines without, and the second with the magnetizing current.

It is evident from the figures that the broadening of the

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[*] Communicated by the Author.

[†] Phil. Mag., March 1897.

[‡] Phil. Mag., Sept. 1892.

[§] Negative ordinates indicate reversal of the fringes.

[||] Phil. Mag., Sept. 1892.

line is relatively insignificant, but that it is separated into two components of equal intensity *. The question at once

Fig. 1.

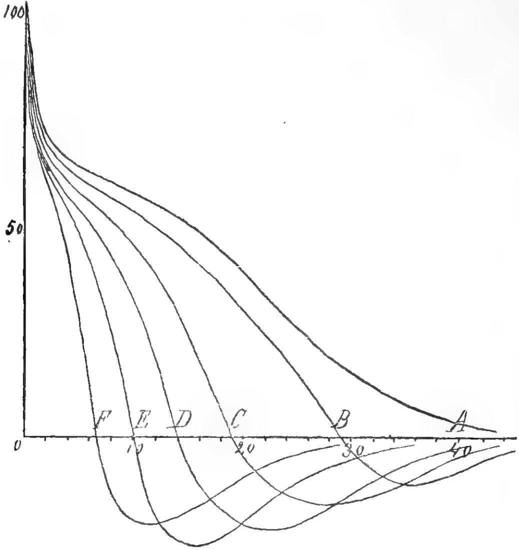
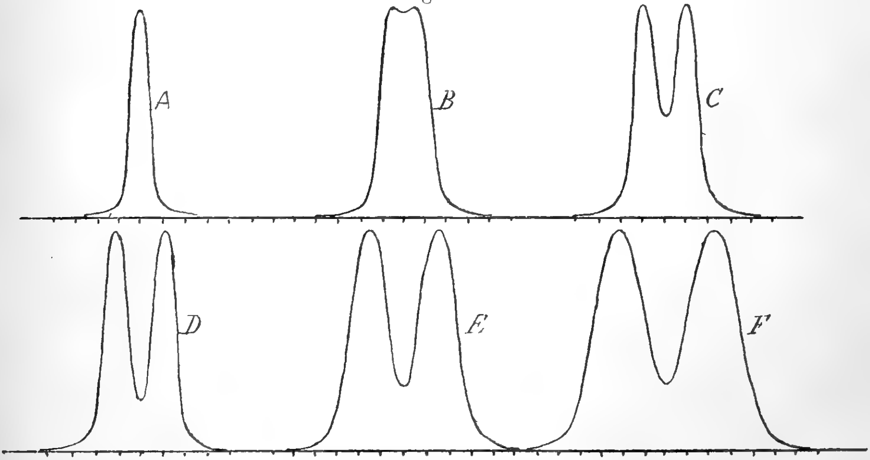


Fig. 2.



arises whether or not this fact lends support to the conclusion

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* A triple line would give a totally different visibility curve.

of Fievez, that the effect of a magnetic field is to produce reversals. The appearance of the figure itself seems to indicate a true separation ; the depression bearing a much larger proportion to the whole area than we are accustomed to observe in reversals ; but this by itself would hardly be considered conclusive. It was thought, however, that if there were a true separation, the distance between the components should vary with the strength of the field, whereas in the case of a reversal one would expect only an increase in the darkness of the absorption.

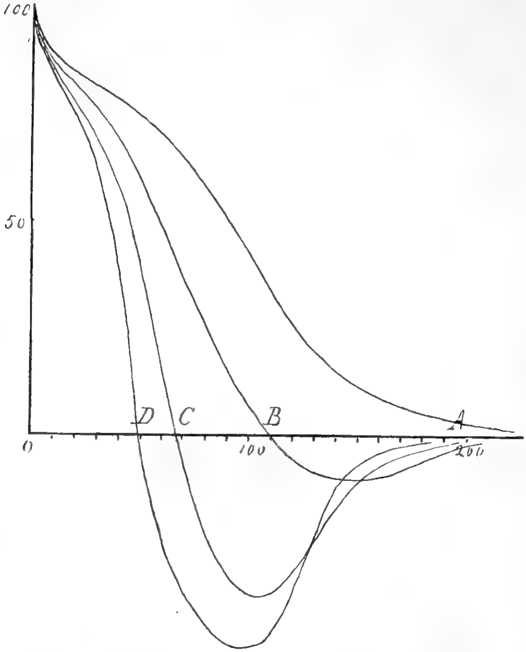
Accordingly a series of observations was made with varying strength of field, the results of which are shown in the visibility curves of fig. 1, and the corresponding intensity curves of fig. 2. The strength of field in the order of the letters was 0, 5, 7, 11, 16, 20. It appears from the figures that up to a strength of field 11, which is about 2000 C.G.S., the principal effect is a doubling of the line ; but beyond this the component lines are broadened as well as separated. It is also clear that the separation is nearly proportional to the strength of field. Thus, assuming this law to be true, the following table shows the agreement between the observed and the calculated distances. F = strength of field ; Δ = difference of path in millimetres corresponding to visibility 50 for single source ; δ = corresponding half-width of the source on a scale of 100 for $D_1 - D_2$; D = period of the coincidences in millimetres due to the doubling, and α = the corresponding distance between the components.

Table I.

	F.	Δ.	δ.	D.	α.	α calc.
A	0	20	·66	∞	0·0	0·0
B	5	20	·66	58	1·0	1·0
C	7	20	·66	38	1·6	1·5
D	11	20	·66	28	2·2	2·3
E	16	14	·94	18	3·3	3·4
F	20	10	1·32	14	4·3	4·2

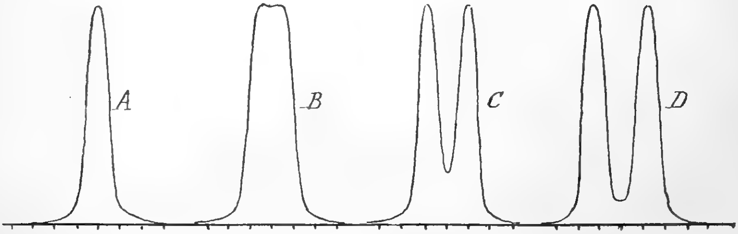
The next substance examined was cadmium. Cadmium filings were enclosed in an end-on vacuum tube which was placed between the poles of the electromagnet. The results are shown in figs. 3 and 4, and prove that there is scarcely any broadening of the red cadmium line with the greatest field-strength employed, which was about 2500 C.G.S. units.

Fig. 3.



The doubling is even more pronounced than in the case of sodium, and the following table shows that the distance

Fig. 4.



between the components is proportional to the strength of field.

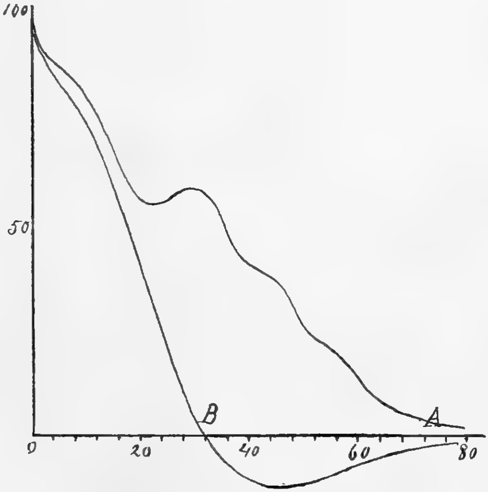
TABLE II.

	F.	Δ.	δ.	D.	α.	α calc.
A	0	90	·15	∞	·00	·00
B	4	90	·15	220	·27	·28
C	6	90	·15	136	·44	·42
D	9	90	·15	102	·59	·63

The results with cadmium are therefore essentially the same as with sodium, and are perhaps even more convincing from the fact that the red cadmium line is almost ideally simple. Further, the fact that the same results are obtained under such very different conditions (metallic cadmium vapour in a vacuum as against sodic carbonate in a blowpipe flame) would seem to furnish additional evidence against the reversal hypothesis.

The light from sodium in a vacuum-tube has a somewhat complicated and variable structure, but in one experiment, the result of which is given in fig. 5, the visibility curve (envelope) is relatively simple, and the corresponding intensity curves (fig. 6) show results almost identical with

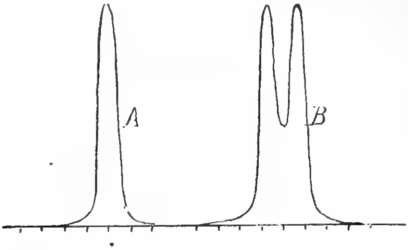
Fig. 5.



A and B (fig. 2). The strength of the magnetic field was approximately the same in the two cases.

The green cadmium line, however, is both separated and broadened, and the blue line more than the green. The green line of mercury is rather interesting on account of its complicated structure, and the results show that the general effect of the magnetic field is to obliterate details of structure, changing the form to a simple doublet as in the other cases. The separation and the broadening are nearly the same as for the green cadmium line.

Fig. 6.



Hydrogen in a vacuum-tube and lithium and thallium in the blowpipe flame are but little affected. These lines are all originally double, and in all three cases the only effect observed in the magnetic field is a slight broadening and a slight increase in the distance between the components.

In all the preceding experiments the light was examined in a direction at right angles with the magnetic field. When sodium light was allowed to pass through cylindrical holes in the pole-pieces so that the pencil was parallel with the field, the same effect of separation of the line into two was observed, and was even more clearly marked than in the transverse direction, but the broadening was inappreciable. This appears from an inspection of figs. 7 and 8.

The fact that broadening occurs only or chiefly when the pencil of light is at right angles with the field may possibly be accounted for by an increase of velocity of the radiating atom in this direction. This is what should be expected if the atom is electrified and in motion; for then a velocity at right angles with the original one would be added, giving a resultant velocity greater than before. The effect of this increased velocity would be a displacement of the corresponding spectral line proportional to the component of the velocity in the line of sight due to the Doppler effect; and as the increased velocity occurs only in the equatorial plane, the broadening would be observed chiefly in this plane *.

Fig. 7.

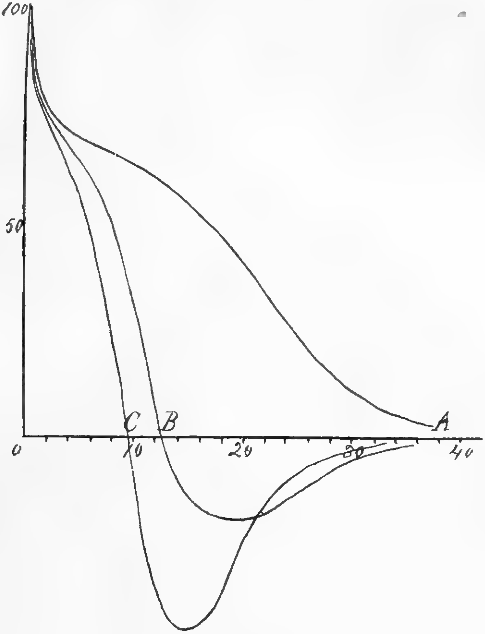
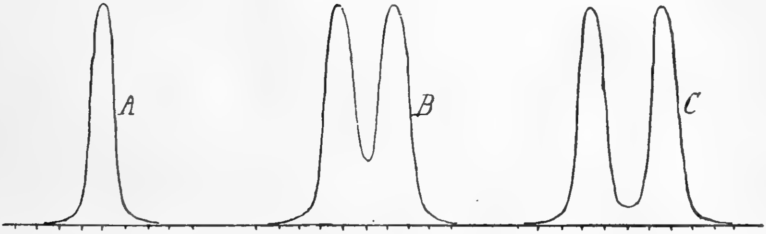


Fig. 8.



According to Zeeman the only effect of the magnetic field is to broaden the spectral lines, and the theoretical investigation fails to account for the doubling which has been observed in almost every case thus far examined.

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* It is worth noting that in almost every instance the magnetic field caused a perceptible increase in brightness.

XV. *An Instrument for the Comparison of Thermometers.* *By* W. Watson, *B.Sc., A.R.C.Sc., Assistant Professor of Physics at the Royal College of Science, London*[*].

IN most investigations where it is necessary to measure temperature it is convenient to use a small thermometer, often of short range, particularly suited for the purpose. The only practicable method of determining the errors of such a thermometer is by comparing its readings with those of a standard thermometer. This comparison is in general rendered very difficult on account of the two thermometers differing both as to the length of the scale and the capacity of the bulb. Hence if, as is essential, the bulbs are placed close together in a water-bath, a considerable length of one of the mercury columns must be out of the bath, involving an uncertain correction for cool column. Again, the difference in size of the bulbs renders the lag of the thermometers different, so that unless the temperature of the bath is kept absolutely constant for some time we cannot be sure that *both* thermometers have reached a steady temperature. These objections make the use of a water-bath at temperatures over about 20° C. both troublesome and unsatisfactory, and this paper is written in order to describe an instrument which the author has designed for comparing some thermometers required for some other work, and which has worked so satisfactorily that it may be of some use to others.

The principle employed is that of using a vapour-jacket in order to obtain different constant temperatures, as recommended by Ramsay and Young (Journ. Chem. Soc. xlvii. p. 640, 1885). A general view of the instrument is shown in fig. 1. A glass tube AB, closed at the top and open below, about 76 cm. long and 2·5 cm. in diameter, passes up inside a second glass tube of about the same length and 4·7 cm. in diameter. The space between these two tubes is at the bottom closed by an indiarubber cork C (figs. 1 and 2), while at the top the inner tube is kept in place by three small pieces of glass rod fused on and forming a triradiate star. The upper end of the outside tube is fused on to a Liebig condenser D, the upper end of the condenser being connected by a rubber joint fitted with a glycerine seal to a manometer E, a stoppered funnel F, and a large glass bottle which acts as a reservoir. This bottle is packed round with cotton-wool and is enclosed in a wooden box G. The

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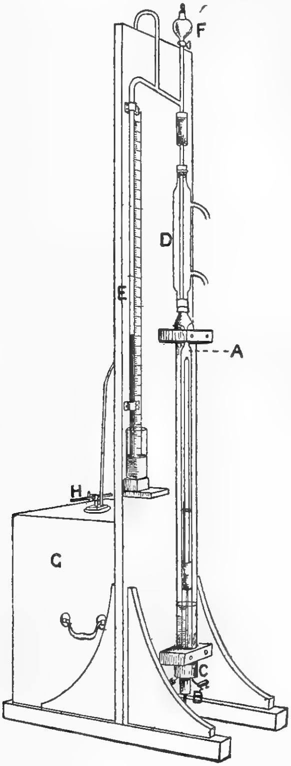
[*] Communicated by the Physical Society: read May 14, 1897.

air can be exhausted from or admitted to this bottle by means of a three-way tap H. Some mercury is placed on the top

Fig. 1.



of the cork C (fig. 2) to prevent the liquid used to form the vapour touching the cork. The liquid is heated by means of a spiral of fine uncovered platinum wire, the terminals being fused into glass tubes which pass down through the mercury and the cork.

Attempts to boil the liquid in an auxiliary bulb were rendered futile on account of the excessive bumping which took place after the liquid had been boiling for a day or two. The heating by a platinum wire, however, entirely obviates all bumping, and might be of use in determining the boiling-point or in distilling liquids which are inclined to bump, so long as they are not electrolytes.

The vapour of the boiling liquid rises between the inside and outside tube, is condensed in the condenser D, and runs down to the bottom again.

Fig. 2.

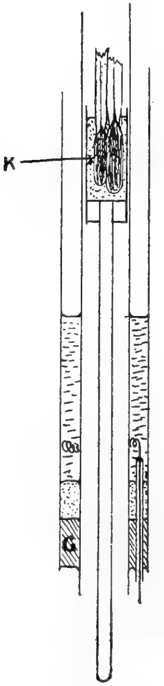
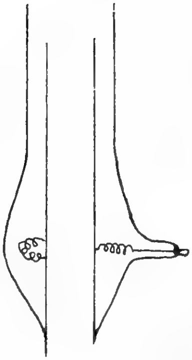


Fig. 3.



The thermometers are fastened together, their bulbs resting in a small glass vessel K (fig. 2) which is filled with mercury. This vessel is supported by a glass rod which rests on the table on which the instrument stands. To remove the thermometers the instrument is drawn to the edge of the table till this rod can be lowered and with it the thermometers. The scales are read by means of two telescopes, it being quite easy to see through the vapour if suitable liquids are used. The three liquids recommended by Ramsay and Young, viz. carbon bisulphide (20° to 46°), ethyl alcohol (46° to 79°), and chlorobenzene (79° to 120°) do very well. Water cannot be used as it forms drops on the inside of the glass so that the thermometers cannot be read. For temperatures higher than 120° the indiarubber cork closing the tube will not do, but the arrangement shown in fig. 3 may be used. The disadvantage of this form is that it can hardly be made by any but a professional glass-blower; the first form, on the other hand, can easily be put together by anyone who can do the most rudimentary glass-blowing.

In order that the indiarubber cork may form a thoroughly air-tight joint, it must be well cleaned with benzene, coated with indiarubber solution, and put in place while the solution is wet. When the solution has got thoroughly dry, which takes four or five days, the joint will be quite air-tight.

With such an arrangement the thermometers can easily be maintained at a temperature constant to within 0°·01 for three or four hours together. The manometer is only used to adjust the temperature to the desired point, a table of vapour-pressures being employed. In about half an hour after altering the pressure, and hence the temperature, the thermometer readings become quite constant. The instrument once started can be left entirely to itself; the only thing the observer has to do is to read the thermometers, then let a little more air in to get the next higher temperature. He may then go away for half an hour, and when he returns will find the temperature constant and can take the new readings.

XVI. *Note on the Effect of Pressure upon the Series in the Spectrum of an Element.* *By* J. S. Ames *and* W. J. Humphreys[*].

It has been known for many years that the spectra of certain elements, notably the alkalies and the alkaline earths, contained series of lines, which obeyed a mathematical law like that giving the distribution of lines in the spectrum of hydrogen. These series have been thoroughly studied by Kayser and Runge, who have classified them according to their physical characteristics as principal, first subordinate, second subordinate. Only a few elements, lithium, sodium, potassium, have in their spectra all three series; while the last two series, the subordinate ones so-called, are common to some ten others; the wave-lengths being different in the spectra of the different elements, the physical properties being the same.

While the shift of the lines in the spectra of the elements was under investigation, it seemed important to study in particular the effect of pressure upon the series of the various elements. To this end, photographs were taken of the arc-spectra of all elements which give series, both at ordinary pressure and at increased pressure; and the shifts were care-

[*] From the Johns Hopkins University Circulars, June 1897.

fully measured of as many of the lines as possible. In certain cases eye-observations were also made. The results for each element may be thus briefly stated:—

1. The lines of any one series of a particular element are shifted alike, *i. e.*, according to the same law, which may be written

$$\Delta\lambda=\lambda\beta(p_1-p_0)\ ;$$

where λ is the wave-length, $\Delta\lambda$ is the shift produced by an increase of pressure p_1-v_0, β is a *constant* for any one series of a definite element.

2. The constant β is different for the different series of the same element, the change being such that, *very nearly*, β for the principal series is one half β for the first subordinate and one quarter that of the second subordinate.

3. The constant β is different for the same series of different elements. Special attention is called to this fact in another note in this Circular. One apparent regularity which demands attention is the fact that, approximately, the value of β for similar elements (*e. g.* zinc, cadmium, mercury) varies as the cube root of the atomic weight.

No satisfactory theory has been advanced to account for these shifts of the spectrum-lines when the arc is under pressure. There is every evidence that it is not due to a temperature effect of any ordinary kind. It would certainly be expected that the outer envelope of an arc would be at a much lower temperature than the core, and that different electric currents might cause different temperatures in the arc; yet no shift due to these variations has been observed. It is our intention to observe the arc under these conditions with a Michelson refractometer as soon as possible, and so to learn what actually occurs. Again, the temperature of the arc should be much greater than that of a Bunsen-flame which is being fed with sodium; but the difference in the wave-length of D_1 under the two conditions is not perceptible with a 21-foot concave grating, 15,000 lines to the inch, *i. e.*, a shift, if any, must be less than 0·002 of an Angström unit. The words "temperature of the arc" are used with considerable hesitation, because so little is known as to the mean condition of the molecules of the vapour which are producing the light.

One can easily understand, however, that if the pressure on a gas is increased, the number of collisions per second must increase; and it is not impossible that this increased internal energy of a molecule, as thus produced, is the immediate cause of a change in the actual size of the molecule. The extent of this change would depend upon the looseness of construction of the molecule apparently; and this quantity is measured to a certain degree by the coefficient of expansion of the element in the solid form. Therefore it might be expected that the measured shift would vary in the same direction as the coefficient of expansion of the solid. This is actually the case, with no exception. Again, since on any theory of emission of waves the wave-length varies directly as the *linear* dimension of the portion of matter producing the waves, it would be expected that the measured shift would vary directly as the coefficient of *linear* expansion, which is found to be the case. If a is this coefficient, it may be stated as an experimental law that for different elements $\beta = ca$, where c is a constant which in some way connects the ordinary changes in size of a solid due to temperature changes with the hypothetical changes in the size of the molecules of the vapour due to pressure changes.

It is not difficult to see that one would expect, as consequences of the above ideas, that the shift would be proportional to the total increase of pressure, regardless of its mode of production; and that it should also vary directly as the wave-length in any one series or group of lines, for in such a case the longer waves indicate greater linear dimensions of the vibrating segments, if the term may be used. These are observed phenomena, as is stated above.

The fact that the shift characteristic of a principal series is less than that of the first subordinate, and this in turn less than that of the second subordinate, would be expected, in accordance with these ideas, *if* the molecules producing the principal series were of a simpler structure than those producing the first subsidiary; and if the molecules producing the second subsidiary were the most complex of all. For, since the shift of a series depends upon the looseness of the molecular structure, it would be expected that, if these molecules split up in any way, the fragments would be more stable and firm than the original molecules, and therefore the shift of the original molecules would be greater than that of the fragments. It is difficult, however, to see any reason why the shift of the different series should vary according to any simple law, or why the shift of the same series of different elements should be in accordance with any formula so simple as that of the cube root of the atomic weight. If it could be assumed that the shift was proportional to the linear dimensions of the segment producing the waves, most interesting deductions might be drawn; but there seems to be no justification for the assumption.

XVII. *The Effects of Tension and Quality of the Metal upon the Changes in Length produced in Iron Wires by Magnetization.* *By* Byron Briggs Brackett [*].

ALL the specimens of iron wire experimented upon were about 2 metres long and 1·25 millim. in diameter. While under test a wire was suspended vertically so that tension could be applied by weights attached to its lower end. About the central portion of the wire was placed the magnetizing solenoid, 84 centim. long and with a free internal diameter of 3·6 centim., allowing no contact between it and the wire or parts of the apparatus attached to the wire. The variations in the length of 70 centim. of the wire within the solenoid was then observed by means of a mechanism, employing a lever and tilting mirror, in such a way as to magnify the actual changes about 25,000 times, and hence to give readings for $\frac{dl}{l}$ multiplied by 1,750,000.

For all tests made, with one exception, the changes in length caused by a definite magnetizing field were read both with the field on and after it had been removed. These readings were made for gradually increasing fields up to the limits of the apparatus, the points taken on each curve being very numerous and especially close together wherever the curve changes its direction rapidly.

All of these tests were followed by careful tests for the magnetic induction in the specimens under conditions identical with those of the elongation tests. From these, the induction in the wire is known for each elongation, and curves are plotted between elongation and induction, as well as between elongation and magnetizing field.

Three different kinds of wire were tested: piano-wire in its natural condition: annealed piano-wire: and very soft annealed iron wire.

Results obtained:—

1. *Natural Piano-wire.*

This wire could not be kept straight without considerable tension; but under the tensions employed (658 and 1949 kg. per sq. cm.) no change of length was observed until the fields were reached at which the magnetization rises rapidly. Changes in length were first observed when the field was

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[*] From the Johns Hopkins University Circulars, June 1897.

removed. This was an elongation, and the length of the wire with the field off continued to increase up to the field strengths at which the induction curve turns towards the horizontal. From that point on, the wires were permanently elongated with the field off. But this value of $\frac{dl}{l}$ did not exceed $3{\cdot}5 \times 10^{-7}$, and could have been produced by the tension caused by adding about 10 grams to the weight already on the wire. At a very short distance beyond the point where the wire begins to lengthen as the current is broken, it shows its first change of length with the field on by beginning to contract. From this point on to the end, the wire contracts with the field on, and the contraction increases apparently in direct proportion to the field until at the end, for a field of 300, $\frac{dl}{l}$ equals about 30×10^{-7}.

The curves for the different tensions differ but little.

2. *Annealed Piano-wire.*

After the piano-wires had been tested in their natural condition they were softened by annealing, and the same wires were tested again. They then showed elongations with the fields both on and off up to the point where the induction curve turns towards the horizontal. The elongations here are greater with the field off than when it is on. But on each curve the elongation increases nearly in direct proportion to the induction. Not far from the turning point of the induction curve, the elongations with the field off cease to show any change, this value of $\frac{dl}{l}$ not exceeding 7×10^{-7}. At about the same point the curve with field on begins to descend along a nearly straight line, just as it did before the annealing. But here the contraction is more rapid, the slope for the straight part of the curves, as plotted to field, being nearly 50 per cent. greater than before annealing.

Increased tension does not affect the general shape of the curve. But it diminishes the amount of the elongation and increases the final contraction with field on, while it causes both elongation and contraction to begin at weaker fields.

3. *Soft Annealed Wire.*

The curves for the very soft wire are in general exactly like those for the annealed piano-wire, except that the changes of length are very much greater, and in some cases for weak fields and slight tension, the wire shortens when the field is removed. But in these cases the residual induction in the wire is always much less than the induction with the field on. And finally the elongation with the field off becomes as great or even greater than the maximum elongation with the field on. After this the length tends to remain constant until the end of the experiment, although there is a very slight shortening for the strongest fields. This permanent lengthening for the soft iron gave a value of $\frac{dl}{l}$ as great as 33×10^{-7} for the least tensions used. The straight portion of the curves with field on shows a contraction that is much more rapid than for annealed piano-wire, and its slope is more than twice as great as that for the natural piano-wire.

Increased tension showed exactly the same effects as upon the annealed piano-wire, though the amount of the changes due to tension was much greater.

The complete series of experiments would tend to confirm Joule's statement that the changes in length caused by magnetization are greater in proportion to the softness of the iron. This is more marked in the initial elongations than in the final contractions.

The elongation curves, both when plotted to magnetizing force and when plotted to induction in the iron, seem to show an *elongation* due to the magnetic induction in the wire and a *contraction* caused by the magnetizing field, directly proportional to the field in value. In all cases for a definite induction the wire was longer with the field off than with the field on.

More than one application of the same magnetizing field gave slightly cumulative effects. Repeating any reading, thus, made the change a very little greater in the direction of the original change, but the increase was not significant.

The changes in length caused by adding and removing a definite weight both before and during magnetization gave no evidence of a change in Young's modulus, due to magnetization. But an examination of all the data of the complete series of experiments suggests an increase in Young's modulus with the intensity of magnetization, the maximum change not exceeding one-half per cent., and nearly all of it having occurred before the magnetization curve turns towards the horizontal. This view fully explains the reduced elongation and greatest final contraction with the larger tensions.

XVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xliii. p. 463.]

April 28th, 1897.—Dr. Henry Hicks, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'Note on a Portion of the Nubian Desert South-east of Korosko.' By Capt. H. G. Lyons, R.E., F.G.S. With Notes on the Petrology by Miss C. A. Raisin, B.Sc. And Water-Analyses by Miss E. Aston, B.Sc.

A general description of the physical structure of the area, which consists mainly of Nubian Sandstone and crystalline rocks, is given. Coming from Korosko to the Murrat Wells, the crystalline rocks are first met with on the east side of Jebel Raft. At Wadi dur Nabadi are ancient gold-workings, indicated by M. de Bellefonds on his map of the Etbai. The crystalline rocks are both massive and schistose. The strike of the latter to the east of Jebel Raft is north and south, while to the south of the hill the strike is a few degrees on either side of east and west. On the crystalline mass of Jebel Raft, and apparently overlain by the Nubian Sandstone, is a very coarse conglomerate containing fragments of crystalline rock, which appears to be older than the Nubian Sandstone.

The Nubian Sandstone has little or no dip, and shows very slight variation in composition. The water-supply of the Nubian Desert is directly dependent on the rainfall, which is very irregular. The wells are sunk in the detritus of the valleys, and contain a large amount of mineral matter in solution, which renders them almost undrinkable; while the second source of supply—the rain-water reservoirs—are deep holes in the ravines which intersect the crystalline hills. These holes must be attributed to water-action; and in the reservoir of Medina in Jebel Raft the spherical stones which assisted in forming the pothole still occur. The author believes that these ravines and reservoirs were formed at an earlier period than the present, when the rainfall was heavier.

Miss Raisin gives accounts of both massive and schistose crystalline rocks, and also of sedimentary rocks. The crystalline rocks described include gneiss, hornblendite, gabbro, often much altered and resembling some of the Alpine gabbros, some allied rocks containing lustre-mottled hornblende, other forms of diabase, quartz-diorite, granite, felsites, certain schistose and a few distinctly fragmental rocks. None of the igneous rocks can be stated with certainty to have originated as a lava-flow. Many of them have undergone much alteration since their consolidation, and the results of this are described. There is clear evidence in many cases of erosion by desert-sand and the formation of a weathered coating. The schistose rocks do not present a very modern facies, and might be late Archæan or early Palæozoic. The massive crystallines may belong to different epochs, but include some rocks (such as the gneiss) resembling Archæan. These seem to mark an eastward extension of the anticlinal axis previously traced by Capt. Lyons to Wady Halfa in the Nile Valley from the west.

Miss Aston gives two tables, one of which shows the actual amounts of substances found in the wells of Murrat, Bir Tilat Abda, and Bir Ab Anaga, while the second shows their approximate constitution.

2. 'On the Origin of some of the Gneisses of Anglesey.' By Charles Callaway, M.A., D.Sc., F.G.S.

The author still maintains the occurrence of two pre-Cambrian groups in Anglesey, the later of Pebidian age. In the present paper a description is given of the production of gneissic structure in rocks of the earlier group occurring in the south of the island. The products of metamorphism are similar to those described by the author in the Malvern area. (i) Simple schists, granite is converted into mica-gneiss, diorite into hornblendic or chloritic gneiss, and felsite into mica-schist; (ii) Injection-schists. In Anglesey felsite plays the part which is taken by granite at Malvern, and certain important differences arise. Gneiss of primary injection is formed by the parallel veining of felsite and diorite, and the author describes cases; he also notices the occurrence of metamorphosed diorite-blocks in grey gneiss, which afford evidence of its eruptive origin. Gneiss of secondary injection is well seen in the Llangaffo cutting, and is explained in a manner similar to that given for an infiltration-gneiss at Malvern, felsite being substituted for granite. The infiltration-products are mainly chlorite, iron oxide and epidote, and out of the chlorite and iron oxide, with the addition of materials from the felsite, black mica has sometimes been constructed. The changes have occurred along planes of discontinuity formed subsequently to the consolidation of the rocks.

May 12th.—Dr. Henry Hicks, F.R.S., President, in the Chair.

The following communications were read:—

1. 'On the Gravels and associated Deposits at Newbury (Berks).' By E. Percy Richards, Esq., F.G.S.

After a general sketch of the geology of the Valley of the Kennet, the superficial deposits at and in the neighbourhood of Newbury are described in detail, from observations made by the author during the progress of the main drainage-works in 1894.

The author classifies the strata which he has examined in five groups:—

1. The Preglacial Southern Drift.
2. The Glacial Drift (Donnington).
3. The Upper River-gravel.
4. The Lower River-gravel.
5. The Neolithic peat-beds (shell-marl, peat, and loam).

1. The first of this series is the flint-gravel constituting the

heathy Commons north and south of Newbury (as of Snelsmore and Greenham). About 6 feet of gravel is exposed at the average level of 440 feet above Ordnance datum. It has been referred to the 'Southern Drift' by the late Sir Joseph Prestwich. At the edges of the Commons some of this gravel slips down and mixes with glacial deposits of later age.

2. North of the town are the Glacial Gravel and Loam of Donnington and the Bath Road (17 feet), ranging from 254 to 300 feet above O.D. At the confluence of the Lamborne and Kennet Valleys these beds have been much denuded. The thickness, character, and composition of these deposits, as well as those of other gravels, etc., described hereafter, have been carefully noted by the author.

3. The River-gravels of the Kennet Valley are of great interest, and have been carefully studied by the author at their several exposures, in Bull's Lane, the Enbourne Road. etc. Here the Upper River-gravel forms fine broad terraces at about 260 feet O.D. Palæolithic flint-implements are not uncommon in this gravel. Bones and teeth of mammoth, common and extinct ox, sheep, pig, horse, and reindeer have also been found in it. Immediately north of Newbury this terrace is cut off locally from the town by an east-and-west fault at Northcroft, but is distinct a little farther to the west.

4. The Lower River-gravel lies in the middle of the valley under the peat-beds of Newbury, and rests on the Chalk. It appears to have been derived from all the other gravels, but with a large percentage of flints direct from the Chalk. It is about 10 feet thick locally, and at some places has a white clay at its base.

The author finds that this gravel forms ridges across the valley, which seem to have been the limits of shallow-water areas, in which peat-bogs, and clear pools with river-shells were locally produced (5), giving rise to patches and alternations of peat and marl. These varied in extent and in thickness, according to the damming-up of the lakes and their being drained by the currents causing gaps in the gravel-barriers, which are more or less easily traced. The peat varies from mere seams to beds 5 or 6 feet in thickness, and the marl from a few inches to 8 feet. The shells and bones found in the peat and shell-marl are enumerated, much as in earlier lists, so also the plant-remains.

Above the peat-beds in the town of Newbury about 4 feet of coarse triturated gravels represent layers of road-material laid down in Roman and Mediæval times; these are blackened with ancient sewage, and contain various metal and other relics. Cannon-balls fired from Donnington in 1644 A.D. occurred only 18 inches below the road-surface; 'macadam' has been accumulated on the top to a thickness of 2½ feet. A pile-structure of fir-wood was discovered in the peat in St. Bartholomew Street.

2. 'The Mollusca of the Chalk Rock: Part II.' By Henry Woods, Esq., M.A., F.G.S.

May 26th.—Dr. Henry Hicks, F.R.S., President, in the Chair.

The following communications were read:—

1. 'On Augite-Diorites with Micropegmatite in Southern India. By Thomas H. Holland, Esq., A.R.C.S., F.G.S., Officiating Superintendent, Geological Survey of India.

This paper deals with a series of basic dykes intersecting the pyroxene-granulites and gneisses of the Madras Presidency, and believed to be of the same age as the lava-flows of the Cuddipah system. These dykes consist essentially of augite (near hedenbergite) and a plagioclase-felspar (near labradorite), between which we find masses of micropegmatitic intergrowths of felspar and quartz, with a micro-miarolitic structure. Around the patches of micropegmatite, chemical changes have frequently taken place in the minerals of the rock. Similar occurrences of micropegmatite in basic rocks have been described by J. A. Phillips and Waller at Penmaenmawr, by Hill and Bonney in Charnwood Forest, by Teall in the Whin Sill and in the Cheviot district, by the last-named author and Harker at Carrock Fell, by Sollas at Carlingford, and by Harker in Skye.

After discussing the chemical constitution of the rock, and of its various constituents, and the relation between the micropegmatite and the surrounding minerals, the author points out that three methods for the formation of the micropegmatite may be conceived of:—

(α) During the primary consolidation of the magma.
(β) By secondary changes induced in the rock.
(γ) By subsequent intrusion of granophyric material into the augite-plagioclase rock.

In opposition to (γ), the author points out the entire absence of granitic intrusions in the neighbourhood. He regards the absence of all proofs of subaerial hydration, and the remarkable freshness of the rocks as precluding the possibility of the micropegmatite having been formed by secondary change. The primary origin of the micropegmatite he believes to be proved by:—(1) The crystallographic continuity of its felspar with that of the normal plagioclase of the rock; (2) the mode of occurrence of the micropegmatite, filling in the angles and spaces between the augite and the plagioclase; and (3) its variation in coarseness of grain agreeing with that of the remaining two constituents of the rock.

These augite-diorites with micropegmatite are then compared with the granophyric gabbros of Barnavave, Carlingford, described by Prof. Sollas, to which the author maintains that his explanation equally applies. He insists that the separation of the acid and basic materials of a magma, which takes place so frequently on a large scale, equally occurs in the midst of a consolidating mass, and combinations such as that described in the paper will be formed. He insists also on the proofs of close genetic relations between the gabbros and 'granophyres,' and of their practical contemporaneity in the various districts in which these phenomena have been noticed.

2. 'The Laccolites of Cutch and their Relations to the other Igneous Masses of the District.' By the Rev. J. F. Blake, M.A., F.G.S.

The author has observed thirty-two domes of various kinds in Cutch, distributed as follows:—(i) those connected with the northern islands; (ii) those of Wagir; and (iii) those along the northern edge of the mainland. They are divisible into four classes: (*a*) those which are so elongated on the line joining adjacent ones that they seem to be mere modifications of anticlinals, though the supposed anticline is not really continuous; (*b*) those which lie in a line, but are not elongated in that direction, and often in no other; (*c*) those which are related to a fault, which cuts them in half; and (*d*) those which are not in any particular relation to each other, or to any other stratigraphical feature.

The domes vary in degree of perfection: some are irregular, while some have the strata running in concentric circles, the outer and newer strata dipping away from the inner and older. In no less than ten of the thirty-two domes igneous bosses are found occupying the centre, and these are distributed amongst all of the above classes. The author gives reasons for maintaining that the domes are the results of intrusion of igneous rocks in the form of laccolites, and are not anticlinal folds which have afterwards been affected by cross-folds. The domes are contrasted with igneous peaks which occur in abundance in a different part of the area, usually at a higher horizon of the strata and at a higher level above sea. These are probably volcanic pipes through which the lava was forced and extruded at the surface.

The author compares the rocks of the bosses with those of the dykes and flows. Both are principally perfectly fresh dolerites, but the former are distinguished by the presence of intergrowths of micropegmatite as the last stage of consolidation, as in the 'Konga diabases.' There is also among them a felsite-breccia with micropegmatite developed in the cracks.

He considers that nearly all the igneous rocks of Cutch have been derived from a single magma, which in a solid condition must have contained large crystals of augite, olivine, and ilmenite in a groundmass of lime-felspars, and have been throughout of a basic character. Such a magma originated in more than one centre. One was possibly not far from the Sindree basin, whence lines of weakness diverged. Along these, owing to the thickness of the strata, there was no extrusion at the surface, and laccolite-domes were formed. Where the rock reached higher levels, it spread out into sheets between the domes and aided in the production of synclinals. Another centre was west of Bhuj, where the rock reached the surface without materially disturbing the sedimentary rocks, and formed the so-called 'stratified traps.' There were also areas here, especially to the south, where escape was impossible and domes were formed.

The author observes that his conclusions, if correct, may be applied to explain the source of the Deccan trap without eruptive centres. It may have been forced out from innumerable orifices as from a sieve, none of these being so much larger than others as to make a definite centre.

June 9th.—Dr. Henry Hicks, F.R.S., President,
in the Chair.

The following communications were read:—

1. 'The Cretaceous Strata of County Antrim.' By W. Fraser Hume, D.Sc., F.G.S.

The paper, which deals with the Irish Cretaceous strata, is divided into four parts.

I. This contains a detailed account of the principal subdivisions, their local distribution, and characteristic fossils. The area occupied by these rocks is separated into five divisions, each marked by special lithological and palæontological features. These are:—(*a*) The Southern Division, extending from Moira to Lisburn. Here the highest zone of the Chalk, that of *Belemnitella mucronata*, rests directly on the Triassic strata, the junction-bed containing large pebbles, associated with the above-mentioned species.—(*b*) The Central Division, mainly including the sections between Lisburn and Belfast. All the main lithological features are here displayed, namely: 1. Glauconitic Sands (a blue-green rock rich in glauconite); 2. Glauconitic Marls; 3. Yellow Sandstones (a light calcareous sandstone); 4. Chloritic Sands and Sandstones of the *Exogyra columba*-zone (yellow-green sands and sandstones); and 5. White Limestone. Under No. 3 Tate and Barrois both included the zone rich in *Inoceramus*-fragments occurring in the next division, but this zone is different in colour and chemical composition, and is practically absent in the Central area, the White Limestone with conglomeratic or nodular chalk at the base resting unconformably on the lower beds. Palæontologically, all these lower beds are rich in large dimyarian bivalves, especially *Trigoniæ*, *Arca*, *Cucullæa*, etc.—(*c*) The Eastern Division, extending along the eastern coast from the neighbourhood of Carrickfergus to north of Carnlough, including Islandmagee. The Glauconitic Sands and Yellow Sandstones with chert are present, but generally thinner than the same beds in the Central area. In them dimyarian bivalves are very rare, brachiopoda of stunted forms, corals, and sea-urchins having been, on the contrary, only found in them in this eastern region. The *Exogyra columba*-zone is barely represented, but the *Inoceramus Crispi*?-zone (which Tate thought to be below that of *Exogyra columba*) attains a thickness of over 20 feet, and evidence is brought forward to show that it overlies the *Ex. columba*-zone. There is a palæontological unconformity between the two zones, the *Inoceramus*-beds passing insensibly into the White Limestone without the intervention of a conglomeratic bed.—(*d*) The Peninsular Division. In the northern portion of county Antrim, the highest zones of the White Chalk rest on Archæan, Carboniferous, and Triassic rocks (which must have formed a peninsula or insular area in the Cretaceous sea), the basal conglomeratic bed being well marked, and old Cretaceous beaches preserved.—Finally (*e*) The Northern Division. This occupies a strip on the northern and north-eastern shores, and includes Rathlin Island. The lower beds are only feebly developed, while the three highest zones of the White Limestone (*Belemnitella vera* to *B. mucronata*) attain a great thickness, and are very pure chalk-beds. Taking the various divisions together, it is concluded that Divisions *b*, *c*, and *e* first underwent depression, area *b* being nearest to the shore-line, Divisions *a* and *d* being higher ground, and only becoming submerged at a late period in Upper Chalk times.

II. Chemical and Micromineralogical examination of the lithological types.—The Glauconitic Sands are characterized by the abundance of glauconite (23 °/$_\circ$ $CaCO_3$), showing evidence of having been formed in the interior of foraminiferal shells; the Glauconitic Marls by an abundance of spheres and rods of pyrites; the Yellow Sandstones by a series of heavy minerals, notably rutile, zircon, tourmaline, kyanite, and perfectly-formed crystals of garnet; the *Inoceramus*-zone contains delicate glauconitic meshworks of hexactinellid sponges and silicified portions of *Inoceramus* and brachiopoda, besides an abundant series of heavy minerals: in the Chloritic Chalk above the Glauconitic sponge-casts become very abundant, associated with delicate casts of foraminifera; the White Limestone itself has scarcely any residue. The analyses show that the percentage of carbonate of lime increases steadily from base to summit, the Glauconitic Marls alone being an exception.

III. This section deals with the stratigraphical conclusions. The Glauconitic Sands, regarded by Tate as equivalent to the lower portion of the Upper Greensand, and by Barrois as synchronous with the upper beds (*Pecten asper*-zone) of the same formation, are regarded as having commenced to be deposited in the earlier portion of that period, and to have been still forming during the *Pecten asper* times in England. The Yellow Sandstones with chert are mainly equivalent to the Warminster Sandstones with chert in the S.W. of England. The above two beds therefore constitute the Upper Greensand in Ireland. The *Exogyra-columba* and *Inoceramus-Crispi*? zones were both considered by Tate as Cenomanian, and by Barrois as Turonian. In the present paper they are considered as absolutely distinct, the *Exogyra columba*-zone being truly Cenomanian (as limited by Jukes-Brown and Hill); the *Inoceramus Crispi*?-zone, on the contrary, being similar in its fauna to the Chalk Rock and lowest Senonian beds in England, and on theoretical and faunal grounds being in preference assigned to the latter.

The *Micraster*-zones of the Senonian have been directly recognized, but the higher ones of *Belemnitella vera* (*Marsupites*) and *B. quadrata* are well represented. The nodular beds at the base of the

White Chalk are of varying age, but generally referable to the above two zones. The conclusion arrived at is that there was a gradual depression followed by a partial elevation, and a subsequent depression of a far more pronounced character. The Central Division being nearer the shore, the deposits of detrital material here formed were thicker, but at the same time during the period of elevation the denudation was more considerable, so that the unconformity is very marked; in the Eastern Division, where the sea is considered to have been deeper, the detrital deposits were not so thick and denudation less pronounced; on depression commencing at the beginning of the Senonian period, limestones containing a Chalk Rock fauna were being laid down here, while denuding influences were still active in the Central area. Speaking generally, there was a deepening northward of the Cretaceous Sea, but the high ground in the north of County Antrim and south of Lisburn was the cause of the very marked beach and conglomeratic features observed in the Peninsular and Southern districts.

IV. *General Questions.*—The author here touches on and discusses (i) the occurrence of certain organic remains in the glauconitic sands and yellow sandstones of the Eastern division, which suggest higher zonal conditions than those indicated by the species with which they are associated; (ii) the great development of the *Exogyra-columba* beds south of Belfast and their feebleness north of that town, while the opposite holds in the case of the strata with *Inoceramus*-fragments and *Spondylus speciosus*; (iii) cases of unconformity and overlap in the series; (iv) the existence of definite beaches of Upper Chalk age; (v) evidence in favour of the influence of currents; (vi) the relation of the Spongiarian bands to the strata above and below; (vii) faunal repetitions; and (viii) the general distribution of the *Rhynchonellæ* and *Terebratulæ*.

2. 'An Account of the Portraine Inlier.' By C. I. Gardiner, Esq., M.A., F.G.S., and S. H. Reynolds, Esq., M.A., F.G.S.

At the north-western corner of the inlier a coarse conglomerate, referred to the Old Red Sandstone, dips in a north-westerly direction, and from beneath it a series of green and purple andesites having a south-easterly dip appear. These andesites have undergone considerable change since their extrusion. They are succeeded by a conglomerate of varied character, of which the matrix is at first an ashy shale, including fragments of ash, andesite, shale, and limestone. Interbedded with the conglomerate are bands of limestone and shale, which have yielded fossils of Middle Bala facies. The authors give reasons for maintaining that this conglomeratic series is not due to earth-movement, but is a sedimentary accumulation, though the case is otherwise with a conglomerate developed along a thrust-plane which separates the volcanic series from an overlying limestone. This conglomerate is compared with one recently described in the Isle of Man, though the alteration which marks the latter is practically absent from the former.

Igneous rocks like those found to the north-west of the outlier are also seen at the southern end.

The general succession of the greatly disturbed Portraine Limestone Series from above downwards is:—3. Thin-bedded limestone, with shaly partings; 2. Beds of compact crystalline limestone, with many fossils; 1. Thin-bedded limestones, with shaly partings: the upper beds being in places crowded with corals. This limestone is comparable with the Chair of Kildare Limestone, and also with the *Staurocephalus*, Keisley, and Sholesbrook Limestones of Great Britain. Succeeding the Limestone Series, and separated from it by a thrust-conglomerate, is a Grit Series, which has yielded no fossils, but which resembles the Balbriggan Grits containing black shales with Birkhill graptolites.

Mr. F. R. C. Reed, M.A., F.G.S., in an appendix, gives a list of the fossils found by the authors, and offers some remarks upon the age of the deposits.

XIX. *Intellience and Miscellaneous Articles.*

MEASUREMENT OF ELECTROLYTIC CONDUCTIVITY BY CONTINUOUS CURRENTS.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN your June number (p. 464) Dr. Neesen writes to say that Dr. Henderson and I made no mention of his name in a paper on "A Satisfactory Method of Measuring the Conductivity of Electrolytes by means of Continuous Currents" published in No. 260 of your Journal.

We must in the first place express great regret at having altogether overlooked Dr. Neesen's paper in Wied. *Ann.* vol. xxiii. p. 482. Some excuse for the omission may be found in the fact that no reference to the paper is to be found in Winkelmann's *Handbuch* (1895) where scores of papers on the measurement of electrolytic conductivity are mentioned. There is, however, as I have just now found, a reference to it among the Nachträge in Wiedmann's *Electricität*, iv. p. 1326. No allusion is made even there to Dr. Neesen having devised any special method of using continuous currents for the measurement of electrolytic resistances.

We willingly concede to Dr. Neesen the credit of having conceived before ourselves the idea which underlies a part of the method which we adopted. Nevertheless we feel that we may claim that the "satisfactoriness" of the method is in large measure due to ourselves. Dr. Neesen succeeded in attaining an accuracy of 1 in 1000 for the special case of solution of sulphate of iron with iron electrodes. We succeeded in attaining a much greater accuracy with Pt plates in dilute sulphuric acid or KCl solution, which Dr. Neesen must admit is quite a different thing.

There are three ways known to us of diminishing the disturbing effects of polarization in the use of continuous currents:—

(1) The use of so-called non-polarizable electrodes.

(2) The use of high potentials so as to drown the polarization E.M.F.

(3) The use of a balancing cell in a Wheatstone bridge as first suggested by Kohlrausch, employed by Tollinger, and greatly improved by Neesen

The first method is of very limited application. The third method by itself we know to be quite unsatisfactory, but combined with the first it will be fairly satisfactory as Neesen found. The combination of the methods (2) and (3) which we regard as being the distinguishing feature of our method leaves nothing to be desired, and is in our opinion much more convenient and accurate than the alternating current method. Dr. Neesen makes no claim in his paper of 1884 to have devised a method superior to the alternating current method.

The last two paragraphs of Dr. Neesen's letter show how little he appreciates the points of our method, one of the main features of which is the employment of high potentials to drown the residual polarization E.M.F. This necessitates high electrolytic resistance, not high resistance of the two equal arms of the bridge.

I am, Gentlemen,
Your obedient servant,
W. STROUD.

SURFACE TENSION OF WATER AND OF DILUTE AQUEOUS SOLUTIONS. BY N. ERNEST DORSEY.

During the past year I have been endeavouring to determine the surface-tension of dilute aqueous solutions by means of the method of ripples. All work previously done on the surface-tension of solutions has been on solutions of about one half normal concentration, or greater, and most of the observers have deduced the surface-tension from the measured rise of the solution in capillary tubes.

For at least two reasons the method of capillary tubes is open to serious objections. First, the height a liquid rises in a tube depends upon the angle between the wall of the tube and the surface of the liquid *where it meets the tube.* This contact angle cannot possibly be measured, since the surface of the liquid lies entirely on one side of the point where we wish to know its inclination; and as we can measure the inclination of a finite surface only, every measured value of the contact angle must be too large.

The second objection is that probably the surface-tension of the solution-glass surface, as well as that of the solution-air surface, varies with the concentration of the solution. If such is the case, the surface-tension found will depend upon two changes which cannot be readily separated, and which render the interpretation of the results difficult.

For these reasons I decided to use the method of ripples, which was first successfully used by Lord Rayleigh, although with his arrangement of apparatus individual observations differ by about

2 per cent. After trying many plans one was finally adopted that gives individual results that agree to about $\frac{1}{3}$ per cent.; and the average departure of single observations from the mean of several seldom exceeds $\frac{1}{7}$ per cent.

The waves were generated by a fork whose frequency was often determined, and was always near 62·87 double vibrations per second. The water and solutions were contained in a porcelain tray 1 by 12 by 14 inches. The wave-length was measured by means of a telescope mounted on a dividing-engine, whose screw had a pitch of 1·0328 millim. The waves were invisible under ordinary conditions, but were observed by Foucault's method for rendering visible small vibrations in plane or spherical surfaces.

The water used was especially distilled by Mr. W. T. Mather from chromic acid and alkaline potassium permanganate, and was condensed in a block-tin condenser; it was the kind used by him for his electrolytic work. The salts were obtained from Eimer and Amend, and were said to be chemically pure.

With this apparatus I have determined the surface-tension of water and of solutions of sodium chloride, potassium chloride, sodium carbonate, potassium carbonate, and zinc sulphate, of concentrations varying from 0·05 normal to normal.

The value found for water is T=75·98 dynes per centimetre at 0° C.; while Sentis (*Journ. de Phys.* (3) vi. p. 183, 1897), working by an entirely different method, found T=76·09 at 0° C., which differs from the other by only 0·14 per cent. These values agree very well with the values given by Lord Rayleigh, Hall, Volkmann, and others, but are much lower than Quincke's value.

It was found that the surface-tensions of dilute aqueous solutions are linear functions of the concentration; so that we may write $T_s = T_w + kC$, where T_s= surface-tension of the solution, T_w= surface-tension of water at the same temperature, k is a constant, C is the concentration in gramme molecules per litre. Below is a table showing the values of k as determined by different observers:—

	Dorsey.	Volkmann.	Quincke.	Rother.
NaCl	1·53	1·59	1·57	1·38
KCl	2·23	1·41	1·57	1·47
$\frac{1}{2}Na_2CO_3$	2·00	0·987	1·57	
$\frac{1}{2}K_2CO_3$	1·77	1·78	1·57	
$ZnSO_4$	1·86			

Volkmann found that the curve for Na_2CO_3 at great dilution becomes steeper than the one for K_2CO_3, which agrees with my results. Quincke's value 1·57 does not agree with his results except for KCl and NaCl. I cannot account for the very high value I found for KCl; but it must be borne in mind that the values given above are not fairly comparable, since my values are for solutions generally less concentrated than $\frac{1}{2}$ normal, while the other are found for solutions of greater concentration.—*Johns Hopkins University Circulars*, June 1897.

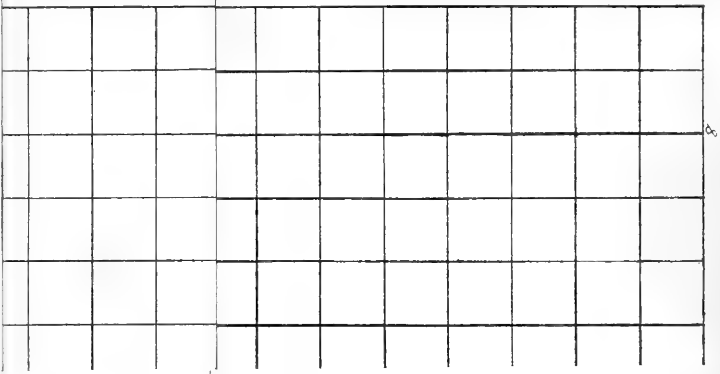
PRELIMINARY NOTE ON THE ENERGY SPECTRUM OF A BLACK BODY. BY C. E. MENDENHALL AND F. A. SAUNDERS.

In the Astrophysical Journal for August, 1895, Dr. H. F. Reid suggested the experimental study of this subject, based upon the theoretical deductions of Stewart and Kirchoff, that the radiation existing in an enclosure with walls at a uniform temperature would be independent of the character of the walls and would be that of an absolutely black body at that temperature, provided only that the walls had at least an infinitesimal absorptive power for all wave-lengths. This work we have been carrying on, and though it is not completed, enough has been obtained to warrant a brief account.

In order to study the radiation in the interior of a closed body it is of course necessary to cut a small slit in its walls, and thus the theoretical conditions cannot be absolutely fulfilled; then, again, it is practically impossible to obtain an absolutely uniform heating of a body of considerable size. If it is found, however, that varying the size of the slit does not change the form of the curve, then we can be reasonably sure that the first of these difficulties is an unimportant one. This point is now under investigation. As regards the non-uniformity of temperature it is reasonable to suppose that any curve will be a mean of the curves corresponding to several adjacent temperatures, and that it will not, in form, depart far from any one of them, if the departure from uniformity is not large.

The bodies we have used are cylinders of iron and of copper, 6 inches high and $4\frac{1}{2}$ inches inside diameter, and the ratio of the area of slit to the total inside area is $\frac{1}{250}$. One arm of the spectrometer reaches to within 3 inches of the furnace and carries the real optical slit which is 1·7 millim. wide. The radiation through this slit falls upon a rock-salt prism with faces 2×3 inches and then upon a lens of the same material, 3 inches in diameter, which focusses it upon the bolometer strips, which are of platinum foil, coated with platinum black, and these subtend an angle of five minutes at the axis of the spectrometer. The galvanometer is a four-coil one, has a resistance of three ohms, connected in parallel, and gives a sensibility of about 5×10^{-10} for 10 sec. complete period and a metre distance.

As far as our preliminary results show there is between 1050° and 500° C., a noticeable shift of the maximum of the energy spectrum towards the blue with rise of temperature. An absorption band, not due to water or to carbon dioxide, falls very near the maxima of our curves and often prevents these from being directly observed; this is probably due to gases filling the black body, entering from the gas-burner by the holes through which pass the platinum-platinum-iridium thermocouples used in measuring the temperature of the black body.—*Johns Hopkins University Circulars*, June 1897.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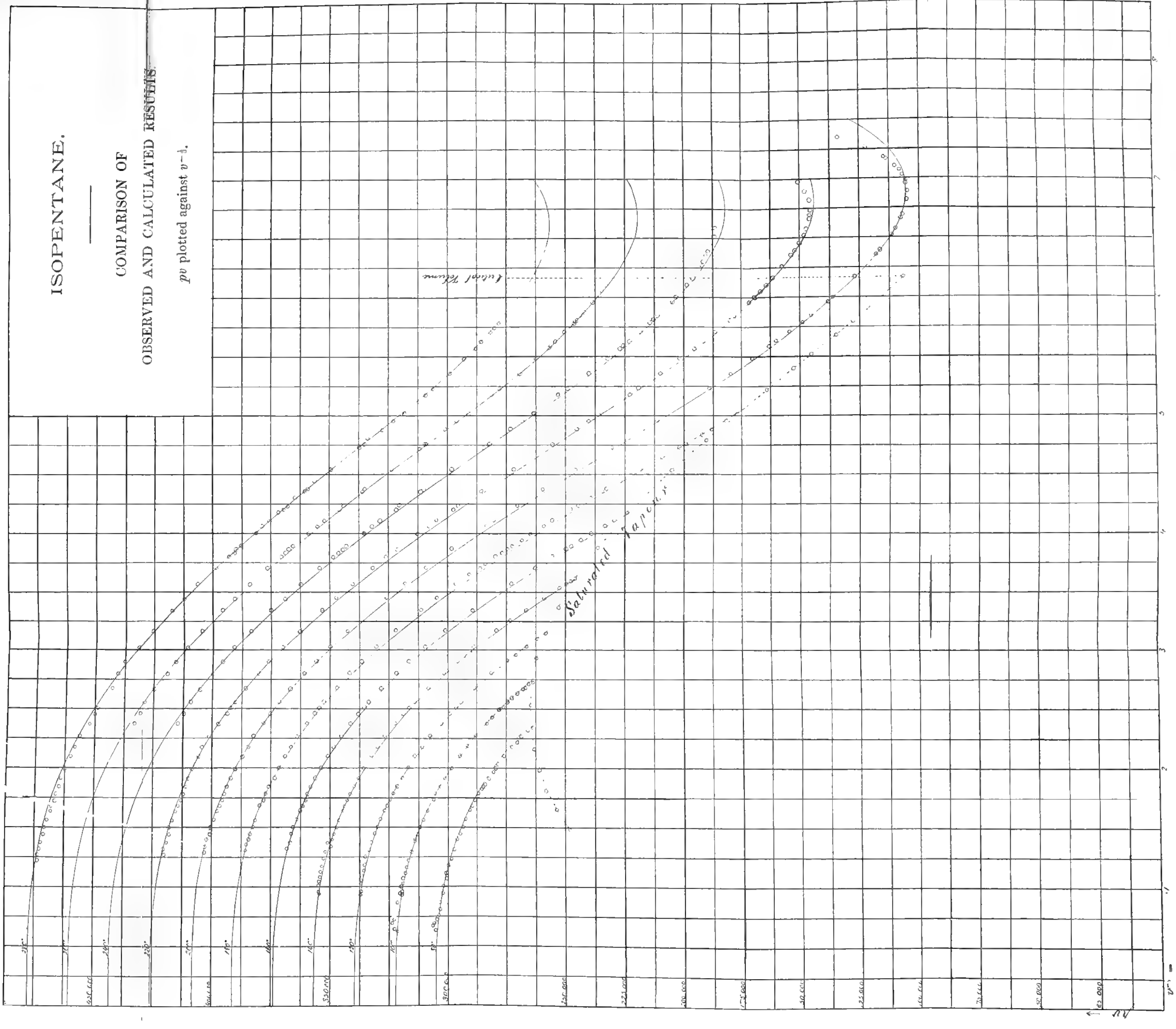
COMPARISON OF OBSERVED AND CALCULATED RESULTS.

pv plotted against v^{-1}.



THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[FIFTH SERIES.]

AUGUST 1897.

XX. *A new Definition of Focal Length, and an Instrument for determining it.* *By* T. H. Blakesley, *M.A.*[*]

I HAVE been frequently struck with the very imperfect way in which the facts of focussing with lenses and lens-combinations are treated in works which purport to deal with such matters in a practical way. In many cases such works are by authors who in other respects give information and rules which are unimpeachable and the result of painstaking study, so that the defect in the particular to which I have alluded must be due either to the inherent inconvenience in the usual formulæ, so invariably and unnecessarily involving the inverse of distances of Object and Image from theoretically fixed but practically undetermined points, and the Focal Length; or to the imperfect way of quoting the rules in the theoretical text-books. If the latter be even partially the cause, and can be remedied, it will be quite unnecessary to inquire if the former is also operative. In addition to the troublesome formula of inverse distances, a difficulty is unnecessarily introduced by defining the focal length as the distance between two points, generally the principal focus and a focal point, or the lens itself if thin, instead of being an abstract length of straight line characteristic of the lens or lens-combination.

To illustrate my meaning, I may point out that the Coefficient of Self-Induction of a coiled conductor is expressed as a length which is fixed so long as the coil remains unaltered in geometrical conditions; but no one would ask between which

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[*] Communicated by the Physical Society: read June 11, 1897.

points is this length to be measured. So the focal length of a lens-combination is simply an abstract length, and not necessarily the distance between two particular points.

But the Royal Society method of finding the focal length of a lens-system, as carried out at the Kew establishment, is based on the definition that the focal length is the distance between a focal centre and the principal focus; and these points having been separately found by processes not free from objection, the distance between them is indirectly measured (*vide* Major Darwin's paper, 'Proceedings of the Royal Society,' 1892).

Various methods have been described in an admirably arranged bibliographical review of the subject by Prof. S. P. Thompson. In this paper the author also describes a method of his own with an apparatus for carrying it out, but that method depends on the same idea that the focal length is a distance between two points. The apparatus fixes those points and measures the distance between them. And the same may be said of all but one of the methods proposed by others, described in that paper.

I have to propose and explain a far more general definition of the focal length of any system of Coaxial Lenses, which leads naturally to more general methods of determining that length than any of those alluded to.

Now any function of the two distances of object and image from their appropriate focal centres and the focal length may be employed in conjunction with the usual formula to eliminate either of those distances; and if that function is at the same time an easy one to determine experimentally, and if the elimination leads to a simple formula for working purposes, a considerable gain in convenience and exactitude may be the result. Such a function is found in the magnification, which I define as the linear relation of Image to Object and which I symbolize under the letter m, and take as positive if the image is erect and is not inverted with regard to the object. Thus in the case of the object and image being at equal distances from, and on opposite sides of a thin double-convex lens of equal radii of curvature, I should describe the magnification as -1. In this notation the negative sign will indicate reversal, the numeral the arithmetic magnification.

A

Let A be the position of one of a pair of conjugate foci on the axis of some coaxial lens-system, and let v be its distance from any fixed point on the axis measured positively in the direction the light is passing. Let m be the magnification, or the linear ratio of an image formed at this point to the object producing it.

Then it is easily proved that $\frac{dv}{dm}=f$ a constant ; and this constant is in fact the focal length.

Hence $$v-v_0=f\,.\,m,$$

where v_0 is the value of v when the magnification is zero, *i. e.* at the principal focus.

It follows from the constancy of $\frac{dv}{dm}$ that dv and dm need not be infinitesimals ; so that if any two positions of A be found for a clear image of an object moved in any way along the axis, the distance between these positions bears to the focal length a ratio which is simply the change in the magnification.

Similarly, if u is the position of the other focus but m retains its first signification,

$$\frac{du}{d\left(\frac{1}{m}\right)} \text{ is constant } = f',$$

and $$u-u_0=f'.\frac{1}{m},$$

where u_0 is the value of u when m is infinite.

Incidentally,

$$\overline{u-u_0}\;\overline{v-v_0}=f\,.\,f',$$

$$\frac{v-v_0}{u-u_0}=\frac{f}{f'}m^2.$$

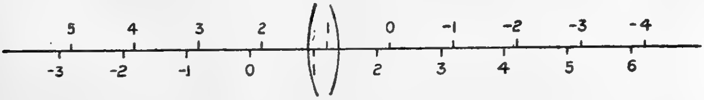
If the end media are alike, $f=f'$, which is the important case

$$\overline{u-u_0}\;\overline{v-v_0}=f^2,$$

$$\frac{v-v_0}{u-u_0}=m^2.$$

The former of these equations is a common one, but the latter is not so well recognized.

m^2 may be called the areal magnification. It clearly has an importance in the question of photographic exposure.



In the sketch let the straight line represent the axis of a lens-system, and let the points marked off at equal distances (f) above the line be numbered in such a way that the number represents the relative size of the Object to the Image, when the Object is placed in any one of the points, *i. e.* for the focus of light before incidence, and let the points numbered below the line, and also at equal distances (f_1), bear a corresponding series of numbers representing the relative size of Image and Object when the Image is at any one of the points.

Then any point represented by p on one of these scales is conjugate to the point represented by $\frac{1}{p}$ on the other. The two principal foci must be marked 0 and the two focal centres must be marked 1.

If the media at the ends are the same, the common distance of the integral points will be the same in the two scales. This condition, as being the common one, I shall assume to exist unless the reverse is especially stated.

The common distance between any two consecutive integral points is the focal length; and this is one important characteristic of the system. Another characteristic is the aberration of the integral points of one scale from those of the other. A third is the order in which the numbers of the scales run with reference to the direction of the light.

It appears to me that two systems to deserve being called equivalent one to the other should have coincidence in all these three respects. If the top series has reference to light before encounter with the system, and light is supposed to come from right to left, then for a convex arrangement the upper series of numbers increases in the same direction, *i. e.* from right to left, and the lower series in the reverse direction; but for a concave arrangement these rules are reversed.

It will be found that one of these diagrams will not be misinterpreted by turning it upside down.

Hence in any combination it is necessary to find:—

(1) The focal length.
(2) Some point of known numerical denomination on one scale.
(3) Ditto for the other scale.

With these three determinations all the facts connected with image-formation as regards position and size may be determined for any position of the object, and *vice versâ.*

The optical bank with which the focal length is determined contains various sliding-pieces in a mahogany groove. Two of them carry pieces for the support of the cells containing the lens-combination, and two carry frames in which can be placed photographs on lantern-plate glasses of evenly-divided scales.

These photographs, being printed off the same negative are exactly alike, and in use are turned with the film-side inwards towards the lenses.

If these photographs are made to occupy the positions of object and image, a comparison of the divisions actually on the plate examined with those of the image of the other scale received upon it will readily give the magnification.

Groups of exactly ten divisions, at ten divisions' interval, are also indicated by dark bars for the convenience of reading. An anatomical magnifier standing behind the scale serves to adjust to coincidence the planes of the actual and the image-scale with great accuracy; and it is also obvious that for any one position of the sliding-pieces the magnifier may be applied at both ends successively, each observation being thus checked by another.

To measure the movement of any sliding-piece a brass ruler is screwed to it at right angles to the motion; a slip of telegraph exchange-paper is fixed by drawing-pins to cork pieces at either end of the bank and the rulers pass immediately over this slip, so that lines can be drawn on the paper by a drawing-pen or pencil at any position, and the range of motion accurately measured.

To employ the instrument for convex lens-combinations, such as photographic lenses and projectors, several methods can be adopted. The one scale and the combination being fixed, search may be made for the image, which is adjusted accurately to the other scale, and a mark made on the paper at the edge at the corresponding ruler, and the magnification determined and checked, if necessary, at the other end.

The first scale may be then moved any distance, and the new position of the image found and adjusted as before by moving the second scale. A new mark is made, and the new magnification found and checked.

If m and n are the two magnifications and l the distance between the marks,

$$\frac{l}{m-n}=f.$$

Or the two scales may remain fixed and the combination may be moved until accurate adjustment is obtained in the two possible positions; the distance between those two positions being obtained by marks made on the paper along the edge of the ruler attached to the combination.

In this case the magnification which is the inverse in one of the positions to what it is in the other may be ascertained by *four* readings. Then obviously $n=\frac{1}{m}$ and $f=\frac{l}{m-\frac{1}{m}}$ $=\frac{l\,.\,m}{\overline{m+1}\cdot\overline{m-1}}$, a very convenient form.

The slips of paper employed may be rolled up and stored, so that reference can be made to them should occasion require it.

The operation of finding the focal length of a concave lens or combination is less important perhaps, but the following general proposition may be of interest, and within limits may be made applicable to any case.

Suppose it possible to change the position of a portion of a combination, so as to recover as in the last operation the same pair of conjugate foci of that part of the combination. Then clearly the focussing of the whole combination will be restored to the same conjugate foci as before, though the magnification will be changed.

Let m be the factor of the total magnification due to the moving portion, and let M be the total magnification : in the first position M may be read off.

In the second position m changes to $\frac{1}{m}$, and M changes to N, say, which may also be determined.

If K be the magnification-factor due to the stationary portion of the combination,

$$Km = M, \qquad \frac{K}{m} = N\,;$$

from which m and K can both be determined, viz.

$$m^2 = \frac{M}{N}, \qquad K^2 = MN.$$

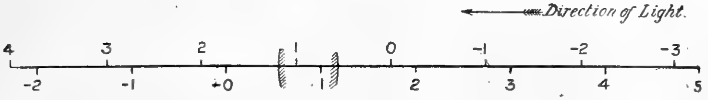
Hence the focal length of the moving part is

$$\frac{lm}{m^2-1} \quad \text{or} \quad \frac{l\sqrt{MN}}{M-N}.$$

I have dealt with the finding of focal lengths before showing how any particular focus may be found, with a view of accentuating how completely independent the two matters are; but it is clear that if any position of a pair of conjugate foci is taken and the magnification is measured, one has only to measure the distance of the scales from some fixed planes of the lens-system (say the edges of the brass cells) in order to allocate not merely the points for the particular magnification, but for all magnifications both above and below the line. Such measurements could easily be made by a sliding rule, accurately to the tenth part of a millimetre; but as the end of the rule would of necessity abut against the photographic film, repeated measurements would be prejudicial to the latter. However, if the negative from which the scales are made is preserved, such scales can be easily and cheaply reproduced.

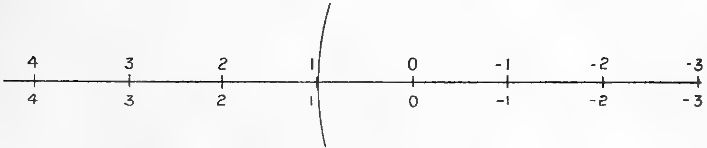
A very accurate way of fixing the principal foci is to place the combination at rest vertically on a plane mirror. Immediately above it hold a sharply-cut point of stiff white paper in a universal holder and adjust its position in the vertical line until its image formed by light passed twice through the lens-system coincides with itself. This adjustment may be most accurately made by a short focal-length hand-lens.

TYPE OF A CONVEX SYSTEM

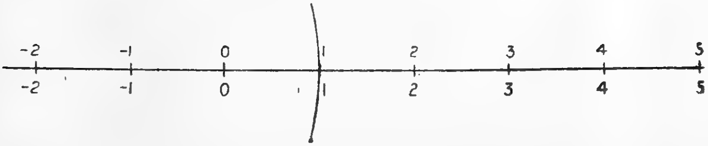


D° CONCAVE SYSTEM

D° CONCAVE MIRROR



D° CONVEX MIRROR



Note.—In all these diagrams light is supposed to come in *from the right hand*; the upper scale applies to the focus for the light before it encounters the system, the under scale to the focus for the light which has emerged from the system. The diagrams hold good if the sheet is turned upside down. Any erecting system of lenses applied to a given combination changes the character of the latter from concave to convex and *vice versâ*.

XXI. *Note on* Mr. Blakesley's *paper, "A new Definition of Focal Length, &c." By* Prof. A. Gray *.

I HAVE read Mr. Blakesley's paper on "A new Definition of Focal Length, &c.," with much interest. It is true, as Mr. Blakesley states, that the treatment of lenses and combinations of lenses in ordinary text-books is frequently faulty, and that much advantage would be gained by a consideration of the question from other points of view than that generally adopted in treatises on geometrical optics. Something of this kind has been achieved by the admirable series of papers on points in physical optics which have appeared from time to time, during the last fifteen years, from the pen of Lord Rayleigh, and the lucid and elementary discussion of the propagation of waves through lenses, and their reflexion from mirrors, published by Dr. S. P. Thompson, in Oct. 1889.

I think further clearness in the presentment of the action of lens systems in important cases would be obtained if more use were made of the notion of the apparent distance of an object seen through a system of lenses. This idea, which is at least as old as Smith's 'Compleat System of Opticks' (Camb. 1738), seems to have been strangely neglected until attention was called to it again by Lord Rayleigh in the Phil. Mag. for June 1886. Yet the formula for the apparent distance of an object situated on or near the common axis of a system of lenses, and viewed along that line, yields at once from its mere form many most valuable theorems: for example, that the interchange of position of the image and object, without change of position of the lens-system, does not affect the magnification; that the magnification of an object, seen through such a lens-system, is equal to the ratio of the real distance of the object from the object-glass to the apparent distance of the object from the eye, or the ratio of the breadth of the pencil at the object-glass to the breadth of the pencil at the eye.

In this mode of discussion attention is, as in Mr. Blakesley's paper, focussed on the magnification produced by the lens-system. Hence the method of determining the focal length of a combination by comparisons of the magnifications for (1) two positions of the object at a measured distance apart, or for (2) two positions of the image at a measured distance apart. If l denote the distance in either case, m, n the magnifications, the focal length is $l\,\dfrac{mn}{m-n}$ in (1), and $\dfrac{l}{m-n}$ in the

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* Communicated by the Physical Society: read June 11th, 1897.

other, or if, as Mr. Blakesley suggests, the two possible positions of the combination for fixed positions of the object and image be taken, and l be the distance through which the combination is moved, the focal length is $\frac{lm}{m^2-1}$.

But this method of determining focal lengths has already been used for several years by Abbe for his optical combinations, and apparatus for the purpose is to be found described by S. Czapski (*Zeitschrift für Instrumentenkunde*, vol. xii. 1892). A discussion of the method and some account of the apparatus will be found in the very valuable treatise on "Optik," by Czapski, lately published as part of the *Handbuch der Physik*, just completed under the editorship of Dr. A. Winkelmann (see vol. ii. pp. 289, 290, *et seq.*).

Though the experimental method of Mr. Blakesley's paper has thus been in the main anticipated, his paper is valuable as a fresh and instructive view of the subject, and as directing attention to methods of focometry as yet apparently not very generally known.

Bangor, June 17, 1897.

XXII. *Attenuation of Electric Waves along Wires and their Reflexion at the Oscillator. By* EDWIN H. BARTON, *D.Sc., F.R.S.E., Senior Lecturer in Physics at University College, Nottingham*[*].

LAST year Mr. Bryan and the present author realized experimentally the absorption by a terminal bridge of the electric waves incident upon it[†]. The first application of this result which suggested itself was the use of such a bridge in determining the attenuation (or rate of decay) of the waves along their parallel leads. For, if one can at pleasure either (1) practically absorb the waves, or (2) reflect them practically undiminished, a suitable electrometer being inserted in the line, we have in the first case a single passage of the wave-train past the electrometer, and in the second a number of passages of the wave-train until it is practically quenched by its repeated losses. Thus, the electrometer in the second case is affected by the sum to infinity of two geometrical progressions, viz., those due to the forward and return waves respectively. Whereas in the first case, when the waves are absorbed at the end of the line, the electrometer is affected by the first term only of one of the above series. Hence the ratio of the electrometer-throws in the two cases is

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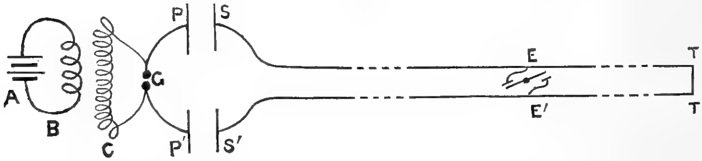
[*] Communicated by the Physical Society: read June 25, 1897.

[†] Proc. Phys. Soc. vol. xv. pp. 23-30, 1897; Phil. Mag. [5] xliii. pp. 39-45, 1897.

a function of the attenuation constant and the dimensions of the line, and would appear to be immediately calculable. But this is not all. For the above is on the assumption that the losses occur only along the wires themselves, whereas something is lost also on reflexion at the primary oscillator where the waves are generated. This fact soon became apparent on the repetition of the experiment with different lengths of line, and the value of this reflexion coefficient was substantially confirmed by further experiments and by the application of this new view of the case to experiments previously carried out in Bonn.

Experimental Arrangement.—The arrangement of the apparatus adopted in the determination is diagrammatically represented in fig. 1. In this figure, A denotes the battery

Fig. 1.



of two storage-cells, B and C are the primary and secondary coils respectively of the induction-coil, of which C has a resistance of 3000 ohms and an inductance of about 20 henries. G is the spark-gap, which was adjusted to 2 mm. The wire PGP′, measured along the semicircle, was about 2 m. PP′ are condenser-plates of zinc 40 cm. diam. placed opposite to and 30 cm. distant from the precisely similar plates S S′. The line is represented by S E T T′ E′ S′, and consists of two parallel copper wires 1·5 mm. diam., and kept 8 cm. apart by wood separators at intervals of about 2 m. E E′ denotes the electrometer, which has a single plane needle, initially uncharged, and suspended by a fine quartz fibre between two disks attached to the line at E and E′. The needle is therefore electrified by induction whenever a wave passes E E′, and its ends are consequently attracted to the disks whatever the sign of their potential-difference. T T′ signifies the several pieces of apparatus successively used at or near the end of the line, and will be described in detail for each case when dealt with. The lengths of the line before and after the electrometer will also be stated in connexion with each experiment. The waves generated by the primary oscillator were about 8·5 m. long.

Theory.—Let δ_1 be the electrometer-throw obtained when a completely absorbing bridge is used at T T′ (fig. 1), and let

δ_2 be the throw when a "no-resistance" bridge is used at TT′, *i. e.*, a piece of short thick copper wire, which therefore reflects the waves without appreciable loss. Then it is required to express the ratio δ_2/δ_1 as a function of the attenuation and reflexion coefficients and then, from the experimental values of the above ratio, to solve the resulting equations for the coefficients sought.

Let the maximum potential-difference of the two wires of the line due to any wave be taken as the amplitude of that wave, let ρ be the factor by which the amplitude of a wave is affected on reflexion from S S′ (fig. 1), and let $e^{-\sigma x}$ be the attenuation factor by which the amplitude is affected in traversing x cms. of the line. For convenience of actual working $10^{-sx'}$ was substituted for the above, x' denoting length traversed in metres, σ was then deduced from s.

Now the throws of the electrometer are proportional to the time-integral of the square of the amplitude of the passing wave-train, *i. e.*, proportional to its energy: hence, if the lengths of the line before and after the electrometer are l_1 and l_2 respectively, and a wave-train of initial energy i leaves the oscillator, we have with the "absorber" at the end of the line

$$ie^{-2l_1\sigma}=k\delta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is the electrometer constant.

Again, with the no-resistance bridge or short circuit at the end, we have two series of impulses at the electrometer, the energy of the forward waves being

$$ie^{-2l_1\sigma}[1+\rho^2e^{-4(l_1+l_2)\sigma}+\ldots],$$

and that of the backward ones $[e^{-4l_2\sigma}]$ times the above. Hence we obtain

$$\frac{ie^{-2l_1\sigma}(1+e^{-4l_2\sigma})}{1-\rho^2e^{-4(l_1+l_2)\sigma}}=k\delta_2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$\delta_2/\delta_1=\frac{1+e^{-4l_2\sigma}}{1-\rho^2e^{-4l\sigma}}=r \text{ say}, \quad . \quad . \quad . \quad . \quad (3)$$

l being written for l_1+l_2.

Two experiments with different values of l_1 and l_2 furnish two equations like (3); these are then cast in the form

$$\rho^2re^{-4l\sigma}=r-1-e^{-4l_2\sigma}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and ρ^2 eliminated between them. The result was an equation in σ of the form

$$A_1e^{-a_1\sigma}+A_2e^{-a_2\sigma}+\ldots=0,$$

or say

$$u=0.$$

Various trial values were then given to σ and plotted as abscissæ of a curve, the corresponding values of u being its ordinates. The value of σ satisfying the equation was thus found graphically as that at which the curve cut the axis of abscissæ.

Experiments.—In the experiments the bridge used as an absorber was placed absolutely at the end of the line, as it is only when in that position that it can absorb all. It consisted of pencil marks on a ground glass disk as previously described, and before use each time was carefully adjusted to the requisite resistance for absorption as mentioned in the paper already referred to. A doubt has been expressed as to the constancy of the value obtained for a resistance of this kind by the Post-Office box, some similar resistances having been said to measure differently according to the voltage used in the test. Probably this might be the case to a slight extent if extreme variations were used, but with the pencil-mark resistances employed in this work no change of value was observable, whether a single Leclanché cell was used or a battery of several storage-cells, so that the resistance was determined with all the accuracy needed for the present purpose.

The other bridge, intended to reflect undiminished the waves incident upon it, was placed a little short of the end of the line, and occupied alternately two positions a quarter of a wave-length apart. This precaution was adopted lest, far as the bridge was beyond the electrometer, the head of the wave-train reflected from the bridge might reach the electrometer before the tail had passed it. In this case a slight interference would arise at the electrometer, and its throws would not be of the values due to simple passages and re-passages of the wave-train. By using the above two positions for the bridge this source of error is obviated, for, if one gave additive interference at the electrometer, the other would give about equal subtractive interference, and their mean would be the result desired as though no interference had occurred.

The length l_2 is of course measured from the electrometer to the point midway between these two positions of the bridge.

Further, to eliminate errors due to the unavoidable irregularity of the sparking in the primary oscillator, a number of readings, about 20, with the absorber only at the end of the line were alternated with a like number of readings with the reflecting bridge interposed. The separate ratios of the electrometer-throws obtained under the two conditions were then taken, and the probable error of their arithmetic mean calculated by the theory of probabilities.

Observations and Results.—Four such sets of readings were taken with different values of l_1 and l_2. One set is shown in Table I., and the data and results of the whole four summarized in Table II.

TABLE I., showing readings for $l_1=117·5$ m. and $l_2=48$ m.

Electrometer-throws with Absorbing Bridge only at end of line.		Electrometer-throws with Reflecting Bridge		Ratios of throws, viz.:—		Mean Ratios.
Actual observations.	Interpolated Means.	at 48 m.$-\frac{\lambda}{8}$ beyond the Electrometer.	at 48 m.$+\frac{\lambda}{8}$ beyond the Electrometer.	$\frac{\text{col. 3}}{\text{col. 2}}$.	$\frac{\text{col. 4}}{\text{col. 2}}$.	
11						
	13	43		3·307		
15						
	16·5		44		2·666	2·986
18						
	17·5	42		2·400		
17						
	18·5		36		1·946	2·173
20						
	18	41		2·277		
16						
	15		44		2·933	2·605
14						
	16	33		2·062		
18						
	15		33		2·200	2·131
12						
	12·5	32		2·560		
13						
	12		28		2·333	2·446
11						
	13·5	27		2·000		
16						
	18·5		46		2·486	2·243
21						
	22	55		2·500		
23						
	22		48		2·181	2·340
21						
	20·5	56		2·731		
20						
	20		43		2·155	2·443
20						
	20	57		2·850		
20						
	21		52		2·476	2·663
22						
	21·5	50		2·325		
21						
	21·5		49		2·279	2·302
22						
		Whence Final Mean Ratio $= 2·43\pm0·05$.				

TABLE II., summarizing Data and Results of Experiments.

Lengths of line before Electrometer $=l_1$.	Lengths of line after Electrometer $=l_2$.	Ratio of Throws $=r$.	The unknown quantities for which the equations were actually solved. s.	The unknown quantities for which the equations were actually solved. ρ^2.	Attenuation Constant. σ.	Reflexion Coefficient. ρ.	Remarks.
metres.	metres.						
117·5	20	2·41 ±0·04	0·000564	0·4776	0·0000130	0·69	As these were the best sets of observations the resulting equations were solved simultaneously.
65·0	20	2·744±0·035					
65·0	48	2·64 ±0·18	0·0005	0·53	0·0000115	0·73	This being the worst set of readings (as seen by the large probable error) was treated alone. Any *number* of pairs of values could, of course, be found that would satisfy the equation.
117·5	48	2·43 ±0·05	0·00043	0·4776	0·000010	0·69	This is the set given in Table I., and is seen to be slightly less trustworthy than those at the head of this table. Two solutions are given of this set, the first seems the more probable.
			0·0005	0·555	0·0000115	0·745	

It may be interesting to regard, in the light of the results in the second table, two sets of observations taken by the writer in Bonn during the summer of 1893 while working under the late Professor Hertz*. In the course of those experiments (directed primarily to an object quite different from that now under discussion) the electrometer was placed almost at the end of the line. By an entirely different method from that now adopted, ratios were obtained for the electrometer-throws due (1) to the passage past it of the sum to infinity of the geometrical progressions formed by the outward and return waves, and that due (2) to the single passage of the initial wave-train. The lengths of the line to the electrometer in the two cases were 160 and 91·5 m. respectively, and the corresponding ratios were 1 : 0·34 and 1 : 0·31. Hence for this case equation (4) takes the form

$$\rho^2 r e^{-4l\sigma} = r - 2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

since $l_2 = 0$.

Solving simultaneously the two equations thus derived from the experimental values of r, we obtain

$$\left.\begin{array}{l} s = 0{\cdot}00027, \qquad \rho^2 = 0{\cdot}476, \\ \sigma = 0{\cdot}0000062 \text{ and } \rho = 0{\cdot}69 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

In these experiments the primary oscillator as regards the parts SPGP′S′ (fig. 1) was precisely the same as in the present paper, and it seems noteworthy that its reflexion coefficient is practically the same. The values of the resistance and inductance of the secondary circuit of the induction-coil used in Bonn cannot be stated, as they were not at the time regarded as of any material consequence and accordingly were not measured.

It will be seen that the attenuation constant in the 1893 case is about half that in the present work. This may possibly be due to the wood separators used along the line being much drier in a German summer than in an English winter and spring.

Comparisons with Heaviside's long-wave theory of reflexion.—After obtaining experimentally the unexpected result that of the wave energy incident upon the primary oscillator only about one-half or less was reflected along the line, it seemed incumbent on one to test this if possible by theory. And in this connexion I have pleasure in acknowledging my indebtedness to Mr. Oliver Heaviside for an explanation of his

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* Proc. Roy. Soc. vol. lvii. pp. 72 & 75 (1895); D.Sc. Thesis for the University of London, 1894, pp. 20–22 and 28–29; Wied. *Ann.* Bd. liii. pp. 525–527 and 528–529 (1894).

theory applied to the case of reflexion from a terminal arrangement prior to its publication *.

On Heaviside's theory developed originally for long waves, say of telephonic frequency, we have for the reflexion coefficient

$$\rho=\frac{Z-Lv}{Z+Lv}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where Z is the resistance operator, L is the inductance per unit length of the line, and v is the speed of light.

Now the resistance operator in the case of condensers, secondary and primary coils is given by

$$Z=\frac{1}{Sp}+(R_2+L_2p)-\frac{M^2p^2}{R_1+L_1p}, \quad . \quad . \quad . \quad (8)$$

where S is the capacity of the condenser, the R's are the resistances and the L's the inductances of the coils, M their mutual inductance, and p the time differentiator.

Of the three terms on the right-hand side of (8), the first only is to be used if the arrangement is short-circuited at the spark-gap, G (fig. 1), the first and second only if the primary circuit of the induction-coil is open, but all three if the primary is closed.

Now in the actual experimental case the sparks at the primary oscillator are started by the break of the primary circuit of the induction-coil, the interruptor remaining open during the greater part of its period (say the $\frac{1}{30}$ of a second). But the time required for the wave-train to pass to the end of the longest line used and back again, say a dozen times (which practically extinguishes it), is only of the order 14 millionths of a second. Hence the form of (8) to be adopted as applicable to the present case is

$$Z=\frac{1}{Sp}+R_2+L_2p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

if indeed the long-wave theory holds at all for waves so short as 8 or 9 m.

But, on working from equation (9) the value of ρ, we obtain a result differing from unity by less than 1 in 10^{10}.

It would appear therefore that the long-wave theory fails to apply to the present case of high-frequency waves and their reflexion from condensers and coils of high inductance, each of which involves of course the time-differentiator.

In the case of a resistance practically devoid of inductance,

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* 'Electromagnetic Theory,' pp. 786–9; 'The Electrician,' April 9, 1897.

on the other hand, as was seen in the last paper on the absorbing bridge, the long-wave theory seems to hold within a close approximation.

Reflexion from Condenser only.—However, in order to test the matter experimentally a little further, a second set of plates like PP′ SS′ (fig. 1) was made and put up at the end of the circuit at TT′, but this set had simply one wire in the form of a semicircle to connect the far plates instead a spark-gap as at G. The induction-coil was also omitted entirely at this end of the line.

Hence the long-wave theory for reflexion for this arrangement gives $Z = 1/Sp$ and

$$\rho = \frac{1/Sp - Lv}{1/Sp + Lv}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

And this equation, applied to an incident wave of the type $\phi_1 \sin nt$, gives for the reflected wave $\phi' \sin(nt+\alpha)$, where

$$\left.\begin{aligned} \phi'/\phi &= \rho = 1, \\ \text{and} \qquad \tan\alpha &= \frac{-2LvSn}{1-(LvSn)^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Here then is an opportunity of comparing simply the behaviour of the short waves with what may rightly be expected of the longer ones. For in this arrangement there is no spark-gap whose heated state in the former case leaves one in some doubt as to how far, when the waves return to it, it acts as a short circuit and so possibly cuts out the induction-coil entirely.

The experiment was now conducted by alternating electrometer readings (1) with the condenser at T T′, fig. 1, and (2) with a no-resistance bridge placed across the wires just before the end of the line.

If then the condenser reflected all, as by equations (10) and (11) must be the case with long waves of the simple harmonic type whatever the capacity of the condenser, we should find the ratio of the electrometer-throws taken under the two conditions to be unity. Whereas the experiment gave the ratio of throws with no-resistance bridge to those with condenser only to be $1{\cdot}26 \pm 0{\cdot}07$, the lengths of the line before and after the electrometer being 117·5 m. and 48 m. respectively. Applying now to this case the theory of the phenomena analogous to that developed in the early portion of this paper, and substituting for ρ^2 and s in the equation so derived the values 0·4776 and 0·000564 as shown in Table II., we obtain for the reflexion coefficient of the condensers alone

$$\rho' = 0{\cdot}82 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This value, while differing but little from that of ρ for the condensers, spark-gap, and induction-coil at the oscillator, differs in the direction that one might anticipate. It also supports the value previously found for ρ, in that ρ' though greater than ρ is still distinctly less than unity. It will be seen from the end of the article in 'The Electrician,' April 9th, 1897, that Mr. Heaviside scarcely expected the long-wave theory to be valid for the high-frequency waves here used.

Conclusion.—The chief results arrived at may be thus summarized :—

1. The value of the attenuation constant σ for the line experimented upon and the waves in use is of the order 0·000013, where the amplitude of the waves is affected by the factor $e^{-\sigma x}$ in traversing x cm. of the line. The energy of the waves is thus reduced to half by traversing about 250 m. of the line.

2. The reflexion coefficient applicable to the primary oscillator and induction-coil as used to generate the waves is of the order 0·69, or rather more than half the energy is lost on reflexion. The reflexion coefficient of a precisely similar arrangement of condenser-plates but devoid of spark-gap and induction-coil is rather greater but still distinctly less than unity. On the long-wave theory the second of these reflexion coefficients would be precisely equal to unity and the first of them practically so.

I have to thank Mr. H. A. Ratcliff for help at the beginning of this work, and especially Mr. R. S. Willows for his valuable assistance at the later experiments and their reduction.

University College, Nottingham,
May, 1897.

XXIII. *The Effect of a Spherical Conducting Shell on the Induction at a Point in the Dielectric outside due to an Alternating Current in a Circular Circuit in the Dielectric inside, the Axis of the Conductor passing through the Centre of the Shell.* By C. S. Whitehead, *M.A.*[*]

(1) It is well known that if there be an alternating current in a primary circuit sounds will be heard in a telephone in a secondary circuit, even if the latter be at a considerable distance from the former. If the sounds are preconcerted, signals may be, and in fact have been, trans-

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[*] Communicated by the Physical Society : read June 11, 1897.

mitted in this manner on land through distances of four or five miles.

It has been suggested that the same plan could be employed to communicate with lightships. The primary is laid on the sea-bottom round the area over which the lightship swings, the ends of the cable being brought to land; the secondary is coiled round the lightship. I have been informed by Mr. S. Evershed that successful experiments were carried out by these means last summer in Dover harbour, the depth of the sea being about 1000 centims. To be able to judge whether a similar experiment will succeed in a greater depth of water, we must calculate the induction through the secondary circuit; sea-water being a conductor the investigation is somewhat complex, the final result, however, comes out in a simple form.

(2) The two following cases are considered:—

CASE I.

A circular circuit carrying an alternating current is placed in the dielectric inside a spherical conducting shell, the axis of the circuit passing through the centre of the shell: to find the normal magnetic induction at any point in the dielectric outside the shell.

CASE II.

The circular circuit is placed on one side of an infinite conducting plate, the plane of the circuit being parallel to the plate: to find the normal magnetic induction at any point on the other side of the plate.

In both cases the following result is arrived at:—

$$\frac{v_0}{u_0} = e^{-q\eta},$$

where v_0 is the maximum value of the normal magnetic induction at any point outside, u_0 the maximum value of the normal magnetic induction due to the current in the primary, supposing the conducting shell or plate absent, at the same point, η is the thickness of the shell or plate,

$$q = \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}},$$

μ is the permeability of the conducting shell or plate, σ its specific resistance, $p = 2\pi$ times the frequency.

Let the frequency be 300, which makes $p = 1885$; σ for sea-water $= 2.10^{10}$ C.G.S. units, $\mu = 1$.

If $\eta = 2000$ centims., which is about the depth of the sea by the North Sand Head Lightship off the Goodwin Sands, then

$$\frac{v_0}{u_0}=\cdot 21, \quad \text{and 79 per cent. is lost.}$$

If $$\eta=1000, \quad p=1885,$$

$$\frac{v_0}{u_0}=\cdot 46, \quad \text{and 54 per cent. is lost.}$$

The above value of σ is from experiments on a sample of sea-water from the North Sea made by Mr. S. Evershed.

(3) Case I.

Take O the centre of the shell for origin, and axis of the circuit for axis of z.

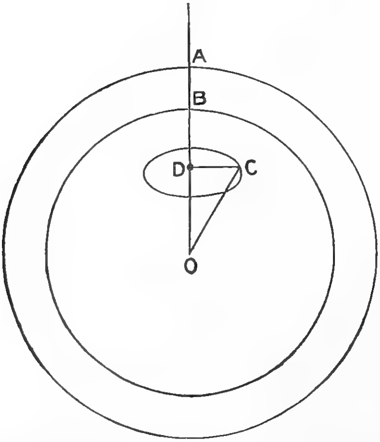
Let D be the centre of the circuit, C any point in it.

OA = external radius of the shell $=a$.

OB = internal radius of the shell $=b$.

OC $=c$, $\angle$ DOC $=\alpha$.

$a-b=\eta$.



Let P, Q, R be the components of the electromotive intensity parallel to the axes of x, y, z respectively, σ the specific resistance of the shell, μ the permeability of the shell, μ' that of the dielectric, K$'$ the specific inductive capacity of the dielectric, r, θ, ϕ the polar coordinates of any point, θ being the colatitude, ϕ the longitude.

Let the current in the circuit be represented by the real part of $\gamma e^{\iota pt}$, $p=2\pi\times$ frequency, $\iota=(-1)^{\frac{1}{2}}$.

By symmetry the induced currents in the shell will flow in circles parallel to the plane of the circuit.

Let Φ be the electromotive intensity tangential to one of these circles;

$$\therefore \; \mathrm{P} = -\sin\phi \,.\, \Phi, \quad \mathrm{Q} = \cos\phi \,.\, \Phi, \quad \mathrm{R} = 0. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Φ is by symmetry independent of ϕ.

Now P satisfies the equation

$$\nabla^2 \mathrm{P} = \frac{4\pi\mu}{\sigma}\frac{d\mathrm{P}}{dt} \text{ in the conductor,}$$

and

$$\nabla^2 \mathrm{P} = \mu' \mathrm{K}' \frac{d^2\mathrm{P}}{dt^2} \text{ in the dielectric.}$$

Assume

$$\mathrm{P} \text{ varies as } e^{\iota pt},$$

and let

$$\lambda^2 = -\frac{4\pi\mu\iota p}{\sigma},$$

$$\lambda'^2 = \mu'\mathrm{K}'p^2.$$

$$\therefore \qquad \nabla^2\mathrm{P} + \lambda^2\mathrm{P} = 0 \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

in the dielectric we must write λ' for λ.

Q and R satisfy equations of the same form.

Assume

$$\left.\begin{aligned} \mathrm{P} &= \psi_n(\lambda r)\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\chi_n \, e^{\iota pt} \\ \mathrm{Q} &= \psi_n(\lambda r)\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\chi_n \, e^{\iota pt} \\ \mathrm{R} &= \psi_n(\lambda r)\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\chi_n \, e^{\iota pt} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \qquad (3)$$

These equations make $\dfrac{d\mathrm{P}}{dx} + \dfrac{d\mathrm{Q}}{dy} + \dfrac{d\mathrm{R}}{dz} = 0$, as should be the case.

χ_n is an arbitrary solid spherical harmonic of degree n.

Substituting in (2) we find

$$\frac{d^2\psi_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\psi_n}{dr} + \lambda^2\psi_n = 0. \quad . \quad . \quad . \quad . \qquad (4)$$

Since $\mathrm{R} = 0$, the last equation of (3) gives

$$\frac{d\chi_n}{d\phi} = 0.$$

$\therefore$ from the first of (3).

$$\mathrm{P} = -\sin\phi \,.\, \psi_n(\lambda r)\frac{d\chi_n}{d\theta} e^{\iota pt}.$$

$$\therefore \; \Phi = \psi_n(\lambda r)\frac{d\chi_n}{d\theta} e^{\iota pt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Let ω = magnetic induction tangential to a meridian,
υ = magnetic induction along a radius.

Then from the theorem that the line integral of the electromotive intensity round a circuit is equal to the rate of decrease of the magnetic induction through the circuit, we obtain

$$\frac{d\omega}{dt} = -\frac{1}{r}\frac{d(\Phi r)}{dr},$$

$$\frac{d\upsilon}{dt} = \frac{1}{r\sin\theta}\frac{d(\Phi\sin\theta)}{d\theta}.$$

But Φ varies as $e^{\iota pt}$,

$$\therefore\ \omega = -\frac{1}{\iota pr}\frac{d(\Phi r)}{dr},$$

$$\upsilon = \frac{1}{\iota pr}\frac{1}{\sin\theta}\frac{d(\Phi\sin\theta)}{d\theta}.$$

Since $\frac{d\chi_n}{d\phi} = 0$, χ_n satisfies

$$\frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\chi_n}{d\theta}\right) = 0,$$

$\therefore$ from (5)

$$\omega = -\frac{1}{\iota pr}\frac{d^2}{d\theta\,.\,dr}\left\{r\psi_n(\lambda r)\chi_n\right\}e^{\iota pt},$$

$$\upsilon = -\frac{1}{\iota pr}\psi_n(\lambda r)\frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right)e^{\iota pt}.$$

But

$$r\frac{d\chi_n}{dr} = n\chi_n,$$

$$\therefore\ \frac{d}{dr}\left(r^2\frac{d\chi_n}{dr}\right) = n(n+1)\chi_n,$$

and

$$\frac{d}{dr}\left\{r\psi_n(\lambda r)\chi_n\right\}$$

$$= \left\{\frac{d\,.\,r\psi_n(\lambda r)}{dr} + n\psi_n(\lambda r)\right\}\chi_n,$$

$$= \phi_n(\lambda r)\chi_n,$$

where

$$\phi_n(\lambda r) = \frac{d\,.\,r\psi_n(\lambda r)}{dr} + n\psi_n(\lambda r), \quad . \quad (6)$$

$$\left.\begin{aligned}\therefore\ \omega &= -\frac{1}{\iota pr}\phi_n(\lambda r)\frac{d\chi_n}{d\theta}e^{\iota pt}\\ \upsilon &= -\frac{n(n+1)}{\iota pr}\psi_n(\lambda r)\chi_n e^{\iota pt}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Again, $\mu' K'$ is very small (it is of the order 10^{-21}), hence, unless p is very large, we may neglect λ'^2. Hence in the dielectric

$$\nabla^2 P = 0,$$

we may therefore assume

$$P = \left(y\frac{d}{dz} - z\frac{d}{dy}\right) X_n e^{\iota pt},$$

where X_n is an arbitrary solid spherical harmonic, so that in the dielectric we must put $\psi_n(\lambda' r) = 1$, and, consequently, $\phi_n(\lambda' r) = n+1$.

Therefore in the dielectric

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr} X_n e^{\iota pt} \\ \omega &= -\frac{n+1}{\iota pr}\frac{dX_n}{d\theta} e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The inducing system is in the dielectric inside the shell. Hence in the dielectric inside the shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr} (X_n + X_{-n-1}) e^{\iota pt} \\ \omega &= -\frac{1}{\iota pr}\left\{(n+1)\frac{dX_n}{d\theta} - n\frac{dX_{-n-1}}{d\theta}\right\} e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

In the dielectric outside the shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr} Z_{-n-1} e^{\iota pt} \\ \omega &= \frac{n}{\iota pr}\frac{dZ_{-n-1}}{d\theta} e^{\iota pt} \end{aligned}\right\}, \quad . \quad . \quad (10)$$

where Z_{-n-1} is an arbitrary solid spherical harmonic of degree $-n-1$.

In the conducting shell

$$\left.\begin{aligned} v &= -\frac{n(n+1)}{\iota pr} \psi_n(\lambda r) \chi_n e^{\iota pt} \\ \omega &= -\frac{1}{\iota pr}\phi_n(\lambda r)\frac{d\chi_n}{d\theta} e^{\iota pt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

The boundary conditions are (1) that the normal magnetic induction, (2) that the tangential magnetic force is to be continuous. The tangential magnetic force $= \frac{\omega}{\mu}$ in the conductor, and $= \omega$ in the dielectric.

$$\left.\begin{aligned}\therefore\ \psi_n(\lambda b)\chi_n &= X_n + X_{-n-1}\\ \psi_n(\lambda a)\chi_n &= Z_{-n-1}\end{aligned}\right\} \quad .\ .\ .\ .\ . \quad (12)$$

$$\left.\begin{aligned}\frac{1}{\mu}\phi_n(\lambda b)\frac{d\chi_n}{d\theta} &= (n+1)\frac{dX_n}{d\theta} - n\frac{dX_{-n-1}}{d\theta}\\ \frac{1}{\mu}\phi_n(\lambda a)\frac{d\chi_n}{d\theta} &= -n\frac{dZ_{-n-1}}{d\theta}\end{aligned}\right\} \quad .\ .\ . \quad (13)$$

From these equations we obtain

$$\frac{X_n + X_{-n-1}}{Z_{-n-1}} = \frac{b^n\psi_n(\lambda b)}{a^n\psi_n(\lambda a)},$$

$$\frac{(n+1)X_n - nX_{-n-1}}{-nZ_{-n-1}} = \frac{b^n\phi_n(\lambda b)}{a^n\phi_n(\lambda a)};$$

$$\begin{aligned}.\ b^n\{(n+1)\phi_n(\lambda a)\psi_n(\lambda b) + n\phi_n(\lambda b)\psi_n(\lambda a)\}Z_{-n-1}&\\ = a^n(2n+1)\phi_n(\lambda a)\psi_n(\lambda a)X_{-n-1}\,.& \quad .\ .\ (14)\end{aligned}$$

Let λa and λb be so large that $\frac{1}{\lambda a}$, $\frac{1}{\lambda b}$ may be neglected.

$\psi_n\ (\lambda r)$ satisfies

$$\frac{d^2\psi_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\psi_n}{dr} + \lambda^2\psi_n = 0.$$

Assume

$$\psi_n = \frac{e^{mr}}{r^{n+1}}u,$$

and let

$$\lambda^2 + m^2 = 0,$$

$$\therefore\ \frac{d^2u}{dr^2} + 2m\frac{du}{dr} - \frac{n(n+1)}{r^2}u = 0.$$

Let

$$u = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \dots$$

Substituting, we find in the usual manner,

$$u = 1 - \frac{n(n+1)}{2mr} + \frac{(n-1)n(n+1)(n+2)}{1\,.\,2\,.\,(2mr)^2} - \dots$$

where

$$m = \pm\iota\lambda.$$

Hence, when $\frac{1}{\lambda r}$ is small,

$$\psi_n(\lambda r) = \frac{e^{\iota\lambda r}}{r^{n+1}} \text{ or } \frac{e^{-\iota\lambda r}}{r^{n+1}}.$$

Now Φ is zero at infinity and must decrease as r increases,

$$\therefore\ \psi_n(\lambda r) = \frac{e^{-\iota\lambda r}}{r^{n+1}},$$

$$\therefore\ \phi_n(\lambda r) = -\iota\lambda\frac{e^{-\iota\lambda r}}{r^n}.$$

Substituting in (14)

$$Z_{-n-1} = \frac{(2n+1)be^{-\iota\lambda\eta}}{(n+1)a+nb}\,X_{-n-1}$$

$$= e^{-\iota\lambda\eta}\,X_{-n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

if both a and b be large compared with their difference.

