



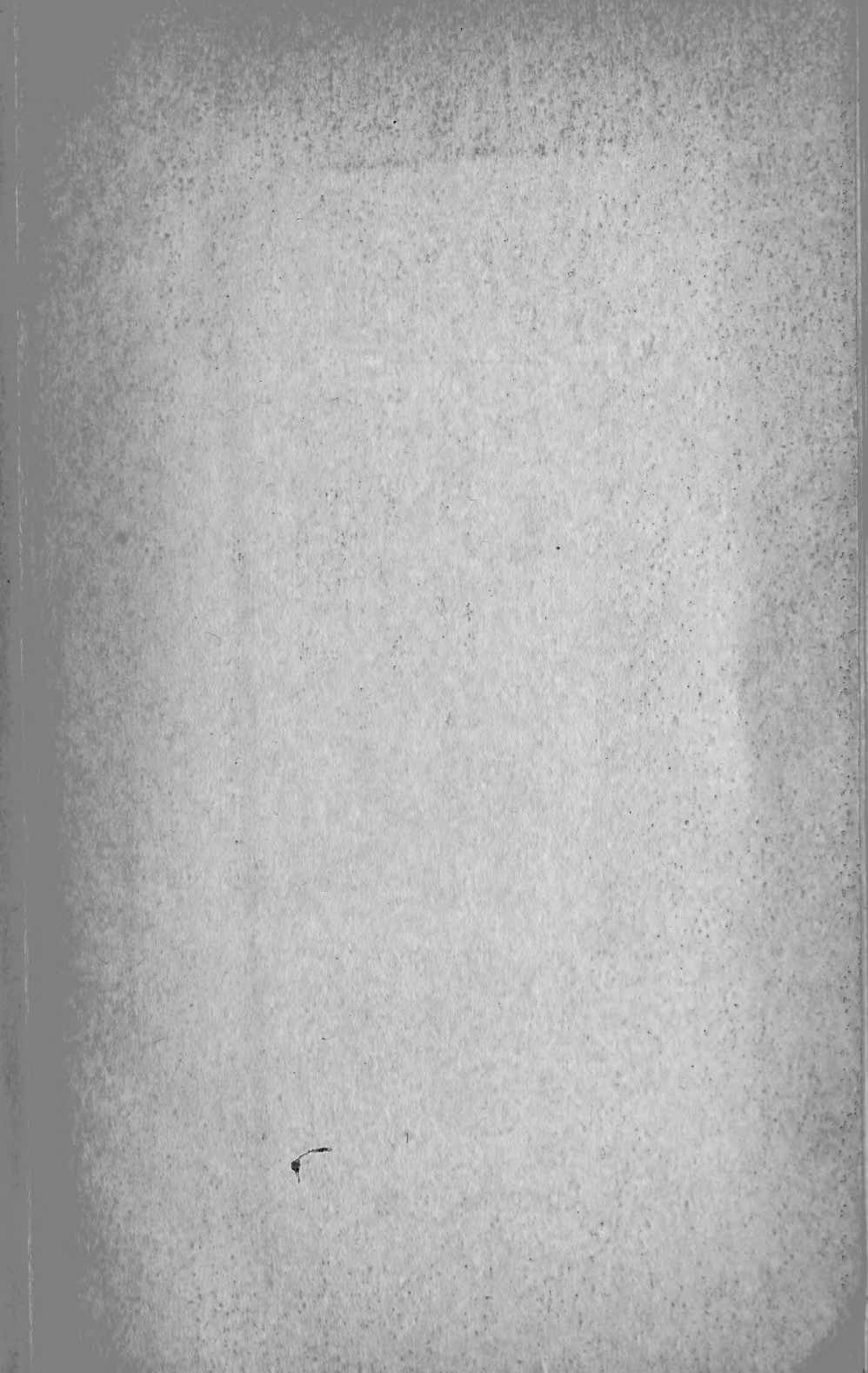
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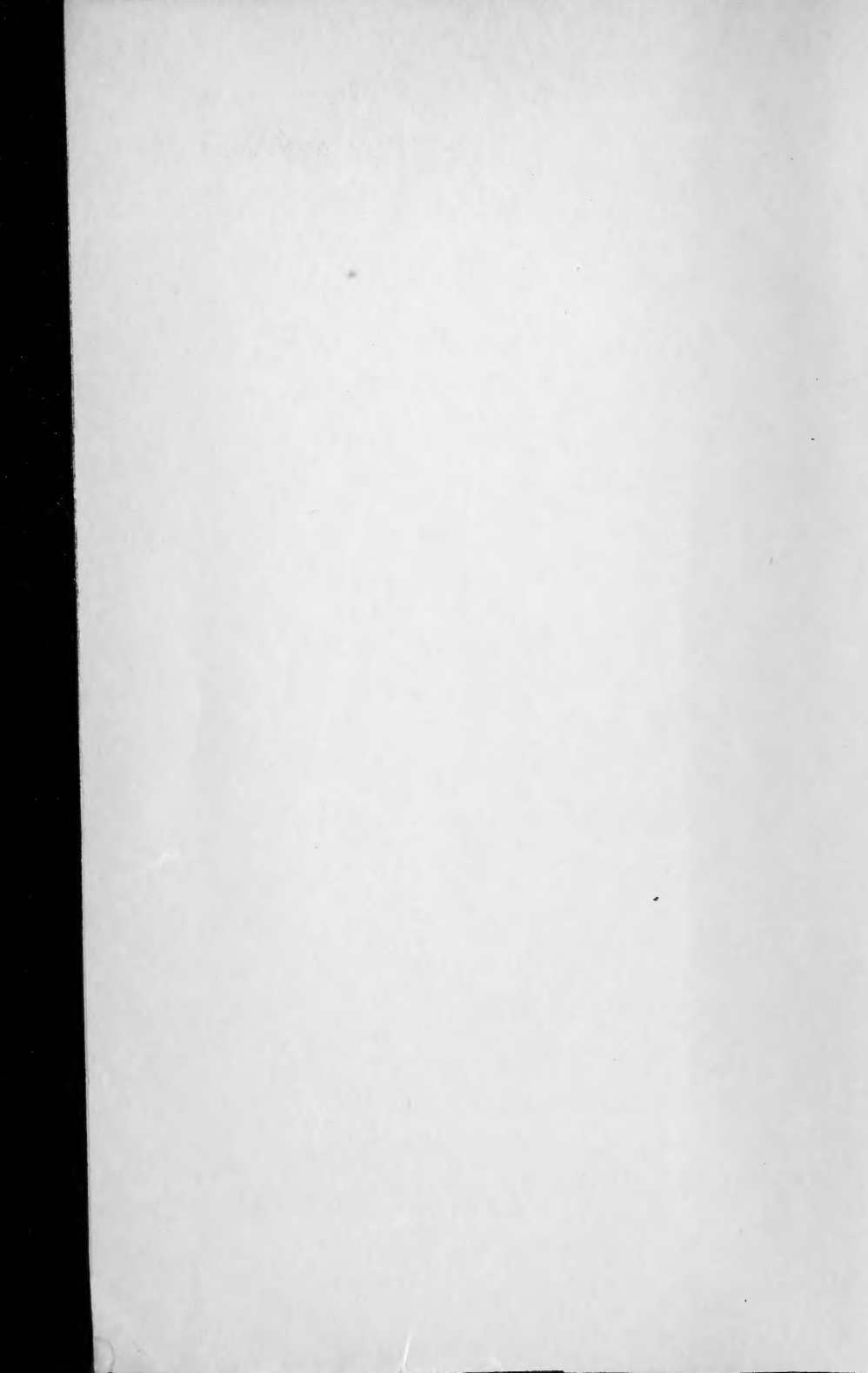
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“Nec araneorum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes.” JUST. LIPS. *Polit. lib. i. cap. 1. Not.*

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VOL. XXI:—SIXTH SERIES.  
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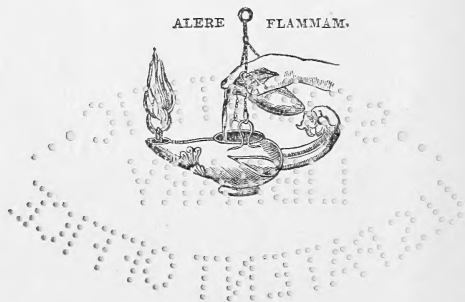
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“Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem.”—*Hugo de S. Victore.*

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—“Cur spirent venti, cur terra dehiscat,  
Cur mare turgescat, pelago cur tantus amaror,  
Cur caput obscura Phœbus ferrugine condat,  
Quid toties diros cogat flagrare cometas,  
Quid pariat nubes, veniant cur fulmina cœlo,  
Quo micet igne Iris, superos quis conciat orbes  
Tam vario motu.”

*J. B. Pinelli ad Mazonium.*



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[SIXTH SERIES.]

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JANUARY 1911.

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I. *On Magnetic Testing.* By JAMES G. GRAY, D.Sc., F.R.S.E., Lecturer on Physics in the University of Glasgow, and ALEXANDER D. ROSS, M.A., B.Sc., F.R.S.E., Lecturer on Natural Philosophy in the University of Glasgow\*.

THE methods in use for the experimental determination of the magnetic constants of materials have been brought to a high state of perfection, and are thoroughly understood by the majority of the investigators engaged in carrying out research work in magnetism. But the precautions which must be taken in order that the results yielded by application of these methods may be accurate, are, however, not so well understood. It is proposed in the present paper to bring forward some facts bearing on this point which have been brought to light by recent work carried out in the Physical Institute of the University of Glasgow.

It is well known, and is pointed out in all the text-books, that the magnetic properties of a test specimen depend on its previous magnetic history. In this connexion Ewing † has drawn attention to the very interesting fact that it is possible to put a specimen—say, a long rod of iron having uniformity of physical structure throughout its length—through a series of magnetic operations such that it is left devoid of residual

\* Communicated by the Authors.

† J. A. Ewing, "Experimental Researches in Magnetism," Trans. Roy. Soc. clxxvi. p. 533.

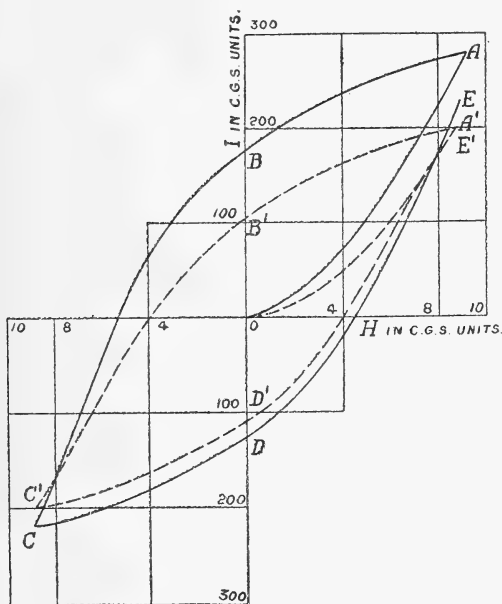
magnetism and in a field of zero intensity, but with vastly different magnetic properties for the two ways in which it may be magnetized in the direction of its length. If we denote the ends of the specimen by A and B, then the permeability of the material in the direction AB differs from that in the direction BA for magnetizing forces which are less than the maximum force employed in the operations. In other words, the chains of molecular magnets, in the closedness of which the external neutrality of the bar consists, are left by the process referred to in an arrangement which is unequally affected by a magnetic field according as it is applied from A to B or from B to A. This difference of response to magnetic force in the two directions is due to the magnetic treatment which the specimen has sustained, and is not removed by annealing unless the temperature is raised above the so-called critical temperature of the material. For this reason it is usual to divest a specimen of its previous magnetic history before submitting it to test. This is accomplished by exposing the specimen to the action of an alternating magnetic field which diminishes gradually from a high value to zero.

But it is not generally known that purely thermal treatment, such as that involved in the process of annealing a specimen, no matter what temperature is reached in the process, develops in the specimen a peculiar state which renders additional precautions necessary. This fact has been disclosed in experiments carried out by the authors, which are described in papers\* read before the Royal Society of Edinburgh. It is there shown that an alteration of the temperature produces what we have called a "Sensitive State" of the material. A magnetization curve yielded by the specimen following upon any change, or cycle of changes, in the temperature, is not what we may here call the *true* magnetization curve of the material. This will be clear from fig. 1, which shows the results obtained on testing, at room temperature by the magnetometric method, a specimen of hard steel which had been heated to 900° C. and allowed to cool slowly. Previous to being heated the specimen had been rendered neutral, and during the process of heating and cooling had been exposed to no magnetizing forces. On submitting the specimen, in the condition brought about by the thermal treatment described, to the action of a magnetizing field which increased gradually from 0 to +9 c.g.s.

\* J. G. Gray and A. D. Ross, "On a 'Sensitive State induced in Magnetic Materials by Thermal Treatment,'" *Proc. Roy. Soc. Edin.* xxviii. pp. 239 & 615 (1908).

units, then diminished gradually from +9 c.g.s. units to -9 c.g.s. units, and finally increased in like manner from -9 c.g.s. units to +9 c.g.s. units, the intensity of magnetization followed the curve O A B C D E.

Fig. 1.



It will be seen that the point E does not coincide with the point A, and that the vertical distance of A above O is much greater than the vertical distance of C below O.

The specimen was now thoroughly demagnetized, and the process repeated, when the curve O A' B' C' D' E' was obtained. It is important to notice that the point E' coincides with A'; the curve is now closed. Further, an inspection of the figure shows that the hysteresis loop is now symmetrical. Cyclic alteration of the magnetizing force between limits  $H = \pm 9$  c.g.s. units resulted in the curve being repeated over and over again.

O A' is the true magnetization curve for the specimen in the annealed condition; it is this curve, and not the curve O A, that is characteristic of the magnetic properties of the material. Table I., which is taken from the figure, shows how serious are the errors involved in neglecting to render the specimen neutral before proceeding to carry out the tests. It will be

seen that the necessity for rendering the specimen neutral is brought about by the thermal treatment, and not by any magnetic treatment to which the specimen has been exposed.

TABLE I.

Value of H in c.g.s. units.	Value of I in c.g.s. units.	
	Given by specimen after thermal treatment.	Given by specimen in neutral condition.
2	22	15
4	67	47
6	136	100
8	220	172
9	275	206

So far, in describing the phenomena of this "Sensitive State," attention has been confined to the case of steel annealed at 900° C. It is not necessary, however, for the production of this state that the temperature change should be great: even a moderate alteration in the temperature is sufficient to bring about the effect to a marked extent. Moreover, it is not necessary that the alteration should consist of a step up in temperature followed by a step down of the same amount. Experiment has shown that if the temperature of a specimen, which is initially neutral, is altered from any one temperature to a second, the specimen assumes the "Sensitive State" at the second temperature. The further fact was established that once the specimen was rendered neutral it remained neutral provided that the temperature was maintained constant. In other words, the "Sensitive State" cannot be induced by prolonged exposure of the specimen to either a high or a low temperature; it is brought on in the process of heating or cooling the specimen.

Reference may here be made to the interesting fact that once the "Sensitive State" has been induced in a specimen, it cannot be got rid of except by submitting the specimen to the demagnetizing process. Thus a specimen of annealed steel was tested after having been laid aside for a period of several weeks, and gave results precisely similar to those yielded by it when in the freshly annealed condition. Further experiments revealed the fact, which is of great importance in connexion with what immediately follows, that once a specimen of steel in the sensitive condition has been exposed to a magnetizing force in one direction, the "Sensitive State"

cannot be got rid of, or even affected, by any magnetic operation or process of operations whatever, unless the sign of the field is changed. Thus if on applying a field of intensity  $+H$  the corresponding intensity of magnetization is  $I_1$ , cyclic variation of the field strength between the limits  $+H$  and  $H-h$ , where  $h$  may have any positive value ranging from 0 to  $H$ , does not result in the value of  $I$  corresponding to the field  $+H$  being diminished. If, however,  $h$  becomes greater than  $H$ , in other words if the field is reversed, then the value of  $I$  corresponding to  $+H$  is reduced. The effect of applying even an extremely small negative field is to bring about a perceptible diminution in the value of  $I$ , that is to reduce, more or less, the "Sensitive State."

A further series of experiments showed that the "Sensitive State" induced by equal augmentations or diminutions of the temperature are of widely different amounts depending on the position of the temperature ranges on the temperature scale. Even a small increment of only  $25^\circ\text{C}$ . in the neighbourhood of  $180^\circ\text{C}$ . produced, in the case of the steel tested, a percentage "Sensitive State" of 10 for a field of 10 c.g.s. units.

Such facts have a very important bearing on magnetic testing. For example, suppose that it is required to test the magnetic properties of a specimen at  $200^\circ\text{C}$ . We shall consider the specimen to be initially neutral and at room temperature. It may now be heated to  $200^\circ\text{C}$ ., and tested at that temperature. Or again it might be heated to say  $250^\circ\text{C}$ ., then cooled to  $200^\circ\text{C}$ ., and tested in like manner at that latter temperature. From what has been said it will be evident that if the specimen has not been rendered neutral at  $200^\circ\text{C}$ . in each of the two cases previous to examination, the results of the two tests will not in general be in agreement.

Or suppose that it is desired to test the magnetic properties of a specimen at temperatures lying between room temperature and the critical temperature of the material. The specimen might be heated up above the critical temperature and allowed to cool very slowly, tests being carried out at different stages of the process. This method has been adopted by many experimenters. If the specimen were composed of steel, it would be heated up to the neighbourhood of  $900^\circ\text{C}$ . and then cooled. A test carried out at  $800^\circ\text{C}$ ., without the specimen having been rendered neutral, would give a certain magnetization curve. The specimen might now be submitted to the process of reversals, allowed to cool to  $700^\circ\text{C}$ ., and then tested in like manner. This procedure,



if carried out at intervals during the cooling process, would result in a series of magnetization curves being obtained, from which it might be supposed that the permeability of the material for all temperatures lying between room temperature and  $800^{\circ}\text{C}$ . might be deduced. The properties so obtained, however, depend not only on the temperature, but on the particular procedure adopted. The fact that the specimen was tested at  $800^{\circ}\text{C}$ . determines to some extent the magnetization curve which is yielded at  $700^{\circ}\text{C}$ .; in other words, if the specimen had been heated to  $900^{\circ}\text{C}$ ., then cooled to  $700^{\circ}\text{C}$ . and tested at that temperature, the magnetization curve obtained would be distinctly different to that yielded by the specimen at  $700^{\circ}\text{C}$ . following upon the test at  $800^{\circ}\text{C}$ .

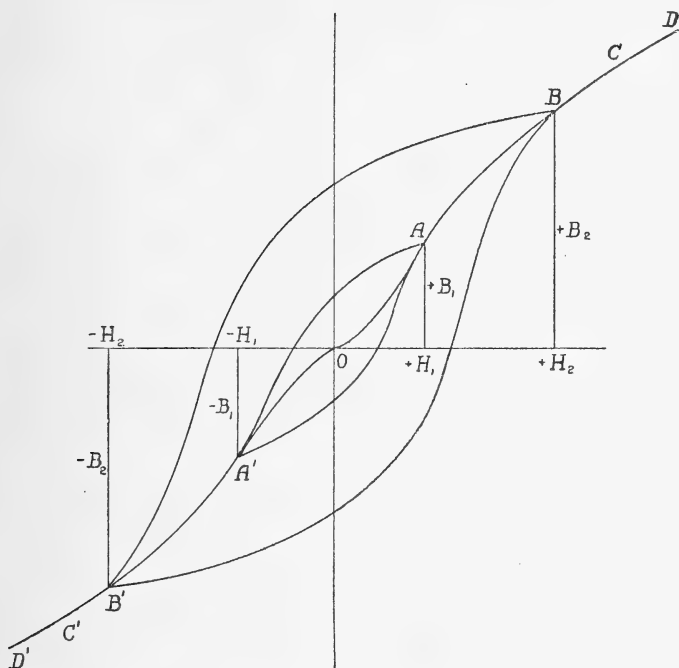
Again, if the specimen were tested at  $850^{\circ}\text{C}$ .,  $750^{\circ}\text{C}$ ., and so on, a new set of curves would result. From the foregoing discussion it will now be evident that the permeability of the specimen at any one temperature for any one particular field as determined from the second set of curves would not in general agree with that obtained from the former set.

If, however, the specimen were submitted to a process of reversals just previous to the tests being carried out at any one temperature, the results would be in strict agreement. For example, if this precaution is observed a magnetization curve obtained say at  $400^{\circ}\text{C}$ . following upon tests carried out at  $800^{\circ}\text{C}$ .,  $700^{\circ}\text{C}$ ., and so on, would coincide with one obtained at  $400^{\circ}\text{C}$ . without intermediate testing.

The magnitude of the errors involved in testing a specimen of annealed steel by the magnetometric method without previously subjecting it to the process of reversals has already been pointed out. It is now proposed to investigate the nature of the errors which would be introduced into the results yielded by a like specimen when tested by Rowland's "Method of Reversals." In this method the magnetizing coil is wound with a secondary coil which is connected up in series with a ballistic galvanometer. Experiment has shown that if the specimen has been divested of its magnetic history in the manner previously described, then, on reversing in turn in the magnetizing coil each of a series of currents of increasing magnitude, the inductions calculated from the corresponding kicks of the galvanometer give, when plotted against the magnetizing forces in the coil, a curve which coincides with the true magnetization curve for the specimen; that is, the curve which would be obtained if the specimen were subjected to the process of reversals previous to the establishment of

each of the currents in the coil. This will be seen from fig. 2, which exhibits some results obtained on testing a specimen of steel. O A B C D is the magnetization curve for the

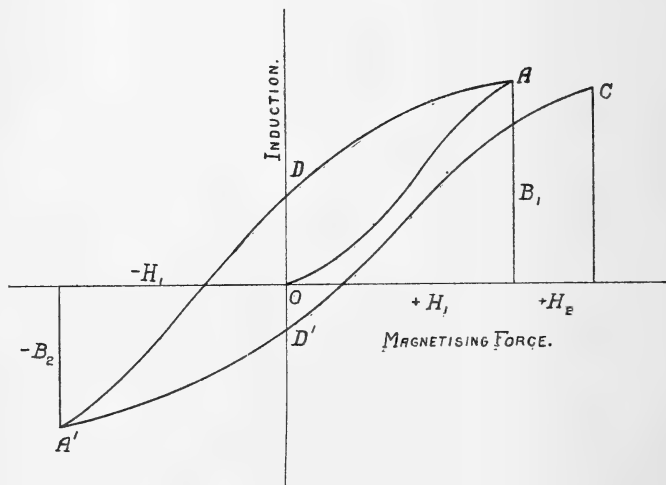
Fig. 2.



material in the neutral condition. If now, starting with the specimen in this condition, the strength of the field is changed from 0 to  $H_1$ , the induction becomes  $B_1$ ; that is to say, the point A on the curve is arrived at. Now let the current in the magnetizing coil be reversed. The induction changes from  $+B_1$  to  $-B_1$ ; the kick of a ballistic galvanometer connected to a secondary coil wound on the specimen would be proportional to  $2B_1$ . If the field is once more reversed the induction becomes  $+B_1$ ; that is, after the double reversal the point A is once more reached. Let now the field be gradually increased from  $+H_1$  to  $+H_2$ , and the induction in the specimen will follow the curve from A to B. If after arriving at the point B the field is changed from  $+H_2$  to  $-H_2$ , the induction changes from  $+B_2$  to  $-B_2$ ; the kick of the galvanometer is therefore proportional to  $2B_2$ . Following this procedure the complete curve may be traced out.

It seems necessary to add that this only holds for a series of fields which are in ascending magnitude, and it is essential that the specimen be initially neutral. For the method to yield accurate results the hysteresis loop must be not only closed but symmetrical. If, however, the neutral condition of the specimen has been interfered with by magnetic or thermal operations, the method is not applicable. This will be apparent from fig. 3, which is intended to show the results

Fig. 3.



obtained on applying the method to the examination of a specimen which had been exposed to thermal treatment. On establishing the field  $+H_1$  the induction became  $+B_1$ ; on reversing the field the induction became  $-B_2$ , and  $B_2$  differs widely from  $B_1$ . The change of induction is  $B_1 + B_2$ , and the kick of the galvanometer is proportional to this change, and hence proportional neither to  $2B_1$  nor to  $2B_2$ . The method of reversals is therefore not applicable unless the specimen is in the strictly cyclic condition. On once more reversing the field and increasing it to  $+H_2$ , the induction followed the curve  $A'D'C$ . If the field  $+H_2$  had been applied in the first place, a widely different value of the induction would have been obtained. Thus the magnetization curve yielded by this method, when applied to a specimen in the condition brought about by thermal treatment, depends largely upon the procedure.

Table II. shows the results obtained on starting and

reversing a number of times, a magnetizing field of 11 c.g.s. units in a solenoid containing a specimen of steel in the freshly annealed condition. It will be seen that the change of induction brought about by the first reversal is 7572 c.g.s. units, whereas that brought about by the tenth is 6248 c.g.s. units. Hence the induction calculated from the first throw of a ballistic galvanometer connected to a secondary coil wound on the helix would be over 20 per cent. in excess of that which is characteristic of the material when in the cyclic condition.

TABLE II.

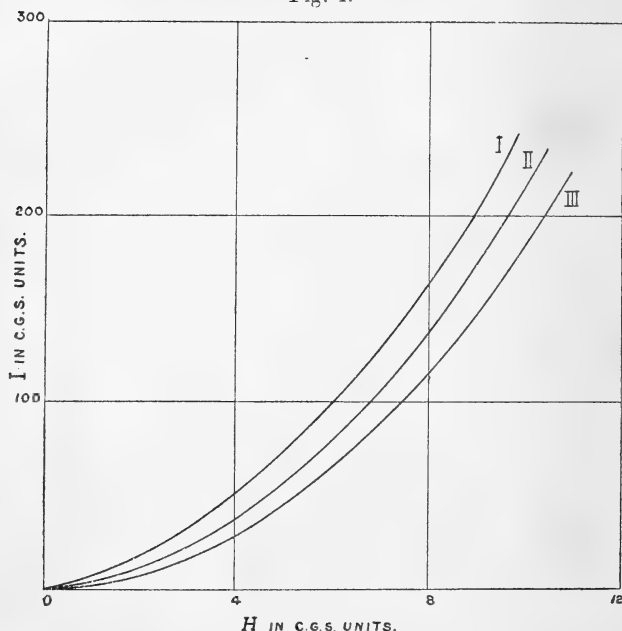
Value of H in c.g.s. units.	Induction in c.g.s. units.	Change of Induction on reversal.
+11	4186	7572
-11	3386	6859
+11	3473	6746
-11	3273	6559
+11	3286	6484
-11	3198	6396
+11	3198	6359
-11	3161	6285
+11	3126	6250
-11	3124	6248
+11	3124	

The table shows that the results of tests which have been carried out on annealed steel are very approximately correct, even though the specimen had not been submitted to the process of reversals, provided that the magnetizing current has been reversed nine or ten times previous to the galvanometer-throw being observed. A magnetization curve so obtained would, however, lie slightly above the curve which is characteristic of the material when in the truly neutral condition.

Fig. 4 shows the magnitude of the "Sensitive State" induced in a specimen of steel by cooling from room temperature to that of liquid air boiling under normal conditions (about  $-190^{\circ}$  C.). Curve I. is the magnetization curve for the material at room temperature when in the neutral condition. This curve having been obtained, the specimen was subjected to the neutralizing process and brought to the

temperature of liquid air. On being tested at this temperature it gave results which are shown in curve II. Finally

Fig. 4.



the specimen was submitted once more to a process of reversals and retested, care being taken to keep the temperature at  $-190^{\circ}\text{C}$ . throughout. The results shown in curve III. were then obtained.

Curves I. and III. are the true magnetization curves for the material at room temperature and at  $-190^{\circ}\text{C}$ . respectively; it is these curves, and not curves I. and II., which must be employed in contrasting the magnetic properties of the materials at the two temperatures. If, for any reason, it is desired to obtain curve II., it is evident that a method of experimenting must be employed which does not involve reversal of the magnetizing force. Tests carried out by the method of reversals upon a specimen in the condition brought about by the cooling alone would yield neither curve II. nor curve III.

In carrying out the tests which furnish the magnetization curve for the specimen at the temperature of liquid air, it is necessary that not only the tests, but also the neutralization process, should be carried out at that temperature. This is



very important, as even a small temperature change is sufficient to produce quite an appreciable "Sensitive State."

Research work in magnetism at temperatures other than ordinary room temperature is somewhat difficult to carry out; low temperature research is moreover very costly; and it seems to the authors of great importance that the special precautions necessary in such work should be widely known.

II. *Some Remarks on Conduction of Heat through Rarefied Gases.* By Prof. M. S. SMOLUCHOWSKI, Ph.D., LL.D.\*

IT has been shown, by researches published more than ten years ago †, that the apparent decrease of thermic conductivity of gases, with progressing rarefaction, is due not to a decrease of the coefficient of conductivity, but to a surface phenomenon, analogous to the sliding of gases discovered by Kundt and Warburg.

There also the kinetic explication of this phenomenon has been given, which leads us to expect the existence of a very simple law of conduction for extreme rarefactions, the quantity of transmitted heat then being proportional to the gas pressure and independent of the thickness of the layer of gas.

Experimental evidence of such relations has been given first in some experiments of C. F. Brush ‡, and recently in a careful investigation by F. Soddy and A. J. Berry §.

The form of the law for extreme rarefactions being established, the question arises as to the value of the constant of proportionality, or as these authors put it: the quantity of heat, referred to unit of hot surface, one degree of difference of temperature and 0.01 mm. of mercury pressure, which they designate by Q.

I had found by a roughly approximative reasoning || (assuming the molecules to be divided in three parts, moving parallel to the axes) that the flux of heat carried for one degree of difference of temperature ought to be of the

\* Part of a paper which will be published in the *Bulletin de l'Acad., Cracovie*, 1910. Communicated by the Author.

† Smoluchowski, *Ann. d. Phys.* lxiv. p. 101 (1898); *Wien. Akad. Sitzgsber.* cvii. p. 304 (1898), cviii. p. 5 (1899); *Phil. Mag.* xlvi. p. 199 (1898); Gehrcke, *Ann. d. Phys.* ii. p. 102 (1900).

‡ *Phil. Mag.* xlv. p. 31 (1898).

§ *Proc. Roy. Soc.* lxxxiii. A. p. 254 (1910).

|| *Wien. Akad. Sitzgsber.* cvii. p. 328 (1898).

order of magnitude  $\frac{\rho sc}{6}$ , where  $\rho$ =density,  $s$ =specific heat,  $c$ =mean molecular velocity. This result applies to the case when every molecule assumes, by its impact on the solid wall, the *vis viva* corresponding to the temperature of the latter; but it ought to be multiplied by  $\frac{1-\beta}{1+\beta}$ , if only a partial equalization of temperatures is taking place, according to the formula

$$\vartheta - \theta_0 = \beta(\theta_M - \theta_0),$$

$\theta_0$ ,  $\theta_M$ ,  $\vartheta$  denoting the temperatures of the wall, of the impinging and the emitted molecules.

Messrs. Soddy and Berry use the same formula, with slight difference of notation, putting

$$Q = \frac{n}{N} \frac{HG}{6},$$

where  $n$ =number of molecules per  $\text{cm.}^3$  at 0.01 mm. pressure,  $N$ =number contained in one gram,  $H$ =molecular heat at constant volume,  $G$ =mean molecular velocity.

Their experiments enabled them to determine the ratio of the observed transport of heat  $K$  to the calculated value  $Q$  for 11 gases, and from these numbers, ranging between 1.09 and 0.25, they attempt to draw conclusions about the factor which in the above has been accounted for by introduction of the coefficient  $\beta$ . These results, however, seemed rather strange, since only values inferior to unity could be expected.

But when exact numbers are in question, the rough estimate referred to above is evidently insufficient, and an exact calculation ought to be substituted instead.

Consider the gas contained between two parallel horizontal plates, the upper one of temperature  $\theta_2$ , the lower one of temperature  $\theta_1$  (supposed to be one degree lower). It is convenient then, instead of making the above supposition about  $\beta$ , to follow Maxwell's supposition\* that the surface of a solid acts as a partial reflector, by reversing the normal velocity component for the fraction  $(1-f)$  of the incident molecules, while the rest are "absorbed" and emitted with the velocity distribution corresponding to the temperature of the wall.

We suppose the gas to be so rarefied that the mean free path of the molecules is much greater than the distance of

\* Maxwell, Phil. Trans. clii. p. 231 (1879).



The quantity of heat lost by the lower plate is

$$Q = \frac{2ms}{\sqrt{6\pi}} [\theta_2 n_2' c_2 + \theta_1 n_1' c_1 - \theta_2 n_2 c_2 - \theta_1 n_1 c_1] = \frac{2ms}{\sqrt{6\pi}} (\theta_2 - \theta_1) f n_1 c_1.$$

Now equations (1) and (4) give

$$n_1 = \frac{n}{2-f} \frac{c_2}{c_1 + c_2},$$

so we have

$$Q = \frac{2fms}{\sqrt{6\pi}(2-f)} \frac{c_1 c_2}{c_1 + c_2}.$$

If we put  $c = \frac{2c_1 c_2}{c_1 + c_2}$ , and  $f = 1 - \beta$ , we get finally

$$Q = \frac{\rho c s}{\sqrt{6\pi}} \frac{1 - \beta}{1 + \beta} \dots \dots \dots (5)$$

This is the exact value for the conduction of heat in highly rarefied gases ; we see that it is greater than the value calculated before, and all the numbers given by Messrs. Soddy and Berry for  $\frac{K}{Q}$  ought to be multiplied by the factor  $\sqrt{\frac{\pi}{6}} = 0.7236$ , which gives the series :

	A.	Ne.	N <sub>2</sub> .	O <sub>2</sub> .	CO.	N <sub>2</sub> O.	C <sub>2</sub> H <sub>2</sub> .	CO <sub>2</sub> .	CH <sub>4</sub> .	He.	H <sub>2</sub> .
$\frac{K}{Q}$	=0.79	0.75	0.68	0.62	0.59	0.56	0.52	0.52	0.49	0.37	0.18

It proves that the coefficient  $\beta$  is never to be neglected, that is to say, that the heat interchange of the molecules on impact is always imperfect. The order of gases suggests the rule that the interchange of energy is worse for the lighter molecules and for the diatomic and polyatomic ones in comparison with the heavier and monatomic ones. The first part of this rule is to be explained by a simple mechanical reasoning, showing the interchange of energy between colliding spheres to be the more imperfect, the greater the difference of their masses (here we have the molecules of the gas colliding with the heavy platinum molecules). Also the second part of this rule seems to be in accordance with other phenomena of conduction of heat, showing intramolecular energy being comparatively less disposed to equalization by impacts than energy of progressive motion.

Lemberg University.

III. *Hamilton's Equations and the Partition of Energy between Matter and Radiation.* By S. B. McLAREN, Assistant Lecturer in Mathematics in the University of Birmingham\*.

§ 1. INTRODUCTION.

IN this paper two things are done. Maxwell's theory of partition is extended to forms of energy not quadratic in the velocities or momenta, and it is applied to the interaction of matter and radiation. As to the form, it is enough if we are assured that for finite energy all momenta are finite. That condition is satisfied by the expressions for energy which arise in the electromagnetic theory of mass, by the formula  $c(p^2 + m_0^2 c^2)^{\frac{1}{2}}$ , for example, which gives the energy of Lorentz's deformable electron. There  $p$  is the momentum,  $m_0$  the mass for an infinitesimal value of  $p$ , and  $c$  the velocity of light.

For the rest (§ 4), I have tried to fill the gap which Larmor (Bakerian lecture) has remarked in the work of Jeans and Lorentz on radiation. With Jeans the radiation is confined to a finite space bounded by reflecting walls. Since within these there is no ordinary matter, all radiation falls at once into the normal vibrations proper to the space enclosed; no redistribution of energy is possible, and the amount of energy to be assigned to each normal mode is fixed once for all by applying Fourier's analysis to the original field of force arbitrarily given.

If Jeans can nevertheless draw conclusions as to the partition of energy, it is because he assumes that there is one dynamical system of which matter and æther are parts, that Maxwell's statistical method can be applied to it, and that in any complete formula for the energy his expression for the radiation will form a part. It is such a formula I give here; but I have not reached it except by assuming the atomic structure of matter. My electron is an invariable distribution of charge free only to move as a whole. It then appears that the whole electromagnetic energy must be regarded as belonging to the radiation, excepting only a term which depends on the position of the electrons at any instant and would be the electrostatic energy if they were at rest. In this division nothing is left for kinetic energy of the electrons, the system of equations cannot be brought to Hamilton's form, and it does not seem that Maxwell's statistical method can be

\* Communicated by the Author.

applied. It may be, however, that there is true material energy as well as electromagnetic; and then Maxwell's reasoning can be resumed, with its old paradoxical conclusion of equipartition.

To many it has always seemed that the method of statistics builds much, and to little purpose, on a very unsure foundation.

I recall its postulates before deducing from them the results just described.

## § 2. THE GENERAL DYNAMICAL METHOD.

(1) The laws of heat are dynamical.

This is fundamental and at once raises the question of reversibility. Fire may freeze a kettle instead of boiling it, only the law of chances favours the more familiar process (Jeans). Let there be a dynamical system with a very large number of coordinates. The equations of motion are in Hamilton's form:

$$\frac{dq_r}{dt} = \frac{dH}{dp_r}, \quad \frac{dp_r}{dt} = -\frac{dH}{dq_r}, \quad (r=1, \dots, n).$$

The  $p$ 's represent momenta; the  $q$ 's are coordinates of position.

Suppose the initial values are regarded as mere matter of chance. Then it may be shown (Jeans' 'Dynamical Theory of Gases') that in the vast majority of cases the distribution of energy and momentum is near what may be called the temperature distribution. It is therefore very long odds that, if we start with any arbitrary heat distribution, the end will be equality of temperature. With this we are brought naturally to the second assumption.

(2) Any set of values of the coordinates and momenta is possible provided it is consistent with the constancy of energy.

(3) All these configurations are of equal probability.

For what is possible by (2) must have attached to it a definite measure of probability. The successive configurations assumed by any one system are equally probable, and so far as we know they have only one thing in common—the fact that their energy is the same. Hence (3), and hence without further assumption the law of equipartition. The only condition to be satisfied by  $H$  has already been stated.

The truth of (2) is certainly not at all obvious. When the number  $n$  is finite it may indeed be allowed very

plausible. Although we cannot deal with single molecules, it seems impossible short of that to fix any limit for the initial distribution. This can hardly be interpreted in terms of dynamics, save by assuming that any values of the co-ordinates may occur. But (2) is certainly not always true when  $n$  is infinite. Thus the vortex theory of matter allows only such values of the variables as satisfy the condition of constant circulation; for that is the essence of the distinction between matter and what is not material. If for any reason (2) be abandoned, it need not be concluded that there is no state of temperature equilibrium. We infer rather that the theory of heat depends upon properties of matter more special than the abstractly dynamical, an inference which is in itself made certain by the observed laws of radiation. These involve an absolute constant, a length in dimensions, which cannot enter into the purely dynamical scheme (Jeans). The conclusion thus forced upon us ought to be as welcome to the physicist as it is distasteful to the mathematician. A deduction of Wien's law from Hamilton's equation adds nothing to our knowledge of the nature of matter; an explanation of it may yet add much

§ 3. THE PARTITION OF ENERGY IN A HAMILTONIAN SYSTEM.

The equations of motion are

$$\frac{dq_r}{dt} = \frac{dH}{dp_r}, \quad \frac{dp_r}{dt} = -\frac{dH}{dq_r} \quad (r=1, 2, \dots n). \quad \dots (1)$$

The kinetic energy belonging to the  $r$ th degree of freedom I define as equal to

$$\frac{1}{2} p_r \frac{dH}{dp_r} \dots \dots \dots (2)$$

This agrees with the ordinary definition when  $H$  is quadratic.

Notice also that if  $L$  is the Lagrangian function, we have

$$L = \sum p_r \frac{dH}{dp_r} - H, \dots \dots \dots (3)$$

$$\frac{1}{2}(L + H) = \sum \frac{1}{2} p_r \frac{dH}{dp_r} \dots \dots \dots (4)$$

$L$  is in the simple case equal to the difference of the kinetic and potential energy;  $H$  is equal to their sum. Thus the kinetic energy again appears in (4) as a sum of terms of the form (2).

We write

$$dN = dp_1 dp_2 \dots dp_n dq_1 dq_2 \dots dq_n. \quad \dots (5)$$

The product on the right-hand side of (5) is known to be a differential invariant.

Let there be systems all satisfying (1) but starting from all initial conditions. Let the number originally having their coordinates within the limits  $dp_r, dq_r$ , and so on be  $dN$ . This distribution is invariant, in consequence of the invariant character of  $dN$ . Such of the systems as have originally identical values of  $H$  will always have the same value for it.

Each of the systems is on the average in a state of temperature equilibrium, and the distribution of energy proper to the temperature exists in it. This distribution of energy can be inferred by averaging over all the systems, since it exists in each.

The theorem of equipartition requires

$$\int p_r \frac{dH}{dp_r} dN = \int p_s \frac{dH}{dp_s} dN. \dots \dots (6)$$

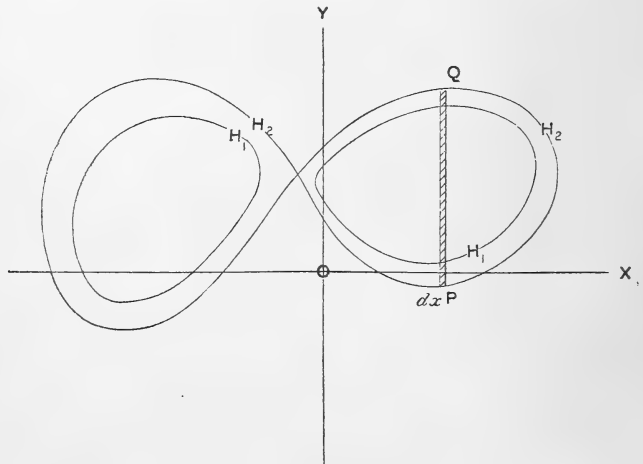
(6) is to hold for all values of  $r$  and  $s$ .

It will be enough to show

$$\int p_r \frac{dH}{dp_r} dp_r dp_s = \iint p_s \frac{dH}{dp_s} dp_r dp_s. \dots (7)$$

From (7) follows (6) by integrating for the other coordinates.

The final result is to apply to systems all of which possess the same total energy. In (7), therefore, the integration may be taken over the region  $H_2 > H > H_1$ , and the difference  $H_2 - H_1$  can be made infinitesimal.



Suppose the curves  $H = H_2$  and  $H = H_1$  are represented in the plane of coordinates;  $p_r$  may be measured along OX and  $p_s$  along OY.



The condition as to  $H$  ensures that these curves have no branches at infinity and are therefore closed. (7) becomes

$$\iint x \frac{dH}{dx} dx dy = \iint y \frac{dH}{dy} dx dy. \dots (8)$$

The integration is to be taken all over the areas enclosed between  $H=H_2$  and  $H=H_1$ . In the figure given the curve  $H=H_2$  has a double point. Now

$$\iint y \frac{dH}{dy} dx dy = \iint_2 y \frac{dH}{dy} dx dy - \iint_1 y \frac{dH}{dy} dx dy. \dots (9)$$

In (9) the right-hand side represents the difference of the integrals taken over the area enclosed by  $H=H_2$  and over the area enclosed by  $H=H_1$ . It is conceivable that the curves may consist of two or more separated portions and that  $H=H_2$  may enclose  $H=H_1$  in one portion and be enclosed by it in the other.

$$\begin{aligned} \iint_2 y \frac{dH}{dy} dx dy &= \iint \frac{d}{dy}(yH) dy dx - \iint_2 H dy dx \\ &= \int H_2(y_Q - y_P) dx - \iint_2 H dy dx \\ &= H_2 A_2 - \iint_2 H dy dx. \dots (10) \end{aligned}$$

$y_Q$  and  $y_P$  are the ordinates where PQ cuts  $H=H_2$ .

In (9)  $A_2$  represents the area enclosed by the curve  $H=H_2$ .

In exactly the same way

$$\iint_2 x \frac{dH}{dx} dx dy = H_2 A_2 - \iint_2 H dy dx \dots (11)$$

From (10) and (11) the truth of (8) follows. There is equal partition of the kinetic energy.

Further important results can be proved. In (8) we may substitute for  $x$  and  $y$  any two of the  $2n$  coordinates. Thus

$$\int q_r \frac{dH}{dq_r} dN = \int p_s \frac{dH}{dp_s} dN, \dots (12)$$

and (12) is true for all values of  $r$  and  $s$ . For the proof it is only necessary that  $H$  considered as a function of the  $q$ 's

should satisfy the same general conditions as before. For a finite value of  $H$  all the  $q$ 's must be finite. (12) is evidently connected with the virial of Clausius, which is, in fact, equal to

$$\sum \frac{1}{2} q_r \frac{dH}{dq_r}.$$

Thus the virial is equally distributed and it is equal to the kinetic energy. In particular, if  $H$  is a quadratic function the virial is identical with the potential energy and the potential and kinetic energies are equal.

One more point may be noticed,

$$\iint x \frac{dH}{dy} dx dy = 0. \dots \dots (13)$$

In (13) the integration is over the same area as in (8).

For

$$\left. \begin{aligned} \iint x \frac{dH}{dy} dx dy &= \iint_2 x \frac{dH}{dy} dx dy \sim \iint_1 x \frac{dH}{dy} dx dy, \\ \iint_2 x \frac{dH}{dy} dx dy &= \int x(H_Q - H_P) dx = 0, \end{aligned} \right\} (14)$$

since  $H_Q = H_P = H_2$ .

Similarly the second term on the right of (14) vanishes and (13) is proved.

For  $x$  and  $y$  in (14) we may substitute any two of the  $p$ 's, or if the necessary conditions hold also for the  $q$ 's, any two of the  $2n$  coordinates. So

$$\int p_r \frac{dH}{dp_s} dN = 0, \dots \dots (15)$$

$$\int q_r \frac{dH}{dq_s} dN = 0, \dots \dots (16)$$

$$\int p_r \frac{dH}{dq_s} dN = 0, \dots \dots (17)$$

$$\int q_r \frac{dH}{dp_s} dN = 0, \dots \dots (18)$$

(15) and (16) hold so long as  $r$  and  $s$  differ; (17) and (18) hold for all values.

§ 4. DISTRIBUTION OF ENERGY IN MATTER AND RADIATION.

For the æther

$$\left(\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}\right)\phi + 4\pi\rho = 0 \dots (19)$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}\right)\mathbf{F} + 4\pi\rho \frac{\mathbf{u}}{c} = 0 \dots (20)$$

$\phi$  and  $\mathbf{F}$  are the scalar and vector potentials.  $\mathbf{u}$  is the vector velocity of electricity of density  $\rho$ .

For any electron

$$\frac{d\mathbf{p}}{dt'} = \int \mathbf{E}'\rho dv \dots (21)$$

In (21)  $\mathbf{p}$  is the true material momentum of the electron, supposed rigid we must remember.  $\mathbf{E}'$  is the whole electric intensity at any point of this moving electron.  $\frac{d}{dt'}$  indicates differentiation at a point moving with the electron. In (21) the integration is all over the volume of the electron

$$\mathbf{E}' = -\frac{1}{c} \frac{d\mathbf{F}}{dt} - \nabla\phi - \frac{1}{c} [\mathbf{u}, \text{Curl } \mathbf{F}] \dots (22)$$

At the perfectly reflecting boundary there is the condition that the tangential component of

$$\mathbf{E} = -\frac{1}{c} \frac{d\mathbf{F}}{dt} - \nabla\phi$$

is zero, and everywhere we have

$$\text{Div } \mathbf{F} + \frac{1}{c} \frac{d\phi}{dt} = 0 \dots (23)$$

Now  $\mathbf{F}$  and  $\phi$  are clearly indeterminate to the extent that we may add to  $\mathbf{F}$  a term of the form  $\nabla\omega$  and to  $\phi$  the term  $-\frac{1}{c} \frac{d\omega}{dt}$ . The electric and magnetic forces remain unaltered, but (23) requires

$$\left(\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}\right)\omega = 0.$$

$\omega$  can be chosen to satisfy this equation and to have any assigned value at the surface: we can therefore so choose  $\omega$  as to make the value of  $\phi$  always zero at the surface. Then the condition that  $\mathbf{E}$  is normal at the surface requires also that  $\mathbf{F}$  is normal.

Suppose  $F = F_1 + \nabla\psi \dots \dots \dots (24)$

and  $\text{Div } F_1 = 0, \dots \dots \dots (25)$

then  $\text{Div } F = \nabla^2\psi$

or  $-\frac{1}{c} \frac{d\phi}{dt} = \nabla^2\psi, \dots \dots \dots (26)$

or by (19)

$$\nabla^2\left(\phi + \frac{1}{c} \frac{d\psi}{dt}\right) + 4\pi\rho = 0. \dots \dots (27)$$

(27) shows that  $\left(\phi + \frac{1}{c} \frac{d\psi}{dt}\right)$  is the potential due to a distribution of density  $\rho$  within the space. If we choose  $\psi$  zero at the surface, then since  $\phi$  is also zero, it follows that  $\phi + \frac{1}{c} \frac{d\psi}{dt}$  is the potential due to charge of density  $\rho$  and the charge it induces on the surface, also at the surface  $\nabla^2\psi$  and  $\psi$  are both zero, the former by (26).

Again at the surface  $F$  is normal. (24) shows that  $F_1$  is also normal since  $\psi=0$ .

In view of (24) and the surface condition,  $F_1$  can be expanded in a series of normal functions (Poincaré, *Am. Journal of Maths.*),

$$F_1 = \sum_{r=1}^{r=\infty} \alpha_r F_r, \dots \dots \dots (28)$$

where

$$\left. \begin{aligned} (\nabla^2 + \kappa_r^2)F_r &= 0 \\ \text{Div } F_r &= 0 \\ F_r \text{ normal at surface} \\ \int F_r^2 dV &= 4\pi \end{aligned} \right\} \dots \dots \dots (29)$$

$dV$  is an element of the volume enclosed, the numbers  $\kappa_r$  are arranged in an ascending series,

The last of (29) is required to determine  $F_r$  absolutely. When  $r$  and  $s$  differ we have the normal property

$$\int F_r F_s dV = 0. \dots \dots \dots (30)$$

It may be in certain cases that  $\kappa_r = \kappa_s$ , but (30) still holds. We can easily see from (28), (29), (30) that

$$\int F_r F_1 dV = 4\pi\alpha_r, \dots \dots \dots (31)$$

also

$$\int F_r \nabla^2 F_1 dV = \int F_1 \nabla^2 F_r dV + \int \left( F_r \frac{dF_1}{dv} - F_1 \frac{dF_r}{dv} \right) dS, \quad (32)$$

$\frac{d}{dv}$  denoting differentiation along the outward drawn normal at the element of surface  $dS$ . At the surface  $F_1$  and  $F_r$  are normal,  $F_1$  and  $F_r$  are their normal components.

From (25) and (29) it can be shown easily that

$$\frac{dF_r}{dv} + F_r \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = 0,$$

$$\frac{dF_1}{dv} + F_1 \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) = 0.$$

Here  $\rho_1$  and  $\rho_2$  are the principal radii of curvature at the surface. These give  $F_r \frac{dF_1}{dV} = F_1 \frac{dF_r}{dV}$ .

Using this and (31) we can reduce (32) to

$$\int F_r \nabla^2 F_1 dV = \int F_1 \nabla^2 F_r dV = -4\pi \kappa_r^2 \alpha_r. \quad (33)$$

We now use the value of  $F$  by (24) in the equations of æther (20) and the equations of motion (21). First in (20) and we have

$$\left( \nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2} \right) F_1 + \nabla \left( \nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2} \right) \psi + 4\pi \rho \frac{\mathbf{u}}{c} = 0.$$

Multiply this equation throughout by  $F_r dV$  and integrate through the volume. By (31) and (33)

$$-4\pi \left( \frac{1}{c^2} \frac{d^2 \alpha_r}{dt^2} + \kappa_r^2 \alpha_r \right) + \int F_r \nabla \left( \nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2} \right) \psi dV + 4\pi \int \rho \frac{\mathbf{u}}{c} F_r dV = 0.$$

The term involving  $\psi$  is easily seen to vanish on partial integration when we recall the surface conditions  $\psi = 0$ ,  $\nabla^2 \psi = 0$ .

We have finally

$$\frac{1}{c^2} \frac{d^2 \alpha_r}{dt^2} + \kappa_r^2 \alpha_r = \int \rho \frac{\mathbf{u}}{c} F_r dV, \quad . . . . \quad (34)$$

and the integral on the right hand of (34) is taken wherever there is electric charge.

Use (24) next to transform (22),

$$\mathbf{E}' = -\frac{1}{c} \frac{d\mathbf{F}_1}{dt} - \nabla \left( \phi + \frac{1}{c} \frac{d\psi}{dt} \right) - \frac{1}{c} [\mathbf{u}, \text{Curl } \mathbf{F}_1],$$

or

$$\mathbf{E}' = -\frac{1}{c} \frac{d\mathbf{F}_1}{dt} - \frac{1}{c} [\mathbf{u}, \text{Curl } \mathbf{F}_1] - \nabla \phi_0,$$

$$\begin{aligned} \text{here} \quad \phi_0 &= \phi + \frac{1}{c} \frac{d\psi}{dt}, \\ \nabla^2 \phi_0 + 4\pi\rho &= 0. \end{aligned}$$

Substitute this value of  $\mathbf{E}'$  in the equations of motion of the  $n$ th electron

$$\frac{d\mathbf{p}_n}{dt'} + \int \left\{ \frac{1}{c} \frac{d\mathbf{F}_1}{dt} + \left[ \frac{\mathbf{u}_n}{c} \text{Curl } \mathbf{F}_1 \right] + \nabla \phi_0 \right\} \rho dv_n = 0;$$

this can be written

$$\frac{d\mathbf{p}_n}{dt'} + \int \left\{ \frac{1}{c} \frac{d\mathbf{F}_1}{dt'} + \nabla \phi_0 - \frac{\mathbf{u}_n}{c} \nabla \mathbf{F}_1 \right\} \rho dv_n = 0.$$

The electron has a movement of translation only, each point of it has the same velocity  $\mathbf{u}_n$ ,  $-\int \nabla \phi_0 \rho dv_n$  is the electrostatic force due to the field of potential  $\phi_0$ .

Let  $V_N$  be the electrostatic energy of this field. It is a function of the positions of the electrons. If  $\mathbf{q}_n$  be the vector position of a point of the  $n$ th electron. Then

$$\int \nabla \phi_0 \rho dv_n = \frac{dV_N}{d\mathbf{q}_n}.$$

$$\text{Also} \quad \int \frac{\mathbf{u}_n}{c} \nabla \mathbf{F}_1 \rho dv_n = \frac{d}{d\mathbf{q}_n} \left\{ \int \frac{\mathbf{u}_n}{c} \mathbf{F}_1 \rho dv_n \right\},$$

on the right the differentiation is partial with respect to  $\mathbf{q}_n$  the function being expressed in terms of the velocity  $\mathbf{u}_n$  and the coordinate  $\mathbf{q}_n$ . The equation of motion becomes

$$\frac{d}{dt'} \left\{ \mathbf{p}_n + \frac{1}{c} \int \mathbf{F}_1 \rho dv_n \right\} - \frac{d}{d\mathbf{q}_n} \left\{ \int \frac{\mathbf{u}_n}{c} \mathbf{F}_1 \rho dv_n - V_n \right\} = 0. \quad (35)$$

In (35) insert the value of  $\mathbf{F}_1$  from (28), and write

$$\frac{1}{c} \int \mathbf{F}_r \rho dv_n = \mathbf{F}_{rn},$$

$\mathbf{F}_{rn}$  is a function merely of the position of the  $n$ th electron,

(34) and (35) can now be written

$$\frac{d}{dt} \left( \frac{1}{c^2} \frac{d\alpha_r}{dt} \right) - \sum_{n=1}^{n=N} \mathbf{u}_n F_{rn} + \kappa_r^2 \alpha_r = 0. \quad (34)$$

$$\frac{d}{dt} \left( \mathbf{p}_n + \sum_{r=1}^{r=\infty} \alpha_r F_{rn} \right) - \frac{d}{d\mathbf{q}_n} \left( \sum_{r=1}^{r=\infty} \mathbf{u}_n \alpha_r F_{rn} - V_n \right) = 0. \quad (35)'$$

Suppose  $\mathbf{p}_n$  is derived from a Lagrangian function  $L_0$  for the purely material energy. Then the whole system of equations (34)' and (35)' has the Lagrangian function  $L$ .

$$L = L_0 - V_N + \sum_{n=1}^{n=N} \sum_{r=1}^{r=\infty} \mathbf{u}_n \alpha_r F_{rn} + \sum_{r=1}^{r=\infty} \left\{ \frac{1}{2c^2} \left( \frac{d\alpha}{dt} \right)^2 - \frac{1}{2} \kappa_r^2 \alpha_r^2 \right\}, \quad (36)$$

remembering that  $F_{rn}$  is a function only of  $\mathbf{q}_n$ .

The corresponding Hamiltonian function is

$$H = \sum_{n=1}^{n=N} \mathbf{u}_n \frac{dL_0}{d\mathbf{u}_n} - L_0 + V_n + \sum_{r=1}^{r=\infty} \left\{ \frac{1}{2c^2} \left( \frac{d\alpha_r}{dt} \right)^2 + \frac{1}{2} \kappa_r^2 \alpha_r^2 \right\}. \quad (37)$$

The "gyrostatic terms" linear in  $\mathbf{u}_n$  and  $\alpha_r$  disappear.

The need for a function  $L_0$  appears at once if we are to transform (34)' and (35)' to Hamilton's form.

If  $L_0$  is zero the momentum  $\frac{dL}{d\mathbf{u}_n}$  does not contain the velocities at all. The  $3N$  material momenta are functions of the coordinates  $\alpha_r$  and  $\mathbf{q}_n$  only. Therefore the velocities  $\mathbf{u}_n$  cannot be expressed as functions involving the corresponding momenta. By (35)' we can derive relations between the various velocities so that these are not independent. The statistical method cannot be applied. If, on the other hand,  $L_0$  is not zero we may apply Maxwell's method in the ordinary way. By the first part of this paper the equipartition of kinetic energy is expressed by the equation

$$\frac{1}{2} \overline{p_r \frac{dH}{dp_r}} = \frac{1}{2} \overline{p_s \frac{dH}{dp_s}},$$

the bars denoting average values.

$$\text{Or} \quad \frac{1}{2} \overline{u_r \frac{dL}{du}} = \frac{1}{2} \overline{u_s \frac{dL}{du_s}},$$

where  $u_r$  and  $u_s$  denote different velocities.

Here put  $\frac{d\alpha_r}{dt}$  for  $u_r$ , and  $u_n$  for  $u_s$ , so that  $u_n$  denotes the component of  $\mathbf{u}_n$  in the direction of the  $x$  axis.

$$\frac{1}{2c^2} \left( \frac{d\alpha_r}{dt} \right)^2 = \frac{1}{2} u_n \frac{dL_0}{du_n} + \frac{1}{2} u_n \sum_{r=1}^{r=\infty} \alpha_r F_{rn},$$

or

$$\frac{1}{2c^2} \left( \frac{d\alpha_r}{dt} \right)^2 = \frac{1}{2} u_n \frac{dL_0}{du_n} + \frac{1}{2} \frac{u_n}{c} \int F_{1x} \rho dt_n. \quad (38)$$

$F_{(1x)}$  denoting the  $x$  component of  $F_1$ . Each degree of æther has the same kinetic energy equal to the kinetic energy of each material degree of freedom. In the latter the gyrostatic term reappears as the second part of (38). Thus the kinetic energy of the electron so far as it is electromagnetic in origin is not any part of the total energy  $H$ .

It will be objected that what appears as radiation in this paper is not the real radiation. The field immediately surrounding an electron ought to be reckoned as part of the electron's energy. If, however, we attempt thus to distinguish radiation from what is not radiation, the result is a set of equations which do not belong to a dynamical system at all. With that I am not here concerned, my object has been to consider the results of the general dynamical method wherever it is adopted.

IV. *On the Ionization of the Atmosphere due to Radioactive Matter.* By A. S. EVE, M.A., D.Sc., McGill University, Montreal\*.

**T**HERE has been quite recently a well-marked advance in the accuracy of the determination of important radioactive constants. This progress is due to improvement in method and technique, and also to the cumulative evidence derived from different types of experiments. Three notable instances may be given: the determinations of the electronic charge, of the rate of discharge of  $\alpha$  particles from a gramme of radium, and of the number of ions produced in air by the various types of  $\alpha$  particle.

It is therefore now possible to estimate, with a reasonable degree of accuracy, the relation in atmospheric electricity between the radioactive agents and the ionizing effects attributable to them. A balance-sheet may be attempted between cause and effect; and although it will not be possible here to review all the valuable work done by many observers

\* Communicated by the Author.



in the past few years, yet we may concentrate the best results together, ascertain the measure of success attained by the radioactive theory of the ionization of the atmosphere, and indicate those outstanding difficulties which require further explanation and investigation.

At the outset it may be stated that the radioactive theory of atmospheric ionization holds the field, and that in most respects it appears sufficient and satisfactory. Yet there are some particulars in which this theory is defective or lacking, so that either there may be causes still undetermined, or our knowledge of the ascertained causes must be incomplete. Also we will consider only normal atmospheric conditions, and not such abnormal ones as thunder-storms—recently explained by Simpson as sometimes due to vertical air currents causing the breaking of water-drops with the well known consequent ionization.

In this paper the value of the electronic charge  $e$  will be taken as  $4.6 \times 10^{-10}$  electrostatic unit, and all numerical results will be given on that understanding. Also the value of  $\alpha$ , the coefficient of recombination of small ions, will be taken as  $3420e$  or  $1.57 \times 10^{-6}$ , at standard temperature and pressure. When the ionization is steady, there will be the well-known relation  $q = \alpha n^2$  between  $q$  the rate of production of ions and  $n$  the number of small ions actually present, in each case per cubic centimetre. So that we have the following table for the corresponding values of  $q$  and  $n$ .

$q$ .	$n$ .	$q$ .	$n$ .
1	800	5	1784
1.57	1000	6	1960
2	1131	6.28	2000
3	1386	9	2400
4	1600	14.13	3000

Unfortunately these simple relations rarely if ever exist in the atmosphere, because many small ions become transformed into large ions owing to the presence of water vapour, dust, mist, smoke, or other physical impurities in the atmosphere. Recombination is thus delayed, and the number of ions present is exceedingly variable and hard to determine. Other difficulties arise when we seek to determine  $q$ , the rate of

production of ions per cubic centimetre, for then an electro-scope must be employed, and at once there is an uncertain factor, namely the ionization due to the intrinsic radiation from the sides of the employed vessel, probably attributable to radioactive matter in the walls of the electro-scope. Moreover we are seeking for the value of  $q$  in free air, while in the electro-scope the ionization is in part due to the secondary rays ( $\beta$  rays) from the electro-scope, and this secondary radiation from the walls is in excess of that from free air. Recollecting all these pitfalls, we will commence with a careful estimate of the rate of production of ions due to the radium products known to be present in the atmosphere.

#### *Alpha Rays in the Atmosphere.*

The most simple and accurate method of measuring the amount of radium emanation in the atmosphere is that of collecting the emanation either by adsorption, using coconut charcoal, or by condensation with liquid air. The results obtained are in fairly good agreement.

Eve * .....	60
Satterley* .....	100
Ashman † .....	89
Mean .....	<u>83</u>

We will select 80 as the mean value, and state that one cubic metre of the atmosphere near the earth's surface contains the amount of radium emanation in equilibrium with  $80 \times 10^{-12}$  gramme of radium‡. . . . . (1)

There are of course considerable fluctuations with time and position, due to barometric pressure, wind, rain, and to the varying radium-contents of the underlying soils, so that it is only an approximation to the mean value which is here quoted. It is marvellous that the properties of this gas are such as to enable us to detect its presence and measure its amount, when it constitutes less than one thousand million billionth ( $1/10^{19}$ ) part of the atmosphere.

From the results of Rutherford and Geiger § we can now calculate the number of ions produced per  $\text{cm}^3$ , due to radium

\* Phil. Mag. Oct. 1908.

† Amer Journ. Sci., Aug. 1908.

‡ i. e.  $80 \times 10^{-12}$  Curies.

§ Proc. Roy. Soc. July 1909.

emanation, A and C, in the atmosphere, namely,

$$80 \times 10^{-12} \times 3.4 \times 10^{10} (1.74 + 1.87 + 2.37) 10^5 \text{ per m.}^3,$$

or  $8 \times 3.4 \times 5.98 \times 10^4 \text{ per m.}^3,$

or  $1.63 \text{ ions per cm.}^3 \text{ per sec, . . . . . (2)}$

*Thorium and Alpha Rays.*

We are not in a position to make any such accurate calculation for thorium. It is known, however, that the active deposit due to thorium on a negatively charged wire is, on an average, about 60 per cent. of that due to radium; and we may perhaps guess, for it is little better than a guess, that the  $\alpha$  rays from the thorium products in the atmosphere will contribute about, and not more than,

$$1.00 \text{ ion per cm.}^3 \text{ per sec. . . . . (3)}$$

*Gamma Rays from RaC in the Earth.*

Next the reasonable assumption will be made that the penetrating radiation, discovered by Rutherford, Cooke, and McLennan, is  $\gamma$  radiation due to the presence of uranium, radium, and thorium. It is noteworthy that the amount of the radioactive products in the soil is of the right order of magnitude to account for the ionization effects due to the penetrating radiation, at least near the earth's surface; also that the amount of radium emanation is of the right order to account for the active deposit which may be collected on a negatively charged wire. These relations encourage the belief that we are dealing with a true cause. It has also been shown repeatedly that the natural leak of an electroscope may be decreased in rate by surrounding it with screens of lead or water. Thus McLennan and Wright at Toronto have found that the water and ice of Lake Ontario form an efficient screen from the rays from the radioactive matter in the land beneath. F. W. Bates has verified this result over the ice on the River St. Lawrence near Macdonald College, and Gockel also has repeated the observation in a boat over a Swiss lake. There is therefore abundant and certain evidence that at least a part of the ionization of the air in an electroscope is due to  $\gamma$  rays from the radioactive matter in the earth. When we come to examine this question in more detail there are several points of considerable importance and interest.

We will first quote C. S. Wright's figures (Phil. Mag. Feb.

1909), obtained at Toronto over the land, and over the ice on Lake Ontario, correcting them for the new value,

$$e = 4.6 \times 10^{-10}.$$

	Lead.	Zinc.	Aluminium.
Over ice, Lake Ontario .....	6.4	4.4	4.8
Over land, Toronto.....	9.8	8.2	7.7
Difference .....	3.4	3.8	2.9

(4)

This indicates a difference of about 3 ions per  $\text{cm}^3$  per sec. within an aluminium electroscope due to the penetrating rays from the earth near Toronto. We have now to consider what would be the effect in free air, where the secondary radiation is less than in a metal electroscope. If radium is placed outside electroscopes of different metals the ionization within is dependent on the nature of the metal employed. Thus four observers have found, to an arbitrary scale:—

	Wright.	Soddy.	Bragg.	Eve.
Lead .....	100	100	100	100
Zinc .....	81.5	75	55	63
Aluminium .....	71.5	57	49	54

wherein the variations between observers are no doubt largely due to different dispositions of apparatus. McLennan found that in lead the secondary radiation was twice that due to the primary, but in aluminium one-half. On the other hand, W. Wilson found that in aluminium the secondary radiation gave rise to six times as much ionization as the primary; but his apparatus was not altogether satisfactory for this determination. Some experiments which I am making, not yet completed, seem to indicate that the ionization in an aluminium electroscope is only ten to twenty per cent. more than in free air; and Bragg finds that the ionization in a cardboard electroscope is about the same as in free air. He advocates the view that the primary  $\gamma$  rays do not ionize at all, but give rise to secondary  $\beta$  rays, and that these alone produce ionization. From Wright's figures we may therefore conclude that in free air the penetrating rays from the radioactive matter in the earth generate about

$$2.5 \text{ ions per } \text{cm}^3 \text{ per sec.} \quad \dots \quad (5)$$

This value is smaller than that found by several observers for the total ionization due to the penetrating radiation as determined by complete screening with lead or water. This fact suggests that a considerable part of the penetrating radiation might come from the radioactive matter in the air. Subsequently it will be shown, however, that such a conclusion appears altogether untenable, according to the present state of our knowledge and information.

*Theoretical Consideration.*

If there be a quantity  $Q$  grammes of *radium* at a point source, then the ionizing effect at a distance  $r$  in free air will be proportional to  $Q/r^2 e^{-\lambda r}$ , where  $\lambda$  is the coefficient of absorption of the  $\gamma$  rays in air. If  $N$  is the rate of production of ions in a cubic centimetre, we may write  $N = KQ/r^2 e^{-\lambda r}$ . This constant  $K$  was measured by me\* with a fair degree of accuracy, and its value inside a thin aluminium electroscope is  $3.4 \times 10^9$ . Its value in free air at standard pressure and temperature is therefore about  $3.1 \times 10^9$ . I am hoping to evaluate shortly this somewhat important constant with more accuracy. . . . . (6)

Integrating from 0 to  $\infty$  we can calculate the total number of ions which one gramme of radium can produce in virtue of the  $\gamma$  rays from the equilibrium amount of RaC. We have

$$\int_0^\infty K4\pi r^2 dr \frac{Q}{r^2} e^{-\lambda r}, \text{ or } 4\pi QK/\lambda. \dots (7)$$

Now  $\lambda/\text{density}$  is equal to .034 according to McClelland and to .04 according to Soddy. Selecting, in the present case, the former value, we have  $\lambda$  for air equal to .000044. Hence the total number of ions due to the  $\gamma$  rays from a gramme of radium in equilibrium is  $8.5 \times 10^{14}$  per sec. . (8)

But from Geiger's† experiment on the ions produced by  $\alpha$  rays, we know that Ra Em, A and C in equilibrium with one gramme of radium produce  $2 \times 10^{16}$  ions per second, or about 23 times as many as the  $\gamma$  rays, so far as the radium products in the *atmosphere* are concerned. Near the earth the effect of the  $\gamma$  rays must be halved, so that we get a ratio of 1 to 46 or

$\alpha$ rays Ra Em, A, C .....	1.63
$\gamma$ rays RaC .....	.035 . . . (9)

The addition for the ionization of the primary  $\beta$  rays is at

\* Phil. Mag. Sept. 1906.  
 † Proc. Roy. Soc. July 1909.

present unknown, but it appears to be about equal to that of the  $\gamma$  rays in effect. Perhaps the distinction is an erroneous one, but in any case we obtain an estimate of this sort—

Rays.	Ra.	Th.	Total.
$\alpha$ .....	1.63	1.00	2.63
$\beta$ .....	.035	.025	.06
$\gamma$ .....	.035	.025	.06
	1.70	1.05	2.75

(10)

Thus 2.7 ions per  $\text{cm.}^3$  per sec. is our value for the mean total ionization, due to all the radioactive matter in the *atmosphere*, near the earth's surface. It will be seen that the  $\alpha$  rays are mainly effective.

We can, however, calculate the ionization due to the  $\gamma$  rays from radium C in a more direct way than above. If  $N$  is the number of ions produced per  $\text{cm.}^3$  per sec.,  $\lambda$  the coefficient of absorption of the  $\gamma$  rays by air,  $Q$  the amount of radium C per  $\text{cm.}^3$  of the atmosphere, expressed in terms of the number of grammes of radium with which it is in equilibrium, then

$$N = \int_0^{\infty} 2\pi r^2 dr K \frac{Q}{r^2} e^{-\lambda r} = 2\pi QK/\lambda. \quad \dots (11)$$

Here  $K = 3.1 \times 10^9$ ,  $\lambda = .000044$ ,  $Q = 8 \times 10^{-17}$ ;

$$\text{whence } N = 2\pi \times 3.1 \times 10^9 \times 8 \times 10^{-17} / 4.4 \times 10^{-5} \left. \vphantom{N} \right\} = .035 \text{ ion per cm.}^3 \text{ per sec.} \quad (12)$$

This agrees with the former result (9).

It is possible to use a formula exactly similar to (11) in order to calculate the rate of production of ions near the earth's surface, due to the radium C in the *earth*. In this case  $K$  has the same value as before, but  $Q' = 1.4 \times 10^{-12} \times 2.7$ , the mean value found for sedimentary rocks by Strutt, and  $\lambda' = .034 \times 2.7$  (McClelland).

Hence

$$\begin{aligned} N' &= 2\pi Q'K/\lambda' \\ &= \frac{2\pi \times 3.1 \times 10^9 \times 1.4 \times 10^{-12} \times 2.7}{.034 \times 2.7} \\ &= \frac{2\pi \times 3.1 \times 1.4}{3.4} \times 10^{-1} \\ &= 0.80 \text{ ion per cm.}^3 \text{ per sec.} \quad \dots (13) \end{aligned}$$

This is the calculated value of the mean ionization at a given locality, near the earth, due to the penetrating rays from the radium C in the earth. It will be nearly constant at any given place, but it will vary from place to place according to the amount of radium present in the surface soils and rocks of the more immediate neighbourhood. It may be noted in particular that the ratio of  $N$  to  $N'$  is equal to that of  $Q/\lambda$  to  $Q'/\lambda'$ , whose accented letters refer to the earth. But  $\lambda$  and  $\lambda'$  are proportional to  $\rho$  and  $\rho'$ , the densities of the earth and soil. Hence

$$N'/N = Q'/\rho' / Q/\rho = \frac{1.4 \times 10^{-12} \times 2.7 \times .0013}{2.7 \times 8 \times 10^{-17}} = \frac{1.4 \times 1.3}{8} \times 100 = 23. \quad (14)$$

The penetrating radiation from the earth is therefore normally about 23 times as great as the penetrating radiation from the atmosphere, at least so far as radium C is concerned. I see no escape from this conclusion\*, and it appears to

\* We may, however, consider an extreme case. Suppose that the active deposit of radium and thorium were distributed uniformly, as near the earth's surface, for a height of 5 kilometres; and suppose, further, that the whole of this active deposit were carried suddenly to the earth by a fall of rain. There would then be on or near the earth's surface  $5 \times 10^5 \times 8 \times 10^{-17}$  or  $4 \times 10^{-11}$  gramme of RaC (expressed in terms of radium). This is equal to the amount of radium C in 30 or 40 centimetres of the soil, in each case per square centimetre. The thin layer of RaC derived from the atmosphere would, however, be by far the more efficient ionizer, because the  $\gamma$  rays would not have to penetrate through, and be absorbed by, the surface layers of the earth. The increase of the penetrating radiation during rain has been observed by Mache, and by Gockel. Such variations, however, were not found by McLennan at Toronto. The motion of RaC earthwards, not carried by a water drop, would be only about 50 metres an hour, due to the potential gradient.

I have calculated the ionization at various altitudes due to this very hypothetical surface layer of radium C, and obtain,

$$N = 2\pi qK \int_0^1 e^{-\frac{\lambda k}{z}} \frac{dz}{z},$$

whence

Altitude.	Ions/cm. <sup>3</sup> sec.
1 m.	1.8
10	1.05
100	.32
1000	negligible

On comparing this table with result (18) it will be seen that, in extreme cases, near the earth's surface the ionization due to the RaC carried down by rain or snow might equal or exceed that due to the RaC contained in the soil and rocks. In general, the active deposit would doubtless enter the soil with the rain.

*negative* the view that the penetrating radiation comes equally from all directions. So far as our knowledge of the distribution of thorium at present goes, we may also predict that a similar statement will be found to hold good for the  $\gamma$  rays of thorium C.

Returning to the result previously stated (13), we will next assume that the work of Blanc, Joly, W. Wilson (Phil. Mag. Feb. 1909), and others indicates that the penetrating radiations from the radium and from the thorium in the earth are about equal, and we can now set forth a complete table.

Type.	Whence.	Ra.	Th.	Total.
$\alpha$ rays .....	Air	1.63	1.00	2.63
$\beta$ " .....	Air	.035	.025	.06
$\gamma$ " .....	Air	.035	.025	.06
$\gamma$ " .....	Earth	.80	.80	1.60
		2.50	1.85	4.35

(15)

This rate of production of ions per cubic centimetre, namely 4.35, corresponds to the presence of 1660 ions per  $\text{cm.}^3$ , supposing that they were all small ions. The maximum value of ions measured in clear weather by an Ebert ion-counter is usually somewhat less than this.

It must be remembered that the numbers quoted in the preceding table are subject to considerable variations both with time and place; and are only intended as a guide to the mean values, which will no doubt in due course of time be determined with an accuracy to which my estimates can of necessity make no claim.

It will be seen that the theoretical value found, 1.6 ions per  $\text{cm.}^3$  per sec. for the penetrating radiation from the earth, is less than that obtained by Wright at Toronto, namely 2.5. It is also considerably less than the number of ions per  $\text{cm.}^3$  per sec. lost when an electroscope is well screened by lead, amounting to about 6. It is true that my calculations do not take account of the feeble and negligible  $\gamma$  radiation from actinium, the distribution of which is unknown. Further, it has been shown by Soddy (Phil. Mag. Oct. 1909) that the  $\gamma$  rays from uranium X are more penetrating than was at first supposed. The addition which should on this account



be made is of an uncertain character\*. An increase in the value of  $q$  will affect the value of  $n$ , which is already in excess of that observed. Yet with every allowance for the value of  $q$ , due to penetrating radiation, the theoretical value appears to be smaller than the observed. We may deduce either that the radioactive matter in the earth has been underestimated, which is improbable, or that the value found for  $K$  is too small, or that the observed values of  $q$  are too great, or that radiation exists of a type not yet expected or discovered proceeding from matter rather than from radioactive matter.

*Ionization over the Ocean.*

It would be interesting to make a table corresponding to that given above (15), but with values for mid-ocean. Unfortunately we have no certain measurements of the mean quantity of radium or thorium emanations over the ocean. From experiments by many observers, it appears that sea breezes on the western side of a continent contain less radium C than do land breezes. It is, however, certain that radium C does exist over the ocean in measurable quantities, and there is little doubt that radium emanation escapes from the ocean with greater facility than from the land, and this fact compensates to some extent for the minuteness of the radium contents of the ocean as contrasted with the land. There is also some uncertainty as to the mean value of the amount of radium in sea water, for Joly (Phil. Mag. Sept. 1909) has found  $1.1 \times 10^{-14}$  gramme of radium per  $\text{cm}^3$  for sea-water from various localities, and I have found  $10^{-15}$  for sea-water from the middle of the North Atlantic. For the present we will take the higher value given by Joly. The ratios of the penetrating radiations from the RaC in land, sea, and air are then, necessarily, near the earth's surface, as

$$Q/\rho : Q'/\rho' : Q''/\rho'',$$

where the  $Q$ 's denote quantities of radium C per  $\text{cm}^3$ , expressed in equivalent grammes of radium, and the  $\rho$ 's denote the respective densities.

\* If equal to radium we get  $3 \times 0.8 = 2.4$ , agreeing with Wright, see (15).

Thus we have:—

Penetrating Radiation, RaC.

	Land.	Sea.	Air.
Q .....	$4.3 \times 10^{-12}$	$1.1 \times 10^{-14}$	$8 \times 10^{-17}$
$\rho$ .....	2.7	1.0	.0013
Q/ $\rho$ .....	$1.4 \times 10^{-12}$	$1.1 \times 10^{-14}$	$6.1 \times 10^{-14}$
Ratios .....	130	1	5.5
or .....	23	.18	1
Ions/cm. <sup>3</sup> sec. ....	.80	.006	.035

It will at once be seen that the penetrating radiation over the ocean from the radium in the sea is negligible, being less than one-fifth of that from the radium C in the atmosphere over land. Since the radium emanation in the atmosphere over the sea is probably less than over the land, it is altogether quite extraordinary that those observers who have used Ebert's ion-counter, and those who have measured the conductivity of the atmosphere over the sea, have found values in each case not much, if any, less than over the land. It is possible that the greater purity of the air over the ocean—the absence of dust and smoke and physical impurities of that kind—may cause the small ions over the ocean to remain small ions, except during time of actual fog, while over the land they tend to become large ions. Yet such an explanation is very doubtful, and in this discrepancy lies at present our chief objection to a purely radioactive theory of atmospheric ionization. It is a question on which further investigation is needed, before any weighty opinion can possibly be hazarded.

We get a sidelight on this interesting and difficult point in the action of Hertzian waves in wireless telegraphy, which appear to have a larger effective radius in the dark, or during mist or fog, and are to a considerable extent absorbed in the presence of sunlight. It is possible that the small ions both render the air a conductor and tend to absorb or disperse the Hertzian waves, and that the sunlight may itself produce small ions. There appears to be no direct evidence of the influence of sunlight increasing the ionization near the earth's surface, but if the electrons were freed a little from the

atoms by sunlight and then returned to the parent atoms, these might account for the greater absorption of the Hertzian waves during the daytime. The ultra-violet light from the sun seems to be mainly absorbed in the upper layers of the atmosphere, so that it is not an important ionizing agent near the earth's surface. Further observations are needed on this point.

*Effect of Altitude.*

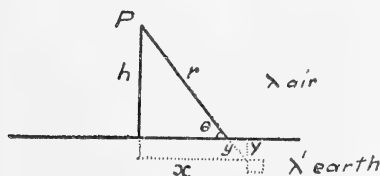
In conclusion we have to discuss the effect of elevation on the radiation from the radium C in the earth. The  $\gamma$  rays are absorbed rapidly in the earth, slowly in the air, and the ratio of the coefficients of absorption is nearly that of the densities, or as 2.7 is to .0013, about 2000 to 1. So that 1 cm. of the soil is as effective as 20 metres of air. In the case of a point source of radium C the law of inverse square of the distance also obtains, so that the radiation at a distance  $r$  from the source varies inversely as  $r^2 e^{\lambda r}$ . Hence if with such a point source the radiations were represented by 100 at 1 metre, there would be a rapid decrease with distance thus

At 1 metre, radiation 100.			
10	"	"	·95
100	"	"	·0064
1000	"	"	·000012

But in the case of the earth we have a *sheet* of radioactive matter, distributed with some approach to uniformity through the surface layers, which alone is effective in the atmosphere. The calculation has not, I think, yet been given, and therefore it is set forth here with some fulness.

Let P (fig. 1) be a point at elevation  $h$  above the earth's

Fig. 1.



surface, assumed plane. Let Q grammes of radium be the contents per cm.<sup>3</sup> in the earth. Consider an elementary ring  $dx$  by  $dY$ , radius  $x$ , at a depth  $Y$  below the surface. Then

the ionization at P will be

$$\iint \frac{2\pi x dx dY QK}{(r+y)^2} e^{-\lambda'y} e^{-\lambda r},$$

where  $y$  is small compared with  $r$ ,

$$= \int_0^\infty \int_0^\pi 2\pi QK \frac{h \cot \theta \cdot h \operatorname{cosec}^2 \theta d\theta}{h^2 \operatorname{cosec}^2 \theta} e^{-\lambda h \operatorname{cosec} \theta} e^{-\lambda'Y \operatorname{cosec} \theta} dY$$

$$= 2\pi QK \int_0^{\pi/2} \frac{\cot \theta d\theta}{\lambda' \operatorname{cosec} \theta} e^{-\lambda h \operatorname{cosec} \theta}$$

$$= \frac{2\pi QK}{\lambda'} \int_0^{\pi/2} \cos \theta e^{-\lambda h / \sin \theta} d\theta$$

$$= \frac{2\pi QK}{\lambda'} \int_0^1 e^{-\lambda h / z} dz. \dots \dots \dots (17)$$

In order to evaluate this troublesome integral I have drawn the curve  $y=e^{-\lambda h/z}$ , and determined the area from  $z=0$  to  $z=1$ , for various values of  $h$ , and substituted the values of  $\lambda, \lambda', Q$ , and  $K$  given in this paper. When  $h$  is zero we have  $2\pi QK/\lambda'$ , agreeing with our former work.

Thus the numbers of ions produced per  $\text{cm.}^3$  per sec. at various altitudes by the penetrating radiation due to radium C in the earth is given in the following table:—

Height in metres.	Penetrating Radiation.	
	Ratio.	Ions per $\text{cm.}^3$ per sec.
0 .....	1.00	.80
1 .....	.98	.78
10 .....	.83	.67
100 .....	.36	.29
1000 .....	.001	.008

(18)

The values in the right column should perhaps be doubled if we add the effect of thorium to that of radium.

In this calculation I have not made any correction for the decrease of density of the air with increase of altitude. This would tend to *augment* the lower figures, because the rays would be less readily absorbed; but, on the other hand, the number of ions produced would be *diminished* with a decrease of pressure.

The results shown in the table (18) are very remarkable. They indicate a decrease of ionization with altitude which is within the limits of detection for an elevation of 100 metres. They show, moreover, that at 1000 metres altitude the penetrating rays from the earth are ineffective as an ionizer. This result cannot well be put to the proof by ascending a mountain, because the radium contents of the ground beneath the observer are changing as he ascends. The experiment may be commended to those who are able to fly kites or captive balloons for 24 hours at a few hundred metres. It would be necessary to have an electroscope with a low natural leak and insulation of a high order fortified with a guard ring. Gockel has made observations in a balloon (not captive) at an altitude of 4000 metres on the penetrating radiation and found but a *moderate* decrease in the saturation current of an electroscope, carried upwards in the balloon. There are, as he points out, difficulties and uncertainties in making such measurements.

It will be seen from this paper that the radioactive theory of the ionization of the atmosphere is gradually approaching what may almost be termed an exact science, but that there are three outstanding problems of difficulty and importance. These are: (1) the high value of the ionization over the ocean is not yet explained ( $n$  is larger than expected,  $q$  is unknown); (2) the rapid decrease of ionization with altitude has not been detected; (3) the theoretical value of the ionization due to the penetrating radiation is *less* than that found when observers completely screen an electroscope with lead.

Until these three questions have been satisfactorily explored we cannot rest wholly content with the theory, but have a vague feeling that all matter may give rise to feeble rays of the Röntgen type, of which the more penetrating may integrate into an effective total. This last view is by no means advocated by the writer.

Last May I attempted some measurements on the North Atlantic, and found that in a cabin of the iron s.s. 'Dominion,' in mid-Atlantic, the leaf of an electroscope closed a little more slowly than on land, indicating a slight decrease of penetrating radiation over the ocean. But I regret to add that the inherent difficulties of observations on shipboard prevented me from reaching any exact or convincing results.

The writer will be grateful for any information or suggestions tending to correct the somewhat tentative values given in this paper.

July 1910.

*Note added Oct. 26, 1910.*

Since this paper was written there have appeared two communications by Wulf (*Le Radium*, June 1910; *Phys. Zeit.* 15 Sept. 1910). These throw much light on the subject of penetrating radiation and confirm some of the conclusions set forth above.

Wulf, with the vessels he used, finds for the penetrating radiation in Holland  $g=10$ , in Paris  $q=6$ , and at a height of 300 m. on the Eiffel Tower  $q=3.5$ .

The reduction in  $q$  over a lake was about 4.9, and a similar reduction was determined at a depth of 12 m. beneath the surface of the water. His interesting experiments seem to establish the terrestrial origin of the penetrating radiation and the diminution of intensity with altitude.

The figures appear to require reduction for values in free air, perhaps on the basis of lead 100, zinc 60, aluminium 52, free air 42.

It will be seen that Wulf's value for  $q$ , even at Paris, is larger than that calculated in this paper, and that a loss of 40 per cent. for an altitude of 300 m. is less than the loss of 64 per cent. which I calculate for an altitude of 100 m. It will no doubt take time to reconcile these points.

At present it would seem that the penetrating radiation both passes through the air more readily, and also produces more ions than laboratory experiments, with Ra C and metal screens, lead us to infer. It will, however, be remembered that no direct measurements of the coefficient of absorption of the  $\gamma$  rays by gases have yet been made. A. S. E.

V. *Note on the Ratios which the Amounts of Substances in Radioactive Equilibrium bear to one another.* By HUGH MITCHELL, M.A.\*

(1) **T**HE law of radioactive change is that the rate of disintegration of an active substance is proportional to the amount of it present. A series of radioactive substances is said to be in equilibrium when the ratio of the amount of one substance to that of any other substance in the series remains unchanged as time goes on. It is known that assuming the disintegration rate of the parent substance to be small in comparison with that of any other in the series, the ratio of the amounts of any two substances in the series is approximately equal to the ratio of their average lives. The present note is intended to show that without

\* Communicated by F. Soddy, M.A., F.R.S.

this assumption the theoretically correct deduction from the above law still remains very simple.

(2) Let  $\xi_n$  be the number of atoms initially and  $x_n$  the number at time  $t$  of a radioactive substance  $S_n$  which changes into a radioactive substance  $S_{(n+1)}$  at the rate of  $\lambda_n x_n$  atoms per unit time, where  $n = 1, 2, 3 \dots n$ . Then the average life of an atom of  $S_n$  is  $\frac{1}{\lambda_n}$  units of time and  $x_n$  satisfies the differential equation

$$\frac{dx_n}{dt} = \lambda_{n-1}x_{n-1} - \lambda_n x_n,$$

in which when  $n = 1$ ,  $\lambda_0 x_0$  is to be taken as zero.

(3) The general solution of this equation giving the quantity  $x_n$  of the  $n$ th substance after time  $t$  when the parent substance is initially free from products is

$$x_n = \xi_1 \lambda_1 \lambda_2 \dots \lambda_{(n-1)} \left\{ \frac{\epsilon^{-\lambda_1 t}}{\Pi(\lambda_i - \lambda_1)} + \frac{\epsilon^{-\lambda_2 t}}{\Pi(\lambda_i - \lambda_2)} + \dots + \frac{\epsilon^{-\lambda_n t}}{\Pi(\lambda_i - \lambda_n)} \right\}$$

where the expression  $\Pi(\lambda_i - \lambda_x)$  is to be taken to mean the product of all terms of the type  $(\lambda_i - \lambda_x)$  when  $\lambda_i$  takes all possible values from  $\lambda_1$  to  $\lambda_n$  except  $\lambda_x$ .

For example,

$$x_1 = \xi_1 \epsilon^{-\lambda_1 t},$$

and

$$x_2 = \xi_1 \lambda_1 \lambda_2 \left\{ \frac{\epsilon^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{\epsilon^{-\lambda_2 t}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \right. \\ \left. + \frac{\epsilon^{-\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} + \frac{\epsilon^{-\lambda_4 t}}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right\}.$$

(4) When the parent substance is the longest lived member of the series, as  $t$  increases,  $\frac{x_n}{x_1}$  approaches the value

$$\frac{\lambda_1 \lambda_2 \dots \lambda_{n-1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \dots (\lambda_n - \lambda_1)}.$$

Now

$$1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} + \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \dots \\ + \frac{\lambda_1 \lambda_2 \dots \lambda_{(n-1)}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \dots (\lambda_n - \lambda_1)} = \frac{\lambda_2 \lambda_3 \dots \lambda_n}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \dots (\lambda_n - \lambda_1)}.$$

Hence

$$\frac{x_1}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \dots + \frac{x_n}{x_1} = \frac{\lambda_n}{\lambda_1} \cdot \frac{x_n}{x_1}$$

That is

$$\frac{x_n}{x_1 + x_2 + x_3 + \dots + x_n} = \frac{\lambda_1}{\lambda_n}$$

Hence in a series of substances in radioactive equilibrium the ratio of the number of atoms of the  $n$ th substance to those of it and all the preceding substances in coexistence with it is equal to the ratio of the average life of the  $n$ th substance to that of the first substance present in the series.

High School, Dundee.

VI. *A Note on a Method of Determining Capacities in Measurements of Ionization.* By NORMAN CAMPBELL\*.

THE ordinary method of determining the capacity of the ionization vessels and the measuring instrument used in observations on ionization consists in comparing the rate of rise of potential of the electrode system for a constant current (1) when a known capacity is put in parallel with the electrode, and (2) when that capacity is absent. The advantages of the method lie in the simplicity of the apparatus required; its disadvantages are connected with the choice of the standard capacity. Standard condensers of not less than .01 m.f. are easily procured, but their capacity is very great compared with that of the usual measuring systems, which is usually less than .0005 m.f. If they are used, either the time for which the rise of potential is observed in (1) must be so long that difficulties are likely to arise from defective insulation, or the time for which the rise is observed in (2) must be so short that it cannot be measured with accuracy. On the other hand, accurate and well-insulated standard condensers, the capacity of which is of the same order as that of the measuring instrument, are not easily procured, and the (unknown) capacity of the conductors used to connect them to the measuring system may be too large a fraction of their capacity to be safely neglected.

The following method, which appears not to be known generally, is free from these disadvantages. Though there is nothing new in its principle, which is sufficiently obvious, it may be worth while to draw attention to it.

\* Communicated by the Author.



Let  $c_1, c_1'$  be the inner and outer coatings of the condenser forming the measuring system, of which the capacity  $C_1$  is to be determined. Let  $c_0, c_0'$  be the coatings of the standard condenser of known capacity  $C_0$ , and  $c_2, c_2'$  those of an "auxiliary" condenser of capacity  $C_2$ . The "inner" coating is that which is usually connected to the measuring system, the "outer" coating that which is usually connected to earth.

Perform the following observations:—(1) Connect and earth  $c_1, c_2$ ; also earth  $c_1', c_2'$ . Insulate  $c_1 + c_2$ , and then raise  $c_2'$  to the potential  $V_1$ , the potential of earth being 0. In virtue of the charge induced the electrometer (or other measuring instrument) indicates a potential  $v_1$ . The charge induced on  $c_1$  must be equal and opposite to that induced on  $c_2$ : hence

$$C_1 v_1 + C_2 (v_1 - V_1) = 0. \quad . \quad . \quad . \quad (1)$$

(2) Connect and earth  $c_1, c_2, c_0$ ; earth  $c_1', c_2', c_0'$ . Insulate  $c_1 + c_2 + c_0$ , and then raise  $c_2'$  to the potential  $V_2$ . Then, as before, if  $v_2$  is the potential indicated by the measuring instrument,

$$(C_1 + C_0) v_2 + C_2 (v_2 - V_2) = 0. \quad . \quad . \quad . \quad (2)$$

We have thus two equations to determine  $C_1/C_0$  and  $C_2/C_0$  in terms of the known quantities  $V_1, V_2, v_1, v_2$ ; and the problem is solved.

Let us now examine the accuracy of the method. For the determination of  $V_1$  and  $V_2$ , and also of the sensitiveness of the electrometer (which gives  $v_1$  and  $v_2$  in terms of the deflexion), a resistance-box arranged as a potentiometer will doubtless be used. The potential (V) of the battery in the circuit need not be known, so long as it is constant throughout the observations. The error in  $V_1$  or  $V_2$  will be due to the deviation of the ratio of the resistances from their nominal value, and need not exceed 1 part in 1000 over a range from V to 1/1000 V. The advantage of this method lies in the fact that potentials, unlike times, can be measured with the same instrument over a wide range with great accuracy. On the other hand, the error in determining  $v_1$  or  $v_2$  from the deflexion is likely to be about 1 part in 100 for a single observation. Accordingly we shall neglect the error in  $V_1$  and  $V_2$  compared with that in  $v_1$  or  $v_2$ .

In practical measurements  $V_1$  and  $V_2$  will be so chosen that  $v_1$  and  $v_2$  are nearly equal. If  $\epsilon$  is the probable error in the observed value  $v$ ,  $\epsilon_1, \epsilon_2, \epsilon_{12}$  those in the deduced values of  $C_1, C_2, C_1 + C_2$ , the ordinary formulæ for the relation of

the probable errors give, when we put  $v_1 = v_2 = v$ ,

$$\frac{\epsilon_1}{C_1} = \frac{\{V_1^2(v - V_2)^2 + V_2^2(v - V_1)^2\}^{\frac{1}{2}} \cdot \epsilon}{(V_1 - v)(V_2 - V_1) \cdot v}, \quad \dots \quad (3)$$

$$\frac{\epsilon_2}{C_2} = \frac{(V_1^2 + V_2^2)^{\frac{1}{2}} \cdot \epsilon}{(V_2 - V_1) \cdot v}, \quad \dots \quad (4)$$

$$\frac{\epsilon_{12}}{C_1 + C_2} = \frac{\sqrt{2} V_2 \cdot \epsilon}{(V_2 - V_1) \cdot v} \dots \dots \dots (5)$$

So far as  $C_2$  is concerned, equation (4) shows that the accuracy increases with the ratio  $\frac{V_2}{V_1}$ , *i. e.* with  $\frac{C_0}{C_1 + C_2}$ .

But for  $C_1$  the relation is more complex and involves the ratio  $\frac{C_2}{C_1}$ . However, we can avoid using  $C_1$  explicitly, by

taking as the "auxiliary" condenser an ionization vessel which forms part of the measuring system.  $C_1$  is then the capacity of the rest of the measuring system, and we are concerned only to know the value of  $C_1 + C_2$ . The accuracy

of  $C + C_2$  increases also steadily with the ratio  $\frac{C}{C_1 + C_2}$ .

Hence, by the method of measurement here proposed, the greatest accuracy is attained, when the standard condenser used has as great a capacity as possible. The use of such large standards is convenient on other grounds. A limit is set to the possible magnitude of  $C_0$  by the condition that both  $V_1$  and  $V_2$  must fall within the limits of measurement by the potentiometer; but a capacity as large as 0.1 m.f. would be convenient for determining the capacity of a system including an ionization vessel of reasonable size and an electrometer.

If the insulation of the electrode is not perfect, a correction will have to be made for the time which elapses between raising  $c_2'$  to  $V_1$  or  $V_2$  and reading the electrometer. The correction is easily determined by watching the decay of the deflexion of the electrometer for a short time; and such observations will give the value of the insulation resistance, a quantity of which the knowledge is important apart from measurements of capacity.

It will be observed that the method is useful for determining the variation of the capacity of the electrometer with its deflexion—a troublesome relation to measure by any of the ordinary methods. For this purpose it is only necessary

to repeat the observations with different values of  $V_1$  and  $V_2$ , and, consequently, different values of  $v$ . The method may also be applied to the determination of the capacity of condensers designed to serve as secondary standards. These condensers can be used as "auxiliary" condensers, and the determination of their capacities will have the same accuracy whatever the ratio of those capacities to that of the measuring system.

It is found in practice that it is quite easy to attain such accuracy that the probable error of the value deduced from a *single* observation is not greater than 1 per cent.

Leeds, September 1910.

VII. *On the Relation between Viscosity and Atomic Weight for the Inert Gases; with its Application to the case of Radium Emanation.* By A. O. RANKINE, D.Sc., Assistant in the Department of Physics, University of London, University College\*.

A FURTHER examination of the data obtained in my measurements of the viscosities of the gases of the argon group † has brought to light a simple relation between viscosity and atomic weight. This relation has some important applications. It may be used, as will be shown, to form an estimate of the critical temperature of neon. Further, in conjunction with another relation previously published, it renders possible the estimation of the viscosity and molecular dimensions of radium emanation, upon the assumption that this gas belongs to the argon group.

The data upon which the present paper is based are given in Table I.

TABLE I.

Gas.	$\eta_0 \times 10^4$ .	C.
He .....	1.879	70
Ne .....	2.981	56
A .....	2.102	142
Kr .....	2.334	188
X .....	2.107	252

\* Communicated by Prof. F. T. Trouton, F.R.S.

† Proc. Roy. Soc. A, vol. lxxxiii. p. 516, and Proc. Roy. Soc. A, vol. lxxxiv. p. 181.

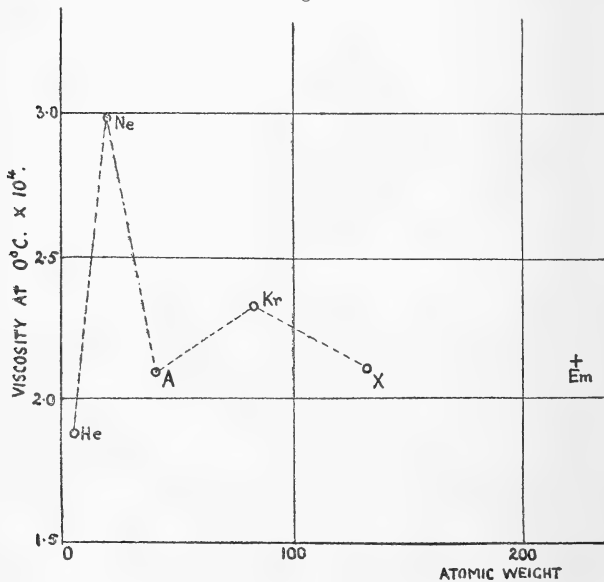
The numbers in the second column are the viscosities in absolute units at  $0^{\circ}$  C. multiplied by  $10^4$ . Those in the third column are the values of the constant  $C$  in Sutherland's\* equation

$$\eta = \eta_0 \left( \frac{T}{T_0} \right)^{\frac{5}{2}} \left( \frac{C + T_0}{C + T} \right),$$

where  $\eta$  is the viscosity at the absolute temperature  $T$ .

The first five points in fig. 1 show the viscosities at

Fig. 1.



$0^{\circ}$  C.  $\times 10^4$  plotted against the atomic weight. The undulatory nature of the curve is apparent, and it is at once recognized that no simple algebraic relationship obtains. Indeed, such a connexion is hardly to be expected in view of the fact that comparison is made of the viscosities of the gases at the same temperature. It seems much more reasonable to perform the comparison at corresponding temperatures—for example, the boiling points or critical temperatures of the respective gases. With the data available this procedure is only possible if the truth of Sutherland's equation be assumed, and the values of the viscosity found by extrapolation. In this manner the viscosities of the gases have been calculated, with the striking

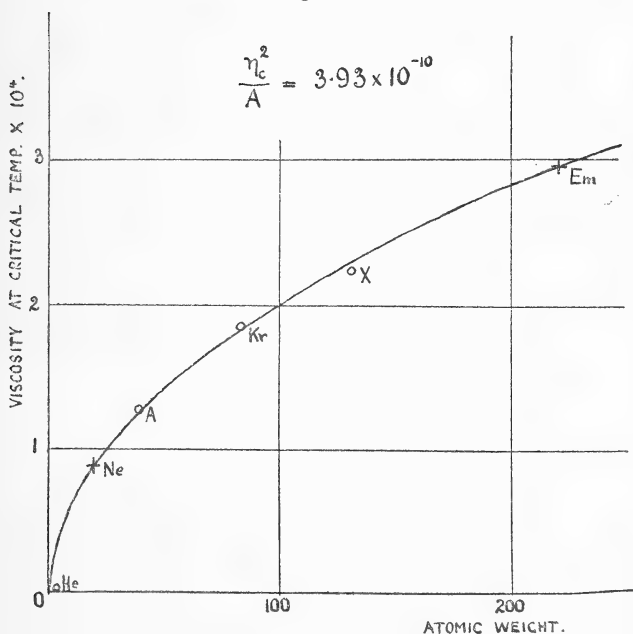
\* Phil. Mag. 1893, vol. xxxvi, p. 507.

result that the viscosity-atomic weight curve becomes smooth, and the square of the viscosity at the critical point is found to be proportional to the atomic weight. This relation has been deduced from the values for argon, krypton, and xenon only, because the critical temperature of neon has not been determined, and that of helium is not known with certainty.

The curve in fig. 2 is the parabola whose equation is

$$\frac{\eta_c^2}{A} = 3.93 \times 10^{-10},$$

Fig. 2.



where  $\eta_c$  is the viscosity at the critical temperature and  $A$  is the atomic weight. The calculated value of  $\eta_c$  for helium (assuming the probable value 5° absolute for the critical temperature) is also shown on the curve. It will be seen that it is considerably removed from the theoretic curve, and it should be pointed out that the uncertainty as to the critical temperature is wholly insufficient to account for the departure.

Table II. shows the numbers relating to the curve.

TABLE II.

Gas.	$T_c$ .	A.	$\eta_c \times 10^4$ .	$\eta_c \times 10^4$ (calculated).	Difference per cent.
He .....	(5)?	3.96	0.021	0.32	...
Ne .....	...	20.03	...	...	...
A .....	155.6	39.92	1.261	1.253	0.6
Kr .....	210.5	83.0	1.828	1.806	1.2
X .....	288	130.7	2.220	2.266	-2.1

The numbers in the fourth column are calculated from Sutherland's equation using  $\eta_0$ , C, and  $T_c$ ; those in the fifth column are deduced from the equation

$$\frac{\eta_c^2}{A} = 3.93 \times 10^{-10}.$$

With the exception of helium this equation is a good fit, and even the exception seems capable of reasonable explanation. In the cases of helium and argon we may avoid the necessity of such extensive extrapolation by using the experimental results recorded by Schmitt\*. These measurements extend from  $-193^{\circ}.2$  C. to  $+183^{\circ}.7$  C., and the value of the viscosity of argon at the critical temperature can be deduced by *interpolation*. The value thus obtained is  $1.25 \times 10^{-4}$ , which is remarkably near the value calculated here, and suggests that in this case the extrapolation is valid. When we turn to helium, however, the results are very different. At a temperature of  $80^{\circ}$  absolute the actual value of the viscosity exceeds that calculated by using Sutherland's equation by 26 per cent., and the divergence *increases* with fall of temperature. The temperature at which the viscosity is required is  $5^{\circ}$  absolute, or  $75^{\circ}$  lower, where we should expect the divergence to be more serious still. In fact, an extrapolation over this range of 75 degrees indicates that  $0.32 \times 10^{-4}$  is a much more probable value of the *actual* viscosity at  $5^{\circ}$  C. than  $0.021 \times 10^{-4}$ . This would bring helium into line with the other gases. It is quite possible, therefore, that helium also conforms with the law here presented, and that the apparent divergence should be attributed to the failure of Sutherland's equation at temperatures so near to absolute zero.

\* *Ann. der Physik*, Bd. xxx. Heft 2, p. 399 (1909).

*The Critical Temperature of Neon.*

It is evident that, working backwards, this relation may be used to estimate the critical temperature of neon, upon the assumption that it applies for this gas. In that case the value of  $\eta_c$  for neon (shown by a cross in the diagram) would be  $0.887 \times 10^{-4}$ . This, according to Sutherland's equation, would be the viscosity of neon at a temperature of  $61^{\circ}.1$  absolute. That is to say, the critical temperature of neon is  $61^{\circ}.1$  absolute.

The author \* has already made an estimate of this temperature based upon another relation, namely, that the critical temperature is proportional to Sutherland's constant. This is expressed by the equation

$$T_c = 1.12 C.$$

Taking the value  $C=56$  obtained from the experiments, this gives  $T_c=62^{\circ}.7$  absolute. The agreement between these two figures  $61.1$  and  $62.7$  is remarkable, and constitutes weighty evidence of the probable accuracy of the estimate.

*Application to Radium Emanation.*

It is generally accepted that the emanation from radium belongs to the same group in the periodic table as the gases previously referred to. There is considerable justification, therefore, for applying the two above-mentioned relations to this case. The fact that the atomic weight and critical temperature of the emanation are now known renders it possible to estimate the viscosity, not only at the critical point, but also at any other temperature, together with the dependent molecular properties.

The values of the atomic weight and critical temperature used are those obtained by Ramsay and Gray †, viz.  $A=222$  and  $T_c=377^{\circ}$  absolute.

In the first place, using the relation

$$T_c = 1.12 C,$$

we obtain  $C=337$ .

Further, using the second relation

$$\eta_c^2 = \frac{3.93}{10^{10}} \cdot A,$$

$\eta_c$  is found to be  $2.954 \times 10^{-4}$ . [This is shown by the cross marked *Em* on figure 2.]

\* Proc. Roy. Soc. A, vol. lxxxiv. p. 190.

† Trans. Chem. Soc. 1909, p. 1073. Brit. Assoc. Report, Sheffield, 1910.

Now, making use of Sutherland's equation

$$\eta_0 = \eta_c \left( \frac{T_0}{T_c} \right)^{\frac{3}{2}} \left( \frac{C + T_c}{C + T_0} \right),$$

as applied to the emanation we obtain

$$\begin{aligned} \eta_0 &= 2.954 \left( \frac{273}{377} \right)^{\frac{3}{2}} \left( \frac{714}{610} \right) \cdot 10^{-4} \\ &= 2.130 \times 10^{-4}. \end{aligned}$$

This is the estimated value of the viscosity of radium emanation at  $0^\circ \text{C}$ . It is recorded by means of the cross on fig. 1. It lies between the values for xenon and krypton, and its position strongly suggests that the up and down distribution of the points is not without significance. The alternations were already regular but of decreasing amplitude before the addition of the point representing the emanation; its addition carries the rule a step further in *both* respects. Of course, it is possible that another unidentified member of this group of gases lies between xenon and radium emanation; but it will be seen that a place could be found for it on the diagram. Indeed, a point higher than X or Em and lower than Kr, placed about midway between X and Em, would be more in keeping with the general trend of distribution than a single step from X to Em.

It is now possible to compare the molecular dimensions of emanation and helium, since the viscosities at the same temperature are known. The connexion according to the kinetic theory is

$$\frac{s_{\text{Em}}^2}{s_{\text{He}}^2} = \frac{\eta_{\text{He}}}{\eta_{\text{Em}}} \sqrt{\frac{\rho_{\text{Em}}}{\rho_{\text{He}}}},$$

where  $s$  is the radius of the molecule and  $\rho$  the density of the gas under constant conditions. This gives

$$\frac{s_{\text{Em}}}{s_{\text{He}}} = 2.57,$$

and consequently

$$\frac{v_{\text{Em}}}{v_{\text{He}}} = 16.97,$$

where  $v$  denotes the molecular volume.

Thus the molecule of emanation is larger than that of any other known gas in the group, as will be seen from the following table. The other figures are copied from a



previous paper for purposes of comparison, helium in all cases being taken as unity.

The last three columns refer to the molecular dimensions corrected according to Sutherland's theory. This is that owing to the increased frequency of collision caused by molecular attractions, the molecules behave as though their dimensions were greater than their real values. If we wish to find the true radius we must diminish that calculated by the simple theory in the proportion  $1 : \left(1 + \frac{C}{T}\right)^{\frac{1}{2}}$ . The numbers in the last three columns are obtained from columns 3 to 5 by the application of this process.

TABLE III.

Gas.	Relative Atomic wt.	Molecular Radius.	Molecular Volume.	Density of the atom.	With Sutherland correction.		
					Molecular Radius.	Molecular Volume.	Density of the atom.
He ...	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Ne ...	5.06	1.19	1.69	2.99	1.21	1.79	2.83
A ...	10.08	1.68	4.74	2.12	1.53	3.59	2.81
Kr ...	20.96	1.91	6.98	3.00	1.65	4.53	4.63
X ...	33.01	2.25	11.37	2.90	1.83	6.11	5.40
<b>Em...</b>	<b>55.5</b>	<b>2.57</b>	<b>16.97</b>	<b>3.28</b>	<b>1.93</b>	<b>7.16</b>	<b>7.75</b>

Finally, using the value  $N = 2.8 \times 10^{19}$  for the number of atoms per c.c. at N.T.P., we obtain for the absolute molecular radius of radium emanation

$$s_{Em} = 2.16 \times 10^{-8} \text{ cm.}$$

This is the uncorrected value, *i.e.* the value obtained without applying the Sutherland diminution, it being still customary to record molecular dimensions in this way.

In order to avoid confusion with regard to these two separate measures of the molecular radius, the following consideration may be found useful.

Rayleigh \* has pointed out that if the molecules mutually attract one another, mutual repulsions must also exist, and that the laws of variation of force with distance are different

\* Phil. Mag. xxx. pp. 285 & 456 (1890).

in the two cases, the variation being necessarily more rapid for the repulsive force. Perhaps from this point of view we should regard the uncorrected radius as referring more particularly to distances from the centre of the atom at which the values of the attractive force become serious, and the corrected or Sutherland radius as the distance at which the forces of repulsion and attraction balance one another.

*Connexion between the Critical Temperature and  
Molecular Radius.*

It will now be shown that a simple relation exists between the critical temperatures of these gases and their molecular dimensions. Starting from the equation

$$\eta = \frac{1}{3}nmV\lambda$$

obtained from the kinetic theory;  $\eta$  being the viscosity,  $n$  the number of molecules per c.c.,  $m$  the molecular mass,  $V$  the root mean square velocity, and  $\lambda$  the effective mean free path. Hence

$$\eta^2 = \frac{1}{9}n^2m^2V^2\lambda^2.$$

Now,  $mV^2$  is proportional to  $T$ , the absolute temperature. Therefore

$$\frac{\eta^2}{m} = K \cdot n^2T\lambda, \dots \dots \dots (1)$$

where  $K$  is a constant not depending on the particular gas. Further,

$$\lambda = \frac{1}{\sqrt{2}n\pi s^2}$$

where  $s$  is the effective radius. According to Sutherland,

$$s^2 = s_0^2 \left(1 + \frac{C}{T}\right),$$

where  $s_0$  is the real radius. Whence

$$\lambda^2 = \frac{1}{2\pi^2 n^2 s_0^4 \left(1 + \frac{C}{T}\right)^2} \dots \dots \dots (2)$$

Combining with (1) we obtain

$$\frac{\eta^2}{m} = q \frac{T}{s_0^4 \left(1 + \frac{C}{T}\right)^2} \dots \dots \dots (3)$$

$q$  being a constant. Now the author has shown that at the

critical temperature  $\frac{\eta^2}{m}$  and  $1 + \frac{C}{T}$  are constant. Therefore, from (3)

$$\frac{T_c}{s_0^4} = \text{constant.}$$

That is, the critical temperature, and therefore C also, is proportional to the fourth power of the true atomic radius. The truth of this law depends on the accuracy of the two laws previously given, and the figures given in column 4 of Table IV. are expected to be constant. The numbers in column 2 are relative to helium.

TABLE IV.

Gas.	$s_0$ .	$T_c$ .	$\frac{s_0}{(T_c)^{\frac{1}{4}}}$ .
He .....	1 00	5?	0.669
Ne .....	1.21	62?	0.433
A .....	1.53	155.6	0.433
Kr .....	1.65	210.5	0.433
X .....	1.83	288	0.444
Ein .....	1.93?	377	0.438

The values to which doubt attaches or which are deduced from previous considerations in this paper are marked with a query. Helium, as one would expect, is an exception. This and the constancy of the remaining ratios are direct consequences of the relations previously given.

VIII. *Note on Bessel's Functions as applied to the Vibrations of a Circular Membrane.* By Lord RAYLEIGH, O.M., F.R.S.\*

**I**T often happens that physical considerations point to analytical conclusions not yet formulated. The pure mathematician will admit that arguments of this kind are suggestive, while the physicist may regard them as conclusive.

The first question here to be touched upon relates to the dependence of the roots of the function  $J_n(z)$  upon the order

\* Communicated by the Author.

$n$ , regarded as susceptible of continuous variation. It will be shown that each root increases continually with  $n$ .

Let us contemplate the transverse vibrations of a membrane fixed along the radii  $\theta=0$  and  $\theta=\beta$  and also along the circular arc  $r=1$ . A typical simple vibration is expressed by\*

$$w = J_n(z_n^{(s)}r) \cdot \sin n\theta \cdot \cos(z_n^{(s)}t), \quad \dots \quad (1)$$

where  $z_n^{(s)}$  is a finite root of  $J_n(z)=0$ , and  $n=\pi/\beta$ . Of these finite roots the lowest  $z_n^{(1)}$  gives the principal vibration, *i. e.* the one without internal circular nodes. For the vibration corresponding to  $z_n^{(s)}$  the number of internal nodal circles is  $s-1$ .

As prescribed, the vibration (1) has no internal nodal diameter. It might be generalized by taking  $n=\nu\pi/\beta$ , where  $\nu$  is an integer; but for our purpose nothing would be gained, since  $\beta$  is at disposal, and a suitable reduction of  $\beta$  comes to the same as the introduction of  $\nu$ .

In tracing the effect of a diminishing  $\beta$  it may suffice to commence at  $\beta=\pi$ , or  $n=1$ . The frequencies of vibration are then proportional to the roots of the function  $J_1$ . The reduction of  $\beta$  is supposed to be effected by increasing without limit the potential energy of the displacement ( $w$ ) at every point of the small sector to be cut off. We may imagine suitable springs to be introduced whose stiffness is gradually increased, and that without limit. During this process every frequency originally finite must increase †, finally by an amount proportional to  $d\beta$ ; and, as we know, no zero root can become finite. Thus before and after the change the finite roots correspond each to each, and every member of the latter series exceeds the corresponding member of the former.

As  $\beta$  continues to diminish this process goes on until when  $\beta$  reaches  $\frac{1}{2}\pi$ ,  $n$  again becomes integral and equal to 2. We infer that every finite root of  $J_2$  exceeds the corresponding finite root of  $J_1$ . In like manner every finite root of  $J_3$  exceeds the corresponding root of  $J_2$ , and so on.

I was led to consider this question by a remark of Gray and Mathews ‡—"It seems probable that between every pair of successive real roots of  $J_n$  there is exactly one real root of  $J_{n+1}$ . It does not appear that this has been strictly proved; there must in any case be an odd number of roots

\* 'Theory of Sound,' §§ 205, 207.

† *L. c.* §§ 88, 92 *a.*

‡ Bessel's Functions, 1895, p. 50.

in the interval." The property just established seems to allow the proof to be completed.

As regards the latter part of the statement, it may be considered to be a consequence of the well known relation

$$J_{n+1}(z) = \frac{n}{z} J_n(z) - J_n'(z). \quad \dots \quad (2)$$

When  $J_n$  vanishes,  $J_{n+1}$  has the opposite sign to  $J_n'$ , both these quantities being finite\*. But at consecutive roots of  $J_n$ ,  $J_n'$  must assume opposite signs, and so therefore must  $J_{n+1}$ . Accordingly the number of roots of  $J_{n+1}$  in the interval must be *odd*.

The theorem required then follows readily. For the first root of  $J_{n+1}$  must lie between the first and second root of  $J_n$ . We have proved that it exceeds the first root. If it also exceeded the second root, the interval would be destitute of roots, contrary to what we have just seen. In like manner the second root of  $J_{n+1}$  lies between the second and third roots of  $J_n$ , and so on. The roots of  $J_{n+1}$  *separate* those of  $J_n$  †.

The physical argument may easily be extended to show in like manner that all the finite roots of  $J_n'(z)$  increase continually with  $n$ . For this purpose it is only necessary to alter the boundary condition at  $r=1$  so as to make  $dw/dr=0$  instead of  $w=0$ . The only difference in (1) is that  $z_n^{(s)}$  now denotes a root of  $J_n'(z)=0$ . Mechanically the membrane is fixed as before along  $\theta=0, \theta=\beta$ , but all points on the circular boundary are free to slide transversely. The required conclusion follows by the same argument as was applied to  $J_n$ .

It is also true that there must be at least one root of  $J_{n+1}'$  between any two consecutive roots of  $J_n'$ , but this is not so easily proved as for the original functions. If we differentiate (2) with respect to  $z$  and then eliminate  $J_n$  between the equation so obtained and the general differential equation, viz.

$$J_n'' + \frac{1}{z} J_n' + \left(1 - \frac{n^2}{z^2}\right) J_n = 0, \quad \dots \quad (3)$$

\* If  $J_n, J_{n+1}$  could vanish together, the sequence formula, (8) below, would require that every succeeding order vanish also. This of course is impossible, if only because when  $n$  is great the lowest root of  $J_n$  is of order of magnitude  $n$ .

† I have since found in Whitaker's 'Modern Analysis,' § 152, another proof of this proposition, attributed to Gegenbauer (1897).

we find

$$\left(1 - \frac{n^2}{z^2}\right) J'_{n+1} + \frac{n}{z^3} (n^2 - 1 - z^2) J'_n + \left(1 - \frac{n^2 + n}{z^2}\right) J''_n = 0. \quad (4)$$

In (4) we suppose that  $z$  is a root of  $J'_n$ , so that  $J'_n = 0$ . The argument then proceeds as before if we can assume that  $z^2 - n^2$  and  $z^2 - n(n+1)$  are both positive. Passing over this question for the moment, we notice that  $J''_n$  and  $J'_{n+1}$  have opposite signs, and that both functions are finite. In fact if  $J''_n$  and  $J'_n$  could vanish together, so also by (3) would  $J_n$ , and again by (2)  $J_{n+1}$ ; and this we have already seen to be impossible.

At consecutive roots of  $J'_n, J''_n$  must have opposite signs, and therefore also  $J'_{n+1}$ . Accordingly there must be at least one root of  $J'_{n+1}$  between consecutive roots of  $J'_n$ . It follows as before that the roots of  $J'_{n+1}$  separate those of  $J'_n$ .

It remains to prove that  $z^2$  necessarily exceeds  $n(n+1)$ . That  $z^2$  exceeds  $n^2$  is well known\*, but this does not suffice. We can obtain what we require from a formula given in 'Theory of Sound,' 2nd ed. § 339. If the finite roots taken in order be  $z_1, z_2, \dots z_s \dots$ , we may write

$$\log J'_n(z) = \text{const.} + (n-1) \log z + \Sigma \log (1 - z^2/z_s^2),$$

the summation including all finite values of  $z_s$ ; or on differentiation with respect to  $z$

$$\frac{J''_n(z)}{J'_n(z)} = \frac{n-1}{z} - \Sigma \frac{2z}{z_s^2 - z^2}.$$

This holds for all values of  $z$ . If we put  $z = n$ , we get

$$\Sigma \frac{2n}{z_s^2 - n^2} = 1, \quad \dots \dots \dots (5)$$

since by (3)

$$J''_n(n) \div J'_n(n) = -n^{-1}.$$

In (5) all the denominators are positive. We deduce

$$\frac{z_1^2 - n^2}{2n} = 1 + \frac{z_1^2 - n^2}{z_2^2 - n^2} + \frac{z_1^2 - n^2}{z_3^2 - n^2} + \dots > 1; \quad \dots (6)$$

and therefore

$$z_1^2 > n^2 + 2n > n(n+1).$$

Our theorems are therefore proved.

\* Riemann's *Partielle Differential Gleichungen*; 'Theory of Sound,' § 210.

If a closer approximation to  $z_1^2$  is desired, it may be obtained by substituting on the right of (6)  $2n$  for  $z_1^2 - n^2$  in the numerators and neglecting  $n^2$  in the denominators. Thus

$$\frac{z_1^2 - n^2}{2n} > 1 + 2n (z_2^{-2} + z_3^{-2} + \dots) \\ > 1 + 2n \left\{ z_1^{-2} + z_2^{-2} + z_3^{-2} + \dots - \frac{1}{n(n+2)} \right\}.$$

Now, as is easily proved from the ascending series for  $J_n'$ ,

$$z_1^{-2} + z_2^{-2} + z_3^{-2} + \dots = \frac{n+2}{4n(n+1)};$$

so that finally

$$z_1^2 > n^2 + 2n + \frac{n^3}{(n+1)(n+2)} \dots \dots (7)$$

When  $n$  is very great, it will follow from (7) that  $z_1^2 > n^2 + 3n$ . However, the approximation is not close, for the ultimate form is \*

$$z_1^2 = n^2 + 1.02684 n^{4/3}.$$

As has been mentioned, the sequence formula

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z) \dots \dots (8)$$

prohibits the simultaneous evanescence of  $J_{n-1}$  and  $J_n$ , or of  $J_{n-1}$  and  $J_{n+1}$ . The question arises—can Bessel's functions whose orders (supposed integral) differ by more than 2 vanish simultaneously? If we change  $n$  into  $n+1$  in (8) and then eliminate  $J_n$ , we get

$$\left\{ \frac{4n(n+1)}{z^2} - 1 \right\} J_{n+1} = J_{n-1} + \frac{2n}{z} J_{n+2}, \dots \dots (9)$$

from which it appears that if  $J_{n-1}$  and  $J_{n+2}$  vanish simultaneously, then either  $J_{n+1} = 0$ , which is impossible, or  $z^2 = 4n(n+1)$ . Any common root of  $J_{n-1}$  and  $J_{n+2}$  must therefore be such that its square is an integer.

Pursuing the process, we find that if  $J_{n-1}$ ,  $J_{n+3}$  have a common root  $z$ , then

$$(2n+1)z^2 = 4n(n+1)(n+2),$$

so that  $z^2$  is rational. And however far we go, we find that

\* Phil. Mag. vol. xx. p. 1003, 1910, equation (8).

the simultaneous evanescence of two Bessel's functions requires that the common root be such that  $z^2$  satisfies an algebraic equation whose coefficients are integers, the degree of the equation rising with the difference in order of the functions. If, as seems probable, a root of a Bessel's function cannot satisfy an integral algebraic equation, it would follow that no two Bessel's functions have a common root. The question seems worthy of the attention of mathematicians.

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IX. *On the Rate of Evolution of Heat by Pitchblende.*

By HORACE H. POOLE\*.

A DESCRIPTION was given in a former paper (Phil. Mag., Feb. 1910) of a determination of the rate of evolution of heat by Joachimsthal pitchblende. As the results obtained were considerably greater than was to be expected, further experiments seemed to be desirable.

The same method and apparatus were again employed, and for a description of them reference must be made to the previous paper. Before insertion in the calorimeter the pitchblende was gently heated for several hours on a metal plate. It was then placed while still hot in the calorimeter in which the flexible thermo-junction had already been inserted, and molten paraffin-wax was run in to completely fill up all the interstices. The calorimeter was gently tapped till air-bubbles ceased to rise to the surface. The neck of the calorimeter was then closed as before and the whole put aside to cool. The calorimeter contained 525 grams of pitchblende. A week later the calorimeter was buried in ice in the usual way. Another couple was placed in the ice vessel with its junctions at different points in the ice, to indicate what temperature differences might occur in the ice itself.