Let Ω denote the solid angle subtended by the circuit at any point;

V the potential due to the circuit;

U the normal magnetic force.

$$\Omega = 2\pi\sin^2\alpha\sum_1^\infty\frac{1}{n+1}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta),$$

if $r > c$. P_n is the nth zonal harmonic.

$$V = \gamma e^{\iota pt}\Omega,$$

$$U = -\frac{dV}{dr},$$

$$= 2\pi\gamma\sin^2\alpha e^{\iota pt}\frac{1}{r}\sum_1^\infty\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta). \quad . \quad . \quad . \quad . \quad (16)$$

Now in the dielectric inside the shell,

$$v = -\frac{n(n+1)}{\iota pr}(X_n + X_{-n-1})e^{\iota pt},$$

$$\therefore\ -\frac{n(n+1)}{\iota p}X_{-n-1} = 2\pi\gamma\sin^2\alpha\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta),$$

$$\therefore\ X_{-n-1} = -\frac{2\pi\gamma\sin^2\alpha\ \iota p}{n(n+1)}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta), \quad . \quad . \quad (17)$$

$\therefore$ from (15)

$$Z_{-n-1} = -e^{-\iota\lambda\eta}\frac{2\pi\gamma\sin^2\alpha\,.\,\iota p}{n(n+1)}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta). \quad . \quad (18)$$

But in the dielectric outside the shell

$$v = -\frac{n(n+1)}{\iota pr}Z_{-n-1}e^{\iota pt},$$

$$\therefore\ v = e^{-\iota\lambda\eta}2\pi\gamma\sin^2\alpha\,e^{\iota pt}\frac{1}{r}\left(\frac{c}{r}\right)^{n+1}P_n'(\alpha)P_n(\theta).$$

Now

$$\lambda^2 = -\frac{4\pi\mu\iota p}{\sigma} = (1-\iota)^2q^2,$$

where

$$q^2 = \frac{2\pi\mu p}{\sigma},$$

$$\therefore \text{ real part of } e^{-\iota\lambda\eta} e^{\iota pt} = \text{real part of } e^{-q\eta} e^{\iota(pt-q\eta)}$$

$$= e^{-q\eta} \cos(pt - q\eta),$$

$$\therefore \ v = e^{-q\eta} 2\pi\gamma \sin^2 \alpha \frac{1}{r}\left(\frac{c}{r}\right)^{n+1} \mathrm{P}_n'(\alpha)\mathrm{P}_n(\theta) \cos(pt - q\eta). \quad . \ (19)$$

Let v_0 be the maximum value of v, U_0 that of U ;
∴ from (16) and (19)

$$\frac{v_0}{\mathrm{U}_0} = e^{-q\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The method employed in this investigation is taken from a paper by Professor H. Lamb, Phil. Trans. Part ii. 1883.

(4) Case II.

The result for this case may be deduced from the preceding by using a particular case of a transformation due to Professor C. Niven (Phil. Trans. Part ii. 1883).

He shows that if P_n denote a zonal harmonic of the nth degree, $s = \sin\theta$, $n = ka$, $s = \frac{\rho}{a}$; then when n and a become infinite, k and ρ remaining finite,

$$\mathrm{P}_n = \mathrm{J}_0(k\rho),$$

$$\mathrm{P}_n' = \frac{ka^2}{\rho}\mathrm{J}_1(k\rho),$$

J_0 and J_1 being Bessel functions.

To find the value of Ω in terms of Bessel's functions. Let P be any point, draw PM perpendicular to the axis of z.

Let CD, the radius of the circuit, $= f$,
$\mathrm{PM} = \rho$, $\angle\mathrm{POM} = \theta$, $\angle\mathrm{COD} = \alpha$, $\mathrm{OC} = c$, $\mathrm{DM} = z$, $\mathrm{OP} = r$.

$$\Omega = 2\pi \sin^2\alpha \sum_1^\infty \frac{1}{n+1}\left(\frac{c}{r}\right)^{n+1} \mathrm{P}_n'(\alpha) \,.\, \mathrm{P}_n(\theta), \quad r > c.$$

Let
$$n = kr = k_1 c,$$

$$\sin\theta = \frac{\rho}{r}, \quad \sin\alpha = \frac{f}{c}.$$

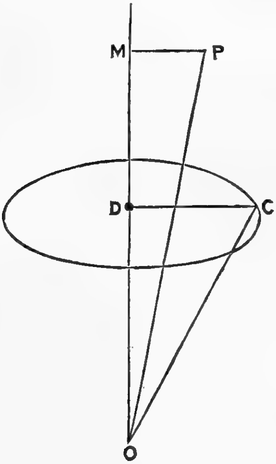
let n, r, c become infinite, k, ρ, f remaining finite ;
∴ ultimately $k = k_1$,

$$P_n(\theta) = J_0(k\rho), \quad P_n'(\alpha) = \frac{kc^2}{f} J_1(kf),$$

$$c = OD = OM - DM$$

$$= r - z \text{ ultimately ;}$$

$$\therefore \left(\frac{c}{r}\right)^n = \left(1 - \frac{z}{r}\right)^{kr} = e^{-kz} \text{ in the limit.}$$



The successive values of n are 1, 2, 3 . . .; let $k + dk$ be the successive values of k ;

$$\therefore n + 1 = (k + dk)r \text{ ;}$$

$$\therefore rdk = 1 \text{ ;}$$

$$\therefore \frac{k}{n+1}\left(\frac{c}{r}\right) = \frac{n}{n+1} dk = dk \text{ ;}$$

$$\therefore \Omega = 2\pi f \int_0^\infty e^{-kz} J_0(k\rho) J_1(kf) dk.$$

This result can also be deduced from the equation

$$\frac{d^2\Omega}{dr^2} + \frac{1}{r}\frac{d\Omega}{dr} + \frac{d^2\Omega}{dz^2} = 0,$$

remembering that

when $z = 0$, $\Omega = 2\pi$ from $r = 0$ to $r = a$.

$\Omega = 0$ from $r = a$ to $r = \infty$.

We thus find

$$v=2\pi\gamma e^{-q\eta}f\cos(pt-q\eta)\int_0^\infty ke^{-kz}J_0(k\rho)J_1(kf)dk,$$

$$U=2\pi\gamma f\cos pt\int_0^\infty ke^{-kz}J_0(k\rho)J_1(kf)dk\,;$$

$$\therefore\ \frac{v_0}{U_0}=e^{-q\eta}$$

as before.

APPENDIX.

It may be useful to add the proof of the transformation used in Case II.

If $\mu=\cos\theta$,

P_n satisfies

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2}-2\mu\frac{dP_n}{d\mu}+n(n+1)P_n=0. \quad . \quad . \quad (1)$$

Let $s=\sin\theta$.

(1) transforms into

$$(1-s^2)\frac{d^2P_n}{ds^2}+\frac{1-2s^2}{s}\frac{dP_n}{ds}+n(n+1)P_n=0.$$

Assuming

$$P_n=a_0+a_1s+a_2s^2+\dots$$

we find in the usual manner

$$P_n=a_0\left\{1-\frac{n(n+1)}{2^2}s^2+\frac{(n-2)n(n+1)(n+3)}{2^2.4^2}s^4-\dots\right\}.$$

But $P_n=1$ when $s=0$;

$$\therefore\ a_0=1.$$

Let $n=ka,\quad s=\frac{\rho}{a},$

and let n and a become infinite, k and ρ remaining finite.

$$\frac{n(n+1)}{2^2}s^2=\frac{k^2\rho^2}{2^2},$$

ultimately

$$\frac{(n-2)n(n+1)(n+3)}{2^2.4^2}s^4=\frac{k^4\rho^4}{2^2.4^2},$$

$$\&c. \quad = \quad \&c.$$

$\therefore$ ultimately

$$P_n=1-\frac{k^2\rho^2}{2^2}+\frac{k^4\rho^4}{2^2.4^2}-\frac{k^6\rho^6}{2^2.4^2.6^2}+\dots$$

$$=J_0(k\rho).$$

Again

$$s^2+\mu^2=1;$$

$$\therefore\ sds=-\mu d\mu,$$

$$\mu^2=1-s^2=1 \text{ ultimately,}$$

$$s=\frac{\rho}{a};\quad \therefore\ ds=\frac{d\rho}{a};$$

$$\therefore\ d\mu=-s\frac{d\rho}{a}=-\frac{\rho d\rho}{a^2};$$

$$\therefore\ \mathrm{P}'_n=\frac{d\mathrm{P}_n}{d\mu}=-\frac{a^2}{\rho}\frac{d\mathrm{J}_0(k\rho)}{d\rho}$$

$$=\frac{ka^2}{\rho}\mathrm{J}_1(k\rho).$$

XXIV. *A Comparison of Rowland's Mercury Thermometers with a Griffiths Platinum Thermometer.* *By* C. W. Waidner *and* F. Mallory*.

THE determinations of the mechanical equivalent of heat by electrical and mechanical methods show a close agreement between the values obtained by different experimenters using the same method of experiment. There appears, however, a very appreciable difference between the values as determined by the two methods, a difference greater than can be accounted for by errors of experiment. This difference can only be explained on the assumption that the results of the different experimenters are based on different thermometric standards or of a still undiscovered error in the system of electric units employed. The recent comparison by Professor Schuster of Joule's thermometers has indirectly furnished a connexion between Professor Rowland's air thermometer and the nitrogen standard of the Bureau International.

This comparison pointed to differences in these two standards as great as ·05° C., but as the details of Joule's comparison of Rowland's thermometer (No. 6166) with his own are not known, this correction is uncertain. It was therefore thought advisable to make another comparison of Professor Rowland's mercurial thermometers *under conditions* as nearly

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* From the Johns Hopkins University Circulars, June 1897.

as possible *similar* to those under which they *were used* by him in his mechanical equivalent determinations. On account of the many advantages offered by a Griffiths platinum thermometer for the standardization of calorimetric thermometers, this instrument was selected for the comparison. In the comparison the platinum and the mercurial thermometers were placed side by side in a calorimeter and the two read simultaneously while the temperature was slowly rising. Readings on mercurial thermometers were taken by means of a micrometer eyepiece.

The stem corrections of the mercurial thermometers were made, as in Rowland's experiment, by surrounding the thermometer stem for a short distance above the calorimeter with a water jacket, the remaining portion of the exposed stem being assumed to be at the temperature of the surrounding air. The mercury thermometers were placed in ice and distilled water for several hours and the zero determined before each comparison. The corrected stem readings obtained in our comparison were then reduced to Rowland's zero and the corresponding temperatures of his air thermometer taken from his tables of comparison of this thermometer with his Baudin thermometers (Nos. 6163, 6166). The platinum thermometer used in this experiment was one constructed by Mr. E. H. Griffiths. It was provided with compensating leads so that the temperature was independent of stem immersion, provided this was great enough to prevent conduction down the leads to the platinum coil.

The resistance measurements were made with a Griffiths resistance box (No. 7) especially designed for the measurement of platinum temperature. We are under great obligations to the University of Chicago for the use of this box, which was kindly loaned us for this work. The construction and calibration of a resistance box similar to the one used by us has been described by Mr. E. H. Griffiths in 'Nature,' November 14th, 1895. The method there described was employed by us. Two independent calibrations of the coils and bridge wire gave practically identical results.

Temperature on the platinum scale is defined by the equation

$$pt = \frac{R - R_0}{R_1 - R_0} \times 100,$$

where pt denotes platinum temperature, R_0 the resistance of the thermometer at 0° C., R_1 the resistance at 100° C., and R the resistance at the temperature pt. Platinum temperature is therefore independent of the unit of resistance employed, and all of our measurements are given in terms of the mean box unit of Box No. 7 (approximately one mean box unit equals ·01 ohm). Griffiths gives as the values of R_1 and R_0 the following:—$R_1=358{\cdot}078$, $R_0=258{\cdot}362$ (in mean box units, Box No. 6). From which the fundamental interval

$$R_1-R_0=99{\cdot}716, \text{ and } \frac{R_1}{R_0}=1{\cdot}38596.$$

To reduce these values to true Board of Trade ohms the factor for this box (No. 6) at 20° C., as given by Griffiths, is 0·010003. We are uncertain as to the absolute value of the mean box unit of our box (No. 7), and hence cannot yet compare our values in the standardization of our thermometer directly with those of Griffiths.

The following values of R_0 were obtained:—

258·471
258·472
258·469
258·471

258·471 mean box units, 20° C.

Each of these values is the mean of five or six measurements with different combinations of coils and bridge wire. It is interesting to note that the last two determinations suggest the possibility that the effect of radiation is appreciable in a room whose temperature is 20° above the thermometer coil; the first of these was made in a double-wall vessel with ice in both compartments, the second immediately after in the same mixture of ice and water with the inner polished vessel removed.

The determination of R_1 gave the following values:—

358·228
358·229
358·235

358·231 mean box units, 20° C.

These determinations were made in a hypsometer in which the thermometer was screened on all sides from radiation effects. Each of the above measurements was made on a different day with great differences of barometric height, and is the mean of five or six readings all in close agreement.

This gives for the value of $\frac{R_1}{R_0}$ 1·38597.

The conversion from the platinum to the air scale is made by the following formula:—

$$t - pt = \delta\left\{\left(\frac{t}{100}\right)^2 - \frac{t}{100}\right\}$$

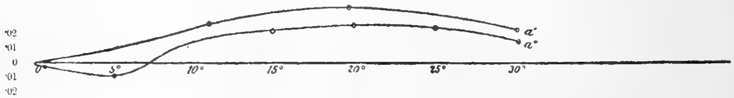
where t is the temperature on the air scale, pt the platinum temperature as defined above, and δ a constant. (For the experimental evidence in favour of this formula, see the elaborate comparison of the air and platinum thermometers by Callendar and Griffiths, Phil. Trans. Roy. Soc. vol. clxxxii. (1891), A). This relation being a parabolic one, only three temperatures are necessary for the complete standardization of a platinum thermometer, *i. e.* for the determination of δ. The identity of our value of $\frac{R_1}{R_0}$ with that obtained by Griffiths in his standardization of our thermometer, made it unnecessary for us to determine the resistance of the thermometer in boiling sulphur (the third point usually employed), and we therefore accepted the value of δ, 1·491, given by Griffiths, as correct.

All auxiliary thermometers used for resistance box, measurement of air temperatures, &c., were compared and reduced to air scale.

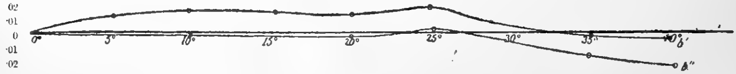
Each of the mercury thermometers was separately compared with the platinum thermometer in two independent series of observations.

The results of the comparisons are best shown by the accompanying curves, in which abscissæ represent temperatures on Rowland's air scale and ordinates the corresponding corrections which must be added to Rowland's scale to reduce to the air scale obtained through the platinum thermometer.

6166.



6163.



The differences in the above comparisons are far greater than can be accounted for by experimental errors, and must be partly due to the nature of the glass used in these thermometers, which seem more dependent upon their previous use than the hard glass thermometers. The depression of the zero in these thermometers is large. Thus in one experiment with 6163 the zero at the beginning, after the thermometer had been at about 20° C. for several weeks, was 60·11 millim.; after heating to 40° C. the zero was 59·77, a depression corresponding to ·038° C.

A similar experiment with Baudin (No. 6165), which was exactly similar to these two thermometers, gave a depression of ·021° C. after being raised to 30° C.

It will be observed that from 14° to 25° the range on the two air scales does not differ by more than a few thousandths of a degree. We must therefore conclude that the difference between the mechanical and electrical determinations of the mechanical equivalent of heat cannot be accounted for by differences in standards of thermometry, but must be sought in the energy determinations.

XXV. *A Recalculation of Rowland's Value of the Mechanical Equivalent of Heat, in terms of the Paris Hydrogen-Thermometer.* *By* W. S. DAY[*].

THE measurement of the mechanical equivalent of heat made by Rowland in 1877-79 (Proc. Am. Acad. xv. p. 75, 1879) is probably the best one in which the heat was produced by the expenditure of mechanical energy. Later determinations made with great care, in which the heat was produced by the expenditure of electrical energy, give results higher by about one part in four hundred. Rowland's measurement of temperature was based on comparisons made between an air-thermometer and three Baudin mercurial thermometers, by which he reduced his measurements to the absolute thermodynamic scale. It was the object of the present investigation to compare his thermometers with the hydrogen scale of the International Bureau of Weights and Measures, at Sèvres, near Paris, and make a recalculation of his value of the mechanical equivalent accordingly.

For this purpose, three Tonnelot thermometers which had been carefully studied at the International Bureau, and compared with their standards at several points of the scale, were obtained and compared with the three principal thermometers used by Rowland in his experiment. These comparisons were made in a horizontal comparison tank, designed and

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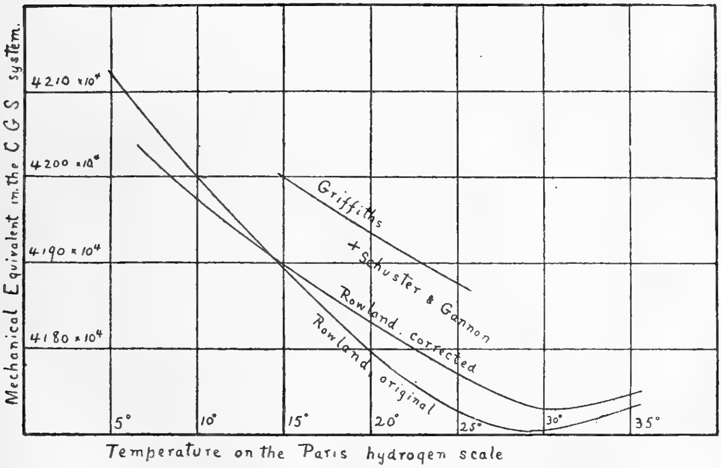
[*] From the Johns Hopkins University Circulars, June 1897.

constructed for the purpose. Rowland's thermometers were originally compared with the air-thermometer in a vertical position, and they were used in the same position in the calorimeter. This made it seem more natural to make the comparisons described here vertically also, but a number of reasons of a practical nature made the author decide to make them horizontally. The chief reason for this was the necessity of taking the zero-points of the Tonnelot thermometers after each reading, this being the method used at the International Bureau, and which experience has shown to be the only accurate one. It was necessary, therefore, to apply a pressure-correction to the readings on Rowland's thermometers to reduce them to what they would have been if the comparison had been made vertically, but this was measured very accurately by an apparatus similar to that used at the International Bureau. In all other respects the attempt was made to use Rowland's thermometers in the way in which he used them in his experiments.

The Tonnelot thermometers had their zeros determined immediately after a measurement at any given temperature. The zero-points were determined by plunging the thermometers into a vessel filled with finely-crushed ice mixed with distilled water. The ice used was very pure artificial ice. The vessel in which the zero-points were taken was constructed so that the thermometers could be immersed in the mixed ice and water to a distance of 7 centim. above the zero-point. They were then read through a small ebonite tube that extended horizontally through the tank from one side to the other, and through which, at its middle point, the thermometer stem passed. By a simple device the water in the tank was prevented from running out. The object of this arrangement was to make sure that all parts of the stem in which the mercury was, as well as the bulb, should be at a temperature of 0° C., and to avoid the presence of dew and of drops of water on the stem, which, in the usual way of taking zero-points, often interfere with the proper reading. The thermometers were always read, in taking zeros, and in the comparison-tank, by means of a micrometer, consisting of a reading-telescope supplied with a micrometer-screw that moved the whole telescope.

From the comparisons made, corrections were obtained for each of Rowland's thermometers, which, when applied to their indications reduced to the absolute scale by the tables given in his paper on the mechanical equivalent, would make them agree with the Paris hydrogen scale. From these corrections Rowland's value of the mechanical equivalent was corrected, taking into account each individual experiment, the thermometers used in it, and the number of observations made with each thermometer. The original values, and the corrected values found in this way, are compared at several temperatures, in the following table:—

Temp.	Old.	Corrected.	Griffiths.	Schuster and Gannon.
6°	4209 × 10⁴	4204 × 10⁴		
10	4200	4197		
15	4189	4189	4199·7 × 10⁴	
20	4179	4183	4193·2	4191 × 10⁴
25	4173	4177	4187·4	
30	4171	4173		
35	4173	4174		



These numbers are in the C.G.S. system, and hydrogen scale. The results given above are also represented by the curves shown in the diagram. The corrected value of the mechanical equivalent found in this way is the same as Rowland's old value at 14°·5 C., at 10° it is lower by one part in 1700, at 20° it is higher by one part in 1200, and at 25° higher by one part in 1000. The heat-capacity of water, as indicated by the corrected values of the mechanical equivalent, varies at about the same rate as that given by the experiments of Griffiths. (Phil. Trans. clxxxiv. A. p. 361, 1893. Phil. Mag. xl. pp. 437 and 447, 1895.) If, therefore, curves were drawn representing the specific heat of water between 15° and 25°, as given by Rowland's experiment, and by Griffiths, taking the value found in each case at 15° as unity, the two curves would be practically identical. This seems to show that the difference between the value of the mechanical equivalent found by Rowland, and that found by Griffiths and by Schuster and Gannon using electrical methods, may not be due to an error in the experiment itself, but to some error in the electrical standards of resistance, or of electromotive force.

These results are in almost absolute agreement with those obtained by Mr. Waidner and Mr. Mallory. (See page 165.)

XXVI. *On the Decomposition of Silver Salts by Pressure.*

By J. E. Myers, *M.Sc., Ph.D., late* 1851 *Exhibition Science Scholar, and* F. Braun, *Ph.D., Professor of Physics in the University of Strassburg i.E.*[*]

As was demonstrated by Carey Lea[†] some years ago, silver salts and others may be decomposed by the application of pressure. By pounding a portion of the salt in a mortar one may readily effect decomposition.

A convincing proof is obtained by compressing a halogen salt of silver in the absence of sunlight and then subjecting the compressed mass to the developing and fixing processes of photography. A black residue of finely divided silver is the result. Exposure of the salt to sunlight, before compression, makes of course this test useless: the difficulty may be overcome by employing a mixture of $AgNO_3$ and KR (R=Cl, Br, or I) in the first instance.

The decomposition is most marked in the case of bromide of silver.

Two different metals immersed in liquid bromine constitute a galvanic element[‡]. The couple Ag, Br, Pt has an E.M.F. of about 0·95 volts. It appeared therefore of interest to determine the E.M.F. developed when AgBr is compressed between electrodes of platinum and silver.

The salt was contained in a so-called diamond mortar, which was placed under a press and subjected to a pressure of 3000–5000 atmospheres. The apparatus did not admit of a direct measurement of pressure, but for purposes of comparison, by using the same quantity of salt and turning the lever to the same position the same pressure (approximately) could be applied. The pestle was insulated from the mortar by means of a thin sheet of mica bent into a cylindrical

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[*] Communicated by the Physical Society: read June 11, 1897.

[†] Phil. Mag. xxxi. p. 323 (1891); xxxiv. p. 46 (1892); xxxvi. p. 351 (1893); xxxvii. pp. 31 and 470 (1894).

[‡] F. Braun, Wied. *Ann.* xvii. p. 610 (1862).

form. A piece of platinum foil beneath the salt, and a short cylinder of silver immediately beneath the pestle, formed the electrodes, which were connected to the quadrants of an electrometer.

The following serves to give a general idea of the run of an experiment :—

(One Daniell element gave a deflexion of 132·1 scale-divisions.)

Ag Br.	Scale-divisions.
On applying pressure, immediate deflexion	1·2
after 2 minutes	2·8
,, 10 ,,	6·8
,, 15 ,,	7·2
,, 40 ,,	9·2
,, 55 ,,	9·0
,, 85 ,,	8·7
,, 95 ,,	8·2
,, 105 ,,	7·0
Pressure removed	5·0

after which the deflexion rapidly sank to zero.

Thus the maximum E.M.F generated was about 0·07 Daniell. The direction of current was from Ag to Pt through the compressed substance.

A higher applied pressure produced an E.M.F. of 0·09 Daniell.

The silver bromide was prepared from pure $AgNO_3$ by precipitation with pure KBr. Before each experiment the salt was carefully dried by heating for two hours or more at a temperature of 150° C., and then allowed to cool in a desiccator. Pure $AgNO_3$ showed no appreciable E.M.F. on compression, neither did KBr : a mixture of the two in equivalent quantities compressed between electrodes of Pt and Ag had an E.M.F. of 0·08 Daniell.

With AgCl and AgI the E.M.F.'s were smaller.

Ag, AgCl, Pt = 0·03 Daniell.
Ag, AgI, Pt = 0·04 ,,

In both cases a marked darkening in colour was noticeable.

Somewhat remarkable is the fact that under high pressure AgBr becomes gradually transparent even when 3–4 millim. thick. At the same time the electrical resistance increases considerably. For instance, in one case the resistance increased gradually from $3{\cdot}65 \,.\, 10^3$ ohms to $14{\cdot}62 \,.\, 10^3$ ohms (measured with alternating current and telephone) in the course of 48 hours, pressure being continuously applied.

Strassburg, Physik. Institut.

XXVII. *Experiments on the Condensation and Critical Phenomena of some Substances and Mixtures.* By Dr. J. P. Kuenen *.

1. THE investigation contained in this paper is a continuation of a research on "The Condensation and the Critical Phenomena of Mixtures of Ethane and Nitrous Oxide," which was read before the Physical Society some time ago †. The results arrived at in the latter paper were stated as follows :—

(1) All mixtures of C_2H_6 and N_2O which contain more than 0·1 of C_2H_6 have critical temperatures lying beneath the critical temperature of C_2H_6.

(2) The vapour-pressures of the mixtures lie partly above those of N_2O, and accordingly show a maximum situated near 0·2 of C_2H_6.

(3) This maximum does not disappear with rise of temperature, but remains up to the critical region; the maximum-curve reaches the plaitpoint-curve.

(4) The mixtures ranging between 0·2 and 0·5 of C_2H_6 have retrograde condensation of the second kind, all the others retrograde condensation of the first kind.

I have now discovered two more instances of mixtures with similar properties, viz. mixtures of ethane and acetylene, and mixtures of ethane and carbonic acid. I will begin by relating some experiments which were made in connexion with the former paper, and afterwards give an account of the results obtained with those mixtures.

A. *On Ethane prepared by different methods.*

2. The ethane used in my former experiments had been prepared from sodium acetate by electrolysis. Its vapour-pressures and critical constants, especially the latter, were different from those given by other observers. The different values are the following, p being expressed in atmospheres :—

	Dewar ‡.	Olszewski §.	Haenlen ‖.	Kuenen ¶.
t_c . . .	35°	34°	34·5°	32·05°
p_c . . .	45·2	50·2	50	48·8

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* Communicated by the Physical Society: read June 25th, 1897.

† Kuenen, Phil. Mag. (5) xl. pp. 173–194.

‡ Phil. Mag. (5) xviii. p. 214.

§ *Bulletin Ac. des Sciences de Cracovie,* 1889, p. 27.

‖ Lieb. *Ann.* cclxxxii. p. 245.

¶ Phil. Mag. (5) xl. p. 181.

Ethane from Ethyl Iodide.

3. In order to ascertain the probable reason of this divergence I prepared some ethane from ethyl iodide, the method used by Haenlen, and investigated its properties. The results have been communicated to the Royal Society of Edinburgh. I will therefore simply state my conclusions here so far as they bear upon the point in question.

Ethane from ethyl iodide contains a variable quantity which may go up to several per cent. of an admixture of higher critical temperature and higher density than ethane, probably butane. This impurity raises both the critical temperature and pressure, and lowers its vapour-pressures. Pure ethane is difficult if not impossible to obtain by this method.

This result accounts for Haenlen's critical values being higher than mine (and probably also for Olszewski's figures), and it shows at the same time the greater purity of my ethane. Even this was not absolutely pure, as shown by some change of the vapour-pressure with volume. But whatever the admixture may be that this was due to, it was probably too small to affect the critical temperature by more than a few tenths of a degree and the critical pressure by more than some tenths of an atmosphere.

Ethane from Sodium Acetate.

4. In order to confirm my results, and at the same time to obtain some more pure ethane for further experiments, I repeated the preparation of ethane from sodium acetate. The anode, consisting of two pieces of platinum foil (joint surface 150 cm.^{2}), was placed inside a good-sized porous pot, care being taken to obtain a high current-density *. A current of about 15 amperes was used. Special precaution was required to prevent the indiarubber stopper which closed the porous pot from being burnt by the heat developed. A glass tube, passing through the stopper, was bent into a U-shape outside and contained some mercury. The contact with the platinum foil was made by platinum wires soldered to the foil.

The gas was washed in sodium hydrate, fuming sulphuric acid, and potash, and collected in a gasometer which was filled with a solution of potash. This gasometer consisted of two carboys, closed with indiarubber stoppers and connected by a wide glass tube. The yield of ethane was very high, very little oxygen being formed. The oxygen was moreover absorbed by introducing a piece of phosphorus into the gasometer.

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* Murray, Journal Chem. Soc. lxi. pp. 10–36.

5. In order to purify the gas further, it was compressed by means of a Natterer pump into a small copper cylinder fitted with high-pressure stopcocks *. In this process it was once more washed in potash and subsequently dried by calcium chloride and phosphorus pentoxide. Pump and cylinder were previously exhausted by a mercury air-pump. During the compression the copper cylinder was placed in solid carbonic acid. The pressure required to liquefy the gas was pretty high: this shows that it contained a considerable quantity of some permanent gas; this may have been air which in some unknown manner had got into the gas. By thoroughly boiling the ethane, as much of this admixture was removed as possible. The results obtained with this ethane show that some admixture was left, of about the same amount as in the gas which I used in my former experiments. It is noteworthy that with other substances, such as carbonic acid and nitrous oxide, I always succeeded in removing practically all impurity by this process; for some reason the same thing seems more difficult with ethane. It is not impossible that ethane forms a mixture of constant boiling-point with the admixture, whatever it may be.

6. The apparatus which was used for filling the compression-tubes with the gases consisted almost of one piece of glass. The mercury air-pump was sealed to the double mercury gasometer which I used for preparing the mixtures (see below). This was connected at the other side with a tube leading to the compression-tube, and another tube through which the purified gas was supplied from the compression cylinder. The connexion with the cylinder was made by means of a copper tube, sealed with sealing-wax into a glass tube, the connexion with the compression tube being made by a ground joint. This was so adjusted, that the tube after being filled could be turned upright without being detached, by which manipulation the tube was shut off at the bottom by some mercury. There are two reasons why I prefer this manner of filling the tubes to sealing them off. In the first place many gases are more or less decomposed at the temperature of softened glass, and in the second place the sealing place is apt to end in a sharp point, while for the proper action of the little iron stirring-rod which I use inside the tubes, a rounded top is absolutely necessary. It need hardly be added that the apparatus was fitted with a

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* Most of my high-pressure apparatus are manufactured by Giltay, Delft, Holland, and are of the same type as those used in the Physical Laboratory at the University of Leyden (Director, Prof. Kamerlingh Onnes).

certain number of safety-tubes dipping in mercury. I found that the amount of spring in the apparatus was quite sufficient, and that special glass springs were superfluous.

7. For the observations properly speaking, I used Ducretet's apparatus. The temperatures of air-manometer and gas-tube were kept constant with the running water of the supply. A constant water-pressure was obtained by means of an overflow arrangement, and the current before passing round the gas-tube was heated by a long coil in warm water. A sufficiently constant temperature was obtained in this manner.

TABLE I.

Ethane from Sodium Acetate (April 1897).

t.	p_b.	p_e.	v_b.	v_e.
5·55	26·90	27·42	231·2	33·0
14·5	32·91	33·52	177·1	35·6
15·0	33·23	33·77	176·1	35·5
23·05	39·74	40·27	130·9	39·0
30·2	46·35	46·72	94·5	46·2
32·2	48·64	C	61·5	

8. Table I. gives the results obtained: p_b and v_b are the pressures and volumes when the condensation begins, p_e and v_e when the whole substance is liquefied; p is expressed in parts of an atmosphere, the v's in this and in all the other tables are expressed in an arbitrary unit, but reduced to the some initial volume at 0° and 1 atmosphere. C indicates the critical point. The experimental determination of the p_e's gives no difficulty if the stirring-rod is continually used. For p_b this is not always the case, because stirring is not so effective when the volume is much larger. In cases of doubt p_b and v_b were therefore determined graphically as the points of intersection in the p-v diagram of the two parts of the isothermal, the part where the substance (mixture) is homogeneous and the part where there are two phases. Another advantage in drawing the latter part of the isothermal is that from its shape and the direction of its curvature some idea may often be obtained as to the nature of the admixture *.

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* Compare a paper on ethane in the Proc. R. S. of Edinburgh, 1897. I thought it unnecessary to publish the p-v diagrams as they offer no special interest.

TABLE II.

Ethane. (April 1895.) 1st sample.

t.	p_b.	p_e.
5·85	27·38	27·92
10·65	30·45	31·11
15·4	33·84	34·48
22·4	39·73	40·25
29·35	45·94	46·25
31·0	47·64	47·95
31 95	48·81 C	

TABLE III.

Ethane. (April 1895.) 2nd sample.

t.	p_b.	p_e.
3·95	26·17	26·89
15·4	34·11	34·66
22·4	39·89	40·32
29·3	45·89	46·19
32·05	48·91 C	

9. Tables II. and III. contain values obtained for two samples of ethane on a former occasion. The pressures as shown by a comparison of the p-t curves are this time slightly lower and the critical temperature a little higher. Absolutely pure ethane would probably have pressures slightly lower even and a slightly higher critical temperature. The p-t curves in the diagrams for ethane and the other single substances are drawn with p_b; as this value is probably nearer the true value of p than p_e.

B. *Mixtures of Carbonic Acid and Acetylene.*

10. In my former paper I gave a first instance of mixtures having critical temperatures below those for the component substances. The only instance of critical temperatures above those of the components seemed to be that of mixtures of carbonic acid and acetylene. According to an observation of Dewar's[*], a mixture of $\frac{1}{2}$ CO_2 and $\frac{1}{2}$ C_2H_2 has a critical temperature of 41° C., those for carbonic acid and acetylene being 31° and 37° respectively. It seemed necessary to try the experiment, because experiments on mixtures have always to be accepted with caution, unless it appears that exceptional

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[*] Proc. R. S. of L. xxx. p. 542.

precautions have been taken to avoid retardation and impurities*. The phenomenon, if confirmed, would have been worth completely investigating. The results which I have obtained contradict Dewar's experiment.

Preparation of Carbonic Acid.

11. Both carbonic acid and acetylene were prepared before I was in a position to apply the method of purification described for ethane by condensation and distillation. I had therefore to resort to chemical methods of purifying the gases. Carbonic acid was made from sodium bicarbonate and sulphuric acid. The acid was contained in a boiling-flask provided with a dropping-funnel with the sodium bicarbonate solution. All solutions used were previously boiled and cooled in a current of carbonic acid. The gas passed through sulphuric acid and phosphorus pentoxide. Before the solution was admitted the whole apparatus had been exhausted with a mercury air-pump.

12. The purity of the carbonic acid was first tested in the compression-apparatus and appeared to be quite satisfactory.

Table IV.

Carbonic Acid.

t.	p_b.	p_e.	Amagat.
14·95	50·02	50·24	50·0
20·5	57·16	......	57·0
25·85	64·92	65·06	64·5
31·1	73·26	C	

Amagat's values for the critical point are 31°·35, 72·9 atm. The agreement is very close, the deviations of my values being probably due to a slight admixture of permanent gas. This would, in fact, make the vapour-pressures and critical pressure higher and the critical temperature lower. Similar deviations will be noticed by comparing some of Andrews's values with Amagat's, the former having been reduced to atmospheres.

t.	Andrews.	Amagat.
13·09	48·05–49·50	47·6
21·46	58·94–61·35	58·2

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* Prof. Dewar, on learning that I had repeated this experiment of his (forming part of an essentially qualitative investigation carried out in 1880), told me that his acetylene contained C_2H_3Br. This accounts for the high critical temperature and low vapour-pressures observed by him.

Andrews's critical constants are $30^\circ{\cdot}9$ and about 73 atmospheres. The deviations are in the same direction, only much more pronounced. Andrews's carbonic acid contained about $\frac{1}{500}$ of air.

Preparation of Acetylene.

13. This gas was prepared in the same apparatus as carbonic acid, from calcium carbide and water. The water had been previously saturated with carbonic acid. The gas was washed in silver nitrate to absorb the phosphoretted hydrogen and sulphuretted hydrogen which it contained. It was freed from carbonic acid by soda-lime and dried in sulphuric acid and phosphorus pentoxide. Two samples were obtained. The first probably contained some air and also some phosphoretted hydrogen. In preparing the gas for the second time, a second washbottle with silver nitrate was introduced, and this bottle was only just beginning to get blackened, when the acetylene was collected. The only possible impurity of this sample is therefore a trace of air.

Some of this sample was used in making the mixture with CO_2, the results for which are communicated below. Some mixtures of the first sample with CO_2 gave, however, similar results. The results obtained with the purer sample are contained in Table V.

TABLE V.

Acetylene.

t.	p_b.	p_e.
14·95	38·09	38 66
20·15	43·18	43·75
26·8	50·29	50·85
27·1	50·55	51·12
30·8	55 05	……
35·25	61·02 C	

14. Evidently the gas contained an admixture of some other substance. If this admixture is air, the vapour-pressures are slightly higher than for pure acetylene, and the critical temperature a few tenths of a degree too low. In this case also the p_b's were used in drawing the vapour-pressure curves in the diagrams.

The pressures agree very well with those given by Villard *,

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* *Ann. de Chimie et Physique,* March 1897.

who finds 37·9 at 15° C., and 42·8 at 20°·2. Ansdell's * values are a good deal different, his pressures being 32·77 at 13°·5, 39·76 at 20°·15, 48·99 at 27°·55, 56·20 at 31°·6. For the critical point he found 68·0 atmospheres and 36°·9 C. As to Pictet's † values, they show no resemblance whatever with the others. His pressures were : 29·0 at 14°·1, 33·5 at 19°·5, 38·5 at 27°·6, and 48·0 at 36°·5 (critical point).

Mixture of Carbonic Acid and Acetylene.

15. The mixture was made in a mercury gasometer consisting of two parallel wide glass tubes of 300 c.c. each, provided at the bottom with a tap, an indiarubber tube, and a funnel with mercury sufficient to fill up the whole of the tubes. At the top the tubes ended in threeway stopcocks communicating together, the whole, as described before, connected on one side with the pump, and at the other side with the rest of the apparatus. The mixture contained $\frac{1}{2}C_2H_2$ and $\frac{1}{2}CO_2$.

16. The results obtained are the following :—

TABLE VI.

Mixture $0{\cdot}5\,C_2H_2 - 0{\cdot}5\,CO_2$.

t.	p_b.	p_e.
7·35	35·98	36·90
14·95	43·46	44·62
20·15	49·54	......
24·75	55·18	56·10
26·15	57·10	57·95
31·55	64·79	65·17
32·55	66·52 C	

The differences between p_b and p_e (fig. 1), the pressures at the beginning of the condensation and at the end, are somewhat larger than for acetylene, as might be expected. It would be possible to make an allowance for the impurity of the component gases, and in that manner obtain approximate corrected values for the pressures. This would, however, be a rather uncertain calculation, and in drawing the border-curve for this mixture I have preferred to use the direct experimental values. Obviously, though the slight impurities of the CO_2, and specially of the C_2H_2, also affect the values for the mixture, the divergence from the true values can only be of the same order of magnitude as for the components.

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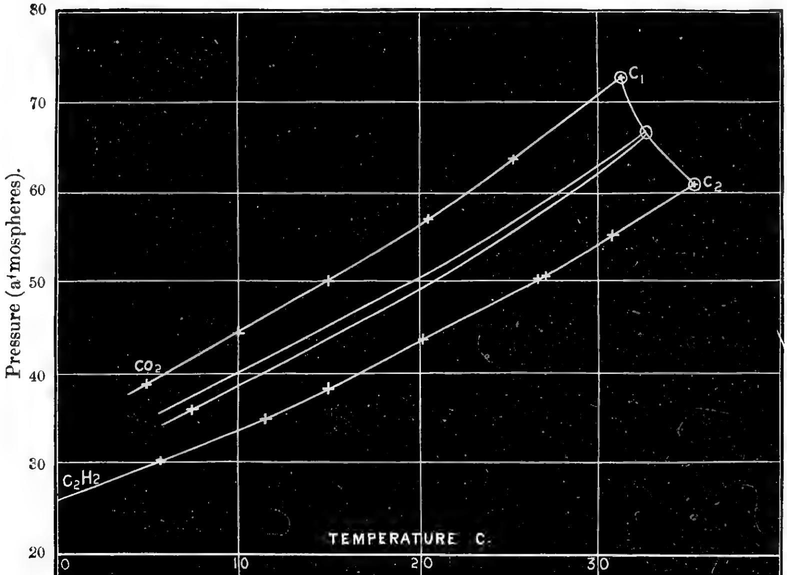
* Proc. R. S. of London, xxix. p. 211 (1879).

† *Zeitschrift Compr. und flüssige Gase,* i. p. 32 (1897).

Fig. 1.

(*Vide* Tables IV., V., VI.)



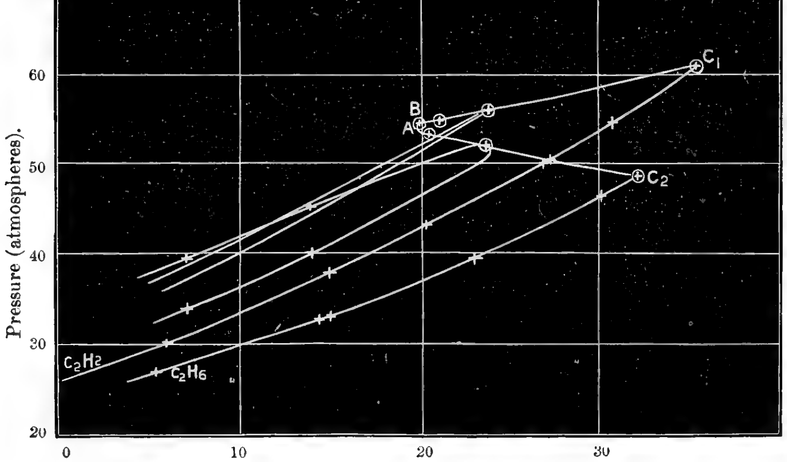


Fig. 2.

(*Vide* Tables I., V., VII.-XI.)

17. The diagram (fig. 1) shows that the plaitpoint curve is a line with small curvature and that mixtures of carbonic acid and acetylene have critical temperatures between those for the pure gases. The border-curve for the mixture investigated is relatively narrow. This entails small differences in composition between the coexisting phases, the vapour containing only a little more carbonic acid than the liquid. It also follows that the critical phenomena will closely resemble those of a pure substance. This was found to be the case, the plaitpoint temperature and critical temperature nearly coinciding.

18. An instance of a critical temperature above those for the components has therefore not been discovered as yet. From the theory it may be deduced that this phenomenon will probably occur for mixtures which have a minimum vapour-pressure at low temperature ; in the same way as the critical temperatures below those for the components seem to occur for mixtures which have a maximum vapour-pressure (as for nitrous oxide and ethane).

The law of the connexion between the two phenomena will be again referred to in connexion with the application of Prof. van der Waals's theory.

C. *Mixtures of Acetylene and Ethane.*

19. The relative position of the vapour-pressure curves and critical points in the *t-p* diagram of these gases was such as to promise interesting results for mixtures of them : in fact the same reasoning holds for this combination which led me originally to the choice of nitrous oxide and ethane. If we consider the curves for the four gases the critical temperatures of which are between 30° and 40°—CO_2, C_2H_6, C_2H_2, and N_2O—we find that besides with the combinations C_2H_6–N_2O and C_2H_6–C_2H_2 we may expect similar phenomena with C_2H_2–N_2O, a combination which I have not been able to try. Results obtained with the couple C_2H_6–CO_2 will be described later on. They agree with those of N_2O–C_2H_6, and C_2H_2–C_2H_6. The case C_2H_2–CO_2 was discussed above. The combination CO_2–N_2O offers no interest *a priori*. Moreover, some time ago I investigated a mixture of these two gases, but did not obtain any remarkable results.

Mixtures of some of these four gases with hydrochloric acid (51°) or methyl fluoride (44°·9) also promise new results. Mixtures of hydrochloric and carbonic acids were investigated by Dewar * and Ansdell †. Their figures differ from each

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* Proc. R. S. of London, xxx. p. 543.

† *Ibid.* xxxiv. pp. 113–119.

other, but the plaitpoint curve does not seem to offer any peculiarities. It is hardly necessary to add, that a number of combinations of substances at other temperatures might be mentioned which would probably display similar phenomena.

20. I began by preparing two mixtures of acetylene and ethane containing 0·68 and 0·27 of the latter gas respectively.

After having got the constants for these two and not having any more acetylene available at the time, I made new mixtures by partly exhausting the compression-tube containing the 0·27 C_2H_6 mixture and introducing fresh quantities of ethane. In this manner four more mixtures were obtained. Though great care was taken not to introduce any traces of air into the tubes by this repeated process, it is not a safe method, and the results seem to show, that some impurity must have got into the mixture. Moreover, the calculation of the composition of each fresh mixture was again somewhat uncertain. The consequence is, that the last mixture prepared in this manner, which, according to the calculations, contained 0·76 C_2H_6, showed higher changes of vapour-pressure ($p_e - p_b$) than the first mixture ($x = 0·68$), and at the same time a slightly lower critical temperature and higher critical pressure. All this does not agree with the position and shape of the plaitpoint curve. The results for this mixture had therefore to be rejected. This also makes the values for the other three mixtures (especially the last one) doubtful. It must, however, be understood that this small uncertainty does not make the general result in the least doubtful, only the curves would be slightly different for the mixtures if pure.

TABLE VII.

Mixture 0·68 C_2H_6–0·32 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
6 97	......	39·48	......	33·9
7·05	34·1	......	172·5	
13·95	40·35	45·25	134·3	37·2
22·2	49 17	51·83	88·2	46·5
23·05	......	52·35	......	50·2
23·15	......	52·40 } M	......	50·7 } M
23·35	......	52·41	......	54·3
23·4	......	52·32 P	......	57·4 P
23·45	51·31	......	72·9	......
23·5	51·33	......	72·9	......
23·55	......	52·29	......	59·1
23·6	......	52·22	......	60·2
23·65	51·72	52·16	67·2	60·3
23·75	52·06	C	65·1	

TABLE VIII.
Mixture 0·61 C_2H_6–0·39 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
8·85	39·6	44·04	136·6	34·3
13·97	44·6	48·42	112·9	37·3
20·15	52·10	53·68 } M	74·6	48·4
20·55	52·61	53·68	70·6	52·4
20·75	53·06	53·47 P	67·5	59·5
20·77	53·4 ?	C	62·5	

TABLE IX.
Mixture 0·51 C_2H_6–0·49 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
8·4	40·87	45·60	130·4	33·1
14·0	46·4	50·50	106·3	36·1
19·35	53·27	55·02 } M	73·2	46·1
19·75	54·10	55·02	65·2	51·2
19·85	......	54·87 P	......	54·4
19·9	54·24	......	64·4	......
19·95	54·57	54·72	60·7	57·7
19·96	54·65 ?	C	58·2	

TABLE X.
Mixture 0·41 C_2H_6–0·59 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
7·95	......	42·70	......	31·4
8·0	39·95	......	136·2	......
14·0	46·1	48·19	107·3	34·6
19·35	52·0	53·67	83·1	40·6
20·95	54·50	55·11 } M	67 ?	47·9
21·0	54·62	55·12	......	50·0
21·05	......	55·12	......	50·6
21·1	......	55·05 P	......	53·7
21·12	......	55·02	......	55·2
21·15	54·99	C	55·1	

TABLE XI.
Mixture 0·27 C_2H_6–0·73 C_2H_2.

t.	p_b.	p_e.	v_b.	v_e.
7·6	38·25	39·61	148·4	29·7
13·95	44·63	45·64	114·5	31·7
22·2	53·9	54·55	75·7	39·0
23·75	56·05	56·27 } M	61·7	47·7
23·8	......	56·25 P	......	51·1
23·85	56·23	C	55·4	

21. The Tables VII.–XI. contain the results for the five mixtures; the diagram (fig. 2) shows the relative position of the vapour-pressure curves for the pure substances, the border-curves for the mixtures, and the plaitpoint curve which envelops the border-curves. Only two of the border-curves, for mixtures 0·68 and 0·27, have been drawn in order not to make the diagram too complicated.

The diagram is qualitatively the same as that for nitrous oxide and ethane: quantitatively there is a difference in the circumstance that the mixture of minimum critical temperature and the mixture of maximum vapour-pressure are much nearer to each other than in the other case. Mixture 0·51 was about the mixture of maximum pressure (point B), while the minimum critical temperature (point A) occurs for a mixture containing only a little more ethane ($x=0·53$ or $0·54$). The same points B and A were much farther apart for N_2O–C_2H_6, where they belonged to mixtures $x=0·2$ and $0·5$ respectively.

22. It was stated in my former paper that the critical phenomena of the mixtures of N_2O and C_2H_6 are little marked. Between C and A, and also between C and B, the condensation should be retrograde of the first kind between the two critical temperatures t_p and t_c; between A and B the condensation should be retrograde of the second kind. The latter phenomenon has never yet been observed owing to disturbing circumstances such as gravitation and small impurities. For C_2H_2–C_2H_6 the chances of observing special critical phenomena between A, B are obviously even smaller than for N_2O–C_2H_6.

I succeeded this time in observing critical phenomena of the first kind with all five mixtures. This is chiefly owing to the temperatures being more constant than formerly. The phenomena were very marked for the mixtures 0·68, 0·61, and 0·51, much less marked with mixtures 0·41 and 0·27. Though the relative position of P (plaitpoint) and C (critical point) is the same along the curves C_1B and C_2A, this is not the case as regards the point of maximum pressure M. Between B and C_1, M is between P and C; along AC_2, P is between M and C. The results with mixtures 0·68, 0·61, and 0·27 are in accordance with this rule; mixture 0·41 shows M to be near P but on the wrong side; mixture 0·51, which is about the mixture of maximum vapour-pressure, should have one critical point; evidently it contained some other admixture. Moreover, the differences between p_b and p_e are considerable for this mixture, while they should be very small, not larger than for the components used. Altogether the results show that at least some of the mixtures contained a small admixture of air.

The figures are therefore less trustworthy than those for N_2O–C_2H_6 or those for CO_2–C_2H_6, the combination to be described presently.

The v–x curves have the same character as those for N_2O–C_2H_6: it seems therefore unnecessary to publish them.

D. *Mixtures of Carbonic Acid and Ethane.*

23. The ethane used in these experiments was the same as for the mixtures with acetylene. The carbonic acid was taken from an ordinary carbonic-acid cylinder, after having been distilled and purified in a small high-pressure cylinder in the way as described for ethane (see § 5).

TABLE XII.

Mixture 0·50 C_2H_6–0·50 CO_2.

t.	*pb.*	*pe.*	*vb.*	*ve.*
8·8	......	47·47	......	32·2
8·85	45·74	......	113·8	......
9·05	45·94	......	111·8	......
9·1	......	47·74	......	32·4
14·95	52·94	54·10	82·6	37·1
17·55	56·63	57·18 } M	62·4	44·8
17·75	56·90	57·24 } P	60·8	51·4
17·85	56·99	57·15	59·9	55·9
17·88	57·10		57·4	

TABLE XIII.

Mixture 0·43 C_2H_6–0·57 CO_2.

t.	*pb.*	*pe.*	*vb.*	*ve.*
8·95	47·27	48·46	106·8	30·7
14·95	54·50	55·30	77·4	35·5
17·55	......	58·42 } M	......	44·8
17·58	58·18	.....	57·9	......
17·62	58·37	P and C	52·7	

TABLE XIV.

Mixture 0·30 C_2H_6–0·70 CO_2.

t.	*pb.*	*pe.*	*vb.*	*ve.*
8·95	48·58	49·07	106·0	28·8
14·95	55·82	56·28	79·8	32·2
17·28	59·02	59·31	66·9	35·0
18·68	......	61·19	......	41·4
18·69	61·14	......	53·3	......
18·73	......	61·28 } M	......	42·8
18·75	61·21	61·31 } M	52·5	46·1
18·77	......	61·29 } P		
18·80	61·26	C	48·5	

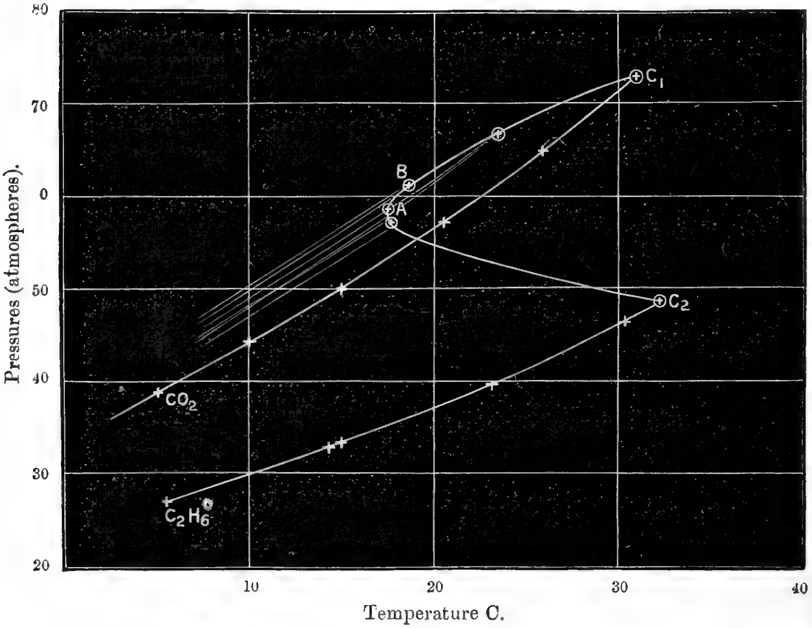
TABLE XV.

Mixture 0·15 C_2H_6–0·85 CO_2.

t.	p_b.	p_e.		v_b.	v_e.
10·35	48·84	49·46		111·2	25·4
16·0	55·72	56·32		90	27·2
23·15	......	66·32		......	37·7
23·2	66·15	......		52·9	......
23·35	66·44	66·52	P	49·6	41·8
23·37	......	66·57	M	......	43·2
23·4	66·54		C	47·6	

24. These mixtures (Tables XII.–XV., fig. 3) show again the phenomena of minimum critical temperature and maximum vapour-pressure which had been found with mixtures of ethane with N_2O and C_2H_2. The minimum critical temperature occurs for a mixture $x = 0·45$ about (A), the maximum vapour-pressure for $x = 0·30$ approximately (B). These compositions are nearer together than for N_2O-C_2H_6, but not so near as for C_2H_2–C_2H_6.

Fig. 3. (*Vide* Tables I., IV., XII.–XV.)



25. Again, the critical phenomena characteristic of mixtures were very little marked. As just stated, mixtures containing ethane between the limits 0·30 and 0·45 ought theoretically to show the critical phenomena of the second kind. For mixture 0·43 points P and C were, however, so near together that I could not observe a difference. For mixture 0·50 P is between M and C; for mixture 0·30 M and P coincide, and very nearly also with C; for mixture 0·15 M seems to lie between P and C. All this agrees with the rules given above. The differences between p_b and p_e are very small, especially for the maximum mixture. This also is in accordance with theory. The results for this combination are more trustworthy than those for C_2H_2–C_2H_6, and as good or better than those for N_2O–C_2H_6.

The *v*-*x* diagram (not published) has the same character as for the two other combinations.

E. *Application of Van der Waals's Theory*[*].

26. If taken as generally as possible, van der Waals's theory of mixtures depends on the assumption of the existence of a relation between p, v, and t for a mixture of the same character as for pure substances. The p-v curve must be supposed to have an unstable part below a certain temperature, and in changing the composition of the mixture gradual changes in the relation between p, v, and t should take place. The laws of the condensation, the critical phenomena, in general of the coexistence of phases are then found by an application of thermodynamics in one of its many forms: Van der Waals used the ψ function ($\psi = \epsilon - T\eta$). (T = absolute temperature.) In fact wherever thermodynamics is used for purposes of this kind, the existence of a similar equation of condition must be presupposed, though the actual form of this equation can only be given in the simplest cases, and even then only qualitatively. It is well known that the theoretical equation for single substances given by van der Waals does not agree with the results of experiments, and the same is true for the analogous equation for mixtures of two substances which he uses in his theory. It is well, however, to bear in mind that a number of the results which he arrives at are independent of the form of the equation. Even those results which are not, still may have a resemblance to actual phenomena in the same manner as for single substances.

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[*] *Archives Néerl.* xxiv. pp. 1–56; *Zeitschrift physik. Chemie*, v. pp. 133–173; *Kon. Akad. van Wet. Amsterdam*, Nov. 24, 1894, pp. 133–137; *ibid.* Mei 25, 1895, pp. 20–30; Juni 29, 1895, pp. 1–12.

In the following account of some of the theoretical results I will only consider the case of the coexistence of vapour and liquid, *i. e.* a separation of the liquid into two will be excluded from the consideration.

I shall sometimes follow a method different from van der Waals's, chiefly for the sake of shortness and simplicity.

27. Let us suppose a set of isothermal curves (p-v diagram) to be drawn for the mixture as determined by observations of p, v, and t in homogeneous states, and completed by partly unstable interpolation-curves joining the curves in the homogeneous gaseous and liquid condition.

Suppose we forget for a moment that the substance is a mixture, and apply the Maxwell-Clausius criterion for finding the coexisting phases in the usual manner. We should then get the border or saturation-curve separating the homogeneous states from those where these are unstable or less stable than the separated states. The diagram thus obtained gives us the condensation-pressures, the vapour- and liquid-densities, and the critical point for the mixture, if supposed to behave like a pure substance, *i. e.* not dividing into phases which have a different composition. The real border-curve can only be found by a more complicated application of thermodynamics: still we may easily see one thing, that the real border-curve would lie entirely outside the hypothetical one just obtained. Instead of the horizontal line cutting off equal areas from the isothermal, we get a sloping line: this line has again to cut off equal areas. Obviously this is only possible by making it start lower and finish higher than the straight line. This is true at all temperatures, and the conclusion is, as stated, that the border-curve will be entirely outside the hypothetical one. From this it follows, *e. g.*, that the real critical temperature is higher than the critical temperature for the undivided mixture; also that the critical isothermal does not touch the border-curve at the top M but at a point to the right C, where $\frac{dp}{dv} < 0$ instead of $=0$ as for a single substance. The volume in the critical point is obviously larger and the pressure smaller than the same quantities at the top of the border-curve.

28. The last fact about the critical pressure (point C) not being the maximum pressure (point M) for the mixture on the border-curve was pointed out above with reference to the p-t diagram. In fact the loops for the mixtures are nothing but the real border-curves represented in a different manner. The third point P which we considered in that case (plait-point), the point of contact of the loop and the enveloping

curve (plaitpoint curve), cannot be found by a simple geometrical construction in the p-v diagrams. The connexion between the latter diagrams for different mixtures is not so simple as the connexion between the loops (border-curves) in the p-t diagram. A point of intersection of two loops gives the pressure and temperature at which the two mixtures to which the loops belong will coexist as vapour and liquid[*]. The enveloping-curve contains the points of intersection of loops infinitely near each other; in other words, it gives the conditions of pressure and temperature of the coexistence of identical phases: at those points the mixtures are therefore in the critical condition, and will display the critical phenomenon. Evidently these points P do not in general coincide with either M or C. At the critical point C the mixtures do not show the critical phenomenon, the latter belonging to P, where the temperature is lower. This was fully explained on former occasions.

29. The properties of the v-x diagram were sufficiently discussed in my previous paper. Also its meaning as projection on the v-x plane of curves on the ψ surface studied by van der Waals. It was found in the first place that the maximum mixture in its critical point behaves entirely as a single substance; this fact would appear in the p-v diagram by the critical isothermal touching the border-curve at the top, and by points M, P, and C coinciding. If the maximum mixture is the same at all temperatures (as is approximately realized by N_2O–C_2H_6 within the limits of temperatures used), the real border-curve coincides entirely with the hypothetical border-curve examined above. If the maximum mixture changes with temperature the two curves coincide at the top only.

30. The consequence of this coincidence is that the critical point of the maximum mixture may be determined in exactly the same way as for a single substance: the conditions being

$$\frac{dp}{dv}=0, \quad \frac{d^2p}{dv^2}=0.$$

The direction of the vapour-pressure curve of the maximum mixture at B (*i. e.* the direction of the maximum-curve at that point) is also the same as for a single substance. This direction may be easily proved to satisfy the equation

$$\left(\frac{dp}{dt}\right)=\left(\frac{\partial p}{\partial t}\right)_v \dagger,$$

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[*] This is not so when a mixture of maximum or minimum pressure is between the two mixtures.

[†] This equation is true for single substances at the critical point, and

the index indicating the quantity which is kept constant in the differentiation. Van der Waals[*] has discovered that for single substances $\frac{T}{p}\frac{dp}{dt}$ is approximately equal to 7 at the critical point. It is an interesting confirmation of the theory that my maximum-curves also give a value about equal to 7 at point B.

31. An important question in connexion with this is, What is the direction of the plaitpoint-curve at B? In my former paper this point was left undecided. In the mean time Prof. van der Waals, to whom I showed my results, has published two papers[†] in which the properties of both points B and A are completely discussed. I will deduce here some of his results, sometimes using his equations in a different manner.

32. For $\frac{dp}{dt}$ of the plaitpoint-curve the following may be written:—

$$\left(\frac{dp}{dt}\right)_P = \left(\frac{\partial p}{\partial t}\right)_{vx} + \left(\frac{\partial p}{\partial v}\right)_{tx}\left(\frac{dv}{dt}\right)_P + \left(\frac{\partial p}{\partial x}\right)_{vt}\left(\frac{dx}{dt}\right)_P.$$

At B both $\frac{\partial p}{\partial v}$ and $\frac{\partial p}{\partial x}$ will be seen to be $=0$. This follows immediately from the conditions of the v-x diagram, $\frac{\partial p}{\partial v}$ being $=0$, because the line of equal pressure always touches the connodal curve at the plaitpoint, and is therefore parallel to the v-axis in this particular case; while $\frac{\partial p}{\partial x}=0$, because the

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for the maximum mixture at B; for other mixtures it would hold at the hypothetical critical point for the homogeneous mixture. The hypothetical vapour-pressure curve for a mixture lies in between the two branches of the loop in the p-t diagram, and comes to an end at this hypothetical critical point at some distance from the plaitpoint-curve. The equation consequently does not hold for a mixture at its plaitpoint. It seems therefore impossible to identify the expression $\left(\frac{\partial p}{\partial t}\right)_{vx}$ with the $\frac{dp}{dt}$ of the hypothetical vapour-pressure curve at the plaitpoint of an arbitrary mixture, although this is done by van der Waals in his interpretation of the equation which gives the difference of the value of $\left(\frac{dp}{dt}\right)_P$ and $\left(\frac{p}{\partial t}\right)_{vx}$ (*Kon. Ak. Amsterdam,* Mei 25, 1895, pp. 13, 30 Juni 29, 1895, p. 3, form. (5)).

[*] "Continuity" &c., Supplementary Notes; *Kon. Ak. Amsterdam,* Mei 25, 1895, p. 27.

[†] *Kon. Ak. Amsterdam,* Mei 25, 1895, pp. 20–30; Juni 29, pp. 1–12.

point has still the properties of a maximum pressure-point. Neither $\frac{dx}{dt}$ nor $\frac{dv}{dt}$ can be ∞ along the plaitpoint-curve at B, because that would mean that the temperature went through a maximum or minimum. The result is that at B

$$\left(\frac{dp}{dt}\right)_{\mathrm{P}} = \left(\frac{\partial p}{\partial t}\right)_{v} = \text{(see above)} \left(\frac{dp}{dt}\right) \text{ along maximum curve,}$$

i. e. the maximum curve touches the plaitpoint-curve at B. The diagram in my former paper is therefore incorrect, as the plaitpoint-curve was drawn with a break at B.

33. As to point A, it may be proved that $\left(\frac{dp}{dt}\right)_{\mathrm{P}}$ is infinite there. We must use for this the conditions which hold for the point where a plait is split up. A plaitpoint is a point of the connodal curve on the ψ surface, but at the same time of the spinodal curve. The connodal curve is the curve traced out by the double tangent-plane: the spinodal curve, which is inside the connodal curve, separates the part of the surface where the substance is stable from the unstable part. The equation of this curve is

$$\mathrm{A} = \frac{\partial^2\psi}{\partial v^2}\frac{\partial^2\psi}{\partial x^2} - \left(\frac{\partial^2\psi}{\partial x\,\partial v}\right)^2 = 0.$$

Inside the curve $\mathrm{A} < 0$, outside $\mathrm{A} > 0$.

Applying the condition that $\mathrm{A} = 0$ in the plaitpoint, we have

$$\left(\frac{d\mathrm{A}}{dt}\right)_{\mathrm{P}} = \left(\frac{\partial\mathrm{A}}{\partial t}\right)_{vx} + \left(\frac{\partial\mathrm{A}}{\partial x}\right)_{vt}\left(\frac{dx}{dt}\right)_{\mathrm{P}} + \left(\frac{\partial\mathrm{A}}{\partial v}\right)_{xt}\left(\frac{dv}{dt}\right)_{\mathrm{P}} = 0.$$

Now at A $\frac{\partial\mathrm{A}}{\partial t}$ is > 0, because just before the division of the plait $\mathrm{A} < 0$ in that point, and just after $\mathrm{A} > 0$*. Moreover we have at point A $\left(\frac{\partial\mathrm{A}}{\partial x}\right)_{vt} = 0$, and $\left(\frac{\partial\mathrm{A}}{\partial v}\right)_{xt} = 0$. This will be seen to be true on considering that in the direction of the x-axis A passes from negative through 0 back to negative, in the direction of v from positive through 0 back to positive. The conclusion we come to is that either $\frac{dx}{dt}$ or $\frac{dv}{dt}$, or both, are infinite at A. The last alternative is the true one; but either of the three is sufficient for our

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* A more rigorous proof may be given by working out $\frac{\partial\mathrm{A}}{\partial t}$: compare van der Waals, *l. c.* Juni 29, p. 2.

purpose. If we put in the general equation for $\left(\frac{dp}{dt}\right)_{\mathrm{P}}$ in (32) $\left(\frac{dx}{dt}\right)_{\mathrm{P}}=\infty$, we find $\left(\frac{dp}{dt}\right)_{\mathrm{P}}=\infty$, because at A neither $\frac{\partial p}{\partial x}$ nor $\frac{\partial p}{\partial v}$ is 0, the direction of the p line not coinciding with either the v or the x axis.

34. From $\left(\frac{dp}{dt}\right)_{\mathrm{P}}=\infty$ at A it follows, that the plaitpoint curve has a vertical tangent at A and the loop for the mixture of minimum critical temperature touches the plaitpoint curve at A in the ordinary manner. In my former paper I drew the plaitpoint curve with a break at A, as suggested by the experimental data, and drew some conclusions from this which now appear to have been erroneous. The two branches of the connodal curve at the moment of the division of the plait intersect as two straight lines. This was not sufficiently brought out in the diagram (fig. 11) *. Moreover the curves were drawn in such a manner, that a line parallel to the v-axis had, under certain circumstances, four points of intersection. This was an unnecessary assumption which, in cases of splitting up of the first plait, is not a probable one. Altogether the paragraphs 40–42 with the figures 7, 11, and 12 should be withdrawn.

35. All these results are entirely independent of the form of the equation between p, v, and T for the mixtures. We will now apply van der Waals's equation for some calculations regarding the mixtures investigated.

The equation between p, v, and T has the form

$$\left(p+\frac{a_x}{v^2}\right)(v-b_x)=\mathrm{RT},$$

in which a_x and b_x depend on the composition of the mixture x in the following manner :—

$$a_x=a_1(1-x)^2+2a_{12}x(1-x)+a_2x^2,$$
$$b_x=b_1(1-x)^2+2b_{12}x(1-x)+b_2x^2 \dagger ;$$

a_1, b_1 being the constants for the first and a_2, b_2 for the second substance ; a_{12} is the attraction-constant between the two substances, b_{12} the volume-constant for the combination. According to the simple suppositions underlying the deduction of the formula, we should have :

$$\sqrt[3]{b_{12}}=\tfrac{1}{2}\{\sqrt[3]{b_1}+\sqrt[3]{b_2}\}.$$

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* Kuenen, *l. c.* p. 192.

† Lorentz, Wied. *Ann.* xii. p. 134.

36. Now the three combinations investigated offer a good opportunity for calculating a_{12} and b_{12}. We saw that the same formulæ hold for the critical point of the max. mixture as for a single substance. From the T and p in the maximum point B we can therefore calculate a_x and b_x and deduce from these the a_{12} and b_{12}. The values of a_{12} will give some idea of the relative attraction of the two kinds of molecules, probably quite as accurate an idea as the ordinary a gives of the mutual attraction of the same molecules. The b_{12} obtained may be compared with the theoretical value.

37. The results of my calculations are the following :—

TABLE XVI.

	t_c.	p_c.	a.	b.
N_2O	309	71·9	0·00753	0·00197
C_2H_2	308·25	61·02	0·00881	0·00231
CO_2	304·1	73·26	0·00714	0·00190
C_2H_6	305	48·8	0 01078	0·00286
			a_{12}.	b_{12}.
N_2O–C_2H_6 (x=0·18) ...	302·8	65·3	0·00859	0·00239
C_2H_2–C_2H_6 (x=0·51) ...	292·96	54·7	0·00789	0·002295
CO_2–C_2H_6 (x=0·30) ...	291·8	61·3	0·008076	0·002357
CO_2–C_2H_2 (x=0·50) ...	305·55	66·5	0·00792	0·00210

I have added to the table calculated values of a_{12} and b_{12} for the combination CO_2–C_2H_2. The mixture of these two gases was proved to have a very narrow border-curve, and consequently the critical point observed cannot have been very much above the hypothetical critical point. The calculation of a_{12} and b_{12} from the critical constants, though not legitimate, must have given values for a_{12} and b_{12} not very far wrong.

The values of b_{12} calculated from the formula in § 35 coincide with the values in Table XVI. in the case of N_2O–C_2H_6 and of CO_2–C_2H_6 : for C_2H_2–C_2H_6 b_{12} is even smaller than b for C_2H_2. a_{12} is in all cases smaller than the mean of the a's, and for C_2H_2–C_2H_6 even smaller than a for C_2H_2. It is chiefly on this smallness of a_{12} that the existence of a minimum critical temperature depends.

38. Knowing a_{12} and b_{12}, we may calculate the critical constants for the other mixtures, if supposed to remain undivided. It is with the temperature that we are chiefly concerned.

Table XVII.

N_2O-C_2H_6.		
x.	$273\frac{8}{27}\frac{a}{b}$.	T_c.
0·18	302·8	302·8
0·43	299·0	299·05
0·55	298·7	299·05
0·76	300·5	300·85
C_2H_2–C_2H_6.		
0·27	296·7	296·8
0·41	293·1	294·15
0·51	293·0	293·0
0·61	293·6	293·8
0·68	294·8	296·75
CO_2–C_2H_6.		
0·15	296·1	296·4
0·30	291·8	291·8
0·43	290·7	290·62
0·50	290·8	290·88

Table XVII. gives the calculated values of the hypothetical critical temperature as compared with the real critical temperature. The former is found to be everywhere lower than the latter (except for mixture 0·43 CO_2–C_2H_6, where the calculated value is 0°·08 above the observed T_c, a difference almost within the limits of experimental errors): for the maximum mixture the two naturally coincide. The differences are very small, especially with N_2O–C_2H_6 and CO_2–C_2H_6. For C_2H_2–C_2H_6 they are somewhat higher for mixtures containing more C_2H_6: this is in agreement with the fact that the critical phenomena were more marked for those mixtures than for the others. In plotting the values of Table XVII. against the x's, two curves are obtained, connecting the same points C_1 and C_2 and touching at a point corresponding to B. The curves are close together and the minimum of a/b should belong to approximately the same value of x as the minimum T_c. The existence of a minimum for a/b depends on a_{12}/b_{12} being smaller than both a_1/b_1 and a_2/b_2*. Table XVIII. gives the values of x for the minimum of a/b and for the

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* Van der Waals, *Arch. Néerl.* xxiv. p. 23.

observed minimum critical temperature (point A). The two sets agree very closely.

TABLE XVIII.

	Min. a/b.	Min. $T_{c'}$: A.	Max. pressure.	
			Calculated.	Observed : B.
N_2O–C_2H_6 ...	$x=0{\cdot}52$	0·50	0·01	0·20
C_2H_2–C_2H_6 ...	$x=0{\cdot}50$	0·53	0·36	0·51
CO_2–C_2H_6 ...	$x=0{\cdot}48$	0·45	0·10	0·30

39. In Table XVIII. are also given the mixtures of maximum pressure as observed and as calculated from the theory. The latter calculation is simple. In coexisting phases we have the three conditions that $\frac{\partial\psi}{\partial v}, \frac{\partial\psi}{\partial x}$, and $\psi - v\frac{\partial\psi}{\partial v} - x\frac{\partial\psi}{\partial x}$ are equal. Now ψ may be found from the equation p v T, because $\frac{\partial\psi}{\partial v} = -p$. This leads to

$$\psi = -\mathrm{RT}l(v-b_x) - \frac{a_x}{v} + \phi(x).$$

The two first conditions mentioned lead immediately to

$$\left(\frac{1}{v_1}+\frac{1}{v_2}\right)\frac{\partial b_x}{\partial x} = \frac{1}{a_x}\frac{\partial a_x}{\partial x}*,$$

remembering that x is the same in the two phases ; v_1 and v_2 are the volumes of the coexisting phases. At the critical point : $v_1 = v_2 = 3b_x$, and the equation becomes

$$\frac{2}{3b}\frac{\partial b_x}{\partial x} = \frac{1}{a_x}\frac{\partial a_x}{\partial x}.$$

There is not a very good agreement between the values calculated from this formula and those observed. The points of agreement are that both sets of x's are smaller than the x's for the min. cr. temp., and that the smaller the x observed, the smaller also is the x calculated.

Considering how insufficient van der Waals's formula is, we should not expect a better agreement ; still the equation seems to give correct indications as regards the relative position of the two points A and B. This connexion may be

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* Van der Waals, *l. c.* p. 23.

better shown in the following manner: the condition for the min. of a_x/b_x may be written thus:—

$$\frac{1}{a_x}\frac{\partial a_x}{\partial x}-\frac{1}{b}\frac{\partial b_x}{\partial x}=0,$$

and the condition for point B

$$\frac{1}{a_x}\frac{\partial a_x}{\partial x}-\frac{2}{3b_x}\frac{\partial b_x}{\partial x}=0.$$

When $\frac{\partial b_x}{\partial x}>0$ (in which case also $\frac{\partial a_x}{\partial x}>0$), as is the case in the three combinations investigated, the first equation gives a higher value of x than the second *.

40. But another question arises—What happens to the maximum at very low temperatures? This can be found from the same equation given above (in 39) by putting v_1 (volume of the liquid) equal to b_x and neglecting $1/v_2$. The result is the same equation as holds for the min. cr. temperature. Probably therefore, if the equation does not mislead us here too much, approximately the same mixture which has a min. cr. temp. will be the mixture of maximum vapour-pressure at low temperatures. We may therefore expect the mixtures of C_2H_6 with N_2O, C_2H_2, and CO_2 which contain about 0·5 of each to have a max. vapour-pressure at low temperatures. By raising the temperature this maximum moves towards the substance of higher vapour-pressures (N_2O, C_2H_2, and CO_2). As we saw for C_2H_2–C_2H_6, this motion has been very small, the points B and A almost coinciding. It is not impossible that for other mixtures which have a max. pressure at low temperature the maximum reaches one of the components at higher temperature and thus disappears. Those mixtures would still have a minimum for a_x/b_x and, if the real critical temperatures are not too much above the hypothetical ones, they would also show a minimum critical temperature.

Instances of maximum vapour-pressures are:—ethyl alcohol and carbon disulphide (Berthelot †), chloroform and carbon bisulphide (Guthrie ‡), ether and carbon disulphide (Guthrie), propyl alcohol and water (Konowalow §). Some of these combinations would be worth investigating at higher temperatures up to the critical condition.

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* Van der Waals, *Kon. Ak.* 29 June, 1895, pp. 11–12.

† Berthelot, *Comptes Rendus*, lxvii. p. 430.

‡ Guthrie, Phil. Mag. (5) xviii. pp. 513.

§ Konowalow, Wied. *Ann.* xiv. p. 34.

41. Similar considerations hold for maximum critical temperature and minimum vapour-pressure. A minimum vapour-pressure at low temperature means, according to the equation, a maximum for a/b; and a maximum of a/b is the same as a maximum of the hypothetical critical temperature, and therefore *a fortiori* of the *real* critical temperature. The minimum pressure will move towards the substance of lower vapour-pressure, with rise of temperature, and may either disappear or else exist up to the critical region. It is easy to understand what the p-t diagram would be like in this case. But the case has not been observed yet. A minimum vapour-pressure was found by Roscoe and Dittmar * for hydrochloric acid and water and for nitric acid and water, by Konowalow † for formic acid and and water, by Friedel ‡ for hydrochloric acid and methyl ether. I hope to have an early opportunity of trying the latter combination, which will not require temperatures as high as the combinations with water.

42. One remark may be added. In the condensation of some substances it is assumed, that an association takes place of molecules to more complicated systems. Substances with which that happens do not obey the laws of corresponding states and van der Waals's formula would be even more inadequate in those cases than otherwise. We may expect similar phenomena to occur in the liquefaction of some mixtures. An association of that kind one would feel inclined to identify with what is called the formation of molecular compounds. This would naturally have a great influence on the vapour-pressure, and it is not unlikely, that rules like those given above and deduced from van der Waals's formula would fail entirely in those cases. But these considerations can only be tested by experiment.

University College, Dundee.

XXVIII. *On the Propagation of Electric Waves along Cylindrical Conductors of any Section.* *By* Lord RAYLEIGH, *F.R.S.* §

THE problem of the propagation of waves along conductors has been considered by Mr. Heaviside and Prof. J. J. Thomson, for the most part with limitation to the case of a wire of circular section with a coaxal sheath serving as a

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* Roscoe & Dittmar, Lieb. *Ann.* cxii. p. 327, cxvi. p. 327.

† Konowalow, Wied. *Ann.* xiv. p. 34.

‡ Friedel, *Comptes Rendus*, lxxxi. p. 152.

§ Communicated by the Author.

return. For practical applications it is essential to treat the conductivity of the wire as finite; but for some scientific purposes the conductivity may be supposed perfect without much loss of interest. Under this condition the problem is so much simplified that important extensions may be made in other directions. For example, the complete solution may be obtained for the case of parallel wires, even although the distance between them be not great in comparison with their diameters.

We may start from the general equations of Maxwell involving the electromotive intensity (P, Q, R) and the magnetic induction (a, b, c), introducing the supposition that all the functions are proportional to $e^{i(pt+mz)}$, and further that $m=p/\mathrm{V}$, just as in the case of uninterrupted plane waves propagated parallel to z. Accordingly $d^2/dt^2=\mathrm{V}^2d^2/dz^2$, and any equation such as

$$\frac{d^2\mathrm{P}}{dx^2}+\frac{d^2\mathrm{P}}{dy^2}+\frac{d^2\mathrm{P}}{dz^2}=\frac{1}{\mathrm{V}^2}\frac{d^2\mathrm{P}}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

reduces to

$$\frac{d^2\mathrm{P}}{dx^2}+\frac{d^2\mathrm{P}}{dy^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

They may be summarized in the form

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)(\mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ a,\ b,\ c)=0. \quad . \quad . \quad . \quad (3)$$

The case to be here treated is characterized by the conditions R=0, $c=0$; but it would suffice to assume one of them, say the latter. Since in general throughout the dielectric

$$dc/dt=d\mathrm{P}/dy-d\mathrm{Q}/dx, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

it follows that P and Q are derivatives of a function (ϕ), also proportional to $e^{i(pt+mz)}$, which as a function of x and y may be regarded as a potential since it satisfies the form (2). Thus $d\mathrm{P}/dx+d\mathrm{Q}/dy=0$, from which it follows that $d\mathrm{R}/dz$ and R vanish. It will be convenient to express all the functions by means of ϕ. We have at once

$$\mathrm{P}=d\phi/dx, \quad \mathrm{Q}=d\phi/dy, \quad \mathrm{R}=0. \quad . \quad . \quad . \quad . \quad (5)$$

Again, by the general equation analogous to (4), since R=0, $ipa=im\,\mathrm{Q}$; so that

$$a=\mathrm{V}^{-1}d\phi/dy, \quad b=-\mathrm{V}^{-1}d\phi/dx, \quad c=0. \quad . \quad (6)$$

Thus the *same* function ϕ serves as a potential for P, Q and as a stream-function for a, b.

The problem is accordingly reduced to dependence upon a simple potential problem in two dimensions. Throughout the dielectric ϕ satisfies

$$d^2\phi/dx^2+d^2\phi/dy^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

At the boundary of a conductor, supposed to be perfect, the condition is that the electromotive intensity be entirely normal. So far as regards the component parallel to z this is satisfied already, since $\mathrm{R}=0$ throughout. The remaining condition is that ϕ be constant over the contour of any continuous conductor. This condition secures also that the magnetic induction shall be exclusively tangential.

It is to be observed that R is not equal to $d\phi/dz$. The former quantity vanishes throughout, while $d\phi/dz$ remains finite, since $\phi \propto e^{i(pt+mz)}$. Inasmuch as ϕ satisfies Laplace's equation in two dimensions, but not in three, it will be convenient to use language applicable to two dimensions, referring the conductors to their sections by the plane xy.

If a boundary of a conductor be in the form of a closed curve, the included dielectric is incapable of any vibration of the kind now under consideration. For a function satisfying (7) and retaining a constant value over a closed contour cannot deviate from that value in the interior. Thus the derivatives of ϕ vanish, and there is no disturbance. The question of dielectric vibrations within closed tubes, when m is not limited to equality with p/V, was considered in a former paper*.

For the case of a dielectric bounded by two planes perpendicular to x we may take

$$\phi=xe^{i(pt+mz)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

giving

$$\mathrm{P}=e^{i(pt+mz)}, \qquad \mathrm{Q}=0, \qquad \mathrm{R}=0, \quad . \quad . \quad . \quad (9)$$

$$a=0, \qquad b=-\mathrm{V}^{-1}e^{i(pt+mz)}, \qquad c=0, \quad . \ (10)$$

in which, as usual, $m=p/\mathrm{V}$. Since $\mathrm{Q}=0$, $\mathrm{R}=0$ throughout, the dielectric may be regarded as limited by conductors at any planes (perpendicular to x) that may be desired.

If the dielectric be bounded by conductors in the form of coaxal circular cylinders, we have the familiar wire with sheath return, first, I believe, considered on the basis of these equations by Mr. Heaviside. We may take, with omission of a constant addition to $\log r$ which has here no significance,

$$\phi=\log r\,.\,e^{i(pt+mz)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

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* Phil. Mag. vol. xliii. p. 125 (1897).

giving

$$\mathrm{P, Q, R} = e^{i(pt+mz)}\left(\frac{x}{r^2},\quad \frac{y}{r^2},\quad 0\right), \quad . \quad . \quad . \quad (12)$$

$$\mathrm{V}(a, b, c) = e^{i(pt+mz)}\left(\frac{y}{r^2},\quad -\frac{x}{r^2},\quad 0\right). \quad . \quad . \quad . \quad (13)$$

And here again it makes no difference to these forms at what points (r_1, r_2) the dielectric is replaced by conductors.

For the moment these simple examples may suffice to illustrate the manner in which the propagation along z takes place, and to show that ϕ is determined by conditions completely independent of p and its associated m. In further discussions it will save much circumlocution to suppose that p and m are zero and thus to drop the exponential factor. The problem is then strictly reduced to two dimensions and relates to charges and steady currents upon cylindrical conductors, the currents being still entirely superficial. When ϕ is once determined for any case of this kind, the exponential factor may be restored at pleasure with an arbitrary value assigned to p and the corresponding value, viz. p/V, to m.

The usual expressions for electric and magnetic energies will then apply, everything being reckoned per unit length parallel to z. It suffices for practical purposes to limit ourselves to the case of a single outgoing and a single return conductor. We may then write

$$\text{Electric energy} = \frac{(\text{charge})^2}{2 \times \text{capacity}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\text{Magnetic energy} = \tfrac{1}{2} \times \text{self-induction} \times (\text{current})^2\,; \quad . \quad (15)$$

and the value of the self-induction in the latter case is the reciprocal of that of the capacity in the former.

Thus, for a dielectric bounded by coaxal conductors at $r=r_1$ and $r=r_2$, we have $\phi = \log r$, and

$$\text{self-induction} = (\text{capacity})^{-1} = 2 \log \frac{r_2}{r_1}. \quad . \quad . \quad . \quad (16)$$

Among the cases for which the solution can be completely effected may be mentioned that of a dielectric bounded by confocal elliptical cylinders.

More important in practice is the case of parallel circular wires. In Lecher's arrangement, which has been employed by numerous experimenters, the wires are of equal diameter; and it is usually supposed to be necessary to maintain them at a distance apart which is very great in comparison with that diameter. The general theory above given shows that there is no need for any such restriction, the manner and velocity of propagation along the length being the same whatever may be the character of the cross-section of the system.

The form of ϕ, and the self-induction of the system, may be determined in this case, whatever may be the radii (a_1, a_2) of the wires and the distance (b) between their centres. If r_1, r_2 are the distances of any point P in the plane from fixed points O_1, O_2, the equipotential curves for which ϕ, equal to $\log(r_2/r_1)$, assumes constant values are a system of circles, two of which can be identified with the boundaries of the conductors. The details of the investigation, consisting mainly of the geometrical relations between the ultimate points O_1, O_2 and the circles of radii a_1, a_2, are here passed over. The result for the self-induction per unit length L, or for the capacity, may be written *

$$L = -2\log\frac{b^2-a_1^2-a_2^2-\sqrt{\{(b^2-a_1^2-a_2^2)^2-4a_1^2a_2^2\}}}{2a_1a_2}. \quad (17)$$

As was to be expected, L vanishes when $b=a_1+a_2$, that is, when the conductors are just in contact.

When a_1, a_2 are small in comparison with b, the approximate value is

$$L = -2\log\frac{a_1a_2}{b^2}\left(1+\frac{a_1^2+a_2^2}{b^2}\right); \quad . \quad . \quad . \quad . \quad (18)$$

or, if $a_1=a_2=a$,

$$L = 4\left(\log\frac{b}{a}-\frac{a^2}{b^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The first term of (19) is the value usually given. The same expression represents the reciprocal of the capacity of the system per unit length.

In the application of Lecher's arrangement to the investigation of refractive indices, we have to consider the effect of a variation of the dielectric occurring at planes for which z is constant. It will be seen that no new difficulty arises in the case of systems for which the appropriate function ϕ in two dimensions can be assigned.

Regarding ϕ as a given function, *e.g.* $\log r$ for the case of a coaxal wire and sheath (compare (11)), we may take as the

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* Compare Macdonald, Camb. Phil. Trans. vol. xv. p. 303 (1894).

solution for any length of uniform dielectric

$$\mathrm{P}, \mathrm{Q}, \mathrm{R} = (\mathrm{A}\,e^{i(pt+mz)} + \mathrm{B}\,e^{i(pt-mz)})\left(\frac{d\phi}{dx}, \frac{d\phi}{dy}, 0\right), \quad . \quad (20)$$

$$\mathrm{V}\mu(\alpha, \beta, \gamma) = (\mathrm{A}\,e^{i(pt+mz)} - \mathrm{B}\,e^{i(pt-mz)})\left(\frac{d\phi}{dy}, -\frac{d\phi}{dx}, 0\right), \quad (21)$$

in which $\mu\alpha = a$, &c., and provision is made for waves travelling in both directions.

At a plane where the dielectric changes, the conditions to be satisfied are the continuity of P, Q and of α, β; and this is secured if

$$\mathrm{A}\,e^{imz} + \mathrm{B}\,e^{-imz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\frac{1}{\mathrm{V}\mu}(\mathrm{A}\,e^{imz} - \mathrm{B}\,e^{-imz}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

are continuous. It will be seen that the conditions are altogether independent of the section of the conductors, being the same in fact as if there were no conductors and we were dealing with infinite plane waves represented by $\phi = x$.

As a particular case we may suppose that waves travelling in the negative direction in the dielectric (V, μ) meet at $z=0$ a dielectric of altered character (V$'$, μ'). The expressions (20), (21) represent the incident (A) and reflected (B) waves. For the second medium it suffices to accent V and μ, writing also A$'$ for A and 0 for B. Thus (22), (23) give

$$\left.\begin{aligned} \mathrm{A} + \mathrm{B} &= \mathrm{A}', \\ (\mathrm{A} - \mathrm{B})/\mathrm{V}\mu &= \mathrm{A}'/\mathrm{V}'\mu', \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

by which B and A$'$ are determined. For the reflected wave

$$\frac{\mathrm{B}}{\mathrm{A}} = \frac{\mathrm{V}'\mu' - \mathrm{V}\mu}{\mathrm{V}'\mu' + \mathrm{V}\mu}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

or if the difference between the dielectrics relate only to the dielectric constants (K, K$'$),

$$\frac{\mathrm{B}}{\mathrm{A}} = \frac{\mathrm{V}' - \mathrm{V}}{\mathrm{V}' + \mathrm{V}} = \frac{\sqrt{(\mathrm{K})} - \sqrt{(\mathrm{K}')}}{\sqrt{(\mathrm{K})} + \sqrt{(\mathrm{K}')}}, \quad . \quad . \quad . \quad (26)$$

in agreement with Young's well-known optical formula.

Whether the dielectric consist of uniform portions with discontinuous changes of character at the boundaries, or whether it be a continuous function of z, the solution of the problem is the same, whatever be the character of the cylindrical conductors. It is only the form of ϕ that is influenced by the latter consideration.

Terling Place, Witham.

XXIX. *Heat produced by Moistening Pulverized Bodies*[*]. *By* TITO MARTINI[†].

THE phenomena discovered by Pouillet[‡], regarding the production of heat by the moistening of porous bodies, are well known, and since Pouillet many experiments have been made by Cantoni[§], Melsens[||], Chappuis[¶], E. Wiedemann[**], and Meissner[††].

Notwithstanding the investigations of these philosophers, however, I believe that it may be of some interest to publish the results of my own researches in this direction, which are more extended than those of the *savants* referred to.

Among the numerous results contained in my paper I will select the most complete, that is to say, those concerning chemically pure animal charcoal, believing that such important manifestations are due to a different method of experimenting adopted by me; consisting of an arrangement whereby the liquid ascended to the powder, which was placed in a glass tube separated from it by a piece of light linen cloth.

TABLE (8th of the original paper).

Name of the liquid.	Temp. of the air.	Temp. of the liquid.	Temp. of the charcoal.	Max. temp.	Increase of the temp.
Distilled water	18°·10	17°·51	17°·92	33°·50	15°·58
Absolute alcohol	18·09	17·82	18·28	35·75	17·47
Sulphuric ether	17·10	16·82	17·20	34·70	17·50
Acetic ether	18·30	18·22	18·48	37·42	18·94
Benzine	18·30	18·52	18·70	36·30	17 60
Bisulphide of carbon...	17·10	17·12	17·50	38·10	20·60

Owing to this mode of conducting the experiment, I was enabled to observe that when the rise of temperature was not

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[*] "Intorno al calore che si sviluppa nella umettazione delle polveri," *Atti del R. Istituto Veneto*, t. viii. serie 7, Maggio 1897.

[†] Communicated by the Author.

[‡] *Annales de Chimie et de Physique*, 2^{me} série, t. xx. p. 411.

[§] *Atti del R. Istituto Lombardo*, Maggio 1866.

[||] *Mémoires de l'Académie royale de Belgique*, t. xxiii. (1883).

[¶] Wiedemann's *Annalen*, t. xix. p. 21 (1883).

[**] *Ibid.* t. xxv. p. 145 (1885).

[††] *Ibid.* t. xxix. p. 114 (1886).

great (in the case of mineral powders) the powder became a soft paste; but that with a great increase of temperature (in the case of charcoal, sawdust, &c.) the powder was transformed into a consistent mass resembling stone.

This curious phenomenon would tend to confirm the opinion that the production of heat is due to the molecular energy being transformed into thermal energy.

In another communication I will give further notice in regard to this interesting question.

Venice, June 1897.

XXX. *On Proofs of a Theorem in Wave-motion.*

To the Editors of the Philosophical Magazine.

Gentlemen,

I CRAVE permission to correct an error on p. 101 of my letter in the July number of the Philosophical Magazine, where I described the waves in each of the equations

$$\xi = F_1(x, y, z, t),$$
$$\eta = F_2(x, y, z, t),$$
$$\zeta = F_3(x, y, z, t)$$

(when extended to all space and expanded by Mr. Preston's valuable corollary to Fourier's theorem) as waves that are non-natural in two respects. In fact, they are non-natural in only one of these respects. The wave-lengths, in the case there dealt with, do not range in these equations from infinity down to cypher as I supposed when writing the letter, but are of one wave-length, as may be proved by a slight modification of the well-known theorem given on p. 435 of my second paper on Microscopic Vision in the Phil. Mag. for November, 1896. The waves of each of the above equations are non-natural, *i. e.*, merely kinematical, in consequence of their transversals standing in impossible positions.

Nevertheless, as is the case with every kinematical resolution when applied to a real motion, the *resultant* of combining the three kinematical components in any one direction, must of necessity give the *real* undulation in that direction if there be any such. That there is a real undulation I had previously shown by MacCullagh's method of proof, and further that it consists of uniform plane waves of the kind which the medium can propagate unchanged.

It is to be observed that this may also be proved analytically—a circumstance not referred to by Mr. Preston. Some proof on this head is essential, because Mr. Preston's extension of Fourier's theorem has the unlimited generality of the original theorem ; it applies indifferently to every motion in space that can in any way be represented mathematically, and takes no notice of whether it is physically or even geometrically possible. When a proof is given that the three impossible waves of Mr. Preston's equations give a resultant which is a possible wave, when his treatment of the physical problem receives other modifications, and when the *whole* of space is included under it, it can be made to furnish an alternative proof of what had previously been ascertained by MacCullagh's more direct method, viz.: that *the radiation from a disturbed portion of a uniform medium which is capable of propagating waves, can be resolved into undulations of uniform plane waves which the medium can transmit forward unchanged.*

This, however, is not what Mr. Preston put forward in the paper in which he announced his important extension of Fourier's theorem. He also introduced two statements to which I reluctantly found myself obliged to demur, as in both he erred with regard to my work. I had, in October 1896, proved a theorem (quoted by Mr. Preston on p. 281 of the April Magazine) concerning the *radiations* from a region of disturbance. About this, Mr. Preston on that page states that Fourier's theorem applied "between assigned limits" to "a disturbance which is a function of a single variable" is in that case the analytical expression of my theorem [which concerns the *radiations* from the disturbance]. This mistake I pointed out in the following number of the Philosophical Magazine, and gave a case on p. 321 in which it is manifestly not true. On this Mr. Preston observes on p. 459 of the June Magazine that after resolving the originating disturbance into one of the numberless resolutions into uniform plane waves which are possible, " nothing more remains to be done. As to what happens outside the disk, this is quite another question." Precisely so: and as it is with this "quite other question" that my theorem is concerned, associated too with *another* of the possible resolutions within the disk, it became necessary to point out that Mr. Preston had here fallen into the fallacy known as *ignoratio elenchi*, where the proof of one thing is mistaken for the proof of another. Moreover, it is important to emphasize, since a mistake might easily be made on this head, that the resolution *within* the disk into a simple Fourier's expansion is *only one of innumerable* ways in which the motion within the disk may be resolved into trains of uniform plane waves, and is the principal one of a definite group among them the members of which are characterized by having no bearing upon the physical theorem.

Again, on p. 283, Mr. Preston gives a *scalar* equation which he speaks of as "the analytical expression of the general theorem enunciated by Dr. Stoney." On p. 371, I pointed out that the first step towards correcting this oversight is to substitute three expressions for ξ, η, and ζ for Mr. Preston's single expression. This Mr. Preston does in his subsequent letter in the June number of the Magazine, p. 458. The further steps necessary to complete the correction are: (1) To add the proof referred to above in the third paragraph of this letter; and (2) to embrace the *entire* of space in the investigation.

This naturally leads to the remark that when Mr. Preston's investigation is confined to a limited area round the originating disturbance it furnishes a *different* resolution. This is a kinematical* resolution; and the kinematical resolution of this Fourier analysis is *only one of innumerable resolutions of a like kind that are possible.* Attention was called to their unlimited number in my first paper on Microscopic Vision.

It is very instructive to trace out what the physicist is to understand by that particular resolution which the Fourier analysis offers to us, when it is made applicable to space of three dimensions and when it is restricted to a defined area. This can be done. Let the whole of space be called A; let the "prescribed area" (which includes the originating disturbance and a limited region about it) be called B; and let the rest of space be called C. Then B+C=A. Let now the radiations from the originating disturbance (which may spread to any distance) be resolved into trains of uniform plane waves of infinite extent laterally and travelling with the velocity proper to the medium. These waves can advance unchanged through the medium, and are a body of uniform plane waves which we shall call (a). This resolution is unique. It is that special one which results from the medium being agitated by the originating disturbance *only.* It may be called the legitimate resolution of radiations from that disturbance. This complete system of undulations may be distinguished into two parts: (a_1) the part of it which falls within the region B, and (a_2) the part which lies outside.

* By a kinematical resolution is to be understood a resolution into undulations (in this case of limited extent) which the medium is incompetent to transmit forward without change, and, moreover, of which no one could be propagated even within the prescribed area, if isolated from the others.

Then $(a)=(a_1+a_2)$. Hence (a_1) is a resolution of the actual motion going on within the region B. Now we find that the Fourier analysis when restricted to the region B, furnishes another resolution, of which all the waves differ from the corresponding waves of (a_1). This Fourier system of plane waves we may call (b). Suppose each wave of the Fourier system to be extended without limit laterally, and let the aggregate of these extensions be called (c). They occupy the space C. Then, although (b) consists of waves which a physical medium cannot propagate, $(b+c)$ is a system of waves which the medium can propagate. The question now is, What motion in the medium, or in space, does $(b+c)$ represent? Let us call it X. We know that if we confine our survey to the region B, $(b+c)$ will represent within that space the same motion as (a) represents. Hence X would result from anything which maintains the waves (a_1) within the space B and the waves (c) outside it: in other words, it is the same motion as would result from maintaining the originating disturbance with its actual radiations throughout an infinite *medium*, and by adding on the kinematical motion $(c-a_2)$ in the space C. It is this artificial complex* which the Fourier analysis represents.

It is further noteworthy that any other arbitrary motions might have been set up within the space C, and that every change made in these will alter the systems of plane waves which sweep across the space B, so that the particular set of undulations furnished by the Fourier analysis is not unique. There are numberless others†.

Another way of viewing the subject is that any system of uniform plane waves of infinite extent laterally, of which the resultants are cypher resultants at all points within the space B whatever they may be elsewhere, may, from a mere kinematical point of view, be superposed upon the legitimate resolution; thus producing an unlimited number of kinematically possible resolutions so long as we do not look beyond the region B. But among these (a) is the only resolution which is *physically* legitimate, in the sense of being the only one which is due to the action of the originating

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* This complex may be presented under an exclusively physical aspect by taking into account the considerations referred to in the footnote on p. 275 of the April Magazine. But to do this would have required so much explanation that, in the text, for brevity, the subject is dealt with in a way which does not carry the analysis so far and leaves some of the motion kinematical.

† These others are not beyond the reach of the Fourier analysis, if artificially manipulated.

disturbance operating *alone* upon an actual physical medium. It is therefore the one which should be employed by the physicist.

Mr. Preston's method of analysis is very much to be welcomed, both as extending Fourier's analysis with all its kinematical merits to two, three, and more dimensions, and as one which, *when correctly handled*, can furnish an alternative proof of an important physical theorem by which it has been possible to unravel the very intricate phenomena of microscopic vision*. It is perhaps disappointing to find that the analytical proof has not as yet added anything to the knowledge of nature which had previously been obtained by the geometrical proof. This from the mathematical standpoint appears to be because the definite integrals of the analytical proof rest upon data in which the character of the originating disturbance and the properties of the medium are not kept separate. If a proof is found in which they can be kept distinct, either by combining a physical theorem with Mr. Preston's extension of Fourier's kinematical theorem, or otherwise; this is the kind of analytical proof which may be expected to add something to our knowledge of nature.

The whole inquiry is a striking example of the great advantage of pushing geometrical and analytical proofs forward side by side. We thus gain, as was distinctly seen during MacCullagh's lifetime (see July Magazine, p. 98), a deeper and more accurate knowledge of what is going on in nature than is within our reach if we restrict ourselves to the analytical method, as is too much the practice. In the present instance the geometrical proof is the more direct and places clearly before us the physical connexion of the actual events with one another. It was therefore not unnaturally the method of proof by which the theorem was discovered. On the other hand the Fourier method, while it leaves these physical relations out of sight, brings new kinematical relations into prominence and forces upon our attention how largely the further progress of events in nature when once set going admits of such purely kinematical expression as links the actual events of nature with events which though physically

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* Typical specimens of all the principal phenomena, except those that concern black-field illumination, are dealt with in three papers on "Microscopic Vision," in the Phil. Mag. for October, November, and December, 1896. When black-field illumination is employed the image is produced in one or other of two distinct ways, of which a study has since been made by the present writer. He hopes soon to have time to publish it, and this will apparently complete the explanation of microscopic vision.

impossible can be more or less clearly imagined by the mathematician.

It must not be forgotten that beside the resolution into plane waves dealt with above, there are an infinite number of resolutions into waves of other forms which are equally possible, and several of which present themselves in optical inquiries.

I am, Gentlemen,
Very faithfully yours,
G. JOHNSTONE STONEY.

8, Upper Hornsey Rise, N.,
July 14, 1897.

XXXI. *Intelligence and Miscellaneous Articles.*

SOME OBSERVATIONS ON A NICKEL STRESS TELEPHONE. BY F. J. JERVIS-SMITH, M.A., F.R.S., MILLARD LECTURER, TRINITY COLLEGE, OXFORD.

AT page 26 of the present volume of the Philosophical Magazine a Nickel Stress Telephone is described by Messrs. Garrett and Lucas. When I was working in 1891 on the magnetization of wires of iron, steel, and nickel, which had received a permanent set, I found that excellent telephonic effects could be produced by means of wires under stress (p. 395, Phil. Mag. 1891, vol. xxxii.). The authors of the paper mention the difficulty of comparing the speaking-powers of different instruments. I expect that a form of electrodynamometer, which was shown in the Electrical Exhibition in Paris of 1881, would be found of use in comparing the relative outputs of different kinds of telephonic apparatus. The instrument was constructed thus:—a small light coil, carried on a balanced arm hanging by a bifilar suspension, moved over a curved solenoid which passed through it. The alternating current generated by a telephone when transmitting a musical note gave a large deflexion of a light straw pointer attached to the suspended arm.

Oxford, July 2, 1897.

ON THE INTERFERENCE, AND THE ELECTROSTATIC DEFLEXION OF KATHODE RAYS. BY PROFESSOR JAUMANN.

Two parallel kathode plates at a distance of 2 cm. from each other are connected by a wire loop 250 cm. in length, and the negative current of an induction machine is passed by means of a sliding contact. By displacing this along the loop, with a suitable adjustment the bright sharp plane is displaced in which the two blue kathode lights interfere without becoming essentially broader. This, being an action of a slight difference in the conducting circuits to the kathodes, cannot be explained otherwise than as an *interference* of the kathode lights. The simple interference surfaces of feeble kathode rays may be doubled or bifurcated if the rays are *strengthened*. This phenomenon may be compared with the bifurcation of the interference surfaces of strong sound waves which was demonstrated by E. Mach.

The author contests the existence of Goldstein's deflexion as being due to an action at a distance of the kathode elements on the kathode rays. The phenomena of deflexion may be explained partly by interference and partly by the electrostatic deflexion of kathodic rays.

Even before the discovery of the electrostatic deflexion of kathode rays, it was inferred from the theory of the author that simple feeble kathode rays follow the maxima and minima lines of the electrostatic field, but are deflected from these lines by magnetic force. This is calculated to explain all the deflexions of the kathode rays hitherto known. Above all, this explains what at first sight is the *unexpected* direction of the electrostatic deflexion of the kathode rays, as well as the deflexion of the starting-point on the kathode. The statement of E. Wiedemann that in the electrostatic deflexion only the starting-point is displaced, and that the rays continue rectilinear and perpendicular to the kathode is experimentally refuted.

The spontaneous stretching of the kathode rays is brought nearer an explanation as being a simple phenomenon of discharge. This incited experiments to compel the kathode rays to travel to the anode. *Anode rays* were obtained, that is kathode rays which proceed from the side of the glass, but do not remain perpendicular, but curve to the anode, and collect in a thin stem. The glass side scarcely fluoresces at all, but there is a bright fluorescent spot on the corresponding prepared anode. These anode rays are of the same nature as the kathode rays and show also corresponding deflexions.—*Wiener Berichte*, July 1897.

ON THE MEASUREMENT OF THE BRIGHTNESS OF DAYLIGHT.

BY H. W. VOGEL.

The author gives the following as the results of a series of observations on this subject by a method which he describes; they were made in Berlin in January of this year in very variable weather, and show the enormous differences of daylight in winter.

Jan. 13, 3 P.M.	Strong fog	2·3	metre-candles
,, 14, 2 P.M.	Slight fog......	79·59	,, ,,
,, 15, 2 P.M.	Grey sky. Fog..	89·55	,, ,,
,, 16, 2 P.M.	Grey sky. Fog..	55·54	,, ,,
,, 22, Sunny day.	Clear, somewhat misty	394·5	,, ,,

Wiedemann's Annalen, June 1897.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE

[FIFTH SERIES.]

SEPTEMBER 1897.

XXXII. *On the Magnetic Properties and Electrical Resistance of Iron as dependent upon Temperature.* By DAVID K. MORRIS, *Ph.D., formerly* 1851 *Exhibition Scholar, University College, London*[*].

[Plates II. & III.]

THE changes brought about by heating in the various physical properties of iron have been, since the first investigations into the nature of the phenomenon of recalescence by Gore and by Barrett, the subject of much experimental work, which, during the last ten years, has been largely upon the relation of the magnetic properties of iron to temperature.