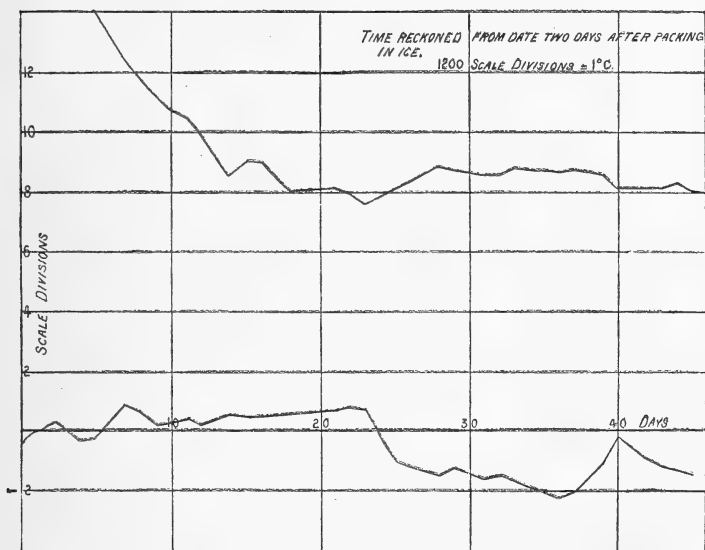
The readings of the couples are shown on the chart (fig. 1), on which the upper line indicates the temperature of the pitchblende and the lower the differences of temperature between the terminals of the second couple. As the latter was slightly more sensitive than that employed with the pitchblende, the readings have been reduced to the same scale, *i. e.* scale-divisions indicated by the pitchblende couple. This and the galvanometer were the same as those previously employed, but the distance from the galvanometer mirror to the scale having been increased from 102 cms. to 107 cms.,

\* Communicated by the Author.



the arrangement was about 5 per cent. more sensitive, so that 1° C. corresponded to about 1200 scale-divisions. The chart begins two days after the ice was packed. It will be seen that for a week after packing the temperature difference

Fig. 1.



indicated by the couple in the ice never exceeded 0°·0004 C., and was sometimes in one direction sometimes in the other ; but as time went on the differences and irregularities became greater and greater, the maximum difference of temperature recorded being nearly 0°·0020 C. There can be little doubt that the irregularities observed in the apparent temperature of the pitchblende are to be ascribed to fluctuations in the temperature of the outer junction. These fluctuations almost invariably increase with the time that has elapsed since packing. These variations of temperature may be due to the action of the water on the zinc of which the vessel is composed, or to regelation effects ; these points will be further considered later.

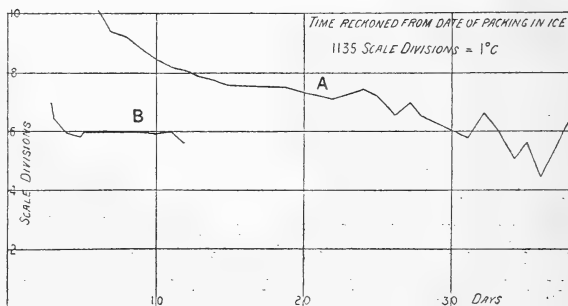
The mean temperature between the pitchblende and the ice for the last 30 days was 8·3 scale-divisions. As the heat escaping from the calorimeter in calories per hour is 6·2 times the temperature difference between the interior and the ice, the heat evolution per gram of pitchblende is  $\frac{8\cdot3 \times 6\cdot2}{1200 \times 525}$

*i. e.*  $8.15 \times 10^{-5}$  calorie per hour. This result being even higher than the previous ones, the calorimeter and pitchblende were put aside unopened for some months so that any chemical action which might be occurring might cease. As a short time previous to this experiment the wax had been melted, it is possible that crystallographic changes may have been occurring in the wax.

During the interval, in the course of some experiments on orangite which will be described in a later paper, a new inner ice vessel was made with double walls and lid, the space between the walls, about  $\frac{1}{4}$  inch, being filled with cotton-wool. This vessel is watertight, and arranged so that water from the outer ice cannot enter round the lid. In these experiments the ice employed was specially frozen by an ice-making firm out of water distilled in the laboratory. As, however, the temperature variations in this ice seemed quite as large as those in the ordinary commercial ice, the latter was again employed in the subsequent experiments. The distilled-water ice attacked the zinc vessel much more than the ordinary ice, as the sides were found to be coated with a thin layer of oxide. This was scrubbed off and the vessel thoroughly dried and coated while warm with vaseline.

Nine months after the insertion of the pitchblende in the calorimeter it was again placed in the ice. The second couple was not employed on this occasion, as there seems to be no doubt that variations in the ice temperature will be detected by the couple used with the pitchblende, and it was thought best to have as few leads entering the ice as possible. The temperature of calorimeter is shown on the chart (A, fig. 2). As the distance from galvanometer-mirror to

Fig. 2.



scale had been reduced to 161 crs.,  $1^{\circ}$  C. corresponds to about 1135 scale-divisions. As usual, the irregularities get

larger as time goes on, finally becoming so large that the experiment was discontinued. The mean for the last 20 days, which, however, is not very reliable, is  $6.25$  scale-divisions, corresponding to a heat evolution of  $6.5 \times 10^{-5}$  calorie per hour per gram.

It is evident that in all previous experiments the first fortnight during which the ice temperature is fairly steady is expended in the cooling of the pitchblende. So after the conclusion of the last experiment, before the pitchblende had warmed up more than  $0^{\circ}.2$  C., the ice vessel was repacked. As will be seen (B, fig. 2) the temperature became steady four days after packing, and remained so for seven days, after which irregularities again made their appearance. The mean  $5.9$  scale-divisions corresponds to a heat evolution of  $6.2 \times 10^{-5}$  calorie per hour.

This figure agrees very well with the mean of the previous results,  $6.1 \times 10^{-5}$ . In this experiment the pitchblende had been in the wax nearly eleven months and had been approximately at  $0^{\circ}$  C. for about two months. The steadiness of the final temperature attained also lends weight to the result.

The variations in the temperature of the ice may be due to traces of zinc oxide which, however, is almost insoluble in water, but, as coating the zinc with vaseline, though it stopped all apparent action, did not effect any improvement, it seems more probable that the variations are in some way caused by regelation. The ice when put in is in very small pieces and almost resembles snow in appearance; but when the vessel is opened the ice is always found to be frozen into a glassy cake evidently pierced by numerous small air spaces. It is hard to say whether these are isolated cavities or whether they may form connected channels through which air might circulate. No connexion could be traced between the temperature variations and the occasions on which the outer ice vessel was repacked.

The three experiments recorded give for the heat evolution—

(1)	$8.15 \times 10^{-5}$	calorie	per hour	per gram.
(2)	$6.5 \times 10^{-5}$	„	„	„
(3)	$6.2 \times 10^{-5}$	„	„	„
Mean	$6.95 \times 10^{-5}$	„	„	„

The mean obtained in the previous paper was  $6.1 \times 10^{-5}$ ; so if we take the mean of all the results we obtain  $6.5 \times 10^{-5}$ . The discrepancies between the various results are so large

that the exact value of their mean is not very important, but the later experiments strongly support the general result before obtained, that the heat evolution is at least  $6 \times 10^{-5}$  calorie per hour per gram, instead of  $4.4 \times 10^{-5}$ , which is the figure obtained on the assumption that each gram of radium generates 110 calories per hour.

Physical Laboratory,  
Trinity College, Dublin.  
October 1910.

X. *On the Bending of Electric Waves round a Large Sphere.*  
III.\* *By J. W. NICHOLSON, M.A., D.Sc.†*

*Determination of the constant  $\beta$ .*

THE present section of the paper takes up the problem of the determination of  $\beta$ , the numerical coefficient which is necessary to a final tabulation of the intensity at any point of the surface of the obstacle. This coefficient is defined by the fact that the first root of

$$\partial/\partial z \cdot z^{\frac{1}{2}} K_m(z) = 0, \quad \dots \quad (102)$$

where the Bessel function involved is regarded as a function of  $m$ , has an imaginary portion  $-iz^{\frac{1}{2}}\beta$ , and a real portion whose most significant term is  $z$ . It was shown before that no root of this equation occurs whose real part has not an order  $z$  at least (if the real part is to be preponderant as supposed), and therefore that the first possible root should be sought within such a region of variation of  $m$  that  $|m-z|$  is of order  $z^{\frac{1}{2}}$ .

Now for values of  $m$  and  $z$  corresponding in this way, a development of  $K_m(z)$ , suitable for the present purpose, can be deduced at once from the results of a previous paper‡ to which reference has already been made. For we may write, if  $\rho = (m-z)(6/z)^{\frac{1}{2}}$ , and if the terms of the series converge,

$$J_m(z) = \frac{1}{3\pi} \left(\frac{6}{z}\right)^{\frac{1}{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \cos \frac{\pi}{6} + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \cos \frac{5\pi}{6} + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) \cos \frac{9\pi}{6} + \dots \right\}$$

$$J_{-m}(z) = \frac{2}{3\pi} \left(\frac{6}{z}\right)^{\frac{1}{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \sin \frac{\pi}{3} \cos\left(m\pi - \frac{\pi}{3}\right) + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \sin \frac{2\pi}{3} \cos\left(m\pi - \frac{2\pi}{3}\right) + \dots \right\}$$

the error involved being of order  $z^{-1}$  at most.

\* Part I., *Phil. Mag.* April 1910. Part II., *Phil. Mag.* July 1910.

† Communicated by the Author.

‡ *Phil. Mag.* August 1908.

In the paper in question, a portion of the proof assumes that  $m$  is real, but the extension to complex values whose real part is large and positive may be shown to be lawful.

By reference to the original definition of  $K_m(z)$ , namely,

$$K_m(z) = \frac{\pi}{2 \sin m\pi} \left\{ J_{-m}(z) - e^{im\pi} J_m(z) \right\} e^{-\frac{1}{2}im\pi},$$

it may be shown that

$$K_m(z) = \frac{1}{6} \left( \frac{6}{z} \right)^{\frac{1}{2}} e^{-\frac{1}{2}im\pi} \left\{ \Gamma\left(\frac{1}{3}\right) \left(1 + e^{-\frac{1}{3}i\pi}\right) + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \left(1 + e^{\frac{1}{3}i\pi}\right) + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) \left(1 + e^{\frac{2}{3}i\pi}\right) + \dots \right\}. \quad (103)$$

Thus the equation

$$\partial/\partial z \cdot z^{\frac{1}{2}} K_m(z) = 0$$

becomes

$$\partial/\partial z \cdot z^{\frac{1}{2}} \psi(\rho) = 0,$$

where

$$\psi(\rho) = \Gamma\left(\frac{1}{3}\right) \left(1 + e^{-\frac{1}{3}i\pi}\right) + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \left(1 + e^{\frac{1}{3}i\pi}\right) + \dots$$

But

$$\begin{aligned} \partial/\partial z \cdot z^{\frac{1}{2}} \psi(\rho) &= z^{\frac{1}{2}} \psi'(\rho) \partial\rho/\partial z - \frac{1}{6} z^{-\frac{5}{6}} \psi(\rho) \\ &= -z^{-\frac{5}{6}} \left\{ \psi'(\rho) 6^{\frac{1}{2}} z^{\frac{2}{3}} + \frac{1}{6} \psi(\rho) \right\}, \end{aligned}$$

since for a value of  $\rho$  of order unity,  $\partial\rho/\partial z = -(6/z)^{\frac{1}{2}}$ , as in a previous section. The first term of the last expression is therefore preponderant, and the equation for the required zero may be treated as

$$\psi'(\rho) = 0$$

to a sufficient order for the present purpose. We must therefore solve the equation

$$\begin{aligned} \left(1 + e^{\frac{1}{3}i\pi}\right) \Gamma\left(\frac{2}{3}\right) + \left(1 - e^{\frac{2}{3}i\pi}\right) \Gamma\left(\frac{4}{3}\right) \frac{\rho^2}{2!} + \left(1 + e^{\frac{1}{3}i\pi}\right) \Gamma\left(\frac{5}{3}\right) \frac{\rho^3}{3!} \\ + \left(1 - e^{\frac{2}{3}i\pi}\right) \Gamma\left(\frac{7}{3}\right) \frac{\rho^5}{5!} + \dots = 0, \end{aligned}$$

certain of the terms vanishing. This may be written in the form

$$\frac{\rho^2}{2!} + \frac{4}{3} \frac{\rho^5}{5!} + \dots = -\frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} e^{\frac{1}{3}i\pi} \left\{ 1 + \frac{2}{3} \frac{\rho^3}{3!} + \frac{2 \cdot 5}{3^2} \frac{\rho^6}{6!} + \dots \right\} \quad \dots \quad (104)$$

and  $-i\beta$  is the imaginary part of the first root of this equation.

In determining the first approximation to this root, we may shorten the equation to

$$\frac{\rho^2}{2!} + \frac{4}{3} \frac{\rho^5}{5!} = -\frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} e^{\frac{1}{3}i\pi} \left\{ 1 + \frac{2}{3} \frac{\rho^3}{3!} \right\},$$

provided that the other terms are really negligible for the ensuing value of  $\rho$ . This is found to be the case. If  $\rho^3 = w$  (substituting the values of the Gamma functions),  $w$  is a root of

$$w^2 \left( 1 + \frac{w}{45} \right)^3 = 27 \cdot 895 \left( 1 + \frac{w}{9} \right)^3, \dots (105)$$

and this is a quintic with real coefficients, and therefore with at least one real root. It is found by the usual method of trial that there is only one real root, given by  $w = -3 \cdot 115$ . With this value, higher approximations to the solution of the actual equation may be deduced by the method of continued approximation, but this is not necessary for the present purpose, as an examination of the neglected terms indicates at once.

Thus

$$(m-z)^3 \frac{6}{z} = -3 \cdot 115,$$

or

$$m-z = -z^{\frac{1}{3}} (1, \omega, \omega^2) (\cdot 5192)^{\frac{1}{3}},$$

where  $\omega$  and  $\omega^2$  are cube roots of unity, given by

$$\frac{1}{2}(-1 \pm i\sqrt{3}).$$

But two of these values may be rejected, for,  $\rho^3$  being real, (104) indicates that  $\rho^2$  must be proportional to  $-e^{\frac{1}{3}i\pi}$ ,  $\rho$  proportional to  $e^{-\frac{1}{3}i\pi}$ , and therefore to  $1 - i\sqrt{3}$  with a positive real part. The only root available is therefore given by

$$m-z = \frac{1}{2} z^{\frac{1}{3}} (1 - i\sqrt{3}) (\cdot 5192)^{\frac{1}{3}}, \dots (106)$$

the others being introduced in cubing (104). Thus for the first root of  $\partial/\partial z \cdot z^{\frac{1}{3}} K_m(z) = 0$ , the imaginary part is on reduction

$$- \cdot 696 iz^{\frac{1}{3}}. \dots (107)$$

This imaginary part is negative, as stated in the previous section, and  $\beta = \cdot 696$ , roughly equal to  $2/3^*$ . With this value, the ratio of disturbed to undisturbed amplitude is given by (101) as  $(8\pi \sin \theta)^{\frac{1}{2}} (ka)^{\frac{1}{2}} \tan \frac{1}{2}\theta \beta^{-1} e^{-(ka)^{\frac{1}{2}} \beta \theta}$ .

\* This was inadvertently quoted as  $1/3$  in the previous section, but no deduction was made from it.

*Application to wireless telegraphy.*

As a typical case, suitable for the tabulation of the formula (101), we may take that of electric waves and the earth, the height of the antennæ being about 260 feet, and the corresponding wave-length therefore a quarter of a mile. In this case,  $ka$  is  $1\cdot01\cdot10^5$ .

When the orientation of the receiver from the oscillator is not greater than about ten degrees, or seven hundred miles, a convenient practical formula for the ratio of the diffracted amplitude to the amplitude corresponding to an undisturbed oscillator is found to be

$$\cdot058 \theta^{\frac{3}{2}} (\cdot574)\theta, \dots (108)$$

where  $\theta$  is in degrees. The value of  $ka$  corresponding to average practice has been inserted. A change of the wave-length, say from a quarter to a fifth of a mile, does not change the order of magnitude, or in general, by more than unity, the most significant figure of this formula. The variation of this function is exhibited in the following table.

TABLE I.

$\theta$ .	Amp. ratio.	Terrestrial miles.	$\theta$ ,	Amp. ratio.	Terrestrial miles.
1°	·033	69	6°	·030	414
2°	·054	138	7°	·022	483
3°	·057	207	8°	·015	552
4°	·050	276	9°	·011	621
5°	·040	345	10°	·007	690

Beyond 10°, the amplitude ratio in the same case may be computed from the formula

$$50\cdot2 \tan \frac{1}{2}\theta \sqrt{\sin \theta} (\cdot574)\theta, \dots (109)$$

where  $\theta$  is again measured in degrees.

This is tabulated below for every degree from 0° to 30°, and for every five degrees from 30° to 90°. The values given by the two formulæ for  $\theta=10^\circ$  are in agreement, and between 10° and 30°, an interpolation method may be used for intermediate cases. The convenient notation  $\cdot0^n m$  has been used for  $\cdot m10^{-n}$ .

TABLE II.

$\theta$ .	Amp. ratio.	Terrestrial miles.	$\theta$ .	Amp. ratio.	Terrestrial miles.
11°	·0047	759	21°	·0448	1449
12°	·0030	828	22°	·0330	1518
13°	·0020	897	23°	·018	1587
14°	·0013	966	24°	·011	1656
15°	·081	1035	25°	·068	1725
16°	·052	1104	26°	·042	1794
17°	·032	1173	27°	·025	1863
18°	·020	1242	28°	·015	1932
19°	·013	1311	29°	·093	2001
20°	·078	1380	30°	·056	2070

TABLE III.

$\theta$ .	Amp. ratio.	Terrestrial miles.	$\theta$ .	Amp. ratio.	Terrestrial miles.
35°	·044	2415	65°	·0166	4485
40°	·034	2760	70°	·01546	4830
45°	·025	3105	75°	·01632	5175
50°	·018	3450	80°	·0722	5520
55°	·0113	3795	85°	·01815	5865
60°	·0194	4140	90°	·01910	6210

These tables are in the form which indicates most readily the amount of shadowing due to the earth at any point of its surface, without confusion with the ordinary fall of intensity due to distance from the oscillator. The shadowing soon becomes very complete, for the ratio of the mean energies per unit volume in the two cases is given by the square of the amplitude ratio.

But in estimating the capacity of diffraction for giving an explanation of the great success of experimenters, other tables are required, for the degree of sensitiveness possible to the receiver plays a determining part. Let us suppose that a given receiver will function at a distance of about seventy miles or one degree from the oscillator. Then the further degree of sensitiveness necessary, in order that it shall detect radiation at an orientation  $\theta$ , may be found by comparing the two cases  $\theta = \theta^\circ$  and  $\theta = 1^\circ$ , without reference



to the undisturbed oscillator. In the following tables this comparison is made both for the amplitudes and the energies:—

TABLE IV.

$\theta$ .	$\frac{\text{Amp. at } \theta^\circ}{\text{Amp. at } 1^\circ}$	$\frac{\text{Energy at } \theta^\circ}{\text{Energy at } 1^\circ}$	Terr. miles.	$\theta$ .	$\frac{\text{Amp. at } \theta^\circ}{\text{Amp. at } 1^\circ}$	$\frac{\text{Energy at } \theta^\circ}{\text{Energy at } 1^\circ}$	Terr. miles.
1°	1	1	69	16°	·0 <sup>8</sup> 96	·0 <sup>8</sup> 93	1104
2°	·812	·659	138	17°	·0 <sup>2</sup> 57	·0 <sup>6</sup> 33	1173
3°	·571	·326	207	18°	·0 <sup>3</sup> 34	·0 <sup>8</sup> 11	1242
4°	·378	·143	276	19°	·0 <sup>3</sup> 20	·0 <sup>7</sup> 39	1311
5°	·243	·059	345	20°	·0 <sup>3</sup> 12	·0 <sup>7</sup> 14	1380
6°	·153	·023	414	21°	·0 <sup>4</sup> 69	·0 <sup>3</sup> 47	1449
7°	·095	·0 <sup>2</sup> 89	483	22°	·0 <sup>4</sup> 40	·0 <sup>3</sup> 16	1518
8°	·058	·0 <sup>2</sup> 34	552	23°	·0 <sup>4</sup> 24	·0 <sup>2</sup> 56	1587
9°	·035	·0 <sup>2</sup> 12	621	24°	·0 <sup>4</sup> 14	·0 <sup>2</sup> 19	1656
10°	·021	·0 <sup>3</sup> 46	690	25°	·0 <sup>8</sup> 81	·0 <sup>1</sup> 66	1725
11°	·013	·0 <sup>3</sup> 16	759	26°	·0 <sup>3</sup> 47	·0 <sup>1</sup> 22	1794
12°	·0 <sup>2</sup> 77	·0 <sup>4</sup> 59	828	27°	·0 <sup>2</sup> 27	·0 <sup>1</sup> 74	1863
13°	·0 <sup>2</sup> 46	·0 <sup>4</sup> 21	897	28°	·0 <sup>2</sup> 16	·0 <sup>1</sup> 26	1932
14°	·0 <sup>2</sup> 27	·0 <sup>5</sup> 75	966	29°	·0 <sup>2</sup> 94	·0 <sup>1</sup> 289	2001
15°	·0 <sup>2</sup> 16	·0 <sup>5</sup> 26	1035	30°	·0 <sup>2</sup> 55	·0 <sup>1</sup> 230	2070

TABLE V.

$\theta$ .	$\frac{\text{Amp. at } \theta^\circ}{\text{Amp. at } 1^\circ}$	$\frac{\text{Energy at } \theta^\circ}{\text{Energy at } 1^\circ}$	Terr. miles.	$\theta$ .	$\frac{\text{Amp. at } \theta^\circ}{\text{Amp. at } 1^\circ}$	$\frac{\text{Energy at } \theta^\circ}{\text{Energy at } 1^\circ}$	Terr. miles.
35°	·0 <sup>7</sup> 37	·0 <sup>11</sup> 13	2415	65°	·0 <sup>14</sup> 27	·0 <sup>29</sup> 74	4485
40°	·0 <sup>8</sup> 24	·0 <sup>17</sup> 59	2760	70°	·0 <sup>15</sup> 17	·0 <sup>31</sup> 30	4830
45°	·0 <sup>9</sup> 16	·0 <sup>19</sup> 25	3105	75°	·0 <sup>16</sup> 11	·0 <sup>33</sup> 12	5175
50°	·0 <sup>10</sup> 10	·0 <sup>21</sup> 11	3450	80°	·0 <sup>18</sup> 69	·0 <sup>36</sup> 47	5520
55°	·0 <sup>12</sup> 66	·0 <sup>24</sup> 44	3795	85°	·0 <sup>19</sup> 43	·0 <sup>38</sup> 19	5865
60°	·0 <sup>13</sup> 43	·0 <sup>26</sup> 18	4140	90°	·0 <sup>20</sup> 27	·0 <sup>41</sup> 73	6210

The orientation taken as a standard of comparison must be such as to make the distance from oscillator to receiver large in comparison with the height of antenna. This condition is satisfied sufficiently by taking 1° as the standard.

It is difficult to draw any conclusions from these tables for small orientations, owing to the lack of a reliable series

of quantitative measurements of the effect at the receiver for different distances from the sending apparatus. But when such measurements are available, it will be possible to decide at once what proportion of the observed effect is due to diffraction round the surface. That diffraction may be a very important factor for small orientations is not contradicted by the above figures, although the investigation by Sommerfeld\* of the effect of conduction through the earth must be borne in mind.

But the smallness of the numbers corresponding to larger orientations shows very clearly that, as already pointed out, diffraction must be a relatively insignificant agency in the success of experiments such as those of Marconi. For example, the ratio of the energy falling in unit time on a receiver at a distance of 2000 miles, to that falling on the same receiver at a distance of 70 miles, if diffraction alone were the agency, would be of order  $10^{-12}$ . It is improbable that any receiver could be sensitive enough to record so small an effect, and we are compelled to seek an explanation of the experiments elsewhere. Two alternatives have been proposed. The first is that of Sommerfeld, who, by taking an infinite plane to represent the surface of the earth for convenience of mathematical analysis, concludes that the finite conductivity of the earth may be sufficient to account for experimental success. But the simplification thus introduced into the analysis may render the results, at the same time, inapplicable to large orientations, and for these great distances a more rigorous investigation is desirable. Unfortunately the method used by Sommerfeld does not appear to be applicable to the case of an obstacle with a spherical boundary. The other alternative is the hypothesis that upper layers of the terrestrial atmosphere may, by being rendered conducting, reflect back to the earth the radiation which they receive from the sending apparatus. This seems to furnish a hopeful line of attack on the problem, and the view is supported by several known experimental results, in particular by the difference experienced in signalling by night and by day. But no investigation of the matter from a theoretical standpoint has yet been published.

The next section of the paper is concerned with the determination of a higher approximation to the effect in the region previously called the "region of brightness."

\* *Ann. der Phys.* 1909, March 16.

XI. *New Determinations of some Constants of the Inert Gases.* By CLIVE CUTHBERTSON, *Fellow of University College, London* \*.

IN a recent paper in the Philosophical Magazine, Dr. G. Rudolf † calculated some of the molecular constants of the inactive gases with the object of seeing how far the values obtained by various methods agreed among themselves. The results were disappointing.

Since that time three papers ‡ on the physical properties of these gases have provided additional data for their study, and a comparison of these figures, obtained by different methods, points to some interesting conclusions.

RELATION BETWEEN DETERMINATIONS OF VOLUME OF MOLECULES FROM VISCOSITY AND REFRACTIVITY.

On certain assumptions it is possible to calculate the volume occupied by the atoms of a gas both from the viscosity and from the refractivity.

We find in Rankine's first paper the first complete set of measurements of the viscosity of the five inert gases. The experiments were made at atmospheric temperature, with the same apparatus under similar conditions. The results are therefore of high value.

Rankine gives the figures shown in the first columns of the table below for the relative viscosity, mean free path, and molecular radius of the five gases, taking the figures for helium as unity in each case.

He adds that, in order to obtain the absolute values of the radii, each number in the third column must be multiplied by  $1.68 \times 10^8$ . The number of molecules per unit volume

\* Communicated by the Author.

† "The Molecular and some other Constants of the Inactive Gases," by G. Rudolf, Ph.D., B.Sc. *Phil. Mag.* June 1909, p. 795.

‡ "On the Viscosities of the Gases of the Argon Group," by A. O. Rankine, B.Sc. *Proc. Roy. Soc.* vol. lxxxiii. p. 516 (1910).

"On the Variation with Temperature of the Viscosities of the Gases of the Argon Group," by A. O. Rankine, B.Sc. *Proc. Roy. Soc.* vol. lxxxiv. p. 181 (1910).

"On the Refraction and Dispersion of Argon and Redeterminations of the Dispersion of Helium, Neon, Krypton and Xenon," by C. & M. Cuthbertson. *Proc. Roy. Soc.* vol. lxxxiv. p. 13 (1910).

TABLE I.

Element.	1. Relative viscosity.	2. Relative mean free path.	3. Relative molecular radius.	4. Absolute molecular radius) $\times 10^8$ .	5. Absolute molecular volume per normal unit vol.	6. $(\mu_\infty - 1) \frac{2}{3}$ .	7. Ratio of column 5 to column 6.
Helium .....	1.000	1.000	1.000	.84	.0000695	.0000231	3.0
Neon .....	1.585	0.704	1.19	.9996	.0001171	.0000444	2.53
Argon .....	1.124	0.354	1.68	1.4112	.0003296	.0001848	1.785
Krypton ...	1.253	0.274	1.91	1.6044	.0004844	.0002791	1.733
Xenon.....	1.136	0.198	2.25	1.890	.0007918	.0004545	1.742

is taken as  $2.8 \times 10^{19}$ . The calculation is based on the relation

$$L^{-1} = \sqrt{2} \cdot N\pi\sigma^2,$$

where  $L$  is the mean free path and  $\sigma$  is taken to be the molecular radius. It is, however, more usual to take  $\sigma$  as the radius of the sphere of action of the molecule and equal to the molecular diameter. Hence the absolute value of the molecular radius is given by multiplying Rankine's figures by

$$\frac{1.68 \times 10^8}{2}.$$

The sum of the volumes of the molecules per unit volume of gas is

$$N \frac{4\pi\sigma^3}{3}.$$

These figures are given in the fourth and fifth columns of the table.

Turning now to the refractivities, we start from Clausius's well-known equation

$$g = \frac{\kappa - 1}{\kappa + 2},$$

where  $g$  denotes the fraction of the volume containing a gas which its molecules actually occupy. This becomes

$$g = \frac{\mu^2 - 1}{\mu^2 + 2} \text{ if } \kappa = \mu^2.$$

In the case of gases, where  $\mu - 1$  is small,

$$g = \frac{2}{3}(\mu - 1) \text{ approximately.}$$

The refractivities of the inert gases given in the paper quoted above are shown in the sixth column of the table, multiplied by  $\frac{2}{3}$ , and in the last column are shown the ratios of the molecular volumes as determined from the viscosities to those derived from the refractivities. The constancy of the ratio in the cases of argon, krypton, and xenon is remarkable, and, considering the dissimilarity of the two methods and the number of assumptions upon which they rest, the concordance between the two sets of values in absolute measure is not less surprising.

RELATION BETWEEN THE NUMBER OF ELECTRONS IN THE ATOM AND THE RADIUS OF THE SPHERE OF ACTION.

Till recently the refraction and dispersion of gases have usually been expressed in terms of Cauchy's formula

$$\mu - 1 = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} \dots$$

But in view of the success which had attended the use of a formula of Sellmeier's type

$$\mu - 1 = \frac{N}{n_0^2 - n^2}$$

in explaining the observed facts of dispersion in other cases, it was thought better to express the refractivities of the inert gases in this form in the paper quoted above.

Here  $N$  is a constant,  $n_0$  is the frequency of the free vibration of the parts of the atom (assuming there to be only one) and  $n$  that of the light whose refractivity is measured. When this equation is transformed into the shape it assumes on the electronic theory, as developed by Drude,  $N$  becomes proportional to the number of electrons in the molecule which are effective in influencing dispersion.

Table II. shows the values of the constants  $N$  and  $n_0^2$  obtained experimentally\*.

\* In the paper quoted above the values for  $N$  were doubled for comparison with similar figures for diatomic gases. The numbers here given are those calculated from the actual observations by the method of least squares.

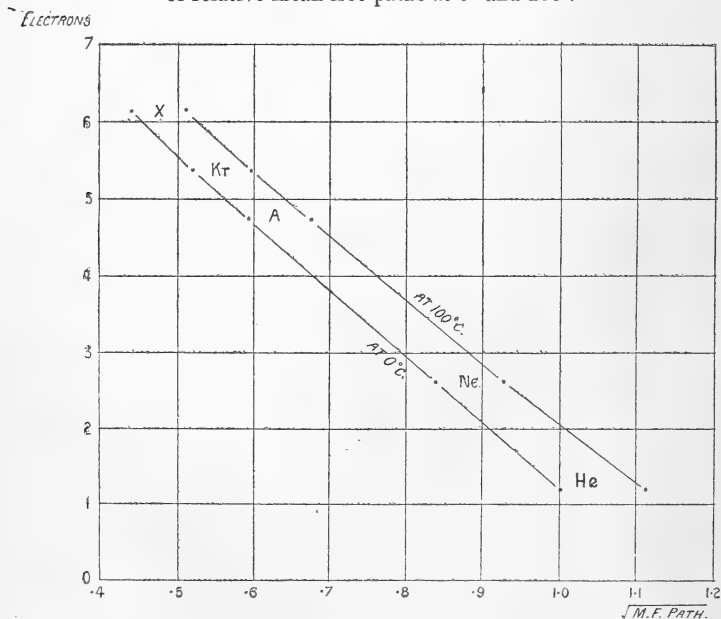
TABLE II.

Element.	1. $N \times 10^{-27}$ .	2. $n_0^2 \times 10^{-27}$ .
Helium .....	1.21238	34992
Neon .....	2.59826	38916
Argon.....	4.71632	17009
Krypton.....	5.3446	12768
Xenon .....	6.1209	8978

If the figures in the first column, which are proportional to the number of "dispersion electrons" in the atom, are plotted against the square roots of Rankine's values for the relative mean free path at  $0^\circ$  and  $100^\circ$  C., we obtain two

Fig. 1.--Inert Gases.

Relative number of dispersion electrons plotted against square roots of relative mean free paths at  $0^\circ$  and  $100^\circ$ .



curves which are shown in fig. 1. The radius of curvature is directed away from the origin, but the lines are so nearly

straight that for present purposes the relation may be considered linear. The numbers used for the mean free paths are given below.

TABLE III.

Element.	Square roots of mean free paths (Rankine).	
	At 0° C.	At 100° C.
Helium .....	1	1.11
Neon .....	.839	.927
Argon.....	.595	.674
Krypton.....	.523	.598
Xenon .....	.445	.515

On the kinetic theory in its simplest form the root of the mean free path is inversely proportional to the radius of the sphere of action. To a first approximation, then, we find in these five gases a linear relation between the number of electrons influencing dispersion and the reciprocal of the radius of the sphere of action as determined from measurements of the viscosity.

Another interesting result is obtained by plotting  $N^2$  against  $\sigma$ , the reciprocals of the numbers given in Table III. These numbers are shown in Table IV.

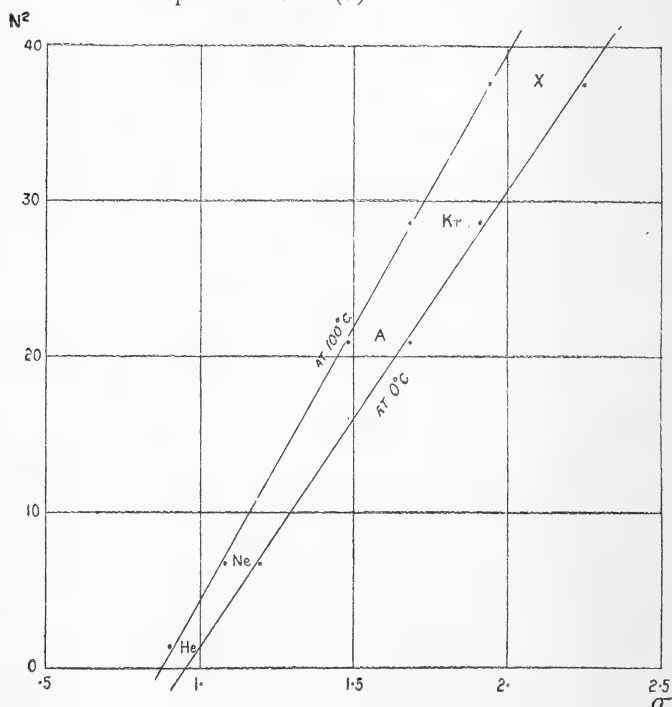
TABLE IV.—Relative radii of the spheres of action of the inert gases at 0° and 100° C.

	At 0° C.	At 100° C.
Helium .....	1.0	.902
Neon .....	1.19	1.082
Argon.....	1.68	1.482
Krypton .....	1.01	1.68
Xenon .....	2.25	1.94

Fig. 2 shows that this relation also is approximately linear\*. It denotes that the squares of the number of

Fig. 2.—Inert Gases.

Relative number of electrons squared ( $N^2$ ) plotted against radii of spheres of action ( $\sigma$ ) at  $0^\circ$  and  $100^\circ$  C.



“dispersion” electrons are proportional to the radii of the spheres of action of the atoms diminished by a constant. This constant is about 95/100 of the radius of the sphere of action of helium at  $0^\circ$  C.

#### RELATION BETWEEN THE NUMBER OF ELECTRONS AND THE CRITICAL TEMPERATURE.

If the squares of the relative numbers of “dispersion” electrons are plotted against the critical temperatures of the gases, the four known points fall nearly on a straight line

\* For obvious mathematical reasons these relations cannot both be exactly linear. But they exhibit the broad relationships of the constants sufficiently well for the present.



which passes near the origin (fig. 3). The numbers employed are given in Table V.

Fig. 3.—Inert Gases.  
Relative number of electrons squared ( $N^2$ ) plotted against critical temperatures.

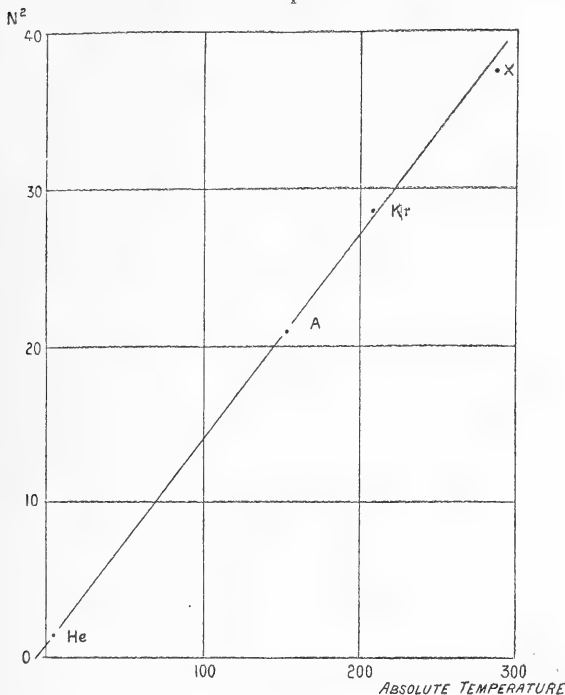


TABLE V.

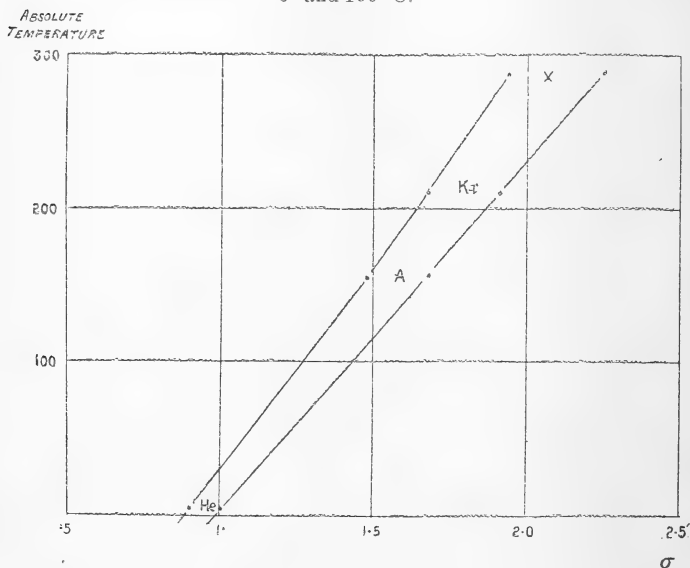
	Relative number of Electrons squared.	Critical Temperature (absolute).
Helium .....	1.46	5
Neon .....	6.7	—
Argon.....	20.97	155.6
Krypton .....	28.57	210.6
Xenon .....	37.5	287.7

The critical temperature of neon is unknown : but calculated from the ordinate it should be in the neighbourhood of 46° A.

Since it has been shown above that a linear relation also exists between the squares of the number of electrons and the radius of the sphere of action, it follows that a similar relation exists between the radius and the critical temperature. This is shown in fig. 4. The agreement is even better than in figs. 2 and 3.

Fig. 4.—Inert Gases.

Critical temperatures plotted against radii of spheres of action at 0° and 100° C.



RELATION BETWEEN THE NUMBER OF ELECTRONS AND THE CONSTANT C IN SUTHERLAND'S FORMULA.

In his second paper Rankine shows that the temperature coefficient of the viscosity may be of great importance. Using Sutherland's formula

$$\eta = KT^{\frac{1}{2}} / \left(1 + \frac{C}{T}\right),$$

where T is the absolute temperature and K and C constants for the gas, he obtains the following values for C from two observations of the viscosity :—

	C.
Helium .....	70
Neon .....	56
Argon .....	142
Krypton .....	188
Xenon .....	252

These numbers he plots against the critical temperature, and obtains a straight line running through argon, krypton, and xenon, which passes very near the origin, but helium is far off the line.

Since the critical temperatures are now shown to be proportional to the squares of the number of electrons, it follows that for argon, krypton, and xenon,  $C$  in Sutherland's equation is nearly proportional to the square of the number of electrons.

#### *Remarks.*

These results throw additional light on Rankine's discovery that the constant in Sutherland's formula for the temperature coefficient of viscosity is intimately connected with the critical temperature; and they suggest that the forces which, by acting between the atoms, cause both these phenomena, are to be found in the electric charges of the electrons which influence dispersion. Further they show that there is an intimate relation between the number of electrons in the atoms of these five gases and the radii of their spheres of action at the same temperature.

One indirect inference may further be noted. Doubts have been felt whether it is permissible to express the refractivity of a gas by a single term of a formula of Sellmeier's type, since this is equivalent to the assumption that the electrons influencing dispersion are all of the same type and have the same free frequency; and it was recognized that such a simplification was only tentative.

The facts that figures derived from this formula can be correlated with others based upon the kinetic theory of gases is evidence that, at least in the case of these gases, the hypothesis is substantially correct.

#### *Application to other Gases.*

The relations exhibited above cannot be extended to hydrogen, oxygen, or nitrogen. The best agreement is found in the case of the relation between the volumes, as determined from measurements of viscosity and refractivity respectively. The ratio in the case of oxygen is the same as in that of argon, krypton, and xenon, while in that of nitrogen it is not far different. But hydrogen departs widely from this value.

It is probable that in the diatomic gases the conditions are more complicated than in the case of the argon group.

XII. *On Magnetostriction.* By R. A. HOUSTOUN, M.A.,  
D.Sc., Ph.D., Lecturer on Physical Optics in the University  
of Glasgow\*.

THE object of this paper is to derive a relation connecting magnetostriction with the change of magnetization produced by stress and to test it by the results of experiment. In substance the relation is not new but in method of statement it is, and the derivation presented here is shorter and simpler than other methods. The paper also proves in a new and simple way a theorem due to Lord Kelvin.

Consider a ferro-magnetic wire hanging vertically inside a vertical solenoid with heating jacket, a pan being attached to the lower end of the wire for holding weights. Then, if hysteresis be neglected, the state of the wire may be regarded at any time as a function of the three independent variables T, F, and H,—temperature, stretching force, and magnetic field intensity. If T, F, and H suffer small changes, then the heat received by the whole wire is given by

$$dq = cdT + bdF + adH.$$

Let B denote the induction in the wire,  $v$  its volume, and  $x$  the vertical displacement of its lower end. Then the work done on the wire when F and H are increased is  $Fdx + vHdB/4\pi$ . Let U be the intrinsic energy of the wire and S its entropy. Then—

$$\begin{aligned} dU &= dq + Fdx + \frac{vHdB}{4\pi} \\ &= \left( c + F \frac{\partial x}{\partial T} + \frac{vH}{4\pi} \frac{\partial B}{\partial T} \right) dT + \left( b + F \frac{\partial x}{\partial F} + \frac{vH}{4\pi} \frac{\partial B}{\partial F} \right) dF \\ &\quad + \left( a + F \frac{\partial x}{\partial H} + \frac{vH}{4\pi} \frac{\partial B}{\partial H} \right) dH, \end{aligned}$$

and

$$dS = \frac{c}{T} dT + \frac{b}{T} dF + \frac{a}{T} dH.$$

Since these are perfect differentials, we have the following six independent relations :—

$$\frac{\partial c}{\partial F} + \frac{\partial x}{\partial T} = \frac{\partial b}{\partial T} \dots \dots \dots (1)$$

$$\frac{\partial c}{\partial H} + \frac{v}{4\pi} \frac{\partial B}{\partial T} = \frac{\partial a}{\partial T} \dots \dots \dots (2)$$

\* Communicated by Prof. A. Gray, F.R.S. Part of the matter of this paper has appeared in "On Two Relations in Magnetism," Proc. Roy. Soc. Edin. vol. xxx. p. 457 (1910).

$$\frac{\partial b}{\partial H} + \frac{v}{4\pi} \frac{\partial B}{\partial F} = \frac{\partial a}{\partial F} + \frac{\partial x}{\partial H} \dots \dots \dots (3)$$

$$\frac{1}{T} \frac{\partial c}{\partial F} = \frac{\partial}{\partial T} \left( \frac{b}{T} \right) = \frac{1}{T} \frac{\partial b}{\partial T} - \frac{b}{T^2} \dots \dots \dots (4)$$

$$\frac{1}{T} \frac{\partial c}{\partial H} = \frac{\partial c}{\partial T} \left( \frac{a}{T} \right) = \frac{1}{T} \frac{\partial a}{\partial T} - \frac{a}{T^2} \dots \dots \dots (5)$$

and 
$$\frac{\partial b}{\partial H} = \frac{\partial a}{\partial F} \dots \dots \dots (6)$$

The differential coefficients of *a*, *b*, and *c* cannot be easily determined experimentally; hence we eliminate them by combining (1) with (4), (2) with (5), and (3) with (6), and obtain

$$\frac{\partial x}{\partial T} = \frac{b}{T},$$

$$\frac{v}{4\pi} \frac{\partial B}{\partial T} = \frac{a}{T},$$

and 
$$\frac{v}{4\pi} \frac{\partial B}{\partial F} = \frac{\partial x}{\partial H}.$$

The first of these is the well-known relation between the coefficient of linear expansion of a wire and the cooling effect produced on stretching it.

The second relation states that if the induction in a wire increases with temperature, *a* is positive, and consequently the temperature will fall if *H* is increased; if the induction diminishes with temperature, the temperature of the wire will rise if *H* is increased. This theorem has been derived before by Lord Kelvin,\* and in quite another way. He states the result differently,—that a substance in which the magnetism diminishes with temperature when drawn gently away from a magnet experiences a cooling effect; a substance in which the magnetism increases with temperature when drawn gently away from a magnet experiences a heating effect. This cooling and heating effect is probably always masked by the irreversible heating due to Foucault currents in the wire, and to the viscous resistance to the motion of the molecular magnets.

The third relation is the one that the paper is mainly concerned with. It states, that if the induction increases when the wire is stretched, the length of the wire increases when it is magnetized, and *vice versa*. The connexion between

\* "On the Thermo-elastic, Thermo-magnetic, and Pyro-electric Properties of Matter," *Phil. Mag.* [5] v. (1878) p. 4.

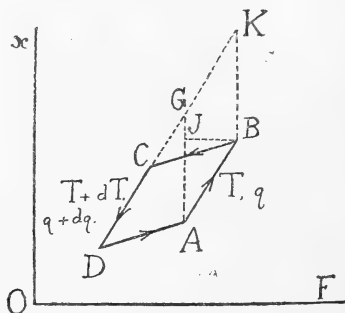
magneto-striction and the effect of stress on magnetism has been worked out several times, amongst others by J. J. Thomson in his 'Applications of Mathematics to Physics and Chemistry.' The result nearest mine is given by A. Heydweiller\*. His equation is

$$\frac{\partial E}{\partial H} = -\frac{E^2}{4\pi} \frac{\partial^2 B}{\partial p^2},$$

$E$  being Young's modulus for the wire and  $p$  the stress per unit area of cross-section. My equation can be put in this form. Heydweiller starts with another term in the expression for the external work done on the wire, namely,  $HBdv/4\pi$ , but he makes approximations afterwards, which are equivalent to neglecting this term. Heydweiller's formula has been proved experimentally by H. Rensing†; and has been attacked by R. Gans‡.

On the suggestion of Dr. C. G. Knott I have tested the third relation with the data of Nagaoka and Honda§. Before stating the results of the test I shall, however, prove the relations in another way.

Fig. 1.



Suppose that the magnetic field intensity remains constant and that the wire is carried through the cycle depicted in the adjoining diagram, AB and CD being isothermals and

\* "Zur Theorie der magneto-elastischen Wechselbeziehungen," *Ann. d. Phys.* [4] xii. (1903) p. 602.

† "Ueber magneto-elastische Wechselbeziehungen in para-magnetischen Substanzen," *Ann. d. Phys.* [4] xiv. (1904) p. 363.

‡ "Magneto-Striktion ferromagnetischer Körper," *Ann. d. Phys.* [4] xiii. (1904) p. 634. "Zur Heydweillerschen Kritik meiner Formeln betreffend Magneto-Striktion ferro-magnetischer Körper," *Ann. d. Phys.* [4] xiv. (1904) p. 638.

§ "On Magneto-Striction." H. Nagaoka and K. Honda. *Phil. Mag.* [5] xlvi. (1898) p. 260.

BC and DA adiabatics. On AB a quantity of heat  $q$  is taken in; on CD a quantity  $q + dq$  is given out. The area of the figure is equal to the work done on the wire in the cycle and this is equal to  $dq$ . Hence

$$dq = AC = AK = AG \times BJ.$$

Also  $q/T = dq/dT$ . Therefore, eliminating  $dq$ ,

$$AG \times BJ = qdT/T.$$

But  $q$ , the heat received in the isothermal AB, is equal to  $b dF$ , *i. e.*  $b \overline{BJ}$ . Substituting,

$$\frac{b}{T} = \frac{AG}{dT}.$$

But  $AG$  is the alteration of  $x$ ,  $F$  constant in going from the one isothermal to the other; hence

$$\frac{b}{T} = \frac{\partial x}{\partial T}.$$

The second relation can be proved in an analogous manner by using the  $B, H$  diagram.

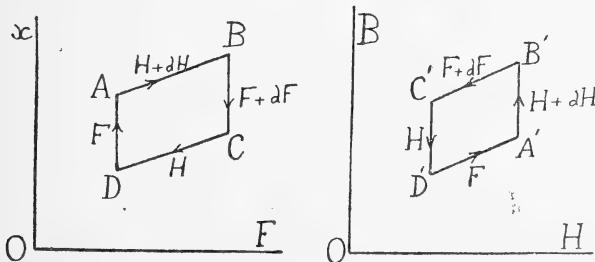
If hysteresis be not disregarded, the cycle becomes irreversible, and in place of  $q/T = dq/dT$  we must use

$$\frac{q}{T} - \frac{q + dq}{T + dT} \leq 0.$$

Also the figure is no longer a parallelogram as the shape of the isothermals and adiabatics is different according to the direction in which the wire is being put through the change.

In order to prove the third relation, assume that the wire is put through the cycle represented in the two following diagrams.

Fig. 2



The temperature is kept constant. The two diagrams represent the same cycle, the points ABCD in the one figure corresponding to the points A'B'C'D' in the other.

The area ABCD represents the work done against the stretching force, and  $v/4\pi$  multiplied by the area A'B'C'D' the work done by the magnetic field intensity during the cycle. Therefore, by the principle of energy

$$ABCD = \frac{v}{4\pi} A'B'C'D'.$$

Now  $ABCD = dF \frac{\partial x}{\partial H} dH$ , and  $A'B'C'D' = dH \frac{\partial B}{\partial F} dF$ .

Hence substituting and dividing out by  $dFdH$ ,

$$\frac{v}{4\pi} \frac{\partial B}{\partial F} = \frac{\partial x}{\partial H}.$$

*Thus the third relation follows solely from the principle of energy; the second law of thermodynamics has not been used at all.* This is not apparent from the former method of proof.

In the paper by Nagaoka and Honda measurements are given on the change in length of an ovoid of iron and a nickel rod of square cross-section when magnetized in the direction of the axis. The major axis of the ovoid was 20 cms. long and its minor axis 0.986 cm.; the length of the nickel rod was 26 cms. and its breadth was 0.514 cm. Measurements are also given on the effect of longitudinal pull on the magnetization of the same nickel rod and on the magnetization of a rod made from the same iron as the ovoid. Since we are dealing with nickel and iron we may write  $B = 4\pi I$ . The relation to be tested then becomes

$$v \frac{\partial I}{\partial F} = \frac{\partial x}{\partial H},$$

$$\text{or} \quad \frac{\partial I}{\partial p} = \frac{\partial a}{\partial H},$$

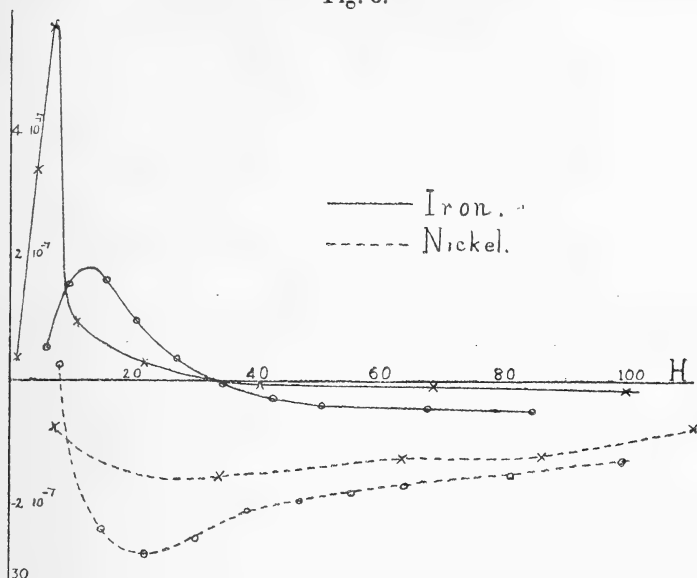
if  $a$  be the elongation per unit length and  $p$  the stretching force per unit area of cross-section. The results are given by the following curves (fig. 3). The curves marked by circles give  $\partial I/\partial p$  and those marked by crosses  $\partial a/\partial H$ . They should of course coincide. The differential coefficients were formed by taking the differences of the successive figures in the tables in the article. The smaller value of  $\delta p$ , 0.19 kg./sq. mm. was taken.

The agreement, such as it is, is as good as that given by Kirchhoff's theory which Nagaoka and Honda tested numerically.



Similar relations can of course easily be derived for other forms of stress, torsion or hydrostatic pressure. We have only

Fig. 3.



to substitute for  $F$  the twisting couple or hydrostatic pressure and for  $dx$  change of angle or of volume. But the change of magnetization produced by hydrostatic pressure is only of the same order as the change of volume produced. Hence in this case no agreement can be expected; the order of magnitude is, however, given correctly.

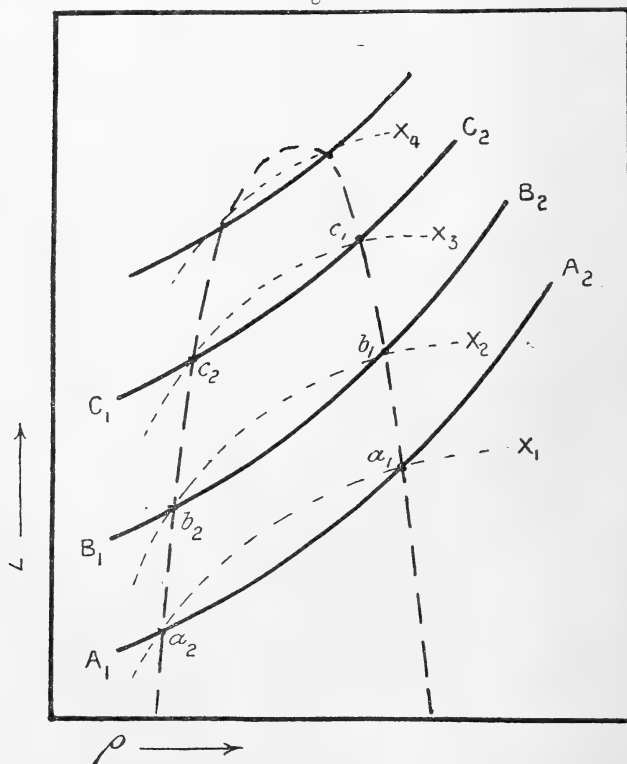
XIII. *An Investigation of the Determinations of the Law of Chemical Attraction between Atoms from Physical Data.*  
By R. D. KLEEMAN, D.Sc., B.A., Mackinnon Student of the Royal Society\*.

**I**N this paper the writer proposes to investigate to what extent the law of chemical attraction can be derived from latent heat, surface tension, and other physical properties of substances; and an endeavour will be made to place the whole subject on a sound mathematical basis. This is highly desirable as different investigators have obtained different laws for the attraction. These laws will be briefly discussed and compared with the results obtained in this paper.

\* Communicated by the Author.

It can be shown strictly mathematically that it is impossible to determine completely the law of attraction between atoms from latent heat or surface tension data ; in other words, the law deduced must contain an arbitrary function of the distance between the attracting molecules and their temperature. Thus, let  $\psi_1(z, T)$  denote the internal heat of evaporation of a liquid into a vacuum, where  $z$  is the distance of separation of the molecules in the liquid and  $T$  is the temperature, which is supposed to be deduced from the true law of molecular attraction on the supposition that the internal latent heat is the work done against the attraction of the molecules on their getting separated\*. Since  $z$  may be expressed in terms of the density of the liquid this expression may be written  $\psi_2(\rho, T)$ . The above statement can most

Fig. 1.



conveniently be proved graphically. Let the curve  $A_1A_2$  in fig. 1 denote the graph of the equation  $L = \psi_2(\rho, T)$  for a

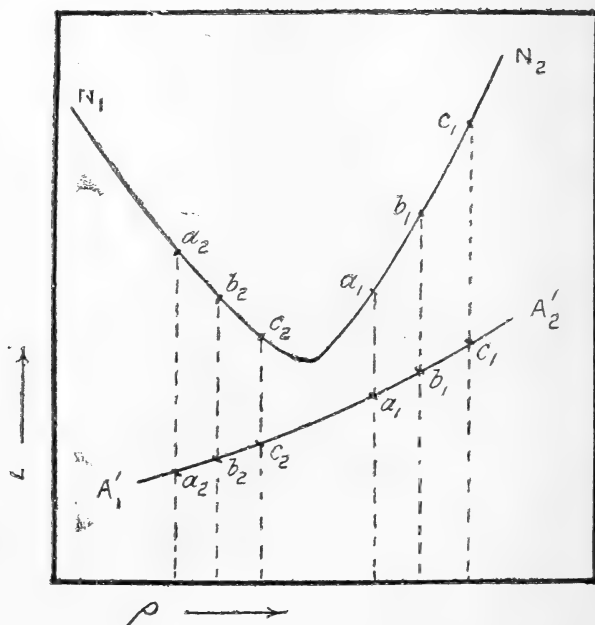
\* We are assuming that the attraction between two molecules depends on their temperature as well as on their distance of separation.

given substance at the temperature  $T_1$ , where  $L$  denotes the internal latent heat, and let  $B_1 B_2$  denote the graph for the temperature  $T_2$ , &c. Let the abscissæ of the points  $a_1, b_1, c_1, \dots$ , denote the densities of the substance in the liquid state in contact with its saturated vapour at the temperatures  $T_1, T_2, \dots$ , and the abscissæ of the points  $a_2, b_2, c_2, \dots$ , denote the corresponding densities of the saturated vapour. The internal latent heat of evaporation of the liquid at the temperature  $T_1$  is then the difference between the ordinates of the points  $a_1$  and  $a_2$ , and so on for the other temperatures. Now the finding of a formula for the latent heat consists in finding an equation for a series of curves  $X_1, X_2, \dots$ , which pass through the points  $a_1, a_2, b_1, b_2, \dots$ , as shown in the figure, but which need not coincide with any other points or parts of the curves  $A_1 A_2, B_1 B_2, \dots$ . It is obvious, then, that an infinite number of sets of such curves can be found to each of which corresponds a formula for the latent heat. And since each formula corresponds to some law of attraction between the molecules (for given a law of attraction and we can at once deduce a formula for the latent heat), an infinite number of laws can be obtained in this way, but we cannot be sure that any one of them represents the true law of attraction without further evidence. It follows, therefore, that the law deduced from latent heat data should contain an arbitrary function.

We have considered the most general case in latent heat in assuming that it is a function of the temperature as well as of  $z$  or  $\rho$ . But the same conclusions hold if we could prove that the attraction between two molecules a given distance apart is independent of the temperature. Let the equation of the latent heat of evaporation of a substance into a vacuum in this case be  $L = \psi_2(\rho)$ , where  $\rho$  denotes its density, and the curve  $A_1' A_2'$  in fig. 2 its graph. Let the difference between the ordinates  $c_1, a_2, b_1, b_2, \dots$ , denote the latent heat of a liquid corresponding to the temperatures  $T_1, T_2, \dots$ . Let the curve  $N_1, N_2$ , possess the property that the difference between the ordinates of  $c_1$  and  $a_2$  is equal to the difference of the ordinates of  $c_1$  and  $a_2$  in the curve  $A_1', A_2'$ , and the difference between the ordinates of  $b_1, b_2$  in the curve  $N_1, N_2$  is equal to the difference of the corresponding ordinates in curve  $A_1', A_2'$ , and so on. It is obvious that the curve  $N_1, N_2$  subject to these conditions may have an infinite variety of shapes. Now the equation of the curve  $N_1, N_2$  gives the latent heat in terms of  $\rho$ , and as the curve may have an infinite variety of shapes there are an infinite number of such equations. If the curve is expressed by a single equation it must contain an arbitrary function; and the law of attraction

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to which it corresponds must therefore also contain an  
arbitrary function.

Fig. 2.

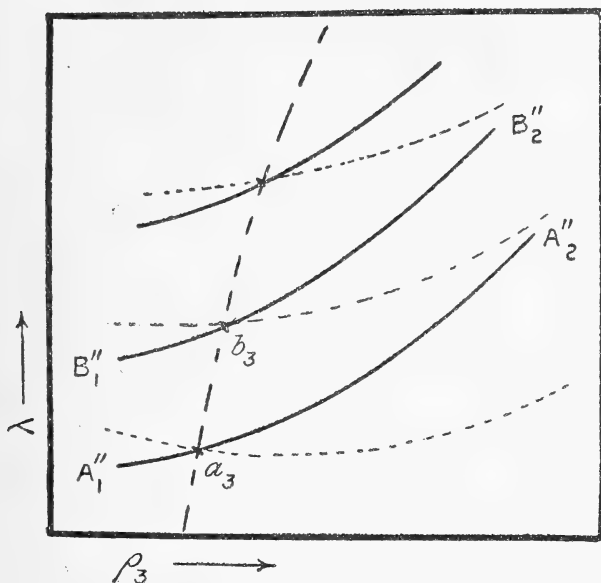


Next let us consider the determination of the law of molecular attraction from surface tension data. The surface tension of a liquid may be defined as the work done against the molecular attraction per  $\text{cm}^2$  of new surface formed in cutting a thick slab of liquid into two parts and separating them by an infinite distance. If the liquid is surrounded by vapour of density  $\rho_2$  we may suppose that an amount of matter of density  $\rho_2$  remains stationary in space when the slabs are separated, which is equivalent to separating two slabs (not surrounded by vapour) each of density  $(\rho_1 - \rho_2)^*$ . Let the equation of the surface tension of a substance for any value of  $(\rho_1 - \rho_2)$  or  $\rho_3$ , deduced from a knowledge of the law of attraction between the molecules, be  $\lambda = \psi_3(T, \rho_3)$ .

\* This is strictly admissible only if matter consists of molecules infinitely small in size, but deviations due to this not being the case will occur only when the liquid and vapour have nearly the same density as occurs near the critical point.

Suppose the curves  $A_1'' A_2''$ ,  $B_1'' B_2''$ , . . . ., in fig. 3 represent the graphs of this equation corresponding to the

Fig. 3.

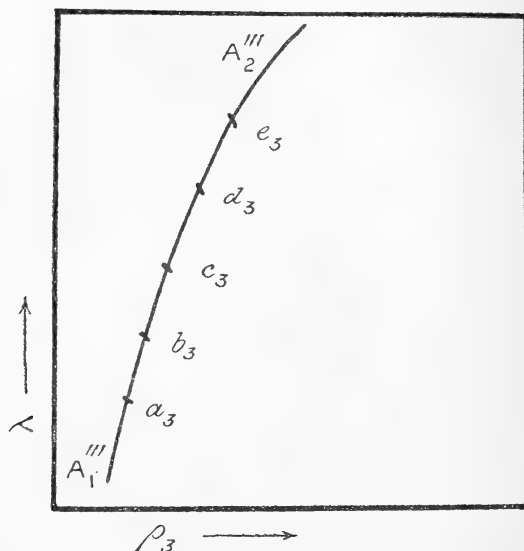


temperatures  $T_1, T_2, \dots$ . Let the abscissæ of the points  $a_3, b_3, \dots$ , denote the values of  $\rho_3$  or  $(\rho_1 - \rho_2)$  of a liquid in contact with its saturated vapour. The ordinates of the points then give the surface tension at different temperatures. The equation of a set of curves which pass through the points  $a_3, b_3, \dots$ , is a formula for the surface tension. An infinite number of such sets of curves can be obtained. It follows, therefore, in the same way as before that the law of molecular attraction deduced from surface tension data should contain an arbitrary function.

Next let us suppose that the attraction between two molecules a given distance apart is independent of the temperature. Let the graph of the equation for the surface tension in this case be represented by the curve  $A_1''' A_2'''$ , in fig. 4, the points  $a_3, b_3, \dots$ , having the same meaning as before. The equation of the surface tension of a liquid in contact with its saturated vapour is the equation of the curve through the points  $a_3, b_3, \dots$ . Now this equation can be approximately found by trial or by the help of the Calculus of Finite

Differences. It will represent the whole curve  $A_1''' A_2'''$  with a degree of closeness depending on the extent of the

Fig. 4.



part of the curve represented by experiment. On the above supposition we could thus approximately determine the law of molecular attraction. But if we are not able to prove that the attraction is independent of the temperature the law deduced should involve an arbitrary function. The above result is, however, of importance, as it will enable us later to determine whether the attraction is a function of the temperature or not.

We have then that an infinite number of formulæ for the surface tension or latent heat of a liquid can be obtained, each of which corresponds to a law of molecular attraction, and that we cannot be sure without further evidence whether any one of the laws happens to be the true law. All these laws are included in the general law that can be deduced which contains an arbitrary function. It is owing to this fact that different investigators of molecular attraction have obtained different results, and in some cases results have been deduced concerning the potential energy of a molecule and the origin and nature of its attractive forces which are quite absurd.

We will now illustrate the result obtained in this paper by some examples. Mills\* has put forward the theory that the molecular attraction which accounts for latent heat &c. is independent of the temperature and varies inversely as the square of the distance between the attracting molecules. This gives for the latent heat the equation

$$L = C (\rho_1^{1/3} - \rho_2^{1/3}), \dots \dots \dots (1)$$

where C is a constant, which was found to agree well with the facts. According to what has gone before it should be possible to discover any number of laws of attraction which will give latent heat formulæ agreeing with the facts. Thus if we assume that the attraction varies inversely as the seventh power of the distance of separation of the molecules we obtain the equation  $L = W(\rho_1^2 - \rho_2^2)$  for the latent heat, where W is a constant. This equation will at once be obtained by substituting  $\frac{K_1}{z^7}$  for  $\phi(z)(\Sigma\sqrt{m_1})^2$  in the general formula for the latent heat given by the writer †. This equation agrees ‡ with the facts as well as that of Mills, and we can therefore with equal justice assume that the attraction varies inversely as the seventh power of the distance of separation of the molecules.

It can be very simply shown that the law of Mills cannot possibly account for the magnitude of the latent heat of evaporation and cannot therefore be true. The attraction between two molecules according to the law of Mills is  $\frac{K_2}{z^2}$  where  $K_2$  is a constant which is the attraction between the molecules at unit distance apart. Substituting this expression for the attraction for  $\phi(z)(\Sigma\sqrt{m_1})^2$  in the general formula for the latent heat quoted above we obtain

$$L = \frac{K_2}{m^{1/3}} (\rho_1^{1/3} - \rho_2^{1/3}) 2.96,$$

where L is the latent heat per unit mass in ergs and m is the mass of a molecule. By means of this equation the value of  $K_2$  can be calculated. In the case of ether at 273° C. we have

$$L = 86.16 \times 4.2 \times 10^7 \text{ ergs, } m = 74 \times 1.61 \times 10^{-24} \text{ gm.,}$$

$$\rho_1 = .7362, \text{ and } \rho_2 = .03827,$$

which on substituting in the equation gives  $K_2 = 8.9 \times 10^{-21}$  dyne. Since the gravitational attraction also varies inversely as the square of the distance of separation of the

\* Journal of Phys. Chemistry, vol. vi. p. 209; vol. viii. p. 383 and p. 593; vol. ix. p. 402; vol. xi. p. 594 and p. 182.

† Phil. Mag. May 1910, p. 801.

‡ Phil. Mag. Oct. 1910, p. 678.

molecules, it should be identical with the attraction we are considering. Now the gravitational attraction of two ether molecules one cm. apart is  $1.84 \times 10^{-51}$  dyne, which is a much smaller quantity than  $K_2$ . Thus the gravitational attraction would have to be about  $10^{31}$  times its actual value to account for the latent heat of evaporation. It follows therefore that the molecular attraction must decrease at a much greater rate with the distance from the molecule than that given by the inverse square law.

That Mills' law cannot be true follows also from surface-tension considerations. If  $\frac{K_2}{z^2}$  is substituted for  $\phi(z)(\sum \sqrt{m_1})^2$  in the general formula for the surface-tension given by the writer\* we obtain  $\lambda = D(\rho_1 - \rho_2)$ , where  $D$  is a constant which varies only with the nature of the liquid. In the case of chlorobenzene we have  $D = 1.84$  at  $423^\circ$  C. and  $D = .89$  at  $533^\circ$  C.  $D$  is thus by no means constant, and the law therefore not true.

Several investigators of the law of attraction between molecules have assumed that it must be such as will make the internal heat of evaporation per gram of liquid independent of the mass of liquid allowed to evaporate completely. But it is not at all necessary that this condition need be fulfilled. Consider a spherical mass of liquid to evaporate till none is left. It is evident then that the energy expended to remove a molecule to infinity depends on the mass of the liquid only when the diameter of the sphere of liquid is equal to the radius of the sphere of action of a molecule, which is taken to be the distance of separation between two molecules for which their potential energy is small in comparison with that corresponding to the distance the molecules are separated in the liquid state. Now the writer† has shown that the radius of the sphere of action of a molecule is of the same order of magnitude as the distance of separation of the molecules in the liquid state. It follows, therefore, that the latent heat will depend on the mass of liquid allowed to evaporate only when it is equal to about  $10^{-24}$  gram, and the latent heat of a gram of liquid therefore independent of external conditions, as has been found to be the case.

The law deduced from surface-tension or latent heat we have seen should contain an arbitrary function. But still the law may contain a good deal of useful information. Thus it might happen that the arbitrary function is an arbitrary function only of certain quantities or of definite functions

\* Phil. Mag. May 1910, pp. 789-792.

† Phil. Mag. June 1910, p. 840.



of certain quantities. Further, the known part of the law may contain quantities which were excluded from the arbitrary function. Now the law of attraction between two molecules of the same kind deduced by the writer\* is

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) \frac{(\sum \sqrt{m_1})^2}{z^5},$$

where  $z$  is the distance between the molecules,  $\sum \sqrt{m_1}$  is the sum of the square roots of the atomic weights of the atoms of a molecule, and  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$  is a quantity which is the same for all molecules at corresponding temperatures. The quantity  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$  must therefore be a function of the ratio of the distance between the molecules to the distance of separation in the liquid state at the critical temperature, and the ratio of the temperature of the molecules to the critical temperature. The function is arbitrary as its form cannot be determined from the data from which the expression for the law was obtained. We see that it does not contain  $\sum \sqrt{m_1}$ , and we therefore have the important result that the "chemical" attraction at a given distance from an atom is proportional to the square root of its atomic weight.

The problem to solve next is to determine the exact form of this function. We have seen † that if we assume it a function of the temperature only, a value for the intrinsic pressure is obtained which agrees very well with that obtained by supposing the matter evenly distributed in space, in which case the intrinsic pressure can be shown to be equal to  $L\rho_1$ . Further, the coefficient of diffusion of one gas into another calculated on this supposition agrees well with the facts. The law of attraction may then be written  $\frac{K}{z^5}$ , where  $K$  is a function of the temperature. It was also shown that the power of  $z$  cannot be less than the fifth but must be rather greater, for otherwise the "chemical" attraction becomes comparable with the gravitational for distances greater than one cm. But we would expect that the function depends strictly both on the distance between the molecules and their temperature, and this can be proved to be the case.

We have seen at the beginning of the paper that if it can

\* Phil. Mag. May 1910, pp. 783-809.

† Phil. Mag. Oct. 1910, pp. 665-670.

be proved that the law of molecular attraction is independent of the temperature, it can be completely determined from surface-tension data. Let us then determine the law of attraction assuming that this can be proved. If we assume that the attraction varies inversely as the eleventh power of the distance between the molecules, we obtain the equation  $\lambda = F(\rho_1 - \rho_2)^4$  for the surface-tension, where  $F$  is a constant which is independent of the temperature but depends on the nature of the liquid. This equation is at once obtained by substituting  $\frac{K_3}{z^{11}}$  for  $\phi(z)(\Sigma\sqrt{m_1})^2$  in the general equation for the surface-tension given by the writer and quoted in a previous part of the paper. According to the equation  $\frac{\lambda}{(\rho_1 - \rho_2)^4}$  should be a constant for each liquid, and this is approximately the case as will appear from Table I. \* Now if the law assumed is true it should also give a formula for the latent heat agreeing with the facts. Substituting  $\frac{K_3}{z^{11}}$  for  $\phi(z)(\Sigma\sqrt{m_1})^2$  in the general equation for the latent heat quoted, we obtain  $L = H(\rho_1^{10/3} - \rho_2^{10/3})$ , where  $H$  is a constant which should depend only on the nature of the liquid. But  $H$  or  $L/\rho_1^{10/3} - \rho_2^{10/3}$  is not constant for the same liquid, as is shown by Table I. The attraction between two molecules a given distant apart cannot therefore be independent of their temperature.

TABLE I.

	Benzene.			Chlorobenzene.			Carbon tetra- chloride.			Methyl formate.		
T .....	353	433	503	423	473	533	363	433	503	303	363	423
$\frac{\lambda}{(\rho_1 - \rho_2)^4}$ ...	46.7	46.7	48.5	21.3	21.1	21.4	40.1	40.3	41.5	27.5	27.4	27.9
$\frac{L}{\rho_1^{10/3} - \rho_2^{10/3}}$ ...	198	210	267	75.4	84.3	102	11.6	13.9	18.0	123	139	174

\* The values of  $\lambda$  contained in the tables in this paper are taken from a paper by Ramsay and Shields, Phil. Trans. of the Royal Society, vol. clxxxiv. p. 647 (1893), and the values of  $L$ ,  $\rho_1$ , and  $\rho_2$ , are taken from the papers by Mills quoted previously, who calculated values of  $L$  by Clapeyron's equation using the density and pressure data of Ramsay and Young.

We have already obtained evidence of this before. The writer \* has deduced a formula for the radius of the sphere of action of a molecule on the supposition that matter does not consist of molecules but is evenly distributed in space. The radius of the sphere of action obtained on this supposition decreases with rise of temperature.

The decrease of the attraction with rise of temperature may be partly direct and partly indirect. Indirectly a change could be, and probably is, produced in the following way. We have obtained some evidence † that the deviation from the additive law for the attraction of a molecule is due to an interaction between the atoms which decreases their attraction. Now the atoms in a molecule are probably in rotation round the centre of gravity of the molecule, equilibrium being maintained between the centrifugal forces and the forces of attraction. The speed of rotation very likely increases with an increase of temperature as this produces an increase of the internal energy, and in order to maintain equilibrium between the forces the molecule must contract. This brings the atoms still further under each other's influence, which decreases the force of attraction of the molecule at an external point.

If we assume that

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$$

is a function of the temperature only so that it may be written  $\phi_2\left(\frac{T}{T_c}\right)$ , the equation for the latent heat becomes

$$L = A_1 \phi_2\left(\frac{T}{T_c}\right) (\rho_1^{4/3} - \rho_2^{4/3}),$$

and that for the surface-tension

$$\gamma = A_2 \phi_2\left(\frac{T}{T_c}\right) (\rho_1 - \rho_2)^2,$$

where  $A_1$  and  $A_2$  are numerical constants. If the assumption is true, the value of the expression

$$\frac{\lambda(\rho_1^{4/3} - \rho_2^{4/3})}{L(\rho_1 - \rho_2)^2}$$

should be independent of the temperature. This is, however,

\* *Phil. Mag.* June 1910, pp. 840-846.

† *Phil. Mag.* May 1910, pp. 798-801.

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 not the case, as is shown by Table II. The expression

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$$

is therefore a function of the distance between the molecules as well as of the temperature.

TABLE II.

	Benzene.			Chlorobenzene.			Carbon tetrachloride.			Methyl formate.		
T.....	353	433	503	423	473	533	363	433	503	303	363	423
L.....	85.6	69.7	50.5	65.8	58.3	49.1	40.6	33.3	23.7	107	85.1	64.0
$\frac{\lambda(\rho_1^{4/3} - \rho_2^{4/3})}{L(\rho_1 - \rho_2)}$ ...	.273	.210	.140	.279	.243	.184	.341	.270	.171	.223	.190	.135

If it could be proved that the above function consists of two factors, one a function of the temperature and the other of the distance between the molecules, say

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) = \phi_3\left(\frac{z}{x_c}\right) \phi_4\left(\frac{T}{T_c}\right),$$

then the form of both

$$\phi_3\left(\frac{z}{x_c}\right) \quad \text{and} \quad \phi_4\left(\frac{T}{T_c}\right)$$

could be completely determined. The formulæ for the latent heat and surface-tension then become

$$L = \phi_4\left(\frac{T}{T_c}\right) \phi_5, \quad \lambda = \phi_4\left(\frac{T}{T_c}\right) \phi_6,$$

where  $\phi_5$  and  $\phi_6$  are known functions of

$$\phi_3\left(\frac{z}{x_c}\right) \quad \text{or} \quad \phi_3\left(\left(\frac{\rho_c}{\rho}\right)^{1/3}\right).$$

Eliminating  $\phi_4\left(\frac{T}{T_c}\right)$  we obtain the equation  $\frac{L}{\gamma} = \frac{\phi_5}{\phi_6}$ , which can be used to determine approximately the form of the function  $\phi_3\left(\frac{z}{x_c}\right)$  by taking a sufficiently large number of equations corresponding to different values of  $\rho$ . The form

of the function  $\phi_4\left(\frac{T}{T_c}\right)$  can then in the same way be determined by means of the first or second of the above three equations. If we further suppose that  $\phi_3\left(\frac{z}{c}\right)$  may be expressed in the form  $\left(\frac{z}{c}\right)^n$ , we have

$$L = \phi_4\left(\frac{T}{c}\right) P_1\left(\rho_1^{\frac{n-1}{3}} - \rho_2^{\frac{n-1}{3}}\right), \quad \lambda = \phi_4\left(\frac{T}{c}\right) P_2(\rho_1 - \rho_2)^{\frac{n+1}{3}},$$

and therefore

$$\frac{L}{\lambda} = P_3 \frac{\left(\rho_1^{\frac{n-1}{3}} - \rho_2^{\frac{n-1}{3}}\right)}{(\rho_1 - \rho_2)^{\frac{n+1}{2}}},$$

where  $P_3$  is a function of  $n$ . Applying this equation to the facts, we find that  $n$  lies between 7 and 11. This makes the value of  $\phi_4\left(\frac{T}{T_c}\right)$  increase with the temperature, in other words, the molecular attraction increases with the temperature. But this is highly improbable; moreover, we will show in a subsequent paper that  $n$  cannot be as large as the above value, and it follows therefore that strictly the arbitrary function cannot be expressed as the product of two factors as we have supposed.

One of the methods used to obtain some information on the law of molecular attraction is based on the change of the coefficient of viscosity of a gas with change of temperature, making the assumptions that the attraction is independent of the temperature and that the molecules behave simply as centres of force. But this cannot give accurate results, as the attraction depends on the temperature, and moreover the results must be influenced by the fact that a molecule has a certain definite volume. If the attraction between two molecules is supposed to be given by the expression  $\frac{K}{z^n}$ , the values of  $n$  vary between the limits 5 and 12 according to the nature of the substance\*. According to the investigations of the writer, the law of attraction is the same for all substances. If the attraction is a function of the temperature, then according to the law obtained by the writer the value of  $n$  found from viscosity data should depend on the extent the temperatures at which the viscosity measurements have been carried out are removed from the critical temperatures. This is probably the explanation why the values of  $n$  increase in the order as the critical temperatures of the

\* Jeans' 'Dynamical Theory of Gases,' p. 257.

substances decrease. On the whole, it appears that the method is of little value to determine the law of molecular attraction.

The arbitrary function in the law of attraction given by the writer consists possibly of the sum of a number of positive and negative terms. In that case the force between two molecules will change in sign a number of times as they are brought nearer to one another. This could be experimentally investigated by allowing a monatomic gas to expand without doing work, and measuring the cooling or heating effect produced (Joule-Thomson effect). If a heating effect is produced there is repulsion for distances of separation of the molecules lying between their distances of separation in the two stages; if a cooling effect is produced the force is one of attraction. It seems best to use a monatomic gas, as we cannot be sure that the frequency of collision of a complex molecule—which changes in the above process—alters the configuration of its atoms and thus causes the effect observed. It is interesting to observe that the distances between two molecules of the same kind for which the force changes sign depends, according to the form of the arbitrary function in the law of attraction, on the nature of the molecule. If two different pairs of molecules are at corresponding temperatures and the molecules of one pair are separated by a distance  $z'$ , and those of the other pair by a distance  $z''$ , the forces will have the same sign when

$$\frac{z'}{x_c'} = \frac{z''}{x_c''},$$

that is, when the distances are to one another as the distances of separation of the molecules at the critical state, for this makes the values of the arbitrary functions assume the same sign in the two cases. It follows, therefore, that the distances between two molecules of a different kind for which the force changes sign are not the same as those when the molecules are of the same kind. The Joule-Thomson effect of a mixture of gases should therefore not be an additive property of the constituent gases, and this has been found to be the case\*. It will also be easily seen that the sign of the Joule-Thomson effect should depend on the density of the gas before and after expansion, and keeping the conditions of expansion constant it should depend on the temperature. When a mass of gas expands

\* Preston's 'Theory of Heat,' p. 811, second edition.

from the density  $\rho_1$  to  $\rho_2$ , the energy  $L$  expended in overcoming the molecular forces is given\* by

$$L = \frac{(\sum \sqrt{m_1})^2}{m} \left\{ \left( \frac{\rho_1}{m} \right)^{4/3} \phi_3 \left( \frac{x_a}{x_c}, \frac{T}{T_c} \right) - \left( \frac{\rho_2}{m} \right)^{4/3} \phi_3 \left( \frac{x_b}{x_c}, \frac{T}{T_c} \right) \right\},$$

where  $x_a$  and  $x_b$  are the distances of separation of the molecules before and after expansion. The inversion of the Joule-Thomson effect occurs when

$$L=0 \quad \text{or} \quad \left( \frac{\rho_1}{\rho_2} \right)^{4/3} \phi_3 \left( \frac{x_a}{x_c}, \frac{T}{T_c} \right) = \phi_3 \left( \frac{x_b}{x_c}, \frac{T}{T_c} \right).$$

Therefore if  $T'$  and  $T''$  are the temperatures of inversion of two different gases, when the experiments are carried out so that the conditions

$$\frac{\rho_c'}{\rho_c''} = \frac{\rho_1'}{\rho_1''}, \quad \frac{\rho_c'}{\rho_c''} = \frac{\rho_2'}{\rho_2''}$$

are satisfied, then

$$\frac{T'}{T_c'} = \frac{T''}{T_c''}$$

from the above equation, or the temperatures of inversion are proportional to the critical temperatures. These conditions have probably been approximately satisfied in the experiments on the subject, since gases with a low critical temperature were found to have a low temperature of inversion. A systematic investigation of the Joule-Thomson effect of a gas at different initial densities would furnish some useful information by means of which the exact form of the arbitrary function in the law of attraction might possibly be determined.

*Particular Cases of the General Law of Attraction*

$$\phi_2 \left( \frac{z}{x_c}, \frac{T}{T_c} \right) \frac{(\sum \sqrt{m_1})^2}{z^5}.$$

We have seen that it is possible to find an infinite number of formulæ for the latent heat and surface tension, each of which corresponds to a definite law of molecular attraction, but none of these laws can be taken to represent the law actually existing without further evidence. All these laws must be particular cases of the general law given above, obtained by giving the arbitrary function in the law definite

\* Phil. Mag. May 1910, p. 794.

forms. This furnishes the means for further tests of the truth of the above law.

We have seen that Mills, assuming that the attraction between two molecules separated by a distance  $z$  is given by  $\frac{K_2}{z^2}$ , where  $K_2$  is a constant, obtained the formula  $L = D(\rho_1^{1/3} - \rho_2^{1/3})$  for the latent heat, which was found to agree well with the facts. To reduce the general law of attraction to this form we must put

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) = \frac{z^3 S}{x_c^3} = \frac{z^3 \rho_c S}{m},$$

where  $\rho_c$  is the critical density and  $S$  is a numerical constant. The value of the constant  $K_2$  is thus  $\frac{S\rho}{m} (\Sigma\sqrt{m_1})^2$ , and the value of  $D$  therefore  $\frac{S\rho_c}{m^{7/3}} (\Sigma\sqrt{m_1})^2$ . These values of  $D$  should agree with those obtained by Mills from latent heat data. This is tested in Table III. for a number of liquids,

TABLE III.

Name of liquid.	$\frac{L}{\rho_1^{1/3} - \rho_2^{1/3}}$	$\frac{12,830\rho_c(\Sigma\sqrt{m_1})^2}{m^{7/3}}$	Name of liquid.	$\frac{L}{\rho_1^{1/3} - \rho_2^{1/3}}$	$\frac{12,830\rho_c(\Sigma\sqrt{m_1})^2}{m^{7/3}}$
Ether .....	104.4	112.7	Benzene .....	109.5	107.6
Di isopropyl .....	98.1	77.57	Hexamethylene.....	103.6	136.9
Di isobutyl .....	86.3	100.3	Fluorobenzene .....	85.6	97.5
Isopentane.....	105.4	119.3	Chlorobenzene .....	81.2	77.3
Normal pentane .	109.9	118.4	Bromobenzene .....	56.1	56.4
„ hexane...	102.8	98.7	Iodobenzene... ..	44.4	41.8
„ heptane .	98.7	85.31	Carbon tetrachloride .	44.1	52.8
„ octane ...	93.0	90.24	Stannic chloride .....	26.0	26.4

which contains the mean values of  $\frac{L}{\rho_1^{1/3} - \rho_2^{1/3}}$  obtained by Mills, and the corresponding values of  $\frac{S\rho_c}{m^{7/3}} (\Sigma\sqrt{m_1})^2$ , putting  $S=12,830$ . The agreement is fairly good.

If we assume that the law of attraction between two molecules is  $\frac{K_1}{z^7}$ , we have that  $K_1$  must be equal to

$$b\left(\frac{m}{\rho_c}\right)^{2/3} (\Sigma\sqrt{m_1})^2,$$



where  $b$  is a numerical constant. The equation for the latent heat then becomes  $L = E(\rho_1^2 - \rho_2^2)$ , where  $E$  is equal to  $b \frac{(\sum \sqrt{m_1})^2}{m^{7/3} \rho_c^{2/3}}$ . This equation we have already deduced previously and found to agree well with the facts\*.

If we give the arbitrary function the form  $u \left(\frac{x_c}{z}\right)^6$ , where  $u$  is a numerical constant, we obtain a very convenient expression for the surface tension. The attraction between two molecules is then given by  $\frac{K_3}{z^{11}}$ , where

$$K_3 = u x_c^6 (\sum \sqrt{m_1})^2 = u \left(\frac{m}{\rho_c}\right)^2 (\sum \sqrt{m_1})^2,$$

an expression for the attraction we have already used in this paper. This gives for the surface-tension the formula

$$\lambda = F(\rho_1 - \rho_2)^4, \quad \text{where } F = u \left(\frac{\sum \sqrt{m_1}}{m \rho_c}\right)^2.$$

In Table IV. (p. 100) the values of  $F$  are given for temperature intervals of  $10^\circ$  for a number of liquids. It will be seen that they are approximately independent of the temperature over the ranges of temperature given in the table. If the general law of attraction is true, the value of  $F$  should be equal to  $u \left(\frac{\sum \sqrt{m_1}}{m \rho_c}\right)^2$ . This is found to be the case, as is shown by

Table V. (p. 101), the value given to  $u$  being determined by the method of least squares to be 32.96.

If the attraction between two molecules is taken to be given by  $\frac{K_5}{z^4}$ , where  $K_5$  depends only on the nature of the liquid, then according to the general law of attraction  $K_5 = S_1 \left(\frac{\rho_c}{m}\right)^{1/3} (\sum \sqrt{m_1})^2$ , where  $S_1$  is a numerical. If this constant expression for the attraction is substituted for  $\phi(z)(\sum \sqrt{m_1})^2$  in the equation † for the intrinsic pressure  $\rho_n$  of a liquid, we obtain  $\rho_n = \frac{F \rho_c^{1/3}}{m^{7/3}} (\sum \sqrt{m_1})^2 \rho^2$ , where  $F$  is a numerical constant. Now van der Waals has proposed the

\* Phil. Mag. Oct. 1910, pp. 686-687.

† Phil. Mag. Oct. 1910, pp. 668-667.

TABLE IV.

Ethyl oxide. Temp. 313°-443°.			Methyl formate. Temp. 303°-433°.			Carbon tetrachloride. Temp. 363°-503°.			Benzene. Temp. 353°-493°.		
$\rho_1 - \rho_2$ .	$\lambda$ .	$\frac{\lambda}{(\rho_1 - \rho_2)^{1/2}}$ .	$\rho_1 - \rho_2$ .	$\lambda$ .	$\frac{\lambda}{(\rho_1 - \rho_2)^{1/2}}$ .	$\rho_1 - \rho_2$ .	$\lambda$ .	$\frac{\lambda}{(\rho_1 - \rho_2)^{1/2}}$ .	$\rho_1 - \rho_2$ .	$\lambda$ .	$\frac{\lambda}{(\rho_1 - \rho_2)^{1/2}}$ .
.6857	14.05	63.56	.9576	23.09	27.46	1.4474	17.60	4.014	.8118	20.28	46.72
.6713	12.94	63.74	.9416	21.56	26.07	1.4240	16.48	4.007	.8005	19.16	46.63
.6580	11.80	62.96	.9250	20.05	27.39	1.4094	15.54	3.947	.7880	18.02	46.76
.6443	10.72	62.22	.9073	18.58	27.44	1.3739	14.32	4.017	.7749	16.86	46.78
.6287	9.67	61.86	.8888	17.55	28.12	1.3478	13.27	4.019	.7616	15.71	46.71
.6102	8.63	62.23	.8698	15.70	27.44	1.3202	12.22	4.019	.7473	14.57	46.72
.5918	7.63	62.82	.8499	14.29	27.38	1.2913	11.21	4.018	.7323	13.45	46.77
.5707	6.63	62.50	.8281	12.90	27.43	1.2617	10.22	4.027	.7166	12.36	46.86
.5471	5.65	63.04	.8049	11.52	27.45	1.2295	9.24	4.052	.7012	11.29	46.71
.5216	4.69	63.39	.7802	10.18	27.74	1.1945	8.26	4.064	.6834	10.20	46.76
.4936	3.77	63.54	.7526	8.86	27.64	1.1567	7.28	4.063	.6657	9.15	46.58
.4624	2.88	62.72	.7226	7.54	27.81	1.1146	6.34	4.105	.6460	8.16	46.85
.4256	2.08	63.40	.6897	6.30	27.86	1.0687	5.40	4.134	.6250	7.17	46.98
.3785	1.33	64.79	.6513	5.06	28.11	1.0187	4.47	4.152	.6011	6.20	47.28
						.9625	3.56	4.148	.5754	5.25	47.89

TABLE V.

	$\frac{\lambda}{(\rho_1 - \rho_2)^4}$	$\frac{(\sum \sqrt{m_1})^2 32.96}{m^2 \rho_c^2}$
Methyl formate .....	27.53	26.93
Carbon tetrachloride .....	3.99	3.33
Benzene .....	46.5	41.76
Ether .....	62.92	68.77

expression  $a\rho^2$  for the intrinsic pressure of a liquid, where  $a$  is a constant which depends only on the nature of the liquid. From the above equation we see that van der Waals' constant  $a$ —which he determines by means of his equation

of state—is equal to  $F \frac{\rho_c^{1/3}}{m^{7/3}} (\sum \sqrt{m_1})^2$ . This is roughly the

case. But the above law of attraction cannot be exactly true since the attraction must be a function of the temperature; moreover it gives a formula for the latent heat which does not agree with the facts\*. However, it is probably a nearer approximation to the actual facts than most of the other definite laws that have been proposed.

At the critical state of a liquid the average kinetic energy of a molecule must be approximately equal to its latent heat of evaporation into a vacuum. If this were not so, then only the molecules having a velocity above a certain limit could escape from the liquid, and it could be in equilibrium with vapour of a different density than the liquid. The internal heat of evaporation of a molecule from a liquid in the critical state into a vacuum is, according to the general law of molecular attraction, equal to

$$A_1 \left(\frac{\rho_c}{m}\right)^{4/3} (\sum \sqrt{m_1})^2,$$

where  $A_1$  is a constant. This is proportional to the critical temperature, and we therefore have

$$T_c = H_2 \left(\frac{\rho_c}{m}\right)^2 (\sum \sqrt{m_1})^2,$$

where  $H_2$  is a constant. This equation the writer has already

\* *Ibid.* p. 677.

obtained previously in a different way \*, and was found to agree well with the facts. Similarly we can equate the expression for the latent heat and the temperature when the former quantity is derived from the general law of molecular attraction giving the arbitrary function a definite form. But in every case the equation should reduce to the above form. For example, Mills obtained in the above way from

his law of molecular attraction that  $\frac{mK_2\rho_c^{1/3}}{T_c} = \text{constant}$ ,

which was found to agree well with the facts. Now we have seen that according to the general law of molecular attraction

$$K_2 = \frac{8\rho_c}{m} (\Sigma \sqrt{m_1})^2,$$

which reduces the equation to the same form as the preceding.

London, Oct. 27, 1910.

#### XIV. *The Radioactivity of the Leinster Granite.*

By ARNOLD L. FLETCHER, B.A.I.†

THE granite of Leinster extends for a distance of 70 miles along the East Coast of Ireland, having an average width of 8 or 10 miles, and covering a land area of some 600 square miles. It extends in some places, near its northern extremity, beneath the Irish Channel, and is by far the largest granitic exposure in the United Kingdom.

Its petrographical and mineralogical characters have formed the subject of many memoirs which deal for the greater number with special matters. The most important treatise dealing generally with the petrology and mineralogy of the granite is the paper by Professor Solias (Trans. R.I.A. xxix. part xiv. 1891).

The Leinster granite is throughout the main chain a true granite, containing both biotite and muscovite micas. This character applies also to most of the outlying exposures of the rock. In two cases, Nos. 10 and 11 in the table on p. 106, biotite granite or granitite is found in such exposures; while in the cases of Nos. 1, 12, and 13 in the table, the mica is mainly of the white variety. In all the granites dealt with, with these exceptions, both micas are found.

\* Phil. Mag. Oct. 1910, p. 688.

† Communicated by the Author.

Throughout the biotite of the Leinster granite pleochroic halos are abundant, and they are occasionally seen in the white mica. In the latter case they are less conspicuous. Primary muscovite often includes biotite flakes in which halos are plentifully developed, while they are absent or only faintly visible in the surrounding muscovite.

Professor Sollas considers the corroded edges of these included plates as evidence for the derivation of the muscovite from originally large crystals of biotite, by a process of magmatic corrosion. "In the Leinster granite, primary muscovite may be regarded as to a great extent the product of the action of the magma on the primarily formed biotite." This view appears to involve an equal distribution of radioactive primary minerals, *e. g.* zircon, allanite, uraninite (?), in both primary muscovite and biotite.

It is interesting to note that the recent advance in our knowledge of the origin of halos bears upon this question of the genesis of the muscovite from the biotite. Thus the fact that a zircon may be found in the muscovite and the segment of a corona, having the zircon for centre, in a closely adjoining flake of biotite, is no proof that biotite once extended to and surrounded the zircon, which would, in fact, have affected the biotite by  $\alpha$  rays sent across the intervening muscovite.

It has been found impossible in the experiments which are cited below to connect definitely the radioactivity of the rock directly with the visible abundance of dark mica present. Indeed, the highest result obtained was upon a somewhat gneissic granite from Glendalough which contained much silvery white mica.

Prof. the Hon. R. J. Strutt has shown (Proc. R. S. lxxviii. A. p. 150) that in a granite, the heavier minerals include most of the uranium-radium elements, and that zircon was the most radioactive of the constituents in the granite examined by him. This result has been recently confirmed by J. W. Waters (Phil. Mag. June 1910) working in Prof. Strutt's laboratory.

Biotite being a mineral of early consolidation, and forming around the still earlier nuclei of the zircon and apatite, comes to possess in this way a greater radioactivity than minerals of later consolidation. From this point of view the radioactivity of the biotite is dependent upon its place in the order of consolidation, and its presence or absence does not affect the radioactivity of the rock. The inferiority of some of the muscovites is readily accounted for in the view advocated by Prof. Sollas, that much of it is derived from the alteration

of felspathic substances, and is thus of quite secondary origin.

The main object of the determinations given below was to ascertain if any considerable variations in radioactivity were present in the mass of the Leinster granite. The experiments, as may be seen, do indeed show a fairly wide range in the quantities detected: from 0.41 gr. per gr. in a specimen of granite taken from the north side of Glenmalure, Co. Wicklow, to  $4.36 \times 10^{-12}$  gr. per gr. in a specimen of granite also from Glenmalure.

The fact that the two specimens of highest and lowest radioactivity measured come from the same locality serves to illustrate the sporadic distribution of the radioactive matter. This is borne out by other results. For example, the determinations  $3.62 \times 10^{-12}$  and  $1.76 \times 10^{-12}$  gr. per gr. were made upon specimens taken from points on Killiney Hill within a distance of half a mile of one another.

Much difficulty has been experienced throughout in preparing solutions which are perfectly clear, or such as will retain their limpidity during their period of storage. In some cases the solutions, while showing no visible precipitate, lacked the sparkling appearance, characteristic of perfect solution.

It did not appear that those determinations made upon solutions containing precipitate fell short of the general mean obtained. In the case of the granite from Aughrim, the acid solution, after a preliminary estimation, was filtered, the precipitate re-fused, and added to the original acid solution, the whole being then reclosed for a further period of storage. A second determination, however, corroborated the first by yielding for the radioactivity nearly the same figure as before. The preparation and estimation of the solutions was carried out in the same manner as before described (Phil. Mag. July 1910, p. 36). The same electroscopes were used and the same constants employed.

In the majority of cases there was no appreciable quantity of radium in the alkaline solutions. As a precaution, however, the alkaline solutions of No. 24—a mica showing a radioactivity of 4.48 gr. per gr. from its acid solution alone—and of Nos. 1 and 5 in the table, having been treated in the electroscope used exclusively for alkaline solutions, were acidified with hydrochloric acid, reclosed and retested as acid solutions. This confirmed the original result by failing to show any increase in the rate of collapse of the gold-leaf.

For purposes of accuracy, and to determine whether there were any tendency to a general falling off in the quantity

of emanation extracted on subsequent ebullition of a solution, a series of thirteen redeterminations were made after the solutions had been stored during a further period of four months. The results were quite satisfactory, no tendency either to a lowering or a raising of the original determinations being apparent. In six of these cases the subsequent experiments yielded figures identical with those originally arrived at. In the case of the granite from Aughrim four determinations yielded the same figure. For the remainder the subsequent results vary but slightly from the original ones with the single exception of No. 17. Every effort was made to expel all the emanation by a vigorous ebullition, which was forced until the steam condensed in the receiver globe. Finely powdered talc spread over the inner surface of the condenser tube served as a perfect indication of the maximum height of condensation.

The following table (p. 106) has been arranged with a view to bringing together results obtained upon specimens from the same locality.

Some attempt was made to determine whether the radioactivity could be assigned to any particular constituent, or whether the distribution was uniform throughout the mass.

A coarse granite from Ballyellin—No. 17 in the accompanying table,—bearing the typically porphyritic muscovite with included biotite, was chosen. The flakes of biotite when they appeared in the muscovite showed pleochroic halos and much irregular radioactive darkening. The granite was coarsely broken and about three grams of mica crystals extracted. An examination of this yielded for the radium present per gram the figure  $4.48 \times 10^{-12}$  gr. The original granite from the same hand specimen yielded a quantity  $2.08 \times 10^{-12}$  gr. per gr., and assuming the mica to be about 20 per cent. of the whole mass (Haughton), it follows that the mica was responsible for about  $1 \times 10^{-12}$  gr. per gr. of the granite, or about  $\frac{1}{3}$  of the whole quantity contained.

Professor the Hon. R. J. Strutt found (Proc. R. S. lxxviii. A. p. 150) that in one gram of Cornish granite over one half (approximately  $\frac{5}{9}$ ) of the radium present, was confined to the brown mica and the heavy minerals contained in it. Generally speaking, this relation therefore seems to hold good in the case of this granite.

An examination of massive quartz, felspar, and muscovite mica, free from biotite inclusions, all from large crystals found in the neighbourhood of the Three Rock Mountain, yielded respectively 0.00, 0.52, and 0.72 gram per gram for their contained radium. The mica having the radium content

Locality.	Radium $\times 10^{-12}$ gr. per gr.	Thorium $\times 10^{-5}$ gr. per gr.	Radium Thorium $\times 10^{-7}$ .
1. Killiney Hill.....	3.62	0.81	4.3
2. " " .....	1.76	1.36	1.3
3. Three Rock Mountain .....	0.90	0.71	1.2
4. " " " .....	1.41	0.60	2.3
5. Ticknock .....	1.00	0.33	3.0
6. Glencullen (3) .....	{ 0.80 1.20 1.14	{ 0.31	2.6
7. Dundrum * (2) .....	{ 1.12 1.29	{ 0.36	3.0
8. Glencree .....	2.05	0.57	3.4
9. Glendalough .....	1.04	0.51	2.0
10. Glenmalure .....	0.41	0.66	0.6
11. " .....	0.65	0.30	2.2
12. " .....	4.36	None detected.	...
13. Ballyknockan (2) .....	{ 1.69 1.92	{ 0.81	2.0
14. Aughrim (5) .....	{ 1.45 1.02 1.02 1.02 1.02	{ 0.71	1.4
15. Ballyellin .....	1.20	0.51	2.3
16. " .....	1.44	0.68	2.1
17. " (2).....	{ 2.08 1.14	{ 0.89]	2.3
18. Bagnalstown .....	1.76	1.50	1.2
19. Newtown .....	4.00	1.25	3.2
20. Blackstairs Mountain (2)...	{ 2.41 2.41 2.57	{ 0.80	3.0
21. Borris (3) .....	{ 2.00 2.15	{ 1.43	1.4
22. Kiltaly .....	1.66	0.67	2.4
23. Geresbridge (2).....	{ 1.00 1.00	{ 0.43	2.3
24. Muscovite with Biotite from granite No. 16 (2.5 grs.)	4.48	None detected.	...
25. Muscovite from Three Rock (5.58 grs.)	0.72	0.91	0.8
26. Felspar ditto (9.3 grs.).....	0.52	None detected.	...
27. Quartz ditto (3 grs.) .....	0.00	None detected.	...
28. Halo-bearing Biotite from Ballyellin (1 gr.)	11.87	None detected.	...
Mean .....	$1.68 \times 10^{-12}$ gr. per gr	$0.70 \times 10^{-5}$ gr. per gr.	

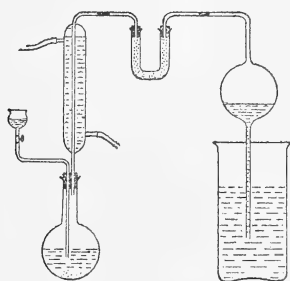
\* From a boring 190 feet below surface level.



of  $11.87 \times 10^{-12}$  gr. per gr. was from a large hexagonal crystal of biotite, with a maximum width of 2.5 cms., enclosed in a crystal of muscovite. It revealed countless pleochroic halos under the microscope, and showed considerable radioactive darkening along cracks, and in the neighbourhood of the junction.

Professor Joly obtained for a specimen of the Ballyknockan granite, using a very limpid solution, a radioactivity of  $5.5 \times 10^{-12}$  gr. per gr. on two closely agreeing experiments on the same hand-specimen ('Radioactivity and Geology,' p. 43). In another experiment on granite from this locality, but using a different method of investigation, the result obtained by Prof. Joly was  $2.6 \times 10^{-12}$  gr. per gr. In this case the melt from the crucible was broken up and closed with some distilled water in a flask for 21 days, when hydrochloric acid was carefully run in, the evolved gases passed over soda-lime, and finally the contents of the flask briskly boiled for fifteen minutes. The steam was returned by a condenser attached to the flask. The gases unabsorbed were collected over a potash solution and finally admitted into the electro-scope. In this procedure, undissolved or precipitated matter remained in the flask (see fig. 1).

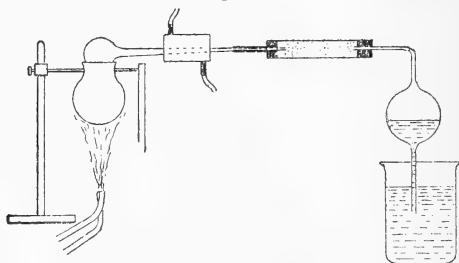
Fig. 1.



A third method was tried upon the same rock. The gases evolved during the decomposition of the rock in presence of the fusion mixture were, after absorption by soda-lime &c., admitted into the electro-scope. In this case the charge, consisting of the powdered rock and the fusion mixture of carbonates, is melted in an air-tight platinum still which can be heated to a high temperature over the blowpipe. A platinum tube leading from the top of the still conveys the

evolved gases to the soda-lime tubes (see fig. 2). This method gave  $2.3 \times 10^{-12}$  gr. per gr. for the contained radium. My own result on a Ballyknockan granite by the solution method used by Professor Strutt yielded a result of  $1.6 \times 10^{-12}$  gr. per gr.

Fig. 2.



The variations in these results, together with other differences in the quantities of contained radium, seem to indicate that a moderately large divergence may exist between closely adjoining specimens, but that the general average fairly approximates to the mean result in any particular district. In no case was the comparatively high result of  $4 \times 10^{-12}$  gr. per gr. maintained in a series of specimens from the same locality. These results have been included therefore in estimating the mean radium content of the whole chain. The thorium was estimated from the same solutions, by the method introduced and described by Prof. Joly (Phil. Mag. July 1909), using his apparatus, and the constants determined by him. I add a description of this apparatus in its latest form.

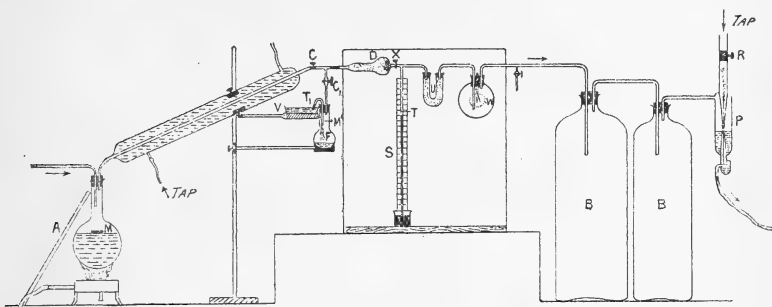
#### *Apparatus for Detection and Measurement of Thorium.*

The electroscope in which the thorium was estimated was made and calibrated by Prof. Joly (Phil. Mag. July 1909) from solutions containing a known quantity of thorite. The vacuum-pump P, which is fed from an overhead tank maintained at a constant level, is first started, the partial vacuum caused in the apparatus being observed by the rise of the oil in the tube T, reading on to an arbitrary scale S behind, which thus serves as an indication of the velocity of the air-current. The vessels B, B serve to steady the influx of air, and thus keep the indicator steady.

The actual rate of flow of the air is about 4 c.c. per sec., and is measured by means of the 100 c.c. flask F. The stopcock C

is first closed and  $C_1$  opened, so that the air is drawn entirely through the measuring flask. The end of the tube  $T_1$  is

Fig. 3.



then dipped below the surface of the water in the vessel  $V$  placed in position for the purpose, and the time of filling to the fixed mark noted. The friction opposing the entry of the water into the measuring flask is compensated for by lengthening the inner limb of the tube at  $M'$ , so that this takes the character of a siphon and a slight static head is given to the entering water. The additional length—about 3 cms.—was found by trial and error, the exact length being hit when no motion of the oil in the gauge was observed to attend the flow of water into the flask.

The air flow was regulated by means of the stopcock  $X$ , which was then allowed to remain untouched throughout the remainder of the work. The standard height of the oil in the gauge was adhered to as far as possible throughout the successive experiments, by regulation of the vacuum-pump. The stopcock  $C$  is of course allowed to remain permanently open, and  $C_1$  closed until it is thought necessary to again test the velocity of influx.

The height of the thoria solution in the flask is marked, and is as far as possible maintained in successive estimations. Flasks of various sizes may thus be used, and standardized to a particular velocity of air-current. It was found possible to maintain a violent ebullition. The current of air entering the electroscopes from the boiling flask is first dried by passage through the tube  $D$ , containing coarsely granulated calcium chloride, and finally through the tube  $U$ , filled with phosphorus pentoxide.

An asbestos card  $A$  was used to prevent the admission of gases from the flame along with the air-current. A small piece of the bulb of the electroscopes was removed and a

window  $w$  of thin cover glass let in for the better observance of the gold leaf.

It was not found necessary, when observing the natural collapse of the gold-leaf, to remove the flask, but merely to discontinue the ebullition.

The thorium content of this granite is very low. In some cases careful and repeated experiment failed to detect with certainty any trace. Efforts were made, but failed to reveal any concentration of thorium existing in any particular constituent, as seems to be the case with radium. Thus one gram of the biotite showing nearly  $12 \times 10^{-12}$  gr. radium per gr., showed no detectable trace of thorium; while 2.5 grams of muscovite with contained biotite picked from No. 17 in the table, also failed to reveal any thorium.

In the case of those solutions containing any quantity of precipitate (either original or subsequently developed during the estimation of the radium), a second experiment was made on the clear solution after the precipitate had been removed by filtration. The effect generally was to slightly increase the collapse of the gold leaf. In the case of No. 1 in the table, an increase in the rate of gain of 2 scale-divisions per hour, and of No. 2 in the table an increase of .5 scale-division per hour was observed.

The mean quantity of thorium found was  $0.70 \times 10^{-5}$  gr. per gr. of the rock. Professor Joly (Phil. Mag. July 1909) obtained on nineteen various primary and secondary rocks a mean thorium content of  $1.07 \times 10^{-5}$  gr. per gr., and on fifty-one rocks, chiefly gneisses of exceptionally high radioactivity from the St. Gothard Tunnel, a mean of  $1.12 \times 10^{-5}$  gr. per gr.

The approximately constant proportions obtaining between the thorium and the radium contained in the granites is a striking feature. The mean ratio borne by the radium to the thorium present is  $2.4 \times 10^{-7}$ . In no less than ten separate specimens, the proportion borne by radium to thorium varies between  $2.0 \times 10^{-7}$  and  $2.6 \times 10^{-7}$ , the three specimens from Ballyellin showing this very closely.

Three remarkable examples, however, are those of Nos. 12, 24, and 28 in the accompanying table, these three specimens showing the highest radium content but no detectable thorium. On comparing these latter results with the others, it would appear that an exceptional rise in the quantity of radium, so far from being attended by any corresponding rise in the quantity of thorium, is actually attended by its absence. It cannot be said, however, that previous work supports so marked an inverse ratio; although it is apparent from the

examination of the published results that exceptional richness in the uranium-radium elements is not attended by a corresponding richness in the thorium series. This is more especially evident in the radium-thorium ratio of Vesuvian lavas, when contrasted with the ratio for other lavas (Joly, *Phil. Mag.* Oct. 1909).

It seems desirable that the investigation of the radium content of rocks should be accompanied by an estimate of the thorium content also; not only to elucidate the question of a possible relationship between the radium and the thorium content, but in the interests of the geological applications of radioactivity. The same solution may be used in both determinations, and the experimental observation is actually more readily made and confirmed by repetition of the test in the case of thorium than in the case of radium.

The results, it may be seen, are obtained from materials taken at points ranging from the most northerly exposure of the granite, and along the western and eastern side of the chain to nearly the southern limit. The specimens from Glendalough and Glenmalure may be regarded as from the central axis.

The specimen from Dundrum—No. 7 in the table—comes from a recent boring at that place, and was taken from a point 190 feet below the surface level.

The experiments therefore probably yield an approximation of say  $1.7 \times 10^{-12}$  gr. per gr. to the mean radioactivity of the mass, so far as it is capable of being determined by the method adopted, and as carefully applied as was found possible.

That there is radium throughout the entire great mass seems quite certain; and it may be safely inferred that in a material so homogeneous a somewhat similar distribution of radioactive materials prevails at all points near the surface; and indeed it is probably allowable to assume that the vertical distribution of radioactive elements throughout the mass is not very different.

In conclusion I have to thank Prof. Joly both for his initiation of the work, and for his continual assistance throughout its performance.

Geological Laboratory,  
Trinity College, Dublin.  
Nov. 24, 1910.

XV. *On the Uniform Motion of a Sphere through a Viscous Fluid.* By Prof. HORACE LAMB, F.R.S.\*

AN important contribution to this subject has recently been made by Prof. C. W. Oseen†, of Upsala, who calls attention to a certain limitation affecting the validity of the accepted solution‡, however small the velocity may be, as regards points at a sufficient distance from the sphere. He proceeds to give an amended solution which appears to be free from this defect, whilst it gives the same distribution of velocity in the immediate neighbourhood of the sphere, and consequently the same value of the resistance, as the older theory.

Prof. Oseen's analysis appears to have more than the immediate object in view, and is for the present purpose somewhat long and intricate. Considering the great interest of the question, it may be permissible to indicate a shorter way of arriving at his results, and to add a few comments dealing somewhat more explicitly than he has done with their interpretation, and with the estimation of the degree of approximation which is attained in different parts of the field.

The problem is most conveniently treated as one of "steady" motion, the fluid being supposed to flow with the general velocity  $U$ , say, parallel to  $x$ , past a fixed spherical obstacle whose centre is at the origin. The distribution of velocity is then given, on Stokes's theory, by the formulæ

$$\left. \begin{aligned} u &= U \left( 1 - \frac{a}{r} \right) - \frac{1}{4} U a (r^2 - a^2) \frac{\partial^2 1}{\partial x^2 \partial r}, \\ v &= -\frac{1}{4} U a (r^2 - a^2) \frac{\partial^2 1}{\partial x \partial y \partial r}, \\ w &= -\frac{1}{4} U a (r^2 - a^2) \frac{\partial^2 1}{\partial x \partial z \partial r}, \end{aligned} \right\} \dots (1)$$

where  $a$  is the radius of the sphere, and  $r$  denotes distance from the centre. These formulæ are based on the assumption that the inertia terms in the hydrodynamical equations, which are of the second order in  $u, v, w$ , may be neglected. They were, in fact, only propounded as a limiting form to which

\* Communicated by the Author.

† "Ueber die Stokes'sche Formel, und über eine verwandte Aufgabe in der Hydrodynamik," *Arkiv för matematik, astronomi og fysik*, Bd. 6, no. 29 (1910).

‡ Stokes, *Camb. Trans.* vol. ix. (1851); *Sci. Papers*, vol. iii. p. 55.

the distribution of velocity may be supposed to tend as  $U$  is diminished indefinitely.

It is known\*, however, that results obtained in this way will accurately satisfy the equations, provided these be modified by the introduction of constraining forces

$$X = w\eta - v\zeta, \quad Y = u\zeta - w\xi, \quad Z = v\xi - u\eta, \quad \dots \quad (2)$$

where

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad \dots \quad (3)$$

These forces have a resultant  $R$  which is normal both to the stream-line and to the vortex-line, and whose magnitude is in the present case

$$R = q\omega, \quad \dots \dots \dots (4)$$

where

$$q = \sqrt{(u^2 + v^2 + w^2)}, \quad \omega = \sqrt{(\xi^2 + \eta^2 + \zeta^2)}. \quad \dots \quad (5)$$

The magnitude of these hypothetical forces, as compared with the viscous forces

$$\nu \nabla^2 u, \quad \nu \nabla^2 v, \quad \nu \nabla^2 w, \quad \dots \dots \dots (6)$$

where  $\nu$  is the kinematic viscosity, gives an indication of the degree of approximation which is attained in formulæ such as (1).

Now from (1) we find

$$\xi = 0, \quad \eta = \frac{3}{2} \frac{Uaz}{r^3}, \quad \zeta = -\frac{3}{2} \frac{Uay}{r^3}, \quad \dots \dots \quad (7)$$

so that for large values of  $r$

$$X = 0, \quad Y = -\frac{3}{2} \frac{U^2 ay}{r^3}, \quad Z = -\frac{3}{2} \frac{U^2 az}{r^3}. \quad \dots \quad (8)$$

For the viscous forces (6) we find

$$\frac{3}{2} \nu Ua \frac{\partial^2 1}{\partial x^2 r}, \quad \frac{3}{2} \nu Ua \frac{\partial^2 1}{\partial x \partial y r}, \quad \frac{3}{2} \nu Ua \frac{\partial^2 1}{\partial x \partial z r}. \quad \dots \quad (9)$$

The ratio of the former to the latter is ultimately of the order  $Ur/\nu$ , which increases indefinitely with  $r$ , however small  $U$  may be. This is, under a slightly different form, the objection which Prof. Oseen raises to the validity of (1)

\* Rayleigh, Phil. Mag. [4] vol. xxxvi. p. 354 (1893); Sci. Papers vol. iv. p. 78. The pressure must be supposed altered by a term,  $\frac{1}{2}\rho(u^2 + v^2 + w^2)$ , which vanishes however at the surface of the sphere, on the hypothesis of no slipping.

at points distant from the sphere. Since, however, both the constraining forces and the viscous forces are in these regions relatively small, it does not necessarily follow that the character of the motion in the immediate neighbourhood of the sphere will be seriously affected. At points near the sphere the constraining forces tend to vanish, whilst the viscous forces are of the order  $\nu Ua/r^3$ .

The innovation made by Oseen in the treatment of the question consists in writing  $U + u$  for  $u$ , and neglecting terms of the second order in  $u, v, w$  only. These symbols now denote the components of the velocity which would remain if a uniform velocity  $-U$  were superposed on the whole system. The hydrodynamical equations accordingly take the forms

$$\left. \begin{aligned} U \frac{\partial u}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u, \\ U \frac{\partial v}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v, \\ U \frac{\partial w}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w, \end{aligned} \right\} \dots \dots (10)$$

with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \dots \dots (11)$$

The inertia terms are thus to some extent taken into account, but it is to be remarked that although the approximation is undoubtedly improved at infinity, where  $u, v, w = 0$ , it is in some degree impaired near the surface of the sphere where we now have  $u = -U$ . This will be a matter for subsequent examination.

The solution of the equations (10) and (11) for the purpose in hand can be effected very simply. In the first place we have

$$\nabla^2 p = 0; \dots \dots (12)$$

and a particular solution is therefore obtained if we write

$$p = \rho U \frac{\partial \phi}{\partial x}, \dots \dots (13)$$

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}, \dots \dots (14)$$

where  $\phi$  satisfies

$$\nabla^2 \phi = 0.$$



The solution is completed if we write

$$u = -\frac{\partial\phi}{\partial x} + u', \quad v = -\frac{\partial\phi}{\partial y} + v', \quad w = -\frac{\partial\phi}{\partial z} + w', \quad (15)$$

where  $u', v', w'$  are solutions of the equations

$$\left. \begin{aligned} (\nabla^2 - 2k \frac{\partial}{\partial x})u' &= 0, \\ (\nabla^2 - 2k \frac{\partial}{\partial x})v' &= 0, \\ (\nabla^2 - 2k \frac{\partial}{\partial x})w' &= 0, \end{aligned} \right\} \dots \dots \dots (16)$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \dots \dots \dots (17)$$

We have here written, for shortness,

$$k = U/2\nu. \dots \dots \dots (18)$$

Since the vortex-lines must be circles having the axis of  $x$  as a common axis, we may assume

$$\xi = 0, \quad \eta = -\frac{\partial\chi}{\partial z}, \quad \zeta = \frac{\partial\chi}{\partial y}, \quad \dots \dots (19)$$

where  $\chi$  is a function of  $x$  and  $\varpi$  (the distance from the axis of  $x$ ) only. It follows from (16) that we must have

$$\left(\nabla^2 - 2k \frac{\partial}{\partial x}\right)\chi = 0, \dots \dots \dots (20)$$

an additive function of  $x$  being obviously irrelevant. Hence

$$\left. \begin{aligned} 2k \frac{\partial u'}{\partial x} = \nabla^2 u' &= \frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y} = -\left(\frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2}\right) \\ &= \frac{\partial^2\chi}{\partial x^2} - 2k \frac{\partial\chi}{\partial x}, \\ 2k \frac{\partial v'}{\partial x} = \nabla^2 v' &= \frac{\partial\zeta}{\partial x} - \frac{\partial\xi}{\partial z} = \frac{\partial^2\chi}{\partial x\partial y}, \\ 2k \frac{\partial w'}{\partial x} = \nabla^2 w' &= \frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x} = \frac{\partial^2\chi}{\partial x\partial z}. \end{aligned} \right\} \dots (21)$$

We thus obtain the solution

$$\left. \begin{aligned} u' &= \frac{1}{2k} \frac{\partial \chi}{\partial x} - \chi, \\ v' &= \frac{1}{2k} \frac{\partial \chi}{\partial y}, \\ w' &= \frac{1}{2k} \frac{\partial \chi}{\partial z}, \end{aligned} \right\} \dots \dots \dots (22)$$

which is easily verified.

The equation (20) may be written

$$(\nabla^2 - k^2) e^{-kx} \chi = 0, \dots \dots \dots (23)$$

the solution of which is well-known, the simplest type being  $e^{-kx} \chi = C e^{-kr}/r$ . Adopting this, we have, finally,

$$\left. \begin{aligned} u &= -\frac{\partial \phi}{\partial x} + \frac{1}{2k} \frac{\partial \chi}{\partial x} - \chi, \\ v &= -\frac{\partial \phi}{\partial y} + \frac{1}{2k} \frac{\partial \chi}{\partial y}, \\ w &= -\frac{\partial \phi}{\partial z} + \frac{1}{2k} \frac{\partial \chi}{\partial z}, \end{aligned} \right\} \dots \dots \dots (24)$$

where

$$\chi = \frac{C e^{-k(r-x)}}{r} \dots \dots \dots (25)$$

Since  $\phi$  must obviously involve only zonal harmonics of negative degree, we write

$$\phi = \frac{A_0}{r} + A_1 \frac{\partial}{\partial x} \frac{1}{r} + A_2 \frac{\partial^2}{\partial x^2} \left( \frac{1}{r} \right) + \dots \dots \dots (26)$$

For small values of  $kr$  we have

$$\chi = C \left( \frac{1}{r} - k + \frac{kx}{r} + \dots \right) \dots \dots \dots (27)$$

which leads to

$$\left. \begin{aligned} \frac{1}{2k} \frac{\partial \chi}{\partial x} - \chi &= -\frac{C}{2k} \left\{ \frac{1}{3} \frac{k}{r} - \frac{\partial}{\partial x} \frac{1}{r} + \frac{1}{3} k r^2 \frac{\partial^2}{\partial x^2} \frac{1}{r} + \dots \right\}, \\ \frac{1}{2k} \frac{\partial \chi}{\partial y} &= -\frac{C}{2k} \left\{ -\frac{\partial}{\partial y} \frac{1}{r} + \frac{1}{3} k r^2 \frac{\partial^2}{\partial x \partial y} \frac{1}{r} + \dots \right\}, \\ \frac{1}{2k} \frac{\partial \chi}{\partial z} &= -\frac{C}{2k} \left\{ -\frac{\partial}{\partial z} \frac{1}{r} + \frac{1}{3} k r^2 \frac{\partial^2}{\partial x \partial z} \frac{1}{r} + \dots \right\}. \end{aligned} \right\} (28)$$

Hence the relations  $u = -U$ ,  $v = 0$ ,  $w = 0$  which are to hold for  $r = a$  will be satisfied, provided

$$C = \frac{3}{2} Ua, \quad A_0 = \frac{3}{2} va, \quad A_1 = -\frac{1}{4} Ua^3, \quad \dots \quad (29)$$

approximately; and it will be noted that the condition for the success of the approximation is that  $ka$ , or  $Ua/\nu$ , should be small.

The equations (24), (25), (26), (29) agree with Prof. Oseen's solution of the problem, obtained by a different process.

To find the distribution of velocity in the neighbourhood of the sphere we may use the formulæ (26) and (28), with the values of the constants given in (29). The result is identical with (1) if regard be had to the altered meaning of  $u$ . The resistance experienced by the sphere has the same value ( $6\mu aU$ ) as on Stokes's theory\*.

In other respects the motion differs widely from that represented by Stokes's formulæ; and the further interpretation is very interesting. In the first place, as pointed out by Prof. Oseen, the stream-lines are no longer symmetrical with respect to the plane  $x = 0$ , the motion being in fact no longer reversible. Again, the (doubled) angular velocity of the fluid elements is

$$\omega = -\frac{\partial \chi}{\partial \varphi} = \frac{3}{2} Ua(1 + kr) \frac{\varpi}{r^3} e^{-k(r-x)}, \quad \dots \quad (30)$$

and is therefore insensible, on account of the exponential factor alone, except within a region bounded, more or less vaguely, by a paraboloidal surface, having its focus at  $O$ , for which  $k(r-x)$  has a moderate constant value. This region may be referred to as the "wake," although it includes a certain space on the upstream side of the sphere. If we superpose a general velocity  $-U$  parallel to  $x$ , the residual velocity tends, for large values of  $r$  and for points outside the wake, to become purely radial, as if due to a simple source of strength  $4\pi A_0$ , or  $6\pi va$ , at the origin. This is compensated by an inward flow in the wake; thus for points along the axis of the wake, to the right, where  $x = r$ , we find

$$u = -\frac{3}{2} \frac{Ua}{r} + \frac{1}{2} \frac{Ua^3}{r^3} \dots \dots \dots (31)$$

This indicates a velocity following the sphere (when the

\* The addition to the pressure (see the footnote on p. 113) is now  $\frac{1}{2}\rho U^2$  at the surface of the sphere. This being uniform, does not affect the resultant.

latter is regarded as moving through a liquid at rest at infinity) which ultimately varies inversely as the distance, instead of as the square of the distance.

It remains to estimate the degree of approximation which the preceding results afford in various parts of the field. For this we have recourse, again, to a comparison of the "constraining" forces, which would be necessary to make the solution exact, with the viscous forces. The former are still given by the formulæ (2), provided  $u$  has its altered meaning.

At distant points, well outside the wake, the terms in (24) which depend on  $\chi$  may be neglected, and we have, ultimately,

$$u = \frac{3}{2} \nu a \frac{x}{r^3}, \quad v = \frac{3}{2} \nu a \frac{y}{r^3}, \quad w = \frac{3}{2} \nu a \frac{z}{r^3} \dots \quad (32)$$

Also, from (30),

$$\xi = 0, \quad \eta = \frac{3}{2} Uka \frac{z}{r^2} e^{-k(r-x)}, \quad \zeta = -\frac{3}{2} Uka \frac{y}{r^2} e^{-k(r-x)}. \quad (33)$$

Hence

$$\begin{aligned} X &= \frac{9}{8} U^2 a^2 \frac{\varpi^2}{r^5} e^{-k(r-x)}, & Y &= -\frac{9}{8} U^2 a^2 \frac{\alpha y}{r^5} e^{-k(r-x)}, \\ Z &= -\frac{9}{8} U^2 a^2 \frac{\alpha z}{r^5} e^{-k(r-x)}, \end{aligned} \quad (34)$$

the resultant of which is

$$R = \frac{9}{8} U^2 a^2 \frac{\varpi}{r^4} e^{-k(r-x)}, \dots \dots \dots (35)$$

in a direction at right angles to the radius vector. The viscous forces may be found from (21) and (25). If we retain only the terms which are most important when  $r$  is large, we find

$$\begin{aligned} \nu \nabla^2 u &= -\nu k^2 C \frac{\varpi^2}{r^3} e^{-k(r-x)}, & \nu \nabla^2 v &= -\nu k^2 C \frac{y(r-x)}{r^3} e^{-k(r-x)}, \\ \nu \nabla^2 w &= -\nu k^2 C \frac{z(r-x)}{r^3} e^{-k(r-x)}. \end{aligned} \quad (36)$$

It follows from (18) that the ratio of the forces is of the order  $(1/kr) \cdot (a/r)$ . The approximation in this part of the field is therefore amply sufficient.

At points well within the wake, where  $k(r-x)$  is small, we have

$$u = -\frac{3}{2} \frac{Ua}{r}, \quad v=0, \quad w=0, \quad \dots \quad (37)$$

and

$$\xi=0, \quad \eta = \frac{3}{2} Uka \frac{z}{r^2}, \quad \zeta = -\frac{3}{2} Uka \frac{y}{r^2}, \quad \dots \quad (38)$$

approximately. These make

$$X=0, \quad Y = \frac{9}{4} U^2ka^2 \frac{y}{r^3}, \quad Z = \frac{9}{4} Uka^2 \frac{z}{r^3} \quad \dots \quad (39)$$

The viscous forces are found to be

$$\nu \nabla^2 u = 2\nu k C \frac{x}{r^3}, \quad \nu \nabla^2 v = 2\nu k C \frac{y}{r^3}, \quad \nu \nabla^2 w = 2\nu k C \frac{z}{r^3}, \quad (40)$$

approximately. The ratio of the magnitudes is of the order  $ka$ .

Near the surface of the sphere we have  $u = -U, v=0, w=0$ , approximately, and therefore, from (2) and (19),

$$X=0, \quad Y = -U \frac{\partial \chi}{\partial y}, \quad Z = -U \frac{\partial \chi}{\partial z}, \quad \dots \quad (41)$$

or, by (27) and (29),

$$X=0, \quad Y = \frac{3}{2} U^2a \frac{y}{r^3}, \quad Z = \frac{3}{2} U^2a \frac{z}{r^3} \quad \dots \quad (42)$$

The resultant is therefore of the order  $U^2/a$ . The viscous forces are obtained from (21) and (28); thus

$$\begin{aligned} \nu \nabla^2 u &= \frac{3}{2} \nu Ua \frac{3x^2 - r^2}{r^5}, & \nu \nabla^2 v &= \frac{3}{2} \nu Ua \frac{3xy}{r^5}, \\ \nu \nabla^2 w &= \frac{3}{2} \nu Ua \frac{3xz}{r^5}, & & \dots \quad (43) \end{aligned}$$

giving a resultant of the order  $\nu U/a^2$ . The ratio of the magnitudes is therefore  $Ua/\nu$ , which has already been assumed to be small. The approximation, although less perfect here than on Stokes's theory, is seen to be adequate.

I may repeat that the object of this note has been merely to give a simpler demonstration of Prof. Oseen's results, and to elucidate a little more fully their scope and significance. I have not touched upon another aspect of the question which is referred to in his paper, and which is apparently to form the subject of a further investigation.

It is of some interest to apply the same method to the two-dimensional problem of the flow past a circular cylinder where, as is well-known, Stokes was led to the conclusion that a steady motion is impossible. It will appear that when the inertia terms are partially taken into account, in the manner above explained, this conclusion is modified, and that a definite value for the resistance is obtained.

The hydrodynamical equations are now satisfied by

$$\left. \begin{aligned} u &= -\frac{\partial\phi}{\partial x} + \frac{1}{2k} \frac{\partial\chi}{\partial x} - \chi, \\ v &= -\frac{\partial\phi}{\partial y} + \frac{1}{2k} \frac{\partial\chi}{\partial y}, \end{aligned} \right\} \dots \dots \dots (44)$$

$$p = \rho U \frac{\partial\phi}{\partial x}, \dots \dots \dots (45)$$

provided  $\nabla_1^2\phi = 0, \dots \dots \dots (46)$

and  $(\nabla_1^2 - 2k \frac{\partial}{\partial x})\chi = 0, \dots \dots \dots (47)$

where  $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \dots \dots \dots (48)$

The appropriate solution of (47) is

$$\chi = C e^{kx} K_0(kr), \dots \dots \dots (49)$$

where \*

$$\begin{aligned} K_0(kr) &= -\left(\gamma + \log \frac{1}{2} kr\right) I_0(kr) + \frac{k^2 r^2}{2^2} + \frac{k^4 r^4}{2^2 \cdot 4^2} \left(1 + \frac{1}{2}\right) + \dots \\ &= \sqrt{\left(\frac{\pi}{2kr}\right)} e^{-kr} \left\{ 1 - \frac{1^2}{8kr} + \frac{1^2 \cdot 3^2}{1 \cdot 2(8kr)^2} - \dots \right\}. \end{aligned} \quad (50)$$

For small values of  $kr$  we have

$$\chi = -C(1+kx)(\gamma + \log \frac{1}{2} kr), \dots \dots \dots (51)$$

whence

$$\left. \begin{aligned} \frac{1}{2k} \frac{\partial\chi}{\partial x} - \chi &= -\frac{C}{2k} \left\{ k \left( \frac{1}{2} - \gamma - \log \frac{1}{2} kr \right) \right. \\ &\quad \left. + \frac{\partial}{\partial x} \log r - \frac{1}{2} kr^2 \frac{\partial^2}{\partial x^2} \log r + \dots \right\}, \\ \frac{1}{2k} \frac{\partial\chi}{\partial y} &= -\frac{C}{2k} \left\{ \frac{\partial}{\partial y} \log r - \frac{1}{2} kr^2 \frac{\partial^2}{\partial x \partial y} \log r + \dots \right\}. \end{aligned} \right\} (52)$$

\* The notation is that of Gray and Mathews, Treatise on Bessel Functions, p. 68.

Hence if we put

$$\phi = A_0 \log r + A_1 \frac{\partial}{\partial x} \log r + \dots, \quad (53)$$

we find that the conditions  $u = -U$ ,  $v = 0$ ,  $w = 0$ , will be satisfied for  $r = a$ , provided

$$C = 2U / (\frac{1}{2} - \gamma - \log \frac{1}{2}ka), \quad A_0 = -C/2k, \quad A_1 = \frac{1}{4}Ca^2, \quad (54)$$

approximately. Hence near the sphere we have

$$\left. \begin{aligned} u &= \frac{1}{2}C \left\{ \gamma - \frac{1}{2} + \log \frac{1}{2}kr + \frac{1}{2}(r^2 - a^2) \frac{\partial^2}{\partial x^2} \log r \right\}, \\ v &= \frac{1}{4}C(r^2 - a^2) \frac{\partial^2}{\partial x \partial y} \log r. \end{aligned} \right\} \quad (55)$$

The vorticity is given by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = Ce^{kx} \frac{\partial}{\partial y} K_0(kr), \quad (56)$$

which for large values of  $kr$  takes the form

$$\zeta = -kC \frac{y}{r} \sqrt{\left(\frac{\pi}{2kr}\right)} e^{-k(r-x)} \quad (57)$$

The general interpretation would follow the same lines as in the case of the sphere.

To calculate the force exerted by the fluid on the cylinder we have to integrate the expression

$$\left(-p + 2\mu \frac{\partial u}{\partial x}\right)x + \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)y,$$

or

$$-px + \mu r \frac{\partial u}{\partial r} + \mu \left(x \frac{\partial u}{\partial x} + y \frac{\partial v}{\partial x}\right), \quad (58)$$

with respect to the angular coordinate ( $\theta$ ) from 0 to  $2\pi$ . The products of plane harmonics of different orders will of course disappear in this process. The first term of (58) gives, when  $r$  is put equal to  $a$ ,

$$-pUA_0 \int_0^{2\pi} \cos^2 \theta \, d\theta = -\pi pUA_0 = \pi\mu C, \quad (59)$$

by (45), (53), (54). The second term contributes, on substitution from (55),  $\pi\mu C$ . The third term vanishes identically, to our order of approximation. The final value for the resistance per unit length is therefore

$$2\pi\mu C = \frac{4\pi\mu U}{\frac{1}{2} - \gamma - \log(\frac{1}{2}ka)} \quad (60)$$

The investigation is subject as before to the condition that  $ka$ , or  $Ua/2\nu$ , is to be small.

XVI. *The Change of Resistance of Nickel and Iron Wires placed longitudinally in Strong Magnetic Fields.* By EDWIN A. OWEN, B.Sc., *University Student, University College of North Wales, Bangor* \*.

NUMEROUS experiments have been carried out on the effect of magnetization on the resistance of ferromagnetic metals; the most recent being those of Gray and Jones †, Barlow ‡, and W. E. Williams §.

The experiments of Gray and Jones were carried out with the object of determining simultaneous values in a specimen of soft iron wire of the magnetizing force, the magnetization, and the change of resistance due to magnetization. They studied in particular the longitudinal effect, when the direction of the electric current in the specimen is parallel to the lines of magnetic force, and found that  $\Delta\phi = aI^4$  represented approximately, in soft iron for fields ranging from 30 c.g.s. to 250 c.g.s. units, the relation between the magnetization  $I$  and the fractional increase of resistance  $\Delta\phi$ , *i. e.*, the increase of resistance divided by the resistance in zero field.

Barlow studied the same effect in nickel and found that the change of resistance showed a decided maximum:—

$$\Delta\phi = 0.0156, \quad H = 2000 \text{ c.g.s. units,}$$

and in higher fields decreased continuously to the value

$$\Delta\phi = 0.0100, \quad H = 18,000 \text{ c.g.s. units.}$$

He suggested that this decrease was due to the end elements of his coil, which were transversely magnetized. The electrical resistance of nickel diminishes when transversely magnetized in fields stronger than 2000 c.g.s. units, and this may therefore explain the diminution in the value of  $\Delta\phi$  observed by Barlow in his experiments.

The object of the experiments described in the present paper was to examine the effect in still stronger fields, and in doing so to employ specimens which were placed entirely longitudinally in the field. For this purpose the specimens examined took the form of very thin straight wires, only about a millimetre long. These, while having sufficient resistance to measure accurately, could be placed longitudinally in the narrow gap between the pole-faces of an electromagnet.

\* Communicated by Prof. E. Taylor Jones, D.Sc.

† Roy. Soc. Proc. 1900, vol. lxvii. p. 208.

‡ Roy. Soc. Proc. 1902, vol. lxxi. p. 30.

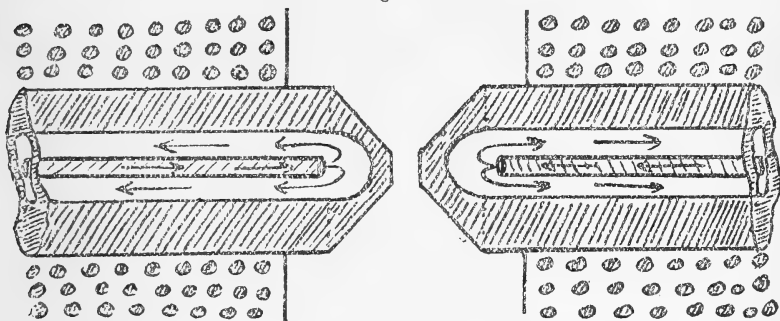
§ Phil. Mag. October 1902, December 1903, January 1905.



## (1) Apparatus.

A portion of the electromagnet by means of which the fields were produced is shown in section in fig. 1. The

Fig. 1.



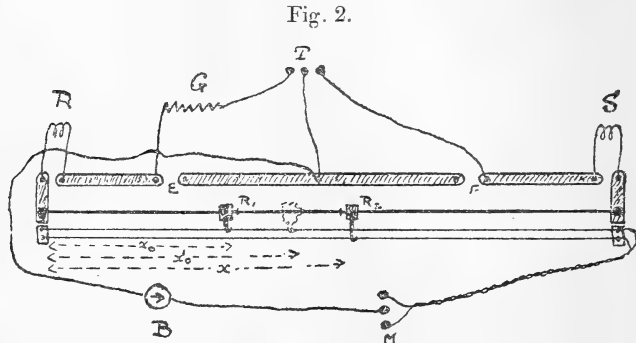
currents used in the magnet ranged from .1 ampere to about 22 amperes. The current was allowed to flow through the magnet for as short a time as possible, only during the few seconds required for obtaining the balancing-point on the resistance bridge. To prevent rise of temperature of the pole-pieces, water was arranged to flow along the hollow cores to within a few millimetres of the surfaces of the pole-pieces; the water was previously passed through a spiral tube which was slightly heated to bring the temperature of the water up to that of the room.

The strengths of the fields were measured directly after each set of readings for the change of resistance. This was done by the ballistic method,—the exploring coil being connected to a ballistic galvanometer provided with a telescope and scale. An arrangement was set up by which the exploring coil could be placed in the same position each time between the poles, and by which it could be suddenly removed from the field. The galvanometer was standardized before and after each set of readings by means of a standard solenoid and a secondary coil.

No attempt was made to measure the magnetization or the field inside the specimen. No form of "Isthmus Method" would in fact be suitable for such a thin specimen.

The change of resistance was measured by means of a slide wire bridge, the metal parts of which were all mounted on ebonite. In order to obtain a large step on the bridge, a thick german-silver wire was used as the bridge-wire, and also two auxiliary coils of german-silver wire were placed in

the gaps R and S. These two coils were immersed in the same oil-bath. Brass springs, attached to two ebonite riders  $R_1$  and  $R_2$  (fig. 2), were always in contact with the bridge-wire. These springs were connected to mercury keys which could be worked from a distance so as not to bring the hands



T: Terminals attached to bridge-table to receive cable. G: Compensating coil. R, S: Auxiliary coils. B: Broca Galvanometer. M: Mercury keys.

near the bridge connexions. In the inner gaps, E and F, of the bridge were placed respectively the specimen examined and a very thin platinum wire used as a comparison resistance. These were mounted on two carriers, and placed in the same trough which had pure paraffin oil flowing slowly through it (see fig. 3 a). They were placed close together so as to be subjected, as nearly as possible, to the same external changes of temperature. In series with the specimen examined was placed a compensating coil of platinoid wire, whose resistance was adjusted so that the temperature coefficient was the same for the two branches E and F.

If G is the resistance of the compensating coil, then its value is given by the formula  $G = N \cdot \frac{\beta - \alpha}{\alpha}$ , where N is the resistance of the specimen;  $\alpha$  and  $\beta$  the temperature coefficient of platinum and the specimen respectively.

A number of different suspended coil galvanometers were tried, but not one was found sensitive enough for the experiment. The galvanometer ultimately used was a Broca galvanometer with coils whose combined resistance was about 90 ohms. This instrument was found to be quite sensitive enough and it worked exceedingly well. It was used with a telescope and scale, the lamps used to light up the scale being immersed in a glass trough through which a

stream of cold water was passing. This precaution was taken to prevent any heat radiation from the lamps to the bridge connexions.

The battery circuit consisted of one small storage-cell in series with a resistance of 150 ohms, and a mercury key. The current was allowed to flow through the bridge for about an hour before any readings were taken, and it was not stopped until the set of readings was over. This was done in order that the rise of temperature due to the passage of the current through the specimen might have attained a steady value when the readings were being taken.

The electromagnet was situated broadside on to the galvanometer, and at a distance of about 45 feet from it. In this position and at this distance, it was found that the field due to the magnet did not affect the galvanometer needle. The specimen examined and the platinum wire were connected to the bridge table by means of three lengths of thick cable twisted together and passing from the table over the beams of the laboratory to three terminals on a wooden upright fixed to the frame of the magnet and insulated from it. The cables were twisted together to prevent inductive effects when the wires moved in the earth's field.

The specimen was held in position in the field as shown in fig. 3 a. A thin piece of ebonite, thickness about 1 mm.,

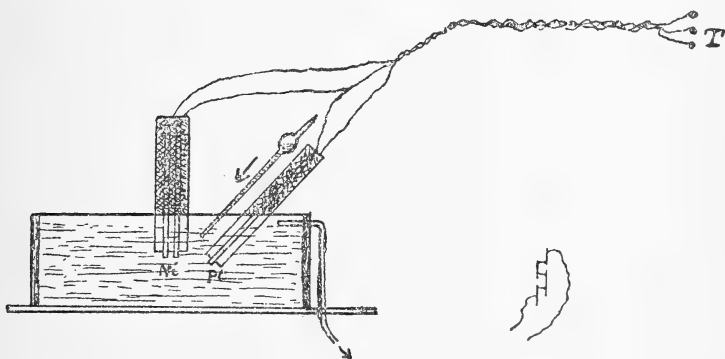


Fig. 3 a.

Fig. 3 b.

T: Terminals on wooden upright attached to frame of magnet.

was cut into a rectangle 6 cm. by 1 cm. Two strips of copper were fixed to each side of the ebonite, and three pieces of the specimen examined were soldered across the ends of these as shown in fig. 3 b. Great care was taken to solder them straight across and parallel to one another. The

comparison wire of thin platinum was mounted in a similar way on another ebonite plate.

The whole was fixed in a trough of mica just wide enough to contain the specimens. The trough was held horizontally in a wooden stand resting on a slab of plate glass which was separated from the fixed table on which it stood, by a thick layer of paraffin wax. The trough, and with it the thin wires, could be withdrawn from between the poles of the magnet to a distance of about 80 cm. by lowering the top part of the stand and turning it in its socket through  $180^\circ$ .

The whole of the apparatus including the cable, bridge-table, galvanometer, specimen, and all connecting-wires were thoroughly insulated from the earth.

### (2) *Method of taking Readings.*

The specimen was demagnetized by reversals, and then removed to a distance of about 80 cm. from the pole-pieces. When in this position, that is, when the specimen was in zero magnetic field, the balancing point was obtained on the bridge wire with the rider  $R_1$  (see fig. 2). The specimen was then turned into the field. Preliminary experiments had shown that the magnetic change of resistance in both nickel and iron reached a maximum value in a certain field, and the current required to produce this field had been carefully determined. This current was now passed through the magnet coils and the balancing point obtained on the bridge with the rider  $R_2$ . The specimen was again demagnetized and the same operation repeated until about a dozen readings were taken. The mean of these readings gave the maximum step on the bridge-wire. If  $x_0$  is the balancing point in zero field, and  $x$  that in the field, then  $(x - x_0)$  gives the maximum step.

The specimen was then permanently placed between the poles. Different currents were sent through the magnet alternately with the current which gave the maximum step, and the balancing points on the bridge wire found in each case, —  $R_2$  for the fixed current, and  $R_1$  for the other currents. The varying currents were not turned on regularly in ascending or descending order of magnitude, but quite irregularly. The mean of two consecutive readings with the fixed current was taken as the balancing point for this current. Let it be  $x$ . If  $x'_0$  is the balancing point for any other current, then  $(x - x'_0)$  is the amount to be subtracted from the maximum step to give the step for that current.

The riders were moved about by means of a long ebonite rod, so as to avoid bringing the hands near the bridge-wire.

Changes of resistance were examined for fields ranging from 500 c.g.s. units to 20,000 c.g.s. units with the pole-

pieces at about 7 mm. apart, but for fields reaching up to 30,000 c.g.s. units, the pole-pieces were about 2 mm. apart.

(3) *Results.*

The fractional change of resistance  $\Delta\phi$  in the specimen examined corresponding to the step  $\Delta x$  on the bridge-wire is calculated from the formula

$$\Delta\phi = \frac{\Delta N}{N} = \frac{R + S}{R \cdot S} \cdot \frac{Q}{N} \cdot \sigma \cdot \Delta x ;$$

where  $\sigma$  = resistance of bridge-wire per unit length ;

N = resistance of the specimen examined.

Q = total resistance in the bridge-gap containing the specimen. This includes the resistance of the compensating coil and the connecting wires and cables ;

R and S = resistances of the auxiliary coils.

Three experiments were carried out :—

Experiment I. with three pieces of nickel wire in series, diam. .0206 mm.\*

Experiment II. with two pieces of nickel wire in series, diam. .0159 mm.

Experiment III. with two pieces of iron wire in series, diam. .0208 mm.

TABLE I.—Particulars of coils, &c. used in the experiments.

	Expt. I. (18°·5 C.)	Expt. II. (17°·5 C.)	Expt. III. (17°·5 C.)
Resistance of specimen in ohms. ....	0·761	0·835	0·916
Resistance of auxiliary coil R .....	0·615	0·615	1·527
Resistance of auxiliary coil S .....	0·506	0·492	1·231
Resistance of compensating coil.....	0·324	0·443	0·444
Total resistance in gap E containing specimen .....	1·240	1·506	1·530
Resistance of 95·5 cm. of bridge wire .	0·0132	0·0132	0·0132
Maximum step in cm. $(x - x_0)$ .....	18·01	16·47	6·28

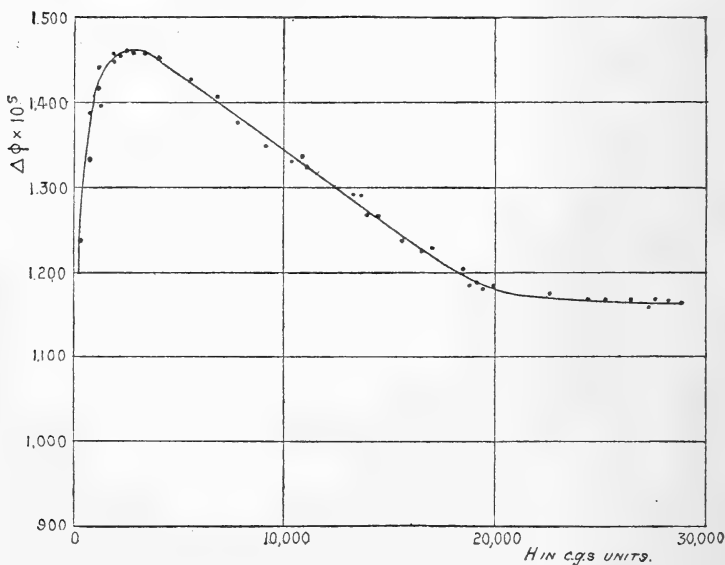
\* The nickel and iron wires were supplied by Messrs. Hartmann & Braun, Frankfurt.

The results obtained are given in figs. 4, 5, and 6. It is observed that there is a decided maximum change of resistance in each case (see Table II.), followed by a diminution in stronger fields\*.

TABLE II.—Maximum values of  $\Delta\phi$ .

	Maximum value of $\Delta\phi$ .	H in c.g.s. units.
Experiment I.—Nickel wire; diam. .0206 mm.	0.01462	2800
Experiment II.—Nickel wire; diam. .0159 mm.	0.01418	2800
Experiment III.—Iron wire; diam. .0208 mm.	0.00222	1900

Fig. 4.—Nickel (.0206 mm. diam.). Temperature = 18°·5 C.



\* It has recently been brought to my notice that a maximum value in the magnetic change of resistance of nickel was also observed by R. Dongier (*Beiblätter z. d. Ann. d. Physik*, Bd. 27, p. 677, 1903), but I have not seen any details of his measurements.

Fig. 5.—Nickel (.0159 mm. diam.). Temperature = 17°·5 C.

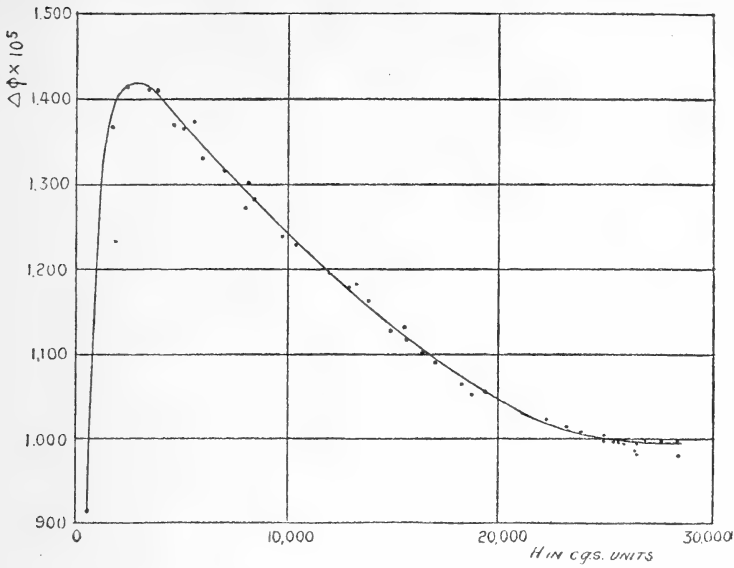
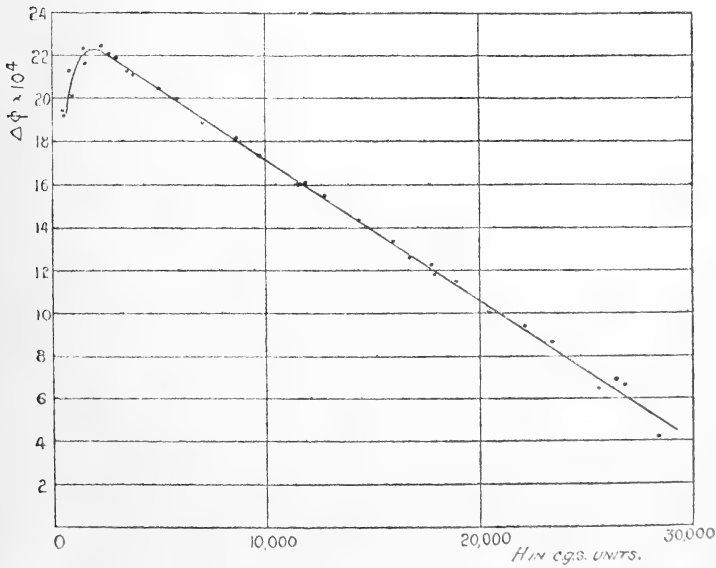


Fig. 6.—Iron (.0208 mm. diam.). Temperature = 17°·5 C.



In the case of nickel, the change of resistance shows a tendency towards a constant minimum which is reached in a field of about 24,000 c.g.s. units. For the first specimen the total decrease in the value of  $\Delta\phi$  after the maximum is reached is 0.0030, and for the second specimen 0.0042.

There is no tendency towards a constant minimum shown in iron in the fields examined, the fractional change of resistance steadily diminishing as the magnetic field increases.

Comparing these results with those of Barlow, we find that the diminution in the value of  $\Delta\phi$  for nickel between the field strengths 2000 c.g.s. and 18,000 c.g.s. units, is about one half that observed by him. It is possible, therefore, that the diminution of  $\Delta\phi$  observed by him in strong fields is not entirely to be accounted for by the end elements of his coil, which were magnetized transversely, but was made up of the decrease due to both longitudinal and transverse effects acting together.

According to the electron theory of metallic conduction as given by Drude\*, the change of resistance due to a longitudinal magnetic force is proportional to the square of the strength of the field in which the electrons are moving. In a magnetic metal this is presumably to be identified with the magnetic induction, and the change of resistance should therefore be proportional to the square of the magnetic induction. The results of the present experiments are thus not in accordance with this form of the electron theory.

All the above experiments were carried out in the Physical Laboratory of the University College of North Wales. In conclusion, I desire to express my thanks to Professor E. Taylor Jones for the interest he has taken in the work, and for much valuable help and advice.

XVII. *The  $\gamma$ -Rays of Thorium and Actinium.* By ALEXANDER S. RUSSELL, M.A., B.Sc., and FREDERICK SODDY, M.A., F.R.S.†

SIMILAR investigations to those described in two previous papers‡ on the  $\gamma$ -rays of uranium X and radium C have been carried out with the  $\gamma$ -rays of actinium C (or possibly it may prove to be actinium D) and with the two types of powerful  $\gamma$ -radiation in the thorium series, namely that given

\* Drude, *Ann. der Phys.* iii. p. 378 (1900).

† Communicated by the Authors.

‡ *Phil. Mag.* 1909 [6] xviii. p. 620; 1910, xix. p. 725. For brevity these two papers will be referred to throughout as I. and II. respectively.



by the short-lived  $\beta$ -ray product between mesothorium 1 and radiothorium, which we shall refer to as mesothorium 2 (Hahn, *Phys. Zeit.* 1908, ix. pp. 245 & 246), and that given by the last known product of the disintegration series, thorium D (Hahn and Meitner, *Phys. Zeit.* 1908, ix. p. 649). The paper conveniently divides itself into three sections. The first deals with the relative intensity of the  $\gamma$ - and  $\beta$ -rays of these substances. In the second section a number of so far unexplained effects in the measurement of the absorption coefficients of the  $\gamma$ -rays are described in detail. In the last section the penetrating power of the actinium and thorium types of  $\gamma$ -rays are compared with that of radium C. A brief indication of the general character of the results may conveniently precede their detailed consideration.

The two thorium products resemble radium C remarkably closely both in their  $\gamma/\beta$  ratio, and in the penetrating power of their  $\gamma$ -rays, and, although interesting differences exist, these are comparatively small. The most penetrating  $\gamma$ -ray known is that given by thorium D, that of mesothorium 2, speaking in a general sense, being about as much less penetrating than that of radium C as that of thorium D is more penetrating\*. These three bodies are sharply distinguished from all the other  $\beta$ - and  $\gamma$ -ray products by their high and similar  $\gamma/\beta$  ratio. At the other extreme are radium B and radium E, which we have not examined, the  $\gamma$ -rays from which are, either not at all, or only barely detectable†. It is perhaps still natural to leave radium B out of the comparison, as its  $\beta$ -rays are excessively feebly penetrating; but the established very low value of the  $\gamma$ -rays of radium E, taken in conjunction with the character of its  $\beta$ -rays, which, although feebly penetrating, are of the same order as those of actinium C and mesothorium 2, furnishes another example of the lack of connexion between the two types of rays. We have to consider actinium C and uranium X, each of which differs in this respect from all the other types. For uranium X, as we have previously found, both  $\beta$ - and

\* Apparently nothing has been previously published with reference to the  $\gamma$ -rays of mesothorium 2; but it should be mentioned that Eve (*Phys. Zeit.* 1907, viii. p. 185), in comparing the  $\gamma$ -rays of a preparation of radiothorium (thorium D) with those of radium, found that they were almost identical in penetrating power, although the measurements indicated that the radiothorium  $\gamma$ -rays were a little the more penetrating. The difference, however, Eve considered to be within the error of the experiments.

† H. W. Schmidt, *Phys. Zeit.* 1906, vii. p. 764; *Ann. Physik*, 1906 [4] xxi. p. 609. Meyer and von Schweidler, *Wien. Ber.* 1906, cxv. II a. p. 697. H. W. Schmidt, *Phys. Zeit.* 1907, viii. p. 361.

$\gamma$ -rays are similar in penetrating power to those of thorium and radium C, but the  $\gamma/\beta$  ratio is of an entirely different and lower order of magnitude. For actinium C, both  $\beta$ - and  $\gamma$ -rays are less penetrating, and the  $\gamma/\beta$  ratio, although of the same order, is distinctly greater than for uranium X. Indeed there is no rule about the matter. From the fact that hard  $\gamma$ -rays usually accompany hard  $\beta$ -rays, it might be supposed that thorium D, which gives the most penetrating  $\gamma$ -ray known, would have also the highest  $\gamma/\beta$  ratio. Whereas it is distinctly lower, both than that of radium C and that of mesothorium 2.

Owing to the importance of the  $\gamma$ -ray method as a standard means of comparison of radioactive substances, we have thought it advisable in the second section to collect a number of observations showing how the most curious changes of penetrating power of the  $\gamma$ -rays are brought about by the slightest change in the experimental disposition, although we have no general explanation of these effects to offer.

SECTION I.—*The Ratio of the  $\gamma$ - to the  $\beta$ -rays of Mesothorium 2, Thorium D, Actinium C, and Radium C.*

In a previous paper (I. p. 629) the ratio of the  $\gamma$ - to the  $\beta$ -rays of uranium X has been accurately compared under the same conditions with the same ratio for radium C. It was found, assuming the rays to be homogeneous, by measuring the  $\gamma$ -rays through 1 cm. of lead and calculating the initial intensities from the known absorption coefficients, that the  $\gamma/\beta$  ratio for uranium X was about 50 times smaller than that for radium C. When the  $\gamma$ -rays were measured through 0.6 cm. of aluminium, the ratio, uncorrected for absorption, was 18 times less. Similar measurements have now been extended to actinium and thorium. It may be pointed out that we are measuring a highly complicated effect in this  $\gamma/\beta$  ratio, and that before any absolute comparison is possible, it would be necessary to have a good deal more data than are at present available. We have been concerned only with the relative order of magnitude of the ratio sought, and have attempted to get at least a rough idea of this order by carrying out the  $\gamma$ - and  $\beta$ -ray measurements respectively for both substances under exactly the same conditions. Considerable differences will be found in the ratio according to the method of measurement employed, as is only to be expected; but the general order of the effects measured is sufficiently correctly indicated by the measurements here given.

For measurement of  $\beta$ -rays a cylindrical brass electroscopie of the ordinary type, 13 cm. high and 10.8 cm. diameter, was used. The thickness of the walls was 0.32 cm. The base consisted of a layer of aluminium foil, 0.095 mm. thick. In the first series of measurements the preparation was placed at a distance of 60 cm. below the base, but later various shorter distances were also used. The apparatus was set up so as to reduce secondary  $\beta$ -radiation to a minimum. The electroscopie was flush with the table, which had a large hole cut in it, and was supported by light steel brackets from the wall. The preparation was supported centrally below the electroscopie on a light framework of brass rods, made in sections screwed together, and attached to the under side of the table, so that any distance between the preparation and base of the electroscopie could be employed. For measurements of  $\gamma$ -rays three different dispositions were used, as follows:—

( $\gamma_1$ ) The active preparation was placed at a distance of 8.6 cm. below an electroscopie of lead of the usual type having wall and base thickness 0.3 cm. of lead;

( $\gamma_2$ ) at a distance of 3.3 cm. below a lead electroscopie whose wall was 0.65 cm. and whose base 0.975 cm. thick.

( $\gamma_3$ ) At a distance of 8.7 cm. below the same electroscopie as in  $\gamma_2$ .

The radium C was prepared by exposing one side of a negatively charged brass disk to a quantity of radium emanation for 20 hours. After the exposure the disk was placed in a brass cell and a piece of 0.095 mm. thick aluminium foil was cemented over the cell to prevent any possible escape of adhering emanation. One hour after the preparation of the film of radium C, alternative measurements of the  $\beta$ - and  $\gamma$ -rays, in the four different dispositions detailed above, were taken over a period of three hours, and from the four decay curves obtained from these measurements the  $\beta$ - and the three different  $\gamma$ -activities could be readily deduced. The four decay curves obtained were found to be very nearly exponential with values of  $\lambda$  very approximately the same, ( $1.39$  (hour) $^{-1}$ ). The relative values therefore of the intensity of the preparations measured under the four different dispositions could be accurately obtained.

For thorium D the procedure of preparation of the disk was exactly the same, the disk being exposed negatively charged to the thorium emanation from a preparation of radiothorium spread out on a shallow dish for 20 hours under conditions, such that only the one side of the disk was coated. The disk was then mounted and covered with the same

thickness of aluminium as the radium C disk. Fifteen minutes after the disk had been removed from the emanation the first measurements were taken. Experiments in each of the four different dispositions were made with thorium D, just as they had been made with radium C, and the relative intensities of the rays at each disposition accurately determined. The mesothorium 2 used in the experiments was separated chemically from a preparation of mesothorium 1 previously prepared from several kilograms of thorianite. The precipitate, which weighed only a few milligrams, was evaporated down on a small watch-glass of diameter exactly the same as that of the disks used for the other two active bodies. The quantity of matter was so small that self-absorption can be neglected. The watch-glass was covered with a piece of aluminium foil, 0.095 mm. thick, and several measurements were made for each disposition over a period of  $3\frac{1}{2}$  hours. The four decay curves were exponential ( $\lambda = 0.12$  (hour) $^{-1}$ ). The residual activity remaining two days later was negligible. In the following table the actual results found for each disposition for each source are given :—

	$\gamma_1/\beta$ .	$\gamma_2/\beta$ .	$\gamma_3/\beta$ .
Radium C .....	1.081	1.849	0.711
Mesothorium 2...	1.222	2.039	0.813
Thorium D .....	0.795	1.420	0.525

Or, taking the radium values as the standard :—

	$\gamma_1/\beta$ .	$\gamma_2/\beta$ .	$\gamma_3/\beta$ .
Radium C .....	1.00	1.00	1.00
Mesothorium 2...	1.13	1.13	1.14
Thorium D .....	0.736	0.768	0.738

For these dispositions therefore mesothorium 2 gave 13 per cent. more  $\gamma$ -rays per  $\beta$ -ray than radium C, and thorium D about 25 per cent. less. Also, the variation of the ratios thorium D to radium C, or mesothorium 2 to radium C, with the thickness of the lead base is slight, if any. This is to be expected because, as will be shown in

another part of the paper, the penetrating powers of all three types of  $\gamma$ -radiation are very similar.

A second series of experiments, which embraced in addition the actinium active deposit, was started in order to investigate various influences which are likely to affect the values of the  $\gamma/\beta$  ratio. These are the difference in penetrability of the hard  $\beta$ -rays of each element and the effect of scattering of such rays by the air between the preparation and the electroscopes. Some metal must necessarily intervene between the preparation and the electroscopes, but the amount used was the minimum, so that the absorption of  $\beta$ -rays by it was small. When the ratio of  $\gamma$ - and  $\beta$ -rays for uranium X was measured in a former experiment it had been necessary to place the preparation at a great distance from the electroscopes in order not to give too large a  $\beta$ -ray effect. This was not essential in the case of active deposits having a quick rate of decay, so that the values of the  $\beta$ -activity of the preparations, placed at five different distances from the electroscopes, has been measured in order to see how the distance affects the values of the  $\gamma/\beta$  ratio of thorium D, mesothorium 2 and actinium active deposit relatively to that of radium C. The distances used were 51.1 cm., 35.9 cm., 20.6 cm., 13.0 cm., and 5.37 cm. These positions are denoted as  $\beta_1, \beta_2, \beta_3, \beta_4,$  and  $\beta_5$  respectively. The disposition  $\gamma_1$ , used before, was used again. The mesothorium 2, separated chemically as before, was this time mounted on a brass disk, exactly the same as those on which the active deposits of the other bodies were collected, placed in a brass cell and covered with the usual aluminium foil. The substitution of the brass for the glass had only a small effect on the value of the  $\beta$ -rays. The active deposits were prepared and mounted as before, the actinium active deposit being obtained in the same manner as that of thorium. The very fine actinium preparation, lent by Messrs. Buchler & Co., and described in Section III., was employed. Measurements were started in the  $\gamma_1$  and  $\beta_1$  positions, and, when the preparation had become too weak to be easily measurable in the latter, it was placed in the  $\beta_2$  position, and so on till the  $\beta_5$  position. The residual activity was also measured in the  $\beta_5$  position; but since the  $\beta$ -rays of radium C and mesothorium 2 reach a minimum and then again increase, owing to the formation of later  $\beta$ -ray products, the correction for the residual activity cannot be directly applied, but must be extrapolated. From the decay curves of all the bodies experimented with in the five different dispositions, corrected, where necessary, for residual activity, the  $\gamma/\beta$  ratios in the

different positions are readily obtainable. The final results are given in the following table, in terms of the ratio for radium C which is taken in each position as unity.

<i>Positions :</i>					
	$\beta_1$ .	$\beta_2$ .	$\beta_3$ .	$\beta_4$ .	$\beta_5$ .
$\gamma_1/\beta$ Mesothorium 2 ...	1.036	0.935	0.676	0.800	0.524
$\gamma_1/\beta$ Thorium D .....	0.690	0.637	0.568	0.510	0.336

For actinium the  $\gamma$ -rays were measured also through 0.95 cm. zinc ( $\gamma_4$ ) as well as through 0.3 cm. lead ( $\gamma_1$ ). This was done because, as was shown previously by Godlewski and confirmed later in this paper, the  $\gamma$ -rays of actinium are abnormally highly absorbed by lead. Radium C, the ratio for which is again taken throughout as unity, was also measured under the same conditions for comparison. The final results are:—

<i>Positions :</i>					
	$\beta_1$ .	$\beta_2$ .	$\beta_3$ .	$\beta_4$ .	$\beta_5$ .
$\gamma_1/\beta$ Actinium C.....	0.077	0.065	0.072	0.065	0.067
$\gamma_4/\beta$ Actinium C.....	0.128	0.103	0.116	0.105	0.109

It will be seen that the general effect of decreasing the distance of the preparation in the  $\beta$ -ray measurements is to decrease the  $\gamma/\beta$  ratio, to about one half, over the range of distance examined, for mesothorium 2 and thorium D; but for actinium C the difference is less marked. No doubt the causes of this are very complex. In the first place it is to be expected that the scattering of the  $\beta$ -rays by the air between the preparation and the electroscopie will be the greater the less penetrating the  $\beta$ -rays, which, in descending order of penetrating power, are—radium C, thorium D, mesothorium 2, actinium C. Then, with diminishing distance and greater angle of the cone of rays entering the electroscopie, the effect of "reflexion" of the rays from the inner sides

of the electroscope will come in. According to the experiments of Kovarik and Wilson (Phil. Mag. 1910, xx. p. 849 & p. 866), this reflexion increases with the penetrating power of the rays up to a maximum, at  $\lambda(\text{cm.})^{-1}$  aluminium  $\approx 20$ , and then again diminishes. This effect for most of the rays therefore would oppose that of scattering. An attempt to evaluate the absorption in the thin aluminium foil covering the preparation led to the unexpected result that the ionization was slightly greater with the foil than without. This was at the time ascribed to a possible generation in the foil of secondary  $\beta$ -rays, an effect which had frequently been looked for previously but never actually obtained. The same effect is recorded by Kovarik (*loc. cit.*), who made a closer examination of it than we have done, and ascribed it to an effect of scattering. None of the effects considered affect the order of the  $\gamma/\beta$  ratio to a serious extent, and they are relatively unimportant. Probably from the theoretical point of view the original ratios obtained with the preparation at a great distance from the electroscope are to be preferred. For practical purposes the  $\beta_4$  position may be selected, as here the distance, 13 cm., is a very usual one. The ratios, relatively to that of radium C, are for these positions:—

	$\gamma_1/\beta_1$	$\gamma_1/\beta_4$	$\gamma_4/\beta_1$	$\gamma_4/\beta_4$
Mesothorium 2...	1.036	0.800		
Thorium D .....	0.690	0.510		
Actinium C .....	0.077	0.065	0.128	0.105

The  $\gamma/\beta$  ratios are thus of the same order for radium C, mesothorium 2, and thorium D; while for actinium C it is only about one-eighth to one-sixteenth of radium C and therefore more nearly approaches the order previously found for uranium X.

*Experiments with Thorium Minerals.*—Before any mesothorium or radiothorium preparations were ready, Mrs. Soddy had carried out a comparison of the  $\gamma/\beta$  ratio for Ceylon thorite (containing about 66 per. cent. of thoria) and Joachimsthal pitchblende (containing about 40 per. cent. of uranium oxide). The former mineral contains practically no uranium, and the latter no thorium, so that they represent respectively the thorium and uranium disintegration series in equilibrium. The result may be briefly referred to here. The  $\gamma/\beta$  ratio of these two minerals was practically the same

Bearing in mind the fact that the uranium mineral contains two products, uranium X and radium E, which contribute  $\beta$ -rays with little or no  $\gamma$ -radiation, this result is in agreement with what is to be expected from the determinations just described of the ratio for the three single products, radium C, mesothorium 2, and thorium D. The thickness of the lead through which the  $\gamma$ -rays were measured also made no practical difference in the ratio. Eve has already shown that the  $\gamma$ -rays from thorium nitrate and uraninite have practically the same penetrating power. The similarity of penetrating power of the  $\gamma$ -rays from thorium minerals is to be expected from the result, detailed in Section III., that the radium C  $\gamma$ -rays are intermediate in penetrating power between the two types of  $\gamma$ -rays given by the thorium series.

There remains one practically important question with reference to the  $\gamma$ -rays of the thorium series. What is the relative intensity of the radiation contributed by the two products mesothorium 2 and thorium D? Mr. Alexander Fleck has done some experiments with Ceylon thorite with a view to obtaining information on this point, and although, so far, only preliminary results are available, these may be given here. The thorite was dissolved in hydrochloric acid, precipitated with excess of ammonia, filtered, and the precipitated hydrates subjected to the same treatment five times, in all, without unnecessary lapse of time. The filtrates were collected, evaporated and ignited together, and sealed up in a box with a thin aluminium lid for  $\beta$ -ray measurement. The filtrate from a sixth precipitation, carried out immediately after the fifth, treated separately in the same way, was inactive, showing that by this treatment the whole of the mesothorium and thorium X had been separated. The  $\beta$ -ray decay curve of the preparation was taken for some weeks. The intensity of the rays rose to a maximum after 2 days (due to thorium D) and then fell, until a constant minimum, due to mesothorium only, was attained. The decay curve for the maximum onward was extrapolated back to an origin corresponding to the time of the last precipitation with ammonia, and from the initial and final value of the intensity of the  $\beta$ -rays, the proportion due to each product could be obtained. The measurements indicate that the thorium D contributes distinctly more  $\beta$ -radiation than the mesothorium 2. The difference is not great, and may be estimated provisionally as from 25 to 50 per cent. Since thorium D is richer in  $\beta$ -rays, relatively to the  $\gamma$ -rays, than mesothorium 2, it follows that the  $\gamma$ -radiation from the two types must be very similar in intensity. This result, although only approximate, will prove



useful in calculating the rise in intensity of the  $\gamma$ -radiation of a mesothorium preparation with time, due to the generation of radiothorium and thorium D, provided that the fraction of the radiation due to radium is known.

SECTION II.—*Variations in the Values of the Absorption Coefficients of  $\gamma$ -rays* \*.

*Effect of distance of the preparation from the electroscope.*—In all this work a lead electroscope with detachable base, similar to that previously described (II. p. 752, fig. 17), has been used. It has been shown to exclude external secondary radiations effectually. Moreover it gives of all metals easily the maximum ionization for a given intensity of  $\gamma$ -radiation (I. p. 642), which Bragg has since explained on the view that the  $\beta$ -ray, produced by the  $\gamma$ -ray, possesses really a greater penetrating power in lead than in equal weights of other metals, though its trajectory is more entangled.

As a result of experiments on the effect of the distance of the preparation from the electroscope, it was found that at a distance above 14 cm. the values of  $\lambda$  became constant, the beam being now practically parallel (II. p. 736). In most experiments the distance employed was therefore 14 cm. The following table gives the value of  $\lambda$  at various distances for radium  $\gamma$ -rays, a thick lead base being used with the electroscope, the absorbing lead being laid directly on the preparation.

Distance (cm.)...	5.0	6.4	8.5	13	14	25	115
$\lambda$ (cm.) <sup>-1</sup> .....	0.574	0.553	0.517	0.500	0.500	0.498	0.498

Similarly, for zinc, the value of  $\lambda$  at 13 cm. was 0.278, and at 115 cm. 0.274, which agree within the experimental error.

*The "Hardening" of  $\gamma$ -rays by passage through Lead.*—Some further experiments were done on this effect. By passage through lead the value of  $\lambda$  for lighter metals is reduced, though other bodies possess the same power as lead in lesser degree. In no case have we observed any "softening." Hardening is more pronounced when the rays pass first through the lead and then through the lighter metal, than *vice versa*. Thus, with a brass base to the electroscope,

\* This section may be regarded as a continuation of Section III. of the previous paper (II, p. 744).

the values of  $\lambda$  for zinc were (1) 0.23, (2) 0.21 when 1.24 cm. of lead were placed (1) between the zinc and the electroscope, (2) between the zinc and the radium. Similarly the values for iron were (1) 0.304, (2) 0.280, 1 cm. thickness of lead being used. These results indicate at once that the values found for  $\lambda$  are to some extent influenced by the nature and thickness of the base employed. Thus the results given, for example, in Table II. (I. p. 644), in which the mean value of  $\lambda/d$  for Class II. bodies ( $\gamma$ -rays of radium) was 0.040, refer to a lead base (0.975 cm.). The mean values for other bases were as follows:—

Base .....	Lead 0.98 cm.	Lead 2.86 cm.	Brass 0.6 cm.	Brass 1.73 cm.
Mean $\lambda/d \times 100$ .....	3.99	3.81	3.92	3.82

An experiment was conducted with zinc to see to what degree the process of hardening could be pushed. The disposition used in this case was different to that of the last, the radium being placed 25 cm. below the usual lead electroscope and the absorbing zinc clamped up to form the base. Seven thicknesses of lead were placed in turn over the preparation and the absorption by the zinc for each thickness of lead measured. The results are given in the following table\*.

Thickness of Lead ...	0	0.124	0.249	0.373	0.64	1.24	2.50	3.75
$\lambda_{2n}(\text{cm.})^{-1}$ .....	0.325	0.322	0.309	0.300	0.291	0.272	0.265	0.258

Thus the final value is 0.79 of the initial, and there does not seem to be a limit to the degree to which the hardening process may be carried. The remarkable point is that the curves are in every case, as nearly as can be seen, exponential with the new value of  $\lambda$ , after from 1 to 0.35 cm. equivalent thickness of zinc has been penetrated, according to the amount of lead covering the preparation. For lead the value of  $\lambda$  is, as we have shown, independent of thickness up to 22 cm. (II. p. 752). Provided there is a base of lead 1 cm. thick

\* Incidentally it may be noticed that the value of  $\lambda$  for the base preparation (0.325) is 7 per cent. greater than the value found in the standard series of measurements given later (Table B, p. 148). The cause of this difference is not yet clear, but may be due to the fact that in this experiment the preparation rested in a groove in a lead disk, which somewhat confined the beam of  $\gamma$ -rays.

and 0.5 cm. of lead over the preparation, the absorbing plates of lead may be placed *anywhere* between the preparation and the base, the value of  $\lambda$  is constant at 0.50 (cm.)<sup>-1</sup>. These two facts are extraordinarily difficult to reconcile. The first indicates that each successive thickness of lead traversed modifies the nature of the radiation remaining, continuously without limit, while the second indicates that the effect of each successive thickness is to absorb a definite fraction of the radiation without modification of the nature of the remainder.

This difficulty suggested the possibility that a heterogeneous beam, consisting of two types of about equal energy but different penetrating power, might be transformed by a layer of the absorbent so as to act subsequently as a homogeneous beam. A heterogeneous beam of  $\gamma$ -rays was made by placing two point sources, one of radium and the other of mesothorium, side by side. The activity of the mesothorium was 40 per cent. of the total, measured through 1 cm. of lead. Lead laid over the preparations was used as absorbent.  $\lambda$  for lead for radium for this disposition is 0.50, and for mesothorium 0.62. The differences are thus of the same order as that produced in the value of  $\lambda$  (zinc) by hardening the  $\gamma$ -rays of radium with lead. The absorption curve for the heterogeneous beam, over the 5 cm. of lead investigated, was not exponential and by no allowance for experimental errors could it be made so, the curve being plainly convex to the origin\*. The theoretical values of the ionization of the beam for the different thicknesses of lead were calculated from the data known, and they agreed perfectly with the observed values. The only way of escape from the difficulty seems to be to deny that an exponential absorption curve necessarily means the stopping by the metal of a definite proportion of the incident radiation without modification of the fraction emerging. A similar state of things exists with regard to the  $\beta$ -rays. It has been fairly conclusively shown that an exponential absorption occurs with a heterogeneous beam, and that the part unabsorbed suffers reduction in velocity and penetrating power. But a true exponential curve for the  $\beta$ -rays has never been obtained by anyone over any considerable range, whereas with the  $\gamma$ -rays under proper conditions the absorption curve never departs measurably from the exponential form.

*Initial Part of the Absorption Curves.*—The deviations

\* This experiment suggests a possible means of detecting the adulteration of radium by mesothorium without opening the tube.

from the exponential curve over the first part of the range (0 to 1 cm. thickness of lead or its equivalent) now call for remark. If we place a source of radium at a distance of 13 cm. below a lead electroscope and clamp up as the base varying thicknesses of absorbing material, we invariably get a similar result. Between a thickness sufficient to absorb all  $\beta$ -rays and a thickness equivalent to 1 cm. of lead the absorption curve is convex to the origin, *i. e.* the value of  $\lambda$  decreases continuously. From about 1 cm. onwards  $\lambda$  is constant. Experiments under identical conditions were carried out with lead, tin, zinc, and aluminium as absorbers. The absorption curves for each consist of two parts, an upper steep curved part and a lower straight part. The greater the density of the body the steeper is this upper part and the longer it takes to join the straight part of the range. If now 1.3 cm. of lead be placed on the radium and the results repeated, the general character of the curves is the same as before. The slopes of the lower part are less, since the rays are hardened by the lead, and the upper part joins with the lower part at a smaller equivalent thickness (0.5 instead of 1.0), thus smoothing out somewhat, but not eliminating, the convexity of the curve. With lead as absorber the placing of 1 cm. of lead over the radium has little effect, the convexity over the first centimetre being only slightly diminished. This shows that the effect has nothing to do with a soft type of  $\gamma$ -rays initially present with the hard type, as has previously been supposed. If we now carry out similar experiments with another disposition (permanent base of lead or brass, absorbing material laid directly on the radium) the results are very different. For lead the upper part of the curve is again convex, whether the base be brass or lead, the convexity being the greater for a brass base: with zinc and aluminium, the upper part is concave, the concavity being about the same with either base. Typical results are given below. The ranges are expressed in actual thicknesses of lead and zinc.

1. *Lead.* Brass base 1.2 cm. thick.

Range (cm.) ...	0 to 0.25	0.25 to 1.27	1.27 to 9.9
$\lambda$ (cm.) <sup>-1</sup> ...	0.875	0.612	0.502

2. *Zinc.* Brass base 1.73 cm. thick.

Range (cm.), ...	0 to 0.66	0.66 to 1.62	1.62 to 6.00
$\lambda$ (cm.) <sup>-1</sup> ...	0.226	0.233	0.255

3. *Lead.* Lead base 1 cm. thick.

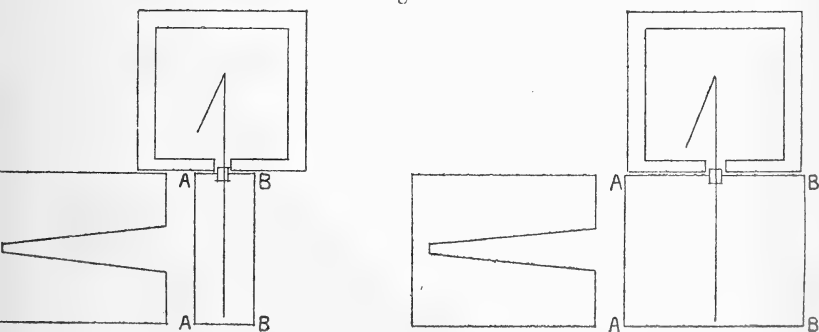
Range (cm.), ...	0 to 0.25	0.25 to 0.62	0.62 to 6.0
$\lambda$ (cm.) <sup>-1</sup> ...	0.608	0.545	0.500

4. *Zinc.* Lead base 1 cm. thick.

Range (cm.), ...	0 to 0.34	0.34 to 1.0	1.0 to 6.0
$\lambda$ (cm.) <sup>-1</sup> ...	0.238	0.251	0.270

But it is possible to find a disposition in which zinc absorbs the  $\gamma$ -rays strictly according to an exponential law with the normal value of  $\lambda$ , from thicknesses sufficient to absorb all the  $\beta$ -rays up to the greatest thickness it was possible to use with the disposition. This was briefly referred to before (II. p. 754) and may now be more fully described. A preparation of radium (7 mg.) was placed at the apex of a cone of length 11.5 cm. and base 3 cm. diameter cut out of a cylinder

Fig. 1.



of lead (length 12.7, diameter 10.5). 2 cm. from the mouth of the cone was placed a short cylindrical ionization chamber of lead, 9.2 cm. inside diameter and 4.2 cm. long, an electrode in

the centre of which communicated with a leaf system contained in a lead electroscop. Fig. 1 (p. 143) shows the disposition. Absorbing screens could be clamped tightly against the ionization chamber in position A, A. BB was a thick lead plate. Experiments were carried out with lead, tin, zinc, and aluminium, first with the short cylindrical ionization chamber and secondly with one three times as long but of the same diameter. This was done to find out if the shape of the ionization chamber had an effect on the values of the coefficients of absorption. Results are given below.

## A. Single Ionization Chamber.

	$\lambda$ .	$100 \lambda/d$ .	Range (eq.cm. of Pb).
Pb.	0.850 to 0.728	7.46 to 6.38	0.4 to 1.7
Sn .	0.355	4.90	0.34 onward
Zn .	0.267	3.77	0.21 onward
Al .	0.080	2.83	0.16 onward

## B. Triple Ionization Chamber.

	$\lambda$ .	$100 \lambda/d$ .	Range (eq.cm. of Pb).
Pb.	0.850 to 0.61	7.46 to 5.33	0.3 to 1.8
Sn .	0.354	4.88	0.34 onward
Zn .	0.264	3.74	0.21 onward
Al .	0.083	2.93	0.20 onward

Thus  $\lambda/d$  increases with  $d$  for this disposition. All the curves are exponential, except that for lead which is convex. The length of the ionization chamber made no practical difference. These dispositions are analogous to those used in Table V. (II. p. 753). For these the rays were confined in the same cone, which however entered directly through the base of an electroscop. The distance between the base of the electroscop and the mouth of the cone was 3.5 cm. as compared with 2.0 for the present dispositions. The results, however, were quite different from those detailed above. Thus zinc had for  $\lambda/d \times 100$ , 3.0 to 4.2 over a range of 0.35 to 2.6, the absorption curve being concave.

These results show how meaningless a value of  $\lambda$  for an absorbing substance is unless a full account of the experimental disposition under which the measurements are made is given.

SECTION III.—*The relative penetrating powers of the  $\gamma$ -rays of the Radio-elements.*

It is convenient to state at the outset that, as a result of the experiments to be described, the relative penetrating power of the  $\gamma$ -rays is, in descending order, thorium D, radium C, mesothorium 2, uranium X, and actinium C. Thorium D thus gives the most penetrating  $\gamma$ -rays known, though the differences between the first four are not great. But although it is easy to arrange the various types of  $\gamma$ -rays in order of their penetrating power it is a more difficult matter to assign accurate values for  $\lambda$  to each as their values as we have shown (II. p. 754–755) depend greatly upon the conditions, and their ratio for different rays is by no means constant when the results for several dispositions are compared. Thus the values of  $\lambda$  for the uranium  $\gamma$ -rays in four combinations, of two different metals with two different dispositions, were respectively 46, 28, 25, and 18 per cent. higher than for those of radium, and similar considerations hold good equally when the thorium and radium  $\gamma$ -rays are compared.

The radioactive preparations used in the following measurements were:—

1. *Radium*: 7 mg. of radium bromide, and in some experiments 0.5 mg. of radium bromide.
2. *Mesothorium*: a single tiny grain of concentrated mesothorium obtained from Knöfler and Co., equivalent in  $\gamma$ -rays to about 0.31 mg. radium bromide measured through 3 mm. of lead.

All these preparations are practically point sources in sealed glass tubes.

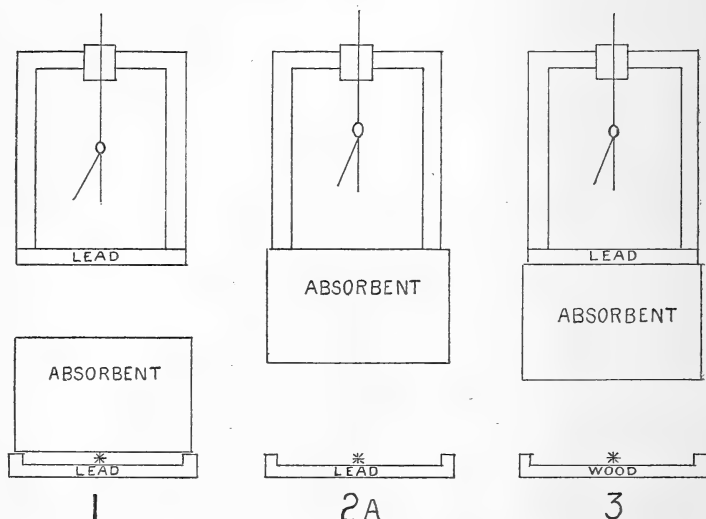
3. *Radiothorium*: this body mixed with moist thorium hydrate was contained in a sealed cylindrical tube about 6 cms. long and about 0.5 cm. diameter. It therefore differed from the others in not being a point source. It was equivalent in  $\gamma$ -rays to about 0.56 mg. radium bromide measured through 0.3 cm. lead. It also was obtained from Knöfler and Co.

*Dispositions.*—Sketches of the three dispositions employed are shown in fig. 2. They hardly need further description. Disposition 1 corresponds to that used in obtaining Tables II. and III. (I. pp. 644 & 646). It is easiest to use in practice

*Phil. Mag. S. 6. Vol. 21. No. 121. Jan. 1911.* L

as the electroscope is not disturbed. Disposition 2 is similar to that used in obtaining Table I. (I, p. 633) except that the electroscope is of lead as in Disposition 1. It offers certain theoretical advantages, but is more difficult to work with.

Fig. 2.



The only reason for using Disposition 3 was that uranium X had once been examined with it. It was not possible to prepare uranium X for the present set of measurements, but many of the previous results can be utilized. Those of Tables II. and III. (I, pp. 644 & 646) only differ from the present Disposition 2 in that a wood stand instead of a lead stand was used for the preparation, but parallel experiments with mesothorium 2 showed that this had no effect on the values of  $\lambda$ . The results with Disposition 1 are tabulated in Table A.

After a thickness equivalent to 1 cm. of lead (total thickness, with base, 2 cm. of lead) all the curves are exponential. Lead is in a class by itself (Class I.) with high value for  $\lambda/d$  throughout. The Class II. bodies have an approximately constant  $\lambda/d$ , but the limit of density down to which this holds is greater for the more penetrating rays than for those less penetrating. The lightest bodies (Class III.) have again higher values for  $\lambda/d$ .



TABLE A.—Disposition 1.

	Thorium D.		Radium C.		Mesothorium 2.		Uranium X.	
	$\lambda$ .	$100\lambda/d$ .	$\lambda$ .	$100\lambda/d$ .	$\lambda$ .	$100\lambda/d$ .	$\lambda$ .	$100\lambda/d$ .
Lead .....	0.462	4.05	0.500	4.38	0.620	5.44	0.725	6.36
Copper .....	0.294	3.34	Mean value 3.99		0.373	4.23	Mean value 4.70	
Brass .....	0.271	3.25			0.355	4.25		
Iron .....	0.250	3.28			0.316	4.15		
Tin .....	0.236	3.26			0.305	4.21		
Zinc .....	0.233	3.30			0.300	4.24		
Slate .....	0.0961	3.37			—	—		
Aluminium .....	0.0916	3.24			0.119	4.21		
Glass .....	0.0886	3.52	0.113	4.48	0.122	4.84		
Magnesia Brick...	0.062	3.23	0.090	4.69	0.0917	4.78		
Sulphur .....	0.066	3.69	0.078	4.38	0.083	4.65	0.0921	5.16
Paraffin Wax.....	0.031	3.61	0.040	4.64	0.050	5.80	0.0433	5.02

The nature of the absorbent has a marked effect on the relative penetrating power of the four types of rays. Taking the values for radium C as unity, this is shown in the following table, mean values for Class II. being employed.

	Thorium D.	Radium C.	Mesothorium 2.	Uranium X.
Lead .....	0.924	1.00	1.24	1.45
Class II.....	0.82	1.00	1.06	1.18

In Disposition 2, with the absorbing plates clamped up to form the base of the electroscopes, two sets of experiments were performed, namely (2 A) with the preparation bare, and (2 B) with the preparation covered with 0.64 cm. lead, to see if the hardened rays produced would be exponentially absorbed by *all* bodies according to the density law. The result shows that this is the case except for lead. Table B (p. 148) shows the results obtained.

All absorption curves for this disposition are also exponential after 1 cm. of equivalent thickness of lead has been traversed. The curves made by plotting equivalent thickness of metal against the logarithms of the ionizations are coincident. So they are also in the case of Disposition I. This holds both for Dispositions 2 A and 2 B. This is opposite to what was previously obtained with a brass electroscope for the absorption by bodies over small initial

TABLE B.—Dispositions 2 A and 2 B.

	RADIUM C.			THORIUM D.			MESOTHORIUM 2.				
	2 A.		2 B.	2 A.		2 B.	2 A.		2 B.		
	$\lambda$ .	100 $\lambda/d$ .	$\lambda$ .	$\lambda$ .	100 $\lambda/d$ .	$\lambda$ .	100 $\lambda/d$ .	$\lambda$ .	100 $\lambda/d$ .		
Lead .....	0.517	4.535	0.500	0.409	3.588	0.408	3.579	0.642	5.632	0.633	5.553
Copper .....	0.377	4.279	0.336	0.327	3.71	0.307	3.49	0.387	4.40	0.358	4.06
Brass .....	0.359	4.299	0.309	—	—	—	—	—	—	—	—
Iron .....	0.322	4.226	0.285	0.285	3.76	0.266	3.49	0.334	4.38	0.289	3.80
Tin.....	0.310	4.280	0.287	0.277	3.82	0.252	3.48	0.315	4.35	0.298	4.10
Zinc .....	0.305	4.314	0.264	0.268	3.79	0.246	3.48	0.315	4.46	0.260	3.68
Aluminium .....	0.124	4.380	0.108	0.106	3.75	0.0971	3.43	0.126	4.45	0.109	3.85
Sulphur.....	0.0903	5.059	0.0663	—	—	—	—	—	—	—	—
Paraffin Wax .....	0.0465	5.152	0.0332	0.0484	5.62	0.0307	3.56	—	—	0.0355	4.11

ranges of thickness of the rays of uranium X (II. p. 729). There the bodies arrange themselves one above the other in order of the density of the material, the lighter substances giving for equivalent thicknesses far more ionization than the denser. The mean values of  $100 \times \lambda/d$  are tabulated below.

	Thorium D.	Radium C.	Mesothorium 2.
Lead, 2 A .....	3.59	4.54	5.63
Lead, 2 B .....	3.58	4.38	5.55
Class II. 2 A...	3.76	4.28	4.41
Class II. 2 B...	3.48	3.78	3.95

It is seen that the values for lead are very near one another but the differences are quite real. For radium the value of  $\lambda=0.517$  has been obtained at least six times for Disposition 2 A. Taking the values for radium as unity the results are summarized in the following table:—

	Thorium D.	Radium C.	Mesothorium 2.	Uranium X.	
Lead... {	2 A	0.79	1.00	1.24	1.27
	2 B	0.82	1.00	1.27	—
Class II. {	2 A	0.88	1.00	1.03	—
	2 B	0.92	1.00	1.045	—

In Disposition 2 A, the values of  $\lambda$  are all about 5 per cent. greater than in Disposition 1. In Disposition 2 B lead still absorbs normally, but both Class II. and Class III. bodies now obey the density law.

Disposition 3 was only used for copper. The following values of  $\lambda$  were obtained:—

	Thorium D.	Radium C.	Mesothorium 2.	Uranium X.
$\lambda$ Copper .....	0.277	0.345	0.357	0.432

or relatively

0.80	1.00	1.035	1.26
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The results are collected in the following table:—

		Thorium D.	Radium C.	Mesothorium 2.	Uranium X.
1.	Lead 1 .....	0.92	1.00	1.24	1.45
2.	Class II. 1.....	0.82	1.00	1.06	1.18
3.	Lead 2 A .....	0.79	1.00	1.24	1.27
4.	Lead 2 B .....	0.82	1.00	1.27	—
5.	Class II. 2 A...	0.88	1.00	1.03	[1.25]*
6.	Class II. 2 B...	0.92	1.00	1.045	—
7.	Copper 3 .....	0.80	1.00	1.035	1.26

\* Not exactly the same disposition.

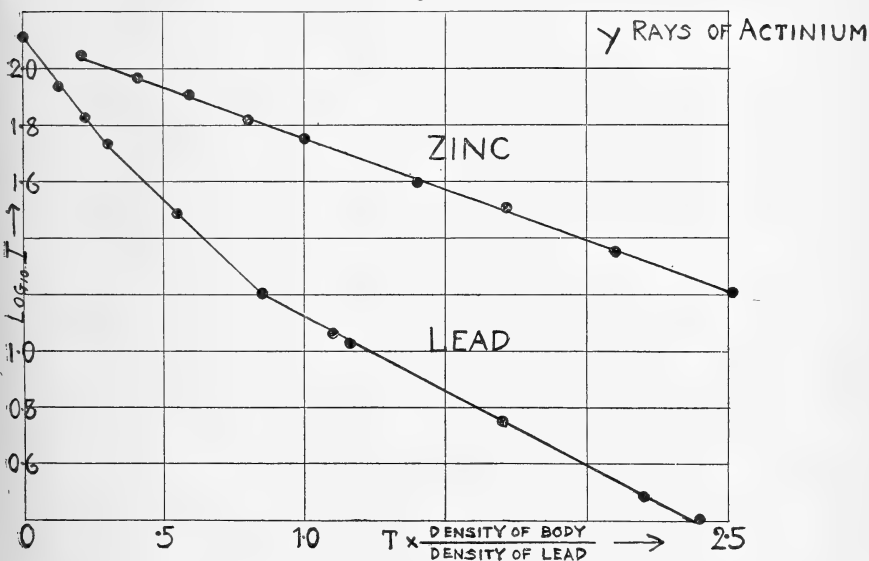
Thus the value of  $\lambda$  for thorium D is from 8 to 21 per cent. less, and for mesothorium 2 from 4 to 25 per cent. greater than for radium C. Disposition 1 might well be adopted as a standard disposition for  $\gamma$ -ray comparison of the intensity of radioactive preparations. Preparations of widely differing activity might be compared by means of carefully prepared lead blocks of known thickness, adopting, for the value of  $\lambda$ ,  $0.500 \text{ (cm.)}^{-1}$ .

*The  $\gamma$ -rays of Actinium.*—Through the kindness of Messrs. Buchler and Co., of Braunschweig, who most generously lent for the purpose a very fine preparation of actinium, weighing 1.5 grams and equivalent in  $\gamma$ -activity through 3 mm. of lead to 0.23 mg. of radium bromide, it has been possible to make a more extended examination of the  $\gamma$ -rays of this substance than has hitherto been done. The preparation was in a sealed tube for these measurements, and subsequently the tube was opened and the preparation used to obtain the active deposit for the determination of the  $\gamma/\beta$  ratio described in Section I. Godlewski (Phil. Mag. [6] x. p. 378, 1905) measured the absorption of the actinium  $\gamma$ -rays over the range from 0 to 3.5 mm. of lead and 0 to 10 mm. of iron and of zinc, and obtained nearly exponential curves, the value of  $\lambda \text{ (cm.)}^{-1}$  for lead being 4.54, for iron 1.23, and for zinc 1.24. He thus showed clearly the markedly less penetrating power of these rays as compared with those of radium and the abnormally high absorption in lead. Eve (*Phys. Zeit.* viii. p. 185, 1907) with a stronger preparation found for  $\lambda$  4.1 over a similar range of lead. At 3 mm. thickness a sudden change to  $\lambda = 2.7$  took place and continued

up to 5.7 mm., the limit of the measurements. We have been able to extend the range up to 2.5 cm. of lead, or including the 3 mm. thickness of lead as base to a total thickness of 2.8 cm.

Owing to the abnormally high absorption of the rays by lead, as noticed by Godlewski, Disposition 1 could not be employed, but Disposition 2 was possible. A new arrangement (Disposition 5) similar to Disposition 1 was used, in which the walls and base of the lead electroscopes were 3 mm. thick and the preparation was mounted 8.3 cm. below. The absorbing lead and zinc were placed directly over the preparation, and parallel experiments were done with radium for the same disposition. The results for lead and zinc are plotted in the curves (fig. 3). The abscissæ are centimetres

Fig. 3.



of thickness multiplied by the relative density of the absorber compared with that of lead. The aluminium curve would coincident with that of zinc if plotted. The figure shows how abnormal lead is. Although the measurements were made through a base of lead 3 mm. thick there is still the point of inflexion in the lead curve at 0.3 cm., noticed by Eve, and a still more decided point of inflexion at 0.85 cm. After that the curve is linear up to 2.5 cms. For zinc and aluminium the curves are exponential. In Disposition 2 the curves for copper, aluminium, and zinc are exponential after

a small initial thickness (0.36 cm. for zinc), but for lead the curve is convex to the axis up to 0.8 cm., as in Disposition 5, and then is straight. The results are tabulated below, the results for lead referring to the straight part of the curve beyond 0.8 cm.

*Disposition 5.*

	Values of $\lambda$ (cm.) <sup>-1</sup> .		$\lambda$ (Act.)/ $\lambda$ (Rad.).
	Actinium.	Radium.	
Lead .....	1.20	0.547	2.20
Zinc .....	0.521	0.269	1.94
Zinc (hardened by 0.6 cm. Pb).	0.420	0.262	1.60
Aluminium .....	0.217	0.115	1.89

*Disposition 2.*

Lead .....	1.85	0.517	3.57
Zinc .....	0.618	0.305	2.02
Zinc (hardened by 0.6 cm. Pb).	0.495	0.264	1.87
Aluminium .....	0.234	0.124	1.90
Copper .....	0.722	0.377	1.91

The separate mean values of  $\lambda$  for different thicknesses of lead are shown below.

*Disposition 5.*

Range (cm.). ...	0-0.124	0.124-0.30	0.30-0.85	0.85-2.40
$\lambda$ (cm.) <sup>-1</sup> .....	3.31	2.65	2.22	1.20

*Disposition 2.*

Range (cm.). ...	0.2-0.3	0.3-0.5	0.5-0.8	0.8-1.7
$\lambda$ (cm.) <sup>-1</sup> ...	4.24	3.26	2.50	1.85

In Disposition 2 the curve is the more continuously convex to the origin. For both dispositions the ratio of the absorption coefficients is about 1 : 1.9 for Class II. bodies, and the effect of hardening the zinc is the same, the value of  $\lambda$  being reduced to 80 per cent. But the actinium rays are, relatively to radium, hardened more in Disposition 5 than in 2. Godlewski concluded that the actinium  $\gamma$ -rays have only one-tenth of the penetrating power of the harder  $\gamma$ -rays of radium. As in the case of uranium X, however, work with more active preparations over greater ranges has reduced this difference, though actinium is still far removed in this respect from the other  $\gamma$ -ray sources examined.

*Some Generalizations with regard to γ-Rays.*

The results for actinium help to accentuate several relationships which exist between the penetrating power of the  $\gamma$ -rays of the radio-elements in solid bodies and other properties they possess. There is obviously an intimate connexion between penetrating power, abnormality of absorption by lead and hardening by lead. The rule is—the greater the penetrating power of the ray, the less abnormal is the absorption by lead as compared with that of Class II. bodies, and further the less hardening effect has lead on the rays subsequently absorbed by Class II. bodies. In Disposition 1, however, thorium D is an exception.

	Thorium D.	Radium C.	Mesotho- rium 2.	Uranium X.	Actinium C.
$\frac{\lambda/d \text{ Lead}}{\lambda/d \text{ Class II.}}$ Disposition 1 ..	1.28	1.10	1.28	1.35	—
$\frac{\lambda/d \text{ Lead}}{\lambda/d \text{ Class II.}}$ Disposition 2A .	0.95	1.07	1.28	—	1.93
$\frac{\lambda/d \text{ Lead}}{\lambda/d \text{ Class II.}}$ Disposition 2 B.	1.03	1.16	1.40	—	—
$\frac{\lambda/d \text{ Class II. (Disposition 2 A)}}{\lambda/d \text{ Class II. (Disposition 2 B)}}$	1.08	1.12	1.12	—	1.25

Again, as regards penetrability, there is far more connexion between the  $\gamma$ -ray and the  $\alpha$ -rays preceding and following it in the disintegration series, than there is between

the  $\gamma$ -ray and the  $\beta$ -ray which accompanies it. It is possible, from the results that have been given, to make a rule connecting the penetrability of the  $\gamma$ -ray with the period of the change in which it occurs, analogous to Rutherford's well-known rule for the range of the  $\alpha$ -particle. Actinium C is an exception throughout, and has been omitted from the comparison. In ascending order the penetrabilities of the  $\gamma$ -rays are :—

Uranium X, Mesothorium 2, Radium C<sub>1</sub>, Thorium D.

This is also the descending order of their periods, and also of their probable atomic weights.

If the conclusions of Hahn and Meitner (*Phys. Zeit.* x. p. 697, 1909) are correct, the  $\beta$ - and  $\gamma$ -rays of radium C come from radium C<sub>1</sub> (half-period 19 minutes), while the  $\alpha$ -rays come from a succeeding product, radium C<sub>2</sub> (half-period, about 2 minutes). This accords with Rutherford's rule, as the still more penetrating  $\alpha$ -ray of thorium C is, in all probability, derived from a change even more rapid than that of radium C<sub>2</sub>. The above order of the penetrabilities of the  $\gamma$ -rays is also that of the  $\alpha$ -rays which precede and follow them, whereas the order of the penetrabilities of the  $\beta$ -rays is quite different. The figures in brackets denote ranges of the  $\alpha$ -rays in centimetres of air.

$\gamma$ -rays .....	Uranium X	Mesothorium 2	Radium C <sub>1</sub>	Thorium D
$\alpha$ -rays preceding:—	Uranium (2·7)	Thorium (3·5?)	Radium A (4·83)	Thorium C (8·6)
$\alpha$ -rays following:—	Ionium (2·8)	Radiothorium (3·9)	Radium C <sub>2</sub> (7·06)	—
$\beta$ -rays .....	Mesothorium 2	Thorium D	Uranium X	Radium C <sub>1</sub>

Neither in penetrability, relative intensity nor homogeneity are the  $\beta$ -rays obviously connected with the  $\gamma$ -rays, whereas there is a certain connexion between the  $\gamma$ -rays and the  $\alpha$ -rays.

Physical Chemistry Laboratory,  
Glasgow University.



XVIII. *The Normal and the Selective Photoelectric Effect.*By R. POHL, *Ph.D.*, and P. PRINGSHEIM, *Ph.D.*\*

THE researches made by different authors on the photoelectric effect lead to some results which seem to be in striking contradiction to each other.

1. E. Ladenburg † and Mohlin ‡ have shown that the number of electrons produced by the unit of ultraviolet light increases with decreasing wave-lengths.

2. Elster and Geitel § found qualitatively that, for the alkali metals, this number has a maximum in the visible part of the spectrum, and T. Braun || proved this to be right quantitatively for Rb and K.

3. On the contrary, Hallwachs ¶ denied the existence of these maxima, and from new measurements he concluded that Ladenburg's and Mohlin's law is true also for the alkali metals.

Contradictions of a similar kind are found in the researches on the effects of polarized light.

4. According to Pohl\*\* the orientation of the vectors of ultraviolet light has an influence upon the number of liberated electrons only so far as the absorption of light depends upon it; this is true for solid surfaces of Cu and Pt ††, as well as for liquid Hg ††.

5. But Elster and Geitel §§ had found previously for the visible part of the spectrum, that with alkali metals the photoelectric current is proportional to the absorbed energy of light at different angles of incidence only if the electric vector swings *at right angles* to the plane of incidence: if the electric vector swings *in* this plane there is only an approximate proportionality between the component of the electric vector normal to the surface, and the photoelectric current due to this component. In this second case the

\* Communicated by the Authors.

† E. Ladenburg, *Verh. d. D. Phys. Ges.* ix. p. 504 (1907).

‡ H. Mohlin, *Akad. Abhandl.*, Upsala, 1907.

§ T. Elster and H. Geitel, *Wied. Ann.* lii. p. 493 (1894).

|| T. Braun, Dissertation, Bonn, 1905.

¶ W. Hallwachs, *Verh. d. D. Phys. Ges.* xi. p. 514 (1909).

\*\* R. Pohl, *ibid.* xi. p. 339 (1909).

†† R. Pohl, *ibid.* xi. p. 339 (1909).

‡‡ R. Pohl, *ibid.* xi. p. 609 (1909).

§§ T. Elster and H. Geitel, *Wied. Ann.* lxi. p. 445 (1897).

factor of proportionality was much larger; hence the ratio of the photo-effect in the two main planes of polarization was found to be sometimes as high as 1:50. This has been confirmed by different authors.

6. Finally, Pohl\* found that these observations of Elster and Geitel are true only in the visible part of the spectrum, while in the ultraviolet the alkali metals behave like the others, the orientation of the electric vector being immaterial.

Now we are going to show how the coexistence of these different facts, though they seem to be absolutely incompatible, may be easily explained by assuming that there is, apart from the *normal* photoelectric effect, the only one known as yet, still a new *selective* one. The full report on this subject has been published in the *Verhandlungen der Deutschen Physikalischen Gesellschaft*, vol. xii.†, and all details may be found there; the numbers in [ ] correspond with the divisions of these papers.

We used a spectroscope with quartz-fluorite lenses, illuminated by a mercury arc-lamp; the energy of the single lines was measured by means of a Rubens thermocouple, and the light was thrown through a quartz polarizer and a window of fused quartz into a tube of the type described by Elster and Geitel, which contained the alkali metal or some alloy. By the aid of this apparatus the photoelectric current could be measured, which was produced by the same energy of *incident* light in the two main planes of polarization and in different parts of the spectrum.

Table I. and fig. 1 show the results for an alloy of mercury and potassium, containing 2.5 atomic percentages of the alkali [11]. These numbers and the corresponding ones for Na prove that at an incidence of  $60^\circ$  the emission of electrons increases continuously with decreasing wave-length, quite independently of the plane of polarization. The effect is larger if the electric vector swings parallel to the plane of incidence ( $E \parallel$ ) than for  $E \perp$ , in consequence of the greater absorption of light which follows from the optical constants [12]. But for the same energy of *absorbed* light the plane of polarization has no influence upon the intensity of the emission. Hence for this photoelectric effect of the alloys,

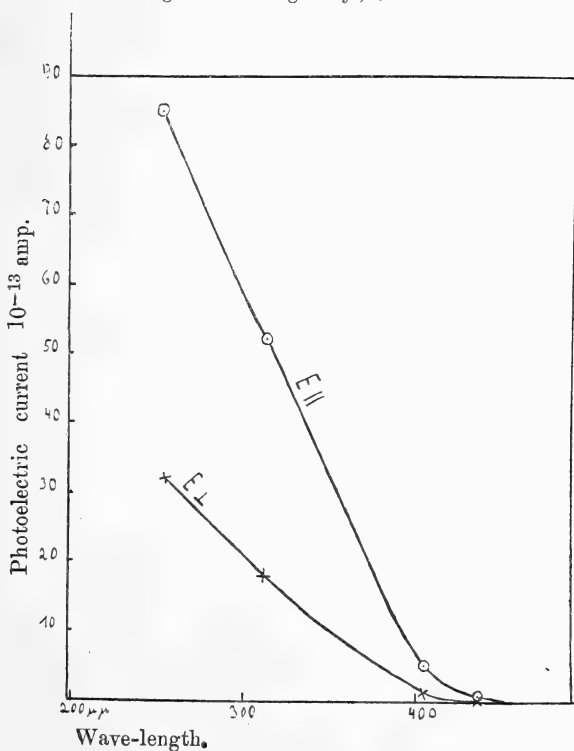
\* R. Pohl, *Verh. d. D. Phys. Ges.* xi. p. 715 (1909).

† R. Pohl and P. Pringsheim, *Verh. d. D. Phys. Ges.* xii. (1910) pp. 215-230, §§ 1-9; pp. 249-260, §§ 10-17; pp. 682-696, §§ 18-27; pp. 697-710, §§ 28-37.

TABLE I.

2.5 atom. per cent. R ; 97.5 atom. per cent. Hg.  $\phi = 60^\circ$ .

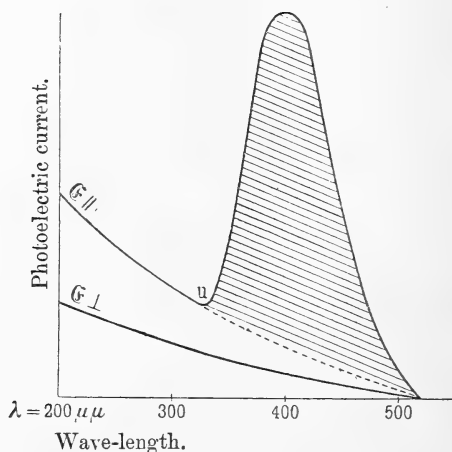
No.	Wave-length $\mu\mu$ .	Photoelectric current.	
		E $\parallel$ $10^{-13}$ amp.	E $\perp$ $10^{-13}$ amp.
95 .....	436	0.45	0.13
90 .....	405	5.0	1.4
92 .....	313	52.5	18.3
93 .....	254	84.4	32

Fig. 1.—K—Hg alloy ;  $\phi = 60^\circ$ .

which is due exclusively to the alkalis, the assertions 1 and 4 are true, even in the visible part of the spectrum. We call it the *normal* effect. All the well-known facts such as the independence of the temperature, the curves of the initial velocity and others refer to this effect, and so it may be explained by the atomistic theory of light according to Planck-Einstein\* [18].

The results are very different for pure alkali metals or Hg alloys of certain concentrations. Also in this case the normal effect exists for  $E \parallel$  and  $E \perp$  as in fig. 1, but for  $E \parallel$  there is a second effect superposed on the normal one,

Fig. 2.



in a short range of the spectrum; it grows rapidly from a small value to a very great intensity, and decreases then again with decreasing wave-length to zero, suggesting a resonance phenomenon; it is the more intense the stronger the component of the electric vector normal to the surface of the metal. The orientation of plane of polarization is of great importance for this effect, which may be called the *selective* one. Fig. 2 shows the superposition of the two effects [18]. Fig. 3 and Table II. contain the results which we obtained with polarized light on a sodium-potassium

\* Cf. R. Ladenburg, *Jahrb. d. Rad. u. El.* vi. pp. 425-487 (1909).

alloy (69.4 per cent. K) [10]. In fig. 3 the ultraviolet end of the selective effect cannot yet be recognized; but

Fig. 3.—K-Na alloy; polarized light.

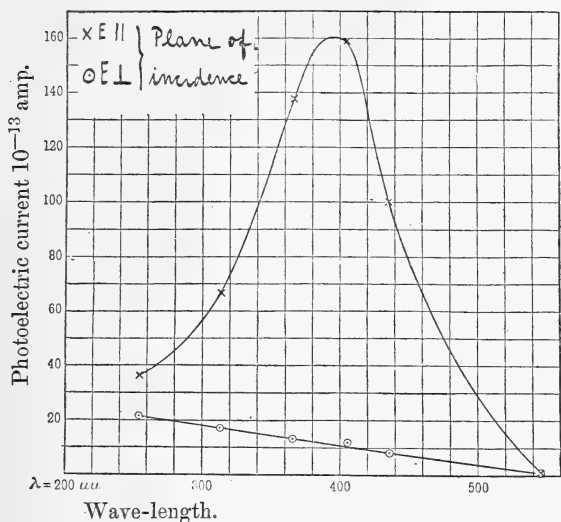
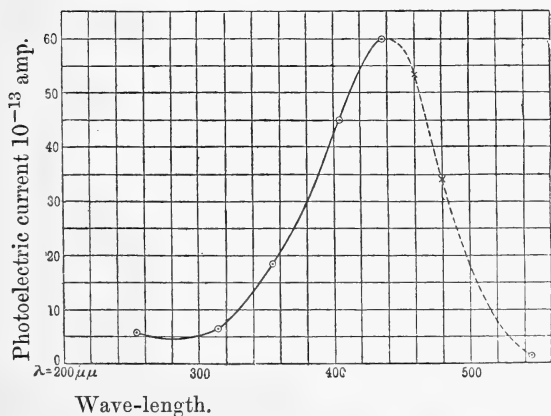


Fig. 4.—Potassium; unpolarized light.



this is easily seen in figs. 4 and 5, which were observed on K [14] and Rb [15] in unpolarized light, and show again

the addition of the two effects; Tables III. and IV. correspond to these figures.

Fig. 5.—Rubidium; unpolarized light.

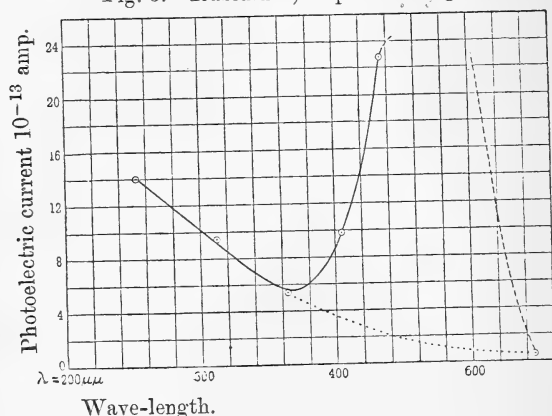


TABLE II.

K-Na alloy.  $\phi = 60^\circ$ .

TABLE III.

Solid K.  $\phi = 60^\circ$ .

TABLE IV.

Solid Rb.  $\phi = 60^\circ$ .

No.	Wave-length $\mu\mu$ .	Photoelectric current.		No.	Wave-length $\mu\mu$ .	Photoelectric current $10^{-13}$ amp.	No.	Wave-length $\mu\mu$ .	Photoelectric current $10^{-13}$ amp.
		$E \parallel$ $10^{-13}$ amp.	$E \perp$ $10^{-13}$ amp.						
1...	546	1.0	0.11	122	546	0.9	135	546	0.47
2...	436	100.0	8.0	123	436	60	134	436	22.9
3...	406	159.0	12.3	124	406	45	133	406	9.8
4...	365	138	13.0	125	365	18.4	132	365	5.4
5...	313	66.7	17.1	126	313	6.3	131	313	9.4
6...	254	36.2	21.0	127	254	5.8	129	254	12.6

There is no more difficulty in explaining the assertions 2, 3, 5, and 6 with the aid of the figs. 2-5.

(2) Elster and Geitel and Braun, using visible light at an oblique incidence, obtained the addition of the two effects, and as the selective one is much stronger their curves had a maximum in the visible part of the spectrum.

(3) In Hallwachs's experiments the light being incident

normally had no electric vector swinging in this direction, and so the author measured only the normal effect which increases continuously with decreasing wave-length.

(5) Elster and Geitel investigated the influence of polarization only in that part of the spectrum in which the K-Na alloy has its selective effect. Consequently they found only for  $E \perp$ , that at different angles of incidence the emission of electrons was proportional to the absorbed energy; and the ratio of the effects in the two main planes of polarization had naturally values, which like 1:50\* are much too large to be only explained by the difference in the absorption of light, and which prove the existence of some special efficacy of  $E \parallel$  [19].

(6) The same was true in Pohl's experiments, in which above the ultraviolet end  $u$  of the selective effect (fig. 2) the singularity of  $E \parallel$  disappears, so that only the normal effect is left.

Inasmuch as all the contradictions which we spoke of in the beginning are explained now, we think that we are justified in saying that a new selective effect exists, apart from the normal one. We do not as yet know much about the true nature of this effect, but from experiments made on K-Hg alloys [37] and on the influence of the angle of incidence [27], we conclude that it is a molecular resonance phenomenon, in which the electrons follow directly the electric vector; at any rate it cannot be explained by the "Quantentheorie" of Planck-Einstein.

We are continuing the researches with the aid of a grant from the Jagor Fund (Berlin), for which we wish to express our best thanks.

Physikalisches Institut der Universität Berlin,  
October 1910.

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XIX. *On the Maintenance of Periodic Motion by Solid Friction.* By ANDREW STEPHENSON †.

1. **T**HE maintenance of periodic motion by solid friction demonstrates that such friction diminishes as the velocity increases through a small range at least. However the friction may vary there is always a position of equilibrium and the small motion about it is evidently of type

$$\ddot{x} + (\kappa - \lambda)\dot{x} + n^2x = 0,$$

\* We got even the ratio 1:300 in K-Na alloys, in which the normal effect was comparatively much smaller than in fig. 3.

† Communicated by the Author.

where  $\lambda$  is a positive quantity proportional to the velocity-rate of change of the frictional force at the value corresponding to the relative velocity in equilibrium,  $v$  say. The position of equilibrium is therefore unstable if  $\lambda$  is greater than  $\kappa$ , and any slight disturbance is secularly magnified in free period.

The equation continues to represent the motion for larger amplitudes if  $\lambda$  is taken to represent the mean velocity-rate of change in the frictional force between 0 and  $\dot{x}$ . So long as  $\dot{x}$  is less than  $v$ ,  $\lambda$  is positive; but when there is no relative motion it is indeterminate, having any value from that given by continuity down to a large negative limit. The motion therefore increases until  $\dot{x}$  reaches the limit  $v$ , if  $\kappa$  is small enough, and the steady motion is determined by the fact that  $\dot{x} = v$  when

$$x = (\lambda - \kappa)v/n^2,$$

where  $\lambda$  has the greatest statical value, which is proportional to the quotient by  $v$  of the difference of the coefficients of friction at the relative velocities zero and  $v$ . Starting from this configuration the motion is approximately simple and of slowly increasing amplitude, so that the velocity  $v$  is again attained when  $x$  has a negative value numerically greater than that given above: the static friction then prevents any further change of velocity until the initial state is reached.

The magnifying action of solid friction was observed by W. Froude in the case of a pendulum suspended from a rotating shaft\*.

2. The instability of the position of equilibrium under friction holds evidently when the system has more than one degree of freedom. The general problem of determining the steady motion would be difficult and of little interest, but happily in the important case of the violin string simplicity is introduced through the infinitude of freedom, and a possible steady motion may readily be determined †.

In the absence of damping any free motion is possible in which the bowed point has no relative motion with the bow and constant speed against it, the mean position being that produced by a constant force equal to the friction in the backward swing. This is evident from the fact that the maintenance of the static state at the point of contact during the forward swing necessitates the constancy of the friction throughout the whole motion.

\* Cited by Lord Rayleigh, 'Sound,' i. § 138.

† The kinematics of the motion were long ago established by Helmholtz in his well known experimental examination.



If the bow is applied at a node dividing the length in the ratio  $p/q$  where  $p$  and  $q$  are integers with no common measure, there are  $p+q-1$  free motions in which the point has constant speeds to and fro, the ratio of the forth and back intervals being  $r/s$  where  $r$  and  $s$  are any integers such that  $r+s=p+q$ . Whatever the position of the node, therefore, two in general distinct motions are always possible, and in these the intervals are proportional to the segments of the string.

When the string is subject to slight air resistance any possible steady motion must approximate to the undamped kind, and we shall prove that with the bowed point at a node that motion is possible in which the ratio of the forward interval to the backward is equal to the ratio of the greater segment to the less.

Let the length of the string be  $\pi$ , and the point of bowing at distance  $a$ ,  $>\pi/2$ , from the end (1); also let  $y_1$  and  $y_2$  be the displacements from the mean positions at distances  $x_1$  and  $x_2$  from the ends, both measured positive in the direction of bowing.

If the maintained motion of the bowed point is of free type

$$y = \Sigma A_r \sin rt,$$

where the summation necessarily does not include components having a node at the bowed point,

$$= \Sigma \frac{A_r}{\sin ra} \{ \sin rx_1 (\sin rt + \kappa a \cot ra \cos rt) - \kappa x_1 \cos rx_1 \cos rt \} \quad \dots \quad (1)$$

$$= \Sigma \frac{A_r}{\sin r(\pi-a)} \{ \sin rx_2 (\sin rt + \overline{\kappa \pi - a} \cot r\overline{\pi - a} \cos rt) - \kappa x_2 \cos rx_2 \cos rt \} \quad \dots \quad (2)$$

terms of order  $\kappa^2$  being neglected.

At the bowed point,

$$\frac{dy_1}{dx_1} + \frac{dy_2}{dx_2} = 2\kappa\pi \Sigma r \frac{A_r}{\sin^2 ra} \cos rt \quad \dots \quad (3)$$

The quantity on the right with the addition of some constant represents the force necessary to maintain the motion. The condition for a steady motion is that this be equal to the frictional force so long as there is relative motion, and be not greater than the statical friction when there is no relative motion.

In the case under examination the velocity is  $v$  and

$-\alpha/(\pi-\alpha)$  for intervals  $-\alpha$  to  $\alpha$  and  $\alpha$  to  $2\pi-\alpha$ , so that

$$A_r = \frac{2v}{\pi-\alpha} \frac{1}{r^2} \sin r\alpha,$$

and the right hand side of (3) becomes

$$\frac{4\kappa\pi v}{\pi-\alpha} \sum \frac{1}{\sin r\alpha} \cos rt.$$

In the simpler cases  $\alpha = \pi/2, 2\pi/3,$  and  $3\pi/4,$  it is readily seen that this represents a constant between  $\alpha$  and  $2\pi-\alpha$ ; and in any example, since the cosecant coefficients of  $\cos rt/r$  recur and all lie below a certain limit numerically, a similar result as to the constancy during the shorter interval may be verified arithmetically. Let this constant value be  $k^*$ , and the greatest value during the interval  $-\alpha$  to  $\alpha, k'$ . Then the force maintaining the motion is not greater than  $P + pk'$  during the forward swing, and is equal to  $P + pk$  in the backward swing,  $p$  being the tension of the string.

If the pressure is  $F, \mu$  the coefficient of statical friction, and  $\mu'$  the coefficient at relative speed  $\pi v/(\pi-\alpha),$  then

$$P + pk = F\mu' \text{ and } P + pk' \succ F\mu.$$

The equation determines the mean position of the string, and from the inequality it is evident that the pressure must not be less than  $p(k' - k)/(\mu - \mu')$ .

We have shown then that the bow when applied at a node maintains that steady free motion which does not contain the components corresponding to the node and in which the point of bowing vibrates with constant speeds during intervals proportional to the segments of the string, travelling with the bow during the longer interval without slipping: and the only condition for the possibility of the motion is that the pressure exceed a certain limit.

All further properties belong to the free vibration defined by these facts: the equation of the motion, in accordance with (1) and (2), is

$$\frac{2v}{\pi-\alpha} \sum \frac{1}{r^2} \sin rv \sin rt, \quad \dots \dots (4)$$

where the summation includes all except the node components.

Thus the amplitude of any component, when it occurs, is

\* If  $p+q=n, \sum \frac{\sin rt}{r^2 \sin ra} = \frac{n^2-1}{n^2} \frac{\pi^2}{6}$  when  $t=a,$  so that

$$k = -\frac{2}{3} \frac{n^2-1}{n^2} \left(\frac{\pi}{\pi-a}\right)^2 \pi\kappa v.$$

proportional to the product of the speed of the bow and the reciprocal of the smaller segment of the string, provided this product is not too large.

If the bow is transferred from a more important node, such as that at a trisection of the string, to one at a distance of only a small fraction of the length, the missing components are restored without other appreciable change, since the components dropped are relatively insignificant. Equations (1) and (2) hold good so long as  $\kappa\alpha/\sin r\alpha$  is small, and the node harmonics are therefore brought to normal amplitude when the fractional displacement is large compared with  $\kappa/\sigma$ , where  $\sigma$  is the lowest of the frequencies involved. The rapid restoration of missing components in passing from a node has been observed aurally.

3. The motions examined experimentally have been found to be of the type (4) : others, however, might occur. These are to be sought for among the known group of undamped motions. A sufficient illustration is afforded by the simple case when the bow is applied at a point one fourth of the length from an end. There are then three undamped motions, and of the two in which the ratio of the intervals is equal to that of the segments, under air resistance only that is possible in which the motion with the bow occupies the longer interval. In the third the intervals are equal and

$$A_r = \frac{4v}{\pi} \frac{1}{r^2} \sin r \frac{\pi}{2}$$

so that the series in (3) is equal to  $4\kappa\pi v$  from 0 to  $\pi/2$ , and to  $-4\kappa\pi v$  between  $\pi/2$  and  $\pi$ . The motion is therefore steady if the pressure is not less than  $8\kappa\pi vp/(\mu - \mu')$ .

In the ordinary case the pressure is not less than

$$16\kappa\pi vp/(\mu - \mu'),$$

so that for the range of pressure between these limits only that motion is maintained in which the times to and fro are equal :

$$y = 4\sqrt{2} \frac{v}{\pi} \sum_0^\infty (-1)^n \left\{ \frac{1}{(1+4n)^2} \sin \overline{1+4n} x \sin \overline{1+4n} t \right. \\ \left. - \frac{1}{(3+4n)^2} \sin \overline{3+4n} x \sin \overline{3+4n} t \right\},$$

a vibration distinguished by the absence of all the odd octaves.

Whether any possible steady motion is set up by suitable bowing remains a matter for practical trial.

October, 1910.

XX. *On a Peculiar Property of the Asymmetric System.*

By ANDREW STEPHENSON\*.

IF an asymmetric system is subject to a direct periodic force the equation of motion is of the form

$$\ddot{x} + \kappa n \dot{x} + n^2(1 + x/a)x = bq^2 \cos qt.$$

$a$  being finite and  $b$  and  $\kappa$  small, the steady motion is

$$x = b \frac{q^2}{n^2 - q^2} \cos (qt + \eta),$$

when terms of the second order are neglected. This motion may, however, be unstable if  $q$  approximates to  $2n$  †. Putting  $q = 2(n + p)$  and  $bq^2 = 3cn^2$ , we find that there is instability if  $c^2 > (\kappa a)^2$  for the range of frequency for which

$$|p| < \frac{1}{2}n\sqrt{(c/a)^2 - \kappa^2}.$$

The steady state of motion is then given by

$$x = -\frac{1}{2}\frac{r^2}{a} + r \cos\{(n+p)t - \alpha\} + \frac{1}{6}\frac{r^2}{a} \cos 2\{(n+p)t - \alpha\} - c \cos 2(n+p)t,$$

where

$$p = \frac{1}{2}\kappa n\sqrt{(c/a)^2 - \kappa^2}, \quad |k| \text{ being } < 1,$$

$$r^2 = \frac{6}{5}ca(1-k)\sqrt{1 - (\kappa a/c)^2},$$

and

$$\sin 2\alpha = \kappa a/c, \quad \text{where } \pi/2 > |\alpha| > \pi/4.$$

Thus if  $c$  sufficiently exceeds  $\kappa a$ , e. g.  $= 2\kappa a$ , the amplitude is large, being of order  $\sqrt{c}$ . Any small deviation from this motion may be analysed into two simple motions one of which is reduced and the other undisturbed (so far as terms of order  $c$  are concerned).

The asymmetric system then is sensitive to direct periodic force of double the natural frequency and of intensity exceeding a certain limit proportional to the motional resistance.

If asymmetric oscillations exist within the molecule there would follow the possibility of monochromatic fluorescence with a frequency of emission half that of incidence.

September, 1910.

\* Communicated by the Author.

† "On the Stability of the Steady State of Forced Oscillation," Phil. Mag. December, 1907.

XXI. *The Reflective Power of Lamp- and Platinum-Black.*

By T. ROYDS, M.Sc., 1851 *Exhibition Science Scholar* \*.

ÅNGSTRÖM found in 1898 † that the reflective power of a thickly sooted platinum-black surface was from 0·82 to 1·25 per cent. for different regions of the spectrum. On the other hand, Féry ‡ concluded from his experiments that as much as 18 per cent. of the radiation from a black body at 100° C. was reflected from a platinum-black surface, and approximately the same amount from a lampblack surface. Since a knowledge of the reflective power of the surfaces usually employed as receivers of heat radiation is important for the determination of the absolute radiation constants, a method suggested by Prof. Paschen § has been employed to determine the reflective power of lamp- and platinum-black for definite wave-lengths in the infra-red.

The method consists in measuring the galvanometer deflexions, first when rays fall directly on a thermopile, and then when they fall on the lamp- or platinum-black, the rays diffusely reflected from it being focussed on a thermopile by means of a polished hemisphere of german-silver. The black surface A (fig. 1, p. 168) was attached alongside the thermopile slit B which was situated immediately in front of the exposed junctions. An external concave mirror cast, through a narrow opening O cut in the hemisphere, an image of an illuminated slit on to the thermopile when placed at C a little to the left of the centre D of the hemisphere. If the thermopile together with the black surface was now displaced relative to the hemisphere until the thermopile came to B at an equal distance on the opposite side of the centre, the slit image would then fall on the black surface and the reflected light would be focussed on the thermopile by the hemispherical mirror. The galvanometer deflexions in these two positions of the thermopile would measure the intensities of the incident and reflected radiation respectively. In order to interchange different black surfaces or to interpose screens in front of the thermopile only, the thermopile with the surface attached could be taken out and afterwards replaced

\* Communicated by the Author. A preliminary communication appeared in the *Physikalische Zeitschrift*, xi. p. 316 (1910).

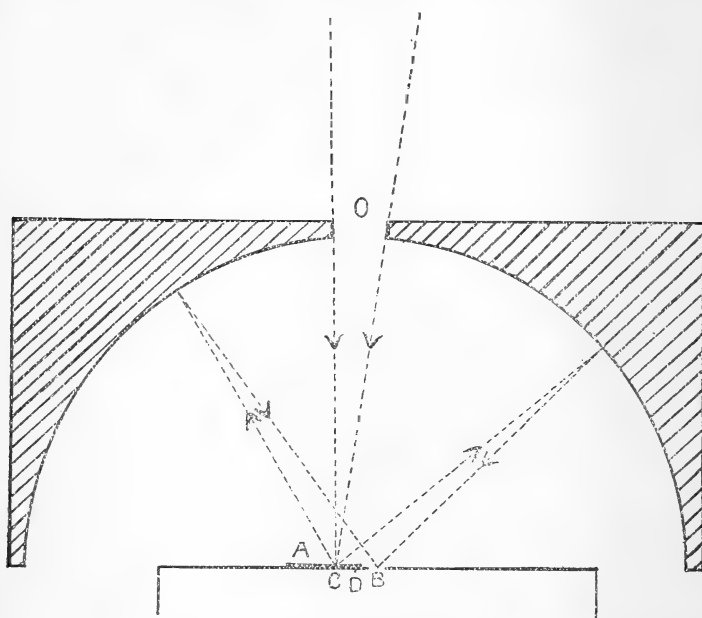
† Ångström, *Öfversigt of K. Vetensk.-Akad. Förhandl. Stockholm*, v. p. 283 (1898).

‡ Féry, *C. R.* 148. p. 777 (1909).

§ Modification of method by which Paschen made his bolometer blacker. See *Ber. Berl. Akad. d. Wiss.* April 27th, 1899.

in exactly the same position relative to the hemisphere. When the external optical arrangement was altered, the

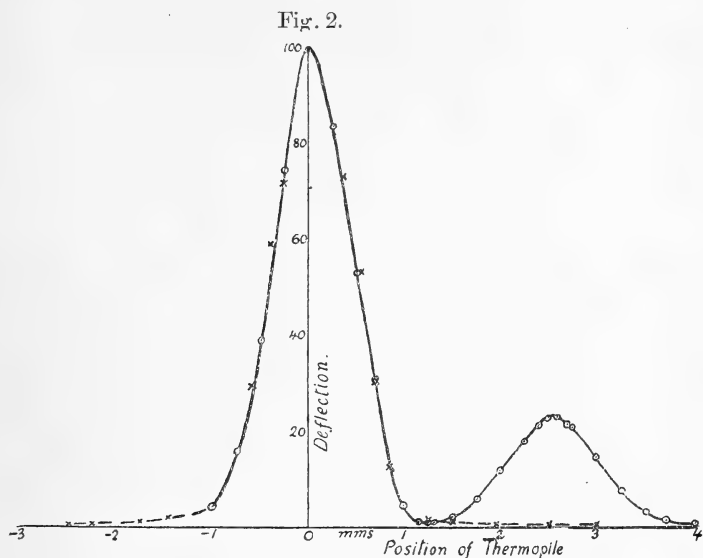
Fig. 1.



thermopile was placed in its first position to receive the light directly, and the external concave mirror so adjusted that the slit image fell on the thermopile slit which could be seen from behind through a window. In this way, it was not necessary to alter the two positions of the thermopile. These two positions were determined once for all by experiment and checked at intervals. The continuous line in fig. 2 shows the deflexions for a matt reflecting surface as the thermopile was gradually displaced in the great circle of the hemisphere. The distance between the incident slit image and its image reflected in the hemisphere was made as small as possible in order to have good definition; this distance amounted to 2.5 mm., the diameter of the hemisphere being 5 cm. The thermopile slit was 1 mm. wide, the image of the illuminated slit falling just within it. A series of Langley diaphragms was placed in front of the opening in the hemisphere in order to avoid strong convection currents.

The lampblack surfaces were deposited on polished silver,  $\frac{1}{10}$  mm. thick, from the fine flame of a small petroleum

lamp. The most satisfactory way of measuring their thickness was to place the surfaces nearly vertical under the



microscope after the conclusion of the experiments and to take the mean thickness of clean edges cut in the lampblack at all parts. The platinum-black surfaces were deposited according to the recipe of Lummer and Kurlbaum for different lengths of time.

A galvanometer deflexion of 1 mm. corresponded to a current of  $0.7 \times 10^{-10}$  ampere; the deflexions were proportional to the current.

The reflective power was determined for the following radiations whose maxima lie in the infra-red:—

1. That part of the radiation from an incandescent mantle which is transmitted by a layer of water 1 cm. thick. The water absorbs wave-lengths larger than  $1 \mu^*$ . The maximum of the transmitted radiation lies at about  $0.8 \mu$ .
2. The Reststrahlen from Selenite. A parallel beam from a slit illuminated by an incandescent mantle was reflected at the surfaces of three plates of gypsum before being focussed on the thermopile. The maximum lies at  $8.7 \mu \dagger$ .

\* Paschen, *Wied. Ann.* lii. p. 209 (1894) and others.

† Aschkinass, *Ann. d. Phys.* i. p. 42 (1900) and others.

3. The Reststrahlen from Fluorspar. The light from a slit illuminated by an incandescent mantle was reflected at three fluorspar surfaces, the second of which was concave. The maximum of these rays lies at  $25\cdot5\ \mu^*$ .
4. The Reststrahlen from Rock-salt. The rock-salt plates were arranged exactly as those of selenite. A more intense source, a Nernst filament, was used. The rays were very impure, but a fourth reflexion rendered the intensity too small. In order to increase the purity a clear rock-salt shutter (4 mm. thick), as applied by Rubens, was used instead of a metal one. The maximum of the pure Reststrahlen lies at  $51\cdot2\ \mu^\dagger$ ; in our case, however, somewhat towards smaller wave-lengths.

The following simple observations serve to indicate roughly the purity of the Reststrahlen employed :—

	Deflexions for Reststrahlen from		
	Selenite.	Fluorspar.	Rocksalt.
Total energy .....	1330	688	100
Energy transmitted through glass (3 mm.) .....	3	0	45
"    "    "    quartz (2·5 mm.)...	7	4	62
"    "    "    rocksalt (2·35 cm.)	1150	0	65

Several corrections are to be applied to the radiation which falls on the thermopile when placed to receive the radiation reflected from the black surface.

Firstly, diffuse radiation from the incident image may fall on the thermopile. To determine its amount the slit image was placed in the centre of the hemisphere, and the energy falling on the thermopile when placed at a distance equal to that in the experiments (2·5 mm.) was measured. The curve of the deflexions so obtained for all distances of the thermopile is shown by the dotted line in fig. 2. It was found that 0·42 per cent. of the energy in the incident image fell on the thermopile.

Secondly, energy is radiated from the black surface since it becomes heated when exposed to the incident radiation.

\* Rubens, *Phys. Z.S.* iv. p. 726 (1903).

† Rubens & Aschkinass, *Wied. Ann.* lxiv. p. 241 (1898).



This radiant energy was estimated in two ways : by measuring the energy received (1) when the radiant energy was absorbed, and (2) when the reflected energy was absorbed. A cover-glass, 0.1 mm. thick, placed in front of the thermopile slit, absorbs the energy radiated from the heated surface, but transmits the radiation which passes through 1 cm. of water. On the other hand, a plate of rock-salt 0.5 mm. thick, transmits the radiant energy, but absorbs a large and measurable portion of the flourspar Reststrahlen.

The following values were found for the heating effect :—

Lampblack (0.205 mm. thick) . . . . .	0.95%	of the incident energy.
Platinum-black (deposited 15 mins.) . .	0.0	" " "
" " ( " 3 " ) . .	0.0	" " "

A part of the reflected radiation escapes through the opening in the hemisphere by which the incident radiation enters. The slit image being near the centre of the hemisphere in order to obtain good definition, a portion of the light regularly reflected escaped in this way. This loss was considerable in the case of thin deposits, as was seen by mounting the black surface so inclined that the regularly reflected radiation could not pass through the aperture. With the thickest layers, however, inclining the surface thus did not cause any appreciable increase in the reflected energy\* ; further than this it was not found possible to estimate the correction for the aperture in the hemisphere.

After all other corrections have been applied, the results are to be multiplied by the reciprocal of the reflective power of german-silver, for before the radiation reflected from the black surface falls on the thermopile it undergoes a reflexion at the german-silver hemisphere †. The reflective power of german-silver has been determined by Paschen ‡, and the corrections are made using the following values :—

$\lambda 0.8 \mu$ . . .	71	per cent.
8.7 $\mu$ . . .	94.5	"
25.5 $\mu$ . . .	100	"
51 $\mu$ . . .	100	"

\* Consequently, the values given in the preliminary communication (*loc. cit.*) remain unaltered for the thickest layers only. For the thinner layer of lampblack, the apparent reflective powers there given are but 56.0 per cent. of their true values. The correction could not be determined for the thin lampblack layers, for they were destroyed in measuring their thickness before this correction was found to be so considerable ; the correction is probably smaller than that found above for the medium thickness and larger for the thinnest layer of lampblack.

† The polish of the hemisphere employed was not all that could be desired ; this fact is of little account for the longer wave-lengths, but the correction to be applied for  $\lambda = 0.8 \mu$  is uncertain for this reason.

‡ Paschen, *Ann. d. Phys.* iv. p. 304 (1901).

The results for the reflective power of lampblack and platinum-black are given in the following table :—

Wave-length.	Lampblack (0.205 mm. thick).		Platinum-black (deposited 15 mins.).		Platinum-black (deposited 3 mins.).	
	Apparent.	True.	Apparent.	True.	Apparent.	True.
0.8 $\mu$ .. .. .	2.17 %	1.13 %	0.54 %	0.17 %	1.34 %	1.30 %
8.7 $\mu$ .. .. .	1.99	0.66	0.98	0.59	5.80	5.70
25.5 $\mu$ .. .. .	2.04	0.67	1.35	0.93	7.50	7.08
51 $\mu$ <sup>1</sup> .. .. .	2.8	1.4	1.21	0.79	7.2	6.8
51 $\mu$ <sup>2</sup> .. .. .	3	1.6	1.5	1.1	7.8	7.4

<sup>1</sup> Rock-salt Reststrahlen using metal shutter.

<sup>2</sup> Rock-salt Reststrahlen using rocksalt shutter.

As a test of the method, the reflective power of polished silver for fluorspar Reststrahlen was determined and the value 94 per cent. was found.

Kurlbaum's determinations of Stefan's constant were not corrected for the reflective power of platinum-black. The radiation from a black body at 100° C. has its maximum at about  $\lambda = 8 \mu$ , where the reflective power has been found to be 0.59 per cent. Hence Kurlbaum's value

$$7.061 \times 10^{-15} \frac{\text{erg}}{\text{cm.}^2 \text{ degrees}^4}$$

becomes

$$7.10 \times 10^{-15} \frac{\text{erg}}{\text{cm.}^2 \text{ degrees}^4}.$$

Planck's value for the elementary quantity of electricity is consequently increased in the same ratio, and becomes, taking Paschen's value for  $b^*$ ,

$$4.624 \times 10^{-10} \text{ E.S. unit.}$$

A reflective power of 1.0 per cent. would bring it to  $4.65 \times 10^{-10}$ , Rutherford and Geiger's † experimental value.

I am greatly indebted to Prof. F. Paschen for suggesting the investigation and for his helpful advice at all stages.

Physikalisches Institut,  
Tübingen University.

\*  $b = 0.292$ , instead of Lummer and Pringsheim's value, 0.294.

† Rutherford & Geiger, Proc. Roy. Soc. A., lxxxi. p. 162 (1908).

XXIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xx, p. 1009.]

May 25th, 1910.—Prof. W. W. Watts, Sc.D., M.Sc., F.R.S.,  
President, in the Chair.

**T**HE following communications were read:—

1. ‘*Dedolomitization in the Marble of Port Shepstone(Natal).*’  
By F. H. Hatch, Ph.D., M.Inst.C.E., F.G.S., and R. H. Rastall,  
M.A., F.G.S.

The Port Shepstone marble is shown by chemical analysis to be a dolomite (the molecular ratio of calcium carbonate to magnesium carbonate being as 3 : 2). It owes its marmorization to thermal metamorphism by an extensive intrusion of granite, which completely surrounds it and penetrates it in broad dykes. This intrusion took place at some time prior to the deposition of the Table Mountain or Waterberg Sandstone, and is therefore pre-Devonian. The dolomite is relegated to the Swaziland Period.

The metamorphism of the dolomite under normal conditions is shown to have produced a saccharoidal marble of coarse texture, consisting almost entirely of carbonates; and the fact that neither periclase nor brucite has been produced in the normal marble is taken to indicate that the high-pressure conditions obtaining during the metamorphism precluded dedolomitization. In those places, however, where the dolomite contains blocks or boulders of earlier granitic rocks, interaction took place between the magnesium and calcium carbonates of the dolomite and the silica and alumina provided by the inclusions, resulting in the production, in the zone of marble immediately surrounding the inclusions, of a number of interesting silicates of magnesium, calcium, and aluminium, such as olivine, forsterite, diopside, wollastonite, and phlogopite, as well as the oxides brucite and spinel. Magnesian compounds predominate, the excess of lime recrystallizing as calcite. A noteworthy feature is the absence of minerals such as garnet and cordierite, which are especially characteristic of low-temperature metamorphism, thus indicating the prevalence of a high temperature during the metamorphism of the dolomite.

The paper concludes with a reference to the occurrence of granite boulders as foreign inclusions in other limestones, and a discussion of the chemical reactions by which the formation of the above-mentioned minerals may be theoretically explained as a result of

dedolomitization. Comparison is made with the dedolomitized Cambrian limestones of Assynt and Skye described by Dr. Teall and Mr. Harker, from which the Port Shepstone occurrence differs in the localization of the affected areas to reaction-rims around foreign boulders, and in the part played by alumina in the formation of new minerals.

2. 'Recumbent Folds in the Highland Schists.' By Edward Battersby Bailey, B.A., F.G.S.

A description is presented of the stratigraphy and structure of a considerable portion of the Inverness-shire and Argyllshire Highlands. The district considered lies south-east of Loch Linnhe, and extends from the River Spean in the north to Loch Creran in the south.

The following conclusions are arrived at :—

(1) The schists of the district are disposed in a succession of recumbent folds of enormous amplitude—proved in one case to be more than 12 miles in extent.

(2) The limbs of these recumbent folds are frequently replaced by fold-faults, or 'slips,' which have given freedom of development to the folds themselves.

(3) The slipping referred to is not confined to the lower limbs of recumbent anticlines, and is therefore due to something more than mere overthrusting. It is a complex accommodation-phenomenon, of a type peculiar perhaps to the interior portions of folded mountain-chains. In fact, the cores of some of the recumbent folds have been squeezed forward so that they have virtually reacted as intrusive masses.

(4) In the growth of these structures many of the earlier formed cores and slips have suffered extensive secondary corrugation of isoclinal type.

June 15th.—Prof. W. W. Watts, Sc.D., M.Sc., F.R.S.,  
President, in the Chair.

The following communications were read :—

1. 'The Natural Classification of Igneous Rocks.' By Dr. Whitman Cross, F.G.S.

The author reviews the various systems of classification which have been proposed. He discusses the origin of the difference of composition of igneous rocks due to: (1) Primæval difference, (2) Magmatic differentiation, (3) Assimilation; and points out

that differentiation and assimilation are in a measure antithetical processes.

If the deep-seated magmas of large volume have acquired their various chemical characters in different ways, it appears at once evident that this primary genetic factor cannot be used in classification, unless the characters of different origin can be distinguished in the rocks.

Classification by geographical distribution of chemically different rocks is considered, and the groupings proposed by various writers are discussed; and it is shown that the rocks of the Pacific zone of North America indicate that they possess provincial peculiarities of interest, but that these are not by any means identical with the features emphasized by Becke and others as characterizing the Pacific kindred.

The factors of magmatic differentiation are then reviewed. The aschistic and diaschistic magmas of Brögger and the 'dyke rocks' of Rosenbusch are discussed; and it is contended that certain dyke rocks of Colorado show a notable exception to the rule postulated by Rosenbusch. The conclusion is reached that the sharp distinction between the two 'dyke rock' groups is a purely arbitrary one, resting on an unproved hypothesis.

A discussion on the classification by eutectics follows, and the writings of G. F. Becker and J. H. L. Vogt on this subject are criticized. The view that graphic, spherulitic, and felsitic textures are characteristically eutectic is considered to be incorrect, and it is contended that magmatic classification by eutectics is fundamentally weak—because it rests on hypothesis, because it does not apply to all rocks, and because it does not allow for the entire magma of most rocks. A classification by eutectics may, in the future, be realized; but it seems inevitable that it must be a classification for a special interest, not for the general science of petrography.

The author considers that the distinction between felspathic and non-felspathic rocks which has been so prominent in current systems is not only unnatural, but is in the highest degree arbitrary.

The use of texture is then discussed, and it is shown that classification by occurrence, as determining texture, or by texture, as expressing the broad phases of occurrence, is based on long disproved generalizations made from limited observation. The 'American Quantitative System of Classification' is then briefly dealt with, and the following general conclusions are formulated:—

'The scientific logical classification of igneous rocks must apparently be based on the quantitative development of fundamental characters, and the divisions of the scheme must have sharp artificial boundaries, since none exist in Nature.

'Chemical composition is the fundamental character of igneous rocks, but it may be advantageously expressed for classificatory purposes in terms of simple compounds, which represent either rock-making minerals or molecules entering into isomorphous mixtures in known minerals. It is probable that the

magmatic solution consists of such molecules, and that the norm of the "Quantitative System" is a fairly representative set of these compounds.

'The actual mineral and textural characters of igneous rocks are variable qualifiers of each chemical unit, and should be applied as such to terms indicating magmatic character.'

2. 'The Denudation of the Western End of the Weald.' By Henry Bury, M.A., F.L.S., F.G.S.

There are two main theories of Wealden denudation :— (1) attributing the removal of most of the Chalk to marine planation ; and (2) denying planation, and relying solely on subaërial denudation. Prof. W. M. Davis's suggestion of a subaërial peneplain forms a sort of connecting link between the two.

The evidence in favour of planation which Ramsay and Topley brought forward is inconclusive, and might plausibly, if it stood alone, be attributed to pre-Eocene causes. On the other hand, Prestwich's arguments against planation are equally weak, while the Chalk plateau to which he draws attention strongly supports Ramsay's views. The distribution of chert is fatal to Prof. Davis's hypothesis, and very difficult to account for, except on the marine theory.

In the case of the River Blackwater it can be proved that, long after the Hythe Beds of Hindhead were uncovered, the river-system remained extremely immature ; and this affords very strong grounds for the acceptance of the marine hypothesis.

The evidence of the other western rivers is less conclusive, though the Wey and the Mole both provide minor arguments pointing in the same direction. The anomalous position of the Arun, at the foot of the northern escarpment of the Lower Greensand on either side of the Wey, is almost certainly due to comparatively recent captures from the latter river, and affords no ground for assuming a river-system of great age matured on a Miocene peninsula.

There is no proof that any of the existing connexions between rivers and longitudinal folds are of a primitive character ; and, on the other hand, there are many alleged examples of transverse disturbances having served as guides to consequent rivers. This again, on the whole, supports the marine hypothesis, especially if, as there are reasons for believing, the longitudinal folds are older than the transverse.

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XXIII. *Hydrodynamical Notes.*  
By Lord RAYLEIGH, O.M., F.R.S.\*

Potential and Kinetic Energies of Wave Motion.—Waves moving into Shallower Water.—Concentrated initial Disturbance with inclusion of Capillarity.—Periodic Waves in Deep Water advancing without change of Type.—Tide Races.—Rotational Fluid Motion in a Corner.—Steady Motion in a Corner of Viscous Fluid.

**I**N the problems here considered the fluid is regarded as incompressible, and the motion is supposed to take place in two dimensions.

*Potential and Kinetic Energies of Wave Motion.*

When there is no dispersion, the energy of a progressive wave of any form is half potential and half kinetic. Thus in the case of a long wave in shallow water, "if we suppose that initially the surface is displaced, but that the particles have no velocity, we shall evidently obtain (as in the case of sound) two equal waves travelling in opposite directions, whose total energies are equal, and together make up the potential energy of the original displacement. Now the elevation of the derived waves must be half of that of the original displacement, and accordingly the potential energies less in the ratio of 4:1. Since therefore the potential energy of each derived wave is one quarter, and the total energy one half that of the original displacement,

\* Communicated by the Author.

it follows that in the derived wave the potential and kinetic energies are equal\*\*.

The assumption that the displacement in each derived wave, when separated, is similar to the original displacement fails when the medium is dispersive. The equality of the two kinds of energy in an infinite progressive train of simple waves may, however, be established as follows.