As long ago as 1879, however, Baur of Zürich [†] described experiments on an iron bar inserted while hot into a long magnetizing helix, which served to determine its permeability at any instant, the temperature being estimated from a knowledge of the law of cooling. This method, although rough, enabled him to show that in weak magnetic fields the permeability of iron rises, and that in strong ones it falls, with rise of temperature; and that as the critical point is reached the magnetic qualities of iron in fields of all strengths very rapidly disappear.

These results have been confirmed and extended by the

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[*] Communicated by the Physical Society: read May 14, 1897. Inaugural Dissertation for the Degree of Ph.D. at the University of Zürich.

[†] C. Baur, "Neue Untersuchungen über den Magnetismus," Wied. *Ann.* 1880, p. 394.

researches of Ledeboer in 1888[*], and by those of Dr. Hopkinson, communicated to the Royal Society in 1889[†]. In the latter, a ring-magnet was used, furnished with primary and secondary coils insulated with asbestos, and capable of withstanding a red heat, which temperature was obtained by means of a gas-furnace. Measurements could thus be made of the permeability of the iron core by the ballistic method, and the temperature was deduced from the resistance of the copper secondary winding.

M. Curie, in a more recent investigation[‡], used a method by which the permeability could be obtained in intense magnetic fields. He experimented on a sample of iron in the interior of an electrically-heated porcelain furnace, and deduced its magnetization from the forces acting on it when placed in a non-uniform magnetic field. A thermo-electric couple served to measure the temperature.

The variations of magnetic hysteresis with temperature have been studied by W. Kunz of Darmstadt[§].

The relation of the electrical resistance of iron to temperature has also received the attention of several experimenters; curves expressing this relation, and extending to beyond the critical temperature, having been given by Kohlrausch in 1887[‖], by Hopkinson in 1889[¶], and by Le Chatelier in 1891[**].

General Description of the Method of Experiments.

By the method used in the present work, measurements of the electrical resistance and of the magnetization of the iron could be made simultaneously.

It consisted in the employment, as in Dr. Hopkinson's experiments, of a ring-magnet, whose insulation was calculated to withstand a white heat. The core of this ring-magnet was formed of a length of *insulated* iron wire or strip, whose ends projected out of the ring, so that the resistance of the iron in the core could be measured; and *in this core* was imbedded an insulated platinum wire, from the resistance of which the temperature of the iron could be deduced.

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[*] Ledeboer, *Journal de Physique* (2) vol. vii. p. 199.

[†] Hopkinson, "Magnetic and other Physical Properties of Iron at a High Temperature," Phil. Trans. Roy. Soc. 1889, A. p. 443.

[‡] P. Curie, "Propriétés magnétiques des corps à diverses températures," *Ann. de Chim. et de Phys.* July 1895, p. 289.

[§] W. Kunz, *Elektrotechnische Zeitschrift*, 1894, p. 194.

[‖] W. Kohlrausch, Wied. *Ann.* vol. xxxiii. p. 42.

[¶] Hopkinson, Proc. Roy. Soc. vol. xlv. p. 457.

[**] Le Chatelier, *Comptes Rendus*, vol. cx. p. 283.

The heating was carried out electrically. For this purpose the ring (which was made quite small, about an inch in diameter) was covered with a non-inductively wound layer of insulated platinum wire. By passing a suitable current through this wire the temperature could be raised to any required extent. The heat was thus generated exactly where wanted; and by wrapping the ring thickly with asbestos the loss of heat by radiation was made small, and the temperature in the interior fairly uniform.

In the experiments of M. Curie, 1500 watts were absorbed in obtaining a temperature of 1350°: by the above method, 73 watts (a current of 4·6 amperes with 16 volts across the terminals) was found sufficient to maintain the ring at 1150°; and 45 watts sufficient at 800°, *i. e.*, above the critical temperature of the iron. Dealing thus with comparatively small currents, the accurate regulation of the temperature was rendered simple; whilst absolute constancy for long periods of the heating current, and therefore of the temperature, was, with a set of accumulators in good order, not difficult to attain.

To avoid oxidation of the iron, the ring-magnet was placed in a glass vessel, and the electrical connexions brought through a well-fitting cork rendered air-tight by a thick layer of sealing-wax poured over it while hot. The oxygen contained in this closed space could, previous to the experiments, be absorbed by an auxiliary coil of bare iron wire, heated to bright redness by an electric current. In the later experiments the jar was, as a preliminary, exhausted by a small air-pump to about a third of an atmosphere; during the heating, the pressure might rise above that of the atmosphere, in which case the excess of heated air was allowed to escape from a glass tube dipping into mercury; and this at other times served as a gauge to show whether the glass vessel was really air-tight.

Details of the various Ring-Magnets.

During the course of the experiments, four ring-magnets were made.

The first came quickly to grief through accidental overheating and partial oxidation of the iron. The oxide seems to have combined with the silicates of the insulating materials, forming a kind of slag, for the ring, on taking to pieces was simply a collection of platinum wires buried in a brown glassy substance with a little iron left in the core.

The experiments with the second ring (whose core was formed of a length of ordinary soft iron wire) were more successful; but the results, though interesting, have not been thought worth including in this paper. The experience gained in the construction and use of the first two ring-magnets was turned to account in the two later ones.

Platinum wires were exclusively employed for the windings of the ring-magnets, as no substance either magnetic or fusible below 1200° was admissible. These wires, after use in one ring, were used again, with fresh insulation, in the next.

Iron cores.—The cores of the third and fourth ring-magnets (referred to in this paper as specimens A and B respectively), were formed from strips, one chosen from each of two groups of iron samples kindly procured for the author by Mr. R. Jenkins from Messrs. Jos. Sankey and Sons, of Bilston. The sample strips were 1 cm. in width; they were described as follows:—

Specimen A.—"Charcoal Iron, No. 4 Quality; thickness ·02 in."

" B.—"Best Transformer Quality ◈; thickness ·014 in." *

These specimens proved, after careful annealing, to be exceptionally good both as regards high permeability and low hysteresis.

Platinum Thermometer Wire. Method of deducing Temperatures.—The specimen of pure annealed platinum wire used in the measurement of temperature was from the firm of W. C. Heräus, of Hanau, near Frankfurt-a.-M. The same piece of wire was used in both the later ring-magnets. Its diameter was ·0453 cm., its specific resistance at 0° was 10,110 C.G.S. units, and its temperature-coefficient, which was remarkably high, was $\alpha_{0^\circ-100^\circ} = {\cdot}00386$.

The resistance-temperature curve was taken by mounting the platinum wire (carefully re-annealed by heating to bright redness in a mass of asbestos) on a small cylindrical mica frame, and measuring its resistance at intervals of about 30° up to 200° C. in a bath of linseed oil. The temperature was measured by a standard thermometer which was carefully checked at 0° and 100°. The oil was vigorously stirred, and observations of resistance were only made when the temperature was very nearly constant.

The experiments of Callendar and Griffiths have established

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* This iron, according to the above firm, gives on analysis 99·925 per cent. of iron by difference, the impurities (·075 per cent.) being distributed between carbon, phosphorus, and silicon, with a trace of manganese. It is as pure Swedish iron as can be produced commercially.

the fact that, for pure annealed platinum wire, if $\frac{1}{100}$th part of the increase of resistance which it experiences in passing from 0° to 100° be called the increment for one "platinum degree," so that the resistance of a platinum wire is a linear function of its "platinum temperature," pt°; then, if t° is the actual temperature on the air-thermometer scale, the difference $t^\circ - pt^\circ$ can be expressed over wide ranges of temperature by the form $a + bt + ct^2$, which in this case must be reducible to the form $t^\circ - pt^\circ = \delta\{(t/100)^2 - t/100\}$.

It was thought that, in the present investigation, an error in the determination of the absolute values of the higher temperatures of even several degrees was of comparatively small moment where the recalescence point for different specimens of iron is known to differ by 50° or more. The resistance-temperature curve for the platinum thermometer wire was not determined directly above 200°, but was extrapolated by the help of Callendar's formula. The value of δ for this wire was 2·10.

The *absolute* values of the temperatures given must therefore be accepted with caution. They may be wrong by as much as 5° in the neighbourhood of 800°, the error rising with the square of the temperature. For the present purpose, however, it is the *relative* accuracy which is of importance, and this is of a much higher order.

To obtain reliable measurements of temperature, it was necessary that the thermometer-wire should satisfy the following conditions:—(1) Its temperature must not differ sensibly from that of the iron which it is intended to measure; (2) its constants must not alter in any way during the heating of the ring; (3) its resistance must be measured with sufficient precision.

(1) The first condition was the most difficult to satisfy: and it was only by actually burying the thermometric wire in the core itself, and by having the temperature very nearly constant before taking any observations, that the temperature of the thermometer-wire and that of the iron core were brought to a satisfactory coincidence.

(2) With regard to the second condition: the resistance-temperature curve of the thermometer-wire, besides being taken previous to experiments, was also taken after those with each of the specimens A and B; in neither case could any alteration either in the resistance-coefficient or in the value of δ be detected. (Alterations of these constants which had taken place with former thermometer-wires were thought to be due to the fact that the insulation in contact with them had not been previously decarbonized (see p. 220).)

One alteration, however, did take place, particularly during the last annealing of specimen B, when a temperature of 1150° was reached and maintained for some time: the resistance of the thermometer-wire rose (on this occasion as much as 1 per cent.); though, as stated above, the form of its resistance curve remained throughout unchanged.

Messrs. Heycock & Neville [*] have repeatedly noticed such changes, though slight in amount, in their resistance pyrometers; and conclude that they were due to strains set up by the permanent expansion of the mica frames on which their wires were supported. It is possible that such increase of resistance may also be due, as Prof. Weber has suggested, to evaporation of the platinum at these high temperatures; none of the author's platinum wires retained, after *prolonged* heating, their bright metallic surface [†].

In reducing the results, it has been assumed that this alteration of resistance, which was generally very small, took place entirely during the time that the ring was at its highest temperature.

(3) In order to satisfy the third condition, it was not only necessary that precise measurements of resistance should be possible ($\frac{1}{20}$ per cent. represents more than $\frac{1}{2}$° at 1000°), but also that the resistance of the thermometer-wire and that of its compensating leads should be measured very nearly simultaneously.

To obtain the nett resistance from a *single observation*, the following arrangement (fig. 1) was employed, which is similar to, though not identical with, that used in Callendar's platinum thermometers.

The platinum thermometer-wire and leads formed one arm of the Wheatstone's bridge, and the compensating leads (made exactly equal to those of the thermometer), plus a portion AC of the slide-wire, formed a second arm equal to

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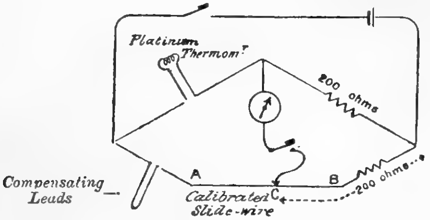
[*] Heycock and Neville, "On the Determination of High Temperatures by means of Platinum Resistance Pyrometers," Trans. Chem. Soc. 1895, p. 160.

[†] Since the above was written, Mr. E. H. Griffiths, F.R.S., has kindly drawn my attention to the fact that his own platinum thermometer wires on mica frames, when heated for considerable periods at temperatures of 1200°–1400°, do not thus lose their former bright surface. It is therefore rather improbable that in my case the dull surface should point to any serious evaporation of the platinum. I have in this connexion one or two observations which I had not given as I was in doubt as to their reliability:—The mean *sectional area* of the platinum thermometer wire, as measured by the specific gravity method, diminished from ·001182 cm.^{2} to ·001162 cm.^{2} during use in Rings A *and* B; the *length* increased from 31·33 to 31·56 cm. during the heating in Ring B, and during this same heating the *weight* of the wire did not alter.—D. K. M.

the first. The other two arms contained equal resistances; the 200-ohm coil of a resistance-box in the one; and in the other the smaller coils of the box (together making 200 ohms) with just so many tenths of an ohm short-circuited as were in the remainder CB of the slide-wire. It is clear then that, if C is the position of balance, the resistance AC is equal to the nett resistance of the thermometer-wire.

Fig. 1.—Form of Bridge for Temperature Measurements.



The resistance of the slide-wire, whose calibration curve was very uniform, was about 1·4 ohms. This was approximately that of the thermometer-wire at 1200°, so that the balancing point for all lower temperatures was within the range of the metre-long slide-wire. The resistance per unit length of the latter was conveniently obtained by inserting a standard ohm in the place of the thermometer-wire.

The above method was also applied to the measurement of the resistance of the iron in the core.

Insulation.—The chief difficulty of the method of these experiments lies in so insulating the wires themselves and the various layers of the ring-magnet from one another, that the insulation shall still remain good at the highest temperatures reached.

The only material which is practically applicable to the insulation of the wires is *asbestos paper*; this paper, which though fairly thin is by no means uniform, was used in the form of long narrow strips wound spirally over the wires to be insulated. After considerable practice, it was found possible to cover wires even as small as ·3 mm. in diameter in this way with a reliable, and fairly thin and uniform covering; for such wires, a strip about 4 mm. wide was found to be most suitable.

Asbestos paper must of necessity contain starch or something similar to hold it together; and experiments proved that, when heated in a non-oxidizing atmosphere, the resulting carbon deposit reduced the insulating power of the asbestos to a very low value. This difficulty was overcome by carefully burning out all the hydrocarbons in one layer of insulation, leaving, so to speak, nothing but the asbestos 'ash,' a white and very brittle covering ; and then cautiously over-covering this with a layer of ordinary asbestos paper.

In the first experiments, the leakage between the heating-coil (with 10 volts across its terminals) and the secondary winding was sufficient, when the ring was at 1000°, to drive the astatic needle of the sensitive galvanometer included in the secondary circuit almost round to its stops; and though the needle could be easily brought back artificially to its zero position again, the leakage was uncertain enough in amount to render ballistic observations at this temperature quite out of the question. The introduction of the above-described improvement in the asbestos insulation reduced this leakage to about $\frac{1}{100}$th part of its former amount; but it still represented a not very constant zero error of several centimetres.

With the object of still further reducing the leakage, *mica* was tried as insulation between the layers. This plan ultimately succeeded very well ; but the difficulty of applying a reliable layer of mica insulation to a surface of "double curvature," such as that of a small ring, without the assistance of any adherent whatever, appeared at first insurmountable. Latterly, however, it was found that if the mica insulation for the "doubly-curved" parts was applied in the form of small suitably-shaped pieces, previously heated in a Bunsen flame * and then bent or continuously broken by pressing over a rounded edge, then, by putting each piece of mica into place as the coil of insulated wire above it was wound, a reliable layer of mica insulation could be obtained without much trouble.

Mica was also used in the later ring-magnets both for the insulation of the iron strip and for that of the thermometer-wire, in a way about to be described ; it was not only put between the turns of the iron strip, but was also placed next to it, beneath the asbestos covering ; since any carbon deposited from the latter would, if in contact with the glowing iron, combine with it and alter its character.

It is easily shown that no appreciable error in the measured resistance of the iron or thermometer-wire would arise from a conductivity of the insulation sufficient to give rise to even a large deflexion of the ballistic galvanometer needle.

Construction of Ring-Magnets.—The rings containing

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* This diminishes the tendency of the mica to split, and seems to do so by burning off the raw cut edges.

specimens A and B were almost identical in construction, differing mainly in the superiority of the insulation of the later ring. They were made in the following way (see figs. 2–4):—

A sample iron strip was chosen, and a narrow band 2 mm. wide was cut by a cold chisel out of the middle, leaving a specimen consisting of two parallel strips each 4 mm. in width joined at one end. The mean section of this double strip was determined by the specific gravity method. It was then insulated with mica and covered with asbestos as shown in fig. 2.

Fig. 2.—Preliminary Stages. Half full size.

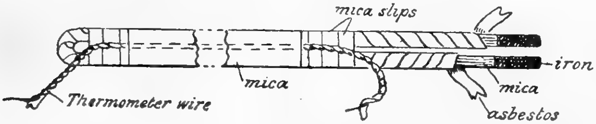
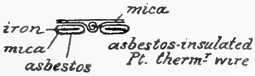


Fig. 3.—Section. Full size.



The insulated thermometer-wire was next laid in the space between the two iron strips, separated from them by slips of mica laid crosswise, and covered over by a plain mica strip (fig. 3). The whole was then rolled up on a temporary wooden form into a ring of four or five turns (fig. 4).

Thus, though asbestos was used in the core as insulation, it was the mica alone that was relied upon.

The ends of the iron specimen were then bent so as to leave the ring at right angles, and form a "stem" about 10 cms. in length. A portion of the same iron specimen resembling this stem both in form and insulation, but only leading just up to the ring and back (see fig. 6), was bound on beside the stem that compensation for its resistance might afterwards be made. The ends of the platinum thermometer-wire, together with a similar compensating resistance formed of a piece of the same platinum wire, were also led up the stem, which was in all parts most carefully insulated with mica.

The next step was to overwind the ring with three layers of platinum wire to serve respectively as magnetizing coil or primary, as heating coil, and as secondary. These wires, each of which, like the thermometer-wire, was doubly-insulated with asbestos in the way previously described, were respectively 110, 150, and 100 cm. in length, and ·6, ·45, and

·3 mm. in diameter. Mica insulation separated the several windings from each other and from the core.

Fig. 4.—Section of completed Ring. Scale ½ full size.

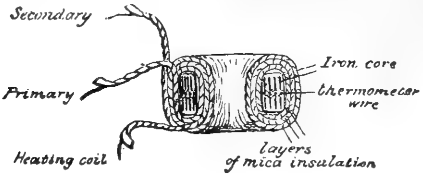
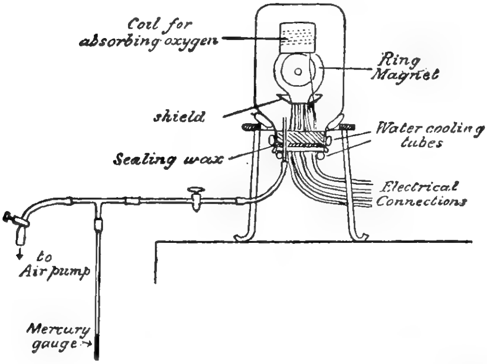


Fig. 5.—Showing Ring in Glass Vessel.



The heating wire, wound back on itself to avoid magnetizing effect, was passed once or twice round the lower part of the stem to prevent that local abstraction of heat to which the stem would otherwise have given rise.

The secondary winding was wound last of all; it being possible in this way to secure better insulation, particularly from the primary, from which even a small leakage into the secondary was much more serious than any leakage from the heating coil; for in the former case every ballistic throw due to a change in the primary current would be associated with a change of zero, while in the latter case no such change would occur.

The ring was afterwards successively covered with a layer of mica, a layer of asbestos, a covering of iron-wire gauze, and, finally, with a jacket consisting of three coverings of soft asbestos millboard. The iron gauze, which became red hot when the ring was being used at high temperatures, was placed there to intercept oxygen on its way in.

The *oxygen-absorbing coil* was, however, quite distinct from this. It consisted of two bare iron wires, connected in parallel, wound side by side on a cylindrical asbestos frame, and surrounded by a loosely-fitting asbestos mantle. Before the experiments they were heated by an electric current to a temperature at which they would readily absorb oxygen.

The coverings of the ring-magnet formed a mass of asbestos about 6 cm. in diameter; a small hole through them enabled one to see what sort of temperature had been reached in the middle of the ring.

The coil for absorbing oxygen was then made fast to the ring-magnet, and the leading wires brought through a cork (previously boiled in linseed oil), which was then inserted into the glass jar and covered with melted sealing-wax (fig. 5). Finally, the terminal wires were soldered to those leading away to the rest of the apparatus.

The following particulars relate to the last two ring-magnets constructed:—

Core of ring .... specimen	A.	B.
Description of iron	Charcoal Iron.	Swedish Transformer Iron.
Number of turns of core	4	5 turns
Density of iron (after experiments)	7·775	7·461
Total sectional area of iron............	·131	·143 cm.^{2}
Mean diameter of ring................	2·35	2·23 cm.
Number of turns of primary	28	29 turns.
„ per cm. of magnetic circuit	3·792	4·143 turns per cm.
Field due to 1 amp. in primary	4·765	5·206 C.G.S. units.
Mean sectional area enclosed by primary.	·72	·83 cm.^{2}
Number of turns of secondary..........	15	17 turns.
Resistance of iron at 18°	·0486	·0477 ohm.

Electrical Arrangements.—Reduction of Results to Absolute Measure.—The various electrical circuits communicating with the ring-magnet are indicated in fig. 6, which needs little explanation, except perhaps with regard to the arrangement for standardizing the ballistic galvanometer.

For this purpose an auxiliary ring-magnet of known dimensions and having a non-magnetic core was used. Its secondary, in series with that of the iron ring-magnet, formed, in circuit with the galvanometer and a coil of 100 ohms, the secondary circuit, as it remained throughout the experiments. On reversing a known current in the primary of the auxiliary coil, the resulting throw of the ballistic galvanometer was a measure of its sensitiveness to induction alterations in that particular secondary circuit, whether taking place in the core of the auxiliary coil, or in that of the iron ring-magnet. Such standardizing throws were observed before the readings at each temperature, and in this way direct compensation was made for alterations in the resistance of the secondary circuit.

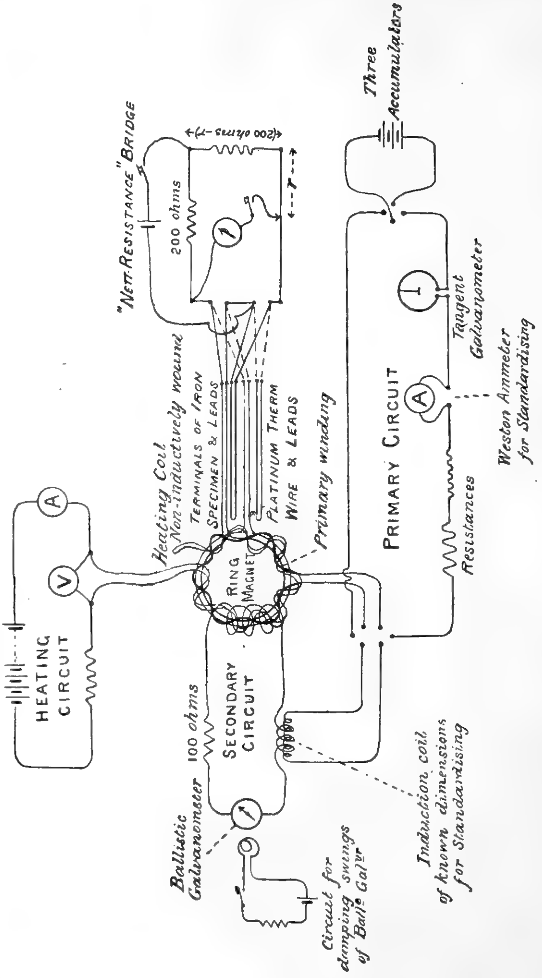


Fig. 6.—General Plan of Connexions.

The standardizing throw and the numerical data of the iron ring-magnet being known, it was possible to obtain from each ballistic observation the change of magnetic induction in the iron core to which it was due; correction being of course made for the induction which did not pass through the core.

The magnetizing current was measured with a tangent galvanometer, whose indications were compared with those of a standard Weston ammeter. The magnetizing force was assumed to be uniform in intensity throughout that part of the section of the ring occupied by the iron; which assumption is, in the present case, but approximately true, though the oblong cross-section favours the uniformity.

Description of the Experiments.

The iron cores in their initial condition may be regarded as unannealed; for the original annealing was destroyed during their preparation.

To study the effects of annealing, measurements were taken during the first heating of specimen A. With this exception, however, all the experiments recorded in this paper were made during the cooling from some definite temperature at which the iron had been thoroughly annealed, that all effects other than those due directly to change of temperatures might be eliminated. The duration of each of the long sets of experiments was about 20 hours.

The ring containing *specimen A* was first heated slowly up to 1050°, stopping at various stages on the way up for observations, and then allowed to cool, when another series of observations was made. It was afterwards heated to 920° to find whether the annealing was complete.

The ring was then taken to pieces, that the cross-section of its iron and the constants of the thermometer-wire might be re-determined.

The arrangements for absorbing the oxygen were with this ring not very perfect. The coil for this purpose had but one iron wire instead of two in parallel; and, becoming locally oxidized, the wire fused before it had absorbed all the oxygen. The cross-section of the iron core on removal from the ring and on detaching the oxide was found to have diminished by as much as 8 per cent. Since, however, no change of section occurred (as deduced from the constancy of the resistance and permeability) during the concluding heating to 920°, it may safely be assumed that the oxidation took place previous to the cooling from 1050°; and that the section during subsequent experiments was the same as that measured at their close.

The ring-magnet, of which *specimen B* was the core, was, after preliminary magnetic measurements, heated at once to 840°, a little above the critical point; at this temperature it was kept for about two hours and then allowed to cool. The following day the ring was again heated to this temperature, it being intended to take a series of observations during cooling. The series had, however, to be postponed, owing to an unfortunate occurrence—the sealing-wax over the cork had slightly softened during the prolonged heating, and the excess of outside pressure began to drive the cork slowly into the jar, rendering it no longer air-tight. The heating current was immediately shut off; but, when cold, the iron core was found to have been partially oxidized by the inrush of air, reducing its section about 4 per cent. To prevent a repetition of this occurrence, an indiarubber tube containing flowing water was arranged to keep the neck of the jar cool.

During the second absorption of the oxygen, the iron-wire coil, which had already been used once for this purpose, burnt out. But it was thought that the absorption might be complete, and the experiments were proceeded with. After heating the ring, therefore, a third time to 840°, a series of measurements was made while the temperature was falling.

The ring was then still further annealed by heating to 1150°; and, during cooling, a set of observations like the previous was taken. On their completion, this ring was also taken to pieces that the cross-section of the core might be re-determined and the constants of the thermometer-wire checked. The last absorption of the oxygen does not seem to have been at all complete, for during the last heating the section diminished by 6 per cent.

Thus, in these experiments, though the method of absorbing the oxygen was shown to be efficient, yet troubles due to oxidation were by no means absent. Irrespective of the diminution of the sectional area of the iron, the presence of oxide in the core might affect the results in a variety of ways:—(1) by the shunting effect due to its conductivity, or (2) by impairing the quality of the insulation, both tending to diminish the apparent resistance of the iron; and (3) by having an appreciable permeability, affecting the magnetic measurements.

(1) The shunting effect is quite negligible. (2) Though the inner surface of the mica was altered by contact with the oxide, the insulation was not seriously injured. (3) With regard to the third possibility:—In small fields, such as those used in the present experiments, the magnetic qualities of soft iron are at ordinary temperatures so very much more intense than even those of the magnetic oxide of iron, that no perceptible alteration of the ballistic galvanometer throws could arise from any oxide present. At temperatures above the critical point, the experiments of M. Curie, already cited,

have shown that though the susceptibility of that oxide rapidly decreases with rise of temperature, yet it may be comparable with that of iron, which is also very feebly magnetic in the neighbourhood of 1000° C. The author's methods were, however, not adapted to the determination of the magnetic qualities of feebly magnetic bodies, and any determinations by them of the permeability of the iron core when $\mu < 10$ have in any case little or no worth; so that errors of such values need hardly be considered.

The only serious effect of the oxidation was to alter the sectional area of the iron. By directly measuring this section both before and after the experiments with each ring-magnet, uncertainties in the absolute values, both of the permeability and resistance measurements, were practically eliminated in all the series of observations, except that taken after annealing specimen B at 840°; and even in this case the alteration of resistance which took place gave the means of determining with but little uncertainty that sectional area which was not directly measured. Whilst cooling from 1150° a pause of half-an-hour was made at 1071°. During this time the resistance of the iron did not increase, and it is therefore concluded that from this time onwards no further change of section due to oxidation took place.

Method of taking Observations at each Temperature.—The experiments with specimen B were as follows (with specimen A they were similar but not so full):—

When the temperature had become quite stationary, and the "standardizing throw" had been observed, the iron core was *de-magnetized* by repeatedly reversing the magnetizing current whilst diminishing its strength.

Observations were then taken, by the method of reversals, of the magnetic induction in the core for each of eight values of the field; beginning with $\mathbf{H} = \cdot 078$ and $\cdot 153$ to get the *permeability under very small forces*, and ending with $\mathbf{H} = 9 \cdot 20$. Next, the iron core was carried through a cyclic process of magnetization between *definite limits of magnetic field* ($\mathbf{H} = \pm 6 \cdot 83$), observations being taken by the "step-by-step" method—sixteen steps round the cycle. This process had, however, been previously gone through several times till the iron had reached a cyclic state. Finally, the iron was carried through a further cyclical magnetization, this time between *definite limits of induction* (*e. g.* $\mathbf{B} = \pm 4550$ lines/cm^2).

The last experiment was by far the most difficult; for the magnetizing currents had to be so adjusted that the *sum* of the ballistic throws during one half of the cycle might add up to a certain previously calculated amount depending on the magnitude of the standardizing throw. The difficulty was much reduced by diminishing the number of observations, though the number taken (just eight all round the cycle) barely suffices to determine the hysteresis loop.

The group of experiments at one temperature was concluded by a measurement of the resistance of the iron.

Throughout the whole of such sets of experiments, and particularly when working immediately below the critical point, where the magnetic properties vary so rapidly, the most scrupulous attention was paid to the temperature that it should not, if possible, be allowed to alter more than a fraction of a degree ; and this attention was the more necessary when working with the larger magnetizing currents, for their heating effect in the primary circuit was quite perceptible.

The time-period of the ballistic galvanometer needle being about eleven seconds, magnetic "creeping" effects lasting much more than a second would not be recorded.

In reducing the observations for each curve of cyclic magnetization, the mean of the corresponding throws taken during the ascending and descending halves of the process has in every case been taken ; though this eliminates any systematic unsymmetry which the curve might have possessed.

Discussion of Results.

The results of the experiments are given in the accompanying Tables and Curves (Plates II. & III. and Woodcuts, pp. 247–254) ; the series being in each case arranged in the order in which they were taken.

I. *Curves of Magnetization by the Method of Reversals.*— The effect of annealing (at 1050°) on specimen A is strikingly shown in the curves marked 1, and in the corresponding permeability curves 9 and 10. Its effect on the *maximum* permeability and corresponding field is shown in Table IX. The ultimate result of the annealing was to increase the maximum permeability at ordinary temperatures three-fold ($\mu=1355$ to 4050) whilst reducing the corresponding field to little more than half its former value (3·72 to 2·02).

The first annealing of specimen B affected the permeability in a similar manner (compare curves No. 3). The experiments on this specimen, however, were carried out with a view to showing especially the *difference between annealing at a red heat and at a low white heat.* The curves of magnetization at different temperatures obtained after annealing respectively at 840° and 1150° are numbered 7 and 8 ; whilst the corresponding permeability-temperature curves for different strengths of magnetic field are shown in the curves 11, 12, and 13.

A reference to these will show that the difference between annealing at these two temperatures is very marked; the iron having by the last annealing been rendered much more susceptible to weak magnetic fields. Annealed at 840°, the maximum permeability at ordinary temperatures was about 4000 in a field of 1·84 C.G.S. units; annealed at 1150° this rose to the remarkably high value of 4680 in the much smaller field of 1·48. The effect of this difference in annealing is, at temperatures immediately below the critical point, even more striking, as is shown by a comparison of the μ-t curves (Nos. 11 & 12) for values of **H** below unity.

An interesting point with regard to specimen B is the *diminution of permeability* which it experiences *in the neighbourhood of* 550° C. This diminution can be traced on the μ-t curves for all but the highest values of **H**, though it is especially noticeable when the magnetizing force is about 1 C.G.S. unit. The effect is more pronounced after the annealing at 1150° than after that at 840°. This peculiarity is hardly, if at all, possessed by specimen A, though it must be borne in mind that the temperature-interval between observations is here much greater, and the effect may thus be disguised. In some earlier experiments made on a ring-magnet having soft commercial iron wire for its core, the minimum of permeability, which occurred at about 500°, was still more marked than in specimen B; the depression being noticeable in the μ-t curves for fields as large as 5 C.G.S. units. It may be mentioned that, in a list of those temperatures at which the physical properties of iron undergo marked alteration, Tomlinson* gives 550° as that at which the internal friction of soft iron begins to rapidly increase, and at which the "specific heat of electricity" for that material changes sign.

The *permeability under small magnetizing forces* ($\mathbf{H} < \cdot 5$ C.G.S. unit) rises at first slowly (·2 to ·3 per cent. per degree) when the temperature is raised; then, in the neighbourhood of 300°, quite rapidly. It remains nearly constant between 400° and 550°, after which it rises with increasing rapidity to an enormously high value ($\mu = 12660$ for $\mathbf{H} = \cdot 078$ is the highest *observed*) only to sink still more rapidly at the critical temperature to a value quite insignificant.

As the magnetic field gets smaller (curves 11 and 12) the

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* H. Tomlinson, Phil. Mag. vol. xxvi. July 1888, p. 21.

maximum of permeability increases, and the temperature at which it occurs approaches the critical temperature, so that in very small fields the fall of the permeability is exceedingly sudden.

An examination of the **B-H** curves at various temperatures (Nos. 6, 7, and 8) shows that as the temperature rises, the initial slope of the curve increases and approaches more and more nearly to the maximum slope. And the critical temperature occurs just when they would coincide, *i. e.* when the **B-H** curve would start from the origin with its maximum slope, and the maximum permeability would occur in an infinitely small magnetic field.

The *critical temperature* is as near as can be given 795° for specimen A, and 780° for specimen B. For the soft iron wire specimen it was about 770°.

Above this temperature the induction is proportional to the magnetizing force (curves 4 and 5) indicating the *same permeability for all forces,* a result in accordance with the experiments of M. Curie. The curve of variation of the permeability with temperature (No. 13, II.) is given with some diffidence as the observations were liable to considerable error (see first part of Tables IV. and V.), and as the results are too remarkable to pass without further corroboration. It indicates a maximum of about $\mu=100$ at 820°, and another of $\mu=17$ at about 1050°; whilst between 920° and 980° the permeability is but 2·3.

II. *Curves of Cyclic Magnetization.*—The variations in the form of these curves due to changes of temperature can be studied in the series of curves Nos. 2, 4, and 5, in which the results recorded in Tables II., VI.–VIII. are plotted.

The process of annealing of specimen A is graphically depicted in the first of these series.

In series No. 5 it will be noticed how, as the temperature rises and approaches 550°, the inclination of the curves to the vertical axis increases, and how with further rise of temperature the curves become more erect and angular. Those taken at 764°·5 in this series are interesting as showing the magnetic condition of the iron almost immediately below the critical temperature. The coercive force required to remove the remanent magnetism was little more than $\frac{1}{10}$ of a C.G.S. unit. For the steep-sloping part of the curve the value of $\frac{d\mathbf{B}}{d\mathbf{H}}$ is about 15000; when **H** is much above 1 unit, its value is about 150. The transition from the steep part of the curve to the gently-sloping part is very sudden.

These values of $\frac{d\mathbf{B}}{d\mathbf{H}}$ are of the same order of magnitude in all the sets of curves of both series taken within 50° of the critical temperature, and they seem to represent an approximately constant property of the iron in this region.

The area enclosed by each hysteresis loop has been summed up and divided by 4π, giving the quantity $\frac{1}{4\pi}\int \mathbf{H}\, d\mathbf{B}$, the work done in ergs per c.cm. of the iron by the magnetic forces during the performance of one cycle. The results are given beside the corresponding temperatures in the Tables numbered X. (*a* and *b*), and are plotted in curves 14–16, which show the way in which the hysteresis diminishes to zero as the iron is heated up to the critical temperature.

No evidence of hysteresis above this temperature could be experimentally obtained, a result which one might be led to expect from the fact that the permeability at these temperatures is the same in all magnetic fields.

A comparison of the two curves No. 15 shows the marked diminution of hysteresis at all temperatures which resulted from annealing at 1150° instead of at 840°. The curves expressing the hysteresis for cycles within definite limits of induction (Curves 16 I. & II.) are not comparable, as the limits during the later series were, owing to oxidation, greater than during the former. In order to render them comparable a third curve has been dotted in, for which the ordinates of I. were multiplied by a constant 1·11 (the assumed ratio of the hysteresis when $\mathbf{B} = \pm 4260$ to that when $\mathbf{B} = \pm 4550$, which, according to results of many observers, cannot at ordinary temperatures be far wrong). Comparing then curve I. with curve III. an idea is obtained of the extent to which the hysteresis can be diminished by annealing at the high temperature 1150°, instead of at the lower one 840°. The comparison is given on Table X. *c*.

Table X. *d*. shows concisely the various stages of annealing of specimen B. The hysteresis at ordinary temperatures for $\mathbf{B} = \pm 4550$ was originally 1480 ergs per cub. centim. per cycle. Prolonged annealing at 840° reduced this to about 800 ergs; whilst annealing at 1150° brought the hysteresis down to 612 ergs.

In a paper read before the Institution of Civil Engineers about a year ago*, Prof. Ewing described some tests which

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* Ewing, "Magnetic Testing of Iron and Steel," Journ. Inst. C. E. May 1896.

he made on a sample of transformer plate of similar thickness and obtained from the same firm as the author's were. The maximum permeability was about 4500, while the hysteresis varied from 740 down to 580 ergs for **B** = ±4000; and attention is drawn to these values as representing the best iron both as regards high permeability and low hysteresis which he had tested. It is therefore of interest to note that specimen B had after annealing at 1150° a maximum permeability of over 4600 at ordinary temperatures, and a hysteresis as calculated for **B** = ±4000 of just under 500 ergs, corresponding to a hysteresis loss of but ·295 watts per lb. at a frequency of 100 ~. The following values are taken from the magnetization curve at 24° C.:—

B.	H.	μ.
2000	·74	2700
4000	1·01	3960
6000	1·33	4520
8000	1·82	4400
10000	2·84	3520
12000	5·63	2130

Between 0° and 200° the *diminution of hysteresis with rise of temperature is ·08 per cent. per degree*, when the limits of induction are kept constant.

W. Kunz [*] concludes from his experiments that for soft iron the hysteresis is expressible as a linear function of the temperature. This is not in accordance with the author's results. The curves expressing this relation (Nos. 16) for specimen B are far from straight, and show a decided dip at about 500° corresponding to the minima which occur near that temperature on the permeability-curves.

III. *Resistance-Temperature Curves.*—The variations of specific resistance with temperature are shown graphically on curves 17 and 18.

By taking a sufficient number of observations (see Tables XI. and XII.) the author has been able to show that the resistance-temperature curve undergoes *no abrupt change of direction* at or near the critical temperature. (Previous observers have given curves all of which were *broken* at this temperature; an examination of the figures given by Hop-

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[*] W. Kunz, '*Elektrotechnische Zeitschrift,*' 1894, p. 196.

kinson and Le Chatelier in the papers already cited shows that in each case they would be more strictly satisfied by a *continuous curve* like those here given.)

The annealing of specimen A has reduced its specific resistance considerably.

With specimen B, the further annealing at 1150° has but slightly altered it from the condition as regards specific resistance in which it was after the annealing at 840°.

The temperature-coefficients and specific resistance at 0° and 1000° are given at the foot of Tables XI. and XII.

As regards the resistance-changes both of specimens A and B, and of the soft commercial iron previously tested :— The resistance rises with rising temperature at a rate which increases fairly uniformly till a specific resistance of about 100,000 C.G.S. units is reached. Here the maximum slope of the curve occurs at a point *but few degrees below the critical temperature.* The rate of increase of resistance then falls off rapidly, so that at 1000° the temperature-coefficient is about ·0024 and the specific resistance 118,000.

With specimen B, which, judging from the low value of its specific resistance at 0° (10,050 C.G.S. units) and from the high value (·0057 at 0°) of its temperature-coefficient, should be a sample of very pure iron, the *maximum coefficient* after annealing at 1150° was ·0204 at about 765°. The number of observations on this resistance-temperature curve (18 II.) make it perhaps permissible to note its irregularities a little more closely. A flat part of the curve will be noticed between 830° and 900°, and the temperature-coefficient, which remains tolerably constant over this range, has at 865° the value ·0068. (Dr. Hopkinson gave ·0067 as the coefficient of iron above the critical temperature.) The points of maximum curvature 800° and 920° seem to be in agreement with the minima of the permeability curve (No. 13) obtained at the same time.

The above experiments were carried out in the laboratories of the Physical Institute of the Polytechnicum, Zürich ; and the author takes this opportunity of expressing his gratitude to the director, Prof. H. F. Weber, for encouragement received throughout the progress of the work.

Zürich, March 1897.

TABLE I.—SPECIMEN A.

Induction and Permeability in various Magnetic Fields taken at different temperatures.

(*a*) Before annealing. (Curves 1 and 9.)

H in C.G.S. units.	20°.		220°.		365°.		532°.		660°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·917	401	438	463	505	506	552	933	1017	1864	2033
2·24	2495	1114	2868	1281	2990	1335	4180	1867	5285	2360
3·94	5325	1352	5345	1357	5145	1306	6380	1620	7215	1830
6·10	7465	1225	7270	1192	6850	1107	8000	1312	8685	1440
8·26	8805	1066								

(*b*) After annealing at 1050° C. (Curves 1, 6, and 10.)

H in C.G.S. units.	774°.		722°.		636°.		506°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·070			292	4170	109	1560	64	915
·140			583	4170	272	1950	166	1190
·343			2475	7220	1188	3465	617	1800
·917	5980	6520	6008	6550	5395	5880	4790	5220
2 24	7795	3480	8360	3730	8630	3850	8820	3940
3·94	8290	2105	9475	2405	9870	2480	10320	2620
6·10	8330	1366	10180	1670	10705	1750	11310	1855

H in C.G.S. units.	324°.		188°.		21°·5.	
	B	μ	**B**	μ	**B**	μ
·070	48	685	29	414	21 ?	300 ?
·140	104	743	68	485	50	357
·343	366	1067	248	723	181	527
·917	3355	3660	1890	2060	1208	1318
2·24	9105	4060	9175	4095	8845	3950
3·94	10925	2775	11420	2900	11435	2904
6·10	11960	1960	12550	2060	12770	2093

(*c*) After re-heating to 920°. (Curves not given.)

H in C.G.S. units.	770°.		759°.		680°.		630°.		22°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·070			158	2260	112	1600	69	985	24	343
·140			391	2800	261	1870	164	1170	53	379
·343			1908	5560	1298	3785	810	2360	168	490
·917	5810	6340	5850	6380	5745	6260	5020	5470	1430	1560
2·24	8045	3590	8120	3625	8560?	3820?	8390	3745	8485	3790
3·94	8940	2270	9465	2405	9945	2525	9825	2495	11380	2890
6·10	9110	1494	10180	1670	10840	1778	10790	1770	12790	2095

TABLE II.—Specimen A.

Data for Cyclic **B-H** *Curves at various Temperatures.*
Limits of Cycle: **H** = ±6·10.

(*a*) Before annealing. (Curves 2.)

H	20°.	224°.	367°.	534°.	665°.
	B	**B**	**B**	**B**	**B**
6·10	7370	7145	6830	7980	8610
3·94	6985	6645	6300	7365	7715
2·24	6380	5945	5530	6500	6570
·917	5620	5090	4660	5410	5240
0	4800	4232	3805	4265	3750
− ·917	3513	2793	2285	1750	− 526
−2·24	−1642	−2365	−2600	−3935	−5148
−3·94	−5410	−5345	−5170	−6380	−7140
−6·10	−7370	−7145	−6830	−7980	−8610
Hysteresis in ergs per cm.³	3900	3210	2800	2670	1850

(*b*) After annealing at 1050° C. (Curves 2 and 14.)

H	765°.	708°.	625°.	506°.	325°.	175°.	21°·5.
	B	**B**	**B**	**B**	**B**	**B**	**B**
6·10	9100	10220	10640	11185	11850	12320	12480
3·94	8790	9630	10085	10615	11400	12000	12190
2·24	8075	8720	9220	9880	10815	11520	11760
·917	6720	7205	7730	8565	9925	10835	11180
0	2985	3930	4780	6190	8170	9565	10210
− ·917	−5830	− 5845	− 5290	− 4290	− 2005	1389	4170
−2·24	−7880	− 8275	− 8400	− 8620	− 9090	− 9145	− 8990
−3·94	−8780	− 9440	− 9750	−10190	−10810	−11210	−11340
−6·10	−9100	−10220	−10640	−11185	−11840	−12320	−12480
Hysteresis in ergs per cm.³	500	920	1345	2060	3090	4110	4570

(*c*) After re-heating to 920°. (Curves not given.)

H	738°.	679°.	630°.	22°.
	B	**B**	**B**	**B**
6·10	10370	10840	10660	12520
3·94	9565	10115	10020	12210
2·24	8515	9120	9085	11790
·917	6880	7510	7505	11185
0	3535	4397	4500	9995
− ·917	− 5610	− 5240	− 4435	2766
−2·24	− 8100	− 8365	− 8080	− 8420
−3·94	− 9450	− 9820	− 9670	−11210
−6·10	−10370	−10840	−10660	−12520
Hysteresis in ergs per cm.³	780	1280	1510	4720

TABLE III.—Specimen B.

Preliminary Measurements at 24° C. (Curves No. 3.)

Induction and Permeability in various Magnetic Fields.

H	Initial condition.		After 2 hours at 840°.		After 5 hours at 840°.	
	B	μ	**B**	μ	**B**	μ
·078	14	178	32	410	35	450
·153	37	242	79	516	84	549
·378	102	270	280	741	307	813
1·017	560	550	2656	2608	2995	2942
2·49	4352	1746	10295	4135	10525	4230
4·40	7265	1652	12395	2816	12405	2820
6·83	9220	1349	13420	1963	13340	1953
9·20	10420	1133	14005	1523	13920	1515

Cyclic **B-H** Curves between Limits **H** = ± 6·83.

H	Initial condition.	After 2 hours at 840°.	After 5 hours at 840°.
	B	**B**	**B**
6·83	9120	13100	13120
4·40	8670	12800	12820
2·49	8005	12425	12430
1·017	7135	11905	11885
0	6165	10890	10740
−1·017	4265	1420	778
−2·49	−3882	−10105	−10380
−4·40	−7270	−12140	−12210
−6·83	−9120	−13100	−13120
Hysteresis in ergs per c. cm.	4740	5020	4625

Cyclic **B-H** Curves at 20° C.

Limits of Cycle : **B** = ± 4020. Limits **B** = ± 4140.

Initial condition.	
H	**B**
2·47	4025
1·007	3520
0	2763
−1·007	870
−2·47	−4025
Hysteresis per cycle	$=1218\frac{\text{ergs}}{\text{cm.}^3}$

After 2 hours at 840°.	
H	**B**
? 1·15	4015
·646	3760
0	2887
− ·646	− 63
−1·15	−4015
Hysteresis $=664\frac{\text{ergs}}{\text{cm.}^3}$	

After 5 hours at 840°.	
H	**B**
1·170	4140
·660	3810
0	2900
− ·660	− 242
−1·170	−4140
Hysteresis $=669\frac{\text{ergs}}{\text{cm.}^3}$	

TABLE IV.—SPECIMEN B.

Induction and Permeability in various Magnetic Fields taken successively during cooling from 840° C.

Above critical Temperature. (Curves 4, 7, and 13.)

H	820°.		816°.		799°.	
	B	μ	B	μ	B	μ
1·017			19	19	40	39
2·49	29	12	35	14	97	39
4·40			78	18	163	37
6·83	84	12	101	15	262	38
9·20			124	13	363	39
Mean Permeability		12	...	16	...	38

H	797°.		788°.		784°·5.		782°·5.	
	B	μ	B	μ	B	μ	B	μ
1·017					124	122		
2·49					273	110		
4·40					518	118		
6·83	349	51	409	60	787	115	825 ?	121 ?
9·20					1035	113		
Mean Permeability		...	...	...	...	116	...	...

Below critical Temperature. (Curves 4, 7, and 11.)

H	770°.		764°.		746°.	
	B	μ	B	μ	B	μ
·078			298	3820	198	2540
·153			663	4330	490	3200
·378			2780	7360	2248	5950
1·017			5675	5575	5830	5730
2·49			6720	2700	8020	3220
4·40	5960	1355	7070	1606	8565	1950
6·83	6500	952	7410	1084	8905	1304
9·20			7870	856	9165	996

H	710°.		648°·5.		567°.		467°.	
	B	μ	B	μ	B	μ	B	μ
·078	131	1680	90	1150	61	780	57	730
·153	322	2100	230	1500	141	920	143	935
·378	1722	4560	1041	2755	645	1710	511	1354
1·017	5955	5850	5685	5585	5110	5020	5195	5100
2·49	8630	3470	9030	3625	8995	3610	9475	3800
4·40	9840	2260	10440	2375	10695	2430	11030	2510
6·83	10580	1550	11350	1662	11770	1725	12090	1770
9·20	10955	1191	11870	1290	12315	1340	12660	1377

Table IV.—(*continued*).

H	360°·5.		251°·5.		136°.		24°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·078	50	640	31	400	30	380	20	250
·153	126	820	70	450	68	440	40	260
·378	571	1510	294	778	318	840	178	470
1·017	5125	5040	4360	4280	3253	3200	1934	1900
2·49	9800	3935	9940	3990	10005	4020	9420	3780
4·40	11580	2630	11980	2720	12140	2760	11835	2690
6·83	12705	1862	12890	1888	13260	1942	12980	1902
9·20	13320	1449	13525	1471	13860	1507	13600	1480

TABLE V.—SPECIMEN B.

Induction and Permeability in various Magnetic Fields taken successively during cooling from 1150° C.

Above critical Temperature. (Curves 5, 8, 12, and 13.)

H	1145°.		1047°.		980°.		975°.		960°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
1·017										
2·49			42	17	15	6				
4·40										
6·83	52 ?	8					17	2·5	17	2·5
9·20										
Mean Permeability ...	...	8	...	17	...	6	...	2·5	...	2·5

H	922°.		895°·5.		851°.		829°·5		816°.		787°·5.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
1·017									108	106	51	50
2·49			129	52	173	69	260	104	260	104	120	48
4·40							434	99	434	99	215	49
6·83	16	2·3	388	57	475	70	650	96	781	114	336	49
9·20											518?	56?
Mean Permeability	}	2·3	...	54	...	69·5	...	100	...	106	...	51

Below critical Temperature. (Curves 5, 8, and 12.)

H	776°·5.		775°.		764°·5.		748°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·078					631	8090	485	6210
·153					1935	12660	1585	10360
·378					4210	11140	4465	11820
1·017	925	908	1353	1330	5020	4930	6720	6600
2·49	1788	718	1964	789	5780	2320	7980	3200
4·40			2442	556	6305	1434	8290	1885
6·83			2798	410	6440	943	8555	1252
9·20			3006	327	6550	712	8645	940

Table V. (*continued*).

H	730°.		695°.		634°.		554°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ
·078	372	4770	243	3120	126	1620	73	935
·153	1161	7600	713	4660	332	2170	172	1120
·378	4165	11020	3565	9430	2128	5630	996	2640
1·017	6850	6730	6845	6720	5780	5680	4275	4200
2·49	8415	3380	8985	3610	8800	3530	7565	3040
4·40	9160	2085	9985	2270	10160	2280	9630	2190
6·83	9495	1390	10670	1562	11050	1620	10940	1603
9·20	9730	1024	11110	1209	11670	1269	11690	1271

H	457°.		352°.		249°.		137°·5.		24°.	
	B	μ	**B**	μ	**B**	μ	**B**	μ	**B**	μ
·078	78	1060	77	980	48	615	43	550	35	450
·153	172	1125	167	1090	113	740	99	650	85	555
·378	989	2620	879	2325	457?	1210?	471	1246	356	942
1·017	5200	5110	5115	5020	4730	4650	4410	4330	4010	3940
2·49	8710	3495	9230	3705	9385	3770	9740	3910	9480	3810
4·40	10460	2380	10930	2485	11145	2535	11560	2625	11400	2590
6·83	11475	1680	11930	1748	12170	1783	12495	1830	12440	1823
9·20	12150	1321	12460	1355	12760	1388	13090	1423	13040	1418

TABLE VI.—SPECIMEN B.

Data for Cyclic **B-H** *Curves* taken successively at diminishing temperatures after annealing at 840° C.

Limits of Cycle: $\mathbf{H} = \pm 6{\cdot}83$. (Curves 4 and 15.)

H	764°.	746°.	710°.	648°·5.	567°.
	B	**B**	**B**	**B**	**B**
6·83	7370	8870	10350	11130	11495
4·40	7060	8545	9730	10500	10935
2·49			8870	9665	10090
1·017	6005	6425	7380	8295	8620
0		3246	4330	5515	5770
−1·017	−5560	−5820	− 6000	− 5555	− 4535
−2·49			− 8555	− 8900	− 8860
−4·40	−7060	−8480	− 9700	−10250	−10480
−6·83	−7370	−8870	−10350	−11130	−11495
Hysteresis in ergs per c. cm.	426	573	1010	1650	2180

Table VI. (*continued*).

H	467°.	360°·5.	251°·5.	136°.	24°.
	B	**B**	**B**	**B**	**B**
6·83	11910	12540	12770	13040	12685
4·40	11400	12110	12390	12720	12360
2·49	10695	11495	11805	12230	11895
1·017	9520	10355	10810	11380	11110
0	7240	8245	8925	9770	9680
−1·017	− 4940	− 4010	− 2608	− 429	1395
−2·49	− 9340	− 9715	− 9820	− 9910	− 9395
−4·40	−10830	−11495	−11735	−11980	−11610
−6·83	−11910	−12540	−12770	−13040	−12685
Hysteresis in ergs per c. cm.	2550	3195	3700	4360	4700

TABLE VII.—Specimen B.

Data for Cyclic **B-H** *Curves* taken successively at diminishing temperatures after annealing at 1150° C.

Limits of Cycle: **H** = ± 6·83. (Curves 5 and 15.)

H	764°·5.	748°.	730°.	695°.	634°.
	B	**B**	**B**	**B**	**B**
6·83	6145	8240	9420	10615	10760
4·40	5815	8030	9100	10035	10060
2·49	5490	7670	8495	9205	9080
1·017	5075	6800	7185	7820	7180
0	1389	2455	2856	3480	3215
−1·017	−5060	−6540	−6835	− 6920	− 5490
−2·49	−5485	−7575	−8320	− 8870	− 8430
−4·40	−5795	−7975	−9040	− 9850	− 9860
−6·83	−6145	−8240	−9420	−10615	−10760
Hysteresis in ergs per c. cm. per cycle	120	328	426	797	1010

H	554°.	457°.	352°.	249°.	137°·5.	24°.
	B	**B**	**B**	**B**	**B**	**B**
6·83	10825	11360	11735	12110	12280	12150
4·40	10025	10810	11230	11730	11950	11810
2·49	8650	9950	10435	11125	11420	11300
1·017	6377	8340	8875	9980	10510	10400
0	3322	5395	6050	7520	8525	8560
−1·017	− 4140	− 4930	− 3960	− 3881	− 3196	− 2327
−2·49	− 7600	− 8710	− 8650	− 9295	− 9570	− 9290
−4·40	− 9575	−10365	−10620	−11100	−11330	−11150
−6·83	−10825	−11360	−11735	−12110	−12280	−12150
Hysteresis in ergs per c. cm. per cycle	1345	2025	2565	3130	3500	3660

TABLE VIII.—SPECIMEN B.

Data for Cyclic **B-H** *Curves between constant limits of Induction* taken successively at diminishing temperatures.

(*a*) After annealing at 840° C.

Limits of Cycle : **B** = ±4260. (Curves 4 and 16.)

764°.		746°.		710°.		648°·5.		567°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·60	4250	·60	4255	·73	4265	·81	4260	·93	4250
·31	3520	·32	3547	·34	3668	·36	3725	·56	3970
0	2093	0	2261	0	2611	0	2860	0	2870
−·31	−1762	−·32	−1359	−·34	− 440	−·36	− 189	−·56	−2263
−·60	−4250	−·60	−4255	−·73	−4265	−·81	−4260	−·93	−4250
Hysteresis per cycle } $=187\ \frac{\text{ergs}}{\text{cm.}^3}$		$216\ \frac{\text{ergs}}{\text{cm.}^3}$		$318\ \frac{\text{ergs}}{\text{cm.}^3}$		$357\ \frac{\text{ergs}}{\text{cm.}^3}$		$398\ \frac{\text{ergs}}{\text{cm.}^3}$	

467°.		360°·5.		251°·5.		136°.		18°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·88	4265	·93	4265	1·05	4265	1·20	4265	1·318	4265
·38	3760	·40	3743	·42	3732	·43	3700	·426	3725
0	3010	0	2930	0	3000	0	3033	0	3115
−·38	− 733	−·40	924	− ·42	1378	− ·43	1630	− ·426	1910
−·88	−4265	−·93	−4265	−1·05	−4265	−1·20	−4265	−1·318	−4265
Hysteresis per cycle } $=396\ \frac{\text{ergs}}{\text{cm.}^3}$		$495\ \frac{\text{ergs}}{\text{cm.}^3}$		$590\ \frac{\text{ergs}}{\text{cm.}^3}$		$656\ \frac{\text{ergs}}{\text{cm.}^3}$		$744\ \frac{\text{ergs}}{\text{cm.}^3}$	

(*b*) After annealing at 1150° C.

Limits of Cycle : **B** = ±4550. (Curves 5 and 16.)

764°·5.		748°.		730°.		695°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·640	4543	·403	4540	·443	4547	·516	4540
·210	3690	·174	3662	·184	3722	·197	3712
0	1660	0	2002	0	2272	0	2564
—·210	− 2665	−·174	−1921	−·184	−1468	− ·197	− 897
−·640	−4543	−·403	−4540	−·443	−4547	−·516	−4540
Hysteresis per cycle } $=81\ \frac{\text{ergs}}{\text{cm.}^3}$		$109\ \frac{\text{ergs}}{\text{cm.}^3}$		$128\ \frac{\text{ergs}}{\text{cm.}^3}$		$178\ \frac{\text{ergs}}{\text{cm.}^3}$	

Table VIII. (*continued*).

634°.		554°.		457°.		352°.	
H	**B**	**H**	**B**	**H**	**B**	**H**	**B**
·730	4540	·935	4550	·898	4553	·881	4542
·336	3928	·373	3770	·371	3840	·361	4002
0	2720	0	2520	0	2827	0	3060
− ·336	−1886	− ·373	−1074	− ·371	− 488	− ·361	239
− ·730	−4540	− ·935	−4550	− ·898	−4553	− ·881	−4542
Hysteresis per cycle $=264\ \frac{\text{ergs}}{\text{cm.}^3}$		$335\ \frac{\text{ergs}}{\text{cm.}^3}$		$379\ \frac{\text{ergs}}{\text{cm.}^3}$		$475\ \frac{\text{ergs}}{\text{cm.}^3}$	

249°.		137°·5.		18°.	
H	**B**	**H**	**B**	**H**	**B**
1·001	4550	1·032	4545	1·114	4545
·382	3952	·388	3980	·398	3972
0	3043	0	3188	0	3184
− ·382	712	− ·388	1163	− ·398	1386
−1·001	−4550	−1·032	−4545	−1·114	−4545
Hysteresis per cycle $=508\ \frac{\text{ergs}}{\text{cm}^3.}$		$555\ \frac{\text{ergs}}{\text{cm.}^3}$		613 ergs per c. cm. per cycle.	

TABLE IX.

Relation of Maximum Permeability and of the Field in which it occurs to Temperature.

(The values are estimated from the curves, and are only *approximate*.)

SPECIMEN A.

(Curves 9 and 10.)

Before annealing.

Temperature.	Maximum Permeability.	Corresponding Magnetic Field.
°C.	μ	**H**
20	1355	3·70
220	1370	3·25
365	1360	2·80
532	1880	2·20
660	2650	1·50

After annealing at 1050°.

Temperature.	Maximum Permeability.	Corresponding Magnetic Field.
°C.	μ	**H**
722	8350	·44
636	5950	·92
506	5450	1·05
324	4860	1·46
188	4370	1·70
21·5	4050	2·02

Table IX. (*continued*).

SPECIMEN B.

(Curves 11 and 12.)

After annealing at 840°.

Temperature.	Maximum Permeability.	Corresponding Field.
°C.	μ	**H**
764	7800	·50
746	7090	58
710	6070	72
648·5	5610	93
567	4900	1·09
467	5190	1·18
360·5	5280	1·13
251·5	4940	1·38
136	4250	1·69
24	3980	1·84

After annealing at 1150°.

Temperature.	Maximum Permeability.	Corresponding Field.
°C.	μ	**H**
764·5	14600	·20
748	13400	·25
730	11700	·33
695	9430	·43
634	6040	·58
554	4240	·99
457	5120	·94
352	5050	·96
249	4820	1·19
137·5	5020	1·36
24	4680	1·48

TABLE X.

Variation of Magnetic Hysteresis with Temperature.

(See Curves 14, 15, and 16.)

(*a*) Cycle between constant Limits of Magnetizing Force.

SPECIMEN A.

Limits: **H** = ±6·10.
After annealing at 1050°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
765	500
708	920
625	1345
506	2060
325	3090
175	4110
21·5	4570

SPECIMEN B.

Limits: **H** = ±6·83.

Annealed at 840°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764	426
746	573
710	1010
648·5	1650
567	2180
467	2550
360·5	3195
251·5	3700
136	4360
24	4700

Annealed at 1150°.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764·5	120
748	328
730	426
695	797
634	1010
554	1345
457	2025
352	2565
249	3130
137·5	3500
24	3660

Table X. (*continued*).

(*b*) Cycle between constant Limits of Magnetic Induction.

SPECIMEN B.

Annealed at 840°. Limits: **B** = ±4260.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764	187
746	216
710	318
648·5	357
567	398
467	396
360·5	495
251·5	590
136	656
18	744

Annealed at 1150°. Limits: **B** = ±4550.

Temperature.	Hysteresis in ergs per c.cm. per cycle.
°C.	
764·5	81
748	109
730	128
695	178
634	264
554	335
457	379
352	475
249	508
137·5	555
18	613

(*c*) SPECIMEN B.

Comparison of Hysteresis after annealing at 840° and 1150°.

(Reduced to a uniform Induction Limit of **B** = ±4550.)

Temperature.	Annealed at 840°.	Annealed at 1150°.
°C.		
20	822	612
200	686	530
400	504	437
600	430	302
700	358	176
750	236	106

(*d*) Hysteresis of SPECIMEN B at 20° and Induction Limits ±4550, showing various stages of annealing.

	Hysteresis in ergs per c.cm. per cycle.
Initial condition	1480
After 2 hrs. at 840°	810
„ 5 hrs. „	780
„ 6 hrs. „ and *very* gradual cooling.	822
After heating to 1150° and *very* gradual cooling.	612

TABLE XI.

IRON SPECIMEN A.

Electrical Resistance at Different Temperatures.

(*a*) Before and during annealing. (Curve 17, I.)

Temperature. ° C.	Specific Resistance in C.G.S. units.	Temperature. ° C.	Specific Resistance in C.G.S. units.
18°	14610	530°·5	61550
127	22150	535·5	61550
189	26300	643	77400
210·5	27680	674	83400
213	27880	752	96700
354·5	41300	763	99700
373	43450		

(*b*) After annealing at 1050°. (Curve 17, II.)

[Critical Temp.: 795°.]

Temperature. ° C.	Specific Resistance in C.G.S. units.	Temperature. ° C.	Specific Resistance in C.G.S. units.
1050°	115800 ?	330°·5	37000
1026·5	115800 ?	201·5	25130
950·5	114200	171·5	23480
936	113100	18	13820
870	109400	16	13580
881	110000		
808·5	101100	After reheating to 920°. (Also plotted for Curve II.)	
805	101100	885·5	110600
786	97200	916	112800
761	92600	736	89000
724 5	86600	686·5	81300
706	84400	681	79750
653	75600	632·5	72400
620	71350	630	72900
504·5	55550	20	13940
506	55700	16·8	13510
327	36760		
321	36280		

		Before annealing.	After annealing. at 1050°.	
Temperature-Coefficient	at 0°..................	·0044	·0045	
	maximum value at about 780°	·0125	·0144	
	at 1000°	......	·00236	
Specific Resistance......	at 0°..................	13600	12500	C.G.S. Units.
	at 1000°	......	115600	

TABLE XII.

IRON SPECIMEN B.

Electrical Resistance at Different Temperatures.

(*a*) After annealing at 840°. (Curve 18, I.)

[Critical Temp. = 782°.]

Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.
826°	108300	679°	83750	358°·5	39800
800	105100	649	78900	314·5	34920
786·5	103000	605·5	71950	252·5	28660
772	100000	555	64400	190·5	23280
764	98600	561	65400	128·5	18440
751	96200	517	59250	101	16410
739·5	94200	466·5	52700	52	13210
714·5	89300	413·5	45900	18	11110

(*b*) After annealing at 1150°. (Curve 18, II.)

[Critical Temp. = 778°.]

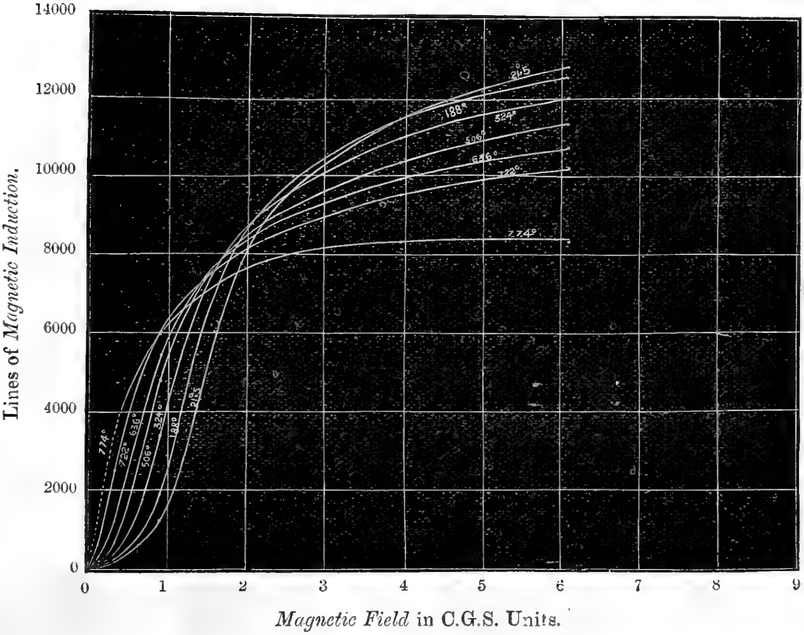
Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.	Temp. ° C.	Sp. Resist. in C.G.S. Units.
1150°	120600 ?	871°	113400	630°·5	76400
1142	121000 ?	858	112500	637	77400
1115·5	120850	844	111400	594	70850
1095·5	120600	824·5	109600	553·5	64850
1071·5	120350	813·5	108400	554	64900
1058·5	120050	792·5	105600	503·5	57850
1036	119650	787	104600	454·5	51770
1014·5	119100	781	103400	406	45720
989·5	118550	770	100900	386	43400
983	118350	765·5	100000	861	40580
977·5	118200	764·5	99850	295·5	34020
972	118100	745·5	96150	252·5	29080
946·5	117200	744	95650	196·5	24020
926·5	116600	725·5	92050	140·5	19410
916·5	116200	697·5	87200	84	15390
891·5	114900	658	80850	18	11140

		After annealing at 840°.	After annealing at 1150°.	
Temperature-Coefficient	at 0°	·0057	·0057	
	maximum value at about 765°	·0189	·0204	
	at 1000°	……	·00244	
Specific Resistance ……	at 0°	10050	10050	C.G.S. Units.
	at 1000°	……	118800	

B-H *Curves at various Temperatures.*

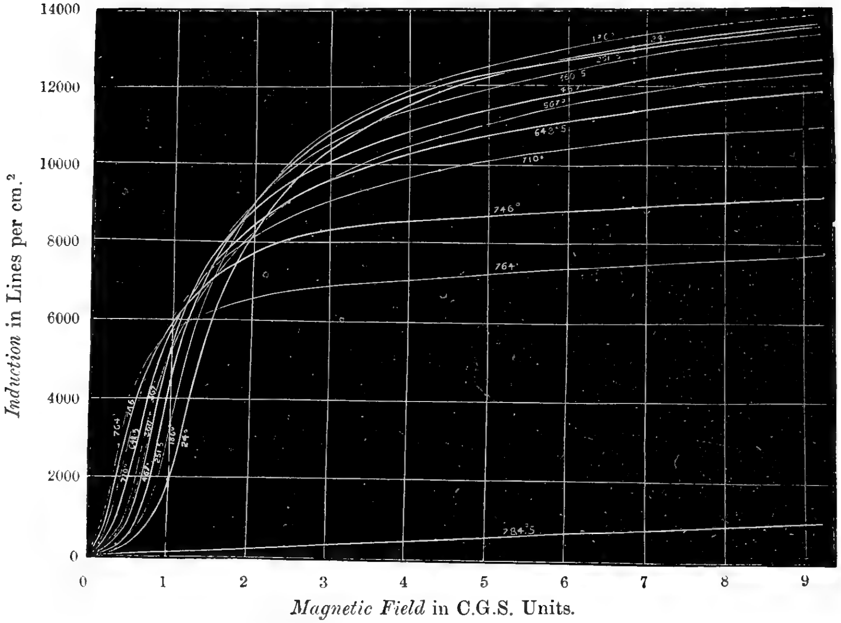
Specimen A after annealing at 1050° C.

6.



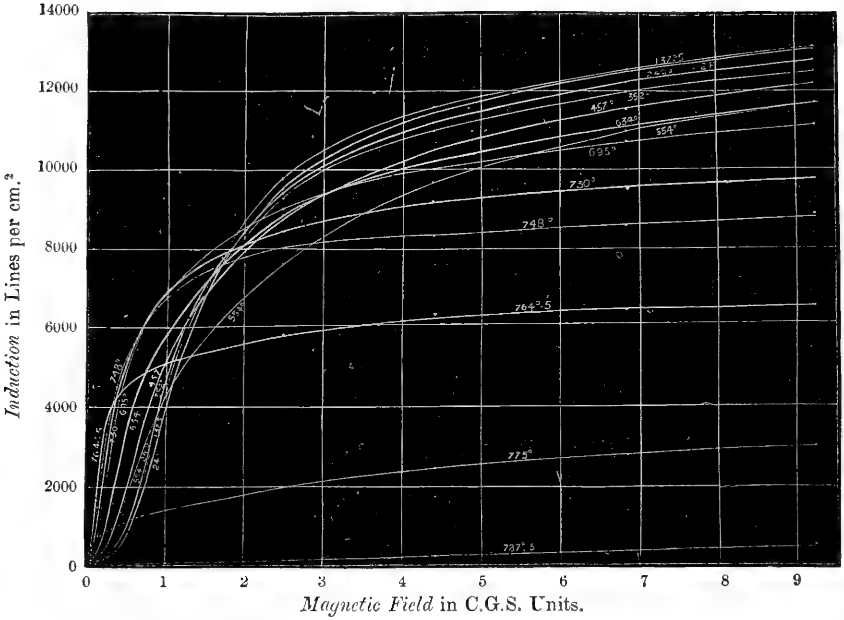
B-H *Curves at various Temperatures.*
Specimen B after annealing at 840° C.

7.



B-H *Curves at various Temperatures.*
Specimen B after annealing at 1150° C.

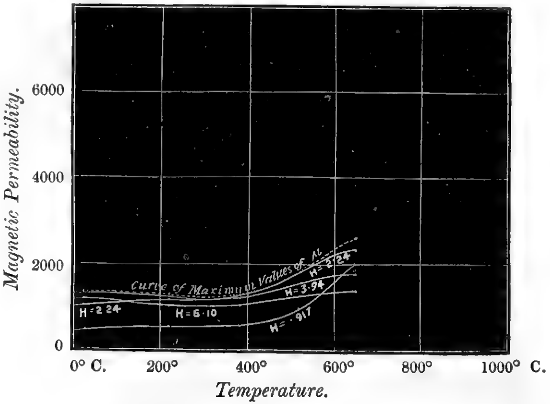
8.



Permeability-Temperature Curves.

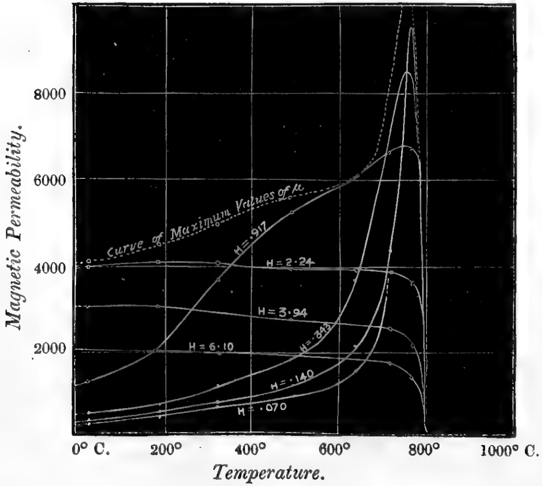
Specimen A before annealing.

9.



Specimen A after annealing at 1050° C.

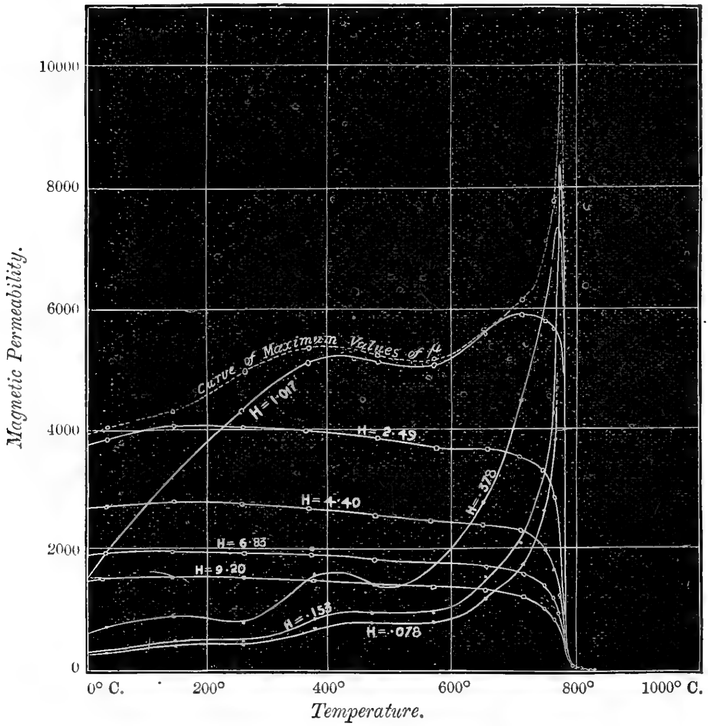
10.



Permeability-Temperature Curves.

Specimen B after annealing at 840° C.

11.

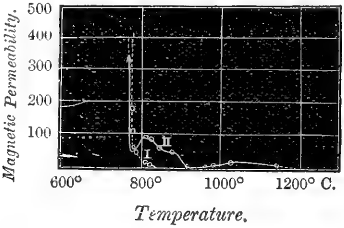


Magnetic Permeability of Iron specimen B

at Temperatures above its Critical Point.

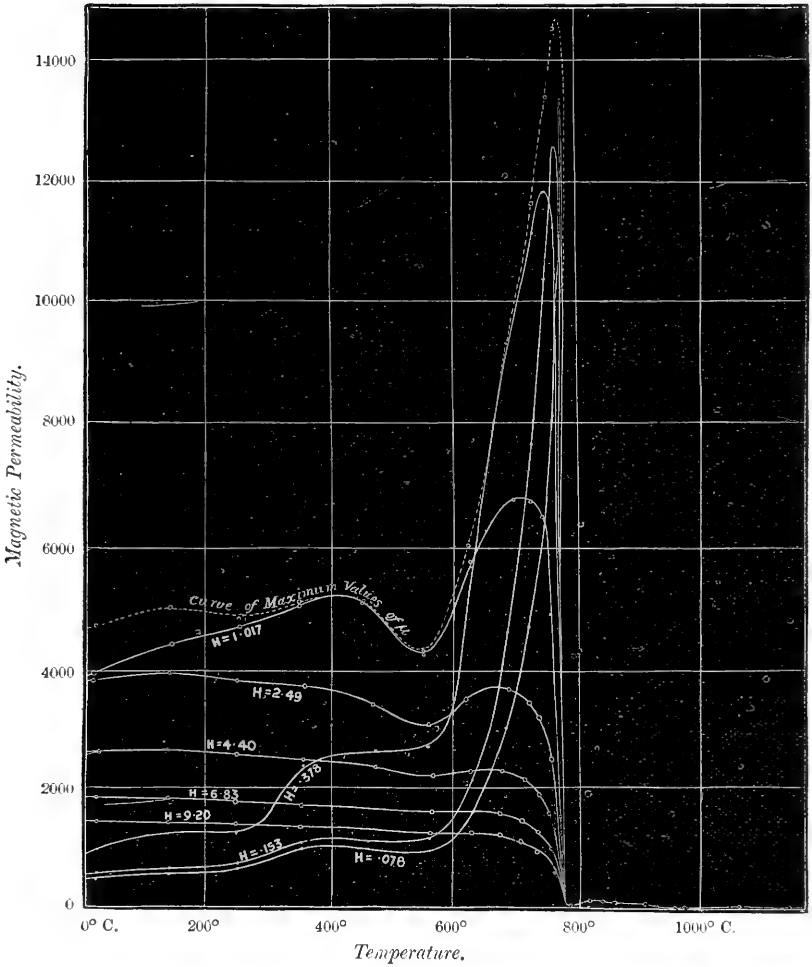
Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.

13.



Specimen B after annealing at 1150° C.

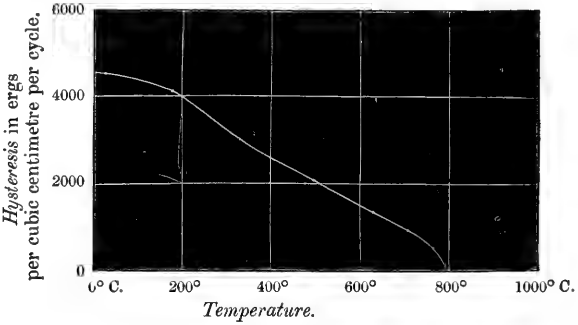
12.



Hysteresis-Temperature Curves.

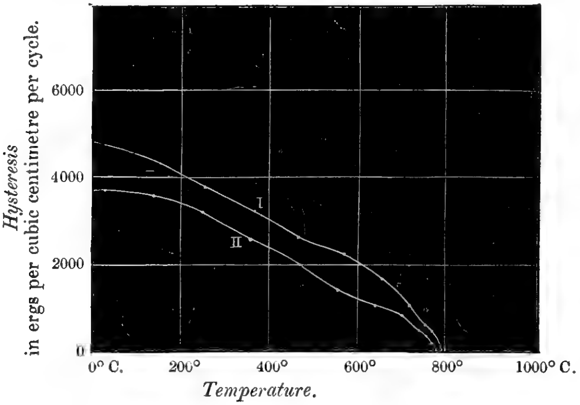
Specimen A. Annealed at 1050° C. Limits of Cycle. $\mathbf{H}=\pm 6{\cdot}10$.

14.



Specimen B. Limits of Cycle. $\mathbf{H}=\pm 6{\cdot}83$ C.G.S. Units:
Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.

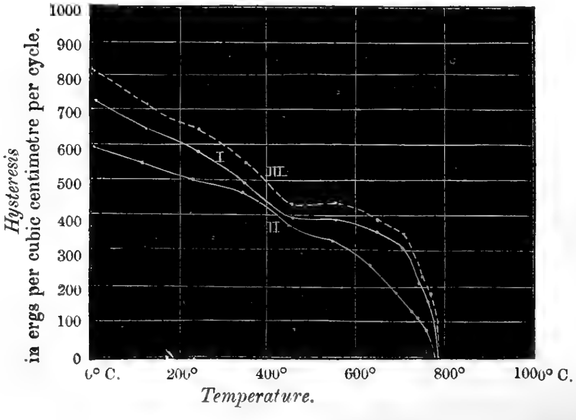
15.



Specimen B.

Limits of Hysteresis Loop.
Curve I. After annealing at 840° C. $\mathbf{B}=\pm 4260$.
„ II. After annealing at 1150° C. $\mathbf{B}=\pm 4550$.
„ III. Calculated from I. for the Limits $\mathbf{B}=\pm 4550$.
(to compare with II.)

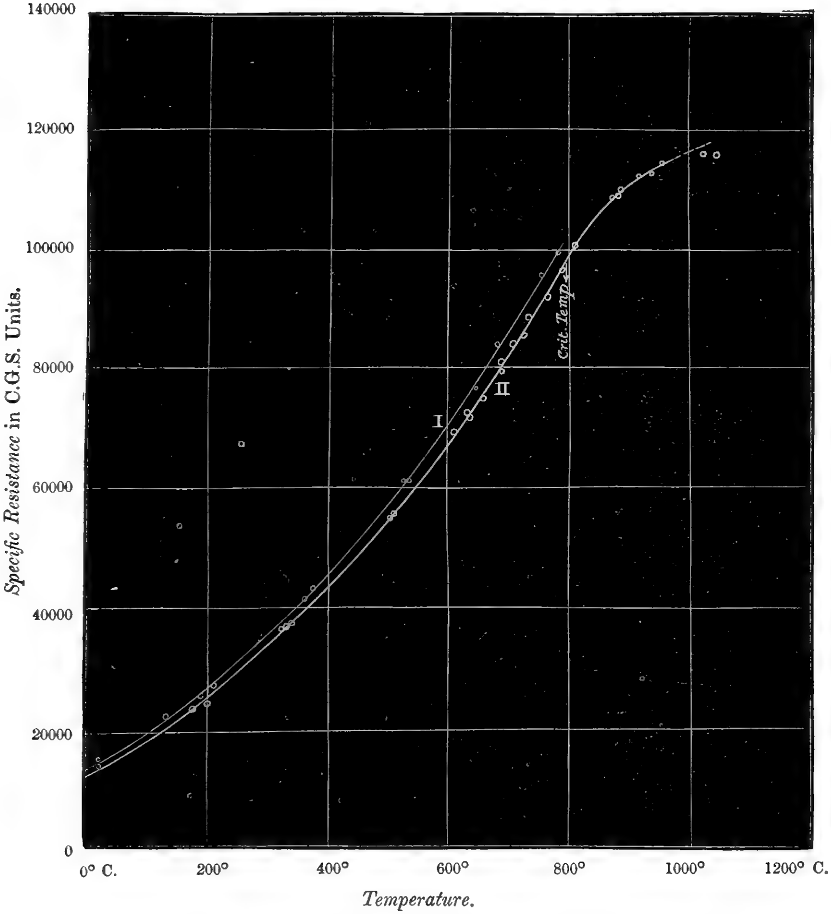
16.



Resistance-Temperature Curves.

17. Iron specimen A. (Crit. temp. 795° C.)

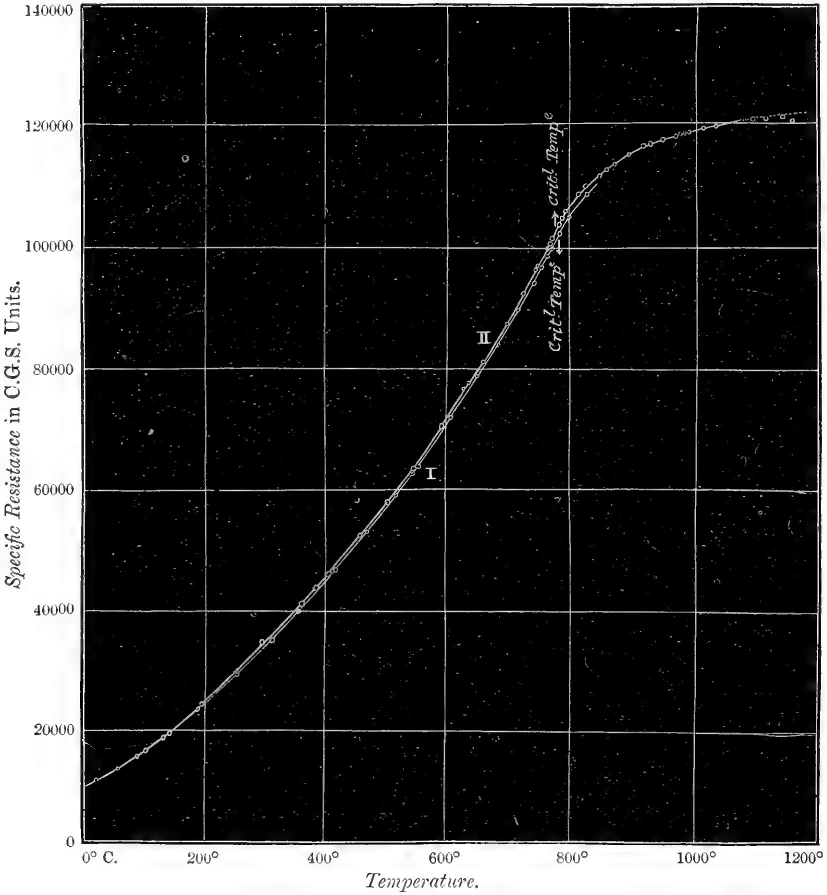
Curve I. Before annealing. Curve II. After annealing at 1050° C.



Resistance-Temperature Curves.

18. Iron specimen B. (Crit. temp. 780° C.)

Curve I. After annealing at 840° C. Curve II. After annealing at 1150° C.



XXXIII. *Doublets and Triplets in the Spectrum produced by External Magnetic Forces.*—(II.)* *By* Dr. P. Zeeman†.

13. *Magnetic Triplet.*—With the grating of § 8 and with a very strong current and rightly-shaped conical poles I succeeded in observing also the pure triplet (§ 5) with cadmium. Again the blue cadmium line was examined which I have used for the other characteristic phenomena. The field used was about 32,000 C.G.S. Using this strong field and looking without a nicol across the lines of force the cadmium line was seen tripled, *i. e.*, broken up into *three* lines, separated by dark spaces. The existence of this triplet demonstrates, as it seems to me, irrefragably, and independently of the examination of the state of polarization, the magnetic nature of the phenomenon.

If now a nicol is placed in the rays with its plane of polarization horizontal, then only the two outer lines of the triplet are seen. A rotation of the nicol over 90° makes appear the central line and quenches completely the light of the outer ones. Hence the central line of the triplet emits plane-polarized light, the plane of polarization being vertical; the outer lines, on the other hand, emit light polarized in a horizontal plane. This result entirely confirms the considerations of § 3 and § 5. The question of § 12 is now answered. New evidence in favour of the interpretation by Lorentz's theory of the magnetization of the spectral lines has been obtained.

14. *Measurement of Magnetic Change.*—The triplet and also the "triplet *a*" of § 6 enable us to measure accurately the magnitude of the magnetic change; on a former occasion‡ I have given only the result of a rough measurement in order to determine the order of magnitude. Using the "triplet *a*" I have now obtained a far more reliable value. Looking across the lines of force and quenching the horizontal vibrations by means of a nicol the vertical vibrations only emerge. Using a grating there are to be seen two separate lines, consisting of vertical vibrations. The distance between the centra of these lines corresponds to the double change of the period. Of course, this distance can be measured far more accurately than the widening of a line. The accuracy of the measurement by means of a micrometer eyepiece is much increased if

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* Continued from page 60. The §§ 13–17 were communicated to the June meeting of the Amsterdam Academy, § 18 is now added.

† Communicated by the Author.

‡ Phil. Mag. March 1897, p. 230.

the grating gives brilliant lines. This quality is possessed by a grating in the possession of the laboratory of the University of Groningen. Its director, Prof. Haga, kindly invited me to make some measurements with his apparatus, which were in full working order. The grating is mounted in a very stable manner, which, of course, is very favourable for accurate measurements.

15. For the particulars of the mounting I refer to Haga's paper (Wied. *Ann.* lvii. p. 389, 1896). The grating (best quality) has a radius of 10 ft. and 10,000 lines to the inch. The source of light used was a piece of asbestos paper soaked with molten salt and introduced into the flame of coal-gas fed with oxygen under high pressure. An image of the sodium flame was formed on the slit by means of a lens. Between the lens and the slit a large nicol was placed; the distance between the slit and the flame being about 50 cm. Care was taken that there were no absorption-lines in the spectrum of the non-magnetized flame.

16. The nicol was placed so that its plane of polarization was horizontal. With the putting on of the current the two lines mentioned in § 14 appear (*cf.* also § 10). The distance between these lines was now measured by means of a micrometer eyepiece. The movable frame carries cross-wires; the cross resembles that of St. Andrew. For spectroscopic measurements this cross is recommended (see *e. g.* Scheiner, *Spectralanalyse der Gestirne*, p. 74). Illumination of the wires was necessary. Now the position of each of the 2×2 lines formed by the D-lines was read. The difference between two readings gives the distance between the centra for D_1 and for D_2 in terms of the divisions of the screw-head (one revolution = 100 divisions). These differences are entered in the following table. The electromagnet soon became very hot by the heat generated by the necessary current (22 amp.) and by the action of the flame. Hence it was impossible to make more than three or four measurements without interrupting the current.

Thirty-eight measurements give for the distance between D_1 and D_2 288 divisions. The probable error of one measurement of the magnetic change is 6·5 divisions for D_1, 4·5 for D_2. The results have the probable errors 1·5 and 1·0 divisions. The magnetic change is the same for the two sodium lines, the difference lying within the limits assigned by the probable errors. The intensity of the field (determined by a bismuth spiral) was 22,400 C.G.S. In this field the positive and negative magnetic change of the period amounts to $\frac{1}{17,800}$. Hence e/m is $1{\cdot}6 \,.\, 10^{-10}$.

Distances between the Centra in divisions of screw-head.

for D_1	for D_2
26	36
18	30
26	32
45	37
25	46
28	36
38	46
42	26
26	33
33	25
35	32
53	28
36	31
51	21
26	34
26	35
25	37
31	25
21	25
Mean $32{\cdot}3 \pm 1{\cdot}5$	$32{\cdot}4 \pm 1{\cdot}0$

On a former occasion I have found by a rough measurement for the magnetic change $\frac{1}{40,000}$, the field being 10,000 C.G.S. Assuming proportionality between change and intensity the present measurements give for the said field $\frac{1}{39,800}$. The close agreement between the result of my rough measurement and the new one is of course the effect of mere chance, for the magnetic change in the case of sodium depends on temperature*, which may have been different in the two cases. Probably it was not very high in the case now considered. The order of magnitude of e/m is entirely the same as the one formerly given.

17. The great brightness of the Groningen grating makes it possible to observe very nicely also with sodium the "triplet *a*," which I described for cadmium. If no nicol is used the exciting of the magnet seems to break up the sodium line into two lines, the phenomenon somewhat resembling the one observed if a nicol is used, but with this difference, that the darker part is not very dark and not narrowed; hence the appearance differs considerably from the one we are accustomed to observe with reversals.

The explanation will be (as was remarked by Prof. Haga

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* Phil. Mag. March 1897, p. 227, §§ 3 and 4.

to the author) that now the three constituents of the triplet partially overlapping one another, the maxima are conspicuous and the inner part appears dark by contrast. Thus there would be no reversal in our case. This really seems to be so, as is confirmed by the following experiment. The above-mentioned hazy dark line being visible a nicol was placed in the beam so as to get rid of light whose plane of polarization is vertical; now only the bright line which emits horizontal vibrations remains visible, but without the slightest trace of a reversal. For this observation it is of course very desirable to use a bright grating. I had no opportunity of obtaining a measurement of the doublet to be seen along the lines of force.

18. A few words may perhaps be said here concerning Prof. Michelson's paper in the July number of the Phil. Mag. Michelson applies his beautiful method of the interferometer to our subject, and there can certainly be but one opinion as to its particular advantages in such cases. Some of his results seem at first sight at variance with mine. Only in one case is there perfect agreement between Michelson's results and mine, viz. when the light is emitted along the lines of force. We both get a doublet in this case. Michelson adds that a broadening is inappreciable. Evidently he means a broadening of the components, which I did not advocate, though it exists in a slight degree in some cases. I referred in my first paper only to a broadening, because I had not yet resolved the doublet or the triplet. It is of course a proof of the superiority of Michelson's method that with relatively weak magnetic forces he could recognize the duplicity, whereas for me it was not so easy.

Looking across the lines of force I have observed a triplet, whereas Michelson also for this case finds a doublet, though less clearly marked than in the other case. This certainly at first sight seems a great difference. I venture, however, to give the following explanation:—

Let us assume, not only that the visibility-curve is practically the same as that due to a doublet, but also that it is undoubtedly proved that a triplet cannot give a practically identical curve in some cases, of which I for myself am not sure at present. Granting that we must assent to a doublet the question arises whether there cannot be assigned another cause for the difference. I think there can. Michelson in making his experiments was yet unacquainted with the particular states of polarization in the triplet. I think that a sufficient reason for the difference mentioned may be found in the perpendicular polarizations of the outer lines and the central one of the triplet (§ 13) or of the central band and the outer edges of triplet *a* (§ 6).

Supposing that the apparatus was arranged in a horizontal plane, then the reflexions (under 45°) from the two plane plates, one unsilvered and one semisilvered with a transparent film of silver, must weaken especially the horizontal vibrations and hence almost annihilate their influence. For unsilvered plates it is easily calculated that the intensity of horizontal vibrations is under the circumstances stated weakened five or six times more than that of the vertical ones. This ratio is of course somewhat changed by the silver film. How much cannot be said without knowledge of further details.

If I have indicated the real cause Michelson has observed a case analogous to my triplet *a*, when a nicol is interposed in such a position as to get rid of the horizontal vibrations. Let us hope that Prof. Michelson will soon give us his opinion.

Amsterdam, July 10.

[*Note added August* 10.]—Prof. Michelson kindly informs me just now that he believes my explanation of the discrepancy in our results to be correct.

XXXIV. *The Oscillatory Discharge of a Large Accumulator.* *By* JOHN TROWBRIDGE[*].

THE discharge from a large number of Planté cells is characterized by a sibilant flame which, by quickly separating the spark-terminals, can be drawn out to a length of several feet. It closely resembles the light produced by passing an electric spark through lycopodium powder. When a photograph of this flaming discharge is examined it is seen to have an intensely bright spark as a nucleus (fig. 1). On account of the flaming discharge it is difficult to examine its character by means of a revolving mirror. By employing, however, two spark-gaps it seemed possible to ascertain whether the discharge is oscillatory or not.

In my experiments the circuit was made at the instant the revolving mirror was in the position to reflect an image of the discharge of the battery upon a sensitive plate. The photographs obtained in this way showed disruptive discharges superimposed upon continuous discharge. The latter, however, masked any appearance of an oscillatory discharge. It was evidently necessary to blow out the flaming discharge in order to see if oscillations followed the pilot discharge. The first experiment was made with 2500 cells arranged in series ; and the flaming discharge was much lessened both by the reduction in the number of cells and by a suitable arrange-

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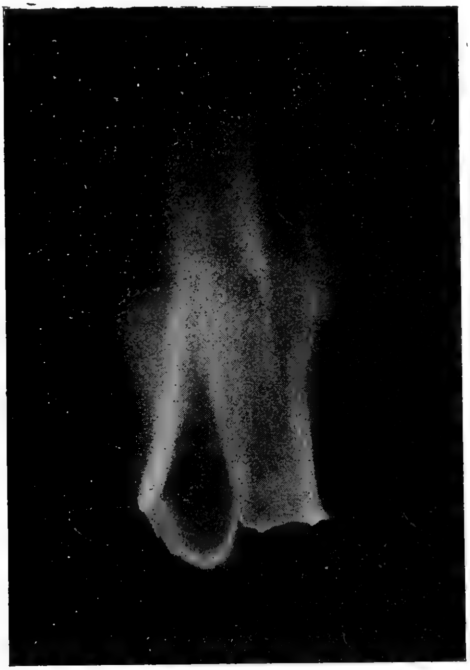
[*] Communicated by the Author.

ment for blowing it out. On developing the photographs it was found that the discharge was an oscillatory one, for as many as five or six clearly defined oscillations followed the first, or pilot discharge. The number of cells was then doubled, and, although more difficulty was experienced with the flaming discharge, oscillations were again obtained.

Fig. 1.



On the supposition that each cell of the battery can be regarded as a leaking condenser, and that it is equivalent in capacity to a condenser shunted by a resistance equal to that of the electrolyte, we can treat such a cell as a conducting condenser under the influence during discharge of a periodic current. The analysis of the well-known case is as follows:—Let ABC and AEC be two circuits*, the current ABC being a shunt to the circuit AEC which contains a condenser E. Let L be the coefficient of self-induction of ABC, R its resistance, C the capacity of the condenser in the current AEC, and r the resistance of the wires leading to the

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* 'Elements of Electricity and Magnetism,' Prof. J. J. Thomson, p. 431.

plates of the condenser. Then if i is the current through ABC, and x the charge on the plate nearest to A,

$$L\frac{di}{dt}+Ri=r\frac{dx}{dt}+\frac{x}{c},$$

since each of the quantities is equal to the electromotive force between A and C.

If $i=\cos pt$, then

$$x=\frac{(L^2p^2+R^2)^{\frac{1}{2}}}{\left(\frac{1}{C^2}+r^2p^2\right)^{\frac{1}{2}}}\sin(pt+A),$$

where

$$A=\tan^{-1}\frac{Lp}{R}+\tan^{-1}\frac{1}{rpc}.$$

Hence

$$-\frac{dx}{dt}=\sqrt{\frac{L^2p^2+R^2}{\frac{1}{C^2p^2}+r^2}}\cos(pt+A).$$

Thus the maximum current along AEC is to that along ABC as $\sqrt{L^2p^2+R^2}$ is to $\sqrt{\frac{1}{C^2p^2}+r^2}$; or if we neglect the resistance r of the leading wires, as $\sqrt{L^2p^2+R^2}:\frac{1}{CP}$; or neglecting L, as $\frac{R}{\frac{1}{Cp}}$.

In the case of one cell of the battery the polarization-capacity is undoubtedly very large. G. M. Gordon* finds that the polarization-capacity of the surfaces of platinum 0·65 cm.^{2}, separated by an interval of 2 mm., amounts to more than 50 microfarads. The cells of my battery consist of lead plates of about 10 cm^2. surface separated by about 6 mm. The layer of peroxide of lead undoubtedly gives a very large polarization-capacity. The resistance of each cell is about one quarter of an ohm. Even with the small value of R, under the effect of rapidly oscillating currents, such as my experiments show arise when the battery discharges through air or gases, a large portion of the oscillating currents pass through the condenser-circuit. Since the electrolyte acts as a semi-insulator, with a very high value of p, no current would pass through the condenser-circuit and the electrolyte, and the cells would therefore act like leyden jars.

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* Wied. *Ann.* No. 5, 1897, p. 28.

In the case I am considering the Planté cells evidently act like leaky leyden jars coupled in series. If C is the apparent capacity of one cell, $\frac{C}{n}$ would be the capacity of n cells.

An examination of the photographs of the oscillations produced by 2500 cells showed an apparent capacity of about 1000 electrostatic units. Five thousand cells gave an apparent capacity of about 500 units. The small apparent capacity C results from the leaking of the condenser due to the conduction through the electrolyte. Since the discharge from an accumulator of a large number of cells is in general oscillatory, I am led to the belief that the discharge from any primary battery is also oscillatory for all cases in which we have to deal with capacity and self-induction. It is evident that a galvanometer or telephone in circuit with a Geissler tube cannot detect the oscillatory discharge, since it is of high period. Moreover, when a Geissler tube is lighted by a large battery with no resistance save that of the tube and the battery in the circuit, and the light is examined in a revolving mirror by the eye, no oscillation or intermittence of light can be perceived on account of the flaming discharge through the rarefied gas.

The oscillatory discharge may be said to be the common occurrence in nature in the case of electrical discharges, and the one-direction discharge the uncommon. This has been expressed by the remark that electricity takes the path of least resistance, this common belief must be modified, however, under certain conditions of resonance; but in general nature avoids a unidirectional discharge.

Jefferson Physical Laboratory,
Harvard University,
Cambridge, Mass., U.S.

XXXV. *A Method of determining Magnetic Hysteresis loss in Straight Iron Strips.* By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London**.

[Plate V.]

THE exact determination of the magnetic hysteresis loss in sheet iron as used for alternating current transformer construction has become of commercial importance of late years. Owing to the obvious necessity for reducing the core loss in transformers to the lowest attainable limit, it is now the custom to carefully test the iron to be used and to express

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* Communicated by the Physical Society: read June 11th, 1897.

the determined hysteresis loss in the iron sheet, in watts per pound per 100 frequency, at a maximum induction during the cycle of 2500 C.G.S. units. Two processes are in ordinary use for effecting this measurement. The most accurate but most tedious is by the employment of a ballistic galvanometer and the graphical delineation of a number of (B, H,) cycles. This method has the advantage that it can be conducted with either small or large samples of iron, but the reduction of the observations to give the required hysteresis loss is very tedious; and including the construction of the necessary ring-coil, the actual observations, and the reductions, a single determination of the hysteresis value of an iron can hardly be carried out, even by a couple of industrious workers, in less than a day.

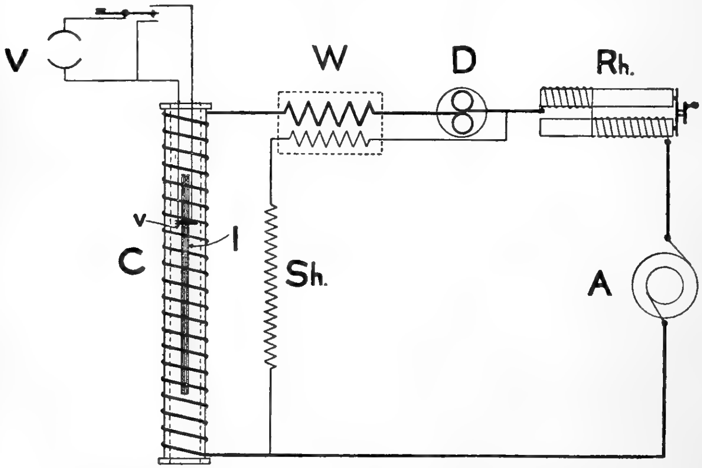
The method employed by Professor Ewing and Mr. F. Holden of rotating a laminated specimen of the iron in a magnetic field, or rotating a field round the specimen and comparing the torque produced on the field-magnet or on the specimen with that produced by a similar sample of iron of known hysteretic value or else measuring it by a calibrated torsion spring, is very much more rapid. This last method is, however, open to the great objection that if used merely to compare a sample of iron with a standard sample, the user of the instrument is at the mercy of his instrument-maker. He receives with it one or more samples of iron which he is told have certain hysteretic values, but he cannot check the truth of the statement. Moreover, if he happens mechanically to injure the standard samples he may seriously change their hysteresis constant. The writer therefore sought for a method which should combine the advantages of both the above processes whilst eliminating their disadvantages, and which should be an absolute method, like the ballistic method, but at the same time a rapid method like the rotating-field method.

The following paper contains a description of a process which is based upon the use of the bifilar reflecting wattmeter and operates upon samples of iron, large or small, in the form of straight strips which are simply slipped into a long magnetizing coil. All the actual practical work and the numerous calculations involved in putting this process to test have been conducted under the author's direction by Mr. A. R. Peart and Mr. W. M. Park working in the Electrical Laboratory of University College, London; and to these gentlemen belongs the full credit for all this portion of the research as well as for the labour required in much tedious numerical and graphical work.

The method is based upon an interesting discovery with regard to the distribution of the induction in a bundle of straight iron strips or iron or steel wires, when placed in a uniform magnetic field.

If a long, straight, magnetizing coil is prepared by winding insulated wire upon a paper or vulcanized fibre tube, and if the length of the solenoid is, say, ten times its mean diameter, then for a distance equal to at least three-quarters of the total length of the coil we have a practically uniform magnetic field in the interior of the bobbin, when a constant current is sent through the wire. If into this uniform magnetic field a bar of iron or straight bundle of iron strips or iron wire is introduced, the length of the iron being not greater than the uniform portion of the interior field of the coil when the iron is not there, this iron has magnetic induction created in it which varies in density from point to point in the bar, but is symmetrical in value about the central point of the bar.

The induction density (B) is greatest in the centre of the bar but falls off towards each end. If the magnetizing coil is traversed by an alternating current, and if the iron is laminated, the maximum value of the induction density in the bar can be easily determined at any point in its length by placing upon the bar a flat bobbin of fine insulated wire which is connected with an electrostatic voltmeter capable of reading from 1 to 20 volts (see diagram). The first portion of this



C. *Magnetizing coil.* I. *Iron strips.* *v. Exploring coil.*
V. *Electrostatic voltmeter.* W. *Wattmeter.* D. *Dynamometer.*
Rh. *Rheostat.* A. *Alternator.*

investigation was concerned with the study of the mode in which this induction density varies from point to point in the length of a bundle of iron strips or iron wire or steel rods of various lengths and diameters, thus placed in the centre of a long magnetizing solenoid.

A magnetizing coil was prepared, wound on a paper tube, 5 7 centimetres in internal diameter and 99 centimetres long. On this tube was wound, in three layers, a coil of 1235 turns of cotton-covered copper wire, ·064 inch diameter or of No. 16 size S.W.G. The resistance of this coil was 1·069 ohms at 15° C.

The number of turns per centimetre length of this coil was 12·47. Hence since the interior central field is $4\pi/10$ times the ampere turns per unit of length of the coil, we have thus the value of the field H in the central regions of the interior of this bobbin given by the equation

$$H = 15{\cdot}66\,A,$$

where A is the ampere current through the coil.

The secondary or exploring coil consisted of a flat bobbin wound up of silk-covered copper wire ·0024 inch in diameter and having 2000 turns ; the thickness or length of the bobbin being 0·7 centimetre and the external diameter about 4·5 centimetres.

The exploring coil was wound on a circular ebonite former having a rectangular hole in it so as to closely fit the bundle of iron strips to be used, but at the same time it could be slid along the bundle.

The voltmeter used with it was one of Professor Ayrton's low-reading electrostatic instruments of the idiostatic type and read from 1 to 25 volts, the reading being taken by a mirror and scale. The voltmeter was calibrated when required by means of standard cells or a potentiometer.

The first experiments were conducted with strips of transformer iron, kindly furnished to us by Messrs. Sankey. This sheet iron was ·0121 of an inch or ·0307 of a centimetre in thickness and the strips were each one inch or 2·54 centimetres wide. The strips used were of various lengths : 60 centimetres, 30 centimetres, 15 centimetres, down to 5 centimetres. A bundle I (see diagram) of a certain number of strips was made and the secondary coil *v* slipped on to it. The magnetizing coil C was then traversed by an alternating current having a frequency of 80 ∼, and the R.M.S. (root mean square) value of the current was read by a dynamometer D or Kelvin balance.