Consider first an infinite series of simple stationary waves, of which the energy is at one moment wholly potential and half a period later wholly kinetic. If  $t$  denote the time and  $E$  the total energy, we may write

$$\text{K.E.} = E \sin^2 nt, \quad \text{P.E.} = E \cos^2 nt.$$

Upon this superpose a similar system, displaced through a quarter wave-length in space and through a quarter period in time. For this, taken by itself, we should have

$$\text{K.E.} = E \cos^2 nt, \quad \text{P.E.} = E \sin^2 nt.$$

And, the vibrations being *conjugate*, the potential and kinetic energies of the combined motion may be found by simple addition of the components, and are accordingly independent of the time, and each equal to  $E$ . Now the resultant motion is a simple progressive train, of which the potential and kinetic energies are thus seen to be equal.

A similar argument is applicable to prove the equality of energies in the motion of a simple conical pendulum.

It is to be observed that the conclusion is in general limited to vibrations which are infinitely small.

#### *Waves moving into Shallower Water.*

The problem proposed is the passage of an infinite train of simple infinitesimal waves from deep water into water which shallows gradually in such a manner that there is no loss of energy by reflexion or otherwise. At any stage the whole energy, being the double of the potential energy, is proportional per unit length to the square of the height; and for motion in two dimensions the only remaining question for our purpose is what are to be regarded as corresponding lengths along the direction of propagation.

In the case of long waves, where the wave-length ( $\lambda$ ) is long in comparison with the depth ( $l$ ) of the water, corresponding parts are as the velocities of propagation ( $V$ ), or

\* "On Waves," *Phil. Mag.* i. p. 257 (1876); 'Scientific Papers,' i. p. 254.



since the periodic time ( $\tau$ ) is constant, as  $\lambda$ . Conservation of energy then requires that

$$(\text{height})^2 \times V = \text{constant}; \quad \dots \dots (1)$$

or since  $V$  varies as  $l^{\frac{1}{2}}$ , height varies as  $l^{-\frac{1}{4}}$ .\*

But for a dispersive medium corresponding parts are not proportional to  $V$ , and the argument requires modification. A uniform regime being established, what we are to equate at two separated places where the waves are of different character is the *rate of propagation of energy* through these places. It is a general proposition that in any kind of waves the ratio of the energy propagated past a fixed point in unit time to that resident in unit length is  $U$ , where  $U$  is the *group-velocity*, equal to  $d\sigma/dk$ , where  $\sigma = 2\pi/\tau$ ,  $k = 2\pi/\lambda$  †. Hence in our problem we must take

$$\text{height varies as } U^{-\frac{1}{2}}, \quad \dots \dots (2)$$

which includes the former result, since in a non-dispersive medium  $U = V$ .

For waves in water of depth  $l$ ,

$$\sigma^2 = gk \tanh kl, \quad \dots \dots (3)$$

whence

$$2\sigma U/g = \tanh kl + kl(1 - \tanh^2 kl). \quad \dots \dots (4)$$

As the wave progresses,  $\sigma$  remains constant, (3) determines  $k$  in terms of  $l$ , and  $U$  follows from (4). If we write

$$\sigma^2 l/g = l', \quad \dots \dots (5)$$

(3) becomes

$$kl \cdot \tanh kl = l', \quad \dots \dots (6)$$

and (4) may be written

$$2\sigma U/g = kl + (l' - l^2)/kl. \quad \dots \dots (7)$$

By (6), (7)  $U$  is determined as a function of  $l'$  or by (5) of  $l$ .

If  $kl$ , and therefore  $l'$ , is very great,  $kl = l'$ , and then by (7) if  $U_0$  be the corresponding value of  $U$ ,

$$2\sigma U_0/g = 1, \quad \dots \dots (8)$$

and in general

$$U/U_0 = kl + (l' - l^2)/kl. \quad \dots \dots (9)$$

\* *Loc. cit.* p. 255.

† *Proc. Lond. Math. Soc.* ix. 1877; 'Scientific Papers,' i. p. 326.

Equations (2), (5), (6), (9) may be regarded as giving the solution of the problem in terms of a known  $\sigma$ . It is perhaps more practical to replace  $\sigma$  in (5) by  $\lambda_0$ , the corresponding wave-length in a great depth. The relation between  $\sigma$  and  $\lambda_0$  being  $\sigma^2 = 2\pi g/\lambda_0$ , we find in place of (5)

$$l' = 2\pi l/\lambda_0 = k_0 l. \dots \dots (10)$$

Starting in (10) from  $\lambda_0$  and  $l$  we may obtain  $l'$ , whence (6) gives  $kl$ , and (9) gives  $U/U_0$ . But in calculating results by means of tables of the hyperbolic functions it is more convenient to start from  $kl$ . We find

$kl$ .	$l'$ .	$U/U_0$ .	$kl$ .	$l'$ .	$U/U_0$ .
$\infty$	$kl$	1.000	.6	.322	.964
10	$kl$	1.000	.5	.231	.855
5	4.999	1.001	.4	.152	.722
2	1.928	1.105	.3	.087	.566
1.5	1.358	1.176	.2	.039	.390
1.0	.762	1.182	.1	.010	.200
.8	.531	1.110	$kl$	$(kl)^2$	$2kl$
.7	.423	1.048	—	—	—

It appears that  $U/U_0$  does not differ much from unity between  $l' = .23$  and  $l' = \infty$ , so that the shallowing of the water does not at first produce much effect upon the height of the waves. It must be remembered, however, that the wave-length is diminishing, so that waves, even though they do no more than maintain their height, grow *steeper*.

*Concentrated Initial Disturbance with inclusion of Capillarity.*

A simple approximate treatment of the general problem of initial linear disturbance is due to Kelvin\*. We have for the elevation  $\eta$  at any point  $x$  and at any time  $t$

$$\begin{aligned} \eta &= \frac{1}{\pi} \int_0^\infty \cos kx \cos \sigma t \, dk \\ &= \frac{1}{2\pi} \int_0^\infty \cos (kx - \sigma t) \, dk + \frac{1}{2\pi} \int_0^\infty \cos (kx + \sigma t) \, dk, \dots (1) \end{aligned}$$

in which  $\sigma$  is a function of  $k$ , determined by the character

\* Proc. Roy. Soc. vol. xlii. p. 80 (1887); 'Math. and Phys. Papers,' iv. p. 303.

of the dispersive medium—expressing that the initial elevation ( $t=0$ ) is concentrated at the origin of  $x$ . When  $t$  is great, the angles whose cosines are to be integrated will in general vary rapidly with  $k$ , and the corresponding parts of the integral contribute little to the total result. The most important part of the range of integration is the neighbourhood of places where  $kx \pm \sigma t$  is stationary with respect to  $k$ , i. e. where

$$x \pm t \frac{d\sigma}{dk} = 0. \quad . . . . . (2)$$

In the vast majority of practical applications  $d\sigma/dk$  is positive, so that if  $x$  and  $t$  are also positive the second integral in (1) makes no sensible contribution. The result then depends upon the first integral, and only upon such parts of that as lie in the neighbourhood of the value, or values, of  $k$  which satisfy (2) taken with the lower sign. If  $k_1$  be such a value, Kelvin shows that the corresponding term in  $\eta$  has an expression equivalent to

$$\eta = \frac{\cos(\sigma_1 t - k_1 x - \frac{1}{4}\pi)}{\sqrt{\{-2\pi t d^2\sigma/dk_1^2\}}}, \quad . . . . . (3)$$

$\sigma_1$  being the value of  $\sigma$  corresponding to  $k_1$ .

In the case of deep-water waves where  $\sigma = \sqrt{gk}$ , there is only one predominant value of  $k$  for given values of  $x$  and  $t$ , and (2) gives

$$k_1 = gt^2/4x^2, \quad \sigma_1 = gt/2x, \quad . . . . . (4)$$

making

$$\sigma_1 t - k_1 x - \frac{1}{4}\pi = gt^2/4x - \frac{1}{4}\pi, \quad . . . . . (5)$$

and finally

$$\eta = \frac{g^{\frac{1}{2}} t}{2\pi^{\frac{1}{2}} x^{\frac{3}{2}}} \cos \left\{ \frac{gt^2}{4x} - \frac{\pi}{4} \right\}, \quad . . . . . (6)$$

the well-known formula of Cauchy and Poisson.

In the numerator of (3)  $\sigma_1$  and  $k_1$  are functions of  $x$  and  $t$ . If we inquire what change ( $\Lambda$ ) in  $x$  with  $t$  constant alters the angle by  $2\pi$ , we find

$$\Lambda \left\{ k_1 + \left( x - t \frac{d\sigma}{dk_1} \right) \frac{dk_1}{dx} \right\} = 2\pi,$$

so that by (2)  $\Lambda = 2\pi/k_1$ , i. e. the effective wave-length  $\Lambda$  coincides with that of the predominant component in the original integral (1), and a like result holds for the periodic

time \*. Again, it follows from (2) that  $k_1x - \sigma_1t$  in (3) may be replaced by  $\int k_1 dx$ , as is exemplified in (4) and (6).

When the waves move under the influence of a capillary tension T in addition to gravity,

$$\sigma^2 = gk + Tk^3/\rho, \dots \dots \dots (7)$$

$\rho$  being the density, and for the wave-velocity (V)

$$V^2 = \sigma^2/k^2 = g/k + Tk/\rho, \dots \dots \dots (8)$$

as first found by Kelvin. Under these circumstances V has a minimum value when

$$k^2 = g\rho/T. \dots \dots \dots (9)$$

The group-velocity U is equal to  $d\sigma/dk$ , or to  $d(kV)/dk$ ; so that when V has a minimum value, U and V coincide. Referring to this, Kelvin towards the close of his paper remarks "The working out of our present problem for this case, or any case in which there are either minimums or maximums, or both maximums and minimums, of wave-velocity, is particularly interesting, but time does not permit of its being included in the present communication."

A glance at the simplified form (3) shows, however, that the special case arises, not when V is a minimum (or maximum), but when U is so, since then  $d^2\sigma/dk_1^2$  vanishes. As given by (3),  $\eta$  would become infinite—an indication that the approximation must be pursued. If  $k = k_1 + \xi$ , we have in general in the neighbourhood of  $k_1$ ,

$$kx - \sigma t = k_1x - \sigma_1t + \left(x - t \frac{d\sigma}{dk_1}\right) \xi - \frac{t}{1.2} \frac{d^2\sigma}{dk_1^2} \xi^2 - \frac{t}{1.2.3} \frac{d^3\sigma}{dk_1^3} \xi^3. \dots \dots (10)$$

In the present case where the term in  $\xi^2$  disappears, as well as that in  $\xi$ , we get in place of (3) when  $t$  is great

$$\eta = \frac{\cos(k_1x - \sigma_1t)}{2\pi \left\{ \frac{1}{6} t \frac{d^3\sigma}{dk_1^3} \right\}^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \cos \alpha^3. d\alpha, \dots \dots (11)$$

varying as  $t^{-\frac{1}{2}}$  instead of as  $t^{-\frac{1}{3}}$ .

The definite integral is included in the general form

$$\int_{-\infty}^{+\infty} \cos \alpha^m. d\alpha = \frac{2}{m} \Gamma\left(\frac{1}{m}\right) \cos \frac{\pi}{2m}, \dots \dots \dots (12)$$

\* Cf. Green, Proc. Roy. Soc. Ed. xxix. p. 445 (1909).

giving

$$\int_{-\infty}^{+\infty} \cos \alpha^2. dx = \sqrt{\left(\frac{\pi}{2}\right)}; \int_{-\infty}^{+\infty} \cos \alpha^3. dx = \frac{1}{\sqrt{3}} \Gamma\left(\frac{1}{3}\right). \quad (13)$$

The former is employed in the derivation of (3).

The occurrence of stationary values of  $U$  is determined from (7) by means of a quadratic. There is but one such value ( $U_0$ ), easily seen to be a minimum, and it occurs when

$$k^2 = \left\{ \sqrt{\frac{4}{3}} - 1 \right\} \frac{g\rho}{T} = 0.1547 \frac{g\rho}{T}. \quad . . . \quad (14)$$

On the other hand, the minimum of  $V$  occurs when  $k^2 = g\rho/T$  simply.

When  $t$  is great, there is no important effect so long as  $x$  (positive) is less than  $U_0 t$ . For this value of  $x$  the Kelvin formula requires the modification expressed by (11). When  $x$  is decidedly greater than  $U_0 t$ , there arise two terms of the Kelvin form, indicating that there are now two systems of waves of different wave-lengths, effective at the same place.

It will be seen that the introduction of capillarity greatly alters the character of the solution. The quiescent region inside the annular waves is easily recognized a few seconds after a very small stone is dropped into smooth water\*, but I have not observed the duplicity of the annular waves themselves. Probably the capillary waves of short wave-length are rapidly damped, especially when the water-surface is not quite clean. It would be interesting to experiment upon truly linear waves, such as might be generated by the sudden electrical charge or discharge of a wire stretched just above the surface. But the full development of the peculiar features to be expected on the inside of the wave-system seems to require a space larger than is conveniently available in a laboratory.

*Periodic Waves in Deep Water advancing without  
change of Type.*

The solution of this problem when the height of the waves is infinitesimal has been familiar for more than a century, and the pursuance of the approximation to cover the case of moderate height is to be found in a well-known paper by

\* A checkered background, *e. g.* the sky seen through foliage, shows the waves best.

Stokes \*. In a supplement published in 1880 † the same author treated the problem by another method in which the space coordinates  $x, y$  are regarded as functions of  $\phi, \psi$  the velocity and stream functions, and carried the approximation a stage further.

In an early publication ‡ I showed that some of the results of Stokes's first memoir could be very simply derived from the expression for the stream-function in terms of  $x$  and  $y$ , and lately I have found that this method may be extended to give, as readily if perhaps less elegantly, all the results of Stokes' supplement.

Supposing for brevity that the wave-length is  $2\pi$  and the velocity of propagation unity, we take as the expression for the stream-function of the waves, reduced to rest,

$$\psi = y - \alpha e^{-y} \cos x - \beta e^{-2y} \cos 2x - \gamma e^{-3y} \cos 3x, \quad (1)$$

in which  $x$  is measured horizontally and  $y$  vertically downwards. This expression evidently satisfies the differential equation to which  $\psi$  is subject, whatever may be the values of the constants  $\alpha, \beta, \gamma$ . From (1) we find

$$\begin{aligned} U^2 - 2y\eta &= (d\psi/dx)^2 + (d\psi/dy)^2 - 2gy \\ &= 1 - 2\psi + 2(1-g)y + 2\beta e^{-2y} \cos 2x + 4\gamma e^{-3y} \cos 3x \\ &\quad + \alpha^2 e^{-2y} + 4\beta^2 e^{-4y} + 9\gamma^2 e^{-6y} + 4\alpha\beta e^{-3y} \cos x \\ &\quad + 6\alpha\gamma e^{-4y} \cos 2x + 12\beta\gamma e^{-5y} \cos x. \quad \dots \quad (2) \end{aligned}$$

The condition to be satisfied at a free surface is the constancy of (2).

The solution to a moderate degree of approximation (as already referred to) may be obtained with omission of  $\beta$  and  $\gamma$  in (1), (2). Thus from (1) we get, determining  $\psi$  so that the mean value of  $y$  is zero,

$$y = \alpha(1 + \frac{5}{8}\alpha^2) \cos x - \frac{1}{2}\alpha^2 \cos 2x + \frac{3}{8}\alpha^3 \cos 3x, \quad \dots \quad (3)$$

which is correct as far as  $\alpha^3$  inclusive.

If we call the coefficient of  $\cos x$  in (3)  $a$ , we may write with the same approximation

$$y = a \cos x - \frac{1}{2}a^2 \cos 2x + \frac{3}{8}a^3 \cos 3x. \quad \dots \quad (4)$$

\* Camb. Phil. Soc. Trans. viii. p. 441 (1847); 'Math. and Phys. Papers,' i. p. 197.

† *L. c. i.* p. 314.

‡ Phil. Mag. i. p. 257 (1876); Sci. Papers, i. p. 262. See also Lamb's 'Hydrodynamics,' § 230.

Again from (2) with omission of  $\beta, \gamma,$

$$U^2 - 2gy = \text{const.} + 2(1 - g - \alpha^2 - \alpha^4)y + \alpha^4 \cos 2x - \frac{4}{3} \alpha^5 \cos 3x. \quad (5)$$

It appears from (5) that the surface condition can be satisfied with  $\alpha$  only, provided that  $\alpha^4$  is neglected and that

$$1 - g - \alpha^2 = 0. \quad (6)$$

In (6)  $\alpha$  may be replaced by  $\alpha,$  and the equation determines the velocity of propagation. To exhibit this we must restore generality by introduction of  $k$  ( $= 2\pi/\lambda$ ) and  $c$  the velocity of propagation, hitherto treated as unity. Consideration of "dimensions" shows that (6) becomes

$$kc^2 - g - \alpha^2 c^2 k^3 = 0, \quad (7)$$

or

$$c^2 = g/k \cdot (1 + k^2 \alpha^2). \quad (8)$$

Formulae (4) and (8) are those given by Stokes in his first memoir.

By means of  $\beta$  and  $\gamma$  the surface condition (2) can be satisfied with inclusion of  $\alpha^4$  and  $\alpha^5,$  and from (5) we see that  $\beta$  is of the order  $\alpha^4$  and  $\gamma$  of the order  $\alpha^5.$  The terms to be retained in (2), in addition to those given in (5), are

$$2\beta(1 - 2y) \cos 2x + 4\gamma \cos 3x + 4\alpha\beta \cos x \\ = 2\beta \cos 2x + 2\alpha\beta (\cos x + \cos 3x) + 4\gamma \cos 3x + 4\alpha\beta \cos x.$$

Expressing the terms in  $\cos x$  by means of  $y,$  we get finally

$$U^2 - 2gy = \text{const.} + 2y(1 - g - \alpha^2 - \alpha^4 + \beta) \\ + (\alpha^4 + 2\beta) \cos 2x + (4\gamma - \frac{4}{3} \alpha^5 - 2\alpha\beta) \cos 3x. \quad (9)$$

In order to satisfy the surface condition of constant pressure, we must take

$$\beta = -\frac{1}{2} \alpha^4, \quad \gamma = \frac{1}{12} \alpha^5, \quad (10)$$

and in addition

$$1 - g - \alpha^2 - \frac{3}{2} \alpha^4 = 0, \quad (11)$$

correct to  $\alpha^5$  inclusive. The expression (1) for  $\psi$  thus assumes the form

$$\psi = y - \alpha e^{-y} \cos x + \frac{1}{2} \alpha^4 e^{-2y} \cos 2x - \frac{1}{12} \alpha^5 e^{-3y} \cos 3x, \quad (12)$$

from which  $y$  may be calculated in terms of  $x$  as far as  $\alpha^5$  inclusive.

By successive approximation, determining  $\psi$  so as to make the mean value of  $y$  equal to zero, we find as far as  $\alpha^4$

$$y = (\alpha + \frac{5}{8}\alpha^3) \cos x - (\frac{1}{2}\alpha^2 + \frac{4}{3}\alpha^4) \cos 2x + \frac{3}{8}\alpha^3 \cos 3x - \frac{1}{3}\alpha^4 \cos 4x, \dots \quad (13)$$

or, if we write as before  $a$  for the coefficient of  $\cos x$ ,

$$y = a \cos x - (\frac{1}{2}a^2 + \frac{1}{2}\frac{7}{4}a^4) \cos 2x + \frac{3}{8}a^3 \cos 3x - \frac{1}{3}a^4 \cos 4x, \dots \quad (14)$$

in agreement with equation (20) of Stokes' Supplement.

Expressed in terms of  $a$ , (11) becomes

$$g = 1 - a^2 - \frac{1}{4}a^4, \dots \quad (15)$$

or on restoration of  $k, c$ ,

$$g = kc^2 - k^3a^2c^2 - \frac{1}{4}k^5a^4c^2, \dots \quad (16)$$

Thus the extension of (8) is

$$c^2 = g/k \cdot (1 + k^2a^2 + \frac{4}{3}k^4a^4), \dots \quad (17)$$

which also agrees with Stokes' Supplement.

If we pursue the approximation one stage further, we find from (12) terms in  $\alpha^5$ , additional to those expressed in (13). These are

$$y = \alpha^5 \left\{ \frac{373}{6.32} \cos x + \frac{243}{128} \cos 3x + \frac{125}{12.32} \cos 5x \right\} \dots \quad (18)$$

It is of interest to compare the potential and kinetic energies of waves that are not infinitely small. For the stream-function of the waves regarded as progressive, we have, as in (1),

$$\psi = -\alpha e^{-y} \cos(x - ct) + \text{terms in } \alpha^4,$$

so that

$$(d\psi/dx)^2 + (d\psi/dy)^2 = \alpha^2 e^{-2y} + \text{terms in } \alpha^5.$$

Thus the mean kinetic energy per length  $x$  measured in the direction of propagation is

$$\begin{aligned} \frac{\alpha^2}{2} \int dx \int_y^\infty e^{-2y} dy &= \frac{\alpha^2}{4} \int dx e^{-2y} = \frac{\alpha^2}{4} \int dx (1 - 2y + 2y^2) \\ &= \frac{\alpha^2}{4} \left\{ x + 2 \int y^2 dx \right\}, \end{aligned}$$

where  $y$  is the ordinate of the surface. And by (3)

$$\int y^2 dx = \left\{ \frac{1}{2}(\alpha^2 + \frac{5}{4}\alpha^4) + \frac{1}{8}\alpha^4 \right\} x.$$



Hence correct to  $\alpha^4$ ,

$$\text{K.E.} = \frac{1}{4}\alpha^2(1 + \alpha^2)x. \quad . \quad . \quad . \quad . \quad (19)$$

Again, for the potential energy

$$\text{P.E.} = \frac{1}{2}g \int y^2 dx = \frac{1}{2}gx(\frac{1}{2}\alpha^2 + \frac{3}{4}\alpha^4);$$

or since  $g=1-\alpha^2$ ,

$$\text{P.E.} = \frac{1}{4}\alpha^2(1 + \frac{1}{2}\alpha^2)x. \quad . \quad . \quad . \quad . \quad (20)$$

The kinetic energy thus exceeds the potential energy, when  $\alpha^4$  is retained.

### *Tide Races.*

It is, I believe, generally recognised that seas are apt to be exceptionally heavy when the tide runs against the wind. An obvious explanation may be founded upon the fact that the relative motion of air and water is then greater than if the latter were not running, but it seems doubtful whether this explanation is adequate.

It has occurred to me that the cause may be rather in the motion of the stream relatively to itself, *e.g.* in the more rapid movement of the upper strata. Stokes's theory of the highest possible wave shows that in non-rotating water the angle at the crest is  $120^\circ$  and the height only moderate. In such waves the surface strata have a mean motion forwards. On the other hand, in Gerstner and Rankine's waves the fluid particles retain a mean position, but here there is *rotation* of such a character that (in the absence of waves) the surface strata have a relative motion backwards, *i.e.* against the direction of propagation\*. It seems possible that waves moving against the tide may approximate more or less to the Gerstner type and thus be capable of acquiring a greater height and a sharper angle than would otherwise be expected. Needless to say, it is the steepness of waves, rather than their mere height, which is a source of inconvenience and even danger to small craft.

The above is nothing more than a suggestion. I do not know of any detailed account of the special character of these waves, on which perhaps a better opinion might be formed.

### *Rotational Fluid Motion in a Corner.*

The motion of incompressible inviscid fluid is here supposed to take place in two dimensions and to be bounded by two fixed planes meeting at an angle  $\alpha$ . If there is no rotation,

\* Lamb's Hydrodynamics, § 247.

the stream-function  $\psi$ , satisfying  $\nabla^2\psi=0$ , may be expressed by a series of terms

$$r^{\pi/\alpha} \sin \pi\theta/\alpha, \quad r^{2\pi/\alpha} \sin 2\pi\theta/\alpha, \dots r^{n\pi/\alpha} \sin n\pi\theta/\alpha,$$

where  $n$  is an integer, making  $\psi=0$  when  $\theta=0$  or  $\theta=\alpha$ . In the immediate vicinity of the origin the first term predominates. For example, if the angle be a right angle,

$$\psi = r^2 \sin 2\theta = 2xy, \quad . . . . . (1)$$

if we introduce rectangular coordinates.

The possibility of irrotational motion depends upon the fixed boundary not being closed. If  $\alpha < \pi$ , the motion near the origin is finite; but if  $\alpha > \pi$ , the velocities deduced from  $\psi$  become infinite.

If there be rotation, motion may take place even though the boundary be closed. For example, the circuit may be completed by the arc of the circle  $r=1$ . In the case which it is proposed to consider the rotation  $\omega$  is *uniform*, and the motion may be regarded as steady. The stream-function then satisfies the general equation

$$\nabla^2\psi = d^2\psi/dx^2 + d^2\psi/dy^2 = 2\omega, \quad . . . . . (2)$$

or in polar coordinates

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \frac{1}{r^2} \frac{d^2\psi}{d\theta^2} = 2\omega. \quad . . . . . (3)$$

When the angle is a right angle, it might perhaps be expected that there should be a simple expression for  $\psi$  in powers of  $x$  and  $y$ , analogous to (1) and applicable to the immediate vicinity of the origin; but we may easily satisfy ourselves that no such expression exists\*. In order to express the motion we must find solutions of (3) subject to the conditions that  $\psi=0$  when  $\theta=0$  and when  $\theta=\alpha$ .

For this purpose we assume, as we may do, that

$$\psi = \Sigma R_n \sin n\pi\theta/\alpha, \quad . . . . . (4)$$

where  $n$  is integral and  $R_n$  a function of  $r$  only; and in deducing  $\nabla^2\psi$  we may perform the differentiations with respect to  $\theta$  (as well as with respect to  $r$ ) under the sign of summation, since  $\psi=0$  at the limits. Thus

$$\nabla^2\psi = \Sigma \left( \frac{d^2R_n}{dr^2} + \frac{1}{r} \frac{dR_n}{dr} - \frac{n^2\pi^2}{\alpha^2 r^2} R_n \right) \sin \frac{n\pi\theta}{\alpha}. \quad . . (5)$$

\* In strictness the satisfaction of (2) at the origin is inconsistent with the evanescence of  $\psi$  on the rectangular axes.

The right-hand member of (3) may also be expressed in a series of sines of the form

$$2\omega = \delta\omega/\pi \cdot \sum n^{-1} \sin n\pi\theta/\alpha, \quad \dots \quad (6)$$

where  $n$  is an *odd* integer; and thus for all values of  $n$  we have

$$r^2 \frac{d^2 R_n}{dr^2} + r \frac{dR_n}{dr} - \frac{n^2 \pi^2 R_n}{\alpha^2} = \frac{4\omega}{n\pi} \{1 - (-1)^n\}. \quad (7)$$

The general solution of (7) is

$$R_n = A_n r^{n\pi/\alpha} + B_n r^{-n\pi/\alpha} + \frac{4\omega \alpha^2 r^2 \{1 - (-1)^n\}}{n\pi(4\alpha^2 - n^2 \pi^2)}, \quad (8)$$

the introduction of which into (4) gives  $\psi$ .

In (8)  $A_n$  and  $B_n$  are arbitrary constants to be determined by the other conditions of the problem. For example, we might make  $R_n$ , and therefore  $\psi$ , vanish when  $r=r_1$  and when  $r=r_2$ , so that the fixed boundary enclosing the fluid would consist of two radii vectores and two circular arcs. If the fluid extend to the origin, we must make  $B_n=0$ ; and if the boundary be completed by the circular arc  $r=1$ , we have  $A_n=0$  when  $n$  is even, and when  $n$  is odd

$$A_n + \frac{8\omega \alpha^2}{n\pi(4\alpha^2 - n^2 \pi^2)} = 0. \quad \dots \quad (9)$$

Thus for the fluid enclosed in a circular sector of angle  $\alpha$  and radius unity

$$\psi = 8\omega \alpha^2 \sum \frac{r^{n\pi/\alpha} - r^2}{n\pi(n^2 \pi^2 - 4\alpha^2)} \sin \frac{n\pi\theta}{\alpha}, \quad \dots \quad (10)$$

the summation extending to all odd integral values of  $n$ .

The above formula (10) relates to the motion of uniformly *rotating* fluid bounded by *stationary* radii vectores at  $\theta=0$ ,  $\theta=\alpha$ . We may suppose the containing vessel to have been rotating for a long time and that the fluid (under the influence of a very small viscosity) has acquired this rotation so that the whole revolves like a solid body. The motion expressed by (10) is that which would ensue if the rotation of the vessel were suddenly stopped. A related problem was solved a long time since by Stokes\*, who considered the *irrotational* motion of fluid in a *revolving* sector. The solution of Stokes's problem is derivable from (10) by mere

\* Camb. Phil. Trans. vol. viii. p. 533 (1847); Math. and Phys. Papers, vol. i. p. 305.

addition to the latter of  $\psi_0 = -\frac{1}{2}\omega r^2$ , for then  $\psi + \psi_0$  satisfies  $\nabla^2(\psi + \psi_0) = 0$ ; and this is perhaps the simplest method of obtaining it. The results are in harmony; but the fact is not immediately apparent, inasmuch as Stokes expresses the motion by means of the velocity-potential, whereas here we have employed the stream-function.

That the subtraction of  $\frac{1}{2}\omega r^2$  makes (10) an harmonic function shows that the series multiplying  $r^2$  can be summed. In fact

$$8\alpha^2 \sum \frac{\sin(n\pi\theta/\alpha)}{n\pi(u^2\pi^2 - 4\alpha^2)} = \frac{\cos(2\theta - \alpha)}{2 \cos \alpha} - \frac{1}{2},$$

so that

$$\psi/\omega = \frac{1}{2}r^2 - \frac{r^2 \cos(2\theta - \alpha)}{2 \cos \alpha} + 8\alpha^2 \sum \frac{r^{n\pi/\alpha} \sin n\pi\theta/\alpha}{n\pi(n^2\pi^2 - 4\alpha^2)}. \quad (11)$$

In considering the character of the motion defined by (11) in the immediate vicinity of the origin we see that if  $\alpha < \frac{1}{2}\pi$ , the term in  $r^2$  preponderates even when  $n=1$ . When  $\alpha = \frac{1}{2}\pi$  exactly, the second term in (11) and the first term under  $\Sigma$  corresponding to  $n=1$  become infinite, and the expression demands transformation. We find in this case

$$\begin{aligned} \psi/\omega = & \frac{1}{2}r^2 + \frac{2r^2}{\pi} (\theta - \frac{1}{4}\pi) \cos 2\theta + r^2 \sin 2\theta \left( \frac{2}{\pi} \log r - \frac{3}{2\pi} \right) \\ & + \frac{2}{\pi} \sum \frac{r^{2n} \sin 2n\theta}{n(n^2-1)}, \dots \dots \dots \quad (12) \end{aligned}$$

the summation commencing at  $n=3$ . On the middle line  $\theta = \frac{1}{4}\pi$ , we have

$$\psi/\omega = \frac{1}{2}r^2 - \frac{3r^2}{2\pi} + \frac{2}{\pi} \left\{ r^2 \log r - \frac{r^6}{3.8} + \frac{r^{10}}{5.24} - \dots \right\}. \quad (13)$$

The following are derived from (13) :—

$r$ .	$-\frac{1}{2}\pi\psi$ .	$r$ .	$-\frac{1}{2}\pi\psi$ .	$r$ .	$-\frac{1}{2}\pi\psi$ .
0.0	·00000	0.4	·14112	0.8	·13030
0.1	·02267	0.5	·16507	0.9	·07641
0.2	·06296	0.6	·17306	1.0	·00000
0.3	·10521	0.7	·16210		

The maximum value occurs when  $r = \cdot 592$ . At the point  $r = \cdot 592, \theta = \frac{1}{4}\pi$ , the fluid is stationary.

A similar transformation is required when  $\alpha=3\pi/2$ .

When  $\alpha=\pi$ , the boundary becomes a semicircle, and the leading term ( $n=1$ ) is

$$\psi/\omega = -\frac{8}{3\pi}r \sin \theta = -\frac{3}{8\pi}y \dots \dots \dots (14)$$

which of itself represents an irrotational motion.

When  $\alpha=2\pi$ , the two bounding radii vectors coincide and the containing vessel becomes a circle with a single partition wall at  $\theta=0$ . In this case again the leading term is irrotational, being

$$\psi/\omega = -\frac{32}{15\pi}r^{\frac{3}{2}} \sin \frac{1}{2}\theta. \dots \dots \dots (15)$$

*Steady Motion in a Corner of a Viscous Fluid.*

Here again we suppose the fluid to be incompressible and to move in two dimensions free from external forces, or at any rate from such as cannot be derived from a potential. If in the same notation as before  $\psi$  represents the stream-function, the general equation to be satisfied by  $\psi$  is

$$\nabla^4\psi=0 ; \dots \dots \dots (1)$$

with the conditions that when  $\theta=0$  and  $\theta=\alpha$ ,

$$\psi=0, \quad d\psi/d\theta=0. \dots \dots \dots (2)$$

It is worthy of remark that the problem is analytically the same as that of a plane elastic plate clamped at  $\theta=0$  and  $\theta=\alpha$ , upon which (in the region considered) no external forces act.

The general problem thus presented is one of great difficulty, and all that will be attempted here is the consideration of one or two particular cases. We inquire what solutions are possible such that  $\psi$ , as a function of  $r$  (the radius vector), is proportional to  $r^m$ . Introducing this supposition into (1), we get

$$\left\{ m^2 + \frac{d^2}{d\theta^2} \right\} \left\{ (m-2)^2 + \frac{d^2}{d\theta^2} \right\} \psi=0, \dots \dots (3)$$

as the equation determining the dependence on  $\theta$ . The most general value of  $\psi$  consistent with our suppositions is thus

$$\psi = r^m \{ A \cos m\theta + B \sin m\theta + C \cos (m-2)\theta + D \sin (m-2)\theta \}, \dots \dots \dots (4)$$

where A, B, C, D are constants.

Equation (4) may be adapted to our purpose by taking

$$m = n\pi/\alpha, \dots \dots \dots (5)$$

where  $n$  is an integer. Conditions (2) then give

$$\begin{aligned} A + C = 0, \quad A + C \cos 2\alpha - D \sin 2\alpha = 0, \\ \frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) D = 0, \\ \frac{n\pi}{\alpha} B + \left(\frac{n\pi}{\alpha} - 2\right) C \sin 2\alpha + \left(\frac{n\pi}{\alpha} - 2\right) D \cos 2\alpha = 0. \end{aligned}$$

When we substitute in the second and fourth of these equations the values of  $A$  and  $B$ , derived from the first and third, there results

$$\begin{aligned} C(1 - \cos 2\alpha) + D \sin 2\alpha = 0, \\ C \sin 2\alpha - D(1 - \cos 2\alpha) = 0; \end{aligned}$$

and these can only be harmonized when  $\cos 2\alpha = 1$ , or  $\alpha = s\pi$ , where  $s$  is an integer. In physical problems,  $\alpha$  is thus limited to the values  $\pi$  and  $2\pi$ . To these cases (4) is applicable with  $C$  and  $D$  arbitrary, provided that we make

$$A + C = 0, \quad B + \left(1 - \frac{2s}{n}\right) D = 0. \dots (5)$$

Thus

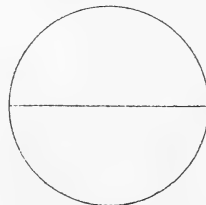
$$\begin{aligned} \psi = Cr^{n/s} \left\{ \cos\left(\frac{n\theta}{s} - 2\theta\right) - \cos\frac{n\theta}{s} \right\} \\ + Dr^{n/s} \left\{ \sin\left(\frac{n\theta}{s} - 2\theta\right) - \left(1 - \frac{2s}{n}\right) \sin\frac{n\theta}{s} \right\}, \quad (6) \end{aligned}$$

making

$$\nabla^2 \psi = 4\left(\frac{n}{s} - 1\right) r^{-2+n/s} \left\{ C \cos\left(\frac{n\theta}{s} - 2\theta\right) + D \sin\left(\frac{n\theta}{s} - 2\theta\right) \right\}. \dots \dots \dots (7)$$

When  $s=1$ ,  $\alpha=\pi$ , the corner disappears and we have simply a straight boundary (fig. 1). In this case  $n=1$  gives a nugatory result. When  $n=2$ , we have

Fig. 1.



$s = 1$

$$\psi = Cr^2(1 - \cos 2\theta) = 2Cy^2, \dots \dots \dots (8)$$

and  $\nabla^2 \psi = 4C$ . When  $n=3$ ,

$$\psi = Cr^3(\cos \theta - \cos 3\theta) + Dr^3(\sin \theta - \frac{1}{3} \sin 3\theta), \quad (9)$$

$$\nabla^2 \psi = 8r(C \cos \theta + D \sin \theta) = 8(Cx + Dy). \quad (10)$$

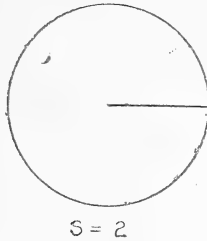
In rectangular coordinates

$$\psi = 4Cxy^2 + \frac{4}{3}Dy^3, \quad (11)$$

solutions which obviously satisfy the required conditions.

When  $s=2$ ,  $\alpha=2\pi$ , the boundary consists of a straight wall extending from the origin in *one* direction (fig. 2). In

Fig. 2.



this case (6) and (7) give

$$\psi = Cr^{\frac{3}{2}n} \left\{ \cos \left( \frac{1}{2}n\theta - 2\theta \right) - \cos \frac{1}{2}n\theta \right\} + Dr^{\frac{3}{2}n} \left\{ \sin \left( \frac{1}{2}n\theta - 2\theta \right) - \left( 1 - \frac{4}{n} \right) \sin \frac{1}{2}n\theta \right\}, \quad (12)$$

$$\nabla^2 \psi = (2n-4)r^{\frac{3}{2}n-2} \left\{ C \cos \left( \frac{1}{2}n\theta - 2\theta \right) + D \sin \left( \frac{1}{2}n\theta - 2\theta \right) \right\}. \quad (13)$$

Solutions of interest are afforded in the case  $n=1$ . The C-solution is ( $C=\frac{1}{4}$ )

$$\psi = \frac{1}{4}r^{\frac{3}{2}}(\cos \frac{3}{2}\theta - \cos \frac{1}{2}\theta) = -r^{\frac{3}{2}} \cos \frac{1}{2}\theta \sin^2 \frac{1}{2}\theta, \quad (14)$$

vanishing when  $\theta=\pi$ , as well as when  $\theta=0$ ,  $\theta=2\pi$ , and for no other admissible value of  $\theta$ . The values of  $\psi$  are reversed when we write  $2\pi-\theta$  for  $\theta$ . As expressed, this value is negative from 0 to  $\pi$  and positive from  $\pi$  to  $2\pi$ . The minimum occurs when  $\theta=109^\circ 28'$ . Every streamline which enters the circle ( $r=1$ ) on the left of this radius leaves it on the right.

The velocities, represented by  $d\psi/dr$  and  $r^{-1}d\psi/d\theta$ , are infinite at the origin.

For the D-solution we may take

$$\psi = r^{\frac{1}{2}} \sin^3 \frac{1}{2}\theta. \quad \dots \quad (15)$$

Here  $\psi$  retains its value unaltered when  $2\pi - \theta$  is substituted for  $\theta$ . When  $r$  is given,  $\psi$  increases continuously from  $\theta=0$  to  $\theta=\pi$ . On the line  $\theta=\pi$  the motion is entirely transverse to it. This is an interesting example of the flow of viscous fluid round a sharp corner. In the application to an elastic plate  $\psi$  represents the displacement at any point of the plate, supposed to be clamped along  $\theta=0$ , and otherwise free from force within the region considered. The following table exhibits corresponding values of  $r$  and  $\theta$  such as to make  $\psi=1$  in (15) :—

$\theta$ .	$r$ .	$\theta$ .	$r$ .
150°	1.00	60°	64.0
150°	1.23	20°	10 <sup>6</sup> × 3.65
120°	2.37	10°	10 <sup>6</sup> × 2.28
90°	8.00	0°	∞

When  $n=2$ , (12) appears to have no significance.

When  $n=3$ , the dependence on  $\theta$  is the same as when  $n=1$ . Thus (14) and (15) may be generalized :

$$\psi = (Ar^{\frac{1}{2}} + Br^{\frac{3}{2}}) \cos \frac{1}{2}\theta \sin^2 \frac{1}{2}\theta, \quad \dots \quad (16)$$

$$\psi = (A'r^{\frac{1}{2}} + B'r^{\frac{3}{2}}) \sin^3 \frac{1}{2}\theta. \quad \dots \quad (17)$$

For example, we could satisfy either of the conditions  $\psi=0$ , or  $d\psi/dr=0$ , on the circle  $r=1$ .

For  $n=4$  the D-solution becomes nugatory ; but for the C-solution we have

$$\psi = Cr^2(1 - \cos 2\theta) = 2Cr^2 \sin^2 \theta = 2Cy^2. \quad \dots \quad (18)$$

The wall (or in the elastic plate problem the clamping) along  $\theta=0$  is now without effect.

It will be seen that along these lines nothing can be done in the apparently simple problem of a horizontal plate clamped along the rectangular axes of  $x$  and  $y$ , if it be supposed free from force\*. Ritz † has shown that the

\* If indeed gravity act,  $w=x^2y^2$  is a very simple solution.

† *Ann. d. Phys.* Bd. 28, p. 760, 1909.



solution is not developable in powers of  $x$  and  $y$ , and it may be worth while to extend the proposition to the more general case when the axes, still regarded as lines of clamping, are inclined at any angle  $\alpha$ . In terms of the now oblique coordinates  $x, y$  the general equation takes the form

$$(d^2/dx^2 + d^2/dy^2 - 2 \cos \alpha d^2/dx dy)^2 w = 0, \quad . \quad (19)$$

which may be differentiated any number of times with respect to  $x$  and  $y$ , with the conditions

$$w = 0, \quad dw/dy = 0, \quad \text{when } y = 0, \quad . \quad (20)$$

$$w = 0, \quad dw/dx = 0, \quad \text{when } x = 0. \quad . \quad (21)$$

We may differentiate, as often as we please, (20) with respect to  $x$  and (21) with respect to  $y$ .

From these data it may be shown that at the origin *all* differential coefficients of  $w$  with respect to  $x$  and  $y$  vanish. The evanescence of those of zero and first order is expressed in (20), (21). As regards those of the second order we have from (20)  $d^2w/dx^2 = 0$ ,  $d^2w/dx dy = 0$ , and from (21)  $d^2w/dy^2 = 0$ . Similarly for the third order from (20)

$$d^3w/dx^3 = 0, \quad d^3w/dx^2 dy = 0,$$

and from (21)

$$d^3w/dy^3 = 0, \quad d^3w/dx dy^2 = 0.$$

For the fourth order (20) gives

$$d^4w/dx^4 = 0, \quad d^4w/dx^3 dy = 0,$$

and (21) gives

$$d^4w/dy^4 = 0, \quad d^4w/dx dy^3 = 0.$$

So far  $d^4w/dx^2 dy^2$  might be finite, but (19) requires that it also vanish. This process may be continued. For the  $m+1$  coefficients of the  $m$ th order we obtain four equations from (20), (21) and  $m-3$  by differentiations of (19), so that all the differential coefficients of the  $m$ th order vanish. It follows that every differential coefficient of  $w$  with respect to  $x$  and  $y$  vanishes at the origin. I apprehend that the conclusion is valid for all angles  $\alpha$  less than  $2\pi$ . That the displacement at a distance  $r$  from the corner should diminish rapidly with  $r$  is easily intelligible, but that it should diminish more rapidly than any power of  $r$ , however high, would, I think, not have been expected without analytical proof.

XXIV. *On the Recent Theories of Electricity.* By LOUIS T. MORE, Ph.D., Professor of Physics, University of Cincinnati\*.

THE theories of matter and electricity which have been recently advanced have aroused the interest of the thinking public generally, and rather startling accounts have appeared concerning a scientific revolution. In a series of essays † I attempted to give a more or less philosophical discussion of these theories; their effect on thought; and their relation to the older atomic theories. During the writing of these essays, I became convinced that the new ideas were sufficiently crystalized to permit of a more technical and critical discussion of their merits and defects. And in spite of an apparent advantage they may have over the older conceptions, it is quite possible that this gain is not permanent, and that the present movement is on the whole harmful to correct scientific procedure because it tends to obliterate the boundary between science and metaphysics. But I should have hesitated to raise any protest if I had not believed that a simple modification in the definition of electricity would reconcile many of the discrepancies between the new and older theories, and make unnecessary the substitution of electricity for matter and electrodynamics for mechanics.

While it is not possible to draw a definite boundary line between the regions of physics and metaphysics, still we may do so in a general way by saying that the domain of physics concerns the discovery of phenomena and the formulation of natural laws based on postulates which are determined by experience and generally accepted as true; the causes of phenomena and the discussion of the postulates of science are the province of the metaphysician. This differentiation in methods of thought cannot be rigidly adhered to since this boundary line is more or less obscure, and is liable to considerable displacement as a science advances; but the acceptance of this principle would prevent much of the confusion which has been introduced into the science of physics by writers who have not recognized this to be a general rule. For example, the principle of relativity is not strictly a physical law but the expression, in mathematical symbols, of the general philosophical law of the finite nature of the human mind which has been accepted for centuries. Again, the discussion of the shape of the atom or electron is not

\* Communicated by the Author.

† L. T. More, *Hibbert Journal*, July 1909 and July 1910.

a physical problem, as it is incapable of verification by experience. This does not mean that such questions should not be discussed, but the method of their discussion and the results obtained are properly the method and results of metaphysics and are not in the category of physical phenomena and laws.

As the purpose of this article is to show that we may retain the idea of the mechanical nature and invariable inertia of matter, and at the same time account for the electromagnetic momentum and other properties of electrified matter by considering electricity as an attribute of matter rather than the converse, it is convenient to state and discuss a few postulates beforehand which bear on the subject.

We shall assume length, mass, and time to be the fundamental units of measure. These quantities and their derivatives are continuous or, at least, indefinitely divisible. The continuity of space and time is generally accepted; without this belief it is impossible to establish the geometrical laws founded on the point, line, and surface or the analytical laws of the calculus. But the divisibility of matter is not usually supposed to be infinite. Indeed, the denial of this assertion is the foundation of all atomic theories. Yet it is difficult to see how mathematics can be anything but abstract logic, or how it can be applied to physical problems unless this third fundamental quantity, which is as it were the connecting link between the abstract and the concrete, be also indefinitely divisible. How, otherwise, can we replace finite bodies by mathematical centres of inertia? In this connexion Professor Sir Joseph Larmor \* says: "The difficulty of imagining a definite uniform limit of divisibility of matter will always be a philosophical obstacle to an atomic theory, so long as atoms are regarded as discrete particles moving in empty space. But as soon as we take the next step in physical development, that of ceasing to regard space as mere empty geometrical continuity, the atomic constitution of matter (each ultimate atom consisting of parts which are incapable of separate existence, as Lucretius held) is raised to a natural and necessary consequence of the new standpoint." This is clearly an attempt to reconcile the two incommensurable antinomies of continuity and discontinuity, which are usually attached to the names of Descartes and Lucretius. This Sir J. Larmor tries to do by postulating the existence of a *true* matter, which is a continuous plenum and imperceptible to our senses, and relegating *sensible*

\* 'Ether and Matter,' p. 76.

matter to the rôle of a mere variation in this otherwise changeless plenum,—making it an attribute rather than an entity. If his theory denies the infinite divisibility of matter, it apparently accepts its indefinite divisibility; the atom, as a variation limited only by our power of observation, must become smaller with each advance in the refinement of our apparatus. Such a plenum must remain a pure creation of the imagination, and its existence is not determinable by physical or experimental methods; it must therefore be classed as a problem for the metaphysician. The distinction between atoms continually diminishing in size and the infinite, or at least indefinite, divisibility of matter is merely a question of words—the definition of what matter is.

As another fundamental principle, we shall postulate the objective reality and conservation of matter. The quantitative measure of this matter is its mass or inertia, which is also to be taken as an invariable factor in the derived quantities, force and energy. M. Hannequin\* expresses this idea well when he says: “Il n'existe donc rien, dans le monde mécanique, que des masses en mouvement ou, pour parler un langage rigoureux, qu'une somme constante d'énergie de mouvement et des masses sur lesquelles elle se distribue.” Although mass is here considered to be infinitely divisible, its scientific unit of measurement is at any time, the least amount of which we have cognizance; at present, this happens to be the electron or corpuscle. Further consideration of this unit is left to the discussion of the atom.

Few things have been brought out more clearly by the work of the school of energetics than that, if we accept the doctrine of the conservation of energy, either of the two quantities, mass or energy, may be considered as the fundamental unit from which the other can be derived. This undoubtedly follows from the fact that we have no conception of mass without energy or of energy without mass. But while it is thus possible mathematically to make either of them a starting point for the explanation of phenomena, the advocates of energetics apparently soon develop a pronounced tendency to prefer the abstract to the concrete and to subtilize objective facts into metaphysical ideas. A science like physics, to be useful and not merely an intellectual gymnastic, should preserve in all its speculations a close touch with the practical and the concrete, a certain common sense. The history of the science shows these advantages have been obtained most frequently by those who maintain

\* *L'hypothèse des atomes*, p. 127.

mass to be a fundamental unit: The failure of the mechanistic school has arisen from the attempt to explain the nature of matter, the cause of its forces, and the properties of atoms. However we may try to reason away the belief in the objective reality of matter, our minds persistently cling to the advantage, and even necessity, of such a postulate; and we consciously or unconsciously endow any substitute of it with all the properties of matter, excepting its name.

Energy therefore, although it is an inalienable property of matter, will remain a derived unit, the total quantity of which is a constant. It is customary and convenient to divide energy into two classes—kinetic and potential. The measure of potential energy is usually taken to be the product of the attractive force of two masses into a power of their distance apart. In the majority of problems we can go no further, but in certain cases, as for instance the pressure of gases, we may express a portion of the energy of the whole mass as due to the kinetic energy of small, or molecular, portions of it. But the internal energy of a gas must still be considered as strictly potential and incapable of further explanation. Kinetic energy is mass into a function of velocity and its formula is  $T = m\phi(v)$ . There is gradually arising a tendency, which is well founded, to distinguish two kinds of kinetic energy. The first is that produced by the momentum of a moving body, the *vis viva* whose measure is  $1/2mv^2$ ; the other is the kinetic energy, which originates in one body and is transferred through space, apparently vacuous, to another body—in other words, the radiant energy known as heat, light, and electricity.

It has been the persistent attempt of physicists for centuries to explain this radiant energy by mechanical analogies. And this effort has fastened on the science an interminable series of impossible fictitious æthers and mechanical atoms. The most indefatigable labours of the greatest minds have been spent to imagine an atom, which would serve satisfactorily as a source and, at the same time, as a receptacle of radiant energy, and an æther which would transfer it. Not one of these models has been even partially adequate: the course of the development has been steadily from the simple to the complex, from the concrete to the abstract, from the physical to the metaphysical, until the most recent atom is a complex more intricate than a stellar cosmogony whose parts are an entity called electricity, and the æther is an abstraction devoid of any mechanical attributes. Out of all this controversy we have gained the following facts:—Heat, light,

and electrical energy, originating in one body, pass through space undiminished and unaugmented to another body. We can also express this energy as kinetic energy while it is associated with matter. In transit, since our experience gives us no clue or criterion, we can assume as a formula for the energy, either a periodic motion of an hypothetical something, called an æther, or a projectile motion of an hypothetical mass-particle. In either case, all we really do is to divide the initial or final material energy into two mathematical quantities, one a mass-factor and the other a velocity-factor, and give to each such a value as to make their product remain a constant. As a rule, we make the mass-factor so small that we can shut our eyes to its existence and imagine anything about it we please. The time relation is fixed by experiment. For the purposes of theory, although this radiant energy appeals to our senses in the three forms of heat, light, and electricity, which in their qualitative respect are each fundamental and not referable one to another, we fortunately find that quantitatively all three are satisfied by one dynamic formula. We have therefore obtained an adequate quantitative knowledge of energy, but not an inkling of the qualitative coefficients in this formula.

The hypothesis of the æther is an attempt to accomplish the impossible. And while it is now generally admitted that we cannot create such a substance as will satisfy the physical requirements of a transmitter of radiant energy, still the æther is claimed to be a useful hypothesis. This utility is said to consist in giving us a crude image, in a mechanical way, of what occurs. In other words, it lessens our innate dislike to confessing complete ignorance, and it provides a set of concrete analogies for abstract statements and equations. Thus Poincaré says in the preface to his *Théorie de la Lumière*: “Peu nous importe que l'éther existe réellement ; c'est l'affaire des métaphysiciens ; l'essentiel pour nous c'est que tout se passe comme s'il existait et que cette hypothèse est commode pour l'explication des phénomènes. Après tout, avons nous d'autre raison de croire à l'existence des objets matériels ? Ce n'est là aussi qu'une hypothèse commode ; seulement elle ne cessera jamais de l'être, tandis qu'un jour viendra sans doute où l'éther sera rejeté comme inutile.” Now the old elastic-solid and mechanical æther did afford us a concrete image and we could speak of it with some intelligence to one another, because everyone has a conception of an elastic solid. To be sure, this solid æther became a grotesque. It permitted the transference of heat and light energy, but only at the

expense of creating a kind of matter entirely outside of, and contradictory to, anything in our experience. We have only to recall the properties ascribed to this æther to find that it operated equally well if it had a density indefinitely great or one indefinitely small; if it were rigid or if it were collapsible, &c. As certainly as one physicist endowed it with a property, another arose who showed that just the opposite property was equally efficient. Yet we might still be staggering along with the conviction that somehow this supposititious stuff was of use to us; at least it gave us a set of words conveying some meaning. But when Maxwell proved mathematically that a third kind of radiant energy of an electrical type should be looked for, and when Hertz demonstrated its existence, no elastic solid would serve for all three kinds; and so, for a time, we were taught simultaneously the properties of two co-existent æthers. An elastic-solid and a so-called electromagnetic æther in one space were not amicable, and the former soon acted as Lord Kelvin had suggested, it really collapsed. Maxwell's idea produced a revolution in the theory of physics; heat and light remained no longer a form of mechanical waves but became electromagnetic waves of special periodicity. By a progressive subtilization we have now arrived at Sir J. Larmor's celebrated definition of æther which will satisfy all forms of radiant energy. The æther\* is "a *plenum* with uniform properties throughout all extension, but permeated by intrinsic singular points, each of which determines and, so to speak, locks up permanently a surrounding steady state of strain or other disturbance." This plenum is continuous, without atomic structure, and absolutely quiescent. Since these points of intrinsic strain are the atoms of matter, "the † ultimate element of material constitution being taken to be an electric charge or nucleus of permanent ætherial strain," it is evident "that ‡ the motion of matter does not affect the quiescent æther except through the motion of the atomic electric charges carried along with it." These ideas evidently reduce matter to an attribute of electricity, and make all forces of the type called electrical forces. But if electricity is everything, we must inevitably some time explain pure mechanical actions in terms of this electrical substance. Sir J. Larmor clearly foresees this, as shown by his statement §: "The electric character of the forces of chemical affinity was an accepted part of the chemical views of Davy, Berzelius, and Faraday; and more

\* 'Æther and Matter,' p. 77.

† L. c. p. 27.

‡ L. c. p. 2.

§ L. c. p. 165.

recent discussions, while clearing away crude conceptions, have invariably tended to the strengthening of that hypothesis. The mode in which the ordinary forces of cohesion could be included in such a view is still quite undeveloped." He thus rather leaves this question in the air by concluding that a complete theory is not necessary. But the history of science shows that we shall soon hypothecate two æthers or try to give properties to one which will include electrical, chemical, and material forces; indeed this latter is already being attempted. Evidently he fears the tendency of explaining things too exactly by mechanical analogies, for he believes \* "all that is known (or perhaps need be known) of the æther itself may be formulated as a scheme of differential equations defining the properties of a *continuum* in space, which it would be gratuitous to further explain by any complication of structure; though we can with great advantage employ our stock of ordinary dynamical concepts in describing the succession of different states thereby defined." Yet it seems that a more complicated structure than the modern molecule, composed of an interminable series of electrical ætherial strains, could hardly be conceived.

If the conception of an elastic-solid æther was admittedly a fiction of the mind, and one impossible to align with any known kind of matter, the electromagnetic æther is so esoteric, so subtilized from all substance, that it merely provides a nomenclature for a set of equations expressing the propagation of radiant energy. Both Sir J. Larmor and Professor Lorentz give the impression in their writings that the least said of the properties of such an æther the better, since the final verdict will be that the process of radiant energy, in transit through space, is best expressed as an equation containing unknowable qualitative coefficients. We may well go still further, for I believe the time is rapidly approaching when all scientific discussion of the nature of the æther will be considered futile. But Sir J. Larmor does lay down in his treatise certain attributes, mostly negative, which he accepts. Thus, the æther is a continuous and quiescent plenum, absolutely unaffected by mechanical energy. Existing in it, are countless places of discontinuity, or electrical strains, which constitute the elements of matter. Its only positive properties are the ability to maintain such strains and to transmit any electromagnetic disturbance with the velocity of light. All chemical and mechanical forces must therefore be attributes of electricity, or else referable to some other distinct cause.

\* *L. c.* p. 78.



Professor Lorentz recognizes the growth of the idea that it is unnecessary for the physicist to dwell on the mechanism of the coefficients introduced into our equations, but he adheres to the view\* that "we cannot be satisfied with simply introducing for each substance these coefficients, whose values are to be determined by experiment; we shall be obliged to have recourse to some hypothesis about the mechanism that is at the bottom of the phenomena." In respect to the æther he is exceedingly vague, and so far as a mechanism of it is concerned, gives nothing. For example, while speaking of the state of this medium when it is the seat of an electromagnetic field, he says †: "We need by no means go far in attempting to form an image of it and, in fact, we cannot say much about it." In agreement with Sir J. Larmor, his æther is a plenum always at rest, capable of maintaining and of transmitting electric strains, and containing electrons, or extremely small particles charged with electricity. Sometimes he gives the impression that these electrons are electricity only. The æther not only penetrates the spaces between the electrons but also pervades them: "We ‡ can reconcile ourselves with this, . . . by thinking of the particles of matter as of some local modifications in the state of the æther." Here also, mechanical forces and attributes are discarded, and he holds: "That § the phenomena going on in some (any?) part of the æther are *entirely* determined by the electric and magnetic force existing in that part." Now we are indebted to Maxwell, and to him alone, for a set of electromagnetic stresses and strains which will satisfy the requirements of the electromagnetic field. In his mind, there is no doubt, these stresses produced a material strain in the æther which was communicated physically to dielectrics existing in the æther and materially distorted them. It is, of course, impossible to test experimentally the existence of these strains in the free æther, but in a long series of papers in this journal I have shown that no distortion is produced by them in dielectrics. These experiments were undertaken when the real existence of the ætherial stresses was generally accepted. I was led to the opposite opinion, because in no other case had any static connexion been found between æther and matter, and because Helmholtz had shown that such stresses would produce motion in the æther, an effect unlikely to be true. Professor Lorentz certainly concurs in this opinion. He states ||: "While thus denying the real

\* 'Theory of Electrons,' p. 8.

† *L. c.* p. 1.

§ *L. c.* p. 26.

‡ *L. c.* p. 11.

|| *L. c.* p. 31.

existence of æther stresses, we can still avail ourselves of all the mathematical transformations by which the application of the formula may be made easier. We need not refrain from reducing the force to a surface-integral, and for convenience sake we may continue to apply to the quantities occurring in this integral the name of stresses. Only, we must be aware that they are only imaginary ones, nothing else than auxiliary mathematical quantities." It is also, I think, safe to say that Sir J. Larmor believes in the fictitious character of the Maxwellian stresses. Does not this also lead to the idea that electrons, which are disembodied electricity and which produce these imaginary stresses, are themselves imaginary?

There is at present a controversy whether these electrons are rigid or deformable. The only consequence of these two views necessary to comment on now is a very pertinent remark of Hr. Abraham\*, that if the electron be deformable, work will be required to effect this deformation, and to avoid contradiction with the law of conservation of energy, the electron must possess internal potential energy. This opinion of Hr. Abraham is almost impossible to avoid. To provide the electron itself with this kind of energy is to deny its character as the fundamental and indivisible unit of matter, for a body having potential energy must contain mutually reacting parts which may themselves be considered as units of a lower order; nor will many approve of M. Poincaré's rather embarrassing suggestion, that the æther may be a great and inexhaustible store-house of energy, drawn on at will by the electron each time it moves. This idea will hardly be taken seriously, as the assumption of unlimited energy existing in a fictitious æther is in no sense a scientific notion; it contradicts the prevailing idea of the inertness of the æther and makes of it a sort of *deus ex machina* which interposes to help us out of difficulties. And indeed the electromagnetic æther, without material properties other than imaginary stresses, is an explanation more difficult to grasp than the phenomena of radiant energy which require explanation.

Such a revolution in the nature of æther requires a like one in our ideas of matter. The most notable effort in theoretical physics, at the present time, is the hypothesis that the ultimate element of matter is not a material atom, a sort of microcosm of sensible matter, but a *free* electrical charge, considered to be an entity for the purpose; added to this are the dependent ideas that inertia and all other properties of

\* *Theorie der Elektrizität*, II. Kap. 3, Leipzig 1905.

matter are attributes of electricity. This hypothesis can mean nothing else than that the Lucretian atom, the centres of force of Boscovich, the vortices of Kelvin and all the atomic models (made of weights and springs and strings), have failed and become useless as aids to the imagination.

Sir J. Larmor defines this new atom as a proton\*, "in whole or in part a nucleus of intrinsic strain in the æther, a place at which the continuity of the medium has been broken and cemented together again (to use a crude but effective image) without accurately fitting the parts, so that there is a residual strain all round the place." This strain is not of the character of mechanical elasticity, since the "ultimate † element of material constitution is taken to be an electric charge or nucleus of permanent ætherial strain instead of a vortex ring: . . . . The molecule is composed simply of a system, probably large in number, of positive and negative protons in a state of steady orbital motion round each other: . . . . And, moreover, we may imagine complex structures composed of these primary systems as units, for example successive concentric rings of positive or negative electrons sustaining each other in position."

Positive and negative electrons differ only in their orbital motion from each other and their forces are all of the electrical type. Prof. Sir J. J. Thomson pictures the atoms of the various chemical elements as nuclei of free positive electricity holding in electrical equilibrium free negative charges, placed in various geometrical designs. The degree of stability is determined by the radioactivity of each element. Professor Lorentz considers the proton to be a small particle charged with electricity and probably a local modification of the æther; but his work on electromagnetic mass leads one to the opinion that he believes electricity to be the real essence of the material universe. Professor Abraham and the modern school of German physicists are frankly endeavouring to give a purely electromagnetic foundation to the mechanism of the electron and to mechanical actions in general.

Now to me, and I believe to many men of science, the chief and indeed only value of an atomic theory is to give a concrete, though crude, image of matter reduced to its simplest conditions. The word electricity gives me no such image of matter; it conveys absolutely no idea of materiality nor even of space or time relations. What the originators of the electrical atom have done is apparently to transpose the words, matter and electricity, tacitly giving to the latter

\* 'Æther and Matter,' p. 26.

† *L. c.* p. 27 *passim*.

all the ideas usually associated with the former. We may as well take the next step at once and raise the objective universe on the Leibnitzian monad or on Schopenhauer's philosophy of "Die Welt als Wille und Vorstellung."

Again, the law of the conservation of matter has been one of the most fertile ideas in science; according to this law at least one attribute, inertia, remains constant however all others may change, thus giving continuity to material bodies as well as to space and time. It is quite possible to imagine an element of this new electric matter to be composed of equal quantities of positive and negative electrons, whose motions are so balanced as to make *all* material attributes vanish and produce a quasi-annihilation of matter.

Lastly, when the statement is made that the electron is merely a local modification of the all-pervading æther, some idea should be given us as to the nature of this modification. If it is of the character of a strain, no meaning is conveyed unless this strain is subject to the laws of static or kinetic mechanics. But we have no knowledge of a static strain which fulfils the requirements of matter, especially that it must be localized at definite points and must be uncreatable and indestructible; of kinetic strains, the only one at present available is the vortex ring of Helmholtz and Kelvin. To imply that matter is electricity and that electricity is a static strain or a vortex ring, is to make an impossible assumption and is reasoning in a circle. If the vortex ring of matter failed chiefly because Maxwell said\*: "That at best it was a mode of motion and not matter as we know it," what chance has this new type to survive criticism?

In accordance with the view taken in this paper, no hypothesis will be made to express properties of an æther, whose existence is itself incapable of scientific proof. It is, at the same time, perfectly proper to distinguish space through which radiant energy passes by a special name, such as the æther. The amount of radiant energy in transit is best given by an equation expressing conservation of energy and containing a velocity and an inertia factor. The velocity factor of this equation, most conveniently, takes the form of a periodic motion, but no assumptions need be made as to the nature of the periodicity or of the inertia factor since they also are not subject to experimental verification.

Although matter appears to us as a continuous quantity or at least as divisible far below our present methods of experimentation, still it is convenient to give to the smallest observable portion of matter some such name as protion.

\* *Encycl. Brit.*: The Atom.

This unit of matter must be reduced in size as refinement of observation increases so that we may always be able to discuss it mathematically in the aggregate only.

At the present time this proton is the electron, and the only attributes necessary to assign to it are inertia in the Newtonian sense, a force of gravitational attraction and a force of electrical attraction, either positive or negative in sign. No causes for these attributes can be given as they are fundamental. If the experiments of Kaufmann, which show that an electrified particle in motion has an apparently increased momentum, are cited as supporting the view that inertia is a function of velocity and should be considered as an attribute of an invariable quantity, the electrical charge, I hope to show that it is possible to accept Kaufmann's results and at the same time the invariability of inertia.

Before proceeding further with this discussion it is convenient to assemble the foregoing ideas in a concise form.

1. The fundamental quantitative units are length, mass, and time. These are continuous functions, or at least indefinitely divisible.

2. Matter has an objective reality and its quantity is measured by its mass or inertia.

3. Mass is an invariable function whose total quantity is conservative.

4. Energy is a conservative function.

5. Energy is divided, for convenience, into three types: potential, kinetic, and radiant energy.

6. Potential energy depends on force and position and is measured by the formula,  $V = m\phi\{f \cdot (l - l')\}$ .

7. Kinetic energy is the energy of a moving body: its formula is  $T = m\phi(v)$ .

8. Radiant energy is the expression for the fact that heat, light, and electromagnetic energy pass through free space. It is not associated with matter and is conveniently expressed as a function of an "inertia" and a "velocity" factor,  $R = \phi(m \cdot v)$ . The velocity factor will be taken to be a periodic motion with a translational velocity of  $3 \times 10^{10}$  centimetres per second. The inertia factor is twice the amount of the energy divided by the velocity squared.

9. The æther is a name given to a fictitious substance whose inertia is the inertia factor of radiant energy.

10. The proton, using this name to avoid confusion with the atom of chemical reactions, is the least portion of matter recognized experimentally. It is the scientific unit of mass and can be dealt with mathematically only in aggregates. At the present time this unit is the electron.

So far these ideas have more or less approval and have been already discussed. Those following are more novel and need to be supported.

11. The electron has an invariable ponderable mass,  $m$ , and a variable electromagnetic mass,  $m_e$ , due to its electrical charge when in motion. Its total effective inertia is therefore,  $M = m + m_e$ .

12. The electrical charge,  $e$ , of an electron is an unexplainable property of matter, measured by its force of electrical attraction. Instead of adopting the hypothesis that the electric charge on an electron is constant, we shall consider quantity of electricity to be a function of the velocity of matter. Electromagnetic mass thus becomes an attribute of matter somewhat analogous to hydrodynamic mass. The difference between positive and negative electricity may depend only on the direction of the orbital motion of the electron.

13. The electron possesses a force of gravitational attraction for other electrons, expressed by the law,

$$F_m \propto mm' \phi(l-l').$$

14. The electron has an additional force of electrical attraction according to the formula,

$$F_e \propto \phi\{ee'. (l-l')\}.$$

So great a revolution in thought as to consider inertia a variable quantity and to substitute electricity for matter as the substance of the universe, would only have been undertaken from necessity. A mere matter of convenience would scarcely warrant the labour of revising the work of the past and of discarding what has been considered, until lately, as definitely established. The need for some such radical change in theory is based on the experimental facts discovered in connexion with the passage of electricity through highly rarefied gases, and with radioactivity.

We may consider it established that the phenomena noted, when electricity is discharged in a high vacuum, are most readily explained by supposing the current due to a stream of electrified particles moving with a velocity comparable to light. The experiments of Sir J. J. Thomson and C. T. R. Wilson go to show that the masses of these projectiles, when charged negatively, are about the one-thousandth part of the mass of a hydrogen atom, provided the charge on each is assumed to be the same and equal to that of the hydrogen electrolytic ion. Those charged positively are comparable to the various chemical atoms.

The Zeeman effect, in its elementary form, is satisfactorily explained by this theory, although the recently discovered complex character of the phenomenon is not accounted for.

Radioactivity, on the whole, is best explained by the projection of positive and negative electrons from a certain class of bodies.

And lastly, Kaufmann has shown by a delicate experiment that the effective inertia of an electron is a function of its velocity. This conclusion has been confirmed by others, although in minor points there is a considerable difference in results and opinions.

It must not be lost sight of that all these experiments deal with quantities of matter, supposing it to exist, too small to be appreciable by either chemical analysis or mechanical apparatus, such as the balance, etc. They are ultimately measured by the force of electrical attraction of an electrical charge. We are, therefore, experimenting with matter which appeals to us through only one of its attributes. Is it not almost inevitable that an exclusive attention paid to this single attribute is likely to exalt it into an undue prominence? We have had, in the past, examples of much the same sort of reasoning. When the phenomena of light were predominantly discussed, physicists drifted into the opinion that this property of matter could be explained only by creating a light substance. Again, this process of reasoning occurred when heat was first investigated: we had the creation of caloric. And now we are asked to do the same thing with electricity. It is safe to predict that history will be repeated again, and that electrical charges and their forces will also sink into the condition of an attribute of matter.

It might certainly be true that two experiments showing equal electrical charges would, if we could measure the amount of matter concerned, provide us with unequal quantities of matter, just as conversely equal quantities of matter might show different quantities of electricity. The hypothesis of equivalence of electrical charge and matter rests solely on an analogy to electrolysis, where matter is in a quite different state and also where the equivalence may be only approximate. Matter, on the other hand, in a solid state shows no connexion between volume and density and electrical charge. In dealing with electricity we should not forget the immense superiority of electrical detectors in delicacy to those for mechanical quantities, so that we can appreciate far smaller quantities of electrified than of neutral bodies.

There is no doubt, from the quotations given, that theorists are basing their work on the assumption of the electron as the

unit of matter. And they give to it the following properties:—its mass is wholly electromagnetic; the motive forces are electric forces; and the laws of mechanics are to be deduced from the laws of electromagnetics.

Professor Abraham defines the electron as a rigid sphere with an electric charge distributed uniformly throughout it in concentric spherical shells. This charge is a constant, and its volume and surface densities are homogeneous. Neglecting the obscurity which occurs when we try to imagine *what the sphere* is, on which the electricity is distributed, and what its measure is if it be not mass, we follow him in the development of his equations for energy and momentum.

The total electric or potential energy is given by the equation

$$U = \frac{e^2}{32\pi R} \left[ \frac{3-\beta^2}{\beta} \log \frac{1+\beta}{1-\beta} - 2 \right];$$

the magnetic or kinetic energy by

$$T = \frac{e^2}{32\pi R} \left[ \frac{1+\beta^2}{\beta} \log \frac{1+\beta}{1-\beta} - 2 \right];$$

and the electromagnetic momentum by

$$|G| = \frac{2T}{v} = \frac{e^2}{16\pi R V} \left[ \frac{1+\beta^2}{\beta^3} \log \frac{1+\beta}{1-\beta} - \frac{2}{\beta} \right];$$

where

$e$  is the electric charge;  
 $R$ , the radius of the sphere;  
 $v$ , the velocity of the electron;  
 $V$ , the velocity of light;  
 $\beta = \frac{v}{V}$ .

Since the electromagnetic mass factor of  $|G|$  is dependent on the velocity, we may separate it into two components—a “longitudinal” mass due to a change in linear velocity alone, and a “transverse” mass due to a change in direction only. The equation of motion thus becomes

$$F = (m_0 + m')f' + (m_0 + m'')f'',$$

where

$F$  is any external force;  
 $m_0$ , the ponderable mass;  
 $m'$ , the longitudinal electromagnetic mass;  
 $m''$ , the transverse electromagnetic mass;  
 $f'$  and  $f''$ , the corresponding accelerations.



From  $|G|$ , he finds that

$$m' = \frac{e^2}{4\pi R V^2} \left\{ \frac{2}{3} + \frac{4}{5}\beta^2 + \frac{6}{7}\beta^4 + \dots \right\}$$

and

$$m'' = \frac{e^2}{8\pi R V^2} \left\{ \left(1 + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{5}\right)\beta^2 + \left(\frac{1}{5} + \frac{1}{7}\right)\beta^4 + \dots \right\}.$$

For a stationary electron,

$$\beta = 0 \quad \text{and} \quad m' = m'' = \frac{e^2}{6\pi R V^2}.$$

If the velocity becomes that of light,

$$\beta = 1 \quad \text{and} \quad m' = m'' = \infty.$$

For intermediate values of  $\beta$ ,  $m' > m''$ .

Kaufmann, by an ingenious experiment, tested the increase of the apparent transverse mass with the velocity of radioactive particles and found such an increase.

If  $m_1''$  and  $m_2''$  are the transverse masses of two particles at different velocities, we may put

$$\frac{m_0 + m_1''}{m_0 + m_2''} = k \quad \text{and} \quad \frac{m_1''}{m_2''} = \kappa.$$

Then

$$\frac{m_1''}{m_0} = \frac{\kappa(k-1)}{\kappa-k}.$$

Now by his experiments  $\kappa$  and  $k$  are equal, within the limits of accuracy; since  $m''$  remains finite until the velocity of light is reached,  $m_0$  must be excessively small and is now taken by writers on theory to be actually zero. Although no experiments have been made on the longitudinal electromagnetic mass, the same is held to be true for it.

Other theories of the nature of the electron are essentially the same. They present, however, some important differences. Lorentz and Einstein consider the electron to be a sphere only when at rest and to deform into an ellipsoid of greater and greater eccentricity with increasing velocity until it becomes a disk when the velocity of light is attained. The volume of this ellipsoid is a variable. Bucherer has suggested the same idea of the deformable electron, but in his hypothesis the volume remains constant at all velocities.

Expressed in similar units, the velocity function of  $m''$  in the three cases is as follows:—

$$\text{Abraham} \dots \dots \dots \Phi(\beta) = \frac{3}{4} \frac{1}{\beta^2} \left( \frac{1+\beta^2}{2\beta} \log \frac{1+\beta}{1-\beta} - 1 \right),$$

$$\text{Lorentz-Einstein} \quad \Phi(\beta) = (1-\beta^2)^{-1/2},$$

$$\text{Bucherer} \quad \dots \dots \dots \Phi(\beta) = (1-\beta^2)^{-1/3}.$$

In simplicity of formula the last two have the advantage, and from general theoretical principles there is little to choose between them. Here, too, the shape and other definite properties of a fundamental element of matter are not subject to proof and become finally a mere question of definition.

If the electron is rigid, we must expect to obtain measurable and positive results in experiments, such as those of Michelson and Morley, made on the mutual relations of the quiescent æther and matter. These are now conceded to be impossible, either from the principle of relativity or from what, I think, is a more fundamental idea: that however finely matter be divided, it maintains all such attributes as potential energy. So if the electron is held to be rigid and without potential energy, that attribute must be given to it from some mutual relation between it and a *plenum* of which it is a modification, such as is expressed by Poynting's energy theorem.

On the other hand, if the electron is deformable, work must be done to produce this deformation. This can come either from internal potential energy due to force actions of its own parts, or from an unlimited reservoir, the æther. In the first case, we are compelled to subdivide the electron, which thus ceases to be a fundamental element of matter; and the second case leads us nowhere.

Nor can the principle of relativity aid us in obtaining positive knowledge on such questions; at best it is a principle of negation, stating in mathematical terms the idea long established in philosophy that all our knowledge is relative, and must be so, from the fact of the finitude of our minds.

If now we turn to the experimental evidence to decide between these two forms of the electron, we come to no definite decision.

Kaufmann\* concludes with the following remarks:—  
 “Die vorstehenden Ergebnisse sprechen entschieden gegen die Richtigkeit der Lorentz'schen und somit auch der Einsteinschen Theorie; betrachtet man diese aber als widerlegt, so wäre damit auch der Versuch, die ganze Physik einschliesslich der Elektrodynamik und der Optik auf das Prinzip der Relativbewegung zu gründen, einstweilen als missglückt zu bezeichnen. Eine Betrachtung der Einsteinschen Theorie zeigt, dass man, um bei Beibehaltung dieses Prinzipes dennoch Uebereinstimmung mit meinen Resultaten zu erhalten, bereits die Maxwell'schen Gleichungen für

\* *Annal. der Physik*, xix. p. 534 (1906).

ruhende Körper modifizieren müsste, ein Schritt, zu dem sich wohl einstweilen schwer jemand wird entschliessen wollen."

Hupka \*, by a different method, arrived at contrary results. His conclusions are:—"Das wichtigste Resultat der beschriebenen Versuche ist jedoch ein Beitrag zur Entscheidung der miteinander im Wettstreit liegenden Theorien von Abraham und Lorentz-Einstein. Die mitgetheilten Messungen sprechen zugunsten der letzteren. Zu demselben Ergebnis ist bekanntlich auch Hr. Bucherer gelangt."

It is not necessary to weigh their contradictory evidence, as it has been done most exhaustively by W. Heil †. He shows, and I think conclusively, that it may be taken as established that the apparent inertia of an electron depends on its velocity; but in no case is the experimental accuracy sufficiently great to decide the question of the nature of the electron.

Let us now turn to the mathematical side of the discussion and examine the expressions for the electromagnetic mass: we shall find contradictions between the two general ideas and further evidence that the founding of mechanics on an electromagnetic base does not harmonize with the other branches of physics. The values of  $U$ ,  $T$ , and  $|G|$  all become infinite when the velocity of the particle equals the velocity of light; yet several physicists have given this velocity to the orbital motion of the electron, and others have attempted to give the same velocity to electrified light-particles. On the other hand, when the particle is at rest,

$T$  and  $|G|$  become zero and  $U = \frac{e^2}{8\pi R}$ . It is proper

that the magnetic energy and the electromagnetic momentum should then vanish, as both of them are kinetic functions of electricity; but how are we to account for the finite values of the electrical energy and of the two forms of electromagnetic mass? To do so would require us to assume that a free electric charge at rest still possesses inertia, a consequence difficult to reconcile with our experimental evidence of static electricity.

It is even more convenient to turn to an analysis of Kaufmann's ‡ equations. He has derived the electric and magnetic deflexions of a charged particle moving through an electric and a magnetic field.

\* *Annal. der Physik*, xxxi. p. 203 (1910).

† *Annal. der Physik*, xxxi. p. 519; xxxiii. p. 403 (1910).

‡ *Loc. cit.*

If  $y$  is the displacement by the electric field,  $E$ ,  
 $z$ , the displacement by a magnetic field,  $M$ ,  
 $\mu_0$ , the electromagnetic transverse mass of an electron for  
velocity zero\*,  
 $V$ , the velocity of light,  
 $e$ , the electric charge on the particle,

then

$$y = \frac{e}{\mu_0} \frac{E}{V^2} \frac{1}{\beta^2 \Phi(\beta)},$$

and

$$z = \frac{e}{\mu_0} \frac{M}{V} \frac{1}{\beta \Phi(\beta)},$$

where as before,

$$\Phi(\beta) = \frac{3}{4} \frac{1}{\beta^2} \left( \frac{1+\beta^2}{2\beta} \log \frac{1+\beta}{1-\beta} - 1 \right), \quad . \quad \text{Abraham}$$

$$\Phi(\beta) = (1-\beta^2)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad \text{Lorentz-Einstein}$$

$$\Phi(\beta) = (1-\beta^2)^{-\frac{1}{4}}. \quad . \quad . \quad . \quad . \quad \text{Bucherer}$$

According to these theories, when  $\beta=1$ ,

$$\Phi(\beta) = \infty, \quad \text{and} \quad y = z = 0;$$

and when  $\beta=0$ ,

$$\Phi(\beta) = 1, \quad \text{and} \quad y = z = \infty.$$

All theories agree, that an electrified particle, moving with the velocity of light, cannot be deflected by an electromagnetic field, and this coincides with the idea that the electromagnetic mass then becomes infinite. But for a small velocity these deflexions become very large and are infinite when the electron is at rest. Even if we should accept this result for the case of the electric displacement, we should still have the difficulty of accounting for any action between an external magnetic field and an electrified particle at rest. These results for a stationary particle are difficult to reconcile and impossible to explain without making special and unlikely hypotheses for the constitution of the electron. It should also be noted here, that this is the only case where a small mass, whose velocity, charge,

\* It is impossible for me to form any conception of this quantity if it has a finite value, and yet it is one of the essential factors of the equations which follow. In the first place, an electron at rest has no electromagnetic field; and secondly, how can an electron with no motion have a transverse mass of any sort, when that is defined as mass due to a change in direction only? We might as well give a finite value to the centrifugal force of a body at rest.

and external attracting force are all finite, attains an infinite momentum and kinetic energy. We are compelled to say that the velocity of light is unattainable by matter, because the æther is then impenetrable or that the electromagnetic mass becomes infinite.

All these theories and experiments are based on the experimental fact that  $\frac{e}{m}$  has been found to be  $1.865 \times 10^7$  approximately. It is then assumed that  $e$  is a constant and equal to the charge on the hydrogen electrolytic ion; with this value, the mass  $m$  is about the one-thousandth part of the mechanical mass of the hydrogen atom. How can we account for this mass when the theorists claim that the mechanical mass is entirely electromagnetic? We know what the transverse electromagnetic mass is from Abraham's formula; it is equal to  $m'' = \frac{|G|}{v}$ , where  $v$  is a constant velocity of changeable direction. Substituting we obtain the formula already given,

$$m'' = \frac{e}{8\pi R\bar{V}} \Phi(\beta).$$

This mass  $m''$  is evidently not the denominator of the ratio,  $\frac{e}{m}$ .

To say that  $e$  is a constant is an assumption based solely on an analogy to the experimental laws of electrolysis; but in electrolysis, when we obtain equal electric charges we also find equivalent masses of matter. In the discharge of electricity through gases and in radioactivity the matter deposited is too small to be measured. This is a fundamental difference, and vitiates an analogy between the two. For example, we measure the amount of current in a vacuum tube by an electrical device, and at the same time we measure the deflexion of the current by an electric and magnetic field; in other words, all quantities and forces are electrical, and we say that equal currents in this case require equivalent quantities of matter. But it has not been shown to be impossible or even improbable that electrons, associated with equal quantities of matter but having different velocities, might show different electrical charges; or that electrons producing equal electrical charges, might deposit different amounts of matter if it were sufficient in quantity to be detected by chemical or mechanical reactions. Thus we may imagine the following experiment:—Suppose all conditions in a vacuum tube to remain the same, except that the velocity

of the cathode particles could be changed, it might be that this change in velocity would alter the electric charge deposited on an obstacle.

As an hypothesis, I propose that, in order to make the ratio  $\frac{e}{m}$  agree with the experimental evidence of its value and to account for electromagnetic mass, we consider  $m$  to be the mass of a particle of matter in the Newtonian sense, of constant and small value, and  $e$ , the electrical charge, to be a force attribute of matter which varies with the velocity of the particle.

However novel this hypothesis may be, I have not been able to find any experimental facts more difficult to explain by it than by any of the other hypotheses which have been recently advanced; and, on the other hand, it apparently accounts for much of the modern work in terms of the older and well-established ideas.

1. Since the quantities,  $e$  and  $m$ , occur in all cases only in the form of a simple ratio, either *à priori* may be considered the variable.

2. From Kaufmann's experiments  $e/m$  decreases as the velocity of the electron increases. This is satisfied if we assume that the electric charge has a maximum value for an electron at rest which decreases with increasing velocity until it attains a value of zero at the velocity of light.

3. The decrease in the value of  $e$  does not become noticeable until a velocity comparable to that of light is reached.

4. This hypothesis supports the theoretical value and experimental ideas of electromagnetic mass.

5. At zero velocity matter would retain its mechanical inertia and electric charge, which permits the function  $U$  to have a finite value while  $T$  and  $|G|$  both vanish.

6. Electrical conductivity increases with diminishing temperature and attains a large value at the absolute zero. This squares with the hypothesis that the electrical charge of matter increases with decreasing velocity.

7. If  $e$  becomes zero with the velocity of light, it is evident that the deflexions by a magnetic or electric field would be zero.

8. We are not compelled to assume an infinite momentum for a body moving with a finite velocity.

9. The difference between positive and negative electricity can still be ascribed to the nature of the orbital motion of the electron.

From the very nature of my conception of the limits which should be imposed on scientific inquiry, I make no

attempt to explain the cause for this electrical property of matter any more than I should for its gravitational attributes. Both are fundamental phenomena to be accepted as experimental facts until we gain contrary knowledge. Indeed, I have ventured to indulge in this speculation rather with the idea of showing that the recent hypotheses for electricity and matter; for the æther, protons, and corpuscular light; for the electromagnetic and other non-Newtonian mechanics, are not necessary. We may still account as adequately for all our experimental facts by a simple addition to the properties of matter and continue to base our theories on mechanical laws.

So long as the measurement of physical quantities becomes ultimately a matter of measuring mechanical forces, it is advisable to express quantitative physical laws in terms of mechanical formulæ. For this reason electricity should be considered a function of mechanical energy rather than the converse. If it be possible to place mechanics on an electrodynamic basis, it is certain that we may always explain electricity in terms of ponderodynamic laws. As both are possible, it seems far more natural and more rational to consider electricity as an attribute of matter than matter as a phenomenon of electricity.

Before this article is closed there is a point which should be discussed. The close analogy between electromagnetic mass and the *apparent* increase in mass in hydrodynamic problems has been pointed out by me in a former paper \*, by Professor Lorentz, and must have occurred to everyone. But Professor Lorentz indicates a difference between the two which seems to me less essential than he considers it. To be exact, I quote from him †. In the problem of a ball moving in a perfect fluid, "we are able to determine the effective mass  $m_0 + m'$  (or  $m_0 + m''$ ), but it would be impossible to find the values of  $m_0$  and  $m'$  (or  $m''$ ) separately. Now, it is very important that, in the experimental investigation of the motion of the electron, we can go a step farther. This is due to the fact that the electromagnetic mass is not a constant, but increases with the velocity." This is, if true, important, as it seems to show an essential difference between these two apparent masses. But it can be shown, as follows, that such an essential difference does not exist. We know that a single electron will maintain a state of rest or of uniform motion in a straight line just as a single ball in an

\* Phil. Mag. vol. xviii. p. 17 (1909).

† 'Theory of Electrons,' p. 40.

infinite perfect fluid will, under similar conditions. To produce any variation in the speed or direction of motion of an electron, other electrified particles must be present to exert an external force upon it. Thus Kaufmann determined  $m''$  by measuring the deviation in path of an electrified particle which moved parallel to an electrified plane surface. But if we now call  $m_0$  the true mass and  $m''$  the apparent, or hydrodynamic, mass of a ball moving in a perfect fluid and parallel to a fixed plane surface, can we not also separate these two masses? The formula for the kinetic energy of a sphere, of density  $\rho$  and radius  $a$ , moving in a perfect fluid parallel to, and at a distance  $h$  from, a fixed plane surface, is \*

$$2T = \frac{2}{3} \pi \rho a^3 \left( 1 + \frac{3}{16} \frac{a^3}{h^3} \right) v^2 = m'' v^2.$$

We also have

$$m_0 = \frac{4}{3} \pi \rho a^3.$$

Since the hydrodynamic mass,  $m''$ , is a function of the variable  $h$  and the true mass,  $m_0$ , is a constant, we may measure the kinetic energy of similar spheres which move parallel to a plane boundary but at different distances from it and so separate  $m_0$  and  $m''$ . Practically, we should experience great difficulties; the effect would be very small; we have no perfect fluids; and we have not yet deduced the equations for spheres moving in a viscous fluid. But these are purely experimental difficulties and show no essential differences between the conceptions we should make for hydrodynamic and electromagnetic mass. Both may be considered as variable quantities of the same character added to the true and constant inertia of matter.

University of Cincinnati,  
August, 1910.

XXV. *On a Revolving Table Method of determining the Curvature of Spherical Surfaces.* By G. F. C. SEARLE, M.A., F.R.S., A. C. W. ALDIS, M.A., and G. M. B. DOBSON, B.A.†

§ 1. **F**OR many optical purposes it is necessary to determine the radius of curvature of a convex or a concave surface with considerable accuracy. The revolving table method, described below, has the advantage that the radius is found directly from two readings on a

\* Lamb, 'Hydrodynamics,' p. 143.

† Communicated by the Authors.



straight uniformly divided scale, without corrections or calculations of any sort. We shall not attempt to give an account of all the mechanical devices which might be profitably employed in attaining the highest possible accuracy, but shall content ourselves with describing the plan of the method and the elementary geometrical principles involved in the application of the method to practical measurements. The simple apparatus which we describe has been in use for some months in one of the classes at the Cavendish Laboratory, Cambridge; the method has been found to interest the students.

§ 2. The principle of the method may be described as follows:—

A table turning truly and without shake about a vertical axis is required. For the most accurate work the fit of the vertical spindle in its bearings must be as perfect as in a good lathe-head. The plane of the top of the table is normal to the axis of revolution, and the top carries a straight scale against which slides a carriage bearing the spherical surface (see fig. 1). We shall assume that the scale is so adjusted on the table that the straight line described by the centre of curvature of the spherical surface when the carriage slides along the scale *intersects* the axis of revolution of the table. The position of the carriage relative to the table-top when the centre lies on the axis of revolution will be called the *first position*. If the table be turned through any angle about the vertical axis when the carriage is in the first position, the only effect of the angular motion is to substitute one part of the surface for another. Hence, if rays from a luminous point fall upon the surface, the reflected rays will be unaffected by the motion. It is not necessary that the rays which enter the eye from the luminous point should have met the surface very nearly normally, or, in other words, that the *geometrical* image of the point should be observed. It is sufficient to use a pencil of rays whose axis is nearly horizontal and strikes the surface not very obliquely, and to observe the vertical focal line of the reflected pencil. If a vertical line be used as an object, the "image" by reflexion will be a vertical line formed by the vertical focal lines corresponding to individual points of the object. If a microscope be used for observing the image of the vertical line, the proper adjustment of the carriage along the scale can be made with great accuracy. The microscope must, of course, be firmly supported and must be furnished with cross-wires or with a micrometer-scale. In the absence of a microscope,

a telescope with cross-wires may be used if an extra converging lens be fitted in front of the objective so as to allow the instrument to be used at short distances.

The carriage is now moved along the scale into a *second position* in which the axis of the table is a tangent line to the spherical surface. If, now, the table be turned about its axis, a grain of lycopodium placed at the point of contact of the vertical tangent line and the surface will remain stationary. By using a microscope for observing the lycopodium, this setting can be made with great accuracy. The table may be turned backwards and forwards through an angle of a few degrees about a mean position in which the perpendicular from the centre of curvature of the surface upon the axis of the table approximately coincides with the axis of the microscope.

The radius of curvature of the surface is now given at once by the difference of the two scale-readings of the carriage in the first and second positions.

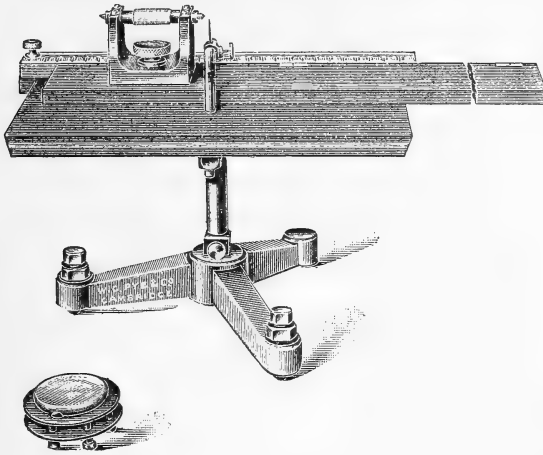
§ 3. When the scale is not in perfect adjustment on the table, the straight line described by the centre of curvature of the spherical surface, as the carriage slides along the scale, will not intersect the axis of revolution of the table. If, however, the shortest distance between these two straight lines be small, and if in both the first and second settings the axis of the microscope point nearly to the centre of curvature of the surface, the error in each of the two settings of the carriage will be very small, provided that, in each case, the table is turned about a mean position in which the scale is parallel to the axis of the microscope. Thus a rough adjustment of the scale on the table is sufficient for any but the most precise work.

§ 4. The adjustments of the surface to be tested are greatly facilitated by the use of a small lathe-head to form the "carriage" mentioned in the preceding paragraphs; and we shall describe the method of making the measurements when the lathe-head is used. The apparatus is shown in figs. 1 and 2.

The revolving table rests upon a tripod stand, and the table-top turns about a vertical rod carried by the tripod. The top of the table carries a scale graduated in millimetres, which can be clamped in any position on the table by the screw seen in fig. 1. Revolving tables of this description are now employed in many laboratories in the determination of the focal lengths of lens-systems by the nodal-point method. The lathe-head and its adjuncts may be regarded as additions made to the revolving table to make it available for the measurement of curvatures.

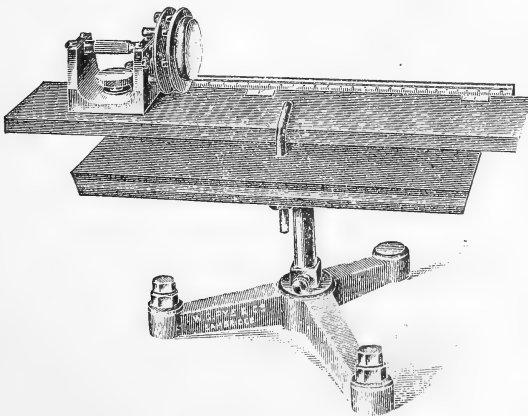
The lathe-head is attached by an adjustable screw to a straight-edged board which rests upon the top of the table

Fig. 1.



with its straight edge in contact with the scale. The spindle of the lathe-head is screwed at one end to fit a brass face-plate furnished with three screws which serve to adjust a

Fig. 2.



second brass plate to which the lens or mirror to be tested is attached by wax. Each end of the spindle is turned to a conical point, and the vertex of each cone lies on the axis of revolution of the spindle. One arm of a steel rod bent at right angles can be secured by a set-screw in a socket

supporting the table-top, and a small clip carrying a pin can be fixed in any position on the other arm of the rod by a set-screw. The bent rod also serves as a means of clamping the board to the table in the manner shown in fig. 2.

§ 5. The first step is to make the axis of the spindle of the lathe-head parallel to the edge of the board. The face-plate is removed from the spindle and the tip of the pin carried by the bent rod is brought into contact with the vertex of one of the conical ends of the spindle, the straight edge of the board being in contact with the scale. The board is then removed from the table and is replaced so that the other conical end of the spindle is near the pin. If, by sliding the board along the scale, this vertex can be made to touch the tip of the pin, the axis of the spindle is parallel to the edge of the board. In order that this may be the case, it is necessary (1) that the axis of the spindle be parallel to the plane of the board, and (2) that the projection of the axis upon the plane of the board be parallel to the edge of the board. The clamping screw enables the lathe-head to be adjusted on the board so that the second of these conditions is secured. In the apparatus shown in the figures no provision has been made for adjusting the axis of the spindle so that the angle between it and the plane of the board may be zero. This angle will, however, be small if the apparatus is constructed with moderate care. If, instead of being zero, the angle be  $\theta$  radians, the error in the measurement of the radius of curvature ( $r$ ) will be  $r(1 - \cos \theta)$  or  $\frac{1}{2}r\theta^2$ , when  $\theta$  is small, and thus the error will generally be negligible.

§ 6. After the adjustment of the lathe-head on the board is complete, the scale must be adjusted on the table-top so that the axis of the lathe spindle *intersects* the axis of the table. The scale and the board are first adjusted roughly by eye so that one of the conical ends of the spindle is not far from the axis of the table. A fixed microscope, with a vertical cross-wire, is then focussed on the vertex of the cone when the axis of the spindle is approximately perpendicular to that of the microscope, the image of the vertex lying on the cross-wire. The table-top is then turned through approximately  $180^\circ$ ; if the image of the vertex does not again lie on the cross-wire, the difference is *halved* by moving the board along the scale, and the microscope is then moved so as to bring the image again to the cross-wire. The table-top is now turned so that the axis of the spindle is approximately parallel to that of the microscope. If the image of the vertex does not lie on the cross-wire, the scale is moved at right

angles to itself until the coincidence is obtained. The vertex of the spindle then lies on the axis of the table. Since the projection of the axis of the spindle on the plane of the board has been made parallel to the edge of the board, the axis of the spindle intersects that of the table for all positions of the board along the scale. These adjustments may, of course, be made once for all, but for purposes of instruction it is best that each student should make them for himself.

§ 7. The face-plate is now attached to the spindle and the lens or mirror to be tested is fixed to the adjustable plate by wax. In the case of a lens, the back surface should, for convenience, be smeared with a mixture of lamp-black and vaseline or be coated with black varnish to stop reflexion at that surface. The spindle is then rotated and some object is observed by reflexion at the spherical surface. If the rotation cause a motion of the image, the three adjusting screws are manipulated until the image remains stationary when the spindle is revolved. The centre of curvature then lies on the axis of the spindle. During the adjustment the table-top is clamped so that it cannot rotate.

A piece of ground glass with a fine vertical line drawn on it forms a convenient object; the glass should be well illuminated. A fixed microscope may be used to facilitate the adjustment. The spindle is turned into the position in which the image of the vertical line is as far to the right as possible. The plate carrying the surface is then made to move as nearly as possible about a *vertical* axis by turning the appropriate screw or screws, so that the image moves to the left through the proper distance. The success of the adjustment is then tested by rotating the spindle; if necessary a second adjustment is made.

The preliminary adjustments are now complete. If they are only nearly but not quite perfect, the errors which they cause in the radius of curvature as found by this method will be very small, since these errors of radius depend on the *second* powers of the errors in the preliminary adjustments.

§ 8. The board carrying the lathe-head is now set into the "first position" in which the centre of curvature of the surface lies on the axis of revolution of the table. The ground glass with the vertical line is set up in such a position that the image of the vertical line which is observed in the microscope is formed by rays which fall *nearly normally* upon the surface. If the image move when the table is turned backwards and forwards about its axis, the board carrying the lathe-head is moved along the scale until the image remains at rest. The centre of curvature then

lies on the axis of the table. The scale-reading of an index mark on the board is then taken and recorded. The mean of several independent readings should be taken.

In the case of some lenses the accuracy of setting is limited by the departure of the surface from perfect sphericity. In these cases it is possible to make the image remain stationary in spite of the motion of the table provided the motion be so small that the part of the surface at which the rays entering the microscope are reflected is not near the edge of the lens. A larger motion of the table causes a rapid motion of the image.

§ 9. The board is next adjusted to the "second position" in which the axis of the table is a tangent line to the reflecting surface. A small patch of lycopodium is placed on the surface and then a sharp-pointed piece of wood is held against the surface while the lathe-spindle is rotated so as to remove all the lycopodium but a circular patch about a millimetre in diameter. If the axis of the spindle be perpendicular to the axis of the table, the tangent plane to the surface at the centre of this patch will be parallel to the axis of the table. The board is then adjusted along the scale so that the central grains of the patch remain stationary when the table is turned backwards and forwards about a mean position in which the axis of the lathe-spindle is *parallel* to that of the microscope. When the adjustment is complete, the scale-reading of the index mark is recorded. The mean of several independent readings should be taken.

The difference between the two mean scale-readings in the first and second positions gives the radius of curvature of the surface.

To allow for the measurement of comparatively large radii of curvature the board is provided with *two* index marks, the distance between them being 15 cm. When one mark is beyond the range of the scale, the reading of the other is taken.

The following experiment serves as a test of the method. Two lenses, one convex and the other concave, are selected so that wide Newton's rings are seen by sodium light when the marked face of the convex lens is placed in contact with the marked face of the concave lens. The radius of the convex surface is determined by the revolving table and the radius of the concave surface is deduced from measurements of the rings. The radius of the concave surface is also determined by the revolving table and the two results for this surface are compared.

XXVI. *Rays of Positive Electricity.*  
By Sir J. J. THOMSON\*.

[Plate I.]

IN the 'Philosophical Magazine' for October 1910 I described some experiments on positive rays which were made with very large discharge-tubes; with such tubes it is possible to work at very low pressures without the difference of potential between the electrodes increasing to such an extent as to endanger the tube by sparking through it. At these low pressures I found that the patterns on the phosphorescent screen produced by the positive rays after they had passed through electric and magnetic fields, were separate parabolic curves; the value of  $e/m$  for the particles striking the curve along one of these parabolas is constant. Parabolas giving values of  $e/m$  corresponding to the hydrogen atom, the hydrogen molecule, the helium atom, the carbon atom, the oxygen atom, and the mercury atom were observed. I had previously, in 1907 (*Phil. Mag.* xiii. p. 561), observed the curves corresponding to the hydrogen atom and molecule, and also to the helium atom. In all these experiments the rays were detected by the phosphorescence they produced on a willemite screen; such a screen, however, does not give a permanent record of the curves traced on it by the positive rays, and accurate measurements of the dimensions and positions of the luminous curves are more difficult than they would be on a photograph. For these reasons I have endeavoured to apply photographic methods to the study of these rays: at first I attempted to photograph the luminosity on the screen, using a very large portrait-lens; I abandoned this method because I found that to get any effect on the photographic plate exposures lasting several hours were required. Apart from the tediousness of this process, it is difficult to keep the conditions in the tube sufficiently constant for so long a time. I see, however, that Dechend and Hammer (*Sitz. Heid. Ak.* 1910) have recently succeeded in photographing the curves corresponding to several gases in this way.

I find that a much more sensitive and expeditious method is to put a photographic plate inside the tube itself and let the positive rays fall directly on the plate instead of on to the willemite screen. The photographic plate is very sensitive to these rays, and the places where they fall are recorded when the plate is developed. The plate is much more sensitive

\* Communicated by the Author.

than the willemite screen, and an exposure of three minutes shows curves on the plate which cannot be detected on the screen\*.

One method I have employed to detect the positive rays by photography is as follows:—The photographic plate is inserted in the lid of a light-tight box, A, fig. 1, so that when

Fig. 1.—Plan.

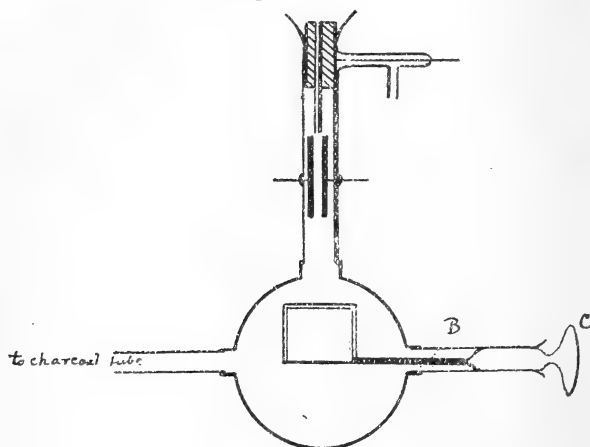
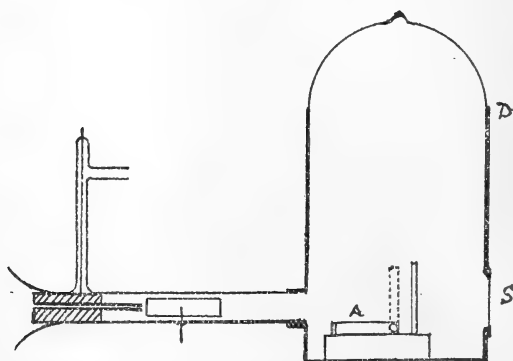


Fig. 1.—Elevation.



the lid is down the plate is protected from light. The bottom of the box is fastened down to the floor of the discharge-tube, while the lid is fastened to a rod, B, which is fixed to a glass tap, C, which rotates in a glass tube; tube and tap are

\* Since this was written I have seen a paper by Koenigsberger & Kelching (*Verh. Deutsch. Physik. Gesellsch.* xii. p. 995, 1910), who have also used the method of putting the plate inside the tube.



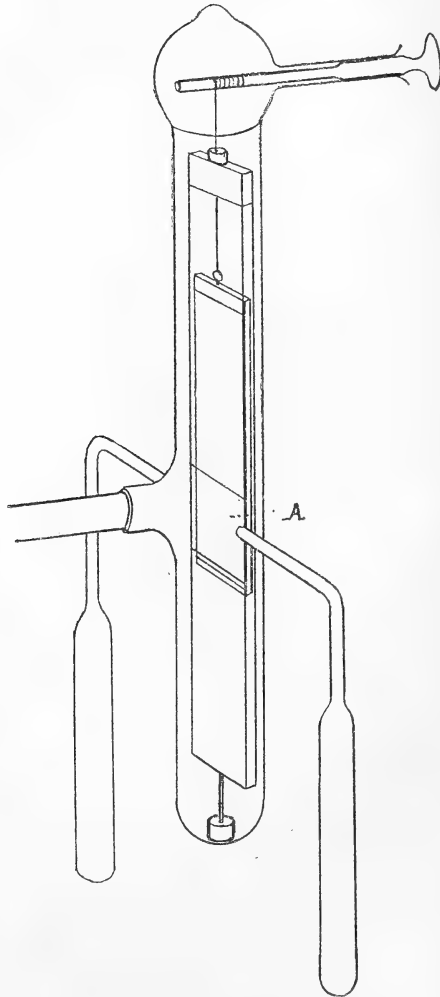
ground so that the tap can be rotated without injury to the vacuum in the discharge-tube. The box containing the plate was placed in front of a willemite screen, S, fastened to the end of the tube; when the lid of the box was down the positive rays could hit the screen; the appearance on the screen showed when the pressure in the tube was such as to give well-developed positive rays and thus indicated when a photograph could be taken with advantage. When this stage was reached the coil or influence machine used to produce the discharge was stopped, the lid of the box lifted by turning the tap, the tap was turned until the lid came against stops placed inside the tube, so that the plate was always in the same position. The tube was surrounded by black velvet so as to prevent light from outside reaching the tube, while the light from the discharge and the phosphorescence of the walls of the tube was prevented from reaching the plate by tightly-fitting stops placed in the neck of the tube. The only way that light could reach the plate was through the long and narrow tube through which the positive rays themselves passed, and this light would produce a small circular patch on the plate coinciding in position with that produced by the positive rays themselves when they were not deflected by electric or magnetic forces.

After the plate had been exposed to the rays for a suitable time the coil was stopped and the tap turned so as to put the plate back again into the box. The lid of the tube, D, which was fastened on with sealing-wax, was taken off, the box taken out of the tube, and the plate developed. The size of the plate was 4 cm.  $\times$  4 cm.

Many photographs have also been taken with an arrangement devised by Mr. F. W. Aston, of Trinity College, shown in fig. 2 (p. 228). In this method the plate is suspended by a silk thread wound round a tap which works in a ground-glass joint; by turning the tap the silk can be rolled or unrolled and the plate lifted up or down. The plate slides in a vertical box of thin metal, light-tight except for the opening A, which comes at that part of the tube through which the positive rays pass; the openings, which are on both sides of the box, are circular and 5 cm. in diameter. When the silk is wound up, the strip of photographic plate in the box is above the opening, so that there is a free way for the positive rays to pass through the opening and fall on a willemite screen placed behind it, so that the state of the tube with respect to the production of positive rays can easily be ascertained. The box is large enough to hold a film long enough for three photographs; by lowering the plate until

the bottom third of the plate comes opposite the opening, taking a photograph, then lowering the plate still further until the middle third comes opposite the opening, taking another

Fig. 2.



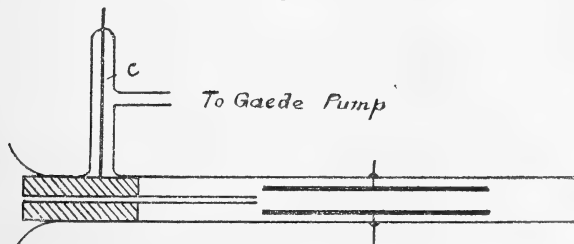
photograph, then lowering the plate until the top third comes opposite the opening, and again taking a photograph, three photographs can be taken without opening the tube. This is very convenient, for the deflexions of the different positive rays vary so much that it is difficult to measure them when

they are all on one plate. For example, the magnetic deflexion of the hydrogen atom is about fourteen times that of the mercury one; thus if the deflexion of the hydrogen atom is within the limits of the plate, that of the mercury atom will be too small to be measured with accuracy. When three photographs are taken, however, we can use a small magnetic force for the first and get the light atoms on the plate, then a larger magnetic field for the second for the study of the atoms and molecules of oxygen and other elements of a similar atomic weight, while we can use a still stronger field for the third for the study of very heavy atoms such as those of mercury.

*Method of exhausting the Tube.*—The tube was exhausted by a Gaede pump and charcoal cooled with liquid air. The narrow tube through which the rays passed fitted at the end next the photographic plate into a plug of ebonite, and the joint was made air-tight with a little wax, the joint between the ebonite plug and the glass tube into which it fitted was also made air-tight in the same way; thus the only way gas could get from the discharge-tube into the space through which the positive rays passed after they left the tube was through this long narrow tube through which the gas only filtered slowly; two large tubes containing charcoal immersed in liquid air led into this part of the apparatus and kept the pressure much lower than it was in the discharge-tube.

The communication with the Gaede pump was made between the cathode and the ebonite plug as shown in fig. 3.

Fig. 3.

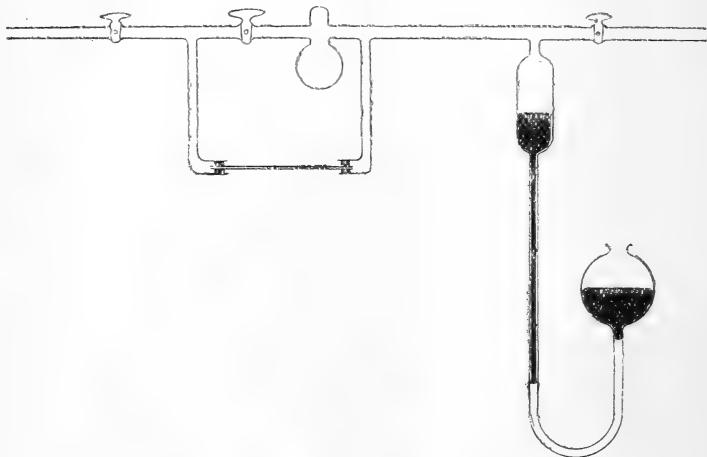


The cathode fits fairly tightly into the tube, so that there may be a considerable difference of pressure between the discharge-tube and the space C.

*Method of filling the Tube with Gases.*—When the rays in some particular gas were under examination a constant stream of that gas was kept flowing through the tube. This was done by fusing on the discharge-tube a glass tube, whose end

was drawn out into an exceedingly fine capillary tube, whose end was left open. This capillary was fastened by an airtight joint into the receptacle containing the gas under examination; this receptacle was practically the space above the mercury in a barometer tube, and by altering the level of the mercury the pressure of the gas in the receptacle could be adjusted so that when the Gaede pump was kept in continuous action the pressure in the discharge-tube was low enough for the positive rays to be well developed. The details of the connexion are shown in fig. 4.

Fig. 4.



*Discussion of the Photographs.*—The appearance of a typical photograph is shown in fig. 5 (Pl. I.). In all these photographs the vertical deflexion is due to the magnetic field, the horizontal to the electrostatic. Many of the photographs were taken with the magnetic field in one direction for half the exposure and in the opposite direction for the other half. It will be seen that the curves on the photograph are of two types.

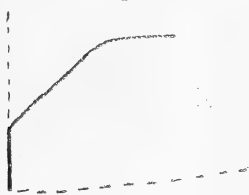
(1) Short parabolic arcs of varying length, having their heads in the same vertical line, showing that the minimum electrostatic deflexion suffered by the particles which produce these curves is the same whatever the nature of the particles may be. The electrostatic deflexion is proportional to  $e/mv^2$ , where  $e$  is the charge,  $m$  the mass, and  $v$  the velocity of the particle. Now if the energy of the particle is due to the fall of its charge through a potential difference  $V$ ,

$$Ve = \frac{1}{2}mv^2 \quad \text{or} \quad e/mv^2 = 1/2V.$$

Hence, as the minimum electrostatic deflexion is the same for all the particles, we conclude that the maximum potential difference through which each set of particles has fallen is the same.

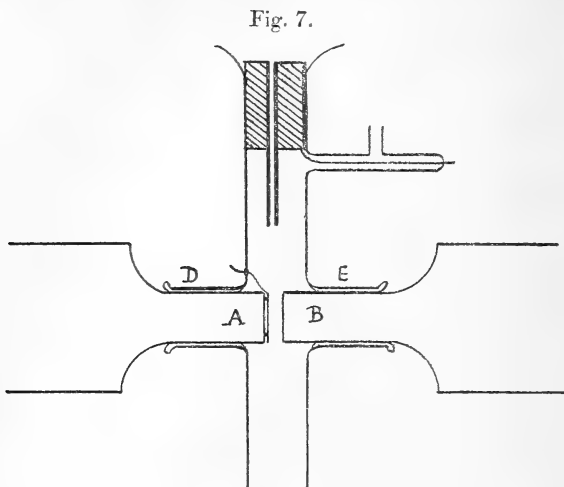
(2) The second type of curve to be seen on the plate are curves which pass through O, the point on the plate struck by the rays when they are not deflected by either electric or magnetic forces. The portions of these curves near O are straight lines when the electric and magnetic fields are coterminous. If the magnetic field overlaps the electrostatic the shape of the curve is that shown in fig. 6. In my paper

Fig. 6.



in the 'Philosophical Magazine' for October 1910 I gave reasons for thinking that these curves represent secondary positive rays produced by the passage of the primary rays through the gas on the way to the plate, and that the parts of the curve near O which are produced by particles which have suffered only small deflexions were produced by rays which had not traversed the whole of the magnetic and electric fields, but had been generated towards the ends of these fields. If this view is right, then, if we shorten these fields, we ought to diminish the intensity of these curves near O. To test this point the magnetic field was reduced from the length of 25 mm., the value it had in most of the experiments, to 6 mm. The arrangement used is represented in fig. 7 (p. 232). The pole-pieces A, B, 6 mm. in diameter and 1 mm. apart, came through sealing-wax joints in the side tubes D and E; the other ends of A and B were screwed into the pole-pieces of a powerful electromagnet. On one of the poles, A, a thin piece of brass was fastened by a thin layer of sealing-wax which insulated it from the pole-piece: the brass was connected with one terminal of a battery of small storage-cells, the other terminal of which was connected with the earth and with the other pole-piece, B. In this way an intense electric field coterminous with the magnetic

could be produced between the pole-pieces. Even with these short pole-pieces there was a very considerable production of

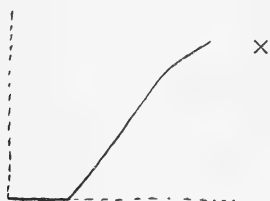


secondary rays, as is shown by the photograph reproduced in fig. 8 (Pl. I.). It ought, however, to be mentioned that the pressure in this case was not very low, as by an accident there was a little grease inside the tube which gave off an appreciable amount of vapour. The photograph is interesting as it shows how intense this production of secondary radiation may be. To test the point further, the length of the magnetic field was reduced to 1 mm. ; this was done by replacing the iron rods A and B in fig. 7 by thin iron plates, and increasing the strength of the electric and magnetic fields sufficiently to produce appreciable magnetic and electric deflexions even though the path of the rays through these fields was only 1 mm. long. In this case all the curves of type (2) completely disappeared, and the only curves on the plate were the parabolas of type (1). A photograph taken with this apparatus is reproduced in fig. 9 ; it will be seen that there is nothing on the plate nearer the vertical than the heads of the bright parabolas.

We should also get rays of type (2) if some of the primary rays, after passing through the cathode, lost their charges as they were passing through the electric and magnetic fields. In this case the parts of the curves near O would be due to rays which had lost their charges near the beginning of the electric and magnetic fields, while in the case we have just been considering this part of the curves would be due to rays

which had been produced near the end of these fields; by the beginning of the field is meant the part of the field nearest the cathode, and by the end of the field the part nearest the photographic plate. This consideration leads to a method by which we can distinguish whether the curves of type (2) originate in one or other of these ways. For suppose that the electric and magnetic fields, instead of being coterminous, are placed one after the other, let us suppose that the magnetic field is the one nearest to the photographic plate. Then if the curves are due to rays produced by the passage of the undeflected rays through the gas, some of these will be produced after the undeflected rays have left the electric and are passing through the magnetic field, the rays so produced will suffer magnetic but not electric deflexions, and the curve they will trace on the photographic plate will be somewhat like that shown in fig. 6 (p. 231). Curves of this kind can be seen on the photographs reproduced in figs. 5 (Pl. I.). If, however, the rays, instead of being produced between the cathode and the photographic plate, are losing their charges as they travel through the electric and magnetic fields, some of them will lose their charges whilst they are in the electric and before they have reached the magnetic field; these will experience an electric but not a magnetic deflexion, and the curves they trace on the photographic plate will be of the kind shown in fig. 11 (Pl. I.).

Fig. 10.



The concavities of the curves in figs. 10 and 11 are turned in opposite directions, and this criterion will enable us to decide as to the way the rays originate. For example, let us take the most prominent curve of type (2) when the rays pass through helium. Fig. 11 represents the curves for helium when the electric and magnetic fields are nearly coterminous; it will be seen that the curve marked with a  $\times$ , which is the one under consideration, is concave to the horizontal axis. When, however, the fields are separated and the magnetic field placed nearer to the photographic plate than the electric, the shape of the curve is as in the photograph

reproduced in fig. 13 ; it is now convex to the origin and of the type shown in fig. 11. The value of  $m/e$  for this curve is 12 times that for the hydrogen atom, and we conclude that the ions which produce it do not start between the cathode and the photographic plate, but are complex ions  $(\text{He}_3)^+$ , which are formed in the dark space in the discharge-tube and are liable to lose their positive charge on their journey from the cathode to the photographic plate.

The secondary rays present some very interesting features. The following are some of the results obtained by the study of a long series of photographs. We shall for brevity call the ratio of  $m/e$  for any ray to the value of  $m/e$  for the atom of hydrogen the electric atomic weight of the particle forming the ray.

Secondary rays consisting of the atoms of hydrogen for which  $e/m=10^4$  are found in all gases when the pressure is not too low. They occur negatively charged as well as positively, and often there is a bright head at the end of the negatively charged rays corresponding to the head of the positively charged ones ; this is shown in the photograph reproduced in fig. 8. It is due, I think, to secondary rays which have been produced before the magnetic field was reached, and which have all, therefore, been exposed to the full magnetic and electric field and, having suffered the same deflexion, strike the photographic plate at the same spot.

Secondary rays consisting of the molecule of hydrogen. Though the primary rays corresponding to the molecule of hydrogen occur on nearly every plate, the secondary radiation due to the hydrogen molecule is generally conspicuous by its absence. An example of this is shown in fig. 5, which represents the appearance when the gases in the tube are the residual gases left after the air which originally filled the tube has been pumped out. The absence of the secondary corresponding to the hydrogen molecule is very marked, while the secondary corresponding to the hydrogen atom is quite distinct. I have found the same thing when the tube contained hydrogen, oxygen, carbonic oxide, marsh-gas, cyanogen, or hydrochloric acid gas instead of the residual gas. When helium is in the tube there is at some pressures, though not at all, secondary radiation corresponding to an electric atomic weight of 2. A photograph of a plate taken with helium in the tube is reproduced in fig. 16, and the secondary radiation of this type is very prominent. I am inclined to regard it in this case as due to an atom of helium with two charges rather than to a molecule of hydrogen with one charge. I had taken so many photographs before I obtained any



indication of secondary radiation of this atomic weight when no helium was present, that I was inclined to doubt whether the hydrogen molecule ever did occur as secondary radiation; quite recently, however, a tube which had got contaminated with grease from one of the taps showed the molecule of hydrogen more strongly than the atom in the secondary radiation; a photograph taken with this tube is shown in fig. 14. There must therefore, I think, be some types of compounds of hydrogen from which the hydrogen molecule is liberated to form the greater part of the secondary radiation due to hydrogen, though in hydrogen itself and in numerous other compounds of hydrogen it is the hydrogen atom which constitutes the larger part of the secondary radiation. It is remarkable that, even in a case like that shown in fig. 14, where the positively-charged molecule largely predominates in the secondary radiation over the positively-charged ion, the negatively-charged hydrogen molecule cannot be detected, while the effects of the negatively-charged hydrogen atom are quite distinct.

In many tubes I have observed a secondary radiation for which the electric atomic weight is about 1.4; the curves on the plate due to it are faint, but they occur with a great variety of gases, and with negative as well as positive charges.

Another secondary radiation, stronger than that just mentioned, but not often very conspicuous, is one with an electric atomic weight of 3; it is specially bright when hydrogen is in the tube, while carbon compounds do not exhibit it to any very marked extent. The electric atomic weight would fit in either with a complex ion  $H_3$  with one charge, or an atom of carbon with four charges. Since it seems not to be specially increased by the introduction of carbon compounds, I think the former hypothesis is the more probable. It occurs with a negative as well as with a positive charge. The radiation mentioned in the preceding paragraph may be due to this ion with a double charge.

Another type of secondary radiation, much stronger than those just mentioned and especially bright in carbon compounds, is one with an electric atomic weight of 6; both this and the preceding type are shown in fig. 15. I have not observed this unless carbon or some of its compounds are present, and so I regard this as due to a carbon atom with a double charge. It is only found with a positive charge.

In compounds not containing carbon there is secondary radiation with an electric atomic weight very near the last,

in fact so near as to make it difficult to distinguish between them, but the measurements I have made of the value of  $m/e$  for the radiation in this case have consistently given me values much nearer 7 than 6, and distinctly higher than those with carbon compounds. If the electric atomic weight is 7, it may possibly be the atom of nitrogen with a double charge, though nitrogen is, as a rule, loth to appear in the positive rays. It only occurs with a positive charge, which makes it unlikely that it is an atom of oxygen with a double charge, as oxygen has a great tendency to appear with a negative charge. In some cases this secondary radiation is the strongest on the plate.

In helium the strongest radiation of the type we are considering is one with an electric atomic weight equal to 12; this is indicated by the continuous line passing through the origin in fig. 11. The experiments already described show that this radiation is due to particles which are charged when they pass through the opening in the cathode, but some of which lose their charges as they pass through the electric and magnetic fields. We have thus here a case of recombination rather than that of a true secondary radiation or dissociation such as that which produces the radiation with atomic weights 1, 3, 6. The radiation with atomic weight 12, when the gas is helium, has not been found with a negative charge. Its atomic weight suggests that the carrier of this radiation is the aggregate  $(\text{He})_3+$ .

In oxygen I have noticed three kinds of radiation of the type we are considering, having respectively the electric atomic weights 16, 48, 96. The first of these corresponds to the atom of oxygen—it is much the strongest of the three; the second to  $(\text{O}_3)_+$ , a molecule of ozone with one positive charge; while the third corresponds to  $(\text{O}_6)_+$ . It is interesting to note that Ladenburg and Lehmann (*Ann. der Phys.* xxi. p. 315) found that a more complex polymer of oxygen than ozone was produced when the electric discharge passed through oxygen at a low pressure. They succeeded in freezing this modification out of the gas with the aid of liquid air, and ascribed to its molecule the composition  $\text{O}_6$ . The secondary radiation corresponding to  $\text{O}_3$  and  $\text{O}_6$  is feeble. I have not yet been able to detect any of it with a negative charge. The secondary radiation with a negative charge corresponding to the oxygen atom is extremely well marked; indeed, after  $\text{H}_-$  it is generally the most pronounced constituent of the negatively charged radiation. In the primary radiation in oxygen the radiation consists of  $\text{O}_+$  and  $(\text{O}_2)_+$ . The rays for oxygen are shown in fig. 16.

Another gas in which the radiation is interesting is mercury vapour. We get radiation corresponding to electrical atomic weight 200, *i. e.* to the atom of mercury in both the primary and secondary radiation. In the secondary radiation we have, in addition, two other kinds, one with an electrical atomic weight of 100, which is probably an atom of mercury with a double charge. The liability of the atom of mercury to take a double charge will explain a peculiar appearance which the line corresponding to the mercury atom in the primary radiation often shows. This appearance is represented in fig. 17. In addition to the bright comma  $\alpha$ , whose head is in the same vertical line as the heads of the other bright primaries, there is, on the same parabola as  $\alpha$ , another bright spot,  $\beta$ , whose horizontal deflexion is only half that of  $\alpha$ . Since  $\beta$  and  $\alpha$  are on the same parabola, the value of  $e/m$  is the same for the particles in  $\beta$  as for those in  $\alpha$ , and since the horizontal deflexion of  $\beta$  is only half that of  $\alpha$ , the kinetic energy of the particles in  $\beta$  is twice that of the particles in  $\alpha$ . This would be just what we should find if the particles in  $\beta$  had had a double charge when they were in the discharge-tube in front of the cathode and exposed to the action of the electric field in the dark space, and lost one of their charges after passing through the cathode and before reaching the electric and magnetic fields.

On some plates there are two bright patches on the parabola corresponding to the hydrogen atom, but in this case the horizontal deflexion of the abnormal spot is twice that of the normal one. The simplest explanation of this spot is that it is due, not to an atom of hydrogen with one charge, but to a molecule of hydrogen with two charges, and that one of these charges has been acquired after the molecule has passed through the cathode. The doubling of the charge, if it arose in this way, would not affect the kinetic energy, but would double the deflexion produced by the electrostatic field, and would put the spot on the curve for which  $e/m$  has the value appropriate to the hydrogen atom with one charge. Fig. 17 A shows a spot of this kind.