The electrostatic voltmeter V being connected to the ends of the secondary coil gave the R.M.S. value of the induced electromotive force set up in the exploring coil. The form of the curve of the primary current was very nearly a true sine curve, and the variation of induction density in the iron at any point closely followed the same law.

In any case, if f is the *form factor* * of the secondary voltage curve, which in these experiments was 1·13, and if

N = the number of windings in the secondary coil,
S = the total cross-section of the iron,
B = the maximum value of the induction density in the iron in C.G.S. units at that point in its length at which the secondary coil is placed, and
n = the frequency of the induction cycle ; then if e_2 is the R.M.S. value of the secondary electromotive force in volts as measured by the voltmeter, we have always

$$10^8 e_2 = 4f\text{NSB}n.$$

Hence the maximum value of the induction density B can be calculated from the known value of all the other quantities. In this manner a large number of measurements of the value of the induction density B were made at different points in the lengths of various bundles of iron strips of different lengths and total cross-sections.

These values were then laid down in a series of induction density curves so drawn that the ordinates represented the maximum value of the induction density in the bar at different transverse sections along the half-length of the bar, taken as corresponding abscissæ.

It has been shown by Steinmetz and fully confirmed by others that for most varieties of iron used for electrotechnical purposes the hysteresis loss in the iron, per cycle of induction, per cubic centimetre varies as a power of the maximum value of the induction density, which is very nearly 1·6, when that maximum value of the induction density has any value below about 9000 C.G.S. units. Hence if the ordinates of the above-mentioned induction curves are all raised to the 1·6th power and a new curve plotted showing the variation of $\text{B}^{1\cdot6}$ along the half-length of the bar, we have another curve which represents the variation of the hysteresis loss per unit of volume along the half-length of the bar from point to point. A large series of such curves showing the variation of B and $\text{B}^{1\cdot6}$ were drawn for bundles of iron strips and steel

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* See 'The Alternate Current Transformer,' Fleming, vol. i., new. ed., p. 583, for a reference to this term.

of different lengths and sections, and the results are collected and represented in the form of curves in the diagrams in Plate V.

The upper curves in each case in the figs. 1, 2, and 3 in Plate V. represent the variation of B along the half-length of the bar, and the lower curves the corresponding curves for the variation of $B^{1\cdot6}$. The first pair of diagrams represent the result of varying the maximum induction density in the centre of a bundle of strips of iron of the same length and section.

The second pair of diagrams give the curves showing the same results for bundles of iron strips of varying lengths but the same section.

The third pair for various samples of iron strips of the same length but different section.

It is clear from these diagrams therefore that such a bar or bundle of iron strips placed in an originally uniform magnetic field has a non-uniform induction density produced in it at its various points, which is symmetrical about the centre ; and a non-uniform hysteresis loss per unit of volume also symmetrical about the centre. The maximum values of each being at the centre of the bar.

If we take the true mean value of the varying distributed hysteresis loss per unit of volume, it is obvious that at some point in the half-length of the bar there must exist an actual induction density B_1 such that the true mean hysteresis loss in the whole bar is proportional to ${B_1}^{1\cdot6}$. Let this value of the induction density be called the *effective value*, and the point in the bar at which it occurs the *effective point*. Then the effective value B_1 must be equal to the 1·6th root of the mean of the 1·6th powers of the actual induction densities taken at equidistant points all along the bar, or

$$B_1 = \sqrt[1\cdot6]{m\,.\,B^{1\cdot6}}$$

where $m\,.\,B^{1\cdot6}$ stands for the mean ordinate of the curve representing the varying values of $B^{1\cdot6}$ all along the half-length of the bar.

If then this effective value B_1 of the induction density is calculated for each of the samples of iron strip used, and the position at which the actual induction has this effective value is marked off on the half-length of the bar (this position is represented by the *cross* on the curves given in Plate V.), we have noticed the remarkable fact that *whatever the length or section of the iron strip may be, the point in the bar at which the actual induction density has a value equal to the effective value, always comes at the same proportional distance from the centre* *of the bar ; and this distance is very nearly equal to ·56 of the half-length of the bar from the centre or at ·22 of the whole length of the bar from one end.*

The following Table I. embodies all the results taken with bundles of annealed transformer iron strip (Sankey's) of very different lengths and sections, and Table II. similar observations on rods of steel and hard iron ; and it will be seen that the effective induction in the case of the iron strip is always found at a point in the bar or bundle which is very nearly 0·56 of the half-length of the bar from the centre. If therefore the secondary coil is placed at that spot and the secondary voltage then observed used to calculate the induction density, the value so obtained is that which corresponds to the true mean value of the varying hysteresis loss per unit of volume all along the bar. In other words, if we suppose the bar magnetized by an alternating current uniformly and with a maximum induction density everywhere equal to that which it actually has under normal conditions at 0 56 of the half-length of the bar from the centre ; then the hysteresis loss in such a bar would be equal to that found in the bar as it is. In all cases where induction density values are given, these are to be understood as being the maximum values during the cycle unless otherwise stated. The frequency of alternations used was 80.

We have therefore in this useful fact regarding the distribution of induction in iron strips in a uniform magnetic field, the foundation of a quick method of determining the hysteretic constant of a sample of iron.

If the iron sample, taken in strips, is placed as above in the interior of the long magnetizing coil, and if a sensitive wattmeter, suitable for use with alternating currents, is connected therewith, we can measure the total power taken up in the iron and in the copper of the coils when any known alternating current is passing through the magnetizing coil. If then the iron strips are withdrawn and another measurement of the same kind made with the current passing through the magnetizing coil adjusted to the same value as in the first case, this last reading gives us the wattmeter-reading corresponding to a known power taken up in the copper of the magnetizing coils. Hence the first wattmeter-reading gives us the iron loss, together with the copper loss, and the second reading gives us the copper loss alone. The iron loss gives us the true mean value of the hysteresis loss in the iron, assuming the iron is so well laminated that eddy-current loss is practically absent.

Table I.

Summary of experiments made to ascertain the "Effective point" in bundles of annealed Transformer Iron Strips (Sankey's) when placed in a uniform alternating magnetic field.

Reference letter to diagram.	Length of specimen in centims.	Sectional area of specimen in sq. cms.	Induction at centre of bar, B. C.G.S. units.	Effective induction, B_1. C.G.S. units.	Ratio of $\frac{B_1}{B}$.	Distance of effective point from centre of bar.	Fraction of half-length of bar from centre at which effective induction exists.
A_1	61	·936	564	420	·744	17·01	·558
A_2	61	·936	1648	1240	·752	16·91	·554
A_3	61	·936	2084	1563	·750	16·92	·555
A_4	61	·936	2536	1900	·749	17·00	·557
A_5	61	·936	3112	2335	·712	17·00	·557
A_6	61	·936	3816	2870	·752	17·10	·561
B_1	30	·936	1597	1192	·746	8·36	·557
B_2	30	·936	4052	3012	·743	8·35	·557
B_3	15	·936	1622	1265	·780	4·11	·548
B_4	15	·936	3430	2660	·775	4·18	·557
B_5	10	·936	1907	1554	·815	2·73	·546
B_6	10	·936	1025	833	·813	2·76	·552
B_7	5	·936	1622	1420	·875	1·39	·556
B_8	5	·936	778	678	·871	1·49	·596
C_1	61	·312	3485	2565	·736	16·84	·552
C_2	61	·312	9673	7250	·749	16·81	·551
C_3	61	·624	2125	1570	·739	17·07	·560
C_4	61	·624	5834	4420	·758	17·60	·577

Mean of last column = ·56.

TABLE II.

Summary of tests made as in Table I. upon Steel and Hard Iron Rods.

Reference letter to diagram.	Length of specimen in centims.	Sectional area of specimen in sq. cms.	Induction in centre of bar, B. C.G.S. units.	Effective induction, B_1. C.G.S. units.	Ratio of $\frac{B_1}{B}$.	Distance of effective point from centre of bar in cms.	Fraction of half-length of bar from centre at which effective induction exists.	Nature of bar.
C_6	61	·970	4499	3525	·783	6·90	·58	Round bar of silver steel $\frac{7}{16}$ inch in diameter.
D_2	32	·178	5997	4784	·798	9·80	·61	Round rod of silver steel $\frac{3}{16}$ inch in diameter.
D_3	32	·534	5555	4244	·764	9·31	·58	Three similar pieces of steel.
D_4	20·45	·645	4746	3587	·756	6·67	·65	Bundle of 60 iron wires of hard iron of rectangular section.
D_5	20·45	·645	2545	1912	·751	6·76	·66	

The above-mentioned law as to the distribution of the induction enables us, then, to assign the corresponding value of the maximum induction density.

Unless this can be done, the mere mean hysteresis loss measurement is no practical use *.

The operation of making the mean hysteresis measurement is effected by a sensitive mirror bifilar electrodynamometer.

This was constructed in the following way:—

The fixed or series-coil consists of 90 turns of No. 16 S.W.G. cotton-covered copper wire wound in five layers on a rectangular block of wood. The movable coil, or shunt-coil, consists of a rectangular coil of 20 turns of No. 24 S.W.G. wire, the turns being fastened together with silk and shellac. This movable coil embraces, without touching, the fixed coil, as in a Siemens dynamometer. The movable coil is suspended by a bifilar suspension consisting of two silver wires ·002 of an inch in diameter and 37 centim. long, fixed about 3 millim. apart. These wires are attached to the ends of the suspended coil, and serve to convey the shunt-current in and out of the coil. The movable coil carries a concave silver mirror by means of which a sharp image of an incandescent lamp-filament is formed upon a scale at a fixed distance of 100 centim. from the mirror. A torsion-head carries the upper ends of the bifilar suspension, and serves to adjust the zero.

In series with the shunt-coil is a high non-inductive resistance of 1000 ohms.

The resistance of the thick-wire fixed coil of the wattmeter is ·133 of an ohm, and the resistance of the shunt-coil and suspension, without the associated inductionless resistance, is 11·7 ohms.

When such a bifilar wattmeter is employed to measure electrical power, it can be shown that the deflexions of the movable coil, when not large, are strictly proportional to the power passing through the wattmeter, so that each scale-division has an assignable power value, which in our case was almost exactly ·01 of a watt per millimetre of scale. The arrangement of circuits was then as shown in the diagram on p. 264. The long magnetizing coil above described was connected in series with the fixed coil of the wattmeter, and also with a sensitive electrodynamometer or ammeter suitable for

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* Methods of measuring hysteresis loss by one observation of an instrument have been suggested by other observers; but the practical value of these methods has been very small, because the investigators have not shown how the corresponding maximum value of the effective induction is to be obtained. The rule above given supplies this deficiency.

measuring small alternating currents of about an ampere or less in value.

The shunt-coil of the wattmeter was joined across the ends of the long coil. Suitable resistances beyond controlled the current.

The process of measurement was then as follows:—A bundle of straight iron strips was fitted with the secondary coil at the effective point, viz. ·56 of the half-length of the bar from the centre. The bundle was slipped into the magnetizing coil, and an alternating current sent through the coil of such a magnitude (generally about ·2 of an ampere) as to give an induction density of not more than about 6000 C. G. S. units at the effective point. The wattmeter deflexion was then read. To interpret this reading the iron strips were removed from the coil and another wattmeter measurement made on the copper coil alone. As the removal of the iron sends up the current strength by decreasing the inductance of the circuit, it is necessary to readjust the current to the same value as before, and to know that value in amperes. The resistance of the coil being known, and also the current passing through it, we can calculate the power being taken up in the copper coil alone for this standard current; and hence know the power in watts producing the observed wattmeter deflexion.

The wattmeter is therefore calibrated on the copper circuit of the magnetizing coil, and from the known copper loss in this coil and the proportionality of its deflexions we are enabled to assign at once the value in watts of its deflexion when used to measure both the iron loss and copper loss together when the iron strip is inserted in the coil.

If the induction is carried up as far as 4000 or 5000 C.G.S. units at the effective point, and if the strips have a thickness over ·01 inch, then it becomes necessary to apply a small correction for the eddy-current loss set up in the iron. This can be at once done by using the following simple practical formula given by the author * for this loss in rectangular sectioned strips.

If t is the thickness of the strip in *mils* (1 mil = ·001 inch), n is the frequency of the current, and B is the maximum value of the induction density during the cycle at any point on the bar, then it can be shown that the eddy-current loss in watts per cubic centimetre at that point in the rectangular

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* See 'The Alternate Current Transformer,' Fleming, vol. ii. p. 490, for proof of this formula.

or sectional strip is closely given by the expression :—

$$\left(\frac{\mathrm{B}}{10^8}nt\right)^2,$$

and hence for a given frequency and thickness the eddy-current loss per pound of iron is very quickly found from a simple parabolic curve representing B^2 in terms of B.

In all the above observations with the iron strip used, which was 12 mils thick, this correction has been applied, and it is sufficient for this purpose to take as B in the above formula the value B_1 of the effective induction.

The eddy-current loss varies from point to point in the bar, being always proportional to the square of the maximum value of the induction density. Hence, properly speaking, the mean value of the eddy-current loss should be obtained by taking the mean square value of the induction all along the half-length of the bar, and using the mean-square value instead of the B^2 in the formula above.

We have, however, found that the point on the half-length of the bar at which the actual induction has a value equal to the root-mean-square value, or to the square root of the mean of the squares of the distributed induction, is always very close to the point at which the 1·6th root of the mean of the 1·6th powers of the distributed induction is found. In other words, the induction at the effective point is the induction corresponding to the mean eddy-current loss as well as to the mean hysteresis loss. This is shown by the curves given in figs. 4 and 5, Plate V., in which the curve marked B is the curve of variation of the maximum value of the induction-density along the half-length of a bundle of iron strips, and the curves marked $\mathrm{B}^{1\cdot 6}$ and B^2 represent (to different scales) the variation of these powers of B. In a bar 5 centim. long (see fig. 5, Plate V.) the points where the actual induction had a value equal to $\sqrt{m\,.\,\mathrm{B}^2}$ and to $\sqrt[1\cdot 6]{m\,.\,\mathrm{B}^{1\cdot 6}}$ were only separated by a fraction of a millimetre. Hence by another curious coincidence we are enabled to apply the proper correction for the eddy-current loss by taking the square of the effective induction B_1 and putting its value into the expression

$$\frac{\mathrm{B}_1{}^2 n^2 t^2}{10^{16}},$$

where n is the frequency and t the thickness of the strip in mils.

In order to put the method to a test, a very careful determination was made by the ballistic method of the hysteresis loss of a sample of the transformer iron strip used. A ring-coil was prepared, and a series of (B-H) cycles of magnetization taken, and a hysteresis loop drawn, and from this loop it was found that for this iron the hysteresis loss was ·25 watts per pound per 100 cycles per second, for a maximum value of the induction density equal to 2500 C.G.S. units.

Straight strips of exactly the same iron were then tested by the method here described, which for shortness will be called the "straight coil" method, and the results of these tests with the iron in different conditions of hardness are shown in the Tables III., IV., and V. (see end).

It will be seen that the results of the "straight coil" test with the wattmeter, as given in Table III., gave exactly the same mean value for the hysteresis loss of the strip when not bent and unbent, as did the calculations made from the ballistic test, viz.: ·25 watts per pound per 100$\sim$ per maximum induction density of 2500. The difference in the time of taking the tests is, however, immense. A very few minutes suffices to take the wattmeter tests, and the necessary calculations can be made in a comparatively short time.

Many dozens of tests of irons can be made by the straight coil method in the time occupied by one ballistic test.

There is another method by which the bifilar wattmeter can be calibrated, but which is not so generally convenient as that above described. If the shunt and series coils of the wattmeter are joined up in series with each other, so as to form a dynamometer, and if a known current I is passed through these coils, causing a deflexion θ in the movable coil; then when the instrument is used as a wattmeter with a total shunt-resistance R (including the added non-inductive resistance) the true power corresponding to a deflexion θ is I^2R watts.

Hence, from a curve showing the deflexion given by various currents, we can at once determine the constant of the instrument, when used as a wattmeter, with given shunt resistance.

If this standardization is taken with continuous currents, then it is necessary to eliminate the effect of the Earth's field by taking right and left deflexions. The curve connecting square of current strength and the deflexions of the movable coil is nearly a straight line, in the case of the bifilar wattmeter.

Standardizing our wattmeter in this way with a continuous current, we then made another set of hysteresis measurements, with 12 strips of perfectly new annealed transformer strips (Sankey's), with the results shown in Table VI. Two other sets of observations for hammered and hardened iron are given in Tables VII. and VIII.

As a matter of fact, in the above experiments in the observations given in Tables III., IV., V., VII., and VIII., the wattmeter was standardized by sending known and measured currents through the copper coil and obtaining a wattmeter constant by finding the deflexions corresponding to known copper losses in the copper coil. Thus are found the following figures, when using the wattmeter to measure power taken up in the magnetizing coil alone :—

Known copper loss (C^2R) calculated from observed current and coil resistance in watts.	Scale-deflexions of wattmeter in millimetres.	Watts per millimetre of scale-deflexion of wattmeter.
·398	39·1	·0102
·292	28·9	·0101
·197	19·0	·0103

In the case of the observations given in Table VI., we employed the method of determining the wattmeter constant by passing known currents through its coils joined up in series and used as a dynamometer, and from the curve showing relation of $(\text{current})^2$ to deflexions we could at any moment find the wattmeter constant.

The method described above has this great advantage that the test can be made on fairly large samples of iron, and it can be made with an alternating current having any desired and practical form factor. The iron is tested under conditions which are exactly similar to those under which it will be used in transformer manufacture, and, in fact, the straight magnetizing coil and iron sample constitute an open circuit transformer of very small power factor.

The above described method effects a great saving of time in comparison with the ballistic method, whilst at the same time it is an absolute method and does not depend upon comparisons with other iron samples of supposed known hysteretic value.

Table III.

Hysteresis Test by "Straight Coil" method of annealed Transformer Iron Strip. Six strips used (new) 61 cm. long, 2·54 cm. wide, and 12 mils thick.

Voltmeter scale-reading in mm.	Effective Induction density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in.	Iron out.					
303·0	6719	87·6	17·5	1·247	·245	1·002	·257	·25 watts per pound per 100 ∼per 2500 B.
225·3	5702	66·0	15·3	·9027	·176	·727	·243	
131·8	4284	40·0	7·8	·5747	·099	·476	·251	
95·1	3623	43·0	8·1	·4377	·071	·367	·251	

TABLE IV.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strip. Same strips used as in Table III., but once bent and unbent.

Voltmeter scale-reading in mm.	Effective Induction density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in Coil.	Iron out of Coil.					
56·9	2786	19·0	4·2	·303	·042	·261	·273	·28 watts per pound per 100 ∼ per 2500 B.
128·8	4244	39·0	8·4	·635	·098	·537	·288	
204·0	5424	58·1	12·4	·948	·160	·788	·285	
273·0	6346	76·0	15·9	1·247	·219	1·028	·289	

Table V.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strip, the same quality as in Tables III. and IV., but bent and unbent several times. Twelve strips used 30 cm. long, 12 mils thick, cut from the same pieces used for the ballistic galvanometer test.

Voltmeter scale-reading in mm.	Effective Induction density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 800∼ and 2500 B.	Mean hysteresis value.
		Iron in coil.	Iron out of coil.					
374·5	3778	135·7	105·1	·539	·078	·461	·296	·28 watts per lb. per 100∼ per 2500 B.
281·4	3219	101·4	78·1	·415	·057	·358	·297	
199·5	2672	71·2	54·8	·277	·038	·239	·268	
111·6	1964	42·1	32·9	·160	·0215	·139	·252	
71·1	1625	57·0	43·1	·125	·017	·107	·282	
50·4	1314	39·2	29·9	·083	·011	·072	·261	

TABLE VI.

Hysteresis Test by "Straight Coil" method of Transformer Iron Strips. Twelve new annealed strips used, never bent. Wattmeter standardized by Potentiometer and Continuous Currents.

Voltmeter scale-reading in mm.	Effective Induction-density, B_1.	Wattmeter scale-reading.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and 2500 B.	Mean hysteresis value.
		Iron in.	Iron out.					
497·0	4742	90·6	26·0	·517	·118	·399	·178	·21 watts per pound per 100 ∼ per 2500 B.
338·8	3576	63·6	17 1	·340	·069	·271	·190	
267·0	3136	50·7	14·7	·295	·053	·242	·210	
204·9	2715	39·8	11·7	·232	·038	·193	·210	
130·8	2138	51·6	14·6	·154	·025	·129	·208	
110·3	1957	44·0	11·9	·133	·022	·111	·205	
77·1	1628	31·0	8·7	·093	·016	·070	·191	
49·5	1295	21·1	6·1	·062	·010	·052	·187	
398·5	3913	71·0	17·5	·466	·083	·383	·234	
282·0	3223	100·9	25·1	·334	·056	·278	·230	
200·5	2679	73·2	18·1	·242	·039	·203	·226	
100·0	1950	42·9	11·0	·118	·021	·097	·181	
77·2	1627	61·4	16·1	·103	·016	·087	·217	
49·2	1302	41·2	11·7	·067	·010	·057	·203	

N.B. The method of standardizing the wattmeter by use as a dynamometer employing a continuous current as above does not give such consistent or accurate hysteresis determinations as when it is standardized on a known copper resistance with known alternating currents.

TABLE VII.

Hysteresis Test by " Straight Coil " method on Hammered Transformer Strip (unannealed). Six strips used 61 centim. long, 2·54 centim. wide, and 12 mils thick.

Voltmeter scale-reading in mm.	Effective induction-density, B_1.	Wattmeter scale-readings.		Total iron loss in watts per lb.	Eddy-current loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and B 2500.	Mean hysteresis value.
		Iron in.	Iron out.					
114·0	3975	54·5	23·5	·562	·086	·476	·284	·290 watts per pound per 100 cycles at B 2500.
160·0	4737	72·9	30·2	·778	·123	·655	·297	
206·0	5438	90·5	37·5	·969	·161	·802	·290	

Calibration Table of Wattmeter.

C^2R.	Wattmeter deflexion.	Watts per millim.
·1875	20·9	·00897
·3263	36·9	·00884
·1195	13·3	·00898

TABLE VIII.

Hysteresis Test by "Straight Coil" method of 18 strips of Iron of inferior quality. Length 61 cms., width ·63 centim., thickness 26·5 mils.

Voltmeter scale-reading in mm.	Effective induction-density, B_1.	Wattmeter scale-reading in mm.		Total iron loss in watts per lb.	Eddy loss in watts per lb.	Hysteresis loss in watts per lb. at 80∼.	Hysteresis loss in watts per lb. at 100∼ and B 2500.
		Iron in.	Iron out.				
83·4	2054	51·9	23·7	·319	·110	·209	·360
136·0	2644	79·7	34·1	·517	·183	·334	·360
176·0	3028	101·0	42·8	·659	·239	·420	·386

Calibration Table of Wattmeter.

C^2R.	Wattmeter deflexion in mm.	Watts per mm.
·272	30·0	·00908
·381	41·4	·00920

In converting hysteresis losses from one maximum induction-density B to another by the aid of Steinmetz's law, it is useful to have the arithmetic values, for different values of B, of the ratios of $B^{1\cdot6}$ to $2500^{1\cdot6}$; since B=2500 C.G.S. units is generally taken as the standard induction-density in reference to hysteresis determinations. The following table (Table IX.) gives these ratios with sufficient accuracy for all practical work. Thus, for instance, if the value of the hysteresis loss in an iron has been determined at a maximum induction-density during the cycle of 5000 units, and we wish to know the value at 2500 units, the observed value has to be divided by 3·031 to obtain the required result.

TABLE IX.

Showing the Values of the Ratio of $B^{1\cdot6}$ to $2500^{1\cdot6}$.

B.	$\frac{B^{1\cdot6}}{2500^{1\cdot6}}$.	B.	$\frac{B^{1\cdot6}}{2500^{1\cdot6}}$.
1000	·231	5500	3·531
1500	·442	6000	4·058
2000	·700	6500	4·614
2500	1·000	7000	5·193
3000	1·338	7500	5·795
3500	1·713	8000	6·430
4000	2·118	8500	7·085
4500	2·563	9000	7·764
5000	3·031	9500	8·466
		10000	9·190

XXXVI. *On an Optical Device for the Intensification of Photographic Pictures.* *By* Lord RAYLEIGH, *F.R.S.**

WHETHER from insufficient exposure or from other causes, it not unfrequently happens that a photographic negative is deficient in density, the ratio of light-transmissions for the transparent and opaque parts being too low for effective contrast. In many cases an adequate

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* Communicated by the Author.

remedy is found in chemical processes of intensification, but modern gelatine plates do not always lend themselves well to this treatment.

The method now proposed may be described as one of using the negative twice over. Many years ago a pleasing style of portrait was current dependent upon a similar principle. A thin positive transparency is developed upon a collodion plate by acid pyrogallol. Viewed in the ordinary way by holding up to the light, the picture is altogether too faint; but when the film side is placed in contact with paper and the combination viewed by reflected light, the contrast is sufficient. Through the transparent parts the paper is seen with but little loss of brilliancy, while the opaque parts act, as it were, twice over, once before the light reaches the paper, and again after reflexion on its way to the eye. For this purpose it is necessary that the deposit, constituting the more opaque parts of the picture, be of such a nature as not itself to reflect light back to the eye in appreciable degree—a condition very far from being satisfied by ordinary gelatine negatives. But by a modification of the process the objection may be met without much difficulty.

To obtain an intensified copy (positive) of a feeble negative, a small source of illumination, *e. g.* a candle, is employed, and it is placed just alongside of the copying-lens. The white paper is replaced by a flat polished reflector, and the film side of the negative is brought into close contact with it. On the other side of the negative and pretty close to it is a field, or condensing, lens of such power that the light from the candle is made parallel by it. After reflexion the light again traverses the lens and forms an image of the candle centred upon the photographic copying-lens. The condenser must be large enough to include the picture and must be free from dirt and scratches; otherwise it does not need to be of good optical quality. If the positive is to preserve the original scale, the focal length of the condenser must be about twice that of the copying-lens.

In carrying this method into execution there are two points which require special attention. The first is the elimination of false light reflected from the optical surfaces employed. As regards the condensing-lens, the difficulty is easily met by giving it a moderate slope. But the light reflected from the glass face of the negative to be copied is less easily dealt with. If allowed to remain, it gives a uniform illumination over the whole field, which in many cases would go far to neutralize the advantages otherwise obtainable by the method. The difficulty arises from the parallelism of the two surfaces of the negative, and is obviated by using for the support of the film a glass whose faces are inclined. The false light can then be thrown to one side and rendered inoperative. In practice it suffices to bring into contact with the negative (taken as usual upon a parallel plate) a wedge-shaped glass of equal or greater area, the reflexion from the adjoining faces being almost destroyed by the interposition of a layer of turpentine. By these devices the false light is practically eliminated, and none reaches the sensitive film but what has twice traversed the original negative.

The other point requiring attention is to secure adequate superposition of the negative and its image in the associated reflector. On account of the slight lateral interval between the copying-lens and the source of light, the incidence of the rays upon the reflector is not accurately perpendicular, and thus any imperfection of contact between the negative film and the reflector leads to a displacement prejudicial to definition. The linear displacement is evidently $2t \sin \theta$, if t denote the interval between the surfaces and θ the angle of incidence, and it can be calculated in any particular case. It is the necessity for a small t that imposes the use of a speculum as a reflector. In practice 2θ can easily be reduced to $\frac{1}{12}$; so that if t were $\frac{1}{50}$ inch, the displacement would not exceed $\frac{1}{600}$ inch, and for most purposes might be disregarded[*]. The obliquity θ could be got rid of altogether by introducing the light with the aid of a parallel glass reflector placed at 45°; but this complication is hardly to be recommended.

The scale of the apparatus depends, of course, upon the size of the negatives to be copied. In my own experiments $\frac{1}{4}$-plates ($4\frac{1}{4}$ in. × $3\frac{1}{4}$ in.) were employed. The condenser is of plate-glass 6 in. diameter and 36 in. focus. The reflector is of silver deposited on glass[†]. The wedge-shaped glass[‡] attached to the negative with turpentine is 4×4 ins. and the angle between the faces is 2°. The photographic lens is of 3 inch aperture and about 18 inch principal focus. It stands at about 36 inches from the negative to be copied.

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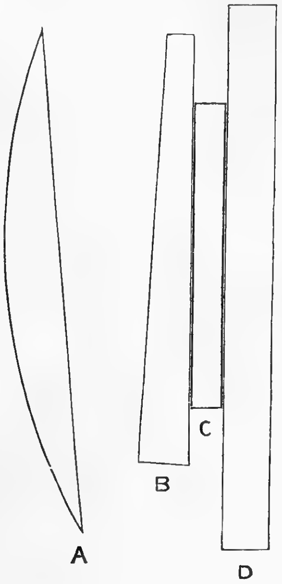
[*] If the glass of the negative were flat, its approximation to the reflector might be much closer than is here supposed.

[†] For a systematic use of the method a reflector of speculum metal would probably be preferable.

[‡] It is one of those employed for a similar purpose in the projection of Newton's rings (Proc. Roy. Inst. March, 1893; 'Nature,' vol. xlviii. p. 212).

The accompanying sketch shows the disposition of some of the parts. It represents a section by a horizontal plane. A is the condensing-lens, B the wedge, C the negative temporarily cemented to B by fluid turpentine, D the speculum.



XXXVII. *Electrical Discharges in Air.*

By JOHN TROWBRIDGE[*].

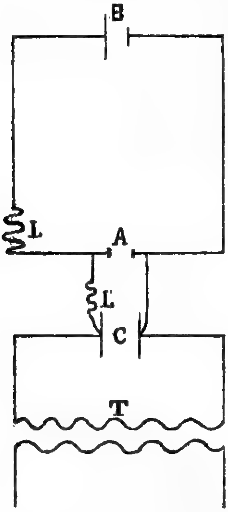
THE flaming discharge from a large accumulator with its nucleus consisting of a dazzling white spark is evidently a form of voltaic arc; and I was interested to discover, if possible, the mechanism, so to speak, of the voltaic arc. Does it follow Ohm's law in respect to resistance, and is there an oscillatory phenomenon? It is well known that electric sparks can be greatly increased in length by interposing a gas-flame between the terminals of a Ruhmkorf coil or by moderately rarefying the air between such terminals. The conditions in the voltaic arc favour a greatly increased length of a disruptive spark between the positive and negative carbons. This can be seen in the photograph of the arc produced by a high-tension accumulator: and doubtless the

[*] Communicated by the Author.

same phenomenon could be observed in the ordinary voltaic arc if it were not so exceedingly brilliant.

I have lately studied the apparent resistance of the voltaic arc in the following manner. In the circuit B (fig. 1) of forty large storage-coils giving 80 volts, was placed a low-resistance choking-coil L, or coil of large self-induction. To the carbon terminals A between which the voltaic arc was produced were led terminals of a condenser C. The latter was charged by a step-up transformer T. The oscillatory discharge of the condenser was thus passed through the voltaic arc: and a spark in a gap in the circuit of the condenser was photographed by the aid of a revolving mirror. The photograph gave the number of oscillations in the circuit containing the arc and the condenser. A curve was then plotted with the number of oscillations as ordinates and the ohmic resistance of the circuit as abscissas. It was thus found that the apparent resistance of the voltaic arc was equivalent, in the case I considered, to a resistance of eight tenths of an ohm (·8 ohm). It was found, moreover, that an arc one quarter of an inch long did not give more resistance than one one half an inch long. The apparent resistance, therefore, of the voltaic arc does not follow Ohm's law. I am led to believe that the mechanism, so to speak, of the voltaic arc is as follows:—A disruptive discharge accompanies a flaming discharge, and serves as a species of pilot-spark. A variable

Fig. 1.



difference of potential is necessary to sustain the disruptive discharge; and this variable difference of potential makes itself evident as an apparent change of resistance. The arc shortens or lengthens in obedience to the mechanism of the lamp which is employed.

A family resemblance may be said to exist between all forms of electrical discharges in air; thus in the voltaic arc we have a disruptive discharge combined with a flaming discharge. In general, this disruptive spark is oscillatory, even in the case where the voltaic arc is produced by a dynamo-machine. When we extend our studies to the forms of electrical discharges which are free, to a great extent, from the flaming discharge, such as the disruptive sparks from electrical machines, Tesla and Thomson transformers, and the Planté rheostatic machine, we are struck by their close resemblance to the ordinary forms of lightning-discharge. I have lately employed in connexion with five thousand Planté cells a Planté machine with thirty condenser-plates made of glass one sixteenth of an inch in thickness, with a coated surface of 15 × 18 inches. Sparks 9 to 10 inches long can be very conveniently studied by means of this apparatus; for a close estimate of the difference of potential is possible, and the spark-terminals do not change their sign during the experiments. To the eye each spark seems to be surrounded by a bright radiance or aureole of which it appears to be the nucleus. In order to ascertain whether this radiance was an actual phenomenon, I employed a portrait-lens of large aperture; and some of the results are exhibited in the accompanying reproductions, which fail, however, to give the details of the negative. Fig. 2 is a photograph of a spark taken with a euryscope-lens, such as is commonly employed for landscape-work; this does not show any detail. Figs. 3, 4, and 5 are photographs taken with a Dallmeyer portrait-lens without a diaphragm, and show, on the negatives, what may be considered an aureole accompanying the spark its entire length; furthermore, the oscillatory nature of the sparks is shown by forked discharges which diverge from the main path of the spark, and which point in opposite directions on the same spark. If a photograph of lightning could be obtained which would show a similar phenomenon, there could be no doubt of the oscillatory nature of ordinary lightning-discharges.

Since one can with a large number of Planté cells in connexion with a rheostatic machine control the sign of the electric charges on the spark-terminals, I was interested to test the question whether the eye can detect any direction in electric sparks. One observer looking through an opening which concealed the spark-terminals, and only revealed the central portion of the sparks, noted down his impression of

Fig. 2.

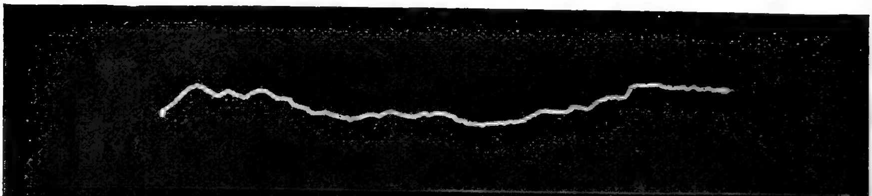


Fig. 3.

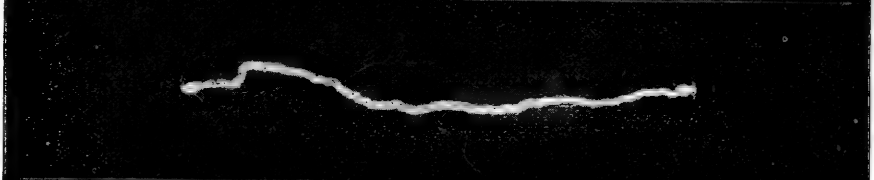


Fig. 4.

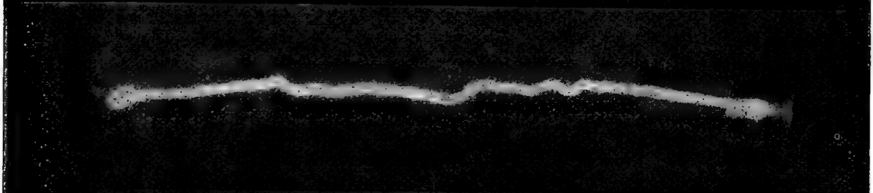
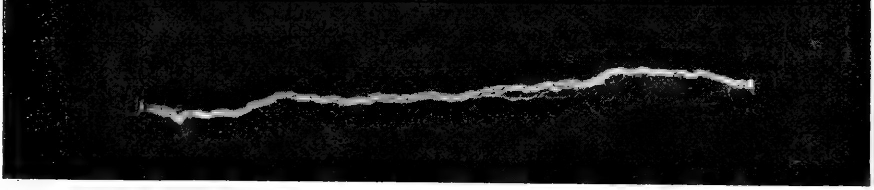


Fig. 5.



the apparent direction of each spark while another observer reversed the poles of the battery which charged the rheostatic machine. On comparing the notes of the observers it was found that there was no agreement in regard to direction. This result was to be expected from the oscillatory nature of the discharges. It may be that in the case of lightning the eye is forcibly impressed by the greater brightness of the positive terminal in the cloud, and the observer concludes that the flash has a unidirectional movement. The phenomenon is psychological rather than physical.

When oscillatory sparks of the nature represented in figs. 3, 4, and 5 are passed through Crookes tubes of the focus-tube pattern, it was found that the photograph could be taken on plates exposed to the inclined surface of the platinum, both when it was made the anode and when it formed the kathode. No difference in definition could be noticed; there was, however, a great actinic difference. Under the oscillatory nature of the leyden-jar discharge the electrodes become alternately positive and negative. Possibly some of the want of definition noticed in Röntgen photographs taken even with the aid of electrical machines may be due to the fact that the oscillatory discharge does not always emanate from the same point on the anode surface. A small anode should therefore give sharper images than one of a large surface.

What is supposed to be a resistance in the case of the voltaic arc, and in the modification of this as seen in discharges from high-tension transformers and in powerful electric sparks, and presumably in lightning-discharges, is a polarization which produces a variable difference of potential at the spark-terminals.

The inconstancy of spark potentials has been shown by Jaumann*. In working with a revolving mirror it is found that the spark-terminals have to be brightened in order to preserve the same spark-length. I was interested also to observe the effect of the surrounding medium upon the spark potentials. Platinum terminals in sodium vapour showed the polarizable condition; but there did not appear to be an appreciable change in resistance apart from this polarization. The same was true when bromine vapour surrounded the spark-terminals. In the case of Crookes tubes it is customary to apply heat if the discharge will not pass through the tube: and some makers provide a connecting receptacle filled with a substance which, on being volatilized, modifies the internal conditions of the tube. This modification is often spoken of as a diminution of resistance of the tube. It should be more properly termed a method of modifying the state of polarization of the electrodes.

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Harvard University,
Cambridge, Mass., U.S.

XXXVIII. *On some New Lines in the Spark-Spectrum of Aluminium.* *By* Gustav A. Hemsalech†.

WHEN doing some spectroscopic work a short time ago I observed some lines in the red, due to aluminium, which I believe have not been recorded before. With a small

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* Wied. *Ann.* 1895, lv. p. 656.

† Communicated by Arthur Schuster, F.R.S.

coil and a small leyden jar only two of them are visible, and they appear as short lines; but when using a large coil giving a 4 to 10-inch spark and a large jar these lines appear long and bright, and are easily seen. One essential point is that the electrodes should be thick and rounded off at the ends.

The wave-lengths of the two brighter lines were measured in two ways, viz. by a concave Rowland grating (25,000 lines per inch) and by the ordinary prism-spectroscope. With the grating the two lines appeared rather weak, and were difficult to measure even in the first-order spectrum. The reference-lines I used were those of potassium, of wave-lengths 6911·2 and 6938·8 (Kaiser and Runge), which are obtained in the arc. The relative distances were measured by means of an eyepiece-micrometer, which was fitted with a pointer. Eight sets of measurements were made, and the wave-lengths calculated by simple proportion. The values obtained were:—

$$\left.\begin{matrix} 7042{\cdot}3, \\ 7056{\cdot}7. \end{matrix}\right\}$$

With a prism of ordinary flint glass having an angle of 60° the aluminium-lines appeared very bright and sharp. For reference I used lithium 6708·1 (Rowland), and the two potassium-lines 7665·6 and 7699·3 (Eder and Valenta). Some of the iron-lines would probably have been more convenient for the purpose, but there was no arc-light available at the time. Greater dispersion than that obtained by one prism could not be employed very well on account of the great distance between the Li and K lines, and therefore the values for the wave-lengths cannot be considered very accurate, but they are near enough for identification. Ten sets of micrometer-readings of the apparent distances were taken on three different days, which agree fairly well with each other. From the mean values of these readings the wave-lengths were determined by the method of inverse squares and were found to be as follows:—

$$\left.\begin{matrix} 7043{\cdot}0 \\ 7058{\cdot}8 \end{matrix}\right\} \text{ with reference to Li } 6708{\cdot}1 \text{ and K } 7665{\cdot}6.$$

$$\left.\begin{matrix} 7042{\cdot}2 \\ 7058{\cdot}3 \end{matrix}\right\} \text{ with reference to Li } 6708{\cdot}1 \text{ and K } 7699{\cdot}3.$$

The value for the first—the brighter one—agrees very well with that obtained by the grating.

For the next three lines a dense prism of Jena glass was employed, and in place of the potassium-lines the newly determined aluminium-lines were used for reference together with Li 6708·1. The greater dispersion—which is more convenient in observing close double lines—made this necessary, since the K lines were not in the field of view at the same time as the Li line. The mean values taken for the wave-lengths of the Al lines used for reference were 7042·5 and 7057·9 respectively. Five sets of measurements were made and the wave-lengths obtained as before. They were found to be :—

6828·3	with reference to Li 6708·1 and Al 7042·5,
6842·6	
6928·3	
6828·2	with reference to Li 6708·1 and Al 7057·9.
6842·5	
6928·2	

The following table gives the mean values for the wave-lengths of the aluminium-lines and their relative intensities approximately, putting the intensity of the lithium-line as it appeared in the same field of view as 10.

Wave-length.	Relative intensity and character.	Oscillation-frequency reduced to vacuo.
6828·2	4 sharp.	14641
6842·6	4 sharp.	14610
6928·2	2 nebulous.	14430
7042·5	6 sharp.	14195
7057·9	5 sharp.	14165

The line 7057·9 is probably double, the two components being very near together, but no accurate measurements were obtained. A large Ruhmkorff-coil by Apps, giving a 10-inch spark, was used in these determinations, also a large leyden jar.

Before concluding I must express my thanks to Professor Schuster for his valuable suggestions and the interest he has taken in the work. My thanks are also due to Mr. A. T. Stanton for the kind assistance he has rendered in working with the grating, and for many useful hints.

The Owens College,
July 15, 1897.

XXXIX. *Intelligence and Miscellaneous Articles.*

OBSERVATIONS ON LATIMER CLARK'S STANDARD CELL WHEN CLOSED. BY THEODOR WULF.

THE Author observes the tension at the terminals of closed normal elements by connecting up a condenser, and then discharging it through a galvanometer. By using Hicke's apparatus these observations can be made for very short intervals after closing the current. The following are the principal results which he obtains:—

1. The internal resistance of the elements used (cylindrical glass vessels 2 cm. in diameter, distance between the electrodes 3–4 cm.) varied at ordinary temperatures between 40 and 80 ohms according to the distance and magnitude of the electrodes.

2. The resistance varied considerably with the temperature, being trebled when it fell from 30° to 5°.

3. No diminution of the internal resistance with the strength and duration of the current could be ascertained.

4. With stronger currents of 0·005 ampere (external resistance 200 ohms) polarization could be perceived even after 0·01 of a second. It increased but little during the first ten minutes, but afterwards more and more rapidly.

5. On opening the circuit the polarization diminished in a few minutes to a small fraction. Repeated short-circuiting through only 50 ohms' external resistance for more than a minute did not permanently injure the element.

6. When an element was closed for 2 hours through 200 ohms, by which its electromagnetic force was reduced almost to one half, in 2 minutes it attained its original strength to within 0·1 p.c. But after this the polarization set in more rapidly and strongly.—*Wiener Berichte*, July 1897.

SPECIFIC HEAT OF DIFFICULTLY FUSIBLE METALS.

BY H. MACHE.

Since the introduction of Moissan's method of reducing difficultly fusible metals by means of the electrical furnace it is possible to obtain these in a compact solid form, while previously they could only be had in powder. By this it is possible to obtain the specific heats with greater accuracy. By the aid of Bunsen's ice-calorimeter, palladium, chromium, vanadium, and tungsten were determined and the specific heats obtained: Pd=0·0549, Cr=0·1208, V=0·1153, Wo=0·0336.—*Wiener Berichte*, July 1897.

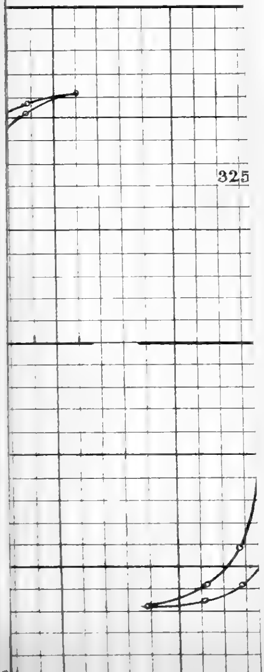
DETERMINATION OF THE COEFFICIENT OF MAGNETIZATION OF LIQUIDS AND THEIR DEPENDENCE ON TEMPERATURE.

BY PROF. DR. GUSTAV JÄGER AND DR. STEFAN MEYER.

The method is that of measuring the ascent in narrow tubes in the magnetic field which was produced by the large electromagnet of the Physical Institute. By taking special precautions considerably greater accuracy could be obtained than heretofore; the position of the meniscus was kept constant by increasing the pressure of a large volume of air. Solutions of magnetic salts in water were principally investigated.—*Wiener Berichte*, July 1897.

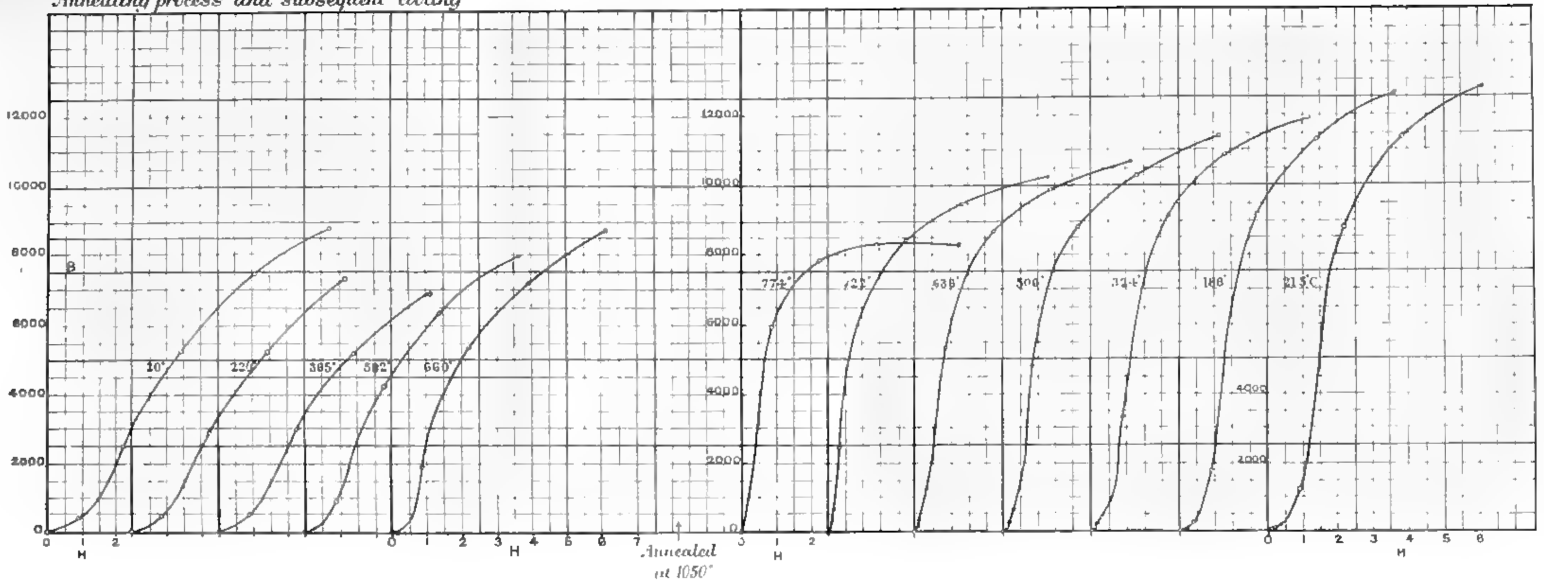
C OBSERVATI

Cr



Curves No. 1. B-H Curves by the method of reversals

Annealing process and subsequent cooling

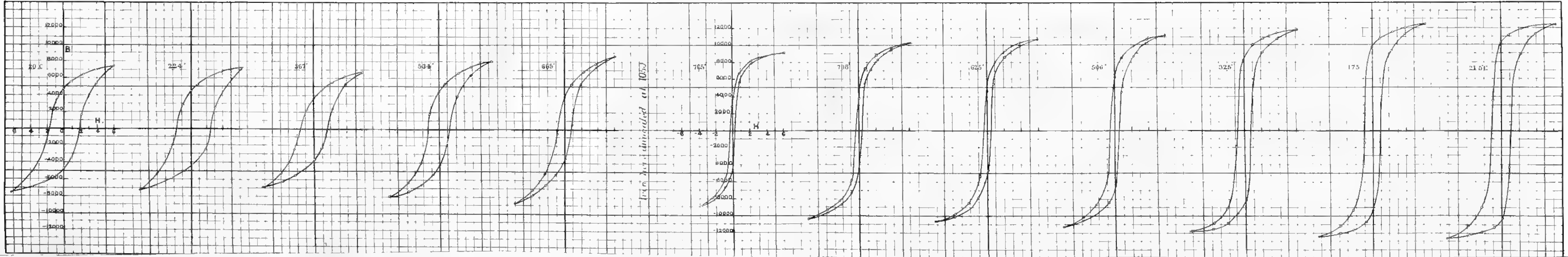


MAGNETIC OBSERVATIONS WITH IRON SPECIMEN A.

Crit. Temp. 795°

Curves No. 2 *Hysteresis Loops*

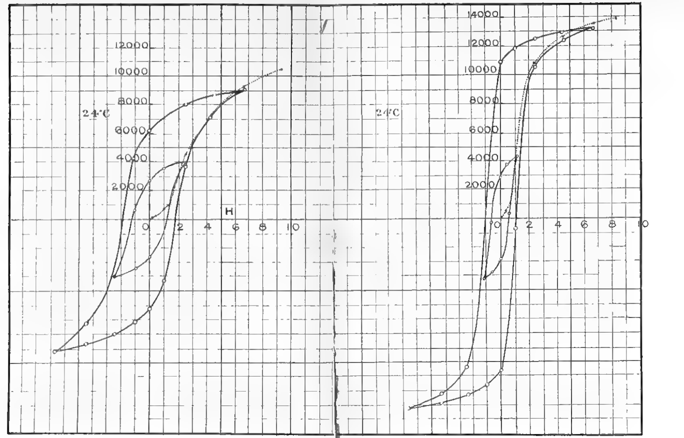
Annealing Process and subsequent cooling



MAGNETIC OBSERVATIONS WITH IRON SPECIMEN B. *(Crit. Temp. 780°.)*

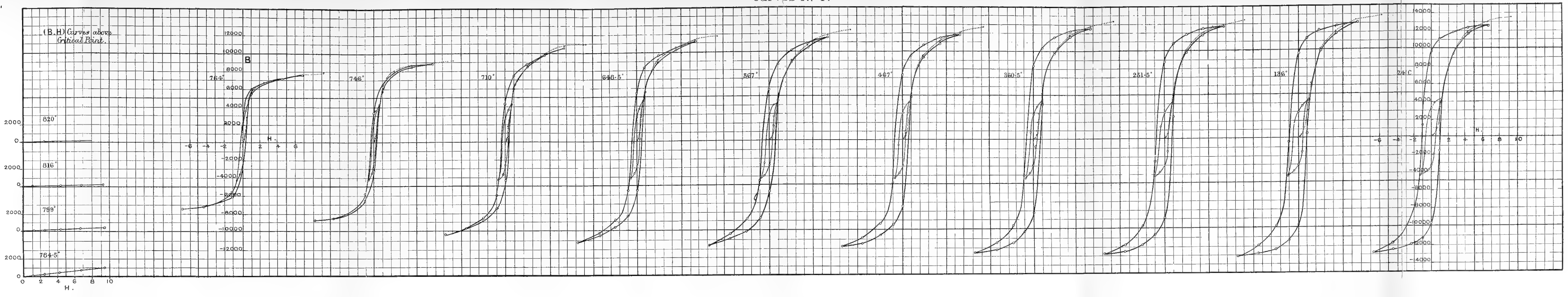
Curves No 3.

Preliminary Measurements before and after Annealing at 840°.



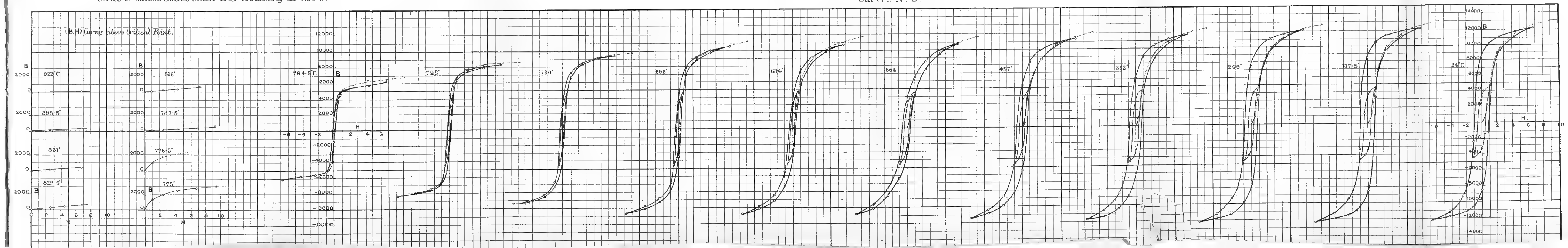
Series taken after annealing at 840°.

Curves No 4.



Series of measurements taken after annealing at 1150°C.

Curves No 5.



Explanation

The larger Hysteresis Loops are all taken exactly within the Limits H = ± 6·83 *CGS Units.*

The smaller loops are within the limits of Induction B = ± 4260 *lines for the 1st series and* B = ± 4550 *lines for the 2nd series*

The dotted curves are the B-H *curves taken by the Method of Reversals.*

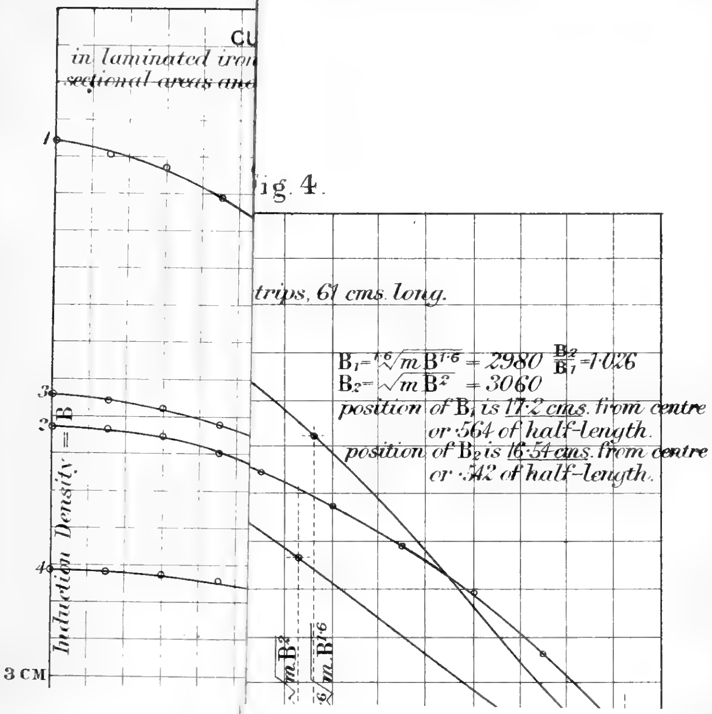

CU
in laminated iron
sectional areas and
trips, 61 cms. long.
B₁ = ¹·⁶√m B¹·⁶ = 2980 B₂/B₁ = 1·026
B₂ = √m B² = 3060
position of B₁ is 17·2 cms. from centre
or ·564 of half-length.
position of B₂ is 16·54 cms. from centre
or ·542 of half-length.
1
3
2
4
Induction Density = B
3 CM
√m B²
¹·⁶√m B¹·⁶

Fig. 4.

Fig. 6.

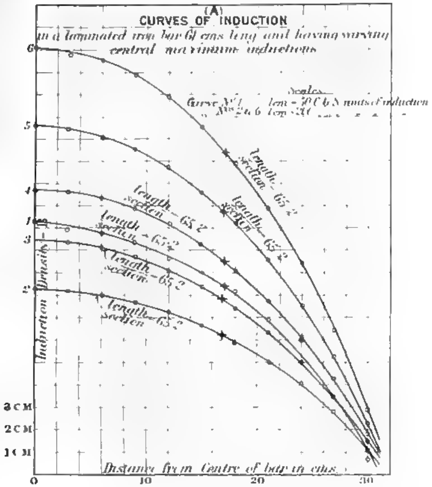


Fig. 7.

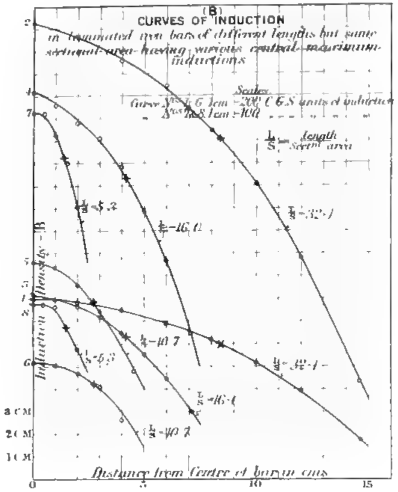


Fig. 8.

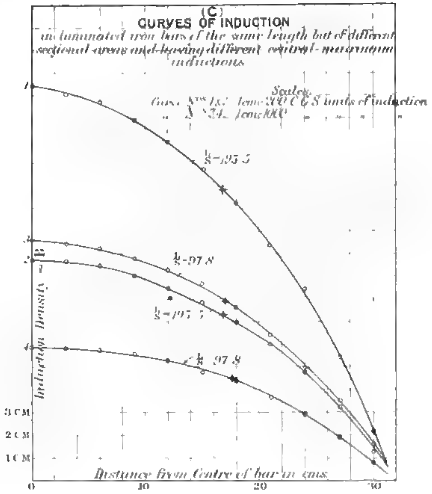
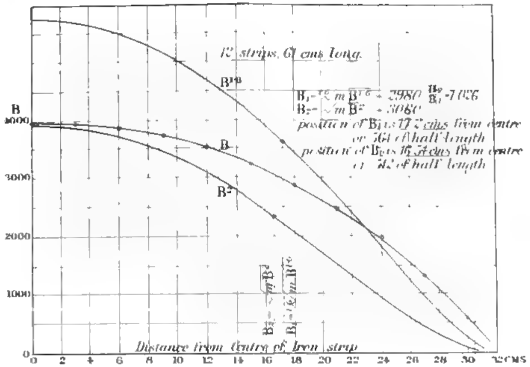


Fig. 4



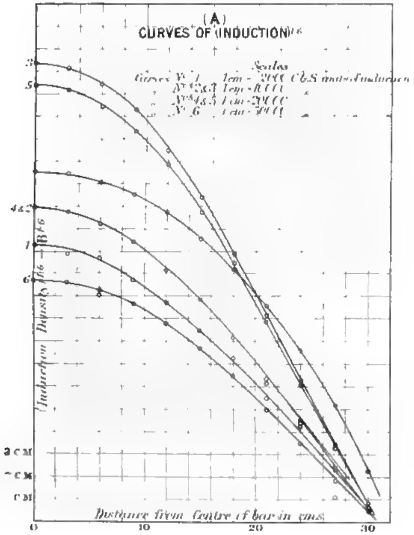


Fig. 1.

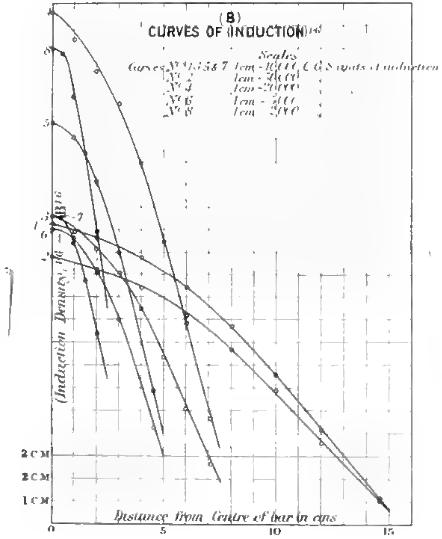


Fig. 2.

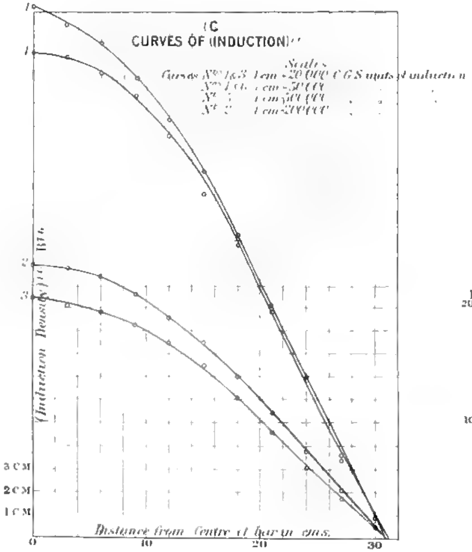


Fig. 3.

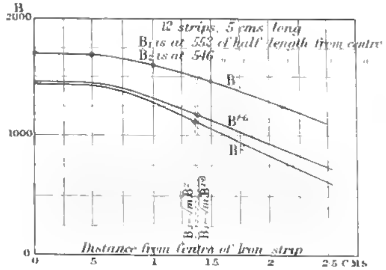


Fig. 5.

Mintern Bros lith.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[FIFTH SERIES.]

OCTOBER 1897.

XL. *Cathode Rays.* *By* J. J. THOMSON, *M.A.*, *F.R.S.*, *Cavendish Professor of Experimental Physics, Cambridge*[*].

THE experiments[†] discussed in this paper were undertaken in the hope of gaining some information as to the nature of the Cathode Rays. The most diverse opinions are held as to these rays; according to the almost unanimous opinion of German physicists they are due to some process in the æther to which—inasmuch as in a uniform magnetic field their course is circular and not rectilinear—no phenomenon hitherto observed is analogous: another view of these rays is that, so far from being wholly ætherial, they are in fact wholly material, and that they mark the paths of particles of matter charged with negative electricity. It would seem at first sight that it ought not to be difficult to discriminate between views so different, yet experience shows that this is not the case, as amongst the physicists who have most deeply studied the subject can be found supporters of either theory.

The electrified-particle theory has for purposes of research a great advantage over the ætherial theory, since it is definite and its consequences can be predicted; with the ætherial theory it is impossible to predict what will happen under any given circumstances, as on this theory we are dealing with hitherto

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[*] Communicated by the Author.

[†] Some of these experiments have already been described in a paper read before the Cambridge Philosophical Society (Proceedings, vol. ix. 1897), and in a Friday Evening Discourse at the Royal Institution ('Electrician,' May 21, 1897).

unobserved phenomena in the æther, of whose laws we are ignorant.

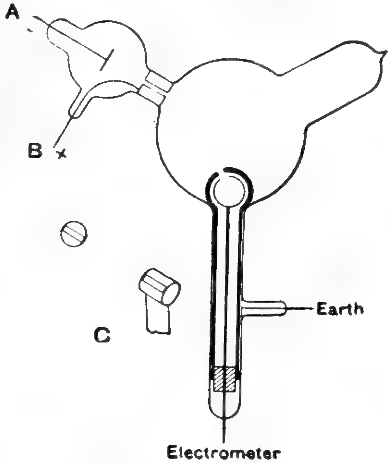
The following experiments were made to test some of the consequences of the electrified-particle theory.

Charge carried by the Cathode Rays.

If these rays are negatively electrified particles, then when they enter an enclosure they ought to carry into it a charge of negative electricity. This has been proved to be the case by Perrin, who placed in front of a plane cathode two coaxial metallic cylinders which were insulated from each other: the outer of these cylinders was connected with the earth, the inner with a gold-leaf electroscope. These cylinders were closed except for two small holes, one in each cylinder, placed so that the cathode rays could pass through them into the inside of the inner cylinder. Perrin found that when the rays passed into the inner cylinder the electroscope received a charge of negative electricity, while no charge went to the electroscope when the rays were deflected by a magnet so as no longer to pass through the hole.

This experiment proves that something charged with negative electricity is shot off from the cathode, travelling at right angles to it, and that this something is deflected by a magnet; it is open, however, to the objection that it does not prove that the cause of the electrification in the electroscope has anything to do with the cathode rays. Now the supporters of the ætherial theory do not deny that electrified particles are shot off from the cathode; they deny, however, that these charged particles have any more to do with the cathode rays than a rifle-ball has with the flash when a rifle is fired. I have therefore repeated Perrin's experiment in a form which is not open to this objection. The arrangement used was as follows:—Two coaxial cylinders (fig. 1) with slits in them are placed in a bulb connected with the discharge-tube; the cathode rays from the cathode A pass into the bulb through a slit in a metal plug fitted into the neck of the tube; this plug is connected with the anode and is put to earth. The cathode rays thus do not fall upon the cylinders unless they are deflected by a magnet. The outer cylinder is connected with the earth, the inner with the electrometer. When the cathode rays (whose path was traced by the phosphorescence on the glass) did not fall on the slit, the electrical charge sent to the electrometer when the induction-coil producing the rays was set in action was small and irregular; when, however, the rays were bent by a magnet so as to fall on the slit there was a large charge of negative electricity sent to the electrometer. I was surprised at the magnitude of the charge; on some occasions enough negative electricity went through the narrow slit into the inner cylinder in one second to alter the potential of a capacity of 1·5 microfarads by 20 volts. If the rays were so much bent by the magnet that they overshot the slits in the cylinder, the charge passing into the cylinder fell again to a very small fraction of its value when the aim was true. Thus this experiment shows that however we twist and deflect the cathode rays by magnetic forces, the negative electrification follows the same path as the rays, and that this negative electrification is indissolubly connected with the cathode rays.

Fig. 1.



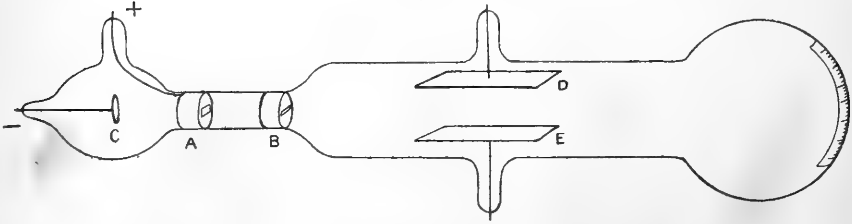
When the rays are turned by the magnet so as to pass through the slit into the inner cylinder, the deflexion of the electrometer connected with this cylinder increases up to a certain value, and then remains stationary although the rays continue to pour into the cylinder. This is due to the fact that the gas in the bulb becomes a conductor of electricity when the cathode rays pass through it, and thus, though the inner cylinder is perfectly insulated when the rays are not passing, yet as soon as the rays pass through the bulb the air between the inner cylinder and the outer one becomes a conductor, and the electricity escapes from the inner cylinder to the earth. Thus the charge within the inner cylinder does not go on continually increasing; the cylinder settles down into a state of equilibrium in which the rate at which it gains negative electricity from the rays is equal to the rate at which it loses it by conduction through the air. If the inner cylinder has initially a positive charge it rapidly loses that charge and acquires a negative one; while if the initial charge is a negative one, the cylinder will leak if the initial negative potential is numerically greater than the equilibrium value.

Deflexion of the Cathode Rays by an Electrostatic Field.

An objection very generally urged against the view that the cathode rays are negatively electrified particles, is that hitherto no deflexion of the rays has been observed under a small electrostatic force, and though the rays are deflected when they pass near electrodes connected with sources of large differences of potential, such as induction-coils or electrical machines, the deflexion in this case is regarded by the supporters of the ætherial theory as due to the discharge passing between the electrodes, and not primarily to the electrostatic field. Hertz made the rays travel between two parallel plates of metal placed inside the discharge-tube, but found that they were not deflected when the plates were connected with a battery of storage-cells; on repeating this experiment I at first got the same result, but subsequent experiments showed that the absence of deflexion is due to the conductivity conferred on the rarefied gas by the cathode rays. On measuring this conductivity it was found that it diminished very rapidly as the exhaustion increased; it seemed then that on trying Hertz's experiment at very high exhaustions there might be a chance of detecting the deflexion of the cathode rays by an electrostatic force.

The apparatus used is represented in fig. 2.

Fig. 2.



The rays from the cathode C pass through a slit in the anode A, which is a metal plug fitting tightly into the tube and connected with the earth; after passing through a second slit in another earth-connected metal plug B, they travel between two parallel aluminium plates about 5 cm. long by 2 broad and at a distance of 1·5 cm. apart; they then fall on the end of the tube and produce a narrow well-defined phosphorescent patch. A scale pasted on the outside of the tube serves to measure the deflexion of this patch.

At high exhaustions the rays were deflected when the two aluminium plates were connected with the terminals of a battery of small storage-cells; the rays were depressed when the upper plate was connected with the negative pole of the battery, the lower with the positive, and raised when the upper plate was connected with the positive, the lower with the negative pole. The deflexion was proportional to the difference of potential between the plates, and I could detect the deflexion when the potential-difference was as small as two volts. It was only when the vacuum was a good one that the deflexion took place, but that the absence of deflexion is due to the conductivity of the medium is shown by what takes place when the vacuum has just arrived at the stage at which the deflexion begins. At this stage there is a deflexion of the rays when the plates are first connected with the terminals of the battery, but if this connexion is maintained the patch of phosphorescence gradually creeps back to its undeflected position. This is just what would happen if the space between the plates were a conductor, though a very bad one, for then the positive and negative ions between the plates would slowly diffuse, until the positive plate became coated with negative ions, the negative plate with positive ones; thus the electric intensity between the plates would vanish and the cathode rays be free from electrostatic force. Another illustration of this is afforded by what happens when the pressure is low enough to show the deflexion and a large difference of potential, say 200 volts, is established between the plates; under these circumstances there is a large deflexion of the cathode rays, but the medium under the large electromotive force breaks down every now and then and a bright discharge passes between the plates; when this occurs the phosphorescent patch produced by the cathode rays jumps back to its undeflected position. When the cathode rays are deflected by the electrostatic field, the phosphorescent band breaks up into several bright bands separated by comparatively dark spaces; the phenomena are exactly analogous to those observed by Birkeland when the cathode rays are deflected by a magnet, and called by him the magnetic spectrum.

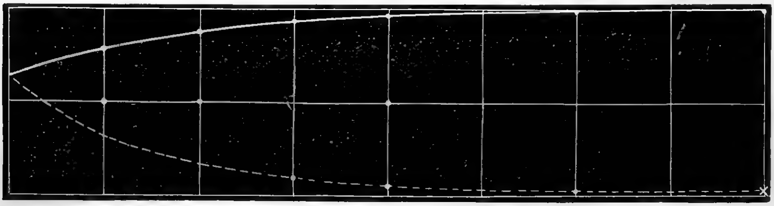
A series of measurements of the deflexion of the rays by the electrostatic force under various circumstances will be found later on in the part of the paper which deals with the velocity of the rays and the ratio of the mass of the electrified particles to the charge carried by them. It may, however, be mentioned here that the deflexion gets smaller as the pressure diminishes, and when in consequence the potential-difference in the tube in the neighbourhood of the cathode increases.

Conductivity of a Gas through which Cathode Rays are passing.

The conductivity of the gas was investigated by means of the apparatus shown in fig. 2. The upper plate D was connected with one terminal of a battery of small storage-cells, the other terminal of which was connected with the earth; the other plate E was connected with one of the coatings of a condenser of one microfarad capacity, the other coating of which was to earth; one pair of quadrants of an electrometer was also connected with E, the other pair of quadrants being to earth. When the cathode rays are passing between the plates the two pairs of quadrants of the electrometer are first connected with each other, and then the connexion between them was broken. If the space between the plates were a non-conductor, the potential of the pair of quadrants not connected with the earth would remain zero and the needle of the electrometer would not move; if, however, the space between the plates were a conductor, then the potential of the lower plate would approach that of the upper, and the needle of the electrometer would be deflected. There is always a deflexion of the electrometer, showing that a current passes between the plates. The magnitude of the current depends very greatly upon the pressure of the gas; so much so, indeed, that it is difficult to obtain consistent readings in consequence of the changes which always occur in the pressure when the discharge passes through the tube.

We shall first take the case when the pressure is only just low enough to allow the phosphorescent patch to appear at the end of the tube; in this case the relation between the current between the plates and the initial difference of potential is represented by the curve shown in fig. 3. In this figure the abscissæ represent the initial difference of potential between the plates, each division representing two volts, and the ordinates the rise in potential of the lower plate in one minute each division again representing two volts. The quantity of electricity which has passed between the plates in

Fig. 3.



one minute is the quantity required to raise 1 microfarad to the potential-difference shown by the curve. The upper and lower curve relates to the case when the upper plate is connected with the negative and positive pole respectively of the battery.

Even when there is no initial difference of potential between the plates the lower plate acquires a negative charge from the impact on it of some of the cathode rays.

We see from the curve that the current between the plates soon reaches a value where it is only slightly affected by an increase in the potential-difference between the plates; this is a feature common to conduction through gases traversed by Röntgen rays, by uranium rays, by ultra-violet light, and, as we now see, by cathode rays. The rate of leak is not greatly different whether the upper plate be initially positively or negatively electrified.

The current between the plates only lasts for a short time; it ceases long before the potential of the lower plate approaches that of the upper. Thus, for example, when the potential of the upper plate was about 400 volts above that of the earth, the potential of the lower plate never rose above 6 volts: similarly, if the upper plate were connected with the negative pole of the battery, the fall in potential of the lower plate was very small in comparison with the potential-difference between the upper plate and the earth.

These results are what we should expect if the gas between the plates and the plug B (fig. 2) were a very much better conductor than the gas between the plates, for the lower plate will be in a steady state when the current coming to it from the upper plate is equal to the current going from it to the plug: now if the conductivity of the gas between the plate and the plug is much greater than that between the plates, a small difference of potential between the lower plate and the plug will be consistent with a large potential-difference between the plates.

So far we have been considering the case when the pressure is as high as is consistent with the cathode rays reaching the end of the tube; we shall now go to the other extreme and consider the case when the pressure is as low as is consistent with the passage of a discharge through the bulb. In this case, when the plates are not connected with the battery we get a negative charge communicated to the lower plate, but only very slowly in comparison with the effect in the previous case. When the upper plate is connected with the negative pole of a battery, this current to the lower plate is only slightly increased even when the difference of potential is as much as 400 volts: a small potential-difference of about

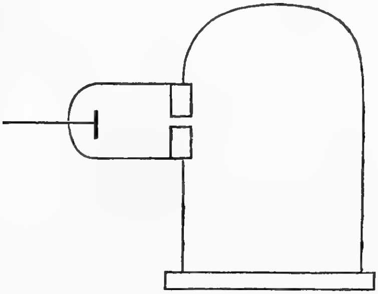
20 volts seems slightly to decrease the rate of leak. Potential-differences much exceeding 400 volts cannot be used, as though the dielectric between the plates is able to sustain them for some little time, yet after a time an intensely bright arc flashes across between the plates and liberates so much gas as to spoil the vacuum. The lines in the spectrum of this glare are chiefly mercury lines; its passage leaves very peculiar markings on the aluminium plates.

If the upper plate was charged positively, then the negative charge communicated to the lower plate was diminished, and stopped when the potential-difference between the plates was about 20 volts; but at the lowest pressure, however great (up to 400 volts) the potential-difference, there was no leak of positive electricity to the lower plate at all comparable with the leak of negative electricity to this plate when the two plates were disconnected from the battery. In fact at this very low pressure all the facts are consistent with the view that the effects are due to the negatively electrified particles travelling along the cathode rays, the rest of the gas possessing little conductivity. Some experiments were made with a tube similar to that shown in fig. 2, with the exception that the second plug B was absent, so that a much greater number of cathode rays passed between the plates. When the upper plate was connected with the positive pole of the battery a luminous discharge with well-marked striations passed between the upper plate and the earth-connected plug through which the cathode rays were streaming; this occurred even though the potential-difference between the plate and the plug did not exceed 20 volts. Thus it seems that if we supply cathode rays from an external source to the cathode a small potential-difference is sufficient to produce the characteristic discharge through a gas.

Magnetic Deflexion of the Cathode Rays in Different Gases.

The deflexion of the cathode rays by the magnetic field was studied with the aid of the apparatus shown in fig. 4. The cathode was placed in a side-tube fastened on to a bell-jar; the opening between this tube and the bell-jar was closed by a metallic plug with a slit in it; this plug was connected with the earth and was used as the anode. The cathode rays passed through the slit in this plug into the bell-jar, passing in front of a vertical plate of glass ruled into small squares. The bell-jar was placed between two large parallel coils arranged as a Helmholtz galvanometer. The course of the rays was determined by taking photographs of the bell-jar when the cathode rays were passing through it; the divisions on the plate enabled the path of the rays to be determined. Under the action of the magnetic field the narrow beam of cathode rays spreads out into a broad fan-shaped luminosity in the gas. The luminosity in this fan is not uniformly distributed, but is condensed along certain lines. The phosphorescence on the glass is also not uniformly distributed; it is much spread out, showing that the beam consists of rays which are not all deflected to the same extent by the magnet. The luminosity on the glass is crossed by bands along which the luminosity is very much greater than in the adjacent parts. These bright and dark bands are called by Birkeland, who first observed them, the magnetic spectrum. The brightest spots on the glass are by no means always the terminations of the brightest streaks of luminosity in the gas; in fact, in some cases a very bright spot on the glass is not connected with the cathode by any appreciable luminosity, though there may be plenty of luminosity in other parts of the gas. One very interesting point brought out by the photographs is that in a given magnetic field, and with a given mean potential-difference between the terminals, the path of the rays is independent of the nature of the gas. Photographs were taken of the discharge in hydrogen, air, carbonic acid, methyl iodide, *i. e.*, in gases whose densities range from 1 to 70, and yet, not only were the paths of the most deflected rays the same in all cases, but even the details, such as the distribution of the bright and dark spaces, were the same; in fact, the photographs could hardly be distinguished from each other. It is to be noted that the pressures were not the same; the pressures in the different gases were adjusted so that the mean potential-differences between the cathode and the anode were the same in all the gases. When the pressure of a gas is lowered, the potential-difference between the terminals increases, and the

Fig. 4.



deflexion of the rays produced by a magnet diminishes, or at any rate the deflexion of the rays when the phosphorescence is a maximum diminishes. If an air-break is inserted an effect of the same kind is produced.

In the experiments with different gases, the pressures were as high as was consistent with the appearance of the phosphorescence on the glass, so as to ensure having as much as possible of the gas under consideration in the tube.

As the cathode rays carry a charge of negative electricity, are deflected by an electrostatic force as if they were negatively electrified, and are acted on by a magnetic force in just the way in which this force would act on a negatively electrified body moving along the path of these rays, I can see no escape from the conclusion that they are charges of negative electricity carried by particles of matter. The question next arises, What are these particles? are they atoms, or molecules, or matter in a still finer state of subdivision? To throw some light on this point, I have made a series of measurements of the ratio of the mass of these particles to the charge carried by it. To determine this quantity, I have used two independent methods. The first of these is as follows:—Suppose we consider a bundle of homogeneous cathode rays. Let m be the mass of each of the particles, e the charge carried by it. Let N be the number of particles passing across any section of the beam in a given time; then Q the quantity of electricity carried by these particles is given by the equation

$$\mathrm{N}e = \mathrm{Q}.$$

We can measure Q if we receive the cathode rays in the inside of a vessel connected with an electrometer. When these rays strike against a solid body, the temperature of the body is raised; the kinetic energy of the moving particles being converted into heat; if we suppose that all this energy is converted into heat, then if we measure the increase in the temperature of a body of known thermal capacity caused by the impact of these rays, we can determine W, the kinetic energy of the particles, and if v is the velocity of the particles,

$$\tfrac{1}{2}\mathrm{N}mv^2 = \mathrm{W}.$$

If ρ is the radius of curvature of the path of these rays in a uniform magnetic field H, then

$$\frac{mv}{e} = \mathrm{H}\rho = \mathrm{I},$$

where I is written for Hρ for the sake of brevity. From these equations we get

$$\frac{1}{2}\frac{m}{e}v^2=\frac{\mathrm{W}}{\mathrm{Q}}.$$

$$v=\frac{2\mathrm{W}}{\mathrm{QI}},$$

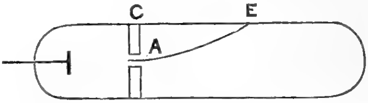
$$\frac{m}{e}=\frac{\mathrm{I}^2\mathrm{Q}}{2\mathrm{W}}.$$

Thus, if we know the values of Q, W, and I, we can deduce the values of v and m/e.

To measure these quantities, I have used tubes of three different types. The first I tried is like that represented in fig. 2, except that the plates E and D are absent, and two coaxial cylinders are fastened to the end of the tube. The rays from the cathode C fall on the metal plug B, which is connected with the earth, and serves for the anode; a horizontal slit is cut in this plug. The cathode rays pass through this slit, and then strike against the two coaxial cylinders at the end of the tube; slits are cut in these cylinders, so that the cathode rays pass into the inside of the inner cylinder. The outer cylinder is connected with the earth, the inner cylinder, which is insulated from the outer one, is connected with an electrometer, the deflexion of which measures Q, the quantity of electricity brought into the inner cylinder by the rays. A thermo-electric couple is placed behind the slit in the inner cylinder; this couple is made of very thin strips of iron and copper fastened to very fine iron and copper wires. These wires passed through the cylinders, being insulated from them, and through the glass to the outside of the tube, where they were connected with a low-resistance galvanometer, the deflexion of which gave data for calculating the rise of temperature of the junction produced by the impact against it of the cathode rays. The strips of iron and copper were large enough to ensure that every cathode ray which entered the inner cylinder struck against the junction. In some of the tubes the strips of iron and copper were placed end to end, so that some of the rays struck against the iron, and others against the copper; in others, the strip of one metal was placed in front of the other; no difference, however, could be detected between the results got with these two arrangements. The strips of iron and copper were weighed, and the thermal capacity of the junction calculated. In one set of junctions this capacity was 5×10^{-3}, in another 3×10^{-3}. If we assume that the cathode rays which strike against the junction give their energy up to it, the deflexion of the galvanometer gives us W or $\frac{1}{2}\mathrm{N}mv^2$.

The value of I, *i. e.*, $H\rho$, where ρ is the curvature of the path of the rays in a magnetic field of strength H was found as follows :—The tube was fixed between two large circular coils placed parallel to each other, and separated by a distance equal to the radius of either; these coils produce a uniform magnetic field, the strength of which is got by measuring with an ammeter the strength of the current passing through them. The cathode rays are thus in a uniform field, so that their path is circular. Suppose that the rays, when deflected by a magnet, strike against the glass of the tube at E (fig. 5), then, if ρ is the radius of the circular path of the rays,

Fig. 5.



$$2\rho = \frac{CE^2}{AC} + AC\,;$$

thus, if we measure CE and AC we have the means of determining the radius of curvature of the path of the rays.

The determination of ρ is rendered to some extent uncertain, in consequence of the pencil of rays spreading out under the action of the magnetic field, so that the phosphorescent patch at E is several millimetres long; thus values of ρ differing appreciably from each other will be got by taking E at different points of this phosphorescent patch. Part of this patch was, however, generally considerably brighter than the rest; when this was the case, E was taken as the brightest point; when such a point of maximum brightness did not exist, the middle of the patch was taken for E. The uncertainty in the value of ρ thus introduced amounted sometimes to about 20 per cent.; by this I mean that if we took E first at one extremity of the patch and then at the other, we should get values of ρ differing by this amount.

The measurement of Q, the quantity of electricity which enters the inner cylinder, is complicated by the cathode rays making the gas through which they pass a conductor, so that though the insulation of the inner cylinder was perfect when the rays were off, it was not so when they were passing through the space between the cylinders; this caused some of the charge communicated to the inner cylinder to leak away so that the actual charge given to the cylinder by the cathode rays was larger than that indicated by the electrometer.

To make the error from this cause as small as possible, the inner cylinder was connected to the largest capacity available, 1·5 microfarad, and the rays were only kept on for a short time, about 1 or 2 seconds, so that the alteration in potential of the inner cylinder was not large, ranging in the various experiments from about ·5 to 5 volts. Another reason why it is necessary to limit the duration of the rays to as short a time as possible, is to avoid the correction for the loss of heat from the thermo-electric junction by conduction along the wires; the rise in temperature of the junction was of the order 2° C.; a series of experiments showed that with the same tube and the same gaseous pressure Q and W were proportional to each other when the rays were not kept on too long.

Tubes of this kind gave satisfactory results, the chief drawback being that sometimes in consequence of the charging up of the glass of the tube, a secondary discharge started from the cylinder to the walls of the tube, and the cylinders were surrounded by glow; when this glow appeared, the readings were very irregular; the glow could, however, be got rid of by pumping and letting the tube rest for some time. The results got with this tube are given in the Table under the heading Tube 1.

The second type of tube was like that used for photographing the path of the rays (fig. 4); double cylinders with a thermo-electric junction like those used in the previous tube were placed in the line of fire of the rays, the inside of the bell-jar was lined with copper gauze connected with the earth. This tube gave very satisfactory results; we were never troubled with any glow round the cylinders, and the readings were most concordant; the only drawback was that as some of the connexions had to be made with sealing-wax, it was not possible to get the highest exhaustions with this tube, so that the range of pressure for this tube is less than that for tube 1. The results got with this tube are given in the Table under the heading Tube 2.

The third type of tube was similar to the first, except that the openings in the two cylinders were made very much smaller; in this tube the slits in the cylinders were replaced by small holes, about 1·5 millim. in diameter. In consequence of the smallness of the openings, the magnitude of the effects was very much reduced; in order to get measurable results it was necessary to reduce the capacity of the condenser in connexion with the inner cylinder to ·15 microfarad, and to make the galvanometer exceedingly sensitive, as the rise in temperature of the thermo-electric junction was in these experiments only about ·5° C. on the average. The results obtained in this tube are given in the Table under the heading Tube 3.

The results of a series of measurements with these tubes are given in the following Table :—

Gas.	Value of W/Q.	I.	*m/e.*	*v.*
		Tube 1.		
Air	$4{\cdot}6 \times 10^{11}$	230	$\cdot57 \times 10^{-7}$	4×10^{9}
Air	$1{\cdot}8 \times 10^{12}$	350	$\cdot34 \times 10^{-7}$	1×10^{10}
Air	$6{\cdot}1 \times 10^{11}$	230	$\cdot43 \times 10^{-7}$	$5{\cdot}4 \times 10^{9}$
Air	$2{\cdot}5 \times 10^{12}$	400	$\cdot32 \times 10^{-7}$	$1{\cdot}2 \times 10^{10}$
Air	$5{\cdot}5 \times 10^{11}$	230	$\cdot48 \times 10^{-7}$	$4{\cdot}8 \times 10^{9}$
Air	1×10^{12}	285	$\cdot4 \times 10^{-7}$	7×10^{9}
Air	1×10^{12}	285	$\cdot4 \times 10^{-7}$	7×10^{9}
Hydrogen	6×10^{12}	205	$\cdot35 \times 10^{-7}$	6×10^{9}
Hydrogen	$2{\cdot}1 \times 10^{12}$	460	$\cdot5 \times 10^{-7}$	$9{\cdot}2 \times 10^{9}$
Carbonic acid	$8{\cdot}4 \times 10^{11}$	260	$\cdot4 \times 10^{-7}$	$7{\cdot}5 \times 10^{9}$
Carbonic acid	$1{\cdot}47 \times 10^{12}$	340	$\cdot4 \times 10^{-7}$	$8{\cdot}5 \times 10^{9}$
Carbonic acid	$3{\cdot}0 \times 10^{12}$	480	$\cdot39 \times 10^{-7}$	$1{\cdot}3 \times 10^{10}$
		Tube 2.		
Air	$2{\cdot}8 \times 10^{11}$	175	$\cdot53 \times 10^{-7}$	$3{\cdot}3 \times 10^{9}$
Air	$4{\cdot}4 \times 10^{11}$	195	$\cdot47 \times 10^{-7}$	$4{\cdot}1 \times 10^{9}$
Air	$3{\cdot}5 \times 10^{11}$	181	$\cdot47 \times 10^{-7}$	$3{\cdot}8 \times 10^{9}$
Hydrogen	$2{\cdot}8 \times 10^{11}$	175	$\cdot53 \times 10^{-7}$	$3{\cdot}3 \times 10^{9}$
Air	$2{\cdot}5 \times 10^{11}$	160	$\cdot51 \times 10^{-7}$	$3{\cdot}1 \times 10^{9}$
Carbonic acid	2×10^{11}	148	$\cdot54 \times 10^{-7}$	$2{\cdot}5 \times 10^{9}$
Air	$1{\cdot}8 \times 10^{11}$	151	$\cdot63 \times 10^{-7}$	$2{\cdot}3 \times 10^{9}$
Hydrogen	$2{\cdot}8 \times 10^{11}$	175	$\cdot53 \times 10^{-7}$	$3{\cdot}3 \times 10^{9}$
Hydrogen	$4{\cdot}4 \times 10^{11}$	201	$\cdot46 \times 10^{-7}$	$4{\cdot}4 \times 10^{9}$
Air	$2{\cdot}5 \times 10^{11}$	176	$\cdot61 \times 10^{-7}$	$2{\cdot}8 \times 10^{9}$
Air	$4{\cdot}2 \times 10^{11}$	200	$\cdot48 \times 10^{-7}$	$4{\cdot}1 \times 10^{9}$
		Tube 3.		
Air	$2{\cdot}5 \times 10^{11}$	220	$\cdot9 \times 10^{-7}$	$2{\cdot}4 \times 10^{9}$
Air	$3{\cdot}5 \times 10^{11}$	225	$\cdot7 \times 10^{-7}$	$3{\cdot}2 \times 10^{9}$
Hydrogen	3×10^{11}	250	$1{\cdot}0 \times 10^{-7}$	$2{\cdot}5 \times 10^{9}$

It will be noticed that the value of m/e is considerably greater for Tube 3, where the opening is a small hole, than for Tubes 1 and 2, where the opening is a slit of much greater area. I am of opinion that the values of m/e got from Tubes 1 and 2 are too small, in consequence of the leakage from the inner cylinder to the outer by the gas being rendered a conductor by the passage of the cathode rays.

It will be seen from these tables that the value of m/e is independent of the nature of the gas. Thus, for the first tube the mean for air is $\cdot 40 \times 10^{-7}$, for hydrogen $\cdot 42 \times 10^{-7}$, and for carbonic acid gas $\cdot 4 \times 10^{-7}$; for the second tube the mean for air is $\cdot 52 \times 10^{-7}$, for hydrogen $\cdot 50 \times 10^{-7}$, and for carbonic acid gas $\cdot 54 \times 10^{-7}$.

Experiments were tried with electrodes made of iron instead of aluminium; this altered the appearance of the discharge and the value of v at the same pressure, the values of m/e were, however, the same in the two tubes; the effect produced by different metals on the appearance of the discharge will be described later on.

In all the preceding experiments, the cathode rays were first deflected from the cylinder by a magnet, and it was then found that there was no deflexion either of the electrometer or the galvanometer, so that the deflexions observed were entirely due to the cathode rays; when the glow mentioned previously surrounded the cylinders there was a deflexion of the electrometer even when the cathode rays were deflected from the cylinder.

Before proceeding to discuss the results of these measurements I shall describe another method of measuring the quantities m/e and v of an entirely different kind from the preceding; this method is based upon the deflexion of the cathode rays in an electrostatic field. If we measure the deflexion experienced by the rays when traversing a given length under a uniform electric intensity, and the deflexion of the rays when they traverse a given distance under a uniform magnetic field, we can find the values of m/e and v in the following way:—

Let the space passed over by the rays under a uniform electric intensity F be l, the time taken for the rays to traverse this space is l/v, the velocity in the direction of F is therefore

$$\frac{\mathrm{F}e}{m}\frac{l}{v},$$

so that θ, the angle through which the rays are deflected when they leave the electric field and enter a region free from electric force, is given by the equation

$$\theta=\frac{\mathrm{F}e}{m}\frac{l}{v^2}.$$

If, instead of the electric intensity, the rays are acted on by a magnetic force H at right angles to the rays, and extending across the distance l, the velocity at right angles to the original path of the rays is

$$\frac{\mathrm{H}ev}{m}\frac{l}{v},$$

so that ϕ, the angle through which the rays are deflected when they leave the magnetic field, is given by the equation

$$\phi=\frac{\mathrm{H}e}{m}\frac{l}{v}.$$

From these equations we get

$$v=\frac{\phi}{\theta}\frac{\mathrm{F}}{\mathrm{H}}$$

and

$$\frac{m}{e}=\frac{\mathrm{H}^2\theta\,.\,l}{\mathrm{F}\phi^2}.$$

In the actual experiments H was adjusted so that $\phi=\theta$; in this case the equations become

$$v=\frac{\mathrm{F}}{\mathrm{H}},$$

$$\frac{m}{e}=\frac{\mathrm{H}^2 l}{\mathrm{F}\theta}.$$

The apparatus used to measure v and m/e by this means is that represented in fig. 2. The electric field was produced by connecting the two aluminium plates to the terminals of a battery of storage-cells. The phosphorescent patch at the end of the tube was deflected, and the deflexion measured by a scale pasted to the end of the tube. As it was necessary to darken the room to see the phosphorescent patch, a needle coated with luminous paint was placed so that by a screw it could be moved up and down the scale; this needle could be seen when the room was darkened, and it was moved until it coincided with the phosphorescent patch. Thus, when light was admitted, the deflexion of the phosphorescent patch could be measured.

The magnetic field was produced by placing outside the tube two coils whose diameter was equal to the length of the plates; the coils were placed so that they covered the space occupied by the plates, the distance between the coils was equal to the radius of either. The mean value of the magnetic force over the length l was determined in the following way: a narrow coil C whose length was l, connected with a ballistic galvanometer, was placed between the coils; the plane of the windings of C was parallel to the planes of the coils; the cross section of the coil was a rectangle 5 cm. by 1 cm. A given current was sent through the outer coils and the kick α of the galvanometer observed when this current was reversed. The coil C was then placed at the centre of two very large coils, so as to be in a field of uniform magnetic force: the current through the large coils was reversed and the kick β of the galvanometer again observed; by comparing α and β we can get the mean value of the magnetic force over a length l; this was found to be

$$60 \times \iota,$$

where ι is the current flowing through the coils.

A series of experiments was made to see if the electrostatic deflexion was proportional to the electric intensity between the plates; this was found to be the case. In the following experiments the current through the coils was adjusted so that the electrostatic deflexion was the same as the magnetic:—

Gas.	θ.	H.	F.	l.	m/e.	v.
Air	8/110	5·5	$1{\cdot}5\times10^{10}$	5	$1{\cdot}3\times10^{-7}$	$2{\cdot}8\times10^{9}$
Air	9·5/110	5·4	$1{\cdot}5\times10^{10}$	5	$1{\cdot}1\times10^{-7}$	$2{\cdot}8\times10^{9}$
Air	13/110	6·6	$1{\cdot}5\times10^{10}$	5	$1{\cdot}2\times10^{-7}$	$2{\cdot}3\times10^{9}$
Hydrogen	9/110	6·3	$1{\cdot}5\times10^{10}$	5	$1{\cdot}5\times10^{-7}$	$2{\cdot}5\times10^{9}$
Carbonic acid...	11/110	6·9	$1{\cdot}5\times10^{10}$	5	$1{\cdot}5\times10^{-7}$	$2{\cdot}2\times10^{9}$
Air	6/110	5	$1{\cdot}8\times10^{10}$	5	$1{\cdot}3\times10^{-7}$	$3{\cdot}6\times10^{9}$
Air	7/110	3·6	1×10^{10}	5	$1{\cdot}1\times10^{-7}$	$2{\cdot}8\times10^{9}$

The cathode in the first five experiments was aluminium, in the last two experiments it was made of platinum; in the last experiment Sir William Crookes's method of getting rid of the mercury vapour by inserting tubes of pounded sulphur, sulphur iodide, and copper filings between the bulb and the pump was adopted. In the calculation of m/e and v no allowance has been made for the magnetic force due to the coil in the region outside the plates; in this region the magnetic force will be in the opposite direction to that between the plates, and will tend to bend the cathode rays in the opposite direction: thus the effective value of H will be smaller than the value used in the equations, so that the values of m/e are larger, and those of v less than they would be if this correction were applied. This method of determining the values of m/e and v is much less laborious and probably more accurate than the former method; it cannot, however, be used over so wide a range of pressures.

From these determinations we see that the value of m/e is independent of the nature of the gas, and that its value 10^{-7} is very small compared with the value 10^{-4}, which is the smallest value of this quantity previously known, and which is the value for the hydrogen ion in electrolysis.

Thus for the carriers of the electricity in the cathode rays m/e is very small compared with its value in electrolysis. The smallness of m/e may be due to the smallness of m or the largeness of e, or to a combination of these two. That the carriers of the charges in the cathode rays are small compared with ordinary molecules is shown, I think, by Lenard's results as to the rate at which the brightness of the phosphorescence produced by these rays diminishes with the length of path travelled by the ray. If we regard this phosphorescence as due to the impact of the charged particles, the distance through which the rays must travel before the phosphorescence fades to a given fraction (say $1/e$, where $e=2{\cdot}71$) of its original intensity, will be some moderate multiple of the mean free path. Now Lenard found that this distance depends solely upon the density of the medium, and not upon its chemical nature or physical state. In air at atmospheric pressure the distance was about half a centimetre, and this must be comparable with the mean free path of the carriers through air at atmospheric pressure. But the mean free path of the molecules of air is a quantity of quite a different order. The carrier, then, must be small compared with ordinary molecules.

The two fundamental points about these carriers seem to me to be (1) that these carriers are the same whatever the gas through which the discharge passes, (2) that the mean free paths depend upon nothing but the density of the medium traversed by these rays.

It might be supposed that the independence of the mass of the carriers of the gas through which the discharge passes was due to the mass concerned being the quasi mass which a charged body possesses in virtue of the electric field set up in its neighbourhood; moving the body involves the production of a varying electric field, and, therefore, of a certain amount of energy which is proportional to the square of the velocity. This causes the charged body to behave as if its mass were increased by a quantity, which for a charged sphere is $\frac{1}{5} e^2/\mu a$ ('Recent Researches in Electricity and Magnetism'), where e is the charge and a the radius of the sphere. If we assume that it is this mass which we are concerned with in the cathode rays, since m/e would vary as e/a, it affords no clue to the explanation of either of the properties (1 and 2) of these rays. This is not by any means the only objection to this hypothesis, which I only mention to show that it has not been overlooked.

The explanation which seems to me to account in the most simple and straightforward manner for the facts is founded on a view of the constitution of the chemical elements which has been favourably entertained by many chemists: this view is that the atoms of the different chemical elements are different aggregations of atoms of the same kind. In the form in which this hypothesis was enunciated by Prout, the atoms of the different elements were hydrogen atoms; in this precise form the hypothesis is not tenable, but if we substitute for hydrogen some unknown primordial substance X, there is nothing known which is inconsistent with this hypothesis, which is one that has been recently supported by Sir Norman Lockyer for reasons derived from the study of the stellar spectra.

If, in the very intense electric field in the neighbourhood of the cathode, the molecules of the gas are dissociated and are split up, not into the ordinary chemical atoms, but into these primordial atoms, which we shall for brevity call corpuscles; and if these corpuscles are charged with electricity and projected from the cathode by the electric field, they would behave exactly like the cathode rays. They would evidently give a value of m/e which is independent of the nature of the gas and its pressure, for the carriers are the same whatever the gas may be; again, the mean free paths of these corpuscles would depend solely upon the density of the medium through which they pass. For the molecules of the medium are composed of a number of such corpuscles separated by considerable spaces; now the collision between a single corpuscle and the molecule will not be between the corpuscles and the molecule as a whole, but between this corpuscle and the individual corpuscles which form the molecule; thus the number of collisions the particle makes as it moves through a crowd of these molecules will be proportional, not to the number of the molecules in the crowd, but to the number of the individual corpuscles. The mean free path is inversely proportional to the number of collisions in unit time, and so is inversely proportional to the number of corpuscles in unit volume; now as these corpuscles are all of the same mass, the number of corpuscles in unit volume will be proportional to the mass of unit volume, that is the mean free path will be inversely proportional to the density of the gas. We see, too, that so long as the distance between neighbouring corpuscles is large compared with the linear dimensions of a corpuscle the mean free path will be independent of the way they are arranged, provided the number in unit volume remains constant, that is the mean free path will depend only on the density of the medium traversed by the corpuscles, and will be independent of its chemical nature and physical state: this from Lenard's very remarkable measurements of the absorption of the cathode rays by various media, must be a property possessed by the carriers of the charges in the cathode rays.

Thus on this view we have in the cathode rays matter in a new state, a state in which the subdivision of matter is carried very much further than in the ordinary gaseous state: a state in which all matter—that is, matter derived from different sources such as hydrogen, oxygen, &c.—is of one and the same kind; this matter being the substance from which all the chemical elements are built up.

With appliances of ordinary magnitude, the quantity of matter produced by means of the dissociation at the cathode is so small as to almost to preclude the possibility of any direct chemical investigation of its properties. Thus the coil I used would, I calculate, if kept going uninterruptedly night and day for a year, produce only about one three-millionth part of a gramme of this substance.

The smallness of the value of m/e is, I think, due to the largeness of e as well as the smallness of m. There seems to me to be some evidence that the charges carried by the corpuscles in the atom are large compared with those carried by the ions of an electrolyte. In the molecule of HCl, for example, I picture the components of the hydrogen atoms as held together by a great number of tubes of electrostatic force; the components of the chlorine atom are similarly held together, while only one stray tube binds the hydrogen atom to the chlorine atom. The reason for attributing this high charge to the constituents of the atom is derived from the values of the specific inductive capacity of gases: we may imagine that the specific inductive capacity of a gas is due to the setting in the electric field of the electric doublet formed by the two oppositely electrified atoms which form the molecule of the gas. The measurements of the specific inductive capacity show, however, that this is very approximately an additive quantity: that is, that we can assign a certain value to each element, and find the specific inductive capacity of HCl by adding the value for hydrogen to the value for chlorine; the value of H_2O by adding twice the value for hydrogen to the value for oxygen, and so on. Now the electrical moment of the doublet formed by a positive charge on one atom of the molecule and a negative charge on the other atom would not be an additive property; if, however, each atom had a definite electrical moment, and this were large compared with the electrical moment of the two atoms in the molecule, then the electrical moment of any compound, and hence its specific inductive capacity, would be an additive property. For the electrical moment of the atom, however, to be large compared with that of the molecule, the charge on the corpuscles would have to be very large compared with those on the ion.

If we regard the chemical atom as an aggregation of a number of primordial atoms, the problem of finding the configurations of stable equilibrium for a number of equal particles acting on each other according to some law of force —whether that of Boscovich, where the force between them is a repulsion when they are separated by less than a certain critical distance, and an attraction when they are separated by a greater distance, or even the simpler case of a number of mutually repellent particles held together by a central force —is of great interest in connexion with the relation between the properties of an element and its atomic weight. Unfortunately the equations which determine the stability of such a collection of particles increase so rapidly in complexity with the number of particles that a general mathematical investigation is scarcely possible. We can, however, obtain a good deal of insight into the general laws which govern such configurations by the use of models, the simplest of which is the floating magnets of Professor Mayer. In this model the magnets arrange themselves in equilibrium under their mutual repulsions and a central attraction caused by the pole of a large magnet placed above the floating magnets.

A study of the forms taken by these magnets seems to me to be suggestive in relation to the periodic law. Mayer showed that when the number of floating magnets did not exceed 5 they arranged themselves at the corners of a regular polygon— 5 at the corners of a pentagon, 4 at the corners of a square, and so on. When the number exceeds 5, however, this law no longer holds: thus 6 magnets do not arrange themselves at the corners of a hexagon, but divide into two systems, consisting of 1 in the middle surrounded by 5 at the corners of a pentagon. For 8 we have two in the inside and 6 outside; this arrangement in two systems, an inner and an outer, lasts up to 18 magnets. After this we have three systems: an inner, a middle, and an outer; for a still larger number of magnets we have four systems, and so on.

Mayer found the arrangement of magnets was as follows:—

1.	2.	3.	4.	5.
1.5 1.6 1.7	2.6 2.7	3.7 3.8	4.8 4.9	5.9
1.5.9 1.6.9 1.6.10 1.6.11	2.7.10 2.8.10 2.7.11	3.7.10 3.7.11 3.8.10 3.8.11 3.8.12 3.8.13	4.8.12 4.8.13 4.9.12 4.9.13	5.9.12 5.9.13
1.5. 9.12 1.5. 9.13 1.6. 9.12 1.6.10.12 1.6.10.13 1.6.11.12 1.6.11.13 1.6.11.14 1.6.11.15 1.7.12.14	2.7.10.15 2.7.12.14	3.7.12.13 3.7.12.14 3.7.13.14 3.7.13.15	4.9.13.14 4.9.13.15 4.9.14.15	

where, for example, 1.6.10.12 means an arrangement with one magnet in the middle, then a ring of six, then a ring of ten, and a ring of twelve outside.

Now suppose that a certain property is associated with two magnets forming a group by themselves; we should have this property with 2 magnets, again with 8 and 9, again with 19 and 20, and again with 34, 35, and so on. If we regard the system of magnets as a model of an atom, the number of magnets being proportional to the atomic weight, we should have this property occurring in elements of atomic weight 2, (8, 9), 19, 20, (34, 35). Again, any property conferred by three magnets forming a system by themselves would occur with atomic weights 3, 10, and 11; 20, 21, 22, 23, and 24; 35, 36, 37 and 39; in fact, we should have something quite analogous to the periodic law, the first series corresponding to the arrangement of the magnets in a single group, the second series to the arrangement in two groups, the third series in three groups, and so on.

Velocity of the Cathode Rays.

The velocity of the cathode rays is variable, depending upon the potential-difference between the cathode and anode, which is a function of the pressure of the gas—the velocity increases as the exhaustion improves; the measurements given above show, however, that at all the pressures at which experiments were made the velocity exceeded 10^9 cm./sec. This velocity is much greater than the value 2×10^7 which I previously obtained (Phil. Mag. Oct. 1894) by measuring directly the interval which separated the appearance of luminosity at two places on the walls of the tube situated at different distances from the cathode.