The other type of secondary radiation in mercury vapour is one with an electrical atomic weight of 800; this would correspond to four atoms of mercury with one electric charge between them. The existence of particles of this kind in the secondary radiation, and of  $O_3$  and  $O_6$  with oxygen, and  $H_3$  with hydrogen, is experimental evidence for the existence of aggregates formed round a charged ion. In one theory of the electrical discharge through gases the existence of such aggregates is assumed in order to explain why the velocity

of the ions under an electric field is not larger than it is. The smallness of the velocity can also be explained as due to the charge on the ion, so that we could not deduce the existence of these aggregates from considerations of the velocity of the ion. It may be pointed out that the radiation due to these aggregates is in every case faint compared with that due to the simple atom, so that at the low pressures at which the experiments on positive rays are made the number of these complex aggregates is very small compared with that of the simple atom.

In marsh-gas there is a secondary radiation with an electrical atomic weight of 36 ; an aggregate of three atoms of carbon with one charge between them would explain this. In other hydrocarbons, as well as in CO and CN, the strongest secondary radiation is one for which the measurements of the magnetic deflexion give values ranging from 72 to 78. This is the strong secondary shown in fig. 18, the gas in the tube being CO. An aggregate of six carbon atoms, or a molecule of benzene,  $C_6H_6$ , with one charge, would give electrical atomic weights agreeing within the errors of experiment with this.

I have tabulated these results for the secondary radiation of the gases I have so far examined in the following Table (p. 239). The first column contains the electrical atomic weight of the particles in the radiation, the second the sign of the charge on these particles (thus + means that they are always positively charged, + and - that they occur with negative as well as positive charges), the third column contains the name of the gas giving the radiation, the fourth the probable composition of the particle, and the fifth whether it is a true secondary, *i. e.* due to dissociation after the rays have passed through the cathode or to recombination.

#### PRIMARY RAYS.

These are the rays which produce on the photographic plate detached parabolic curves not passing through the origin. The primary rays found in the gases investigated up to the present are as follows :—

##### *Hydrogen.*

There are two kinds of primary rays having respectively the electric atomic weights 1 and 2 corresponding to the atom and molecule of hydrogen. The relative brightness of the curves due to the atom and molecule varies greatly with the circumstances of the discharge, in some cases that due

Table of Secondary Radiation.

Electrical atomic weight.	Sign of charge.	Gases in which they are found.	Remarks.	Nature of radiation.
1 .....	+ and -	All gases.	Hydrogen atom.	Dissociation.
1·4 .....	+ and -	Generally faint; brightest in hydrogen.	(H <sub>3</sub> ) <sub>±±</sub> ?	"
2 .....	+	In helium at certain pressures; also in certain hydrocarbons.	He <sub>++</sub> in first case, (H <sub>2</sub> ) <sub>+</sub> in second.	"
3 .....	+ and -	Especially bright in hydrogen; not exceptionally bright in hydrocarbons.	More likely to be (H <sub>3</sub> ) <sub>±</sub> than C <sub>±±±±</sub>	"
6 .....	+	Especially bright in carbon compounds.	C <sub>++</sub>	
7 .....	+	Found most frequently when carbon compounds are absent.	N <sub>++</sub> ?	
12 .....	+	Found in helium; generally the brightest line of this class in that gas.	(He) <sub>3</sub> +	Combination.
16 .....	+ and -	In oxygen; one of the brightest lines on the negative side.	O <sub>±</sub>	
26 .....	+ and -	In cyanogen; also bright on negative side.	(CN) <sub>±</sub>	
36 .....	+	In CH <sub>4</sub> .	(C <sub>3</sub> ) <sub>+</sub>	
48 .....	+	In oxygen.	(O <sub>3</sub> ) <sub>+</sub> faint.	
72-78 ...	+	In most hydrocarbons; also in CO, CN.	(C <sub>6</sub> ) <sub>+</sub> or (C <sub>6</sub> H <sub>6</sub> ) <sub>+</sub>	
96 .....	+	In oxygen.	(O <sub>6</sub> ) <sub>+</sub> faint.	
100 .....	+	In mercury vapour.	(Hg) <sub>++</sub>	
200 .....	+	In mercury vapour.	Hg <sub>+</sub>	
800 .....	+	In mercury vapour.	(Hg <sub>4</sub> ) <sub>+</sub> faint.	

to the molecule is exceedingly faint, while in others it is much the brighter of the two. I have never detected rays consisting of negatively charged molecules, while those due to negatively charged atoms are invariably present unless the pressure of the gas in the discharge-tube is exceedingly low.

### Oxygen.

The primary rays characteristic of oxygen have electric atomic weights 16 and 32, and are due respectively to the

atom and molecule of oxygen. Rays with these atomic weights and having a negative charge are very conspicuous, their only rival on the negative side being the hydrogen atom. A copy of the photograph taken with oxygen at a very low pressure in the tube is given in fig. 16.

#### *Carbonic Oxide.*

The electric atomic weights of the characteristic primary rays for carbonic acid are 12, the carbon atom with one charge, 16 the oxygen atom, 32 the oxygen molecule, and one with an atomic weight about 47, this may be a molecule of ozone which would have an atomic weight 48, or possibly one of carbonic acid which would have an atomic weight of 44. It would thus appear that the molecule of CO does not appear among the positive rays when the discharge passes through this gas. Fig. 18 is a photograph for this gas.

#### *Marsh-Gas.*

The electric atomic weights are 16, the molecule  $\text{CH}_4$  more probably than the atom of oxygen, as it does not appear on the negative side, and 28. This may possibly be a molecule of acetylene  $\text{C}_2\text{H}_2$ , which would give an atomic weight of 26. It is remarkable that there were no lines with an atomic weight of 12 on this plate, though it was present in the other carbon compounds which were investigated, carbonic oxide and cyanogen. Fig. 19 is the photograph for this gas.

#### *Cyanogen.*

The electric atomic weights for the primary rays are 12, the atom of carbon, and 26, the molecule CN. It is remarkable that the latter is also found with a negative charge. Fig. 20 represents the photograph for this gas.

#### *Helium.*

The line corresponding to the electric atomic weight 4, due to the atom of helium, is exceedingly strong, and can be detected when there is only a very small quantity of helium in the discharge-tube. The most prominent line on fig. 21 is due to helium, the tube contained other gases in addition.

#### *Hydrochloric Acid Gas.*

The primary rays were the atom and molecule of hydrogen, and a very faint one presumably due to the chlorine atom as

it had an electric atomic weight of about the right magnitude ; it was too faint to admit of accurate measurement. It was, however, stronger on the negative side. The oxygen line was exceptionally strong in this gas.

One very remarkable fact which appears from the study of these rays is the ease with which the atom of hydrogen acquires a negative charge ; this does not harmonize well with the usual view as to the electro-positive character of hydrogen. With this exception the negative charge, as far as my observations have gone, is assumed by, and only assumed by, those ions which have distinctly electro-negative chemical properties, thus leaving out H. The gases which appear on the negative side are O, Cl, CN, all well recognised electro-negative ions. It would thus seem that here we have direct experimental evidence that the atoms of the electro-negative elements can acquire a negative charge when under the same circumstances the atoms of the other elements do not do so.

*Long and Short Lines.*—The length of the parabolas on the photographs varies very much from one curve to another ; the least deflected parts are all on the same vertical line, indicating, as we have seen, that the maximum potential difference through which the particles have fallen is the same for all the different kinds of particles which give rise to these lines ; this potential difference is probably the potential difference between the cathode and the negative glow. Though the minimum horizontal deflexion is the same for all the curves, the maximum is very different even when the pressure of the gas in the tube is exceedingly low. As will be seen from the reproduction of the photographs, some of the curves are exceedingly short, so that the horizontal deflexion of any one of the particles producing it is not much greater than the minimum, showing that all these particles have practically fallen through the maximum potential difference, and have therefore probably been produced near the confines of the negative glow. On the other hand, there are on the same photograph some curves of great length where the maximum horizontal deflexion is at least five or six times the minimum ; the particles which have suffered the maximum deflexion have therefore fallen through a potential difference of less than one-fifth of that between the cathode and the negative glow. Since the electric force increases rapidly near the cathode there will be, at a distance from the cathode much less than one-fifth the thickness of the dark space, a difference of potential amounting to one-fifth that through the whole of the dark space ; hence we

conclude that the particles which suffer these large deflexions must have been generated quite near to the cathode, so that the production of these particles must be going on through nearly the whole of the dark space. It is rather remarkable that on some of the photographs all the lines are quite long, and there are no indications of any particles which are produced exclusively at the junction of the dark space and the negative glow. The very sharp line of demarcation, which in general characterizes the junction of the dark space and the negative glow, would lead us to expect that this region would be associated with some form of chemical change which does not exist in the rest of the dark space; this, however, seems not necessarily to be the case, as there are several gases in which no short lines can be detected: one of these is oxygen, and this is also one of the gases where the contrast between the negative glow and the dark space is specially well marked.

With some few exceptions which will be considered later, the brightest part of the parabolas corresponding to the primary rays is the part which has experienced the least electrostatic deflexion, these rays which produce this part of the parabola are those which have the greatest kinetic energy and would naturally produce a greater effect than the same number of particles with less kinetic energy; the decay in the intensity as we recede from the head of the parabola seems in some cases to be larger than can be accounted for by the falling off in the kinetic energy, and seems to indicate a considerable diminution in the number of the particles. Now in the dark space ionization is produced (1) by the cathode particles, these move away from the cathode, and (2) by positively charged particles, these move towards the cathode; there are in addition other sources of ionization, such as ultra-violet light and soft Röntgen radiation, which we shall leave out of consideration for the present. The positive particles starting from the negative glow will not at its boundary have acquired any energy from the electric field, and will therefore not be likely to produce ions in that neighbourhood, while this is just the place where the cathode particles are most numerous and most energetic. Now the greater part of the particles forming what we have called the primary rays come from the neighbourhood of the negative glow; we conclude that they represent the ions produced by the collision of rapidly moving cathode particles against the molecules in the discharge-tube. On the other hand, we should expect that the ions produced by the positive particles would show the characteristics of what we have



called secondary rays, for these particles would have their maximum velocity when they reached the cathode, and after passing through the cathode would continue to produce rays of the same type while passing through the electric and magnetic fields.

Though some secondary rays are probably produced in this way, yet I am of opinion that the rays which produce the curves on the photographic plate arise in a different way, for the ions produced by the impact of one molecule against another would probably start off in different directions; this would make the curves on the photographic plate very fuzzy, while, as a matter of fact, they are very often beautifully sharp. Again, if the ions forming these rays came from the dissociation of the molecules of the gas in the deflexion chamber, we should expect that the brightness of the curves due to the mercury ions, for example, would be much more dependent on the presence of mercury vapour in the deflexion-chamber than in the discharge-tube. Experiments on the behaviour of the secondary radiation due to mercury show that this is not the case; on the contrary, if we abstract the mercury vapour from the discharge-tube by charcoal cooled with liquid air, we produce a much greater diminution in the brightness of the secondary lines due to mercury than we do if we abstract by the same means the mercury vapour from the deflexion-chamber.

The evidence is, I think, in favour of the view that the secondary rays are due to the dissociation of systems in the undeflected Canalstrahlen, rather than to the dissociation of the gas in the deflexion-chamber through which the Canalstrahlen pass. The dissociation of the systems in the Canalstrahlen is produced by the collision of these systems with corpuscles and not with ordinary molecules; for if the collision were with ordinary molecules, the direction of motion of the systems and their velocities would change when the collision took place, and the result would be that the lines on the plate would be very fuzzy and ill defined, a collision with a body having as small a mass as that of a corpuscle would leave the direction of motion and the velocity of a body as massive as a molecule practically unaltered. The corpuscles when struck by the Canalstrahlen may be regarded as practically at rest in comparison with the systems which strike against them; for though in the space between the parallel plates used to produce the electrostatic deflexion of the rays there is an electric field strong enough to make the corpuscles move with very great velocities, yet as the path of the Canalstrahlen is the region where the corpuscles

are produced and where they have not had an opportunity of acquiring a high velocity from the electric field, they will only be moving slowly when struck by the Canalstrahlen.

Experiments on the ionization produced by the collision of corpuscles against molecules at rest, or rather moving very slowly in comparison with the corpuscles, have shown that for the corpuscle to ionize the molecule the velocity of the corpuscle must be greater than a certain value: thus, according to the results obtained by Townsend and H. A. Wilson, if a corpuscle is to ionize a molecule of air by colliding against it, the velocity of the corpuscle must exceed that which it would acquire by falling through a potential difference of two volts. It would probably not require so large a velocity to ionize the constituents of the beam of Canalstrahlen, as some of these are probably much more loosely connected systems than a molecule of air; we should expect, too, that the velocity would vary from one constituent to another. Let us suppose that to ionize a particular constituent requires a potential difference of  $n$  volts. If now the corpuscle is at rest and the molecule moving, the relative velocity when there is ionization must be the same as in the previous case; but since the molecule has a much greater mass than the corpuscle, the molecule to acquire the same velocity must fall through a much greater potential difference; if it requires  $n$  volts to give this velocity to the corpuscle, it will require  $w \times n \times 1.7 \times 10^3$  volts to give this velocity to a system whose electric atomic weight is  $w$ ; we have taken the mass of the hydrogen atom as  $1.7 \times 10^3$  times that of the corpuscle. Thus the greater the electric atomic weight of any constituent of secondary radiation the greater is the potential difference through which it must have fallen in the discharge-tube in order that it should be dissociated in the deflexion-tube. Thus the secondary radiation of least electric atomic weight, that of the atom of hydrogen, would require to fall through a smaller potential difference than any other. This explains why at comparatively high pressures, when the potential difference in the discharge-tube is small, the hydrogen atom is the only type of secondary radiation to be seen; the others are, as it were, latent in the beam of Canalstrahlen, but the potential difference in the discharge-tube was not sufficient to make them move quickly enough to be ionized when they came into collision with a corpuscle.

For the uncharged Canalstrahlen to be dissociated, and thus give rise to secondary rays, they must travel with a velocity greater than a certain critical velocity, and to acquire this velocity they must fall through a potential

difference not smaller than the value just found, this gives a minimum value for the potential difference required to produce the secondary radiation. There will, however, be a maximum as well as a minimum value, for the uncharged Canalstrahlen are produced by the combination of a positively charged particle and a negatively charged corpuscle, and in consequence of the small mass of the corpuscle the velocity of the uncharged system will practically be that of the positively charged particle before it combined with the corpuscle. If, however, the velocity of the particle exceeded a certain limit, it could not combine with a corpuscle even though it passed quite close to it, its velocity would be sufficient to carry it far away from the corpuscle, and it would not retain it as a satellite. If A and B represent respectively the particle and the corpuscle, then if when they are at a distance  $r$  apart their relative velocity is  $v$ , they will separate to an infinite distance of  $v^2 > 2 \cdot \frac{e}{m} \cdot \frac{e}{r}$ , where

$e$  is the charge in electrostatic measure and  $m$  the mass of a corpuscle. Even with the smallest admissible value of  $r$  a corpuscle would acquire a velocity great enough to satisfy the preceding inequality by the fall through a potential difference of two or three volts; hence it is only exceedingly close to the cathode, perhaps only inside the hole through which the rays pass, that the corpuscles are still enough to allow of any combination with a positive particle to take place. Let us take as the case most favourable to recombination the one where the corpuscles are at rest and the relative motion is entirely due to the positive particles, then if  $v$  is the velocity of the positive particle for combination to take place,  $v^2 < 2 \cdot \frac{e}{m} \cdot \frac{e}{r}$ . The smallest value of  $r$  permissible will depend upon the size of the particle: it will be of atomic dimensions. Let it equal  $b \times 10^{-8}$  cm.

$$\text{Putting } e/m = 1.7 \times 10^7 \times 3 \times 10^{10}, \quad e = 4.5 \times 10^{-1}$$

$$\text{we find } v^2 < \frac{4.6}{b} \times 10^{16}.$$

If the velocity is due to the fall of the charged body through  $V$  volts, then if  $w$  is the electric atomic weight of the particle, the preceding relation is equivalent to

$$v < 2.3 \frac{w}{b} \times 10^4.$$

Thus for the secondary radiation due to the dissociation of the uncharged rays,

$$V > w \times n \times 1.7 \times 10^3,$$

$$V < 2.3 \frac{w}{b} \times 10^4.$$

Thus the velocity of each kind of secondary ray must be between certain limits which do not depend on the potential difference between the electrodes in the discharge-tube. If these limits are very close together the velocity of the secondary ray will be very nearly constant. I have found that this is the case for the secondary rays corresponding to the hydrogen atom.

The curves on the photographic plate which pass through the origin may arise either from the dissociation of the uncharged Canalstrahlen—the change from an uncharged particle to a positive ray, or from the reverse process, the change of a positive ray while in the electric and magnetic fields into an uncharged particle by the coalescence with it of a negatively charged corpuscle; a method of distinguishing between these cases is given on p. 233. For such a coalescence to take place, however, the velocity of the positive ray must be below a certain value; if the velocity is greater than this the ray behaves like a primary one. When the difference of potential between the electrodes in the discharge-tube is much greater than is necessary to produce this velocity any secondary ray must have fallen through only a fraction of the potential difference in the tube, and must therefore have been produced near the cathode. Now it is just in this neighbourhood that the positive ions in the dark space have their greatest velocity and are most likely to produce fresh ions by collision. Thus it is probable that among the ions in the secondary rays there are some which have been produced by the collision of positive ions with the molecules of the gas in the dark space, while the primary rays which have fallen through the whole potential difference have been produced by the collision of cathode particles with these molecules.

There are a few, but only a few, ions which occur both as primary and secondary; the positive atom of hydrogen with one charge is the most conspicuous example of this class; others are the atoms of mercury and oxygen and the molecule of hydrogen. In most cases, however, the ions are quite distinct. On looking at the list of ions due to secondary radiation given on page 239 it will be seen that, with the

exception of the hydrogen molecule, there is not one in which the molecule is intact, while many molecules are found among the ions in the primary rays; for example, the molecules of hydrogen, oxygen, marsh-gas, not to mention those of the monatomic gases like helium and mercury vapour. Again, the ions in the secondary rays carry in many cases more than one unit of charge; thus, for example, we have  $C_{++}$ ,  $N_{++}$ ,  $Hg_{+++}$ ,  $He_{++}$ , suggesting that the positively charged particles when they collide with a molecule in many cases detach more than one corpuscle from it.

It is very interesting to find that in the primary rays several different types of ions are found even in elementary gases like hydrogen or oxygen, for, as we have seen, we find both the atoms and the molecules amongst the primary rays in these gases. These ions are supposed to be due to the bombardment of the molecules by the cathode rays. How is it, then, that when we expose the molecules of a gas to bombardment by cathode rays we get two types of ions, atoms as well as molecules in the gases we have just mentioned? It is true that in the dark space next the cathode we have cathode rays with very different velocities; and one way of explaining the two types of ions would be to suppose that when the energy of the cathode rays exceeds a certain value, they split the molecule into atoms when they impinge against it, while the slower cathode rays only succeed in knocking a corpuscle out of the molecule without impairing the cohesion between the atoms. On this view the charged atoms would be produced by the fast cathode rays, the charged molecules by the slower ones. If this were the case, however, we should expect that the lines corresponding to the atom would be shorter than those corresponding to the molecule, as the minimum energy required to produce them is greater than that required for the molecule, and the place where the atoms are produced would therefore be further from the cathode than the corresponding place for the molecules; as a matter of fact, the lines for the atom are often, though not invariably, longer than those for the molecule.

Another way in which the different kinds of ions might arise is as follows:—Let us suppose that a diatomic molecule is made up of two atoms A and B, and that A is positively and B negatively charged. When the cathode rays pass through the gas they may strike either A or B, and detach a corpuscle. If A is struck, then, after the collision, A has two positive and B one negative charge, together they form a system with a total positive charge of

one unit ; the attraction, however, between A and B due to their electric charges has been increased, so that the system AB is less likely to break up into atoms than it was before the collision took place. Suppose, however, that B and not A is struck by the cathode particle, then after the collision B will be uncharged, while A has one unit of positive charge, the total charge on the system is again one unit of positive electricity ; but as B has been deprived of its charge the electrical attraction between A and B is very much less than it was before the collision took place, so that the system will be much more likely to break up into separate atoms and supply us with a charged atom A. The negatively electrified atom B will be a little less likely to be struck by a negatively electrified cathode particle than the positively electrified one A. On the other hand, when a collision did take place, it would be easier to detach a corpuscle from the negatively electrified B than from the positively electrified A.

Similar considerations would apply to compounds as well as to elements. We might in certain cases get some of the atoms of a compound molecule liberated and not others. Thus, if the hydrogen atoms in marsh-gas  $\text{CH}_4$  are negatively charged, and if one of them is struck by a cathode particle, it would lose its charge and be easily detached, the other hydrogen atoms which retained their charge would cling to the carbon atom.

The photographs hitherto described were made with discharge-tubes whose volume was considerably greater than 1000 c.c. ; I have also made some photographs when the tube was very much smaller, the diameter being about 2 cm. The feature of these photographs is, that unless the pressure is reduced very low, when the potential difference between the electrodes is very large, almost the only thing to be seen on the plate, whatever gas may be in the tube, is the secondary radiation, negatively as well as positively charged, corresponding to the hydrogen atom ; the negative portion is not infrequently almost as bright as the positive.

When the tube is filled with air, a very faint curve corresponding to the oxygen atom can with difficulty be detected on the plate ; it is, however, much too faint for reproduction from a photograph. Though this line is so faint, it is remarkable that it is generally the first to appear when the photograph is developed ; the hydrogen line, though so much stronger in the end, takes a much longer time to develop. It appears as if the hydrogen atoms had penetrated much more deeply into the film than the oxygen ones, but that close to the

surface of the film the oxygen atoms produce more effect than the hydrogen ones. The relative intensities on the photographic plate do not always seem the same as on the willemite screen.

I find from the photographs that the slope of the straight part of the curves corresponding to the secondary radiation due to the hydrogen atom, negative as well as positive, for the slopes of the two are the same, does not vary appreciably with the potential difference between the electrodes in the discharge-tube. I have taken photographs with the tube in states for which the equivalent spark gap in air varied from .7 to 4 cm., and found only very slight alterations in the slopes of the curves. As the velocity of the particles in the rays is proportional to the tangent of the angle of slope, this implies that the velocity of the particles in the secondary rays is almost constant, a result I had previously arrived at by the willemite screen.

I wish to express my thanks to Mr. F. W. Aston, of Trinity College, and Mr. E. Everett, for the assistance they have given me with these experiments.

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## XXVII. *Focal Isolation of Long Heat-Waves.*

*By H. RUBENS and R. W. WOOD\*.*

THE isolation of very long heat-waves, which is usually accomplished by selective multiple reflexions (Reststrahlen method) can be accomplished also by selective refraction. It was shown in 1899 by Rubens and Aschkinass that it was possible to separate very long heat-waves from the radiation of an incandescent source by means of quartz prisms of small angle\*. This method, involving the use of a spectrometer, did not however prove to be very efficient, on account of the large loss of energy, and the isolated radiation disappeared almost entirely if a quartz plate of any considerable thickness was interposed in the path of the rays.

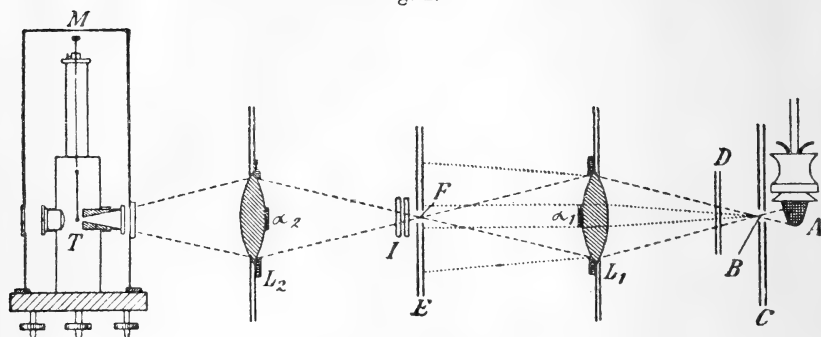
The method which will be presently described is free from these objections, and has enabled us to obtain heat-waves of greater wave-length than any hitherto observed and with sufficient intensity to make accurate measurements of their properties possible. Like the other method, it depends upon the selective refraction of quartz, the separation being accomplished by means of lenses however.

\* Communicated by the Authors.

† H. Rubens and E. Aschkinass, *Wied. Ann.* lxxvii, p. 459.

The arrangement of our apparatus is shown in fig. 1. The radiations from a Welsbach light A pass through a

Fig. 1.



circular aperture 15 mm. in diameter, B, in a screen made of two large sheets of tin-plate, and then in succession through the quartz lens  $L_1$ , a second aperture F, and the quartz lens  $L_2$  which focusses them upon the thermocouple of the radio-micrometer M.

The lenses have a diameter of 7.3 cm. and a thickness of .8 cm. in the middle and .3 cm. at the edge: their focus for light rays is 27.3 cm. The central zones of these two lenses are covered by circular disks of black paper  $a_1$  and  $a_2$ , 25 mm. in diameter. A movable shutter is placed between C and  $L_1$  to cut off the radiation at will.

The operation of this arrangement of apparatus is easily understood. The distances of the lenses from the circular apertures are so proportioned, that a sharp image of the circular source B is focussed upon F only for radiations for which the refractive index of the quartz is 2.14, the square root of the dielectric constant for slow oscillations.

Inasmuch as the refractive index of quartz is between 1.55 and 1.43 for the shorter heat and light waves (fine dotted lines), which are able to pass through it, these rays form a divergent cone after passing through the lens  $L_1$ . This cone is intercepted by the screen E, the circular aperture being shielded by the disk of black paper. As a result, the only rays which can pass through the aperture are the very long heat-waves (coarse dotted lines), which converge upon it, as a result of the high value of the refractive index for these radiations. The second quartz lens, acting in the same way, still further purifies the radiation, and eliminates completely the shorter heat waves, scattered or diffused by the surfaces of the first lens.



Details regarding the micro-radiometer will be found in previous papers\*.

In the present case the instrument was protected by an air-tight metal helmet provided with a quartz window. The sensitiveness was such that a candle at a distance of 2 metres gave a deflexion of 700 scale-divisions, in spite of the quartz window.

The great advantage of this method of focal-isolation is that large angular apertures can be used (in the present case F 3·5), and the radiation is weakened only by reflexion from and absorption by the two quartz lenses. Its only disadvantage is the circumstance that a rather wide spectral range is transmitted, for all radiations for which the refractive index is in the neighbourhood of 2·14 are passed through the apertures.

For wave-length  $63\ \mu$  the refractive index is 2·19, decreasing to 2·14 as a limiting value, with increasing wave-length †. The very powerful absorption of quartz for radiations ranging from 60–80  $\mu$  prevents the shorter waves from getting through the system. Beyond 80  $\mu$  the quartz begins to show stronger transparency, 17 per cent. of the residual-rays from potassium iodide passing through a quartz plate of 18 cm. thickness, corresponding to the amount of quartz in the path of our rays ‡.

This circumstance gives to our transmitted radiation an energy curve very steep on the side towards the shorter wave-lengths, while on the other side it slopes down gradually, at the rate determined by the energy curve of the source of light, decreasing with the 4th power of the wave-length, if the Welsbach mantle has no selective properties in this region, as is probably the case. The increasing transparency of the quartz with increasing wave-length will make the face of the energy curve still more gradual, and we should expect our isolated radiation to have an unsymmetrical energy curve with a maximum shifted towards the shorter wave-lengths with respect to its centre of gravity, and to extend from 80 towards the longer wave-lengths over a range of more than an octave. This expectation was fulfilled, as the results of our experiments will show.

The 40 mm. deflexion of the micro-radiometer was reduced to zero by interposing a plate of rock-salt, 3 mm. thick, into the path of the rays, while a quartz plate of 4 mm.

\* H. Schmidt, *Ann. der Phys.* xxix. p. 1003; H. Rubens and H. Hollnagel, *Phil. Mag.* [6] xix. p. 764.

† H. Rubens and E. F. Nichols, *Wied. Ann.* lx. p. 418.

‡ Rubens and Hollnagel, *loc. cit.*

reduced the deflexion to only 20 mm. To determine the wave-length of the isolated radiation and the energy distribution, we employed the same quartz interferometer which had been used in the investigation of the residual rays from potassium, iodide and bromide, a description of which will be found in the paper already referred to. The plates of the instrument J were placed close to the diaphragm F, and the distance between them varied by turning the screw of the interferometer. The micro-radiometer showed periodic increase and decrease of energy as the thickness of the air-film between the plates was increased.

If we draw a curve representing the observed deflexions as a function of the corresponding thicknesses of the air-film, we can compute the mean wave-length of the radiation from the position of the maxima and minima, and from the "damping" obtain evidence of the spectrum range with which we are dealing.

It was found expedient to use quartz plates of much greater thickness than those used in the former investigation. Having better surfaces and being less easily bent, the opposed faces could be brought into much closer proximity than those previously used. The following investigations were made with pairs of plane parallel plates of 2 mm. and 7.3 mm. respectively.

It was found that the readings of the graduated wheel of the interferometer did not give very reliable indications of the thickness of the air-film, especially when the plates were in close proximity. The distance between the plates was accordingly determined in every case, by observing the interference fringes formed by reflecting the light of a sodium flame normally, from the quartz plates. At the beginning of each series of observations, the quartz plates were brought as nearly as possible into contact, by observing the black spot in the interference fringes obtained with white light. The black spot only appeared when the plates were pressed together with the fingers, the distance between them at the beginning of the series being determined by counting the number of fringes which passed a given point as the pressure was relieved. This usually amounted to 7 or 8 fringes only, or a very small fraction of the wave-length under investigation. The deflexion of the micro-radiometer was now taken, and the plates slowly separated by turning the screw, until 20 more fringes had crossed the mark. This corresponded to an increase in thickness of  $10\lambda_D = 5.89\mu$ . The energy was again measured, and the plates again separated by ten sodium wave-lengths, as long as the maxima and minima could be followed.

The results of two such series of observations selected from eight series, all of which were in good agreement, are reproduced in figs. 2 and 3. Curve 2 was obtained with the

Fig. 2.

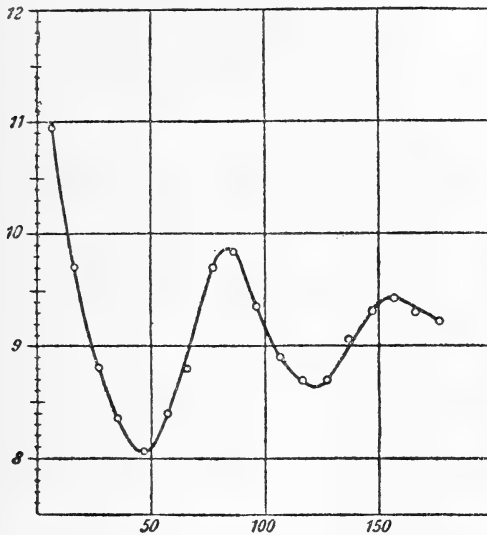
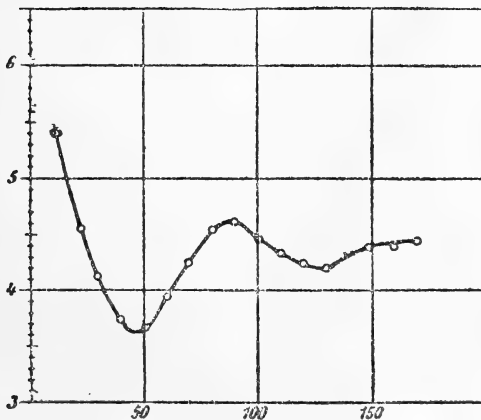


Fig. 3.



thin quartz plates (2 mm.), and curve 3 with the thick ones (7.3 mm.). The ordinates are the micro-radiometer deflexions in millimetres, and the abscissæ are the thicknesses of the air-film, in sodium wave-lengths.

Curve 2 shows a minimum at  $46\lambda_D$ , a maximum at  $85\lambda_D$ , and a second minimum at  $122.5\lambda_D$ . Curve 3 has its first minimum at  $47.5\lambda_D$ , a maximum at  $90\lambda_D$ , and a second minimum at  $128\lambda_D$ . If we calculate the mean wave-length of the radiation complex, in both cases from the first minimum, we obtain for the experimental series of fig. 2,  $\lambda_1 = 46 \times 0.589 \times 4 \mu = 108.2 \mu$ . For fig. 3,  $\lambda_1' = 47.5 \times 3.589 \times 4 \mu = 111.8 \mu$ . The values calculated from the positions of the first maxima in each case are  $\lambda_2 = 100 \mu$  and  $\lambda_2' = 106 \mu$ , and from the second minima,  $\lambda_3 = 96.3 \mu$  and  $\lambda_3' = 100 \mu$ .

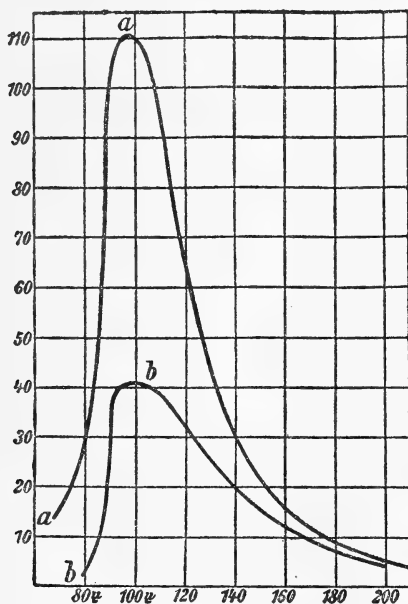
We thus see that we obtain slightly different values for the mean wave-length according to which maximum or minimum we use in the calculation. If we consider the incidence angle as  $6^\circ$  instead of normal (owing to convergence), these values are to be reduced by about  $0.6 \mu$ . In both cases the calculated wave-length decreases with increasing "order" of the maxima and minima. The explanation of this lies in the unsymmetrical character of the energy curve. The first minimum gives us the value of the wave-length corresponding to the centre of gravity of the energy curve, while the following maxima and minima approach more and more near to the wave-length of the maximum of the energy curve. The energy curves in the two cases differ, not only in the position of the wave-length of the centre of gravity, but also in the position of the maximum and the degree of asymmetry. It was foreseen that the asymmetry of the energy curve would be greater for the thin plates than for the thick ones, for in the latter case the radiation was obliged to pass through an additional thickness of  $10.6$  mm. of quartz, and the short waves would be more strongly absorbed than the long ones, for which quartz is very transparent. The rising slope of the energy curve will be displaced towards the longer wave-lengths and made less steep by increasing the thickness of the quartz, while the descending slope will be influenced but little.

It is possible to get an idea of the approximate form of the energy curve of the radiation isolated by the quartz lenses, by a trial and error method. We know the position of the top of the curve approximately, and the terminus on the short wave-length side (determined by the quartz absorption), and we can draw the curve on the long wave-length side, by assuming that our source of radiation is non-selective, and that the intensity decreases with the fourth power of the wave-length. This curve we may now divide in

elementary vertical strips, each one of which represents nearly homogeneous radiation. We now draw the interference curves (sine curves) of the various strips, and the superposition of all these curves should give us a curve identical with the curve obtained with the interferometer, if our energy curve has been correctly figured.

Curves *a* and *b* (fig. 4) examined in this way gave curves

Fig. 4.



resembling very closely the interference curves of figs. 2 and 3, and may be considered as representing the approximate distribution of energy in our radiation. It is thus apparent that we have experimental evidence of the presence of heat-waves certainly  $150\ \mu$  and probably  $200\ \mu$  in length.

#### *Absorption and Reflexion of the Isolated Radiation.*

Notwithstanding the wide spectral range of our isolated radiation, it seemed of interest to determine the reflecting and absorbing power of a variety of substances for these rays, which belong to a region of the spectrum about which we know nothing, and have, up to the present time, been unable to investigate with homogeneous rays. We commenced with the investigation of transparent solids, placing

the flat plates, together with a 4 mm. quartz plate, in front of the diaphragm F. The introduction of the quartz plate was necessary to secure for our radiation the same energy curve as that determined by the interferometer. The following table gives our results not corrected for surface reflexion.

TABLE I.

Material.	Thickness $d$ .	Percentage transmitted $D$ .
	mm.	per cent.
Paraffin .....	3.03	57.0
Mica .....	0.055	16.6
Hard rubber .....	0.40	39.0
Quartz .....	2.00	62.6
Fused quartz.....	3.85	0
Rock-salt .....	0.21	21.5
Fluorite .....	0.59	5.3
Diamond .....	1.26	45.3

The great transparency of the diamond is worthy of notice. About 30 per cent. of the incident radiation is reflected from the surfaces of the plate, and a portion is lost as a result of the circumstance that the plate was not perfectly flat, so that we cannot say with certainty that the plate exercised any appreciable absorption. Plates of rock-salt were absolutely opaque until reduced to thin films by carefully dipping them into water until they were upon the point of falling to pieces.

The part played by the selective absorption of quartz in the isolation of our long waves, made the determination of the absorption of plates of different thickness a matter of great importance. It turned out, as was to be expected, that the absorption-coefficient  $q$ , as expressed by the equation  $\frac{D'}{100} = e^{-qd}$ , decreases rapidly as the thickness  $d$  increases.

$D'$ , in the above expression, represents the transmission in per cents., corrected for reflexion by the two surfaces. The values are given in the following table, in which  $q$  is calculated by substitution of the observed values of  $d$  and  $D'$  in the equation, and  $q_1$  by making use of the difference in thickness of two plates successively investigated and the ratio of their transparencies. From this it is evident that  $q_1$  must decrease more rapidly than  $q$ .

TABLE II. Quartz perpendicular to axis.

<i>d.</i>	<i>D'</i>	<i>q.</i>	<i>q</i> <sub>1</sub>
mm.	per cent.		
2.00	81.3	0.103	0.100
4.03	66.4	0.102	0.089
7.26	49.8	0.096	0.075
11.74	35.5	0.088	0.069
14.66	29.0	0.084	0.062
18.69	22.6	0.079	

The mean wave-length of the radiation under the conditions which obtained in the experiments on reflexion and transmission was  $108 \mu$ . These were made with a 4 mm. quartz plate always in the path of the rays. From Table II. we find that, for such a plate  $q_1 = .089$  for our  $108 \mu$  rays, while for residual-rays of various wave-lengths previous observations have given the following values:—Sylvin ( $6 \mu$ )  $q_1 = .281$ , potassium bromide ( $82 \mu$ )  $q_1 = .216$ , potassium iodide ( $97 \mu$ )  $q_1 = .104$ —.

We find also that black paper, and even black cardboard are partially transparent for the long waves, while films of smoke, so thick that the sun cannot be seen through them, are almost perfectly transparent. The smoke was deposited on a thin film of mica, the absorption and reflexion of the mica being eliminated by making observations with the covered and uncovered portions. The results are given in Table III. for: 1, thin black tissue-paper (.025 mm.) nearly opaque to light; 2, thick black paper such as is used for packing photographic plates (.11 mm.); 3, black card-

TABLE III.

Radiation.	Wave-length $\lambda$ .	Transmission D in per cents.			
		Paper I.	Paper II.	Cardboard.	Smoke.
	$\mu$ .	per cent.	per cent.	per cent.	per cent.
Isolated by spectrometer...	2	0	0	0	0.5
	4	0.9	0	0	8.6
	6	1.7	0	0	16.0
	12	8.2	1.4	0	37.6
Residual rays from fluorite...	26	24.2	3.2	0	76.7
Residual rays from rock-salt.	52	46.0	15.1	0	91.3
Focal isolation .....	108	61.5	33.5	1.6	91.5

board (0.4 mm.); 4, smoke film, 1.8 mg. of carbon per square cm. (determined with Nernst's micro-balance).

The great transparency of the smoke film for the long waves is of interest as showing that the sensitiveness of our radiometer was not much increased by smoking the metal surface. We have increased the deflexions of the instrument used in the present work by covering the thermo-couple with a thin film of water glass, which is very opaque to the longer waves. The best material is probably a mixture of water glass and finely divided carbon (soot deposited by flame), which dries without cracking, and sticks firmly to glass or metal surfaces. The transmission of fluids and vapours was investigated by employing quartz cells of known thickness, or large brass tubes closed by thin quartz plates. The fluid cells are made by pressing the quartz plates against thin rings of lead foil or glass, except in the case of the cell used for the investigation of water which was found to be quite opaque when contained in our thinnest cell (0.158 mm.). To determine the transmission of water we once more inserted the interferometer in the path of the rays, and, having separated the plates a given distance, measured in sodium fringes, introduced a drop of water between them. The water crawled in by capillary attraction, and the sodium fringes were watched throughout the process to make sure that no change in the distance between the plates resulted.

The tube used for the investigation of vapours was 20 cm. in length and 6 cm. in width, an observation being made first with the tube filled with air, and then with air saturated with the vapour, obtained by allowing a slow stream of air to bubble through two bottles filled with the fluid. The absorption of water vapour we measured with a tube 40 cm. long and 9 cm. in diameter, open at both ends, a current of steam flowing constantly into it through a side tube placed near the centre. The walls of the large tube we kept at a temperature of  $150^{\circ}$ , by a coil of wire electrically heated. This prevented the formation of clouds, condensation not taking place until the superheated vapour had risen several centimetres above the open ends of the tube. The results are given in Tables IV. and V. The values observed with the fluids are all a little too high since the introduction of a fluid between the quartz plates more or less completely eliminates the reflexion from the two inner surfaces. A knowledge of the refractive index of the fluid for the long waves would be necessary for the correction of this error.

Water and water vapour are still highly absorbing for



this spectral range, though their transparency is greater here than in many other regions. Water turns out to be more transparent for these rays than for the  $82\ \mu$  residual rays from KBr, while the opposite is true for water vapour.

TABLE IV. (fluids).

Material.	Thickness D.	Percentage transmission D.
	mm.	per cent.
Benzol .....	1.00	56.8
Ethyl alcohol .....	0.158	7.9
Ethyl ether .....	0.158	37.1
Castor oil .....	0.158	46.1
Water ..	0.029	25.8
Water .....	0.044	13.6

TABLE V.

Material.	Thickness of vapour layer $d$ .	Corresponding thickness of fluid layer $\delta$ .	Percentage transmission D.
	mm.	mm.	per cent.
Alcohol vapour.....	200	0.023	88
Ether vapour .....	200	0.350	35.5
Benzol vapour .....	200	0.0627	100
Water vapour .....	400	0.21	19.6
CO <sub>2</sub> .....	200	.....	100

It is possible, however, that the addition of 10 per cent. of glycerine to the water used for preparing the films used in the previous work with the KBr rays diminished its transparency somewhat.

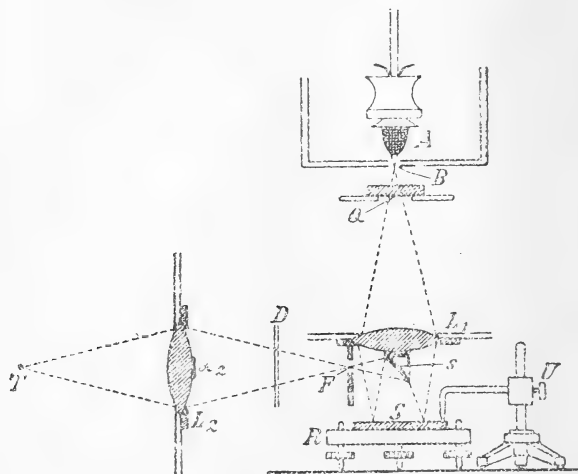
Both benzol and its vapour are exceedingly transparent, which is in agreement with earlier work upon the optical properties of this substance.

From Table V. it is apparent that for an equal number of molecules in the path of the rays, the absorption of the fluid is greater than that of the vapour, especially in the case of water.

For the study of the reflecting power of various substances for these long waves, we modified the apparatus somewhat in order to make perpendicular incidence possible.

The arrangement is shown in fig. 5. The rays are converged by the horizontal quartz lens upon the flat surface S of the

Fig. 5.



substance under investigation.  $s$  is a small elliptical mirror (made by silvering a flat spectacle-glass) mounted with wax on a cork attached to the under surface of the lens. This mirror serves not only to reflect the converging pencil of rays to the diaphragm  $F$ , but acts at the same time as the central stop of the lens.

The observations were made by measuring the reflexion first with a plate of silvered glass at  $S$ , and then with the fluid or solid surface under investigation. Care was of course taken to insure that the reflecting surfaces were level and at the same distance from the lens, as shown by the movable marker  $U$ .

In Table VI. are given the reflecting powers of various substances not only for the  $108\mu$  waves obtained by focal isolation, but also for the  $82\mu$  residual rays from  $KBr$ . In the last two columns are given the dielectric constant and the value of the reflecting power for infinitely long waves calculated from it by the formula  $R_{\infty} = 100 \left( \frac{1 + \sqrt{K}}{1 - \sqrt{K}} \right)^2$ .

The calculated and observed values are in close agreement for rock-salt, fluorite, and glass, while for sylvite the value observed with the  $108\mu$  waves is much nearer the

TABLE VI.

Substance.	Reflecting power, observed for		K.	$R_{\infty}$ .
	residual rays from KBr $\lambda = 82\mu$ .	focal isolation $\lambda = 108\mu$ .		
	per cent.	per cent.		
Iceland spar .....	.....	47.1	8.5	24.0
Marble .....	.....	43.8	6.14	18.1
Rock-salt .....	.....	20.3	6.29	18.5
Sylvite .....	36.0	19.3	4.94	14.3
KBr .....	82.6	31.1	...	...
KI .....	29.6	35.5	...	...
Fluorite .....	19.7	20.2	6.8	19.7
Glass .....	.....	19.2	6.66	19.5
Water .....	9.6	11.6	81	64
Alcohol .....	.....	1.6	25	44.5
Castor oil .....	.....	4.3	4.78	13.9

value calculated for  $\lambda = \infty$  than the value previously found with  $82\mu$  radiation. Iceland spar and the fluids examined showed no such agreement however. The refractive indices of water, alcohol, and castor oil are of the same order of magnitude in this region as in the visible spectrum. The high reflecting power of water probably results from the presence of one or more absorption bands in the region under investigation.

XXVIII. *The Resonance Spectra of Iodine.* By R. W. WOOD,  
*Professor of Experimental Physics, Johns Hopkins University* \*.

AS I have shown in previous papers, the complicated channeled (band) absorption spectrum and the corresponding banded emission-spectrum of sodium vapour, rendered fluorescent by white light, can be analysed into many simple series of (nearly) equidistant lines, by exciting the fluorescence with monochromatic light of various wavelengths obtained from metallic arcs or vacuum-tubes. Such spectra I have named resonance spectra. I have also shown that the fluorescent spectrum of iodine resembles that of sodium, and have now found that it can be excited by monochromatic light and made to yield resonance spectra in the same manner. Thus far but a single source of light has been

\* Communicated by the Author.

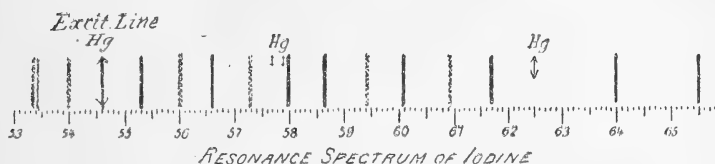
tried for the excitation, namely, the mercury arc, the active line of which is the bright green one at 5460, as can be shown by interposing various ray-filters between the lamp and the bulb which contains the iodine vapour. I now use very large bulbs, 15 to 20 cm. in diameter, made from round-bottomed flasks. The neck is drawn down in the blast-lamp, a few small crystals of iodine are introduced, and the bulb highly exhausted with a Gaede pump. The neck is then sealed. If the light of the arc is focussed at the centre of the bulb with a condensing lens, preferably the condenser of a large projecting lantern, the fluorescence is so bright that it can be seen from the back of the large lecture-hall. No heating of the bulb is necessary.

When illuminated with sunlight condensed by means of a large Voigtländer portrait objective (F. 2·3), I have found that the intrinsic intensity of the fluorescent light in the green part of the spectrum was nearly  $\frac{1}{3}$  of that of the Welsbach light. These experiments will be described in detail in a subsequent paper, the object of the present communication being merely to record the discovery of the resonance spectra of the vapour.

The quartz mercury arc is mounted at such a distance from the bulb that the conjugate foci are equal. It is important to use a very large condenser of very short focus, and it is better to have the mercury arc tube perpendicular to the line joining it with the bulb (i e. *not* end on). In this way we illuminate a broad sheet of the vapour, and the fluorescence, when viewed from the side from a point lying in the plane of the illuminated sheet, has a considerable intensity. Its colour is distinctly reddish, and if we examine it with a small spectroscope designed for faint spectra, we shall find that, instead of having an enormous number of lines forming the characteristic bands of the iodine spectrum, we have a beautiful series of isolated lines separated by distances varying from 65 to 80 Ångström units. The wave-lengths have thus far been determined only with a Hilger "wave-length" spectroscope, and can be considered correct only to within perhaps 5 Ångström units. It will be necessary to photograph the spectrum with high dispersion, as has been done in the case of sodium vapour. The general appearance of the spectrum is shown in fig. 1. The lines indicated by arrows are the lines of the mercury arc, which usually appear, as a result of diffused light, unless great precautions are taken. The green line always appears as it is the exciting line, and the waves are in part re-emitted without change of

wave-length. Such a re-emission I have named "Resonance radiation." The other lines, together with the 546 green

Fig. 1.



line, form the "resonance spectrum." In it we find two lines of shorter wave-length than that of the exciting line (one of them being double, however), and two strong and two very weak lines, between the green and double yellow lines of the mercury spectrum, and eight or ten lines in the red and orange. One line coincides almost exactly with the longer of the two yellow Hg lines which are absent in the resonance spectrum if all diffused and reflected light is cut off.

It seems probable that we shall be able to do more with iodine vapour than has been done with sodium vapour, as the work can be done at room temperature with glass bulbs. I intend to continue the work, using other monochromatic sources of light for the excitation of the vapour, and study the behaviour of the resonance spectra in a magnetic field. It will be extremely interesting as well to observe the probable changes in the resonance spectra when the exciting source of light is placed in a strong magnetic field.

Berlin, Dec. 1st, 1910.

*Supplementary Note added January 16th.*

Since the preparation of the above note I have been able to make further experiments through the courtesy of Prof. Cotton, of the École Normal, who very kindly placed his laboratory at my disposal. The wave-lengths of the lines have been re-determined, and a number of very faint ones, not shown in fig. 1, added to the list.

In the following table the small numerals denote intensities, and the letters *a* and *b* indicate the lines which probably belong to the same series. The approximate constancy of wave-length differences is apparent at once, and we are

reminded of the resonance spectra of sodium previously studied:—

<i>a</i> SERIES.		$\lambda$ .	<i>b</i> SERIES.	
$\lambda$ Differences.			$\lambda$ Differences.	
		5325	1	
		<i>a</i> 5330	1	
65	.....	<i>a</i> 5395	4	
60	.....	<i>a</i> 5460	10	EX. LINE.
65	.....	<i>a</i> 5525	10	
		5545	2	
68	.....	<i>b</i> 5567	2	
		<i>a</i> 5593	5	
67	.....	<i>b</i> 5635	3	..... 68
		<i>a</i> 5660	10	..... 70
70	.....	<i>b</i> 5705	3	
		<i>a</i> 5730	8	..... 69
		<i>b</i> 5774	5	
65	.....	<i>a</i> 5796	10	..... 63
		<i>b</i> 5837	1	
69	.....	<i>a</i> 5865	7	
		5962	2	
		6017	3	

In the earlier work it appeared as if the green 546 line was the only one operative in stimulating fluorescence. I have since found that if the light of the mercury arc is filtered through an eosine screen, which removes the green light completely, we still have a fairly bright fluorescence of a red-orange colour. The spectrum has not yet been photographed, but visual observations with a Hilger wave-length spectroscope show that it is apparently made up of the *a* and *b* series. Two yellow lines, coinciding with the exciting yellow lines, are very conspicuous (resonance radiation); they are of equal intensity, while in the resonance spectrum excited by the total radiation of the lamp the longer line is very much brighter. Bordering these two yellow lines, and symmetrically placed to the right and left, are several very faint pairs of lines, the components of each pair apparently separated by a distance equal to the distance between the yellow exciting lines. It is probable that one of the lines of the "*a*" series excited by the green line coincides with the longer of the two yellow Hg lines, at least within the limits of resolution of the spectroscope used. The "*a*" series can thus be regarded as stimulated at two points, 546 and 5796, when we employ the total radiation. The "*b*" series we can provisionally ascribe to excitation by the shorter of the two yellow lines. By employing a screen of neodymium chloride, we can remove

the yellow lines from the light of the arc. The colour of the fluorescence becomes at once greener. The 5796 line is still visible in the resonance spectrum, but the 5774 line has disappeared, which is what we should expect. The spectra are so faint, with any considerable dispersion, that visual observations are unsatisfactory. As soon as I am able to photograph them with my large three-prism spectrograph, we shall be able to tell better just what we have. In studying the spectra I now place a plane mirror behind the bulb, in such a position that the fluorescent sheet of vapour and its reflected image are seen superposed. Much could be gained by employing two mercury arcs, one on each side of the bulb.

Paris, Jan. 16, 1911.

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XXIX. *Transformation of a Resonance Spectrum into a Band-Spectrum by Presence of Helium.* By R. W. WOOD and J. FRANCK\*.

[Plate II.]

AS has recently been shown by Wood, the band-spectrum of iodine vapour, brought to a state of fluorescence by excitation with white light, can be analysed into simple series of very sharp and widely separated lines by monochromatic excitation. With the mercury arc, the green line being the only one capable of exciting fluorescence, we obtain a series of 15 lines, about 70 A.E. apart, and of very variable intensity. Each band of the channeled spectrum is made up of a large number of fine lines, and we have, roughly speaking, one line of the resonance spectrum for each one of the bands. As we have already shown, the reduction in the intensity of the fluorescent light by the presence of helium, is much less than that produced by any other gas. In helium at 80 mms. pressure, the fluorescence can still be seen though rather feeble and quite red in colour, in marked contrast to the green colour shown by the vapour *in vacuo*. In chlorine, however, at a pressure of 4 or 5 mm. the fluorescence has disappeared entirely.

If we have our iodine vapour in a bulb containing helium at a pressure of only 2 or 3 mm., there is no apparent change in the intensity or colour of the fluorescence, regardless of whether we excite it with white light or monochromatic light. The spectroscope shows, however, that a most remarkable change has taken place. Exciting with the green

\* Communicated by the Authors.

Hg line we find that the presence of the helium has caused the *complete band-spectrum to appear*, the isolated lines of the resonance spectrum being still visible though greatly reduced in intensity. In helium at 10 mm. there is very little trace of the resonance spectrum left, the fluorescence spectrum being almost identical with that excited by white light.

In chlorine, however, which, as we have shown, reduces the intensity of the fluorescence by its electro-negative properties, we do not find this effect. The weakening of the lines of the resonance spectrum is not accompanied by the appearance of the band-spectrum. Photographs of the spectra are shown in fig. 1 (Pl. II.). They were made with a Hilger wave-length spectroscope. Fig. 1 *a* shows a portion of the resonance spectrum of the iodine vapour *in vacuo* excited by the green mercury line; immediately below it is the Barium arc for comparison. Fig. 1 *b* is the same as "*a*" except that it was made with a slightly wider slit, and with a plate having its maximum of sensibility in the yellow-green. On this plate we find a fairly strong line which is wholly absent on plate *a*, which was sensitized for the red and probably had a minimum of sensitiveness at the wave-length in question.

Fig. 1 *c* shows the effect of introducing helium at 2 mm. pressure into the bulb. The band-spectrum comes out distinctly with the lines of the resonance spectrum superposed on it. Just below, fig. 1 *d*, we have the spectrum excited by white light with the iodine *in vacuo*. The duration of the exposures were, 1 hour for the white light fluorescence, and 15 hours for the excitation with the mercury lamp. As will be seen, there is a slightly different distribution of intensity in the band-spectrum excited by white light with the iodine *in vacuo*, and the spectrum excited by the green mercury light with the iodine in helium, apart from the presence of the lines of the resonance spectrum. It is probable that similar effects will be found with sodium vapour, the numerous resonance spectra of which were discovered and studied by Wood.

If we regard each one of the many resonance spectra as due to the vibration of one system of electrons, the effect of the collisions with, or proximity of, the helium molecules, is to couple all of the systems together so to speak. An isolated molecule can perhaps be thought of as containing a large number of electron systems. Monochromatic radiation excites one of these electrons which emits radiation of the same wave-length as that of the exciting light (resonance radiation), and in addition communicates a disturbance to



the other members of the system, each one of which emits a radiation of its own frequency (resonance spectrum). The disturbance is not, however, communicated to the other systems which remain quiet. With white light excitation all of the systems are of course disturbed, and we get the band-spectrum which can be perhaps regarded as a composite photograph of all of the resonance spectra. Now the collisions with the helium molecules appear to destroy the equilibrium of the forces within the molecule, which is necessary for the isolation of the electron systems, and results in a transfer of energy from the excited electron not only to all the members of its own system, but also to the members of the other systems. Electro-negative gases are unable to accomplish this since they practically destroy the fluorescence if they approach near enough to the iodine molecules to affect them at all.

It is to be clearly understood that the collision with the helium molecule does not excite the vibrations, but merely brings about a condition which enables a transfer of energy from one vibrating system of electrons to the other systems to take place. The total amount of light emitted by the iodine vapour *in vacuo* and in helium at 2 mm. pressure is the same: in the former case it is concentrated in the 15 resonance spectrum lines, in the latter it is distributed among the hundreds of lines which make up the band-spectrum. The two spectra can be beautifully shown with the smallest direct-vision spectroscop. It is best to use a bulb 15 or 20 cm. in diameter, and to illuminate the vapour with the light issuing from the *side* of a quartz mercury lamp, brought to a focus at the centre of the bulb with a large lantern condenser. In this way we obtain a broad sheet of illuminated vapour, which when viewed from the side has a considerable intensity. It is obvious that one effect of the collision with the helium molecule is to reduce the amount of resonance radiation (unchanged wave-length) in proportion to the total radiation. This explains perfectly why the helium reduces the intensity of the green portion of the fluorescent spectrum excited by white light more rapidly than the red, as we have shown in a previous paper. The fluorescence is excited chiefly by the green rays, the fluorescence spectrum extending from wave-length 500 to the extreme red. The region of the spectrum chiefly operative in exciting the fluorescence is from 500 to 580. Orange and red light is wholly inoperative.

In the complete band-spectrum excited with white light we may therefore consider the green portion as made up

largely of resonance radiation, the red and yellow region being analogous to the lines of the resonance spectrum. The resonance radiation is weakened by the helium, the energy going over to the red and yellow region, consequently in helium at higher pressures the colour of the emitted light is red. This total intensity of the fluorescent light is reduced as well by helium at pressures above 10 mm., and it is the combination of the two effects that gives us the observed results. This wholly new effect of molecular collisions upon the nature of spectra makes an excellent problem for the theoretical physicists to solve.

Berlin : Phys. Inst. der Universität,  
December 1910.

### XXX. *Electric Dust Figures.*

By JAMES ROBINSON, *M.Sc., Ph.D.\**

**I**N a recent communication to this magazine Mr. Richmond† described some dust figures obtained by the passage of electric sparks at the end of a Kundt tube. Richmond thought that possibly there was some connexion between the distance apart of two ripples and the frequency of the electric oscillation of the spark, but he failed to find any such direct connexion. Such a view seems plausible, as it has been shown‡ that an electric spark gives rise to short waves in the atmosphere of the same frequency as the electric oscillation. The following experiment shows, however, that the ripples obtained in this way are not formed at the nodes of stationary vibrations of these short waves, and that their explanation must be sought in another direction.

If the ripples do form at the nodes of such stationary vibrations in the tube, then the distance apart of two ripples ought to be independent of the intensity of the spark, and of the diameter of the particles of the powder, when the electric frequency is kept constant. An arrangement was suggested to me by Prof. H. Stroud by means of which the intensity of the spark could be varied without altering the frequency.

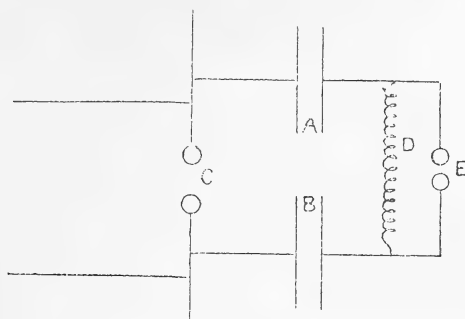
A wire was taken from each of the spark-balls C of a Wimshurst machine to one plate of each of two condensers A and B. The other plates were joined through an inductance D and also through a spark-gap E. On working the machine, sparks passed simultaneously at C and E. The

\* Communicated by the Author.

† Phil. Mag. November 1909.

‡ W. Altberg, *Ann. der Phys.* xxiii, p. 267 (1907).

frequency in the discharge circuit of the jars A and B through E was constant. D is simply a conductor of high



inductance for charging the jars. The energy of the discharge at E depends, however, on the length of the primary spark-gap C. As C increases the energy of the discharge at E, and therefore the intensity of the sound of the spark, increases. This spark E was shielded from the primary spark C, and was used to produce the figures.

The distance between two consecutive ripples was measured for four different lengths of the primary spark, the results of which experiment are shown in the following table. A large 16-plate Wimshurst machine presented by Lord Armstrong to the Armstrong College was used.

Length of the primary spark.	Distance apart of two consecutive ripples.
1 inch	0.893 mm,
1½ inches	1.04 ,,
2 ,,	1.14 ,,
2½ ,,	1.18 ,,

From this table it is seen that, although the frequency of the electric oscillation is kept constant, the distance apart of two ripples varies, and in fact increases as the intensity of the spark increases.

Taken in conjunction with a result from Richmond's experiments that the distance apart of two consecutive ripples increases as the diameter of the powder increases\*, one must

\* Richmond, *loc. cit.*

reject the electrical theory of the phenomenon. It seems more probable that the cause of the ripples lies in the sound of the spark, and that they resemble the ripples between the nodes in a Kundt tube, rather than the nodes themselves. As evidence in support of this conclusion, it might be mentioned that similar results have been obtained for the ripples in a Kundt tube\*, *i. e.* that the distance apart of the ripples increases as the intensity of the sound increases, and that the diameter of the particles has some influence on the distance apart of the ripples.

I have pleasure in recording my best thanks to Prof. H. Stroud for his advice in connexion with this experiment, and also for the use of the necessary apparatus.

The University of Sheffield,  
December 15, 1910.

XXXI. *The Distribution of Secondary X-rays and the Electromagnetic Pulse Theory.* By C. G. BARKLA, M.A., D.Sc., Professor of Physics, and T. AYRES, B.Sc., King's College, London †.

**E**XPERIMENTS on the phenomena of scattering of X-rays furnish what appears to be the strongest evidence in support of the electromagnetic pulse theory. It has been shown by one of us that, as was to be expected on the pulse theory, the primary beam of X-rays proceeding from an X-ray tube in a direction perpendicular to that of propagation of the cathode stream is partially polarized, that the scattered radiation proceeding in a direction perpendicular to that of propagation of the primary radiation is polarized in the manner predicted on the same theory, and that the ratio of the intensities of radiation scattered in directions approximately opposite and perpendicular to the primary beam is within about 5 per cent. of the value calculated on this theory. Each of the experimental results was predicted before it was demonstrated, and it may be added that each has been verified by various observers. Further evidence for the theory is afforded by the facts that rays varying considerably in penetrating power are scattered in the same

\* Robinson, *Phys. Zeit.* 1908, p. 807.

† Communicated by the Authors. The expenses of this research have been partially covered by a Government Grant through the Royal Society. A preliminary announcement of the results of this investigation was made at the Congress of Radiology held in Brussels last year, but a report has not yet been published.

proportion, and that where scattering occurs the scattered radiation differs inappreciably in penetrating power from the primary radiation, that is to say there is no appreciable degradation accompanying the process of scattering. These and other experimental results are in perfect agreement with the theory of scattering, as first given by Sir J. J. Thomson.

A further test was naturally suggested by the first three of these results,—that of making a detailed investigation of the distribution of the scattered X-radiation around the radiating substance.

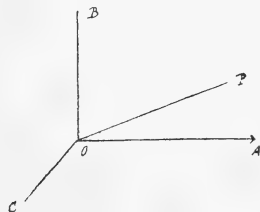
The secondary Röntgen radiation which has been found to proceed from light elements when traversed by a beam of X-rays is such as would be produced by the acceleration of electrons in the direction of electric force in the primary pulses during their passage over these electrons.

If the primary beam be unpolarized, these accelerations are on the average uniformly distributed in direction over a plane perpendicular to the direction of propagation of the primary radiation. Each accelerated electron is the source of a radiation in which—assuming a uniform distribution of electric polarization—the electric force is given by the expression  $\frac{\mu e f \sin \beta}{r}$ \*, where  $e$  is the charge,  $f$  the acceleration

of the electron,  $r$  the distance of the pulse from the origin,  $\mu$  the permeability of the medium, and  $\beta$  the angle between the direction of acceleration of the electron and that of radiation considered. The intensity of radiation in any direction is thus proportional to  $\sin^2 \beta$ .

Thus if OA be the direction of propagation of a primary pulse, OB and OC the directions of the electric and magnetic

Fig. 1.



fields in the pulse, then an electron at O is the source of a radiation the intensity of which in any direction OP is proportional to  $\sin^2 \text{POB}$ .

\* The uniform distribution of tubes of electric force around an electron is, of course, not essential to the calculation. What is essential is the transverse wave theory.

Calling the three angles POA, POB, POC,  $\alpha$ ,  $\beta$ , and  $\gamma$  respectively,

$$\sin^2 \beta = 1 - \cos^2 \beta$$

$$\text{and } \cos^2 \beta = 1 - \cos^2 \alpha - \cos^2 \gamma,$$

and as the average value of  $\cos^2 \beta =$  average of  $\cos^2 \gamma$

$$\therefore \text{average } \cos^2 \beta = \text{average } \frac{1 - \cos^2 \alpha}{2}$$

$$\therefore \text{average } \sin^2 \beta = \text{average } \frac{1 + \cos^2 \alpha}{2}.$$

It follows that the intensity of scattered radiation in any direction is proportional to  $1 + \cos^2 \alpha$ , and may be written

$$I_a = I_{\pi/2}(1 + \cos^2 \alpha),$$

the suffix denoting the angle between the direction of radiation considered and that of propagation of the primary radiation.

This expression gives also, as was shown by Lord Rayleigh, the distribution of monochromatic light scattered by fine particles, though we are not aware of any experimental verification in the case of light. As previously stated, the relative intensities in the two principal directions—those in which  $\alpha = \frac{\pi}{2}$  and as nearly  $\pi$  as practicable—have been studied by one of us\*.

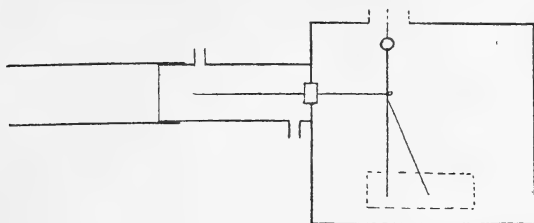
In order to test the theory more completely it seemed desirable to make a series of measurements at various angles.

Carbon was used as the scattering substance, because when it is subject to a beam of moderate penetrating power, the secondary radiation at a distance of several centimetres in air is principally if not entirely scattered radiation. If a true secondary X-radiation—fluorescent X-radiation—is emitted by carbon and other light elements, it is either very easily absorbed and does not penetrate that distance in air, or is so penetrating that it is not excited by a moderately penetrating primary radiation. It seems probable that there are both types of secondary radiation, as with heavier elements. To avoid complications due to this fluorescent radiation the X-ray tube was kept fairly soft throughout the experiments. The secondary beam studied was thus entirely a scattered X-radiation.

\* Phil. Mag. Feb. 1908, pp. 288-296.

The ionization chamber (fig. 2) was cylindrical in shape, 6 cm. long and 2.3 cm. in diameter, with an axial electrode,

Fig. 2.



one end of which passed through an insulating plug into the gold-leaf electroscope, where contact was made with the gold-leaf system. At the other end the cylinder projected beyond the ionization chamber itself, so as to limit the obliquity of rays entering the ionization vessel. By means of slide tubes the length of the protecting tube could be varied to suit each particular experiment. The outer end of the ionization chamber was of thin aluminium and the inner end of lead. Inlet and outlet tubes were provided for this chamber. The sensitiveness of the arrangement was greatly increased by using air saturated at  $0^{\circ}\text{C}$ . with methyl iodide as the gas to be ionized. It was thought that this would be preferable to the tilted electroscope as it was necessary to move the whole system frequently.

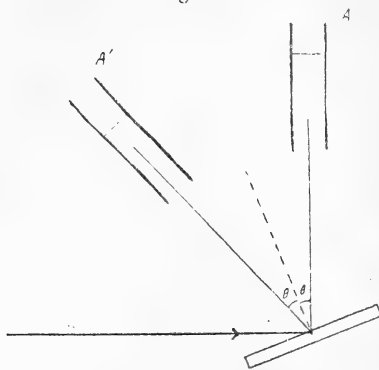
The ionization chamber, electroscope, and observing microscope were placed on a small wooden table at the end of an arm which was capable of rotating round a vertical axis. The axis of the ionization chamber was horizontal and radial, the end of the chamber being about 15 centimetres from the centre of the carbon. This distance was varied slightly for different experiments.

A square slab of carbon 8 cm.  $\times$  8 cm.  $\times$  .8 cm. was held in a vertical plane, with its centre on the axis of rotation of the apparatus, and approximately on the axis of the ionization chamber. The carbon also was capable of rotation round the same vertical axis.

A beam of X-rays rectangular in section was directed horizontally on to the carbon plate. The intensities of secondary radiation proceeding in two directions making equal angles with the normal to the plate were compared by observing the rates of deflexion of the gold-leaf when the axis of the ionization chamber was in the corresponding

positions—as A and A' in fig. 3. A second electroscope was used to standardize the intensity of the radiation in a fixed

Fig. 3.



direction. As the distances in both carbon and air through which the measured secondary radiations passed were the same in the two positions, the ionizations observed were proportional to the intensities of the radiation proceeding from the atoms themselves, without subsequent absorption, in these two directions.

Taking the directions in pairs, the following results were obtained for the ratio  $\frac{I_a}{I_{\pi/2}}$ , the suffix denoting the angle between the direction of propagation of the central primary ray and the central secondary ray in each case. To determine the intensity in directions making angles 20, 30, and 40 degrees with the direction of primary radiation, the secondary radiation from the *face of emergence* of the primary beam was studied. In all other cases the radiation emerging from the face of incidence was studied.

The radiation scattered from the air was measured by direct experiment, and correction was made for this in determining the intensities of radiation from the carbon. This correction was very small in all the experiments except for the angles 20° and 30°.

In Table I., *column 1* gives the angle between the central primary and central secondary rays;

*Column 2* gives the theoretical values of the ratio  $\frac{I_a}{I_{\pi/2}}$  if the primary and secondary beams were narrow pencils coinciding with the axes of the beams actually experimented upon;



TABLE I.

(1)	(2)	(3)	(4)	(5)
Angle $\alpha$ .	Calculated for narrow pencils $\frac{I_\alpha}{I_{\pi/2}} = (1 + \cos^2 \alpha)$ .	Calculated approximately for the actual conditions of experiment $\frac{I_\alpha}{I_{\pi/2}}$ .	Observed $\frac{I_\alpha}{I_{\pi/2}}$ .	Percentage difference between calculated and observed values of $\frac{I_\alpha}{I_{\pi/2}}$ .
20°	1.88	1.90	3.7	+95
30	1.75	1.76	2.1	+19
40	1.59	1.59	1.58	-0.6
50	1.41	1.40	1.43	+2
60	1.25	1.23	1.27	+3
70	1.12	1.09	1.02	-6
80	1.03	1.03	1.07	+4
90	[1.00]	[1.00]	[1.00]	—
100	1.03	1.03	1.06	+3
110	1.12	1.09	1.11	+2
120	1.25	1.23	1.27	+3
130	1.41	1.40	1.41	+1
140	1.59	1.59	1.51	-5
150	1.75	1.76	1.69	-4
160	1.88	1.90	1.84	-3
170	1.97	2.00	1.99	-0.5
-50°	1.41	1.40	1.48	+6
-120	1.25	1.23	1.24	+1
-150	1.75	1.76	1.70	-3

Column 3 gives the values of the ratio calculated for the actual conditions of the experiments as nearly as possible,—taking into account the slight polarization of the primary beam used, and the obliquity of the rays for the position

$\alpha = \frac{\pi}{2}$ . (In other positions of the axis of the ionization

vessel, the errors due to obliquity of rays on each side of this axis approximately cancelled, and were neglected except for the positions  $\alpha = 80^\circ$  and  $100^\circ$ , in which it was estimated the outstanding error would be of the order of 3 per cent. The others would be less. Accordingly the ratio calculated for these positions was increased that amount.) The axis of the cathode stream in the X-ray tube in these experiments was horizontal, and the tube was in a condition which in previous experiments had been found to give a partially polarized radiation, the behaviour of which could be explained by the presence of about 5 per cent. of plane polarized radiation, the direction of electric force being horizontal. Also the obliquity of secondary rays to the axis in the

position  $\alpha = 90^\circ$ , gave an average value of  $\cos^2 \alpha$  for the rays which was estimated as  $\cdot 035$ , consequently the observed intensity was not truly  $I_{\pi/2}$  but  $1\cdot 035 I_{\pi/2}$ . The values  $\frac{I_\alpha}{I_{\pi/2}}$  were thus obtained from the expression  $\frac{1 + \cos^2 \alpha + \cdot 1 \cos^2 \alpha}{1\cdot 035}$ , the third term in the numerator being due to polarization.

Column 4 gives the observed ratio  $\frac{I_\alpha}{I_{\pi/2}}$ ;

Column 5 gives the percentage difference between calculated and observed values.

The observed values in column 4 are the mean of several readings, but in only very few observations did the reading differ by more than 4 per cent. from the value given. Except in the case of angles  $20^\circ$  and  $30^\circ$  the agreement between theory and experiment is remarkably good. In these cases the correction for scattering from air was large, and the results were much less reliable. The fact that the discrepancy appears only for these small angles is significant. It seems just possible that these results are vitiated by an irregular refraction effect. Experiments are being made to test these further. We are not yet in a position to say definitely if for the scattered radiation there is here an appreciable discrepancy between theory and experiment.

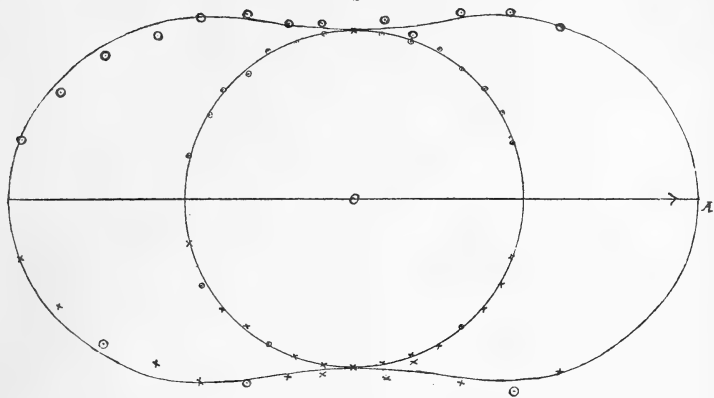
It is interesting to compare the intensity distribution of the scattered radiation with that found for a true secondary radiation (the homogeneous fluorescent X-radiation) from a substance such as copper. In this case the intensity is uniform in all directions as shown by Table II.

TABLE II.

Angle $\alpha$ .	$\frac{I_\alpha}{I_{\pi/2}}$	Angle $\alpha$ .	$\frac{I_\alpha}{I_{\pi/2}}$
$20^\circ$	$\cdot 99$	$120^\circ$	$1\cdot 03$
$30$	$1\cdot 02$	$130$	$\cdot 97$
$40$	$1\cdot 01$	$140$	$1\cdot 02$
$50$	$\cdot 99$	$150$	$\cdot 99$
$60$	$1\cdot 02$	$165$	$1\cdot 01$
$70$	$\cdot 98$		
$80$	$\cdot 99$	$-50^\circ$	$\cdot 99$
$90$	$1\cdot 00$	$-120$	$\cdot 99$
$100$	$1\cdot 00$	$-150$	$1\cdot 03$
$110$	$\cdot 99$		

The distribution in both cases is shown graphically in fig. 4, where OA represents the direction of propagation of

Fig. 4.



The outer curve shows the theoretical distribution of *scattered X-radiation*; the inner curve shows the distribution of *fluorescent X-radiation*; observations are indicated by small circles and duplicated by crosses as symmetry on the two sides was proved on a number of directions.

the primary radiation, and the vector OP represents the intensity of radiation in the direction OP. The outer curve shows the theoretical distribution of the scattered radiation. The inner curve, a circle, exhibits the uniform distribution of the homogeneous secondary (fluorescent) X-radiation. The actual observations are denoted by the positions of the small circles. The crosses merely duplicate the actual observations, as the two sides were found by a number of experiments to be similar.

It will be observed that only a few observations were actually taken on one side of the primary beam, these being compared with the position  $\alpha = -\frac{\pi}{2}$ . The equality in the intensities of the secondary scattered rays making equal angles on the two sides of the primary beam has been shown by one of the writers in a previous paper\*. It was also found to be the case with the homogeneous (fluorescent) X-radiation from copper.

It should be pointed out that it was only by using a soft primary beam that this theoretical distribution was obtained in the case of the scattered radiation.

It has been previously shown by one of us that under other

\* Phil. Trans. A. vol. cciv. 1905, pp. 467-479.

conditions the relative intensities in two directions changes considerably, the ratio  $\frac{I_a}{I_{\pi/2}}$ , dropping from nearly 2 to about 1.5. It is, however, not yet known if the radiation in these cases gave other indications of being a purely scattered radiation. The reasons for the use of a soft beam in these experiments were that possible complications due to the fluorescent radiation were avoided, and because only by using such a beam have experiments on absorbability, and polarization of primary and secondary beams indicated the greatest perfection of scattering. The phenomena need further investigation together, to explain the results under varied conditions.

The experiments, however, show such remarkable agreement between the pulse theory and experiment, under limited conditions, that it can only be concluded the apparent variations are capable of special explanation. They have not yet been fully investigated by the writers.

The experiments on the fluorescent X-radiation (homogeneous secondary radiation) more completely confirm previous experiments showing their uniform distribution, and different origin.

Our thanks are due to Mr. G. H. Martyn, B.Sc., for his valuable help with these experiments.

### XXXII. *Notices respecting New Books.*

*Mysticism in Modern Mathematics.* By HASTINGS BERKELEY.  
Henry Frowde: Oxford University Press, 1910.

**T**HIS book is divided into three parts. The first is devoted to a brief discussion of Thought and its Symbolic Expression; the second is occupied by a critical examination of Imaginary Quantities in Algebra and Imaginary Loci in Geometry; and the third and largest part attacks the philosophical foundations of Non-Euclidean Geometry. The mathematician who follows Labatschewsky and Riemann into the mystical realms where straight lines assume properties which the matter-of-fact Euclidean denounces as not-straight will not probably be diverted from his pursuit by Mr. Berkeley's logic. Nor will the modern analyst or searcher for complex algebras cease to manipulate his imaginaries although he cannot find an explanation of  $\sqrt{-1}$  which can be stated in simple intelligible language. It may be true, as the author suggests, that the modern mathematician has allowed his symbolism to run away with him, and that logic demands a reasonable interpretation of this symbolism. What is an imaginary point? What are imaginary loci? Boole, Cayley, Chrystal, Henrici, Whitehead, among others, are questioned; but apparently no

real light is found. Similarly as regards the axioms of Geometry, the expositions of meta-geometry by Helmholtz, Klein, Clifford, Cayley, and Poincaré are examined and found wanting. The fundamental question here is the nature of the axiom of parallels; and Mr. Berkeley shows that Euclid's axiom of parallels can be broken up into two propositions, which exhibit a remarkable analogy in thought process with the axioms of magnitude. He concludes that the ordinary notion of direction has no logical foothold in non-Euclidean space-conceptions. Here, however, the whole difficulty comes to the front again in the question, what is involved in the ordinary notion of direction? In reasoning about those fundamental *almost a priori* notions, we seem to come round again to our starting point—like a “straight” line in elliptic space. It is quite permissible to define a straight line as such that any part of it is the shortest distance between its end points. If then we proceed to say that such a “straight” line never changes direction, we get a definition of direction which satisfies the familiar experimental notion of direction in Euclidean space. But it does not seem necessarily to follow that the meta-geometer is using this ordinary notion in his discussions of the properties of elliptic and hyperbolic space geometric. This, however, is the accusation made by Mr. Berkeley against these mathematical mystics. No doubt they will find their own defence. Whether or not we agree with the author in his conclusions, there is no doubt that he has given us a book worthy of close study on the part of all interested in these meta-geometrical questions.

### XXXIII. *Proceedings of Learned Societies.*

#### GEOLOGICAL SOCIETY.

[Continued from p. 176.]

November 9th, 1910.—Prof. W. W. Watts, Sc.D., M.Sc., F.R.S.,  
President, in the Chair.

THE following communications were read:—

1. ‘The Rhætic and Contiguous Deposits of West, Mid, and Part of East Somerset.’ By Linsdall Richardson, F.R.S.E., F.L.S., F.G.S.

This paper contains a detailed account of the Rhætic strata of Somerset, with the exception of a small area bordering upon Bristol. The magnificent sections at Blue Anchor and Lilstock are described in detail, and correlated with those on the opposite Glamorgan coast. The record by Prof. Boyd Dawkins of characteristic Rhætic mollusca in the top portion (uppermost 14 feet or so) of the Grey Marls is confirmed, and the contention for their recognition as Rhætic is fully substantiated. The deposit between the top of the fossiliferous Grey Marls or ‘Sully Beds’ and the main Bone-Bed at Blue Anchor measures 22 feet, and teems with interesting Rhætic fossils, such as *Pteromya crowcombeia* Moore. The beds above the Bone-Bed agree very well with those

occupying the same stratigraphical position in Glamorgan, and include the 'Upper Rhætic,' the equivalent of the White Lias proper, and the 'Watchet Beds.' The now obscured magnificent sections, that were temporarily to be seen in the railway-cuttings at Langport and Charlton Mackrell, briefly noticed by Mr. H. B. Woodward, are described in detail (the records being made in company with Mr. E. T. Paris, F.C.S.). Here huge boulder-like masses of rock were noted at the top of the Black Shales, and the White Lias proper, with a well-marked Coral-Bed, totalled 25 feet in thickness. The classic sections of Snake Lane, Dunball (Puriton), Sparkford Hill (Queen Camel), Shepton Mallet, and Milton (Wells), have been re-investigated and brought into line; and the interesting thin Rhætic deposits in Vallis Vale, at Upper Vobster, and sections in the Radstock district, and on the Nempnett and neighbouring outliers, are described. In addition to the record of many new or imperfectly-known sections, this investigation has also shown that the *Microlesites* Marls are equivalent to the Sully Beds: that the Wedmore Stone occurs well below the Bone-Bed; that Moore's 'Flinty Bed' at Beer Crowcombe is probably on the horizon of the *Pleurophorus* Bed (No. 13); that the Upper Rhætic (generally with Cotham Marble or its equivalent) is as persistent as usual, if not quite so thick; that the White Lias proper is of restricted geographical extent; and that on the Bristol Channel littoral are marls, 'Watchet Beds,' above the White Lias. Around Queen Camel, Moore's 'Insect and Crustacean Beds' appear to come in at a horizon which lies between the Watchet Beds and the *Ostrea* Limestone.

The following classification of the Rhætic Series is suggested, and the succession of maxima of the characteristic fossils is given in the paper:—

LIAS.		HETTANGIAN.	<i>Ostrea</i> Beds, etc.	Thicknesses in England.	
RHÆTIC	SOMER- SETIAN	{	I. Watchet Beds ('Marly Beds of the White Lias') .....	} 0 to 7 ft. 7 ins.	
			II. Langport Beds (White Lias proper) .....		} 0 to 25 ft.
			III. Westbury Beds ('Upper Rhætic') .....		
	RHÆTIAN	{	IV. Lilstock Beds (Black Shales)	} 1 to (?) 47 ft.	
			V. Sully Beds (Fossiliferous Grey Marls) .....		} 0 to 14 ft.
KEUPER	KEUPER- IAN	{	Tea-green and Grey Marls .....	} 111 ft. (max.).	
			Red Marls.		

The sudden lithic and faunal changes in the contiguous divisions are held to be the expression of oscillatory movements and interrupted sequential deposition. The fauna of the Rhætic is decidedly Swabian in facies, and the general conclusion to be derived from the study of the beds is in entire agreement with Suess's view, that while the dominant movement was one of subsidence and not local but extended, it was, nevertheless, 'oscillatory and slow.'

2. 'Jurassic Plants from the Marske Quarry.' By the Rev. George John Lane, F.G.S.







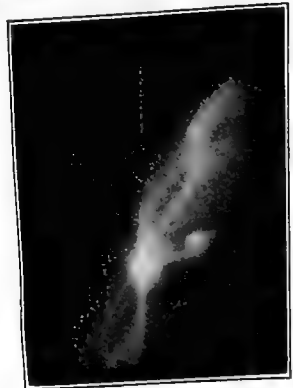
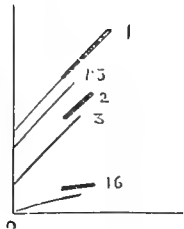


FIG. 5.



Key to fig. 5.—The figures represent the values of  $m/e$  relative to the hydrogen atom.

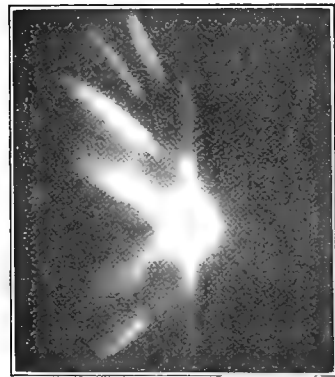


FIG. 12.



FIG. 14.



FIG. 15.



FIG. 17 A.

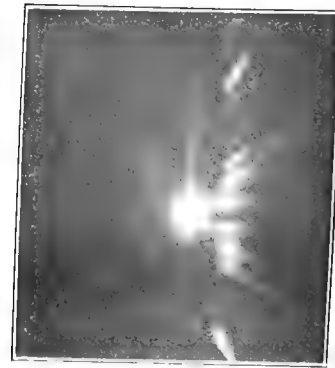


FIG. 20.

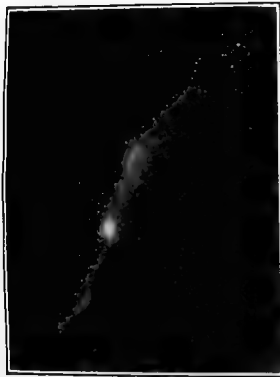


FIG. 8.

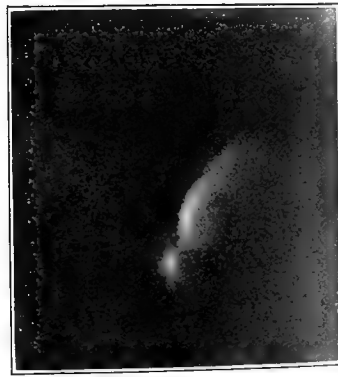


FIG. 9.

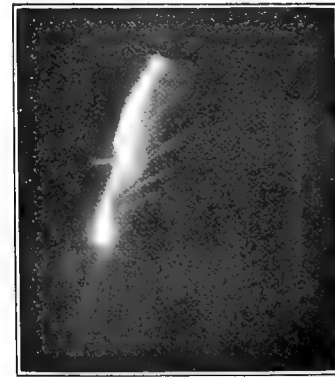
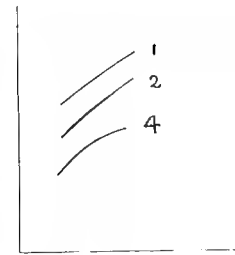


FIG. 13.



Key to fig. 13.

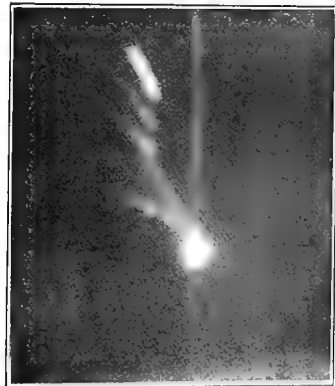
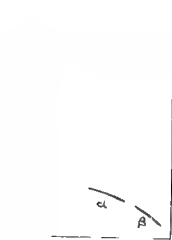


FIG. 17.



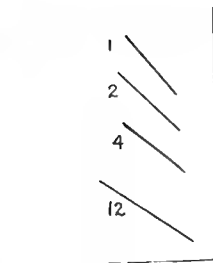
Key to fig. 17.



FIG. 21.



FIG. 11.



Key to fig. 11.

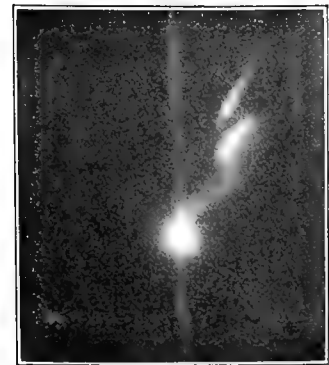
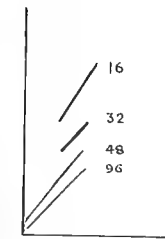


FIG. 16.



Key to fig. 16.

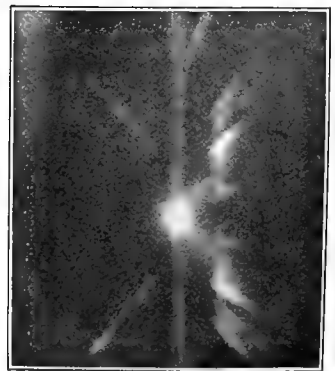


FIG. 18.

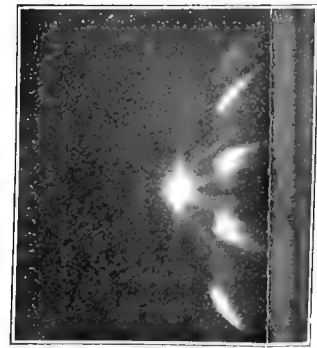


FIG. 19.



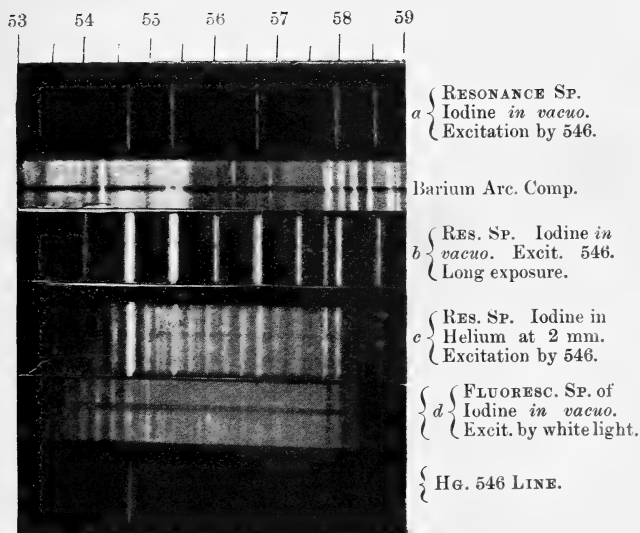


FIG. 1.



N. G. O.

[SIXTH SERIES.]

MARCH 1911.

XXXIV. *On the Bending of Electric Waves round a Large Sphere.*—IV. By J. W. NICHOLSON, M.A., D.Sc.\*

*The evaluation of a type of integral.*

THE analysis necessary for the determination of a second approximation valid for points within the region of brightness, is somewhat involved, and it is convenient, in the first place, to collect certain formulæ from earlier sections for ready reference.

In the previous notation, the value of the magnetic force is given by

$$\gamma\rho = \frac{i \sin^2 \theta}{k\alpha^2} \sum_1^{\infty} m(R_n R_{nr})^{\frac{1}{2}} (1 + e^{2i\chi_n}) e^{i(\phi_n - \phi_{nr})} \frac{d\rho_n}{d\mu}. \quad (110)$$

By the use of the Mehler Dirichlet integral for the zonal harmonic, if  $g(\theta)$  is an operation defined by

$$g(\theta)\omega = \frac{2i \sin^2 \theta}{k\alpha^2 \pi} \frac{d}{d\mu} \int_0^\theta \frac{\omega d\phi}{\sqrt{2} \sqrt{\cos \phi - \cos \theta}}, \quad (111)$$

then by (39)

$$\begin{aligned} \gamma\rho &= g(\theta) \sum_1^{\infty} m(R_n R_{nr})^{\frac{1}{2}} (1 + e^{2i\chi_n}) e^{i(\phi_n - \phi_{nr})} \cos m\phi \\ &= g(\theta) \sum_1^{\infty} u(e^{i v_1} + e^{i v_2}), \quad \dots \dots \dots (112) \end{aligned}$$

where

$$\begin{aligned} u &= \frac{1}{2} m(R_n R_{nr})^{\frac{1}{2}} (1 + e^{2i\chi_n}) \\ (v_1, v_2) &= \phi_n - \phi_{nr} \pm m\phi. \end{aligned}$$

\* Communicated by the Author.

The notation has been slightly changed in the last formula. Moreover,  $x$  being  $m/z$ , where  $m = n + \frac{1}{2}$ , and the accent denoting  $\partial/\partial x$ ,  $v_1'$  is never zero within the range of variation of  $m$ , and  $v_2'$  is zero at a certain "zero point" given by

$$x = \sin \phi / (1 - 2c \cos \phi + c^2)^{\frac{1}{2}}, \quad c = a/r. \quad (113)$$

These results are exact. If, on the other hand, it be possible to replace the zonal harmonic by its asymptotic expansion throughout the series, the zero point is given by  $x = r \sin \theta / R$ , or

$$x = \sin \theta / (1 - 2c \cos \theta + c^2)^{\frac{1}{2}}. \quad (114)$$

These results are to be found in the earlier section headed "Vanishing of the derivate of an exponent." The problem in hand is the determination of a second approximation to the sum of a series of this type, where the exponent has only one zero point. The summation may be replaced at once by an equivalent integral as before, and the main difficulty is therefore the evaluation of this integral to a higher order.

Let  $S$  be a series of the type customary to this paper, given by

$$S = \sum_{n=0}^{\infty} u e^{izv}, \quad (115)$$

where  $z$  is large, and  $u$  and  $v$  are expressed as functions of  $x = (n + \frac{1}{2})/z$ . Then by the previous summation formula

$$S = z \int_{\epsilon}^{\infty} dx \left( U_0 + \frac{U_1}{z} + \frac{U_2}{z^2} + \dots \right) e^{izv},$$

where  $U_0, U_1, \dots$  are given in slightly different notation by (34) and  $\epsilon = 1/2z$  corresponding to  $n=0$ . As was indicated previously, the harmonic term of zero order in  $\gamma$  is zero, so that  $n=0$  may be taken as starting point instead of  $n=1$ . Writing

$$U_0 + \frac{U_1}{z} + \dots \equiv U. \quad (116)$$

Then

$$S = z \int_{\epsilon}^{\infty} dx U e^{izv}, \quad (117)$$

and it is to be assumed that  $v' = 0$  at one, and only one point in the range, namely,  $v = v_0$  corresponding to  $x = x_0$ . Moreover,  $U$  is not oscillatory. In view of definiteness,  $v_0''$  will be given a positive, and  $v_0$  a negative sign. This has already been seen to be the case in the series most important in the

region of brightness, the first approximation being sufficient for a proof. With this convention,  $v_0$  is a minimum value of  $v$ , and since  $v_0$  is negative,  $v - v_0$  must always be positive, there being no other stationary points of  $v$ .

Let a new variable  $\omega$  be defined by

$$\omega = (v - v_0)^{\frac{1}{2}} \dots \dots \dots (118)$$

Then  $\omega$  must always be real. The sign of the ambiguity in  $\omega$  is to be identical with that of  $x - x_0$ , which is of necessity a factor of  $\omega^2$ . Let  $\omega = -\delta$  correspond to  $x = \epsilon$ . Then, since  $v'dx = 2\omega d\omega$ ,

$$S = 2z e^{izv_0} \int_{-\delta}^{\infty} d\omega V e^{iz\omega^2}, \dots \dots (119)$$

where  $V = U\omega/v'$  expressed as a function of  $\omega$ . This function will not be oscillatory. In the physical problem, of course, the value of  $\delta$  may be found on substitution of the proper values of  $\phi_n$  and  $\phi_{nr}$  in  $v$ . It is sufficient for our purpose to observe that it is not small.

$V$  is finite when  $\omega = 0$ , corresponding to  $x = x_0$ . For  $U$  is finite, and the limiting value of  $\omega/v'$  is that of

$$\{\frac{1}{2}v_0''(x - x_0)^2\}^{\frac{1}{2}} / (x - x_0)v_0'' \quad \text{or} \quad (2v_0'')^{-\frac{1}{2}},$$

which is also finite, since  $v_0''$  is not zero. This limit is of course also real, for  $v_0''$  is positive.

But the selection of infinity as the upper limit of the integration with respect to  $\omega$  calls for further remarks when the results are applied afterwards to the actual case. For when  $x$  is infinite,  $\phi_n$  and  $\phi_{nr}$  both vanish, so that  $v - v_0$  becomes  $-\theta x - v_0$ , which is negatively infinite. It follows that another point has occurred, beyond  $x = x_0$ , where  $v'$  is zero, so that the present assumption of only one zero point in the range will not strictly correspond to the case for which its application is intended. But it was shown in an earlier section that no second point of this kind, in addition to  $x_0$ , can occur when  $m$  is such that  $\phi_n$  and  $\phi_{nr}$  are both of order  $z$ , or when both are of lower order than  $z$ , whether in the range of values for which they are approximately  $\frac{1}{2}\pi$  or beyond. It must therefore occur when the smaller only is of lower order than  $z$ , and not the larger. In other words, it occurs for a value of  $m$  such that  $kr - m$  is positive and of order  $kr$ , but at the same time either (1)  $z - m$  or  $m - z$  is only of order  $z^{\frac{1}{2}}$  or less, or (2)  $m - z$  is positive and of order higher than  $z^{\frac{1}{2}}$ .

Now let attention be restricted to points of space not close

to the sphere, or to the region of transition previously defined. Since  $\phi_n$  is of low order in comparison with  $\phi_{nr}$ , the second zero point must effectively satisfy

$$-\partial\phi_{nr}/\partial n - \theta = 0,$$

or as (40) is here applicable,

$$-\sin^{-1} \frac{ax}{r} + \frac{\pi}{2} - \theta = 0,$$

so that

$$m = zx = kax = kr \cos \theta,$$

and

$$m - z = z \left( \frac{r \cos \theta}{a} - 1 \right), \dots \dots (120)$$

and this only tends to be of lower order than  $z$  in the region of space in which  $r \cos \theta$  is nearly equal to  $a$ . This has already been shown to be the "transitional" region, of which an investigation has been given. Thus in the "region of brightness" proper, (1) is excluded, and the second zero point is therefore such that  $m - z$  is positive and of higher order than  $z^{\frac{1}{2}}$ .

Now at a zero point so defined, the non-oscillatory portion  $R_n^{\frac{1}{2}} (1 + e^{2ix_n^*})$  of the multiplier of the exponential in the series (110), which does not involve  $kr$ , but only  $z$ , is small in comparison with any power of  $z^{-1}$ . This follows quickly from some results in the section "On the harmonic terms of infinite order." Accordingly, the portion of the sum dependent on this zero point may be completely ignored\*.

This justifies the use of infinity as the upper limit of the integral (119), for it is not necessary even to regard the limit as of an order so low as  $z$ . Further justification is of course supplied on physical grounds by the fact that, in ignoring the second zero point in the earlier work, the effect of a plane reflector was obtained, as it should be, for the first approximation in the region of brightness. Finally, therefore, we may write, with validity of subsequent application,

$$S = 2z e^{izv_0} \int_{-\delta}^{\infty} d\omega V e^{iz\omega^2}.$$

We have seen that  $V$  is finite when  $\omega = 0$ .

\* In the investigation of Phil. Mag. April 1910, this zero point was ignored, but the above analysis is desirable for completeness.



If  $V_n = \left[ \frac{d^n V}{d\omega^n} \right]_{\omega=0}$

then by Maclaurin's theorem

$$V = V_0 + V_1\omega + \frac{V_2}{2!}\omega^2 + \dots, \dots \quad (121)$$

and

$$\begin{aligned} S e^{-\iota z \omega_0/2z} &= \left\{ \int_{-\delta}^0 + \int_0^{\infty} \right\} d\omega V e^{\iota z \omega^2} \\ &= 2 \int_0^{\infty} d\omega \left( V_0 + \frac{V_2}{2!}\omega^2 + \dots \right) e^{\iota z \omega^2} - \int_{\delta}^{\infty} d\omega V(-\omega) e^{\iota z \omega^2}. \end{aligned}$$

Now by integration in the plane of a complex variable round a contour consisting of two lines at an angle  $\frac{1}{4}\pi$  through the point  $\delta$  on the real axis, and an arc at infinity, it is readily shown that

$$\begin{aligned} &\int_{\delta}^{\infty} d\omega V(-\omega) e^{\iota z \omega^2} \\ &= e^{\iota z \delta^2 + \frac{1}{4}\iota\pi} \int_0^{\infty} dr V(-\delta - re^{\frac{\iota\pi}{4}}) e^{z(-r^2 + 2ir\delta e^{\frac{\iota\pi}{4}})}. \end{aligned}$$

The latter integral may be evaluated in a series of powers of  $z^{-1}$  by parts. The leading term of the series becomes

$$V(-\delta) e^{\iota z \delta^2} / 2\iota z \delta. \dots \dots \quad (122)$$

But  $V(-\delta)$  is the value of  $V$  when  $\omega = -\delta$ , that is, the value of  $V$  when  $x = \epsilon$ . Thus

$$V(-\delta) = \left[ U\omega/v' \right]_{x=\epsilon}$$

Consider then the case in which the exact formula (112) is used. The leading term of  $U$  is of the same order as  $u$ , and when  $x = \epsilon$ , this order is, since  $m = \frac{1}{2}$ , zero in  $z$ . Moreover,  $\delta$  is not small, and therefore  $V(-\delta)/2\iota z \delta$  has an order  $z^{-1}$  at most. On the contrary, in the integral

$$\int_0^{\infty} d\omega V_0 e^{\iota z \omega^2} = V_0 \int_0^{\infty} d\omega e^{\iota z \omega^2}$$

$V_0$  has an order  $z$ , being the value at the zero point where its factor  $m$  has this order. The resulting order of the integral is  $z^{\frac{1}{2}}$ . Thus  $V(-\delta)/2\iota z \delta$  may be neglected, with an error at most of order  $z^{-\frac{3}{2}}$  compared with the main order retained. This will be found sufficient to justify its neglect in a determination of the second approximation, and therefore the matter is not investigated further. But in reality,

the possible error is much less, and by a method used in the case of the shadow round the sphere, it may be shown that the contribution to the magnetic force made by this neglected series, in company with the other series which has no zero point, continues to vanish term by term.

We therefore write finally

$$S = 4z e^{izv_0} \int_0^\infty d\omega \left( V_0 + V_2 \frac{\omega^2}{2!} + \dots \right) e^{iz\omega^2}, \quad (123)$$

and only the first two orders in  $z$  will be retained. To these the above proof applies.

The integrals concerned here are well known. Quoting their values,

$$S = 2(\pi z)^{\frac{1}{2}} \left\{ V_0 - \frac{V_2}{4iz} \right\} e^{izv_0 + \frac{1}{2}i\pi} \quad \dots \quad (124)$$

to a second approximation. If  $v_0''$  were negative, the sign of  $\frac{1}{2}i\pi$  would be negative, and in  $V_0, V_2$  as calculated later,  $(v_0'')^{\frac{1}{2}}$  would become  $(-v_0'')^{\frac{1}{2}}$ .

#### Calculation of the functions $V_0, V_2$ in general.

The calculation of  $V_0$  and  $V_2$  is somewhat laborious, successive stages being shown below. We recall that if  $U$  is defined as a function of  $x$  as in (116), then  $V = U\omega/v'$  expressed as a function of  $\omega$ .  $V_0$  is the value of this function at  $x = x_0$ , and  $V_2$  is its second derivate with respect to  $\omega$  at this point. Moreover, the relation between  $\omega$  and  $x$  is  $\omega = \sqrt{v - v_0}$ ,  $v$  being a function of  $x$ , so that  $x = x_0$  corresponds to  $\omega = 0$ . The suffix zero applied to any quantity denotes that its value at the zero point must be taken.

Let  $y = x - x_0$ , so that  $\partial/\partial x = \partial/\partial y$ , and let either of these operations be denoted by the accent. Then,  $v_0'$  being zero,

$$U = U_0 + yU_0' + \frac{y^2}{2!}U_0'' + \dots$$

$$v = v_0 + \frac{y^2}{2!}v_0'' + \dots$$

$$v' = yv_0'' + \frac{y^2}{2!}v_0''' + \dots,$$

and on reduction,  $v_0''$  being positive, and  $\omega$  having the sign of  $y$  by its definition,

$$\omega = (v - v_0)^{\frac{1}{2}} = y \sqrt{v_0''/2} \{ 1 + Ay + By^2 + \dots \}, \quad (125)$$

where

$$A = v_3/6v_2$$

$$B = (3v_2v_4 - v_3^2)/72v_2^2,$$

and where it has been found convenient to denote  $v_0^{(n)}$  by  $v_n$ .

Again, we may at once derive

$$\omega/v' = (2v_2)^{-\frac{1}{2}} \left\{ 1 + Ay + By^2 + \dots \right\} \left\{ 1 + \frac{y}{2} \frac{v_3}{v_2} + \frac{y^2}{6} \frac{v_4}{v_2} + \dots \right\}^{-1}$$

$$= (2v_2)^{-\frac{1}{2}} \left\{ 1 + \alpha y + \beta y^2 + \dots \right\}, \dots \dots \dots (126)$$

where

$$\alpha = -v_3/3v_2$$

$$\beta = (11v_3^2 - 9v_2v_4)/72v_2^2.$$

Thus

$$V = U\omega/v' = V_0' + yV_1' + y^2V_2' + \dots,$$

where on reduction from (125) and (126),

$$V_0' = U_0(2v_2)^{-\frac{1}{2}}$$

$$V_1' = U_0(2v_2)^{-\frac{1}{2}} \left\{ \frac{U_0'}{U_0} - \frac{1}{3} \frac{v_3}{v_2} \right\}$$

$$V_2' = U_0(2v_2)^{-\frac{1}{2}} \left\{ \frac{1}{2} \frac{U_0''}{U_0} - \frac{v_3}{3v_2} \frac{U_0'}{U_0} - \frac{1}{8} \frac{v_4}{v_2} + \frac{11}{72} \frac{v_3^2}{v_2^2} \right\}. \quad (127)$$

In addition, we have by Maclaurin's theorem,

$$V_0' = (V)_{y=0} = (V)_{\omega=0} = V_0$$

$$V_1' = (\partial V/\partial x)_0, \quad 2V_2' = (\partial^2 V/\partial x^2)_0.$$

Again, by (125),

$$(\partial\omega/\partial x)_0 = (v_2/2)^{\frac{1}{2}}, \quad (\partial^2\omega/\partial x^2)_0 = 2A(v_2/2)^{\frac{1}{2}},$$

whence

$$(\partial x/\partial\omega)_0 = (2/v_2)^{\frac{1}{2}}, \quad (\partial^2 x/\partial\omega^2)_0 = -4A/v_2 = -2v_3/3v_2^2,$$

and

$$V_2 = \left( \frac{\partial^2 V}{\partial\omega^2} \right)_0 = \left\{ \frac{\partial V}{\partial x} \frac{\partial^2 x}{\partial\omega^2} + \frac{\partial^2 V}{\partial x^2} \left( \frac{\partial x}{\partial\omega} \right)^2 \right\}_0$$

$$= \frac{2U_0}{v_2} (2v_2)^{-\frac{1}{2}} \left\{ \frac{U_0''}{U_0} - \frac{v_3}{v_2} \frac{U_0'}{U_0} - \frac{1}{4} \frac{v_4}{v_2} + \frac{5}{12} \frac{v_3^2}{v_2^2} \right\}, \quad (128)$$

whereas

$$V_0 = U_0 (2v_2)^{-\frac{1}{2}}. \quad \dots \dots \dots (129)$$

Restating the result in simpler notation, it has been shown that if a series dependent on the lower limit is negligible in comparison, the second approximation to the integral

$$S = z \int_{\epsilon}^{\infty} dx U e^{zv}, \quad . . . . (130)$$

where  $z$  is large, and  $v$  has a single minimum in the range, and no other turning point,  $U$  being finite and non-oscillatory everywhere, is given by

$$S = 2(\pi z)^{\frac{1}{2}} \left\{ V_0 - \frac{V_2}{4tz} \right\} e^{t'zv + \frac{i\pi}{4}}, \quad . . . (131)$$

where, all values of functions relating to the zero point where the minimum occurs, and suffixes or accents on the right denoting differentiations at the zero point with respect to  $w$ ,

$$V_0 = U(2v_2)^{-\frac{1}{2}},$$

$$V_2 = \frac{2U}{v_2} (2v_2)^{-\frac{1}{2}} \left\{ \frac{U''}{U} - \frac{v_3 U'}{v_2 U} - \frac{1}{4} \frac{v_4}{v_2} + \frac{5}{12} \frac{v_3^2}{v_2^2} \right\}. \quad (132)$$

*Application to Series Summation.*

The sum of a series of the usual type, with our customary notation,

$$S = \sum_{n=0}^{\infty} u \left( \frac{m}{z} \right) e^{zv \left( \frac{m}{z} \right)}$$

is

$$S = z \int_{\epsilon}^{\infty} dx U e^{zv},$$

where

$$U = U_0 + \frac{U_1}{z} + \frac{U_2}{z^2} + \dots,$$

and by (34),

$$U_0 = u\psi_0, \quad U_1 = u'\psi_1 + \frac{1}{2}uw''\psi_2,$$

$$U_2 = \frac{1}{2}u''\psi_2 + \frac{1}{2}u(v'u' + \frac{1}{3}v'''u)\psi_3 - \frac{1}{8}uv''^2\psi_4, \quad . \quad (133)$$

where

$$\psi_s = \left[ \frac{d^s}{d\omega^s} \frac{\omega}{e^\omega - 1} \right]_{\omega=iv},$$

so that

$$\psi_s' = w''\psi_{s+1}, \quad . . . . . (134)$$

the accent denoting differentiation with respect to  $x$ . Let us suppose that the significant zero points can be reduced to one. Then the sum is given to a second approximation

by (131) and (132) if the series dependent on the limit  $\epsilon$  can be ignored. We have seen that both these conditions are satisfied for the series whose sum is intended, and we therefore apply the formulæ in question. The process of calculation of  $V_0$  and  $V_2$  is as follows:—

In the first place, the functions  $\psi$  at the zero point are merely numerical. For

$$\frac{\omega}{e^\omega - 1} = 1 - \frac{1}{2}\omega + \frac{B_1}{2!}\omega^2 - \dots,$$

and by the definition of  $\psi$ , since  $\omega = v\omega' = 0$  at the zero point,

$$\frac{\omega}{e^\omega - 1} = \psi_0 + \omega\psi_1 + \frac{\omega^2}{2!}\psi_2 + \dots$$

Accordingly, at the zero point,

$$\psi_0 = 1, \quad \psi_1 = -\frac{1}{2}, \quad \psi_2 = B_1 = \frac{1}{6}, \dots \quad (135)$$

Secondly, it is sufficient, to the requisite order, to write

$$U = U_0 + \frac{U_1}{z},$$

and we require, for  $V_2$ , the first two derivates of  $U_0$  at the zero point, denoted by  $U'$  and  $U''$  in (132). Now

$$U_0' = \partial/\partial x (u\psi_0) = u'\psi_0 + uv''\psi_1,$$

$$U_0'' = u''\psi_0 + 2u'v''\psi_1 + uv'''\psi_1 - uv''^2\psi_2,$$

differentiating the functions  $\psi$  according to their rule. The derivates of  $U_1$  are not needed explicitly. Thus at the zero point, where suffixes denote differentiations of  $u$  and  $v$ ,

$$U_0 = u, \quad U_1 = -\frac{1}{2}u_1 + \frac{1}{12}uv_2,$$

$$U_0' = u_1 - \frac{1}{2}uv_2, \quad U_0'' = u_2 - uv_1v_2 - \frac{1}{2}uv_3 - \frac{1}{6}uv_2^2. \quad (136)$$

For the final result, two significant orders must be retained in  $V_0$  of (132), but only one in  $V_2$ , for the function needed is  $V_0 - \frac{V_2}{4\mu z}$ . Thus in calculating  $V_2$ ,  $U$  may be identified

with  $U_0$ , but this may not be done in calculating  $V_0$ . This indicates that derivates of  $U_1$  are not needed. Finally, in  $(V_0, V_2)$  we may write, in  $V_0$ ,

$$U = u - \frac{1}{z} \left( \frac{1}{2}u_1 - \frac{1}{12}uv_2 \right),$$

and in  $V_2$ ,

$$U = u, \quad U' = u_1 - \frac{1}{2}uv_2, \\ U'' = u_2 - uv_1v_2 - \frac{1}{2}uv_3 - \frac{1}{6}uv_2^2.$$

Using these values, after some reduction, we find

$$V_0 - \frac{V_2}{4iz} = (2v_2)^{-\frac{1}{2}}u + (2v_2)^{-\frac{1}{2}}\frac{\mu i}{2z},$$

where

$$\mu = \frac{v_2}{v_2} - \frac{u_1 v_3}{v_2^2} - \frac{1}{4} \frac{uv_4}{v_2^2} + \frac{5}{12} \frac{uv_3^2}{v_2^3}, \quad \dots \quad (137)$$

and the sum of the series becomes

$$S = \left(\frac{2\pi z}{v_2}\right)^{\frac{1}{2}} \left\{ u + \frac{i\mu}{2z} \right\} e^{zv + \frac{1}{4}i\pi}, \quad \dots \quad (138)$$

all values being taken at the zero point.

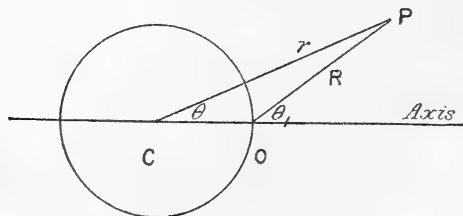
The analysis of this section and that immediately preceding does not apply to the present problem alone. It is generally applicable to all similar problems in which a second approximation is desired in regions of space for which incident and reflected waves are concerned, provided that an harmonic series may be obtained.

*Application to the special case.*

We proceed to apply these results to the special problem in hand. In this problem, in the region of brightness, as in an earlier section,

$$zv = \phi_n - \phi_{nr} - m\theta, \quad \dots \quad (139)$$

where  $\theta$  is replaced by  $\phi$  if the Mehler Dirichlet integral must be used. But when points not close to the axis are concerned, the important harmonics are those of order  $z$ , for the effect depends upon the values of the various functions,



by the formulæ just proved, at the zero point, which is given by  $x = r \sin \theta / R$  in the notation of the figure, and this corresponds to a value of  $m$  of order  $z$ . The zonal harmonic may therefore be expanded asymptotically, if sufficient terms are retained. The usual formula of course fails, and we proceed to obtain the necessary modification.

When  $\sin \theta$  is not too small, Hobson \* has shown that

$$P_n(\mu) = \sqrt{\frac{4}{\pi}} \frac{\varpi(n)}{\varpi(n+\frac{1}{2})} \left\{ \frac{\cos \cdot (n+\frac{1}{2})\theta - \frac{\pi}{4}}{(2 \sin \theta)^{\frac{1}{2}}} + \frac{1^2}{2 \cdot 2n+3} \frac{\cos \cdot (n+\frac{3}{2})\theta - \frac{3\pi}{4}}{(2 \sin \theta)^{\frac{3}{2}}} + \dots \right\}, \quad (140)$$

the asymptotic expansion generally quoted being the first term. We require the first two significant orders in  $m$  or  $n+\frac{1}{2}$ . These may be obtained from the terms

$$P_n(\mu) = \sqrt{\frac{4}{\pi}} \frac{\varpi(m-\frac{1}{2})}{\varpi(m)} \left\{ \frac{\cos \left( m\theta - \frac{\pi}{4} \right)}{(2 \sin \theta)^{\frac{1}{2}}} + \frac{1}{4m} \cdot \frac{\cos \left( m\theta + \theta - \frac{3\pi}{4} \right)}{(2 \sin \theta)^{\frac{3}{2}}} \right\}$$

Now by Stirling's theorem,

$$\varpi(m) = e^{-m} m^m \sqrt{2m\pi} \left\{ 1 + \frac{1}{12m} + \dots \right\}$$

$$\varpi(m-\frac{1}{2}) = e^{-m+\frac{1}{2}} (m-\frac{1}{2})^{m-\frac{1}{2}} \sqrt{(2m-1)\pi} \left\{ 1 + \frac{1}{12m-6} \dots \right\},$$

and therefore to two significant orders in  $m^{-1}$ ,

$$\frac{\varpi(m-\frac{1}{2})}{\varpi(m)} = \left( \frac{e}{m} \right)^{\frac{1}{2}} \left( 1 - \frac{1}{2m} \right)^m.$$

The hyperbolic logarithm of this function may be expanded in the form

$$\frac{1}{2}(1-\log m) - m \left( \frac{1}{2m} + \frac{1}{8m^2} + \dots \right) = -\frac{1}{8m} + \dots - \frac{1}{2} \log m,$$

and therefore

$$\frac{\varpi(m-\frac{1}{2})}{\varpi(m)} = m^{-\frac{1}{2}} \left( 1 - \frac{1}{8m} \right),$$

so that the second approximation to the asymptotic value of  $P_n(\mu)$  when  $n$  is large becomes

$$P_n(\mu) = \sqrt{\frac{2}{m\pi \sin \theta}} \left( 1 - \frac{1}{8m} \right) \left\{ \cos \left( m\theta - \frac{\pi}{4} \right) + \frac{\cos \left( m\theta + \theta - \frac{3\pi}{4} \right)}{8m \sin \theta} \right\},$$

which may be reduced to

$$P_n(\mu) = \sqrt{\frac{2}{m\pi \sin \theta}} \left\{ \cos \left( m\theta - \frac{\pi}{4} \right) + \frac{\cot \theta}{8m} \sin \left( m\theta - \frac{3\pi}{4} \right) \right\} \quad (141)$$

\* Phil. Trans. 1896, A.

to the same order of approximation, and similarly, it may be shown that

$$\frac{dP_n}{d\mu} = \sqrt{\frac{2m}{\pi \sin^3 \theta}} \left\{ \sin\left(m\theta - \frac{\pi}{4}\right) + \frac{3 \cot \theta}{8m} \cos\left(m\theta - \frac{\pi}{4}\right) \right\}. \quad (142)$$

For the purpose of the present problem, we require the portion of  $dP_n/d\mu$  which involves  $-im\theta$  in the exponent, for this alone leads to a series with a zero point. This portion becomes

$$-\sqrt{\frac{m}{2\pi \sin^3 \theta}} \left\{ 1 - \frac{3i \cot \theta}{8m} \right\} e^{-im\theta - \frac{i\pi}{4}}, \quad \dots \quad (143)$$

and this may be substituted for  $dP_n/d\mu$  in the summation.

Second order values of  $R_n$ ,  $\phi_n$ ,  $\phi_{nr}$ , and  $\chi_n$  are also required, and to these we proceed. General expressions for  $R_n$  and  $\phi_n$  were given in (21-24), and isolating the most significant terms when  $m$  is of order  $z$ , and equal to  $z\alpha$ ,

$$R_n = (1 - \alpha^2)^{-\frac{1}{2}}, \quad \dots \quad (144)$$

the next term being of relative order  $z^{-2}$ . Thus the value of  $R_n$  used in the first approximation is sufficient. The second approximation to  $\phi_n$  is

$$\begin{aligned} \phi_n = \frac{1}{4}\pi + z \left\{ (1 - \alpha^2)^{\frac{1}{2}} + \alpha \sin^{-1} \alpha - \frac{\pi \alpha}{2} \right\} \\ + \frac{1}{8z} \left\{ (1 - \alpha^2)^{-\frac{1}{2}} + \frac{5}{3} \alpha^2 (1 - \alpha^2)^{-\frac{3}{2}} \right\}. \end{aligned}$$

Three orders are here retained, for it must be remembered that we are in this case dealing with an exponential. Let

$$\beta_n = \frac{1}{8z} \left\{ \frac{1}{(1 - \alpha^2)^{\frac{1}{2}}} + \frac{5}{3} \frac{\alpha^2}{(1 - \alpha^2)^{\frac{3}{2}}} \right\}, \quad \dots \quad (145)$$

and similarly,  $c$  being  $a/r$  as before,

$$\beta_{nr} = \frac{c}{8z} \left\{ \frac{1}{(1 - c^2 \alpha^2)^{\frac{1}{2}}} + \frac{5}{3} \frac{c^2 \alpha^2}{(1 - c^2 \alpha^2)^{\frac{3}{2}}} \right\}, \quad \dots \quad (146)$$

and the second order value of  $e^{i\phi_n - i\phi_{nr}}$  becomes

$$e^{i(\phi_n - \phi_{nr} + \beta_n - \beta_{nr})},$$

where, in the second expression, the  $\phi$ 's denote the old first order values. This may be written, since the  $\beta$ 's are of order  $z^{-1}$ ,

$$(1 + i\beta_n - i\beta_{nr}) e^{i(\phi_n - \phi_{nr})}.$$

In a series involving the exponential of argument  $i(\phi_n - \phi_{nr})$  we may therefore leave the  $\phi$ 's unaltered from their first



approximations, and multiply the non-oscillatory factors by  $1 + \iota\beta_n - \iota\beta_{nr}$  or

$$1 + \frac{\iota}{8z} \left\{ \frac{1}{(1-x^2)^{\frac{1}{2}}} - \frac{c}{(1-c^2x^2)^{\frac{1}{2}}} + \frac{5}{3} \frac{x^2}{(1-x^2)^{\frac{3}{2}}} - \frac{5}{3} \frac{c^3x^2}{(1-c^2x^2)^{\frac{3}{2}}} \right\}. \quad (147)$$

Finally,  $\chi_n$  is required. In the first approximation in the region of brightness, we wrote  $\chi_n = 0$ . The more accurate value is given by (27) as

$$\chi_n = x^2/2z(1-x^2)^{\frac{3}{2}},$$

so that to the same order,

$$1 + e^{2\iota\chi_n} = 2 \left\{ 1 + \frac{\iota x^2}{2z(1-x^2)^{\frac{3}{2}}} \right\}. \quad (148)$$

Using finally the asymptotic formula for the zonal harmonic and the values of the various functions derived in this section, the series to be summed may be regarded, to the necessary order, as identical with

$$S = \sum_1^{\infty} u e^{\iota z v},$$

where

$$z v = \phi_n - \phi_{nr} - m\theta \quad (149)$$

with the same value and derivatives as in the first approximation, and where by (110), and (143) *et seq.*

$$\begin{aligned} u &= -\frac{\iota x \sin^2 \theta}{ka^2} (R_n R_{nr})^{\frac{1}{2}} (1 + \iota\beta_n - \iota\beta_{nr}) (1 + e^{2\iota\chi_n}) \\ &\quad \times \left( \frac{zx}{2\pi \sin^3 \theta} \right)^{\frac{1}{2}} \left( 1 - \frac{3\iota \cot \theta}{8zx} \right) e^{-\frac{\iota\pi}{4}} \\ &= \frac{\sigma x^{\frac{3}{2}}}{(1-x^2 \cdot 1-c^2x^2)^{\frac{1}{4}}} \left\{ 1 + \frac{\iota\lambda}{z} \right\}, \quad (150) \end{aligned}$$

provided that

$$\sigma = -\sqrt{\frac{2k \sin \theta}{a\pi}} \cdot e^{\frac{\iota\pi}{4}},$$

$$\begin{aligned} \lambda &= \frac{x^2}{2(1-x^2)^{\frac{3}{2}}} - \frac{3 \cot \theta}{8x} + \frac{1}{8} \left\{ \frac{1}{(1-x^2)^{\frac{1}{2}}} - \frac{c}{(1-c^2x^2)^{\frac{1}{2}}} \right. \\ &\quad \left. + \frac{5}{3} \frac{x^2}{(1-x^2)^{\frac{3}{2}}} - \frac{5}{3} \frac{c^3x^2}{(1-c^2x^2)^{\frac{3}{2}}} \right\}. \quad (151) \end{aligned}$$

The derivatives of  $u$  are only wanted to the first order, and become

$$u' = \frac{\sigma x^{\frac{3}{2}}}{2(1-x^2 \cdot 1-c^2x^2)^{\frac{1}{4}}} \left\{ 1 + \frac{1}{1-x^2} + \frac{1}{1-c^2x^2} \right\},$$

$$u'' = \frac{\sigma x^{-\frac{1}{2}}}{4(1-x^2 \cdot 1-c^2x^2)^{\frac{3}{4}}} \{ 3 + 2x^2(1+c^2) - 18c^2x^4 + 6c^2x^6(1+c^2) - c^4x^8 \}.$$

If in each of these we write  $x = r \sin \theta / R$ , the value of  $\gamma\rho$  is given by (138) on reduction. But the general result is too complicated to be of use, and we shall confine attention to a special case, namely, that in which  $r/a$  is large, so that we are investigating the scattering of the waves at a great distance from the sphere. In this case, we may write  $c=0$ ,  $x = \sin \theta$ , and deduce

$$v_2 = (1-x^2)^{-\frac{1}{2}} = \sec \theta.$$

$$v_3 = x(1-x^2)^{-\frac{3}{2}} = \sec^2 \theta \tan \theta.$$

$$v_4 = (1+2x^2)(1-x^2)^{-\frac{5}{2}} = (1+2 \sin^2 \theta) \sec^5 \theta.$$

$$v_3/v_2^2 = \tan \theta.$$

$$v_4/v_2^2 - \frac{5}{3} v_3^2/v_2^3 = \frac{1}{3}(3 + \sin^2 \theta) \sec^3 \theta. \dots (152)$$

$$u_1/u = \frac{1}{2}(1+2 \cos^2 \theta) \sec^2 \theta \operatorname{cosec} \theta.$$

$$u_2/u = \frac{1}{4}(3+2 \sin^2 \theta) \sec^4 \theta \operatorname{cosec}^2 \theta.$$

Write  $C = \cos \theta$ ,  $S = \sin \theta$  for brevity. Then by (137)

$$\begin{aligned} \frac{\mu}{u} &= -\frac{1}{4} \left( \frac{v_4}{v_2^2} - \frac{5}{3} \frac{v_3^2}{v_2^3} \right) - \frac{v_3}{v_2^2} \frac{u_1}{u} + \frac{1}{v_2} \frac{u_2}{u} \\ &= \frac{9-15S^2+11S^4}{12C^3S^2}. \dots (153) \end{aligned}$$

Moreover, by (151),

$$\begin{aligned} \lambda &= \frac{S^2}{2C^3} - \frac{3C}{8S^2} + \frac{1}{8C} + \frac{5}{24} \frac{S^2}{C^3} \\ &= -\frac{9+21S^2+5S^4}{24S^2C^3}, \dots (154) \end{aligned}$$

and the second approximation to  $u$  is

$$u = -\left( \frac{2k}{a\pi \cos \theta} \right)^{\frac{1}{2}} \sin^2 \theta e^{\frac{i\pi}{4}} \left\{ 1 + \frac{e\lambda}{z} \right\}.$$

The sum is by (138),

$$\begin{aligned} \gamma\rho = S &= \left(\frac{2\pi z}{v_2}\right)^{\frac{1}{2}} \left\{ u + \frac{\mu l}{2z} \right\} e^{izv + \frac{v\pi}{4}} \\ &= -2\mu k \sin^2 \theta e^{-ikR} \left\{ 1 + \frac{i\delta}{z} \right\}, \dots \quad (155) \end{aligned}$$

where

$$\delta = \lambda + \frac{1}{2}\mu/u = (3 + 8 \sin^2 \theta)/12 \cos^3 \theta. \dots \quad (156)$$

Thus, superposed on the effect of the oscillator in the presence of a plane reflector, there is an additional vibration of relative order  $z^{-1}$  for which the magnetic force is

$$\gamma = \frac{1}{6a} \cdot \sec^3 \theta \sin \theta (3 + 8 \sin^2 \theta) \frac{e^{-ikR}}{r}, \dots \quad (157)$$

when points in the region of transition and points close to the axis are excluded. This corresponds to an oscillator which, when undisturbed, gives a magnetic force

$$\gamma = \frac{\partial}{\partial \rho} \frac{e^{-ikR}}{R}.$$

The resulting amplitude at any point in the region is only altered to an order  $z^{-2}$  by taking account of this vibration. For we may write

$$1 + \frac{i\delta}{z} = \left(1 + \frac{\delta^2}{z^2}\right)^{\frac{1}{2}} e^{\frac{i\delta}{z}};$$

but this approximation does not determine the amplitude, for terms of relative order  $z^{-2}$  have already been neglected. A determination of the second order terms could, however, readily be made. The phase at a distant point is changed from  $kR$  to  $kR - \delta/z$ . When the first approximation was determined, it was stated that in a case in which  $ka = 10^6$ , an error of relative order  $10^{-12}$  only was involved in the assumption that the region of brightness is determined by the plane reflector effect. This statement is now justified.