In my earlier experiments the pressure was higher than in the experiments described in this paper, so that the velocity of the cathode rays would on this account be less. The difference between the two results is, however, too great to be wholly explained in this way, and I attribute the difference to the glass requiring to be bombarded by the rays for a finite time before becoming phosphorescent, this time depending upon the intensity of the bombardment. As this time diminishes with the intensity of bombardment, the appearance of phosphorescence at the piece of glass most removed from the cathode would be delayed beyond the time taken for the rays to pass from one place to the other by the difference in time taken by the glass to become luminous; the apparent velocity measured in this way would thus be less than the true velocity. In the former experiments endeavours were made to diminish this effect by making the rays strike the glass at the greater distance from the cathode less obliquely than they struck the glass nearer to the cathode; the obliquity was adjusted until the brightness of the phosphorescence was approximately equal in the two cases. In view, however, of the discrepancy between the results obtained in this way and those obtained by the later method, I think that it was not successful in eliminating the lag caused by the finite time required by the gas to light up.

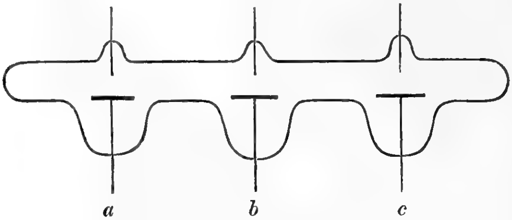
Experiments with Electrodes of Different Materials.

In the experiments described in this paper the electrodes were generally made of aluminium. Some experiments, however, were made with iron and platinum electrodes.

Though the value of m/e came out the same whatever the material of the electrode, the appearance of the discharge varied greatly; and as the measurements showed, the potential-difference between the cathode and anode depended greatly upon the metal used for the electrode ; the pressure being the same in all cases.

To test this point further I used a tube like that shown in fig. 6, where *a, b, c* are cathodes made of different metals, the anodes being in all cases platinum wires. The cathodes were disks of aluminium, iron, lead, tin, copper, mercury, sodium amalgam, and silver chloride ; the potential-difference between the cathode and anode was measured by Lord Kelvin's vertical voltmeter, and also by measuring the length of spark in air which, when placed in parallel with the anode and cathode, seemed to allow the discharge to go as often through the spark-gap as through the tube. With this arrangement the pressures were the same for all the cathodes. The potential-difference between the anode and cathode and the equivalent spark-length depended greatly upon the nature of the cathode. The extent of the variation in potential may be estimated from the following table :—

Fig. 6.



Cathode.	Mean Potential-Difference between Cathode and Anode.
Aluminium	1800 volts.
Lead	2100 ,,
Tin	2400 ,,
Copper	2600 ,,
Iron	2900 ,,

The potential-difference when the cathode was made of sodium amalgam or silver chloride was less even than that of aluminium.

The order of many of the metals changed about very capriciously, experiments made at intervals of a few minutes frequently giving quite different results. From the abrupt way in which these changes take place I am inclined to think that gas absorbed by the electrode has considerable influence on the passage of the discharge.

I have much pleasure in thanking Mr. Everitt for the assistance he has given me in the preceding investigation.

Cambridge, Aug. 7, 1897.

XLI. *On the Construction of Models and Diagrams to Illustrate the Propagation of Light in Biaxals. By* J. H. VINCENT, *B.Sc., A.R.C.Sc., Assistant Demonstrator in Physics at the Royal College of Science, London, S.W.*[*]

IN the following paper it is proposed to furnish such data as will render it possible to construct some models or diagrams illustrating the optics of biaxal crystals, with the minimum expenditure of trouble.

The diagrams of Fresnel's Wave Surface which appear in the text-books are restricted to the sections of the surface by the principal planes, and, although solid models of the surface are procurable, they scarcely lend themselves to educational purposes so well as sectional models which students can readily construct for themselves out of millboard.

The Ellipsoid of Elasticity.

The quadric

$$a^2x^2+b^2y^2+c^2z^2=1$$

is the first surface which a student encounters in approaching the subject from Fresnel's standpoint. It is the ellipsoid of elasticity (the Optical Indicatrix of Fletcher) having

$$\frac{1}{a}, \quad \frac{1}{b}, \quad \frac{1}{c}$$

as its semi-axes. The equation to the ellipsoid of elasticity may also be written

$$\frac{x^2}{\mu_1{}^2}+\frac{y^2}{\mu_2{}^2}+\frac{z^2}{\mu_3{}^2}=1$$

where μ_1, μ_2, μ_3 are the principal indices of refraction. Let the quantities μ_1, μ_2, μ_3 be taken proportional to the numbers 3, 4, 6.

The principal sections of the ellipsoid of elasticity are shown in figs. 1, 2, 3. In fig. 1 the full straight lines through the centre are the traces of diametral planes cutting the quadric in circular sections. These lines are inclined at an angle of 49° 51′ to the positive and negative directions of the axis of x. The broken lines are the optic axes and are perpendicular to the circular sections. The semi-angle

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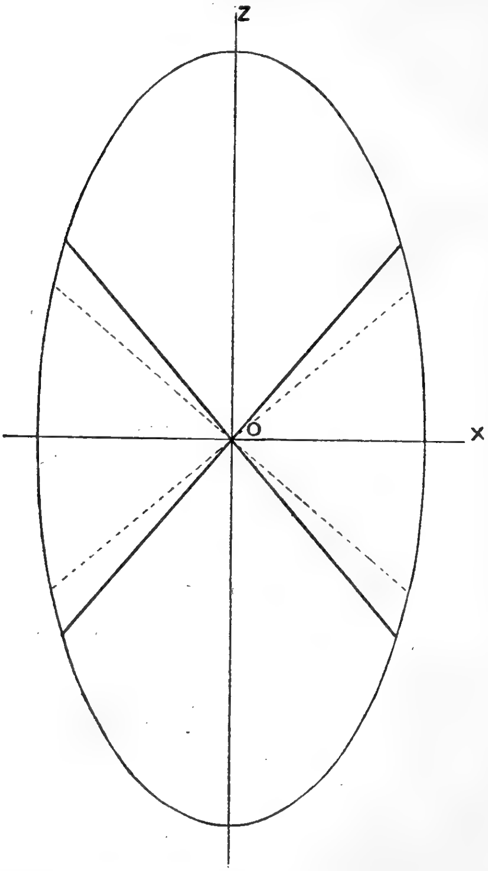
[*] Communicated by the Author.

between the optic axes is 49° 50′ and the radius of the circular section is 4.

Fig. 1.



There is no difficulty in making a cardboard model of the principal sections and the circular sections of this quadric. The ellipses can be drawn by the ordinary mechanical device or by elliptic trammels. If the latter are not available it will be probably found more speedy and accurate to set out the curves on squared paper from the following tables (p. 320).

Fig. 2.

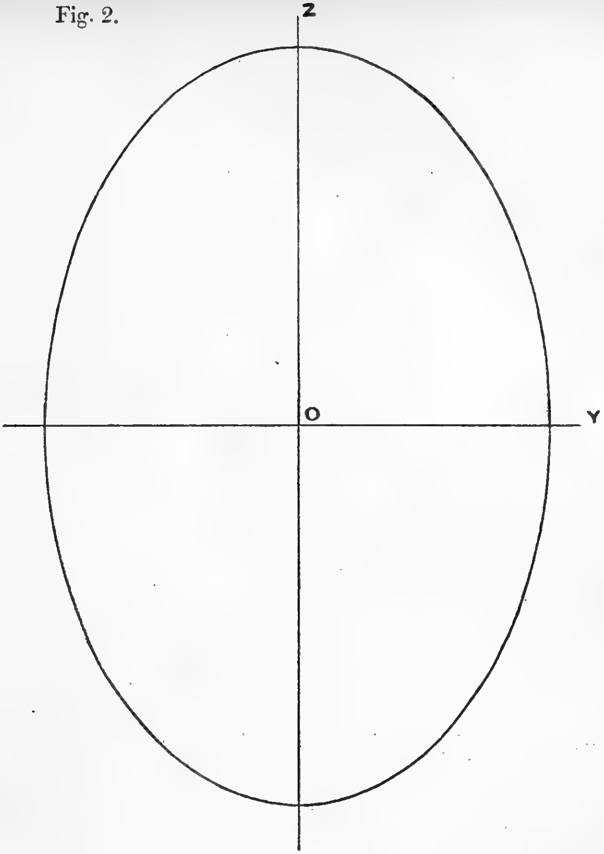
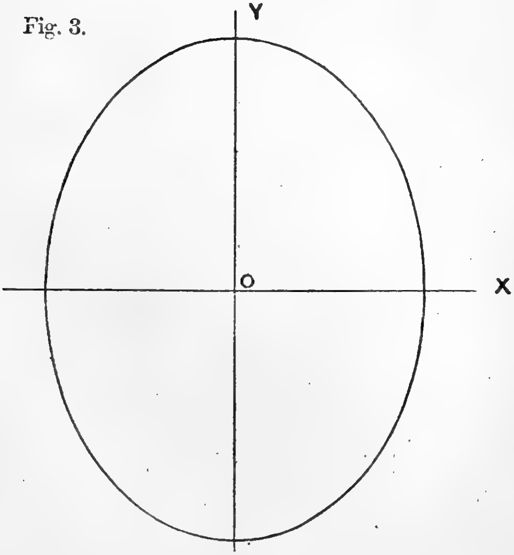


Fig. 3.



Ellipsoid of Elasticity.

Section by XOY *plane*:		*by* ZOX *plane*:		*by* ZOY *plane.*	
x.	y.	x.	z.	y.	z.
0·00	4·00	0·00	6·00	0·00	6·00
1·00	3·77	0·50	5·92	0·50	5·95
1·98	3·00	1·00	5·66	1·00	5·81
2·00	2·98	1·50	5·20	1·50	5·56
2·60	2·00	1·66	5·00	2·00	5·20
2·90	1·00	2·00	4·47	2·21	5·00
3·00	0·00	2·24	4·00	2·50	4·68
		2·60	3·00	2·98	4·00
		2·83	2·00	3·00	3·97
		2·96	1·00	3·46	3·00
		3·00	0·00	3·50	2·91
				3·77	2·00
				3·94	1·00
				4·00	0·00

Fresnel's Wave Surface.

The quantities a, b, c in the equation

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

the usual form in which the equation is met with, are to be taken proportional to the reciprocals of μ_1, μ_2, μ_3. This will be the case if a, b, c are proportional to the numbers 4, 3, 2. A model or diagram illustrating this surface will show the properties of a body whose ellipsoid of elasticity has the same shape as that constructed above.

Fig. 4 shows a series of curves which are the sections of the surface by planes parallel to the XOY plane. If $z=0$ the sectional curves are the ellipse

$$a^2x^2+b^2y^2=a^2b^2$$

and the circle

$$x^2+y^2=c^2.$$

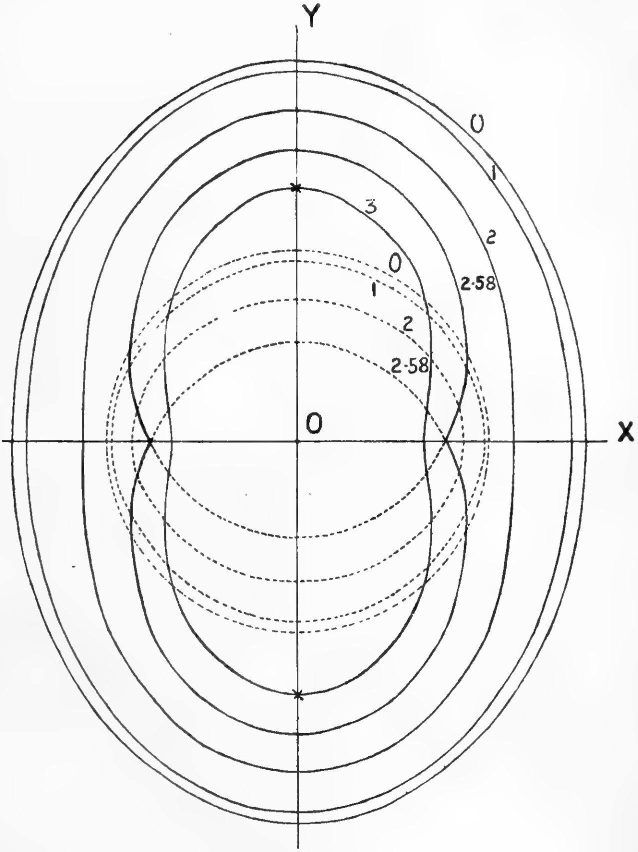
The radius of the circle is 2. The foci of the ellipse are indicated by crosses; the ellipse may be set out from the table given above for the section of the ellipsoid of elasticity by the XOY plane. It need hardly be remarked, that the ellipses in the same principal section of the indicatrix and wave-surface are similar.

In this figure the sections of the inner surface are shown by broken lines.

The ellipse and circle are marked O.

The curves 1, 1 are obtained by putting $z=1$ in the equation to the surface.

Fig. 4.



Section by Plane $z=1$.

Inner.		*Outer.*	
x.	*y.*	*x.*	*y.*
0·00	1·89	0·00	3·87
0·50	1·82	0·50	3·81
0·84	1·68		
1·00	1·61	1·00	3·63
1·35	1·35	1·53	3·28
1·53	1·15	1·61	3·21
		1·80	3·00
		2·00	2·77
		2·30	2·30
		2·44	2·00
1·64	1·00	2·74	1·00
1·94	0·00	2·83	0·00

When $z=2$ we have the curves 2, 2.

Section by Plane $z=2$.

Inner.		*Outer.*	
x.	y.	x.	y.
0·00	1·48	0·00	3·47
0·50	1·42	0·50	3·39
1·00	1·19	1·00	3·17
1·11	1·11	1·23	3·00
		1·53	2·70
		1·96	2·00
		1·98	1·98
		2·00	1·90
1·25	1·00	2·21	1·00
1·53	0·66		
1·61	0·50	2·24	0·50
1·73	0·00	2·24	0·00

The next section is taken through the singular points. The curves are marked 2·58; the co-ordinates of a singular point are

$$y=0,$$
$$z=2{\cdot}58,$$
$$x=1{\cdot}53.$$

Section by Plane parallel to XOY *and through the Singular Points.*

Inner.		*Outer.*	
x.	y.	x.	y.
1·53	0·00	1·53	0·00
1·20	0·50	1·72	0·50
1·00	0·67		
0·81	0·81		
0·50	0·94		
0·26	1·00	1·74	1·00
		1·64	1·64
		1·52	2·00
		1·00	2·69
		0·50	2·97
0·00	1·02	0·00	3·06

If we put $z=3$ in the equation to the surface we obtain the curve marked 3. The section plane touches the inner surface on the axis of z. It will be noticed that the projection of this curve upon the XOY plane passes through the foci of the ellipse in the principal section. This is not a mere coincidence, but is a geometrical property of the wave-surface *.

Section by Plane $z=3$.

Inner section is a point, the origin.

Outer.

x.	y.
0·00	2·65
0·50	2·52
1·00	2·10
1·09	2·00
1·35	1·35
1·39	1·00
1·38	0·69
1·37	0·50
1·35	0·34
1·32	0·13
1·32	0·00

Fig. 4 is a contour-map of the surface, the curves being marked according to their distance from the principal plane XOY.

In fig. 5 we have the section of the surface by the ZOX plane.

The curves of section are the circle

$$x^2+z^2=b^2$$

and the ellipse

$$a^2x^2+c^2z^2=a^2c^2.$$

The radius of the circle is 3.

---

* From the ease with which this relationship follows from the equation to the surface it is improbable that it has not been noticed previously; but it is not, perhaps, well known.

Writing the equation to the surface in the form:—

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2 \\ -c^2(a^2+b^2)z^2+a^2b^2c^2=0$$

and substituting values $z=b$, and $x=0$ we obtain

$$(y^2+b^2)(b^2y^2+c^2b^2)-b^2(c^2+a^2)y^2-b^2c^2(a^2+b^2)+a^2b^2c^2=0,$$

or

$$y^2+b^2-a^2=0,$$

whence

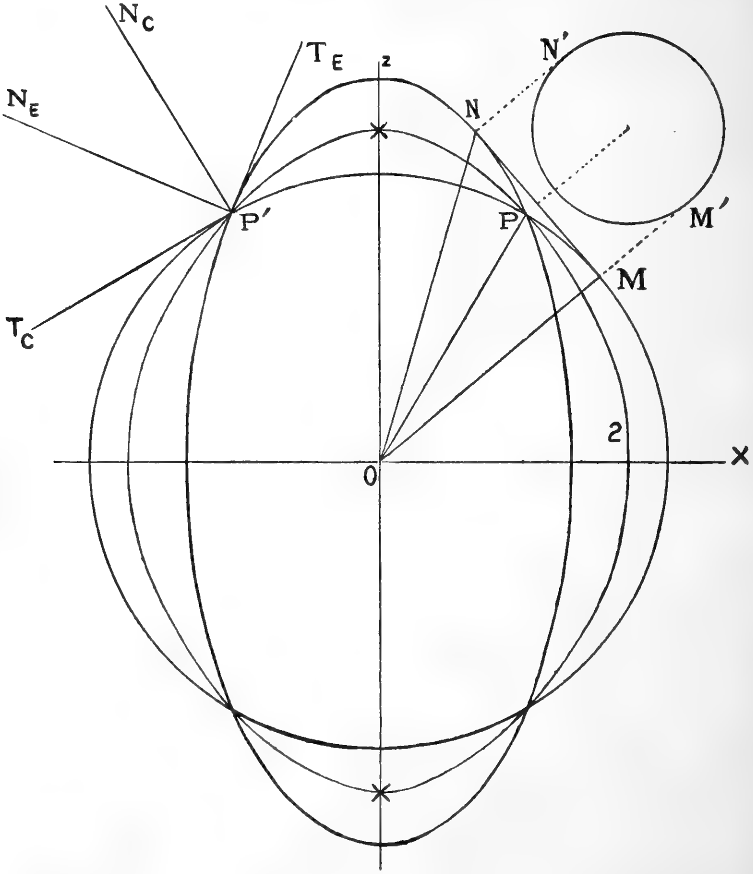
$$y=\pm\sqrt{a^2-b^2},$$

which proves the proposition.

From the symmetry of the equation to the surface, it follows that the two other analogous propositions are also true.

The ellipse may be set out from the table given above for the section of ellipsoid of elasticity by the ZOX plane by reducing the figures in the ratio of 3 : 2.

Fig. 5.



The foci of the ellipse are indicated by crosses. The curve marked 2 is the section of the outer surface by the plane $y=2$. This plane touches the inner sheet on the axis of y and thus (see footnote above) curve 2 goes through the foci of the ellipse.

Section of Surface by Plane $y=2$.

Inner section a point, the origin.

Outer.

x.	y.
0·00	3·46
1·09	3·00
1·50	2·60
1·53	2·58
1·96	2·00
2·44	1·00
2·60	0·00

The projection of the curve appears to go through the singular points, but does not actually do so. It cuts the circle in the point $x=1{\cdot}50$, $z=2{\cdot}60$ and not in the singular point.

The line OM is an optic axis, the angle $\mathrm{MOX}=\tan^{-1}{\cdot}845=40^\circ\ 10'$; the angle ZOM, the semi-angle between the optic axes, $=49^\circ\ 50'$ as we saw before from the ellipsoid of elasticity.

The line OP is an axis of single-ray velocity, the angle $\mathrm{XOP}=\tan^{-1}1{\cdot}69=59^\circ\ 20'$.

The circle N′M′ is the circle of contact, its diameter is 1·97. The co-ordinates of the point M are

$$x=2{\cdot}29,$$
$$z=1{\cdot}94,$$
$$y=0{\cdot}00.$$

The lines $\mathrm{N_EP'}$, $\mathrm{N_CP'}$ are normal to the ellipse and circle respectively, while $\mathrm{T_EP'}$ and $\mathrm{T_CP'}$ are the tangents. The equations to the lines $\mathrm{N_EP'}$, $\mathrm{N_CP'}$ with reference to P′ as origin are:

$$z=\frac{a}{c}\sqrt{\frac{b^2-c^2}{a^2-b^2}}\,x$$

and

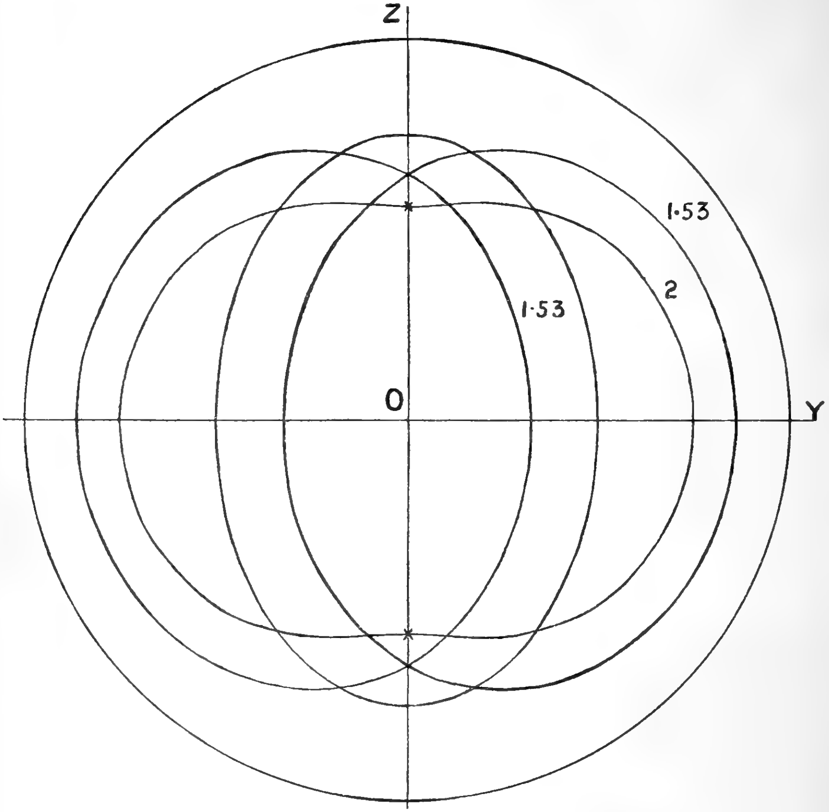
$$z=\frac{c}{a}\sqrt{\frac{b^2-c^2}{a^2-b^2}}\,x$$

respectively. The tangent of the angle which $\mathrm{N_CP'}$ makes with the negative direction of axis of x is thus ·423, and the angle is 22° 50′. The angle which $\mathrm{N_CP'}$ makes with the same direction has been found above to be 59° 20′. The angle $\mathrm{N_EP'N_C}$ is thus 36° 30′ and the angle $\mathrm{T_EP'T_C}$ is 143° 30′.

Fig. 6 gives the ellipse and circle in principal section by the ZOY plane. The radius of the circle is 4. The foci of the ellipse are indicated by crosses; the ellipse has half the linear dimensions of the section of the ellipsoid of elasticity by the ZOY plane.

The curve 2 is a section of the surface by a plane parallel to ZOY and at a distance 2 from it.

Fig. 6.



Section by Plane $x=2$.

Inner section a point, the origin.

Outer.

x.	*y*.
0·00	2·98
0·50	2·93
1·00	2·77
1·50	2·47
2·00	1·90
2·16	1·50
2·27	1·00
2·27	0·50
2·24	0·00

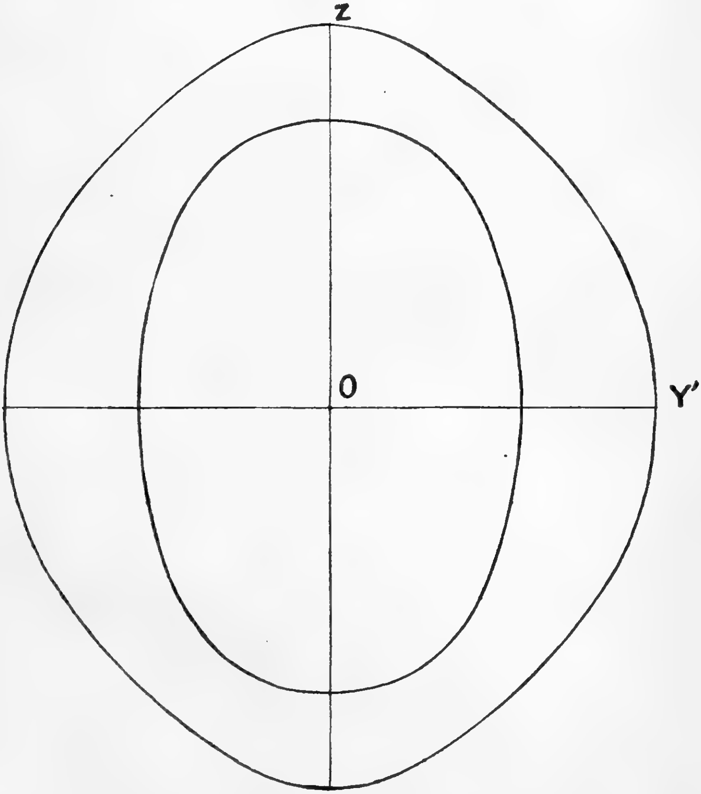
The curves 1·53, 1·53 are the sections by a plane through the singular points, parallel to the ZOY plane.

Section through Singular Points parallel to the XOY *plane.*

Inner.		*Outer.*	
z.	*y.*	*z.*	*y.*
0·00	1·29	0·00	3·44
1·00	1·15	1·00	3·28
1·44	1·00		
2·00	0·66	2·00	2·71
		2·83	1·00
2·58	0·00	2·58	0·00 and 1·98

A section taken through the axis of z and at 45° to the axes of x and y is given in fig. 7. These curves could have been obtained by means of fig. 4, but as a check on the

Fig. 7.



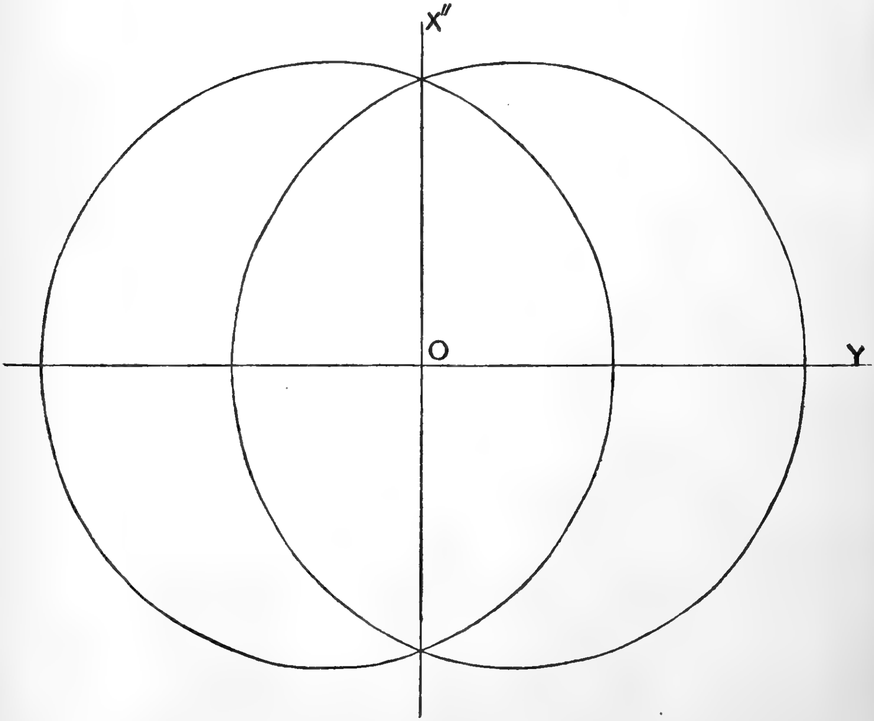
accuracy of the arithmetic they were calculated independently and the results were compared with those from fig. 4.

Section by a Plane at 45° to axes of x *and* y *and passing through axis of* z.

Inner. y'.	z.	*Outer.* y'.	z.
0·00	3·00	0·00	4·00
1·00	2·70	1·00	3·70
1·57	2·00	1·91	3·00
1·90	1·00	2·00	2·91
2·00	0·00	2·79	2·00
		3·00	1·63
		3·25	1·00
		3·39	0·00

Another section of the surface which is of interest is that taken through a singular point and containing the axis of Y. The curves thus obtained are shown in fig. 8. If the axis of

Fig. 8.



single-ray velocity be taken as the axis of x'' we have

$$\left(\frac{a^2c^2}{b^2}x''^2+b^2y^2\right)(x''^2+y^2)-2a^2c^2x''^2-b^2(c^2+a^2)y^2+a^2b^2c^2=0$$

as the equation to the curves.

Section containing the axis of single-ray velocity and the axis of y.

Inner.		*Outer.*	
x''	y.	x''.	y.
0·00	2·00	0·00	4·00
1·00	1·85	1·00	3·85
1·16	1·80	1·16	3·80
1·59	1·59		
2·00	1·33	2·00	3·32
2·32	1·05	2·32	3·03
		2·65	2·65
2·37	1·00	3·18	1·00
2·75	0·50	3·14	0·50
3·00	0·00	3·00	0·00 and 1·97

Angles are given to the nearest 10′, and the other numbers to the second decimal place.

XLII. *On the Steady Motion of an Electrified Ellipsoid.* *By* G. F. C. Searle, *M.A., Demonstrator in Experimental Physics, Cavendish Laboratory, Cambridge* [*].

At the Meeting of the Royal Society on 19th March, 1896, I read a paper on "Problems in Electric Convection." The first part of the paper is printed in the 'Philosophical Transactions of the Royal Society'[†], and contains the principles which are required in the solution of any problems about moving charges. The second part of the paper, which deals with the motion of a charged ellipsoid, was not published by the Royal Society. A few of the results are, however, stated in an abstract published in the 'Proceedings'[‡]. By the permission of the Royal Society I now publish my results for a moving ellipsoid. As frequent reference to my paper in the 'Philosophical Transactions' will be necessary, I shall use the notation {§ 5} and {(9)} to indicate the paragraph or equation referred to in that paper.

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[*] Communicated by the Physical Society: read June 25, 1897.

[†] Phil. Trans. vol. 187 (1896) A. pp. 675–713.

[‡] Proc. Roy. Soc. vol. 59, p. 343.

When any system of electric charges moves with uniform velocity through the æther, the electromagnetic field, when referred to axes moving forwards with the charges, can be completely defined by means of a quantity Ψ, as was first shown by Prof. J. J. Thomson *. The electric force **E** and the magnetic force **H** are simple functions of Ψ. But besides **E** and **H** there is another vector of great importance, viz. the mechanical force **F** experienced by a unit charge moving with the rest of the system. The value of **F** I have shown {§ 10} to be given by the vector equation

$$\mathbf{F}=\mathbf{E}+\mu \mathrm{V}\mathbf{uH}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equations of the field are {§ 4}

$$\text{curl } \mathbf{F}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathbf{H}=\mathrm{K}\mathrm{V}\mathbf{uE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $v=\frac{1}{\sqrt{\mathrm{K}\mu}}$ is the velocity of light, and if α stand for $1-\frac{u^2}{v^2}$, then when the motion takes place parallel to the axis of x, we have {§ 4}

$$\mathrm{F}_1=-\frac{d\Psi}{dx} \qquad \mathrm{F}_2=-\frac{d\Psi}{dy} \qquad \mathrm{F}_3=-\frac{d\Psi}{dz}, \quad . \quad . \quad . \quad (4)$$

$$\mathrm{E}_1=-\frac{d\Psi}{dx} \qquad \mathrm{E}_2=-\frac{1}{\alpha}\frac{d\Psi}{dy} \qquad \mathrm{E}_3=-\frac{1}{\alpha}\frac{d\Psi}{dz}, \quad . \quad (5)$$

$$\mathrm{H}_1=0 \qquad \mathrm{H}_2=\frac{\mathrm{K}u}{\alpha}\frac{d\Psi}{dz} \qquad \mathrm{H}_3=-\frac{\mathrm{K}u}{\alpha}\frac{d\Psi}{dy}. \quad . \quad . \quad (6)$$

From these equations, since **E** has no divergence,

$$\alpha\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here, and throughout the paper, the axes are supposed to move forward with the same velocity as the electrical charges.

Prof. W. B. Morton has considered the motion of an ellipsoid in a paper read before the Physical Society on 27th March, 1896 †. He obtains the two following results, viz.: (1) that the distribution of electricity is the same as if the ellipsoid is at rest, and (2) the value of Ψ when the ellipsoid moves along one of its axes.

Prof. Morton obtains his result by the assumption first

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* Phil. Mag. July 1889.

† Proc. Phys. Soc. No. 71, August 1896, p. 180; Phil. Mag. xli. p. 488.

made by Mr. Oliver Heaviside, F.R.S. [*], that a distribution of electricity on the surface of a charged body such as to give zero disturbance at all points inside the surface is an equilibrium distribution. Since **F** satisfies curl $\mathbf{F}=0$ and **F** vanishes inside the surface, it follows that on the outside of the surface **F** is perpendicular to the surface. This implies that Ψ is constant over the surface. But as neither the electric force **E** nor the mechanical force experienced by each part of the charged surface (calculated from the Maxwell stress) is normal to the surface, I felt unable to accept the validity of Mr. Heaviside's assumption until I discovered {§ 15} that **F** is the mechanical force on an isolated moving unit charge, and that the term $-\mathrm{V}\mathbf{GD}$, which appears in the expression for the force experienced by the surface, has no influence in causing convection of electricity from one part of the surface to another. Here **D** is the electric displacement, and **G** the "magnetic current" $\mu\frac{d\mathbf{H}}{dt}$.

Since Ψ is a true potential for the mechanical force **F**, I have called Ψ the "electric convection potential."

When there has been established the boundary condition that Ψ is constant over the surface, with its consequence that there is zero disturbance within the surface, it is very easy to show that the distribution on an ellipsoid is the same for motion as for rest. Suppose the ellipsoid to have the same distribution as when it is at rest, so that $\sigma=qp/4\pi abc$, where q is the charge, a, b, c the axes of the ellipsoid, and p the perpendicular from the centre upon the tangent-plane at the point. Through any internal point M as vertex draw a slender double cone intercepting two areas N, N′ on the surface. Now the electric force due to a moving point-charge is still radial and still varies inversely as the square of the distance, although it alters with change of direction of the radius vector. Thus it follows just as in electrostatics, since $\sigma \propto p$, that the effects at M of N and N′ are exactly equal and opposite. The whole surface can be treated in the same manner, and thus it follows that $\mathbf{E}=0$ at all internal points. Hence $\mathbf{H}=0$ also. Thus the assumed distribution is in equilibrium and is therefore the actual distribution. Thus the motion has no influence upon the distribution, and this result is true whatever the direction of motion with respect to the axes of the ellipsoid.

In order to find the state of the field near a charged ellipsoid moving with velocity u parallel to the axis of x, it is necessary to find a value of Ψ which shall be constant over

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[*] 'Electrical Papers,' vol. ii. p. 514.

the surface of the ellipsoid, shall vanish at infinity, and shall satisfy (7). We see at once that if $f(x, y, z)$ satisfies $\nabla^2 f=0$, then $f(x/\sqrt{\alpha}, y, z)$ satisfies (7). Now from electrostatics we know that

$$\Phi=\int_\lambda^\infty \frac{d\lambda}{\sqrt{(a'^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where λ is connected with x, y, z by the relation

$$\frac{x^2}{a'^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

satisfies $\nabla^2\Phi=0$.

Hence

$$\Psi=\int_\lambda^\infty \frac{\mathrm{A}\,d\lambda}{\sqrt{(a'^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \quad . \quad . \quad . \quad (8)$$

where λ is connected with x, y, z by the relation

$$\frac{x^2}{(a'^2+\lambda)}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \quad . \quad . \quad . \quad (9)$$

satisfies (7).

Writing a^2 for $\alpha a'^2$, (8) and (9) become

$$\Psi=\int_\lambda^\infty \frac{\mathrm{A}\,d\lambda}{\sqrt{(a^2+\alpha\lambda)(b^2+\lambda)(c^2+\lambda)}}; \quad . \quad . \quad (10)$$

$$\frac{x^2}{a^2+\alpha\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1. \quad . \quad . \quad . \quad (11)$$

This value of Ψ is constant over the surface of the ellipsoid a, b, c, for $\lambda=0$ at all points of this surface; it also vanishes at infinity, and it satisfies (7). It is therefore the value of Ψ required. To find the constant A we make σ have its proper value $q/4\pi bc$ at the end of axis a.

Now

$$\sigma=\frac{\mathrm{K}}{4\pi}\mathrm{E}_n=\frac{\mathrm{K}}{4\pi}\mathrm{E}_1$$

at the end of the axis.

But by (5) $$\mathrm{E}_1=-\frac{d\Psi}{dx}.$$

Again, at $x=a$, $y=z=0$ we have $d\lambda/dx=2a/\alpha$ and consequently

$$\frac{d\Psi}{dx}=\frac{d\Psi}{d\lambda}\frac{d\lambda}{dx}=-\frac{\mathrm{A}}{abc}\cdot\frac{2a}{\alpha}.$$

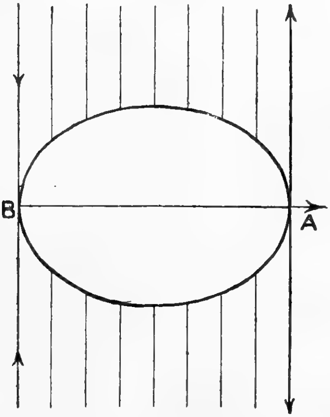
Hence $$\mathrm{A}=\frac{q\alpha}{2\mathrm{K}}.$$

Thus, as Prof. Morton has also shown by the same method,

$$\Psi=\int_{\lambda}^{\infty}\frac{q\alpha\, d\lambda}{2\mathrm{K}\sqrt{(a^2+\alpha\lambda)(b^2+\lambda)(c^2+\lambda)}}. \quad . \quad . \quad (12)$$

Now I have shown {§ 21} that if there is a surface A carrying a charge q, and any surface B is found for which Ψ is constant, then a charge q placed upon B and allowed to acquire an equilibrium distribution will produce at all points not inside B the same effect as the charged surface A.

Fig. 1.



Hence the ellipsoid (11) when carrying a charge q produces at all points not inside itself exactly the same disturbance as the ellipsoid a, b, c with the same charge.

If we make $a=b=c=0$, the surfaces of equal "convection potential" are the ellipsoids given by

$$\frac{x^2}{\alpha}+y^2+z^2=\lambda.$$

They are therefore all similar to each other. Thus the ellipsoid of this form produces exactly the same effect as a point-charge at its centre, and thus an ellipsoid of this form takes the place of the sphere in electrostatics. An ellipsoid with its axes in the ratios $\sqrt{\alpha}:1:1$ I have called a Heaviside Ellipsoid, since Mr. Heaviside* was the first to draw attention to its importance in the theory of moving charges. Whatever be the ratios $a:b:c$, the equipotential surfaces

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* 'Electrical Papers,' vol. ii. p. 514.

approximate to Heaviside ellipsoids as λ is made very great. The value of Ψ at the surface λ is $\frac{q\sqrt{\alpha}}{\sqrt{\lambda}}$.

Putting $c=b$ so that we have an ellipsoid of revolution, the axis of revolution being the axis of x, we see by taking $\lambda=-b^2$ that a uniformly-charged line of length $2\sqrt{a^2-b^2\alpha}$ lying along the axis of x produces exactly the same effect as the ellipsoid a, b, b. It may therefore be called its "image." When $b=a$ this length becomes $2au/v$. Thus, when a charged sphere is at rest it produces the same effect as a point-charge at its centre. When the sphere is in motion it produces the same effect as a uniformly-charged line whose length bears to the diameter of the sphere the same ratio as the velocity of the sphere bears to the velocity of light. When $u=v$, so that the sphere moves with the velocity of light, the line becomes the diameter of the sphere; and the same is true for an ellipsoid. Since when $u=v$ each element of the charged line produces a disturbance which is confined to the plane through the element perpendicular to the direction of motion {(46)}, it follows that the disturbance is entirely confined between the planes $x=\pm a$. Between them the electric force is radial to the axis of x and has exactly the same value, viz. $q/a\mathrm{K}\rho$, as if the line had been of infinite length and had had the same line-density $q/2a$. Here ρ stands for $\{y^2+z^2\}^{\frac{1}{2}}$. The magnetic force is by (3) $qu/a\rho$. Hence the field between the planes $x=\pm a$ is independent of x. There are therefore no displacement-currents except in the two bounding-planes. There is an outward radial current in the front plane and an inward current in the back plane, the total amount of current in each case being qu, equal in amount to the convection-current carried by the ellipsoid.

It appears, however, that at the velocity of light *any* distribution on *any* surface is in equilibrium. For the value of Ψ at any point near a moving point-charge is {(43)}

$$\Psi=\frac{q\sqrt{\alpha}}{\mathrm{K}\sqrt{x^2/\alpha+y^2+z^2}},$$

and this vanishes when $u=v$ (so that $\alpha=0$), even when $x=0$. Thus the value of Ψ for a point-charge vanishes, and the value of Ψ for *any* distribution being derivable from that for a point-charge by integration, it follows that Ψ has the constant value zero everywhere. Hence the charge is in equilibrium however it may be distributed. The same result follows from the expression {§ 19} for the force between two moving charges. When they move parallel to each other with the speed of light the force between them vanishes.

If the ellipsoid is more oblate than Heaviside's the limiting internal surface of ellipsoidal form, whose action is the same as that of the ellipsoid, is a disk of radius $\sqrt{b^2-a^2/\alpha}$, the axis of the disk coinciding with the axis of x.

The form of the lines of the electric force **E** due to an ellipsoid of revolution is easily found. Putting ρ^2 for y^2+z^2, the equilibrium surfaces are given by

$$\frac{x^2}{a^2+\alpha\lambda}+\frac{\rho^2}{b^2+\lambda}=1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Now the mechanical force **F** is normal to this surface, and therefore

$$\frac{\mathrm{F}_\rho}{\mathrm{F}_1}=\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)},$$

where

$$\mathrm{F}_\rho{}^2=\mathrm{F}_2{}^2+\mathrm{F}_3{}^2.$$

But by (5),

$$\mathrm{E}_1=\mathrm{F}_1 \quad \text{and} \quad \mathrm{E}_\rho=\mathrm{F}_\rho/\alpha\,;$$

so that

$$\frac{\mathrm{E}_\rho}{\mathrm{E}_1}=\frac{1}{\alpha}\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Now consider the conic

$$\frac{x^2}{a^2+\alpha\nu}+\frac{\rho^2}{b^2+\nu}=1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

The tangent of the angle which the geometrical tangent makes with the axis of x is

$$-\frac{x(b^2+\nu)}{\rho(a^2+\alpha\nu)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

But if the point x, ρ lies on both (13) and (15), it follows that

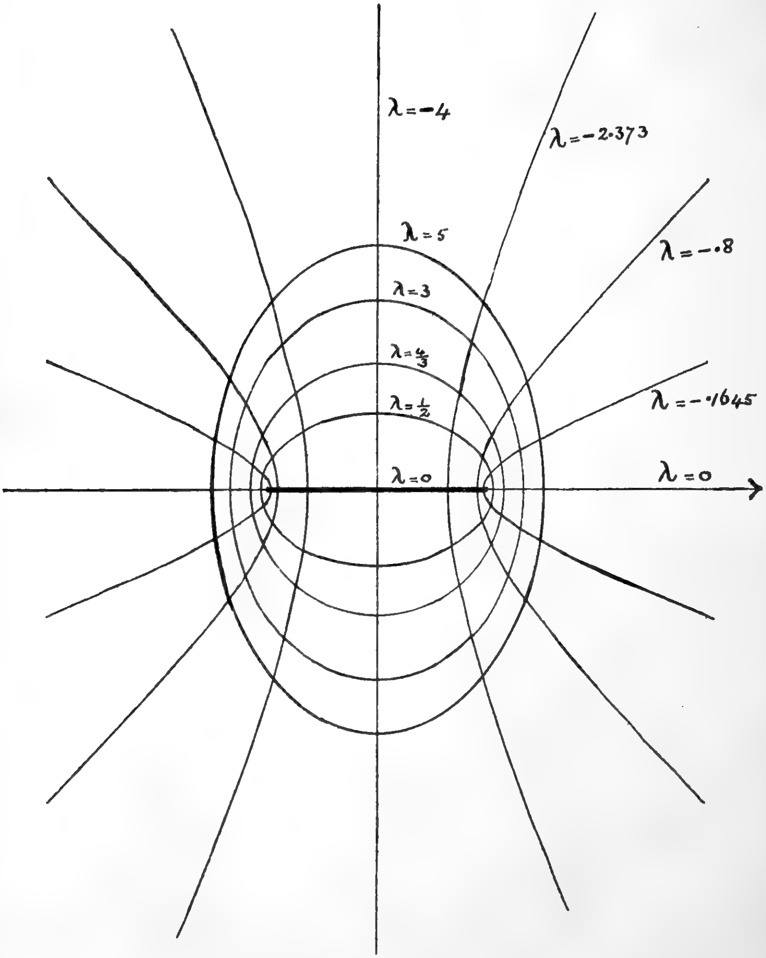
$$-\frac{x(b^2+\nu)}{\rho(a^2+\alpha\nu)}=\frac{1}{\alpha}\frac{\rho(a^2+\alpha\lambda)}{x(b^2+\lambda)}.$$

Hence by (14) and (16) the electric force is always tangential to the conic (15). But this conic has exactly the same equation as the equilibrium surfaces. Thus the single equation (13) represents both the equilibrium surfaces and the lines of electric force.

If any point x, ρ be taken, there are two values of λ which will satisfy (13) considered as a quadratic in λ. One value corresponds to an ellipsoidal equilibrium surface; the other to a hyperbolic surface whose lines of intersection with planes passing through the axis of x are the lines of electric force. The lines of electric force for a charged sphere in motion are not radial but form a series of hyperbolas.

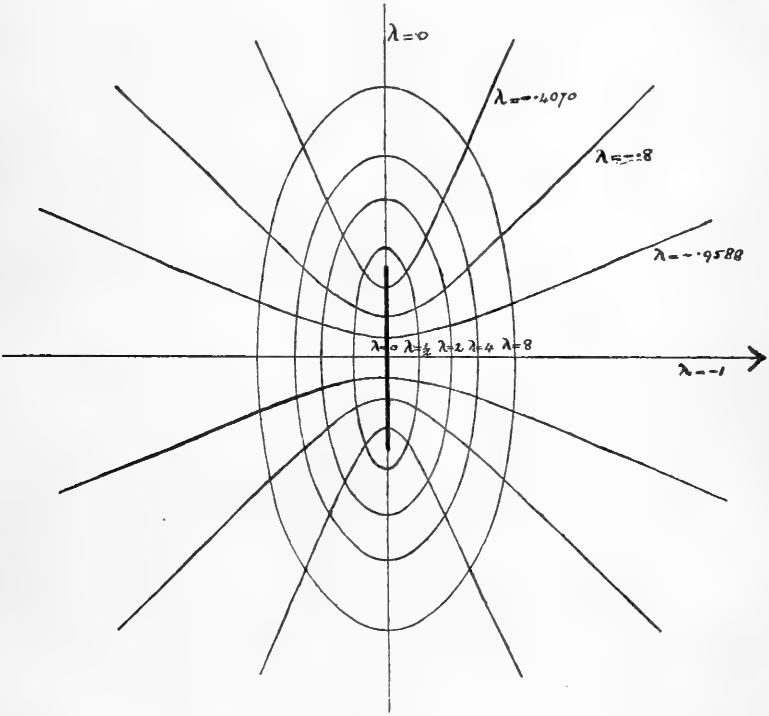
Figs. 2 and 3 show the forms of the equilibrium surfaces and of the lines of electric force, for a line and a disk respectively, when $\alpha=\frac{1}{4}$ so that $u/v=\cdot 866$. In fig. 2 the curve marked $\lambda=\frac{4}{3}$ is a circle, a section of the sphere of which the

Fig. 2.



line marked $\lambda=0$ is the "image." The semi-length of the line and the radius of the disk are each taken as unity.

Fig. 3.



I have attempted to find the lines of the mechanical force **F**, these being everywhere perpendicular to the equilibrium surfaces. But the process involved an impracticable integration, and thus led to no result.

I will now write down the values of E and H at any point near the ellipsoid of revolution with axes a, b, b. Instead of λ it will be convenient to take as the parameter of any one of the equilibrium surfaces its x axis and to denote this by h. Thus

$$h^2 = a^2 + \alpha\lambda\,;$$

and consequently if we put l^2 for $a^2 - \alpha b^2$, so that l is the semi-length of the line which is the "image" of the ellipsoid, we have

$$b^2 + \lambda = \frac{h^2 - l^2}{\alpha}.$$

The value of Ψ in terms of h thus becomes

$$\Psi = \frac{q\alpha}{\mathrm{K}} \int_h^\infty \frac{dh}{h^2 - l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Equation (11) now becomes

$$\frac{x^2}{h^2} + \frac{\rho^2\alpha}{h^2 - l^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

so that instead of the cylindrical coordinates x and $\rho (= \sqrt{y^2 + z^2})$ we can take h and ϕ where

$$x = h \cos\phi, \qquad \rho = \frac{\sqrt{h^2 - l^2}}{\sqrt{\alpha}} \sin\phi. \quad . \quad . \quad . \quad (19)$$

From (18) we have in terms of h and ϕ

$$\frac{dh}{dx} = \frac{(h^2 - l^2)\cos\phi}{h^2 - l^2\cos^2\phi}, \qquad \frac{dh}{d\rho} = \frac{h\sqrt{h^2 - l^2}\sin\phi\sqrt{\alpha}}{h^2 - l^2\cos^2\phi}.$$

Hence

$$\mathrm{E}_1 = -\frac{d\Psi}{dh} \cdot \frac{dh}{dx} = \frac{\alpha q \cos\phi}{\mathrm{K}(h^2 - l^2\cos^2\phi)}, \quad . \quad . \quad . \quad . \quad (20)$$

$$\mathrm{E}_\rho = -\frac{1}{\alpha}\frac{d\Psi}{dh} \cdot \frac{dh}{d\rho} = \frac{qh\sin\phi\sqrt{\alpha}}{\mathrm{K}\sqrt{h^2 - l^2}\,(h^2 - l^2\cos^2\phi)}, \quad . \quad (21)$$

$$\mathrm{H} = \mathrm{K}u\mathrm{E}_\rho = \frac{quh\sin\phi\sqrt{\alpha}}{\sqrt{h^2 - l^2}(h^2 - l^2\cos^2\phi)}. \quad . \quad . \quad . \quad . \quad (22)$$

I now pass on to calculate the total energy possessed by the ellipsoid when in motion along its axis of figure. In making the calculation I shall suppose that $a^2 > \alpha b^2$, *i. e.*, that l^2 is positive. The case in which $a^2 < \alpha b^2$ can be deduced by the appropriate mathematical transformation.

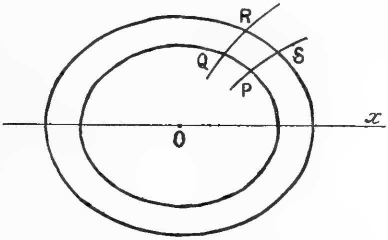
I have shown {§ 22} that the total energy, viz. the volume integral of $\frac{\mathrm{K}\mathrm{E}^2 + \mu\mathrm{H}^2}{8\pi}$, due to the motion of a charge on any surface, is

$$\mathrm{W} = \tfrac{1}{2}q\Psi_0 + 2\mathrm{T},$$

where Ψ_0 is the value of the convection-potential at the surface of the body, and T is the magnetic part of the energy, viz., the volume integral of $\mu\mathrm{H}^2/8\pi$.

Now in fig. 4 let the ellipsoid PQ be determined by h, and the ellipsoid RS by $h+dh$. Let the angular coordinate of

Fig. 4.



P and S be ϕ, and let that of Q and R be $\phi+d\phi$. Then the area PQRS

$$= \frac{d(x,\rho)}{d(h,\phi)}\,dh\,d\phi = \left(\frac{dx}{dh}\frac{d\rho}{d\phi} - \frac{dx}{d\phi}\frac{d\rho}{dh}\right)dh\,d\phi$$

$$= \frac{h^2-l^2\cos^2\phi}{\sqrt{a}\;\sqrt{h^2-l^2}}\,dh\,d\phi.$$

Now if the area PQRS revolve about the axis Ox the volume of the ring traced out is

$$2\pi\rho\,\frac{d(x,\rho)}{d(h,\phi)}\,dh\,d\phi = \frac{2\pi(h^2-l^2\cos^2\phi)\sin\phi}{a}\,dh\,d\phi.$$

Thus for the magnetic part of the energy we have

$$\mathrm{T} = \iint \frac{\mu\mathrm{H}^2}{8\pi}\,2\pi\rho\,\frac{d(x,\rho)}{d(h,\phi)}\,dh\,d\phi$$

$$= \frac{\mu q^2u^2}{4}\iint \frac{h^2\sin^3\phi\,dh\,d\phi}{(h^2-l^2)(h^2-l^2\cos^2\phi)}.$$

Since λ goes from 0 to ∞ h goes from a to ∞. The limits of ϕ are 0 and π.

Now

$$\int_0^\pi \frac{\sin^3\phi\,d\phi}{h^2-l^2\cos^2\phi} = -\frac{1}{l^2}\left[\cos\phi - \frac{h^2-l^2}{2hl}\log\frac{h+l\cos\phi}{h-l\cos\phi}\right]_0^\pi$$

$$= \frac{1}{l^2}\left\{2 - \frac{h^2-l^2}{hl}\log\frac{h+l}{h-l}\right\}.$$

Hence

$$\mathrm{T} = \frac{\mu q^2u^2}{4l^2}\int_a^\infty \left(\frac{2h^2}{h^2-l^2} - \frac{h}{l}\log\frac{h+l}{h-l}\right)dh$$

$$= \frac{\mu q^2u^2}{4l^2}\left[h - \frac{h^2+l^2}{2l}\log\frac{h+l}{h-l}\right]^\infty.$$

When h is large the quantity in [ ]

$$=h-l\left(\frac{h^2}{l^2}+1\right)\left(\frac{l}{h}+\frac{l^3}{3h^3}\dots\right)$$

vanishing when $h=\infty$.

Thus, making use of $\mu\mathrm{K}v^2=1$ we have for the magnetic energy

$$\mathrm{T}=\frac{q^2u^2}{4l\mathrm{K}v^2}\left\{\frac{a^2+l^2}{2l^2}\log\frac{a+l}{a-l}-\frac{a}{l}\right\}.$$

Now by (17) we have at the surface of the ellipsoid

$$\Psi_0=\frac{q\alpha}{\mathrm{K}}\int_a^\infty\frac{dh}{h^2-l^2}=\frac{q\alpha}{2\mathrm{K}l}\log\frac{a+l}{a-l}.$$

Hence the total electromagnetic energy of the ellipsoid is

$$\mathrm{W}=\tfrac{1}{2}q\Psi_0+2\mathrm{T}=\frac{q^2}{4\mathrm{K}l}\left\{\left(1+\frac{u^2a^2}{v^2l^2}\right)\log\frac{a+l}{a-l}-2\frac{u^2a}{v^2l}\right\}. \tag{23}$$

Here we must remember that $l^2=a^2-\alpha b^2$.

(A) *Energy of Heaviside Ellipsoid.* If we put $a/l=\mathrm{S}$ and make S large we have

$$\mathrm{W}=\frac{q^2\mathrm{S}}{2\mathrm{K}a}\left\{\left(1+\frac{u^2\mathrm{S}^2}{v^2}\right)\left(\frac{1}{\mathrm{S}}+\frac{1}{3\mathrm{S}^3}+\dots\right)-\frac{u^2\mathrm{S}}{v^2}\right\}$$

$$=\frac{q^2}{2\mathrm{K}a}\left(1+\frac{1}{3}\frac{u^2}{v^2}\right)\text{ when }\mathrm{S}=\infty. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{24}$$

This corresponds to the Heaviside ellipsoid, for when $\mathrm{S}=\infty$ $a^2=\alpha b^2$. The energy of the same ellipsoid at rest is

$$\frac{q^2\sqrt{\alpha}}{2\mathrm{K}a}\cdot\frac{v}{u}\sin^{-1}\frac{u}{v}.$$

(B) *Energy of a Sphere.* Putting $b=a$ we have $l=au/v$, and thus

$$\mathrm{W}=\frac{q^2}{2\mathrm{K}a}\left(\frac{v}{u}\log\frac{v+u}{v-u}-1\right). \quad . \quad . \quad . \quad . \tag{25}$$

If u is small compared with v we have

$$\mathrm{W}=\frac{q^2}{2\mathrm{K}a}\left(1+\frac{2}{3}\frac{u^2}{v^2}+\dots\right).$$

It will be found that as far as u^2/v^2 the magnetic energy is

$$\frac{q^2u^2}{3\mathrm{K}av^2}=\frac{\mu q^2u^2}{3a}$$

as has been found by Mr. Heaviside*. It follows from this

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* 'Electrical Papers,' vol. ii. p. 505.

that as far as terms in u^2/v^2 the electric part of the energy is unaltered by the motion.

(C) *Energy of a very slender Ellipsoid.* When the ellipsoid is so slender that b^2/a^2 may be neglected in comparison with unity we have

$$W=\frac{q^2}{2Ka}\left\{\left(1+\frac{u^2}{v^2}\right)\log\frac{2a}{b\sqrt{1-\frac{u^2}{v^2}}}-\frac{u^2}{v^2}\right\}. \quad . \quad (26)$$

When u/v is small, this becomes

$$W=\frac{q^2}{2Ka}\left\{\left(1+\frac{u^2}{v^2}\right)\log\frac{2a}{b}+\frac{1}{2}\frac{u^2}{v^2}\right\}.$$

(D) *Energy of a Disk.*

When $a^2<\alpha b^2$ the ellipsoid is more oblate than Heaviside's, and l^2 becomes negative. In this case let us write

$$r^2=b^2-\frac{a^2}{\alpha},$$

so that r is the radius of the disk which is the "image" of the ellipsoid a, b. Then writing $\sqrt{-1}=i$ we have from (23)

$$W=\frac{q^2}{4Kir\sqrt{\alpha}}\left(1-\frac{u^2a^2}{v^2r^2\alpha}\right)\log\frac{1+i\sqrt{\alpha}\,r/a}{1-i\sqrt{\alpha}\,r/a}+\frac{q^2u^2a}{2Kv^2r^2\alpha}.$$

But

$$\frac{1}{i}\log\frac{1+xi}{1-xi}=2\left(x-\frac{x^3}{3}+\frac{x^5}{5}\dots\right)=2\tan^{-1}x,$$

so that (23) becomes

$$W=\frac{q^2}{2Kr\sqrt{\alpha}}\left\{\left(1-\frac{u^2a^2}{v^2r^2\alpha}\right)\tan^{-1}\frac{r\sqrt{\alpha}}{a}+\frac{u^2a}{v^2r\sqrt{\alpha}}\right\}. \quad (27)$$

When $a=0$ we find for the energy of a disk of radius r moving along its axis

$$W=\frac{q^2\pi}{4Kr\sqrt{\alpha}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In all these cases it will be found that when $u=v$ the energy becomes infinite, so that it would seem to be impossible to make a charged body move at a greater speed than that of light.

XLIII. *Observations on Light Propagated in a Dielectric Normal to the Lines of Force.* By D. B. Brace, *Ph.D., Professor of Physics, University of Nebraska*[*].

[Plate IV.]

THE following experiments were made to determine the effect upon polarized light of a transparent medium in a magnetic field.

The observation of an effect upon polarized light in a transparent medium in an electric field, suggests a similar condition in a magnetic field. The state of polarization of such media should affect, to a greater or less extent, the propagation of polarized light. In magnetically polarized media, the Faraday effect is always present except for propagation exactly at right angles to the lines of force, a condition impossible to realize for any finite stream of light. Even with very small divergence of the rays this might mask any new effect. In any case the Faraday effect must be eliminated.

The magnet used was forged from Norway iron. The cores are 8 in. in diameter and 36 in. long. These are bolted to a horizontal bed-plate. The movable pole-pieces are 8 × 8 × 13 in. and drilled to receive various pole terminals. Each core is surrounded with three coils, each having about 1100 turns of No. 10 wire, making approximately 650 pounds and 21 ohms resistance. The six coils may be thrown into any combination by a commutator. The total weight of the magnet approximates 4000 pounds.

A Lippich half-shade polarizer was used, an image of the sun being thrown upon it and observed with a telescope through the analyser. The image, which appeared elliptical, was kept in the middle of the field so that it was bisected. While the difficulty of maintaining these conditions was considerable, the sensibility was several times that of any other source.

In the first experiment, a plate of Faraday glass, index $1{\cdot}76_D$ and 166 × 170 × 15 mm. was used. The ends were silvered so that a ray could be reflected any number of times.

This was placed between the poles of the magnet and adjusted to allow a varying number of reflexions of the ray, making from 5 to 11 passages, or a total distance of from ·8 m. to 1·8 m.

With the plane of polarization at 45° to the lines of force a

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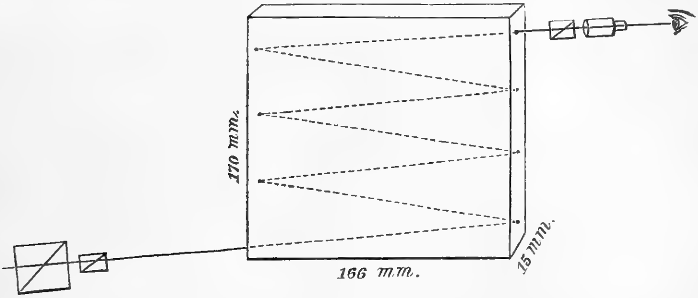
[*] Communicated by the Author.

pronounced effect was observed when the magnet was excited. This effect did not disappear when the plane of polarization was shifted 45° on either side of this position, indicating the Faraday effect. This result showed that even for a very narrow beam of light, the divergence was sufficient to introduce this effect, notwithstanding the direction was shifted so as to make the mean direction of the rays as near normal to the field as possible. Difficulty was experienced both from depolarization at the sides of the plate and from imperfect annealing of the glass, so that this form of the experiment had to be abandoned.



An estimate of the Faraday effect for a small bundle of rays will show the impossibility of eliminating this factor in this way. Assume for Faraday glass, $n_D = 1{\cdot}76$, and a rotary power of $\phi_D = 2 \times 10^{-5}$ radians per cm. If $H = 5250$ the rotation per cm. is $\rho_0 = 6°$ approximately. Allowing 200 cm. for the total path of ray and 1 cm. aperture, we have, from Verdet's law, the total rotation for the most divergent ray

$$\rho = 6° \times 200 \times {\cdot}005 = 6°,$$

or an average of 3°. Taking the sensibility of the instrument at ·01°, the aperture must evidently be ·003 cm. to eliminate the Faraday effect in this arrangement.

It is evident from Verdet's law, $\rho = \rho_0 \cos \theta$, that ρ changes sign when θ passes through $\pi/2$ and $3\pi/2$, hence if a ray is reflected by a plane parallel to the lines of force, ρ will change sign at each reflexion and the total Faraday effect will be zero for an even number of passages. Hence this effect can be eliminated for divergent rays.

This condition was realized in the following experiment. Two cylinders of Faraday glass, $n_D = 1{\cdot}76$, each 22 cm. long and 2·4 cm. in diameter were placed end to end in the field and polarized light was sent through the system and reflected back by a mirror, adjusted so as to be parallel to the lines of force. The analyser showed the presence of depolarization in the glass cylinders due to accidental double refraction, although the same had been carefully selected and annealed. No change in the two halves of the field of view could be detected when the magnet was excited. The difficulty of obtaining the highest sensibility with glass on account of depolarization led to the following experiment with a liquid.

In Plate IV., DE is the tube 48 cm. long, containing the liquid to be examined, surrounded with a water-jacket mounted between the poles of the electromagnet. The faces of the latter were 50·7 cm. by 8·7 cm., accurately planed, and 7·3 cm. apart and placed as near parallel as possible, so as to insure a uniform field of parallel lines of force. The optical system and tube was mounted on an arm A, 70 in. long, pivoted at P to a frame support resting on the bed-plate of the magnet. The other end of the arm carrying the analyser and telescope rested on the carrier S, which could be moved laterally by the micrometer screw M. Sunlight passing through the lens R was brought to a focus at L after being reflected by the right-angle prism P. The ray was then passed through the tube four times by reflexion at the mirrors E and D, making a total distance of 198 cm.; and finally through the telescope T, which formed an image of the half-shade a little in front of the analyser N. This image was observed with the small telescope *t*, which could be moved in the direction of the ray by the micrometer screw *m*. The "Lippich" L could be rotated so that the ray *rr* could be polarized in any plane with respect to the magnetic field. The divided circle of the analyser could be read by the microscopes *cc* to ·003°. On account of diffused light and internal reflexion, the ray was cut down by successive diaphragms to about 1 cm. diameter.

In some of the experiments the optical arrangements shown at R_1 and R_2 were used. In the former the ray, after passing through the Lippich L, was reflected by a double right-angle prism in the direction *r*. This prism was found, however, to depolarize slightly, and did not give as great a sensibility as R. In R_2 the return ray passed within the mounting of the Lippich, close to the nicols. This was used with small tubes of liquids with only two passages of the ray, and with less than half the air-gap between the poles. Better definition was obtained with this system, but the arrangement R, on account of the greater length of the ray, gave the highest sensibility.

The liquid used was a saturated solution of iodide of potassium and red iodide of mercury ($n_D = 1{\cdot}69$ approximately). The transmitted light was of a deep orange colour. In order to obtain good definition, water from the water-mains was allowed to run through the jacket for a couple of days, and the tube was taken out and rolled at intervals.

Under the most favourable conditions of sunlight and homogeneity of the liquid, four observations were obtained during the course of a month, in which the sensibility of the apparatus was such that ·01° change in the plane of polarization could with certainty be detected. In other observations ·02° could usually be observed.

In making the observations, the heliostat was adjusted until the image of the sun appeared in the centre of the field of the polarizer, and the analyser was then adjusted until each half of the field appeared of the same intensity. The magnet was then excited and, if any change of intensity occurred, the arm A was shifted by the screw M until the two halves became of the same intensity again. The current was then reversed and the same adjustments made until no effect could be detected when the current was reversed. In this case the mirror E was evidently made parallel to the lines of force.

By rotating the polarizing combination, the same observations could be made with the plane of polarization making different angles with the lines of force. Observations were made over a range of 90°, but the chief ones were those at 45° with the lines of force; of these four were obtained with a sensibility of ·01°. The current used was generally 9 amperes, but in one case it reached 40 amperes, but this probably did not increase the field by more than 50 per cent.

In no case could an effect be observed after the neutral position was obtained, indicating that there is no double refraction within the limits of the observations.

During each experiment a determination of the sensibility was made. This was effected in two different ways. The first was to determine the rotation of the analyser which would just produce a perceptible change in the two halves of the field. The second was to shift the arm A when the current was on and read the micrometer screw M. The pitch of this screw was 1/32 of an inch. 1/40 of a turn of M could certainly be detected during several of the most favourable observations. These two independent methods serve as a check on each other. An iris diaphragm placed in front of the telescope T allowed the aperture to be varied to give the best results. The following schedule illustrates the varying conditions, the sensibility being determined by one or both of the two methods :—

				Sensibility.	Aperture.	
Jan. 25,	1894.	9 Amp.	....	2/40	10 mm.	No effect.
„ 27,	„	„	Poor Sun.	·02° certain.	„	„
„ 29,	„	„	....	·02°	„	„
„ 30,	„	„	....	·01° certain.	„	„
Feb. 1,	„	„	....	·01° to ·02° 2/40	„	„
„ 8,	„	„	....	1/40 to 2/40	4 mm.	„
„ 12,	„	„	....	·01° 1/40 certain.	5 mm.	„
„ 13,	„	„	....	1/40 certain.	8 mm.	„
„ 24,	„	„	....	·01°	8 mm.	„
„ 26,	„	„	....	1/40 certain.	6 mm.	„

Knowing the specific rotary power ϕ_D of the liquid and the intensity of the magnetic field, the rotation ρ by the latter method may be calculated from Verdet's law. ϕ was determined by wrapping some 25 turns of wire about the tube, and correcting for the ends (·992) and observing the double rotation (1° 24′) of the plane of polarization produced by a reversal of the current (11·73 amp.), the ray being from sodium light and traversing the tube once.

Substituting in Verdet's equation, we have

$$\phi_D = \frac{84'}{4\pi \times 25 \times 1{\cdot}173} \times \frac{1}{{\cdot}992} \times \tfrac{1}{2} = {\cdot}115'.$$

The intensity of the field H was measured with a spiral of bismuth whose constant was known. With an exciting current of 9 amp., H was found to be 2600 C.G.S. units. One turn of the screw M corresponds to an angle whose sine is ·000446, since 1/40 × 1/32 × 1/70 = ·00001116, the length of the arm A being 70 in. and the pitch 1/32 in.

The total length of the liquid column was 196·64 cm., and that of the glass caps of the tube 3·2 cm. = 1·4 cm. of liquid, making a total equivalent length of 198 cm. Hence we have

$$\rho = \frac{{\cdot}115'}{60} \times 198 \times {\cdot}00001116 \times 2600 = {\cdot}011^\circ,$$

the observed value being ·01° by the first method of determining the sensibility.

It is assumed that the transmitted light was homogeneous and that of sodium, which it was not, but appeared a deep orange-colour.

Incidentally a few observations were made on Verdet's law near the neutral plane. The arm A could be moved from this plane by about 10 turns of the screw M. As homogeneous light could not be used, the results are only approximate for the constant, but should be correct for the cosine variation. Through the entire range the tint of the two halves of the field did not change materially. After each turn of the screw the analyser was set to equalize the intensities of the two halves of the field. The readings of the circle are given directly. The aperture was 3 mm.

Rotations of M.	Readings of circle of analyser.	Angular deviation from neutral plane.	Differences or rotations.
0	42°·16	0′·00	0°·00
1/40	42·17	0·038	0·01
1	42·55	1·5	0·39
2	42·96	3·0	0·80
3	43·37	4·5	1·21
4	43·80	6·0	1·64
5	44·23	7·5	2·07
Current reversed. −5	40·09	−7·5	−2·07
+5	44·25	+7·5	+2·09
−5	40·08	−7·5	−2·08
+5	44·23	+7·5	+2·07
0	42·16	0·00	0·00

Second Set of Observations.

(In this set the magnetic field diminished 2 per cent. by the end of the experiment.)

Rotations of M.	Readings of circle of analyser.	Angular deviation from neutral plane.	Differences or rotations.
0	42°·16	0′·00	0°·00
1	42·55	1·5	0·39
2	42·98	3·0	0·82
3	43·37	4·5	1·21
4	43·81	6·0	1·65
5	44·22	7·5	2·06
6	44·63	9·0	2·47
7	45·02	10·5	2·86
8	45·41	12·0	3·25
9	45·78	13·5	3·62
10	46·13	15·0	3·97
11	46·50	16·5	4·34
0	42·16	0·0	0·00

These results show that Verdet's law holds up to 90°, the Faraday effect changing sign for greater angles near the neutral plane, as the law requires.

To compare the intensity of the field calculated from this formula with the direct measurement, a mean of several observations on the rotation were taken for a given measured magnetic field. This mean rotation of the plane of polarization for five turns of M was 2°·075. Assuming mean $\phi=\phi_D$,

$$H=\frac{2^\circ{\cdot}075}{\frac{\cdot115}{60}\times198\times\cdot0022}=2500 \text{ C.G.S. units, approximately.}$$

The value obtained with the bismuth spiral was 2600 C.G.S. units.

The discrepancy was probably due to the assumption of the mean wave-length being that of sodium, while in fact it was greater.

A slight calculation will show the greatest variation in the mean refractive index which could have occurred without being detected.

If v is the difference in phase between the two component vibrations, and θ the angle of a component with the resultant, then if

$$x=r\cos\left(2\pi\left(\frac{t}{T}-\frac{Z}{\lambda}\right)-v\right),$$

$$y=r\sin 2\pi\left(\frac{t}{T}-\frac{Z}{\lambda}\right),$$

$$\therefore\quad \tan\theta=\cot 2\pi\left(\frac{t}{T}-\frac{Z}{\lambda}\right)-\sin v,$$

since v is small, and

$$\frac{dv}{d\theta}=\frac{\cos v}{\cos^2\theta}=\frac{1}{\cos^2\theta}=\frac{100}{49{\cdot}9}=2 \text{ when } \theta \text{ is } 45^\circ.$$

Since $d\theta<\cdot01^\circ$, $dv<\cdot02^\circ$, or $<\cdot000055\lambda$ for a field of 2600 C.G.S. units and a distance of 198 cm. If λ_1 and λ_2 are the wave-lengths of the components, and n_1 and n_2 the reciprocals of the velocities, we have the following equation:—

$$n_2-n_1=n_1\frac{\lambda_1-\lambda_2}{\lambda_2}.$$

Knowing dv for 198 cm., we can find the same for one wave-length, or $\lambda_1-\lambda_2$. Substituting in the above equation, we have

$$n_2-n_1<2{\cdot}8\times10^{-7} \text{ for 2600 C.G.S. units.}$$

If the effect on each component was of an opposite nature, as the nature of the polarizing effect would lead us to suppose, the change in the refractive index could not have been greater than in the eighth decimal place, if any.

A variation of this order would be much smaller than that already observed by Kerr and others in a dielectric under electrostatic strain. Quincke* has observed the effect in the case of carbon bisulphide. The relative retardation which he obtained for the two components in an electrostatic field of one C.G.S. unit was $3\cdot2\times10^{-7}\lambda$ per centimetre. In this case the effect is proportional to the square of the intensity of the field. If we assume the same law for a magnetic field and reduce the value $5\cdot5\times10^{-5}\lambda$—the limit of sensibility for 198 cm. and 2600 C.G.S. units—we have $4\cdot1\times10^{-14}\lambda$ as the relative retardation per centimetre in one C.G.S. unit field which could have been observed.

In the one case we have used the electrostatic unit, and in the other the electromagnetic unit, since we are comparing dynamic effects and must refer them to the dynamic units in each case.

We may also conclude, according to Maxwell's view of the state of polarization and stress in such a medium, that the pressure at right angles to, and the tension along, the lines of force $\left(\text{equal in both cases to } \frac{H^2}{8\pi}\right)$ affect the propagation of light by an amount less than $2\cdot0\times10^{-14}\lambda$ for a C.G.S. unit of intensity per centimetre.

Physical Laboratory, University of Nebraska,
Lincoln, Neb., U.S.A., June 19, 1897.

XLIV. *Change of Phase on Reflexion at the Surface of Highly-absorbing Media.* By Helen G. Klaassen, *Associate and Lecturer, Newnham College* †.

§ 1. If a layer of aniline dye is deposited upon glass by dropping a little of the dye dissolved in alcohol upon the glass and allowing the alcohol to evaporate, just at the edge of the layer so deposited we often see a film, which in reflected light is of quite a different colour from the rest of the dye. If, for instance, we deposit fuchsine, which has a green surface-colour, we often seen a straw-coloured film round the edge. The contrast in colour is more marked if instead of looking directly at the fuchsine we turn over the glass and look at it from the other side. The light reflected at the surface of separation of glass and fuchsine is bluish green, while the film round the edge is red or orange.

I attempted to make larger patches of this straw-coloured film, and succeeded in obtaining thin films of continuous

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* Wied. *Ann.* xix. p. 729 (1883).

† Communicated by the Author.

thickness, which exhibit not only this colour but a series of different colours.

The best method I have yet found of making such films is to pour a little alcoholic solution of required concentration (found by trial) upon a piece of glass, then to wave the glass through the flame of a Bunsen burner. The fuchsine does not melt; but the alcohol is rapidly removed and leaves an amorphous film of the dye. If the concentration and quantity are rightly chosen, the film has the appearance of a brilliant mirror of variegated colour. The following table gives the colour of the film as seen by reflected light at normal incidence. The colours are given in order of increasing thickness of film. The Roman numerals are used for reference only; they are not a measure of the thickness of the film:—

TABLE A.

Thickness of film.	I. Colour of film as seen directly.	II. Colour of the film as seen from the back of the glass on which it is deposited.
I.	Like glass.	Like glass.
II.	Greenish.	Bluish neutral colour, gradually changing with increasing thickness to deep red.
III.	Straw-colour.	Bright orange.
IV.	Blue.	Blue.
V.	Yellowish green.	Pink.
VI.	Green.	Bluish green.

§ 2. *Examination of Film in Monochromatic Light reflected at Normal Incidence.*

Red Light.—Fuchsine is most transparent to red light. Numbers of interference-bands can be seen in this colour.

In Table A:

III. is a thickness which gives maximum brightness in red light;

IV. is a thickness which gives minimum brightness in red light;

V. is a thickness which gives maximum brightness in red light.

The maxima and minima appear to be in the same positions if we look at the film directly or if we see it through the glass on which it is deposited.

Yellow Light.—Only a few maxima and minima can be distinctly seen. If we take an extremely thin film in which the gradient of thickness is very small, looking directly at the film, the intensity rises at first with increase of thickness, reaching a maximum in thickness III. near the corresponding maximum for red light. If, however, we look at the film through the glass on which it is deposited, the intensity first diminishes with increase of thickness, reaching a minimum in the bluish neutral colour of II.; it then rises to a maximum in the bright orange of III.

Green Light.—Fuchsine is very opaque to green light. No maximum or minimum can be seen if we look directly at the film. Seen from the back of the glass, there is one minimum which occurs in the dark red of II., that is at a somewhat greater thickness than the first minimum for yellow light.

Blue Light.—In blue light, for which fuchsine is considerably more transparent than for green light, there is a marked difference in the position of the maxima and minima as seen directly and through the glass. Looking directly, there is a maximum near III. and a minimum between IV. and V. nearer to IV. than to V.

The following is a table showing roughly the result of these observations. The Roman numerals refer to the same thicknesses as in Table A:—

TABLE B.

Thickness.	I. Looking at film directly. Incident Light.				II. Looking at the film from back of the glass. Incident Light.			
	Red.	Yellow.	Green.	Blue.	Red.	Yellow.	Green.	Blue.
Zero						Min.		
							Min.	
								Min.
III.	Max.	Max.		Max.	Max.	Max.		
IV.	Min.	Min.			Min.	Min.		
				Min.				Max.
V.	Max.	Max.			Max.	Max.		

§ 3. *Table* A *and Table* B.

The appearance of the film (Table A) is different according as the film is seen directly or through the glass. This difference is partly due to the fact that the selective reflexion at the surface of a highly absorbing substance depends upon the refrangibility of the neighbouring transparent medium *. The surface-colour of a *thick* layer of fuchsine seen through the glass on which it is deposited is much more blue in tint

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* E. Wiedemann, Pogg. *Ann.* cli.

than the surface-colour of fuchsine in air, but a comparison of the two columns of Table A shows that this is not the complete explanation of their difference.

From Table B we see that the maxima and minima in blue light do not occur at the same thickness of film as seen directly or through the glass. In a transparent film there is a maximum or minimum at zero thickness. This is not the case with fuchsine films observed in yellow, green, or blue light. The peculiarities of Table B can be expressed by saying that the interference-bands as seen from the glass side are displaced towards the thicker end of the film, the displacement increasing as we go from the red to the blue end of the spectrum. The second column shows the displacement of the first minimum in yellow and green light and of the first minimum and maximum in blue light.

If the film is viewed directly the bands in blue light are displaced in the opposite direction. As the displacement is towards the thin end of the film the first minima disappear.

Table B shows a very large wave-length for blue light in fuchsine compared with that for red.

This is in agreement with the recent measurements of Pflüger[*], who finds a much larger anomalous dispersion than Wernicke[†].

This is no doubt due to the fact that my films, like those of Pflüger, are very free from alcohol. Some films which I have kept for two years show no sign of change.

§ 4. The suggestion might be made that the maxima and minima observed in these highly absorbing films are not mainly due to the reflexion at the back surface. To test this I smeared some oil on the film and viewed it from the back of the glass on which the fuchsine is deposited. In the very thin films there is no longer a diminution of illumination with increase of thickness. The minima of Table B and the peculiar colouring of Table A, 2nd column, thickness II. disappear. The maxima and minima in blue light are also displaced towards the thin end of the film. If we turn over the glass and view the film through the oil, the maxima and minima are also displaced, but in the opposite direction. The extent of the displacement is such as to make the maxima and minima occur at approximately the same thickness of

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[*] Wied. *Ann.* lvi. p. 424.

[†]

	Refractive Index.
A	1·73
B	1·81
C	1·90
g	1·31
H	1·54

film as seen from either side. If we may judge from such very rough observations that this is the case, it seems that a thin plate of fuchsine placed between two transparent media of equal refrangibility is free from the peculiarities of Table B.

The change of phase on reflexion absorbent–transparent
= that on reflexion transparent–absorbent $\pm\pi$ approx.

Theoretically this relation should hold if the change of phase on refraction is small compared with that on reflexion.

§ 5. *The Change of Phase on Reflexion: Glass–Fuchsine.*

Wernicke[*] and Potier[†] have exhibited the change of phase which occurs on reflexion from glass to fuchsine. Potier[‡] gives the following measurements:—

δ = retardation on reflexion from crown-glass to fuchsine.

	D.	E.	*b*.	F.	*g*.
$-\dfrac{\delta}{2\pi}$	·41	·30	·25	·18	0

Wernicke's[§] results are not in exact agreement, but he also finds a gradual increase of acceleration as we pass from the blue to the red end of the spectrum, starting from nearly 0 at the blue end and amounting to π at the red end.

Potier made similar measurements, using a flint-glass of high index, 1·96, which he says is higher than the index for any ray in fuchsine. He finds a maximum acceleration $-\dfrac{\delta}{2\pi} = \cdot 34$ in the green between D and E, falling off to 0 at either end of the spectrum. Potier points out that his observations are a qualitative confirmation of the theoretical formula

$$\tan\delta = \frac{2n_0 g}{n^2+g^2-n_0^2};$$

n_0 = refractive index of transparent medium,
n = refractive index of absorbing medium,
g = coefficient of absorption $\times \dfrac{\lambda}{4\pi}$.

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[*] Pogg. *Ann.* clix. p. 198.

[†] *Comptes Rendus*, cviii. p. 995.

[‡] I have the permission of Prof. Potier for inserting the minus sign before $\dfrac{\delta}{2\pi}$.

[§] See second note on page 352. Wernicke's measurement of refractive index.

Pflüger (*loc. cit.*) gives a much higher maximum index. Sirk finds the value 2·44 for C., using melted fuchsine.

Pflüger attributes the disagreement between his and Wernicke's measurements to the fact that Wernicke's fuchsine was not free from alcohol.

§ 6. *Change of Phase on Reflexion: Air–Fuchsine.*

I have observed roughly the change of phase which occurs on reflexion from air to fuchsine and to other dyes in the following way:—

Press a lens on to a layer of dye and examine in monochromatic light. The layer of dye must be deposited so as to have a very level surface, shown by its bright reflecting power. It must also be so thick that the light coming from the lower surface is negligible. In consequence of the change of phase on reflexion, air-dye, we do not get a black centre to Newton's rings.

By examination of the central spot in light of different colours we can make an estimate of the change of phase. The following substances were examined, and each showed a maximum abnormal change of phase in the coloured light which is most absorbed by the dye,—fuchsine, methyl aniline violet, acid green, eosin.

This effect is most marked in methyl aniline violet, where the central spot is very bright in yellow and green light. The direction of the change is an increased retardation where the absorption is great.

If n^2+g^2 for fuchsine were greater than 1 for all rays, these observations would be in qualitative agreement with the formula

$$\tan\delta=\frac{2n_0g}{n^2+g^2-n_0^2};$$

both exhibit a maximum retardation in the rays for which the absorption is greatest, falling off to π at either end of the spectrum.

Pflüger has shown that for light between F and g the index of refraction in fuchsine <1.

If we view the rings produced by pressing a lens on to fuchsine in blue light we might expect to find a bright centre.

On the contrary, the centre was dark in blue light and, as as far as I could tell, the change of phase differed very little from π.

§ 7. *Colours of thin Plates of Fuchsine. Table* A *and Table* B.

A change of phase differing from 0 or π has been observed on reflexion at the surface of fuchsine, the amount of this change depending on the index of refraction of the adjacent transparent medium. If a film of fuchsine is situated between air and glass we should therefore expect to find a displacement of the interference-bands.

Change of Phase on Reflexion.

Glass–fuchsine (§ 5).—An increasing acceleration as we go from the violet to the red end of the spectrum = approx. 0 at the violet end and approx. π at the red end.

Air–fuchsine (§ 6).—A retardation reaching a maximum in the green and falling off to approx. π at either end of the spectrum.

Fuchsine–glass (§ 4).—Assume = glass–fuchsine $\pm\pi$.

Fuchsine–air (§ 4).—Assume = air–fuchsine $\pm\pi$.

Table B, *1st Column.*

Towards the violet end of the spectrum the light reflected dye–glass is abnormally retarded relatively to that reflected air–dye. That is to say, the light coming from the second surface is abnormally retarded relatively to that coming from the first. This causes the maxima and minima when the film is viewed directly to be displaced towards the thin end of the film, § 3.

Table B, *2nd Column.*

Towards the violet end of the spectrum the light reflected dye–air is abnormally accelerated relatively to that reflected glass–dye. That is to say, the light coming from the second surface is abnormally accelerated relatively to that coming from the first. This causes the maxima and minima when the film is viewed through the glass to be displaced towards the thick end of the film, the displacement being greatest towards the violet end of the spectrum.

Thin films of acid green, and of methyl aniline violet, show a concordant displacement of interference-bands.

As it is impossible to regulate the gradient of thickness in these thin films, no attempt has been made at accurate measurement.

XLV. *On the Propagation of Waves along connected Systems of Similar Bodies.* *By* Lord RAYLEIGH, *F.R.S.*[*]

FOR simplicity of conception the bodies are imagined to be similarly disposed at equal intervals (a) along a straight line. The position of each body, as displaced from equilibrium, is supposed to be given by *one* coordinate, which for the rth body is denoted by ψ_r. A wave propagated in one direction is represented by taking ψ_r proportional to $e^{i(nt+r\beta)}$. If we take an instantaneous view of the system, the disturbance is periodic when $r\beta$ increases by 2π, or when ra increases by $2\pi a/\beta$. This is the wave-length, commonly denoted by λ; so that, if $k=2\pi/\lambda$, $k=\beta/a$. The velocity of propagation (V) is given by $V=n/k$; and the principal object of the investigation is to find the relation between n or V and λ.

The forces acting upon each body, which determine the vibration of the system about its configuration of equilibrium, are assumed to be due solely to the neighbours situated within a limited distance. The simplest case of all is that in which there is no mutual reaction between the bodies, the kinetic and potential energies of the system being then given by

$$T=\tfrac{1}{2}A_0\Sigma\dot{\psi}_r^2, \qquad P=\tfrac{1}{2}C_0\Sigma\psi_r^2, \quad . \quad . \quad . \quad . \quad (1)$$

similarity requiring that the coefficients A_0, C_0 be the same for all values of r. In this system each body vibrates independently, according to the equation

$$A_0\ddot{\psi}_r+C_0\psi_r=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$n^2=C_0/A_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The frequency is of course independent of the wave-length in which the phases may be arranged to repeat themselves, so that n is independent of k, while V equal to n/k varies inversely as k, or directly as λ. The propagation of waves along a system of this kind has been considered by Reynolds.

In the general problem the expression for P will include also products of ψ_r with the neighbouring coordinates . . . ψ_{r-2}, ψ_{r-1}, ψ_{r+1}, ψ_{r+2} . . ., and a similar statement holds good for T. Exhibiting only the terms which involve r, we may write

$$T= \ldots +\tfrac{1}{2}A_0\dot{\psi}_r^2-A_1\dot{\psi}_r\dot{\psi}_{r-1}-A_1\dot{\psi}_r\dot{\psi}_{r+1}$$
$$-A_2\dot{\psi}_r\dot{\psi}_{r-2}-A_2\dot{\psi}_r\dot{\psi}_{r+2}-\ldots, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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[*] Communicated by the Author.

$$P = \ldots + \tfrac{1}{2}C_0\psi_r^2 - C_1\psi_r\psi_{r-1} - C_1\psi_r\psi_{r+1}$$
$$- C_2\psi_r\psi_{r-2} - C_2\psi_r\psi_{r+2} - \ldots, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where A_1, A_2, ... C_1, C_2, ... are constants, finite for a certain number of terms and then vanishing. The equation for ψ_r is accordingly

$$A_0\ddot{\psi}_r - A_1\ddot{\psi}_{r-1} - A_1\ddot{\psi}_{r+1} - A_2\ddot{\psi}_{r-2} - A_2\ddot{\psi}_{r+2} - \ldots$$
$$+ C_0\psi_r - C_1\psi_{r-1} - C_1\psi_{r+1} - C_2\psi_{r-2} - C_2\psi_{r+2} - \ldots = 0. \quad (6)$$

In the other equations of the system r is changed, but without entailing any other alteration in (6). Since all the quantities ψ are proportional to e^{int}, the double differentiation is accounted for by the introduction of the factor $-n^2$. Making this substitution and remembering that ψ_r is also proportional to $e^{ir\beta}$, we get as the equivalent of any one of the equations (6)

$$n^2(A_0 - A_1e^{-i\beta} - A_1e^{i\beta} - A_2e^{-2i\beta} - A_2e^{2i\beta} - \ldots)$$
$$= C_0 - C_1e^{-i\beta} - C_1e^{i\beta} - C_2e^{-2i\beta} - C_2e^{2i\beta} - \ldots,$$

or

$$n^2 = \frac{C_0 - 2C_1\cos ka - 2C_2\cos 2ka - \ldots}{A_0 - 2A_1\cos ka - 2A_2\cos 2ka - \ldots}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which β is replaced by its equivalent ka. By (7) n is determined as a function of k and of the fundamental constants of the system.

In most of the examples which naturally suggest themselves A_1, A_2, ... vanish, so that T has the same simple form as in (1). If we suppose for brevity that A_0 is unity, (7) becomes

$$n^2 = C_0 - 2C_1\cos ka - 2C_2\cos 2ka - \ldots \quad . \quad . \quad . \quad (8)$$

When the waves are very long, k approximates to zero. In the limit

$$n^2 = C_0 - 2C_1 - 2C_2 - \ldots \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we call the limiting value C, we may write (8) in the form

$$n^2 = C + 4C_1\sin^2(\tfrac{1}{2}ka) + 4C_2\sin^2(ka) + \ldots . \quad . \quad (10)$$

In an important class of cases C vanishes, that is the frequency diminishes without limit as λ increases. If at the same time but one of the constants C_1, C_2, ... be finite, the equation simplifies. For example, if C_1 alone be finite,

$$n = 2C_1^{\frac{1}{2}}\sin(\tfrac{1}{2}ka). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In any case when n is known V follows immediately. Thus from (10) with C evanescent, we get

$$\frac{V^2}{a^2} = C_1\frac{\sin^2\dfrac{ka}{2}}{\left(\dfrac{ka}{2}\right)^2} + 4C_2\frac{\sin^2 ka}{k^2a^2} + 9C_3\frac{\sin^2\dfrac{3ka}{2}}{\left(\dfrac{3ka}{2}\right)^2} + \ldots . \quad (12)$$

A simple case included under (11) is that of a stretched string, itself without mass, but carrying unit loads at equal intervals (a)*. The expression for the potential energy is

$$P = \ldots + \frac{T_1}{2a}(\psi_r - \psi_{r-1})^2 + \frac{T_1}{2a}(\psi_{r+1} - \psi_r)^2 + \ldots, \quad . \quad (13)$$

T_1 representing the tension. Thus by comparison with (5)

$$C_0 = 2T_1/a, \qquad C_1 = T_1/a, \qquad C_2 = 0, \quad \&c.;$$

so that by (8)

$$n^2 = \frac{2T_1}{a} - \frac{2T_1}{a}\cos ka,$$

$$n = \sqrt{\left(\frac{T_1}{a\mu}\right)} \, . \, 2\sin\left(\tfrac{1}{2}ka\right), \quad . \quad . \quad . \quad . \quad (14)$$

μ being introduced to represent the mass of each load with greater generality. The value of V is obtained by division of (14) by k. In order more easily to compare with a known formula we may introduce the longitudinal density ρ, such that $\mu = a\rho$. Thus

$$V = \frac{n}{k} = \sqrt{\left(\frac{T_1}{\rho}\right)} \, . \, \frac{\sin\left(\frac{1}{2}ka\right)}{\frac{1}{2}ka}, \quad . \quad . \quad . \quad (15)$$

reducing to the well-known value of the constant velocity of propagation along a uniform string when a is made infinitesimal. Lord Kelvin's wave-model ('Popular Lectures and Addresses,' vol. i. 2nd ed. p. 164) is also included under the class of systems for which P has the form (13).

Another example in which again C_2, $C_3 \ldots$ vanish is proposed by Fitzgerald †. It consists of a linear system of rotating magnets (fig. 1) with their poles close to one another and disturbed to an amount small compared with the distance apart of the poles. The force of restitution is here proportional to the sum of the angular displacements (ψ) of contiguous magnets, so that P is proportional to

Fig. 1.

$$\ldots + (\psi_r + \psi_{r-1})^2 + (\psi_r + \psi_{r+1})^2 + \ldots .$$

Here $C_1 = -\frac{1}{2}C_0$, and (8) gives $n^2 = C_0(1 + \cos ka)$,

or
$$n = n_0 \, . \cos\left(\tfrac{1}{2}ka\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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* See 'Theory of Sound,' §§ 120, 148.

† Brit. Assoc. Report, 1893, p. 689.

if n_0 represent the value of n appropriate to $k=0$, *i.e.* to infinitely long waves. Here $n=0$, when $\lambda=2a$. In this case $\psi_{r+1}=-\psi_r$.

Fitzgerald considers, further, a more general linear system constructed by connecting a series of equidistant wheels by means of indiarubber bands. "By connecting the wheels each with its next neighbour we get the simplest system. If to this be superposed a system of connexion of each with its next neighbour but two, and so on, complex systems with very various relations between wave-length and velocity can be constructed depending on the relative strengths of the bands employed." If the bands may be crossed, the potential energy takes the form

$$\begin{aligned} K=&\tfrac{1}{2}\gamma_1(\psi_r\pm\psi_{r-1})^2+\tfrac{1}{2}\gamma_1(\psi_r\pm\psi_{r+1})^2 \\ &+\tfrac{1}{2}\gamma_2(\psi_r\pm\psi_{r-2})^2+\tfrac{1}{2}\gamma_2(\psi_r\pm\psi_{r+2})^2 \\ &+\ldots, \quad \ldots\ldots\ldots\ldots \quad (17) \end{aligned}$$

which is only less general than (5) by the limitation

$$\tfrac{1}{2}C_0\pm C_1\pm C_2\pm\ldots=0. \quad \ldots\ldots \quad (18)$$

Prof. Fitzgerald appears to limit himself to the lower sign in the alternatives, so that C in (10) vanishes. This leads to (12), from which his result differs, but probably only by a slip of the pen.

If we take the upper sign throughout, (8) becomes

$$-\tfrac{1}{4}n^2=C_1\cos^2\frac{ka}{2}+C_2\cos^2\frac{2ka}{2}+C_3\cos^2\frac{3ka}{2}+\ldots \quad . \quad (19)$$

It may be observed that Prof. Fitzgerald's system will have the most general potential energy possible (5), if in addition to the elastic connexions between the wheels there be introduced a force of restitution acting upon each wheel independently.

As an example in which C_2 is finite as well as C_1, let us imagine a system of masses of which each is connected to its immediate neighbours on the two sides by an elastic rod capable of bending but without inertia. Here

$$\begin{aligned} P=\ldots&+\tfrac{1}{2}c(2\psi_{r-1}-\psi_{r-2}-\psi_r)^2+\tfrac{1}{2}c(2\psi_r-\psi_{r-1}-\psi_{r+1})^2 \\ &+\tfrac{1}{2}c(2\psi_{r+1}-\psi_r-\psi_{r+2})^2+\ldots \quad \ldots \quad (20) \end{aligned}$$

A comparison with (5) gives

$$C_0=6c, \qquad C_1=4c, \qquad C_2=-c,$$

so that

$$C=C_0-2C_1-2C_2=0.$$

Accordingly by (10),

$$n^2 = 16c\sin^2(\tfrac{1}{2}ka) - 4c\sin^2 ka = 16c\sin^4(\tfrac{1}{2}ka),$$

or

$$n = 4c^{\frac{1}{2}}\sin^2(\tfrac{1}{2}ka) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Thus far we have considered the propagation of waves along an unlimited series of bodies. If we suppose that the total number is m and that they form a *closed* chain, ψ must be such that

$$\psi_{r+m} = \psi_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

from which it follows that

$$\beta = ka = 2s\pi/m, \qquad . \quad . \qquad (23)$$

s being an integer. Thus (8) becomes

$$n^2 = C_0 - 2C_1\cos(2s\pi/m) - 2C_2\cos(4s\pi/m) - \ldots. \quad . \qquad (24)$$

When the chain, composed of a limited series of bodies, is open at the ends instead of closed, the general problem becomes more complicated. A simple example is that treated by Lagrange, of a stretched massless string, carrying a finite number of loads and fixed at its extremities*. The open chain of m magnets, for which

$$P = \tfrac{1}{2}(\psi_1+\psi_2)^2 + \tfrac{1}{2}(\psi_2+\psi_3)^2 + \ldots + \tfrac{1}{2}(\psi_{m-1}+\psi_m)^2, \quad . \qquad (25)$$

is considered by Fitzgerald. The equations are

$$\left.\begin{array}{l} \psi_1(1-n^2) + \psi_2 = 0, \\ \psi_1 + \psi_2(2-n^2) + \psi_3 = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \psi_{r-1} + \psi_r(2-n^2) + \psi_{r+1} = 0, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \psi_{m-2} + \psi_{m-1}(2-n^2) + \psi_m = 0, \\ \psi_{m-1} + \psi_m(1-n^2) = 0, \end{array}\right\} \quad . \quad . \qquad (26)$$

of which the first and last may be brought under the same form as the others if we introduce ψ_0 and ψ_{m+1}, such that

$$\psi_0 + \psi_1 = 0, \qquad \psi^{m} + \psi_{m+1} = 0. \quad . \quad . \quad . \quad . \qquad (27)$$

If we assume

$$\psi_r = \cos nt \sin(r\beta - \tfrac{1}{2}\beta), \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

the first of equations (27) is satisfied. The second is also satisfied provided that

$$\sin m\beta = 0, \quad \text{or} \quad \beta = s\pi/m. \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

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* 'Theory of Sound,' § 120.

The equations (26) are satisfied if

$$2\cos\beta+2-n^2=0,$$

that is, if

$$n=2\cos\frac{s\pi}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

In (29), (30) s may assume the m values 1 to m inclusive. In the last case $n=0$, and $\beta=\pi$; and from (28),

$$\psi_r=-(-1)^r\cos nt.$$

The equal amplitudes and opposite phases of consecutive coordinates, *i. e.* angular displacements of the magnets, gives rise to no potential energy, and therefore to a zero frequency of vibration. In the first case ($s=1$) the angular deflexions are all in the same direction, and the frequency is the highest admissible. If at the same time m be very great, n reaches its maximum value, corresponding to parallel positions of all the magnets. If we call this value N, the generalized form of (30), applicable to all masses and degrees of magnetization, may be written

$$n=\mathrm{N}\cos\frac{s\pi}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

If m is great and s relatively small, (31) becomes approximately

$$n=\mathrm{N}\left(1-\frac{s^2\pi^2}{8m^2}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

so that as s diminishes we have a series of frequencies approaching N as an upper limit, and are reminded (as Fitzgerald remarks) of certain groups of spectrum lines. A nearer approach to the remarkable laws of Balmer for hydrogen* and of Kayser and Runge for the alkalies is arrived at by supposing s constant while m varies. In this case, instead of supposing that the whole series of lines correspond to various modes of one highly compound system, we attribute each line to a different system vibrating in a given special mode. Apart from the better agreement of frequencies, this point of view seems the more advantageous as we are spared the necessity of selecting and justifying a special high value of m. If we were to take $s=2$ in (31) and attribute to m integral values 3, 4, 5, . . . , we should have a series of frequencies of the same general character as the hydrogen series, but still differing considerably in actual values.

There is one circumstance which suggests doubts whether

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* Viz. $n=\mathrm{N}(1-4m^{-2})$, with $m=3, 4, 5$, &c.

the analogue of radiating bodies is to be sought at all in ordinary mechanical or acoustical systems vibrating about equilibrium. For the latter, even when gyratory terms are admitted, give rise to equations involving the square of the frequency; and it is only in certain exceptional cases, *e. g.* (31), that the frequency itself can be simply expressed. On the other hand, the formulæ and laws derived from observation of the spectrum appear to introduce more naturally the *first* power of the frequency. For example, this is the case with Balmer's formula. Again, when the spectrum of a body shows several doublets, the intervals between the components correspond closely to a constant difference of frequency, and could not be simply expressed in terms of squares of frequency. Further, the remarkable law, discovered independently by Rydberg and by Schuster, connecting the convergence frequencies of different series belonging to the same substance, points in the same direction.

What particular conclusion follows from this consideration, even if force be allowed to it, may be difficult to say. The occurrence of the first power of the frequency seems suggestive rather of kinematic relations* than of those of dynamics.

XLVI. *On the Relations between the Coaxial Minors of a Determinant.* *By* E. J. NANSON, *M.A.*†

1. IT has been shown by Major MacMahon‡ that the coaxial minors of any determinant of order n are connected by 2^n-n^2+n-2 relations, the determinant itself being included under the term coaxial minor. In this Journal Dr. Muir§ has given a simple proof of this theorem and, in the case of an inversely symmetrical determinant, has obtained one of the two relations which connect the coaxials of a determinant of the fourth order.

In the present communication it is proposed, first, to find in several forms the second relation between the coaxials of the special determinant considered by Dr. Muir; and second, to find the relations between the coaxials of the general determinant of the fourth order.

2. The special determinant to be considered may be written

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* *E. g.* as in the phases of the moon.

† Communicated by the Author.

‡ Phil. Trans. clxxxv. (1894) p. 146.

§ Phil. Mag. Dec. 1894, p. 537.

$$\begin{vmatrix} 1 & h & \frac{1}{g} & 1 \\ \frac{1}{h} & 1 & f & 1 \\ g & \frac{1}{f} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Denoting this determinant by Δ and its coaxial first minors by

$$2A-2, \quad 2B-2, \quad 2C-2, \quad 2D-2,$$

we have

$$f+\frac{1}{f}=2A, \quad g+\frac{1}{g}=2B, \quad h+\frac{1}{h}=2C, \quad . \quad . \quad (1)$$

$$fgh+\frac{1}{fgh}=2D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\Delta=\frac{1}{fgh}(1-f)(1-g)(1-h)(1-fgh) \,; \quad . \quad . \quad (3)$$

and the relations in question are to be found by eliminating the three quantities f, g, h from the five equations (1), (2), (3).

3. Let

$$f=e^{i\alpha}, \quad g=e^{i\beta}, \quad h=e^{i\gamma} \,;$$

then from (1), (2) we have

$$\cos\alpha=A, \quad \cos\beta=B, \quad \cos\gamma=C \,; \quad . \quad . \quad . \quad (4)$$

$$\cos(\alpha+\beta+\gamma)=D \,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and from (3) we find

$$\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)=k, \quad . \quad . \quad (6)$$

where

$$k=A+B+C+D-\tfrac{1}{2}\Delta-1. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now let

$$\xi=\sin\alpha, \quad \eta=\sin\beta, \quad \zeta=\sin\gamma \,;$$

then from (5) we get

$$A\eta\zeta+B\zeta\xi+C\xi\eta+\lambda=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\lambda=D-ABC \,;$$

and from (6) we have

$$\eta\zeta+\zeta\xi+\xi\eta+\mu=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\mu=k-\mathrm{BC}-\mathrm{CA}-\mathrm{AB}.$$

Multiply (8) by $\eta\zeta$, $\zeta\xi$, $\xi\eta$, and we find

$$\left.\begin{array}{l} \lambda\eta\zeta+m\mathrm{C}\zeta\xi+n\mathrm{B}\xi\eta+mn\mathrm{A}=0, \\ l\mathrm{C}\eta\zeta+\lambda\zeta\xi+n\mathrm{A}\xi\eta+nl\mathrm{B}=0, \\ l\mathrm{B}\eta\zeta+m\mathrm{A}\zeta\xi+\lambda\xi\eta+lm\mathrm{C}=0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

where

$$l=1-\mathrm{A}^2, \quad m=1-\mathrm{B}^2, \quad n=1-\mathrm{C}^2.$$

Eliminating $\eta\zeta$, $\zeta\xi$, $\xi\eta$ from (8), (9), (10), we obtain the required relations in the form

$$\begin{vmatrix} \lambda & m\mathrm{C} & n\mathrm{B} & mn\mathrm{A} \\ l\mathrm{C} & \lambda & n\mathrm{A} & nl\mathrm{B} \\ l\mathrm{B} & m\mathrm{A} & \lambda & lm\mathrm{C} \\ \mathrm{A} & \mathrm{B} & \mathrm{C} & \lambda \\ 1 & 1 & 1 & \mu \end{vmatrix}=0.$$

4. This result may be reduced to a more symmetrical form. Substituting for l, m, n, λ their values, multiplying the fourth row by BC, CA, AB, and adding to rows 1, 2, 3, we find

$$\begin{Vmatrix} \mathrm{D} & \mathrm{C} & \mathrm{B} & \mathrm{A}+\mathrm{BCD}-\mathrm{A}(\mathrm{B}^2+\mathrm{C}^2) \\ \mathrm{C} & \mathrm{D} & \mathrm{A} & \mathrm{B}+\mathrm{CAD}-\mathrm{B}(\mathrm{C}^2+\mathrm{A}^2) \\ \mathrm{B} & \mathrm{A} & \mathrm{D} & \mathrm{C}+\mathrm{ABD}-\mathrm{C}(\mathrm{A}^2+\mathrm{B}^2) \\ \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D}-\mathrm{ABC} \\ 1 & 1 & 1 & \mu \end{Vmatrix}=0.$$

Now substitute for μ its value, multiply columns 1, 2, 3 by BC, CA, AB, and add to the last column; thus

$$\begin{Vmatrix} \mathrm{D} & \mathrm{C} & \mathrm{B} & \mathrm{A}+2\mathrm{BCD} \\ \mathrm{C} & \mathrm{D} & \mathrm{A} & \mathrm{B}+2\mathrm{CAD} \\ \mathrm{B} & \mathrm{A} & \mathrm{D} & \mathrm{C}+2\mathrm{ABD} \\ \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D}+2\mathrm{ABC} \\ 1 & 1 & 1 & k \end{Vmatrix}=0, \quad . \quad (11)$$

where

$$k=\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}-\tfrac{1}{2}\Delta-1. \quad , \quad . \quad . \quad . \quad (7)$$

5. On expanding the determinant formed with the first

four rows and replacing A by $\frac{1}{2}A+1$ &c. we get the result given by Dr. Muir*.

Again, rejecting in turn each of the first four rows in (11), we obtain four different formulæ each expressing Δ as a rational function of the coaxial first minors.

6. The relation connecting Δ with any three of the four quantities A, B, C, D may readily be found. Thus from (8), (9) we have

$$(1-A)\eta\zeta+(1-B)\zeta\xi+(1-C)\xi\eta+\theta=0, \quad . \quad . \quad (12)$$

where

$$\begin{aligned}\theta&=\mu-\lambda\\&=k-BC-CA-AB-D+ABC\\&=(A-1)(B-1)(C-1)-\tfrac{1}{2}\Delta.\end{aligned}$$

Now from (12) we deduce, by the method previously applied to (8), that

$$\begin{vmatrix}\theta & m(1-C) & n(1-B) & mn(1-A)\\ l(1-C) & \theta & n(1-A) & nl(1-B)\\ l(1-B) & m(1-A) & \theta & lm(1-C)\\ 1-A & 1-B & 1-C & \theta\end{vmatrix}=0.$$

This is the equation connecting Δ, A, B, C. It is of the fourth order in Δ and symmetrical in A, B, C. Three similar relations are found by replacing any one of the letters A, B, C, by D.

7. Thus nine different relations have been found between the coaxials of an inversely symmetric determinant of the fourth order. Five of these are unique and connect A, B, C, D; B, C, D, Δ; C, A, D, Δ; A, B, C, Δ respectively. The remaining four each express Δ as a rational function of A, B, C, D. The nine relations are of course equivalent to not more than two independent equations.

8. Next consider the general determinant of the fourth order,

$$\begin{vmatrix}a & h & g' & x\\ h' & b & f & y\\ g & f' & c & z\\ x' & y' & z' & d\end{vmatrix}$$

Denoting this determinant by Δ and its coaxial minors by

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* Phil. Mag. Dec. 1894, p. 540.

(ab), (abc), &c., let

$$2\mathrm{A}=(bcd)+2bcd-b(cd)-c(bd)-d(bc)$$
$$2\mathrm{B}=(cad)+2cad-c(ad)-a(cd)-d(ca)$$
$$2\mathrm{C}=(abd)+2abd-a(bd)-b(ad)-d(ab)$$
$$2\mathrm{D}=(abc)+2abc-a(bc)-b(ca)-c(ab)$$
$$\mathrm{P}=bc-(bc)\qquad \mathrm{Q}=ca-(ca)\qquad \mathrm{R}=ab-(ab)$$
$$\mathrm{L}=ad-(ad)\qquad \mathrm{M}=bd-(bd)\qquad \mathrm{N}=cd-(cd)$$
$$k=\mathrm{A}a+\mathrm{B}b+\mathrm{C}c+\mathrm{D}d-abcd$$
$$-\tfrac{1}{2}\left\{\Delta+(bc)(ad)+(ca)(bd)+(ab)(cd)\right\},$$

so that A, B, C, D, P, Q, R, L, M, N, k are all functions of coaxials. Then we find that

$$ff'=\mathrm{P}\qquad gg'=\mathrm{Q}\qquad hh'=\mathrm{R}$$
$$xx'=\mathrm{L}\qquad yy'=\mathrm{M}\qquad zz'=\mathrm{N}$$
$$fy'z+f'yz'=2\mathrm{A}$$
$$gz'x+g'zx'=2\mathrm{B}$$
$$hx'y+h'xy'=2\mathrm{C}$$
$$fgh+f'g'h'=2\mathrm{D}$$
$$ghyz'+g'h'y'z+hfzx'+h'f'z'x+fgxy'+f'g'x'y=2k.$$

In accordance with MacMahon's theorem it must be possible to eliminate in two different ways the twelve quantities $f, g, h,\ f'\ g'\ h',\ x, y, z,\ x'\ y'\ z'$ from the eleven equations last written and so obtain two equations connecting the coaxials.

9. In order to effect this elimination let

$$\frac{fy'z}{f'yz'}=e^{2i\alpha},\qquad \frac{gz'x}{g'zx'}=e^{2i\beta},\qquad \frac{hx'y}{h'xy'}=e^{2i\gamma},$$

then we find that

$$\mathrm{A}=\sqrt{\mathrm{MNP}}\cos\alpha,\quad \mathrm{B}=\sqrt{\mathrm{NLQ}}\cos\beta,\quad \mathrm{C}=\sqrt{\mathrm{LMR}}\cos\gamma,$$
$$\mathrm{D}=\sqrt{\mathrm{PQR}}\cos(\alpha+\beta+\gamma),$$
$$k=\sqrt{\mathrm{QRMN}}\cos(\beta+\gamma)+\sqrt{\mathrm{RPNL}}\cos(\gamma+\alpha)$$
$$+\sqrt{\mathrm{PQLM}}\cos(\alpha+\beta),$$

and so the elimination to be performed is practically the same as before. The result is seen to be

$$\begin{vmatrix} \mathrm{DL} & \mathrm{CQ} & \mathrm{BR} & \mathrm{AQRL}+2\mathrm{BCD} \\ \mathrm{CP} & \mathrm{DM} & \mathrm{AR} & \mathrm{BRPM}+2\mathrm{CAD} \\ \mathrm{BP} & \mathrm{AQ} & \mathrm{DN} & \mathrm{CPQN}+2\mathrm{ABD} \\ \mathrm{AL} & \mathrm{BM} & \mathrm{CN} & \mathrm{DLMN}+2\mathrm{ABC} \\ 1 & 1 & 1 & k \end{vmatrix}=0,$$

and is at once verified by multiplying columns 1, 2, 3 by $\frac{1}{2}(ghyz'+g'h'y'z)$, $\frac{1}{2}(hfzx'+h'f'z'x)$, $\frac{1}{2}(fgxy'+f'g'x'y)$, and subtracting from the last column.

Thus in the case of any determinant of the fourth order we have, first, a relation involving the coaxials (a), (ab), (abc), &c. symmetrically; and, second, four different formulæ each expressing the determinant in terms of the coaxials.

Melbourne, July 23, 1897.

XLVII. *Notices respecting New Books.*

Introductory Course in Differential Equations for Students in Classical and Engineering Colleges. By D. A. Murray, *Ph.D.* (Longmans, 1897, pp. xv+234.)

THE title sufficiently indicates the aim of the book. Students who wish for a thorough discussion of the various points which turn up must go to such treatises as those of Boole, Forsyth, and Johnson. To these classics Dr. Murray gives ample references, as well as to numerous other works and original memoirs. The work, which we have read with considerable interest, assumes in the reader little more than a knowledge of the fundamental formulæ of integration, and brings in many practical applications well adapted for the class of students for whom it is intended. The rigorous proofs of many of the theorems are relegated to an appendix to be read when some familiarity with the subject has been acquired. An interesting feature are the numerous historical and biographical notes scattered throughout the text, and there are full indexes of Names and of Subjects which add to its utility. There is a fair number of well selected examples, many of which are worked out. We have not tested these exercises, but in the text we have come across some sixteen only of clerical errors, easily seen to be such: a remarkable result for a first edition of a work of this kind.

The elementary part of a treatise on the Dynamics of a System of Rigid Bodies, being Part I. of a treatise on the whole subject, with numerous examples. By E. J. Routh, *Sc.D., F.R.S.* (London, Macmillan, pp. xiv+424.)

The *sixth edition* of a work by Dr. Routh on a subject which he has made so essentially his own needs no recommendation. It is sure to be up to date. We are only called upon to note any new features in it. The author himself puts in the foremost place the fact that this edition contains an Index, which he hopes will be of use to the advanced student, "not merely by enabling him to find the theorems he is looking for, but also by presenting the subject in another light." Like theorems are grouped together, and a comparison can be made of the different ways of discussing the same thing. The many additions and improvements which have been made, he writes, will be at once apparent to readers already acquainted with the book. Some of the old references have been corrected and many new ones added. These we notice have in many cases quite confirmed our statement made at the outset. We take our leave of this beautiful book with the hope that Dr. Routh may yet bring out other editions even more perfect, if it be possible, than this.

XLVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 133.]

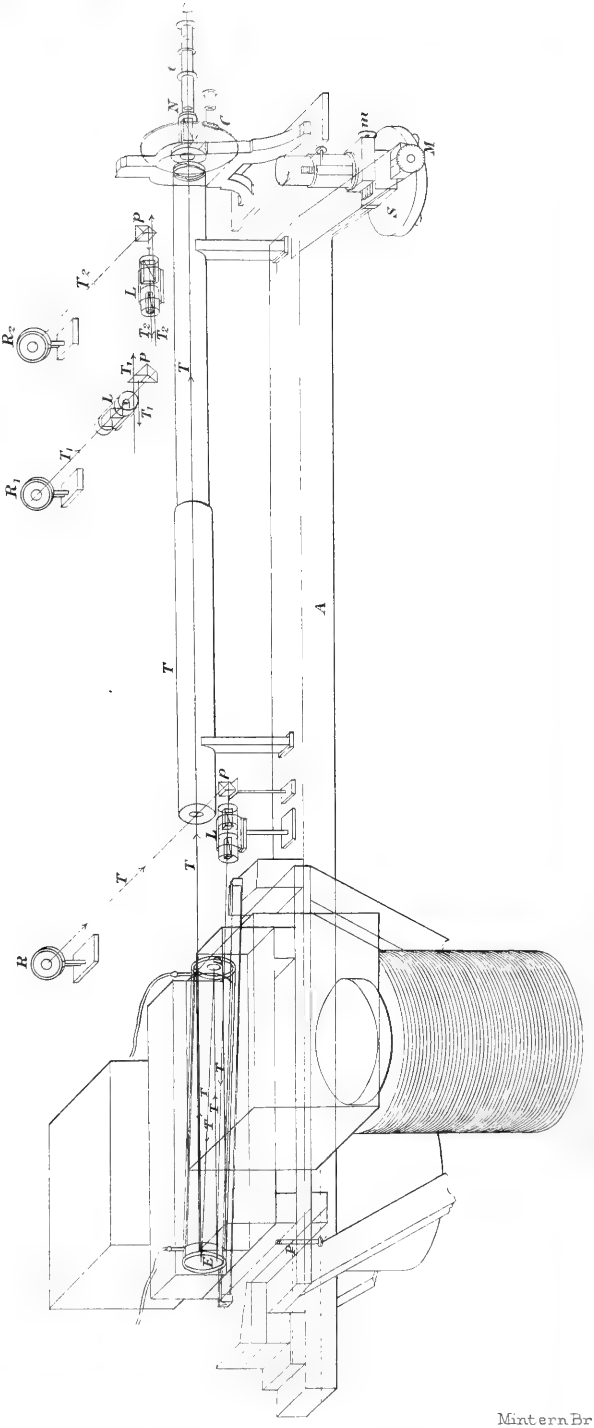
June 9th (*con.*).—Dr. Henry Hicks, F.R.S., President, in the Chair.

3. 'Some Igneous Rocks in North Pembrokeshire.' By J. Parkinson, Esq., F.G.S.

The acid rocks described in this communication are situated at the east end of the Prescelly Hills. Organisms have occasionally been found, and some discovered $\frac{1}{2}$ mile west of Crymmych, at the west end of the area, point to an Upper Arenig or Lower Llandeilo age for the deposits.

The masses of acid rock which occur at Foel Trigarn and Carn Alw show all the characteristics of a true lava-flow. The rock often shows beautifully developed flow-structure. Some of the rocks are very markedly spherulitic and axiolitic. The axiolitic rocks display an interesting structure which the author believes to be best explained by concluding that during the formation of the axiolitic growths a slight movement occurred in the unconsolidated magma. He gives a full account of the variations which mark the spherulitic and axiolitic structures, and also describes nodular bodies produced by flow-brecciation accompanied by very feeble radial growth, and showing irregular but more or less central quartzose areas.

After describing a brecciated rock, which on the whole is best accounted for by a process of flow-brecciation, due to a second lava breaking up the first while it was still in a plastic condition, the author describes true pyroclastic rocks, and concludes with a description of the conspicuous intrusive diabases, which, in part at all events, seem to be of the nature of a laccolitic intrusion.



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THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE

[FIFTH SERIES.]

NOVEMBER 1897.

XLIX. *The Surface-Tension of certain Dilute Aqueous Solutions, determined by the Method of Ripples.* *By* N. ERNEST DORSEY, *Ph.D.*[*]

Introduction.

PREVIOUS observers have found that the surface-tension of most aqueous solutions is a linear function of the concentration; but they have worked on no solutions more dilute than about one-half normal. (A normal solution is one that contains in one litre of the solution as many grams of the salt as there are units in the molecular weight of the salt, the molecular weight of hydrogen being taken as two.) It seems therefore desirable to determine, if possible, the surface-tension of more dilute solutions; and this work was undertaken at the suggestion of Prof. J. S. Ames in the spring of 1895, for the purpose of studying solutions as dilute as possible. Most of the time since then has been spent in studying the various methods, and in perfecting the method of ripples, which appeared to offer fewer theoretical objections than any other.

The arrangement of apparatus finally adopted gives an average departure from the mean of several observations of not more than one-seventh of one per cent. Though different samples of water may give results that differ among themselves by more than this amount, still in the entire series of twenty-one samples of water the average departure of a single observation from the mean was but one-fifth of one per cent. This compares very favourably with the results obtained by any other method.

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[*] Communicated by the Author.

This spring determinations of the surface-tensions of a few solutions were made at concentrations varying from one-tenth normal to normal, and the values found lie upon straight lines starting from the value for pure water.

Historical and Critical.

Among those who have worked on the surface-tension of aqueous salt solutions may be mentioned Frankenheim[1], Buliginsky[2], Valson[3], Quincke[4], Lippmann[5], P. Volkmann[6], O. Rother[7], Traube[8], Röntgen and Schneider[9], Klupathy[10], Goldstein[11], Jäger[12], N. Kasanskine[13], Canestrini[14], Sentis[15].

Excepting Sentis, Jäger, and Klupathy, they all used the method of capillary tubes, and Quincke supplemented this with his method of large bubbles. All determinations of the surface-tension from the measured rise of the liquid in capillary tubes involve an assumption in regard to the contact-angle—*i.e.* in regard to the angle included between the wall of the tube and the tangent to the liquid surface at the *point of contact with the tube.* This angle cannot be measured; for by any possible method we must measure the inclination of a finite arc, and as the surface extends on one side only of the point at which we desire the inclination, the finite arc whose inclination we measure must lie entirely on one side of the point of contact. Hence every measured value of this angle will necessarily be too large. This is probably the reason why Prof. Quincke obtains finite contact-angles for liquids that wet the glass in contact with them.

Another objection to measuring the surface-tensions of solutions at the line of intersection of three bodies—solution, air, and glass—is that probably the tensions of both the

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[1] Pogg. *Ann.* xxxvii. p. 409 (1836).

[2] Pogg. *Ann.* cxxxiv. p. 440 (1868); *Ann. chim. phys.* [4] xx. p. 361 (1870).

[3] *Compt. Rend.* lxxiv. p. 103 (1872).

[4] Pogg. *Ann.* clx. pp. 337 & 560 (1877).

[5] Wied. *Ann.* xi. p. 316 (1880).

[6] Wied. *Ann.* xvii. p. 353 (1882).

[7] Wied. *Ann.* xxi. p. 576 (1884).

[8] *Journ. f. pr. Chem.* xxxi. p. 192 (1885).

[9] Wied. *Ann.* xxix. p. 165 (1886).

[10] *Math. u. naturwiss. Ber. aus Ungarn,* v. p. 101 (1887); Wied. *Ann. Beibl.* xii. p. 750 (1888).

[11] *Zeitschr. f. phys. Chem.* v. p. 233 (1890).

[12] *Acad. d. Wiss. in Wien,* 100 2A. p. 493 (1891).

[13] *J. de la Soc. phys. chem. Russe,* xxiii. p. 468 (1891); *Journ. de Phys.* [3] i. p. 406 (1892).

[14] *Riv. Sc. ind.* p. 33 (1892); Wied. *Ann. Beibl.* xvi. p. 335 (1892).

[15] *Journ. de Phys.* [2] vi. p. 571 (1887); *Thesis* published at Paris, Feb. 1897; *Journ. de Phys.* [3 vi. p. 183 (1897).

solution-air and the solution-glass surfaces change with the concentration of the solution, so that the results obtained involve two changes which cannot be separated. This renders the interpretation of the results very difficult.

Jäger used a method depending on the relation between the diameter of a capillary tube and the distance it must be immersed in the liquid in order that the same air-pressure may cause steady streams of bubbles of the same size to issue from it and from another tube of known diameter, immersed to a fixed depth. He determined this relation empirically from experiments on water; and he accepted the results of Brunner and of Wolf as correct.

Quincke's work on flat bubbles has been discussed in several papers, and seems to be not very satisfactory. From this we see that, excepting the work done by Sentis and by Klupathy, the determinations of the surface-tensions of solutions are not all that can be desired.

Sentis employed an entirely different method, and one that appears to be very free from objections. He draws the liquid up into a capillary tube; then removes the tube from the liquid, and allows the liquid to run out so as to form a drop around the end of the tube. This drop supports a liquid column of a certain length in the tube. He then measures the maximum diameter of the drop, focusses a microscope on the meniscus in the tube, and slowly raises below the tube a beaker of the liquid. When the liquid in the beaker reaches the drop, the column in the tube falls; the position of the beaker is now noted, and it is then again slowly raised until the meniscus in the tube has come back to its previous position, and the position of the beaker is again noted. The vertical distance between these two positions of the beaker is (after certain corrections are made) the height the liquid will rise in a tube for which the contact-angle is zero, and whose radius is equal to that of the drop. His individual determinations may differ by one-half of a per cent., but generally they agree very well.

In the present work it was desired to use a method entirely independent of the mutual action between the liquid and a solid; and the only methods so far devised that completely satisfy this condition are Lenard's* method of vibrating falling drops, the method of Eötvös†, that employed by Sentis, and the method of ripples which was first successfully used by Lord Rayleigh‡. Sentis's method, as described in his first paper§, did not seem very satisfactory; and of the others

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* Wied. *Ann.* xxx. p. 209 (1887).

† *Ibid.* xxvii. p. 448 (1886).

‡ Phil. Mag. [5] xxx. p. 386 (1890); Theory of Sound, vol. ii. p. 344.

§ *Journ. de Phys.* [2] vi. p. 571 (1887).

mentioned Lord Rayleigh's appeared to be the most promising, and is the one finally adopted.

Method of Ripples.

This method is based on the relation between the surface-tension of a liquid and the velocity of propagation on its surface of a train of small waves of given period. This was deduced by Lord Kelvin * for the case of infinitely small waves on the surface of a perfect liquid of given depth. His formula is

$$V^2 = \left\{ \frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda} \right\} \tanh \frac{2\pi h}{\lambda}.$$

Here V is the velocity of propagation; λ is the wave-length; T is the surface-tension; ρ is the density; h is the depth of the liquid; and g is the acceleration of gravity. If n is the frequency of the waves, we can write this equation thus:

$$T = \rho \left\{ \frac{\lambda^3 n^2}{2\pi} \coth \frac{2\pi h}{\lambda} - \frac{g\lambda^2}{4\pi^2} \right\}.$$

Now

$$\coth \frac{2\pi h}{\lambda} = \frac{1+\epsilon^{-\frac{4\pi h}{\lambda}}}{1-\epsilon^{-\frac{4\pi h}{\lambda}}} = 1 + 2\epsilon^{-\frac{4\pi h}{\lambda}} \text{ approximately}$$

if h is large. In this work h is never less than 1·1 cm., and λ is never as great as 0·5 cm. Hence $\frac{4\pi h}{\lambda} \nless 25$;

$$\therefore \quad \coth \frac{2\pi h}{\lambda} \ngtr 1 + 2\epsilon^{-25}.$$

This shows that we may consider

$$\coth \frac{2\pi h}{\lambda} = 1,$$

and may write the equation

$$T = \rho \left\{ \frac{\lambda^3 n^2}{2\pi} - \frac{g\lambda^2}{4\pi^2} \right\}.$$

Lamb† gives the more exact formula for the case of a liquid of infinite depth,

$$V = \left\{ \frac{\rho-\rho'}{\rho+\rho'} \frac{g\lambda}{2\pi} + \frac{2\pi T}{(\rho-\rho')\lambda} \right\};$$

or

$$T = (\rho+\rho') \left\{ \frac{\lambda^3 n^2}{2\pi} - \frac{\rho-\rho'}{\rho+\rho'} \frac{g\lambda^2}{4\pi^2} \right\},$$

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* Phil. Mag. [4] xlii. p. 375 (1871)

† Hydrodynamics, p. 446.

where ρ' is the density of the air in contact with the surface. For the accuracy obtained in measuring λ, the first and simpler formula is sufficiently exact.

Of the terms on the right of these equations, the first is much the larger for small waves. Hence the value calculated for T will be proportional to the cube of the wave-length and to the square of the frequency. Here is the first objection to this method. The value found for T will be only one-third as accurate as the measurement of the wave-length, and one-half as accurate as the determination of the frequency. Again, since the formulæ are deduced for waves of infinitely small amplitudes, we must use very moderate vibrations, and these are difficult to measure.

Several, among whom we need mention only Matthiessen, Riess, and Ahrendt, have attempted to test experimentally the extreme accuracy of Lord Kelvin's formula.

When a needle is placed so that its point dips slightly into a jet of water, a series of stationary waves is formed behind the needle, and their length is equal to that of waves whose velocity is equal to the velocity of the surface of the jet at that point. This is the method employed by Ahrendt*. He found that the value of the surface-tension calculated from the observed wave-length is too great, but that it decreases as the needle is removed from the orifice. This proved that the fault lay in assuming that the surface of the jet moves with the velocity calculated for the ideal case. In reality, the surface has a smaller velocity, probably owing to friction with the edge of the orifice and with the air. Hence he proved nothing, so far as the formula is concerned.

Both Riess † and Matthiessen ‡ failed to satisfy two of the conditions which were assumed by Lord Kelvin in obtaining his formula. One of these is that the amplitude of the waves shall be very small in comparison with the wave-length; and the other is that the surface of the liquid may be represented by an equation of the form

$$y = a \sin (mx + nt);$$

i. e., the formula is deduced for the case of a single series of very small plane waves. The waves employed by these observers were of such an amplitude that they could be easily seen with the unaided eye: this alone might very probably introduce a discrepancy between the observed and the cal-

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* Exner's *Rep. der Phys.* xxiv. p. 318 (1888).

† Exner's *Rep. der Phys.* xxvi. p. 102 (1890).

‡ Pogg. *Ann.* cxxxiv. p. 107, cxli. p. 375; Wied. *Ann.* xxxii. p. 626, xxxviii. p. 118.

culated values for the wave-length. This applies to all measurements made by them. Most of their observations were made on the interference-waves produced on the surface of a liquid when a vibrating fork is supported vertically above it so that two needles, one attached to either prong, may dip into the liquid. This gives two series of circular waves, and so the second condition mentioned above is not fulfilled. I do not know how much this would affect the results; but until it is proved to have no effect the results must be considered nugatory. Riess * says:—"Der Schlüssel für die definitive Lösung ist in einer minutiösen Messung der Amplituden zu suchen;" and he goes on to say that the only sure method is an optical or photographic one. Matthiessen found that the formula gives results in accord with his observations except in the neighbourhood of the minimum value of the velocity.

C. M. Smith † employed this method for determining the surface-tension of mercury, but obtained very erratic results. Lord Rayleigh ‡ thinks he obtained waves due to the suboctave of his fork, and these are almost sure to have too large an amplitude.

W. Ochsé § attempted to determine the surface-tension of solutions by means of ripples. He measured the interference-pattern of two series of circular waves, as Riess and Matthiessen had done; his waves had a large amplitude, he assumed that the density did not enter into the equation, and he used rather concentrated solutions. His measurements of the wave-lengths are quite rough, so that the results have but little value even when corrected for the density.

From this it is seen that the formula has not been disproved; and the fact that Lord Rayleigh obtained the same value for the surface-tension of water with two forks of different frequencies, and that the result obtained for water in this work agrees within his experimental error with Lord Rayleigh's, seems to justify the application of Lord Kelvin's formula to these two cases.

All the work so far described has been with waves easily visible, and these very probably have an amplitude too large to allow of the application of Lord Kelvin's formula. Lord Rayleigh was the first to use waves invisible under ordinary conditions. His method was to have the waves generated by means of a plate of glass which was attached to the lower

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* Exner's *Rep. der Phys.* xxvi. p. 131 (1890).

† Proc. Roy. Soc. Edinb. xvii. p. 115.

‡ Phil. Mag. [5] xxx. p. 386 (1890).

§ Exner's *Rep. der Phys.* xxvi. p. 641 (1890).

prong of a vibrating fork, and which dipped into the liquid to be experimented upon. Then, "in order to see the waves well, the light was made intermittent in a period equal to that of the waves, and Foucault's optical method, for rendering visible small departures from truth in plane or spherical reflecting-surfaces, was employed." The liquid was contained in a shallow porcelain tray, and its surface was cleaned by a flexible brass hoop wider than the depth of the liquid. This was placed in the liquid so as to include the plate that generated the waves, but otherwise in as contracted a position as possible. It was then expanded to its maximum extent. This gave a fresh surface as clean as the body of the liquid. The mean of his results for water is $T=74$ dynes per centimetre at 18° C., and this is probably correct to about one per cent.

Description of Apparatus.

I shall now describe the apparatus used in this work. It is simply a natural development of Lord Rayleigh's although it contains many new features.

The arrangement of apparatus is represented diagrammatically in fig. 1. S is a sixteen-candle-power incandescent electric lamp; S_2 consists of two narrow strips of thin copper, one of which is fastened to each prong of the large tuning-fork F_1, and each has near its upper end a small hole so placed that when the fork is at rest these holes shall lie one directly behind the other, in a horizontal line. F_2 is the fork which generates the ripples by means of the plate of glass (P) 2·5 by 9·7 cm. L_1 and L_2 are two small double-convex lenses. D is the vessel for holding the liquid to be experimented upon; it is a porcelain developing-tray 2·3 by 29 by 36 centimetres. E is a dividing-engine furnished with a screw whose pitch is 1·0328 millim. M_1 and M_2 are silver-on-glass mirrors. T is a telescope. To the carriage of the dividing-engine are rigidly fastened a wooden arm (B) and a brass tube (A); the tube is parallel to the wooden arm and is fastened to it by means of V. To the end of B is fastened the mirror M_2 by means of brass mountings that allow it to be turned and clamped in any position desired. Through the tube A passes a brass rod (R) to which is fastened the mirror M_1. This allows M_1 to be rotated about the axis of the tube, and it is clamped in the necessary position by means of the screws I and K. W is an iron weight hung from a rod fastened to the carriage, and is intended to counterbalance the rod, mirrors, etc.

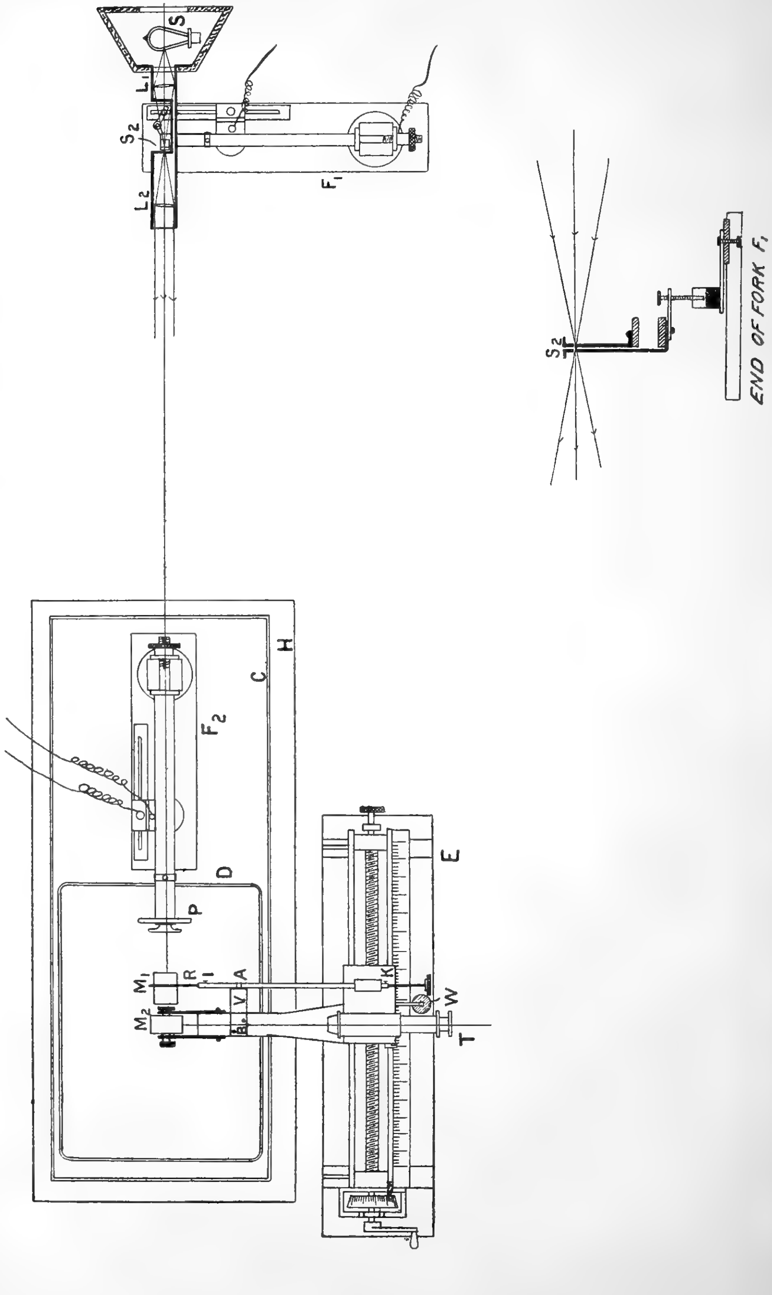


Fig. 1.

To protect the liquid as well as possible from dust and other impurities in the air, the tray (D) and the fork (F_2) were placed in the galvanized iron box (C), which was closed by a tightly-fitting lid provided with a window of good, *plane, parallel* plate-glass, through which all readings were taken.

The fork F_1 is an ordinary electrically driven tuning-fork provided with a mercury-platinum contact-maker. F_2 is also an electrically driven tuning-fork; it is tuned in unison with F_1, and is driven by it by means of a shunt circuit.

By means of L_1 the light from S is focussed on a point midway between the two holes in S_2; it then passes through L_2 which renders it parallel, then it is reflected by M_1 to the surface of the liquid, from this it is reflected to M_2, and from M_2 into the telescope T. If now the forks are started and the telescope is focussed for the light reflected from the crests of the waves, the entire field of the telescope, which is dimly illuminated, will be crossed by a series of parallel bright lines which correspond to the crests of the waves. These lines appear stationary, and the distance between them is approximately one-half the wave-length, since each wave is seen in two positions (once when the prongs of F_1 pass through their positions of equilibrium and are approaching one another, and again when they are separating). The crests seen in one position do not lie exactly midway between those seen in the other.

The plan of operation is to set the spider-line of the telescope on one of these lines, read the position of the carriage, then move it along several centimetres, and set on another line of the same series. The difference of these two readings divided by the number of waves passed over gives the wave-length.

Sources of Error, and Adjustments.

The first objection that presents itself is that, as the carriage is moved, the mirrors will not remain parallel to their original positions. Owing to the length of the arms supporting the mirrors (30 cm.) and to the distance of the liquid below them (45 cm.), any slight displacement of the mirrors will produce a great error in the determination of the wave-length. These errors may be introduced either by irregularities in the ways of the engine, or by a motion of the arms with respect to the carriage. The latter is simply a matter of rigid connexions and can be easily remedied, so we shall first consider the former.

To test the accuracy of the ways, the rotation of the carriage was tested about three rectangular axes—one along the screw, another horizontal and perpendicular to the screw, and the third perpendicular to the plane of the other two. To do this a mirror was mounted on the carriage perpendicular to the length of the screw, and a telescope (placed about ten feet from it) was focussed on the image in the mirror of a vertical scale fixed to the telescope. If the carriage rotates about a horizontal axis perpendicular to the screw, the image of the scale will be displaced vertically with respect to the spider-lines of the telescope. The carriage was moved several times over the entire length of the ways, but no displacement of the scale was noticed except what might be due to changing my position with respect to the stand on which the telescope was placed.

The rotations about the other axes were tested in a similar manner and with the same result. This proved that the irregularities of the ways are at most very slight.

To test the rigidity of the arms and mirror-clamps, the carriage was screwed along until the spider-line in the telescope on the carriage coincided with a well-defined crest. It was then slipped back, leaving the nut in place on the screw, and again moved up by hand to the nut. The line again coincided with the crest. This proved that moving the carriage by hand produced no effect on the relative positions of the carriage and its appurtenances. The carriage was then again slid from the nut, and while held with one hand it was struck several sharp, horizontal taps with a block of soft wood. It was then moved back against the nut. If the telescope and mirrors are clamped tightly the line again coincides with the crest. This proves that slight jars of the screw will introduce no error into the results if the telescope and mirrors are well clamped.

Having settled this point, the other adjustments, which will now be described, were made; and then the ways were again tested by a method which will be described further on, and which is much more delicate than the one first used.

Leaving aside the accuracy of the screw and the motion of the carriage, there are six fundamental adjustments to be made:—

1. The carbon filament and the axes of the lenses L_1 and L_2 must lie in a straight line.

2. The beam of light must be composed of parallel light, and

3. It must be horizontal before reflexion by M_1.

4. The ways of the engine must be horizontal and at such a height that the beam of parallel light may strike the mirror M_1; and

5. They must be parallel to the beam of light.

6. The fork F_2 must be parallel to the ways of the engine. This last is easily accomplished by using a plumb-line hung from the rod R.

1. The lenses L_1 and L_2 are placed in a tin tube against diaphragms perpendicular to the axis of the tube. This makes their axes practically coincide. The lamp S is then adjusted so that the light (from the filament that is to be used) after passing through L_1 covers the centre of L_2. Then the filament and the axes of L_1 and L_2 are in a straight line.

2. The lens L_1 is moved until the light is focussed on S_2, and then L_2 is moved until, on reflecting the beam of light back along its original path, it appears to converge to the hole in S_2 through which it originally came. The position of L_2 is then adjusted until the light is as nearly parallel as one can detect with a small telescope focussed for infinity.

3. The entire system L_2, L_1, and S is then inclined by levelling-screws until the beam of light is as nearly horizontal as can be detected by means of a water-level two metres long.

4. The engine is placed on a slab of slate mounted on levelling-screws and at the desired height. The horizontality of the ways is tested by means of a delicate spirit-level.

5. If two strips of paper are pasted on M_1, so that their edges touch opposite sides of the part of M_1 that is covered by the beam of parallel light; and if the engine is moved until the light remains on this spot while the carriage is moved over the entire length of the ways (about 50 cm.), we may be sure that the ways are quite approximately parallel to the beam of parallel light.

If water is now placed in the tray, if the mirrors and the telescope are turned so that the light is thrown down the tube of the telescope, and if cross-threads are stretched across L_2, a dark cross will be seen in the field of the telescope when it is focussed for a point about one metre below the surface of the water (which is the focal length used throughout the work). The cross-threads of the telescope are made to coincide with this cross, and the carriage is moved along the ways.

If the ways are accurate, and if the other adjustments have been well made, these crosses will coincide throughout the length of the screw. But, owing to slight irregularities in the ways, it is found that the crosses are relatively very slightly shifted, though from scale-reading 15 to scale-reading 29 they remained coincident, except from 22·6 to 24·3 where one bearing of the carriage rests upon a small dent in one of the ways. Hence all settings were taken between 16·5 and 22, and between 25 and 27. In these regions I could detect not the slightest shift; so I think the readings are entirely free from any error due to inaccuracies in the ways. Any error in the other adjustments also causes a shift in the positions of the crosses; but this is uniform throughout the length of the ways, and so can easily be distinguished from the non-uniform displacements due to faults in the ways.

Extraneous Disturbances.

Having disposed of these difficulties we come to another that gave great trouble. This is the disturbance of the liquid by extraneous vibrations. After trying many plans I finally placed the box containing the tray of water and the fork F_2 upon an iron slab (H) weighing about 150 pounds, and supported by long steel springs hung from a joist whose ends rested in the walls of the building but which was free from the floor above. It was hung so that when weighted for work it was about three or four centimetres above the top of a brick pier, and the intermediate space between it and the pier was loosely filled with cotton batting. If the cotton is packed neither too tightly nor too loosely, this arrangement cuts out most extraneous disturbances. The dividing-engine was supported on this pier. The fork which generated the ripples, and the vessel of water were each supported on pieces of rubber tubing, so that the water might not be disturbed by any slight vibration of the stand of the fork.

Influence of Glass Plate.

Another point of prime importance is to have the glass plate which generates the waves well wetted by the liquid. If it is not, the crests will be so irregular and so distorted that no correct setting can be made. A microscope-slide was first used, but it was very difficult to get its surface in such a condition that water would wet it readily; and one lot of slides I was entirely unable to use. I tried to clean them with caustic potash, sulphuric acid, nitric acid, chromic acid; I even boiled them in chromic acid and still water would not readily wet them, and never under any condition did I succeed in wetting them uniformly. I finally used a piece of German thin plate mirror from which the silvering was removed. This was easily cleaned, and there was no trouble in keeping it in such a condition that water and salt-solutions would wet it readily and uniformly.

When I began this work I always took one reading within two centimetres of the fork, in order that I might run the fork with as small an amplitude as possible, so as to avoid reflected waves. The other reading was taken some five or ten waves further from the fork.

The results obtained by this method were very erratic. The wave-lengths found on any one day would agree fairly well among themselves, but those obtained on different days seldom agreed at all. Later on it was found that the apparent wave-length depended upon how many waves were included between the settings.

After trying every plan I could devise to find the trouble, I at last decided to measure each individual wave and see if these separate determinations would throw any light on the difficulty. In this series of measurements I went as close to the fork as I could measure, and I found (although these separate determinations were very rough) that the apparent wave-length *increased* as the fork was approached. As a last resort I attributed this effect to the rise of the liquid against the plate which generates the ripples, although I did not think that this could affect waves two centimetres distant. To test this experimentally, the plate that was well wetted was replaced by one coated with paraffin. It was now found that the waves apparently *decreased* as we approach the fork. This proved that, contrary to expectations, the effect of the meniscus makes itself felt two or three centimetres from the fork. Two typical curves showing this effect are given in figs. 2 & 3; the abscissæ denote distances from the plate, and the ordinates represent the apparent wave-lengths magnified ten times. The numbers denote centimetres.

Fig. 2.

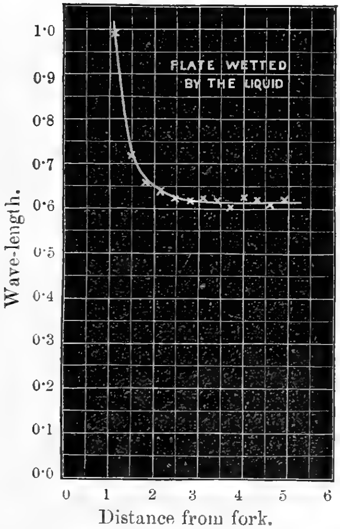
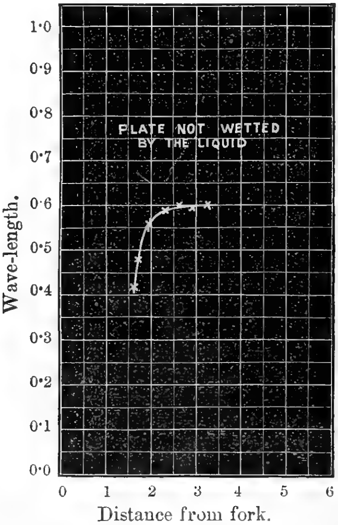


Fig. 3.



The magnitude of the error introduced in this way depends upon the amplitude of the waves, and this is probably the reason why the results obtained on different days were not comparable; for I frequently changed the amplitude of my fork so as to alter the distinctness of the field in the telescope. Many weeks were spent in locating this error; and after finding it no reading was taken nearer the fork than four centimetres. Under these conditions the measured wave-length does not depend upon the number of waves measured, nor upon the distance of the waves from the fork, nor upon the amplitude of the fork (so long as it is small). All these points were tested experimentally. I was very seldom troubled with reflected waves.

The mirrors M_1 and M_2 must be good plane metal mirrors. Owing to the double reflexions ordinary glass-front mirrors cannot be used, since in my arrangement the images overlap and distort one another. The silver surface obtained by depositing silver by Brashear's process* on good thin plate German mirror-glass was used in this experiment.

Periods of Forks.

The fork F_2 is a large one with a mirror and counter-weight screwed into the ends of its prongs, and belongs to a set of König forks used to show Lissajous's figures. The mirror was replaced by a second counter-weight, and the plate of glass which generated the waves was fastened by screwing down the weight until the plate was clamped tightly between it and the end of the prong of the fork. This allowed the plate to be removed and cleaned without danger of wetting the fork. Sliding on the prongs were light brass weights that could be clamped wherever desired.

F_1 was also provided with sliding weights; and after putting on the strips of copper (S_2) I tuned it (by means of a vibration-microscope) to unison with F_2 while the weights on the latter were near the centre of its prongs, and while the plate of glass was dipping into water.

By means of the smoked glass and pendulum method the frequency of the vibration-microscope, which was an octave above F_1, was determined. It made 125·8 vibrations per second. Then as a check the vibration-microscope was again compared with F_1, and it was found that the ratio of their frequencies was still two to one. The vibration-microscope rather than F_1 was compared with the pendulum because it was more readily handled than the larger fork.

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* Astrophysical Journal, i. p. 252 (1895).

This method, however, cannot be depended upon except to give the whole number of vibrations per second. To determine the exact frequency Michelson's method * was employed; but the results were carried to but two decimal places, as the frequency of a fork driven by a mercury contact does not seem to be more constant than this approximation. Now, in order to use this method, the clock must give exactly equal intervals of time; Prof. Michelson says that the intervals must differ by less than 0·002 second in rating a fork of 128 complete vibrations per second. The clock at my disposal had a mercury contact, and I found that working alone I could not adjust this so accurately in the centre of the arc of the pendulum as is required for consistent results. In any case, as Prof. Michelson states, a mercury contact is not very good unless it is quite narrow in the direction of the swing of the pendulum.

After trying many contacts, and working on the subject for some time, I finally devised one that is very simple and has so far given perfect satisfaction. A contact is desired that can be easily adjusted so as to give absolutely equal intervals of time; so it seemed that the best method would be to use the complete period of the pendulum instead of the half period, as is ordinarily adopted. This would obviate the difficulty of centering the apparatus.

The construction of the contact is as follows:—A steel watch-spring about 8 cm. long is soldered to a brass collar which is fastened by a screw to one end of a brass plate 4 cm. wide and 10 cm. long. The collar can be rotated about the axis of the screw, and can be clamped tightly at any angle desired. Perpendicular to the flat surface of the spring and about 5 cm. from the collar is soldered a platinum point which, when the spring is slightly depressed, dips into mercury in a cup attached to the brass plate but insulated from it. At the same place and perpendicular both to the platinum point and to the length of the spring is soldered the point of a fine steel sewing-needle. This needle-point must be quite fine and well polished. At the further end of the spring is fastened a vane of mica, which dips into oil contained in another cup fastened to the brass plate and is intended to damp the vibrations of the spring. The brass plate to which these parts are attached is fastened to the back of the clock near the top of the pendulum, and in such a position that the needle-point comes near the centre of the arc of the pendulum.

To the pendulum rod is fastened a light brass collar, on the back of which is soldered a short piece of steel watch-spring

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* Phil. Mag. [5] xv. pp. 84–87 (1883).

which projects horizontally from the rod, and whose plane is inclined to the rod at an angle of about 50° or more. The lengths of the needle-point and of this short piece of spring are such that when they are in place and the pendulum is at rest they overlap by an amount just sufficient to ensure the depressing of the needle-point at each complete vibration of the pendulum. Both sides of this piece of spring are carefully polished, and the collar is so adjusted that the needle-point comes just below the upper edge of the spring. The electric circuit is completed through the long spring and the mercury cup. Then, when the pendulum swings in one direction the platinum point is depressed into the mercury closing the circuit, and it immediately rises and breaks the circuit as the pendulum swings on; on the return swing the platinum point is slightly raised. If the damping vane works properly the vibrations of the spring will be practically dead-beat, and we shall have contact made once only during each complete vibration of the pendulum.

Dr. J. F. Mohler kindly compared the clock I used with the astronomical one, and found that each single swing of the pendulum corresponds to 1·000397 seconds; the mean solar day being the unit of time. Hence we are justified in taking the complete period of the pendulum as two seconds.

As any change in the position of the blades S_2 affects the period of the fork, its frequency was often determined so as to avoid errors due to careless handling.

Amplitude of Waves.

We have seen that Lord Kelvin's equation applies to waves of infinitely small amplitude, so it is well to consider the amplitude of the waves used in this work. I have no means of measuring its size directly, but a general idea of its smallness may be obtained by considering the fact that throughout the work the telescope was focussed for a point one metre below the surface of the liquid. Since the beam of incident light is parallel, this means that in the neighbourhood of a crest the radius of curvature is about two metres. This, together with the fact that these waves are only 0·48 centim. long, shows that the amplitude must be extremely small. We can also see that the amplitude is very small from the effect of the meniscus upon the apparent length of the waves at such a great distance as two centimetres from the plate (p. 385).

Effect of Viscosity.

Before going further it is well to consider another point.

How does the viscosity of the liquid affect the apparent wave-lengths? Prof. Tait* has shown that viscosity does not affect the true wave-length, but simply introduces a damping factor of the form $\epsilon^{-\alpha x}$, where α depends upon the wave-length and upon the viscosity of the liquid. Hence the apparent wave-length is the distance between the consecutive corresponding points where the curve $y = \mathrm{A}\epsilon^{-\alpha x} \sin mx$ has a constant slope c. Hence it is necessary to find the roots of the equation

$$\frac{dy}{dx} = -\mathrm{A}\alpha\epsilon^{-\alpha x} \sin mx + \mathrm{A}m\epsilon^{-\alpha x} \cos mx = c,$$

or

$$\mathrm{B} \cos\,(mx+\theta) = \frac{c}{\mathrm{A}}\epsilon^{\alpha x}, \quad \text{where} \quad \mathrm{B} = \sqrt{\alpha^2 + m^2}\,; \ \tan\theta = \frac{\alpha}{m}.$$

Since α is very small for water and for dilute solutions, B is approximately equal to m. The maximum value of c is at the point $x=0$ where $c = \mathrm{A}m$. Hence $\frac{c}{\mathrm{BA}} \ngtr 1$, and the roots of the equation are the abscissæ of the points of intersection of curves

$$z = \cos\,(mx+\theta), \text{ and}$$
$$z = \mathrm{E}\epsilon^{\alpha x} = \mathrm{E}(1+\alpha x) \text{ approximately};$$

and $\mathrm{E} \equiv \frac{c}{\mathrm{BA}} \ngtr 1$. By setting near a crest E can be made very small, and hence the slope of the line $z = \mathrm{E}\epsilon^{\alpha x}$ will also be small, practically zero, and the apparent wave-length must be very nearly $\frac{2\pi}{m}$, which is the true wave-length. If we could make c exactly zero, the apparent and the real wave-lengths would be exactly equal.

To test experimentally the error that might be introduced if c were not zero, the telescope was turned so as to look on one side of the crests at points having as great a slope as could be observed, and the apparent wave-length was measured. It was then turned so as to look at corresponding points on the other side of the crests, and the wave-length was again determined. Each time I measured over a distance equal to twenty waves, which is the interval usually employed in the rest of the work. The maximum difference of the two series amounted to one-fifth of one per cent. The apparent field swept over by the spider-line in turning the telescope

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* Proc. Roy. Soc. Edinburgh, xvii. pp. 110–115 (1890).

from one of these positions to the other is over two centimetres; and I can easily adjust the line to the centre of this field to within one millimetre; *i. e.* one-twentieth of the total angle through which I turned the telescope. If the error was proportional to the angle this would correspond to an error of 0·01 per cent.; but the error varies almost as the cosine of the angle. Hence the error introduced is negligible.

Effect of Particles.

In the same article Prof. Tait shows that the effect of a "uniform film of oxide or dust, in *separate* particles which adhere to and move with the surface" is to increase the wave-length if there is no viscosity. If terms involving the first power of the viscosity must be taken into account, the surface-layer will not affect the wave-length, but will aid viscosity in causing the waves to subside as they advance.

Water and Salts.

The water used in these experiments was especially distilled by Dr. W. T. Mather, and was the kind he used for his electrolytic work. He distilled it from chromic acid, and from potassium permanganate, and condensed it in a block-tin condenser. It was collected in a bottle used for no other purpose, which was kept always covered or stoppered. It was transferred from this to a larger bottle that had a hole bored near its bottom so that the water could be drawn off through a piece of black rubber tubing and a glass nozzle. This tubing and nozzle always remained filled with water which protected their inner surface from impurities in the air. The glass nozzle was removed and cleaned before drawing the water. In this way it was hoped to get water as pure as possible. I might mention here that the water and solutions were not free from air; but Röntgen and Schneider[*] have shown that the effect of dissolved air is entirely negligible at ordinary pressures.

I used Eimer and Amend's chemically pure salts.

The temperatures of the water and of the solutions were determined by means of a Bender and Hobein thermometer of "Normalglas" graduated to tenths of a degree Centigrade.

The densities of the solutions of sodium chloride, of sodium carbonate, and of zinc sulphate were calculated from the values given by Kohlrausch and Hallwachs[†]; those for potassium chloride from the results obtained by Bender[‡];

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[*] Wied. *Ann.* xvii. p. 207 (1882).

[†] Wied. *Ann.* liii. p. 14 (1894).

[‡] Wied. *Ann.* xxii. p. 179 (1884).

and those for potassium carbonate from Gerlach's values as given in Landolt and Börnstein's *Physikalisch-Chemische Tabellen*. The densities for water were taken from Marek's values *.

Method of Observation.

Before each series of readings the tray, the glass plate, the brass hoop for cleaning the surface of the liquid, the thermometer, and the stirring rod were washed, first with tap-water, then with an alcoholic solution of caustic potash, then with running tap-water, then with chromic acid, and, finally, they were well rinsed in running tap-water. Then everything was placed in position in the iron box, water was run into the tray until the lower edge of the glass plate dipped just under the surface. I then expanded the hoop, put the lid on the box, and brought the iron slab to rest as soon as possible. Then the forks were set in vibration, the spider-line of the telescope was brought into coincidence with a crest near the middle of the pan, and the amplitude of the vibrations was increased until the crest was almost out of focus. As stated before, the waves appear stationary. Then the shunt circuit by which the driven fork (F_2) was run was broken and the fork was allowed to come to rest at its free period. The driving-fork (F_1) being still vibrating, the crest will have an apparent motion unless the free period of the driven fork coincides with the running period of the driving one. If the two did not coincide, the weights on the driven fork were moved until they did. By care the depth of the water can be adjusted so that this correction is seldom necessary ; but unless the two forks are in unison the crests are almost sure to oscillate, on account of the imperfection of the mercury contact. The plate which generates the waves should never leave the liquid, but it should dip below the surface as little as possible. If it dips deeply into the liquid the waves always appear distorted.

Having brought the forks to unison, the spider-line is brought successively into coincidence with three consecutive crests near scale-division 17. Then the carriage is screwed along and readings are taken on the fortieth, forty-first, and forty-second crest beyond the one on which the first setting was made. The thermometer is then read, the hoop is lifted from the water, contracted, and replaced in the water, and expanded, and another series of six readings is taken. Then a known weight of salt is added, and the solution is stirred until the salt is dissolved. Then readings are taken as for

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* Wied. *Ann.* xliv. p. 171 (1891).

water; then more salt is added, and so on until readings have been taken on water and on solutions of (as a rule) four different concentrations. Then the volume of the solution is measured. I measure the volume last so that the purity of the water may not be affected by any trace of grease that might be on the measuring vessel.

Since each crest is seen in two positions, measuring over forty crests gives the length of twenty waves. Such a series of readings as I have just described occupies about two hours; and to make sure that the surface in this length of time does not become contaminated so as to affect my results, a check series was taken on water. The first readings gave $T_{18°\cdot 9}=73\cdot 10$, or $T_{18°}=73\cdot 24$; the last, about two and one-half hours later, gave $T_{19°\cdot 3}=72\cdot 91$, or $T_{18°}=73\cdot 18$. Hence the results obtained from such a series of readings that extend over about two hours should be not appreciably affected by the contamination of the surface by the air.

By this plan of work we obtain for each concentration two sets of three independent determinations of the wave-length. If the averages of these two sets agree fairly well, the surface-tension is calculated from their mean. If they do not agree well another series is taken, and the surface-tension is calculated from the weighted mean of the three series.

Temperature Reduction.

To reduce the results to the standard temperature, 18 degrees Centigrade, I used the formula $T_t=T_0(1-\alpha t)$, and took $\alpha=0\cdot 0020$, which is the average of the following values:—

Brunner............	0·0019	Pogg. *Ann.* lxx. p. 481.
Braun	0·0020	Winklemann's *Handb. d. Phys.* i. p. 467.
Frankenheim	0·0018	Pogg. *Ann.* lxxii. p. 177 (1847).
Wolf	0·0018	*Ann. chim. et phys.* [3] xlix. p. 269 (1857).
Volkmann	0·0018	Wied. *Ann.* xvii. p. 353 (1882).
Timberg	0·0022	Wied. *Ann.* xxx. p. 545 (1887).
Jäger	0·0023	*Acad. d. Wiss. in Wien,* c. p. 245 (1891).
Cantor	0·0021	Wied. *Ann.* xlvii. p. 399 (1892).
Humphreys & Mohler	0·0020	Phys. Review, ii. p. 387 (1895).
Sentis	0·0020	*J. de Phys.* [3] vi. p. 183 (1897).
	0·0020	

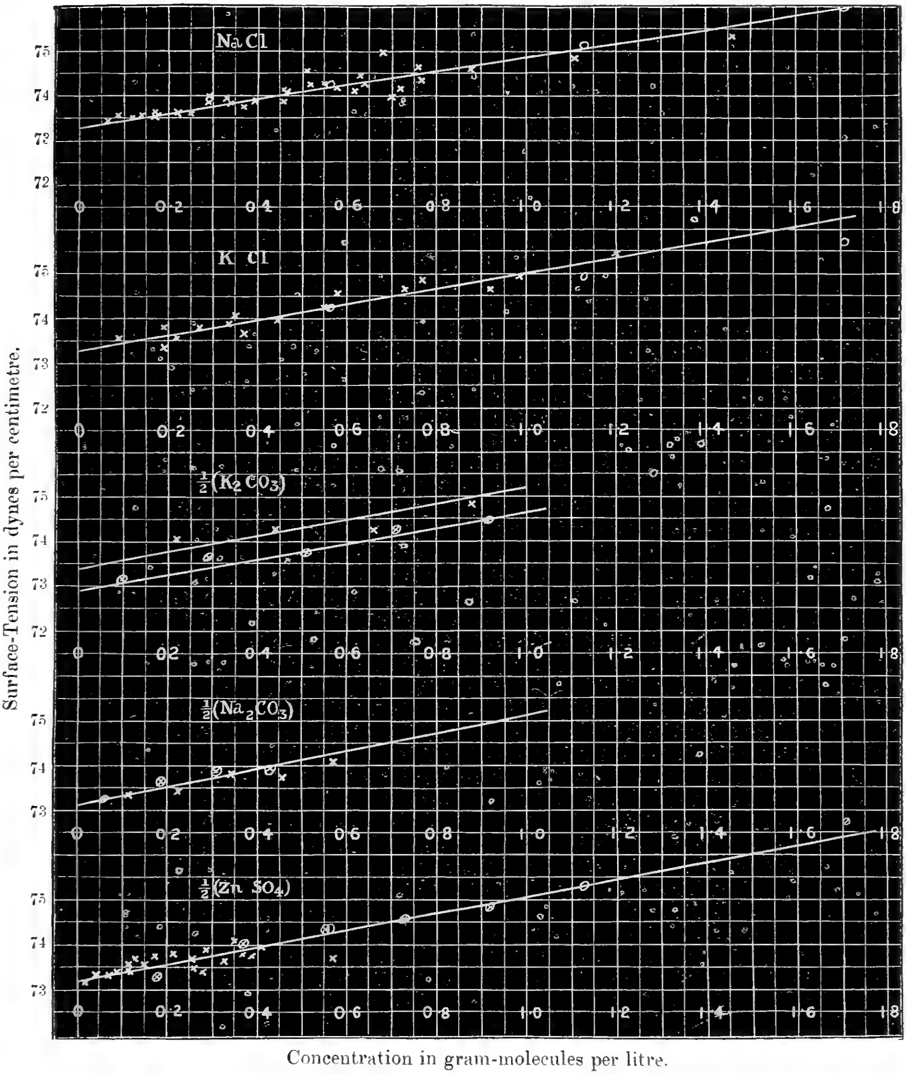
In this formula t is the temperature in degrees Centigrade; T_t is the surface-tension at $t°$ C.; and T_0 is the surface-tension at 0° C.

Since the salt solutions used were very dilute I used the same temperature-coefficient (0·0020) for them; Sentis says that in any case the temperature-coefficient for aqueous salt solutions differs from that for water by an amount less than his experimental error, which is about the same as mine.

Results.

The results obtained are given in the tables and are represented graphically in fig. 4.

Fig. 4.



In the tables the concentrations are expressed in gram-molecules per litre, and are given in the column marked $\frac{\text{gm.-mol.}}{\text{litre}}$; t denotes the temperature in degrees Centigrade; n is the frequency of the fork; ρ denotes the density; and T_t and T_{18} are the surface-tensions at t° and 18° C. respectively, both are expressed in dynes per centimetre; λ_1 is the observed wave-length in terms of the scale-divisions of the dividing engine, one division = 1·0328 centim.

$\frac{\text{gm.-mol.}}{\text{litre}}$.	t.	n.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.
NaCl.						
0	16·0	63·07	0·4858	73·72	73·64	73·34
0·067	17·2	"	0·4852	73·44	73·56	73·44
0·14	17·8	"	0·4849	73·29	73·58	73·55
0·22	18·2	"	0·4843	73·01	73·57	73·60
0·29	18·6	"	0·4843	73·01	73·78	73·87
NaCl.						
0	16·3	63·07	0·4858	73·72	73·64	73·38
0·12	16·7	"	0·4852	73·44	73·70	73·50
0·29	17·4	"	0·4849	73·29	74·08	73·99
0·46	17·7	"	0·4840	72·88	74·16	74·12
0·63	18·5	"	0·4837	72·73	74·37	74·45
NaCl.						
0	13·1	63·07	0·4865	74·03	73·99	73·25
0·17	13·5	"	0·4859	73·76	74·23	73·55
0·33	14·1	"	0·4855	73·57	74·52	73·92
0·51	15·0	"	0·4854	73·52	75·00	74·54
0·68	19·0	"	0·4841	72·92	74·83	74·99
NaCl.						
0	16·4	62·96	0·4863	73·67	73·59	73·35
0·088	17·0	"	0·4860	73·54	73·72	73·57
0·17	17·3	"	0·4855	73·30	73·73	73·62
0·25	17·4	"	0·4849	73·01	73·68	73·59

Table (*continued*).

$\frac{\text{gm.mol.}}{\text{litre}}$.	t.	n.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.
NaCl.						
0	18·6	62·96	0·4854	73·25	73·14	73·23
0·37	19·3	"	0·4839	72·56	73·55	73·75
0·72	19·8	"	0·4825	71·91	73·91	74·17
1·11	20·1	"	0·4815	71·45	74·53	74·86
1·46	20·0	"	0·4805	70·99	75·02	75·33
NaCl.						
0	19·3	62 96	0·4842	72·69	72·57	72·77
0·397	20·3	"	0·4837	72·47	73·53	73·88
0·468	20·9	"	0·4837	72·47	73·72	74·17
0·549	21·3	"	0·4833	72·28	73·76	74·27
0·62	21·4	"	0·4824	71·86	73·54	74·06
0·703	21·3	"	0·4817	71·55	73·46	73·97
0·768	21·2	"	0·4821	71·73	73·83	74·32
NaCl.						
0	15·9	62·96	0·4860	73·54	73·47	73·15
0·34	16·2	"	0·4852	73·16	74·11	73·84
0·40	16·5	"	0·4850	73·06	74·19	73·96
0·46	16·8	"	0·4845	72 76	74·06	73·88
0·52	16·9	"	0·4847	72·92	74·40	74·23
0·58	17·1	"	0·4841	72·65	74·30	74·16
0·64	17·2	"	0·4840	72·56	74·38	74·26
0·76	17·3	"	0·4840	72·56	74·72	74·61
0·88	17·4	"	0·4831	72·19	74·69	74·60
KCl.						
0	18·6	62·87	0·4854	73·02	72·91	73·00
0·09	18·6	"	0·4861	73·35	73·49	73·58
0·19	18·4	"	0·4858	73·21	73·76	73·82
0·27	18·3	"	0·4852	72·93	73·75	73·80
0·35	18·2	"	0·4853	72·98	74 05	74·08
0·58	18·1	"	0·4855	73·07	74·52	74·54
KCl.						
0	18·7	62·78	0·4855	72·84	72·73	72·84
0·338	19·3	"	0·4851	72·66	73·68	73·88
0·557	19·9	"	0·4849	72·57	73·96	74·25
0·771	20·2	"	0·4839	72·11	74·55	74·89
0·986	20·3	"	0·4827	71·56	74·62	74·98
1·204	20·3	"	0·4823	71·37	75·11	75 48

Table (*continued*).

$\frac{\text{gm.-mol.}}{\text{litre}}$.	*t*.	*n*.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.
			KCl.			
0	18·0	62·87	0·4863	73·43	73·33	73·33
0·19	17·3	,,	0·4851	72·88	73·44	73·33
0·37	16·8	,,	0·4847	72·69	73·86	73·68
0·55	16·4	,,	0·4847	72·69	74·46	74·21
0·73	16·0	,,	0·4844	72·55	75·00	74·69
0·92	15·8	,,	0·4836	72·19	74·96	74·62
			KCl.			
0	21·1	62·78	0·4860	73·08	72·81	73·28
0·221	20·6	,,	0·4848	72·52	73·14	73·54
0·446	20·3	,,	0·4843	72·29	73·64	73·99
			$\frac{1}{2}(K_2CO_3+2\ aq.)$.			
0	18·3	62·86	0·4862	73·38	73·28	73·33
0·22	19·2	,,	0·4850	72·82	73·85	74·04
0·44	19·9	,,	0·4833	72·04	73·97	74·27
0·66	20·4	,,	0·4813	71·13	73·90	74·27
0·88	20·8	,,	0·4802	70·61	74·41	74·84
			$\frac{1}{2}(K_2CO_3+2\ aq.)$.			
0	13·7	62·86	0·4864	73·46	73·41	72·76
0·096	13·7	,,	0·4860	73·28	73·81	73·16
0·294	13·6	,,	0·4851	72·86	74·29	73·62
0·512	13·5	,,	0·4836	72·17	74·43	73·74
0·713	13·5	,,	0·4826	71·70	74·95	74·26
0·915	13·5	,,	0·4812	71·07	75·14	74·45
			$\frac{1}{2}(Na_2CO_3+10\ aq.)$.			
0	18·5	62·85	0·4857	73·11	73·00	73·08
0·061	18·5	,,	0·4855	73·02	73·20	73·28
0·187	18·1	,,	0·4853	72·92	73·61	73·63
0·310	17·6	,,	0·4849	72·74	73·95	73·89
0·432	17·4	,,	0·4840	72·33	74·04	73·95
			$\frac{1}{2}(Na_2CO_3+10\ aq.)$.			
0	17·9	62·86	0·4857	73·14	73·04	73·03
0·115	17·6	,,	0·4855	73·05	73·42	73·36
0·228	17·1	,,	0·4848	72·73	73·54	73·40
0·344	16·6	,,	0·4848	72·73	74·02	73·81
0·458	16·3	,,	0·4838	72·27	73·99	73·73
0·574	15·9	,,	0·4837	72·22	74·40	74·08

Table (*continued*).

$\frac{\text{gm.mol.}}{\text{litre}}$.	*t*.	*n*.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.
$\frac{1}{2}(ZnSO_4 + 7\text{ aq.})$.						
0	18·1	62·78	0·4867	73·40	73·30	73·31
0·129	18·7	,,	0·4850	72·65	73·32	73·43
0·257	18·7	,,	0·4840	72·16	73·58	73·69
0·39	18·7	,,	0·4822	71·47	73·64	73·75
$\frac{1}{2}(ZnSO_4 + 7\text{ aq.})$.						
0	17·7	62·78	0·4864	73·26	73·16	73·12
0·283	18·2	,,	0·4832	71·79	73·36	73·39
0·567	18·4	,,	0·4804	70·51	73·65	73·71
$\frac{1}{2}(ZnSO_4 + 7\text{ aq.})$.						
0	18·6	62·78	0·4867	73·49	73·12	73·21
0·115	19·6	,,	0·4853	72·76	73·33	73·58
0·173	19·7	,,	0·4851	72·66	73·48	73·74
0·216	19·7	,,	0·4845	72·38	73·54	73 80
0·263	19·7	,,	0·4832	71·79	73·22	73·48
$\frac{1}{2}(ZnSO_4 + 7\text{ aq.})$.						
0	18·4	62·78	0·4866	73·35	73·24	73·31
0·0435	19·5	,,	0·4857	72·94	73·07	73·30
0·0887	20·2	,,	0·4847	72·47	72·88	73·38
0·133	20·7	,,	0·4850	72·61	73 28	73·69
0·35	20·5	,,	0·4832	71·79	73·71	74·09
$\frac{1}{2}(ZnSO_4 + 7\text{ aq.})$.						
0	18·3	62·78	0·4868	73·45	73·34	73·39
0·0199	18·4	,,	0·4860	73·08	73·11	73·17
0·066	18·8	,,	0·4856	72·89	73·18	73·30
0·108	18·9	,,	0·4853	72·76	73·30	73·44
0·152	18·7	,,	0·4850	72·61	73·42	73·52
0·286	18·7	,,	0·4841	72·21	73 80	73·90
0·33	18·5	,,	0 4830	71·71	73·54	73·62
0·372	18·4	,,	0·4829	71·65	73·75	73·79
0·414	18·3	,,	0·4827	71·55	73·87	73·91

Table (*continued*).

$\frac{1}{2}(ZnSO_4)$.

$\frac{\text{gm.-mol.}}{\text{litre}}$.	*t*.	*n*.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.
0	14·8	62·85	0 4870	73·74	73·68	73·20
0·18	15·2	,,	0·4848	72·70	73·72	73·30
0·37	15·5	,,	0·4839	72·31	74·43	74·05
0·555	15·9	,,	0·4822	71·53	74·67	74·35
0·736	16·2	,,	0·4804	70·71	74·82	74·54
0·919	16·4	,,	0·4789	70·03	75·06	74·81

H_2O.

t.	*n*.	λ_1.	$\frac{T_t}{\rho}$.	T_t.	T_{18}.	T_0.
16·0	63·07	0·4858	73·72	73·64	73·34	
16·3	63·07	0·4858	73·72	73·64	73·38	
13·1	63·07	0·4865	74·03	73·99	73·25	
16·4	62·96	0·4863	73·64	73·59	73·35	
18·6	62·96	0·4854	73·25	73·14	73·23	
19·3	62·96	0·4842	72·69	72 57	72·77	
15·9	62·96	0·4860	73·54	73·47	73·15	
18·6	62·87	0 4854	73·02	72·91	73·00	
18·7	62·78	0·4855	72·84	72·73	72·84	
21·1	62·78	0·4860	73·08	72·81	73·28	
18·0	62·87	0·4863	73·43	73·33	73·33	
18·3	62·86	0·4862	73·38	73·28	73·33	
13·7	62·86	0·4864	73·46	73·41	72·76	
17·9	62·86	0·4857	73·14	73·04	73·03	
18·5	62·85	0·4857	73·11	73·00	73·08	
18·1	62·78	0·4867	73·40	73·30	73·31	
17·7	62·78	0·4864	73·26	73·16	73·12	
18 6	62·78	0·4867	73·40	73·12	73·21	
18·4	62·78	0·4866	73·35	73·24	73 31	
18·3	62·78	0·4868	73·45	73·34	73·39	
14·8	62·85	0·4870	73·74	73·68	73·20	
17·44				73·25	73·17	75·90

Omitting T_{18}=72·77; 72·84; 72·76 we find the following values:—

17·48	...	...	...	73·32	73·24	75·9

In the first series for potassium chloride the two forks were not in unison when the surface-tension of the water was determined, but were afterwards adjusted. This accounts for the low value found for water. In the case of the carbonates, for which but two series of determinations were made, the two sets of points are denoted on the figure by crosses and by crosses in circles respectively. It will be noticed that the two series for potassium carbonate do not coincide but lie parallel to one another. If there were any impurity in the body of the water that decreased its surface-tension, it could not be removed by the flexible hoop, and probably it would lower the whole curve as we observe in this case. Those points for zinc sulphate which are represented by crosses in circles were obtained from a series of readings on solutions of the anhydrous salt. This contained quite a little of the sulphide, which floated around in the solution but apparently produced no effect on the wave-length of the ripples. The values obtained by Sentis are represented by circles.

It is evident that these results can be best represented by straight lines. If we write $T_S = T_W + \kappa c$, where T_S is the surface-tension of the solution containing c gram-equivalents per litre, and T_W is the surface-tension of water at the same temperature, and κ is a constant, we can find κ from the plotted results. The values found for κ by various observers are given in the following table; those assigned to Sentis were calculated from his observations on solutions of 0·56 and 1·13 normal concentrations.

	Volkmann.	Quincke.	Sentis.	Röther.	Dorsey.
NaCl; $\kappa=$	1·59	1·57	1·57	1·38	1·53
KCl; $\kappa=$	1·41	1·57	1·41	1·47	1·71
$\frac{1}{2}(Na_2CO_3)$; $\kappa=$	0·987	1·57	...	...	2·00
$\frac{1}{2}(K_2CO_3)$; $\kappa=$	1·78	1·57	...	...	1·77
$\frac{1}{2}(ZnSO_4)$; $\kappa=$	...	...	1·78	...	1·86

All of Volkmann's curves appear to become steeper as they approach zero concentration; and his curves for K_2CO_3 and Na_2CO_3 cross near the origin, so that the curve for Na_2CO_3 for dilute solutions becomes steeper than the one for K_2CO_3.

The comparison of the results of this work with those obtained by other experimenters must not be pushed too far, because they have worked exclusively on solutions of greater concentration than half-normal, while most of my observations are on solutions less concentrated than that.

The average of all observations taken on water gives $T_{17^\circ \cdot 44} = 73\cdot 25$, or $T_{18^\circ} = 73\cdot 17$, which is rather low. If we omit the values $T_{18^\circ} = 72\cdot 77$, 72·84, and 72·76, which are certainly too low, we find the averages given in the last line of the table; viz., $T_{17^\circ \cdot 48} = 73\cdot 32$, which gives $T_{18^\circ} = 73\cdot 24$ and $T_0 = 75\cdot 98$. This is lower than the values found by Quincke, but agrees very well with the results obtained by Volkmann, Hall, Sentis, Rayleigh, and others.

In conclusion I wish to thank Professor Rowland and Dr. Ames for their suggestions and encouragement throughout the entire course of this work.

Johns Hopkins University,
June 1897.

L. *On Determinant Notation.* *By* E. J. Nanson, *M.A.**

THE notation for a determinant introduced in 1861 by H. J. S. Smith has been almost universally adopted on account of its extreme simplicity and clearness. That notation is that the determinant

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

is denoted by

$$| \; a_{pq} \; | \qquad (p, q=1, \ldots n).$$

Now just as it is convenient to have this simple notation for the determinant, so also is it convenient to have a simple notation for the array of symbols used to form the determinant, or, more generally, for an array in which the number of rows is not necessarily the same as the number of columns.

It is therefore proposed to use the notation

$$(a_{pq}) \qquad (p=1, \ldots m; \; q=1, \ldots n);$$

for the array

$$\begin{matrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{m1} & a_{m2} \cdots a_{mn}. \end{matrix}$$

This notation is particularly convenient in dealing with determinants which can be divided by horizontal and vertical lines into blocks, such that all elements in the same block follow the same law. For instance, the bordered determinant

$$\begin{vmatrix} a_{11} \cdots a_{1m} & b_{11} \cdots b_{1n} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ a_{m1} \cdots a_{mm} & b_{m1} \cdots b^{mn} \\ c_{11} \cdots c_{1m} & \\ \vdots \quad\quad \vdots & \\ c_{n1} \cdots c_{nm} & \end{vmatrix}$$

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* Communicated by the Author.

may be denoted by

$$\begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & \end{vmatrix} \quad \binom{p,q=1,\ldots m}{r,s=1,\ldots n}.$$

As an example take the theorem given long ago by Sylvester *, as an illustration of the power of this umbral notation.

In the notation now suggested that theorem may be expressed as follows :—

If $$e_{rs} = \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & d_{rs} \end{vmatrix} \quad (p,q=1,\ldots m);$$

then

$$|\, e_{rs} \,| = |\, a_{pq} \,|^{n-1} \cdot \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & (d_{rs}) \end{vmatrix} \quad (r,s=1,\ldots n).$$

Proofs of this theorem have been given by Scott †, Frobenius ‡, and Netto §. But in no one of these proofs is the fundamental simplicity of the theorem made evident.

In the first place, by the general law of extension due to Muir ‖, the theorem is at once seen to be true, because it is merely the extensional of the obvious identity

$$|\, d_{rs} \,| = \begin{vmatrix} (d_{11}) & (d_{12}) & \ldots & (d_{1n}) \\ (d_{21}) & (d_{22}) & \ldots & (d_{2n}) \\ \vdots & \vdots & & \vdots \\ (d_{n1}) & (d_{n2}) & \ldots & (d_{nn}) \end{vmatrix}$$

In the second place, the theorem may be proved directly by the following elementary method.

Multiply columns 1 to m of the determinant Δ, where

$$\Delta = \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & (d_{rs}) \end{vmatrix} \quad \binom{p,q=1,\ldots m}{r,s=1,\ldots n},$$

by $\lambda_{s1}, \ldots \lambda_{sm}$, add to column $m+s$, and do this for all values of s. Then, provided the multipliers λ are taken so that

$$\Sigma_q\, a_{pq}\lambda_{sq} + b_{ps} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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* Phil. Mag. April 1851, p. 297.

† Proc. Lond. Math. Soc. vol. xiv. p. 92.

‡ *Crelle*, vol. cxiv. p. 189.

§ *Acta Math.* vol. xvii. p. 202.

‖ Trans. R. S. E. xxx. p. 1; 'Theory of Determinants,' p. 213.

we get

$$\Delta = \begin{vmatrix} (a_{pq}) & \\ (c_{rq}) & (f_{rs}) \end{vmatrix}$$

$$= | a_{pq} | \,.\, | f_{rs} | , \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where f_{rs} is given by

$$\Sigma_q \, c_{rq} \lambda_{sq} + d_{rs} = f_{rs} . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Now, eliminating λ from (1), (3), we get

$$\begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & d_{rs} - f_{rs} \end{vmatrix} = 0 ;$$

whence f_{rs} is given by

$$| a_{pq} | f_{rs} = \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & d_{rs} \end{vmatrix}$$

$$= e_{rs}$$

by definition of e_{rs}. Therefore, by (2)

$$| a_{pq} |^{n-1} \Delta = | a_{pq} |^{n} \,.\, | f_{rs} |$$

$$= | e_{rs} | ,$$

which is the theorem to be proved.

Sylvester speaks of this theorem as "one of the most prolific in results of any with which I am acquainted," and further says: "It is obvious that without the aid of my system of umbral or biliteral notation this important theorem could not be made the subject of statement without an enormous periphrasis, and could never have been made the object of distinct contemplation or proof."

The theorem just proved is a particular case of another theorem given by Sylvester in the same paper. Before giving a statement of this second theorem it is necessary to explain a notation for a minor determinant and also for what may be called a minor array.

If θ, ϕ be ν-ads from $1, \ldots n$, then the minor formed from the ν rows θ and the ν columns ϕ of the determinant

$$| \; a_{pq} \; | \qquad (p, q = 1, \ldots n)$$

may be denoted by $| \, a_{\theta\phi} \, |$.

In like manner, if θ be a μ-ad from $1, \ldots m$ and ϕ a ν-ad from $1, \ldots n$, then the array formed by the elements common to the μ rows θ and the ν columns ϕ of the array

$$(a_{pq}) \qquad (p=1,\dots m;\ q=1,\dots n)$$

may be denoted by $(a_{\theta\phi})$. In particular the array formed by all the elements in the μ rows θ may be denoted by $(a_{\theta q})$.

With this notation Sylvester's second theorem may be stated as follows:—

If θ, ϕ be ν-ads from $1,\dots n$ and

$$E_{\theta\phi}=\begin{vmatrix} (a_{pq}) & (b_{p\phi}) \\ (c_{\theta q}) & (d_{\theta\phi}) \end{vmatrix} \quad (p, q=1,\dots m),$$

then

$$|\,E_{\theta\phi}\,| = |\,a_{pq}\,|^{\lambda}.\begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & (d_{rs}) \end{vmatrix}^{\lambda'} \quad (r, s=1,\dots n),$$

where

$$\lambda=\frac{(n-1)(n-2)\dots(n-\nu+1)}{1.2\dots\nu},$$

$$\lambda'=\frac{(n-1)(n-2)\dots(n-\nu)}{1.2\dots(\nu-1)}.$$

Proofs of this theorem have been given by Reiss*, Picquet†, Scott‡, and Van Velzer§. Scott deduces the second theorem from the first in the following simple way:—

By the first theorem we have

$$A^{\nu-1}E_{\theta\phi}=|\,e_{\theta\phi}\,|,$$

where $A=|\,a_{pq}\,|$. Hence the determinant

$$|\,A^{\nu-1}E_{\theta\phi}\,|,$$

which is of order $\lambda+\lambda'$, is the ν-th compound of $|\,e_{rs}\,|$ and is therefore, by Franke's theorem, equal to $|\,e_{rs}\,|^{\lambda'}$. But by the first theorem we have $|\,e_{rs}\,|=A^{n-1}\Delta$, and hence we readily find $|\,E_{\theta\phi}\,|=A^{\lambda}\Delta^{\lambda'}$.

Van Velzer remarks that this proof "seems to leave nothing to be desired, either in simplicity or rigour." Sylvester's second theorem is, however, merely the extensional of Franke's theorem already quoted. It is certainly remarkable that a very general theorem enunciated without proof by Sylvester

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* *Beiträge zur Th. der Det.* p. 34 (1867).

† *Journal de l'Ecole Pol. Cah.* xlv. p. 216.

‡ Proc. Lond. Math. Soc. xiv. p. 93.

§ Am. Journ. Math. vi. p. 168.

in 1851 contains as its simplest special case the theorem of Franke, which was not discovered until 1862.

As another example in illustration of the block notation take the determinant identities associated with the name of Schweins. These are found, after Cayley*, by expanding the determinant

$$\begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & \end{vmatrix},$$

where $p=1, \ldots m$; $q=1, \ldots n$; $r=1, \ldots n-k$; $s=1, \ldots m-k$, in two different ways and equating the results. This process gives, by Laplace's theorem,

$$\Sigma_\theta(-1)^\lambda \begin{vmatrix} (a_{\theta q}) \\ (c_{rq}) \end{vmatrix} \cdot \left| b_{\theta' s} \right| = \Sigma_\phi(-1)^\mu \left| (a_{p\phi})(b_{ps}) \right| \cdot \left| c_{r\phi'} \right|,$$

where θ, ϕ are k-ads from $1, \ldots m$; $1, \ldots n$ respectively; θ', ϕ' are the sets complimentary to θ, ϕ respectively; and λ, μ are the sums of the numbers in the sets θ, ϕ respectively.

The general extensional of this identity is

$$\Sigma_\theta(-1)^\lambda \begin{vmatrix} (a_{\theta q}) & (d_{\theta u}) \\ (c_{rq}) & (d_{ru}) \\ (e_{tq}) & (f_{tu}) \end{vmatrix} \cdot \begin{vmatrix} (b_{\theta' s}) & (d_{\theta' u}) \\ (e_{ts}) & (f_{tu}) \end{vmatrix}$$

$$= \Sigma_\phi(-1)^\mu \begin{vmatrix} (a_{p\phi}) & (b_{ps}) & (d_{pu}) \\ (c_{t\phi}) & (e_{ts}) & (f_{tu}) \end{vmatrix} \cdot \begin{vmatrix} (c_{r\phi'}) & (d_{ru}) \\ (e_{t\phi'}) & (f_{tu}) \end{vmatrix},$$

where $t, u=1, \ldots z$.

Muir † tells us that "the notation which in our time would almost certainly be chosen for the statement of such identities" as those of Schweins, "is the umbral notation of Sylvester."

The object of this short note is to show that the umbral notation is not always the most convenient, and that in some cases at all events the block notation conduces to brevity and lucidity.

Melbourne, July 31, 1897.

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* C. M. P. 9 and 676.

† Phil. Mag. Nov. 1884, p. 422.

LI. *On the Absorption of Röntgen Radiation.*

By W. J. Humphreys[*].

THE present investigation was undertaken for the purpose of determining whether the absorption of Röntgen radiation depends only upon the kind of elements and their amounts passed through, or whether it is dependent to some extent also upon the manner in which these elements are combined. That is, to determine whether the absorption depends at all upon the grouping of the atoms into molecules or only upon the atoms themselves; in short, whether the phenomenon is an atomic or a molecular one.

As everyone knows, the transparency or opacity of a substance to ordinary light cannot be predicted from the properties in this respect of its constituents. Even allotropic forms of a single element, as admirably illustrated by graphite and the diamond, may differ to the greatest extent in respect to their powers of transmitting ordinary light, and therefore it seems certain that the absorption of light-rays is at least largely a molecular phenomenon. The absorption of the Röntgen radiation, on the contrary, judging from the experiments presently to be described, is chiefly, if not entirely, an atomic phenomenon, and consequently the absorption of Röntgen rays due to a compound differs but little, if at all, from the sum of those of its constituents.

The experiments were arranged as follows:—Above a protected dry plate was placed a slab of uniform thickness of some compound, and close beside this slab, on top of each other, slabs of equivalent thicknesses of its constituent elements. Above these in turn, or rather directly above a point halfway between them, was arranged the source of the Röntgen radiation. According to this arrangement, the average amount of any element met by a ray in passing through the compound is the same as that met by a ray (equally inclined to the normal) in passing through the pile of its constituents. Consequently if, after an exposure of suitable length, the photographic plate is unequally affected beneath the compound and the equivalent pile of its elements, one is forced to conclude that for some reason the absorption in the one case was different from that in the other, provided of course that the radiation was equally intense in the two directions, and also, as is probably always the case, there was but little loss due to

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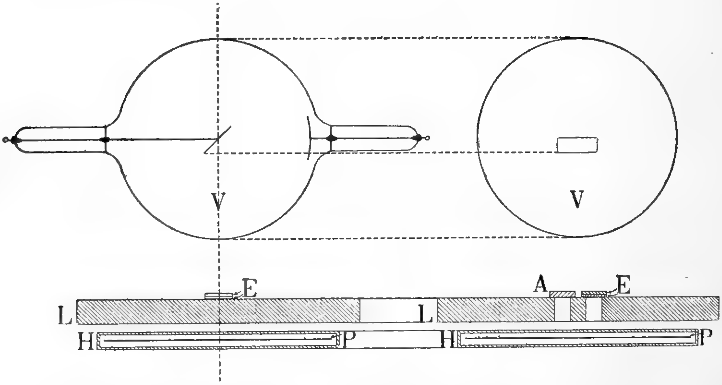
[*] Communicated by the Author.

any cause other than simple absorption. If, however, as it was invariably with my experiments, the plate is equally affected in the two places mentioned, it is clear that the absorptions were equal, or at least that their difference was less than the experimental errors.

To explain fully what is meant by the term "equivalent thickness":—Let C be a compound of the two elements A and B, let d be its density and t the thickness of a slab cut from it, and let a be that fraction which A is of C. Further, let d' be the density of A in the elementary or uncombined form, and t' the thickness of a slab cut from the element A. Then the two slabs have equivalent thicknesses, so far as A is concerned, when $adt=d't'$. Similarly for B, and also for any number of elements of which a compound might be made up.

The arrangement of the experiment can be easily understood by aid of the accompanying sketch—two sectional drawings, at right angles to each other, of the apparatus used.



V is the vacuum-tube, L a thick lead plate used to protect the dry plate P, which of course was protected from light also by an ordinary plate-holder H. The compound A and the elements E were placed as shown over holes bored through the lead plate. The diameters of these holes were one centimetre each and their axes two centimetres apart. The thickness of the lead plate was one and one half centimetres.

The current, approximately four and one half amperes, was taken from a storage-battery of about 118 volts, and used in connexion with an excellent coil and rotary break of the Willyoung pattern. Very high penetrative power was desired from the first, and consequently one of Queen's adjustable tubes was selected for the work.

To reduce the errors due to inaccurate grinding as far as possible, the slabs were made so thick that the time of exposure was never less than half an hour, and in some instances more than an hour. In no case was there an over exposure, and in all cases the images or shadows as formed beneath the compound and its constituents appeared simultaneously and remained of equal intensity during the process of developing.

The substances tested in this way were :—

Zinc blende	ZnS.
Stibnite	Sb_2S_3.
Chalcocite	CuS.
Galena	PbS.
Speculum metal	68 parts copper, 32 parts tin.
Regulus of Venus ...	58 parts copper, 42 parts antimony.
Aluminium bronze ...	92½ parts copper, 7½ parts aluminium.
Brass	65 parts copper, 35 parts zinc.

It would be very desirable of course to work with many more substances ; but, apart from alloys, those compounds, all of whose constituents in the uncombined form are solid and suitable to work with, are by no means numerous. However, from this small number of substances examined I judged that possibly the phenomenon of Röntgen-ray absorption was an atomic one, and concluded that if so all compounds none of whose elements were highly absorptive of Röntgen rays would be transparent to them, while those containing one or more elements which in the free state were opaque to this radiation would themselves be opaque to it. This supposition was fully borne out in every test made—some forty or more compounds of the most varied nature being examined.

Owing to the complex nature of Röntgen radiation, quantitative measurements of the coefficients of absorption of different elements may not be very reliable ; but it seems that the absorption is a function, though probably not a linear one, of atomic weight. About this and certain other points I hope to have something to say in a future communication.

In closing I wish to thank Professor Rowland for allowing me the use of the physical laboratory of the Johns Hopkins University, where this work was done during July of the present year. I wish also to thank Dr. Chandler of Baltimore for kindly lending me his splendid Röntgen-ray outfit, and Dr. Mallet of the University of Virginia for a number of alloys.

University of Virginia,
September 6, 1897.

LII. *The Specific Inductive Capacity of Dielectrics in slowly and rapidly changing Electric Fields.* By Ervin S. Ferry*.

EVER since Maxwell pointed out that the electromagnetic theory of light requires a certain definite relation to hold between the index of refraction and the dielectric constant of a medium transmitting light, persistent attempts have been made to determine the dielectric constants of various optical substances, in the hope that such determinations would afford an explanation of anomalous reflexion and refraction, and the peculiarities observed when light is transmitted through crystalline media that are heated or are under mechanical, electrostatic, or magnetic stress. The results obtained by different experimenters vary through a wide range, due largely to the great difficulties in the determination of the dielectric constant, and also in many cases to lack of knowledge of the periodicity of the electrical disturbance employed in making the measurements. While with our present limitations regarding the production of measurable electromagnetic oscillations of a period approaching that of visible radiance, it may be impossible to obtain quantitative data bearing upon anomalous reflexion and dispersion, still the piezo-optical, electro-optical, magneto-optical, and pyro-optical problems appear of perfectly definite solution. It was with regard to the study of this class of phenomena that the present series of experiments was begun.

The object of the present investigation is the perfection of a null method of determining the specific inductive capacity of crystals and other optical substances that does not require large masses of the substance; that does not require the dimensions of the substance to be known or the specimen to be of any particular shape; that is independent of the nature of the surface; that is sensitive to small variations of the quantity measured; in which the labour of computation of results is reduced to a minimum; and in which the periodicity of the oscillations used can be easily experimentally obtained.

Three different properties of dielectrics have been utilized to determine the specific inductive capacity of substances. Cavendish discovered that the capacity of a condenser depends upon the medium between the plates. The ratio of the capacity of a condenser having any given dielectric between its plates to its capacity with air as the dielectric, was called by Faraday the Specific Inductive Capacity of the given sub-

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* Communicated by the Author.

stance, and by Maxwell the Dielectric Constant. If two similar condensers are electrified from the same source they will attain the same potential, and the charges they receive will be proportional to their capacities. By determining the ratio of these charges the ratio of the capacities is determined; and if air be one of the dielectrics, this ratio is the specific inductive capacity of the other dielectric. This is the basis of the majority of the methods for determining dielectric constants. Again, the force of attraction exerted between two plates charged to different potentials is proportional to the specific inductive capacity of the dielectric separating them; so that, by weighing the force of attraction between two charged plates the specific inductive capacity of the intervening dielectric can be determined. And finally, methods have been employed based upon the fact that the period of the electrical waves sent out when an electrical system is discharged is proportional to the square root of the capacity and self-induction of the electrical system.

The method used probably more than any other is a modification of Thomson's induction balance first employed by Gordon[*]. It consists of five parallel disks, three forming a primary system and two a secondary. The primary system consists of the two outside plates connected together, and the middle plate. When a difference of potential is established between the outside plates and the middle plate, the other two smaller intermediate plates can be so placed in the resulting field of force that there will be no potential-difference induced upon them. The introduction of a dielectric between any two of the plates will disturb the equilibrium, and in order to again bring the small plates to equal potential, one or both must be moved with regard to the primary system of large plates. The amount of the change of position of the small plates required to again secure equilibrium after the introduction of the dielectric gives the means of computing the change of capacity of the system produced by the introduction of the dielectric, and hence the specific inductive capacity of the dielectric. A successful modification of this method has been used by Blondlot[†], who, instead of securing equilibrium by moving the secondary plate, introduced between one of the primary and one of the secondary plates such a thickness of another dielectric of known specific inductive capacity that the spark ceased in a micrometer air-gap placed between the secondary plates. In Blondlot's experiment he used two thin wedges of sulphur arranged like

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[*] Phil. Trans. Roy. Soc. 1879, part 1, p. 417.

[†] *C. R.* cxii. p. 1058 (1891).

a Babinet Compensator, so that the thickness of sulphur introduced could be varied at pleasure.

By comparing the force of attraction exerted on a small sphere of a dielectric when placed in a field of known intensity with the force exerted on a metallic sphere of equal size placed in the same field, Boltzmann* has measured the dielectric constants of several amorphous solids, and also the dielectric constants of sulphur in the direction of the optical axis.

The law of the proportionality between the force of attraction exerted between two plates charged to different potentials to the specific inductive capacity of the dielectric separating them, has recently been utilized by several experimenters to determine dielectric constants of electrolytes. Rosa † measured the force exerted between a fixed and a movable electrode immersed in the liquid dielectric under investigation with a form of apparatus similar to that used by Boltzmann. One electrode was stationary and one was suspended on the end of a light horizontal rod by means of a thin fibre, in a manner analogous to the Cavendish balance. Lefevre ‡ substituted a beam-balance for the torsion balance. One pan was replaced by an electrode, and the other electrode was fixed a short distance beneath. Trouton and Lilly § have suggested a method consisting in measuring the force exerted by a charged condenser tending to draw a dielectric in between the plates. The apparatus consists of the alternate quadrants of a quadrant-electrometer charged to different potentials, the other two quadrants being removed. The specimen whose dielectric constant is to be determined is made in the form of an electrometer-needle and is suspended between the quadrants.

J. J. Thomson ‖ and A. Perot ¶ have found the constants of various dielectrics by measuring the wave-length of the electrical vibrations given out by a discharging condenser when the plates are separated by air and when they are separated by the dielectric under investigation. The ratio of the capacity of the condenser when air separates the plates to the capacity when another dielectric is inserted, equals the ratio of the squares of the wave-lengths produced when the condenser is discharged under the two conditions. So that by measuring the capacity of the air-condenser and the wave-lengths of the electrical vibrations in the two cases, the specific inductive

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* *Sitz. Akad. Wiss. Wien*, lxviii. p. 81 (1873); lxx. p. 342 (1874).

† Phil. Mag. [5] xxxi. p. 188 (1891).

‡ *Journal de Physique*, June 1892.

§ Phil. Mag. [5] xxxiii. p. 529 (1892).

‖ Proc. Roy. Soc. xlvi. p. 292 (1889).

¶ *C. R.* cxiv. p. 1528 (1892).

capacity of a dielectric is determined. Thwing* has developed the method into one requiring only the measurement of the capacity of an air-condenser. In this method a Hertzian resonating circuit is placed parallel to, and at a short distance from, a similar exciting circuit connected to an induction-coil. By changing the capacities the two circuits can be put into resonance with each other, resonance being determined by noting the maximum deflexion of a dynamometer in the secondary circuit. In determining the dielectric constant of an electrolyte, a sample is placed between the plates of the condenser in the secondary circuit and the capacity of the primary circuit changed till resonance is attained. Then replacing the condenser containing the electrolyte by an air-condenser, resonance is again secured by changing the distance separating the plates of the latter. This gives the capacity of the condenser containing the electrolyte. By knowing the capacity of the same condenser when empty the specific inductive capacity of the electrolyte is determined. This method is probably the most rapidly and most easily operated of any of equal accuracy yet described.

The method now to be described consists in getting two electrical systems of equal self-induction in resonance with a third system in which rapid electrical oscillations are set up by means of an induction-coil and spark-gap. The periods of the electrical oscillations in the two secondary circuits will therefore be equal to each other, and hence the capacities in these two circuits will be equal. The capacities in one of the two secondary circuits is produced by an adjustable air-condenser, and the capacity in the other circuit consists of a condenser with plates separated by a fixed distance. By securing resonance of the two secondary circuits with the primary, first when the empty condenser with fixed plates is in one of the secondary circuits, and then again when a dielectric is inserted between its plates, the specific inductive capacity of the dielectric is determined by simply computing the capacity of the adjustable air-condenser in the two cases.

The method and apparatus employed may be rendered clearer by reference to fig. 1. *mh* is a rectangle of about 70 centim. per side, connected to an air-condenser c''', induction-coil, and spark-gap as shown. This constitutes the primary circuit. The secondary circuits consist of two rectangles of equal dimensions, *pb* and *gd*, connected respectively to a standard air-condenser c', and the condenser with fixed plates c''. In order to determine when the two secondary circuits

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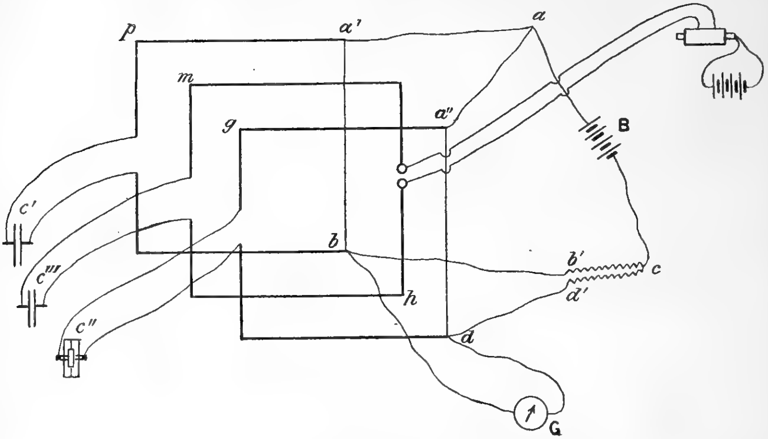
* Phys. Rev. ii. p. 35 (1894).

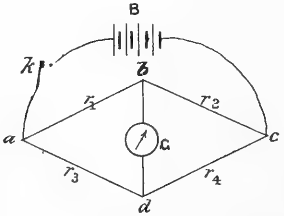
are in resonance with the primary, and hence with each other, the principle of the bolometer is employed.

Fig. 1.



In the conventional diagram of the Wheatstone bridge shown in fig. 2, if *bc* and *dc* are of the same resistance, the galvanometer will show no deflexion on pressing the key *k* when the resistance of *ab* equals the resistance of *ad*. If *ab* and *ad* are of equal resistance at any given temperature and have the same mean temperature-resistance coefficient, the resistance of one will equal the resistance of the other, whatever the temperature. In other words, if no deflexion of the galvanometer-needle is produced on pressing the key *k*, *ab* and *ad* are at the same temperature, so long as the equality of resistance of the other two arms of the bridge remains unaltered. This latter equality is secured by winding the two wires *bc* and *dc* close together and parallel to each other on the same spool. In fig. 1, one side of each of the secondary circuits *a'b* and *a''d* consists of a piece of No. 36 iron wire,

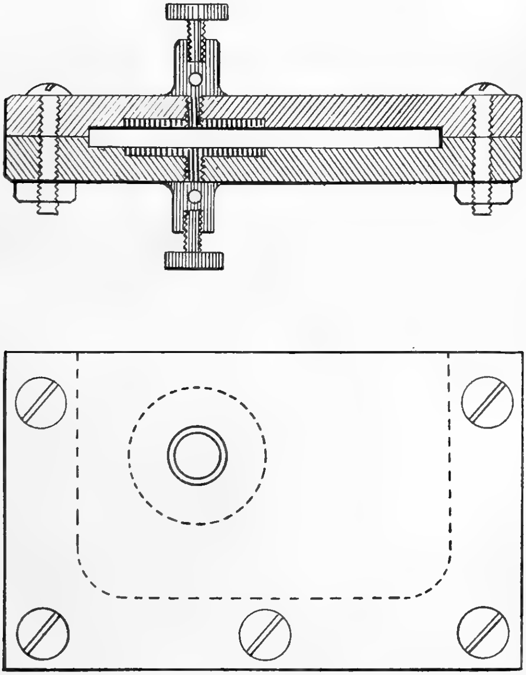
Fig. 2.



forming the two bolometer arms corresponding to ab and ad in fig. 2; and cb' and cd', together, constitutes a double coil of wire, the two parts of which are of as nearly the same resistance as it is possible to construct.

The balancing arms of the bridge cb' and cd' consist of two No. 22 insulated copper wires, each of about two ohms' resistance, wound side by side on a brass tube. The lengths were adjusted till the difference of resistance of the two wires was less than one one-hundred-thousandth of an ohm, as indicated by the Carey-Foster method. The bolometer arms of the bridge were selected by trial of several pieces of No. 36 iron wire till two were found that, when connected up as a Wheatstone bridge with the balancing arms already constructed, gave no deflexion of the needle of a sensitive low-resistance galvanometer when they were immersed in water at either 20° or 100° C. The condensers c' and c''' are ordinary adjustable air-condensers of about 12 and 16 centim. diameter respectively. The condenser c'' is shown in fig. 3. It consists of two brass disks of about 3·2 centim. diameter imbedded

Fig. 3.



in the sides of a hard rubber cell, with a fixed distance of about 0·3 centim. separating the disks.

The first adjustment of the apparatus required is to secure resonance of the two secondary circuits with the primary. With circuits of fixed self-induction placed in inductive relation with each other, resonance can be produced by simply altering the capacity in one of the circuits. The adjustment is known to be that giving the most perfect resonance when the current oscillating back and forth in the secondary is a maximum. In this experiment the current flowing in the secondary circuits is determined by measuring the change in resistance of a portion of the circuit produced by the heating effect of the current. If the two secondaries are singly in resonance with the primary, they are in resonance with each other. The condenser that is to contain the specimen is connected to one primary circuit, the condenser removed from the other secondary, and then the capacity in the primary circuit adjusted till the galvanometer-needle gives a maximum deflexion. Now this secondary is in resonance with the primary. Then the condenser is removed from this secondary circuit and inserted in the other secondary. If the two secondaries have the same self-induction, this latter arrangement will also give a maximum galvanometer deflexion and in the opposite direction to that in the former arrangement, but the deflexions in the two cases may not be exactly equal. These maximum deflexions can be made equal by slightly adjusting the distance between the rectangular circuits. In these experiments the three rectangular circuits were placed about 10 cm. apart. The two condensers now in circuit are allowed to remain unaltered, the standard air-condenser is inserted in the remaining secondary circuit and its capacity altered until the galvanometer deflexion is zero. The condensers in the secondary circuits may now be interchanged without altering the galvanometer deflexion. The capacity of the standard condenser is now the capacity of the condenser in the other secondary circuit.

All parts of the bolometer must be carefully screened from heating effects. Air-draughts and similar sudden changes can be guarded against by thick coverings of cotton-wool. If for any reason the temperature of different parts of the bolometer-circuit is different at one time from that at another time, the galvanometer-needle may be brought back to the same zero-point by shunting a large resistance around one of the balancing arms. In these experiments the shunt was seldom less than fifteen thousand ohms.

The diameter of the plates and the distance separating the plates of the standard condenser substituted in Kirchhoff's formula gives, with a minimum amount of computation, the capacity of the condenser containing the specimen. By determining the capacity when the specimen condenser is empty and when it is filled with a dielectric, the specific inductive capacity of the dielectric is determined.

In this way the specific inductive capacity was determined of several oils as follows:—Petroleum 1·99; cotton-seed oil 3·00; olive oil 3·02; castor oil 4·49.

The period of the electrical oscillations used in charging the dielectrics was determined by a method due to J. J. Thomson*. Two bare copper wires of about 20 metres length were stretched parallel to each other about the room with one end of each wire attached to one of two small copper disks placed close to the plates of the standard air-condenser. The remaining two ends of the long copper wires were left free. A sensitive telephone was connected by wires of equal length to the long conductors. When one telephone terminal is attached to one end of one of the long conductors and the other terminal made to slide along the other conductor, the sound in the telephone will pass through a series of intensities from minimum to maximum. The linear distance between any two points of the conductor giving a minimum sound is one half of the wave-length of the electrical vibrations charging the condenser containing the dielectric. The wave-length was found to be about 9·10 metres, which corresponds to a vibration frequency of 33,000,000 per second. Since it is only the order of magnitude of the vibration frequency that is required, and as the accuracy of this determination is necessarily uncertain, this value was taken to apply to the case of all the dielectrics examined. This is the more justifiable as the range of these specific inductive capacities extends only from 1·99 to 4·60.

This method is also applicable to the determination of the specific inductive capacity of solids when large specimens can be obtained of known dimensions, and when the constants of the condenser are known. In these experiments, however, these requirements were eliminated by the simple device of using in the condenser such a mixture of liquid electrolytes that the galvanometer-needle would be unaffected by the introduction of the solid being investigated. Thus the specific inductive capacity of the mixture in the condenser equals that of the solid.

When the condenser-cell was filled with a mixture of petroleum and castor oil of specific inductive capacity 2·32,

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* Proc. Roy. Soc. xlvi. p. 1.

no deflexion of the galvanometer-needle was produced when a sheet of ebonite was introduced between the plates of the condenser or when the ebonite was removed. The specific inductive capacity of ebonite is consequently taken to be 2·32.

A number of sections of quartz of different thicknesses, from 1 millimetre to 3 millimetres, were cut from several clear crystals. Sections were made both in the direction of the optic axis and normal to the axis. Some sections had approximately parallel faces and some were visibly wedge-shaped; some were polished, some simply smoothed, and some left just as they came from the saw. The mean value of the specific inductive capacity of the sections cut normal to the optic axis was found to be 4·34, and of those cut parallel to the axis 4·27. The greatest difference between either of these mean values and of any value actually observed would change simply the figure in the second decimal place.

In order to determine the change of specific inductive capacity with change of period of the electrical vibration to which the dielectric is subjected, the spark-gap in the primary circuit was opened so wide that no sparks crossed, and hence the electrical oscillation would be produced simply on the make and break of the interrupter of the induction-coil. But with this arrangement there was not sufficient energy induced on the secondary circuits to produce readable galvanometer deflexions even when the condenser was removed from one of the secondary circuits.

In consequence, an entirely different method was employed to determine the specific inductive capacity with slowly vibrating fields. The method is a very simple one, and no serious trouble was found in its operation. It is simply a Wheatstone bridge with four capacities substituted for resistances, the secondary of a very small induction-coil for the battery, and a sensitive telephone for the galvanometer. The same three condensers of the preceding method were used in this, with the addition of another exactly like the standard air-condenser used before. The induction-coil as used would give a spark of about 2 millimetres. It was enclosed in a box, and this box packed in "excelsior" inside of another closed box, so that no sound of the vibrating end was audible. The loudest note in the telephone should be similar to the hum of a mosquito. The ear appears to be most sensitive to changes in this sort of a note.

The theory of the method is entirely analogous to that of the Wheatstone bridge. If in fig. 2 it be imagined that the four resistances are replaced by four capacities k_1, k_2, k_3, k_4, and if at any moment the potentials at a and c are v and v_1, there will be no displacement of electricity along *bd* when the potential at *b* is

$$\frac{k_1v+k_2v_1}{k_1+k_2} \text{ and that at } d \text{ is } \frac{k_3v+k_4v_1}{k_3+k_4}.$$

That is, when $k_1k_4=k_2k_3$, *b* and *d* are at the same potential and the telephone is silent.

This method is only applicable to electrolytes that are poor conductors. In case the electrolyte is a conductor, the telephone will have no place of silence on account of the difference in phase of the electrical waves in the two branches of the circuit. But if the electrolyte conducts but slightly, silence can be produced in the telephone by shunting a high resistance around the other condenser in the same branch of the circuit as that containing the electrolyte. In this experiment the resistance employed consisted of two fine vertical glass tubes connected at the bottom by means of a bottle, and the whole filled with water. By means of wires that could be inserted to different depths in the tubes, a wide range of resistance can be obtained. The problem of the Wheatstone bridge using an alternating current when the circuit contains resistance, capacity, and self-induction has been worked out by Oberbeck*.