The corresponding problem for points near the axis requires a different treatment, and will be investigated later. It will be shown that, as in the first approximation, the type of solution does not change near the axis.

XXXV. *Note on the Derivation from the Principle of Relativity of the Fifth Fundamental Equation of the Maxwell-Lorentz Theory.* By RICHARD C. TOLMAN, Ph.D., Instructor in Physical Chemistry at the University of Michigan\*.

IF we consider two systems of "space time coordinates" S and S' in relative motion in the X direction with the velocity  $v$ , any kinematic phenomenon which occurs may be described in terms of the variables  $x, y, z$  and  $t$  belonging to the system S or  $x', y', z'$  and  $t'$  belonging to the system S'. The Einstein theory of relativity has led to the following equations for transforming the description of a kinematic phenomenon from one set of coordinates to the other †.

$$t' = \frac{1}{\sqrt{1-\beta^2}} \left( t - \frac{v}{c^2} x \right) \dots \dots \dots (1)$$

$$x' = \frac{1}{\sqrt{1-\beta^2}} (x - vt) \dots \dots \dots (2)$$

$$y' = y \dots \dots \dots (3)$$

$$z' = z \dots \dots \dots (4)$$

(where  $c$  is the velocity of light and  $\beta$  is substituted for the fraction  $\frac{v}{c}$ ).

The content of these equations may be expressed in words, by saying that an observer in the moving system S' (S having been arbitrarily taken as at rest) uses a metre stick which, although the same length as a stationary metre stick when held perpendicular to the line of relative motion of the two systems, is shortened in the ratio of  $\sqrt{1-\beta^2}:1$  when held parallel to OX, that clocks in the moving system beat off seconds which are longer than those of stationary clocks in the ratio  $1:\sqrt{1-\beta^2}$ , and that a clock in the moving system which is  $x'$  units to the rear of the one at the centre of coordinates is set ahead by  $x' \frac{v}{c^2}$  seconds, although the two clocks appear synchronous to the moving observer. A simple non-analytical derivation of these relations has been given in another place ‡.

Let us now take the Maxwell-Hertz equations for the

\* Communicated by the Author.

† Einstein, *Ann. d. Physik*, xvii. p. 891 (1905); *Jahrbuch der Radioaktivität*, iv. p. 411 (1907).

‡ Lewis and Tolman, *Proc. Amer. Acad.* xlv. p. 711 (1909); *Phil. Mag.* xviii. p. 510 (1909).

electromagnetic field

$$\text{curl } \mathbf{H} = 4\pi\rho\mathbf{u} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \dots \dots \dots (5)$$

$$\text{curl } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \dots \dots \dots (6)$$

$$\text{div. } \mathbf{E} = 4\pi\rho \dots \dots \dots (7)$$

$$\text{div. } \mathbf{H} = 0. \dots \dots \dots (8)$$

If these equations are true descriptions of electromagnetic phenomena, it is evident then by the first postulate of relativity that similar equations

$$\text{curl } \mathbf{H}' = 4\pi\rho'\mathbf{u}' + \frac{1}{c} \frac{\partial \mathbf{E}'}{\partial t'}, \dots \dots (9)$$

$$\text{curl } \mathbf{E}' = -\frac{1}{c} \frac{\partial \mathbf{H}'}{\partial t'}, \dots \dots \dots (10)$$

$$\text{div. } \mathbf{E}' = 4\pi\rho', \dots \dots \dots (11)$$

$$\text{div. } \mathbf{H}' = 0, \dots \dots \dots (12)$$

must hold when the phenomena are described by an observer moving with the system S'.

It has been shown by Einstein that the following equations, together with the kinematic relations (1-4) already given are necessary and sufficient for transforming equations (5, 6, 7 and 8) into (9, 10, 11 and 12)\*.

$$E_x' = E_x \dots \dots \dots (13)$$

$$E_y' = \frac{1}{\sqrt{1-\beta^2}} \left( E_y - \frac{v}{c} H_z \right) \dots \dots \dots (14)$$

$$E_z' = \frac{1}{\sqrt{1-\beta^2}} \left( E_z + \frac{v}{c} H_y \right) \dots \dots \dots (15)$$

$$H_x' = H_x \dots \dots \dots (16)$$

\* For the purposes of this transformation, it is necessary to use not only the simple kinematic relations (1-4), but also the following relations which can be directly derived from them:—

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \quad u_y' = \frac{u_y \sqrt{1-\beta^2}}{1 - \frac{u_x v}{c^2}}, \quad u_z' = \frac{u_z \sqrt{1-\beta^2}}{1 - \frac{u_x v}{c^2}},$$

$$\rho' = \frac{\partial E_x'}{\partial x'} + \frac{\partial E_y'}{\partial y'} + \frac{\partial E_z'}{\partial z'} = \frac{\left(1 - \frac{u_x v}{c^2}\right) \rho}{(1-\beta^2)}$$

$$H_y' = \frac{1}{\sqrt{1-\beta^2}} \left( H_y + \frac{v}{c} E_z \right) \dots \dots (17)$$

$$H_z' = \frac{1}{\sqrt{1-\beta^2}} \left( H_z - \frac{v}{c} E_y \right) \dots \dots (18)$$

Thus at a given point in space, we may distinguish between the electric vector  $\mathbf{E}$  as measured by a stationary observer and the vector  $\mathbf{E}'$  as measured in units of his own system by an observer who is moving past the stationary system with the velocity  $v$  in the X direction. If  $\epsilon\mathbf{E}$  is the force acting on a small stationary test charge of magnitude  $\epsilon$ , then  $\epsilon\mathbf{E}'$  will be the force acting on the same test charge or electron when it is moving through the point in question with the velocity  $v$ , the force  $\epsilon\mathbf{E}'$  being measured in units of the moving system\*.

We are more particularly interested, however, in the vector  $\mathbf{F}$  which determines the force  $\epsilon\mathbf{F}$  that acts on the moving charge but which is measured in "stationary units," thus determining the equations of motion of the test charge  $\epsilon$  with respect to stationary coordinates. Since, however, it is possible to obtain relations between the units of force used by stationary and moving observers, a method is presented of calculating  $\mathbf{F}$  from the values of  $\mathbf{E}'$  already given by the transformation equations (13-18). As a matter of fact the expression for  $\mathbf{F}$  which can thus be obtained is identical with the fifth fundamental equation of the Maxwell-Lorentz theory.

*Relation between the Units of Force used in Moving and Stationary Systems.*

Consider a body having the mass  $m_0$  when at rest and moving with the same velocity  $v$  as a system of coordinates  $S'$ . Evidently its acceleration with respect to those coordinates is determined by Newton's laws of motion, and its acceleration with respect to stationary coordinates can be found by making the proper substitutions, giving us

$$F_x' = m_0 \dot{u}_x' = m_0 \frac{\dot{u}_x}{(1-\beta^2)^{3/2}} \dots \dots (19)$$

$$F_y' = m_0 \dot{u}_y' = m_0 \frac{\dot{u}_y}{(1-\beta^2)} \dots \dots (20)$$

$$F_z' = m_0 \dot{u}_z' = m_0 \frac{\dot{u}_z}{(1-\beta^2)} \dots \dots (21)$$

\* It should be noticed that according to the first postulate of relativity, if the charge of a stationary electron, for example a hydrogen ion, is  $\epsilon$ , then when the electron is in motion it must still appear to have the charge  $\epsilon$  to an observer who is moving along with it, otherwise the possibility would be presented of distinguishing between relative and absolute motion. This justifies us in taking  $\epsilon\mathbf{E}'$  as the force acting on the moving electron and measured in the moving system.

The substitutions

$$\dot{u}_x' = \frac{\dot{u}_x}{(1-\beta^2)^{\frac{3}{2}}}, \quad \dot{u}_y' = \frac{\dot{u}_y}{(1-\beta^2)} \quad \text{and} \quad \dot{u}_z' = \frac{\dot{u}_z}{(1-\beta^2)}$$

are an obvious consequence of the relations between the units of length and time used in the two systems. For example, if a body has an acceleration in the Y direction, of magnitude  $\dot{u}_y$  when measured in the system S, evidently its acceleration  $\dot{u}_y'$  as measured in the system S' will be greater because the units of time used in that system are "lengthened" in the ratio  $1 : \sqrt{1-\beta^2}$ . Remembering that the units of length in the Y direction are the same in both systems, and noticing the time enters to the second power in the expression for acceleration, the relation  $\dot{u}_y' = \frac{\dot{u}_y}{(1-\beta^2)}$  is evident. The other relations may be obtained in a similar way.

If now we define force as the increase in momentum per second we shall have, as has already been pointed out by Lewis\*,

$$F = \frac{d}{dt}(mu) = m \frac{du}{dt} + u \frac{dm}{dt},$$

where a possible change in mass as well as a change in velocity is allowed for. It has, moreover, been shown by Professor Lewis and the writer†, that the two postulates of relativity, themselves, combined simply with the principle of the conservation of momentum are sufficient for a proof that the mass of a body is increased when set in motion in the ratio  $1 : \sqrt{1-\beta^2}$ , so that in general the mass of moving body  $m = \frac{m_0}{\sqrt{1-\beta^2}}$ . Substituting in the equation above, we have

$$F = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \frac{du}{dt} + u \frac{d}{dt} \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

\* Lewis, *Phil. Mag.* xvi. p. 705 (1908).

† Lewis and Tolman, *loc. cit.*

or in the case where  $u_x = v$ ,  $u_y = u_z = 0$ :-

$$F_x = \frac{m_0}{\sqrt{1 - \frac{u_x^2}{c^2}}} \frac{du_x}{dt} + u_x \frac{d}{dt} \frac{m_0}{\sqrt{1 - \frac{u_x^2}{c^2}}} = \frac{m_0}{(1 - \beta^2)^{\frac{3}{2}}} \dot{u}_x,$$

$$F_y = \frac{m_0}{(1 - \beta^2)^{\frac{3}{2}}} \dot{u}_y,$$

$$F_z = \frac{m_0}{(1 - \beta^2)^{\frac{3}{2}}} \dot{u}_z,$$

and by the further substitution of equations (19-20-21) we obtain

$$F_x = F_x' \dots \dots \dots (22)$$

$$F_y = \sqrt{1 - \beta^2} F_y' \dots \dots \dots (23)$$

$$F_z = \sqrt{1 - \beta^2} F_z', \dots \dots \dots (24)$$

which are the desired relations connecting measurements of force in the two systems.

#### *The Fifth Equation.*

Returning now to the consideration of an electron which is moving with the same velocity as the system  $S'$ , we see that the transformation equations (13-14-15) together with the above equations lead to the relation

$$F_x = F_x' = E_x' = E_x$$

$$F_y = \sqrt{1 - \beta^2} F_y' = \sqrt{1 - \beta^2} E_y' = \left( E_y - \frac{v}{c} H_z \right)$$

$$F_z = \sqrt{1 - \beta^2} F_z' = \sqrt{1 - \beta^2} E_z' = \left( E_z + \frac{v}{c} H_y \right),$$

which is the desired equation :

$$\mathbf{F} = \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H}.$$

This result agrees with that obtained by Einstein in his second treatment (*Jahrbuch der Radioaktivität*, iv. p. 411, 1907), where instead of defining force as equal to mass times acceleration, he defined it by the equations

$$F_x = \frac{d}{dt} \frac{m_0 u_x}{\sqrt{1 - \beta^2}}, \quad F_y = \frac{d}{dt} \frac{m_0 u_y}{\sqrt{1 - \beta^2}}, \quad F_z = \frac{d}{dt} \frac{m_0 u_z}{\sqrt{1 - \beta^2}},$$

which agree with our definition of force as equal to the rate of increase of momentum.



The special purpose of this note is to make clear that the fifth fundamental equation of electromagnetic theory may be derived from the four field equations and the principle of relativity without making any arbitrary convention as to the mass of a moving body. It is quite unnecessary to place the transverse mass of a moving body equal to  $\frac{m_0}{\sqrt{1-\beta^2}}$  and the longitudinal mass equal  $\frac{m_0}{(1-\beta^2)^{\frac{3}{2}}}$ . The simple relation for the mass of a moving body  $m = \frac{m_0}{\sqrt{1-\beta^2}}$ , which was derived directly from the principle of relativity by Lewis and Tolman (*loc. cit.*) and from ideas of light pressure by Lewis (*loc. cit.*) is sufficient.

The fact that the fifth equation can be derived by combining the principle of relativity with the four field equations is one of the chief pieces of evidence which support the theory of relativity.

University of Michigan,  
Ann Arbor, Mich.,  
November 11, 1910.

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XXXVI. *A Note on the Measurement of the Refractive Index of Liquids.* By O. W. GRIFFITH, B.Sc., A.R.C.S.\*

FOR some years past the author has been setting his students, as a laboratory exercise, to determine the refractive index of water by using an ordinary spherical flask filled with water as a convergent lens. The results have always been strikingly concordant, the error in the values obtained by different observers being small and fairly constant. It was therefore thought that an inquiry into the best conditions for the experiment might prove interesting, and this paper contains the results of such an investigation and indicates two very simple methods of determining the index of refraction of liquids. It will be seen that these methods are capable of giving accurate and reliable values.

As a rule the problem of refraction through a sphere either receives very meagre treatment in the ordinary text-book or is relegated to the collection of mathematical exercises at the end of the book. It is, however, usually demonstrated that the principal points of a sphere are coincident with its centre. So that if U and V are the reciprocals of the distances of conjugate foci measured from the centre of a transparent

\* Communicated by the Author.

sphere, and  $R$  is the curvature of the sphere whose refractive index is  $\mu$ , then (with the usual modern convention as to signs)

$$U + V = 2R \cdot \frac{\mu - 1}{\mu}.$$

We may call the right-hand side of this equation the converging power of the sphere. If then  $F$  is the power,

$$F = 2R \cdot \frac{\mu - 1}{\mu}.$$

Hence, since  $F$  and  $R$  can be easily measured,  $\mu$  for the sphere may be as easily calculated.

In the case of a spherical flask filled with a liquid, there are two disturbing factors which vitiate the values of  $\mu$  obtained from the above equation directly. They are (a) the thickness of the glass, and (b) the spherical aberration. It will be an advantage to consider these two sources of error separately.

(a) *Effect of the thickness of the glass.*

Consider the refraction of a narrow axial pencil through a sphere of index  $\mu$  enclosed in a concentric spherical shell of index  $\mu'$ . Let the pencil diverge from a point in air the reciprocal of whose distance from the centre of the sphere is  $U$ . Let  $V_1, V_2, V_3, V$  be the corresponding quantities for the successive conjugate points after refraction at the several surfaces; and let  $R_1$  and  $R_2$  be the external and internal curvature of the shell respectively. Then we have the following equations for the refraction at the different surfaces,

$$\mu'U + V_1 = R_1(\mu' - 1)$$

$$\mu V_1 + \mu' V_2 = R_2(\mu - \mu')$$

$$\mu' V_2 + \mu V_3 = R_2(\mu - \mu')$$

$$V_3 + \mu' V = R_1(\mu' - 1)$$

whence 
$$U + V = 2 \left( R_1 \frac{\mu' - 1}{\mu'} + R_2 \frac{\mu - \mu'}{\mu \mu'} \right) \dots \dots (1)$$

If  $F$  is the power of the system and  $K = R_2 - R_1$ ,

then 
$$F = 2R_2 \frac{\mu - 1}{\mu} - 2K \frac{\mu' - 1}{\mu'} \dots \dots (2)$$

In the case of a thin spherical flask containing a liquid, equation (2) shows that the effect of the glass is to decrease

the power of the system, acting as a thin divergent lens placed at the centre of the sphere and of power  $-2K \frac{\mu' - 1}{\mu'}$ .

Putting  $\delta\mu$  for the error in the value of  $\mu$  introduced by an error  $\delta F$  in measuring  $F$ , we find

$$\delta F = 2R_2 \frac{\delta\mu}{\mu^2}$$

or 
$$\delta\mu = \frac{\mu^2 \delta F}{2R_2} \dots \dots \dots (3)$$

But the numerical value of  $\delta F$  due to the thickness of the glass is given by

$$\delta F = 2K \cdot \frac{\mu' - 1}{\mu'}$$

Hence the error introduced by the glass envelope is

$$\delta\mu = \frac{\mu^2 K}{R_2} \cdot \frac{\mu' - 1}{\mu'}$$

Let  $t$  = the thickness of the glass shell, and  $r_1$  the external radius, then

$$K r_1 = t R_2,$$

whence

$$\delta\mu = \frac{\mu^2 t}{r_1} \cdot \frac{\mu' - 1}{\mu'} \dots \dots \dots (4)$$

To calculate the magnitude of the error, assume the following values:

$$\mu = \frac{4}{3}, \mu' = \frac{3}{2}, t = \cdot 05 \text{ cm.}, \text{ then } \delta\mu = \frac{\cdot 03}{r_1};$$

and if  $t$  be taken =  $\cdot 04$  cm.,

$$\delta\mu = \frac{\cdot 02}{r_1}.$$

So that in the case of a flask 20 cms. in diameter, and of thickness  $\cdot 05$  cm., the error would be  $\cdot 003$ , and this is of the order observed when using a flask of the size mentioned. Even were this factor of no account, it appears from equation (2) that an additional small error is introduced owing to the difficulty of measuring accurately the internal radius of the flask. It would be a great advantage if it were possible to express the value of  $F$  in terms of the external radius. By slight re-arrangement of the terms in the right-hand side of equation (1) we get

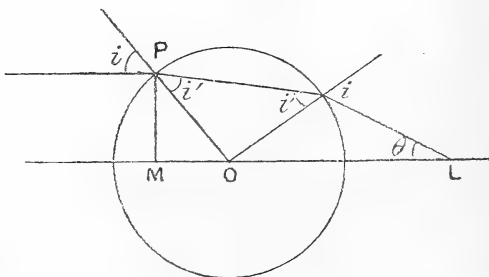
$$F = 2R_1 \frac{\mu - 1}{\mu} - 2K \left( \frac{1}{\mu} - \frac{1}{\mu'} \right), \dots \dots \dots (5)$$

which states that the system is equivalent to a water sphere of radius  $r_1$  and a divergent lens of power equal to the second term. The value of this form of result will be evident when we have computed the effect of spherical aberration.

(b) *Effect of Spherical Aberration.*

Fig. 1 illustrates the refraction of a parallel beam through a sphere.

Fig. 1.



The focal length, measured from the centre, is

$$OL = \frac{OP \sin i}{\sin \theta}.$$

Also  $\theta = 2(i - i')$  and  $\sin i = \mu \sin i'$ . Substituting for  $i'$  and expanding in terms of  $\sin i$  by the Binomial Theorem, neglecting all terms above the second power, we get

$$\sin \theta = 2 \frac{\mu - 1}{\mu} \left[ 1 + \frac{3\mu - \mu^2 - 1}{2\mu^2} \cdot \sin^2 i \right] \sin i.$$

Putting  $PM = a$ ,  $OP = r$ , then if  $f$  is the focal length,

$$f = \frac{\mu r}{2(\mu - 1)} - \frac{a^2}{r} \cdot \frac{3\mu - \mu^2 - 1}{4\mu(\mu - 1)}. \quad \dots \quad (6)$$

The second term on the right-hand side of equation (6) represents the error due to spherical aberration, and its effect is to increase the converging power of the sphere. Applying this to the case of a thin glass flask filled with water, we may

assume that the greater part of the aberration is produced by the water sphere. Assigning the value  $4/3$  to  $\mu$  in the aberration term, we see that the error introduced in the value of  $f$  is numerically  $\frac{11 a^2}{64 r}$ .

Now, reverting to equation (5) we see that the value of  $f$ , when the thickness of the glass is taken into account and the aberration is negligible, is given by

$$f = \frac{\mu r_1}{2(\mu - 1)} + \frac{K r_1^2 (\mu' - \mu) \mu}{2(\mu - 1)^2 \mu'}$$

or, since  $K = \frac{t}{r_1 r_2}$ ,

$$f = \frac{\mu r_1}{2(\mu - 1)} + \frac{t r_1 (\mu' - \mu) \mu}{2 r_2 (\mu - 1)^2 \mu'} \dots \dots \dots (7)$$

Substituting  $\mu = 4/3$  and  $\mu' = 3/2$  in the second term on the right-hand side of equation (7), we find that the numerical value of that term is

$$\frac{2}{3} \cdot t \cdot \frac{r_1}{r_2}.$$

Hence the thickness of the glass tends to increase the focal length by an amount

$$\frac{2}{3} t \cdot \frac{r_1}{r_2},$$

and the spherical aberration tends to decrease it by

$$\frac{11 a^2}{64 r_2}.$$

Hence it is possible by adjustment of the diameter of the aperture of the entrant beam to arrange that these errors should just compensate. This will be the case when

$$\frac{2t}{3} \cdot \frac{r_1}{r_2} = \frac{11 a^2}{64 r_2},$$

or  $\frac{a^2}{r_1} = \frac{128}{33} t = 4t$ , approximately. . . . (8)

The following table gives the diameter of apertures that might be used with spheres of different radii.

Radius of sphere in cm.	Diameter of aperture in cm.	
	$t = .04$ cm.	$t = .05$ cm.
1	.80	.90
2	1.14	1.28
3	1.39	1.56
4	1.60	1.80
5	1.80	2.00

Small glass flasks of 6 cm. diameter and thickness .04-.05 cm., painted over with dull black varnish except an aperture of about 1.5 cm. in diameter, seem to answer the purpose very well, and good values of the index of refraction of a liquid can be obtained by their use. It is essential that the exposed parts of the flask should be spherical, and this may be tested by measuring the radius of curvature at the aperture with a spherometer and comparing it with the distance between the two diametrically opposite apertures, as measured with a vernier calipers. The parallel beam is obtained by means of a collimator, and the position of the focus determined by the optical bench eyepiece, or by means of a travelling microscope, the source of light being the slit of the collimator illuminated with radium light. By using a short length of platinum wire heated with an electric current as a source and a thermopile or bolometer as detector, the method can be applied to obtain a rough estimate of the refractive index for heat radiation of quartz in the form of a sphere, or of a liquid (such as a solution of iodine in carbon bisulphide), enclosed in a fused silica flask.

It is beyond the scope of this note to discuss the general question of spherical aberration, but the accompanying diagrams are interesting in that they compare the aberrations produced in a sphere and in the usual types of convergent lenses. The effect of the aberration may be represented in two ways. If  $F$  is the power of the system,  $f$  its focal length,  $R$  the curvature of a surface and  $a$  the radius of the aperture, and  $\mu$  the index of refraction, then the error produced by

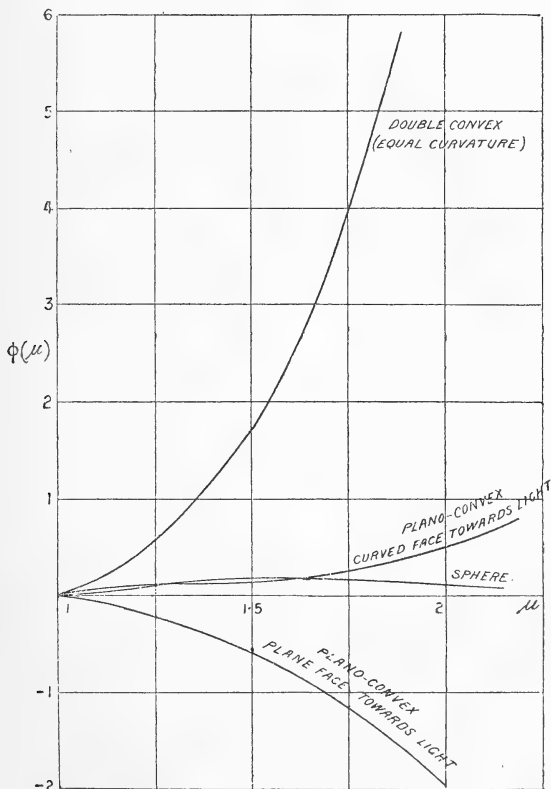
the spherical aberration of a parallel beam may be written

$$\Delta F = R^3 a^2 \phi(\mu), \dots \dots \dots (9)$$

$$\Delta f = -R a^2 \psi(\mu), \dots \dots \dots (10)$$

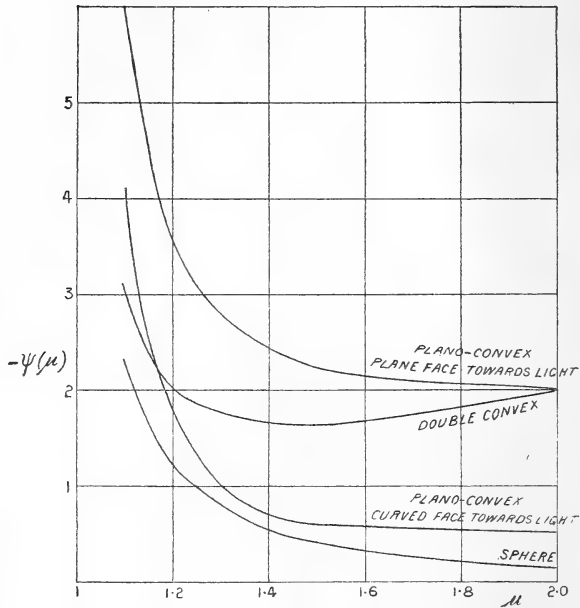
where  $\phi(\mu)$  and  $\psi(\mu)$  are certain functions depending on the type of lens employed. The curves are drawn for (1) a sphere, (2) a plano-convex lens with light incident on curved face, (3) a plano-convex lens with light incident on plane face, (4) double convex lens with faces of equal curvature. Fig. 2 represents the graph of equation (9) for values of  $\mu$  between

Fig. 2.



1 and 2, and  $R^3 a^2$  being unity in all cases. Fig. 3 gives the graph of equation (10) for values of  $\mu$  between 1 and 2, the value of  $R a^2$  being unity in all cases. It is noticeable that the errors in the case of the sphere are considerably less than they are for any of the other lenses.

Fig. 3.



*Second method of determining  $\mu$  for a liquid.*

Arrange two plano-convex lenses so as to form a telescopic system of minimum aberration, that is so that a parallel incident beam may emerge as a parallel beam. Let the outer faces of the lenses be of equal curvature  $R$ , and let  $t$  be the thickness of each lens, and  $\mu'$  the refractive index of its material. Then the equations representing the successive refraction of a parallel beam through the two surfaces of the first lens are

$$\mu'V_1 = -R(\mu' - 1),$$

$$\mu V_2 = \mu'V_1 / (1 + V_1t),$$

$V_1$  and  $V_2$  having the usual meanings, and  $\mu$  being the refractive index of the medium adjoining the plane surface of the lens. This gives

$$\mu V_2 = -\frac{\mu'(\mu' - 1)R}{\mu' - Rt(\mu' - 1)}.$$

Hence, if  $d =$  distance between the plane-faces of the lenses,

$$\frac{2\mu}{Rd} = \frac{\mu'(\mu' - 1)}{\mu' - Rt(\mu' - 1)}.$$



Let  $d=d'$  when  $\mu=1$ .

Then evidently  $d=\mu d'$ .

For glass lenses of 10 cm. focus,  $d=20$  cm.,  $d' \doteq 27$ , when  $\mu=4/3$ . Therefore it is quite possible to measure both of these lengths to a high degree of accuracy. In the apparatus used, the two lenses are fitted on the ends of two tubes sliding one within the other through a stuffing-box. The apparatus is mounted on the table of a spectrometer whose telescope and collimator have been previously focussed for parallel light. The distance between the lenses is adjusted so that a clear image of the collimator slit is observed in the telescope with the tube (a) full of air, (b) full of water or other liquid. Index marks are placed on the tubes and the distance between these may be measured with vernier calipers, or by means of a travelling microscope if great accuracy is desired.

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XXXVII. *The Destruction of the Fluorescence of Iodine and Bromine Vapour by other Gases.* By R. W. WOOD\*.

[Plate III.]

AN extended study of the fluorescence of sodium, potassium, mercury and iodine vapour has shown that the intensity of the emitted light is greatly reduced if air, or some other chemically inert gas, is present. A quantitative study of the phenomenon, showing the relation between the intensity of the fluorescence and the pressure and molecular weight of the foreign gas, is much to be desired as a means of testing any hypothesis which may be made regarding the action of the gas upon the radiating molecules. The vapour of iodine is especially suited to the work, since its fluorescence can be observed at room temperature in glass bulbs, and the conditions of pressure, density, &c. can be accurately determined, which is nearly or quite impossible with sodium vapour.

A satisfactory theory of the phenomenon should not only explain the destruction of the fluorescence by the inert gas, but also the failure of bromine to show any trace of fluorescence when under the same conditions as iodine vapour. Its absorption spectrum is very similar, and yet it usually remains quite dark even under the most powerful excitation.

Some years ago I suggested the hypothesis that the molecule might be capable of storing up a certain amount of

\* Communicated by the Author.

energy without the emission of light, but that a saturation point must be reached eventually, after which there will be an emission of radiation. If we assume that on collision with another molecule the energy absorbed by the molecule is transformed into heat, the internal energy dropping back to its original value, and the molecular velocity increasing in proportion, it is clear that if the mean free path is traversed before the saturation point is reached, there will be no fluorescence. On this hypothesis we should explain the failure of bromine to fluoresce by ascribing to the bromine molecule a greater capacity for storing energy. In other words, the path cannot be increased sufficiently to allow the saturation point to be reached before a collision occurs. It seemed possible to test this theory by experiment. By sufficiently increasing the length of free path, we ought to be able to observe fluorescence, provided that a sufficient number of molecules remain to produce a visible illumination. A small amount of bromine vapour was introduced into a bulb, and condensed upon the wall by the application of solid carbon dioxide and ether. The bulb was then exhausted to the highest possible degree and sealed. On warming it to room temperature the bromine vaporized, and though it was so highly rarefied that it showed no colour, no fluorescence could be detected.

Sunlight was now concentrated at the centre of the bulb by means of a portrait lens having a ratio of focus to aperture of 2.3. Even in a dark room with careful screening off of diffused light, no fluorescence could be detected. The outside of the bulb was now touched with a piece of solid carbon dioxide, which gradually condensed the bromine upon the wall. In two or three seconds a faint green fluorescence appeared, which vanished almost immediately, owing to the complete removal of the bromine vapour. There appears then to be one density at which bromine shows a visible fluorescence. At higher densities collisions destroy it, at lower, there are too few molecules present. This appears to be in accord with our hypothesis regarding absorption of energy, saturation point, &c.; but more recent work, made in collaboration with J. Franck, has shown that another factor comes into play. The question will be considered again in the paper immediately following the present one.

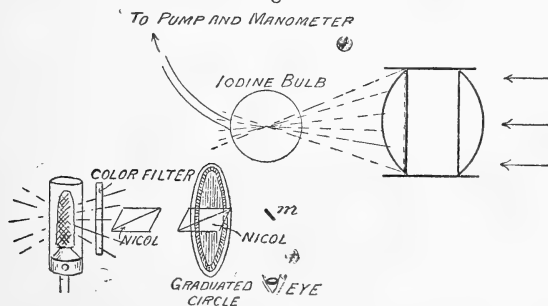
The method of observing the iodine fluorescence has been so improved that it is now possible to demonstrate it to the largest audience. A large bulb 15 or 20 cms. in diameter is prepared by drawing down the neck of a round-bottomed flask, which should be most carefully cleaned with aqua regia

and distilled water, avoiding the use of alcohol and ether for drying. A few small crystals of iodine are now introduced into the flask, the neck drawn down to a 1 mm. capillary at one point, and the flask thoroughly exhausted with a Gaede or other mercurial pump. It is most important to have a very perfect vacuum (less than .01 mm.), and it is usually necessary to keep the pump running for 15 or 20 minutes to secure this, if the capillary is narrow. The flask is now sealed, and can be used for demonstration purposes at any time. It requires no heating, for the iodine fluorescence is brightest at the pressure which the vapour has at room temperature. We have only to hold the bulb in the converging beam furnished by the condensers of a large projecting lantern, or the condensed beam from a heliostat. If a lantern is used it is best to throw the beam upwards, as the reflexions give less trouble. The intensely brilliant cone of yellowish-green fluorescent light can be seen from the back of the largest lecture hall. The aberration of the spherical lenses is well shown as well.

The bulbs used in the photometric work were smaller, having diameters of about 7 cm.

The fluorescence was excited by sunlight reflected into a dark room by a large heliostat, and concentrated to a focus at the centre of the bulb by a Voigtländer portrait objective of 12 cm. aperture and 27 cm. focus (F. 2.3). The work was done only on very clear days when the sky was free from haze and clouds. A Welsbach light was used as a standard, its colour being brought to a match with that of

Fig. 1.



the iodine fluorescence by a combination of pale cobalt glass and a dilute solution of bichromate of potash. The arrangement of the apparatus is shown in fig. 1. Between the eye and the brightest part of the fluorescent cone is mounted a small mirror *m*, measuring about 1 × 3 mm., made by silvering a

piece of plate glass and breaking off thin scales by tapping the edge with a hammer, striking the blow in a direction nearly parallel to the silvered surface. The silvered scales obtained in this way usually have one edge of razor sharpness, and this edge forms the vanishing line of the photometer. The silvered mirror reflects to the eye the light from the comparison source, which is placed in such a position that the glass sliwer cuts across the fluorescent cone at its brightest point. By rotating the graduated nicol prism the intensities may be perfectly matched, the sharp edge of the illuminated mirror vanishing. The intensity of the fluorescence is measured by the square of the cosine of the angle through which the nicol has been turned, measured from the position of complete extinction.

The bulb containing the iodine was in communication with a Gaede pump, a manometer, and a reservoir of the gas under investigation. The bulb was first highly exhausted and the intensity of the fluorescence measured. Air or some other gas was now introduced until the manometer showed a pressure of 1 mm., and the intensity again measured. The pressure of the gas was increased by progressively small steps, the intensity being measured for each pressure. Plotting the results, with the intensities as ordinates and the pressures as abscissæ, gives us a curve showing the rate at which the intensity of the fluorescence decreases with increasing gas pressure. Experiments were made with air, hydrogen, carbon dioxide, and ether vapour. Several series were made with each gas, and the results were in good agreement.

The curves are reproduced in Plate III. together with curves obtained with other gases, which will be discussed in the following paper.

An examination of these curves showed that the hypothesis of free path and saturation point would not represent the facts, and that some other factor must be taken into consideration.

The effectiveness of the gas in destroying the fluorescence appeared to increase with its molecular weight, but was by no means proportional to it. For example, the intensity of the fluorescence was reduced from 45 to 8.5 by 3 mm. of ether vapour, 7 mm.  $\text{CO}_2$ , 11.5 mm. air, and 24 mm. of hydrogen. It seems clear from these results that some other property of the gas than its molecular weight must be operative.

In the case of the fluorescence of anthracene, Elston found that the presence of hydrogen and nitrogen was almost with-

out influence upon the intensity of the emitted light. Oxygen and CO<sub>2</sub>, on the other hand, reduced its intensity very rapidly with increasing pressure. This I have provisionally ascribed to "incipient chemical action" which of course in reality means nothing at all. It was found that no permanent chemical change took place, for on cooling the bulbs the anthracene condensed upon the walls, and none of the oxygen had disappeared. At high temperatures, however, the oxygen acts upon the anthracene, and it seemed possible that the first stage of the process might occur at the lower temperatures used in the fluorescence experiments, the process reversing as soon as the bulbs were cooled. This is what I called "incipient chemical action." The probable real nature of the action will be given in the subsequent paper.

The only difficulty found in measuring the intensity of the fluorescent light resulted from the slight change of colour which occurred when the intensity was considerably reduced. The colour always became slightly reddish at the higher pressures, though it was only conspicuous when seen in the photometer, the colour match not remaining perfect.

The gases apparently weaken the green portion of the fluorescent spectrum to a greater extent than the red. This is a very interesting and important matter, which will be discussed in the following paper.

Experiments were also made with a highly exhausted bulb, the vapour pressure of the iodine being varied by immersing a side tube in freezing mixtures of various temperatures, or heating the entire bulb in a water-bath. The values found below room temperature are given in the following table, and are shown in graphical form on Plate III. (small inset):—

Temp.	Intensity.	
19	43	The relation is very nearly linear.
6°	29	
0°	22	
-10°	11	
-15	6	
-20	2	

Above room temperature no perceptible increase could be detected. The intensity remained about the same up to 30°, after which it gradually decreased. At high pressures the fluorescence disappeared entirely

It is probable that increased absorption compensates very nearly for any increase that may occur above room temperature. Accurate measurements were impossible on account of the change of colour due to the absorption of the iodine vapour between the fluorescent cone and the eye.

XXXVIII. *The Influence upon the Fluorescence of Iodine and Mercury of Gases with different Affinities for Electrons.*  
By J. FRANCK and R. W. WOOD\*.

[Plate III.]

WARBURG † has shown that in nitrogen, helium, argon, and hydrogen, which have been very carefully freed from all traces of oxygen, the current obtained with the negative point discharge is much greater than when traces of oxygen are present. To explain this circumstance he made the hypothesis that in the pure gases the negatively charged electrons move with a higher velocity. Small traces of oxygen, by condensation on the electrons, increase their mass and reduce their velocity.

Many other phenomena of the discharge of electricity through gases are influenced by traces of oxygen ‡.

Finally J. Franck made direct measurements of the mobility of the electrons in argon, nitrogen, and helium (in the latter case in collaboration with G. Gehloff—unpublished), and showed that small traces of more or less electro-negative gases operated in the manner assumed by Warburg.

The affinities for electrons—*i. e.*, the forces acting between neutral molecules and electrons—decrease as we proceed from strongly electro-negative to the inert gases argon, helium, &c. In the latter the forces appear to be nil. The possible existence of a relation between the affinities for electrons and an effect upon the emission of spectrum lines has been suggested in the paper referred to.

For the investigation of such a possible effect the fluorescence of iodine and mercury is especially adapted, for the excitation is caused by a single factor only, namely the impact of light-waves, which is uninfluenced by the admixture of other gases, whereas in the case of electrical excitation, potential gradient, current strength, and density are all affected by small traces of other gases.

The reduction in the intensity and the final destruction of the fluorescence of iodine vapour by the presence of other gases has been investigated by Wood and described in the preceding paper.

Hydrogen showed the least influence, and taken in increasing degree, air, CO<sub>2</sub>, and ether vapour.

This sequence is not opposed to the above hypothesis, for

\* Communicated by the Authors.

† Warburg, *Wied. Ann.* vol. xl. p. 1 (1896).

‡ See Summary by J. Franck, *Verh. der Deut. Phys. Ges.*, July 1910.

the affinity for electrons increases for these gases in the order mentioned. It is not a proof of the theory, however, for the molecular weights increase in the same order. If we accept Lorentz's hypothesis that a damping factor results from collisions, we can assume that the damping factor is a function of the molecular weight, and we have no means of knowing whether a second factor, the affinity for electrons, is superposed on it. The results previously obtained appeared to point towards the existence of some other factor however.

In the present paper we shall show that gases which interfere with the motions of the free electrons, by their affinity for them, interfere as well with the motions of the bound electrons, the vibrations of which give rise to the spectral lines.

It may here be pointed out that Pohl and Pringsheim have come to the same conclusion with regard to the natural frequency and damping of the resonance electrons in the selective photo-electric effect.

To settle the question in the case of the emission of fluorescent light we must compare a heavy gas of small or zero affinity for electrons with a light gas of strong affinity. We have therefore examined the influence of helium, argon, nitrogen, oxygen, and chlorine upon the fluorescence.

The method was identical with that described in the preceding paper, except that the light from the crater of a right-angle arc lamp was used in place of sunlight, which was not available on account of unfavourable weather conditions. An image of the arc was projected upon the wall of the room by means of a lens, the position of the carbons giving maximum illumination ascertained, and their outlines on the wall marked with a pencil. By frequent hand regulation they were kept always in the same relative position.

The helium gas, carefully purified, was furnished by the firm of Siemens and Halske, through the courtesy of Dr. Holm. The argon was prepared in the Institute. The nitrogen was obtained from commercial bombs, and was not very pure, and the chlorine and oxygen prepared by heating gold chloride and potassium permanganate respectively.

The results are given in the curves (Plate III.), in which the ordinates are the intensities of the iodine fluorescence and abscissæ the pressures in mms. of the admixed gases. The values obtained in the previous work with air, hydrogen, CO<sub>2</sub>, and ether are given on the same plate. Check observations were also made in the present work, with air, to make sure that the results were comparable with the earlier ones.

One sees at once that helium, in spite of the fact that its molecular weight is double that of hydrogen, is much less detrimental to the fluorescence, and that the argon curve runs nearly in coincidence with that of hydrogen, though its molecular weight is twenty times that of hydrogen. Above the air curve we have a point for nitrogen, and below it one for oxygen, exactly as we should expect, for nitrogen is nearly neutral, while oxygen is strongly electro-negative.

The lowest curve of all is that for chlorine, with an atomic weight of 70, much lower than the curve for ether with an atomic weight of 75. It is clear that the affinity of a gas for electrons is a powerful factor in suppressing the fluorescence.

For helium, we have given two curves, one for the green portion of the fluorescence spectrum, the other for the red. The measurements were made in the two cases by suitably selected colour filters.

This separation was necessary in the case of helium, for it was found that the colour of the fluorescence changed rapidly from green to reddish orange, as more and more helium was added, which made photometric measurements with a standard of fixed colour impossible.

As is apparent, the two curves cross, resulting from the circumstance that *in vacuo* the green portion of the fluorescent spectrum is much stronger than the red, but that it is rapidly reduced in intensity by the addition of helium, while the red portion is reduced to a much less degree.

A similar effect, though to a much lesser degree, is observed with hydrogen and argon. This change of colour is probably a pure collision effect, for with electro-negative gases, such as chlorine, it is not noticeable. The very feeble fluorescence observed when we have 3 or 4 mms. of chlorine in the bulb has practically the same colour as when the iodine is *in vacuo*. The cause of the colour change will be given in a subsequent paper dealing with the resonance spectrum of iodine recently discovered by Wood. It seems probable that when an iodine molecule is near enough to a chlorine molecule to be influenced at all, its fluorescence is practically destroyed. The fluorescence observed when chlorine is present probably comes from those iodine molecules which at the moment happened to be beyond the sphere of action of any chlorine molecule. Their number will be fewer and fewer as the pressure of the chlorine increases.

We have also investigated qualitatively the effects of these gases upon the fluorescence of mercury vapour, which has already been extensively studied by Wood\*. Mercury

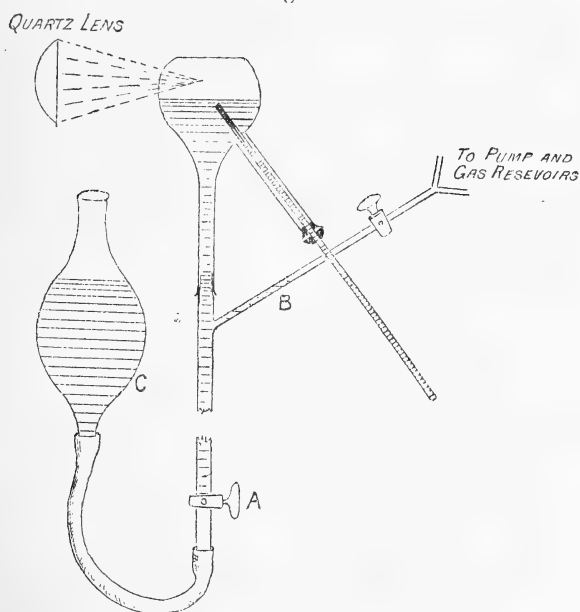
\* Phil. Mag. vol. xviii. p. 240 (Aug. 1909).



vapour is strongly electro-positive, and its fluorescence should therefore be very sensitive to the presence of an electro-negative gas.

Our results confirm this view, for in oxygen at a pressure of only 3 mms. we obtained no fluorescence at all, though the quartz bulb was heated to a temperature sufficient to give us mercury vapour at 30 cms. pressure. The apparatus is shown in fig. 1. A quartz bulb with a double neck was used, one sealed to a vertical tube with sealing-wax, the other serving for the introduction of the high temperature thermometer, also sealed in with wax. The quartz necks of the flasks were kept cooled with wet cotton, and the portion of the bulb containing the mercury heated with a burner.

Fig. 1.



The temperature of the mercury, the coolest part of the system, was measured, and this gave the pressure of the vapour. At the beginning of the experiment the stop-cock A was opened and the mercury allowed to sink below the lateral tube B, by lowering the mercury reservoir C.

The stop-cock A was then closed and the bulb thoroughly exhausted. The cock A was then opened, D being closed, and the mercury allowed to rise until the bulb of the thermometer was just covered. The cock A was now closed

and the fluorescence, caused by condensing the light of a cadmium spark to a focus at the centre of the bulb, observed at different temperatures. Small amounts of any foreign gas were admitted by lowering the mercury below the junction of the tube B, and allowing the gas to enter from a reservoir through the tube B, the pressure being read with a manometer. This pressure was about doubled by raising the mercury to its original level. By raising the mercury to a higher level still greater density of the gas can be obtained, its pressure being easily determined by measuring the difference of level of the mercury surfaces in the flask and in the reservoir B, when the cock A is open. Closing the cock seals off the bulb, and heating then causes no change of level.

While oxygen at 3 mms. destroys the fluorescence, helium at a pressure of 1 *atmosphere* scarcely affects the intensity at all!

The fact previously observed by Wood, that the maximum intensity of the fluorescence, in the absence of any foreign gas, occurs at a pressure which is very different according to the nature of the fluorescent gas, can probably be explained by our hypothesis. In order to obtain a visible fluorescence we must have a sufficient number of molecules present. Their number must not, however, be so great as to cause them to disturb each other. In a strongly electro-negative gas the vibration electrons in one molecule are influenced by the presence of neighbouring molecules. In the case of bromine, therefore, which is more strongly electro-negative than iodine, we have fluorescence only at very low pressures, probably less than  $\cdot 001$  mm., while in the case of iodine the maximum intensity of the fluorescence occurs at a pressure of about  $\cdot 2$  mm. (tension at room temperature), and in the case of the strongly electro-positive mercury vapour the maximum intensity is not reached until we have a pressure of several atmospheres. The intensity of the fluorescence of mercury vapour at high pressures should be quantitatively investigated. It will not be easy, for as the pressure increases the fluorescence is confined to a layer of decreasing thickness covering the wall through which the light enters the bulb.

Two facts seem to have been established by this investigation: first, that the bound electrons, which emit the fluorescent light, are effected in much the same way by the presence of electro-negative gases as are the free electrons which carry the current in vacuum tubes; secondly, that the pressure at which the maximum intensity of the fluorescence of a gas occurs depends upon the electrical character of the molecule.

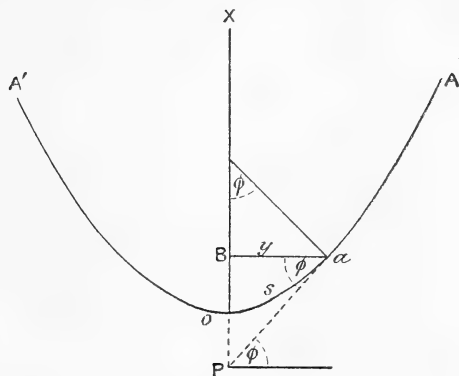
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XXXIX. *The Problem of Uniform Rotation treated on the Principle of Relativity.* By H. DONALDSON, B.A., B.Sc., Scholar of Sidney Sussex College, Cambridge, and G. STEAD, B.A., late Scholar of Clare College, Cambridge\*.

IN a previous paper (Phil. Mag. July 1910) we commenced an investigation of the above problem, dealing only with the changes in the dimensions of a rotating disk. We showed that the contractions demanded by the relativity theory, due to the motion of the disk, are fulfilled if the disk buckle into a cuplike form, whose section by a vertical plane is an epicycloid of intrinsic equation

$$s = \frac{2}{\sqrt{3}} \frac{c}{\omega} \sin \frac{\phi}{2}.$$

Fig. 1.



In the figure shown we have

$$y = s(1 - v^2/c^2)^{\frac{1}{2}} = s \left( 1 + \frac{s^2 \omega^2}{c^2} \right)^{-\frac{1}{2}},$$

where  $v$  is the linear velocity of any point on the disk ( $=y\omega$ ).

We shall find that this cuplike form of the disk gives a most useful method of visualising the processes going on during the rotation on relativity hypotheses. For, if the disk were considered as remaining plane, we should have, owing to the lessening of the circumference and the invariability of the radius, a change in the value of the "constant"  $\pi$ , whereby we are transformed to a "real

\* Communicated by the Authors.

curved space." This cuplike representation of our disk is a method of representing the contracted disk in ordinary space, and the results obtained by considering it will still hold good, even if the disk really remain plane.

Let us now consider an observer on a fixed disk regarding the moving disk. The number of revolutions performed by the moving disk relative to the fixed disk as measured by the observer on the fixed disk will obviously be the same as the number of revolutions of the fixed disk relative to the moving disk as measured in the same period of time by an observer supposed placed on the moving disk. That this will be so follows from the fact that a number is of no dimensions in length, mass, or time. But to perform one revolution a point on the moving disk has to traverse a linear distance which is less than that traversed by the corresponding point of the fixed disk in the ratio

$$(1 - v^2/c^2)^{\frac{1}{2}} : 1.$$

Hence, in order that the two numerical measures should be the same, it is necessary that the time unit on the moving system should be *greater* than that on the fixed system in the ratio  $(1 - v^2/c^2)^{-\frac{1}{2}} : 1$ . This is the result which has previously been deduced by other writers in a variety of ways for systems in linear motion.

We shall now proceed to calculate the kinetic energy of our rotating disk, and to show how this leads us to the necessity for a change in the mass of the system due to its rotation.

Since a ring element of area  $2\pi s \cdot \delta s$  on the fixed disk becomes a ring element of area  $2\pi y \cdot \delta s$  on the moving disk, and since

$$y = s(1 - v^2/c^2)^{\frac{1}{2}},$$

we shall have

$$\sigma' = \sigma (1 - v^2/c^2)^{-\frac{1}{2}},$$

where  $\sigma'$  and  $\sigma$  are the surface densities of the fixed and moving disks respectively.

Now  $v = y\omega$ , the units being those of the fixed system, and therefore, since

$$y = s(1 - v^2/c^2)^{\frac{1}{2}} = s \left( 1 + \frac{s^2\omega^2}{c^2} \right)^{-\frac{1}{2}},$$

we have

$$\underline{\sigma' = \sigma \left( 1 + \frac{s^2\omega^2}{c^2} \right)^{\frac{1}{2}} = \sigma \cdot \frac{s}{y}.}$$

The moment of inertia of the disk about its axis of rotation can now be obtained immediately, for, taking our ring element, we have

$$\delta I = 2\pi y \cdot \delta s \cdot \sigma \left(1 + \frac{s^2 \omega^2}{c^2}\right)^{\frac{3}{2}} y^2$$

and 
$$y = s \left(1 + \frac{s^2 \omega^2}{c^2}\right)^{-\frac{1}{2}};$$

$$\therefore \delta I = 2\pi s^3 \left(1 + \frac{s^2 \omega^2}{c^2}\right)^{-1} \sigma \cdot \delta s,$$

$$\begin{aligned} \therefore I &= 2\pi \sigma \int_0^r s^3 \left(1 + \frac{s^2 \omega^2}{c^2}\right)^{-1} ds \\ &= 2\pi \sigma \left[ \frac{c^2}{2\omega^2} \left(1 + \frac{\omega^2 s^2}{c^2}\right) \left\{ \log \left(1 + \frac{\omega^2 s^2}{c^2}\right) - 1 \right\} \right]_0^r \\ &= 2\pi \sigma \left[ \frac{r^2 c^2}{2\omega^2} - \frac{c^4}{2\omega^4} \log \left(1 + \frac{r^2 \omega^2}{c^2}\right) \right]. \end{aligned}$$

When  $\omega$  is small this reduces, as a first approximation, to the ordinary expression for the moment of inertia of a circle about an axis through its centre perpendicular to its plane.

The kinetic energy of the disk is given by  $\frac{1}{2} I \omega^2$ , that is, by

$$\frac{\pi \sigma r^2 c^2}{2} - \frac{c^4}{2\omega^2} \log \left(1 + \frac{r^2 \omega^2}{c^2}\right).$$

When  $\omega$  is infinite this becomes

$$\frac{1}{2} M c^2 - \frac{c^4}{2} \text{Lt}_{\omega=\infty} \frac{\log \left(1 + \frac{r^2 \omega^2}{c^2}\right)}{\omega^2}$$

The limit term is seen, in the ordinary way, to be zero, so that the maximum kinetic energy possible for the disk is  $\frac{1}{2} M c^2$ .

Now, when the disk possesses this energy its volume has become zero, as shown in the preceding paper, and therefore we arrive at the seeming absurdity that a *finite* expenditure of energy is able to reduce the volume of the disk to zero.

This conclusion can be avoided in two ways, or by a combination of the two. The possibilities are, first, an increase in the mass of the moving disk, due to its velocity, and, secondly, an increase in the internal ("potential")

energy of the disk. Let us consider, first of all, the increase of mass. To calculate this increase we have to make some assumption, and we shall adopt as our hypothesis the most general principle of mechanics, namely, the conservation of linear and angular momentum. In our particular case this means that the angular momentum of the moving disk, relative to the fixed disk, as measured by an observer on the fixed disk will have the same numerical value as the angular momentum of the fixed disk relative to the moving disk as measured by an observer on the moving disk.

Now, angular momentum is given by terms of the form  $mr^2\omega$ , and  $r$  is measured in the direction perpendicular to the direction of motion of the ring element considered, and is therefore the same on both systems of units. This leaves us with the fact that

$$\left| \frac{m}{m_0} \right| = \left| \frac{\omega_0}{\omega} \right| = \left| \frac{t}{t_0} \right|,$$

where the letters without suffixes denote units on the moving system and those with the zero suffix denote the units of the same quantities on the fixed system.

Now, we have shown that the time unit on the moving system is greater than the time unit on the fixed system in the ratio  $(1 - v^2/c^2)^{-\frac{1}{2}} : 1$ , and therefore we have

$$m = m_0(1 - v^2/c^2)^{-\frac{1}{2}},$$

which is the formula previously deduced by other writers for uniform motion, and experimentally confirmed for such motion by the researches of Kauffmann and Bucherer on moving corpuscles. If now we assume, as a second fundamental principle, the constancy of Total Energy of our moving system; that is, that the Total Energy of the contracted disk is the same as that of an uncontracted disk moving with the same angular velocity, we can calculate an expression for the change in the internal energy of the disk due to dimensional and mass changes which have occurred in it on the relativity theory.

We have that the kinetic energy of the disk, corresponding to an angular velocity  $\omega$ , is

$$E = \frac{1}{2}I\omega^2,$$

and therefore

$$dE = I\omega d\omega + \frac{1}{2}\omega^2 dI.$$

If we consider our curved disk we shall be able to regard  $\omega$  as on the fixed system of units and shall be able to proceed

directly. Thus, when  $\omega$  becomes  $\overline{\omega + d\omega}$ , the increase in kinetic energy differs from what it would be if  $I$  were constant by  $\frac{1}{2}\omega^2 dI$ , and hence, if we consider that the change in internal energy of the disk is  $\frac{1}{2}\omega^2 dI$ , we conserve our total energy.

Hence if we write  $P_0$  as the internal energy of the uncontracted disk moving with angular velocity  $\omega$ , and  $P$  as the internal energy of the contracted disk having the same angular velocity, we have

$$P - P_0 = \int_0^\omega \frac{1}{2}\omega^2 \frac{dI}{d\omega} d\omega.$$

Now for the value of  $I$  for our rotating contracted disk, taking into account the fact that the mass of each element is altered in the ratio

$$1 + \frac{s^2\omega^2}{c^2} : 1,$$

we have

$$\begin{aligned} \delta I &= 2\pi y \cdot \delta s \cdot \sigma \left(1 + \frac{s^2\omega^2}{c^2}\right)^{\frac{1}{2}} \left(1 + \frac{s^2\omega^2}{c^2}\right)^{\frac{1}{2}} \cdot y^2 \\ &= 2\pi s^3 \left(1 + \frac{s^2\omega^2}{c^2}\right)^{-\frac{1}{2}} \sigma \cdot \delta s; \\ \therefore I &= 2\pi\sigma \int_0^r s^3 \left(1 + \frac{s^2\omega^2}{c^2}\right)^{-\frac{1}{2}} ds \\ &= 2\pi\sigma \left[ \frac{1}{3} \frac{c^4}{\omega^4} \left(\frac{\omega^2}{c^2} s^2 + 1\right)^{\frac{1}{2}} \left(\frac{\omega^2}{c^2} s^2 - 2\right) \right]_0^r \\ &= 2\pi\sigma \left[ \frac{1}{3} \frac{c^4}{\omega^4} \left(\frac{\omega^2}{c^2} r^2 + 1\right)^{\frac{1}{2}} \left(\frac{\omega^2}{c^2} r^2 - 2\right) + \frac{2}{3} \frac{c^4}{\omega^4} \right]. \end{aligned}$$

To shorten our calculations, we will obtain  $P - P_0$ , not from the integral expression for it given above, but from the fact that

$$P - P_0 = E_0 - E,$$

where  $E_0$  and  $E$  are kinetic energies corresponding to  $P_0$  and  $P$ , since

$$\int_0^\omega dE = E \quad \text{and} \quad \int_0^\omega I\omega d\omega = E_0.$$

Now

$$E = \frac{\pi\sigma c^4}{3\omega^2} \left[ \left( \frac{\omega^2}{c^2} r^2 + 1 \right)^{\frac{1}{2}} \left( \frac{\omega^2}{c^2} r^2 - 2 \right) + 2 \right]$$

from our above expression for I, and, to a first approximation, this becomes

$$E = \frac{\pi\sigma}{4} \left\{ \omega^2 r^4 - \frac{1}{3} \frac{\omega^4 r^6}{c^2} \right\}$$

But

$$E_0 = \frac{1}{4} \pi r^4 \sigma \omega^2,$$

and therefore

$$\begin{aligned} \underline{E_0 - E} &= \frac{1}{12} \pi \sigma_0 \frac{\omega^4 r^6}{c^2} \\ &= \frac{1}{3} \frac{\omega^2 r^2}{c^2} \underline{E_0}. \end{aligned}$$

Hence, to a first approximation, the change in internal energy due to the contraction of the disk is given by

$$P - P_0 = \frac{1}{3} \frac{\omega^2 r^2}{c^2} E_0.$$

In conclusion we consider that this investigation shows that the relativity theory involves no contradictions when applied to the case of uniform rotation, and also that the case of uniform rotation gives a very sound and direct method of proceeding from length to time units, depending only on the fact of a number having no dimensions in mass, length, or time. Again, the maximum kinetic energy possible for the disk shows the necessity for some change in mass, though it does not give, of course, any result which is *uniquely* satisfied by the relativity change of mass-unit. It will be noticed that we have given no proof of the change in the length-unit of the moving system, the reason for the omission being that the change can be directly deduced on the method given by Messrs. Lewis and Tolman (Phil. Mag. Oct. 1909) for uniform rectilinear motion.

Cambridge.

October 31, 1910.



XL. *Relations between the Density, Temperature, and Pressure of Substances.* By R. D. KLEEMAN, D.Sc., B.A., Mackinnon Student of the Royal Society\*.

THE writer† has shown in a previous paper that an infinite number of equations can be found connecting the surface tension or latent heat of evaporation of a liquid with its temperature and density and density of the saturated vapour. These equations correspond to laws of attraction between the molecules, but none of which is necessarily the law that actually exists. Each of these laws can be obtained by giving a definite value to the arbitrary function contained in the general law of attraction that can be deduced from surface tension or latent heat data. This general law of attraction between two molecules of the same kind is  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) \cdot \frac{(\sum \sqrt{m_1})^2}{z^5}$ , where T denotes the temperature of the molecules and  $z$  their distance of separation,  $x_c$  denotes their distance of separation in the liquid state at the temperature  $T_c$ ,  $\sum \sqrt{m_1}$  denotes the sum of the square roots of the atomic weights of the atoms in a molecule, and  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$  denotes an arbitrary function of  $\frac{T}{T_c}$  and  $\frac{z}{x_c}$ .

There exists, therefore, an infinite series of equations of the form  $L = \psi_1(\rho_1, \rho_2, T) = \psi_2(\rho_1, \rho_2, T) = \dots$ , where L denotes the internal heat of evaporation of a liquid at the temperature T, and  $\rho_1$  and  $\rho_2$  denote the density of the liquid and that of the saturated vapour respectively. It is obvious that we can obtain from these equations an infinite number of equations containing T,  $\rho_1$ , or  $\rho_2$  only. Now each of the equations thus obtained must obviously be either an identity or be an equation which has an infinite number of real positive roots lying between certain limits. In practice, however, the equations found in the above way (usually by trial) do not exactly satisfy these conditions. But this is of no consequence, as the equations deduced, containing any two of the quantities  $\rho_1$ ,  $\rho_2$ , T, usually agree very well with the facts. It is the object of this paper to point out and discuss some of these equations.

Mills‡ has shown that the internal latent heat of evaporation is very approximately given by the equation  $L = D(\rho_1^{1/3} - \rho_2^{1/3})$ , where D is a constant depending only on

\* Communicated by the Author.

† Phil. Mag. Jan. 1911, p. 83.

‡ Journ. of Phys. Chem. vol. viii. p. 405 (1904).

the nature of the liquid. The writer\* has shown that this equation can be deduced if  $\phi_2\left(\frac{v_c}{z}, \frac{T_c}{T}\right)$  is put equal to  $\left(\frac{z}{x_c}\right)^3$  in the general law of attraction between molecules. We then obtain  $D = \frac{S\rho_c}{m^{1/3}}(\Sigma \sqrt{m_1})^2$ , where  $m$  denotes the molecular weight of the liquid,  $\rho_c$  the critical density, and  $S$  a numerical constant. This value of  $D$  agrees fairly well with the facts. The latent heat is also given by the equation †

$$L = \frac{K_4 RT}{m} \log\left(\frac{\rho_1}{\rho_2}\right),$$

where  $K_4$  is a numerical constant equal to about 1.75; which also corresponds to a particular case of the general law of attraction between molecules. Equating these two different equations for the latent heat we obtain

$$B(\rho_1^{1/3} - \rho_2^{1/3}) = T \log\left(\frac{\rho_1}{\rho_2}\right),$$

where

$$B = \frac{S\rho_c}{K_4 R m^{4/3}} (\Sigma \sqrt{m_1})^2.$$

Another expression for  $B$  which is convenient can be obtained by determining  $B$  at the critical temperature from the first of the above two equations. Writing  $\rho_2 = x\rho_1$  we have

$$B = \left[ \frac{T \log \frac{1}{x}}{\rho_1^{1/3}(1-x^{1/3})} \right]_{\text{Lt } x=1} = \frac{3T_c}{\rho_c^{1/3}};$$

and the above equation becomes

$$\frac{3T_c}{\rho_c^{1/3}}(\rho_1^{1/3} - \rho_2^{1/3}) = T \log\left(\frac{\rho_1}{\rho_2}\right). \quad \dots \quad (1)$$

If the two different expressions obtained for  $B$  are equated we get

$$T_c = \frac{S}{3K_4 R} \left(\frac{\rho_c}{m}\right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

an equation which has already been discussed in previous papers.

Equation (1) has many useful applications, and it has therefore been tested over considerable ranges of temperature for a number of liquids in Table I., which contains the values of  $\frac{10^4}{B}$  calculated by means of this equation. The

\* *Loc. cit.*

† *Phil. Mag.* Oct. 1910, p. 688.

TABLE I.

Ethyl oxide, C <sub>2</sub> H <sub>5</sub> O.		Pentane, C <sub>5</sub> H <sub>12</sub> .		Stannic chloride, SnCl <sub>4</sub> .		Octane, C <sub>8</sub> H <sub>18</sub> .		Benzene, C <sub>6</sub> H <sub>6</sub> .		Heptane, C <sub>7</sub> H <sub>16</sub> .	
T.	10 <sup>4</sup> / B.	T.	10 <sup>4</sup> / B.	T.	10 <sup>4</sup> / B.	T.	10 <sup>4</sup> / B.	T.	10 <sup>4</sup> / B.	T.	10 <sup>4</sup> / B.
273	4.36	273	4.21	373	4.96	273	3.02	273	3.76	273	3.35
293	4.42	313	4.29	393	4.99	393	3.42	353	3.95	353	3.60
313	4.46	333	4.29	413	5.01	413	3.44	373	3.96	373	3.63
333	4.48	353	4.29	433	5.02	433	3.44	393	3.97	393	3.65
353	4.49	373	4.29	453	5.02	453	3.46	413	3.96	413	3.66
373	4.42	393	4.28	473	5.02	473	3.46	433	3.95	433	3.67
393	4.48	413	4.27	493	5.01	493	3.47	453	3.93	453	3.67
413	4.46	433	4.27	513	4.99	513	3.48	473	3.92	473	3.68
433	4.47	453	4.27	533	5.04	533	3.48	493	3.91	493	3.68
453	4.48	463	4.29	553	4.99	552	3.52	513	3.90	513	3.68
460	4.54	468	4.33					533	3.91	533	3.74
466	4.56	469	4.31					553	3.93	539	3.78
				Hexane, C <sub>6</sub> H <sub>14</sub> .		Carbon tetrachloride, CCl <sub>4</sub> .		Di-isobutyl, C <sub>8</sub> H <sub>18</sub> .		Ethyl acetate, C <sub>4</sub> H <sub>8</sub> O <sub>2</sub> .	
Iodo-benzene, C <sub>6</sub> H <sub>5</sub> I.		Bromo-benzene, C <sub>6</sub> H <sub>5</sub> Br.		273	3.74	273	4.15			273	3.41
303	3.39	403	3.31	343	3.92	373	4.94	273	3.26	363	4.09
473	3.63	533	3.84	363	3.94	393	4.91	373	3.56	383	4.11
563	3.80	553	3.85	383	3.94	413	4.93	393	3.60	383	4.11
583	3.80	573	3.85	403	3.94	433	4.91	413	3.62	403	4.13
603	3.80	593	3.85	423	3.94	453	4.89	433	3.62	423	4.14
623	3.81	613	3.85	443	3.94	473	4.87	453	3.63	443	4.15
643	3.81	633	3.85	463	3.94	493	4.85	473	3.62	463	4.15
				483	3.95	513	4.83	493	3.62	483	4.16
				499	3.97	533	4.84	513	3.64	503	4.19
				506	4.01	553	4.87	533	3.66	518	4.28
				Acetone, C <sub>3</sub> H <sub>6</sub> O.		Chloroform, CHCl <sub>3</sub> .		Fluor-benzene, C <sub>6</sub> H <sub>5</sub> F.		Hexa-methylene, C <sub>6</sub> H <sub>12</sub> .	
273	3.34	243	8.30	273	3.93	293	4.89			273	3.70
413	3.70	263	8.38	293	3.98	313	4.94	273	3.89	353	3.87
433	3.71	283	8.30	313	4.02	333	4.98	373	4.13	383	3.89
453	3.71	298	8.44	333	4.05			393	4.14	403	3.89
473	3.71	Sulphur dioxide, SO <sub>2</sub> .		Ethyl formate, C <sub>3</sub> H <sub>6</sub> O <sub>2</sub> .		Ethyl propionate, C <sub>5</sub> H <sub>10</sub> O <sub>2</sub> .		413	4.15	423	3.88
493	3.71	263	6.12	333	4.35	383	3.85	433	4.15	443	3.87
513	3.70	283	6.18	353	4.38	403	3.88	453	4.14	463	3.85
533	3.69	303	6.19	393	4.40	433	3.90	473	4.13	483	3.84
Methyl formate, C <sub>2</sub> H <sub>4</sub> O <sub>2</sub> .		323	6.26	413	4.40	453	3.91	493	4.12	503	3.83
303	4.66	343	6.21	433	4.40	473	3.92	513	4.11	523	3.83
323	4.72	363	6.15	453	4.40	493	3.93	533	4.11	543	3.84
363	4.73	383	6.15	473	4.40	513	3.94	553	4.15	552	3.88
403	4.72	403	6.16	493	4.41	533	3.98				
443	4.71	418	6.19	507	4.47	541	4.00				

complete data used for these calculations and others in this paper can be obtained from the tables of density and pressure data of Ramsay and Young given in papers by Mills\*; part of the data (taken from this source) can be obtained from a previous paper by the writer†. It will be seen that the value of  $\frac{10^4}{B}$  is very approximately independent of the temperature for each liquid. Table II. contains the mean values

TABLE II.

Name of substance.	$\frac{B}{10^3}$ .	$\frac{3T_c}{10^3\rho_c^{1/3}}$ .	$\psi_1(T)$ .	$\frac{\rho_c^{1/3}}{3}(3-\log\rho_c)$ .	E.	$\frac{T_c}{\rho_c^2 \times 2.303}$ .
Ethyl oxide .....	2.24	2.20	.930	.931	1507	1496
Pentane .....	2.33	2.29	.912	.916	1951	1940
Stannic chloride ...	2.00	1.96	1.08	1.29	237	233.4
Octane .....	2.88	2.78	.916	.914	2393	2282
Benzene .....	2.54	2.51	.950	.939	1357	1316
Heptane .....	2.72	2.63	.914	.915	2246	2138
Iodo-benzene .....	2.64	2.59	1.08	1.22	463.4	462.9
Bromo-benzene .....	2.60	2.56	1.03	1.23	621.0	617.7
Hexane .....	2.53	2.47	1.07	.914	2094	2011
Carbon tetrachloride	2.05	2.03	1.02	1.27	403.7	388.2
Di-isobutyl .....	2.76	2.64	.918	.915	2247	2109
Ethyl acetate .....	2.40	2.35	.952	.938	1239	1266
Chlorobenzene .....	2.70	2.66	.983	.977	1049	1118
Carbon dioxide .....	1.20	1.43	.991	.972	322.7	306.8
Acetone .....	2.51	...	.953	...	...	...
Chloroform .....	2.02	...	1.09	...	...	...
Fluor-benzene .....	2.42	2.37	.872	.953	1001	968.7
Hexamethylene .....	2.61	2.54	1.23	.929	1661	1604
Sulphur dioxide ...	1.61	...	1.05	...	...	...
Ethyl formate .....	2.13	2.23	1.00	.942	1092	1102
Ethyl propionate ..	2.62	2.48	.933	.934	1410	1447
Methyl formate .....	2.12	2.07	.969	.951	893.9	869.0

of  $\frac{B}{10^3}$  of each liquid and the corresponding values of

$\frac{3T_c}{10^3\rho_c^{1/3}}$ . The agreement between the two sets of values is very good. The values of  $\rho_c$  and  $T_c$  used to calculate the value of the above expression are given in Table IV.

Several results of interest and usefulness may be deduced from equation (1). It may be expressed as two equations thus :

$$\rho_1^{1/3} - \frac{\rho_c^{1/3}T}{3T_c} \log \rho_1 = \psi_1(T), \quad \rho_2^{1/3} - \frac{\rho_c^{1/3}T}{3T_c} \log \rho_2 = \psi_1(T),$$

where  $\psi_1(T)$  is an appropriate function of  $T$ . Now it was found

\* Journ. Phys. Chem. vol. viii. p. 405 (1904).

† Phil. Mag. Oct. 1910, pp. 679-681.

that  $\psi_1(T)$  is approximately a constant for each substance, and that it varies very little from substance to substance. This is shown by Table III., which contains the values of

TABLE III.

Ethyl acetate.		Hexa-methylene.		Carbon tetrachloride.		Bromo-benzene.		Heptane.		Benzene.	
T.	$\psi_1(T)$ .	T.	$\psi_1(T)$ .	T.	$\psi_1(T)$ .	T.	$\psi_1(T)$ .	T.	$\psi_1(T)$ .	T.	$\psi_1(T)$ .
363	.977	273	1.02	373	1.05	533	1.05	353	.933	353	.961
383	.969	363	1.07	393	1.04	553	1.04	373	.926	373	.955
403	.962	383	1.17	413	1.04	573	1.03	393	.921	393	.951
423	.957	403	1.24	433	1.03	593	1.03	413	.917	413	.950
443	.953	423	1.27	453	1.03	613	1.02	433	.915	433	.950
463	.949	443	1.30	473	1.02	633	1.01	453	.914	453	.951
483	.945	463	1.32	493	1.02			473	.914	473	.951
503	.940	483	1.32	513	1.01			493	.913	493	.950
518	.934	503	1.32	533	1.00			513	.911	513	.948
522	.932	523	1.30	553	.988			533	.907	533	.945
		543	1.26					539	.904	553	.940
		552	1.22								
Fluor-benzene.		Octane.		Stannic chloride.		473	1.17	Hexane.		Di-isopropyl.	
353	.995			373	1.14	563	1.08	343	1.12	323	.922
373	.987	393	.928	393	1.12	58.5	1.07	363	1.11	343	.915
393	.982	413	.922	413	1.10	603	1.06	383	1.10	363	.916
413	.976	433	.919	433	1.09	623	1.06	403	1.09	383	.916
433	.974	453	.916	453	1.08	643	1.05	423	1.08	383	.915
453	.971	473	.915	473	1.06			443	1.06	403	.916
473	.967	493	.913	493	1.06	Chloro-benzene.		463	1.05	423	.919
493	.967	513	.911	513	1.05	413	.995	483	1.04	443	.918
513	.963	533	.910	533	1.04	433	.990	499	1.02	463	.919
533	.959	553	.906	553	1.02	453	.985	506	1.00	483	.917
553	.952					473	.982			493	.916
						493	.979	Pentane.		Di-isobutyl.	
Methyl formate.		Ethyl propionate.		Ethyl oxide.		513	.976	313	.913		
323	.991	383	.950	293	.945	533	.973	353	.911		
403	.968	423	.937	353	.929			373	.911	323	.936
483	.949	463	.930	413	.928	Acetone.		393	.913	343	.925
		500	.926	460	.919	273	.971	433	.914	363	.919
		538	.920			330	.935	453	.913	393	.916
								463	.912	423	.915
								469	.909	453	.914
										483	.911
										498	.906
Ethyl formate.		Carbon dioxide.		Sulphur dioxide.		Chloroform.		Propyl formate.			
333	1.04			283	1.06	273	1.13	373	.968		
393	1.01	243	1.01	323	1.03	330	1.06	433	.955		
433	1.00	298	.973	363	1.02			493	.947		
507	.961			403	1.00						

$\psi_1(T)$  at different temperatures for a number of liquids, the latter of the above equations being used for calculating  $\psi_1(T)$ . The slight deviations of  $\psi_1(T)$  from constancy are regular, there being a slight decrease in its value with the temperature.

The value of  $\psi_1(T)$  in terms of the critical constants, obtained by substituting for the quantities  $T$ ,  $\rho_1$ , and  $\rho_2$  in either of the equations their critical values, is

$$\frac{\rho_c^{1/3}}{3} (3 - \log \rho_c).$$

The above equations may thus be written:

$$\left. \begin{aligned} \rho_1^{1/3} - \frac{\rho_c^{1/3} T}{3T_c} \log \rho_1 &= \frac{\rho_c^{1/3}}{3} (3 - \log \rho_c), \\ \rho_2^{1/3} - \frac{\rho_c^{1/3} T}{3T_c} \log \rho_2 &= \frac{\rho_c^{1/3}}{3} (3 - \log \rho_c). \end{aligned} \right\} \dots (2)$$

Table II. contains the mean value of  $\psi_1(T)$  for each liquid contained in Table III., and the corresponding values of

$$\frac{\rho_c^{1/3}}{3} (3 - \log \rho_c).$$

The agreement between the two sets of values is fairly good.

Since the equations (2) are the same in form, for a given temperature each must have at least two real positive roots, one being equal to the value of  $\rho_1$  and the other to the value of  $\rho_2$ .

An expression for  $\frac{d\rho_1}{dT}$  is obtained by differentiating the first of equations (2), which gives

$$\frac{d\rho_1}{dT} = \frac{\rho_1 \rho_c^{1/3} \log \rho_1}{T_c \rho_1^{1/3} - T \rho_c^{1/3}}.$$

It will be seen that at the critical temperature  $\frac{d\rho_1}{dT} = \infty$ , a result that has already been established by thermodynamics.

At the absolute zero  $\frac{d\rho_1}{dT}$  is finite. For intermediate temperatures the equation gives values of  $\frac{d\rho_1}{dT}$  which agree roughly with the facts. That a good agreement is not obtained is due to  $\psi_1(T)$  being only approximately a constant. A better agreement would be obtained by writing  $\psi_1(T) = a - Tb$ , when we obtain

$$\frac{d\rho_1}{dT} = \frac{\rho_1 \rho_c^{1/3} \log \rho_1 - 3T_c \rho_1 b}{T_c \rho_1^{1/3} - \rho_c^{1/3} T}.$$

The value of  $b$  is best obtained from this equation by applying it to a case when all the quantities it contains are known except  $b$ .

We may substitute  $\rho_2$  for  $\rho_1$  in the above equations. It can then be easily shown that at the absolute zero  $\frac{d\rho_2}{dT} = -\infty$ .

Let us express the quantities  $\rho_1$ ,  $\rho_2$ , and  $T$  in terms of their critical values thus,  $\rho_1 = \rho_c n_1$ ,  $\rho_2 = \rho_c n_2$ ,  $T = T_c n_3$ , and substitute in equation (1). The equation reduces to

$$\frac{n_3}{3} \log \frac{n_1}{n_2} = n_1^{1/3} - n_2^{1/3}, \quad \dots \dots \dots (3)$$

which, it will be seen, is independent of the nature of the liquid under consideration. Since  $\rho_1$  and  $\rho_2$  are each a function of  $T$ , and  $n_1$  and  $n_2$  therefore each a function of  $n_3$ , it follows that for equal values of  $n_3$  for a number of substances,  $n_1$  and  $n_2$  will each have equal values. The equation thus demonstrates the theory of corresponding states, and gives a relation between the quantities  $n_1$ ,  $n_2$ ,  $n_3$ . The corresponding state of substances, it will be shown later, has its fundamental reason in the occurrence of the function in the general law of attraction between molecules which has the same value for all substances at corresponding temperatures.

In a previous paper\* we have established the relation  $E(\rho_1^2 - \rho_2^2) = T \log \frac{\rho_1}{\rho_2}$ , where  $E$  is a constant depending only on the nature of the liquid. This equation, like equation (1), represents two different formulæ for the latent heat equated. It was found to agree very well with the facts. The value of  $E$ , when the logarithm in the equation is taken to the base 10, was shown to be given by

$$\frac{258 \cdot 8 (\sum \sqrt{m_1})^2}{m^{4/3} \rho_c^{2/3}}$$

A different expression for  $E$  can be obtained in the same way as was obtained for  $B$ . Writing  $\rho_2 = x\rho_1$ , the value of  $E$  at the critical temperature is

$$\left[ \frac{T \log \frac{1}{x}}{\rho_1^2 (1-x^2)} \right]_{T=T_c} = \frac{T_c}{2\rho_c^2}, \text{ or } \frac{T_c}{\rho_c^2 \cdot 2 \times 2 \cdot 303},$$

\* Phil. Mag. Oct. 1910, pp. 686-687.

if the logarithm is taken to the base 10. The equation may now be written

$$\frac{T_c}{\rho_c^{2/2}}(\rho_1^2 - \rho_2^2) = T \log \frac{\rho_1}{\rho_2} \dots \dots \dots (4)$$

The mean values of E for a number of substances, given by the above equation, and the corresponding values of  $\frac{T_c}{\rho_c^{2/2} \times 2.303}$ , are given in Table II. The agreement between the two sets of values is fairly good.

From equations (1) and (4) we have

$$(\rho_1^2 - \rho_2^2) = 6 \rho_c^{5/3} (\rho_1^{1/3} - \rho_2^{1/3}) \dots \dots \dots (5)$$

If we, as before, express the quantities in this equation in terms of their critical values, we have

$$n_1^2 - n_2^2 = 6 (n_1^{1/3} - n_2^{1/3}) \dots \dots \dots (6)$$

Equations (1) and (5) express approximately the relation between the quantities  $\rho_1, \rho_2$ , and T, and equations (3) and (6) the relation between the quantities  $n_1, n_2$ , and  $n_3$ .

Equations (1) and (5) are very useful. From equation (5) we have  $\rho_c = \left( \frac{\rho_1^2 - \rho_2^2}{6\rho_1^{1/3} - 6\rho_2^{1/3}} \right)^{3/5}$ , which may be used to calculate

TABLE IV.

Name of liquid.	T.	$\rho_1$ .	$\rho_2$ .	$\rho_c$ (Landolt & Börnstein).	$\left\{ \frac{\rho_1^2 - \rho_2^2}{6\rho_1^{1/3} - 6\rho_2^{1/3}} \right\}^{3/5}$	$\frac{T_c}{\rho_c}$ (Landolt & Börnstein).	$\frac{B\rho_c}{3}$ <sup>1/3</sup>
Ethyl oxide .....	293	.7135	.00187	.2604	.2661	467.4	484.8
Octane .....	333	.6168	.0033	.2327	.2243	569.2	592.4
Ethyl propionate ...	383	.7823	.004739	.2860	.3011	545.4	580.1
Pentane .....	313	.6062	.00339	.2323	.2505	470.0	489.8
Benzene .....	353	.8145	.002722	.3045	.3071	561.5	569.4
Heptane .....	353	.6311	.001996	.2341	.2343	539.9	570.7
Hexane .....	343	.6122	.00337	.2344	.2347	507.8	524.7
Carbon tetrachloride	373	1.4343	.01026	.5576	.5563	556.1	555.0
Iodo-benzene .....	473	1.7079	.04400	.5814	.5755	721	729.3
Bromo-benzene .....	433	1.2994	.005255	.4853	.4919	670	684.4
Di-isobutyl .....	373	.6236	.002967	.2366	.2125	543.8	558.4
Ethyl acetate .....	363	.8112	.004673	.2993	.3076	522.5	550.3
Fluor-benzene .....	353	.9496	.002885	.3541	.3559	559.5	575.0
Carbon dioxide .....	263	.9560	.0725	.464	.4606	304.3	307.0
Hexamethylene .....	363	.7106	.003759	.2735	.2721	553	558.0
Chloro-benzene .....	413	.9723	.004316	.3654	.3695	633	646.8
Methyl formate ...	303	.9598	.002225	.3489	.3567	487	477.7
Ethyl formate .....	333	.8639	.003356	.3150	.3360	503	525.5



the critical density of a liquid from any available density data. The critical densities of a number of liquids have been calculated by means of this equation and are contained in Table IV., which contains also the data used. The agreement between these values and the critical densities given in Landolt and Börnstein's Tables, 5th edition, which are also given in Table IV., is quite good. The formula should always give a fairly accurate value of  $\rho_c$ , since its form is such that the percentage error in the value of  $\rho_c$  caused by errors in the values of  $\rho_1$  or  $\rho_2$  is not in most cases larger than the percentage error in the latter quantities.

Equation (1) gives the value of B or  $\frac{3T_c}{\rho_c^{1/3}}$ , which enables us to calculate  $T_c$  if  $\rho_c$  is known. The values of  $T_c$  or  $\frac{B\rho_c^{1/3}}{3}$  have been calculated in this way for a number of substances, and are given in Table IV., using the values of  $\rho$  previously found, and calculating B by means of the data used for calculating  $\rho_c$ . The agreement of these values of  $T_c$  with those taken from Landolt and Börnstein's Tables is as good as can be expected. It will be observed that since the cube root of  $\rho_c$  occurs in the expression for  $T_c$ , the percentage error in the value of  $\rho_c$  introduces a much smaller percentage error in the value of  $T_c$ .

Further relations of interest can be deduced. We have seen that

$$L = 1.75 \frac{RT}{m} \log \frac{\rho_1}{\rho_2},$$

and we therefore have from equation (1) that the value of D in the equation

$$L = D (\rho_1^{1/3} - \rho_2^{1/3})$$

may be written 
$$\frac{5.25RT_c}{m\rho_c^{1/3}}.$$

Substituting the value of L from the latter equation in Clapeyron's equation we have

$$\frac{dP}{dT} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) T = P \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + \frac{5.25RT_c}{m\rho_c^{1/3}} (\rho_1^{1/3} - \rho_2^{1/3}),$$

where P denotes the pressure of the saturated vapour.

We may express this equation in the form of two thus :

$$\left. \begin{aligned} \frac{dP}{dT} \frac{T}{\rho_1} &= \frac{P}{\rho_1} + \frac{5 \cdot 25 RT_c}{m \rho_c^{1/3}} \rho_1^{1/3} + \psi_2(T), \\ \frac{dP}{dT} \frac{T}{\rho_2} &= \frac{P}{\rho_2} + \frac{5 \cdot 25 RT_c}{m \rho_c^{1/3}} \rho_2^{1/3} + \psi_2(T). \end{aligned} \right\} \dots (6)$$

Now it was found that the value of  $\psi_2(T)$  is approximately constant for the same substance. This is shown for two substances in Table V.,  $\psi_2(T)$  being calculated by means of the latter of the above equations from the data given in the table.

TABLE V.

PENTANE.					HEXANE.				
T.	$\rho_2$ .	P in mm. of Hg.	$\frac{dP}{dT}$ .	$\psi_2(T)$ .	T.	$\rho_2$ .	P in mm. of Hg.	$\frac{dP}{dT}$ .	$\psi_2(T)$ .
313	·003390	865·3	28·81	93·2	343	·00337	784·8	24·73	89·7
333	·006024	1601·8	46·19	90·1	363	·00585	1409	38·80	87·2
353	·01013	2742·1	69·37	93·5	383	·00952	2358	57·20	86·8
373	·01627	4409·1	99·05	91·9	403	·01502	3723	80·55	85·9
393	·02503	6740·5	136·02	92·1	423	·02299	5606	109·2	85·2
413	·03861	9890	181·1	91·4	443	·03472	8123	144·0	84·2
433	·05910	14032	235·4	91·0	463	·05155	11407	186·3	84·1
453	·09354	19362	300·2	90·6	483	·07900	15619	237·4	83·6

The constancy of  $\psi_2(T)$  can be tested more conveniently by means of the equation obtained by eliminating  $\left(\frac{dP}{dT} - P\right)$  from equations (6). We thus obtain

$$C(\rho_1^{4/3} - \rho_2^{4/3}) = \rho_1 - \rho_2, \text{ where } C = \frac{5 \cdot 25 RT_c}{m \rho_c^{1/3} \cdot \psi(T)}.$$

It follows, therefore, that if it is found that C is independent of the temperature for a substance, then  $\psi_2(T)$  must possess the same property. Now this is approximately realized, as will appear from Table VI., which contains the values of C at different temperatures for a number of substances.

We can find another expression for the value of C, which enables us to obtain an expression for the value of  $\psi_2(T)$ . Let us write  $\rho_2 = x\rho_1$  in the above equation, and the value of

C is given by

$$C = \left[ \frac{1}{\rho_1^{1/3}} \left( \frac{1-x}{1-x^{4/3}} \right) \right]_{Lix=1} = \frac{4}{3\rho_c^{1/3}}$$

The above equation may then be written

$$\frac{3}{4\rho_c^{1/3}} (\rho_1^{4/3} - \rho_2^{4/3}) = \rho_1 - \rho_2 \quad \dots \quad (7)$$

We have also

$$\frac{3}{4\rho_c^{1/3}} = \frac{5 \cdot 25 RT_c}{m \rho_c^{1/3} \psi_2(T)}, \text{ and hence } \psi_2(T) = \frac{7 RT_c}{m}$$

In the case of pentane and hexane we obtain for  $\psi_2(T)$  from this equation the values 90.3 and 90.6 respectively, which agree fairly well with 91.7 and 85.8, the mean values of  $\psi_2(T)$  obtained from Table V.

The mean value of  $\frac{1}{C}$  for each substance is given at the bottom of Table VI. The agreement with the corresponding values of  $\frac{4}{3}\rho_c^{1/3}$  is fairly good. But it will be noticed that equation (7) does not on the whole agree so well with the facts as either equation (1) or (5).

TABLE VI.

Methyl butyrate.		Heptane.		Di-isobutyl.		Hexane.		Hexa-methylene.		Carbon tetrachloride.	
T.	$\psi_2(T)$ .	T.	$\psi_2(T)$ .	T.	$\psi_2(T)$ .	T.	$\psi_2(T)$ .	T.	$\psi_2(T)$ .	T.	$\psi_2(T)$ .
383	.930	273	.694	363	.861	333	.857	363	.897	273	1.18
403	.923	353	.702	383	.855	353	.849	383	.890	373	1.14
423	.916	373	.707	403	.848	373	.845	403	.884	393	1.13
443	.909	383	.707	423	.843	393	.837	423	.876	413	1.12
463	.903	393	.705	443	.836	413	.831	443	.874	433	1.11
483	.898	413	.701	463	.832	433	.826	463	.869	453	1.11
503	.894	433	.698	483	.827	453	.823	483	.866	473	1.10
523	.890	453	.677	503	.823	473	.820	503	.863	493	1.10
543	.890	473	.662	523	.822	493	.818	523	.861	513	1.09
551	.888			543	.822	503	.820	543	.862	533	1.13
Mean	.924		.695		.837		.833		.874		1.12
$\frac{3}{4}\rho_c^{1/3}$	.884		.822		.825		.822		.866		1.38

## General Considerations.

We will now deduce some further results of a general nature from the law of attraction between molecules quoted at the beginning of the paper.

It can be easily shown that the theory of corresponding states follows from the law quoted. It will first be proved in the case of a liquid in contact with its saturated vapour. The equation for the internal latent heat of evaporation deduced from the law\* is

$$L = A_1 \frac{\rho_1^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2 - A_2 \frac{\rho_2^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2,$$

where

$$A_1 = \phi_3 \left( \frac{x_a}{x_c}, \frac{T}{T_c} \right), \quad A_2 = \phi_3 \left( \frac{x_b}{x_c}, \frac{T}{T_c} \right),$$

and  $x_a, x_b$  denote the distances of separation of the molecules in the liquid and gaseous state respectively. The form of  $A_1$  and  $A_2$ , it should be noticed, depends only on the form of the expression  $\phi_2 \left( \frac{z}{x_c}, \frac{T}{T_c} \right)$  occurring in the law of attraction. We have seen that an infinite number of formulæ for the latent heat can be obtained which correspond to different forms of  $\phi_2$ , and there exists, therefore, a series of equations of the form

$$\begin{aligned} L &= A_1 \frac{\rho_1^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2 - A_2 \frac{\rho_2^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2 \\ &= A_1 \frac{\rho_1^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2 - A_2 \frac{\rho_2^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2 = \dots, \end{aligned}$$

or

$$\phi_4 \left( \frac{x_a}{x_c}, \frac{T}{T_c} \right) - \phi_5 \left( \frac{x_a}{x_c}, \frac{T}{T_c} \right) = \left\{ \phi_4 \left( \frac{x_b}{x_c}, \frac{T}{T_c} \right) - \phi_5 \left( \frac{x_b}{x_c}, \frac{T}{T_c} \right) \right\} \left( \frac{\rho_2}{\rho_1} \right)^{4/3} = \dots,$$

where  $\frac{x_a}{x_c} = \frac{\rho_1}{\rho_c}$ , and  $\frac{x_b}{x_c} = \frac{\rho_2}{\rho_c}$ .

It will be readily seen that if we express the quantities  $\rho_1, \rho_2$ , and  $T$  in terms of their critical values in the same way as before, the equations will contain  $n_1, n_2$ , and  $n_3$ , only. Since it must be possible to express  $n_1$  and  $n_2$  in terms of  $n_3$ , therefore if  $n_3$  for a number of substances has equal values,  $n_1$  and  $n_2$  will each have equal values. This proves the theory of corresponding states from the general law of attraction for the quantities  $\rho_1, \rho_2$ , and  $T$ .

\* Phil. Mag. May 1910, pp. 793-794.

The above series of equations are the general fundamental equations connecting  $T$ ,  $\rho_1$ , and  $\rho_2$ , for a liquid in contact with its saturated vapour. Since they also contain  $\rho_c$  and  $T_c$ , they are the fundamental equations for determining these quantities from any convenient data. The equations we have been discussing, obtained by equating different formulæ for the internal latent heat, will readily be recognized as belonging to this type of equations.

Making use of the above result, the equation for the latent heat may be written

$$L = B^2 \frac{\rho_1^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2,$$

when  $B$  is a constant which is the same for all liquids at corresponding states. It can then be easily shown \* by means of this equation that  $L = L_0 \eta$ , where  $L_0$  is the latent heat at the absolute zero, and  $\eta$  is a constant which is the same for all liquids at corresponding states; which establishes the required result in the case of the internal latent heat.

By means of the foregoing results and thermodynamics the law of corresponding states can be established for the pressure  $p$  of the saturated vapour of a liquid. If the value of  $L$  from the former of the above two latent heat equations is substituted in Clapeyron's equation, it can be reduced to the form

$$W (\Sigma \sqrt{m_1})^2 \left( \frac{\rho_1}{m} \right)^{7/3} = p - \frac{T}{T_c} p_c \dagger,$$

where  $W$  is the same for all liquids at corresponding temperatures. At the critical point the right-hand side becomes zero, and since  $\rho_1$  is finite at the critical point,  $W$  must also

become zero. The limiting value of the ratio  $\frac{p - T p_c / T_c}{W}$

may be written  $p_c \cdot V_c$ , where  $V_c$  is a numerical constant, and the equation at the critical point becomes

$$p_c = \frac{1}{V_c} (\Sigma \sqrt{m_1})^2 \left( \frac{\rho_c}{m} \right)^{7/3}.$$

Combining this equation with the above equation we have

$$V_c W p_c \left( \frac{\rho_1}{\rho_c} \right)^{7/3} = p - \frac{T p_c}{T_c},$$

or  $p = n_4 p_c$ , where  $n_4$  is the same for corresponding temperatures, which is the required result.

\* Phil. Mag. June 1910, p. 845.

† Phil. Mag. Dec. 1909, p. 903.

The equation  $p = n_4 p_c$  is of theoretical importance, as  $n_4$  is a function of  $n_1, n_2, n_3$ , and the equation therefore gives the relation between  $p, \rho_1, \rho_2$ , and  $T$ . It could at once be written out if the exact form of the arbitrary function in the law of attraction were known. It will be of interest to develop the equation as far as possible. The value of  $n_4$  is  $(V_c W n_1^{7/3} + n_3)$ , from one of the above equations. Referring again to the paper quoted above, we have  $W = N - \frac{T}{T_c} N_c$ , where  $N_c$  is a numerical constant, and

$$N = \frac{n_3}{n_1^{7/3}} \int \frac{B^2 n_1^{2/3}}{a n_3^2} \cdot dn_3,$$

where

$$a = \frac{1}{n_2} - \frac{1}{n_1} = \frac{n_1 - n_2}{n_1 n_2},$$

changing some of the symbols to those used in this paper. The quantity  $B^2$  which occurs in the equation for the latent heat is a function of  $n_1, n_2$ , and  $n_3$ ; its form would be known if that of the arbitrary function in the law of attraction were known. From the way it is obtained it is likely to be a series\*. Writing  $B^2 = \psi_3(n_1, n_2, n_3)$ , the above equation reduces to

$$p = \left\{ V_c \int \frac{\psi_3(n_1, n_2, n_3) \cdot n_2 n_1^{5/3}}{(n_1 - n_2) n_3^2} \cdot dn_3 + 1 - N_c V_c n_1^{7/3} \right\} n_3 p_c.$$

This can be changed into an equation involving  $\rho_1, \rho_2$ , and  $T$ , by means of the equations  $\rho_1 = n_1 \rho_c$ ,  $\rho_2 = n_2 \rho_c$ ,  $T = n_3 T_c$ . Thus the part of the right-hand side of the equation not under the integral sign at once reduces to

$$\frac{T p_c}{T_c} \left( 1 - N_c V_c \left( \frac{\rho_1}{\rho_c} \right)^{7/3} \right).$$

The quantities  $n_2, n_1$  under the integral sign can be expressed in terms of  $n_3$  by means of the results already obtained and the expression integrated, and  $\frac{T}{T_c}$  then substituted for  $n_3$ .

This cannot yet, however, be effected in practice since we do not yet know the exact form of the function  $\phi_2$ . Without performing this operation, the equation connecting the

\* Phil. Mag. May 1910, pp. 793-794.

quantities  $p, \rho_1, \rho_2, T$ , may be written

$$p = \frac{T}{T_c} p_c \left( 1 - N_c V_c \left( \frac{\rho_1}{\rho_c} \right)^{7/3} \right) + \frac{T p_c}{\rho_c^{5/3}} \int \frac{\psi_3 \left( \frac{\rho_1}{\rho_c}, \frac{\rho_2}{\rho_c}, \frac{T}{T_c} \right) \cdot \rho_2 \rho_1^{5/3} \cdot dT}{(\rho_1 - \rho_2) T^2}.$$

The law of corresponding states can also be proved for all possible states of matter. From thermodynamics we have

$$\left( \frac{dQ}{dp} \right)_{T \text{ const.}} = T \left( \frac{dv}{dT} \right)_{p \text{ const.}}$$

where  $dQ$  is the amount of heat given out by a mass of matter of volume  $v$  at a temperature  $T$ , when the pressure changes by  $dp$ . Now  $Q = pv + u$ , where  $u$  is the internal energy of the substance. The internal energy we will take (as usual) equal to the internal heat of evaporation at constant temperature into a vacuum. According to the formula for the latent heat used in a previous part of the paper (deduced from the law of molecular attraction), this is for unit mass equal to

$$\phi_3 \left( \frac{x_d}{x_c}, \frac{T}{T_c} \right) \cdot \frac{\rho^{4/3}}{m^{7/3}} (\Sigma \sqrt{m_1})^2,$$

where  $x_d$  is the distance of separation of the molecules of the matter and  $\rho$  its density. This expression for the internal energy we will write for shortness  $H C \rho^{4/3}$ , where

$$C = \frac{(\Sigma \sqrt{m_1})^2}{m^{7/3}}.$$

We have then

$$\begin{aligned} \left( \frac{dQ}{dp} \right)_T &= v + p \frac{dv}{dp} + C \left( \frac{dH}{dp} \rho^{4/3} + \frac{4}{3} H \rho_{1/3} \frac{d\rho}{dp} \right) \\ &= \left[ v + \left( \frac{dv}{dp} \right) \left\{ p + C \left( \frac{dH}{dv} \frac{1}{v^{4/3}} - \frac{H}{3v^{7/3}} \right) \right\} \right]_T = T \left( \frac{dv}{dT} \right)_p. \end{aligned}$$

Let us express the quantities  $p, v$ , and  $T$  in terms of their critical values thus

$$p = \phi p_c, \quad v = \alpha v_c, \quad T = \beta T_c.$$

The above equation may then be written

$$\left[ \alpha + \phi \cdot \left( \frac{d\alpha}{d\phi} \right) + \frac{C}{p_c v_c^{7/3}} \left( \frac{d\alpha}{d\phi} \right) \frac{1}{\alpha^{7/3}} \left( \frac{dH}{d\alpha} \cdot \alpha - \frac{4}{3} H \right) \right]_{\beta} = \beta \left( \frac{d\alpha}{d\beta} \right)_{\phi},$$

or  $\frac{CD}{p_c v_c^{7/3}} = E$ , where  $D$  and  $E$  are functions of  $\alpha, \beta$ , and  $\phi$ .

At the critical point, when  $\alpha=1$ ,  $\beta=1$ ,  $\phi=1$ , let the numerical limiting values of D and E be denoted by  $D_c$  and  $E_c$ , so that  $\frac{CD_c}{p_c v_c^{7/3}} = E_c$ . Eliminating  $v_c$ , C, and  $p_c$ , we have  $\frac{D}{D_c} = \frac{E}{E_c}$ , an equation which contains the quantities  $\alpha$ ,  $\beta$ , and  $\phi$  only. This proves the law of corresponding states in general.

Meslin has shown (*C. R.* 116, 135, 1893) that if the equation of state contains as many constants as there are variables (*i. e.* three—volume, temperature, pressure), it can be reduced to an equation involving  $\alpha$ ,  $\phi$ , and  $\beta$  only, or the law of corresponding states applies in that case to substances. Since we have established the law in a different way, we deduce that the equation of state must contain three constants.

The equation 
$$\frac{CD_c}{p_c v_c^{7/3}} = E_c$$

on substituting for  $p_c$ ,  $V_c$ , and C, from the equations

$$p = \phi p_c, \quad v = \alpha v_c, \quad \text{and} \quad c = \frac{(\sum \sqrt{m_1})^2}{m^{7/3}},$$

may be written

$$p = M^2 \left( \frac{\rho}{m} \right)^{7/3} (\sum \sqrt{m_1}),$$

where  $M^2$  is the same for all substances at corresponding states. This equation we have previously obtained for a liquid in contact with its saturated vapour. We now see that it applies in general.

The equation

$$T_1 = H^2 \left( \frac{\rho_1}{m} \right)^{4/3} (\sum \sqrt{m_1})^2,$$

where  $H^2$  has the same value for all substances at corresponding states, was also previously shown to apply to a liquid in contact with its saturated vapour\*. Now if we substitute for the quantities  $T_1$ ,  $\rho_1$ , from the equations  $\rho_1 = a\rho$ ,  $T_1 = bT$ , where T and  $\rho$  refer to any given state of matter, we obtain the equation

$$T = H_a \left( \frac{\rho}{m} \right)^{4/3} (\sum \sqrt{m_1})^2,$$

\* The equation was deduced with the help of thermodynamics from the general law of attraction quoted at the beginning of the paper.



where  $H_a$  is the same for corresponding states. This follows since the above two equations may be written

$$n_1 p_c = a x \rho_c, \quad \text{and} \quad n_2 T_c = b \beta T_c,$$

and therefore

$$a = \frac{n_1}{\alpha}, \quad \text{and} \quad b = \frac{n_2}{\beta},$$

or  $b$  and  $a$  have the same values at corresponding temperatures. Thus the equation in question is proved for all states of matter.

From this and the above equation we obtain  $pv = \frac{IRT}{m}$ ,

which applies to all states of matter, where  $I$  is the same at corresponding states.

The general equation of state can be deduced from one of the foregoing equations, but it is of a form which is not of any use. We have obtained

$$v + (p + k) \frac{\delta v}{\delta p} = T \frac{\delta v}{\delta T},$$

where

$$k = C \left( \frac{dH}{dv} \frac{1}{v^{4/3}} - \frac{4}{3} \frac{H}{v^{7/3}} \right),$$

and is therefore a function of  $v$  and  $T$ . This is a linear partial differential equation, and the Lagrangian subsidiary equations are therefore

$$\frac{dv}{v} = \frac{dT}{T} = \frac{dp}{p+k}.$$

An independent integral of these equations is  $\frac{v}{T} = A$ , and

let the other be denoted by  $\psi_5(T, v, p) = B$ . Then an integral of the partial differential equation is given by

$$\psi_6 \left( \frac{v}{T}, \psi_5(T, v, p) \right) = 0,$$

where  $\psi_6$  is an arbitrary function. One of the conditions which determines the form of the arbitrary function is that the equation must approximate to  $pv = \frac{RT}{m}$  when the matter is in the gaseous state and the pressure is lowered.

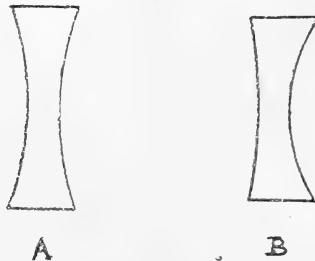
London, December 19, 1910.

XLI. *The Problem of the Uniform Rotation of a Circular Cylinder in its Connexion with the Principle of Relativity.*

By W. F. G. SWANN, D.Sc., A.R.C.S., Assistant Lecturer in Physics at the University of Sheffield\*.

EHRNFEST (*Phys. Zeit.* Nov. 1909) considers the problem of the uniform rotation of a circular cylinder in its relation to the principle of Relativity. He remarks that according to that principle, each element of the circumference moving with linear velocity  $v$  should contract in the ratio  $\left(1 - \frac{v^2}{C^2}\right)^{\frac{1}{2}}$ ,  $C$  being the velocity of light, while since each element of the radius is moving in a direction perpendicular to its length, the radius should not alter in dimensions. He points out that the co-existence of these two conditions is impossible. In the *Philosophical Magazine* (ser. 6, vol. xx. no. 115, p. 92), Stead and Donaldson have suggested that the explanation is to be found, at any rate in the case of a thin disk, by assuming the disk to become cup-shaped when rotating, and they have calculated the shape of the cup as a function of the velocity of rotation which is necessary to ensure non-violation of the above conditions. They suggest, moreover, that in the case of a cylinder of appreciable length, the cylinder more probably becomes strained. It must be observed, however, that the assumption of the cup-shaped body involves a displacement of the particles of the disk in a direction perpendicular to its initial plane, the relative displacement of the particles varying with the distance from the axis of rotation; and it seems that this displacement of the particles would be as inconsistent with the assumed principle of relativity as a contraction of the radius would be. Further, suppose that instead of considering a flat disk, we consider a disk which when not

Fig. 1.



rotating has a shape the section of which is shown in fig. 1, A. If this disk is set in rotation, and there is to be

\* Communicated by the Author.

no alteration of its dimensions perpendicular to its median plane, and if say the right side is to buckle so as to satisfy the assumed principle of relativity, the left side will have to take the shape shown in section in fig. 1, B, so that the condition of the particles on the left side will not be consistent with the principle. Again, the solution of the problem is not to be found in the disk becoming strained, for a strain would involve the alteration of the radius, and the final state of the disk would be one inconsistent with the conditions assigned above, which conditions are required by the principle of relativity if this problem is one to which the principle of relativity can be applied in the form which it has been applied.

The object of the present paper is to show that there is no fundamental difficulty in the question as far as its applications to ordinary matter is concerned, and in fact that the apparent inconsistencies arise from neglect of considering all the phenomena involved.

In the first place we notice that though each point of the circumference of the cylinder is moving with constant velocity, the problem is not one of constant velocity in a straight line: each particle has an acceleration towards the centre. As a matter of fact, we know that if the disk were set in rotation, it would not contract at all, it would expand owing to centrifugal force, to an extent depending on the elastic properties of the material. It would seem useless to endeavour to limit the case to that of a rigid body, for the very alterations of dimensions which we are discussing are alterations which are necessary to secure equilibrium of the internal forces and motions when the body is set in motion as a whole, *and the alteration in dimensions due to centrifugal force is as necessary for the maintenance of internal equilibrium as is the contraction of which we are speaking.* If we look upon all internal forces, cohesion included, as being of electromagnetic origin, a complete electromagnetic solution of the problem, if it could be obtained for our rotating cylinder, should show us that the cylinder actually does expand, the centrifugal force phenomenon being in fact involved in the electromagnetic scheme. It would be impossible, however, to obtain such a solution, unless we knew the nature of the electronic distributions and motions in the cylinder when at rest, because the solution would, as we know from experience, involve the elastic properties of the material, which on our view are totally determined by the nature of the electronic distributions and motions. I think the real solution of the apparent inconsistency is to be found in the following.

If we set a piece of matter in motion in a straight line with constant velocity, the electromagnetic theory leads to the conclusion that whatever may be the nature of the internal electronic, atomic, and molecular motions, the system as a whole will contract in the ratio  $\left(1 - \frac{v^2}{C^2}\right)^{\frac{1}{2}}$  in the direction of motion in such a way to insure that at corresponding times the  $x$  component of any electron shall be  $\left(1 - \frac{v^2}{C^2}\right)^{\frac{1}{2}}$  times the  $x$  component of the electron in the fixed system \*. Of course, so far as we are concerned in our notions of the constitution of matter as founded on the electromagnetic theory, all differences of elastic properties, &c. are to be attributed to differences of the strengths, distribution, and motions of the electrons. The curious thing is that the above contraction is absolutely independent of the nature of the elastic properties of the matter considered, *i. e.*, it is independent of the nature of the electrons and of their motions in the system †. Now when we come to consider problems of motion other than motion in a straight line with uniform velocity, it may be, and in fact certainly will be, that the final condition of affairs will depend not only on the motion imparted to the system but also on the motions which the electrons had in the system when at rest.

It is interesting to attempt to treat the problem of uniform rotation on the principle adopted by Sir Joseph Larmor for uniform translatory motion; for in spite of the fact that the principle of relativity is a principle which is postulated independently of its complete verification from the electromagnetic theory, it must nevertheless be looked upon as being suggested by direct argument, and it is of interest to trace the course of an analogous line of argument for the case of uniform rotation.

Let us first briefly review the problem of uniform translatory motion in the manner given by Larmor ('Æther and Matter,' pages 167-177).

\* Strictly speaking this is not true: the electromagnetic theory really only leads to this conclusion when the system discussed is one in which all the electrons in the fixed system are absolutely devoid of motion (see final paragraph of this paper).

† Really this fact is not as curious as it seems when we realize that the conclusion as derived directly from the electromagnetic theory only strictly holds for one particular kind of system, *viz.*, one in which the electrons have no orbital motions. It is the postulation of the principle as holding for all systems which makes the result seem so startling.

Suppose our system is moving parallel to the axis of  $x$  with velocity  $v$ . Let  $f, g, h$  be the ætherial displacement vectors,  $a, b, c$  the magnetic induction vectors, and  $C$  the velocity of light. The electromagnetic equations when referred to a system of axes at rest in the æther are of the form

$$4\pi \frac{\partial f}{\partial t} = \frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}, \quad -(4\pi C^2)^{-1} \frac{\partial a}{\partial t} = \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \quad (1)$$

with similar equations for  $\frac{\partial g}{\partial t}, \frac{\partial h}{\partial t}, \frac{\partial b}{\partial t}, \frac{\partial c}{\partial t}$ .

When we transform to axes  $x', y', z'$  moving with the system so that

$$x', y', z' = (x - vt, y, z) \quad t' = t,$$

we obtain, in view of the fact that  $\frac{d}{dt}$  becomes  $\frac{d}{dt'} - v \frac{d}{dx'}$ ,

$$\left( \frac{\partial f}{\partial x'} + \frac{\partial g}{\partial y'} + \frac{\partial h}{\partial z'} = 0 \text{ and } \frac{\partial a}{\partial x'} + \frac{\partial b}{\partial y'} + \frac{\partial c}{\partial z'} = 0 \right),$$

a set of equations of which a typical pair are

$$\left. \begin{aligned} 4\pi \frac{\partial f}{\partial t'} &= \frac{\partial c}{\partial y'} - \frac{\partial(4\pi v g)}{\partial y'} & -(4\pi C^2)^{-1} \frac{\partial a}{\partial t'} &= \frac{\partial h}{\partial y'} + \frac{\partial\left(\frac{vb}{4\pi C^2}\right)}{\partial y'} \\ -\frac{\partial b}{\partial z'} - \frac{\partial(4\pi v h)}{\partial z'} & & -\frac{\partial g}{\partial z'} + \frac{\partial\left(\frac{vc}{4\pi C^2}\right)}{\partial z'} & \end{aligned} \right\} (2)$$

Now it will be seen that the terms on which  $\frac{\partial}{\partial y'}$  operates may be grouped together, a similar remark applying respectively to the terms on which  $\frac{\partial}{\partial x'}$  and  $\frac{\partial}{\partial z'}$  operate in the other equations. On writing

$$\left. \begin{aligned} a', b', c' &= (a, b + 4\pi v h, c - 4\pi v g) \quad . \quad . \quad . \\ f', g', h' &= \left( f, g - \frac{vc}{4\pi C^2}, h + \frac{vb}{4\pi C^2} \right) \quad . \quad . \quad . \end{aligned} \right\} (3)$$

our equations take the typical form

$$4\pi \frac{\partial f}{\partial t'} = \frac{\partial c'}{\partial y'} - \frac{\partial b'}{\partial z'} \quad -(4\pi C^2)^{-1} \frac{\partial a}{\partial t'} = \frac{\partial h'}{\partial y'} - \frac{\partial g'}{\partial z'}. \quad (4)$$

In virtue of the relations (3) the elimination of  $f, g, h$  from the equations may be completed, for example

$$g = g' + \frac{v}{4\pi C^2} (c' + 4\pi v g),$$

so that on writing  $\epsilon^{-1}$  for  $1 - \frac{v^2}{C^2}$  we obtain

$$\epsilon^{-1} g = g' + \frac{vc'}{4\pi(C^2)} \dots \dots \dots (5)$$

Finally, on writing

$$a_1, b_1, c_1 = \epsilon^{-\frac{1}{2}} a', b', c'; \quad f_1, g_1, h_1 = \epsilon^{-\frac{1}{2}} f', g', h';$$

$$t_1 = \epsilon^{-\frac{1}{2}} \left( t' - \frac{v}{C^2} \epsilon x' \right)$$

our equations revert to the original type (1), the typical equation becoming

$$4\pi \frac{\partial f_1}{\partial t_1} = \frac{\partial c_1}{\partial y_1} - \frac{\partial b_1}{\partial z_1}, \quad -(4\pi C^2)^{-1} \frac{\partial a_1}{\partial t_1} = \frac{\partial h_1}{\partial y_1} - \frac{\partial g_1}{\partial z_1} \dots (6)$$

The remainder of the argument then follows as given in 'Æther and Matter' (pages 174-177), the above being as much as we need abstract for the purpose in hand.

Now let us turn to the problem of the uniform rotation of a cylinder, and let us choose coordinates  $\theta, r, z$ . Let  $f, g, h, a, b, c$  be the corresponding ætherial displacement, and magnetic vectors. Maxwell's equations may then be written down in terms of  $\theta, r, z$ . It will only be necessary for our purpose to write down one pair

$$4\pi \frac{\partial f}{\partial t} = \frac{\partial c}{\partial r} - \frac{\partial b}{\partial z} \quad -(4\pi C^2)^{-1} \frac{\partial a}{\partial t} = \frac{\partial h}{\partial r} - \frac{\partial g}{\partial z}$$

If we transform to a system of coordinates  $\theta', r', z'$ , moving with the cylinder, so that  $\frac{\partial}{\partial t}$  becomes  $\frac{\partial}{\partial t'} - \omega \frac{\partial}{\partial \theta'}$ ,  $\omega$  being the angular velocity, then in view of the fact that

$$\frac{\partial f}{r' \partial \theta'} + \frac{\partial g}{\partial r'} + \frac{\partial h}{\partial z'} = 0 \quad \text{also} \quad \frac{\partial a}{r' \partial \theta'} + \frac{\partial b}{\partial r'} + \frac{\partial c}{\partial z'} = 0,$$

the above equations give

$$4\pi \frac{\partial f'}{\partial t'} = \frac{\partial c}{\partial r'} - r' \frac{\partial}{\partial r'} (4\pi v g) \quad - (4\pi C^2)^{-1} \frac{\partial a}{\partial t'} = \frac{\partial h}{\partial r'} + r' \frac{\partial}{\partial r'} \left( \frac{vb}{4\pi C^2} \right)$$

$$- \frac{\partial b}{\partial z'} - r' \frac{\partial}{\partial z'} (4\pi v h) \quad - \frac{\partial g}{\partial z'} + r' \frac{\partial}{\partial z'} \left( \frac{vc}{4\pi C^2} \right)$$

corresponding equations of course holding for

$$\frac{\partial g}{\partial t'}, \frac{\partial h}{\partial t'}, \frac{\partial b}{\partial t'}, \frac{\partial c}{\partial t'}$$

When we reached this stage in the case of uniform rectilinear motion (see equations 2), we were able to put

the terms on which  $\frac{\partial}{\partial y'}$  operated together, and similarly for

the terms on which  $\frac{\partial}{\partial x'}$  and  $\frac{\partial}{\partial z'}$  operated. Considering

the corresponding case in our present problem, however, we

see that we cannot for instance put  $\frac{\partial c}{\partial r'} - r' \frac{\partial}{\partial r'} (4\pi v g)$  in the

form  $\frac{\partial c'}{\partial r'}$ , unless we know  $g$  as a function of the variables ;

and similarly for the other equations it would be necessary to know  $f$  and  $h$ ,  $a$ ,  $b$ , and  $c$ , as functions of the variables in order to accomplish this, *i. e.*, it would be necessary to know the strengths, distributions, and motions of the electrons in the system. Again, even supposing that these were known so that we could put our equations in the form corresponding to set (4),  $a'$ ,  $b'$ ,  $c'$ ,  $f'$ ,  $g'$ ,  $h'$ , being, however, functions of  $r$ ,  $\theta$ ,  $z$ ,  $t$ , which involve the strengths, distribution, and motions of the electrons in the system, we should, nevertheless, be unable in general to perform the step indicated by (5) in such a way as to obtain for example  $g$  in the form

$$g = Ag' + Bc',$$

where  $A$  and  $B$  are *constants*.  $A$  and  $B$  would have to be functions of the variables depending on the conditions peculiar to the particular system we were studying. If  $A$  and  $B$  cease to be constants, of course the whole endeavour to perform the step analogous to that performed in (6) breaks down, and the equations cannot in general revert to the type for the fixed system of coordinates. It may be remarked that even in the case of uniform rectilinear motion

treated by Larmor, it is necessary, in order to show that the moving system which he discusses is the system which the fixed system becomes when set in motion, to restrict the problem to the case in which all the electrons in the system at rest are absolutely devoid of motion. Such a system could not of its own accord be in equilibrium, without imposed constraints, and the introduction of such constraints of non-electromagnetic origin presents a serious difficulty to the mind, in view of the fact that the whole theory rests on the assumption of the completeness of the electromagnetic scheme. The difficulty of imagining the constraints may to a certain extent be overcome, and this fact, combined with the definite experimental fact to hand in the Michelson and Morley experiment, provides us with the confidence we need to postulate the principle of relativity for uniform rectilinear motion; but it would seem that in view of the difficulties encountered in this case, the postulation of the principle for non-rectilinear motion can hardly be expected not to lead us into mutually contradictory conclusions.

XLII. *On the Value of the Pitot Constant.* By J. D. FRY, Lecturer and Demonstrator in Physics, and A. M. TYNDALL, M.Sc., Lecturer in Physics in the University of Bristol\*.

THE method of measuring the velocity of a stream of gas by means of a Pitot tube is well known.

A "pitot" is generally a fine tube of which one end is open and directly facing the stream, and the other end is connected to a pressure-gauge. A pressure is thereby set up in the pitot, the value of which is a function of the velocity ( $v$ ) of the stream at its end.

If  $P$  is the excess pressure over the static pressure at the open end of the pitot (referred to in what follows as the "pitot pressure") and if  $\rho$  is the density of the gas, then

$$v = K \sqrt{\frac{2P}{\rho}}$$

$K$  is the Pitot Constant, and previous experiments have shown that its value is not far from unity and constant over a considerable range in velocity. Thus among the more recent determinations of  $K$  are those by Threlfall †, who

\* Communicated by the Authors.

† Threlfall, Proc. Inst. Mech. Eng. 1904.



obtained 0.974 between the limits 10 and 60 ft. per sec., and Stanton\*, whose value was 1.03. Stanton's number was obtained for use in his work on the resistance of plane surfaces in a uniform current of air, and was known by him to be a few per cent. too high.

During 1903-05 a long series of determinations of  $K$  in air by two different methods was carried out by the authors; but as discordant results were obtained, which at the time received no explanation, the work remained unpublished †.

The authors have since reinvestigated the question, and it will be shown below that by applying certain corrections the results by the two methods may be brought into agreement.

In the first method, the pitot was moved at constant speed through stationary air. For this purpose, it was fixed to the end of a revolving arm and connected to one side of a pressure-gauge by a tube leaving the arm at the centre of rotation. The pitot and the arm were at right angles to one another so that the former always pointed along the direction of motion.

Due to the relative motion of the air and the pitot there will be a pitot pressure  $P$  given by

$$v = K \sqrt{\frac{2P}{\rho}},$$

where  $v$  is in this case the velocity of the pitot.

At the same time, however, due to centrifugal force the hydrostatic pressure in the arm at the centre of rotation will be below that at the pitot end by an amount  $p$ .

If  $\frac{dp}{dr}$  is the slope of the pressure in the arm at a radius  $r$ , then

$$dp = \rho \omega^2 r dr,$$

where  $\omega$  is the angular velocity of the rotating arm whence

$$p = \rho \omega^2 \int_0^R r dr = \rho \frac{v^2}{2}$$

where  $v$  has the same meaning as before. The resultant

\* Stanton, Proc. Inst. Civil Eng. 1903.

† These experiments were first undertaken in collaboration with Professor A. P. Chattock, who identified himself so much with them that his name is omitted from this publication only at his express wish. The authors take this opportunity of thanking him for the part that he took in the earlier work and for his guidance in its more recent stages.

350 Messrs. J. D. Fry and A. M. Tyndall on  
 pressur:  $P - p$  indicated by the gauge is therefore

$$\frac{\rho v^2}{2K^2} = \frac{\rho v^2}{2}.$$

From this  $K$  may be calculated. It will be noticed that if its value is exactly unity there will be no gauge deflexion, and for this reason this method possesses an advantage over the second method.

In the second method the pitot was placed parallel to the axis of a pipe of radius  $R$  through which air was flowing at a constant rate, and a series of pressure readings obtained by moving the pitot along a diameter of the pipe.

If  $p_r$  is the pitot pressure and  $v_r$  the velocity of flow at a distance  $r$  from the centre of the pipe, then as before

$$v_r = K \sqrt{\frac{2p_r}{\rho}}.$$

The quantity of air flowing per second through a ring of thickness  $dr$  at this radius is therefore given by

$$dq = K \sqrt{\frac{2p_r}{\rho}} 2\pi r dr.$$

Integrating this across the section of the pipe, we have

$$Q = \int_0^R K \sqrt{\frac{2p_r}{\rho}} 2\pi r dr,$$

where  $Q$  is the total quantity of air flowing through the pipe per second.

Unless  $K$  has a constant value at all velocities it cannot be taken outside the integral, since the velocity of the air in the pipe varies at different points along a diameter. If  $K$  is not constant and this is done, the values of  $K$  obtained are purely artificial and depend not only on the quantity of flow through the pipe but also on the distribution of that flow across the section.

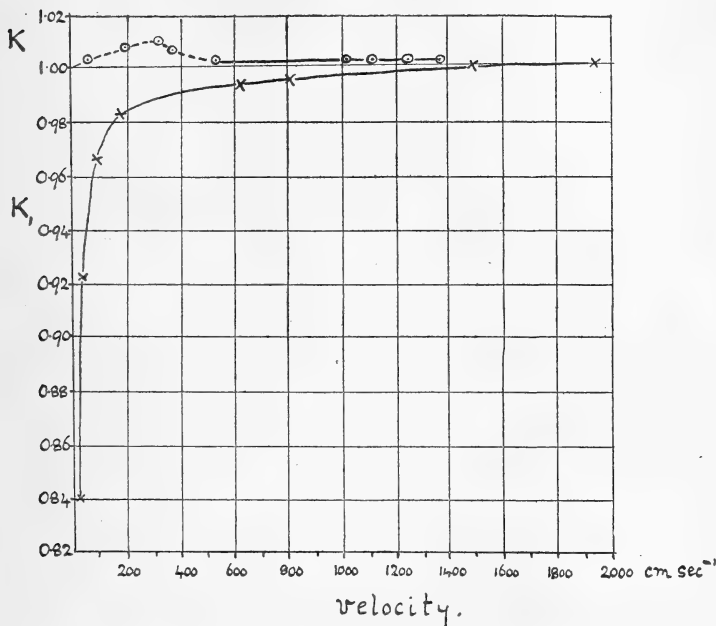
The symbol  $K_1$  will be used in what follows for the artificial values obtained in this way.

The only justification for using this method to measure  $K$  itself lay in the fact that previous experimenters had found that  $K$  had a constant value over a limited range, and there appeared to be no theoretical reason why any marked change should occur when the range was extended.