The substances previously examined, when subjected to this slowly changing field having a period of about five hundredths of a second, gave the following values for the specific inductive capacity:—Castor oil 4·65; olive oil 3·13; cotton-seed oil 3·09; petroleum 2·05; ebonite 2·55; quartz cut normal to the optic axis 4·46; quartz cut parallel to the optic axis 4·38. For purposes of comparison, the values of the dielectric constants for rapidly and for slowly changing fields are arranged side by side in the annexed table.

Substance.	Period, 30,000,000 per second.	Period, 20 per second.
Castor oil …………	4·49	4·65
Olive oil …………	3·02	3·13
Cotton-seed oil ……	3·00	3·09
Petroleum ………	1·99	2·05
Ebonite ……………	2·32	2·55
Quartz ⊥ to axis …	4·34	4·46
Quartz ‖ to axis …	4·27	4·38

The specimens of quartz used were cut from positive crystals and the values for the specific inductive capacity in the two directions were found to differ in the same sense as the index

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* Wied. *Ann.* xvii. p. 816 (1882).

of refraction, in accordance with the electromagnetic theory. The value of the dielectric constant in the two directions, however, is nearly twice as large as the square of the corresponding index of refraction.

The values in the preceding table show that in the case of every substance examined, there is a difference of the dielectric constant for rapidly changing and slowly changing fields. It is to be expected that the amount of this difference should be proportional to the electrical absorption of the substance. If electrical absorption is due to a heterogeneous structure of the medium, as developed in the theories of Clausius and Maxwell, the minimum amount of electrical absorption would be exhibited by crystals; and consequently, the difference in the value of the dielectric constant of a crystal for rapidly changing and slowly changing fields would be less than that of an amorphous substance. Rowland and Nichols * have measured the electrical absorption of several substances, and find crystals to exhibit very much less absorption than amorphous substances. Thus is explained the fact that the dielectric constant of quartz suffers less change than the other substances examined when subjected to fields of various periodicities. For the same change of periodicity of charge, ebonite shows the greatest difference in specific inductive capacity of any of the substances examined. The insulating oils give differences nearly equal to each other.

It is interesting to note in this connexion the behaviour of some plates of quartz that in polarized light showed in one part of the section beautiful Airy's spirals. With these specimens it was difficult to obtain the specific inductive capacity, because the change in position of the specimen with respect to the plates of the oil condenser gave slightly different values of the constant being determined. But this much is certain, that the difference in the specific inductive capacity in rapidly changing and slowly changing fields is greater in the part of the crystal that shows Airy's spirals in polarized light, than is this difference in crystals that do not exhibit Airy's spirals. This observation is in accord with Maxwell's theory that electric absorption is produced by a laminated structure of the dielectric.

Physical Laboratory,
University of Wisconsin.

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* Phil. Mag. [5] xi. p. 414 (1881).

LIII. *The Measurement of Small Gaseous Pressures.*

By CHARLES F. BRUSH*.

PRIOR to the invention of the McLeod vacuum gauge, the measurement of even moderately small gaseous pressures was difficult, and subject to large errors. The introduction of the McLeod gauge, however, early in the seventies, seemed to solve the problem. In its ordinary form, and for most purposes, this beautiful instrument admirably serves the purpose for which it is designed. But when *very* accurate measurements of pressures as small as a few millionths only, of atmospheric pressure are desired, its performance is extremely unsatisfactory and vexatious. As is well known, the chief cause of the difficulty is the unequal and variable capillary depression of the two small columns of mercury, whose difference in height indirectly serves as the measure of pressure. Accurate measurement of this capricious difference obviously avails nothing.

Three or four years ago, I was engaged in an investigation requiring frequent and simultaneous measurements of slight, but different pressures, in two large glass globes connected by a capillary tube. For this purpose I constructed and carefully callibrated two large McLeod gauges. The internal diameter of the mercury tubes was about 3 millimetres, and they were made from contiguous parts of the same glass tube selected for uniformity of bore. These gauges were often compared by measuring the same vacuum with both; but they rarely gave concordant results. Indeed it was not uncommon at high exhaustions, for one or the other of them to indicate a negative vacuum; that is to say, less than no pressure at all. The case of these two gauges is cited because of the opportunity they afforded for comparison. In prior work I had, like most experimentalists, used but one gauge; and while always suspicious of its indications, had no means of knowing how large its errors might be.

The phenomenon which I next desired to investigate, was the spontaneous evolution of gas from glass, and other surfaces in high vacua. For this purpose an accurate and entirely reliable means for measuring very small pressures was necessary, because I could not afford to wait months or years for the evolution of sufficient gas to be detected with certainty by the old gauges. To meet these requirements I designed, constructed, and learned how to use the modified form of

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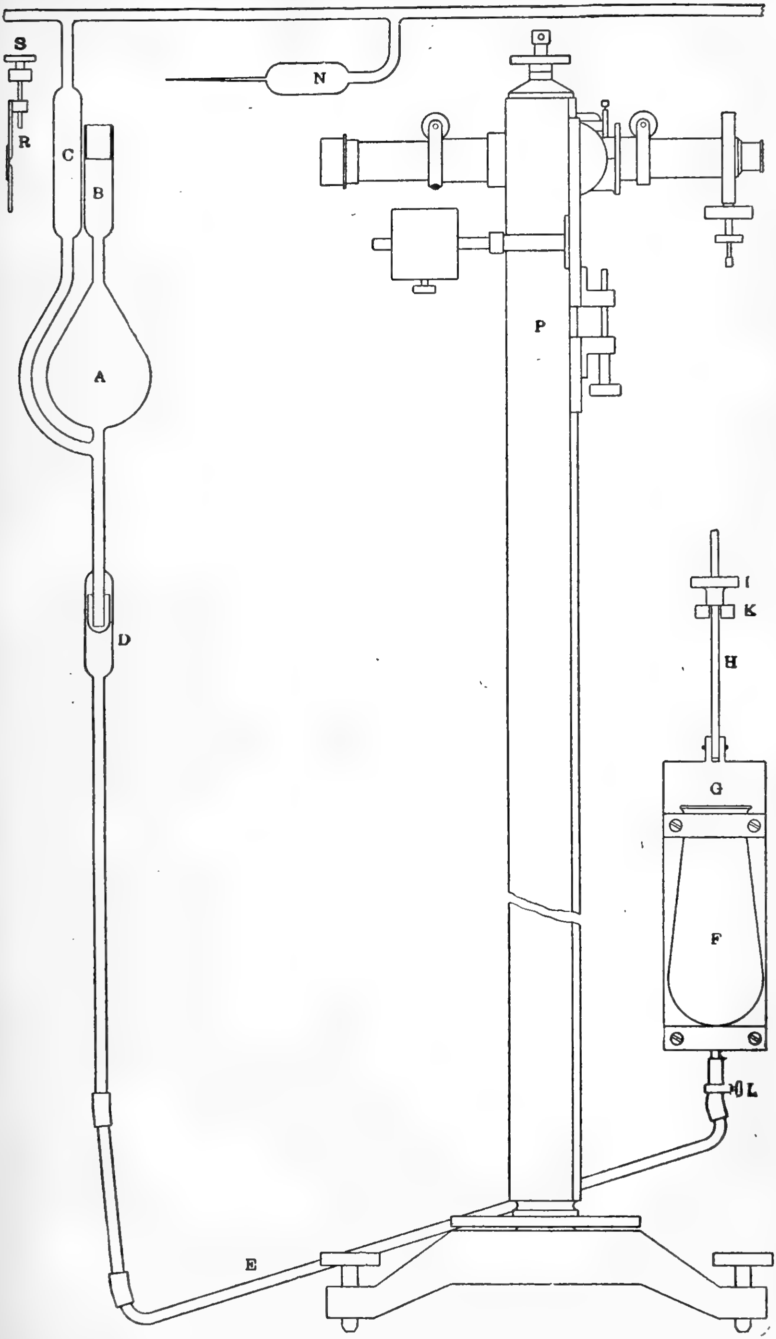
* Communicated by the Author, having been read before the American Association for the Advancement of Science, August 12, 1897.

McLeod gauge which it is the purpose of this paper to discuss. It was deemed necessary to provide an apparatus capable of easy use, and giving results having a probable error not greater than a fiftieth part of a millionth of atmospheric pressure. My hopes of attaining this high ideal were far more than realized, as the sequel will show. Instead of having to wait days or weeks, in the course of my investigations, for the evolution of a measurable quantity of gas as had been expected, the progress of evolution could be noted from hour to hour; and the increase in the *rate* of evolution due to a rise of one degree in the temperature of the laboratory was unmistakable.

The diagram herewith shows the essential parts of my apparatus. The bulb A, of the gauge, is made conical in its upper part to avoid adhesion of gas bubbles when the mercury rises. This bulb holds about eleven pounds of mercury. B and C are the gauge-head and comparison tube respectively. They are nearly 20 millimetres inside diameter, and are made from contiguous parts of the same carefully selected tube. D is the usual air-trap, and E is a long glass tube, with flexible pure rubber connexions to the lower end of the gauge stem and the mercury cistern F. The latter is mounted on a carriage G, which moves vertically on fixed guides. The height of the carriage is adjustable, at the upper end of its range of motion, by means of the screw H, thumb-nut I, and forked support K. The screw is pivoted to the carriage, so that it may swing out of the fork when the carriage is lowered. L is a pinch-cock with screw, for regulating the flow of mercury, or stopping it altogether, while pumping out the trap D. N is a bulb containing phosphorus pentoxide, to keep the interior of the gauge and other parts of the apparatus perfectly dry. P is a very elaborate cathetometer for observing the mercury columns in B and C. This beautiful instrument has a revolving column with vertical scale, and vernier with microscope, reading to hundredths of a millimetre. The eyepiece micrometer reads directly to hundredths of a millimetre, and the divisions on the revolving head of the screw are so open, that tenths of divisions are easily and certainly estimated by an experienced eye; thus permitting the micrometer to be read directly to thousandths of a millimetre. Of course the cathetometer is perfectly located, not as shown, but with the objective of its telescope equally distant from the axes of the tubes B and C when it is alternately directed to them; and at such a distance that its micrometer readings correspond to a millimetre scale. The whole apparatus is located in a basement room, on a stone floor, whereby vibrations are reduced to a minimum.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The most important part of the gauge is the head B. The purpose of its great diameter is the reduction of capillary depression in its mercury column. But its size necessitates a very close approach of the mercury to its upper end, in order to sufficiently reduce its capacity. Yet the remaining space must be measurable by the cathetometer with the utmost precision. Hence the glass must not be distorted by heating, and the closed end just over the mercury must be sharply defined. In constructing this part of the apparatus, I selected a piece of heavy tubing which would just slip inside of B, with the least possible clearance. One end of this tube was closed as squarely as possible by fusion, and then ground with fine emery and a suitable tool to a convex spherical surface of long radius. Care was taken to make the centre of curvature lie in the axis of the tube, and the ground surface was left unpolished to facilitate observation. A suitable length of the closed end of the tube was then cut off, slipped into B, and both tubes were fused together at their open ends, as shown.

For callibrating the head B, a ground-glass stopper with a capillary duct was fitted to its neck, before the latter was sealed to the bulb A. The head was then filled with mercury by boiling, thus completely filling the small space between its wall and the cap. After cooling, the stopper was inserted to expel all excess of mercury, and the whole weighed. Next the head was emptied, and the mercury in the annular space distilled out. Again the head was very nearly filled with mercury, without allowing any to get into the annular space, and weighed as before; and the space between the top of the mercury and the convex end of the head was very carefully measured by the cathetometer. This process of weighing and measuring was repeated several times, with less mercury each time. Thus the capacity of a vertical millimetre of the head was ascertained, as well as the capacity that would remain, if the top of the meniscus of mercury just touched the convex end of the gauge above it. Finally the neck was sealed to the bulb A, and the capacity of head, neck, and bulb combined was found by weighing them empty, and again filled with mercury.

For lighting the top of each mercury column, a narrow horizontal slit in an opaque screen R is used. The slit is covered with a strip of ground glass, and obliquely illuminated by an electric lamp. The screen and slit are vertically adjusted by a thumb-screw S. The heat of the lamp is prevented from reaching the mercury columns, and head B, by a thick screen. This is very necessary.

In order to get the best results from the apparatus, many precautions are necessary. After filling A and B with mercury, time must be allowed for the compressed gas to cool. The effect of changing barometric pressure is nearly eliminated by so regulating the quantity of mercury in F that its surface is in the small tube at the bottom of the cistern, when the gauge is properly filled. Its area is then very small, as compared with that in B and C. The height of the meniscus in both tubes is easily adjusted sensibly equal by a little manipulation. I *always* raise the mercury above the point at which readings are to be taken, and then lower it, so as to read on a falling meniscus. This is highly important.

Some trouble was occasionally experienced at first from electrostatic induction between the mercury in B and the glass above it. This was shown by distortion of the meniscus when it was brought very near the glass. The difficulty was partially, but not wholly remedied by putting mercury in the outside open end of the gauge-head, and connecting it by a flexible conductor with the mercury in the cistern F. A complete remedy was effected by moistening the inside of the gauge-head with a dilute solution of phosphorus pentoxide. This became completely dried by the anhydrous phosphorus pentoxide in N, but was of course not dehydrated; and hence always remains conducting, and dissipates the static charge.

Large pressures, up to a thousand millionths or more, are readily measured with this apparatus by finding with the cathetometer the distance between the mercury in B and the end of the head above it; from this is quickly calculated the necessary multiplier for the number of millimetres difference in height between the columns in B and C, also measured by the cathetometer, in order to express the result in millionths. For very small pressures, the micrometer wires are set at such a distance apart as to give a convenient constant (usually 2); and the column in B is adjusted this distance away from the glass; careful allowance being made for the thickness of the wires. Then the micrometer is used for repeated measurements of the difference in height of the mercury in B and C. The disturbing effect of *bias* is entirely eliminated by giving the micrometer screw a partial turn after each reading. Thus the next measurement is made without any knowledge of its difference from the preceding one, until the eye is removed from the telescope.

In my early experience with the apparatus, unusually careful measurements of very small pressures were often made to determine how far its indications might be relied upon. In this connexion I quote as follows from my notes, under date of Feb. 20, 1895, concerning the last one of a series of pressure determinations:—"Following is the last reading in detail, showing the extreme accuracy of these measurements:

	·432	M.	·438	M.	·441	M.
	·441	„	·4335	„	·429	„
	·4335	„	·4275	„	·4305	„
	·426	„	·450	„	·435	„
	·4335	„	·4425	„	·432	„
	·4395	„	·432	„	·4185	„
	·4305	„	·435	„	·435	„
	·441	„	·432	„	·453	„
	·435	„	·4215	„	·4425	„
	·435	„	·4245	„	·438	„
Means...	·4347	„	·43365	„	·43545	„

"Mean of all the readings, ·4346 M.

"Readjusted zero-point of micrometer before each reading of each set. Partially emptied gauge and readjusted capillary depression before each set of readings. The first series has no known source of error. The second and third series were made during wind-squall, and surface of mercury was often tremulous. In the third series, capillary depression was perceptibly, though very slightly, unequal, in direction to make readings too high."

In the above quotation, "M" means millionths of atmospheric pressure. The calculated probable error of the thirty readings taken together is only ninety-two hundredths of a unit in the third decimal place; that is to say, less than a thousandth part of a millionth of atmospheric pressure! The probable error of the three mean results, considered as single readings, is only eleven hundredths of a unit in the third decimal place of millionths. The net result may be expressed as follows, in terms of atmospheric pressure:—Considered as thirty measurements:

$$0·000\,000\,434\,60 \pm 0·000\,000\,000\,92.$$

Considered as three measurements:

$$0·000\,000\,434\,60 \pm 0·000\,000\,000\,11.$$

Here we have the measurement of a total quantity of less than half a millionth of atmospheric pressure, with a probable error of only about a fifth of one per cent. of the quantity measured.

To show how small is the effect of variable capillary depression in the large mercury columns, the following measurements were made July 25, 1897. No correction was made of accidental capillary differences, but the columns were always observed with a falling meniscus. The zero of the micrometer was freshly adjusted for each reading; and before each of the six sets of readings the mercury was lowered, and then readjusted to the proper height in the gauge-head.

	M.	M.	M.	M.	M.	M.
	2·210	2·203	2·209	2·198	2·198	2·202
	·204	·195	·202	·203	·204	·198
	·209	·198	·204	·208	·200	·196
	·203	·204	·210	·200	·196	·208
	·203	·192	·202	·198	·196	·203
Means	2·2058	2·1984	2·2054	2·2014	2·1988	2·2014

Calculating the probable errors, we have:

Six mean readings...... 2·20187 M. ±0·00073 M.
All readings .... 2·20187 „ ±0·00059 „

The effect of not equalizing the capillary depression is very apparent when these results are compared with the earlier ones quoted. But on account of increased skillfulness of observation, due to long experience, the individual readings of each set are more uniform than before; so that the net result is better.

In this example we have the measurement of about two millionths of atmospheric pressure, with a probable error of only one part in three thousand of the quantity measured.

From the foregoing we may safely conclude that with the apparatus described, small gaseous pressures may be easily measured with a probable error of less than a thousandth part of a millionth of atmospheric pressure.

The smallness of this fraction is difficult to realize. It is comparable with a thousandth part of a milligram, in a thousand grams; or a single kernel of wheat in two thousand bushels; or an inch and a half in the circumference of the earth; or the thickness of a sheet of tissue paper in 16 miles.

LIV. *The Velocity and Rate of Recombination of the Ions of Gases exposed to Röntgen Radiation.* *By* E. RUTHERFORD, *M.A., B.Sc.,* 1851 *Exhibition Science Scholar, New Zealand University, Trinity College, Cambridge**.

AIR which has been exposed to Röntgen radiation preserves the power of discharging positive and negative electrification a short time after the rays have ceased. It has been shown (J. J. Thomson and McClelland, Proc. Roy. Soc. lix. 1896) that a plate can be discharged some distance from the direct line of radiation from the Crookes' tube by blowing the Röntgenized air towards the plate. In this way it is possible to discharge electrification after the air which has been exposed to the rays has passed through a tube several yards in length (J. J. Thomson and E. Rutherford, Phil. Mag. Nov. 1896).

In the following paper the duration of the *after*-conductivity of air and other gases has been investigated, and from the data thus obtained the velocity of the ions through various gases has been determined.

Two distinct methods of determining the duration of the conductivity were used, both of which gave consistent results, viz.:—

(1) By blowing air at a known velocity along a tube, and testing the conductivity at different distances from the point of action of the rays.

(2) By applying an electromotive force to the gas at definite intervals after the rays have ceased, and measuring the quantity of electricity that passed through the gas.

In the first method an aluminium cylinder was connected in series with a long brass tube 3 cm. in diameter and a metre long. The aluminium cylinder was placed over an opening in the metal tank in which the bulb and coil were placed in order to completely screen off all external electrostatic disturbances. All radiation to external points was carefully screened off by a thick lead covering over the aluminium cylinder. The aluminium tube was connected in series with a gasometer filled with air, the pressure of which could be regulated by weights placed on top. The air before reaching the aluminium tube passed through a tube filled with glass wool to remove the dust from the air. Three equal and similar insulated electrodes were placed at known distances

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* Communicated by the Author.

apart along the brass tube. The brass tube was connected to earth, and one of the electrodes was connected to the electrometer. Both quadrants were connected together, and the whole was charged up to a high potential (in practice about 100 volts). The quadrants were then insulated from each other. When the rays were turned on, there was no effect on the electrometer until a current of air from the gasometer passed along the tube, when the movement of the electrometer needle showed that the air passing the electrodes was conducting. The rate of leak for each of the electrodes was tested in turn, and since the capacity in the electrometer circuit was nearly equal in the three cases, the rates of leak were proportional to the conductivity of the air at the electrodes.

By noting the volume of air which passed from the gasometer in a given time, and knowing the diameter of the metal tube, the mean velocity of the current of air could be readily calculated, and therefore also the intervals of time taken by the current of air to pass from one electrode to the other. By altering the weights on the gasometer the velocity of the blast of air could be varied at will.

The following table shows the way in which the conductivity of the air varied with the length of time after exposure to the rays. The first column gives the times taken by the current of air to pass from one electrode to the other, and the second column gives the ratio of the rates of leak of the electrodes.

Time in seconds.	Ratio of the rates of leak.
·13	·75
·22	·61
·28	·57
·65	·39
2·4	·11

It will be seen that the conductivity falls off rapidly with the time, and after 2·4 seconds is only $\frac{1}{9}$ of the original value.

It is probable that some of the conducting particles give up their charge to the sides on their passage down the tube, but the correction for this is probably very small as there is no force acting on the charged gas tending to repel itself to the sides as is the case when free electrification is present.

In a previous paper (J. J. Thomson and E. Rutherford, Phil. Mag. Nov. 1896) it has been shown that when a gas is acted on by the Röntgen rays a steady state is reached when the rate of production of the ions by the rays is equal to their rate of recombination. If q be the number of conducting particles per c.c. produced per second by the rays, and N the final number, then when a steady state is reached

$$q = \alpha N^2,$$

where α is a constant for any particular gas but varies for different gases. When the rays are stopped the rate of diminution of the number of conducting particles is given by

$$\frac{dn}{dt} = -\alpha n^2,$$

or if N is the maximum number and n the number after an interval t,

$$\frac{1}{n} - \frac{1}{N} = \alpha t.$$

The time T for the number of conducting particles to fall to half their total number is given by

$$\frac{1}{N} = \alpha T.$$

Now the rates of leak at the electrodes in the tube are proportional to n, since an electromotive force is applied to the gas sufficient to completely saturate the gas. From the experimental data of the rates of leak at different intervals we may compare the experimental results with those obtained from the formula

$$\frac{1}{n} - \frac{1}{N} = \alpha t.$$

This can be best shown by plotting curves whose ordinates represent conductivities and abscissæ time intervals.

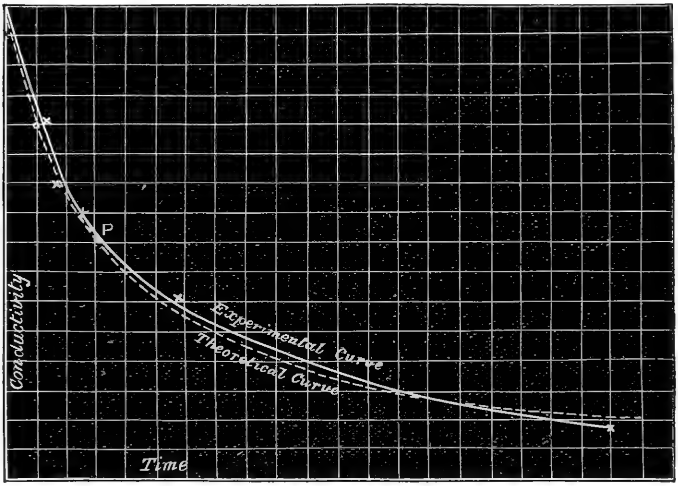
Fig. 1 shows such curves where the continuous curve represents the relation between the conductivity and time as determined by experiment, and the dotted curve the theoretical relation which is deduced. In order to compare the curves, one point on the experimental curve is chosen also as a point on the theoretical curve. In fig. 1, the common point is P, where the conductivity has fallen to half its value.

It will be observed that the two curves are in close agreement, and the differences between them are well within the limits of experimental error. This shows that the formula

$$\frac{dn}{dt} = -\alpha n^2$$

represents very closely the law of the rate of recombination of the ions. This agreement has been tested for a large number of experiments in which the intensity of the radiation varied widely, but in all cases the rate of decay was found to be in close agreement with theory.

Fig. 1.



Second Method.

The method just described of determining the duration of the conductivity by blowing air along a tube, could only be used for air on account of the large volume of gas required for a series of observations.

The following method could be used for testing the rate of recombination of the ions for different gases, and for widely different values of intensity of radiation.

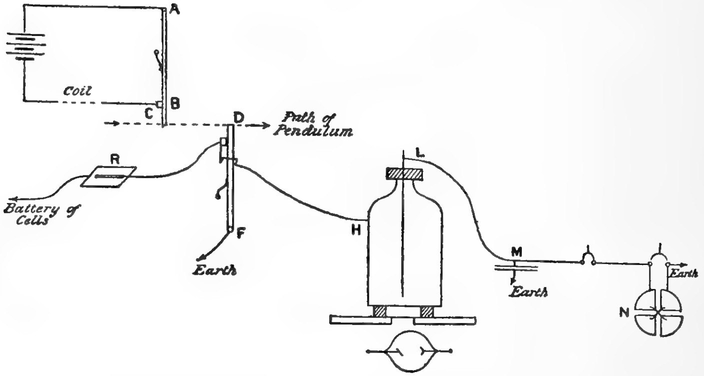
A glass bell-jar was taken, the bottom of which was covered with a plate of thin ebonite. A central electrode which reached nearly to the bottom of the bell-jar was insulated by passing through a paraffin stopper in the mouth of the jar. The outside of the bell-jar was coated with tinfoil which had a metallic connexion with the inside. The bell-jar was placed on insulating blocks over a hole in the metal tank, which was covered with aluminium, and the Crookes' tube was placed in position beneath.

In order to determine the after-conductivity the coil was turned on for a few seconds and then turned off. At definite intervals after the cessation of the radiation a large E.M.F. was applied to the outside coating of the bell-jar and the quantity of electricity which was given up to the central electrode was determined.

A pendulum interrupter was used to break the battery circuit and to apply the E.M.F. at varying intervals. A heavy iron ball was suspended by a wire 315 cm. long, and the contacts were broken by a rod fixed to the bottom of the ball.

Fig. 2 represents the arrangement of the experiment. The induction-coil primary circuit was completed through the brass lever AB, which pressed against a copper support C.

Fig. 2.



When the pendulum struck the lever AB was knocked away from C and the current broken. In order to apply a high E.M.F. at a definite instant, recourse was had to a shunt method which worked very well in practice. A battery of one hundred small accumulators had one pole connected to earth and the other pole through a high carbon resistance R to the copper support E. A wire passed from E to the tinfoil on the outside of the conducting vessel H. The lever DF, which was kept pressed against E by a spring, was connected to a good earth. When the lever was in position, therefore, the vessel H was very nearly at zero potential, for the resistance of the lever DF and the earth connexions was very small compared with the carbon resistance R. When the pendulum struck the lever DF the earth connexion was broken and the vessel H was immediately charged up to the potential of the battery. By altering the distance between the two levers the time between breaking the battery current and applying the E.M.F. could be varied within limits.

Immediately after the passage of the pendulum the lever DF was replaced, thus reducing H to zero potential again. In order to prevent a sudden deflexion of the electrometer when the E.M.F. was applied to and removed from H, the electrometer was not directly connected to L until the lever had been replaced. The capacity M was introduced in order to prevent the potential of the circuit LM rising to any considerable extent when the E.M.F. was applied to H.

It was found that applying and removing the E.M.F. alone produced no effect on the electrometer; but when the rays had acted on the gas inside the vessel L, there was always a deflexion in the electrometer, showing that the electrode L had received a charge of the same sign as the pole of the battery.

Since the E.M.F. applied was usually over 200 volts, a value sufficient to completely saturate the gas, the quantity of electricity that passed through the gas was proportional to n, the number of ions present in the gas at the instant the E.M.F. was applied, while an immediate application of the E.M.F. after the rays had ceased gave the total number N.

The rate of recombination of the ions could thus be completely determined, as the times taken by the pendulum to pass from one lever to the other could be very approximately calculated.

The following table shows the way in which the deflexion of the electrometer varied with the time after the rays had ceased, when the bell-jar was filled with dust-free air.

Time in seconds.	Deflexion of electrometer.
·004	184
·08	183
·45	106
2	37
4	19

The value of the deflexion is practically constant for nearly $\frac{1}{10}$ second after the rays have ceased. After an interval of 4 seconds the air still possesses appreciable conductivity.

Since the gases have widely different conductivities under the X rays, it is to be expected that the rate of recombination is different for the various gases.

The following table gives the times T for the number of ions to fall to half their original number. The intensity of the radiation was sensibly constant for all the gases.

Gas.	T in seconds.	Conductivity compared with air=1.
Hydrogen	·65	·5
Air	·3	1
Hydrochloric Acid Gas	·35	11
Carbon Dioxide	·51	1·2
Sulphur Dioxide	·45	4
Chlorine	·18	18

There seems to be no close connexion between the values of T and the conductivities, although as a general rule it may be taken that the value of T diminishes with increase of conductivity.

It was found, however, that the rate of recombination was not always the same for the same gas with the same intensity of radiation, but depended largely on the amount of dust suspended in the gas as will be shown later in the paper.

Effect of Intensity of Radiation on the value of T.

The value of T for the same gas was found to depend largely on the intensity of the radiation. This is to be expected from theoretical considerations, for

$$q=\alpha N^2,$$

and

$$\frac{1}{N}=\alpha T,$$

therefore

$$T=\frac{1}{\sqrt{q\alpha}}.$$

If then α is a constant for the same gas in the same state, T varies inversely as $\sqrt{q}$. This relation was found to hold experimentally, for q is proportional to the intensity of the radiation, which varies inversely as the square of the distance from the bulb.

For example, for a definite intensity of radiation the value of T was ·25 second. On placing a thick aluminium sheet below the conducting vessel, which cut down the intensity of the radiation to $\frac{1}{6}$ of its former value, the value of T rose to ·6 second.

For weak radiation the values of T are much greater than for strong radiation.

Time in seconds.	Deflexions.
·004	174
·45	139
2	107
4	54
8	30
16	16

The above table shows the variation of the after-conductivity with time for very weak radiation. The value of T is about 3 seconds, and even after 16 seconds $\frac{1}{11}$ of the original number of ions are still uncombined.

Air distant about a metre from an ordinary Crookes' tube possesses quite a measurable proportion of its conductivity for over a minute after the rays have ceased.

Effect of finely suspended Particles in a Gas on its Rate of Loss of Conductivity.

It was found that the value of T varied greatly for the same gas for the same intensity of radiation. When, for example, chlorine was first passed into the testing-vessel the value of T was ·19 sec.; after standing for an hour the value of T rose to ·3 sec., although the conductivity of the gas as tested by the usual method was found to be unaltered. Freshly made gases were found, in all cases, to lose their conductivity more rapidly than when they had stood undisturbed for some time.

The cause of this effect was not at first clear, but later experiments on the influence of dust in the air led to the conclusion that it was due to the presence of finely divided matter, liquid or solid, in the freshly prepared gas. The value of T was found to be greater for a gas that passed through a long tube filled with cotton-wool than if the cotton-wool were removed. This is probably due to the fact that the cotton-wool would not allow the small particles to pass through its pores.

The presence of dust in air was found to very greatly affect the duration of the after-conductivity. As an example of the effect of dust we may give the following experiment:—

The cylinder was filled with air which had passed through a plug of glass-wool, and then allowed to stand all night, and the value of the after-conductivity was taken next day without disturbing the gas.

The quantity of electricity that passed through the gas after the rays had ceased gave a deflexion in the electrometer of 70 divisions, and the value of T was 1 second. A blast of dusty air from a bellows was then sent into the cylinder and the deflexion due to the after-effect fell immediately to 15 divisions, with a value of T of about ·15 sec. When the air was allowed to stand the after-effect gradually increased again to 35 divisions, with a value of T of ·5 sec. after an interval of about 10 minutes. Several hours elapsed before the after-effect rose to 60 divisions. This experiment shows what a variable quantity T is for the same gas, depending as it does on the amount of suspended matter in the gas.

The effects observed in air and other gases seem to point to the conclusion that freely suspended particles greatly assist a gas to lose its conducting property after the rays have ceased.

Since the dust-particles are very large compared with the ions, an ion is more likely to strike against a dust-particle, and give up its charge to it or to adhere to the surface, than to collide with an ion of opposite sign. A positive ion striking a dust-particle gives it a positive charge, and this is neutralized by a charge from a negative ion, and in this way the rate of loss of conductivity is much more rapid than if the loss of conductivity were due to collisions between the ions themselves. It seems probable that if a gas could be obtained completely dust-free the rate of recombination which would be due entirely to molecular collisions would be very much slower than for ordinary air.

When the rays act upon a gas the number of ions per c.c. increases until a definite stage is reached, when the rate of production is equal to the rate of recombination. It is of interest to find the time that elapses after the radiation has commenced before this maximum is reached. In most of the experiments there were generally 50 breaks per second in the induction-coil, so that for the sake of calculation we may very approximately suppose that the bulb was giving out rays uniformly, corresponding to the production of q ions per c.c. per second. The rate of increase of n is given by

$$\frac{dn}{dt}=q-an^2.$$

Solving this equation it is easily seen that the time t required for the production of n ions per c.c. in the gas is given by

$$t=\frac{1}{2\sqrt{qa}}\log_e\left(\frac{\sqrt{\frac{q}{a}}+n}{\sqrt{\frac{q}{a}}-n}\right).$$

When the maximum number N is reached

$$\frac{dn}{dt}=0, \qquad \text{and} \quad q=a\text{N}^2.$$

Therefore

$$t=\frac{1}{2a\text{N}}\log_e\frac{1+r}{1-r},$$

$$\text{if} \quad r=\frac{n}{\text{N}}.$$

Now if T be the time taken for the number of ions to fall to half their number when the rays have ceased

$$1=a\text{NT},$$

therefore

$$t=\frac{\text{T}}{2}\log_e\frac{1+r}{1-r},$$

or

$$r=\frac{e^{\frac{2t}{\text{T}}}-1}{e^{\frac{2t}{\text{T}}}+1}.$$

The following table gives the values of r deduced for different values of t from the above equation:—

$\frac{t}{\text{T}}$.	r.
·125	·123
·25	·245
·5	·462
1	·736
2	·878
4	·998

For example, in the case of air close to the Crookes' tube, T=·3 sec. Therefore, in ·15 sec. the number of ions is nearly one-half of their final value. After an interval of one second the number of ions has practically reached the maximum value. For air distant about a metre from the source of radiation the value of T is much larger, and several seconds will elapse before the number approximates to the final value.

Velocity of the Ions.

It is a question of considerable interest to determine the velocity with which the ions travel through a gas under the influence of an electromotive force, as it indirectly gives us some information in regard to the nature and size of the carrier of electricity in conduction under the Röntgen rays.

The method of determining the velocities was based on an investigation given in a previous paper (J. J. Thomson and E. Rutherford, Phil. Mag. Nov. 1896), where it is shown that

$$\frac{\iota}{\sqrt{\mathrm{I}(\mathrm{I}-\iota)}}=\frac{\mathrm{EUT}}{l^2},$$

where ι is the current between two parallel plates, l cms. apart, when there is a potential-difference E acting between the plates.

I = the maximum current through the gas, when a saturating E.M.F. is applied.

U = sum of the velocities of the positive and negative ions for a potential gradient of one volt per cm.

T = time taken for the number of ions to fall to half its original value after the rays have ceased.

If E is chosen so that $\frac{\iota}{\mathrm{I}}$ is small this equation very approximately reduces to

$$\frac{\iota}{\mathrm{I}}=\frac{\mathrm{EUT}}{l^2}.$$

Now all the quantities in this equation can be measured, so that the velocity is readily calculated. In practice the value of E was taken of such an order that the value of the current ι corresponding to it was about $\frac{1}{10}$ of the maximum current.

The ratio $\frac{\iota}{\mathrm{I}}$ is not a constant for the same gas for the same potential gradient, as it depends on the intensity of the radiation, and also on the rate of recombination of the ions, which is in turn largely dependent, for the same intensity of radiation, on the amount of dust and other solid matter in the gas.

Although these quantities are variable, the velocity U is a constant, for its value is quite independent of the value of T or of the intensity of the radiation.

The steepness of the curves showing the relation between the current and the E.M.F. for air and other gases, which are given in a previous paper (*loc. cit.*), is very variable, depending as it does on whether the gas is freshly made or has stood undisturbed for some time.

For a determination of the velocity of the ions the value of T requires to be known with accuracy; but it is a difficult matter to determine T accurately with a pendulum interrupter, and moreover the intensity of the radiation from the Crookes' tube, on which the value of T largely depends, is very liable to change over a long range of experiments. Recourse was therefore had to a simpler method of determining T, which was found to give very consistent and reliable results. It has been shown that the final number of ions N is given by

$$q = a\mathrm{N}^2,$$

and also

$$\frac{1}{\mathrm{N}} = a\mathrm{T},$$

therefore

$$\mathrm{T} = \frac{\mathrm{N}}{q}.$$

Now if a very large E.M.F. is applied to the testing-vessel the instant after the rays have ceased, the deflexion of the electrometer is proportional to N, and q is proportional to the rate of leak per second for a saturating E.M.F. The ratio of these two quantities is thus readily determined and the value of T known.

Arrangement of the Experiment.

The testing-vessel consisted of two parallel plates, the lower being of aluminium and the upper of sheet lead. These plates were separated by blocks of paraffin, which were melted together and formed the sides of the testing-vessel, and at the same time insulated the top from the bottom plate. In order to measure the after-conductivity with accuracy fairly large plates were required. In the apparatus used the plates were 22 cm. square and 4·7 cm. apart. The rays passed through the lower aluminium plate and made the gas inside the vessel a conductor, but were completely stopped by the top lead plate. Care was taken that the radiation fell only on the central portion of the plate where the electrostatic field was sensibly uniform.

The arrangement of the apparatus was the same as that given in fig. 2, with the exception that the bell-jar was replaced by the vessel with parallel plates. One pole of a battery of small accumulators was connected to the lower plate, and the electrometer to the upper.

To determine N an E.M.F. of 200 volts was applied to the lower plate the instant after the rays had ceased. The two shunt-levers were close together, and the method of breaking the current and applying the E.M.F. has been explained in the earlier part of the paper (p. 426). The value of q was determined by noting the leak per second for a saturating E.M.F. of 200 volts. The value of ι was obtained by applying an electromotive force of two or more volts to the lower plate, and determining the rate of leak.

The following table shows the values of T, $\frac{\iota}{I}$, U obtained for the various gases. T is expressed in seconds, U in centimetres per sec. The values of $\frac{\iota}{I}$ are given for a potential gradient of $\frac{1}{10}$ of a volt per cm. between the plates. The values of $\frac{\iota}{I}$ are only approximate, and are deduced from the observed values on the assumption that Ohm's law holds for electromotive forces small compared with the saturating values. The ratios of $\frac{\iota}{I}$ were determined for different electromotive forces in the different gases. For example, the value of $\frac{\iota}{I}$ for hydrogen was ·32 for 1·4 volts acting between the plates, while the value of $\frac{\iota}{I}$ for sulphur dioxide was ·066 for 9·3 volts acting between the plates.

Table of Velocities.

Gas.	T.	$\frac{\iota}{I}$.	Velocity, U.
			cms.
Hydrogen	·4	·108	10·4
Oxygen	·4	·021	2·8
Nitrogen	·31	·019	3·2
Air	·29	·019	3·2
Carbonic Acid Gas	·34	·015	2·15
Sulphur Dioxide	·17	·0033	·99
Chlorine.....................	·21	·0085	2
Hydrochloric Acid Gas...	·18	·01	2·55

It will be seen from the above table that the velocities of the ions follow the inverse order of the densities with the exception of chlorine gas. The velocity of the hydrogen ion through hydrogen is nearly four times as fast as the velocity of the oxygen ion in oxygen. The ions of sulphur dioxide gave the slowest velocity, being only about $\frac{1}{10}$ of that of hydrogen.

The velocity of the ions through a gas was found to be independent of the amount of ionization of the gas. The velocities deduced from the two different sets of experiments, when the intensity of radiation in one case was six times that of the other, were found to be the same. In the one case, therefore, six times as many ions per c.c. were present as in the other, but the velocity remained unaltered. There is thus no correction to be applied for the velocity of the ions in a gas like chlorine, whose ionization is large compared with that of air. The fact that the velocity under a given small electromotive force is independent of the number of ions per c.c. in the gas, shows that the movement of the positive and negative ions does not produce any resultant electrostatic field between the plates.

In the method which has been used for determining the velocity, it has been assumed that the conductivity of the gas is purely due to volume ionization of the gas. Perrin (*Comptes Rendus*, March 1st, 1897) has, however, recently shown that the rate of leak between two plates is made up of two parts, one due to the volume ionization of the gas, and the other due to surface ionization at the surface of separation of the metal plate, on which the radiation impinges, and the surrounding gas. The rate of leak due to the surface action is quite comparable with that due to the volume ionization, when the plates are 1 cm. apart, especially in the case when the electrodes are of silver, gold, or zinc.

It seems probable that the gas close to the surface of the plate on which the radiation falls has a much greater density of ionization than the gas between the plates, and since this increase is confined to a very thin layer close to the electrode, that the rate of recombination for the surface-ions is far more rapid than for the Röntgenized air some distance from the surface. The existence of this effect would tend to increase the rate of leak q; while the number of ions, N, as determined by the application of an E.M.F. a short interval after the rays had ceased, would not be appreciably affected, since the ions near the surface probably recombine with great rapidity. The value of T which is obtained from the equation $T=\frac{N}{q}$ would thus be too small. The correction for the surface ionization is probably, however, very small, for the lower plate of the testing-vessel was of aluminium—a metal which does not appreciably exhibit the phenomena of surface ionization—and the upper of lead, in which the effect is slight. In addition to this the plates were nearly 5 cms. apart, so that the volume effect was very large compared with that due to the surface.

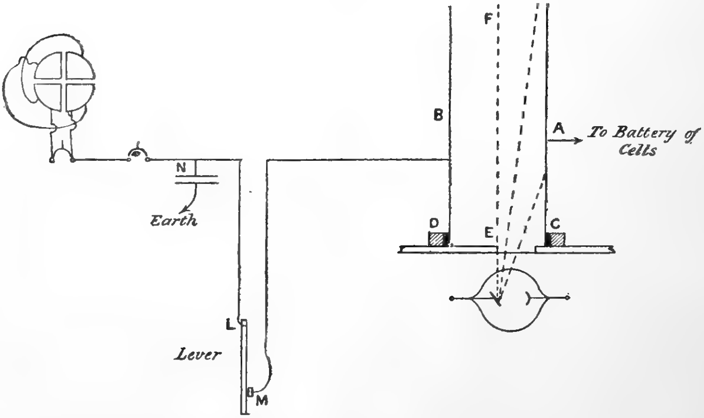
The velocities which have been determined are the sum of the velocities of the positive and negative ions, but we have so far given no direct experimental evidence to show whether the velocities of the positive and negative ions are the same.

In the case of air, an experiment which will now be described seems to show that the velocities of the two ions are equal or very approximately so.

In previous determinations the calculation of the velocity of the ions has depended on the truth of an equation which has been experimentally verified as far as possible, but in the case of air the velocity may be obtained by a method not involving any theory depending on the rate of recombination of the ions.

Two large plane plates, A and B (fig. 3), were placed parallel to one another, 16 cm. apart, on insulating blocks C and D. The bulb was so arranged, in regard to the plates A and B, that the radiation fell on the plate A and half of the volume of air between A and B. No radiation reached the air to the left of the dotted line EF in the figure, which was 8 cm. from either plate. The plate A was connected to one terminal of a large battery of storage-cells, the other

Fig. 3.



pole being connected to earth. The plate B was connected through a contact-lever, LM, mounted on an insulating block, to one pair of quadrants of the electrometer, the other being connected to earth. The pendulum interrupter was so arranged as to *make* the current in the primary of the induction-coil, to *break* the electrometer-circuit by knocking away the lever LM, and then to *break* the battery-circuit shortly afterwards. For this two other shunt-levers were required which are not shown in the figure. A condenser, N, was introduced into the electrometer-circuit to increase its capacity. A steady difference of potential of 220 volts was applied between the two plates.

When the bulb was excited the ions on one side have to travel over a distance of 8 cm. before they reach the plate B. The object of the experiment was to determine the interval between the starting of the rays and the arrival of the ions at the plate B. It was found that there was only a small deflexion of the electrometer until after a definite interval had elapsed, when the deflexion increased rapidly. This was taken as the instant when the ions had reached the electrode. The deflexion of the electrometer was proportional to the quantity of electricity that had passed from A to B during the time between making the current and breaking the electrometer-circuit. The electrometer itself was not connected with its circuit until after the contacts had been broken.

It was found that after the rays had been acting for ·36 of a second the electrometer deflexion commenced to increase rapidly. A potential-difference of 220 volts was acting between the plates, so that the electromotive intensity was 13·75 volts per cm. In the time ·36 of a second, therefore, under a potential gradient of 13·75, the ions have travelled over a distance of 8 cm. This gives a velocity of the ion of 1·6 cm. a second, and corresponds to the velocity of a positive or a negative ion, for the time taken was found to be independent of the sign of the ion. The sum of the velocities of the positive and negative ions is therefore 3·2 cm. per second—a result agreeing with that determined for air by a completely independent method. The close agreement between these two results affords strong evidence of the truth of the theory by the aid of which the velocities of the ions of different gases have been obtained.

The velocities with which the ions move through the gas are immensely greater than the velocity of the ions in the electrolysis of liquids. The velocity of the hydrogen ion in nearly pure water is 1·08 cm. per hour under a potential gradient of 1 volt per cm.; so that the velocity of the hydrogen ion in conduction under the Röntgen rays (10·4 cm. per sec.) is over 36,000 times as great.

From considerations based on the kinetic theory of gases the velocity of a small charged body moving through the gas under the influence of an electric field may be determined.

Let e be the charge on the positive ion,
$-e$ on the negative ;
m_1 m_2 the masses of the positive and negative ions ;
χ electromotive intensity ;
κ_1 κ_2 the quotient of pressure by density for the positive and negative ions.

Then, assuming that the partial pressure of the dissociated gas is small, the velocity u_1 of the positive ion is given by

$$u_1=\frac{\chi e}{m_1 \kappa_1} \mathrm{D}_1,$$

where D_1 is the coefficient of interdiffusion of the positive ions and the undissociated gas. (J. J. Thomson, Brit. Assoc. Report, 1894, and Art. "Diffusion," *Encyclopædia Brittanica.*) The velocity u_2 of the negative ion is given by

$$u_2=\frac{\chi e}{m_2 \kappa_2} \mathrm{D}_2,$$

where D_2 is the coefficient of interdiffusion of the negative ions and the gas.

The sum of the velocities of the ions is thus given by

$$\begin{aligned} u&=u_1+u_2 \\ &=\frac{\chi e}{m_1 \kappa_1} \mathrm{D}_1+\frac{\chi e}{m_2 \kappa_2} \mathrm{D}_2. \end{aligned}$$

When the ions are of equal mass

$$u=\frac{2\chi e}{m\kappa} \mathrm{D}.$$

We have no means of determining D, the coefficient of interdiffusion of the ions into the gas, nor the mass and charge of the carrier. If, however, we assume that the ion carries the same charge that it does in the electrolysis of liquids, we can theoretically deduce the velocity with which a *molecule* carrying the atomic charge would move through the gas. D then becomes the coefficient of interdiffusion of a gas into itself, and is given by the relation

$$\mathrm{D}=1{\cdot}5435 \frac{\mu}{\rho},$$

where μ is the coefficient of viscosity of the gas and ρ its density.

For hydrogen $D=1{\cdot}7$ and the value of $\frac{e}{m}=10^4$ approximately.

Therefore $\frac{e}{m\kappa}=10^{-6}$,

and $\chi=10^8$ for 1 volt per cm.

therefore $u=340$ cm.

The velocity of a molecule of hydrogen through hydrogen and carrying an atomic charge is thus 340 cms. per second, while the experimentally determined value is only 10·4 cms. per second. The disagreement of theory and experiment seems to point to the conclusion that either the charge is less than the charge carried by an ion in ordinary electrolytes, or that the carrier is larger than the molecule. We have not sufficient experimental evidence to decide between the two suppositions, but some experiments on the velocity of the ions in gases like hydrochloric acid and sulphuretted hydrogen seem to point to the conclusion that the carrier is larger than the molecule. We may suppose that when an ion is liberated it becomes a centre of an aggregation of molecules. This cluster of molecules is in stable equilibrium under the attractive force due to the charge on the ion, and the size of the cluster is determined by the intensity of the bombardment of the molecules of the gas on its surface. Such an hypothesis would explain the observed fact that the positive and negative ions of gases like hydrochloric acid and sulphuretted hydrogen have equal velocities; for the size of the aggregation is dependent only on the charge on the ion for the same gas, and is therefore the same for the positive and negative ions. In the electrolysis of hydrochloric acid we know that the velocity of the hydrogen ion is much greater than that of the chlorine ion, so that there is an essential difference between the carriers in the two cases.

Proceeding on this assumption of the formation of clusters round a central nucleus, we may readily determine the diameter of the clusters to give the observed velocity of the carrier. If D is the coefficient of interdiffusion of a molecule of diameter σ_1, the coefficient of interdiffusion of the carrier of diameter σ_2 through the gas is given by

$$\frac{\sigma_1^2}{\sigma_2^2}D.$$

Therefore

$$u=\frac{\chi e}{m\kappa}\left(\frac{\sigma_1}{\sigma_2}\right)^2 \mathrm{D}.$$

The value $\frac{e}{m\kappa}$ is the same for the molecule as for the cluster. For hydrogen, then, the ratio of the diameter of the cluster to the diameter of the molecule is given by

$$\sqrt{\frac{340}{10\cdot4}}=5\cdot7.$$

For oxygen D=·21 and, assuming its charge is double that of hydrogen, we get a theoretical value of the velocity of 85 cms. per second. This corresponds to a cluster of the diameter of 5·5 molecules.

In the case of chlorine, assuming the charge on the ion is the same as that on hydrogen, we get a cluster of diameter 2·7 molecules. We see, therefore, that to explain the observed results the carrier need not be greater than 5 times the radius of the molecule.

Further experiments, which are not yet completed, have been made to find the velocity of the ions of a gas conducting under the influence of the radiation given out by uranium and its salts. It can be shown that the velocity of the ions in the conducting gas is the same as when the gas is acted on by Röntgen radiation, so that the carrier in the two cases is identical. Further results, however, must be reserved for a future paper.

In conclusion I desire to express my thanks to Professor J. J. Thomson for many valuable suggestions during the course of this investigation.

Cavendish Laboratory,
July 19, 1897.

LV. *Notices respecting New Books.*

The Potentiometer and its Adjuncts. *By* W. Clark Fisher, *A.M.I.C.E.* London, "The Electrician" Publishing Co.

THE usefulness of the potentiometer in the measurement of electrical quantities, and more especially in connexion with continuous-current systems, is now so fully recognized that a treatise on the instrument and its applications is in no wise superfluous. The author of the present volume has endeavoured to increase the value of such a treatise by the addition of chapters on standards of electromotive force and resistance, on galvanometers, and on platinum thermometers; in doing so he has collected much information which, partly on account of its detailed character, partly by reason of its novelty, has not found its way into ordinary text-books. The form of instrument described is that of Crompton, apparently because Mr. Fisher is more familiar with it than with any other; the descriptions given are, however, such as will enable an electrician to perform measurements with other types of instrument. The second part of the volume deals with earlier forms of potentiometer and leads naturally to the discussion of Wheatstone's Bridge and its modifications. The final chapter treats of Lorentz's method of determining the unit of resistance, undoubtedly the most important investigation in which the potentiometer method has been employed. J. L. H.

Eine Kritik der Nernst'schen thermodynamischen Anschauungen. By Dr. ALFRED H. BUCHERER. Freiberg, Saxony: Craz und Gerlach, 1897.

THIS is a small brochure of thirty pages, in which the author replies to criticisms by Nernst of his work on the thermodynamic theory of electrochemical forces. In addition, he criticizes Nernst's application of thermodynamics to the theory of migration of ions. J. L. H.

Elektrische Ströme. By EMIL COHN, *Extraordinary Professor of Physics in the University of Strassburg.* Leipsic: S. Hirzel, 1897.

THIS volume contains the substance of ten lectures delivered in Strassburg, and attended by engineers, architects, and business men. The lectures aimed at giving a popular, though strictly scientific, account of the generation, distribution, and measurement of the electric current supplied from a central station; they treated of the magnetic, chemical, and heating effects of a current, including induction and alternating currents. The last lecture consists of a short account of polyphase motors and generators. Much compression is of course necessary in dealing with such a comprehensive subject within 180 small pages, but the author, by giving great prominence to experiments illustrating principles and by suppressing all unnecessary detail, has compiled a very readable book which may prove serviceable to readers possessing technical, even though not electrical, knowledge. J. L. H.

Physikalisches Praktikum mit besonderer Berücksichtigung auf physikalisch-chemische Methoden. By E. WIEDEMANN *and* H. EBERT. Third and enlarged edition. Brunswick: Vieweg, 1897.

ON the appearance of the second edition of this work, four years ago, we had occasion to call attention to it as a suitable text-book for first-year laboratory work. The authors have now greatly extended the section on electricity and magnetism, which was previously the weakest part of the book. Some new experiments on capillarity, notably Quincke's determination of surface-tension by measuring the sizes of bubbles and globules, and a few notes on manipulation, have also been inserted, thus adding to the completeness and usefulness of the volume. J. L. H.

LVI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 368.]

June 23rd, 1897.—Dr. Henry Hicks, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'Notes on a Collection of Rocks and Fossils from Franz Josef Land, made by the Jackson-Harmsworth Expedition during 1894–96.' By E. T. Newton, Esq., F.R.S., F.G.S., and J. J. H. Teall, Esq., M.A., F.R.S., V.P.G.S.

A large collection of rocks and fossils, obtained by the members of the Jackson-Harmsworth Expedition, chiefly from the neighbourhood of Cape Flora, on the south-west of Northbrook Island, but also from more distant localities visited during boat- and sledge-journeys, have been sent to the Director-General of the Geological Survey, and examined by the authors.

After a summary of what was previously known of the geology of Franz Josef Land, an account of the new specimens is given. The rocks are for the most part basalts, and are described in detail; they are usually formed of labradorite, augite, and interstitial matter which is sometimes represented by palagonite containing a large percentage of iron-oxide. This palagonite is regarded as the hydrated representative of the residual magma left after the separation of labradorite and augite; and the conclusion is reached that in this case progressive crystallization has resulted in the concentration of iron-oxide in the mother liquor.

Most of the fossils have been collected around Cape Flora. The presence of *Ammonites macrocephalus*, *A. modiolaris*, and *Belemnites Panderi* indicate the presence of rocks of Lower Oxfordian or Callovian age; while, apparently above these, a plant-bed was met with in which the genus *Ginkgo* is conspicuous, and this is believed to be of Upper Jurassic age.

The oldest fossiliferous bed yet found occurs about 20 miles to the west of Cape Flora, and also contains plant-remains, which, it is thought, may be Lower Jurassic and possibly of the age of the Great Oolite.

These plant-beds and numerous indications of layers of lignite seem to show that these Jurassic strata are to a great extent of estuarine or freshwater origin.

The general structure of the country appears to be typified by what occurs at Cape Flora, where cliffs of sedimentary strata some 600 feet high (for the most part hidden by talus) are overlain by 500 feet of basalt. At some other localities, however, the basalt is found at the sea-level.

It is pointed out that the islands, which make up the archipelago of Franz Josef Land, are fragments of a formerly extensive region of plateau-basalts, similar to that of which the Færöe and the Western Isles of Scotland must have formed a part.

2. 'Deposits of the Bajocian Age in the North Cotteswolds.—I. The Cleeve Hill Plateau.' By S. S. Buckman, Esq., F.G.S.

This paper deals with a portion of the Northern Cotteswolds termed the Cleeve Hill plateau; and, like a previous communication on the Mid Cotteswolds, it is concerned with the deposits which intervene between the Upper Freestone and the Upper *Trigonia*-grit. The author shows that in the Cleeve Hill plateau there is, beneath the Upper *Trigonia*-grit, a further series of beds which have not been found elsewhere in the Cotteswolds, and by placing them in their order he is able to show the sequence of the Cleeve Hill 'Inferior Oolite' rocks, about which there has been so much misconception. A map is given to show the positions of the different quarries, with notes as to the strata which they exhibit.

The paper also gives further information concerning the Bajocian denudation, and with it is presented a map showing the areas of the different rocks upon which the Upper *Trigonia*-grit reposes non-sequentially. Notice is also taken of a water-bearing bed, and of its economic importance. Some remarks are made upon the ancient geography of the Cleeve Hill plateau in regard to streams. Their probable courses are marked upon a map.

3. 'Pleistocene Plants from Casewick, Shacklewell, and Grays.' By Clement Reid, Esq., F.L.S., F.G.S.

4. 'An Explanation of the Claxheugh Section (Co. Durham).' By D. Woolacott, Esq., M.Sc.

The section of which an explanation is offered in this communication occurs about 2 miles west of Sunderland, and has been noticed by Messrs. King & Howse, and Prof. Lebour. The base shows the Permian Yellow Sands, which are succeeded at the west end of the section by the Marl Slate, thin-bedded limestones, and at the top crystalline limestones without any trace of bedding. At the east end the Marl Slate and thin-bedded limestone are absent, and except when a breccia intervenes the crystalline limestone rests on the Yellow Sands, though the thin-bedded limestone and Marl Slate show no signs of thinning-out. There are also minor complications. The author suggests that the section may be explained by supposing that denudation occurred in a cavern, the roof of which afterwards fell in, and that disturbances were also produced by 'creep'-movements.

LVII. *Intelligence and Miscellaneous Articles.*

HERTZ WAVES AND METALLIC ENCLOSURES.

To the Editors of the Philosophical Magazine.

GENTLEMEN,

AMONG other misstatements made in this country I observe one to the effect that electric waves of moderate length can penetrate a complete metallic enclosure and affect a coherer inside. As shown by me in 1894*, they can readily get in by insulated wires, or by chinks or other interruption of metallic conductivity, but into a cavity really bounded by a conducting wall of fair thickness they do not go.

The statement that they do is a purely misleading one, likely to cause eminent continental physicists to surmise that there may after all be some discovery involved in those sensational newspaper accounts which have not scrupled to use the absurd phrase "Marconi Waves" and to speak of them as if they were novelties unknown to science.

Yours faithfully,
OLIVER J. LODGE.

ON THE VISCOSITY OF INSULATING LIQUIDS IN A CONSTANT ELECTRICAL FIELD. BY G. QUINCKE.

The results of this investigation are as follows:—

I. If solid spheres of insulating material are made to oscillate in an insulating liquid between condenser plates, either parallel or at right angles to the lines of force, the oscillations will be the more strongly damped, the greater is the charge on the plates of the condenser.

II. The difference of the logarithmic decrement of the oscillations with a charged and an uncharged condenser, $\lambda-\lambda_0$, is a measure of the increase of viscosity due to electric force, or a measure of the electric viscosity of the liquid at right angles or parallel to the electric lines of force.

III. The electric viscosity at right angles to the lines of electrical force is for ether, for carbon bisulphide, for a mixture of equal volumes of carbon bisulphide and oil of turpentine, and for benzole, nearly proportional to KP/a, where K is the dielectric constant of the liquid, P the difference of potential, and a the distance between the condenser plates.

IV. Electric viscosity parallel to the electric lines of force is 1·5 to 6 times smaller than electrical viscosity at right angles to the lines of force.—Wiedemann's *Annalen*, No. 9, 1897.

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* 'The Work of Hertz and his Successors,' published by the Electrician Co., Salisbury Court, Fleet Street, London.

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

DECEMBER 1897.

LVIII. *On Platinum Temperatures.*

By J. D. Hamilton Dickson, *M.A., F.R.S.E.*[*]

VARIOUS formulæ have been suggested connecting the electric resistance of a metallic wire with its temperature. The earliest of these was given by Clausius[†], in 1858, on the results of Arndtsen's experiments. Neglecting iron, which then as now seems anomalous, he found that Arndtsen's results were nearly represented by the formula

$$R = R_0(1 + \cdot 00366t), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R is the resistance at t° C., and R_0 is the resistance at the freezing-point of water. The constant in this formula being the same as that in Gay-Lussac's law for the relation between the volume and the temperature of a gas at constant pressure, he added that this might probably be the real value of the constant for pure metals, and in that case their resistance would vanish at the absolute zero. This formula held for all Arndtsen's metals except iron.

Siemens[‡], in the Bakerian Lecture for 1871 (only a summary of which was published by the Royal Society), on theoretical grounds gave the formula

$$R = R_0 + a\sqrt{t} + bt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a and b are constants.

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[*] Communicated by the Author.

[†] Pogg. *Ann.* vol. civ. (1858).

[‡] Proc. Roy. Soc. 1870–71.

In 1887 Professor Callendar[*], as the result of an elaborate and careful experimental investigation of the possibility of indicating temperature by the variation of the resistance of platinum, devised the idea of "platinum-temperatures," given by the formula

$$\varpi = \frac{R - R_0}{R_1 - R_0} 100, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ϖ is the "platinum-temperature" corresponding to the resistance, R, of a given wire whose resistances are R_1 and R_0 at 100° C. and 0° C. To this formula he added the correction-formula

$$t - \varpi = \delta \left\{ \left(\frac{t}{100} \right)^2 - \frac{t}{100} \right\}, \quad . \quad . \quad . \quad . \quad (4)$$

by which ϖ is to be reduced to "normal air-temperature" t, δ being a constant to be determined once for all for each wire.

Further researches since then by Mr. E. H. Griffiths[†], by Prof. Callendar and Mr. Griffiths[‡], by Professors Dewar and Fleming[§], by Messrs. Heycock and Neville[||], and by Messrs. Holborn and Wien[¶], go to show that, for temperatures ranging from −200° C. to 1300° C., however roughly a given platinum wire may be treated as regards rapid or excessive variations of temperature or long immersions in regions of either very low or very high temperatures, provided it be carefully annealed, it undoubtedly always has the same resistance at the same temperature.

Mr. E. H. Griffiths further found (Gr.) that two different platinum wires, although giving different values (each on its own scale) of the platinum-temperature corresponding to a given normal air-temperature, nevertheless agreed, when properly corrected, in giving the same value for the normal air-temperature.

Finally, Professors Dewar and Fleming[**] showed experimentally that for pure metals the resistance vanished at or near the absolute zero, but that the temperature-coefficient, which Clausius hoped might be ·00366 for all, varied in each case.

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[*] Trans. Roy. Soc. 1887 A; in the present paper referred to as (Call.).

[†] Trans. Roy. Soc. 1891 A; in the present paper referred to as (Gr.).

[‡] Trans. Roy. Soc. 1891 A; in the present paper referred to as (C. and Gr.).

[§] Phil. Mag. 1895, vol. xl. p. 97, § 4.

[||] Trans. Chem. Soc. 1895.

[¶] Wied. *Ann.* vol. lix. (1896).

[**] Phil. Mag. vol. xxxiv. (1892).

Professor Callendar gave (Call.) the formula, convenient for some cases,

$$R = R_0 e^{\frac{\alpha t}{1+\beta t}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where α and β are constants: and Mr. E. H. Griffiths used (Gr.) the more elaborate formula

$$t - \varpi = a\varpi + b\varpi^2 + c\varpi^3 + d\varpi^4, \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where a, b, c, d are constants.

Formula (1) is, in the light of later and more accurate experiments, only a first approximation: and Professor Callendar (Call.) found that (2) was unsuitable for the results he obtained in his experiments.

Formulæ (3) and (4), on eliminating ϖ, take the form (given by Professor Callendar as well as others) of

$$\frac{R}{R_0} = 1 + \alpha t - \beta t^2,$$

where α and β are positive constants. This is the equation of a parabola whose axis is perpendicular to the axis of t; and (1) it leads to a maximum value for R at $t = \alpha/2\beta$, and (2) at lower temperatures any given resistance corresponds to *two* temperatures. Both of these statements indicate physical conditions which we have no reason to suppose exist; and therefore, however closely the formulæ for any particular set of experiments correspond with observations within the limits of the experiments, it is obvious that they can only be an approximation. Professor Callendar gives the values (Call. p. 220)

$$\alpha = \cdot 003{,}448{,}0\,; \quad \beta = \cdot 000{,}000{,}533\,;$$

whence

$$t = \frac{\alpha}{2\beta} = 3234^{\circ}\cdot 5, \text{ and } R = 6\cdot 576\, R_0.$$

No experiments have been published over a range of temperature above the freezing-point of water so great as to increase the resistance to this extent. However, in a series of experiments by Messrs. Holborn and Wien extending to 1610° C., in the course of which a part of the apparatus was cracked, thus leading to a smaller increase of resistance as the temperature rose than was to be expected, they found a resistance nearly equal to $6\,R_0$, while throughout the range of the 1600° of their experiments the temperature-coefficient had varied very slowly and steadily and only to the amount of some 20 or 25 per cent. of its initial value. It is therefore not unfair to believe that 6·576 R_0 might have been within the range of attainable values in their experiments had not the apparatus unfortunately broken down.

On p. 139 of C. and Gr. data are given by which δ-formulæ (4) may be calculated for three thermometers named E, M_1, and M_2. There are three observations for E, six for M_1, and five for M_2. Making use of these we get the formulæ,

$$\left.\begin{aligned}
\varpi_E &= 108{\cdot}155\,R - 331{\cdot}149\,; & t-\varpi_E &= 1{\cdot}7564\left\{\left(\frac{t}{100}\right)^2-\frac{t}{100}\right\}\\
\varpi_{M_1} &= 69{\cdot}315\,R - 295{\cdot}973; & t-\varpi_{M_1} &= 1{\cdot}56394\left\{\left(\frac{t}{100}\right)^2-\frac{t}{100}\right\}\\
\varpi_{M_2} &= 69{\cdot}793\,R - 295{\cdot}750; & t-\varpi_{M_2} &= 1{\cdot}56447\left\{\left(\frac{t}{100}\right)^2-\frac{t}{100}\right\}
\end{aligned}\right\}\quad(7)$$

whence the platinum-temperatures of the absolute zero (R=0) are −331·149, −295·973, −295·750; or, employing the corrections given by the δ-formulæ of (7) (namely +25·077, +18·329, +18·311), they become −306°·072, −277°·644, and −277°·439 on the normal air-thermometer.

To formula (5), for which Professor Callendar found (though only as an "odd coincidence," Call. p. 196) that the values $\alpha = {\cdot}003{,}425{,}9$, $\beta = {\cdot}001{,}529$ "agree best with the observations" (Call. p. 190), there are serious objections. According to it, R vanishes only when $1+\beta t$ is *positive* and infinitely small, that is, when t has a negative value indefinitely little *less* numerically than $1/\beta$ or 654·022, —in other words, R vanishes indefinitely near the (quasi) temperature of −654°·022 but above it, while immediately below it R is infinite. On the other hand, there is an asymptotic value for R as t increases to infinity, namely $R_0 e^{\alpha/\beta}$, or about 9·4 R_0. In the experiments of Professors Dewar and Fleming we have evidence that the resistances of all pure metals tend to zero at or near the absolute zero; and there is no reason at present known why there should be an upper limit to the resistance with the increase of temperature. Hence this formula is generally inadmissible.

Mr. Griffiths's formula (6), having four disposable constants, is, so far, better than one with a less number, throughout the range over which it is used, namely from 0° to 400° C. But again the objections to it are not easily overcome. In table XII. on p. 59 of (Gr.) he gives the values of a, b, c, d for three thermometers named E, F, G; and on pp. 65, 67, and 68 he gives ϖ for each of these thermometers, namely

$$\left.\begin{aligned}\varpi_E &= \frac{R-3{\cdot}0570}{{\cdot}9277}\times 100 = 107{\cdot}8\,R - 329{\cdot}52,\\ \varpi_F &= \frac{R-4{\cdot}7662}{1{\cdot}5053}\times 100 = 66{\cdot}43\,R - 316{\cdot}63,\\ \varpi_G &= \frac{R-4{\cdot}9543}{1{\cdot}5856}\times 100 = 63{\cdot}07\,R - 312{\cdot}46.\end{aligned}\right\}\quad . \quad (8)$$

Assuming that the absolute zero is where R vanishes, these formulæ give for it the respective platinum-temperatures −329·52, −316·63, −312·46. Applying the corrections deduced by (6) and table XII. (Gr.), these become on the normal air-thermometer −245°·99, −253°·06, −259°·50. This formula is therefore not trustworthy for extrapolation.

For each of the formulæ suggested (I have passed over those which have been already rejected by others) there appears therefore to be some objection not easily overborne; but naturally there are close agreements, otherwise they would not have been proposed. Probably the direction in which investigation has been made has tended to the expression of the resistance in terms of the temperature; but when the problem is reversed and we seek to determine temperature by resistance it is equally natural to write

$$t = a' + b'R + c'R^2 + \dots,$$

where a', b', c'... are constants.

These and other circumstances have led to the consideration of a formula of the form

$$(R+a)^2 = p(t+b), \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where a, p, and b are constants, as being more representative of the connexion between temperature and resistance than any formula hitherto proposed, and at least as simple as any.

It evades many, if not all, of the difficulties appearing in the formulæ already employed. Beginning at the zero value of R, t and R increase together indefinitely; there is no maximum value for R beyond which it cannot increase, and there is always one and only one value of t for each value of R. [It may be said, however, that there are two values of R for each value of t. The answer is doubly against this contention; (1) such a value of R would be negative, which is inconceivable; and (2) even if a negative value of R were conceivable, the double values would only arise below the absolute zero, which is again inconceivable.] As to its positive merits, it, too, has three disposable constants, like (2), (3, 4), and (5); and if it is still thought desirable to retain platinum-temperatures, instead of converting them at once into normal air-temperatures, an equally simple δ-formula like (4) exists, (19). There only remains one point; does the proposed formula represent the results of experiment as well as the others? To answer this question, I have carefully calculated formulæ, by the method of least squares, from the best experimental results obtainable. I give the details of such calculations for Prof. Callendar's observations, for those of Professors Dewar and Fleming, of Mr. E. H. Griffiths, and of Messrs. Holborn and Wien at low temperatures. I have also calculated formulæ for Messrs. Holborn and Wien's high-temperature results*, but it is unfortunately impossible to base much upon them after the break-down of the apparatus. Prof. Witkowski† gives a rough set of results below 0° C., but without details. I have calculated a formula with equal care for them, but it leads so far astray that I presume he only meant his results to be taken as showing the trend of a possible series of experiments.

Professor Callendar's Observations.—Professor Callendar is at first less concerned with finding an absolute connexion between platinum degrees and the readings of an air-thermometer, than with determining the prior question whether there is a strict connexion between the two. The painstaking and repeated efforts made to overcome, as he has done successfully, the difficulties that arose during the investigation fully entitle him to claim the demonstration of his proposition, and at the same time make his observations of great value. For my present purpose I deal only with those observations contained in tables A.–v. on pp. 213–216 of (Call.), which he considers the most free from error. I have selected the observations chosen, for the following reasons:—Prof. Callendar found that his air-thermometer readings at constant volume gave higher values of $t-\varpi$ than those at constant pressure, the difference in one case being as much as 1° C. at 540° C. (p. 221): the gauge used was specially designed for readings at constant pressure (p. 209): readings at constant pressure have less chance of error through dispensing with the manometer (pp. 126, 129 of Call. and Gr.): the temperature employed is defined to be "the temperature by normal air-thermometer at constant pressure, and is nearly equal to the absolute temperature on the thermodynamic scale" (p. 218 Call.). Hence I have used only the observations recorded at *constant pressure*. Of these Professor Callendar considers

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* Wied. *Ann.* vol. lvi. p. 360 (1895).

† Phil. Mag. vol. xli. p. 314 (1896).

Nos. 5, 11, 13 less trustworthy than the remaining twelve. These twelve are given in Table I. In this table the 1st column is the number of Professor Callendar's observation as given in his table A.–v.; the 2nd and 3rd columns are the corresponding Centigrade and platinum temperatures respectively, also from table A.–v.; the 4th column contains

TABLE I.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
14	624°·2	572°·9	19·674	57·933	57·936	57·935	steady.
31	599·8	552·6	19·668	56·558	56·563	56·56	2·75
24	532·2	496·0	19·668	52·805	52·784	52·795	steady.
30	440·9	417·1	19·668	47·516	47·517	47·52	1·2
12	391·8	373·5	19·674	44·619	44·619	44·62	2·12
6	360·9	345·8	19·674	......	42·769	42·77	0·43
23	288·7	280·6	19·668	38·416	38·403	38·41	3·23
7	278·4	270·9	19·674	37·770	37·767	37·77	steady.
10	259·5	253·3	19·674	36·602	36·591	36·60	steady.
9	255·5	249·5	19·674	36·342	36·337	36·34	3·00
35	232·90	228·08	19·668	34·894	34·896	34·895	steady.
36	225·71	221·14	19·668	34·427	34·433	34·43	steady.

the values of R_0 which he used in calculating ϖ; the 7th column is the resistance of the platinum wire from which ϖ was obtained, taken as the mean of columns 5 and 6; the 8th column indicates the *steadiness* of the temperature during each experiment. The 7th column was obtained from Professor Callendar's numbers in two ways; (1) by going through the process exemplified at the foot of p. 218 of (Call.), and (2) by calculating backwards from the formula, p. 218 of (Call.),

$$\varpi = \frac{R/R_0 - 1}{\cdot 0033947},$$

the two processes giving respectively the 5th and 6th columns, of which the 7th column was the mean. In the 8th column an experiment was considered to be at *steady* temperature when the *observed* resistances of the spiral sometimes increased and sometimes decreased. In the case of a *varying* temperature the number in this column is the number of hundredths of an ohm by which the final observation in the set exceeded or fell short of the first observation in the set, divided by the number of minutes during which the set lasted.

These twelve observations were carefully plotted, and I finally employed the numbers in Table II., columns 2 and 3, from which to construct a formula connecting temperature and resistance for this particular wire.

TABLE II.

(1)	(2)	(3)	(4)	(5)	(6)
14	57·935	624·2	624·13	+·07	·0049
31	56·56	599·8	599·40	+·40	·16
24	52·795	532·2	532·58	−·38	·1444
30	47·52	440·9	441·13	−·23	·0529
12	44·62	391·8	391·93	−·13	·0169
6	42·77	360·9	360·95	−·05	·0025
mean of 9, 10	36·47	257·5	257·78	−·28	·0784
mean of 35, 36	34·66	229·3	228·81	+·49	·2401
	26·345	100·0	99·55	+·45	·2025
	19·671	0·0	·35	−·35	·1225

This formula is

$$(\mathrm{R}+140{\cdot}1185)^2=21{\cdot}951052(t+1162{\cdot}813), \quad . \quad . \quad (10)$$

which gives, for the resistances observed, the temperatures calculated in column 4. The 5th column gives the excess, v, of the temperature observed over the temperature calculated; and the 6th column contains v^2, whence the probable divergence between an observed and a calculated temperature is

$$\cdot 6745\sqrt{\frac{\Sigma(v^2)}{9}}=\cdot 23,$$

or a little over a fifth of a degree on temperatures ranging over 600°—a degree of closeness quite comparable with the observations themselves.

This formula gives for the absolute zero (R=0) the temperature −268°·4 C.

*Observations of Profs. Dewar and Fleming**.—Profs. Dewar and Fleming have investigated the alteration of the resistance of various metals at low temperatures; and among these they give their observations of the alteration of the resistance of a certain platinum wire with alteration of temperature. In view of their later researches on the properties of matter at low temperatures, and in those cases where they record careful numerical results in terms of platinum-temperatures, it is of much importance to know† that this wire is the actual piece of wire forming the platinum-thermometer afterwards referred to by them as "P_1".

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* Phil. Mag. 1893, vol. xxxvi. p. 282.

† I am indebted for this information to Messrs. Petavel and Morris, who so ably assisted them in these researches.

Their researches (Table III.) are given in ohms and Centigrade degrees down to 0°·60, below this point the temperatures are only indicated. From these I have chosen two as being perhaps the best known,—for (*a*) which was "taken in solid carbonic acid and ether" I use −78°·2 C., and for (*b*) which was "taken in liquid oxygen boiling at 760 millim." I use −182°·5 C.

TABLE III.

R, in ohms.	Temp. C. Observed.	Temp. C. Calculated.	*v*.	v^2.
5·2171	195°·2	195°·9	−·7	·49
4·6345	141·0	140·2	+·8	·64
4·1876	98·5	98·4	+·1	·01
4·1813	98·05			
3·9982	80·90	80·9	0	·00
3·7695	60·0	59·9	+·1	·01
3·5440	39·60	39·4	+·2	·04
3·3108	18·45	18·46	−·01	·00
3·3043	17·90			
3·2991	17·20			
3·2739	14·90			
2·2487	12·80			
3·2450	12·60			
3·1136	0·60	0·9	−·3	·09
⋮				
2·2085	(*a*)− 78°·2	− 77·9	−·3	·09
⋮				
0·9473	(*b*)−182°·5	−182·6	+·1	·01

Of the observations given in Table III. I have employed the ten indicated in the 3rd column from which to calculate, by the method of least squares, the formula

$$(\mathrm{R}+20{\cdot}529023)^2={\cdot}53270015(t+1048{\cdot}4396). \quad . \quad . \quad (11)$$

By means of this formula I have calculated the temperatures in the 3rd column, whence the 4th and 5th columns give (as before) the divergences, and their squares, leading to the probable value of the divergence between a calculated and an observed temperature being

$${\cdot}6745\sqrt{\frac{\Sigma(v^2)}{9}}={\cdot}26,$$

or about a quarter of a degree over a range of nearly 400°,—a result within the acknowledged errors of observation. The absolute zero given by this formula (R=0) is −257°·3 C.

*Observations of Messrs. Holborn and Wien**.—Messrs. Holborn and Wien give five (or six) series of observations of

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* Wied. *Ann.* vol. lix. (1896).

the resistance of a certain platinum wire at low temperatures, the wire being at various times immersed in liquid air, cooled paraffin, ether, and a mixture of carbonic acid and alcohol. There were altogether ninety-seven observations. Arranging these, irrespective of their series, in order of temperature, the lowest being −190°·2 and the highest 0° on the *hydrogen* thermometer (with Centigrade scale), I adopted Table IV. from which to calculate, by the method of least squares, the formula

$$(\mathrm{R}+293{\cdot}28615)^2=119{\cdot}49237(t_{\mathrm{H}}+976{\cdot}8586). \quad . \quad . \quad (12)$$

Table IV.

R, in ohms.	Temp. (H). Observed.	Temperature. Calculated.	v.	v^2.	Remarks.
48·37	0·0	0·0	0	·00	Obs. 97.
45·22	− 18·4	− 17·9	−·5	·25	Avge. of 2 obs. lying within 0°·6.
41·59	− 38·4	− 38·4	0	·00	,, 2 ,, ,, 0°·6.
38·39	− 56·1	− 56·2	+·1	·01	,, 3 ,, ,, 1°·6.
34·12	− 78·9	− 79·8	+·9	·81	,, 8 ,, ,, 1°·0.
28·87	−108·5	−108·3	−·2	·04	Obs. 77.
25·14	−128·2	−128·3	+·1	·01	,, 56.
21·66	−147·0	−146·8	−·2	·04	,, 46.
16·29	−175·1	−174·8	−·3	·09	Avge. of 8 obs. lying within 0°·6.
13·34	−189·9	−190·0	+·1	·01	,, 8 ,, ,, 0°·6.

From this formula the results in the 3rd column of Table IV. were obtained, leading to the divergences in the 4th column, and their squares in the 5th column, whence the probable value of the divergence between a calculated and an observed temperature is found to be

$$\cdot 6745\sqrt{\frac{\Sigma(v^2)}{9}}=\cdot 25,$$

or a quarter of a degree over a range of nearly 200°. The hydrogen-temperature of the absolute zero (R=0) given by this formula is −257° 0.

Messrs. Holborn and Wien give the following formula connecting temperature and resistance "true to about 1°,"

$$t=-258{\cdot}3+5{\cdot}0567\mathrm{R}+0{\cdot}005855\,\mathrm{R}^2, \quad . \quad . \quad (13)$$

and they add, as examples of the closeness between observed and calculated temperatures, these three results :—

R.	t observed.	t calculated.	v.
27·56	115°·1	114°·5	+·6
23·13	139·1	138·2	+·9
21·87	145·5	144·9	+·6

so that (12) is preferable to (13).

For the agreement between hydrogen-temperatures and temperatures on the air-thermometer, they give

−189°·9 hydrogen-temperature = −190°·6 air-temperature,
−186°·4 " " " = −186°·9 " "

or, near the temperature of liquid air the reading of their hydrogen-thermometer was ·6 of a degree higher than the reading of the normal air-thermometer.

Mr. E. H. Griffiths's Observations.—In two papers published in the Transactions of the Royal Society for 1891 (A), the first (Gr.) by Mr. E. H. Griffiths, and the second (C. and Gr.) by Messrs. Callendar and Griffiths, a series of most careful observations on the change of the resistance of platinum with temperature has been recorded in connexion with "platinum-temperatures."

In (Gr.) $t-\varpi$ is plotted as a function of ϖ by means of a formula giving $t-\varpi$ in powers of ϖ as far as the fourth power. For example, for a particular platinum-thermometer called G, the formula is

$$t-\varpi=-{\cdot}022{,}915{,}5\,\varpi+{\cdot}000{,}260{,}768\,\varpi^2$$
$$-{\cdot}000{,}000{,}401{,}3\,\varpi^3+{\cdot}000{,}000{,}000{,}85\,\varpi^4. \quad . \quad . \quad (14)$$

This is the equation of a curve somewhat like a parabola, but, with the given constants, a little flatter in one part than a parabola would be, without the curvature at any point ceasing to be continuous.

Mr. Griffiths was led to this curve by the consideration of eight observations, namely, the temperatures of melting ice, and of the boiling-points (at 760 millim. pressure) of water, aniline, naphthalene, methyl salicylate, benzophenone, mercury, and sulphur. An attempt was made to "draw a smooth curve" through these points when plotted; but it was found that while such a curve could be drawn through five of the points, those representing mercury, methyl salicylate, and aniline lay quite off the curve. The five points were therefore used as "fixed points" from which formulæ like (14) were constructed for eight different and very carefully made platinum-thermometers. There does not seem sufficient cause to choose *only* the five points, especially when by the method of least squares each of the eight observations could have been made to contribute its quota towards getting that relation which, by all mathematical methods at our disposal, is nearest to the truth as displayed by the observations.

Of the eight observations given in (Gr.) I am unable to find data to enable me to use methyl salicylate. From the other seven as given in tables I. and X. (Gr.) for thermometer G (taking the mean of the two values of the boiling-point of mercury in table I.), I have, by least squares, obtained the formula

$$(R+22{\cdot}897523)^2={\cdot}91924237(t+843{\cdot}3048). \quad . \quad (15)$$

In the first line of Table V., after the first three headings, appear the names of the five substances used by Mr. Griffiths, with the respective boiling-points he adopted placed below them in the 2nd line. The 3rd line contains the temperatures calculated by (15) from the resistances in Mr. Griffiths's table X.; and the 4th line shows the divergences between the 2nd and 3rd lines, from which the probable value of a divergence, calculated as before, is 0°·43. The absolute zero (R=0) given by this formula is −272°·95.

Table V.

Absolute zero.	Ice.	Steam.	Aniline.	Naphthalene.	Benzophenone.	Mercury.	Sulphur.
	0°	100°	184°·41	218°·06	306°·08	357°·72	448°·38
−272°·95	0·57	99·39	183·74	217·87	306·80	358·55	447·73
	−·57	+·61	+·67	+·19	−·72	−·83	+·65
−272·45	0·45	99·25	*183·66*	217·82	306·88	*358·74*	448·11
	−·45	+·75	*+·75*	+·24	−·80	*−1·02*	+·27
−278·66	0·44	99·66	*183·85*	217·79	305·94	*357·06*	444·86
	−·44	+·34	*+·56*	+·27	+·14	*+·66*	−·33

Note.—Numbers in *italics* were not employed in calculating the corresponding formulæ, but these results are entered in order to complete the table.

I also used Mr. Griffiths's five fixed points from which to develop a formula like the above, namely

$$(R+22{\cdot}4407292)^2={\cdot}90473803(t+829{\cdot}05810). \quad . \quad (16)$$

The 5th line of Table V. contains the temperatures calculated by this formula, and the 6th line gives the divergences between them and the temperatures in the 2nd line, leading to the probable value of a divergence as 0°·37. The absolute zero ($R=0$) given by this formula is −272°·45 C.

Noting the extreme care employed in the second paper (C. and Gr.), and although reluctant to give up Regnault's determination of the boiling-point of sulphur 448°·38 C. (as used in Gr.; the correct value, however, being given on p.146 of C. and Gr. as 447°·48 "at 760 mm. at 0° in lat. 45°") I calculated another formula, from the five fixed points, adopting Messrs. Callendar and Griffiths's new value 444°·53C. for the boiling-point of sulphur, namely

$$(R+27{\cdot}03473)^2=1{\cdot}0477427(t+976{\cdot}2342). \quad . \quad . \qquad (17)$$

The 7th and 8th lines of Table V. give the temperatures calculated by this formula, and their divergences from the values in the 2nd line, leading as before to the probable value of a divergence being 0°·22. The absolute zero ($R=0$) of of this formula is −278°·66 C.

In Table VI. I have collected the results given by the different thermometers named A, B, C, D, E*, E, F, G, and a "new" unnamed one, in the two papers. For instance, in the 3rd line of the table, taken from p. 54 of (Gr.), the thermometer A gives the boiling-points of aniline, methyl salicylate, and mercury as 184°·22, 223°·08, and 357°·61 respectively. Lines 3 to 10 are taken from (Gr.), lines 11 to 15 from (C. and Gr.). The (Gr.) values are calculated with 448°·38 as the boiling-point of sulphur, the (C. and

TABLE VI.

	Thermometer.	Aniline.	Naphthalene.	Methyl Salicyl.	Benzophenone.	Triph.-Methane.	Mercury.	Sulphur.
From Table I. (Gr.)	......	184°·41	218°·06	222°·88	306°·08	......	357°·25 358·2	448°·38
Gr. p. 54 ...	A	184·32	......	223·08	......	......	357·61	
	B	184·27	......	223·12	......	......	357·59	
	C	184·29	......	223·16	......	......	357·65	
p. 55 ...	D	184·21	......	......	......	......	357·54	
p. 58 ...	E*	184·24	......	......	......		......	
pp. 60, 61...	E	184·20	......	223·18	......	357°·38	357 64	
	F	184·23	......	223·21	......	......	357·68	
	G	184·23	......	......	......	357·33	357·62	
C. and Gr. new value.		......	......	......	......	......	......	444·53
C. and Gr. p. 148...	L, M_1, M_2	......	......	......	......	......	356·60	
p. 150......	E	184·11	217·88	222·98	305·82	356·47	356·74	
	F	184 13	217·96	223·08	305·87	......	356·82	
	G	184·14	217·98	......	305·78	356·41	356·71	
p. 152......	"new"	......	218·04	......	......	......	356·73	

Gr.) values with 444°·53 for this temperature. The various observations for aniline agree in giving 184° for its boiling-point, but they vary among themselves by 0°·3; for naphthalene, similarly, the boiling-point is 218° with a variation of 0°·2; for the boiling-point of methyl salicylate we have 223° with a variation of 0°·3; and for benzophenone 306° with a variation of 0°·3; triphenyl methane is about 356° or 357°, but this is at the unusual pressure of 770 millim.; and the boiling-point of mercury lies between 356½° and 358°, the variations shown ranging over 1°·6. The results given by the formulæ (15), (16), and (17) are therefore fairly comparable with these in accuracy.

From these calculations we are therefore justified in taking the proposed formula (9) as competently satisfying all the requirements of an empirical formula, within the greatest limits of temperature in which experiments may be made, where platinum is employed to determine temperature. The variation of the constants in the different numerical formulæ given above corresponds with the variation of δ in the δ-formula.

If it be still thought desirable to preserve "platinum-temperature," a formula similar to the δ-formula may be easily devised. Equation (9) can obviously be written in the form

$$(\varpi+\mathrm{A})^2=\mathrm{P}(t+b), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

giving

$$(100+\mathrm{A})^2=\mathrm{P}(100+b), \text{ and } \mathrm{A}^2=\mathrm{P}b,$$

whence

$$100+2\mathrm{A}=\mathrm{P}.$$

Thus (18) becomes, writing—by an obvious extension of arithmetical notation,—$\cdot 0\varpi$ for $\varpi/100$,

$$t-\varpi=\delta'(\cdot 0\varpi^2-0\varpi), \quad . \quad . \quad . \quad . \quad (19)$$

where

$$\mathrm{P}\delta'=10{,}000,$$

or, on reduction,

$$\sqrt{\delta'}=\sqrt{(b+100)}-\sqrt{b}. \quad . \quad , \quad . \quad . \quad (20)$$

For a first approximation, since b is about 1000, we have

$$4b\delta'=10{,}000.$$

Although it has been demonstrated that platinum is a suitable substance for determining temperatures over a very wide range, not much different probably from 2000° C., nevertheless seeing that each platinum thermometer needs at least to have its constant specially and carefully determined, not by three, but by a series, of observations, it cannot be too strongly urged that this work should in each case accompany the record of results, when expressed in platinum-temperatures; and no one will deny that to have these results expressed at once in terms of the normal air-thermometer will permanently enhance the value of the work in such a manner as to amply recompense the extra labour. With the view of helping towards this desirable end I have endeavoured to show that the simple formula (9) is superior to the joint formulæ (3) and (4).

Peterhouse, Cambridge,
August 1897.

LIX. *On Real and Apparent Freezing-Points and the Freezing-Point Methods. By* MEYER WILDERMANN, *Ph.D.*[*]

1. IN the *Zeitschrift für phys. Chem.* xv. p. 682, Professor Nernst gives us a series of interesting considerations from which he further draws the conclusion that the mercury of the thermometer comes to rest, not at the freezing-point T_o, but at a different temperature t' which he calls the apparent freezing-temperature. Professor Nernst's considerations, which, however, require modification, are of great importance as leading the way to an understanding of the real meaning of an equilibrium at a freezing-temperature. I shall try now to ascertain theoretically the cardinal points upon which a good freezing-point method depends, and to give a theoretical explanation to a series of rules which have previously been found empirically by my late friend P. B. Lewis and myself (Trans. Chem. Soc. 1894 &c.).

Before proceeding to the detailed consideration of the processes which take place between ice and water or solutions, I quote Professor Nernst's views on this point. He writes as follows:—"From researches hitherto made into the phenomena of the solution of solid bodies, and especially from the work of Boguski[†], we can regard the rate of solution of the solid substance as proportional to the remoteness from the condition of equilibrium; *i. e.* since the heat absorbed is proportional to the mass dissolved, we can assume that the change of temperature at any moment is proportional to the said remoteness from the freezing-point."

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[*] Communicated by the Author. Part III. of paper communicated to the Royal Society, January 13, 1896. See Proceedings of the Royal Society, 1896, *Zeitschrift für physik. Chem.* (1896), xix. p. 63.

[†] On metals in acids.

"Thus, $dt = K(T_o - t)\,dz$. We need not consider the significance of K further; it is sufficient to state that K is directly proportional to the total surface of the solid solvend, and to latent 'heat of fusion'."

He further proceeds to apply Newton's equation

$$dt = K'(t_o - t)\,dz$$

to the cooling of a solution when no ice has been separated, depending upon the convergence-temperature of the given experimental conditions, and comes to the conclusion that the real change of temperature is expressed by the superposition of the two equations, i. e. $\frac{dt}{dz} = K(T_o - t) + K'(t_o - t)$; the thermometer-thread comes to rest when

$$\frac{dt}{dz} = K(T_o - t') + K'(t_o - t') = 0,$$

and therefore

$$t' = T_o - \frac{K'}{K}(t' - t_o);$$

i. e. the temperature of rest of the mercury-thread is not at T_o (the real freezing-temperature), but at the more or less different one t' which he calls the apparent freezing-temperature.

Prof. Nernst further thinks that as $t_o - t'$ is positive when $t_o > t'$, and negative when $t_o < t'$, therefore the above equation is to be applied equally for the case when the convergence-temperature is above as when it is below the freezing-temperature; *i. e.* that, under all conditions, the deviations are to be explained from the process of ice-melting; an assumption with which I cannot quite agree.

Boguski investigated the action of acids on metals; the processes and equilibrium being purely chemical they have little to do with the phenomenon of ice-melting and "perfect" equilibrium. In the following considerations I start instead from the idea of perfect equilibrium, and draw the conclusions for the freezing-point method which necesarily follow from the chief properties of this equilibrium.

As known, the equilibrium between ice and water or ice and a solution is a "perfect" one; and this kind of equilibrium is characterized by this fact, that at the smallest change of its temperature (the freezing-temperature) one of the two parts of the heterogeneous system must disappear. Above the freezing-temperature the solid part (the ice), not the liquid, disappears; and below the freezing-temperature the liquid, not the solid: therefore, if the convergence-temperature be above the freezing-temperature of a given liquid, the heterogeneous system, at the freezing-temperature, will be continuously warmed by the arrangements of the experiment (by the air- or liquid-bath, stirring &c.), and the ice-melting process must take place. On the other hand, if the convergence-temperature be below the freezing-temperature the heterogeneous system is continuously cooled, at the freezing-temperature, by the arrangements of the experiment, and the process of ice-separation takes place.

In order to know better the processes which take place at equilibrium and to get *the chief conditions for a good freezing-point method, as far as it concerns the heterogeneous system at equilibrium*, let us now consider the following processes:—

(1) The cooling of the liquid by the air-bath (in Lewis's and my method, and in the methods of Jones and Abegg) or by the liquid-bath (Loomis, Raoult, Pickering, Eyckman) when the liquid is not stirred, *i. e.* is not otherwise warmed.

(2) The warming of the liquid by stirring (*i. e.* by the stirrer and the air drawn in).

(3) The process of melting of ice or of solid solvends in liquid solvents or solutions.

(4) The process of separation of ice or of solid solvends from overcooled liquids.

2. *As regards* (1) *and* (2) if the temperature of the air-bath be T_A, of the ice-bath T_I, the temperature read off at a given time z be t, we ought to have, for the rate of cooling of the liquid, Newton's equation $\left(\frac{dt}{dz}\right)=C(T_A-t)$, if no warming through stirring is taking place.

Here C is inversely proportional to the mass and to the heat-capacity, and directly proportional to the total surface of the liquid. If at the same time the liquid is warmed by regular stirring, the same amount of heat will be conveyed to it in the unit of time, and we have for this single effect $\left(\frac{dt}{dz}\right)'=K'$, where K' is inversely proportional to the heat-capacity and to the mass of the liquid. The actual rate of warming or cooling the liquid is therefore

$$\frac{dt}{dz}=C(T_A-t)+K'. \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

The point at which $\frac{dt}{dz}=0$ is the convergence-temperature (t_g) in the given arrangement of the experiment. It follows from the equation $\frac{dt}{dz}=C(T_A-t_g)+K'=0$ that

$$T_A=t_g-\frac{K'}{C}, \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

and putting this value t_A in the equation (I.) we get

$$\frac{dt}{dz}=C\left(t - \frac{K'}{C} - t\right)+K'=C(t_g-t);$$

i. e. we can with Prof. Nernst bring the velocity of cooling of the liquid into connexion with the convergence-temperature t_g. Since K′ is positive, $C(T_A-t_g)$ must be negative; *i. e.* the convergence-temperature is always above the temperature of the air-bath.

3. *As regards the process of ice-melting*: Prof. Nernst makes the assumption, on the basis of Boguski's investigations, that the velocity of ice-melting is to be expressed, as above quoted, by the equation $\frac{dt}{dz}=K(t_0-t)$. Since the conclusions obtained greatly depend upon the form of the equation, the latter must be first ascertained. Neither Boguski nor Spring, who later investigated the same subject, could succeed in getting a simple expression for the velocity of the processes with which they dealt, because of the unfavourable and very difficult experimental conditions. But besides this, as mentioned above, their reactions do not belong to those under consideration here (processes before perfect equilibrium). I should like, however, to give another explanation why the above equation must nevertheless represent the velocity of ice-melting. We investigate the velocity of ice-melting by measuring the velocity of the cooling of the liquid by the melting ice. Now this may be regarded as a phenomenon which is regulated by Newton's law of conduction. The convergence-temperature is that at which cooling of the liquid no longer takes place; *i. e.* the freezing-temperature at which ice and liquid are in equilibrium, and the ice plays the role of the heat-absorbing medium. Should the ice not dissolve, its surface remaining constant, Newton's equation $dt=K(t_0-t)dz$ would express the process of ice-melting, where t_0 is the freezing-temperature, and t is the temperature of the liquid at the time z. Since the melting of the ice takes place through its surface, the conduction is directly proportional to the surface of ice in contact with the liquid; *i. e.* K is directly proportional to the quantity of ice present in the liquid at the time z. It is obvious that the surface of the ice is a function of the temperature of the liquid, since the latter is conditioned by the amount of melted

ice, therefore our equation is $dt = K_{,}(t_0 - t)f(t)dz$. I tried to investigate the velocity of the melting of separated solid solvends, *e. g.* of ice, in liquid solvents or solutions, of a temperature higher than the freezing-temperature, as carefully as possible. The experimental difficulties are here very much greater than in delicate freezing-point determinations themselves, as will be shown in a future paper, "On the Velocity of Reaction before Perfect Equilibrium takes place." This is the reason why I was unable to obtain in this way even rough experimental evidence for the above equation. Only by investigating cube-shaped pieces of ice in warmer water or solutions could I succeed in this. This also furnishes a method for accurate determination of the *relative* velocities with which ice melts in water and aqueous solutions of different concentrations. Having proved the above equation to be correct, we can approximately assume for the velocity of melting of ice when separated from overcooled liquids the equation

$$\frac{dt}{dz} = K_{,}(t_0 - t)\left[(t_0 - t_{ov}) - (t_h - t)\right];$$

where t_{ov} is the temperature to which the liquid was overcooled below the freezing-temperature t_0, and t_h is the temperature of the taken warmer liquid, with which the separated ice is brought in contact. If $t_h - t$ is very small, we may assume

$$\frac{dt}{dz} = K_{,}(t_0 - t)(t_0 - t_{ov}).$$

4. *The Process of Ice-separation.*

Careful measurements have shown that the velocity of the formation of ice in overcooled liquids finds expression in the equation $\frac{dt}{dz} C''(t_0 - t)(t - t_{ov})$; in which t_0 is the freezing-temperature, t the read temperature of the overcooled liquid at the time z, C'' is a constant which is directly proportional to the latent heat of fusion and inversely proportional to the specific heat of the liquid, and t_{ov} is the temperature to which the liquid was overcooled, Since the rise of temperature of an overcooled liquid is directly proportional to the quantity of separated ice, the above equation means that the *velocity of ice separation from overcooled liquids is directly proportional to the remoteness from the freezing-temperature and to the surface of the ice in contact with the liquid.* Here both parts of the heterogeneous system are of the same, or nearly of the same temperature during the formation of ice, and the above equation represents the first case in which it is, in its form but not in its purpose, identical with Newton's equation. A full account of the reactions of ice-melting and ice-separation and the experimental proof of the above equations will be given in the paper "On the Velocity of Reaction before Perfect Equilibrium takes place." (See Report Brit. Assoc. Liverpool, 1896.)

Let us now bring into connexion those processes which take place simultaneously before and after equilibrium :—

A. When the Convergence-Temperature is above the Freezing-Temperature.

Here we have to deal with the process of *ice-melting*.

The general equation is $\frac{dt}{dz}=\mathrm{K}_{,}(t_o-t_{fl})(t_o-t_{ov})$ (*ice-melting process*) + $\mathrm{C}(t_g-t_{fl})$ (*experimental arrangements*). Since t_o-t_{ov} is always positive, we have:—

If $t_{fl}>t_g$, t_g is also $>t_o$ (*i. e.* if the temperature of the liquid is above the convergence-temperature) ; $\mathrm{C}(t_g-t_{fl})$ is then negative (the liquid is cooled by the arrangements of the experiments); and $\mathrm{K}_{,}(t_o-t_{fl})$ is also negative (*i. e.* the liquid becomes cooler while the ice is melting).

If $t_{fl}>t_o$ and $<t_g$ (*i. e.* if the temperature of the liquid is below the convergence-temperature and above the freezing-temperature), then $\mathrm{C}(t_g-t_{fl})$ is positive (the liquid is warmed by the arrangements of the experiment), $\mathrm{K}(t_o-t_{fl})$ is negative (the ice-melting process takes place and the liquid cools down). If $t_{fl}<t_o<t_g$ (*i. e.* if the liquid is below the freezing-temperature), $\mathrm{C}(t_g-t_{fl})$, as well as $\mathrm{K}(t_o-t_{fl})$, is positive; the meaning of it is:—the liquid is warmed by the arrangements of the experiment ; but $\mathrm{K}(t_o-t_{fl})$ does not mean that the ice-melting process is taking place, since through the latter the liquid can only be cooled, not warmed ; and indeed the lowest limit, where the process of ice-melting no longer comes into consideration, is t_o. If $t_{fl}=t_o$ and $<t_g$, then $\frac{dt}{dz}=\mathrm{C}(t_g-t_{fl})$, and $\mathrm{C}(t_g-t_{fl})$ is positive; *i. e.* the liquid and ice cannot be in equilibrium at the temperature t_o, and the liquid must become warmer. If equilibrium takes place at the temperature t', we have

$$\frac{dt}{dz}=\mathrm{C}(t_g-t')+\mathrm{K}(t_o-t')(t_o-t_{ov})=0, \quad . \quad . \quad . \quad \text{(A)}$$

and

$$t'=t_o+\frac{\mathrm{C}}{\mathrm{K}_{,}}\frac{(t_g-t')}{t_o-t_{ov}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A')}$$

and t', as can be seen, must be $>$ than t_o and $<t_g$, for since

$K_{,}(t_0-t')$ is unavoidably negative, $C(t_g-t')$ must unavoidably be positive; that is to say, in the ice-melting process, the process of cooling the liquid *must* be compensated by the *counter-reaction* of warming by the experimental arrangements in order that $\frac{dt}{dz}$ can $= 0$.

When the convergence-temperature is above the freezing-temperature, the temperature of rest of the mercury-thread is not at the real freezing-temperature t_0 *but at a temperature* t', *the apparent freezing-temperature, which differs more or less from* t_0, *and the temperature* t *will be obtained between the real freezing-temperature* t_0 *and the convergence-temperature* t_g.

I shall now, using the equation (A′), explain a series of rules for the freezing-point method which have been previously found empirically and communicated in the paper of my late friend P. B. Lewis and in my paper, "Zur Bestimmung des Gefrierpunkts des Wassers," *Zeitsch. phys. Chem.* xv. p. 538.

In order that the apparent freezing-temperature t' may coincide with the real freezing-temperature as far as possible it is necessary that $\frac{C}{K_{,}}\,\frac{t_g-t'}{t_0-t_{ov}}$ be kept as small as possible.

How are we to obtain this by the freezing-point method?

(*a*) C is (p. 461) inversely proportional to the quantity as well as to the heat-capacity of the liquid used and directly proportional to the conducting surface of the liquid.

The first condition is therefore that the quantity of *liquid used for the experiment be as large as possible* in order that C may be kept as small as possible. Lewis and I used 1250 c.c., Jones 1000 c.c., Loomis* 70 c.c., and Abegg† 100 c.c. &c. The second condition is to arrange the overcooling so that the same quantity of the liquid, at the same value of t_g-t', be cooled by the arrangements of the experiments as little as possible. For this *an air-bath* (Lewis, Wildermann, Jones, Abegg) *is much preferable to a liquid-bath* (Loomis*, and others). The value of C is in a liquid-bath about 20 times greater than in an air-bath.

(*b*) K must be kept as large as possible: in the equation $\frac{dt}{dz}=K(t_0-t)$ (the velocity of ice-melting) K is directly proportional to the total surface of the separated ice. It follows from this that *the overcooling* t_0-t_{ov} *of the liquid before the separation of the ice must be sufficiently great* (Lewis and I,

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* Wiedem. *Ann.* vol. li.

† *Zeitschrift für physik. Chemie,* vol. xv. p. 682.

Jones and Abegg* use pretty much the same amount of overcooling, viz. about 1°, Loomis† only 0°·3; Loomis‡, in his new method, and others have an exceedingly small value of $t_o - t_{ov}$, since they determine the freezing-temperature with ice, when it is almost completely melted); *the ice must be separated into as fine needles as possible*, and for this it is necessary that the liquid be overcooled *in a bath which is not too cold, that it may not freeze of itself* § (since in this case large flakes are formed instead of fine films); *the ice must not be allowed to form a conglomerate mass*, nor to settle on the sides or at the bottom of the beaker. The ice, in a word, must always be separated into fine films and in sufficient quantity throughout the whole of the liquid.

(*c*) $t_g - t'$ must be kept as small as possible; *i. e.* the convergence-temperature must be brought as near as possible to the freezing-temperature. (The value of $t_g - t'$ can be made $= {\cdot}1^\circ$ or ${\cdot}2^\circ$, or even less. In some of the methods $t_g - t'$ amounts to several degrees.) For freezing-point depressions, however, the more important point is that $t_g - t'$ and $t_g - t''$ in two liquids may be equal; i. e. *the ice-bath must be kept within very narrow limits of temperature*. I use the expression "as much as possible," since we are practically limited in bringing about the ideal conditions we require; we are not able to have an infinite quantity of ice, *i. e.* to make $t_o - t_{ov} = \infty$, and so also to make $t_g - t' = 0$. Our object is therefore to arrange the freezing-method so that $t_o - t'$ is reduced to a value which, under given circumstances, can be quite neglected.

Attention has been already drawn, when the equation (A′) was under consideration, to the fact that if the convergence-temperature is above the freezing-temperature, the temperature t' of the liquid is at equilibrium higher than the real freezing-temperature t_o (and below the convergence-temperature t_g).

According to our conception of perfect equilibrium, the solid part (ice) cannot exist above the temperature t_o: therefore the ice will not be warmed by conduction to the higher temperature of the liquid, but will melt and remain at the freezing-temperature t_o. It has been already empirically found (see paper mentioned above, "Zur Bestimmung," &c.) that if the ice-cap be well closed, about the same values are obtained for the *difference between the read freezing-temperature with ice-cap and that without it*, and also ‖ that in the case of ice-bands, network of ice,

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* *Zeitschrift für Physik. Chemie*, xv. p. 682.

† Wied. *Ann.* li. ‡ Wied. *Ann.* 1896.

§ Measurements of the velocity of ice-melting in all these cases will be given in the paper "On the Velocity," &c.

‖ See Phil. Mag. July 1895; *Zeitschr. f. phys. Chem.* xv.

different values are obtained at the same value of $t_g - t_o$. We now know why it *must* be so. As the liquid has the temperature t', we always obtain the apparent freezing-temperature if the ice-cap is removed, and, if the mercury-bulb is covered by ice, the latter cools the thermometer according to the amount of surface covered and tends to bring it to t_o, and in the presence of an ice-cap the freezing-temperature must therefore be obtained between t_o and t'.

I pass now to the consideration of the question which is of most importance to us, the *freezing-point depressions* when the convergence-temperature is above the freezing-temperature.

We have

$$t' = t_o + \frac{C}{K_{\prime}} \frac{t_g - t'}{t_o - t_{ov}} \qquad \text{(Water)},$$

$$t'' = t_o' + \frac{C'}{K_{\prime}'} \frac{t_g' - t''}{t_o' - t_{ov}'} \qquad \text{(Solution)},$$

and

$$t' - t'' = t_o - t_o' + \left[\frac{C}{K_{\prime}} \frac{t_g - t'}{t_o - t_{ov}} - \frac{C'}{K_{\prime}'} \frac{t_g' - t''}{t_o' - t_{ov}'} \right], \quad . \quad (A'')$$

where $t' - t''$ is the *apparent* freezing-point depression as obtained from the *apparent* freezing-temperatures, and $t_o - t_o'$ is the *real* freezing-point depression as obtained from the real freezing-points. In order that $t' - t''$ may be as equal as possible to $t_o - t_o'$, the expression

$$\frac{C}{K_{\prime}} \frac{t_g - t'}{t_o - t_{ov}} - \frac{C'}{K_{\prime}'} \frac{t_g' - t''}{t_o' - t_{ov}'}$$

must be brought as nearly as possible to zero. This becomes more easy when $C = C'$ and $K_{\prime} = K_{\prime}'$, $t_o - t_{ov} = t_o' - t_{ov}'$.

(*a*) C and C′: The velocity of overcooling of water and of very dilute aqueous solutions must be regarded as pretty much the same.

(*b*) $K_{\prime}$ and $K_{\prime}'$: From the paper "On the Velocity of Reaction before Perfect Equilibrium takes place" it will be seen that, going over from more dilute to more concentrated solutions, the velocity of ice-melting probably decreases but very slowly, so that we can in dilute solutions assume the same value for K. Then the amount of overcooling under the t_o, i. e. $t_o - t_{ov}$ and $t_o' - t_{ov}'$, must be in all dilute and in more concentrated solutions as nearly as possible the same *. Because of this, all the methods in which the freezing-point is

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* Since in $dt = K(t_o - t)dz$, K is directly proportional to the total surface of the ice.

determined with ice, when it has almost completely melted, belong to the less accurate ones. Not only is the value of $t_o - t_{ov}$ thus made at the freezing-temperature exceedingly small, and consequently the experimental error exceedingly great, but the remaining quantity of ice cannot be strictly regulated, and this means $t_o - t_{ov}$, and with it the experimental error, can easily be in one experiment a multiple of the other. Moreover, already with a quantity of ice even $=0^\circ{\cdot}1$ or $0^\circ{\cdot}2$ overcooling, no good equilibrium between ice and the liquid can be practically established, *i. e.* no *constant* apparent freezing-point can be obtained. For the correction for the separated ice see P. B. Lewis, Trans. Chem Soc. 1894.

Putting for dilute solutions $C = C'$, $K_{/} = K_{/}'$, and arranging $t_o - t_{ov} = t_o' - t_{ov}'$, we obtain from (A″)

$$t' - t'' = t_o - t_o' + \frac{C}{K_{/}} \frac{[(t_g - t_g') - (t' - t'')]}{t_o - t_{ov}}; \quad . \quad (A'')$$

and from this it follows :—

(1) If t_g is put $= t_g'$, then we obtain

$$t' - t'' = t_o - t_o' - \frac{C}{K_{/}} \frac{(t' - t'')}{t_o - t_{ov}}$$

or

$$\left(1 + \frac{C}{K_{/}(t_o - t_{ov})}\right)(t' - t'') = t_o - t_o';$$

that is to say, at a constant convergence-temperature the apparent freezing-point depressions will be in all concentrations $\dfrac{1}{1 + \dfrac{C}{K_{/}(t_o - t_{ov})}}$ smaller than the real.

(2) Now t_g is practically always more or less different from t_g'. The variations of the ice-bath lead through this to experimental error in the more as well as in the less dilute solutions, which is $= \dfrac{C}{K_{/}} \dfrac{(t_g - t_g')}{t_o - t_o'}$.

It is clear that in dilute solutions in which $t' - t''$ is small compared with $t_g - t_g'$, the latter is only to be taken into account in our consideration of the experimental error, and, going over to more concentrated solutions where the depressions $t' - t''$ become of greater importance, $t_g - t_g' - (t' - t'')$ must be kept as nearly as possible equal to 0; *i. e., in dilute solutions* we have the following rule for the freezing-point method :—*the same convergence-temperature, the same temperature of the bath at the same temperature of the room*; and, *in more concentrated solutions,* again, the other rule becomes necessary:—the ice-bath must be arranged at temperatures successively lower so that the convergence-temperature of the solution falls below the convergence-temperature of water by the amount of the freezing-point depression of the given solution.

I should like to draw attention to one point. It has been said above that in dilute solutions $t'-t''$ differs from t_o-t_o' by $\frac{C}{K_{\prime}}\frac{(t_g-t_g')}{t_o-t_{ov}}$. This difference between the real and apparent freezing-point depressions *is only an ideal one. The absolute values,* t_g-t_g', C, and $K_{\prime}$ *are of importance, not only because of the value of the correction* $\frac{C}{K_{\prime}}\frac{(t_g-t_g')}{t_o-t_{ov}}$, *but also and chiefly because of the variations of the obtained freezing-temperature.*

Since equilibrium has always to be established and brought to the above theoretical condition (apparent freezing-temperature), the experimental error in the freezing-point method may be considerably greater than can be concluded from the value $\frac{C}{K_{\prime}}\frac{t_g-t_g'}{t_o-t_{ov}}$.

When the convergence-temperature is *above* the freezing-temperature it is easy to see why, *calculating from a dilute solution which does not form an ice-cap, we get van't Hoff's constant, and why, calculating from water containing an ice-cap, this constant is not obtained, but is so if the ice-cap be avoided**. This is to be explained by the fact that we can determine freezing-point depressions only from the apparent freezing-temperatures and not from the real ones, which in reality we never get even in the presence of a well-closed ice-cap. As in solutions which have a depression of 0°·02 ice does not settle on the mercury-bulb, the apparent freezing-temperature is obtained; therefore, in order to get correct freezing-point depressions, we are obliged to prevent its settling on the bulb in the case of pure water and the most dilute solutions, which form an ice-cap, so as to get the apparent freezing-temperature in these cases also.

The ice-cap is interesting for us (working with the convergence-temperature above the freezing-temperature), not only because by avoiding it we have succeeded in confirming van't Hoff's constant (in a second way) directly from water and have been enabled to prove Arrhenius's generalization experimentally, but it is to be regarded as an important proof that we are justified in our conception of *the equilibrium at the*

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* See Phil. Mag. July 1895.

apparent freezing-temperature as a DYNAMIC *one, where the processes which took place before equilibrium continue.* From the above equation for the velocity of melting or separation of solid solvends,

$$\frac{dt}{dz}=\mathrm{C}_{\prime}''(t_o-t)(t-t_{ov}) \quad \text{or} \quad \frac{dt}{dz}=\mathrm{K}(t_o-t)(t_o-t_{ov}),$$

it follows that when t' becomes equal to t_o, $\frac{dt}{dz}$ becomes $=0$ and no further reaction takes place. *At the real freezing-temperature the equilibrium is therefore a* STATIC *one* ; but as the reaction of ice-melting or ice-separation never takes place alone, we never reach the real freezing-temperature, *and the dynamic, not the static, in reality takes place in nature.* (See Rep. Brit. Assoc., Liverpool, 1896.)

From this conception of equilibrium at the apparent freezing-temperature, it follows for the freezing-point method :—Since at the freezing-temperature the liquid continues to be warmed by the arrangements of the experiment and, correspondingly, the ice continues to melt, the warming of the liquid must be kept as small as possible ; *i. e.* C and t_g-t must be, also for this reason, correspondingly small (the conditions are given above). This is especially necessary in the case of solutions in which the melting of the ice is accompanied by change of concentration and therefore of the freezing-temperature. From the data, given below, it will be seen that the velocity of warming or cooling the liquid by the arrangements of the experiment, if $t_g-t'=0^\circ{\cdot}4$ or $=-0^\circ{\cdot}4$ (in our method), is less than about $0^\circ{\cdot}0012$ per minute ; during the time of 15 minutes, which a freezing-point determination requires, the quantity of melted or separated ice corresponds therefore to $0^\circ{\cdot}018$, and the solution becomes after 15 minutes of the experiment less dilute or more concentrated by about ·025 per cent. This is in complete accordance with the direct experimental proof of the same question (see footnote to the paper of P. B. Lewis, *Zeitschr. f. phys. Chem.* vol. xv.).

B. When the Convergence-Temperature is below the Freezing-Temperature.

Here we have to deal with the process of *ice-separation.*

The general equation is $\frac{dt}{dz}=\mathrm{C}_{\prime}''(t_o-t_{fl})(t_{fl}-t_{ov})$ *(separation of ice)* $+\mathrm{C}(t_g-t_{fl})$ *(arrangements of experiment)*. Since $t_{fl}-t_{ov}$ is always positive, we have :—

If $t_{fl} < t_g < t_o$ (*i.e.* if the temperature of the liquid be below the convergence-temperature and the latter under the real freezing-temperature), $C_{\prime}''(t_o - t_{fl})$ is positive (*i.e.* the process of ice-separation takes place and the liquid is warmed by the freed latent heat of melting), $C(t_g - t_{fl})$ is also positive (the liquid is warmed by the arrangements of the experiment). If $t_{fl} > t_g$ and $< t_o$, $C_{\prime}''(t_o - t_{fl})$ is positive (the process of ice-separation and warming of the liquid takes place), $C(t_g - t_{fl})$ is negative (*i. e.* the liquid is, on the other hand, at the same time cooled by the arrangements of the experiment). If $t_{fl} = t_o$ and $> t_g$, we get $\frac{dt}{dz} = C\,(t_g - t_{fl})$; that is to say, the liquid and the separated ice cannot be in equilibrium at zero, and the liquid will be cooled by the arrangements of the experiment.

If $t_{fl} > t_g > t_o$, $C_{\prime}''(t_o - t_{fl})$ is negative and $C\,(t_g - t_{fl})$ is also negative. The meaning of this is as follows:—The liquid is cooled by the arrangements of the experiment, but $C_{\prime}''(t_o - t_{fl})$ no longer means a process of ice-separation, since the latter can only warm and not cool the liquid; and, indeed, the highest limit for the process of ice-separation is t_o. (We cannot use one equation for the processes of ice-separation and of ice-melting, since it has been found that the constants $C_{\prime}''$ and K are different.) If equilibrium is attained at t', we have

$$\frac{dt}{dz} = C_{\prime}''(t_o - t')(t' - t_{ov}) + C(t_g - t') = 0, \quad . \quad . \quad \text{(B)}$$

and

$$t' = t_o + \frac{C}{C_{\prime}''}\frac{(t_g - t')}{t' - t_{ov}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B')}$$

in this t' must be $> t_g$ and $< t_o$. Because as $C_{\prime}''(t_o - t')(t' - t_{ov})$ in the process of ice-separation is necessarily positive, $C(t_g - t')$ must be negative; *i.e.* in order that $\frac{dt}{dz}$ can be $= 0$, the process of warming the liquid *must* be *compensated* by the process of cooling it. *The real freezing-temperature when the convergence-temperature is below the freezing-temperature will be obtained therefore not at the real freezing-temperature* t_o *but at the temperature* t', *which is more or less different from* t_o, *and the obtained temperature* t' *will be between the real temperature* t_o *and the convergence-temperature* t_g.

In order that the apparent freezing-temperature t' may be as near as possible to the real freezing-temperature t_o, $\frac{C}{C_{\prime}''}\frac{t_g - t'}{t' - t_{ov}}$ must also be kept, in this case, as small as possible;

the conditions are the same as when the convergence-temperature is above the freezing-temperature.

For the *freezing-point depressions* we have :—

$$t' = t_o + \frac{C}{C_{\prime}''}\frac{t_g - t'}{t' + t_{ov}} \qquad \text{(Water)},$$

$$t'' = t_o' + \frac{C}{C_{\prime\prime}''}\frac{t_g' - t''}{t'' - t_{ov}'} \qquad \text{(Solution)},$$

and

$$t' - t'' = t_o - t_o' + \left[\frac{C}{C_{\prime}''}\frac{t_g - t'}{t' - t_{ov}} - \frac{C}{C_{\prime\prime}''}\frac{t_g - t''}{t'' - t_{ov}'}\right]. \qquad (B'')$$

Since it has been found that $C_{\prime\prime}''$ in dilute solution can be put equal to $C_{\prime}''$, and conducting the overcooling so that $t' - t_{ov} = t'' - t_{ov}'$ or $t_o - t_{ov} = t_o' - t_{ov}'$, we have therefore also in this case

$$t' - t'' = t_o - t_o' + \frac{C}{C''}\left[\frac{(t_g - t') - (t' - t'')}{t' - t_{ov}}\right]$$

or

$$t' - t'' = t_o - t_o' + \frac{C}{C''}\frac{t_g - t_g'}{t_o - t_{ov}},$$

and we get for the freezing-point method, in this case, the same conclusions for dilute and concentrated solutions as before.

We have a further convenience here that we are able to determine the freezing-point of water and of the most dilute solutions with great simplicity and certainty *without* the necessity of *avoiding* the ice-cap :—As the apparent freezing-temperature in this case is below the real, and the ice can only melt above the real freezing-temperature, the separated ice can obtain the temperature of the liquid by conduction, and in reality does so. (See "ice-caps," "net-work of ice," "well-closed ice-caps," &c. in the paper "On the Determination of the Freezing-point in Dilute Solutions to ·4° depression," which will soon be published.) It is hardly necessary to add that the equilibrium, when the convergence-temperature is below the freezing-point, is also a *dynamic* one, and that overcooling by the arrangements of the experiment, as well as the separation of ice, continues to take place at equilibrium.

The principles laid down in this paper are of a general nature, i. e. *they give the leading points for the arrangements of the equilibrium of the heterogeneous system for all solvents or solutions and for all temperatures, whether we determine the freezing-point of water, of a metal, or of a liquefied gas.*

It is clear that there is no reason whatever to think that one method is better than the other because a ·001° instead of a ·01° or a platinum instead of a Hg-thermometer, &c. is used as long as we do not know *what* they measure. Moreover the study of the registering instrument itself cannot properly begin till a good equilibrium of the heterogeneous system itself has been secured, *e.g.* many of the so-called "properties" of the Hg-thermometer do not exist at all and come from the incorrect interpretation of the observed variations of temperature. For the determination of molecular depressions in very dilute solutions, a very fine and accurate registration is required. But at very high or very low temperatures, where a ·01° or ·001° error or more does not come much into consideration, the arrangements for accurate registration can be easily made, and the correct determination of temperature depends almost entirely upon how the equilibrium of the heterogeneous system has been established *. The convergence-temperature being here very far removed from the measured temperature, the observed results can easily be affected by very great errors. If the principles given above be carried as far as possible, very important advance will be made in these regions also.

Experimental Proof.

I. *Determination of* C *and* t_g *in the method I use.*

From $\frac{dt}{dz}=C(t_g-t)$ it follows that $C(z_2-z_1)=\log\frac{t_g-t'}{t_g-t''}$. The arrangements of the experiment, the quantity of liquid used (1250 c.c.), the stirring arrangements, &c., &c. have here been the same as in a freezing-point determination.

I investigated distilled water (the value of C can be put equal for water and dilute solutions). I used for the measurements a $\frac{1}{100}$° thermometer.

While measuring the velocity of overcooling, the cork must always be continuously tapped, the stirring must be strictly regulated and the experiment so conducted that the time required for the reading of the thermometer does not influence the obtained results considerably (I made one reading in each five minutes). The reading of the $\frac{1}{100}$° thermometer has been made to ·001°. Since the liquid is not only warmed by stirring but also by the warm air which is injected by it, the convergence-temperature depends not only on the number of movements of the stirrer in the unit of time, but also on

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* A more detailed theoretical discussion of the conditions necessary for accurate measurements at high temperatures will be soon given in one of the subsequent articles on the freezing-point method.

the temperature of the room (through this, the necessity of using a colder bath at a higher temperature of the room).

The convergence-temperature is, in the first instance, to be investigated at the temperature of the room at which we carry out our experiments. By determining the convergence-temperature under successive changes of the temperature of the ice-bath, we can get to the temperature of the bath at which the convergence-temperature is at the given temperature of the room very near to the freezing-temperature, *i.e.* to the condition when $t_g - t_0$ is small. The accurate determination of the convergence-temperature from measurements of the velocity of overcooling is by no means so easy as might at first glance be thought. Even if the greatest care be taken, we cannot determine it with much greater accuracy than to ·1°. We are obliged to make a series of readings: the falling of temperature must be so slow that the reading-error does not exceed ·001° or ·002°; we are therefore obliged to keep the ice-bath constant during a very long time. From Table I A. it will be seen that $C = \cdot 003°$, the velocity of cooling of my beaker is therefore only ·0003° per minute when $t_g - t = \cdot 1°$. Thus we come into regions where correct measurements of temperature even at considerable intervals of time are no longer possible and where observations are of no use, since no air-chamber can be arranged with greater accuracy than to ·01°, and a change of temperature of the air-chamber is followed by a still greater change of the convergence-temperature, i. e. *this has no constant value.* But the chief difficulty lies in getting the convergence-temperature from calculation of the observed results. This is the reason why we are unable to get $t_g - t' = 0$. Happily, as will be seen, we do not need to heap difficulties and expenses on the experimenter, and we are perfectly satisfied with knowing the convergence-temperature to ·1° or even ·3° as long as we do not require a greater accuracy than ·0001°.

The convergence-temperature we calculate thus:

$$\frac{dt}{dz} = C(t_g - t).$$

$$C(z_2 - z_1) = \log(t_g - t_2) - \log(t_g - t_1).$$

Again,

$$C(z_3 - z_1) = \log(t_g - t_3) - \log(t_g - t_1),$$

$$\therefore \frac{t_g - t_3}{t_g - t_1} = \left(\frac{t_g - t_2}{t_g - t_1}\right)^{\frac{z_3 - z_1}{z_2 - z_1}}.$$

This equation is of a higher order, but we can bring it to a very simple expression by making $\frac{z_3-z_1}{z_2-z_1}=2$. We get $(t_g-t_3)(t_g-t_1)^2=(t_g-t_1)(t_g-t_2)^2$ or

$$t_g=\frac{t_1t_3-t_2^2}{t_1+t_3-2t_2}.$$

Therefore we require to make readings at equal intervals of time: $z_2-z_1=z_3-z_2=z_4-z_3$ and so on. But the reading-error is here of great importance if the temperature-intervals are not very great. Because of this we have to use the greater temperature-intervals and to make many readings. Table I. gives the convergence-temperature as obtained from the first column of Table I A.

One is helped in obtaining the value of t_g by plotting out the results. In $\frac{dt}{dz}=\mathrm{C}(t_g-t)$, t becomes $=t_g$, when $dt=0$. Taking the dt as ordinates, the t as abscissæ, we get an almost straight line which cuts the axis of abscissæ at the convergence-temperature. The observations must be made not very far from the convergence-temperature, otherwise it may be determined very erroneously. In this way I obtained for a convergence-temperature the results given in Table I A.:—

TABLE I.

By calculation.	By plotting.
+1·17	+1·1
+0·54	+0·6
+0·45	+0·25?
−0·07	−0·1
−0·21	−0·1
−0·05	−0·03

Having the value of t_g, we calculate the value of C (see Table I A.). As almost all my experiments for over two years were carried out at a temperature of about 18–22°, I investigated the convergence-temperature at these temperatures of the room.

As we see, the lowering of the temperature of the ice-bath by ·1° is therefore accompanied by a lowering of the convergence-temperature by about ·15°. Passing from 40 to 45 stirring movements to 35 per minute and from temperature of room 21°·5 to 17°, we find, at same temperature of the bath of −·2°, that the convergence-temperature has fallen by about ·28°–·35°.

TABLE I A.

	Readings on the 1/100° thermometer above 0°.	Difference in 1/1000°.	Time, in minutes.	
1st SERIES. Room temperature, 21°·6 at the beginning; 21°·6 at the end. Ice-bath, −1°·55. Barom. 766·5 mm. 40–45 movements of stirrer in the minute.	1. 2°·044	35	5 min.	1–2 : c=0·00319
	2. 2·009	(34)	,,	2–3 : c=0·00322
	3. 1·975	31	,,	3–4 : c=0·00304
	4. 1·944	(29)	,,	4–5 : c=0·00295
	5. 1·915	29	,,	5–6 : c=0·00305
	6. 1·886	28	,,	6–7 : c=0·00305
	7. 1·858	27	,,	7–8 : c=0·00305
	8. 1·831	25	,,	8–9 : c=0·00292
	9. 1·806		,,	
The Convergence-temperature is at +1°·1.				C=0·0030 min.^{-1}
2nd SERIES. Room temperature, 21°·6; 21°·6. Ice-bath, −1°·8. Barom. 766·5 mm. 40–45 movements of stirrer in the minute.	1. 1·748	39	5 min.	1–2 : c=0·00300
	2. 1·709	37	,,	2–3 : c=0·00295
	3. 1·672	(34)	,,	3–4 : c=0·00280
	4. 1·638	34	,,	4–5 : c=0·00290
	5. 1·604	33	,,	5–6 : c=0·00291
	6. 1·571	33	,,	6–7 : c=0·00291
	7. 1·539	31	,,	7–8 : c=0·00292
	8. 1·508	30	,,	8–9 : c=0·00292
	9. 1·478		,,	
The Convergence-temperature is at +0°·6.				C=0·0029 min.^{-1}
3rd SERIES. Room temperature, 21°·6; 21°·5. Ice-bath, −2°·0. Barom. 766°·5. 40–45 movements of stirrer in the minute.	1. 1·355	41	5 min.	1–2 : c=0·00328
	2. 1·314	39	,,	2–3 : c=0·00324
	3. 1·275	38	,,	3–4 : c=0·00328
	4. 1·237	36	,,	4–5 : c=0·00323
	5. 1·201	34	,,	5–6 : c=0·00316
	6. 1·167	32	,,	6–7 : c=0·00309
	7. 1·135	31	,,	7–8; c=0·00310
	8. 1·104		,,	
The Convergence-temperature is at +0°·25?				C=0·0032 min.^{-1}

TABLE I A. (*continued*).

	Readings on the 1/100° thermometer above 0°.	Difference in 1/1000°.	Time, in minutes.	
4TH SERIES.	1. 0°·685	30	5 min.	1–2: $c=0·00338$
	2. 0·655	28	”	2–3: $c=0·00328$
	3. 0·627	27	”	3–4: $c=0·00328$
	4. 0·600	(25)	”	4–5: $c=(0·00316)$
	5. 0·575	25	”	5–6: $c=0·00328$
	6. 0·550	24	”	6–7: $c=0·00327$
	7. 0·526	23	”	7–8: $c=0·00325$
	8. 0·503	22	”	8–9: $c=0·00325$
	9. 0·481			$C=0·0033$ min.$^{-1}$

The Convergence-temperature is at $-0°·1$.

	Readings on the 1/100° thermometer above 0°.	Difference in 1/100°.	Time, in minutes.	
1ST SERIES. Room temperature, 17°; 17°·5. Ice-bath, $-1°·95$. Barom. 763 mm. About 35 movements in the minute.	1. 0·788	30	5 min.	1–2: $c=0·00228$
	2. 0·758	29	”	2–3: $c=0·00299$
	3. 0·729	28	”	3–4: $c=0·00298$
	4. 0·701	27	”	4–5: $c=0·00298$
	5. 0·674	(27)	”	5–6: $c=0·00308$
	6. 0·647	25	”	6–7: $c=0·00296$
	7. 0·622		”	$C=0·0030$ min.$^{-1}$

The Convergence-temperature is at $-0°·1$.

2ND SERIES.	1. 0·759	30	5 min.	1–2: $c=0·00339$
	2. 0·729	29	”	2–3: $c=0·00340$
	3. 0·700	(27)	”	3–4: $c=0·00330$
	4. 0·673	27	”	4–5: $c=0·00343$
	5. 0·643	25	”	5–6: $c=3·00330$
	6. 0·621		”	$C=0·0033$ min.$^{-1}$

The Convergence-temperature is at $-0°·03$.

TABLE II.

1st Series.—*Dichloracetic acid* (0·03312 norm.).

	Temperature of the ice-bath.	Read fr. temp. on the $\frac{1}{1000}$° therm.		Temperature of the ice-bath.	Read fr. temp. on the $\frac{1}{1000}$° therm.
Barom. at the beginning, 750·7 mm. Temp. of room, 20°. Overcooling, 0°·72.	−2°·2	0·2548			0·2563
		0·2548			0·2563 (4)
		0·2548			0·2563 (4)
		0·2550			0·2563
		0·2548			0·2563
		0·2550		——	——
				(−0°·4)	(0·2568
	−1°·6	0·2554			0·2569
		0·2554 (3)	The exper. lasted about 30 minutes.		0·2570
		0·2554			0·2570
		0·2554			0·2571
			Barom. at the end, 750·8 mm.		0·2572)
	−1°·1	0·2557			
		0·2557 (8)	Temperature of room, 20°.	(−0°·25)	(0·2578
		0·2557			0·2578
		0·2557 (8)			0·2578
		0·2557			0·2580
	−0°·6	0·2563	At end of exper. sufficient ice in the beaker.		0·2580
					0·2580)

2nd Series.—*Dichloracetic acid* (0·0502 norm.).

Barom. at the beginning, 750·5 mm. Temp. of room, 20°.	−2°·1	0·2009		−1°·1	0·2020 (1)
		0·2009 (10)			0·2020
		0·2009			0·2020
		0·2009 (10)		——	——
		0·2009		(−0°·4 to −0°·5)	(0·2024
		0·2009			0·2024
					0·2025
	−1°·6	0·2016	The exper. lasted ca. 20 minutes.		0·2024
		0·2016			0·2025
		0·2016 (7)	Barom. at the end, 750·5 mm.		0·2026
		0·2016			0·2026
		0·2016 (7)	Temperature of room, 20°.		0·2027
		0·2016			0·2027
	−1°·1	0·2020			0·2028
		0·2020	Very much ice in beaker.		0·2028)

II. *The Value of* C *in other Methods.*

A beaker with 1250 c.c. water of the temperature 9°·6 was placed in a large liquid-bath (water) of the temperature 9°·65.

The liquid in the beaker was stirred by the porcelain stirrer about 30 times per minute, and the temperature was measured by the fine ·01°-thermometer. The liquid-bath was stirred from time to time to get a uniform temperature. Hot water was introduced into the beaker till the temperature of the liquid rose about 1°·2, and the time required for the passing of the Hg-thread from one point of the thermometer-scale to the other was measured to $\frac{1}{5}$ second. I thus found for 1250 c.c. water in a liquid-bath, C to be about 20 times greater than in the air-bath. Now the velocity of cooling of a given quantity of a liquid is inversely proportional to its quantity or volume and directly proportional to its conducting surface. So *e. g.* 100 c.c. cool about 2·32 times quicker than 1250 c.c.

III. *The Determination of the Temperature of the Air-bath.*

This was determined and found to be about ·4° higher than that of the ice-bath. The measurements have been made in such a way that the $\frac{1}{100}$° thermometer was at first overcooled and allowed to rise in the air-bath to its maximum, then it was allowed to fall from a higher temperature to the air-bath temperature; thus the same values have been obtained for the temperature of the bath.

At the time when this paper was communicated to the Royal Society, I was only able to give the *minimum* values for the velocity of ice-melting and ice-separation. The reason was: the velocity with which my ·01°-thermometer assumed the temperature of the liquid was found to be 3° min.^{-1} when the Hg-thread falls, and 5° min.^{-1} when it rises. Again, when the thermometer was in a liquid in which ice separated or melted, about the same values of 3° min.^{-1} and 5° min.^{-1} were obtained. It followed from this that the velocity of ice-separation or of ice-melting expressed in degrees of temperature could not be smaller than 3° min.^{-1} and 5° min.^{-1}, but could be greater. I have since continued my investigations of the velocity of ice-melting and ice-separation.

IV. *On the Velocity of Crystallization of overcooled Liquids and Solutions.*

A beaker with 1250 c.c. of liquid is placed in a zinc vessel which serves as an air-bath. This placed in a liquid-bath, the temperature of which is regulated to the given temperature of the room, so that the cooling of the liquid stirred is so slow that for the time of the reaction it may be neglected, or only a small correction required. A very effective stirrer is required which does not allow the separated solid to settle on the beaker, but keeps its particles thoroughly mixed with the liquid. A very fine sensitive ·01°-thermometer is of great importance. I had a thermometer made for the purpose with a long bulb, so as to have a large surface in contact with the liquid; the glass of the bulb was made as thin as possible, and was protected from the stirrer by a cage. I investigated water, aqueous solutions, solutions of water and acetic acid. From the obtained results, I find that the equation

$$\frac{1}{t_o - t_{ov}} [\log n\ (t_2 - t_{ov}) - \log n\ (t_1 - t_{ov}) + \log n\ (t_o - t_1) - \log n\ (t_o - t_2)] = C'(z_2 - z_1)$$

holds good, where t_{ov} is the temperature to which the liquid was overcooled, t_o is the freezing-point, t_1 and t_2 the temperatures of the liquid at the times z_1 and z_2. From this we obtain for the velocity of reaction the differential equation $\frac{dt}{dz} = C'(t_o - t)\,(t - t_{ov})$, i. e. *the velocity with which an overcooled liquid crystallizes, while approximating to its freezing-temperature, is, at the given time* z, *directly proportional to the remoteness of its temperature from the freezing-temperature and to the total surface of the solid in contact with the liquid.* The equation, it is clear, can hold good only for *heterogeneous* systems and not for homogeneous, *i. e.* $t - t_{ov}$ cannot be put $=0$. The value of C' for $t - t_{ov} = 1°$, *i. e.* for one degree of overcooling, is found to be about 14 or 15.

V. *On the Velocity of Melting of the Separated Ice in the Warmer Water or Solution.*

The measurement of the velocity of melting of ice-cubes in water and aqueous solutions enabled me to show (1) that the equation $\frac{dt}{dz} = K_1(t_o - t)f(t)$ holds good, i. e. *the velocity of ice-melting is directly proportional to the remoteness from the freezing-temperature and to the surface of the ice in contact with the liquid:* (2) since the ice-cubes melt in water and dilute aqueous solutions with nearly the same velocity, provided the surface in contact with the liquid is the same, it follows that the value of K_1 can be assumed to be in all cases the same also when ice is separated from overcooled liquids. The velocity of melting of separated ice cannot be measured with any degree of accuracy, since in warming the liquid an essential part of the fine ice disappears; the value of t_h also cannot be determined with accuracy. Though there is no possibility of calculating the equation, we are able to ascertain the *minimum* value of K. The velocity with which my thermometer in all these measurements approached the freezing-temperature was in no case smaller than 6, consequently K is > 6.

There is an indirect way of determining K:

We have:

$$t' - t_o = \frac{C}{K} \frac{t_g - t'}{t' - t_{ov}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(I.)}$$

C is known; we arrange the convergence-temperature over the freezing-temperature; t_g is known. We overcool the liquid sufficiently and register t_{ov}. After the separation of the ice we get the apparent freezing-temperature t'; K and t_o remain unknown. After the freezing-temperature observed on the ·001°-thermometer has remained for two or three minutes constant, we change the temperature of the ice-bath and with it the convergence-temperature. We read the new apparent freezing-point, when it remains constant for two or three minutes. We have

$$t'' - t_o = \frac{C}{K} \frac{t_g' - t''}{t'' - t_{ov}}, \qquad \text{(II.)}$$

where t_o, C, K, t_{ov} are the same as in (I.), t_g', t'', C, t_{ov} are known, t_o, K unknown.

From (I.) and (II.) we get:

$$t' - t'' = \frac{C}{K(t_o - t_{ov})} [(t_g - t') - (t_g' - t'')]$$

or, since $t' - t''$ is very small,

$$t' - t'' = \frac{C(t_g - t_g')}{K(t_o - t_{ov})}.$$

Thus we obtain

$$K = \frac{C(t_g - t_g')}{t_o - t_{ov}} \cdot \frac{1}{t' - t''}.$$

These experiments can be carried out only in narrow limits of temperature, as we soon get to conditions when the freezing-point can no longer be obtained constant. In a good freezing-point method the value of $t' - t''$ is very small; so that the reading-error, even when it is, as in our case, ·0001° or ·0002° for each point, makes a very great part of the total value observed. In Table II. (p. 478) are given such observations. From them we get in a very rough way the result that the value of k is under my experimental conditions about 5 or 6. It may be remarked that we cannot use conditions different from those of the freezing-point method in order to get greater values of $t'-t''$, since under new conditions the value of K changes in different ways (the surface of the separated ice becomes different, &c., &c.) In our subsequent calculations we shall assume for K the *minimum* value found above, this being more reliable.

From the above experimental data it follows :—

A. *If the Convergence Temperature is above the Freezing Temperature.*

TABLE III.

Room Temperature.	No. of movements in the min.	Temperature of the ice-bath.	$\frac{C}{K}\frac{(t_g-t')}{t_o-t_{ov}}$ (at 1° over cooling).	$\frac{C}{K}\frac{(t_g-t_g')}{t_o-t_{ov}}$ at $t_g-t_g'=$ 0°·1.	$\frac{C}{K}\frac{(t_g-t_g')}{t_o-t_{ov}}$ at $t_g-t_g'=$ 0°·2.	$\frac{C}{K}\frac{(t_g-t_g')}{t_o-t_{ov}}$ at $t_g-t_g'=$ 0°·3
21·5°	40–45	−1·55	$\frac{0\cdot0031}{6}(-1^\circ\cdot1)$ *i. e.* −0°·00055	$\frac{0\cdot0031}{76}(0^\circ\cdot1)$ *i. e.* 0°·00005	$\frac{0\cdot0031}{6}(0^\circ\cdot2)$ *i. e.* 0°·0001	$\frac{0\cdot0031}{6}(0^\circ\cdot3)$ *i. e.* 0°·00015
"	"	−1·8	$\frac{0\cdot0031}{6}(-0^\circ\cdot6)$ *i. e.* −0°·0003	"	"	"
"	"	−2·0	$\frac{0\cdot0031}{6}(-0^\circ\cdot25)$ *i. e.* −0°·00012	"	"	"
"	"	−2·2	$\frac{0\cdot0031}{6}(-0^\circ\cdot1)$ *i. e.* −0° 00005	"	"	"
17–17°·5	ca. 35	−1·95	$\frac{0\cdot0031}{6}(-0^\circ\cdot1)$ *i. e.* −0°·00005	"	"	"
		−2·0	$\frac{0\cdot0031}{6}(-0^\circ\cdot03)$ *i. e.* −0°·000015	"	"	"

In the above experiments $\frac{C}{K}\frac{(t_g-t')}{t_o-t_{ov}}$, $\frac{C}{K}\frac{(t_g-t_g')}{t_o-t_{ov}}$, the value of C was assumed to be equal to 0·0031 (the average from the values 0·0030 min.$^{-1}$, 0·0029 min.$^{-1}$, 0·0032 min.$^{-1}$, 0·0033 min.$^{-1}$, 0·0030 min.$^{-1}$, 0·0033 min.$^{-1}$). For K, assuming the overcooling to be equal 1°, we put the *minimum value* 6. From the above table is to be seen that the difference between the real and apparent freezing-temperature $\frac{C}{K}\frac{(t_g-t')}{t_o-t_{ov}}$ obtains different values according to the temperature of the ice-bath, of the room, and to the number of stirring movements in the unit of time.

From the values $\frac{C}{K}\frac{(t_g-t')}{(t_o-t_{ov})}$ in the above table, it follows that the *absolute values* of the obtained freezing-temperatures can be easily made so accurate as to be affected by errors *smaller* than ·00005° to ·0001° if the convergence-temperature is determined with an accuracy to ·1° or ·2° and t_g-t' is ·1° or ·2°.

Again, from the values of $\frac{C}{K}\left(\frac{t_g-t_g'}{t_o-t_{ov}}\right)$ in the above table we have *that in the method I use the obtained freezing-temperatures for one and the same liquid by repetition of the experiment, as well as the real and apparent freezing-point depressions, cannot differ one from another by more than* ·00005° to ·0001°, *as far as the equilibrium in the* heterogeneous system is concerned, since $K>6$.

B. *When the Convergence Temperature is below the Freezing Temperature.*

TABLE IV.

Room Temperature.	No. of movements in the min.	Temperature of the ice-bath.	$\frac{C''}{C}\frac{(t_g-t')}{t_o-t_{ov}}$.	$\frac{C''}{C}\frac{(t_g-t_g')}{t_o-t_{ov}}$ when t_g-t_g' $=0°{\cdot}1$.	$\frac{C''}{C}\frac{(t_g-t_g')}{t_o-t_{ov}}$ when t_g-t_g' $=0°{\cdot}2$.	$\frac{C''}{C}\frac{(t_g-t_g')}{t_o-t_{ov}}$ when t_g-t_g $=0°{\cdot}3$.
21°·5	40–45	−22°	$\frac{0{\cdot}0031}{15}(-0°{\cdot}1)$ *i. e.* −0°·00002	$\frac{0{\cdot}0031}{15}(0°\ 1)$ *i. e.* 0°·00002	$\frac{0{\cdot}0031}{15}(0°{\cdot}2)$ *i. e.* 0°·00006	$\frac{0{\cdot}0031}{15}(0°{\cdot}3)$ *i. e.* 0°·00006
17–17°·5	ca. 35	−2°·0	$\frac{0{\cdot}0031}{15}(-0°{\cdot}1)$ *i. e.* −0°·00002			

It is to be seen from the above table that, if the convergence-temperature is by ·1°, ·2°, below the freezing-temperature, *the difference between the real and apparent freezing-temperatures is not greater than* ·00002°–·00004° as far as $\frac{C}{C''}\frac{(t_o-t')}{(t_o-t_{ov})}$ is concerned, and so also when the variations of the convergence-temperature are in a series of experiments only ·1°, ·2°, ·3°, *the obtained freezing-temperature for one and the same liquid, as well as the apparent and real freezing-point depressions, cannot differ from one another more than by* ·00002°, ·00004°, *or* ·00006°, as far as the equilibrium of the heterogeneous system is concerned, as C'' = about 15.

The method with the convergence-temperature below the freezing-temperature is to be preferred to that with it above, for the following reasons :—

(1) The ice-cap need not be avoided.
(2) Since C'' is essentially greater than K, we are able more easily to reduce the experimental error and the variations of temperature, as far as the equilibrium of the system is concerned.

I will now show how by the use of the results given above to calculate the experimental error in other methods. *1st example*: in a given method 70 c.c. liquid and a liquid-bath at −·3° are used, and the quantity of ice present in the liquid at equilibrium is equal to ·3° overcooling. Then we have :—The value of C is found for 1250 c.c., about 20 times greater if a liquid instead of an air-bath is used. For 70 c.c. C is 2·7 times greater than for 1250 c.c., *i.e.*, C is 54 times greater than mine. Again t_o-t_{ov} is 3 times smaller than mine. Therefore $\frac{C}{C''}\frac{(t_g-t')}{(t_o-t_{ov})}$ is $\frac{0{\cdot}003\,.\,54\,.\,0{\cdot}1}{15.0{\cdot}3}$ = about 0°·004 for $t_g-t'=0{\cdot}1$, 0°·008 for $t_g-t'=0^\circ{\cdot}2$, &c., *i.e.*, already about 20 × 2·7 × 3, or about 160 times greater than mine for the same value of t_g-t_g'. In the same way $\frac{C}{C''}\frac{(t_g-t_g')}{(t_o-t_{ov})}=$ 0° 004 for each 0°·1 of t_g-t_g', *i.e.*, the obtained apparent freezing-point of one and the same liquid by repeating the experiment as well as the apparent molecular depressions are already affected by errors of 0°·004 for each $t_g-t_g'=0^\circ{\cdot}1$, *i.e.*, 160 times greater than in my method for the same value of t_g-t_g'. We see how very easily the error in a freezing-point method may become great.

2nd example: in a given method the quantity of the liquid used or some other data, which enable us to calculate the values of C, t_g-t_g', t_g-t', are not given. Assume, the only data we have, are: that during 5 minutes of the experiment ·05 to ·15 per cent. ice separates, and that the quantity of the ice present in the liquid at equilibrium is $= {\cdot}3^\circ$. From the equation $t' = t_o + \frac{C}{C'} \frac{(t_g - t')}{(t_o - t_{ov})}$, it follows that *when* $t_g - t' = 0$, $t' = t_o$, and $C(t_g - t') = 0$, *i.e.*, no cooling by experimental arrangements, *no further separation of ice, takes place.* Therefore the separation of ice first of all shows that $t_g - t'$ is not equal 0, and that the convergence-temperature is in the given method below the freezing-temperature. After this we can easily calculate the *comparative* error in the methods. In my method $C = 0^\circ{\cdot}003$, keeping $t_g - t' = 0^\circ{\cdot}2$, $C(t_g - t') = 0^\circ{\cdot}006$, in 5 minutes, $0^\circ{\cdot}003$. To one degree overcooling, a quantity of separated ice $= 1{\cdot}25$ per cent. corresponds. This gives about $0^\circ{\cdot}004$ per cent. separated ice in my method during 5 minutes. In the given method the experiment lasts 5 minutes, the value of 0·05 per cent. to 0·15 per cent. shows that the total value of $C(t_g - t')$ in the given method is about 12 to 36 times greater than in mine. Now $t_o - t_{ov}$ is three times smaller than in mine, therefore $\frac{C}{C''} \frac{(t_g - t_g')}{(t_o - t_{ov})}$, *i.e.*, the difference between the real and apparent freezing-points, is from 36 to 110 times greater than mine, if my $t_g' - t' = 0^\circ{\cdot}2$. Again, the fact that the quantities of separated ice in two different experiments were 0·05 per cent., and 0·15 per cent. $=$ 0·1 per cent. difference, shows that the value of $\frac{C}{C''} \frac{(t_g - t_g')}{(t_o - t_{ov})}$, *i.e.*, the apparent freezing-point of one and the same liquid obtained by repeating the experiment, and also the apparent freezing-point depressions, are affected by errors 150 times greater than in my method, if my $t_g - t_g' = 0^\circ{\cdot}1$, &c.

CONCLUSION.

The quintessence of this paper is to show that, assuming all the investigators had an absolutely perfect instrument for registration of temperature, and that other sources of error, which have nothing to do with the method, do not exist, the results obtained by the different methods are still affected by errors according to the conditions of the established equilibrium. Neither does a constant apparent freezing-temperature give us any evidence as to the absolute value of the obtained freezing-temperature, nor does a depression obtained from two observed constant apparent freezing-points as to the correct value of the depression we ought to get. A repetition of an experiment under the same conditions must lead to nearly *the same result, but this does not give any assurance as to its correctness. Only the investigation of the values* $C(t_g-t)$, t_0-t_{0v}, C'', K, *gives us information as to how far our results are reliable.* As we see from the above, the correct establishment of the equilibrium is the most important side of the question, since otherwise we run easily into several thousandths or even hundredths of a degree experimental error, with methods which at first sight seem careful and accurate, and even where we do not work with very high, or very low, temperatures. This quite accounts for the discrepancies which we find between the observations of different investigators for the same substances.

June 1897.

LX. *The Secular Softening of Cold Hard Steel, with Remarks on Electrical Standards. Second Report.* *By* CARL BARUS[*].

TWELVE years ago I began a series of experiments on the gradual softening of the temper of glass-hard steel in the lapse of time, at ordinary atmospheric temperatures. The results which I published[†] in 1888 covered an interval of but three years, so that with the present data I am able to add materially to the inferences previously drawn. The work was originally undertaken with a view to its bearing on the construction of permanent magnets. Prof. V. Strouhal and I had shown[‡] that the breakdown of chemical configurations during the tempering of hard steel is accompanied by an accentuated breakdown of the magnetic configurations which may be present. In spite of its high coercitive force, glass-hard steel is therefore unsuitable for use when (as in all instruments) the magnets are to preserve exceptional permanence in the lapse of time. I was induced to resume the present measurements in consequence of certain discouraging results mentioned in a paper by Profs. J. Viriamu Jones and W. E. Ayrton[§], in which the discrepancies encountered from the molecular changes of resistance-standards were serious enough to vitiate the results of refined measurement. It appeared from this that the laws subject to which the temper of any metal (whether imparted by sudden cooling or mechanically) changes with time and temperature are not generally known.

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[*] Presented to the U. S. National Academy of Sciences. Communicated by the Author.

[†] This Magazine, [5] xxvi. p. 397 (1888).

[‡] Bulletin U. S. Geological Survey, No. 14, p. 151 *et seq.* (1885); or Wied. *Ann.* xx. p. 662 (1883).

[§] Paper read at the Toronto Meeting of the British Association, 1897, on the determination of the ohm &c.

By way of introduction I will therefore compute a table of the resistances of tempered steel from data obtained by Prof. Strouhal and myself seventeen years ago *, as it will aid materially in the interpretation of the present results.

2. Steel is a well chosen metal for the purpose, since its electrical resistance may by the process of tempering from soft to hard be made to increase more than *threefold*. One is thus able to trace the secular variations on this enormous scale with certainty, and without reference to concomitant secular changes of the resistance of the german-silver standards with which the steel rods are compared. The resistance of these standards cannot have varied more than a fraction of 1 per cent., whereas the steel rods show a change of about 15 per cent. during the interval of observation. Of course the same standards and the same steel rods were used throughout. For details as to their mass constants, homogeneity, methods, &c., the reader may consult the previous paper (*l. c.*).

The effect of a drawn strain imparted to a metal on its electrical resistance is fortunately of very much smaller order as compared with temper in steel. Nevertheless where electrical standards are in question these small changes are a serious discrepancy. Much of this can be removed by judicious treatment I dare say, since I showed that the mechanical strain or any molecular change varies in the lapse of time and with temperature under like conditions with temper in steel †.

3. The following Table I. contains the results for the change of the specific resistance, s_0, of originally glass-hard Stubbs' steel rods with time and temperature. These data are the means of closely accordant values of three rods for each temperature. Instead of reducing the values for resistance directly, the object in view was more easily attained by using the thermo-electric chart which Prof. Strouhal and I constructed, inasmuch as we showed that the thermo-electric power of steel is subject to similarly phenomenal variations with temper; and that the relation of the thermo-electric power of steel referred to silver and the specific electrical resistance of steel (over ninety rods were examined) is expressed by the equation

$$y=15{\cdot}176-0{\cdot}4123\chi,$$

where y denotes thermo-electric power in microvolts and

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* Wied. *Ann.* xi. p. 930 (1880); Bull. U. S. Geolog. Survey, No. 14 p. 55 (1885).

† This Magazine, Feb. 1889, p. 155; Bulletin U. S. Geolog. Survey, No. 94, p. 17 *et seq.* (1892).

degrees centigrade relative to silver, and χ the specific resistance of steel (microhms, cubic centim. 0° C.), so that 15·176 would be the thermo-electric power of a steel rod whose resistance is zero.

Table I.—Change of Specific Resistance of Glass-hard Stubbs' Steel Rods with the Temperature and Time of Tempering.

Temperature. ° C.	Time. Minutes.	s_0. Microhm, 0° C.	Temperature. ° C.	Time. Minutes.	s_0. Microhm, 0°C.
60	0	40·8	185	0	39·1
	60	39·4		10	26·7
	120	38·8		30	25·4
	180	38·3		60	24·7
				120	24·3
100	0	39·0		180	23·7
	10	36·4			
	30	34·5	330	0	37·7
	60	33·2		1	17·2
	120	31·8		30	16·8
	180	31·1		90	16·7
			1000	...	15·0

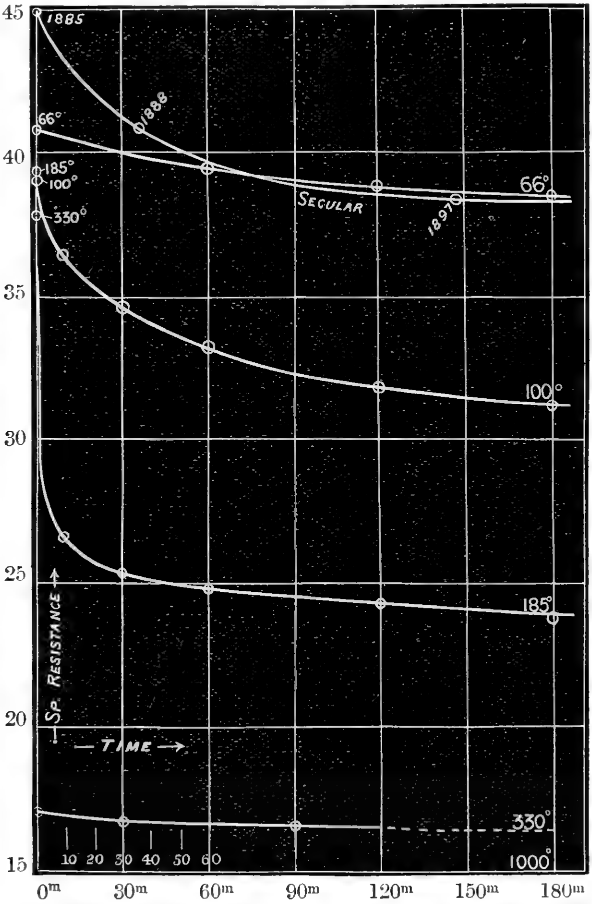
Thus in Table I., s_0 is the specific resistance of glass-hard steel, exposed to the temperature stated in the first column, continually during the time t in minutes given in the second column. The value s_0 is reduced to zero centigrade.

The results are graphically given in the accompanying chart, where the abscissas are minutes of time of exposure, and the ordinates the corresponding resistances s_0, when the rod is tempered at the temperature marked on the curve.

Clearly these curves (all exponential in character) are the expression of a single law giving the relation of resistance to time and temperature; but with the investigation of this function I am not further concerned here. The features to be noted are the occurrence of a single and definite degree of soft temper for each temperature. This limit of softness is approached more rapidly in proportion as the temperature to which the hard rod is continually exposed is higher.

4. In correlative experiments Prof. Strouhal and I further showed that a hard rod which has been tempered to the limit of softness for any temperature, say the boiling-point of water, is then molecularly stable at a lower temperature, and this in a more marked degree in proportion as the latter temperature falls below the temperature at which the rod is

tempered. So much of the excess of glass-hard temper vanishes even at 100°, while the rod retains hardness and coercive force, that we based our method of making permanent magnets* on these observations. It is upon the



Curves 66°, 100°, 185°, 330°, 1000° show the reduction of the specific resistance of steel due to continued tempering at these temperatures. The abscissas are minutes of exposure.

Curve "secular" shows the corresponding secular softening of steel. The abscissas are months.

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* Temper the glass-hard steel rod to the limit for 100°; then magnetize to saturation; finally temper the magnet to the magnetic limit for 100°.

TABLE II.—Resistance per linear centim., r_t, and specific resistance s_0, of cold glass-hard steel in the lapse of time. Effective lengths of rods about 23 centim.

No.	Diameter $\times 10^3$.	Resistance, r_t, and Temperature, t.						Specific Resistance at 0° Centigrade.				
		r_t June 1885.	t.	r_t July 1888.	t.	r_t Sept. 1897.	t.	s_0 June 1885.	s_0 July 1888.	δs_0.	s_0 Sept. 1897.	δs_0.
	centim.	microhm.	°C.	microhm.	°C.	microhm.	°C.					
1.	80·6	9260	27·5	8460	28	7960	23	46·0	41·7	−4·2	39·5	−6·5
2.		9250	27·8	8430	28	7910	23	46·0	41·7	−4·3	39·2	−6·8
3.		8870	28·3	8010	28	7480	23	43·7	39·3	−4·4	36·8	−6·9
4.		8810	27·0	7950	28	7440	23	43·8	39·3	−4·4	37·0	−6·8
5.		8840	27·2	8010	28	7470	23	43·7	39·4	−4·3	36·8	−6·8
6.		9260	26·8	8460	28	7960	23	46·1	42·0	−4·1	39·5	−6·6
7.		9200	27·0	8400	28	7870	23	45·8	41·5	−4·3	39·0	−6·8
8.		9260	26·3	8510	28	7960	22·5	46·0	41·9	−4·1	39·3	−6·7
9.		8720	26·3	7920	28	7430	22·5	43·6	39·2	−4·3	37·0	−6·6
10.		8950	25·1	8170	28	7670	22·5	45·1	40·8	−4·3	38·4	−6·7
Mean	80·6	9042	26·9	8232	28·0	7717	22·5	44·98	40·68	−4·30	38·25	−6·73
11.	44	30670	25·2	27860	29	26130	22	45·1	40·5	−4·6	38·2	−6·9
12.		30320	25·3	27710	29	25950	22·5	44·4	40·3	−4·2	37·9	−6·5
13.		30620	24·8	28000	29	26200	22	45·0	40·7	−4·3	38·3	−6·7
14.		30080	25·5	27490	29	25720	22	44·1	40·0	−4·1	37·6	−6·5
15.		30990	26·8	28400	29	26560	22	45·4	41·3	−4·1	38·8	−6·5
Mean	44	30536	25·5	27892	29·0	26112	22·1	44·80	40·56	−4·24	38·16	−6·64
16.	126·5	3400	26·4	*2979	29	2767	22	41·0	*35·4	−5·6	33·2	−7·8
17.		3382	26·4	2972	29	2757	22	41·0	35·5	−5·5	33·2	−7·8
19.		3643	26·7	3248	29	3037	22	44·3	39·2	−5·1	36·7	−7·6
20.		3427	27·2	2995	29	2787	22	41·5	35·8	−5·7	33·6	−7·9
Mean	126·5	3463	26·7	3048	29·0	2837	22·0	41·95	36·46	−5·49	34·18	−7·77

* An error in the preceding report of these data is herewith corrected.

same principles that a method for constructing invariable electric standards of resistance will have to be based, though the desideratum of hardness and resilience (whatever metal may be chosen) is not of special moment here.

5. Returning from this brief digression to the special results of the present paper, I will refer to Table II., in which the softening effect of atmospheric temperature acting through prolonged time is exhibited. Twenty rods (No. 18 broken) of different diameters (·0806 centim., ·044 centim., ·126 centim. in the different batches) were selected. Columns 3 to 8 show the resistances, r_t, in microhms per linear centimetre, at the temperature t° and on the dates stated at the heads of the columns. Columns 9–13 give the corresponding specific resistances, s_0, in microhms between opposed faces of a centim^3., at zero centigrade, for the same dates. The reduction to zero was made by aid of the relation investigated for steel*

$$(s+\cdot 78)(\alpha-\cdot 0001435)=\cdot 0682,$$

wherein an essential dependence of the temperature-coefficient α_0 and corresponding specific resistance s for any state of temper is evidenced. Finally, the columns δs_0 denote the resistance decrements since the first measurement (1885). This was made only a few days after hardening in the electrical apparatus†. It will be seen that there is a fair order of agreement between the corresponding values of δs_0, quite as much as one may expect from the small resistances and the long interval between measurements.

The table also contains *mean* values for each group, showing that the thickest rods seem at first to have softened more rapidly than the thinner rods. The mean values of s_0 are inscribed in the chart forming the curve marked "secular," the ordinates being specific resistances as above, and the abscissas the time elapsed since hardening, in *months* instead of minutes. It will be seen that the present rods (from a perfected mode of tempering) are initially much harder (greater s_0) than the rods used in studying temper. I do not as yet know what effect the initial hardness‡ has upon the limiting softness of a rod tempered at any temperature. I can merely note therefore that atmospheric temperature acting

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* Bulletin U.S.G.S. No. 14, p. 15, *et seq.*; Wied. *Ann.* xx. p. 525 (1883); furthermore, Bull. U.S.G.S. No. 54, p. 160 (1889); American Journal, xxxvi. p. 427 (1888).

† Bulletin U.S.G.S. No. 14, p. 29; Bull. U.S.G.S. No. 73, p. 7 (1891).

‡ For the same steel one would expect this effect to be *nil*; but the experiments have not been conclusive.

for a period of 180 *months* has produced about the same degree of hardness as 66° acting for the period of 180 *minutes*. One may infer with certainty, however, that 100° C. acting for a period of minutes produces much greater softening than atmospheric temperature acting for a like period of months. In order to estimate the secular change of a resistance-standard of any metal (*cf.* § 2), one should previously expose the wire to the tempering effect of 100°, for a number of hours. One may then estimate *the secular change of resistance of the untempered standard for any period of months to be very much less than the change of the tempered wire* (100°) *for a like period of minutes*. Two hours of annealing at 100° would thus furnish the superior limit of change of the resistance-standard within ten years. The same rule is applicable if the resistance-standard (as is preferable) is made of wire tempered at 100° or more, as far as the limiting degree of softness*. This tempering should be applied even after coiling, in order that all molecular instability may be wiped out as far as 100° at least.

6. I will conclude with a word on the causes of the changes of temper set forth in the chart.

The simplest explanation would assume that in hardening a strain has been imparted to the steel rod. This strain vanishes through infinite time more rapidly in proportion as the viscosity of the metal is more fully reduced by temperature. At each temperature, therefore, there is a limiting or residual strain corresponding to the reduced value of the viscosity of the rod at this temperature.

Again, in consequence of the occurrence of recognizable chemical *débris* within the rod as the result of tempering, we may interpret these curves with reference to Wilhelmy's law, and conclude that they are an expression of increased rate of reaction or of chemical decomposition with increased temperature. There is some difficulty, however, in assuming that the limiting softness for each temperature is a case of chemical equilibrium: for whereas the hard molecule during tempering passes easily into the soft molecule, it is not probable that the soft molecule can at low temperatures again change into the hard molecule.

If, however, the occurrence of the hard molecule is conditioned by the strain imparted to steel by sudden cooling, and vanishes contemporaneously with this strain, the explanation in terms of the viscosity of the metal is again at hand, and is now in such shape as not to ignore the chemical

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* Tempered at too high a temperature, the wire may receive fresh strain in cooling.

processes within the metal. Since, however, the viscosity of steel *decreases* in marked degree from soft to hard* one would first have to show that the viscosity in cold hard steel is greater than the viscosity in the hot tempered steel, in any comparison.

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LXI. *The Causes of Osmotic Pressure and of the Simplicity of the Laws of Dilute Solutions.* *By* WILLIAM SUTHERLAND†.

WHILE the new school of physical chemists has been busy pushing on the theory of dilute solutions to all its available conclusions, physicists have been more concerned about the secure establishment of the premises on which all these conclusions are based. The feeling of physicists has been well voiced by FitzGerald in the last Faraday lecture at the Chemical Society of London, and Poynting's paper on "Osmotic Pressure" (Phil. Mag. 1896, October) is one attempt at securing a sound physical foundation for the theory of solutions. The present paper is another. The reasons usually assigned for the remarkable fact that the laws of the osmotic pressure of a solute in a solvent are the same as if the solvent were annihilated and the solute left as a perfect gas, are quite inadequate, because they would apply to any selected lot of the molecules of the solvent, and therefore to all the molecules, with the result that the laws of a perfect gas would have to hold for all fluids down to and in the liquid state. The essence of the whole matter lies in this :—Why should the laws of gases hold for the osmotic pressure of the solute, when the solvent is in a condition involving enormous departure from the laws of gases? The answer is to be found in a closer study of the semipermeable membrane. If we seek to picture to ourselves how a membrane allows water molecules to pass, but not sugar molecules, our simplest conception of its structure is that of a mesh, amidst the threads of which the water molecules are packed in such a manner as to give way before one another almost as in ordinary water, while the sugar molecules are held back by the mesh. Thus the mesh forms a solid or quasi-solid framework through which water can pass with high viscous resistance, while the sugar molecule is absolutely blocked. Now if the framework turns back the sugar molecules, it must take the force of their

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* Barus, Phil. Mag. [5], xxvi. p. 183 (1888); *cf.* 'Nature,' xli. p. 369 (1890).

† Communicated by the Author.

blows and shield the water molecules from them. If then we suppose a semipermeable membrane separating water and a dilute solution of sugar in water, the sugar molecules are to be regarded as replacing some water molecules, but their collisions on the water in the membrane are rendered inoperative by the shielding action of the framework, so that the water molecules in the membrane receive more impacts on the side of the pure water than on the side of the solution, and therefore water flows through the membrane, until in the solution there is enough excess of hydrostatic pressure established to compensate for the inoperative impacts of the sugar molecules; this inequality of pressure which can be hydrostatically balanced is the osmotic pressure. Next as to its laws. Let us find the number of molecules that in a second cross unit area of a plane in a collection of molecules. We know that in the case of a gas with n molecules per unit volume and mean velocity v the number is $nv/6$. Next consider a number arranged in cubical order, the edge of each cube being d, and each molecule oscillating through a distance a smaller than d in a fixed direction parallel to one set of edges; then a molecule crosses a plane within distance a of its central position $v/2a$ times a second in the same direction, and a plane at greater distance than a no times a second. If then a plane is placed at random at right angles to the vibrating molecules its chance of being traversed by molecules is a/d, therefore an area d^2 similarly placed at random will on the average be crossed $(v/2a)(a/d)$ times a second; so that unit area is on the average crossed in the same direction $v/2d^3$ times a second by molecules, that is to say, $nv/2$ times. If the molecules were vibrating in all directions and we assumed that a third of them vibrated in any direction, and the remaining two thirds in directions at right angles to it, we should then have the average number of times unit plane is crossed as $nv/6$, the same as in the case of the perfect gas. We can easily see after these two special cases that the result is general, no matter how crowded the molecules may be, so long as the time of collision of a molecule may be neglected; because as each molecule keeps moving on with the same average velocity, if it is diverted by collision with another, then on the average the only effect of collision is to alter the actual path of each molecule, but not the average space traversed: thus in any collection of molecules moving at random, whether as compact as in a liquid or as free as in a gas, the number of times a unit plane is crossed by molecules per second is $nv/6$.

If then our unit area is taken on the surface of the semipermeable membrane in contact with the dilute sugar solution, each molecule of sugar in being turned back receives momentum $2mv$, so that the whole pressure taken by unit area of the meshes of the membrane is $nmv^2/3$, where n is the number of molecules of sugar per unit volume. Thus for the osmotic pressure we have arrived at exactly the same expression as for the gas pressure of the sugar molecules if the water were annihilated and the sugar remained as a perfect gas. By the nature of the argument, we are not involved in any difficulties in the way of its leading us to assert Boyle's law as holding down to and in the liquid state. We assert that what holds down to and in the liquid state for any fluid is that the number of times a unit plane is crossed in the same direction by molecules per second is $nv/6$, and as the result of this and of the property of semipermeability the identity of the laws of osmotic pressure with those of perfect gases comes to pass, notwithstanding that the complete equation for the pressure of the pure water of the dilute sugar solution involves all the complications of collisions and powerful attractions with absolute departure from Boyle's law and Charles' law.

The reason why we cannot assert that a liquid having n molecules to unit volume exerts a pressure $nmv^2/3$ on the solid wall containing it because of collisions, is that a unit area of the wall cannot be regarded as a random unit area, but a very special one with the special condition that no molecule can pass it. Now the meshes of our semipermeable membrane in turning back the molecules of sugar give the unit plane containing them a special property, but when the solution is dilute and the numerous water molecules pass freely, the meshes cause what is only a slight derangement from the conditions of a random plane, and in our first approximation we have neglected this derangement on account of its slightness. Molecules of sugar can be turned back by molecules of water which project beyond the plane of the mesh by any fraction of their diameter.

As the surface of a solid wall bounding liquid cannot be treated as an average plane we have to take a special method of calculating the total collisional pressure on it, as for instance that in "The Kinetic Theory of Solids" (Phil. Mag. [5] xxxii.). The essential distinction then between a semipermeable membrane and a solid wall in relation to a dilute solution is that while the membrane has special relation to only the few dissolved molecules, the solid wall has a special relation to all the solvent as well as the dissolved molecules.

A derivative result of some importance from our argument is the identity of the mean kinetic energy of the molecules of liquids and gases at the same temperature; for it is involved in the result that the equation for the osmotic pressure of the dissolved sugar molecules comes out as if they were in the gaseous state: many lines of investigation have already made this identity tolerably certain, but none so simply and clearly as van't Hoff's discovery of the identity of the osmotic pressure laws with those of perfect gases.

Van't Hoff has shown how, given semipermeability, it is possible by thermodynamics to deduce the laws of dilute solutions in a simple manner; the difficulty has been to get a conception on molecular grounds of the rationale of osmotic pressure and the simple laws of dilute solutions. But with a direct molecular explanation for osmotic pressure we can then adopt van't Hoff's thermodynamical extensions and treat the whole matter as placed on a molecular basis; but for the sake of the illustration we will essay a direct molecular explanation of another typical simple law of dilute solutions, namely, that of the lowering of vapour-tension.

In a gas the average velocity and the average distance apart of the molecules are such that, although molecules are continually approaching quite close to one another, they seldom do so under conditions which would cause a pair to remain within finite orbits relatively to one another; even when two molecules collide, the relative path after the collision is generally part of a curve extending to infinity, just as the paths of comets are sometimes hyperbolas; the parabola is a possible orbit for a comet, and marks the transition from the infinite hyperbola to the finite ellipse of certain comets and the planets. So in the case of a gas which has molecular velocity lowered by cooling to the critical temperature, and mean distance apart reduced by compression up to the critical pressure, the relative orbits of colliding molecules become restricted to a finite range, one pair of molecules that have a hold on one another entangles another similar pair, and so on; the conditions are those of incipient liquefaction. When the temperature is lowered, liquefaction becomes decided, because of the more decided entanglement, and gravitation separates out the entangled portions into a continuous mass with horizontal surface, leaving more space to those which had high enough velocities to escape entanglement, and therefore still further increasing their chances of freedom, which they enjoy as vapour in contact with the liquid. Solidification occurs when each molecule is imprisoned by its neighbours, or if on the surface, is so strongly attracted that it cannot escape. But suppose a liquid surrounding a solid, then if the molecular attraction between solid and liquid is such that the potential energy becomes less when a molecule of the solid is surrounded by those of the liquid than by its own sort, the conditions are favourable for the escape of the surface-molecules of solid, that is to say, the solid dissolves in the liquid and its molecules share the relative freedom of those of the liquid. Now the vapour-pressure of most solids at ordinary temperature is so low that it has not been measured, that is to say, that the molecular attraction between their molecules is so great that very few can retain their freedom; for instance, if common salt were vaporized by heat and then suddenly cooled to ordinary temperature, the molecules would capture one another so thoroughly that almost a vacuum would be created by the precipitation of the solid salt from the space filled with vapour. If, then, we try to picture what occurs at the surface of a solution where the solvent is free to evaporate into a region that ultimately becomes saturated, we must imagine the molecules of solute to be behaving practically as kinetic substitutes for those of the solvent which they displace, and participating with them in chances favourable to escape at the surface. Now, in the vapour the molecules of solvent that approach one another have speed enough also to escape from one another, whereas the strongly attracting molecules of solute freely capture one another (or possibly sometimes the molecules of solvent) with immediate entanglement and prompt return to the liquid mass, so that the region of evaporation and condensation of the solute is a very limited one near the free surface. If there are n molecules of solute to N of solvent, then the difference between a surface of the pure solvent and one of the solution is that for every $n+\mathrm{N}$ molecules of solvent liberated at the pure surface, there are N of solvent and n of solute set free at the surface of the solution; the latter capture one another very promptly, just as they would do if present alone, and return to the solution, while the N solvent molecules pass on to the region of vapour. Thus equilibrium at the two surfaces will be maintained in the same manner, namely, when for every $\mathrm{N}+n$ molecules that leave either, $\mathrm{N}+n$ return; but in the case of the pure solvent the vapour has to return $\mathrm{N}+n$ molecules for every N returned by the vapour above the solution, that is to say, the vapour-pressure over the solution is to be less than that over the pure solvent in the ratio of N to $\mathrm{N}+n$, which of course is the fundamental experimental law whereby molecular masses are determined from lowering of vapour-pressure*.

[* An essentially similar explanation of the effect of dissolved substances in lowering the vapour-pressure of solvents was given by Mr. F. G. Donnan in this Magazine, vol. xxxiv. pp. 411-414 (1892).—Ed., Phil. Mag.]

Thus a direct molecular explanation of the molecular lowering of vapour-pressure in solutions is available, and might be made the starting-point of the molecular theory of solutions.

It may be urged against the theory here outlined that it makes no provision for the facts of the "dissociation" of salts in watery solution, so beautifully discovered by Arrhenius, and worked out so consistently by Ostwald and the other enthusiasts of the new Physical Chemistry; but it is easy to provide for the facts of dissociation by stipulating that the word molecule is to include any mass whose translational kinetic energy is on the average that of a molecule of a gas at the same temperature. The electrically charged ion, if it is really a separate kinetic entity, must, according to the kinetic theory, be a molecule under the above stipulation; and thus the requirements of the dissociation theory of watery solutions of salts are met. Of course we should expect such dissociated ions as Cl and Na to be very mobile, and to escape with ease from the surface of a solution of NaCl; but as soon as they get free from the liquid water which is the cause of their dissociation, they will combine with avidity to form NaCl, which then returns at once to the solution in the manner described above. Thus dissociation presents no difficulty in this connexion. Evidently the very marked dissociating power of water is associated with its generally exceptional character; if water were not exceptional, it ought to be a somewhat difficultly liquefiable gas, whereas its high molecular force constrains it to liquefy and so produce a dense aggregation of swiftly moving molecules; that these should tear apart the atoms of salts with their known electrical contrast is a strange but fruitful discovery in physical chemistry. Whatever the actual cause of the dissociating power of water may be, its existence has been established almost beyond cavil, and offers no difficulty in osmotic theory, as here presented.

Such interesting cases of association as those investigated by Prof. S. Pickering would, according to the theory above outlined, need to be treated, each on its own merits, as belonging to those exceptions which, accepted grudgingly at first, are ultimately found to afford the best opening into new regions of law.

Melbourne, Sept. 1897.

LXII. *Concerning Accidental Double Refraction in Liquids.*

By J. E. ALMY, *A.M., Fellow in Physics, University of Nebraska, Lincoln, Nebraska, U.S.A.*[*]

THE question whether the condition of strain which occurs in a mass of liquid, deformed by friction, gives rise to double refraction of light, as does such condition in the case of solids, has frequently been raised.

Mach [†], Maxwell [‡], Quincke [§], and Kundt [||] have made observations of the phenomenon in liquids of different types: Mach with extremely viscous substances, semi-solids; Maxwell with Canada balsam; and Kundt with the more nearly fluid liquids, such as oils and solutions of gelatine, glue, collodium, and balsam.

Of Kundt's conclusions the following are significant:—

1. "The magnitude of the internal friction (viscosity) of liquids is not primarily sufficient to determine the occurrence of double refraction."

2. The liquids and solutions which give double refraction are oils, or belong to that class of substances known as colloids.

3. Solutions of crystalline substances, without exception, give no double refraction. "However, it is not to be inferred," he adds, "that with improved means of investigation, the effect may not be obtained with solutions of crystalline substances."

It has been the purpose of this work to find whether double refraction may be shown in liquids not oils, or with solutions of crystalline substances. And, while double refraction has not been obtained in either case, it has been possible to extend the limit of accuracy with which the subject has been tested by a factor of over 3×10^2 in case of water, and by 10^2 in the case of solutions of crystalline substances.

Apparatus.

The apparatus used for producing the condition of strain consists of two brass cylinders, each 10·8 cm. in diameter, 15 cm. in length, mounted the one above the other, their axes parallel, the surfaces of the two being about 0·4 cm.

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[*] Communicated by the Author.

[†] Mach, *Optic-Akustisch Versuche.* Prag, 1873.

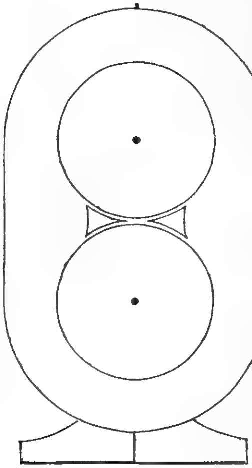
[‡] Maxwell, Proc. Roy. Soc. xxii. p. 46 (1874).

[§] Quincke, Wied. *Ann.* x. p. 536 (1880).

[||] Kundt, Pogg. *Ann.* cliii. p. 10 (1874); and Wied. *Ann.* xiii. p. 110 (1880).

apart at the nearest point. In the widening space between them two concave wedges are placed in such way as to prevent the formation of eddies or vortices in the part of the liquid to be examined optically. The cylinders are surrounded by a cast-iron chest, which was filled with the liquid to be examined. The axles of the cylinders pass through packings to bearings outside; packings and bearings are adjustable, so as to admit of variation of the distance between the cylinders. A 4-H.P. electric motor furnished power for running the apparatus. The beam of light, from an oxy-hydrogen calcium light, passes through the polarizing Nicol and Lippich of the polariscope, through the liquid between the cylinders, parallel to the axes of the cylinders, and out through the analyser. The polariscope, a "Lippich half-shade" instrument, has an adjustable Lippich so that the amount of light admitted can be varied according as the liquid is more or less opaque.

Cross-section of Apparatus. Scale 2 : 10.



Theory of the Experiment.

According to the development by Stokes * for the case of two concentric cylinders immersed in a liquid, the inner one rotating, the outer one fixed in position, there is on each element of volume of the liquid between the cylinders a shearing stress, tangential to the cylinder. The magnitude

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* Stokes, Math. and Phys. Papers, i. p. 102 (1880), or Trans. of the Cambridge Soc. viii. (1845).

of the force on one face of unit cube is proportional to the expression

$$\frac{2av}{2ad+d^2}\cdot\left(\frac{a+d}{r}\right)^2,$$

in which

v is the velocity of the moving cylinder ;
a is the radius of that cylinder ;
d is the distance between cylinders ;
r is the distance of the face considered from the axis of the cylinder.

In the case of the apparatus used, the concave wedges take the place of the fixed cylinder. There is, midway between the two cylinders, an infinitesimal layer of the liquid which has zero velocity ; hence, the distance between the fixed and the moving surface is $d/2$, and the stress on unit area tangential to the cylinder is proportional to

$$\frac{2av}{ad+\frac{d^2}{4}}\cdot\left(\frac{a+d/2}{r}\right)^2;$$

or, since $d/2$ is small compared to a and r, approximately $2v/d$. Assuming for liquids that which is found true in case of solids, namely, that the accidental double refraction due to a stress is proportional to the stress, we have, that the double refraction varies directly as the velocity and inversely as the distance between the surfaces.

As a test of the correctness of this formula for the apparatus used, observations were made with a solution of gelatine, and the results obtained compared with those of other observers* with two concentric cylinders. The results obtained agree with those given by the other apparatus, so the experiment confirms the correctness of the formula for the apparatus used.

Observations.

With Water.—With water the most favourable conditions are to be obtained. The sensitiveness attained with the polariscope, with some training of the eye, is remarkable. A rotation of the analyser which corresponds to a retardation of one component with respect to the other of 0·000025 λ could readily be detected. Then water is typical of liquids not oils, quite unlike colloids ; its chemical composition is simple, its physical properties well known.

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* Umlauf, *Ann. der Phys. u. Chem.* xlv. p. 304 (1892).

The dimensions of the apparatus, the velocity attained, and the sensibility of the optical system, together with those of Kundt, are given in the following table:—

		Kundt.
Speed of cylinders — revolutions per second	70 and 75	0 and 150
Relative linear velocity—cm. per second	4915	2356
Length of cylinders—cm. . . .	15	15
Distance between cylinders—cm. .	0·38	0·4
Least relative retardation which would be detected by the optical system	0·000025 λ	0·0038 λ*.

Hence, assuming that the double refraction produced in liquids is proportional to the relative velocity of the cylinders, by having doubled the relative velocity and increased the sensibility of the optical system by a factor of 150, the limit to which the non-occurrence of accidental double refraction has been proved is extended by a factor of 3×10^2.

With Crystalline Solution.—As a type of crystalline solutions a saturated solution of sodium hyposulphite was examined. The density of the solution at 20° C. was 1·40, its coefficient of viscosity 0·043. With the limited power at hand, it was not possible to obtain as great speed as in the case of water. The solution had a tendency to settle into layers of different density; also, after a time, there formed in the solution small solid particles which interfered with the optical accuracy.

A phenomenon which was, for a time, very confusing, was an apparently "semistatic" double refraction, corresponding to a retardation of 0·0001 λ, which was undoubtedly due to the small solid particles suspended in the solution. This double refraction occurs when the cylinders are given any slight rotation, and disappears when the velocity increases above a certain rather small velocity. It is independent of the direction of motion of the cylinders—in fact it occurs when the cylinders rotate in opposite directions so that the relative velocity of the two surfaces is zero.

Eliminating these disturbances no double refraction was obtained with the higher speeds. The speed and other data of the experiment are:—

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* No statement is made in Kundt's paper in reference to the accuracy obtained, but his apparatus was duplicated and its sensibility carefully determined, with the result given.

Relative linear velocity of surfaces. . .	3040 cm. per sec.
Distance between cylinders	0·38 cm.
Least relative retardation which could have been detected	0·00005 λ.

Assuming the speed and sensibility attained by Kundt in this case the same as that in case of water—it could not have been greater—the limits of the experiment with solution of sodium hyposulphite are extended by a factor of 100.

LXIII. *On the Theory of the Magnetic Influence on Spectra; and on the Radiation from moving Ions.* *By* J. Larmor, *D.Sc., F.R.S.*[*]

A THEORETICAL analysis of somewhat general character can be developed in connexion with Zeeman's phenomenon[†], which may help to throw light on the nature of the electric vibrations in the molecule. It will be convenient to begin with a simple case.

1. Consider a single ion e, of effective mass M, describing an elliptic orbit under an attraction to a fixed centre proportional to the distance therefrom. The equations of motion will be $(\ddot{x}, \ddot{y}, \ddot{z}) = -a^2(x, y, z)$; and the frequency of oscillation in any direction will be $a/2\pi$. Now suppose that a uniform magnetic field H, in a direction (l, m, n), is introduced: the equations of motion will become

$$\ddot{x} = -a^2x + \kappa(n\dot{y} - m\dot{z})$$

$$\ddot{y} = -a^2y + \kappa(l\dot{z} - n\dot{x})$$

$$\ddot{z} = -a^2z + \kappa(m\dot{x} - l\dot{y})$$

where $\kappa = e\mathrm{H}/\mathrm{M}c^2$, in which c is the velocity of radiation[‡]. To obtain the frequencies $(p/2\pi)$ of the oscillations thus modified, we make as usual (x, y, z) proportional to $e^{\iota pt}$. This gives, after easy reduction, an equation for p,

$$(p^2 - a^2)^3 - \kappa^2p^2(p^2 - a^2) = 0.$$

Thus, corresponding to each original period represented by $p = a$, there are three modified ones represented by $p = a$ and $p^2 \pm \kappa p - a^2 = 0$; when, as in practice, κ is very small the two

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[*] Communicated by the Author.

[†] Zeeman, Phil. Mag. March and July 1897; Michelson, Phil. Mag. May 1897; Lodge, Proc. Roy. Soc. Feb. and June, 1897.

[‡] *Cf.* Phil. Trans. A, 1895, p. 718. These equations only apply strictly (*infra*, § 10) when the velocity of the ion is small compared with c.

latter will be approximately $p = a + \kappa^2/8a \pm \frac{1}{2}\kappa$, or with sufficient accuracy $p = a \pm \frac{1}{2}\kappa$. Each vibration period will therefore be tripled: and the striking feature is that the modification thus produced is the same whatever be the orientation of the orbit with respect to the magnetic field.

An inquiry into the cause of this feature enables us to generalize the result. Suppose that the original orbit is referred to a system of axes (x, y, z) that are themselves revolving with angular velocity ω round an axis of which the direction is (l, m, n). The component velocities (u, v, w) referred to this moving space are $\dot{x} - y\omega n + z\omega m, \ldots, \ldots,$ and the component accelerations are $\dot{u} - v\omega n + w\omega m, \ldots, \ldots$. Thus the component acceleration parallel to x is

$$\ddot{x} - 2\omega(n\dot{y} - m\dot{z}) - \omega^2 x + \omega^2 l(lx + my + nz).$$

If, then, we take ω equal to $\frac{1}{2}\kappa$, and so can neglect ω^2, the equations of the original orbit referred to this revolving space are identical with those of that orbit as modified by the magnetic field. In other words, the oscillation thus modified will be brought back to its original aspect if the observer is attached to a frame which revolves with angular velocity $\frac{1}{2}\kappa$ or $e\mathrm{H}/2\mathrm{M}c^2$ round the axis of the magnetic field. In a circular orbit described one way round this axis the apparent rotation will in fact be retarded, in one described the other way round it will be accelerated, in a linear oscillation along the axis there will be no alteration: hence the three periods found above.

2. Now the argument above given still applies, whatever be the number of revolving ions in the molecule, and however they attract each other or are attracted to fixed centres on the axis, provided κ has the same value for them all. In any such case the actual oscillation in the magnetic field is identical with the unmodified oscillation as seen from a revolving frame; or, more simply, the modification may be represented by imparting an opposite angular velocity $\frac{1}{2}\kappa$ to the vibrating system. Thus the period of a principal oscillation of the system will be affected by the magnetic field in the opposite way to that of its optical image in a plane parallel to the field; and these two oscillations, previously identical as regards period, will be separated on account of their right-handed and left-handed qualities. An oscillation which does not involve rotation round an axis parallel to the field will, however, present the same aspect to the field as its image, and will not be affected at all. This latter type of oscillation, in a compound system, will be a very special one; and when a crowd of vibrators indifferently orientated are

considered, radiation of this kind will usually be practically nonexistent. Thus *each* spectral line of the vibrator will be split up into two with righthanded and lefthanded circular polarizations when seen along the axis, and plane-polarized with phase difference of half a wave-length when seen at right angles to it, and with differences of frequency the same for all lines in the spectrum, as in the special case above. This simple statement applies to all systems in which the electric charge of each mobile ion in the vibrator is proportional to its effective mass, which implies that the charges of the mobile ions are all of the same sign.

3. The characters of the three principal oscillations in § 1 may be determined in the usual manner by substituting in the equations of motion $(x, y, z) = (x_0, y_0, z_0)e^{\iota pt}$ and determining (x_0, y_0, z_0) from the resulting system of linear equations. But algebraic reductions will be avoided by taking the magnetic field to be along the axis of z, so that $(l, m, n) = (0, 0, 1)$, as might in fact have been done from the beginning. The equations of motion are then

$$\ddot{x} = -a^2x + \kappa\dot{y}, \qquad \ddot{y} = -a^2y - \kappa\dot{x}, \qquad \ddot{z} = -a^2z.$$

They show at once that the unmodified principal vibration is a linear oscillation parallel to the z-axis. As regards the others, writing $(x, y) = (x_0, y_0)e^{\iota pt}$ we have

$$(a^2 - p^2)x = \iota\kappa py, \qquad (a^2 - p^2)y = \iota\kappa px\,;$$

thus $p^2 \pm \kappa p - a^2 = 0$, or very approximately $p = a \pm \frac{1}{2}\kappa$ as before; and separating the real parts of this solution

$$x = \frac{\mathrm{A}}{a^2 - p^2}\cos pt, \quad y = \frac{\mathrm{A}}{\kappa p}\sin pt,$$

which to our order of approximation represents motion round a circle in the plane of (x, y) righthanded or lefthanded according to the value of p that is taken*. The character of the radiation from such a vibrator is thus precisely independent of the orientation of its orbit with respect to the magnetic field. With a large number of such vibrators, orientated indifferently, every spectral line seen in a direction at right angles to the magnetic field would be split up into three lines, each of the same breadth as the original, the middle one plane-polarized at right angles to the magnetic field, the outer ones in the direction of the field: as the aggregate light must be

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* This is no doubt the analysis recently indicated by Prof. FitzGerald in 'Nature,' Sept. 1897.

unpolarized, the intensity of the middle line would be twice that of either of the outer ones. But viewed along the field the middle line would disappear, as the exciting vibration would be end on to the observer and could not therefore send out transverse radiation: the other lines (equally sharp as before) would be circularly polarized, and their directions of polarization would, as Zeeman remarks, determine whether the vibrator involves a positive or a negative electron.

4. A view has been enunciated that it is only one kind of ions, namely the negative ones, that are mobile and free to vibrate in the atom or molecule, the other kind being fixed to the matter and immobile. On such an hypothesis, if the charges of these negative ions are proportional to their effective masses, for example if they are simple electrons without inertia other than that of the electric charge, the intervals (measured in difference of frequency) between magnetic doublets and the outside lines of magnetic triplets in the spectrum should be the same for all lines. Moreover, they should be the same in different spectra. Thus an hypothesis of that kind can be definitely put to the test.

5. When there are ions of different kinds describing orbits in the molecule, these exact results no longer hold: but even then we can assert that the difference of frequency between the lines of a magnetic doublet is of the order $eH/2\pi Mc^2$, and the order of magnitude of e/M can be thence derived. Thus Zeeman concludes from his experiments that the effective mass of a revolving ion, supposed to have the full unitary charge or electron, is about 10^{-3} of the mass of the atom. This is about the same as Professor J. J. Thomson's estimates of the masses of the electric carriers in the cathode rays. If we took these carriers to be simply electrons, as their constancy under various environments tends to indicate, there would thus be about 10^3 electrons in the molecule.

6. In view of the above considerations, the circumstance that in a magnetic field certain lines, viewed transversely, are divided into sharp triplets with perfect plane polarizations, which has been described by Zeeman and assented to by Lodge and by Michelson*, is an important clue to the character of the principal oscillations which emit those lines. In an oscillating molecule undisturbed by a magnetic field there must be three types of vibration which all have the period belonging to that line, namely two types which differ only by involving rotations in opposite directions round the axis of that magnetic field and would naturally have the same

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* In Cornu's experiments, *Comptes Rendus*, Oct. 18, the application of the polarizing apparatus seems to have been required to divide the lines.

period, and a third type which does not involve any rotation with respect to that axis. Now that extraneous axis may have any direction with reference to the molecule. Hence a principal oscillation which is thus magnetically tripled must be capable of being excited with reference to any axis in the molecule: otherwise there would be merely hazy broadening or duplication instead of definite triplication.

7. A system of electrons or ions of the same sign, confined to a surface over which they are free to move and constituting an electric charge on it, is an artificial vibrator whose periods illustrate these results. The free periods of such a vibrating system in which the forces acting on the electrons work against the inertia of the moving electrons, would be only theoretically different from the free periods of an actual electric charge on a metallic conductor; although in the latter case the forces acting on the ions work mainly against the ohmic diffusive resistance to their transfer either actual or electrolytic through the crowd of neighbouring molecules, which is far greater than the reaction arising from inertia alone unless it is rapid optical vibrations that are dealt with. In either case the forces acting on the ions are so great compared with the possible kinetic reactions to their motion that their distribution is at each instant practically in equilibrium on the surface, so that there is no electric force along it: and either set of conditions simply reduces to the condition that there shall be no electric or magnetic field in the interior. It follows that the oscillations of an electric charge on any conductor of the form of a surface of revolution are modified by the introduction of a magnetic field along the direction of the axis of the surface, just as if the angular velocity $\frac{1}{2}\kappa$ above given were imparted to the vibrating system. All the free periods except those of zonal oscillations would be duplicated in the manner above explained, the interval measured in difference of frequencies between the components of the doubled vibration being the same for all. In the special case of zonal oscillations on a sphere they would be triplicated, because the period of a zonal vibration along the axis would not be modified: but the middle line would be very weak compared with the flanking ones*.

8. This analysis gives a hint as to one way in which a

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* The radiation from a continuously distributed electric charge is, however, known to be so great as to make its oscillations dead-beat; hence these conclusions could only be applied if (i.) there are only a limited number of discrete ions moving on the surface, or (ii.) there are material forces other than electric imagined to act between the ions, whose energy could maintain the vibration for a large number of periods. (See § 10.)

series of double lines in a spectrum, with equidistant frequencies, might be originated. Suppose, as a very rough illustration, that a polar molecule is constituted of a system of positive electrons around one pole and a system of negative ones around the other, the two systems being so far apart as to have practically separate sets of periods for their orbital motions, each of course disturbed by the presence of the other. Each of these systems moves in the magnetic field, more or less constant, arising from the other; and the effect of this disturbing field will, as above, be to duplicate all the periods of that set in the above regular way.

9. It is desirable to precisely formulate the relation between the motions of the electrons and the radiation emitted by them, which has been tacitly employed in the foregoing discussion. The specification of that radiation may be readily assigned by a summation over the different elements of the paths of the oscillating ions. Suppose that an ion e is at A and after a time δt is at B, where $AB = v\delta t$, v being its velocity; the effect of its displacement is the same as that of the creation of an electric doublet AB of moment $ev\delta t$; thus we have only to find the influence propagated from such a doublet, and then integrate along the paths of the ions of the molecule.

Consider now such a doublet at the origin, lying along the axis of z; for it, or indeed for any distribution symmetrical with respect to that axis, the lines of magnetic force will be circles round the axis, and the force will be specified by a single variable, its intensity H. The current, whether in dielectric or in conducting media, will circulate in wedge-shaped sheets with their edges on the axis, and may be specified by a stream function, as in fact will appear below. If we employ cylindrical coordinates ρ, θ, z, and apply the Amperean circuital relation (viz: circulation of magnetic force equals 4π times current) to the faces of the element of volume $\delta\rho.\rho\delta\theta.\delta z$, we obtain for the components P, R of the electric force

$$\frac{dP}{dt} = -c^2 \frac{dH\rho}{\rho dz}, \quad \frac{dR}{dt} = c^2 \frac{dH\rho}{\rho d\rho},$$

so that $H\rho$ plays the part of a stream function; while by the circuital relation of Faraday

$$\frac{dP}{dz} - \frac{dR}{d\rho} = -\frac{dH}{dt}.$$

Thus the characteristic equation for H is

$$\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho\mathrm{H}+\frac{d^2\mathrm{H}}{dz^2}=c^2\frac{d^2\mathrm{H}}{dt^2},$$

which is $$(\nabla^2-\rho^{-2})\mathrm{H}=c^2d^2\mathrm{H}/dt^2,$$

where ∇^2 is Laplace's operator. But a more convenient reduction comes on substituting $\mathrm{H}=d\mathrm{Y}/d\rho$, and then neglecting an irrelevant operator $d/d\rho$ along the equation: this gives

$$\nabla^2\mathrm{Y}=c^2d^2\mathrm{Y}/dt^2.$$

We can now express the disturbance emitted by an electric doublet situated along the axis of z at the origin, and vibrating so that its moment M is an arbitrary function of the time. As regards places at a finite distance, the doublet may be treated as a linear current-element of strength $d\mathrm{M}/dt$. Close up to such an element in its equatorial plane, the magnetic force H due to it is $-r^{-2}d\mathrm{M}/dt$. The appropriate solution for Y for this simplest case is $\mathrm{Y}=r^{-1}f(t-r/c)$, so that

$$\mathrm{H}=-\sin\theta\left\{\frac{f(t-r/c)}{r^2}+\frac{f'(t-r/c)}{cr}\right\},$$

giving when θ is $\frac{1}{2}\pi$ and r is very small, $\mathrm{H}=-r^{-2}f(t)$: thus $d\mathrm{M}/dt=f(t)$. That is, if the moment of the oscillating doublet is given in the form $d\mathrm{M}/dt=f(t)$, the magnetic force thus originated at the point (r, θ) is

$$\mathrm{H}=-\sin\theta\left\{\frac{f(t-r/c)}{r^2}+\frac{f'(t-r/c)}{cr}\right\},$$

or $$\sin\theta\frac{d}{dr}r^{-1}f(t-r/c).$$

The second term is negligible for movements of slow period, as it involves the velocity c of radiation in the denominator. The components of the magnetic field due to a vibrating doublet M at the origin whose direction-cosines are (l, m, n) are then

$(mz-ny,\ nx-lz,\ ly-mx)r^{-1}d/dr\,r^{-1}f(t-r/c)$, where $d\mathrm{M}/dt=f(t)$;

and the components of the magnetic field, and therefore of the radiation emanating from any system of electric oscillators vibrating in any given manner can thence be expressed in a general form by integration. At present we only want the effect of suddenly establishing the doublet $\mathrm{M}=ev\delta t$ at the origin. This comes by integration over the very small time of establishment; there is a thin spherical shell of magnetic force propagated out with velocity c, the total force integrated across the shell being exactly $-\mathrm{M}r^{-2}\sin\theta$ whatever be its radius, for the integral of the second term in H vanishes because dM/dt is null at the beginning and end of the operation. The aggregate amount of magnetic force thus propagated in the spherical sheet is the same as the steady magnetic force due to a permanent steady current-element of intensity equal to $M/\delta t$, or ev: it is clear, in fact, that this must be so, if we consider a sudden creation of this current-element and remember that its magnetic field establishes itself by spreading out ready formed with the velocity of radiation.

The magnetic force at a point at distance r due to a moving ion thus depends on the state of the ion at a time r/c previously; for near points it is in the plane perpendicular to r, at right angles to the projection v of the velocity of the ion on that plane, and equal to evr^{-2}. For vibrations whose wave-length in free æther is very great compared with the dimensions of the molecular orbit, if we interpret magnetic force as velocity of the æther, the vibration-path of a point attached to the æther, and close to the vibrator, will be in the plane transverse to r, and similar to the projection of the orbit of the electron on that plane when turned round through a right angle *. If the condition of wave-length very large compared with molecular magnitude were not satisfied, phase-differences would sensibly disturb this result, and in effect each spectral line would be accompanied, more or less, by its system of harmonics.

As the vibration of a near point in the æther is thus similar to the projection on the wave-front of the vibration of the electrons in the molecule, it is verified that the free periods of the radiation are those of the system of ions.

10. It might appear also at first sight that every steady orbital motion must rapidly lose its energy by radiation just as vibrations on the conductor in § 7 would do if the ions on it formed a continuous charge. On the other hand it might be argued that what we have really been calculating is the amount added on to the previous motion in the medium by the successive displacements of the electrons; and in the cases of steady motion it is just this amount that is needed to maintain the permanency of the motion in the æther, which of itself has a tendency to be carried away. Thus in the parallel case of the movement of a very long stretched cord when an end of it has a steady circular motion imparted to it, an analysis in the ordinary way leads to a train of circular waves running along the cord; but there

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* For a different treatment of similar topics *cf.* H. A. Lorentz, "La Théorie Electromagnetique de Maxwell," §§ 112–119, *Archives Néerlandaises*, 1892; "Versuch ...," 1895; quoted by Zeeman, Phil. Mag. March, 1897.

exists a steady motion in which the cord whirls round bodily, and which will be generated when the velocity of the motion imposed on the end is gradually increased from a very small initial value to its final amount.

The difficulty is, however, not thus surmounted; for this steady motion which does not involve radiation is really a state of stationary undulation arising from the superposition of a wave-train travelling outwards on another travelling inwards, and the genesis of the latter one would have to be accounted for. We might assume that these non-radiating vibrations consisted of stationary waves reflected backwards and forwards between two vibrating molecules, or between two ions in the same molecule; but even that would not be satisfactory. As a matter of fact, however, no explanation of this kind is needed. The effective electric inertia of an ion e by itself is $\frac{2}{3}e^2a^{-1}$ *, where a is the radius of its nucleus supposed spherical: the rate at which it loses energy by radiation is proportional to e^2, and involves its motion, but does not depend on a at all. The kinetic reaction to change of its velocity which is connected with loss of energy by radiation can thus be made negligible in comparison with the kinetic reaction arising from its inertia. In fact, the energy of the æthereal motion carried along by the moving ion depends on the first term in H, involving r^{-2}, and the radiated energy depends on the second term, involving r^{-1}. But in types of oscillation in which there are crowds of ions moving close together in step, loss of energy by radiation is an important feature in the dynamics of free vibrations.

These considerations can be developed by aid of the analysis of § 9 above. In consequence of the stream-function property of $\mathrm{H}\rho$, the components of d/dt of the electric force, along δr and along $r\delta\theta$, are respectively

$$\frac{c^2}{\rho}\frac{d\mathrm{H}\rho}{rd\theta} \quad \text{and} \quad -\frac{c^2}{\rho}\frac{d\mathrm{H}\rho}{dr},$$

ρ being $r\sin\theta$; thus they are

$$-2c^2\cos\theta\left\{\frac{f(t-r/c)}{r^3}+\frac{f'(t-r/c)}{cr^2}\right\},$$

and

$$-c^2\sin\theta\left\{\frac{f(t-r/c)}{r^3}+\frac{f'(t-r/c)}{cr^2}+\frac{f''(t-r/c)}{c^2r}\right\};$$

and the electric force is obtained by integrating with respect to t.

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* Phil. Trans. A, 1894, p. 812. This inertia is no longer quite constant when the velocity of the ion is considerable compared with that of radiation. In that case also the simple computation of the radiation here given would not be exactly applicable; and the problem would have to be treated by continuous differential analysis after the manner of Phil. Trans. A, 1895, p. 718.

At a very great distance the electric force (as well as the magnetic force) is thus perpendicular to r, and is equal to $-r^{-1}\sin\theta f'(t-r/c)$; and the flow of energy is thus by Poynting's principle radial. For the case of an ion e moving with velocity v, $f(t)$ is equal to ev; and in $f(t-r/c)$ the value of the function f belongs to the position of the molecule at a time r/c previous, where r is its distance at that time. The rate of loss of energy by radiation may be computed by Poynting's formula as $(4\pi)^{-1}$ times the product of the above electric and magnetic forces integrated over an infinite sphere: it is thus

$$(4\pi r^2 c)^{-1}\{f'(t-r/c)\}^2\int\sin^2\theta\,d\mathrm{S}, \text{ or } \tfrac{2}{3}e^2c^{-1}\dot{v}^2.$$

In the process of getting up a velocity v of the ion from rest, there is a loss of energy equal to $\frac{2}{3}e^2c^{-1}\int\dot{v}^2\,dt$. In motion with uniform velocity there is no loss; during uniformly accelerated motion the rate of loss is constant.

As the electric and magnetic forces at a great distance are each proportional to the acceleration of the ion and do not involve its velocity, and as we can combine the components of its motion in fixed directions, it follows generally that the rate of loss of energy by radiation is $\frac{2}{3}e^2c^{-1}\times(\text{acceleration})^2$.

The store of kinetic energy belonging to the ion is $\frac{2}{3}e^2a^{-1}v^2$. Thus the loss of energy by radiation from an undisturbed vibrating molecule would not be sensible compared with its whole intrinsic kinetic energy, when the velocities of the ions are not of the order of magnitude of that of radiation: while for higher velocities the importance of the radiation is, in part at any rate, counteracted by the increase of the inertia coefficient.

11. Finally, it is to be observed that the law of the magnetic vibration excited by a moving ion is stated in § 9 only for the case in which r is small compared with the wave-length. Further away from the ion the law of variation of the magnetic force with distance is $ev/r^2+e\dot{v}/cr$ instead of ev/r^2. Thus at a distance of a large number of wave-lengths, the vibration-curve of the radiation proper is similar to the projection of the hodograph of the orbit of the ion on the wave-front, instead of the projection of the orbit itself.

It would thus appear that when the steady orbital motions in a molecule are so constituted that the vector sum of the accelerations of all its ions or electrons is constantly null, there will be no radiation, or very little, from it, and therefore this steady motion will be permanent. But this is just the condition which holds good so long as the molecule is free from extraneous disturbance.

St. John's College, Cambridge.
Nov. 8, 1897.

LXIV. *On the External Field of Helically Magnetized Rings.*

By W. M. MORDEY*.

ON the subject of some experiments by the present writer relating to drag on armature conductors, enclosed or nearly enclosed in iron, Prof. H. du Bois contributed an article in the *Elektrotechnische Zeitschrift* of August 19 last (a translation of which appeared in the 'Electrician' of August 27), in the course of which attention was particularly directed to a phenomenon of ring magnets. Prof. du Bois writes:—*A ring-magnet experiences a side thrust when in an external field whose lines of force are in the same plane as the ring; and conversely it exerts a thrust in the opposite direction upon the supporter of the external field.*" "This deduction," he adds, "so astonishing at first sight, I have proved to be verified by experiment." As its author gives the emphasis of italics to this passage, it may be well to submit what appears to be the explanation, as the present writer chanced to come across the effect in question a good many years ago. It is, as Prof. du Bois says, rather astonishing at first sight, but the explanation seems to be quite simple. The external field is caused by the advancing or longitudinal constituent of the helix. Magnetically a helix traversed by current acts as a series of rings, but in addition it has a minor magnetizing effect, at right angles to that of the rings, due to the advance of the conductor from end to end of the helix. Thus, a helix bent into a ring or wound about an annular core, forms an endless or ring-magnet when traversed by current. But in addition to this main effect, the faces of the ring, regarded axially, present N. and S. polarity respectively, the whole closed helix acting externally (so far as this subsidiary effect is concerned), and as regards external fields, as a simple loop or ring of conductor carrying current. This may be illustrated by a lines-of-force figure, taken from a ring helically wound with a single layer of winding. In such a figure the lines of force radiate from the centre or axis of the ring at right angles to the axis of the spiral—they represent portions of closed lines of force which enclose the axis of the spiral. In a helix wound with two layers, one forward, the other back, the external magnetizing effect due to this cause is *nil*. In a single layer (or simple) helix, this external effect may be neutralized by returning the wire along the axis of the helix.

A simple straight helix acts externally as a straight conductor.

A flat spiral acts like a disk or wheel, or as a number of radial conductors in one plane traversed by current passing from or to the centre.

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* Communicated by the Author.

LXV. *Notices respecting New Books.*

The Principles of Chemistry. By D. MENDELÉEFF. *Translated from the Russian* (*sixth edition*) *by* GEORGE KAMENSKY, *A.R.S.M. Edited by* T. A. LAWSON, *B.Sc., Ph.D.* In two volumes. Longmans, Green, & Co., 1897.

THE first English edition of this work, which was published in 1891, was noticed in these pages, and during the period that has elapsed since that time, it may be fairly said that Mendeléeff's 'Principles of Chemistry' has taken its position in English scientific literature as a recognized classic. Of this fact the publication of a new edition may be taken as proof, not only because the work is somewhat costly as compared with the rank and file of text-books which our students are in the habit of using, but still more because the leading facts and principles of the science are not presented in the usual cut and dried form required by the syllabus of some examining board. It is essentially a book for the philosophical student of chemistry, and as such it occupies a unique position.

With respect to the present edition, it will suffice to say that so far as concerns the general arrangement it is unchanged, the periodic law being still retained as the basis of the author's treatment. It is, in fact, this particular treatment which gives the special value to Prof. Mendeléeff's work and which has raised it above all other treatises to its present position. That the law of periodicity is a true law of matter is evident on every page, and finds justification in the developments which the principles laid down by the great Russian chemist have undergone in the hands of later investigators. All these developments will be found duly chronicled in the two volumes under consideration, and more especially may attention be called to the table facing Vol. II., in which is set forth the "periodic dependence of the composition of the simplest compounds and properties of the simple bodies upon the atomic weights of the elements."

Apart from the use of this work as a text-book by students, for which purpose it cannot be too highly commended, the working and thinking chemists of the day will naturally turn to these volumes to see what influence the modern views held by a certain school of physical chemists have exerted on the mind of the author during the interval between the fifth and sixth Russian editions. On the question of the nature of solutions for example, in which field Mendeléeff has himself laboured, his position is unchanged, and in the preface he says:—

"Although all aspects of the simplest chemical relations are as far as possible equally developed in this book, yet on looking back I see that I have, nevertheless, given most attention to the so-called indefinite compounds examples of which may be seen in solutions. . . . My own view is that a solution is a homogeneous liquid system of unstable dissociating compounds of the solvent with the substance dissolved. But although such a theory explains much to me, I cannot consider my opinion as proved, and therefore give it with some reserve as one of several hypotheses. . . . If in solutions I see and can frequently prove distinct evidences of the existence of those definite compounds which form the more generalized province of chemical data, I could not refrain from going into certain details respecting solutions; otherwise there would have remained no trace of that general idea, that in them we have only a certain instance of ordinary definite or atomic compounds, subject to Dalton's laws."

In accordance with this statement many allusions to the current theories of solution will be found throughout the work, generally in the form of foot-notes. A summary of the two aspects of the solution question, the physical and the chemical, is given in a lengthy note, running through pages 64–67 of the first volume, and from this it will be seen that the author does not regard these apparently conflicting views as irreconcilable. At the same time he correctly points out that solution cannot be a purely physical phenomenon since it "does not take place between any two substances, but requires a special and particular attraction or affinity between them." In other words, the purely chemical aspect of solution cannot be ignored.

In noticing the first English edition some six years ago we had occasion to call attention to the suggestiveness of the foot-notes, and a re-examination of the present edition from this point of view will serve to bring out even more forcibly the extraordinary breadth of the author's grasp of his subject. All the fundamental philosophical conceptions which have been introduced into chemical science appear to have been considered, not simply in the perfunctory "professorial" sense, but with the calm deliberation of an original mind given to weigh doctrines as well as to enunciate them in the lecture-room. Many of the notes are in fact of the nature of disquisitions on special doctrines, and are often complete from the historical point of view as well as concise in literary treatment. It remains only to indicate that there is a marked improvement in the translation of the present edition, and the ambiguities which were observable here and there in the text of the earlier edition have been removed. The third appendix of the former edition is now replaced by a paper on Argon written in 1895, which is extremely interesting as indicating that the author still had a lingering idea that this gas might be a polymeric condition of nitrogen formed under natural as distinguished from artificial laboratory conditions. Helium was too recent a discovery to be treated of at length, and one can only regret that we have not the advantage of Mendeléeff's latest and most matured views on the influence of the discovery of these inert gases on the periodic arrangement of the elements. In confirming the favourable opinion which was formerly expressed with respect to the value of this treatise—an opinion which the new edition enhances in a very high degree—we may add that both translator and editor have done their work remarkably well.

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END OF THE FORTY-FOURTH VOLUME.

Printed by Taylor and Francis, Red Lion Court, Fleet Street.