The observed values of  $K$  obtained by the centrifugal method and of  $K_1$  by the pipe method are shown in

Curves I, where  $K$  (circles) and  $K_1$  (crosses) are plotted with the velocity of the pitot relative to the air, the velocity in the case of the crosses being the mean rate of flow of the air along the pipe.

CURVES I.



The results of the centrifugal method show that except for a 1 per cent. variation at low velocities, where the experimental difficulties were considerable,  $K$  was 1.002 at all the velocities investigated.

$K_1$ , on the other hand, although about 1.00 at higher velocities, rapidly decreased with decreasing rate of flow.

Now  $K_1$  only differs from  $K$  when the latter is not constant, so that assuming that the centrifugal value of  $K$  is correct the two curves should be identical. There must, therefore, have been some source of error in the pipe method which was not present in the centrifugal method.

No information could be obtained on this point from the work of others, because the velocities at which the disagreement was appreciable were far below those which had been previously investigated. In view of this fact and of the general importance of the question of the flow of air through pipes, a reinvestigation of the pipe method was undertaken.

*Experimental Details.*

*Centrifugal Method.*—One of the chief difficulties of putting this method into practice arises from the fact that the rotating arm inevitably produces a vortex in the air of the laboratory, with the result that error is introduced into the observations in two ways :—

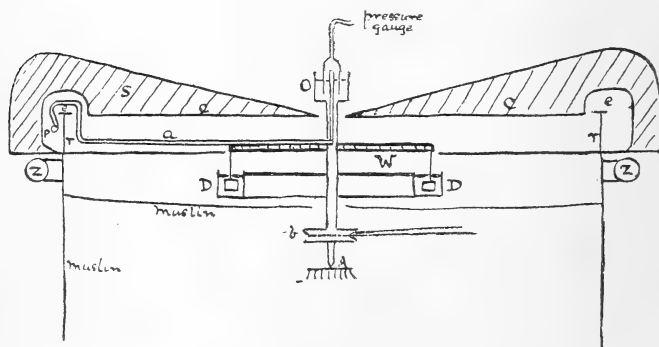
(a) Owing to the motion of the air, the relative speed of pitot and air is not so great as the rate of rotation of the arm would imply.

(b) The static pressure of the air falls towards the centre of rotation, and the pressure at the open end of the pressure-gauge is not necessarily the same as the static pressure along the path of the pitot.

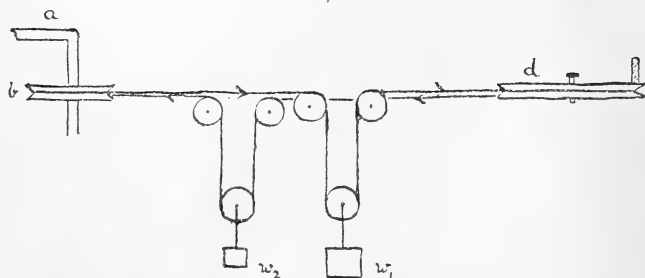
It was, therefore, important (a) to bring the air to rest along the path of the pitot, and (b) to connect the open end of the gauge in such a way that the pressure upon it was the average of the static pressures encountered by the pitot.

Fig. 1.

(a)



(b)



As far as could be ascertained after many trials, the following arrangement fulfilled these two conditions (fig. 1, a and b).

The pitot  $P$  was fixed on the end of a tube leading along an arm  $a$  and passing at the centre of rotation through an oil joint  $o$  to one limb of a sensitive pressure-gauge. The oil vessel was mounted so that the arm  $a$  and the pitot could be rotated in a horizontal plane. The pitot was horizontal and at right angles to the arm and moved round in a circle of 5 feet radius close to a cardboard rim  $r$ , the outer surface of which was covered with a series of paper flutings.

Above  $a$  and over the greater part of the apparatus was a horizontal cardboard sheet  $c$  to which were attached at equal intervals along radii 16 vertical cardboard structures  $s$ ; the pitot passed through holes in these during its rotation. Consistent with free motion of the pitot these holes were made as small as possible. The flutings and screens were effective in breaking up the air-flow in the neighbourhood of the pitot. A horizontal band of card  $e$ , 3 inches wide was placed round the rim  $r$  to stop as much as possible motions under the sheet  $c$  from affecting the air outside  $r$ . For this reason also muslin curtains were hung from  $r$  down to the ground and horizontally across beneath the rotating arm  $a$ . The other end of the pressure-gauge was connected to a static pressure tube, which in this case was a 2-inch cardboard tube  $z$  encircling the apparatus and fitted with 16 openings. In this way the static pressure taken was the average of the static pressures around the pitot's path.

In order to obtain as uniform a motion of the arm as possible, a viscous resistance to the motion was supplied and the arm subjected to a constant turning couple. In the figure,  $W$  is a heavy iron wheel from which the metal vanes  $D$  were supported; these vanes dipped into an annular dash-pot containing oil and supplied the necessary friction.

The device for obtaining a constant couple is shown in fig. 1, *b*. Each half of the belt from the pulley  $b$  passed to the turning table  $d$  over two fixed pulleys between which was a movable pulley carrying a weight. These two floating weights  $w_1$  and  $w_2$  were unequal, and the table was rotated at such a rate as to keep them suspended freely at the same level. When they were balanced in this way the pitot moved at a constant velocity; the rate of turning could be varied by altering the difference between the weights.

The pressure-gauge employed was of the Chattock-Fry type used in the work of Stanton and described in his paper (*loc. cit.*)\*. This is a form of tilting U-tube gauge in which

\* An improved form of this gauge but modified so that its water surfaces are trapped by mercury, has also been described in *Phil. Mag.* [6] xix. p. 451.

a bubble of water in benzene is used as an indicator, the line of separation between the two liquids being viewed through a microscope and kept coincident with the cross wire in the centre of the field of view by motions of a screw. It is not advisable to allow the indicator surface to be much displaced, because, although it is restored to its original position by tilting, slight changes in surface tension cause it to take a temporary set. Hence it is necessary to insert in the gauge a clamping-tap which is not in general opened until the gauge has been set at the anticipated reading. The particular gauge employed when working well would indicate a pressure difference of one ten-thousandth of a millimetre of water.

The results already summarized in Curves I show that for velocities 600–1400 cms. per sec.,  $K$  is quite constant and equal to 1.002. At lower velocities there was a one per cent. variation in  $K$  which the authors never succeeded in eliminating by modification of screens and so forth. There was, however, always a certain very small amount of pumping action at the oil joint during rotation due to imperfections in the pivot  $A$  upon which the rotating system turned, and it is possible that this was responsible for the observed change in  $K$ . The pressure effects of such action would be proportional to the first power of the velocity of rotation, and hence compared with pitot readings would only be appreciable at low velocities. It seems highly probable, at any rate, that this change was due to experimental errors, and that one is, therefore, justified in taking 1.002 as the true value of  $K$ .

*Pipe Method.*—It is by this method that the recent re-investigation has been made, but except in the case of very small mean velocities the apparatus only differed from that of 1903–05 in a few particulars.

In this method one is met with the difficulty that the mean pressure of the gas is above or below the atmospheric pressure. This has been met by other experimenters in various ways, the one adopted by the authors being to use a differential pressure-gauge, one limb of which went to the pitot and the other to a tube in the side of the pipe, and in the same plane as the end of the pitot. It was assumed that if this side tube ended flush with the inner surface of the pipe, it would measure the static pressure of the gas in this plane, except when the velocity was considerable or when the pitot and side tube were too near to an end of the pipe.

In the earlier experiments the pitot and its static pressure tube were placed near the mouth of the pipe, and it was

possible as a result that the static pressure was not eliminated from the recorded pitot pressures owing to a curvature in the equipressure surfaces across the pipe. The pitot in the recent work was placed as in fig. 2 A, 8 inches down a

Fig. 2 B.

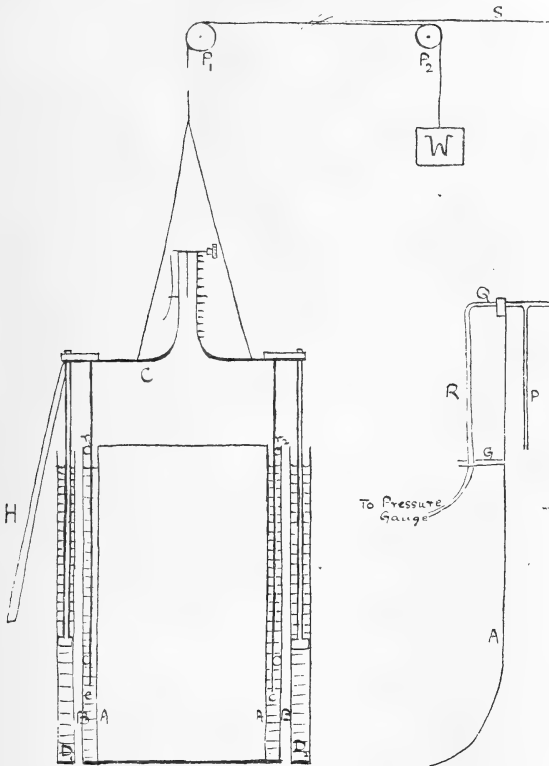
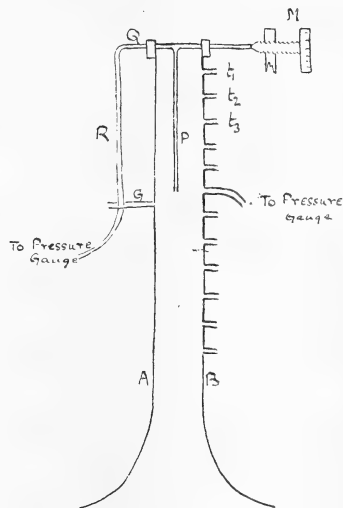


Fig. 2 A.



pipe AB. The pipe, which was 2 inches in diameter and 18 inches long was fitted with 12 side tubes  $t_1, t_2, \&c.$ , and was screwed to a funnel at its lower end, the joint being made as smooth as possible on the inside. The static pressure slope down the pipe could thus be obtained by connecting a gauge to any two of the side tubes, the rest being kept closed. It was found that at the highest mean velocities used, this slope was only uniform down the pipe when a piece of wire gauze was introduced across the section of the pipe at its base. For work at low velocities the gauze was not necessary.

The pitot P could be set at any position along a diameter

by a micrometer head M, and was kept on that diameter by the arm R passing through a guide G.

The same type of pressure-gauge as above was used for low velocities, but for higher velocities a much less sensitive tilting gauge of a simple U type was employed.

The pipe and its funnel were fitted to the top of a gasometer consisting of three zinc cylinders A, B, and C, each 4 ft. 6 ins. in height and arranged as in fig. 2 B, so that C could slide on rollers R<sub>1</sub>, R<sub>2</sub>, &c., in the annular space between A and B. This space was filled with oil. The diameter of C was 1 ft. 6 ins., those of A and B differing from this by about an inch.

The gasometer could be raised about 4 feet and then allowed to fall, its velocity being varied by altering the balancing weight W, which was attached to C over the pulleys P<sub>1</sub> and P<sub>2</sub>.

Piston rods immersed in dash-pots D<sub>1</sub> and D<sub>2</sub> containing treacle tended to render the fall uniform, and the gasometer could also be accelerated or retarded on its downward path by hand application to the rod H.

When a uniform velocity had been attained, the air was expelled through the experimental pipe at a uniform rate which could be determined from the dimensions of the gasometer and its rate of fall.

The latter was measured by clipping to the suspending wire a paper strip S which passed under a recording pen. This was made to mark seconds on the paper by an electromagnetic device connected with a standard clock.

To eliminate those parts of the fall during which the velocity was not correct, a second recording pen was placed side by side with the first. It was arranged that by completing an electric circuit, the observer could indicate with this, those parts of the fall which were uniform enough to be used in calculation. The procedure was then as follows:—

The pitot was placed in a known position and the gauge was clamped and set at a reading which by the process of trial and error had been calculated to be right for the velocity required, and for the particular position of the pitot on a diameter. The gasometer having been drawn up was set free; when it had reached a more or less constant velocity, the gauge was opened and the line kept on the cross wire either by accelerating or retarding the gasometer by means of the handle H.

The experiment was then repeated with the pressure gauge commutated.

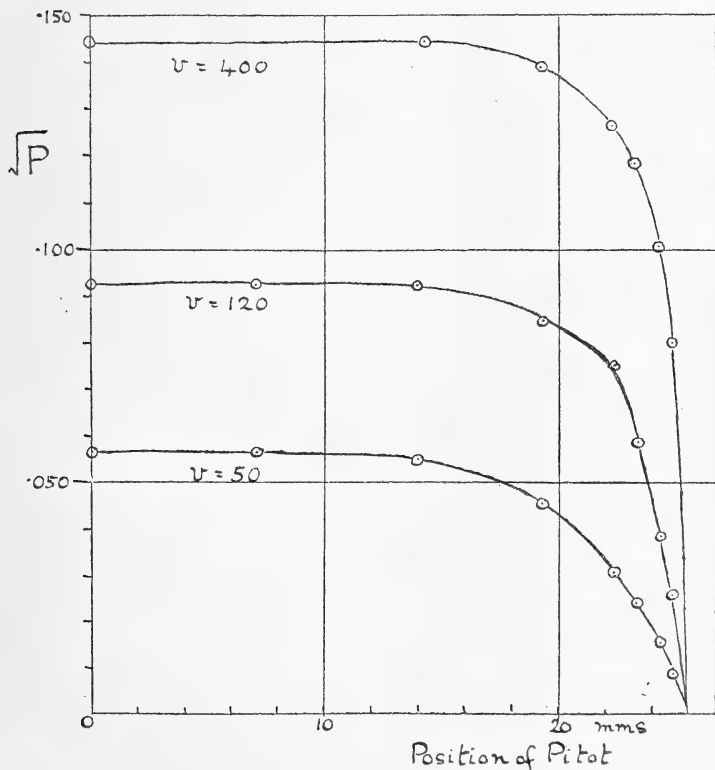
After a little practice it was generally possible to keep the



gasometer at the right velocity for the greater part of its fall, and the required rate of this was given by the average distance between the seconds marks on the paper strip.

*Determination of  $K_1$ .*—Having determined by the process of trial and error the distribution of pitot pressure across a diameter for a given mean rate of flow, the value of  $K_1$  at that velocity can be determined from curves drawn with  $\sqrt{P}$  as ordinates and positions of the pitot along a radius as abscissæ, as mentioned above. Some examples of these are given in Curves II for mean velocities 400, 120, and 50 cms.

CURVES II.



per second respectively,  $P$  being measured in centimetres of water. The values for the ordinates of  $v = 120$  and  $v = 50$  are doubled and trebled respectively for convenience in plotting.

The pitot in these cases was a circular metal tube of radius 0.59 mm. The results verified the general form of

the curve given by crosses in Curves I, and it appeared, therefore, that assuming that there was a source of error in the original work with this method, it had not been eliminated.

It being desirable to check these readings in some way, attempts were made to obtain them at still lower mean velocities when the motion in the pipe would no longer be eddy but stream line.

With stream-line motion the distribution of velocity across a diameter is parabolic. Assuming then that  $K$  is constant and of value 1.00 (as obtained by the centrifugal method), the curve of root pitot pressure across a diameter must also be a parabola. The forms of these parabolas of root pressure for various mean velocities (referred to in what follows as "theoretical parabolas") may be obtained by calculation and compared with the experimental curves.

Pitot pressure readings, however, for mean velocities lower than 50 cms. per second could not be obtained with any accuracy, because the motion of the gasometer caused an irregular bending of the rubber connexions from the pipe to the pressure-gauge, which at low velocities resulted in fluctuations of the observed pressures comparable with the readings themselves. In the hope of obtaining stream-line flow for such velocities a one-inch pipe was substituted for the two-inch, but it was found that the motion of the gasometer was then too slow to be maintained regularly.

The falling gasometer method was, therefore, abandoned, and the stream of air obtained by fixing the pitot and a surrounding one-inch pipe to the top of a 50-gallon tank and gradually filling the latter with water from the mains.

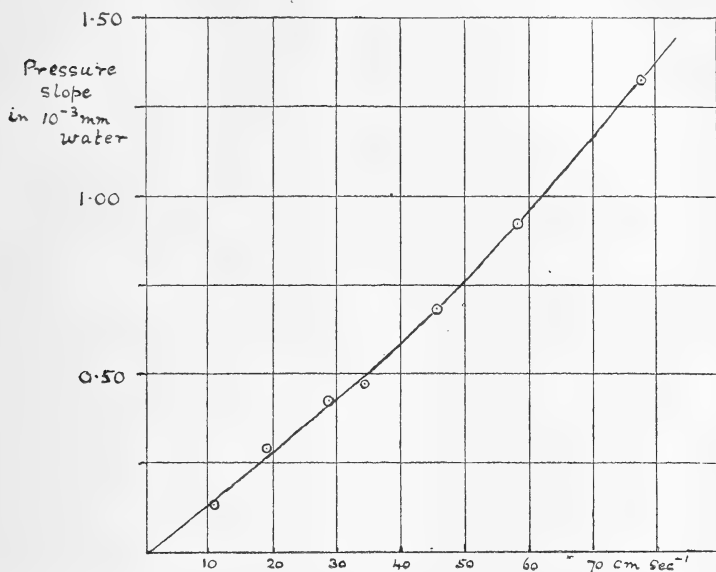
The rate of flow of the air-stream was obtained from the rate of change of the water-level in the tank, and this remained fairly constant provided no water was being drawn elsewhere in the building. The pitot pressures were obtained by the change in gauge reading on commutation. The absence of moving rubber and the more constant mean stream velocity increased the working sensitiveness of the gauge over fifty-fold, so that fairly accurate readings could be obtained to as low a mean velocity as 6 cms. per second.

It was not possible now to allow for the presence of water vapour with certainty in calculating the density of the air in an experiment, because it was probably not accurate to assume that the air was saturated or that its humidity was that of the atmosphere; this, however, was a small correction and negligible compared with the discrepancy investigated.

As it was necessary to know whether the motion was really stream-line at these low velocities, the static pressure

slope along the length of the pipe was measured for different mean rates of flow by means of the side tubes  $t_1, t_2$  &c. shown in fig. 2 A. The results are given in Curve III.

CURVE III.



It is not easy to say when the curve leaves the straight, that is to say when stream-line motion ceases, but it seems justifiable to state that of the various rates used stream-line motion was certainly present at  $v=6.3$  cms. per second, possibly present at  $v=28.5$ , and not present at  $v=76$ . It is now possible to compare the experimental  $\sqrt{P}$  curves with the theoretical parabolas for these three mean rates of flow. This is shown in Curves IV, where  $\sqrt{P}$  is plotted with the position of the pitot on a diameter.

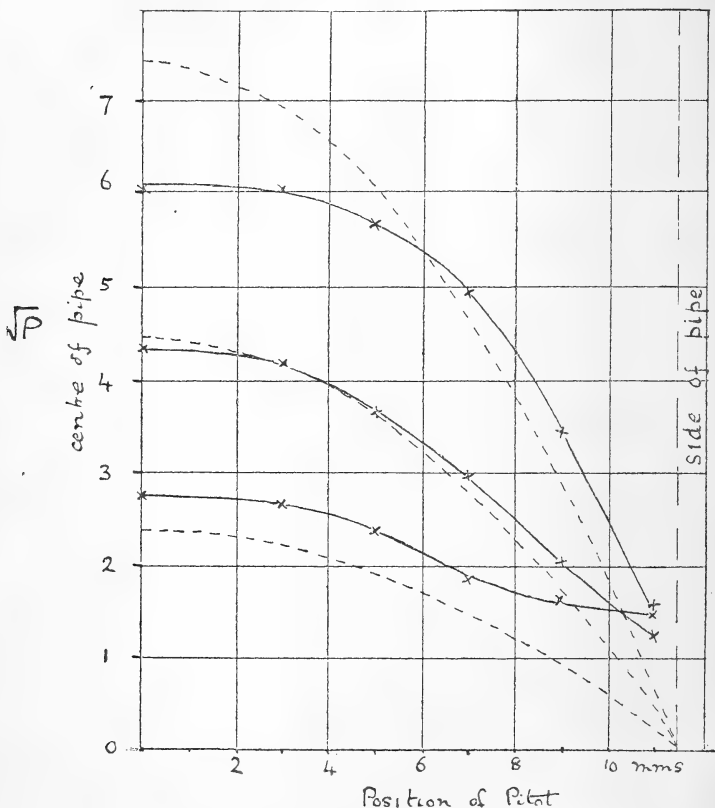
The crosses are the experimental points and the dotted lines are the three corresponding theoretical parabolas obtained by assuming that  $K=1.00$  and constant.

It will be seen that in general there is no coincidence of an experimental curve with the corresponding parabola; in fact in the former it is only by neglecting the readings near the side of the pipe that a smooth curve through the origin can be obtained at all, and there seems to be no reason why these readings should be neglected.

The results at  $v=6.3$  cms. per second, where the motion was certainly stream-line, show that the pitot pressures across

a diameter do not fit the theoretical values, and are moreover too high to give a pitot constant of 1.00. The same is probably true of the other velocities, though the curves for  $v=76$  cms. per second give no information on this point because the flow was evidently eddy.

CURVES IV.



When working at the lowest velocities it was important to eliminate any errors due to temperature effects which might be comparable with the small pressures observed. Thus if the connecting tubes leading from the pressure-gauge to the pitot are not at the same temperature, a spurious pressure will be present which will only be eliminated by commutation if such takes place close to the pitot; at first it was assumed that uniform temperature was obtained when the connecting tubes were twisted together, but in all the later work the commutator was placed as near the pitot as possible

(about 6 inches beneath it). This change had, however, no effect on the results.

Referring to the part PQR of the pitot tube and its connexions in fig. 1, an uncommutated pressure could arise, if the arm P was at a different temperature to the arm R. Such a difference was possible because the air in contact with the water in the tank could quite conceivably differ in temperature from the general air of the room, in which case the arm P would then be affected during the flow. This was tested by taking direct and reversed zero readings of the pressure-gauge immediately after the flow had ceased, and before the temperature of the arm P could have changed appreciably. Sometimes a slight effect seemed to be present, but it was variable in its direction. It might possibly have caused some of the fluctuations which always occurred in the readings.

*Analysis of Results.*

Curves IV as they stand do not bring out the essential nature of the discrepancies between the observed and the ideal pitot pressures; a simplification is introduced if, instead of taking the square roots of the pitot pressures, the actual pitot pressures themselves are compared with the squares of the corresponding points on the theoretical parabola—that is to say, with an ideal set of pressures for  $K = 1.00$  and constant.

The results of this analysis are shown in the following table,

		$r=0$	$r=3$	$r=5$	$r=7$	$r=9$	$r=11$	$\bar{x}$
$v=6.3$ cms. per sec.	$P_1$ .....	1.4	1.3	1.0	0.6	0.5	0.3	
	$P_2$ .....	1.0	1.0	0.7	0.4	0.2	0.0	
	$x$ .....	0.4	0.3	0.3	0.2	0.3	0.3	
$v=11.0$ .	$P_1$ .....	3.5	3.2	2.5	1.5	0.9	0.5	
	$P_2$ .....	3.1	2.7	2.0	1.2	0.5	0.0	
	$x$ .....	0.4	0.5	0.5	0.3	0.4	0.5	
$v=15.6$ .	$P_1$ .....	6.4	5.9	4.7	2.9	1.5	0.8	
	$P_2$ .....	6.1	5.3	4.0	2.3	0.9	0.0	
	$x$ .....	(0.3)	0.6	0.7	0.6	0.6	0.8	
$v=22.0$ .	$P_1$ .....	12.2	11.2	8.6	5.6	2.7	1.2	
	$P_2$ .....	12.4	10.8	8.1	4.8	1.9	0.1	
	$x$ .....	(-0.2)	(0.4)	(0.5)	0.8	0.8	1.1	
$v=28.3$ .	$P_1$ .....	19.5	17.4	13.7	9.0	4.1	1.5	
	$P_2$ .....	20.3	17.5	13.3	7.9	3.0	0.2	
	$x$ .....	(-0.8)	(-0.1)	(0.4)	1.1	1.1	1.3	

where  $P_1$  stands for pitot pressures measured in thousandths of a millimetre of water,  $P_2$  squares of corresponding parabolic pressures, and  $x$  the difference between them for five different mean velocities. These are arranged in vertical columns, each giving the values for a given position ( $r$ ) of the pitot measured from the centre of the pipe along a radius in millimetres.

For the two lowest velocities, in which it will be seen from Curve III that the flow is stream-line, the value of  $x$  is roughly constant for all positions of the pitot and increases as the mean rate of flow through the pipe increases. This is true also for  $v=15.6$  cm. per sec. (if the reading at  $r=0$  is omitted). For  $v=22.0$  and  $x=28.3$  this does not appear to be the case, although it may be so for positions close to side.

Now Curve III shows that the static pressure velocity line leaves the straight gradually, that is to say the transition from stream-line to eddy motion is also gradual. It would appear, therefore, that with increasing velocities of flow, eddy motion, which must first appear at the centre of the pipe's section, gradually spreads towards the walls. Consequently it is reasonable to suppose that, at the velocity  $v=15.6$  cms. per sec. stream-line motion existed in the pipe except near its centre, but that at  $v=22$  and  $r=28.3$  eddy motion had extended to a radius of at least 5 mms. This effect would account for the low and sometimes negative readings of  $x$  in that region. In these cases, therefore, we may assume that the value of  $x$  near the side is that which would have held across the pipe had stream-line persisted. The average values of  $x$  ( $\bar{x}$ ) for different velocities are given in the last column, those values of  $x$  in brackets being omitted on the eddy motion assumption in taking means. The relation between  $\bar{x}$  and the mean rate of flow of air through the pipe is shown in Curve V, where the values of  $\bar{x}$  are plotted as ordinates and rates of flow as abscissæ.

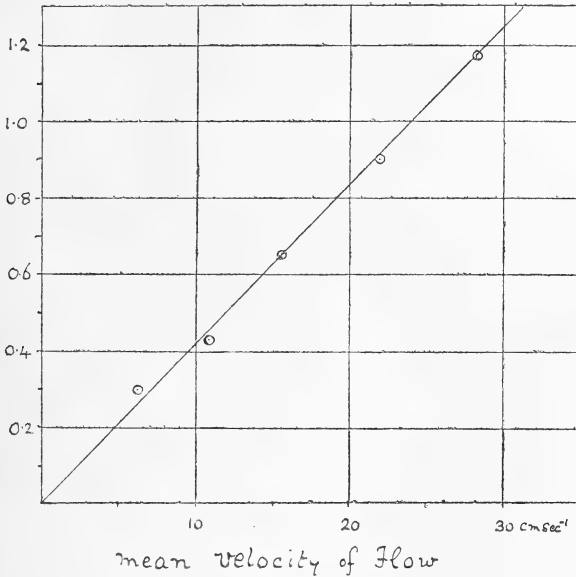
Considering the extreme smallness of  $x$ —the largest value obtained being only just over a thousandth of a millimetre of water—the results are fairly definite and show that  $\bar{x}$  is proportional to the mean rate of flow—at any rate at these low velocities.

The authors have not succeeded in obtaining a satisfactory explanation of this phenomenon, but the following considerations present themselves :—

1. The  $x$  effect could not be accounted for by an uncommutated temperature effect such as is mentioned on page 361 because the difference of pressure which such would produce would, unlike  $x$ , be constant for all velocities of flow.

2. When a pitot is placed in a non-uniform flow the pitot pressure which is measured is an average taken over the

CURVE V.



area of the end of the pitot, and differs from that at its centre by an amount depending upon the area of the pitot and the distribution of velocity over its end. The amount of this error was calculated assuming that the velocity changed linearly across a diameter of the pitot, but it was found to be quite negligible at low rates of flow for the size of pitot used.

3. It seemed feasible that superposed on the main disturbance of the flow in the neighbourhood of the pitot there might be a smaller secondary disturbance which, if it depended on a power of the velocity less than the square, might become comparable with the observed pressure when the velocity was very low, even though it were negligible at higher velocities as seemed likely from the work of other experimenters.

Definite evidence, however, has been obtained against this view for the case of the round pitot used in the above. Thus the pressure readings were unaltered by threading over the pitot a piece of rubber tubing so that the whole pitot stem was thickened about 4 diameters.

4. Lastly, the presence of  $x$  might be the result of a slight suction at the static pressure tube. It is well known that if a piece of thin-walled tubing is placed in a stream so as to lie with its long axis perpendicular to the flow, such a suction does occur, and it may even amount to as much negative pressure as a pitot placed at the same point would give of positive pressure. It has, however, generally been supposed that this is removed by either providing the tube with a wide flange at its mouth or by placing it, as in the present work, with its end flush with the inner surface of the pipe. In support of this there is the work of Heenan and Gilbert (Proc. Inst. Civil Eng. vol. cxxiii.) and Threlfall (*loc. cit.*), and also the fact that in the present work  $K$  is approximately the same as  $K_1$  within the range of velocity used by most experimenters.

It is conceivable, however, that suction is not completely eliminated in these ways. If, for instance, a residual suction is present which is a function of a power of the velocity less than the square, it might easily only be comparable with the pitot pressures at very low velocities, and might therefore have escaped previous detection.

For instance, making the very arbitrary assumption that  $x$  is proportional to  $v$  at all velocities—the values of  $K$  calculated by correcting all the pitot pressures accordingly does not differ more than 1 per cent. from 1.00 throughout the whole range from 6 to 2000 cms. per second.

On the other hand, if there were suction at the static pressure tube, a change in the size and shape of that tube might be expected to affect the value of  $\bar{x}$ . But no difference of pressure was observed when the gauge was connected to any two of the following holes made at the same level in the side of the tube—circular holes 1 mm. and 4 mm. diameter, rectangular holes  $1.5 \times 3$  and  $3 \times 1.5$  mm. respectively. On the whole, therefore, the suction hypothesis does not seem feasible, though the fact that  $\bar{x}$  is proportional to the rate of flow drives one to locate the effect at the wall of the pipe and not at the pitot itself. It is possible that experiments on the pitot pressures in pipes of different sizes may throw some light on the subject.

#### *Pitots of different shapes.*

Before it was realized that the pressure readings across a diameter were not in accordance with theory, an attempt was made to examine the motion of the air at the side of the pipe. Even when eddy motion is present the flow close to the



walls must still be stream-line and must break up into eddies at some unknown distance from the walls. It seemed that the place where this change occurred and its relation to the mean velocity of the air, could be determined by using a pitot which was narrow enough in section to approach the walls very closely. The above results show this is not possible unless the pressure readings are corrected for the  $x$  effect, which in practice is impossible to do very accurately; but at the same time, the attempt brought out a new and interesting fact.

The results of previous experimenters with various pitot tubes have led to the generalization that the size and shape of a pitot has no effect upon the value of the pitot constant  $K$ . The authors found, however, that a very fine pitot does not show the same pitot pressures as a larger one say 1 mm. in diameter. For instance, values of the pitot constant at various velocities have been taken with a very fine rectangular pitot 0.2 mm. by 2.0 mm., and at high velocities this gives a mean value for  $K$  of about 1.10. This pitot was obtained by flattening the end of a brass tube and grinding down its edges to a thickness of 0.05 mm.

A pitot with about the same opening 0.1 mm. by 2.0 mm. but thicker walls each 0.2 mm. gave, however, a normal value  $K=1.00$  at high velocities.

A circular glass pitot 1.77 mm. diameter and with walls 0.27 mm. in thickness, also gave a value of  $K$  several per cent above unity.

Now close to the edges of the pitot, the pressure due to the current of air will fall off rapidly and will be considerably less than that at its centre. This edge effect has been thoroughly investigated by Stanton (*loc. cit.*) for the case of plates placed in uniform currents of air. In an ordinary pitot it has no effect on the observed pressure, which is the average pressure over the internal area of the end of the pitot tube, because either the walls of the pitot are relatively thick or its area is considerable. In the above pitots with high constants, the area influenced by this edge effect was evidently a measurable fraction of the whole area and the resultant pitot pressures were, therefore, all too small.

In support of this view the authors found that if a small piece of mica was fitted on flush with the end of one of these pitots so that the pitot was then a circular plate 2 mm. diameter with a slot at its centre, the value of  $K$  obtained was normal 1.00.

Further, when the wedge pitot was touching the side of the pipe, the pitot pressure was actually greater than when it

was 0.1 mm. from the side. This was evidently due to a change in the disturbance of the flow ; in the former case the flow was restricted to one side of the pitot only, while in the latter it was more uniform. This effect was still more marked in the case of a pitot formed by making a hole in the side of a tube stretching across a diameter of the pipe : the flow past such a pitot close to the wall of the pipe is still further modified.

Owing to the constriction introduced into one side of the gauge connexions by these narrow pitots, it was difficult to obtain accurate results because the two sides of the gauge never filled up at equal rates ; a small disturbance, such as a slight change in velocity or a sudden draught, thus often produced a temporary displacement of the gauge-surfaces which, at low velocities, was comparable with the pitot pressure itself. In fact all the low velocity work, even with the large pitot, could only be undertaken on days of atmospheric calm. The disturbing effects were decreased, but never eliminated, (1) by placing a capillary tube in the circuit on the other side of the gauge, and (2) by making the volumes of the connecting tubes and of the air spaces in the gauge as small as possible. Many more readings will, however, be necessary before a quantitative discussion of the readings with wedge pitots can be presented, but the results as yet obtained seem to show a similar apparent drop in  $K$ , with decreasing velocity.

#### SUMMARY.

1. The value of the pitot constant obtained by a centrifugal method is 1.002 and is constant between velocities 60 and 1400 cms. per second.
2. The method of determining the pitot constant by measuring the distribution of pitot pressure across a pipe is unreliable at small velocities.
3. If a small correction proportional to the mean rate of flow through the pipe is added to all the pitot pressures across a section of the pipe, this method also leads to a value 1.00 between the limits 6-2000 cms. per second.
4. A very small pitot possessing very thin walls gives in general a pitot constant several per cent. above unity. This is readily explained in terms of effects known to be present at the edges of plane surfaces placed normal to a current of air.

XLIII. *On Restricted Lines and Planes of Closest Fit to Systems of Points in any Number of Dimensions.* By E. C. SNOW, M.A.\*

STATEMENT OF THE PROBLEM.

1. **T**HE theory of the lines and planes of closest fit to systems of points when no restriction is placed upon those lines and planes has been developed by Prof. Pearson in various papers † and is of frequent application. The connexion of these lines and planes with the formulæ of the theory of multiple correlation is indicated in those papers. If the criterion of "closest fit" is that the sum of the squares of the deviations from the line or plane *measured in the direction of the "dependent" variable* is to be a minimum, the equation of the line or plane is identical with the corresponding multiple correlation formula. If the sum of the squares of the deviations *measured at right angles to the line or plane* is a minimum (and this, from the purely geometrical point of view, is the more satisfactory criterion), the result is not of such a simple form, but the determinant from which the mean square residual is obtained is similar to the multiple correlation determinant.

While working on certain vital statistics, it was desired to obtain a formula connecting the "dependent" variable with the "independent" variables when the values of all the variables were known at the beginning and end of a certain range, and the correlation between "dependent" and "independent" variables at all intermediate points was a maximum. Thus, if  $x_0$  denote the "dependent" variable, and  $x_1, x_2, \dots, x_n$  the "independent" ones, we require to make the correlation of  $x_0$  with  $x_1, x_2, \dots, x_n$  a maximum, with the condition that when  $x_1, x_2, \dots, x_n$  take up the values  $p_{11}, p_{21}, \dots, p_{n1}, p_{12}, p_{22}, \dots, p_{n2}$  respectively,  $x_0$  is to take the values  $p_{01}$  and  $p_{02}$ .

A similar problem occurs in certain branches of Physics, especially in connexion with solutions and alloys. A property—*e. g.*, the freezing-point—of a pure substance may be definitely known, and it is required to investigate the behaviour of that property as certain amounts of some other substance or substances are added. Fixed conditions will be imposed upon the law which is to be investigated by the known properties of the pure substance. The law, then,

\* Communicated by Prof. Karl Pearson, F.R.S.

† See Phil. Mag. Nov. 1901, pp. 559 *et seqq.*; Phil. Trans. vol. clxxxvii, A, pp. 301 *et seqq.*

has to give the best fit to the observations made of the property as definite quantities of the other substances are added. Two examples of such cases are given from the figures of certain alloys (§7 below).

The idea is capable of generalization, and the theory for the general case will be investigated. Looked at from the point of view of correlation, we shall require to assume a linear law connecting  $x_0$  with  $x_1, x_2, \dots, x_n$ , and shall make the sum of the squares of the deviations of the actual observations from this linear law *measured in the direction of*  $x_0$  a minimum, making use of the exact conditions which are imposed on the law. This will be first investigated. But a better geometrical fit to the observations will be obtained by measuring the deviations *perpendicular* to the "plane" given by the linear law, and this will be worked out subsequently.

#### ANALYTICAL INVESTIGATION.

##### *First Method.*

2. Let there be  $n$  "independent" variables and  $(k+1)$  conditions connecting them with the "dependent" variable.  $(k+1)$  is necessarily less than  $n$ . Measuring from one of these fixed conditions, we can assume our law is

$$x_0 = a_1x_1 + a_2x_2 + \dots + a_nx_n, \quad \dots \quad (1)$$

with the  $k$  conditions

$$\left. \begin{aligned} p_{01} &= a_1p_{11} + a_2p_{21} + \dots + a_np_{n1}, \\ p_{02} &= a_1p_{12} + a_2p_{22} + \dots + a_np_{n2}, \\ &\dots \dots \dots \\ p_{0k} &= a_1p_{1k} + a_2p_{2k} + \dots + a_np_{nk}. \end{aligned} \right\} \dots \quad (2)$$

Then we want to make

$$V = S(x_0 - a_1x_1 - a_2x_2 - \dots - a_nx_n)^2,$$

the sum of the square of the deviations in the direction of  $x_0$ , a minimum, subject to the above conditions.

Hence we must have

$$0 = S(x_0 - a_1x_1 - a_2x_2 - \dots - a_nx_n)(x_1 \cdot da_1 + x_2 \cdot da_2 + \dots + x_n \cdot da_n), \quad (3)$$

with the conditions

$$0 = p_{1s} \cdot da_1 + p_{2s} \cdot da_2 + \dots + p_{ns} \cdot da_n. \quad (s = 1, 2, \dots, k.) \quad (4)$$

Multiply equations (4) by  $\lambda_s$  ( $s=1, 2, \dots, k$ ), and add to (3). Then, by the ordinary theory of maximum and minimum values, we know that the coefficient of  $da^t$  ( $t=1, 2, \dots, n$ ) in the equation so obtained must vanish. This gives the  $n$  equations

$$\begin{aligned} \lambda_1 p_{t1} + \lambda_2 p_{t2} + \dots + \lambda_k p_{tk} \\ = Sx_t(x_0 - a_1 x_1 - a_2 x_2 - \dots - a_n x_n) \\ = R_{0t} - a_1 R_{1t} - a_2 R_{2t} - \dots - a_n R_{nt}, \quad \dots \quad (5) \end{aligned}$$

if  $R_{uv} = S(x_u x_v) = R_{vu}$  for all positive integral values of  $u$  and  $v$ .

(5) gives  $n$  equations connecting the  $(n+k)$  unknowns,  $a_1, a_2, \dots, a_n, \lambda_1, \lambda_2, \dots, \lambda_k$ . (2) gives  $k$  other equations between the  $a$ 's, in which, however, the  $\lambda$ 's are absent. Thus we have the following set of equations from which to determine the  $a$ 's and the  $\lambda$ 's:—

$$\left. \begin{aligned} a_1 R_{1t} + a_2 R_{2t} + \dots + a_n R_{nt} + \lambda_1 p_{t1} + \lambda_2 p_{t2} + \dots + \lambda_k p_{tk} &= R_{0t}. \\ (t=1, 2, \dots, n.) \\ a_1 p_{1s} + a_2 p_{2s} + \dots + a_n p_{ns} &= p_{0s}. \\ (s=1, 2, \dots, k.) \end{aligned} \right\} (6)$$

Let  $\Delta$  denote

$$\begin{vmatrix} R_{00} & R_{10} & \dots & R_{n0} & p_{01} & p_{02} & \dots & p_{0k} \\ R_{01} & R_{11} & \dots & R_{n1} & p_{11} & p_{12} & \dots & p_{1k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ R_{0n} & R_{1n} & \dots & R_{nn} & p_{n1} & p_{n2} & \dots & p_{nk} \\ p_{01} & p_{11} & \dots & p_{n1} & 0 & 0 & \dots & 0 \\ p_{02} & p_{12} & \dots & p_{n2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{0k} & p_{1k} & \dots & p_{nk} & 0 & 0 & \dots & 0 \end{vmatrix},$$

a determinant of the order  $(n+k+1)$ .

Then the solutions of equations (6) are :

$$a_t = -\frac{\Delta_{t0}}{\Delta_{00}} \quad (t=1, 2, \dots, n) \quad \dots \quad (7)$$

and

$$\lambda_s = -\frac{\Delta_{(u+s),0}}{\Delta_{0,0}} \quad (s=1, 2, \dots, k) \quad \dots \quad (8)$$

$\Delta_{uv}$  being the first minor of the constituent of the  $(u+1)$ th column and  $(v+1)$ th row, and being positive or negative according as  $(u+v)$  is even or odd.

Thus the coefficients in the required formula can be found at once by the evaluation of a number of determinants of order  $(n+k)^*$ .

*Particular Cases.*

3. The simplification of the above results in a few particular cases will be useful.

(A)  $k=0$ .—In this case we have a plane passing through a single fixed point and closest fitting to a system of points. Here all the  $\lambda$ 's disappear, and the equation of the plane becomes

$$x_0 = a_1x_1 + a_2x_2 + \dots + a_nx_n,$$

where

$$a_t = -\frac{\Delta_{0t}}{\Delta_{00}},$$

$\Delta$  being

$$\begin{vmatrix} R_{00} & R_{10} & \dots & R_{s0} & \dots & R_{n0} \\ R_{01} & R_{11} & \dots & R_{s1} & \dots & R_{n1} \\ R_{02} & & & & & \vdots \\ \vdots & & & & & \vdots \\ R_{0n} & \dots & \dots & \dots & \dots & R_{nn} \end{vmatrix},$$

$R_{uv}$  being equal to  $R_{vu}$ , and is the sum of the products  $x_u \cdot x_v$  taken throughout the system of points.

If  $\bar{x}_s$  and  $\sigma_s$  be the mean and standard deviation of  $x_s$ , and  $r_{uv}$  the correlation between the coordinates  $x_u$  and  $x_v$ , we have

$$R_{uv} = S(x_u x_v) = N\sigma_u\sigma_v r_{uv} + N\bar{x}_u\bar{x}_v \quad \text{if } u \neq v,$$

and  $R_{uv} = N\sigma_u^2 + N\bar{x}_u^2 \quad \text{if } u = v.$

The analogy between  $\Delta$  and the determinant used in the

\* It is not difficult to show that, by first finding the  $a$ 's in terms of the  $\lambda$ 's from the first  $n$  of equations (6), and substituting these values in the last  $k$  of the same set—thus giving  $k$  equations for the  $\lambda$ 's,—the  $a$ 's can be found in a form involving only determinants of order  $n$ , though the number of determinants it is necessary to evaluate is increased. If  $k$  is large, this increase is considerable. The general result in this form will not be given, but it is exemplified in a particular case below (§ 3).

theory of multiple correlation is now clear, for  $a_t$  can be written  $-\frac{\Delta'_{t0}}{\Delta'_{00}}$ , where

$$\Delta' \equiv \begin{vmatrix} 1 + \frac{\bar{x}_0^2}{\sigma_0^2} & r_{01} + \frac{\bar{x}_0\bar{x}_1}{\sigma_0\sigma_1} & \dots & r_{0n} + \frac{\bar{x}_0\bar{x}_n}{\sigma_0\sigma_n} \\ r_{01} + \frac{\bar{x}_0\bar{x}_1}{\sigma_0\sigma_1} & 1 + \frac{\bar{x}_1^2}{\sigma_1^2} & & \vdots \\ r_{02} + \frac{\bar{x}_0\bar{x}_2}{\sigma_0\sigma_2} & \dots & 1 + \frac{\bar{x}_2^2}{\sigma_2^2} & \vdots \\ \vdots & & & \vdots \\ r_{0n} + \frac{\bar{x}_0\bar{x}_n}{\sigma_0\sigma_n} & \dots & \dots & 1 + \frac{\bar{x}_n^2}{\sigma_n^2} \end{vmatrix}$$

Thus this determinant can be derived from the multiple correlation determinant by increasing  $r_{st}$  in the latter by  $V_s \cdot V_t$ , and by increasing the constituents of the leading term by the corresponding  $V_s^2$ , where  $V_s$  and  $V_t$  are the coefficients of variation of the coordinates  $x_s$  and  $x_t$ . If the fixed point is at the mean of the system of given points,  $\Delta'$  becomes at once the ordinary multiple correlation determinant.

Putting  $n=1$ , we derive the two-dimensional case of a line passing through the origin and giving closest fit (*measuring in the direction of y*) to a system of points. The equation of the line is easily seen to be

$$y = \frac{R_{10}}{R_{11}} x = \frac{S(x \cdot y)}{S(x^2)} x \dots \dots \dots (\alpha)$$

Putting  $n=2$ , we reach the three-dimensional case of a plane passing through the origin and giving closest fit (*measured in the direction of z*) to a system of points. Its equation is

$$\begin{aligned} Z &= \frac{R_{12}R_{02} - R_{01}R_{22}}{R_{11}R_{22} - R_{12}^2} \cdot x + \frac{R_{12}R_{01} - R_{02}R_{11}}{R_{11}R_{22} - R_{12}^2} y \\ &= \frac{S(xy) \cdot S(yz) - S(xz) \cdot S(y^2)}{S(x^2) \cdot S(y^2) - \{S(xy)\}^2} x + \frac{S(xy) \cdot S(xz) - S(yz) \cdot S(x^2)}{S(x^2) \cdot S(y^2) - \{S(xy)\}^2} y. \quad (\beta) \end{aligned}$$

For values of  $n > 2$  it is more convenient to derive the coefficients direct from the determinant, and there is no need to write them in full.

(B)  $k=1$ .—Here we have the case of a plane—in  $n$  dimensions—passing through two fixed points and fitting most

closely a system of other points. We have in this case

$$\Delta = \begin{vmatrix} R_{00} & R_{10} & \dots & R_{n0} & p_0 \\ R_{01} & R_{11} & \dots & R_{n1} & p_1 \\ \vdots & \vdots & & \vdots & \vdots \\ R_{0n} & R_{1n} & \dots & R_{nn} & p_n \\ p_0 & p_1 & \dots & p_n & 0 \end{vmatrix},$$

$p_0, p_1, \dots, p_n$  being the coordinates of one fixed point relative to the other, which is taken as origin.

The coefficients in the required equation can be found from the above in the form of determinants of order  $(n+1)$ .

But in this case the first  $n$  equations of (6) become

$$a_1 R_{1t} + \dots + a_n R_{nt} = R_{0t} - \lambda p_t \quad (t=1, 2, \dots, n).$$

Solving these for  $a_1, a_2, \dots, a_n$  in terms of  $\lambda$ , we have

$$a_t = -\frac{\delta_{0t}'}{\delta_{00}},$$

where

$$\delta \equiv \begin{vmatrix} R_{00} & R_{01} & \dots & R_{0n} \\ R_{01} - \lambda p_1 & R_{11} & \dots & R_{1n} \\ \vdots & \vdots & & \vdots \\ R_{0n} - \lambda p_n & R_{1n} & \dots & R_{nn} \end{vmatrix} \\ = \delta' - \lambda \delta'',$$

where

$$\delta' \equiv \begin{vmatrix} R_{00} & R_{01} & \dots & R_{0n} \\ R_{01} & R_{11} & \dots & R_{1n} \\ \vdots & \vdots & & \vdots \\ R_{0n} & R_{1n} & \dots & R_{nn} \end{vmatrix}$$

and

$$\delta'' \equiv \begin{vmatrix} 1 & R_{01} & \dots & R_{0n} \\ p_1 & R_{11} & \dots & R_{1n} \\ p_2 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p_n & R_{1n} & \dots & R_{nn} \end{vmatrix};$$

so that

$$a_t = -\frac{\delta_{0t}' - \lambda \delta_{0t}''}{\delta_{00}} \quad (t=1, 2, \dots, n).$$



When these values are substituted in the equation giving the fixed condition, viz.

$$p_0 = a_1 p_1 + \dots + a_t p_t + \dots + a_n p_n,$$

$\lambda$  can be found, and from this the  $a$ 's can be completely determined. This method will be used in an example below.

No other particular cases of this method need be worked out in detail. We see that it is always possible to obtain a plane in  $n$ -dimensional space to pass through any number (less than  $n$ ) of fixed points and to be such that the sum of the squares of the deviations of any number of other points from the plane *measured in a fixed direction* is a minimum.

*Second Method.*

4. We have now to investigate the equation of the corresponding plane when the criterion for "closest fit" is that the sum of the squares of the deviations from the plane *measured at right angles to the plane* is the least possible. From a purely geometrical point of view this will give a closer plane to the system of points, but it will not give the regression of one variable on the others.

Let  $l_1 \dots l_n$  be the generalized direction-cosines of the plane, and take one of the points through which the plane has to pass as origin.

Then the equation of the plane is

$$l_1 x_1 + \dots + l_n x_n = 0, \dots \dots \dots (9)$$

the total number of variables being taken as  $n$  for convenience of notation. There will also be the conditions

$$l_1^2 + \dots + l_n^2 = 1 \dots \dots \dots (10)$$

and

$$\left. \begin{aligned} l_1 p_{11} + l_2 p_{21} + \dots + l_n p_{n1} &= 0, \\ l_1 p_{12} + l_2 p_{22} + \dots + l_n p_{n2} &= 0, \\ \dots \dots \dots \dots \dots \dots \dots &\dots \dots \dots \\ l_1 p_{1k} + l_2 p_{2k} + \dots + l_n p_{nk} &= 0, \end{aligned} \right\} \dots \dots \dots (11)$$

( $k + 1$ ) being the total number of fixed conditions.

The criterion for closest fit is

$$V = S(l_1 x_1 + \dots + l_n x_n)^2$$

to be minimum, subject to the above conditions.

Hence

$$0 = S(l_1 x_1 + \dots + l_n x_n)(x_1 dl_1 + \dots + x_n dl_n),$$

together with

$$\begin{aligned} l_1 dl_1 + \dots + l_n dl_n &= 0, \\ p_{11} dl_1 + \dots + p_{n1} dl_n &= 0, \\ \dots \dots \dots \dots \dots &\dots \dots \dots \\ p_{1k} dl_1 + \dots + p_{nk} dl_n &= 0. \end{aligned}$$

The conditions for a minimum give

$$\left. \begin{aligned} X_1 + \mu l_1 + \mu_1 p_{11} + \mu_2 p_{12} + \dots + \mu_k p_{1k} &= 0, \\ & \dots \dots \dots \\ X_n + \mu l_n + \mu_1 p_{n1} + \mu_2 p_{n2} + \dots + \mu_k p_{nk} &= 0, \end{aligned} \right\} \quad (12)$$

where  $\mu, \mu_1, \dots, \mu_k$  are undetermined constants, and

$$\begin{aligned} X_s &= S(l_1 x_1 + \dots + l_n x_n), x_s \\ &= l_1 R_{1s} + \dots + l_n R_{ns}, \end{aligned}$$

where, as before,

$$R_{uv} = S(x_u \cdot x_v) = R_v \cdot u.$$

Multiply equations (12) by  $l_1, l_2, \dots, l_n$  respectively, and add. Remembering the conditions (10) and (11), we at once obtain

$$\mu + l_1 X_1 + l_2 X_2 + \dots + l_n X_n = 0,$$

where  $l_1 \dots l_n$  have the values which make  $V$  a minimum. But then  $l_1 X_1 + l_2 X_2 + \dots + l_n X_n$  becomes  $V_m$ , the minimum value of  $V$ .

It follows that

$$\mu = -V_m.$$

Substituting this value of  $\mu$  in (12), we shall have with (11)  $(n + k)$  equations between  $l_1 \dots l_n, \mu_1 \dots \mu_k$ , and  $V_m$ . Hence we can eliminate the  $l$ 's and the  $\mu$ 's and obtain an equation to determine  $V_m$ .

Since  $X_s = l_1 R_{1s} + \dots + l_n R_{ns}$ ,

this eliminant can be written in the determinantal form

$$D \equiv \left| \begin{array}{ccccccccc} R_{11} - V_m & R_{21} & \dots & R_{n1} & p_{11} & p_{12} & \dots & p_{1k} \\ R_{12} & R_{22} - V_m & \dots & R_{n2} & p_{21} & p_{22} & \dots & p_{2k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ R_{1n} & R_{2n} & \dots & R_{nn} - V_m & p_{n1} & p_{n2} & \dots & p_{nk} \\ p_{11} & p_{21} & \dots & p_{n1} & 0 & 0 & \dots & 0 \\ p_{12} & p_{22} & \dots & p_{n2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{1k} & p_{2k} & \dots & p_{nk} & 0 & 0 & \dots & 0 \end{array} \right| = 0.$$

This is an equation of the  $(n - k)$ th degree in  $V_m$ . Its roots are necessarily positive (being the sum of a number of

squares of real quantities), and the smallest of these must be taken. When this value of  $V_m$  is substituted in (11) and (12),  $(n+k)$  homogeneous equations in  $l_1 l_2 \dots l_n, \mu_1 \mu_2 \dots \mu_k$  are obtained,  $(n+k-1)$  of which, together with (10), suffice to determine all the  $l$ 's and the  $\mu$ 's. As before, this can be done in determinant form, the order of the determinants involved being  $(n+k-1)$ .

*Particular Cases.*

5. Useful particular cases of the general formula are obtained by taking  $k=0$  and  $k=1$ .

(A') If  $k=0$ , we derive the case analogous to (A) above. The equation to determine  $V_m$  takes the well-known form

$$\begin{vmatrix} R_{11}-V_m & R_{21} & \dots & R_{n1} \\ R_{12} & R_{22}-V_m & \dots & R_{n2} \\ \vdots & \vdots & & \vdots \\ R_{1n} & R_{2n} & \dots & R_{nn}-V_m \end{vmatrix} = 0.$$

Putting  $n=2$ , we have the two-dimensional case, and  $V_m$  is the least root of the quadratic

$$\begin{vmatrix} R_{11}-V_m & R_{21} \\ R_{12} & R_{22}-V_m \end{vmatrix} = 0;$$

so that

$$2V_m = (R_{11} + R_{22}) - \frac{\{(R_{11} - R_{22})^2 + 4R_{12}^2\}^{1/2}}{\text{since } R_{12} = R_{21}},$$

and

$$2(R_{11} - V_m) = (R_{11} - R_{22}) + \{(R_{11} - R_{22})^2 + 4R_{12}^2\}^{1/2}.$$

Put

$$\frac{R_{11} - R_{22}}{\cos \theta} = \frac{2R_{12}}{\sin \theta} = \rho.$$

Then

$$2(R_{11} - V_m) = \rho(1 + \cos \theta) = 2\rho \cos^2 \frac{\theta}{2}$$

and

$$2R_{12} = 2\rho \sin \frac{\theta}{2} \cos \frac{\theta}{2}.$$

The first of equations (12) now gives

$$l_1 \cos \frac{\theta}{2} + l_2 \sin \frac{\theta}{2} = 0,$$

since the  $p$ 's vanish when  $k=0$ .

The equation of the line, therefore, is

$$x \sin \frac{\theta}{2} + y \cos \frac{\theta}{2} = 0,$$

where

$$\begin{aligned} \tan \theta &= \frac{2R_{12}}{R_{11} - R_{22}} \\ &= \frac{2S(xy)}{S(x^2) - S(y^2)}, \end{aligned}$$

and the notation is altered to agree with the usual form.

This value of  $\tan \theta$  gives rise to two values of  $\frac{\theta}{2}$ , each less than  $180^\circ$ . In a particular numerical example, however, it is not difficult to pick out the value required; while it can be verified that the other value corresponds to the "worst-fitting" line.

In cases of  $n > 2$  it is better to substitute the values of the  $R$ 's direct in the determinant above, and to find  $V_m$  by the usual methods of approximating to the roots of an equation.

(B') If  $k=1$ , the equation to determine  $V_m$  is

$$d \equiv \begin{vmatrix} R_{11} - V_m & R_{21} & \dots & R_{n1} & p_1 \\ R_{12} & (R_{22} - V_m) & \dots & R_{n2} & p_2 \\ \vdots & \vdots & & \vdots & \vdots \\ R_{1n} & R_{2n} & \dots & (R_{nn} - V_m) & p_n \\ p_1 & p_2 & \dots & p_n & 0 \end{vmatrix} = 0,$$

the origin and the point  $(p_1 p_2 \dots p_n)$  being the fixed points.

In this case, equations (12) take the form

$$\begin{aligned} l_1(R_{11} - V_m) + l_2 R_{21} + \dots + l_n R_{n1} &= -\lambda p_1, \\ &\dots \dots \dots \\ l_1 R_{1n} + l_2 R_{2n} + \dots + l_n (R_{nn} - V_m) &= -\lambda p_n. \end{aligned}$$

Hence  $l_i$  is proportional to  $d_i$ , the first minor of the constituent in the  $i$ th column and bottom row of  $d$ , and  $V_m$  is given the value which is the least root of  $d=0$ . Using (10), the actual values of the  $l$ 's can be found.

6. It will be seen from the foregoing analysis that the work involved in determining the "closest fitting" plane by

the second criterion is much greater, at any rate in all cases of  $n > 2$ , than that necessitated by the first criterion. In the second method, if  $(n - k)$  is three or more, the smallest root of an equation of the third or higher degree has to be approximated to. This in itself is no light task, and is not necessary in the first method. The methods do not necessarily lead to results at all alike (see Example 3, below), and only the terms of the particular question in hand can decide which method is to be used. The second gives the best geometrical fit, considered in a direction perpendicular to the plane. The first gives the "regression" plane—*i. e.*, the most probable value of one variable in terms of the others. This is the most frequently needed in practical cases, as is exemplified in Examples 1 and 2.

### 7. ILLUSTRATIONS.

I. The second column in Table I.\* gives the temperatures (Centigrade) at solidification of a series of alloys of iron and

TABLE I.

<i>x.</i> Percentage of Carbon present.	<i>y.</i> Temperature at end of solidification.	Temperature by 1st method to nearest degree.	Temperature by 2nd method to nearest degree.
·02	1470	1501	1501
·12	1470	1483	1483
·16	1465	1476	1476
·17	1450	1474	1474
·24	1448	1461	1461
·38	1416	1436	1436
·53	1404	1409	1409
·61	1394	1395	1394
·80	1351	1360	1359
·81	1351	1358	1357
1·31	1286	1268	1267
1·51	1244	1231	1230
1·85	1179	1171	1169
2·12	1110	1122	1120
2·21	1107	1105	1103

carbon. The percentage of carbon in the various alloys is given in the first column. The solidifying temperature of pure iron is 1505° C. Any curve, therefore, which attempts

\* The table is taken from a paper on "The Range of Solidification and the Critical Ranges of Iron-Carbon Alloys," by H. C. H. Carpenter, M.A., and B. F. E. Keeling, B.A., in the Journal of the Iron and Steel Institute, No. 1, for 1904.

to express a relationship between the percentage of carbon present and the solidifying point of the alloy should pass through the fixed point (0 %, 1505° C.). Up to 2 % of carbon a straight line seems to be the most likely fit. The two methods will therefore be applied to get a line to pass through the point (0 %, 1505° C.) and to fit the series of observations. It will be seen from the figure that up to a percentage of carbon of 0.5 % the results are irregular, but from that point up to 2 % the irregularities are small and within the limits of experimental error.

Measuring  $x$  positively from zero and  $y$  negatively from 1505, we find

$$S(x^2) = 19.0176,$$

$$S(xy) = 3438.41,$$

$$S(y^2) = 624986.$$

By the first method the equation of the line (measured from 0 % and 1505° C.) is

$$y = \frac{S(x \cdot y)}{S(x^2)} \cdot x \quad \text{or} \quad y = 180.801x.$$

The relationship between the temperature of solidification and percentage of carbon present is therefore

$$T = 1505 - 180.801x.$$

If we apply the second method, we find

$$\tan \theta = -0.0110035,$$

whence

$$\frac{\theta}{2} = 89^\circ 41' 5''.23$$

and

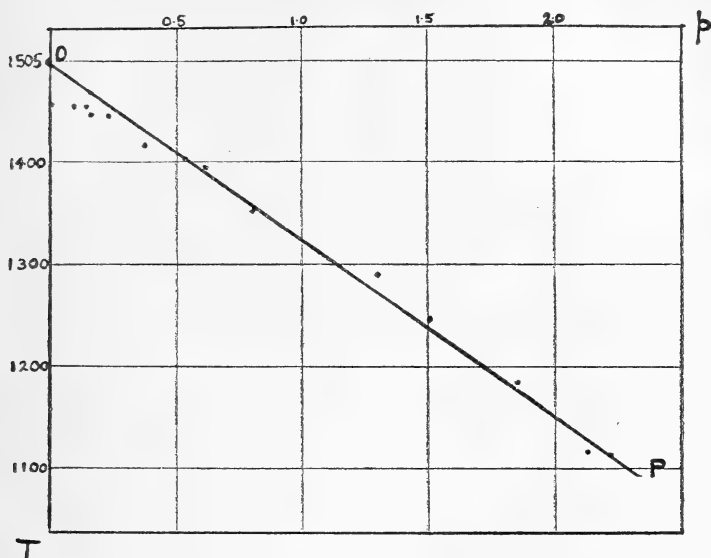
$$\tan \frac{\theta}{2} = 181.80843,$$

which gives the relationship

$$T = 1505 - 181.808x.$$

The actual temperatures obtained from the two formulæ are given in columns 3 and 4 of Table I. We see that, to the nearest degree, there is no difference in the results up to 0.5 % of carbon, and the difference beyond that point is small. The two lines cannot be distinguished on a diagram of the size shown. The line OP in the diagram represents,

therefore, the best fit to the system of points of a line through O by both methods.



II.\* In this case we will take the figures of an alloy of three metals—copper, aluminium, and manganese. The percentage of manganese present varies from 0 to 10.24, and of aluminium from 0 to 7.40. It is not possible to tell from the mere figures if the distribution is approximately coplanar or not, but the material seemed good enough to work upon. The freezing-point of pure copper is 1084° C.; in this case, therefore, we require to find a relationship of the form

$$T - 1084 = p \cdot x + q \cdot y,$$

where T is the temperature at solidification of an alloy containing  $x\%$  of aluminium and  $y\%$  of manganese. Taking our origin at zero percentages of aluminium and manganese and 1084° C., we require to obtain a plane through the origin and fitting most closely the observations in the second, third, and fourth columns of Table II.

\* The figures for this example were taken from Table 43 (p. 229) of the "Ninth Report to the Alloys Research Committee" to the Inst. of Mechanical Engineers, by Dr. W. Rosenhain and Mr. F. C. A. H. Lantsberry.

TABLE II.

Actual Freezing-Point, ° C.	Deviation of Temp. from 1084° C.	Percentage of Aluminium present.	Percentage of Manganese present.	Temp. by Formula ( $\gamma$ ).	Deviation from Actual Temp.	Temp. by Formula ( $\epsilon$ ).	Deviation from Actual Temp.
1077	7	1.19	1.14	1067.8	- 9.2	1067.7	- 9.3
1045	39	1.37	2.75	1053.2	+ 8.2	1052.9	+ 7.9
1051	33	1.04	4.92	1036.6	-14.4	1036.6	-14.4
1023	61	1.42	5.38	1030.7	+ 7.7	1030.2	+ 7.2
1015	69	0.94	6.48	1024.0	+ 9.0	1023.4	+ 8.4
1007	77	1.43	7.72	1010.8	+ 3.8	1010.8	+ 3.8
1011	73	0.91	8.16	1012.1	+ 1.1	1009.2	- 1.8
985	99	1.91	10.24	987.2	+ 2.2	985.9	+ 0.9
1075	9	2.69	.....	1069.3	- 5.7	1069.2	- 5.8
1057	27	2.36	1.95	1054.6	- 2.4	1054.4	- 2.6
1045	39	2.31	3.82	1039.0	- 6.0	1038.6	- 6.4
1010	74	2.27	7.80	1006.5	- 3.5	1004.2	- 5.2
996	88	2.37	9.76	988.4	- 7.6	987.5	- 8.5
1059	25	3.26	0.97	1058.0	- 1.0	1057.8	- 1.2
1042	42	3.71	2.97	1038.6	- 3.4	1038.2	- 3.8
1043	41	3.93	2.99	1037.2	- 5.8	1036.8	- 6.2
1024	60	3.29	4.80	1025.4	+ 1.4	1023.1	+ 0.9
1022	62	3.57	5.64	1016.7	- 5.3	1016.1	- 5.9
1015	69	3.08	6.88	1008.9	- 6.1	1008.2	- 6.8
995	89	3.95	7.95	995.1	+ 0.1	994.3	- 0.7
1067	17	4.67	.....	1058.5	- 8.5	1058.4	- 8.6
1054	30	4.14	1.77	1046.4	- 7.6	1046.1	- 7.9
1031	53	4.62	3.26	1031.1	+ 0.1	1030.7	- 0.3
1036	48	4.48	3.86	1026.8	- 9.2	1027.4	- 9.6
987	97	4.51	7.74	993.8	+ 6.8	993.0	+ 6.0
975	109	4.12	9.60	980.2	+ 5.2	979.2	+ 4.2
1050	34	5.21	0.98	1047.3	- 2.7	1047.0	- 3.0
1035	49	5.66	2.58	1031.2	- 3.8	1030.8	- 4.2
1018	66	5.21	4.86	1014.4	- 3.6	1013.8	- 4.2
1001	83	5.86	6.00	1001.2	+ 0.2	1005.5	- 0.5
998	86	5.19	6.72	998.7	+ 0.7	998.0	0.0
986	98	5.62	8.50	982.7	- 3.3	980.4	- 5.6
976	108	5.99	9.50	970.8	- 5.2	969.8	- 6.2
1025	59	6.88	0.98	1038.1	+13.1	1037.8	+12.8
1030	54	6.54	1.98	1031.5	+ 1.5	1031.1	+ 1.1
1022	62	6.29	3.50	1020.0	- 2.0	1019.5	- 2.5
995	89	6.62	4.82	1007.0	+12.0	1006.4	+11.4
978	106	6.26	8.02	981.9	+ 3.9	981.0	+ 3.0
957	127	6.91	9.06	969.5	+12.5	968.5	+11.5
1042	42	7.30	.....	1043.6	+ 1.6	1043.4	+ 1.4

If  $z$  denote deviation of the temperature from 1084° C., we find

$$S(x^2) = 779.1338, \quad S(yz) = 15421.230,$$

$$S(y^2) = 1336.0155, \quad S(zx) = 10617.960,$$

$$S(z^2) = 190296.0, \quad S(xy) = 751.017.$$



Using equation ( $\beta$ ) above, we quickly reach

$$z = 5.4605x + 8.4732y,$$

and therefore

$$T = 1084 - 5.4605x - 8.4732y. \quad \dots \quad (\gamma)$$

The values of  $T$  obtained by this formula are given in column 5 of Table II. The differences between these values and the experimental results are shown in the next column. It will be seen from the figures that the fit is a good one except at the ends of the range. Had the last seven observations been omitted, *i. e.* had the amount of aluminium present in the alloy been less than 6%, a linear law such as the above one would have agreed quite well with the observed results. As the authors of the original paper state that "the precise temperatures given in the table possess no very great significance," it seems quite reasonable to assume that the observations, up to 6% of aluminium, follow a linear law.

The sum of the squares of the deviations from the observed temperatures in this case is 1641.0210, and the "root mean square" is 6.41\*. The sum of the squares of the deviations measured perpendicular to the plane can be obtained from the above figure by dividing by  $\frac{1}{2}(5.186)^2 + (8.392)^2 + 1$ , *i. e.* 102.6120. It is found to be 15.9924.

When the second method is used, the equation in  $V_m$  is

$$\begin{vmatrix} 779.134 - V_m & 751.017 & 10617.96 \\ 751.017 & 1336.015 - V_m & 15421.23 \\ 10617.96 & 15421.23 & 190296 - V_m \end{vmatrix} = 0.$$

This when expanded becomes

$$V_m^3 - 192411 V_m^2 + 52425892 V_m - 786607941 = 0.$$

We want the least root of this cubic. It is quickly seen to be in the neighbourhood of 15, and by successive approximations is found to be

$$V_m = 15.9362,$$

very nearly.

\* The second decimal place was taken into account in finding this figure. This was done in order to compare with the results of ( $\epsilon$ ), which do not greatly differ from ( $\gamma$ ).

If  $lx + my + nz = 0$

is the equation of the required plane, the equations to find the ratios of  $l$ ,  $m$ , and  $n$  are

$$763\cdot198l + 751\cdot017m + 10617\cdot960n = 0,$$

$$751\cdot017l + 1320\cdot079m + 15421\cdot230n = 0.$$

From these we find

$$\frac{l}{5\cdot4908} = \frac{m}{8\cdot5582} = \frac{n}{-1}.$$

The equation of the plane is

$$z = 5\cdot4908x + 8\cdot5582y$$

and

$$T = 1084 - 5\cdot4908x - 8\cdot5582y. \dots (\epsilon)$$

The temperatures given by this formula are shown in column 7 of the table, and the deviations from the observed values in column 8. They do not differ greatly from the results given by ( $\gamma$ ). The sum of the squares of the deviations in the table is 1658·9718, which is, of course, greater than the corresponding number given by the first method. The "root mean square" is 6·44, not greatly different from the first method value. Also  $l^2 + m^2 + n^2$  becomes 104·3923. The actual sum of the squares of the deviations perpendicular to the plane is therefore 15·8917, which is less than the value given by the first method, as it should be, but is not a very great improvement on it. Thus in this example, as in the last, the two methods lead to very similar results.

III. For a third example we will take the case of a plane in three dimensions to pass through two fixed points and to be closest fitting to a series of other points. The data for this case are taken from a railway time-table. The two fixed points are two terminal stations, and the variables are  $x$ , the distance (in miles) from one of these stations to some other station;  $y$ , the scheduled time (in minutes) allowed for a train between those stations; and  $z$ , the first-class single fare (in pence) between the stations. The figures are:—

$x$ .	$y$ .	$z$ .
30	49	69
52	80	117
60	97	135
69	115	156
81	136	182
100	164	224

the corresponding figures up to the other terminus being 114, 187, and 244 respectively. The figures should be expected to be fairly coplanar, and any formula obtained to represent them ought to give a good "fit." Four cases can be worked out here, viz. those obtained by making the sum of the square of the deviations in the directions of  $x$ ,  $y$ ,  $z$  and perpendicular to the plane respectively a minimum. We find :

$$\begin{aligned} S(x^2) &= 28526, & S(yz) &= 105264, \\ S(y^2) &= 76827, & S(zx) &= 64160, \\ S(z^2) &= 144311, & S(xy) &= 46801. \end{aligned}$$

For the best fit in the direction of  $z$ , the equation of the plane will be

$$z = ax + by,$$

with the condition

$$224 = 114a + 187b. \quad \dots \quad (\eta)$$

In this case we have

$$\Delta = \begin{vmatrix} 1 & 64160-114\lambda & 105264-187\lambda \\ 64160-114\lambda & 28526 & 46801 \\ 105264-187\lambda & 46801 & 76827 \end{vmatrix}.$$

Using the relations

$$a = -\frac{\Delta_{10}}{\Delta_{00}}, \quad b = -\frac{\Delta_{20}}{\Delta_{00}},$$

we obtain

$$\begin{aligned} a &= 2.2376 + .00526\lambda, \\ b &= .00706 - .00077\lambda. \end{aligned}$$

Substituting in  $(\eta)$ ,  $\lambda$  becomes  $-27.2290$ , and therefore

$$\begin{aligned} a &= 2.0943, \\ b &= .0281, \end{aligned}$$

and the best fitting plane in the direction of  $z$  is

$$z = 2.0943x + .0281y. \quad \dots \quad (\theta)$$

In a similar manner we find the best fitting plane in the direction of  $y$  is

$$y = 1.6327x + .0036z, \quad \dots \quad (\zeta)$$

and in the direction of  $x$  it is

$$x = .5031y + .0817z. \quad \dots \quad (\xi)$$

When the deviations are measured in a direction perpendicular to the plane, the equation to determine  $V_m$  (the least sum of the squares of these deviations) is, by § 5 above,

$$\begin{vmatrix} 0 & 114 & 187 & 244 \\ 114 & 28526 - V_m & 46801 & 64160 \\ 187 & 46801 & 76827 - V_m & 105264 \\ 244 & 64160 & 105264 & 144311 - V_m \end{vmatrix} = 0,$$

the determinant being reversed for convenience in evaluation, *i. e.*

$$107501V_m^2 - 19413930V_m + 191936824 = 0,$$

giving

$$V_m = 10.49664.$$

Then

$$d \propto \begin{vmatrix} 1 & 114\mu & 187\mu & 244\mu \\ 114\mu & 28515.5 & 46801 & 64160 \\ 187\mu & 46801 & 76816.5 & 105264 \\ 244\mu & 64160 & 105264 & 144301.5 \end{vmatrix}$$

and

$$l_1 = -\frac{d_{10}}{d_{00}}, \quad l_2 = -\frac{d_{20}}{d_{00}}, \quad l_3 = -\frac{d_{03}}{d_{00}},$$

where

$$l_1x + l_2y + l_3z = 0$$

is the equation of the required plane. Since only the ratios of  $l_1$ ,  $l_2$ , and  $l_3$  are required, it is sufficient to find  $d_{10}$ ,  $d_{20}$ , and  $d_{30}$  (each of which contains  $\mu$  as a factor). When we find these ratios we must divide each by  $\{l_1^2 + l_2^2 + l_3^2\}^{\frac{1}{2}}$  in order to have the sum of their squares unity. In this way we find the equation of the plane is

$$.8843x - .4632y - .0582z = 0. \quad . . . (\phi)$$

The deviations of the results given by the formulæ ( $\theta$ ), ( $\zeta$ ), ( $\xi$ ), and ( $\phi$ ) from the actual values are (the deviation being positive when the formula gives a value greater than the actual value) :—

( $\theta$ )	( $\zeta$ )	( $\xi$ )	( $\phi$ )
Deviation in direction of $z^*$ .	Deviation in direction of $y$ .	Deviation in direction of $x$ .	Deviation perpendicular to plane.
-4.795	+ .227	+ .285	- .181
-5.850	+5.319	-2.200	+2.122
-6.619	+1.445	- .178	+ .275
-8.265	-1.787	+1.592	-1.325
-8.544	-3.101	+2.281	-1.952
-9.966	+ .070	+ .796	- .563

The sum of the squares of these deviations are

$$(\theta) \quad 341.6570,$$

$$(\zeta) \quad 43.2458,$$

$$(\xi) \quad 13.3240,$$

$$(\phi) \quad 10.4942 \text{ (the exact value here should be } 10.4966, \text{ the value of } V_m \text{ above).}$$

The sum of the squares of the deviations given by ( $\theta$ ), ( $\zeta$ ), and ( $\xi$ ) in directions perpendicular to those planes are found to be (by dividing the above values by the sum of the squares of the coefficients of the various equations) 63.4239, 11.7969, and 10.5767 respectively, all these, of course, being greater than the corresponding value given by ( $\phi$ ).

Equation ( $\phi$ ) can be written in the three forms :

$$z = 15.2048x - 7.9644y \quad . \quad . \quad . \quad (\theta')$$

$$y = 1.9091x - .1256z \quad . \quad . \quad . \quad (\zeta')$$

$$x = .5238y + .0658z \quad . \quad . \quad . \quad (\xi')$$

These equations should be compared with ( $\theta$ ), ( $\zeta$ ), and ( $\xi$ ) respectively. The sum of the squares of the deviations

\* At first sight it seems remarkable that all the deviations given by ( $\theta$ ) are of the same sign, but a moment's consideration will show that this is quite possible. For a line in the plane is fixed, and the plane can only swing about this line. All the points may be on one side of the plane, but on either side of the line. Swinging the plane about the line to become closer (measured in a particular direction) to some of the points, therefore, may take it farther from some others. To verify ( $\theta$ ) the results given by the planes  $z=2.1404x$  and  $z=2x+.0856$ , one on either side of ( $\theta$ ), were found. The sum of the squares of the deviations given by these formulæ were 341.6991 and 341.8112, both greater than the corresponding number for ( $\theta$ ). Thus ( $\theta$ ) gives a true minimum.

given by these equations in the directions of  $z$ ,  $y$ , and  $x$  respectively are

$$3102.9568, \quad 48.9069, \quad \text{and} \quad 13.4189.$$

Comparing these with the results given by  $(\theta)$ ,  $(\zeta)$ , and  $(\xi)$  above brings out clearly the fact that the plane which satisfies one criterion for closest fit may give a very bad fit if measured by another criterion. This is particularly the case with  $(\theta)$  and  $(\theta')$ , though  $(\xi')$  is not greatly inferior to  $(\xi)$  as the best fitting plane in the direction of  $x$ .

The Sir John Cass Technical Institute,  
London, E.C.  
December 1910.

XLIV. *A Method of Calibrating Fine Capillary Tubes.*  
By THOMAS RALPH MERTON, *B.Sc. (Oxon.)*\*

THE methods commonly used in the determination of the bore of capillary tubes are direct optical measurement of the bore at the orifice, or weighing a drop of mercury which occupies a known length in the capillary. When very fine capillaries, having an internal diameter of the order of  $\cdot 1$  mm., are to be measured, these methods present serious difficulties.

For many purposes it is necessary to obtain a value of the mean bore, and as no glass capillary is uniform for any considerable length, a measurement of the bore at the orifices is liable to serious error. The weight of a column of mercury 10 cm. long contained in a capillary tube of  $\cdot 1$  mm. bore is about  $0.01$  gm., so that to obtain an accuracy of 1 per cent. the weighing must be correct to  $0.1$  mgrm.

The following experiments were performed with the object of investigating the accuracy with which a measurement of the electrical resistance of a fine glass capillary filled with mercury can be made. From this a mean value of  $r^2$  (where  $r$  is the internal radius) can be calculated.

The first series of experiments was conducted in a large water-bath, containing about 30 litres, kept at  $18^\circ\text{C}$ . by an electric-filament lamp which was governed by a large spiral toluene regulator; and other experiments were performed in a bath kept at  $25^\circ$  by a small gas-flame governed by a fluted toluene regulator. In both baths the temperature could be kept constant to  $0.01^\circ\text{C}$ .

\* Communicated by the Author.

The measurements of resistance were made with a metre bridge with resistances lengthening it to 10 metres and a resistance-box that had been carefully calibrated. The accuracy of the measurement far exceeded the accuracy with which the resistance could be kept constant.

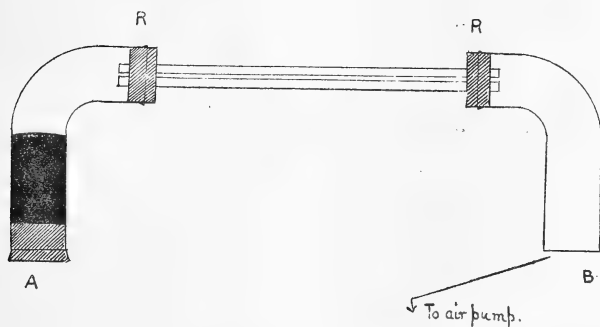
The arrangement of the capillary tube is shown in fig. 1.

Fig. 1.



An airtight joint with the side-tube A and B is made by means of the two rubber plugs, R R. To fill the tube, the limb A is half-filled with mercury and closed with a rubber plug. It is then turned upside down, fig. 2, and exhausted

Fig. 2.



through B by means of an air-pump. On inverting the tube and removing the plug from A, the mercury was forced through the capillary into B.

In making the measurements, connexion is made by means of stout copper wires dipping into the arms A and B, which are half-filled with mercury. The two rubber plugs R R were dispensed with in later experiments by grinding the tubes roughly together, and then cementing them with a small quantity of Schœnbeck enamel.

The tubes were cleaned by drawing through them concentrated sulphuric and chromic acids, followed by distilled water, and were dried by heating to about 100° C. and drawing air through them.

The results obtained for three tubes are given in the following tables.

Tube A at 18° C. Length of Capillary = 14·0889 cm.

	Comparison Resistance.	Resistance of Capillary Thread.
1st Filling .....	18·509 ohms	18·614 ohms
	18·609	18·613
	18·709	18·611
2nd Filling.....	18·709	18·612
	18·609	18·611
	18·509	18·613
3rd Filling .....	18·509	18·605
	18·609	18·605
Reset * .....	18·609	18·604
	18·709	18·602
	18·509	18·604
4th Filling .....	18·609	18·621
	18·709	18·620
Reset .....	18·709	18·616
Reset .....	18·709	18·616
5th Refilling .....	18·609	18·583
	18·509	18·584
	18·709	18·585
	18·609	18·625
	18·709	18·623
	18·509	18·625
Mean Resistance.....		18·609 ohms

\* This consists in drawing a little mercury from one limb to the other.

Tube A was fitted with rubber plugs. Tubes B and C were cemented to the side-limbs and the results obtained with them were more concordant.



Tube B at 25° C. Length of Capillary = 176.14 mm.

	Comparison Resistance.	Resistance of Capillary Thread.
1st Filling .....	12.103 ohms	12.256 ohms
	12.203	12.259
2nd Filling .....	12.203	12.262
Reset .....	12.103	12.259
3rd Filling .....	12.203	12.263
	12.103	12.259
Reset .....	12.303	12.266
Mean Resistance ..... 12.260 ohms		

Tube C at 25° C. Length of Capillary = 210.96 mm.

	Comparison Resistance.	Resistance of Capillary Thread.
1st Filling .....	15.403 ohms	15.432 ohms
	15.303	15.425
Reset .....	15.503	15.434
2nd Filling .....	15.503	15.436
	15.403	15.434
Reset .....	15.303	15.428
3rd Filling .....	15.303	15.430
	15.403	15.435
	15.503	15.438
4th Filling .....	15.503	15.436
	15.403	15.433
	15.303	15.426
Mean Resistance ..... 15.432 ohms		

The mean resistances found for tubes B and C are probably correct to 0.03 per cent.

$r^2$  can be calculated from the mean resistance according to the formula

$$r^2 = \frac{l + 2dr}{\pi k f},$$

where  $r$  is the mean radius,  $l$  the length,  $k$  the specific conductivity of mercury at the temperature of the experiment,  $f$  the resistance of the capillary (found). The term  $2dr$  is a correction to the measured length  $l$  for the stream-lines at the end of the capillary, the value of  $d$  being 0.082 (see Jeans's *Electricity and Magnetism*, 1908, p. 347).

The values thus found for the three tubes were:—

	$r^2$ (sq. mm.).	$r$ (mm.).
Tube A .....	0.002304	0.04780
„ B .....	0.004406	0.06637
„ C .....	0.004192	0.06474

The error in  $r^2$  is probably .000002 sq. mm. A measurement of  $r$  for tube C with a microscope and micrometer eyepiece gave the value of .064 mm.

The only difficulty which arises consists in keeping the tubes clean and free from moisture.

In conclusion I should like to express my thanks to Mr. H. B. Hartley and Mr. D. H. Nagel for their kind assistance and advice in these experiments which were begun at the Balliol and Trinity College Laboratory, Oxford.

18 Grosvenor Street, W.

#### XLV. *Notices respecting New Books.*

*Traité de Radioactivité.* By Mme. P. CURIE.

Two Vols. Paris: Gauthier-Villars, 1910.

THE course in Radioactivity given by Mme. Curie at the Sorbonne forms the basis of these two volumes, and the thousand pages they contain testify to the energy and thoroughness with which the new science is being pursued in the country of its origin. When Mme. Curie was appointed to the position left vacant by M. Curie's untimely death, cut off in the flowing tide of their joint discovery, probably most people imagined that the position, as regards academic duties, was an honorary one, intended to secure to the surviving partner the uninterrupted pursuit and completion of the investigations on radium. How well and continuously the latter object has been served is best known to those who follow most closely the progress of the subject, but the appearance of this work will dispel any illusion that may have been entertained as to the duties of the Professor of Radioactivity at the Sorbonne. It is evident that research and the training of investigators goes on alongside of a full course of lectures covering completely and in detail

every branch of what is now a science in itself. How much the science covers and the manner of its development could not be better told than in the eight pages which introduce this work, where, it is gratifying to note, generous and unreserved acknowledgment is made of the part played by workers in this country and the services rendered by Rutherford's treatise.

The book itself appears to have been in preparation for a considerable time, and both in the scheme of presentation and the order of the subject-matter reflects the growth of ideas as it took place in France, where the view that radioactivity was due to atomic disintegration, now everywhere adopted, was for a considerable time not accepted. Indeed in this respect it is largely autobiographical, as is perhaps only natural, the discovery and investigation of the powerfully radioactive elements, and their "induced" activity and emanations, occupying the first volume and preceding the detailed study of the radiations and of the successive members of the various disintegration series, which form the chief contents of the second. In treatment and scope the book is thoroughly comprehensive, and practically all the important physical and chemical work which has been published is referred to. It will be welcomed in this country particularly as containing an authoritative account of the work of M. and Mme. Curie, much of which, owing to the prevailing custom in France, has hitherto only been very briefly published.

*Les Compteurs électriques à Courants continus et à Courants alternatifs.* By L. BARBILLION and G. FERROUX. Paris: Gauthier-Villars, 1910.—*Notions fondamentales sur la Télégraphie.* By A. TURPAIN. *Téléphonie.* By A. TURPAIN. Paris: Gauthier-Villars, 1910.

THESE three small treatises deal each with a particular branch of technical electricity. The first is a specially clear and elementary account of electric meters adapted to measure either continuous or alternating currents. A very large number of different patterns are described and the theory of them sufficiently outlined. The discussion of meters depending upon rotary fields is especially good. The book is profusely illustrated with figures and diagrams.

In the second book, Professor Turpain outlines the historical development of electrical telegraphy, and the volume will be found useful to all those interested in this development. No theory, in the ordinary sense, is given; attention is concentrated on the mechanical devices, such as relays &c., rather than on the theoretical side. The details of the instruments used are very clearly drawn.

The third volume, also by Professor Turpain, is of a similar scope. The main portion is taken up with a description of central stations. The author has a vigorous style, and he grows sarcastic at theoretic attempts to explain the action of a telephone. One chapter is headed: "L'explication précise du phénomène téléphonique paraît impossible."

XLVI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 280.]

November 23rd, 1910.—Prof. W. W. Watts, Sc.D., M.Sc., F.R.S.,  
President, in the Chair.

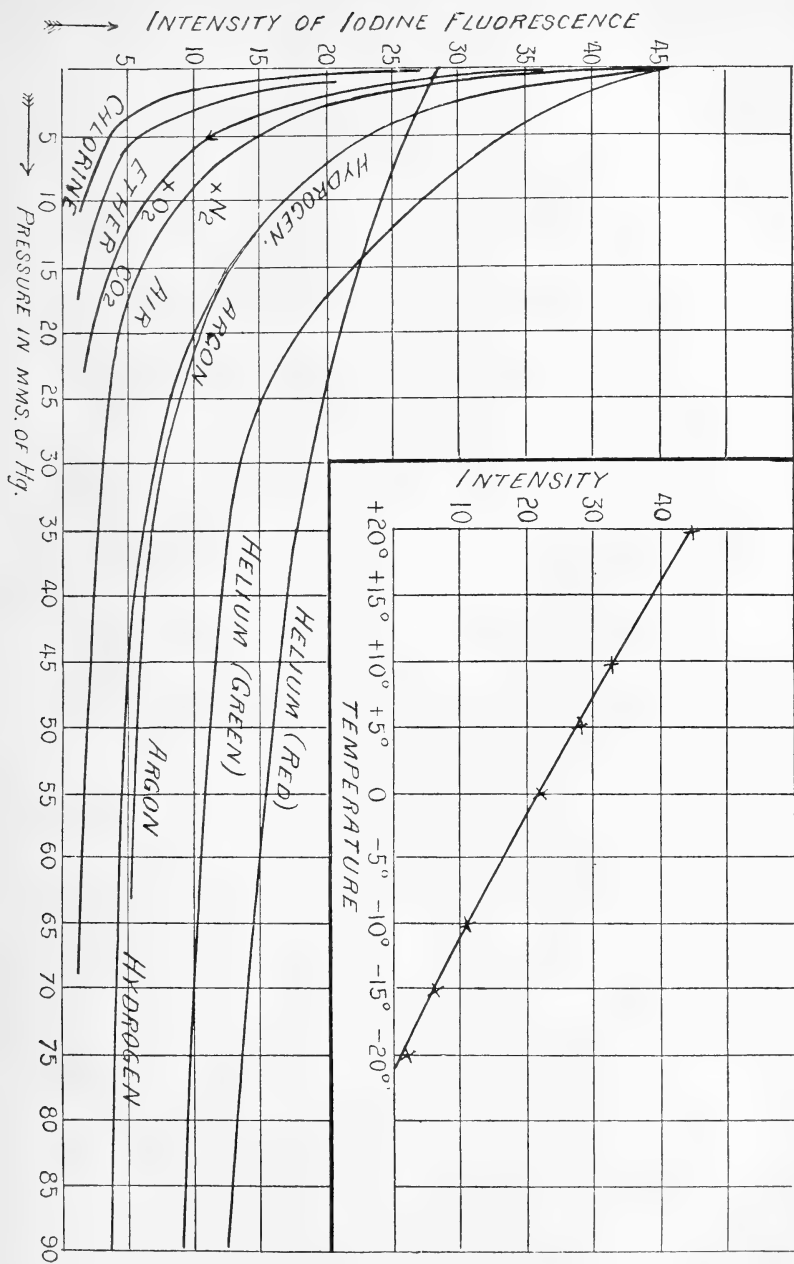
THE following communications were read:—

1. 'The Effects of Secular Oscillation in Egypt during the Eocene and Cretaceous Periods.' By William Fraser Hume, D.Sc., F.G.S., Director of the Geological Survey of Egypt.

2. 'The Origin of the British Trias.' By A. R. Horwood.

It is maintained that, though desert conditions prevailed locally during the Triassic Period in Britain, deposition was brought about solely by the action of water; and the British Trias is a delta-system.

For, during Carboniferous, Permian, and Triassic times deposition was mainly in the same area. There is, moreover, a gradual gradation from the Bunter to the Rhætic, from coarse sediments to fine. The Bunter is known to be of fluvial origin, since Prof. Bonney first showed it to be so; and there is a continuity from Lower to Upper Trias, with an unconformity due to the new mode of formation and change in sedimentation. Oscillation and overlapping, which occur in the Trias, are admittedly due to aqueous agency. The Triassic outcrop and the delta-area of the River Mississippi again are closely similar. The alternations of pebbles and sands, sandstone and marl, etc., are due to those seasonal changes which are characteristic of deltas. Coloration is original, from below upwards, and not coincident with bedding. The thickness of the Bunter is an argument for a subsiding area. The ferruginous types in the Carboniferous, Permian, and Trias are alike due to delta conditions. The Trias is horizontal now, as originally, away from any ancient hills which it covers, and 'radial dip' is merely 'angle of rest.' It is only the skerries, furthermore, that are rippled. Serees, too, occur mainly to the south-west of submerged hills. Sandstones thin out eastward, marls westward, and the skerries are on the hills. The surface-features of ancient hills once covered by Trias are quite unaffected; and desert conditions are merely marginal, limited to granitic or syenitic knolls at one horizon, while in the surrounding area such conditions are absent. Rock-salt and gypsum are also horizontal and continuous in a linear direction. The Keuper gradually merges into the Rhætic phase, and the latter into the Lias. Since the Bunter sediments came from the north-west into the Midlands, so probably did the Upper Trias. Triassic sediments and those of the Nile are similar, but the first have been acted upon chemically, the latter mechanically. Local metamorphic and volcanic rocks may have provided some of the heavier minerals, but as a whole their source was more distant. The flora and fauna can be grouped in provinces around the delta-head of the Trias. These considerations all point to an aqueous mode of sedimentation in a moist and equable climate; and desert conditions only prevailed locally.





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THE  
LONDON, EDINBURGH, AND DUBLIN  
PHILOSOPHICAL MAGAZINE  
AND  
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

APRIL 1911.

XLVII. *On the Ultra-Violet Light from the Mercury Arc.*  
By A. LL. HUGHES, M.Sc., B.A., Research Scholar of  
Emmanuel College, Cambridge\*.

1.

THE velocities of electrons emitted from surfaces illuminated by ultra-violet light have been investigated by Lenard †, Ladenburg ‡, and Hull §. Ladenburg discovered that the maximum velocity of the electrons was proportional to the frequency of the light used. The range over which he worked was from  $\lambda 2600$  to  $\lambda 2010$ . Hull was able to show that the law held down to  $\lambda 1230$ .

In all these experiments, the source of light and the illuminated plate were separated by a window of quartz or fluorite, so that the shortest wave-length which could be employed was determined by the transparency of the window. The maximum velocity of the electrons can only be obtained when the illuminated plate is in a very good vacuum. As the sources of light usually employed (the spark, or the discharge in hydrogen) require the presence of some gas, the vessel containing the illuminated plate must be separated from the source of light by a window of quartz or other transparent material. The shortest wave-length transmitted

\* Communicated by Professor Sir J. J. Thomson.

† Lenard, *Ann. der Phys.* viii. p. 149 (1902).

‡ Ladenburg, *Phys. Zeits.* ix. p. 821 (1908).

§ Hull, *Phys. Zeits.* x. p. 537 (1909).

by thin quartz is  $\lambda 1450$ , while some specially selected specimens of fluorite transmit as far as  $\lambda 1230$ .

It was thought that if the ultra-violet light could be produced in the same vacuum as the plate, much shorter wavelengths might be obtained, since there would be nothing to absorb the light. They would, assuming Ladenburg's law, show their presence by producing electrons of greater speed than have hitherto been obtained from surfaces illuminated by ultra-violet light. If we take the velocities of electrons emitted from a surface as an indication of the type of the radiation falling upon it, we see that there is a large gap between the softest X-rays and ultra-violet light of the shortest wave-length known. The potential required to stop the electrons produced by the former radiation is of the order of 1000 volts, by the latter it is of the order of only 3 volts. All attempts to investigate this gap, either from the X-ray side or from the ultra-violet side, appear to be frustrated by the opacity of all known materials.

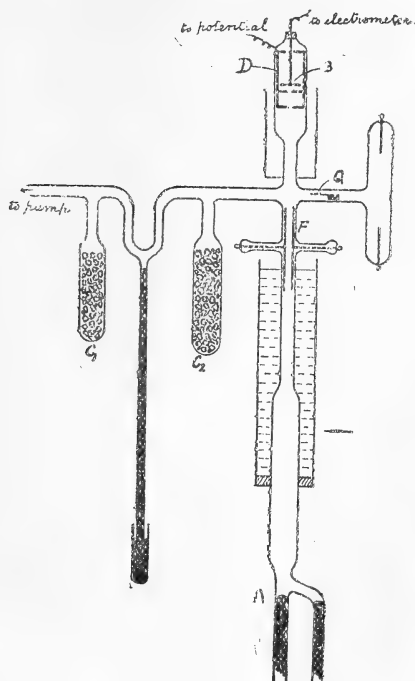
The condition that the illuminated plate must be in a very good vacuum and also in the same apparatus as the source of light, practically restricts the choice of the source of light to the mercury arc. The pressure of mercury vapour in the arc may rise to several centimetres but falls off very rapidly with the distance from the arc. The apparatus used was so designed that the pressure of mercury vapour in the neighbourhood of the illuminated plate was quite inappreciable. A preliminary experiment showed that, after the arc had been produced several times, there was no further evolution of gas from the mercury or the walls of the tube.

The apparatus used is shown in fig. 1. The arc is struck by raising one of the two barometer columns at A. The potential used is about 50 volts, and the current through the arc is kept at the same value (2-4 amps.) throughout a series of experiments. At a distance of 90 cms. from the arc is the illuminated plate B. The velocity of the electrons emitted by the plate B is measured by observing the (negative) potential required by the surrounding case D to stop the leak. The plate B and the case D are covered with soot by means of an acetylene flame in order to reduce the reflexion of light to a minimum. When the case D is at different positive potentials, the leak increases with the potential except when the vacuum is very high. Both Ladenburg and Hull found that with these slow-moving electrons it was necessary to obtain a very good vacuum, otherwise the leak increased with the potential, indicating that ionization by collision was taking place.



The vacuum was obtained as follows:—The apparatus is exhausted by a Toepler pump down to a pressure of about  $\cdot 05$  mm., then filled with pure oxygen and re-exhausted.

Fig. 1.



Meanwhile both the large charcoal tubes  $C_1$  and  $C_2$  were heated. The charcoal tube  $C_1$  is then cooled by liquid air, while  $C_2$  is surrounded by a heating jacket. After four or five hours, during which the arc is repeatedly struck to get rid of any gases in the mercury, the communication between the charcoal tubes is cut off and liquid air is brought up around  $C_2$ . After the liquid air has been in position for about three hours, readings are taken. To diminish as far as possible the pressure of mercury vapour at the top of the tube, a water-jacket surrounds the middle portion and the upper part is kept cool by  $\text{CO}_2$  snow. A discharge tube (about 20 cm. by 4 cm.) serves to indicate the state of the vacuum. When consistent readings were obtained the alternative spark-gap was always greater than 6 or 7 inches.

To obtain the distribution of velocities of the electrons, the leak is measured with the case D at different potentials ranging from +40 volts to -8 volts. The arc is struck, and in a few seconds after which the current through it has become approximately steady, the earth connexion of the electrometer is broken and the leak is measured over an interval of time of a minute. It was not safe to run the arc for much longer as the heating appeared to produce a little gas, making the readings somewhat unsteady.

Readings were also taken with the quartz plate Q interposed in the path of the light. The following are the results obtained for the leaks from the illuminated plate B with and without the quartz in the path of the light.

TABLE I.  
Source of Light—Mercury Arc.

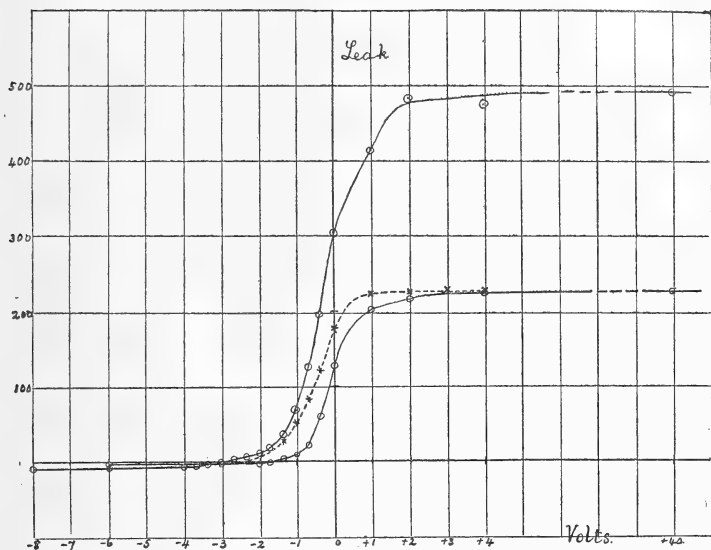
Potential of D.	Leak without quartz.	Leak with quartz.
+40 volts	+490 ( $=1.06 \times 10^{-11}$ amp.)	+226
+ 4	472	225
+ 2	482	217
+ 1	412	202
0	305	127
- .34	195	60
- .66	127	21
-1.00	67	9
-1.34	34	+1.3
-1.66	19	-1.3
-2.00	12	-2.0
-2.34	9	...
-2.66	+ 5	...
-3.00	0	-2.2
-3.34	- 3	...
-3.66	- 5	...
-4.00	- 7	...
-6.00	- 9	-2.7
-8.00	-10	...

To determine the potential at which the leak was zero, the capacity was cut out of the electrometer system and the sign of the leak was observed for different potentials of D. When the potential was -3.0 volts or less the leak was always positive, when it was -3.1 or more it was always negative. With potentials in between these limits the sign of the leak for a given potential was sometimes negative and sometimes positive, and therefore the potential for zero leak could only be indicated by the above limits.

With quartz interposed in the path of the light, the leak was zero for a potential between  $-1.42$  volts and  $-1.44$  volts.

The results of Table I. are plotted in fig. 2.

Fig. 2.



To make sure that B did not charge up positively as a result of positive ions coming up from the arc, the beam of light was passed through an electric field at F. Two brass plates 10 cm. long and .7 cm. apart were maintained at a difference of potential of 200 volts. Assuming  $\frac{e}{m} = 10^3$  for a positive ion, the field would stop all ions moving more slowly than about  $4 \times 10^7$  cm./sec. Ions faster than this would get through the field and communicate a charge to the plate B. However, it was found that the leak was entirely stopped when B was in a magnetic field of 200 gauss parallel to the surface. In such a field, positive ions of velocity  $4 \times 10^7$  cm./sec. would move in a circular path of 50 cms. radius. Consequently, the point of impact of the positive ions on B would be shifted very slightly and the change in the leak would be inappreciable. Hence the charge acquired by B must be due, not to positive ions coming from the arc, but entirely to the loss of electrons under the action of the ultra-violet light.

On comparing the results in Table I. for the light which had passed through quartz with Hull's results, it was found (1) that the zero leak occurred at  $-1.43$  volts, whereas in Hull's experiments it occurred at  $-2.33$  volts; and (2) the ratio of fast electrons to the total number was much less than in Hull's experiments. To make sure that this was not something due to the particular piece of quartz used, or to the state of the blackened surface, the experiments were repeated using a discharge in hydrogen at 2 mm. pressure as the source of light and transmitting it through the same piece of quartz on to the same plate. The results obtained are given in Table II. and plotted in fig. 2 (dotted line); the scale is taken such that the saturation portion of the curves for the quartz effects with the different sources coincide.

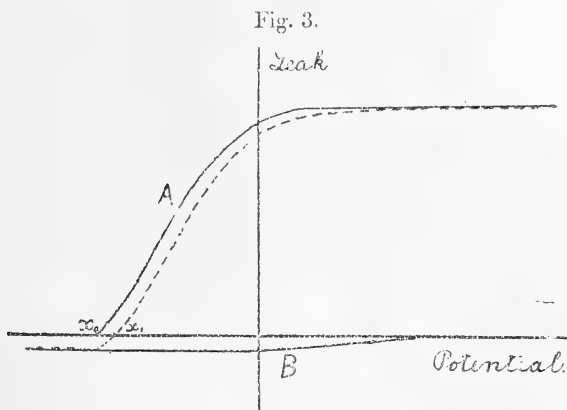
TABLE II.

Source of Light—Discharge in  $H_2$  at 2 mm.

Potential of D.	Leak.
+4 volts	+113 ( $=3.04 \times 10^{-11}$ amp.)
+3	115
+2	111
+1	111
0	88
- .34	61
- .66	41
-1.00	25
-1.34	13.7
-1.66	7.0
-2.00	+1.0
-2.34	-1.8
-2.66	-3.3
-3.00	-3.8
-4.00	-4.0

The zero leak occurred at  $-2.18$  volts, which is less than Hull's value, viz.  $2.33$  volts. The difference is probably due to the greater proportion of reflected light in these experiments. If we suppose that no reflexion takes place, then a curve of the form A (fig. 3) will be obtained, and the point  $x_0$  denotes the potential required to stop the fastest electrons. If, however, a certain fraction of the light is reflected to the case D, the curve B for the leak from the case will be similar to A but reversed and flatter. What is actually observed in the experiments is the sum of the two curves indicated by the dotted line cutting the abscissa in  $x_1$ . The difference between  $x_0$  and  $x_1$  increases with the amount of reflected

light. For a given plate, when the amount of reflected light is small, it seems safe to assume that the different values of  $x_1$  are proportional to the corresponding values of  $x_0$ , and therefore that the ratios of various  $x_1$ 's may be used instead of the ratios of  $x_0$ 's.



In Hull's experiments, the leak due to reflected light was  $\frac{1}{35}$  of the leak due to the light incident on the plate; in these experiments it was about  $\frac{1}{28}$ , hence the potential  $x_1$  required for zero leak is rather less in these experiments than in Hull's. This accounts, in part at any rate, for the difference between the values 2.18 and 2.33 volts.

Comparing the effects of the light from the two sources when transmitted through quartz, it is seen that the hydrogen discharge produces a greater proportion of faster electrons than the arc. Also the maximum velocity of the electrons due to the light from the hydrogen discharge and transmitted through quartz, corresponds to a potential of 2.18 volts; while for the light which comes from the arc through quartz, the potential is 1.43 volts. According to Lyman\*, the shortest wave-length transmitted by thin quartz from a hydrogen discharge is  $\lambda$  1450, which corresponds in these experiments to the potential 2.18 volts. Ladenburg's law states that the maximum velocity of electrons due to ultra-violet light is proportional to the frequency of the light. The velocity is proportional to the square root of the potential required to stop the electrons, so that we have  $\sqrt{V} \propto \lambda^{-1}$ . The wave-length corresponding to 1.43 volts will therefore be  $\lambda$  1780. The experiments show that there is no appreciable radiation from the mercury arc in the region between

\* Lyman, *Astrophysical Journal*, xxv. p. 45 (1907).

$\lambda$  1450 and  $\lambda$  1780. The greatest velocity of electrons produced by the light from the arc corresponds to 3.0 or 3.1 volts, which means that the shortest appreciable wave-length in the spectrum of the mercury arc is  $\lambda$  1230.

It is perhaps necessary to justify the application of Ladenburg's law to this investigation. From considerations of Planck's theory of radiation, one would expect the frequency of the light and the velocity of the electrons to be connected linearly. Ladenburg expressed his results in the form  $\sqrt{V}\lambda = \text{const.}$ ; but Joffé\* has shown that if Ladenburg's results are plotted, the curve showing the relation between the frequency and the velocity is a straight line which, however, does not pass through the origin. As the linear relation has the better support in theory, we may regard Ladenburg's law  $\sqrt{V}\lambda = \text{const.}$  as an empirical relation which is satisfied over a limited range. The work of Ladenburg and Hull shows that this relation is true over the range  $\lambda$  1230 to  $\lambda$  2600. Since the wave-lengths dealt with are found to be within the above limits, the application of Ladenburg's law to this investigation is justified. The only substance available whose transparency was known was thin quartz, and this sufficed to determine the constant in the equation  $\sqrt{V}\lambda = \text{const.}$ , while in Joffé's form there are two constants to determine.

The experiments of Professor Lyman show that the spectrum of hydrogen extends as far as  $\lambda$  1030. He suggests, however, that perhaps still shorter wave-lengths may be emitted, but that either the photographic plates used in the experiments are not sensitive to shorter wave-lengths, or that the grating used for giving the spectrum does not act efficiently beyond about  $\lambda$  1000. The electrical method of detecting the shortest wave-length has the advantage that the shorter the wave-length, the more sensitive is the test. The experiments described above show that the mercury spectrum ends at about  $\lambda$  1230, while Lyman's work shows that the hydrogen spectrum extends further, viz. to  $\lambda$  1030.

## 2.

The following is a short account of an experiment on the nature of the photoelectric effect. Several views as to the mechanism of the effect may be considered.

1. Light may be regarded as molecular in structure and

Joffé, *Ann. der Phys.* xxiv. 5, p. 939 (1907).

each unit of light releases an electron from a molecule in the surface upon which the beam is incident. The velocity of the electron depends solely upon the energy in the unit and consequently upon its frequency.

2. The photoelectric effect may be of the nature of resonance. There may be many different atomic systems in matter each of which can be rendered unstable by light of suitable frequency\*. Here, as before, the velocity of the ejected electron is determined by the frequency of the light.

3. The electrons emitted from a surface may be in temperature equilibrium with the beam of light. The greater the frequency of the light, the higher is its radiation temperature. This also leads to the known experimental result that the greater the frequency of the light, the greater is the velocity of the electrons emitted.

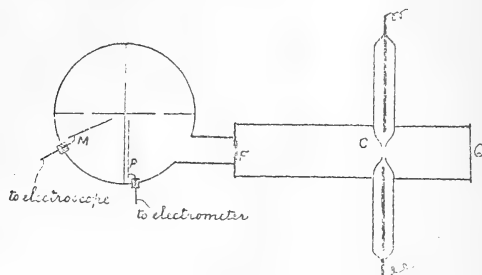
The principle of the experiment is as follows. The distribution of velocities of electrons released by light of the shortest available wave-length is measured, then an intense beam of ultra-violet light of longer wave-length is superposed and the distribution of velocities is again obtained. On the first and second views of the photoelectric effect indicated above, the final distribution of velocities would simply be the sum of the distributions due to the components of the beam. On the third view, however, this would not be the case. The temperature of the mixed beam would be higher than that of its long wave-length component and lower than that of its short wave-length component, and the final distribution of velocities would not be obtained by the simple addition of the distributions due to the two components.

The simplest method of carrying out the experiment is to measure the maximum velocity of the electrons due to a beam of short wave-length ultra-violet light; then superpose longer wave-lengths, and examine whether the maximum velocity is the same as before or less. This was done, using the upper portion of the apparatus in fig. 1 to measure the maximum velocity. An apparent diminution in the maximum velocity was found, but this could be explained, to some extent, by considerations of the same kind as those brought up in the discussion of fig. 3. It was therefore necessary to devise a method in which the disturbing effect of reflected light was completely avoided. In the method

\* Sir J. J. Thomson, *Phil. Mag.* xx, p. 238 (1910).

adopted a magnetic field was used to measure the velocities. The apparatus is shown in fig. 4.

Fig. 4.



A flat cylindrical box is divided into quadrants by thin brass partitions. In three of these, slits 10 mm. by 2 mm. were cut. The radius of the circle passing through the centres of the slits was 1.1 cm. The illuminated plate P was connected to an electrometer and capacity by means of which the total leak from the surface could be measured when required. The surfaces inside the box were all covered with soot.

The source of light of short wave-length was a discharge in hydrogen in the tube C. This was separated from the brass box by a fluorite window F which transmitted light down to  $\lambda$  1330. The superposed long wave-length ultra-violet light was obtained from a mercury arc in a fused quartz tube placed close to the quartz window Q. The shortest wave-length transmitted by the fused quartz was  $\lambda$  1930.

The distribution of velocities for each of the two sources of light was first obtained. This was done by measuring the charge acquired by M for different magnetic fields perpendicular to the plane of the box. Although the light was as intense as could be obtained under the conditions, yet the charge communicated to M was always small. In some cases the leak was measured with the tilted electroscope at a sensitiveness of 600 divisions per volt.

The velocities were obtained by the formula

$$Hev = \frac{mv^2}{R},$$

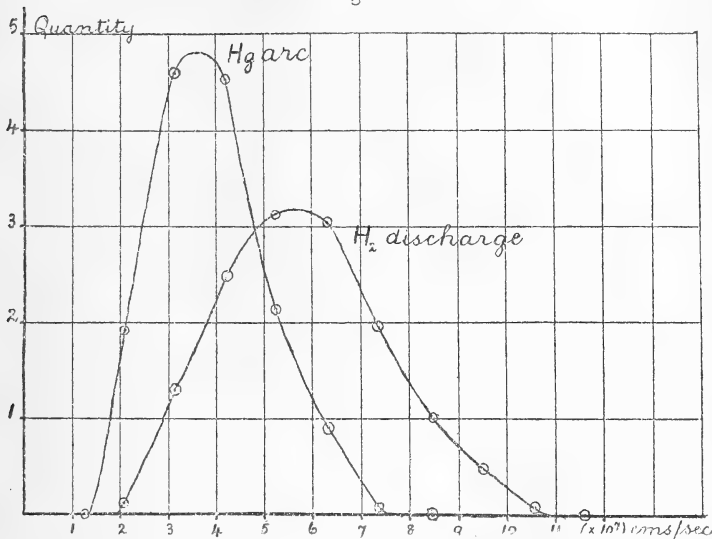
$$\frac{e}{m} = 1.76 \times 10^7, \quad R = 1.1 \text{ cm.}$$

The distributions of velocities are given in fig. 5, the total



leak from the surface P being the same for both sources of light. It is seen that the electrons are on the whole considerably slower when produced by the light from the

Fig. 5.



mercury arc than when produced by the light from the hydrogen discharge. The result of the superposition experiment is given in the following table.

TABLE III.

H: effect due to  $H_2$  discharge alone.  
 A: " " " Hg arc alone.  
 H+A (add.): sum of two previous results.  
 H+A (exp.): effect due to  $H_2$  discharge and Hg arc superposed.

Velocity of electrons.	H.	A.	H+A (add.).	H+A (exp.).
$6.33 \times 10^7$ cm./sec.	34.3	11.8	46.1	46.3
7.38 " "	26.6	1.8	28.4	28.1
8.44 " "	15.6	0	15.6	16.2
9.49 " "	8.4	0	8.4	8.9
10.5 " "	2.8	0	2.8	3.2

The last two columns agree within the limits of experimental error.

It was unnecessary to carry out the experiment for smaller velocities as the effect, if it existed, would scarcely show itself in this region. A few experiments were also tried when the total leak due to the arc was twice as great as that due to the  $H_2$  discharge, but they still led to the same result.

This result points to the conclusion that each electron emitted from a surface illuminated by ultra-violet light is associated with only one frequency in the beam of light, and its velocity is quite independent of the presence or absence of waves of different frequency in the beam.

#### *Summary.*

1. The ultra-violet spectrum of mercury, investigated electrically, extends to about  $\lambda$  1230.

2. There is no appreciable radiation from the mercury arc between  $\lambda$  1450 and  $\lambda$  1780.

3. The hydrogen discharge has relatively more energy in the short wave-lengths than the mercury arc.

4. The velocity of the electrons due to one wave-length is independent of the presence of other wave-lengths in the beam of light.

I have much pleasure in thanking Professor Sir J. J. Thomson for his suggestions and interest during the course of the work.

Cavendish Laboratory, Cambridge.  
Jan. 9th, 1911.

XLVIII. *The Heat liberated during the Absorption of Electrons by Different Metals.* By O. W. RICHARDSON, Professor of Physics, and H. L. COOKE, Assistant Professor of Physics, Princeton University\*.

**I**N a recent paper† under a similar title the authors showed that when slow moving electrons were received by a platinum strip, part of the heat developed was independent of the kinetic energy of the electrons. This production of heat was explained as the thermal equivalent of the difference in the potential energy of the electrons inside and outside the metal. The present paper describes the results of similar

\* Communicated by the Authors.

† Phil. Mag. July 1910.

observations on strips of the following metals and alloys :— gold, nickel, copper, silver, palladium, aluminium, phosphor bronze, and iron.

The method of experimenting was practically identical with that already described (*loc. cit.*). The only important change made was in the method of balancing the disturbing effect of the thermionic current. This was made simpler, both in theory and practice, by the following arrangement. The resistance  $R_3$  (fig. 1, *loc. cit.*) was replaced by a constantan wire of resistance 4.7 ohms wound on a circular drum and placed in an oil-bath. A movable contact-maker enabled the thermionic current to be tapped off at any point of the wire. The contact-maker was connected directly to the point E in the figure referred to and the resistances  $R_4$  and  $R_5$  were done away with. From the principles of the method of balancing the thermionic current already described, it is clear that if it is tapped off at the right point of the resistance  $R_3$  it will produce no effect on the galvanometer G. To balance the effect of the thermionic current it was also necessary first to oppose the two batteries  $C_1$  and  $C_2$  so as to destroy the electromotive force in the Wheatstone's bridge circuit, and rotate the contact-arm until the galvanometer spot was in the same position with the thermionic current "on" or "off." The reliability of the method was tested by dummy experiments, using the electromotive force from a battery in a manner precisely similar to that used in testing the original method, and it was found quite satisfactory.

#### *Effect of Pressure.*

In the former paper it was suggested that the results might perhaps be affected by the pressure of the residual gas present in the apparatus. At that time experiments were made to test the point. So far as they went they seemed to indicate that, if the pressures, such as occurred, exerted an influence on the results, it was an unimportant one. It was felt, however, that these experiments were not very satisfactory; so fresh experiments have been made under better conditions. In these experiments an iron grid was used, and the osmium filaments had been heated continuously for a long time. The conditions were generally very steady.

On account of the continuous evolution of gas by the hot metal, it was necessary so use a continuous pump in these experiments. This made it somewhat difficult to control the pressure. The obvious way of producing a desired change in the pressure is to vary the speed of the pump. This was

found to be no good with the Gaede pump used, as, in order to obtain a pressure appreciably higher than that given by the pump when working at its most efficient speed, it was necessary to work it so slowly that the changes in the temperature of the apparatus caused by the periodic variations of pressure entirely precluded any attempt at measuring the effect.

Several methods were tried, and that finally adopted consisted in connecting the apparatus to a side tube provided with an indiarubber joint which leaked slightly. The connexion was made through a good ground-glass cock. In this way two different pressures were available, namely, the limiting values obtained when the side tube was, and was not, connected with the apparatus. These pressures were about  $2 \times 10^{-3}$  and  $5 \times 10^{-3}$  mm. respectively, and were quite free from the slow periodic variations which had previously been found so objectionable.

With this arrangement several experiments were made, of which the following are typical examples. With 8 volts applied potential-difference, and the side tube shut off, the pressure both before and after one set of observations was  $20 \times 10^{-4}$  mm. The effect in scale-divisions per unit thermionic current under these conditions was found to be 1.680. After connecting the side tube to the apparatus the pressure at the beginning of an experiment with 8 volts applied potential-difference was  $56 \times 10^{-4}$  mm., and at the end  $46 \times 10^{-4}$  mm. The effect in this case, in the same units as at the lower pressure, was 1.683. Thus changing the pressure from  $20 \times 10^{-4}$  mm. to  $51 \times 10^{-4}$  mm. produces no change in the effect per unit thermionic current.

The same was true at other voltages. Thus with 18 volts the pressure was  $20 \times 10^{-4}$  mm. at the beginning of an experiment with the side tube shut off and  $19 \times 10^{-4}$  mm. at the end. The effect per unit thermionic current was 3.01 scale-divisions. After connecting, the pressure was  $46 \times 10^{-4}$  mm. at the beginning and  $42 \times 10^{-4}$  mm. at the end of an experiment under the same potential-difference. In this case the effect was found to be 2.98 in the same units as before. This is identical with 3.01 within the limits of observational error.

These experiments prove conclusively that the measurements are quite unaffected by small fluctuations in the pressure of the surrounding gas; so that such irregularities as have been found must be attributed to other causes.

#### *Experiments with the Different Metals.*

A general review of the results which have been obtained shows that they are much more inconsistent than those yielded

by the former experiments on platinum. We believe that we have discovered the main cause of this inconsistency but, so far, unfortunately, we have not succeeded in regulating it. The cause seems to lie in an instability in the thermionic emission of osmium itself, and we are investigating the phenomenon in detail in the hope of being able to control it. The experiments which follow were carried out, so far as we are able to judge, in the same way and under the same conditions as those previously made with platinum. In view of the considerable range of the value of the effect for each one metal, and also of the peculiar effects first observed when experimenting with iron (see below), we shall only give the final numbers obtained in each case and shall omit the details of the measurements which led to them.

In every case, the strips used were of the purest specimens of the metal obtainable. The gold, silver, palladium, copper, and nickel were obtained as pure from Messrs. Johnson, Matthey & Co., London. This specimen of copper was compared with one rolled from commercial magnet wire, and did not exhibit any notable difference. The aluminium was rolled from commercial aluminium wire and then cut by hand, and the phosphor bronze was a strip such as is used for galvanometer suspensions.

As a rule, a considerable number of experiments were made on each material. The final results of all those which appeared to be satisfactory are given in the following table:—

Metal.	Corrected Values of $\phi$ . Volts.	Mean. Volts.	Weighted mean. Volts.
Gold .. .. .	6.54, 7.16, 6.83, 8.04, 7.36, 6.16.	7.01	7.26
Nickel .. .. .	5.19, 5.38, 5.61.	5.39	5.3
Copper .. .. .	7.36, 7.16, 6.86, 6.86, 5.76, 7.01.	6.84	7.1
Phosphor Bronze ...	5.82.	5.82	—
Palladium .. .. .	6.04, 5.44.	5.72	5.6
Silver .. .. .	5.06, 5.83, 4.16, 5.46, 5.26.	5.15	—
Aluminium .. .. .	7.6, 6.9, 5.3, 7.9, 8.6.	7.4	—
Iron (low) .. .. .	4.95, 4.18, 4.32, 4.85, 5.54, 5.44.	4.88	—
Iron (high) .. .. .	6.27, 6.58, 6.49, 7.52.	6.72	—

The weighted means were judged by inspection of the experimental points. It is questionable whether they are much more reliable than the others, as it is our opinion that the experiments are affected by causes, which we are not able to control, which may remain constant during a single set of experiments. In the case of silver and aluminium the results of the experiments seemed less trustworthy than in the other cases. Often it was impossible to get the same value twice, for the same thermionic current and the same potential-difference, owing to some change with time which was going on. Moreover, in several experiments with these metals the heating effect did not turn out to be a linear function of the applied potential-difference. This was probably due to some parts of the grid being different from others, and the heating current moving from one part to the other as the potential-difference was changed. Probably most of the difference from one part of the grid to another was due to the effect on it of the heating and sputtering. This suggestion is supported by the appearance of the grids. The aluminium ones, after they had been used a few times, were quite changed in texture and were so much altered that they crumbled to pieces as they were unwound. They were also very badly discoloured.

The experiments made with iron grids led to a discovery which we think likely to account for a considerable part of the irregularities which have been noted with all the metals. It seems that under the conditions of the experiments on iron, and probably on the other metals, there is a kind of instability in the thermionic emission of the osmium filaments. The nature of this instability is best described by considering the way the thermionic emission changes as the temperature of the filament is altered. Starting at a comparatively low temperature, the emission increases rapidly with increase of temperature following the usual inverse exponential law. This goes on until temperatures of a certain value are reached, when the current shows a tendency to sag off. If the temperature is now raised and maintained at a certain value, there is a sudden drop in the thermionic emission. In favourable cases the new current may be as little as one-thirtieth of what it previously was at the same temperature. When the temperature is raised further the current is found to be stable again, and increases with temperature according to an inverse exponential law.

The behaviour of the emission as the temperature is lowered is the reverse of what has been described. We have first a quick but regular decrease in current following the inverse

exponential law. At a certain stage there is a sudden increase in the emission. This is followed by a third region in which the current again diminishes with decreasing temperature according to the regular law.

There are thus two ranges of temperature in which the thermionic current is stable. These are separated by a region of instability. We shall refer to the two stable ranges as the low-temperature range and the high-temperature range respectively. The heating effect in the case of iron has been examined, both when the osmium was on the low-temperature range and also when it was on the high-temperature range. The numbers are given in the last two rows of the table. The mean of the six values for the low-temperature range is 4.88 volts, and of the four values for the high-temperature range 6.72 volts. There is thus a difference of nearly two volts in the effect given by the two ranges.

We next attempted to make experiments upon the other metals under such conditions that we knew whether the osmium was on the low- or the high-temperature range. At the outset we obtained values for platinum in the neighbourhood of 7 volts as against the value 5.5 volts obtained in our previous investigation. We believed at the time that we were working on the high-temperature range; but that particular filament burnt out, and later on we found it impossible to get a filament which would develop the two ranges. This situation compelled us to desist from the direct line of attack for the moment; as it is necessary to make a more thorough examination of the thermionic properties of osmium, in order to be able to control the conditions which determine its thermionic emission. We hope to be able to report on this matter at an early date.

If we return for a moment to the table on page 407 it is a striking fact that four of the mean values, viz.: gold, 7.26, copper 7.1, aluminium 7.4, and iron (high) 6.72, are equal, within the range of experimental error, to a common mean value 7.11; whilst the other five, viz.: nickel 5.3, phosphor bronze 5.8, palladium 5.6, silver 5.15, and iron (low) 4.9, are equal within the same limits to the common mean value 5.35. It looks as though the values for gold, copper, and aluminium had all been obtained with the osmium on the high range, and the values for nickel, phosphor bronze, palladium, and silver with the osmium on the low range. It would follow from this that the value of the heating effect is independent of the nature of the metal which receives the electrons, being determined almost entirely by the metal which emits them.

For the reasons which have been stated we have found it impossible to test this question directly, up to the present. The present theory of these phenomena requires that the heating effect should be practically independent of the metal receiving the electrons, as one of the authors has already pointed out\*. It is satisfactory to know that our experimental results, so far as they go, do not conflict with the theory.

In our previous paper we omitted to take into account the possibility of the existence of an electric field between the osmium and the platinum arising from an intrinsic electromotive force; and this omission led us to identify the heating effect with the work done when the electrons escape from hot platinum. The identification should be with the work done when the electrons escape from hot osmium. This quantity has not been measured yet, but we propose to measure it in the course of our investigation of the thermionic properties of osmium already referred to.

The most important facts which we have so far established are:—

(1) The heating effect due to the difference of the potential energy of an electron inside and outside of a conductor, which we previously established for platinum, occurs in the other metals.

(2) The effect is of the same order of magnitude in all cases, the measured values ranging from about 4.5 to 7.5 volts.

(3) The values are influenced very considerably by the nature and state of the thermionic emitter. (The experiments do not preclude the possibility that the true effect is almost independent of the metal receiving the electrons.)

(4) The measured effect is not influenced by changes in the pressure of the residual gas in the apparatus, provided this be reasonably low.

(5) Under certain conditions, not yet completely determined, the thermionic emission from osmium becomes unstable; and there are two ranges of stability, one at low and the other at high temperatures.

We are glad to take this opportunity of thanking Messrs. Baldwin, Carter, Critchlow, Fenger, Frederick, and Gibbs for again assisting in taking the very numerous observations.

Palmer Physical Laboratory,  
Princeton, N.J.

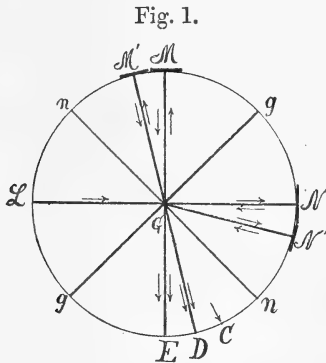
\* O. W. Richardson, *Rapports du Congrès Int. de Radiologie*, Bruxelles, 1910.



XLIX. *Interferometry with the Aid of a Grating.* By CARL BARUS, Ph.D., LL.D., Professor of Physics at Brown University, Providence, U.S.A.\*

Part I.—INTRODUCTION.

1. *Remarks on the Phenomena.*—In the earlier papers † I described certain of the interferences obtained when the oblique plate *gg*, fig. 1, of Michelson's adjustment, is replaced



by a plane diffraction-grating on ordinary plate glass. Some explanation of these is necessary here. In the figure, L is the source of white light from a collimator. Such light is therefore parallel relative to a horizontal plane, but convergent relative to a vertical plane; M and N are the usual silver mirrors. A telescope adjusted for parallel rays in the line GE must, therefore, show sharp white images of the slit. As the grating is usually slightly wedge-shaped, there will be (normally) four such images, two returned by M after reflexion from the front (white) and rear face (yellowish) of the plate *gg*, and two due to N. There will also be two other, not quite achromatic slit images from N or M, respectively, due to *double* diffraction before and after reflexion. These will be treated below. In the direction GD there will thus be a corresponding number of diffraction spectra, more or less coincident in all their parts, and therefore adapted to interfere in pairs throughout their extent. If

\* Communicated by the Author.

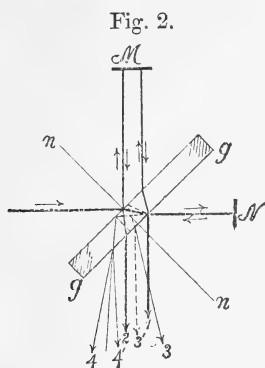
† Abridged from a Report to the Carnegie Institution of Washington, U.S.A. See also Am. Journ. Sci. xxx. 1910, pp. 161-171; Science, July 15, 1910, p. 92; and C. & M. Barus, Phil. Mag. July, 1910, pp. 45-59.

the two white and the two yellow images of the slit be put in coincidence and the mirrors M and N are adjusted for the respective *reduced* or virtual path difference zero, the interferences obtained are usually eccentric; *i. e.* the centres of the interference ellipses are not in the field of view. The effective reflexion in each of these cases takes place from the front and rear face of the grating at the same time. Hence the interference pattern includes the prism angle of the grating plate and is not centred. The air-paths of the component rays are here practically equal. In addition to the ellipses, this self-compensating position also shows revolving linear interferences, and (as a rule) a double set is in the field at once, consisting of equidistant symmetrically oblique crossed lines, passing through horizontality in opposite directions together, when either mirror M or N is suitably displaced.

If either pair of the white and yellowish images of the slit be placed in coincidence when looking along EG, the interference pattern along DG is ring-shaped, usually quasi-elliptic and centred. The light returned by M and N is in this case reflected from the *same* face of the grating, either from the face carrying the grating or the other (unruled) face. The corresponding air-paths of the rays are in this case quite unequal, because the short air-path is compensated by the path of the rays within the glass plate. Hence these adjustments are very different, in one instance GM, in the other GN, being the long path. For the same motion of the micrometer-screw, the fringes as a whole are displaced in opposite directions. In one adjustment there may be a single family of ellipses; in the other there may be two or even three families, nearly in the field at once.

If the grating were cut on optical plate glass, the adjustment for equal air-path would probably be best. But with the grating cut as usual on ordinary plate, or in case of replica gratings on collodium or celluloid films, the adjustment for unequal paths is preferable. Here, again, one of the positions is much to be preferred to the other, owing to the occurrence of multiple slit images from one of the mirrors, as above specified. In fig. 2, for instance, where the grating face is to the rear, there are but two images, 1 and 2, from M if the plate is slightly wedge-shaped; but from N, in addition to these two normal cases (not necessarily coinciding with 1 and 2), there are two other images, 3' and 4', if 3 and 4 are spectrum rays, resulting from double diffraction, with a deviation  $\theta$ , and angle of incidence I, respectively  $\theta < I$  and  $\theta > I$ , in succession; or the reverse.

As the compensation for colour cannot here be perfect, the two slit images obtained are very narrow, practically linear



spectra, but they are strong enough to produce interferences like the normal images of the slits with which they nearly agree in position. Other very faint slit images also occur, but they may be disregarded. The doubly-diffracted slit images are often useful in the adjustments for interference.

Among the normal slit images there are two, respectively white and yellowish, which are remote from secondary images. If these be placed in coincidence both horizontally and vertically along  $EG$ , fig. 1, the observation along  $DG$  through the telescope will show a magnificent display of black, apparently confocal ellipses, with their axes respectively horizontal and vertical, extending through the whole width of the spectrum, from red to violet, with the Fraunhofer lines simultaneously in focus. The vertical axes are not primarily dependent on diffraction, and therefore of about the same angular length throughout; the horizontal axes, however, increase with the magnitude of the diffraction, and hence these axes increase from violet to red, from the first to the second and higher orders of spectra, and in general as the grating space is smaller. It is not unusual to obtain circles in some parts of the spectrum, since ellipses which have long axes vertically in one extreme case, have long axes horizontally in the other extreme case. The interference figure occurs simultaneously in all orders of spectra, and it is interesting to note that even in the chromatic slit images shown in fig. 2, needle-shaped vertical ellipses are quite apparent.

It is surprising that all these interferences may be obtained with replica or film-gratings, though not, of course, so sharply

as with ruled gratings, the ideal being an optical plate. With thin films two sets of interferences are liable to be in the field at once, and I have yet to study these features from the practical point of view. If the film is mounted between two identical plates of glass, rigorously linear, vertical and movable interference fringes, as described \* by my son and myself, may be obtained.

2. *Cause of Ellipses.*—The slit at L (fig. 1) furnishes a divergent pencil of light due (at least) to its diffraction, the rays becoming parallel in a horizontal section after passing the strong lens of the collimator. But the vertical section of the issuing pencil is essentially convergent. Hence, if such a pencil passes the grating the oblique rays relatively to the vertical plane pass through a greater thickness of glass than the horizontal rays. The interference pattern if it occurs is thus subject to a cause for contraction in the former case that is absent in the latter. Hence also the vertical axes of the ellipses are about the same in all orders of spectra. They tend to conform in their vertical symmetry to the regular type of circular ring-shaped figure, as studied by Michelson and his associates, and more recently by Feussner †.

On the other hand, the obliquity in the horizontal direction which is essential to successive interferences of rays is furnished by the diffraction of the grating itself, as the deviation here increases from violet to red. In other words, the interference which is latent or condensed in the normal white linear range of the slit, is drawn out horizontally and displayed in the successive orders of spectra to right and left of it. The vertical and horizontal symmetry of ellipses thus follows totally different laws, the former of which have been thoroughly studied. The present paper will therefore be devoted to the phenomena in the *horizontal* direction only.

At the centre of ellipses the *reduced* path-difference is zero; but it cannot increase quite at the same rate toward red and violet. Neither does the refractive index of the glass admit of this symmetry. Hence the so-called ellipses are necessarily complicated ovals, but their resemblance to confocal ellipses is nevertheless so close that the term is admissible. This will appear in the data.

If either mirror or the grating is displaced parallel to itself by the micrometer-screw, the interference figure *drifts*

\* Phil. Mag. *l. c.*

† See Prof. Feussner's excellent summary in Winkelmann's *Handbuch der Physik*, vol. vi. p. 958 *et seq.* (1906).

as a whole to the right or to the left, while the rings partake of the customary motion toward or from a centre. The horizontal motion in such a case is of the nature of a coarse adjustment as compared with the radial motion, a state of things which is often advantageous. The large divisions of the scale are not lost, in other words. Moreover, the displacements may be used independently.

The two motions are coordinated inasmuch as violet travels toward the centre faster in a horizontal direction, *i. e.* at a greater angular rate, than red. Hence the ellipses drift horizontally but not vertically. Naturally in the two positions specified above for ellipses, the fringes travel in opposite directions for the same motion of the micrometer-screw. As the thickness of the grating is less the ellipses will tend to open into vertical curved lines, while their displacement is correspondingly increased. With the grating on a plate of glass about  $e = .68$  cm. thick, and having a grating space of about  $D = .00351$  cm., at an angle of incidence of about  $45^\circ$ , the displacement of the centre of ellipses from the D to the E line of the spectrum corresponded to a displacement of the grating parallel to itself of about .006 cm. It makes no difference whether the grating side or the plane side of the plate is toward the light or which side of the grating is made the top. If the grating in question is stationary and the mirror N alone moves parallel to itself along the micrometer-screw, a displacement of  $N = .01$  cm. roughly moves the centre of ellipses from D to E, as before. This displacement varies primarily with the thickness of the grating and its refraction. It does not depend on the grating constant. Thus the following data were obtained with film gratings (on different thicknesses  $e$  of glass and different grating spaces  $D$ ) for the displacements,  $N$ , of the mirror at N, to move the ellipses from the D to the E line, as specified:—

Glass grating, ruled .....	$e = .68$ cm.	$\lambda/D = .168$	$N = .010$ cm.
Film on glass plate .....	$e = .57$ „	.352	.008 „
Film on glass plate .....	$e = .24$ „	.352	.003 „
Film between glass plates {	$e = .48$ „	.352	.003 „
	$e' = .24$ „		

Reduced linearly to  $e = .68$  cm., the latter data would be  $N' = .010$  and  $N' = .009$ , which are of the same order and as close as the diffuse interference patterns of film gratings permit. The large difference in dispersion, together with

some differences in the glass, has produced no discernible effect.

An interesting case is the film grating between two *equally* thick plates of glass. With this, in addition to the elliptical interferences above described, a new pattern of vertical interferences identical with those discussed in a preceding paper\* were obtained. These are linear, persistently vertical fringes, extending throughout the spectrum and within the field of view, nearly equidistant and of all colours. Their distances apart, however, may now be passed *through* infinity when the virtual air-space passes through zero; and for micrometer displacements of mirror in a given direction, the motion of fringes is in opposite directions on different sides of the null position of the mirror. I have not been able, however, to make them as strong and sharp as they were obtained in the paper specified.

3. *The Three Principal Adjustments for Interference.*—To compute the extreme adjustments of the grating when the mirror N is moved, fig. 5 (below) may be consulted. Let  $y_g$  be the air-path on the glass side, whereas  $y_a$  is the air-path on the other,  $e$  the thickness of the grating, and  $\mu$  its index of refraction for a given colour. Then for the simplest case of interferences, in the first position N of the mirror, if I is the normal angle of incidence and R the normal angle of refraction for a given colour,

$$y_g + e\mu/\cos R = y_a$$

for equal paths. Similarly, in the second position of the mirror,

$$y'_a = y'_g + e\mu/\cos R.$$

Hence, if the displacement at the mirror be N, as the figure shows,

$$y_a - y'_g = 2e \tan R \sin I,$$

$$N = 2e\mu \cos R.$$

The measured value of this quantity was about 1.88 cm. The computed value would be  $2 \times .68 \times 1.53 \times .71 = 1.84$  cm. The difference is due to the wedge-shaped glass which requires a re-adjustment of the grating for the two positions.

The corresponding extreme adjustments, when the grating  $gg$  instead of the mirror N is moved over a distance  $z$ , are in like manner found to be

$$z = \frac{e\mu}{\cos I \cos R} - e \tan I \tan R.$$

\* Phil. Mag. *l. c.*

The observed value of  $z$  for the two positions was about 1.3 cm. The computed value for  $I=45^\circ$  was the same.

On the other hand, when reflexion takes place from the same face of the grating while the latter is displaced  $z$  cm. parallel to itself, the relations of  $y$  and  $z$  for normal incidence at an angle  $I$  are obviously

$$y \cos I = z.$$

For an oblique incidence  $i$ , where  $i - I = \alpha$ , a small angle, the equation is more complicated. In this case

$$y = z \left( 1 + 2 \frac{\sin \alpha}{\cos i} \sin I \right) / \cos I.$$

This equation is also true for a grating of thickness  $e$ , whose faces are plane parallel. For the direction of the air-rays in this case remains unchanged.

Finally, a distinction is necessary between the path difference  $2y$  and the motion of the opaque mirror  $2N$  which is equivalent to it, since the light is not monochromatic. This motion  $2N$  is oblique to the grating, and if the rays *differ in colour* further consideration is needed. For the simplest type of interferences, in which the glass path difference, as in fig. 5, is  $e\mu/\cos R$ , and for normal incidence at an angle  $I$ , let the upper ray  $y_M$  and the grating be fixed. The mirror moving over the distance  $\Delta N$  changes the zero of path difference from any colour of index of refraction  $\mu_D$  to another of index  $\mu_E$ , while  $y_{ND}$  passes to  $y_{NE}$ . Then a simple computation shows

$$\Delta N = e(\mu_D \cos R_D - \mu_E \cos R_E).$$

This difference belongs to all rays of the same colour difference, or for two interpenetrating pencils.

If reflexion takes place from the lower face the rays are somewhat different, but the result is the same. If the plate of the grating is a wedge of small angle  $\phi$ , the normal rays will leave it on one side at an angle  $I + \delta$ , where  $\delta$  is the deviation

$$\delta = (\mu - 1)\phi.$$

The mirror  $N$  will also be inclined at an angle  $I + \delta$ , to return these rays normally. We may disregard  $d\delta = \phi d\mu$ , if  $\phi$  is small.

Part II.—DIRECT CASES OF INTERFERENCE :  
DIFFRACTION ANTECEDENT.

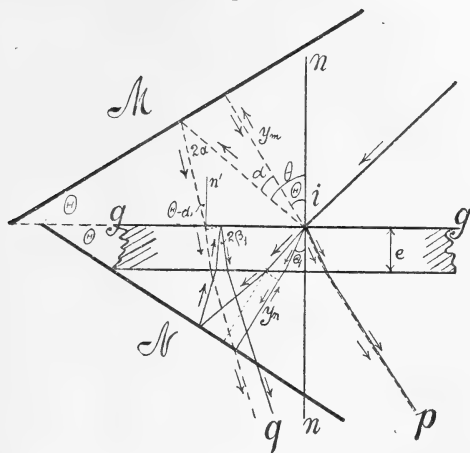
4. *Diffraction before Reflexion.*—If in fig. 1 the diffracted beams (spectra) be returned by the mirrors  $M'$  and  $N'$  to be reflected at the grating, interference must also be producible along  $GD$ . Again there will be three primary cases: if reflexion takes place from both faces of the grating at once, the air-paths must be nearly equal, the grating itself acting as a compensator. The interference pattern is ring-shaped, but, as usual, very eccentric. If the reflexion of both component beams takes place from the same face of the grating, the interference pattern is elliptical and centred, and the air-paths are unequal. The case is then similar to the preceding §§ 1, 2; but there is no direct or normal image of the slit, as  $M$  and  $N$  are absent. In its place there may be *chromatic* images of the slit (linear spectra) at  $C$  due to the specified double diffraction of each component beam in a positive and negative direction, successively. But these chromatic images are nevertheless sharp enough to complete the adjustments for interference by placing the slit images in coincidence. It is not usually necessary to put the spectrum lines into coincidence separately (both horizontally and vertically), as was originally done, both spectra being observed. Again, along  $GE$  (fig. 1) approximately, there must be two successive positive diffractions of each component beam, which would correspond closely to the second order of diffraction. The advantage of this adjustment lies in the fact that there are but two slit images effectively returned by  $M'$  and  $N'$ , and hence these interferences were at first believed to be stronger and more isolated. As a consequence I used this method in most of my early experiments, before finding the equally good adjustment described in the preceding sections.

5. *Elementary Theory.*—To find the path differences fig. 3 may be consulted. the grating face is shown at  $gg$ , the glass plate being  $e$  cm. in thickness below it, and  $n$  is the normal to the grating.  $M$  and  $N$  are two opaque mirrors, each at an angle  $\Theta$  to the face of the grating. Light is incident on the right at an angle  $i$  nearly  $45^\circ$ . In both figures the rays  $y_m$  and  $y_n$  (air-paths) diffracted at an angle  $\Theta$  in air, are reflected normally from the mirrors  $M$  and  $N$  respectively, and issue toward  $p$  for interference. The rays  $y_n$  pass through glass. Both figures also contain two component rays diffracted at an angle  $\theta$  in air, where  $\theta - \Theta = \alpha$ ,



and reflected obliquely at the mirrors M and N, thus enclosing an angle  $2\alpha$  in air and  $2\beta_1$  in glass and issuing

Fig. 3.



toward  $q$ . These component rays are drawn in full and dotted respectively. There may be two incident rays for a single emergent ray, or, as in fig. 3, a single incident ray for two emergent rays interfering in the telescope. The treatment of the two cases is different in detail; but as the results must be the same they corroborate each other.

The notation used is as follows:—Let  $e$  be the normal thickness of the grating,  $e'$  the effective thickness of the compensator when used. Let distance measured normal to the mirror be termed  $y$ ,  $y_n$  and  $y_m$  being the component air-paths passing on the glass and on the air side of the grating  $gg$ , so that  $y_m > y_n$ . Let  $y = y_m - y_n$  be the air-path difference. Similarly let distances  $z$  be measured normal to the grating, so that  $z$  may refer to displacements of the grating.

Let  $i$  be the angle of incidence,  $r$  the angle of refraction, and  $\mu_r$  the index of refraction for the given colour, whence

$$\sin i = \mu_r \sin r.$$

Similarly, if  $\Theta$  is the angle of diffraction in air of the  $y$  rays and  $\Theta_1$  the corresponding angle in glass,

$$\sin \Theta = \mu_\Theta \sin \Theta_1; \quad . . . . . (1)$$

and if  $\theta$  is the angle of diffraction of any oblique ray in air and  $\theta_1$  the corresponding angle in glass,

$$\sin \theta = \sin (\Theta + \alpha) = \mu_\Theta \sin \theta_1 . . . . . (2)$$

Here  $\theta > \Theta$ , so that  $\alpha = \theta - \Theta$  is the deviation of the oblique ray from the normal ray in air. Finally, the second reflexion of the oblique ray necessarily introduces the angle of refraction within the glass, such that

$$\sin(\Theta - \alpha) = \mu_\theta \sin \beta_1 \dots \dots \dots (3)$$

If D is the grating space, moreover,

$$\sin i - \sin \theta = \lambda_\theta / D \quad \text{and} \quad \sin i - \sin \Theta = \lambda / D \dots (4)$$

6. *Equations for the Present Case.*—From a solution of the triangles preferably in the case for single incidence, as in fig. 8, the air-path of the upper oblique component ray at a deviation,  $\alpha$ , is

$$2y_m \cos \Theta / \cos(\Theta - \alpha).$$

The air-path of the lower component ray is

$$2 \cos \Theta (y_n - e \sin \Theta (\tan \theta_1 - \tan \Theta_1)) / \cos(\Theta - \alpha).$$

The optic path of this (lower) ray in glass as far as the final wave-front in glass at  $\beta_1$  is

$$\mu_\theta e (1 / \cos \theta_1 + 1 / \cos \beta_1).$$

The optic path of the upper ray as far as the same wave-front in glass is

$$\mu_\theta \sin \beta_1 \left\{ 2 \frac{\sin \alpha}{\cos(\Theta - \alpha)} (y_m - y_n + e \sin \Theta (\tan \theta_1 - \tan \Theta_1)) - e (\tan \theta_1 - \tan \beta_1) \right\} \dots (5)$$

Hence the path difference between the lower and the upper ray as far as the final wave-front is, on collecting similar terms, if the coefficients of  $y_m - y_n = y$  (where  $y$  is positive) and of  $e$  be brought together, as far as possible, and if the path difference corresponds to  $n$  wave-lengths  $\lambda_\theta$ ,

$$n\lambda_\theta = -2y \cos \alpha + 2e\mu_\theta \cos \alpha (1 / \cos \Theta_1 - \cos(\theta_1 - \Theta_1) / \cos \theta_1) + e\mu_\theta (1 + \cos(\theta_1 - \beta_1)) / \cos \theta_1, \dots (6)$$

which is the full equation in question for a dark fringe. It is unfortunately very cumbersome and for this reason fails to answer many questions perspicuously. It will be used below in another form.

The equation refers primarily to the horizontal axes of the ellipses only, as  $e$  increases vertically above and below this line. The equation may be abbreviated, and in case of a parallel compensator of thickness  $e'$  ( $y$  being the path difference in air) may be written

$$n\lambda_\theta = -2y \cos \alpha + (e - e') (Z_\Theta - Z_\theta) \dots (7)$$

If  $e=e'$ , i. e. for an infinitely thin plate  $e=0$ ,

$$n\lambda_\theta = 2y \cos \alpha \dots \dots \dots (8)$$

Again, for normal rays  $y_m$  and  $y_n$ ,  $\alpha=0$  and  $\Theta_1=\theta_1=\beta_1$ . Hence, for a parallel compensator of thickness  $e'$ ,

$$n\lambda_\Theta = -2y + 2(e-e')\mu_\Theta / \cos \Theta_1 \dots \dots \dots (9)$$

If  $\mu_\Theta=1=\mu_\theta$ , equation (6) reduces to

$$n\lambda = 2 \cos \alpha (y + e' / \cos \Theta),$$

clearly identical with the case  $e=0$  for a different  $y$ .

In view of the complicated nature of equation (6) I have computed path differences for a typical case of light crown glass †, as shown in Table I., for the Fraunhofer lines B, D, E, F, G, supposing the E ray to be normal ( $\alpha=0$ ). The

TABLE I.

Table for Path Difference,  $-2(y_m - y_n) \cos \alpha + eZ_\Theta + eZ_\theta$ ,  
 $= -2y \cos \alpha + eZ$ . Light crown glass. E rays normal to mirrors. Grating space  $D=0.000351$  cm.  $I=45^\circ$ .  
 $\Theta=33^\circ 51'$ .

Spectrum lines...	=	B.	D.	E.	F.	G.	
	$\lambda \times 10^6 =$	68.7	58.93	52.70	48.61	43.08	cm.
	$\mu_\theta =$	1.5118	1.5153	1.5186	1.5214	1.5267	
	$\theta =$	$30^\circ 46'$	$32^\circ 38'$	$33^\circ 51'$	$34^\circ 40'$	$35^\circ 46'$	
	$\theta_1 =$	$19^\circ 47'$	$20^\circ 51'$	$21^\circ 31'$	$21^\circ 57'$	$22^\circ 30'$	
	$\alpha =$	$-3^\circ 5'$	$-1^\circ 13'$	$0^\circ 0'$	$0^\circ 49'$	$1^\circ 55'$	
	$\theta_1 - \theta =$	$-1^\circ 44'$	$-0^\circ 40'$	$0^\circ 0'$	$0^\circ 26'$	$0^\circ 59'$	
	$\beta_1 =$	$23^\circ 25'$	$22^\circ 17'$	$21^\circ 30'$	$21^\circ 13'$	$20^\circ 49'$	
	$\theta_1 - \beta_1 =$	$-3^\circ 38'$	$-1^\circ 26'$	$0^\circ 0'$	$0^\circ 44'$	$1^\circ 41'$	
	$2 \cos \alpha =$	1.9970	1.9996	2.0000	1.9998	1.9996	cm.
	$Z_\Theta =$	.0383	.0176	.0000	-.0095	-.0219	cm.
	$Z_\theta =$	3.2100	3.2404	3.2648	3.2805	3.3058	cm.
Path Difference*	$n\lambda = \left\{ \begin{array}{l} -1.9970 \\ +3.2483 \end{array} \right.$	-1.9970	-1.9996	-2.0000	-1.9998	-1.9996	} cm.
$y=1$ cm.; $e=1$ cm.							
The same for $2y=3.2648$ cm.; $e=.68$ cm.	$2\Delta y_0 =$	-.0079	-.0042	$\pm 0$	+ .0040	+ .0120	cm.
	$2\Delta N_0 =$	-.0107	-.0056	$\pm 0$	+ .0049	+ .0156	cm.

\* The purpose of these data is merely to elucidate the equations.  $\Delta N$  refers to the displacement of either opaque mirror, M or N.

† Taken from Kohlrausch's 'Practical Physics,' 11th edition, 1910, p. 712.

Table is particularly drawn up to indicate the relative value of the functions  $Z_{\Theta}$  and  $Z_{\theta}$ . It will be seen that  $Z=Z_{\Theta}+Z_{\theta}$ , in terms of  $\lambda$ , is a curve of regular decrease having no tendency to assume maximum or minimum values within the range of  $\lambda$ , whereas  $y \cos \alpha$  passes through the usual flat maximum for  $\alpha=0$ . The path difference, which is the difference of the ordinates of these curves, thus passes through zero for a definite value of  $e$  and  $y$ , and this would at first sight seem to correspond to the centre of ellipses. That it *does not* so correspond will be particularly brought out in the next section. It is obvious that for a given  $e$  the value of  $y$  (air-path difference) which makes the total path difference zero, varies with the wave-length, hence on increasing  $y$  continuously the ellipses must pass through the spectrum. It is also obvious that if the grating is reversed, the path difference will change sign, *cut. par.*, and the ellipses will move in a contrary direction, for the same displacements  $y$  of the micrometer-screw, at the mirror M or N respectively.

If  $e=1$  cm. and  $y=3.2648$  cm., the path difference will be zero for the E ray. The Table shows the residual path differences in the red and blue parts of the spectrum. The ellipses will be larger as  $e$  is smaller, the limit of enormous ellipses being reached for  $e=0$  or  $e=e'$  of the compensator.

The same result may be obtained to better advantage by reconstructing equation (6), and making the path difference equal to zero for the normal ray. This determines

$$y=e\mu_{\theta}/\cos \Theta$$

in terms of  $e$ , and the path difference is now free from  $y$ .

7. *Interferometer.*—If, in equation (6),  $y$  alone is variable with the order of fringe  $n$ , while  $\alpha$ ,  $\theta$ ,  $\Theta$ ,  $\theta_1$ ,  $\Theta_1$ ,  $\beta_1$ ,  $e$ ,  $\lambda$ ,  $\mu$ , are all constant; *i. e.* if the number of fringes  $n$  cross a given fixed spectrum line like the D line, when the mirror is displaced over a distance  $y$ , it appears that

$$\frac{dy}{dn} = \frac{\lambda_{\theta}}{2 \cos \alpha} = \frac{dN}{dn}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

where  $\alpha$  refers to the deviation of  $\lambda_{\theta}$  from  $\lambda_{\Theta}$  normal to the mirror. If  $\alpha=0$ ,  $dy/dn=\lambda/2$ , the limiting sensitiveness of the apparatus, which appears for the case of normal rays.

The displacement per fringe  $dy/dn$  or  $dz/dn$ , varies with the wave-length. Hence if the ellipses are nearly symmetrical on both sides of the centre, *i. e.*, if the red and violet sides

of the periphery are nearly the same, or the fringes nearly equidistant, the smaller wave-length will move faster than the larger for a given displacement of mirror. It is at first quite puzzling to observe the motion of ellipses as a whole in a direction opposed to the motion of the more reddish fringes when these are alone in the field.

8. *Discrepancy of the Table.*—The data of Table I. are computed supposing that the path difference zero corresponds to the centre of ellipses. This assumption has been admitted for discussion only and the inferences drawn are qualitatively correct. Quantitatively, however, the displacement of about  $\Delta N = .003$  cm. should move the centre of ellipses from the D to the E line of the spectrum, whereas observations show that a displacement of either opaque mirror of about  $\Delta N = .01$  cm. is necessary for this purpose; *i. e.* over three times as much displacement as has been computed. The equation cannot be incorrect; hence the assumption that the centre of ellipses corresponds to the path difference zero is not vouched for and must be particularly examined. This may be done to greater advantage in connexion with the next section, where the conditions are throughout simpler, but the data of the same order of value.

### Part III.—DIRECT CASE : REFLEXION ANTECEDENT.

9. *Equations for this Case.*—If reflexion at the opaque mirrors takes place before diffraction at the grating, the form of the equations and their mode of derivation is similar to the case of paragraph 6, but the variables contained are essentially different. In this case the deviation from the normal ray is due not to diffraction but to the angle of incidence, and the equations are derived for homogeneous light of wave-length  $\lambda$  and index of refraction  $\mu$ .

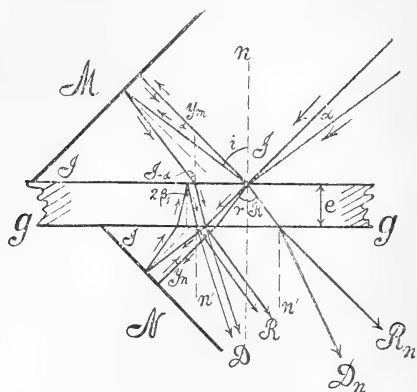
In fig. 4, let  $y_n$  and  $y_m$  be the air-paths of the component rays, the former first passing through the glass plate of thickness  $e$ . Let the angle of incidence of the ray be  $I$ , so that  $y_m$  and  $y_n$  are returned normally from the mirrors M and N respectively, these being also at an angle  $I$  to the plane of the grating. Let  $i$  be the angle of incidence of an oblique ray, whose deviation from the normal is  $i - I = \alpha$ . Let  $R$ ,  $r$ , and  $\beta_1$  be angles of refraction such that

$$\sin i = \sin (I + \alpha) = \mu \sin r ; \quad \sin I = \mu \sin R ;$$

$$\sin (I - \alpha) = \mu \sin \beta_1.$$

The face of the grating is here supposed to be away from the incident ray, as shown at *gg* in fig. 4.

Fig. 4.



Then it follows, as in paragraph 6, *mut. mut.*, that the path difference is (if  $y = y_m - y_n$ )

$$\begin{aligned}
 n\lambda &= -2y \cos \alpha + \mu e \left\{ \frac{2 \cos \alpha}{\cos R} - \frac{2 \cos \alpha \cos (r - R)}{\cos r} + \frac{1 + \cos (r - \beta_1)}{\cos r} \right\} \\
 &= -2y \cos \alpha + e(Z_1 - Z_2 + Z_3), \dots \dots \dots (11)
 \end{aligned}$$

where  $n$  is the order of interference (whole number). This equation is intrinsically simpler than equation (6), since  $\mu_r = \mu_R$  as already stated, and since  $\alpha$  is constant for all colours, or all values of  $\mu$  and  $\lambda$  in question.  $R$  and  $r$  replace  $\Theta_1$  and  $\theta_1$ .

In most respects the discussion of equation (11) is similar to equation (6) and may be omitted in favour of the special interpretation presently to be given. If  $\mu = 1$ , equation (11) like equation (6) reduces to the case corresponding to  $e = 0$ , with a different  $y$  normal to the grating.

All the colours are superposed in the direct images of the slit,  $R_n$  and  $R$  (fig. 4), seen in the telescope, and the slit is therefore white. This shows also that prismatic deviation due to the plate of the grating (wedge) is inappreciable. The colours appear, however, when the light of the slit is analysed by the grating in the successive diffraction spectra,  $D_n$  or  $D$ , respectively. In equation (11),  $\mu$  is a function of  $\lambda$ , and hence of the deviation  $\theta$  produced by the grating, since

$$\sin (I - \alpha) - \sin \theta = \lambda / D.$$

The values of equation (11) for successive Fraunhofer lines and for  $\alpha=3^\circ$  have been computed in Table II. for the

TABLE II.

Path Difference,  $-2(y_m - y_n) \cos \alpha + eZ_1 - eZ_2 + eZ_3$ . Light crown glass.  $\alpha = 3^\circ$  throughout. Grating space 0.000351 cm.  $I=45^\circ$ .  $e=1$  cm.

Spectrum lines...	=	B.	D.	E.	F.	G.	
	$\lambda \times 10^6 =$	68.7	58.93	52.70	48.61	43.08	cm.
	$\mu =$	1.5118	1.5153	1.5186	1.5214	1.5267	
	$R =$	$27^\circ 53'$	$27^\circ 49'$	$27^\circ 45'$	$27^\circ 42'$	$27^\circ 35'$	
	$r =$	$29^\circ 26'$	$29^\circ 22'$	$29^\circ 18'$	$29^\circ 14'$	$29^\circ 8'$	
	$\beta_1 =$	$26^\circ 16'$	$26^\circ 12'$	$26^\circ 8'$	$26^\circ 5'$	$26^\circ 0'$	
	$\alpha =$	$3^\circ 0'$	$3^\circ 0'$	$3^\circ 0'$	$3^\circ 0'$	$3^\circ 0'$	
	$r - R =$	$1^\circ 33'$	$1^\circ 33'$	$1^\circ 33'$	$1^\circ 33'$	$1^\circ 33'$	
	$r - \beta_1 =$	$3^\circ 10'$	$3^\circ 10'$	$3^\circ 10'$	$3^\circ 9'$	$3^\circ 8'$	
	$Z_1 =$	3.4160	3.4218	3.4272	3.4318	3.4403	cm.
	$Z_2 =$	656	714	767	808	896	cm.
	$Z_3 =$	689	747	799	841	928	cm.
Path Difference, $\alpha=3^\circ$ ; $e=1$ cm. ; $y=1$ cm.	$n\lambda \left\{ \begin{array}{l} -1.9992 \\ +3.4193 \end{array} \right.$	$\left\{ \begin{array}{l} -1.9992 \\ +3.4252 \end{array} \right.$	$\left\{ \begin{array}{l} -1.9992 \\ +3.4304 \end{array} \right.$	$\left\{ \begin{array}{l} -1.9992 \\ +3.4352 \end{array} \right.$	$\left\{ \begin{array}{l} -1.9992 \\ +3.4436 \end{array} \right.$		

The data are merely intended to elucidate the equations. The values  $Z$  are nearly equal. So also the cases for  $\alpha=3^\circ$  and  $\alpha=0^\circ$ .

same glass treated in Table I., the data being similar and in fact of about the same order of value. The feature of this Table is the occurrence of nearly constant values,  $\alpha=i-I$ ,  $r-R$ , and  $r-\beta_1$ , throughout the visible spectrum. Hence if the following abbreviations be used:

$$A = \cos \alpha = .9986, \quad B = \cos (r - R) = .9996, \\ C = 1 + \cos (r - \beta_1) = 1.9984,$$

$A, B, C$  are practically functions of  $\alpha$  only and do not vary with colour or  $\lambda$ . Furthermore, if the path difference is annulled at the E line, equation (11) reduces to

$$n\lambda = 2eA(\mu/\cos R - \mu \cos R_E) + e(C - 2AB)\mu/\cos r,$$

where  $2eA$  and  $e(C - 2AB)$  are nearly independent of  $\lambda$  and  
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the last quantity is relatively small. The two terms of this equation for  $e=1$  cm. show about the following variation :—

Line...	B.	D.	E.	F.	G.
—	·0111	— ·0052	·0	+ ·0048	·0132
—	·000009	— ·000004	·0	+ ·000005	·000013

Hence for deviations even larger than  $\alpha=3^\circ$ , the path difference does not differ practically from the path difference for the normal ray. Thus it follows that the equation

$$n\lambda = -2y + 2e\mu/\cos R$$

is a sufficient approximation for such purposes as are here in view.

Finally, for  $e=.68$  cm. the actual thickness of the plate of the grating,  $y$  and  $N$ , the semi-path difference and the displacement of the opaque mirror, will be :—

	B.	D.	E.	F.	G.
$y_0 =$	1·1630	1·1650	1·1668	1·1686	1·1713
$\Delta y_0 =$	— ·0038	— ·0018	·0	+ ·0016	+ ·0045
$\delta N_0 =$	+ ·0014	+ ·0007	·0	— ·0005	— ·0018
$\Delta N_0 =$	— ·0052	— ·0025	·0	+ ·0021	+ ·0063

where  $\delta N_0$  is the colour correction of  $\Delta y_0$ , and  $\Delta N_0 = \Delta y_0 - \delta N_0$  determines the corresponding displacements of the opaque mirror ; or, more briefly,

$$N_0 = y_0 - e\mu \tan R \sin R = 2e\mu \cos R.$$

The data actually found were (the B line being uncertain) :—

Lines.....	B.	D.	E.	F.
$\Delta N_0 =$	— ·0153	— ·0078	·0	+ ·0057

These results are again from 2 to 4 times larger than the computed values. True the glass on which the grating was cut is not identical with the light crown glass of the tables ; but nevertheless a discrepancy so large and irregular is out of the question. It is necessary to conclude, therefore, that here, as in paragraph 8, the assumption of a total path difference zero for the centre of ellipses is not true. In other words, the equality of air-path difference and glass-path does not correspond to the centres in question. It is now in place to examine this result in detail.

10. *Divergence per Fringe.*—The approximate sufficiency of the equation

$$n\lambda = 2e\mu/\cos R - 2y \dots \dots (12)$$

makes it easy to obtain certain important derivatives among



which  $d\theta/dn$  (where  $\theta$  is the angle of diffraction corresponding to the angle of incidence  $I$ , and  $n$  the order of interference) is prominent.

If  $e$  and  $y$  are constant,  $n$ ,  $\lambda$ ,  $\mu$ , and  $R$  variable, the differential coefficients may be reduced successively by the following fundamental equations,  $D$  being the grating space,  $\mu$  and  $r$  corresponding to wave-length  $\lambda$  and angle of diffraction  $\theta$  :—

$$dR = -\tan R \cdot d\mu/\mu, \quad . . . . (13)$$

$$d\lambda = -D \cos \theta \cdot d\theta, \quad . . . . (14)$$

$$d\mu/\mu = -a d\lambda/\lambda, \quad . . . . (15)$$

where (as a first approximation)  $a = \cdot 015$  is an experimental correction, interpolated for the given glass. Incorporating these equations it is found that

$$\frac{d\theta}{dn} = \frac{\lambda^2}{2D \cos \theta} \frac{\cos R}{e\mu - y \cos R + ae\mu(1 - \tan^2 R)}. \quad . (16)$$

If the path difference  $n\lambda$  is annulled,

$$\frac{d\theta_0}{dn} = \frac{\lambda^2}{2D \cos \theta} \frac{\cos R}{ae\mu(1 - \tan^2 R)}, \quad . . . (17)$$

which is the deviation per fringe, *supposedly* referred to the centre of ellipses.

These equations indicate the nature of the dependence of the horizontal axes of ellipses on  $1/D$  (hence also on the order of the spectrum),  $1/e$ ,  $1/\mu$ , and  $1/a$ , where the meaning of  $a$ , here an important variable, is given in equation (15). If instead of the path difference  $y$  the displacement  $N$  of either opaque mirror is primarily considered (necessarily the case in practice), the factor  $(1 - \tan^2 R)$  vanishes.

Table III. contains a survey of data for equations (16) and (17). The results for  $d\theta_0/dn$  would be plausible, as to order of values. The data for  $d\theta/dn$ , however, are again necessarily in error, as already instanced above, paragraph 8. They do not show the maximum at  $E$ , and the  $\lambda$ -effect is overwhelmingly large.

Since equation (17) is clearly inapplicable, giving neither maxima nor counting the fringes, it follows that in this equation  $y > e\mu/\cos R$ ; *i. e.*, the centres of ellipses are not in correspondence with the path difference zero. In other words, the air-path difference is larger than the glass-path difference in such a way that  $d\theta/dn$  is equal to  $\infty$  for centres (here at the  $E$  line), but falls off rapidly toward both sides of the spectrum.

There is another feature of importance which must now be accentuated. In case of different colours, and stationary mirrors and grating,  $y$  is not constant from colour to colour, whereas

$$2N = 2y - 2e\mu \sin R \tan R$$

is constant for all colours, as has been shown above. Thus the equation to be differentiated for constancy of adjustment but variable colour loses the variable  $y$  and becomes

$$n\lambda = 2e\mu \cos R - 2N.$$

$N$  here is the difference of perpendicular distances to the mirrors,  $M$  and  $N$ , from the ends of the normal in the glass plate, at the point of incidence of the white ray. Performing the operations:

$$\frac{d\theta}{dn} = \frac{\lambda^2}{2D \cos \theta} \frac{\cos R}{e\mu(1+a) - y \cos R} \quad \dots \quad (18)$$

Hence maximum  $d\theta/dn = \infty$  at the centres of ellipses which occur for

$$y = \frac{e\mu(1+a)}{\cos R}$$

The value of  $d\theta/dn$  for the different Fraunhofer lines above, if  $a = .015$  is still considered constant and  $e = .68$ , is given in Table III.

TABLE III.

Values of  $d\theta/dn$  and displacements of mirror,  $N_c$ .  $e = .68$  cm.  $N_c$  and  $y_c$ , etc., refer to centres of ellipses.

Spectrum lines...	=	B.	D.	E.	F.	G.
Equation 17:	$d\theta/dn =$	1' 27"	1' 5"	53"	46"	35"
„ 16:	$d\theta/dn =$	3' 4"	1' 52"	1' 20"	59"	33"
„ 18:	$d\theta/dn =$	-6' 53"	-10' 38"	$\infty$	+8' 47"	+3' 51"
	$-d\mu/d\lambda =$	281	445	623	793	1140
„ 19:	$d\theta/dn =$	-1' 54"	-1' 56"	$\infty$	+2' 21"	+40"
„ 20:	$d\theta/dn =$	-1' 44"	-1' 44"	$\infty$	+2' 11"	+36"
„ 21:	$y_c =$	1.1779	1.1811	1.1921	1.1981	1.2090
	$\Delta y_c =$	-.0142	-.0110	$\pm 0$	+0.0060	+0.0169
	$N_c =$	.9235	.9274	.9391	.9456	.9578
	$\Delta N_c =$	-.0156	-.0117	$\pm 0$	+0.0065	+0.0187
(observed)	$\Delta N_c =$	-.0153	-.0098	$\pm 0$	+0.0057	...
Equation 20:*	$d\theta/dn =$	...	-1' 54"	$\infty$	2' 21"	48"

\* Interpolated between D and G by  $\mu = a + b\lambda + c\lambda^2$ , where  $b = -.00273$ ,  $c = .0000197$ .

These results show that the distance apart of fringes on the two sides of the centre of ellipses is not very different, though they are somewhat closer together in the blue than in the red end of the spectrum, as observed. There is thus an approximate symmetry of ovals, and  $d\theta/dn$  falls off very fast on both sides of the infinite value at the centre.

The observed angle between the Fraunhofer lines D and E for the given grating was  $4380''$ . The number of fringes between D and E would thus be even less than  $4380''/638'' = 6.7$  only, which is itself about 4 times too small. The cause of this is then finally to be ascribed to the assumed constancy of  $-a = (d\mu/\mu)/(d\lambda/\lambda)$ , a discrepancy still to be remedied. We may note that  $a$  does not now enter as directly as appeared in equation (16).

By replacing  $a$  by its equivalent, equation (18) takes the form

$$\frac{d\theta}{dn} = \frac{\lambda^2}{2D \cos \theta} \frac{1}{\frac{e}{\cos R} \left( \mu - \lambda \frac{d\mu}{d\lambda} \right) - y}, \quad \dots \quad (19)$$

and a definite series of values may be obtained by computing  $d\mu/d\lambda$ ; but as all experimental reference here is, practically, not to path differences but to displacements of the movable opaque mirror N, the form of the equation applicable is

$$\frac{d\theta}{dn} = \frac{\lambda^2}{2D \cos \theta} \frac{1}{e \left( \mu \cos R - \frac{\lambda}{\cos R} \frac{d\mu}{d\lambda} \right) - N}. \quad \dots \quad (20)$$

To make the final reduction, I have supposed that for the present purposes a quadratic interpolation of  $\mu$  between the B and the  $y$  lines of the spectrum would suffice. Taking the E line as fiducial, I have therefore assumed an equation for short ranges, corresponding to Cauchy's in simplified form,

$$\mu - \mu_E = b / (1/\lambda^2 - 1/\lambda_E^2),$$

in preference to the more complicated dispersion equations. From the above data for light crown glass we may then put roughly,  $b = 4.56 \times 10^{-20}$  and  $d\mu/d\lambda = -2b/\lambda^3$ . Thus I found the remaining data of Table III. The results for  $d\theta/dn$  agree as well with observations as may be expected. The ovals resemble ellipses, but are somewhat coarser on the red side, as is the case.

The centres of ellipses are thus defined by the semi air-path equation

$$y_c = \frac{e}{\cos R} \left( \mu - \lambda \frac{d\mu}{d\lambda} \right) = \frac{e}{\cos R} (\mu + 2b/\lambda^2), \text{ nearly,} \quad (21)$$

or the corresponding equation in terms of  $N_c$ . The trend data for  $\Delta N_c$  agree fairly well with observation, except at the D line, which difference is very probably referable to the properties of the glass, since the grating was not cut on light crown.

The number of fringes between the D and E lines now comes out plausibly, being *less* than  $4380''/104''=42$ . It is difficult to count these fringes without special methods of experiment; but the number computed is a reasonable order of values, about 25 to 30 lines being observed.

Some estimate may finally be attempted as to the mean displacement of mirror  $\delta N$  per fringe, between the D and E lines. As their deviation is  $\theta=1^\circ 13'$  and the displacement from D to E  $\Delta N_c$ ,

$$\delta N = \frac{\Delta N_c}{\theta/(d\theta/dn)} = \frac{d\theta}{dn} \frac{\Delta N_c}{\theta} > \frac{\lambda^2}{2D \cos \theta} \frac{1}{\Delta N_c} \frac{\Delta N_c}{\theta},$$

if the value of  $d\theta/dn$  for the D line be taken. Thus

$$\delta N > \lambda^2/(2D \cos \theta) = \lambda^2/D \sin 2\theta, \text{ nearly.}$$

Hence  $\delta N$  is independent of the thickness,  $e$ , of the plate of the grating, as I showed\* by using a variety of different thicknesses of compensator. Since

$$\lambda = 000059 \text{ cm., } D = 000351 \text{ cm., } \delta N > 00031 \text{ cm.}$$

The values found were between 00033 and 00039, naturally difficult to measure, but of the order required.

11. *Case of  $d\lambda/dy$ , and  $d\theta/dy$ , etc.*—If, in equation (12),  $e$  and  $n$  are constant while  $\mu$ ,  $R$ ,  $y$ , and  $\lambda$  vary, the micrometer equivalent of the displacement of fringes may be found. Here

$$d\mu/dy = (d\mu/d\lambda) \cdot (d\lambda/dy)$$

and

$$dR/dy = (dR/d\mu) \cdot (d\mu/d\lambda) \cdot (d\lambda/dy),$$

which coefficients are given by equations (13), (14), and (15).

Centres correspond to

$$N_c = e\mu \cos R - \frac{e\lambda}{\cos R} \frac{d\mu}{d\lambda}.$$

\* American Journal of Science, xxx. 1910, p. 170.

TABLE IV.

Values of  $d\lambda/dN$ ,  $d\theta/dN$ .  $e = \cdot 68$  cm.

Spectrum lines...	=	B.	D.	E.	F.	G.	
Equation 22: *	$d\lambda/dN =$	...	-0055	$\infty$	+0081	+0031	cm.
„ 22:	$d\lambda/dN =$	-0044	-0050	$\infty$	+0075	+0023	cm.
„ 22:	$d\theta/dN =$	-14.6	-17.0	$\infty$	+26.0	+8.1	rad.

\* Constants interpolated between D and G by  $\mu = a + b\lambda + e\lambda^2$ ,  
where  $b = -00273$ ,  $c = 0000197$ .

Thus

$$\frac{d\lambda}{dN} = \frac{\lambda}{N - e\mu \cos R + e\lambda(d\mu/d\lambda)/\cos R}; \quad \dots \quad (22)$$

so that if  $d\lambda/dN = \infty$ , the maximum at the centres of ellipses, the simultaneous effect at  $\lambda'$  will be (as the mirror has not moved)

$$\frac{d\lambda'}{dN} = \frac{\lambda'}{e(\mu \cos R - \mu' \cos R') - e\left(\frac{\lambda}{\cos R} \frac{d\mu}{d\lambda} - \frac{\lambda'}{\cos R'} \frac{d\mu'}{d\lambda'}\right)}$$

If the centre of ellipses is at the E line the values of Table IV. hold. The motion on the blue side of the E line is thus larger than the simultaneous motion on the yellow side, conformably with observation.

12. *Interferometry in Terms of Radial Motion.*—Either by direct observation or combining the equations (20) and (22) for  $d\lambda/dn$  and  $d\lambda/dN$ , the usual equation for radial motion again results:

$$\frac{dN}{dn} = \frac{\lambda}{2'}$$

where N is the displacement of mirror per fringe. This equation is best tested on an ordinary spectrometer by aid of a thin compensator of microscope glass revolvable about its axis and placed parallel to the mirror M. The change of virtual thickness  $e'$  for a given small angle of incidence  $i$  may then be written:

$$de' = e' \frac{\sin R dR}{\cos^2 R} = \frac{e'}{2} \frac{dI^2/\mu^2}{1 - I^2/\mu^2}, \text{ nearly.}$$

If  $I=0$ ,  $dI^2=(dI)^2$ . Therefore  $2de'=e'dI^2/\mu^2$ . In a rough trial for  $e'=.0226$  cm.,  $dI=.053$  radian,  $\mu=1.53$ , one fringe reappeared. Hence

$$dN/dn = \mu de' = \frac{1}{2} \times 1.53 \times 27 \times 10^{-6} = 21 \times 10^{-6} \text{ cm.},$$

which is of the order of half the wave-length used.

13. *Interferometry by Displacement.*—In a similar experiment the displacement of ellipses due to the insertion of the above glass,  $e'=.0226$  cm., was from the D line to about the G line. If  $\Delta N$  is the displacement of the mirror N, to bring the centre of ellipses back to the same line, D or E, we may write  $\mu = 1 + \Delta N/e'$ . I found

at the E line,  $\mu = 1.53$ ,

at the D line,  $\mu = 1.53$ .

Special precautions would have to be taken to further determine these indices.

Thus there are two methods for measuring  $\mu$ , either in terms of the radial motion of the fringes, or, second, in terms of the displacement of the fringes as a whole. Moreover, the preceding paragraph 10 may be looked upon, reciprocally, as a method for measuring  $d\mu/d\lambda$ , directly.

#### Part IV.—INTERFERENCES IN GENERAL AND SUMMARY.

14. *The Individual Interferences.*—In figs. 5, 6, 7,  $gg$  is the face of the grating, M and N the opaque mirrors, and I the incident ray.

As the result of reflexion from the top face, the available air-path being  $y_m$  and  $y'_m$ , there must be two images of the slit seen in the telescope directly, viz.  $a$  and  $c$  (fig. 5). Of these  $c$  will be more intense than  $a$ , which is tinged by the long path in the glass. These two rays together, on diffraction, will produce stationary interferences whose path corresponds to the equation

$$n\lambda = 2e\mu \cos R.$$

The optical paths of the two rays are

reflected-refracted, I',  $2y_m + e\mu(2 \cos R - \sec R)$ ,

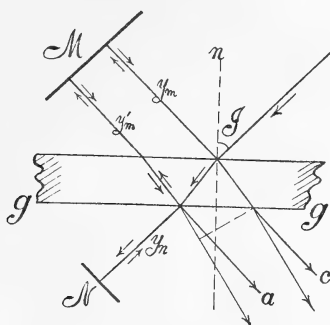
refracted-reflected, II',  $2y'_m + 3e\mu \sec R = 2y_m + e\mu(3 \sec R - 4 \sin R \tan R)$ .

If the plate of the grating were perfectly plane parallel, the slit images  $a$  and  $c$  would obviously coincide.

The directly transmitted rays, however, after reflexion from N give rise to four images of the slit. In case of a

slightly wedge-shaped plate, the one at  $a$  (fig. 5) being white, that at  $c$  yellowish, the distances apart being the same as in

Fig. 5.



the preceding case. Besides this there are two images of the slit at  $b$ , figs. 6, 7, which result from double diffraction

Fig. 6.

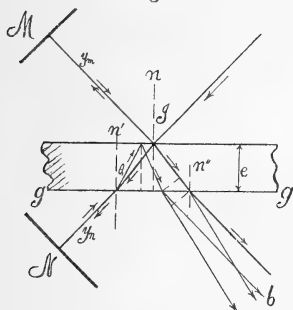
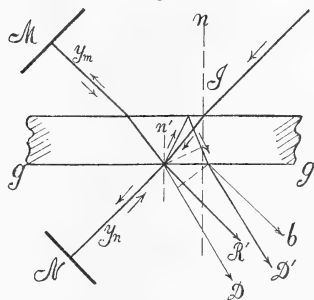


Fig. 7.



at the lower face, the case shown corresponding to  $\theta < i$  first with  $\theta > i$  thereafter, while the other image corresponds to  $\theta > i$  and  $\theta < i$  in the two successive diffractions. These slits show strong chromatic aberration, but are nevertheless linear and useful. The optical paths for the three lower rays are :

- transmitted, I,  $2y_n + e\mu/\cos R$  ;  
 transmitted-refracted, II,  $2y_n + e\mu/\cos R + 2e\mu \cos R$  ;  
 transmitted-diffracted, III,  $2y_n + e\mu/\cos R + 2e\mu \cos \theta_1$ .

These three equations determine the three stationary interferences

$n\lambda = 2e\mu \cos R$  ;  $n\lambda = 2e\mu \cos \theta_1$  ;  $n\lambda = 2e\mu(\cos \theta_1 - \cos R)$ ,  
 discussed in the preceding paper \*. If the grating is between two identical plates of glass, these stationary fringes may

\* C. & M. Barus (*l. c.*).

become movable; for, inasmuch as there is complete symmetry between the rays to be reflected from the mirrors M and N, the stationary fringes become identical for both plates of glass, from either of which they may be reflected before or after diffraction. Hence the motion of either opaque mirror changes the phase and moves the fringes, which are now linear, vertical, and nearly equidistant. They may also be regarded as confocal ellipses of infinite size, the visible parts of their peripheries lying close together in the field of view. They pass through infinity in this field when the virtual path difference is zero.

15. *Continued.*—The five equations given, when the first group I' and II' is combined with the second group I, II, III, lead to six other interferences, all of them of the movable type and useful for interferometry. If for simplicity the path difference is zero, they may be written, if  $y = y_n - y_m$  and  $y' = y_n - y'_m$  and  $y_0$  corresponds to the glass-path difference, N as usually referring to the displacement of the opaque mirror,

Reflexion at		
IV, bottom, equation I-II',	$y'_0 = e\mu / \cos R,$	$N = e\mu \cos R.$
V, bottom, equation II-I',	$-y_0 = e\mu / \cos R,$	$N = e\mu \cos R.$
VI, top, equation III-I',	$-y_0 = e\mu (\cos \theta + \sin R \tan R),$	$N = e\mu \cos \theta_1.$
VII, top and bottom, equation II-II',	$y'_0 = e\mu \sin R \tan R,$	$N = 0.$
VIII, " " " III-II',	$y'_0 = e\mu (\sec R - \cos \theta_1),$	$N = e\mu (\cos R - \cos \theta_1).$
IX, bottom and top, equation I-I',	$-y_0 = e\mu \sin R \tan R,$	$N = 0.$

Hence the fringes of the interference VII and IX are identical throughout the spectrum, when the mirror N moves. The remaining fringes are elliptic and eccentric, because reflected from two faces of the thin wedge. Nos. VII and IX are parallel lines which pass symmetrically from negative to positive obliquity, or the reverse, respectively, through horizontality, in opposed directions.

The results of these equations have been computed for light crown glass, but the treatment being less important may be omitted here.

My thanks are due to Professor Joseph S. Ames, of Johns Hopkins University, for his kindness in lending me the glass diffraction grating by which the above equations were tested. I hope at some other opportunity to work with a grating whose refraction is known throughout the spectrum and also to endeavour to obtain the phenomenon as clearly from film gratings (replicas), as has been possible for the linear series in the preceding paper (*l. c.*). Thus far the above phenomena as obtained from film gratings are not strong and sharp enough for measurements of precision.



L. A Problem in Age-Distribution.

By F. R. SHARPE, Ph.D., and A. J. LOTKA, M.A.\*

THE age-distribution in a population is more or less variable. Its possible fluctuations are not, however, unlimited. Certain age-distributions will practically never occur; and even if we were by arbitrary interference to impress some extremely unusual form upon the age-distribution of an isolated population, in time the "irregularities" would no doubt become smoothed over. It seems therefore that there must be a limiting "stable" type about which the actual distribution varies, and towards which it tends to return if through any agency disturbed therefrom. It was shown on a former occasion † how to calculate the "fixed" age-distribution, which, if once established, will (under constant conditions) maintain itself.

It remains to be determined whether this "fixed" form is also the "stable" distribution: that is to say, whether a given (isolated) population will spontaneously return to this "fixed" age-distribution after a small displacement therefrom.

To answer this question we will proceed first of all to establish the equations for a more general problem, which may be stated as follows:—

"Given the age-distribution in an isolated population at any instant of time, the 'life curve' (life table), the rate of procreation at every age in life, and the ratio of male to female births, to find the age-distribution at any subsequent instant."

1. Let the number of males whose ages at time  $t$  lie between the limits  $a$  and  $a+da$  be  $F(a, t)da$ , where  $F$  is an unknown function of  $a$  and  $t$ .

Let  $p(a)$  denote the probability ‡ at birth that a male shall reach the age  $a$ , so that  $p(0) = 1$ .

Further, let the male birth-rate (*i. e.* the total number of males born per unit of time) at time  $t$  be  $B(t)$ .

Now the  $F(a, t)da$  males whose age at time  $t$  lies between  $a$  and  $a+da$  are the survivors of the  $B(t-a)da$  males born  $a$  units of time previously, during an interval of time  $da$ . Hence

$$F(a, t)da = B(t-a)p(a)da$$

$$F(a, t) = p(a)B(t-a) \dots \dots \dots (1)$$

\* Communicated by the Authors.

† A. J. Lotka, *Am. Journ. Science*, 1907, xxiv. pp. 199, 375; 'Science,' 1907, xxvi. p. 21.

‡ As read from the life table.

2. Let the number of male births per unit time at time  $t$  due to the  $F(a, t)da$  males whose age lies between  $a$  and  $a + da$  be  $F(a, t)\beta(a)da$ .

If  $\gamma$  is the age at which male reproduction ends, then evidently

$$B(t) = \int_0^\gamma F(a, t)\beta(a)da$$

$$= \int_0^\gamma B(t-a)p(a)\beta(a)da. \quad \dots (2)$$

Now in the quite general case  $\beta(a)$  will be a function of the age-distribution both of the males and females in the population, and also of the ratio of male births to female births.

We are, however, primarily concerned with comparatively small displacements from the "fixed" age-distribution, and for such small displacements we may regard  $\beta(a)$  and the ratio of male births to female births as independent of the age-distribution.

The integral equation (2) is then of the type dealt with by Hertz (*Math. Ann.* vol. lxx. p. 86). To solve it we must know the value of  $B(t)$  from  $t=0$  to  $t=\gamma$ , or, what is the same thing, the number of males at every age between 0 and  $\gamma$  at time  $\gamma$ . We may leave out of consideration the males above age  $\gamma$  at time  $\gamma$ , as they will soon die out. We then have by Hertz, *loc. cit.*,

$$B(t) = \frac{\sum_{h=1}^{\infty} \alpha_h^t \int_0^\gamma \{ B(a) - \int_0^a \beta(a_1)p(a_1)B(a-a_1)da_1 \} \alpha_h^{-a} da}{\int_0^\gamma a\beta(a)p(a)\alpha_h^{-a} da}, \quad (3)$$

where  $\alpha_1, \alpha_2, \dots$  are the roots of the equation for  $\alpha$ ;

$$1 = \int_0^\gamma \beta(a)p(a)\alpha^{-a}da. \quad \dots (4)$$

The formula (3) gives the value of  $B(t)$  for  $t > \gamma$ , and the age-distribution then follows from

$$F(a, t) = p(a)B(t-a). \quad \dots (1)$$

4. From the nature of the problem  $p(a)$  and  $\beta(a)$  are never negative. It follows that (4) has one and only one real root  $r$ , which is  $\geq 1$ , according as

$$\int_0^\gamma \beta(a)p(a)da \begin{matrix} \geq \\ \leq \end{matrix} 1. \quad \dots \dots \dots (5)$$

Any other root must have its real part less than  $r$ . For if  $r_1 (\cos \theta + i \sin \theta)$  is a root of (4),

$$1 = \int_0^\gamma \frac{\beta(a)p(a)}{r_1^a} \cos a\theta da. \quad \dots \dots \dots (6)$$

It follows that for large values of  $t$  the term with the real root  $r$  outweighs all other terms in (3) and  $B(t)$  approaches the value

$$B(t) = Ar^t. \dots \dots \dots (7)$$

The ultimate age-distribution is therefore given by

$$F(a, t) = Ap(a)r^{t-a} \dots \dots \dots (8)$$

$$= Ap(a)e^{r'(t-a)}. \dots \dots \dots (9)$$

Formula (9) expresses the "absolute" frequency of the several ages. To find the "relative" frequency  $c(a, t)$  we must divide by the total number of male individuals.

$$c(a, t) = \frac{F(a, t)}{\int_0^\infty F(a, t)da} = \frac{Ap(a)e^{r'(t-a)}}{Ae^{r't} \int_0^\infty e^{-r'a} p(a) da} = \frac{p(a)e^{-r'a}}{\int_0^\infty e^{-r'a} p(a) da}$$

$$= be^{-r'a} p(a), \dots \dots \dots (10)^*$$

where

$$\frac{1}{b} = \int_0^\infty e^{r'a} p(a) da. \dots \dots \dots (11)^*$$

The expression (10) no longer contains  $t$ , showing that the ultimate distribution is of "fixed" form. But it is also "stable;" for if we suppose any small displacement from this "fixed" distribution brought about in any way, say by temporary disturbance of the otherwise constant conditions, then we can regard the new distribution as an "initial" distribution to which the above development applies: that is to say, the population will ultimately return to the "fixed" age-distribution.

\* Compare Am. Journ. Science, xxiv. 1907, p. 201.

It may be noted that of course similar considerations apply to the females in the population. The appended table shows the age-distribution calculated according to formula (10) for England and Wales 1871-1880. The requisite data (including the life table) were taken from the Supplement to the 45th Annual Report of the Registrar General of Births, &c. The mean value of  $r'$  (mixed sexes) for that period was  $\cdot 01401$ , while the ratio of male births to female births was  $1\cdot 0382$ .

It will be seen that at this period the observed age-distribution in England conformed quite closely to the calculated "stable" form.

TABLE.

AGE (Years).	MALES.		FEMALES.		PERSONS.	
	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
0-5...	139	139	136	132	138	136
5-10...	118	123	115	117	116	120
10-15...	107	110	104	104	106	107
15-20...	97	99	95	95	96	97
20-25...	88	87	87	91	87	89
25-35..	150	144	148	149	149	147
35-45...	116	112	116	115	116	113
45-55...	86	84	88	87	87	86
55-65...	57	59	62	61	59	59
65-75...	30	31	35	35	33	33
75-∞	11	12	15	15	13	13

LI. *On the Damping of the Vibrations of a Dielectric Sphere, and the Radiation from a Vibrating Electron.* By J. W. NICHOLSON, M.A., D.Sc.\*

THIS note is supplementary to two short papers in the Philosophical Magazine for October and November last, dealing with the initial motions of conducting and dielectric charged spheres. The object of these papers was chiefly to indicate that if an electron could be regarded as not subject to contraction when in motion, the most useful property to

\* Communicated by the Author.

ascribe to its interior would be that of a dielectric of high specific inductive capacity, for other specifications in terms of conduction would introduce difficulties of an indeterminate type into the discussion of its initial motion under the action of a small force, when the Newtonian mass tends to zero. The present note is devoted to an examination of the rate of decay of the free vibrations of a movable dielectric sphere in the general case in which its motion starts from rest and is not large at any instant considered. Certain conclusions of a negative character are drawn with respect to the radiation from a non-deformable electron possessing the dielectric property, when plane harmonic waves are incident upon it. Like the papers mentioned above, which with some of the present considerations were communicated to the British Association at the Sheffield meeting, this note consists, to a great extent, of a detailed examination of certain points raised by the recent memoir of G. W. Walker, a memoir which constitutes the most comprehensive and successful attempt yet made to set the theory of the accelerated motion of electrified systems upon a rigorous dynamical foundation, without an appeal to the method of the quasi-stationary principle, or to special assumptions, such as that of rigidity of electrification, which cannot be formally justified when the motion of the system ceases to be uniform.

When a dielectric sphere, whether small or not, is fixed or uncharged, or when its Newtonian mass is very large in comparison with that of electromagnetic origin, the period equation for its free vibrations is that given by Lamb\*. With a change of notation, it can be written in the form

$$(\tanh \kappa^{\frac{1}{2}}\lambda)/\kappa^{\frac{1}{2}}\lambda = 1 + \kappa\lambda^2(1-\lambda) / \{(\kappa-1)(1-\lambda) + \kappa\lambda^3\}, \quad (1)$$

where  $\kappa$  is the dielectric constant, and if  $\mu + i\nu$  be any root of this equation in  $\lambda$ , the corresponding vibration has a period  $2\pi(a/\nu c)^{\frac{1}{2}}$  and contains a factor  $e^{-kt}$ , where  $k = \mu C/a$ . The radius of the sphere is  $a$ , and  $C$  is the velocity of light in the free æther outside.

Lamb has discussed this equation when the sphere is of atomic size, and  $\kappa$  is extremely large, but does not give the decrement of the vibrations explicitly. Walker, in his paper, gives a formula for the first root as

$$\lambda = \pm 4.493 i\kappa^{-\frac{1}{2}} + (4.493)^4\kappa^{-5}, \quad . \quad . \quad . \quad (2)$$

from which, for values of  $\kappa$  of the magnitude  $\kappa = 10^6$  used by

\* Camb. Phil. Trans., Stokes Commem. Volume.

Lamb, the vibrations would be very persistent when started. Walker quotes the result without proof, and perhaps only a misprint has occurred, for the true formula, as will appear below, is of the form

$$\lambda = \pm 4.493 i \kappa^{-\frac{1}{2}} + (4.493)^4 \kappa^{-3}, \dots \dots (3)$$

and the resulting decrement is thus increased in the ratio  $10^{12}$ , so that the free vibrations would not have such a pronounced degree of permanence.

When the sphere is capable of free motion while the vibrations on it are taking place, and has a surface charge distributed over it, the field will set it into motion, and the electromagnetic inertia, given for small motions by  $m' = 2e^2/3ac^2$ , where  $e$  is the charge, will enter into the question. The more general formula valid in this case is found by Walker to be

$$(\tanh \kappa^{\frac{1}{2}} \lambda) / \kappa^{\frac{1}{2}} \lambda = 1 + \kappa \lambda^2 \left( 1 + \frac{m'}{m} - \lambda \right) / \left\{ (\kappa - 1) \left( 1 + \frac{m'}{m} - \lambda \right) - \kappa \lambda \frac{m'}{m} + \kappa \lambda^3 \right\}; \dots \dots (4)$$

and if  $m'$  is negligible in comparison with  $m$ , this is Lamb's formula (1). We shall calculate the decrement corresponding to any root of this equation when  $\kappa$  is large. In this case, the main term of  $\lambda$ , expressed as a series of inverse powers of  $\kappa^{\frac{1}{2}}$ , is of order  $\kappa^{-\frac{1}{2}}$ . The roots are therefore, to a first approximation, those of

$$\tanh \kappa^{\frac{1}{2}} \lambda = \kappa^{\frac{1}{2}} \lambda, \dots \dots \dots (5)$$

whose first root is well known to be given by  $\kappa^{\frac{1}{2}} \lambda = \pm 4.493 i$ , being purely imaginary. Let  $\kappa^{\frac{1}{2}} \lambda = \pm i\rho$  denote any pair of roots, and let  $\kappa^{\frac{1}{2}} \lambda = \pm i\rho + \sigma$  be the corresponding roots of the equation (4),  $\sigma$  denoting merely the leading term of the real portion. That  $\sigma$  must be positive is known from the physical consideration that the vibration must die out ultimately, and the equation being real, the roots must occur in pairs in this way. It is evident that  $\sigma$  will be of lower magnitude than  $\rho$ . Then

$$\frac{\tanh(i\rho + \sigma)}{i\rho + \sigma} = 1 + \frac{(i\rho + \sigma)^2 \left( 1 + \frac{m'}{m} - \frac{i\rho + \sigma}{\kappa^{\frac{1}{2}}} \right)}{(\kappa - 1) \left( 1 + \frac{m'}{m} - \frac{i\rho + \sigma}{\kappa^{\frac{1}{2}}} \right) (i\rho + \sigma) \frac{m'}{m} \kappa^{\frac{1}{2}} + (i\rho + \sigma)^3 \kappa^{-\frac{1}{2}}}, (6)$$

where  $\rho$  is of no order in  $\kappa^{-\frac{1}{2}}$ . Expanding in powers of  $\sigma$ ,  
if  $m'/m = n$ ,

$$i \tan \rho + \sigma \sec^2 \rho = i\rho + \sigma - i\rho^3(1+n - i\rho\kappa^{-\frac{1}{2}}) / \{(\kappa-1)(1+n - i\rho\kappa^{\frac{1}{2}}) - in\rho\kappa^{\frac{1}{2}} - i\rho^3\kappa^{-\frac{1}{2}}\},$$

rejecting  $\sigma^2$ , and even  $\sigma$  in the last term on account of the smallness of the denominators. This is justified by the final result. If

$$\alpha = (\kappa-1)(1+n), \quad \beta = \rho(\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}} + n\kappa^{\frac{1}{2}} + \rho^2\kappa^{-\frac{1}{2}}); \quad (7)$$

then

$$i(\tan \rho - \rho) + \sigma \tan^2 \rho = -i\rho^3(1+n - i\rho\kappa^{-\frac{1}{2}}) / (\alpha - i\beta).$$

But the same equation must be true for  $-\rho$ , so that

$$-i(\tan \rho - \rho) + \sigma \tan^2 \rho = i\rho^3(1+n + i\rho\kappa^{-\frac{1}{2}}) / (\alpha + i\beta);$$

whence on addition

$$2\sigma \tan^2 \rho = \rho^3(\beta\alpha_1 - \alpha\beta_1) / (\alpha^2 + \beta^2), \quad \dots (8)$$

where  $\alpha_1 = 1+n$ ,  $\beta_1 = \rho\kappa^{-\frac{1}{2}}$ . Writing now  $\tan^2 \rho = \rho^2$ , and retaining only the most significant orders in  $\kappa$ ,

$$\alpha^2 + \beta^2 = (\kappa-1)^2(1+n)^2 + \rho^2\{\kappa^{\frac{1}{2}}(1+n) + \kappa^{-\frac{1}{2}}(\rho^2-1)\}^2 \\ = \kappa^2(1+m'/m)^2,$$

$$\beta\alpha_1 - \alpha\beta_1 = (1+n)\rho\{\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}} + n\kappa^{\frac{1}{2}} + \rho^2\kappa^{-\frac{1}{2}}\} - \rho(1+n)(\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}}) \\ = (1+m'/m)(\rho^2/\kappa + m'/m)\rho\kappa^{\frac{1}{2}}$$

(the rejection of lower orders was unsafe before), and therefore

$$\sigma = \rho^2(m' + m\rho^2/\kappa) / (m + m')\kappa^{\frac{3}{2}}, \quad \dots (9)$$

from which

$$\lambda = \pm i\rho\kappa^{-\frac{1}{2}} + \rho^2(\kappa m' + \rho^2 m) / (m + m')\kappa^3, \quad \dots (10)$$

giving the formula already quoted when  $m'$  tends to zero.

For the case treated in the earlier paper, when there is no Newtonian mass,  $m = 0$ , so that

$$\lambda = \pm i\rho\kappa^{-\frac{1}{2}} + \rho^2/\kappa^2. \quad \dots (11)$$

Thus the vibration for the fixed sphere contains a factor  $e^{-k_1 t}$  where  $k_1 = \rho^4 C / \alpha \kappa^3$ , and that for the charged and movable sphere with no Newtonian mass contains  $e^{-k_2 t}$  where  $k_2 = \rho^2 C / \alpha \kappa^2$ .

It will be sufficient in applications of these results to restrict attention mainly to the first vibration, for which  $\rho = 4.493$ .

We apply the results in the first place to the model atom used by Lamb to illustrate selective absorption. In that model, it is found that in order to obtain a fundamental vibration which shall fall in the ultra-violet, with a sphere of atomic dimensions,  $\kappa$  must be of order  $10^6$ . The actual values taken by Lamb are  $\kappa = 5 \cdot 10^6$ ,  $a = 1 \cdot 3 \cdot 10^{-8}$ , in c.g.s. units. In this case, it is found from the above formula (3) that  $k_1 = 1 \cdot 5$ , which is not small, and the fundamental vibration is decreased in a ratio  $1/e$  in  $2/3$  of a second. The vibrations are therefore not very permanent. With the uncorrected formula of the decrement given by Walker, the ensuing value of  $k_1$  is of order  $10^{-12}$ , leading to great permanence. The difference in the results is therefore considerable. But the comparatively rapid dissipation of the free vibrations is perhaps not sufficiently rapid to impair the efficiency of Lamb's suggested model of a gas exhibiting selective absorption.

The positive particle may be supposed to be of atomic size. Moreover, we may write for this particle,  $m = 10^{-24}$ ,  $e/c = 10^{-20}$  as approximate values, where  $e$  is its charge. Its electromagnetic mass for small motions is therefore  $m' = 2e^2/3ac^2$  or  $5 \cdot 10^{-33}$ , so that  $m'/m = 5 \cdot 10^{-9}$ . Thus with  $\kappa = 5 \cdot 10^6$ , the value required to bring its vibration also within the visible spectrum, it is possible, with  $\rho$  equal to 5 approximately for the fundamental free vibration, to ignore  $m'/m$  altogether in the expression  $(m' + m\rho^2/\kappa)/(m + m')$  of (10). Accordingly, the decrement may be given its value for the uncharged fixed sphere of atomic size, and is again  $k = 1 \cdot 5$ . This is the decrement of the fundamental free vibration of the positive particle if it can be regarded as a superficially charged dielectric of constant form and of such a character that this vibration comes within the visible spectrum.

The decrement of the higher vibrations is of course greater on account of  $\rho$ . The second root of  $\tanh \kappa^{\frac{1}{2}}\lambda = \kappa^{\frac{1}{2}}\lambda$  is given by  $\kappa^{\frac{1}{2}}\lambda = \pm 7 \cdot 725i$ , and  $m'/m$  being negligible more and more in the higher vibrations, the decrement is proportional to  $\rho^4$ , and becomes  $k' = 13 \cdot 1$ . For the third vibration,  $\rho = 10 \cdot 904$ , leading to  $k'' = 52 \cdot 2$ . The increase in  $k$  is therefore rapid.

Proceeding to the case with which we are at present more immediately concerned, of a hypothetical spherical electron without deformation, we may write, in accordance with current estimates of approximate size,  $a = 10^{-13}$ . As Walker has remarked, in order that a vibration from a sphere of this size shall appear in the visible spectrum, the dielectric constant must be of order  $10^{16}$ . This appears at once from the expression for the period as  $2\pi(a/\nu c)^{\frac{1}{2}}$ , where  $i\nu$  is the imaginary



part of  $\lambda$ . The period is accordingly  $2\pi a^{\frac{1}{2}}\kappa^{\frac{1}{2}}/(\rho c)^{\frac{1}{2}}$ , where  $\rho = 4.493$ , and this leads to the value in question. We may now consider two cases: firstly, that in which the mass is entirely of electrical origin; and, secondly, that in which  $m$  and  $m'$  have an ordinary finite ratio. The second is the case favoured by Walker from his analysis of the results of Kaufmann's experiments.

In the first place, when the mass is wholly electrical, the decrement reduces from (10) merely to  $k_2 = \rho^2 c / a \kappa^2$  as in (11). With  $a = 10^{-13}$ , and  $\kappa = 10^{16}$ , this gives  $k_2 = 6.10^{-7}$ , indicating a very permanent vibration. This vibration would inevitably persist throughout the time during which the equations for the motion of the sphere under an applied force can be regarded as furnishing good approximations to that motion. An exception is of course presented to this statement when the applied force is of a periodic character, so that the forced motion is vibratory, and the sphere never deviates far from its initial position, and therefore the equations for the displacement  $\xi$  of the sphere at any subsequent time, and the function defining the state of things outside, never tend to become less accurately representative. They only do so, for example, in the problems discussed in the earlier papers, of motion under a uniform force, on account of their assumption that the sphere remains approximately at the origin. But in the face of this consideration, the periodic applied force must not be regarded as exceptional. For, the decrement being of order  $10^{-7}$ , it is a matter literally of months, and not seconds, before the free vibrations could be neglected, and therefore all kinds of new agencies would have introduced further free vibrations in the meantime. We may conclude, therefore, that it is never possible to regard the free vibrations as in any way less important than the forced motion.

This persistence is rather more pronounced on Walker's view of the negative electron, for if  $m'$  and  $m$  are of the same order, we may ignore  $m\rho^2/\kappa$ , but not  $m$ , in comparison with  $m'$ , so that  $k_2 = \rho^2 m' / \kappa^2 (m + m')$ . The effect of including  $m$  is therefore to decrease  $k_2$  in the ratio  $m' / (m + m')$ . For example, the actual ratio of  $m$  to  $m'$  derived by Walker is about unity for Kaufmann's second set of experiments, so that  $k_2$  is halved. In other words, the oscillations may be said to be twice as permanent as they would be if the mass were wholly electrical.

Finally, then, we see that if a free period of the negative electron, regarded as dielectric, is to come within the visible spectrum at all, it is necessary to suppose that its vibrations are extremely permanent, and therefore that a constant force

can never produce a constant acceleration in an electron. This result is, however, not necessary in the case of the positively charged particle, nor in the case of an atom consisting of an agglomeration of electrons. The essential basis of this conclusion is the large value of the dielectric constant which is forced upon the electron. A small dielectric sphere could have free vibrations which would vanish very rapidly for extremely large values of the inductive capacity, if its periods were not in the visible spectrum. It would therefore speedily develop a constant acceleration under a constant force. For example, with an electron of the same size, and a value of  $\kappa$  equal even to  $10^{12}$ , we should have  $k_2=60$ , indicating rapid damping. But the necessity for  $\kappa=10^{16}$  determines the matter.

For a conductor, on the other hand, there is only one vibration, which has a period  $4\pi a/c(3+4m'/m)^{\frac{1}{2}}$  and a modulus  $c/2a$  of decay. This modulus is extremely great (of order  $10^{24}$ ). For a dielectric of large inductive capacity this vibration is, as Walker pointed out, an approximation to an isolated vibration of the dielectric not mentioned by Lamb, but when the Newtonian mass is zero it is absent, as appeared in the earlier paper. Its presence in other cases does not of course interfere with the argument above regarding the proper series of vibrations, the first of which we called the fundamental. Their amplitudes must be of the same order as that of the forced vibration under a periodic exciting force.

The foregoing considerations of the damping effect have certain important consequences, and more particularly in the theory of the radiation from an electron executing forced vibrations under an incident periodic force. A small sphere in vibratory motion is usually understood to emit radiation, as the Poynting vector indicates. Walker showed in his paper that the ordinary neglect of the exciting field in the determination of that vector is not justifiable, as well as an assumed relation  $\chi=e\xi/c$  between the quantities  $\chi$  and  $\xi$  below, and that the proper expression from which to determine the radiation is a dissipation function D given by

$$D=c\dot{\chi}(\dot{\chi}-e\dot{\xi}/c)/3a^2, \quad . \quad . \quad . \quad (12)$$

where  $\dot{\xi}$  is the velocity of the vibrating sphere, and  $\chi$  is a function determining the external field at points of space which the vibrations have had sufficient time to reach. The radiation should then be  $2D$ . A calculation which Walker makes in the case of the perfect conductor verifies that the result thus obtained is in accordance with that derived by the Poynting flux method, when the proper relation between

$\chi$  and  $\xi$  is maintained. This of course confirms Larmor's formula for the radiation from a vibrating electron, and the force being mechanical and equal to  $F \cos nt$ , the mean rate of radiation is

$$\frac{1}{3} \frac{e^2 F^2}{m^2 c^3} \left/ \left\{ \left( 1 + \frac{m'}{m} - \frac{a^2 n^2}{c^2} \right)^2 + \frac{a^2 n^2}{c^2} \right\} \right., \quad \dots \quad (13)$$

or merely  $e^2 F^2 / 3c^3 (m + m')^2$  since  $an/c$  can ordinarily be neglected.

But we notice that this is the case of a perfectly conducting electron, and we have just seen that the decrement coefficient of the free vibration is of order  $10^{24}$ . There is no trouble, therefore, in neglecting this vibration in the determination.

Consider now the radiation from a dielectric electron, where  $\kappa$  is of the order already found necessary. In this case Walker has found, by the same method, that for the whole range of periods possible to an harmonic vibration falling on the electron, there are regions in which the forced vibration of the electron leads to absorption of radiation, separated by others in which it leads to emission. Thus for a negative particle, "If  $m'/m$  is greater than 2, there is emission from infinite wave-length to very far out in the ultra-violet. If  $m'/m$  is less than 2, there is absorption from infinite wave-length to a certain wave-length which depends on the closeness of  $m'/m$  to 2. Unless  $m'/m$  is very nearly 2, it will be in the ultra-violet."

These results are obtained by supposing that the free vibrations of the electron have died away, and we have seen that the decrement factor  $k_2$  is of order  $10^{-7}$  for any relative values of electrical and Newtonian mass of the same order of magnitude, the most favourable case being that in which the mass is entirely electrical. Accordingly, the free vibrations must not be ignored, and the expression for the radiation must be greatly modified, and will probably lose its special characteristics.

A precise determination of the matter is difficult for two reasons. In the first place, the distribution of the free vibrations among their several periods presents analytical difficulties, and moreover, on account of the smallness of the damping factor, many vibrations are of equal importance with the fundamental, for in the equation (10) the value of  $\rho$  in the approximate root  $\lambda = \pm i\rho\kappa^{-\frac{1}{2}}$  must be such as to make the damping coefficient  $\rho^2 m' / (m + m') \kappa^2$  small, or in other words,  $\rho$  must be of order  $\kappa$ , or  $10^{16}$  at least. As the successive values of  $\rho$  only differ by about 3, for example,

the tenth is only 29·812, the number of important vibrations is enormous, and an analytical solution by the method of Walker does not appear to be possible, and what the exact result would be cannot apparently be predicted. But one assertion may be made with certainty. The amplitudes of the free vibrations of  $\chi$  and  $\xi$  are of the same order as those of the forced vibrations, in any time for which the periodic force would not be disturbed by other agencies, and it is quite likely that the dissipation function defining the radiation may not be capable of a negative value, with absorption as a consequence instead of emission.

These remarks apply of course only to the dielectric electron with such a coefficient that its free fundamental vibration shall be in the visible spectrum. For a conducting electron, the expression for the radiation under a periodic force, with its consequence of absorption for a certain range, is not affected. But arguments against the possibility of the non-deformable conducting electron can be found on other grounds.

It seems probable that when  $\kappa$  is nearly infinite, the sum of the amplitudes of the forced vibrations (excepting that corresponding to the conductor) would tend to zero, leading to the results for a conductor. But when the vibration is constrained to be in the visible spectrum, no such conclusion can be drawn.

LII. *Homogeneous Röntgen Radiation from Vapours.* By J. CROSBY CHAPMAN, B.Sc., *Layton Research Scholar of the University of London (King's College); Research Student of Gonville and Caius College, Cambridge* \*.

ALL bodies when exposed to Röntgen radiation emit secondary X-rays. It has been shown † that these secondary rays consist of two types—a scattered radiation having the same penetrating power as the primary beam and resembling it in that it is heterogeneous and an X radiation characteristic only of the element used as radiator and independent of the penetrating power of the exciting primary beam.

The elements belonging to the group with atomic weights from hydrogen to sulphur have been shown to give out, when excited, a great preponderance of the first type of radiation termed scattered radiation ‡, while those in the group from

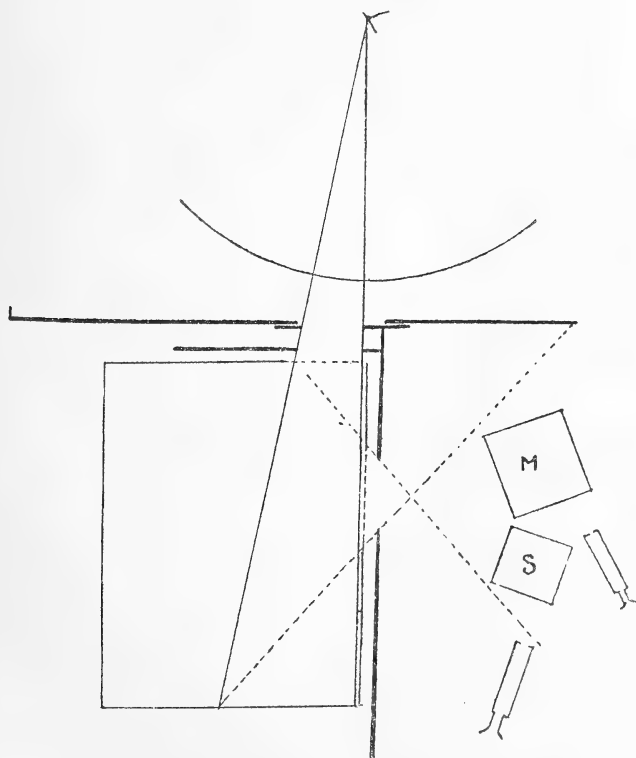
\* Communicated by Prof. C. G. Barkla, M.A., D.Sc.

† Barkla & Sadler, *Phil. Mag.*, Oct. 1908.

‡ Barkla, *Phil. Mag.*, June 1903.

chromium onwards emit almost wholly the characteristic radiation which, on account of its homogeneity, suffers equal percentage absorptions when transmitted through equal thicknesses of aluminium. By determining these percentage absorptions the values of  $\frac{\lambda}{\rho}$  (where  $I=I_0e^{-\lambda x}$  and  $\rho$ =density of aluminium) have been obtained for the different elements giving characteristic radiations. Previously, in experiments performed for determining the coefficient  $\frac{\lambda}{\rho}$ , the elements used have been, for purposes of convenience, in the solid state, either pure or in the form of compounds. The following

Fig. 1.



experiments were undertaken at the suggestion of Prof. Barkla with a view to showing that the same type of homogeneous radiation is emitted by the elements, whether they are in the solid state or in the form of vapour.

The apparatus used to determine the nature of the Secondary Radiation emitted by the vapours when exposed to the rays, consisted of an iron box which contained the gas: this was fitted with aluminium windows and was placed as shown in the diagram (p. 447), so that the radiations reaching the electroscopes could come only from the vapour inside the chamber. That such is the case with this arrangement is indicated by the dotted lines in the figure which mark the path taken by the extreme scattered rays.

The process of the experiment was as follows. The radiation from the vapour inside the chamber was allowed to pass into both electroscopes, which were of the ordinary gold-leaf type described by Prof. Barkla. The deflexion in the electroscope M was observed, while there was a certain deflexion in the standardiser S. An aluminium sheet of required thickness was then placed in front of the electroscope M, and the deflexion again read while the standardising electroscope suffered the same deflexion as before. In this way the percentage of the radiation which had been absorbed could be determined, and thence a value for  $\frac{\lambda}{\rho}$ .

Owing to the difficulties in obtaining and working with gases containing elements of atomic weight greater than 52, in the experiment it has been impossible to determine the coefficients  $\frac{\lambda}{\rho}$  for more than two of the elements. The vapours of ethyl bromide and ethyl iodide were employed on account of their, comparatively speaking, high vapour pressures at the temperature of the experiment.

*Secondary Radiation from the Vapour of Ethyl Bromide and from Solid Bromine Compounds.*

Air which had previously been bubbled through ethyl bromide and afterwards passed through a glass spiral immersed in water at 4° C., so that it was saturated at a temperature lower than that of the room, was drawn through the chamber by means of a filter-pump and a steady condition was obtained. The X-ray bulb, which was permanently connected to a pump, was kept at a suitable degree of hardness in order to obtain a maximum intensity of radiation for measuring purposes. The following typical results were obtained with the bromide, proceeding as previously described.

Radiation from Vapour of Ethyl Bromide.  
(Time of observation 2 to 3 minutes.)

Percentage absorption by Aluminium previous to absorption in other column.	Percentage absorption by Aluminium (.00626 cm.) after absorption in column 1.
0	24.1
24	24.4
42	24.1
56	23.6
67	24.0
75	24.7
Mean value.....	24.2

$$\frac{\lambda}{\rho} = 16.4.$$

As the value of  $\frac{\lambda}{\rho}$  for solid bromine had not previously been determined, the gas-chamber was replaced by a plate of sodium bromide obtained by sticking the powder to a thin aluminium sheet. The same observations were repeated with

Radiation from Solid Bromine Compounds.

Percentage absorption by Aluminium previous to absorption in other columns.	Percentage absorption by Aluminium (.00626 cm.) after absorption in column 1.	
	Radiation from Sodium Bromide.	Radiation from Bromyl Hydrate.
0	24.8	24.0
24	24.3	24.5
42	23.4	24.2
56	24.0	24.3
67	24.1	24.1
75	23.6	—
Mean value....	24.0	24.1

$$\frac{\lambda}{\rho} \text{ (for NaBr)} = 16.2. \quad \frac{\lambda}{\rho} \text{ (for Br(OH))} = 16.3.$$

the radiation from the plate as with that from the bromide vapour. A plate of bromyl hydrate was also used in this way.

It will be observed that the value of  $\frac{\lambda}{\rho}$  obtained from the vapour has, within the limits of error, the same value as that from the solid, thus showing that the two radiations are identical in character. In order to find the nature of the curve connecting the atomic weights with  $\frac{\lambda}{\rho}$ , in the neighbourhood of bromine the values of  $\frac{\lambda}{\rho}$  for the radiations from selenium strontium, molybdenum, were determined.

Element used as radiator.	Value of $\frac{\lambda}{\rho}$ for radiation.
Selenium . . . . .	18.5
Strontium . . . . .	11.1
Molybdenum . . . . .	4.88

Using these values combined with others known before to plot  $\frac{\lambda}{\rho}$  against atomic weights, a smooth curve results, on which the value of  $\frac{\lambda}{\rho}$  for bromine lies, showing that the latter both in the solid and vapour state gives out a characteristic radiation the absorption coefficient of which follows the law determined for solid elements.

*Radiations from Vapour of Methyl Iodide and from Solid Iodine.*

The apparatus was similar to that used with the ethyl bromide with the exception that, in this case, a quantity of the iodide almost sufficient to saturate the space inside the chamber was poured into an aluminium dish in the box; the filter pump and saturation bottle were dispensed with. In addition the X-ray bulb was hardened by abstracting a little air with the pump, in order that the rays given off might excite the hard iodine radiation.

The intensity of the rays from the ethyl iodide was very much less than that from ethyl bromide owing to only a



small part of the incident primary being sufficiently penetrating to excite the characteristic iodine radiation. This combined with the fact that hard rays do not ionize to any large extent made the times of observation much longer than in the other case.

Radiation from Vapour of Methyl Iodide.  
(Time of observation 15 to 20 minutes.)

Percentage absorption by Al previous to absorption in column II.	Percentage absorption by Al (.0377 cm.) after absorption in column I.
22	21.6
39	20.8
52	22.5
Mean value .....	21.6

$$\frac{\lambda}{\rho} = 2.3.$$

A plate of solid iodine was constructed and the radiation from it examined in the same manner as the bromide plate, with the following results:—

Radiation from Solid Iodine.

Percentage absorption by Al previous to absorption in column II.	Percentage absorption by Al (.0377 cm.) after absorption in column I.
22	20.3
39	21.7
52	20.7
Mean value .....	20.9

$$\frac{\lambda}{\rho} = 2.3.$$

Again, the value of  $\frac{\lambda}{\rho}$  for the solid and vapour was the

same. By continuing the curve before mentioned, it will be seen that this value lies approximately on it.

Although it has only been proved in these two cases that elements in the solid and vapour state emit the same type of radiation, yet it is safe to conclude that what applies here holds generally; especially considering that the atomic weights of bromine and iodine are well separated. It is evident that this similarity of character in the radiations is what would follow from the fact that the phenomena of secondary X-rays are atomic in their nature.

#### *Bombardment of Atoms by Ejected Corpuscles.*

In a previous paper\* facts have been brought forward indicating that the characteristic secondary radiation does not result from the subsequent bombardment of atoms by ejected corpuscles. A slight adaptation of the above experiment shows this. For if carbon dioxide and hydrogen are used separately under the same conditions as the gas in which the vapour of ethyl bromide is passed into the chamber, in the former case it is the carbon dioxide gas which is chiefly bombarded by ejected electrons, while with hydrogen it is the ethyl bromide itself which has for the most part to stop the expelled corpuscles. Therefore, if subsequent bombardment causes the characteristic radiation, we should expect greater intensity with the hydrogen than with the carbon dioxide as the gas. This point can be investigated experimentally.

The apparatus used was practically identical with that previously described. The tin box was moved farther back and the primary rays were cut down by a lead tunnel in place of the slits then used. At the same time, the electro-scope S was moved into a position to receive radiations from a thin stick of selenium so placed in this tunnel that a small part of the exciting radiation from the X-ray bulb fell on it. The electro-scope M was brought nearer to the box, this was possible owing to the alteration in breadth of the primary beam.

Since the atomic weight of selenium is 79 while that of bromine is 80, the intensity of secondary radiation from the stick of selenium in the tunnel standardizes the power of the incident rays in exciting the homogeneous bromine radiation. In the first part of the experiment, the carbon dioxide gas obtained from a cylinder was saturated with ethyl bromide at  $3^{\circ}.5$  C., and was passed through the chamber till a steady state was reached. The deflexion in the electro-scope receiving radiations from the chamber, while the other

\* Chapman & Piper, *Phil. Mag.* June 1910.

electroscope suffered a certain deflexion, was noticed. The carbon dioxide cylinder was replaced by a hydrogen "kip," and hydrogen saturated at the same temperature was passed under similar conditions through the box, and the deflexion of the chamber electroscope, while the standardizer underwent the same deflexion, was noticed. The results are shown :—

Intensities of Radiations in the two cases.  
Temperature of saturation = 3°·5 C.

Deflexion of standardizing electroscope.	Deflexion of chamber electroscope with carbon dioxide as saturated gas.	Deflexion of chamber electroscope with hydrogen as saturated gas.
50	38·0	38·6
50	37·8	38·4
50	38·2	38·0
50	38·0	38·4
Mean values.....	38·0	38·4

$$\frac{\text{Amount of secondary radiation with H}_2 \text{ as gas saturated with C}_2\text{H}_5\text{Br}}{\text{Amount of secondary radiation with CO}_2 \text{ as gas saturated with C}_2\text{H}_5\text{Br}} = 1·08.$$

This slight difference in the intensities is of the order of magnitude which would result from the excess of absorption in the heavier carbon dioxide gas.

Knowing the vapour pressure at 3°·5 C., some relative idea of the magnitude of the difference can be deduced. For, assuming that the absorption of the  $\beta$  particles varies directly as the absorbing mass, we get :—

Case I.

$$\frac{\text{Amount of corpuscular radiation absorbed by ethyl bromide}}{\text{Amount of corpuscular radiation absorbed by CO}_2} = \cdot 83.$$

Case II.

$$\frac{\text{Amount of corpuscular radiation absorbed by ethyl bromide}}{\text{Amount of corpuscular radiation absorbed by H}_2} = 18·8.$$

Thus if the expelled electrons do by bombarding the bromine atom make it emit its characteristic radiation, the above calculations show that there must be a most noticeable difference in the intensities of radiation in the two cases.

The experimental results obtained show, however, that there is practically no difference in the intensities for the two gases, which proves that the bombardment theory is quite untenable.

In conclusion my best thanks are due to Professor Barkla for his interest and encouragement during the carrying out of these experiments.

Wheatstone Laboratory,  
King's College.

LIII. *The Direct Measurement of the Peltier Effect.*

By F. W. JORDAN, A.R.C.S., B.Sc.\*

THE Peltier coefficient may be measured directly by the calorimetric methods of Le Roux† and Jahn‡, or it may be deduced from the thermoelectric power by using the thermodynamic relation

$$P = T \frac{dE}{dT},$$

where  $P$  = Peltier coefficient,

and  $\frac{dE}{dT}$  = the thermoelectric power at the absolute temperature  $T$ .

The methods of Le Roux and Jahn are tedious, and can only be applied when one of the junctions is isolated thermally from the other. The apparatus described in this paper was designed in 1909 for the direct measurement of the Peltier coefficient between copper and a short specimen of crystallized bismuth in the limited space between the poles of an electromagnet. Pellat§, in 1901, suggested a somewhat similar method, but he does not appear to have made an experiment to test its accuracy. He discusses the case of a compound bar of iron and zinc traversed by a current of 20 amperes in such a direction that heat is absorbed at the junction. He suggested that the Peltier absorption of heat might be compensated by the heat evolved by a current through a fine insulated wire embedded in the iron close to the junction. The Peltier effect will produce in each bar a temperature

\* Communicated by the Author.

† Le Roux, *Ann. de Chimie et de Phys.* i. p. 201 (1867).

‡ H. Jahn, *Wied. Ann.* xxxiv. p. 755 (1888).

§ Pellat, *Comptes Rendus*, cxxxiii. p. 921 (1901).

gradient towards the junction, and therefore, to compensate this, the current through the fine wire would be adjusted so that the temperature gradient, as indicated by insulated thermojunctions, vanishes in each bar. The Joule effects in the bar and leads 4 mm. thick were considered to be negligible, and to have no disturbing effect on the temperature gradients in the bar.

If  $I$  = the current through the bar,

$i$  = the current through the heating coil,

$e$  = the potential difference at terminals of heating coil.

Then

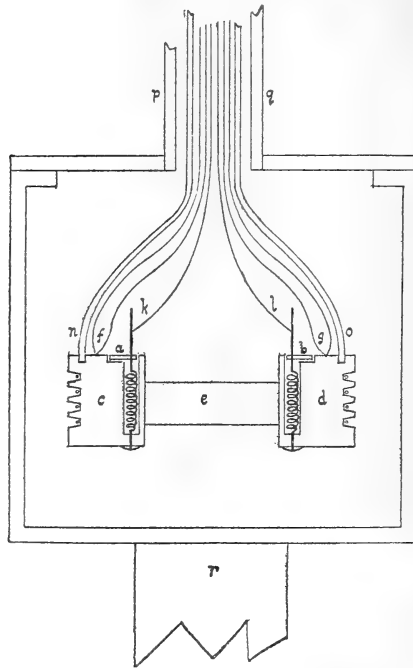
$$PI = ei.$$

At the ordinary temperature of the air, the heat evolved per cm. length of the iron lead is approximately one-third of the heat absorbed at the junction for a current of 20 amperes, and therefore the neglect of the Joule effect is scarcely justifiable. The dimensions of the apparatus are large for an experiment in a limited space, and owing to the large thermal capacity of the bar a considerable interval would be required before the steady state and the final adjustment of the compensating current were attained.

In the following method the two junctions of the copper with the bismuth could not be isolated, and the apparatus is equivalent to a duplication of Pellat's arrangement with many subsequent advantages.

If a current be sent round a circuit of two metals, then the temperature difference between the junctions, arising from the Peltier effects, can be made to vanish by a continuous supply of heat to the cooler junction. In the ideal case, when the junctions are equal in all respects, the rate at which heat must be supplied to the cooler junction is equal to twice that at which it is evolved at the warmer junction. The thermal conductivity of copper is much greater than that of bismuth, and therefore it is preferable to supply the heat by the passage of a current through an insulated fine wire embedded in the copper close to each junction. The presence of a heating coil at each junction renders it possible to eliminate from the final result the electrical resistances and the different thermal emissivities of the junctions. The apparatus was also dimensioned so that, at each junction, the Joule heat for the average current of one ampere was about one-tenth of the Peltier heat. In the experiments of Jahn with a Bunsen ice-calorimeter the Peltier heat was only a small part of the measured quantity of heat.

The bismuth rod *e* was cut from a crystallized mass, prepared by Dr. Lownds, so that its axis was parallel to the principal cleavage plane of the metal. Two copper cylinders *c*, *d*, were bored out to receive the heating coils, and their



curved surfaces were planed down to within a millimetre of the coils. The end of the bismuth rod was pressed against and fused to the flat surface of the cylinder. After solidification the excess of fused bismuth around the rod was removed. Each heating coil consisted of a spiral of fine double-silk-covered eureka wire, which was soldered at each end to thicker copper wire. The upper copper lead was soldered to a circular copper disk *a*, and the lower one was attached to the cylinder by electroplating with copper. A copper wire *k* soldered to the upper lead served as a potential lead for measuring the resistance of the coil, and also as a lead for the heating current. The coil and copper disk were insulated from the cylinder by paraffin wax. The object of securing the coil in this way was to maintain as far as possible an equality of temperature between the copper cylinder and the terminals of the coil, and so minimise the loss of heat by conduction along the leads.

The difference of temperature between the cylinders was indicated by four thermocouples of copper and constantan wires. These junctions were laid in grooves and insulated from the cylinders by thin strips of mica. The grooves were filled in with cotton-wool, and this together with the junctions was tied to the cylinders with silk thread. The available galvanometer was of the suspended-coil type having a resistance of 45 ohms, a period of about 10 seconds, and a figure of merit of 350 mm. scale-divisions per microampere. The galvanometer, when connected to the thermo-junctions, gave a deflexion of about 10 mm. per 0.01 ampere from one cylinder to the other.

The difference of temperature between the copper-bismuth junctions may also be observed by interrupting the currents through the rod and heating coil and connecting the coppers with the galvanometer. Stray thermoelectric forces in the use of this method would be relatively more important since the thermoelectric power of this junction is only about one-third of that of the four copper-constantan couples. This method would be suitable with a galvanometer of low resistance and short period. The temperature of each cylinder was measured by a copper-constantan junction *f, g*, soldered to the copper.

The various leads were insulated from one another in glass tubes and passed through a copper tube, *p q*, which was soldered to the cover of the enclosure. The constant-temperature enclosure was a small rectangular copper vessel with walls 2 mm. thick. This was designed for the small gap between the poles of an electromagnet, and in this position the temperature could be reduced by immersing the copper rod extension *r* in ice and water.

The currents through the junctions and the heating coil were both sent through the same copper lead, *n, o*, and therefore, if necessary, could be easily interrupted simultaneously. The currents were supplied from two separate batteries of accumulators and could be varied almost independently. Each of these currents could be measured to about 1 part in 500 by moving coil ammeters.

In making an experiment the current through the heating coil was kept constant, and that through the junctions was adjusted so that the thermo-junctions gave a small steady deflexion. The exact compensating current through the junctions was deduced from the change of deflexion of the galvanometer, produced by a small change of current through the junctions. Since both the currents traverse the same copper lead to cylinder, it is necessary to reverse the heating

current through the coil in order to eliminate its Joule effect in the lead. This was done, and the mean compensating current through the junctions was observed. The unequal emissivities of the two copper cylinders were eliminated by reversing the current through the junctions and passing the heating current through the other coil.

To obtain the temperature of each junction, one of the thermo-junctions  $f$  was connected in series with another and the galvanometer. The second junction was immersed in water, and this was warmed until the galvanometer was undeflected. The temperature of the water, as read on a thermometer, was then the temperature of the copper cylinder and each junction.

The compensating current through the junctions was also determined by using the copper-bismuth junction as an indicator of the temperature difference between the cylinders. It was found that, in the two methods, the compensating currents differed by 0.005 ampere approximately for a given heating current. It follows that the insulated thermocouples could be relied upon to indicate the temperature difference between the cylinders.

It is assumed, in calculating the Peltier coefficient, that the temperature of each copper cylinder is practically uniform throughout, and that possible Peltier and Thomson effects external to the surface of contact are negligible. There is no doubt that the fusing of the bismuth rod to the copper damages the crystalline structure at the end of the rod, and so produces a thin transition layer through which the Peltier effect is distributed. The sum of the Peltier effects across this thin layer is measured in this experiment, and according to the laws of the thermoelectric circuit this total Peltier effect is equal to the Peltier effect between the copper and the crystallized bismuth.

Let  $Q$  = rate of supply of energy by heating coil ;

$C$  = mean compensating current through junctions for a given current in one heating coil ;

$P$  = Peltier coefficient ;

$r$  = effective resistance of the metals about a junction ;

$k$  = rate of loss of energy from a copper cylinder per degree excess of temperature above the enclosure ;

$t$  = excess of temperature of each cylinder above the enclosure.

Then the following equations hold for a current  $C_1$  through



the junctions during the steady state :

$$Q_1 + C_1^2 r_1 - PC_1 = k_1 t_1 \quad \dots \quad (1)$$

$$C_1^2 r_2 + PC_1 = k_2 t_1 \quad \dots \quad (2)$$

and for a reversed current  $C_2$  :

$$C_2^2 r_1 + PC_2 = k_1 t_2 \quad \dots \quad (3)$$

$$Q_2 + C_2^2 r_2 - PC_2 = k_2 t_2 \quad \dots \quad (4)$$

These equations give

$$P = \frac{k_2 Q_1 + k_1 Q_2}{(C_1 + C_2)(k_1 + k_2)} + (C_1 - C_2) \left\{ \left( \frac{k_1 - k_2}{k_1 + k_2} \right) r_1 + \frac{k_1}{k_1 + k_2} (r_2 - r_1) \right\} \dots \quad (5)$$

In the experiment  $Q_1$  was made nearly equal to  $Q_2$ . In this case the maximum value of the second term, as deduced from the observations, was about 0.001 of the first term, and is therefore negligible. The second term may be made to vanish by making  $C_2 = C_1$ ; but this necessitates an approximate value of the ratio of  $k_1$  to  $k_2$  to evaluate the first term.

Thus

$$P = \frac{Q_1 + Q_2}{2(C_1 + C_2)} \dots \dots \dots (6)$$

The tables below contain some of the results of experiments with different currents and at slightly different temperatures. The thermoelectric powers of the copper-bismuth junction as calculated from the relation

$$P = T \frac{dE}{dT}$$

are also given.

Heating Current.	Junction Current.	Mean Temp. of junction.	Peltier Coefficient.	Thermoelectric Power.
$a + 0.125$ amp. $a - 0.125$ $b + 0.130$ $b - 0.130$	$c - 1.466$ amp. ,, 1.431 $d - 1.366$ ,, 1.330	22° 4 C.	0.01590 volt	$53.8 \times 10^{-6}$ volt
$a + 0.070$ $a - 0.070$ $b + 0.072$ $b - 0.072$	$c - 0.461$ ,, 0.455 $d - 0.426$ ,, 0.411	20° 6	0.01573 volt	$53.6 \times 10^{-6}$ volt
$a + 0.100$ $a - 0.100$ $b + 0.102$ $b - 0.102$	$c - 1.040$ ,, 1.018 $d - 0.892$ ,, 0.868	8° 1	0.01462 volt	$52.0 \times 10^{-6}$ volt

The resistance of coil  $a = 2.780$  ohms.

„ „  $b = 2.692$  „  
2 H 2

As an instance of the calculation of P :—

At 22°·4 C.,

$$P = \frac{0\cdot125^2 \times 2\cdot78 + 0\cdot130^2 \times 2\cdot692}{1\cdot466 + 1\cdot431 + 1\cdot366 + 1\cdot332}$$

P = 0·01590 volt from bismuth to copper.

In this case the excess of temperature of cylinder above enclosure = 3° nearly.

The resistance of the bismuth rod between copper cylinders =  $1730 \times 10^{-6}$  ohm.

Length of rod = 1·76 cm.

Mean cross-sectional area of rod = 0·154 sq. cm.

Specific resistance at 19°·5 =  $151 \times 10^{-6}$  ohm.

The effective resistance of the copper lead to cylinder was deduced from the change of the compensating current through the junction on reversing the current through the heating coil.

Let  $C_1$  and  $C_2$  be the values of the junction currents ;  
 $c$  the current through the heating coil ;  
 $r_3$  the effective resistance of the copper lead which is traversed by both the junction and heating currents ;  
 $r_4$  the effective resistance of the bismuth rod at this junction ;  
 $r_2$  the total effective resistance of the copper lead and bismuth rod at the other junction.

Then the following equations hold for currents  $C_1$  and  $c$  in the same direction through the copper lead :—

$$Q + (C_1 + c)^2 r_3 + C_1^2 r_4 - PC_1 = k_1 t_1 \quad \dots \quad (7)$$

$$C_1^2 r_2 + PC_1 = k_2 t_1 \quad \dots \quad (8)$$

and for a reversed current  $c$  :—

$$Q + (C_2 - c)^2 r_3 + C_2^2 r_4 - PC_2 = k_1 t_2 \quad \dots \quad (9)$$

$$C_2^2 r_2 + PC_2 = k_2 t_2 \quad \dots \quad (10)$$

These equations give

$$r_3(C - C_2 + 2c)(C_1 + C_2) = \frac{k_1 + k_2}{k_2} \{ r_2(C_1^2 - C_2^2) + P(C_1 - C_2) \} - (r_4 + r_2)(C_1^2 - C_2^2).$$

To obtain an approximate value of  $r_3$ ,  $\frac{k_1 + k_2}{k_2}$  may be written = 2; and since the results in the table show that the

effective resistances of the two junctions are nearly the same,  $r_2$  may be written  $= r_3 + r_4$ . Then

$$r_3 = \frac{P(C_1 - C_2)}{c(C_1 + C_2)} \dots \dots \dots (11)$$

Taking the results from the first part of the table,

$$r_3 = \frac{0.0159 \times 0.035}{0.125 \times 2.9}$$

$$r_3 = 0.0015 \text{ ohm nearly.}$$

The maximum possible resistance of each junction will be  $r_3$  together with half the resistance of the bismuth rod.

In the complete equation for P (5),

$$r_1 = r_2 = 0.0015 + 0.0009 = 0.0024 \text{ ohm nearly.}$$

The approximate value of  $\frac{k_1}{k_2}$  as obtained by substitution in (1) and (2) is 1.08. Substituting these values in equation (5) the second term

$$\begin{aligned} & (C_2 - C_1) \left\{ \left( \frac{k_1 - k_2}{k_1 + k_2} \right) r_1 + \frac{k_1}{k_1 + k_2} (r_2 - r_1) \right\} \\ &= -(1.45 - 1.35) \left\{ \left( \frac{1.08 - 1}{1.08 + 1} \right) 0.0024 + 0 \right\} \\ &= -9.2 \times 10^{-6} \text{ volt nearly.} \end{aligned}$$

Since  $P = 15900 \times 10^{-6}$  volt by equation (6), it follows that the second term in (5) is much less than the errors of observation and can be neglected. Thus the calculation of P by equation (6) is correct to 1 part in 1000, even when  $Q_2$  differs from  $Q_1$  by 4 per cent. When errors of observation are taken into account the final result is probably correct to 1 part in 200, and this order of accuracy was aimed at in designing the apparatus.

The bismuth, which was supplied by Griffin and known as Kahlbaum's pure bismuth, was carefully crystallized by Dr. Lownds in the following way. A crucible was overwound with a coil of eureka wire and surrounded on all sides with sand and asbestos. About 1000 gms. of bismuth were melted in the covered crucible, and the whole mass was very slowly cooled by gradually diminishing the current through the coil. The whole operation of cooling lasted 36 hours. The crucible was broken and the lump of bismuth removed. A blow with a hammer near the upper edge

divided the lump along a plane inclined to the axis of figure of the crystallized mass. The brilliant surfaces of cleavage could be traced by chipping and cutting to the middle of the lump, and were found to be nearly plane and parallel to each other. The rod of bismuth in this experiment was cut from the middle of the crystallized mass so that its axis was parallel to the principal cleavage plane of the bismuth. The surfaces of contact of the bismuth rod with the copper were at right angles to the principal cleavage planes.

This method of preparing the specimen of crystallized bismuth is, with the exception of the electrical heating, similar to that employed by Perrot\*. He prepared several crystals from the same mass of bismuth and measured for each the thermoelectric power with copper in two different positions. In one of these positions, the surfaces of contact with the copper were parallel to the principal cleavage plane of the crystal, and in the other position they were at right angles to this plane. The first of these positions was designated by the symbol  $\parallel$  and the second by the symbol  $\perp$ . He found that the thermoelectric power in the position  $\parallel$  was approximately twice as great as in position  $\perp$  at a temperature of  $55^{\circ}$  C. The thermoelectric powers at a temperature of  $55^{\circ}$  C. for four of his crystals in position  $\perp$  are given here for the purpose of comparison with the value derived from the Peltier coefficients in the table above.

	Thermoelectric Power at $55^{\circ}$ C.
Crystal P $\perp$ ...	$53.4 \times 10^{-6}$ volt per degree.
"    G $\perp$ ...	51.1   "   "   "
"    A $\perp$ ...	58.3   "   "   "
"    M $\perp$ ...	55.5   "   "   "

The lengths of these crystals varied from 19 mm. to 30 mm. In the table above the thermoelectric power as derived from the Peltier coefficient would at  $55^{\circ}$  C. be equal to  $57.6 \times 10^{-6}$  volt per degree. Considering the widely different results that Perrot obtained, this value is quite acceptable. The bismuth used by Perrot was analysed by three chemists and found to be pure, with the exception of an undetermined trace of iron. The different results obtained by him are probably due to slight irregularities in the structure of the crystals. The bismuth rod in this apparatus has not yet been analysed, and consequently no accurate comparison of the results can be made.

\* F. L. Perrot, *Arch. des Sciences Phys. et Nat.* Aug. 1898.

In Perrot's experiments a large temperature-gradient was established between the two contact surfaces, and the thermoelectric power would include possible thermoelectric forces arising from slight changes in the crystalline structure. In the apparatus, described in this paper, the mean of the Peltier forces near the ends of the crystallized rod is measured, and therefore the derived value of the thermoelectric power might differ from that of Perrot's. Differences, arising from impurities in the bismuth rod, are also possible.

To apply this method to the measurement of the Peltier coefficient for any two metals, the copper leads  $n, o$  would be replaced by one metal and the bismuth rod  $e$  by the other metal. To dissipate the Joule effect and to reduce the effective resistance of each junction, it is preferable to take each metal in the form of a thin strip. Each strip should be approximately dimensioned so that the total flow of heat per unit temperature-gradient is the same as for the bismuth rod. In this case the resistance of the junction would be much less than that of the copper-bismuth junction in this apparatus. An additional lead for the current through each heating coil must be soldered to each copper cylinder. The same sensibility can be attained, in the case of small Peltier effects, by increasing the number of thermo-junctions attached to each cylinder, and also by reducing the size of the copper cylinders. This modified apparatus would be suitable in the case where only a short length of one of the metals was available. The Peltier effect could also be measured at various temperatures by varying the temperature of the enclosure.

An attempt was made to measure the thermoelectric power of the copper-bismuth junction by passing a current through one of the coils and measuring the E.M.F. of the junctions. The temperature difference thus produced between the junctions was small and could not be measured with sufficient accuracy owing to a defective attachment of the junctions  $f, g$ . One junction of the two wires projected about 1 mm. from the cylinder, and although the other appeared to be satisfactory the results showed that this was slightly defective. Owing to the brittleness of the bismuth rod, these defects were not remedied. The following results indicate the existence of these defects.

$$\text{At } 22^{\circ}.5 \text{ C. } \frac{dE}{dt} = 64.8 \times 10^{-6} \text{ volt per degree,}$$

with faulty junction at the higher temperature.

When the temperature-difference was reversed,

$$\frac{dE}{dt} = 56.3 \times 10^{-6} \text{ volt per degree,}$$

with faulty junction at the lower temperature.

The fault of the junction arose from the temperature gradient along the wires (0.1 mm. diam.). This could be remedied by soldering a junction of thinner wires in a groove and winding the insulated leads around the cylinder, in order to minimize the temperature gradient. A junction was attached in this way to a similar copper cylinder. The defective junction and the insulated junction were also imitated approximately. The latter was laid in a groove and separated from the cylinder by a thin strip of mica 0.02 mm. thick. The rest of the groove was packed with cotton-wool and overwound with silk thread. The cylinder was heated electrically inside a copper enclosure, and the following results were obtained:—

Excess temperature of cylinder over enclosure = 8° nearly.	}	= 0°.25 C.
Temperature difference between the perfect and defective junctions .....		

The temperature difference between the perfect and insulated junction varied from 0°.3 to 0°.45, according to the tightness of the filling of cotton-wool in the groove.

The results indicate the precautions that must be taken in fixing the thermo-junctions to the cylinder for this particular measurement. It is to be noted that this slight defect in the junctions *f, g* will not disturb the temperatures in the third column of the table by more than 0°.1. The high results for the direct measurement of the thermoelectric power are just what are to be expected when the junctions are faulty and the rod is a short one.

It is intended to conduct experiments with this apparatus in a magnetic field to determine how the Peltier coefficient depends on the strength and direction of the field across the surface of contact of the bismuth with the copper.

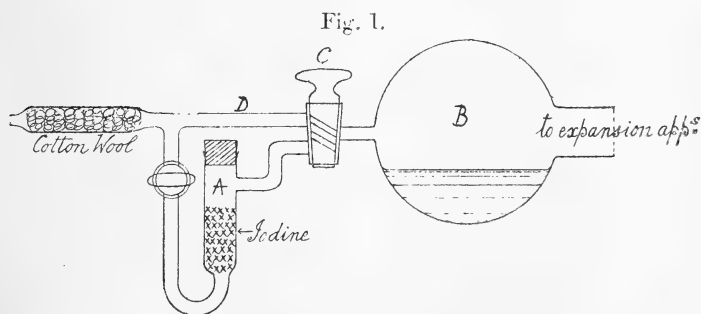
South-Western Polytechnic, Chelsea, S.W.  
September 18, 1910.

LIV. *On Condensation Nuclei produced by the Action of Light on Iodine Vapour.* By GWILYM OWEN, M.A., D.Sc., and HAROLD PEALING, M.Sc., University of Liverpool\*.

FROM experiments made by G. Owen and A. Ll. Hughes (Phil. Mag. Oct. 1907, June 1908) it seems that the gas evolved by a solidified mass of carbon dioxide previously condensed in a dry and dust-free condition contains large numbers of nuclei, the presence of which can be shown by their ability to act as centres for the condensation of super-saturated water-vapour. This fact suggests that the sublimation of solid  $\text{CO}_2$  consists not merely in the escape of separate gaseous molecules, but also in the liberation of large numbers of relatively large molecular aggregations. The question arises as to whether the same is true for other subliming substances, and it was during the course of experiments designed to test this point that the effects described in the present paper were observed.

Experiments were made with camphor, naphthalene, benzoic acid, and iodine—the method being to pass a current of dust-free air through a tube containing one of these volatile substances, and then to test the air for nuclei by suddenly expanding it in a bulb containing distilled water.

In fig. 1, A is the glass tube containing the substance to



be tested. The air swept through this tube was first rendered dust-free by a plug of cotton-wool as shown. The cloud-chamber B was a glass bulb some 6 cms. in diameter and was sealed on to a Wilson expansion apparatus †, the piston of which worked in distilled water. The whole apparatus to the right of the tap C was made of glass. Any clouds produced in B were rendered visible by focussing on the bulb the light from a Nernst lamp. If we take as a

\* Communicated by the Authors.

† C. T. R. Wilson, Camb. Phil. Soc. Proc. ix. p. 323 (1897).

measure of the expansion the pressure-fall—that is, the difference between initial and final pressure of the gas in the apparatus,—then, as is well known\*, there is no condensation in the body of the gas when the pressure-fall is less than 15 cms. of mercury. When the expansion is between 15 cms. and 20 cms. a few scattered drops are observed (“rain-like” condensation), the nuclei in this case being the few ions always present in the gas. On subjecting the gas to a pressure-fall over 20 cms., a dense fog is obtained. The nuclei on which this fog forms are generally regarded as being minute drops of water continually being formed from the saturated water-vapour. In the present paper we shall refer to the above effects as the “Wilson effects.”

*Results of the Tests made with Camphor, Naphthalene,  
Benzoic Acid, and Iodine.*

With camphor, naphthalene, benzoic acid, the showers or clouds obtained on expansion were identical with the usual Wilson effects. Evidently, then, these substances do not sublime in the form of molecular aggregations sufficiently large to act as condensation nuclei. C. Barus † had previously obtained a similar result with a somewhat different apparatus for camphor and naphthalene. With iodine, however, we obtained very marked effects, as is shown in the following table. The figures in the columns marked

TABLE I.

Cloud-chamber filled with pure moist dust-free air.		Cloud-chamber filled with air which had passed over iodine.	
Pressure-fall in cms.	Observation. (Ordinary Wilson effects.)	Pressure-fall in cms.	Observation.
15.0	0	15.0	Few drops.
16.5	Few drops.	16.5	Thin shower.
17.5	Thin shower.	17.5	Good shower.
18.5	Good shower.	18.5	<u>Tinted cloud.</u>
19.5	Very dense shower.		
20.5	Fog.		

\* C. T. R. Wilson, *Phil. Trans. A.* vol. cxxxix. (1897).

† C. Barus, ‘Condensation of Vapor as induced by Nuclei and Ions.’ (Carnegie Institution of Washington, May 1907.)



“pressure-fall” are centimetres of mercury, and, as already explained, may be taken as a measure of the expansion. For the purpose of comparison the Wilson effects obtained with our apparatus are first given.

The above table shows that the presence of the iodine vapour in the cloud-chamber produces a considerable increase in the density of the clouds obtained, and that the effect of the iodine is specially marked for pressure-fall of 18·5 cms.

The influence of the iodine was found to be very persistent, sweeping in fresh air through the by-path D for several hours failed to get rid of the effect. In fact, it was found necessary to take the apparatus down and wash it thoroughly to make it give once more the ordinary Wilson effects. Later experiments (described below) showed that the influence of the iodine ultimately disappears if the apparatus is allowed to stand in bright light for three or four days.

When the above effect was first obtained, we were naturally led to believe that iodine, in contradistinction to the other substances tried, does sublime in the form of particles sufficiently large to act as condensation nuclei. But this conclusion was upset by the following modification of the experiment. Between the iodine reservoir and the cloud-chamber a tube containing a long plug of glass-wool was inserted, in order to see if the iodine particles could be trapped and prevented from reaching the cloud-chamber. After sweeping air and iodine vapour through this tube for two minutes, the clouds obtained with a pressure-fall of 18·5 were now much denser than before. Further, the passage of the iodine vapour was observed to produce a discoloration of the wool, and, finally, when the latter was coloured through its whole length, heavy clouds were obtained for quite small expansions corresponding to a pressure-fall of less than 10 cms. This last effect was at first regarded as showing that when the iodine is dispersed in the form of minute crystals in the interstices of the wool the mechanical action of the air-current results in small solid particles of iodine being dislodged from these crystals and carried over into the bulb. This view, however, was shown to be erroneous by the following subsequent observations:—

- (1) The glass-wool very soon lost its power of giving rise to the larger nuclei.
- (2) On expanding *without previously illuminating* the gas in the cloud-chamber the results were just the same as the Wilson effects.

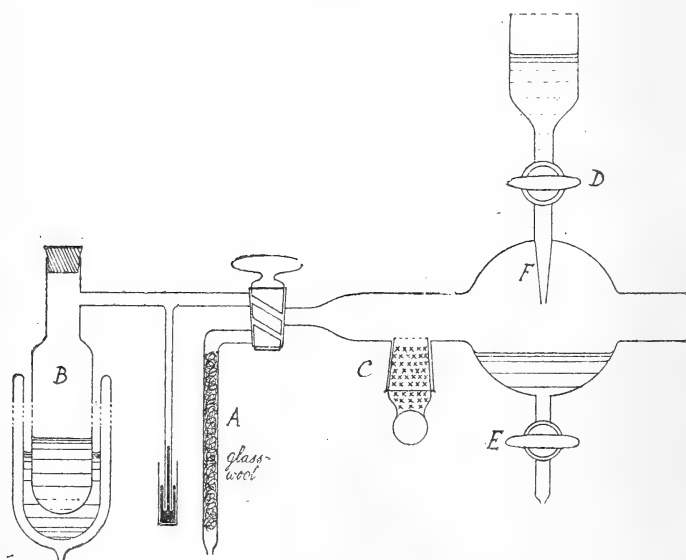
Thus the effects obtained with the iodine are not mechanical at all, but *photochemical*.

A large number of observations have since been made (a) when the iodine vapour was placed inside the cloud-chamber itself; (b) when iodine vapour was swept through glass-wool before admission into the apparatus. It will be convenient to discuss the two cases separately, although, as will be shown, they are probably intimately connected.

*Experiments with Iodine placed in the Cloud-Chamber itself.*

When studying the effect of iodine placed inside the cloud-chamber, the apparatus shown in fig. 2 was found convenient.

Fig. 2.



The apparatus could be filled either by filtered laboratory air drawn in through A or by air from boiling liquid air stored in B. As a matter of fact, both these sources of dust-free air gave practically the same results. The apparatus was first calibrated with pure air and then iodine vapour was introduced by placing a few crystals in the hollow tap C. The taps D, E allowed fresh distilled water to be admitted when desired. In order to study the effect of light, the cloud-chamber and expansion apparatus were wrapped in black cloth. The covering of the cloud-chamber was provided with two vertical slits, one on each side; the one slit

admitted the light, and the other permitted an inspection of the cloud obtained. The slits were provided with flaps, which could be closed when it was desired to keep the apparatus in complete darkness.

Table II. shows the effect of light. The results given in the last column were obtained by producing the expansion in the dark and then admitting the light immediately afterwards.

TABLE II.

No Iodine in the Apparatus. Light on Bulb continuously.		Iodine in the Apparatus. Light on Bulb continuously.		Iodine in Apparatus. Expansion made in the dark. Light put on immediately afterwards.	
Pressure-fall.	Observation.	Pressure-fall.	Observation.	Pressure-fall.	Observation.
.....	.....	14.5	Few drops.	14.2	0
15.5	Few drops.	15.5	Fair shower.	.....	.....
17.5	Thin shower.	17.5	Dense shower.	17.5	Thin shower.
18.5	Good shower.	18.5	<u>Coloured cloud.</u>	18.5	Good shower.

The above table shows clearly :—

- (a) That the addition of the iodine causes an immense number of fresh nuclei to appear.
- (b) That the majority of the nuclei are small and consequently require a large expansion corresponding to a pressure-drop of 18.5 cms. to catch them.
- (c) That all these nuclei are wholly produced by the action of light, for when the expansions are made in the dark the effects observed on admitting the light immediately afterwards are the same as in the absence of iodine.

These effects were obtained with a Nernst lamp as the source of illumination. Other modes of illumination—arc lamp, fishtail burner, Nernst light screened by red glass, and diffused daylight—were tried with the same results. In the last case the clouds were observed by illuminating the bulb with a Nernst lamp *after* the expansion had been made.

*Experiments on the Growth of the Nuclei in the Light and their Decay in the Dark.*

Table III. shows how the density of the cloud depends upon the duration of the illumination.

TABLE III.

Duration of illumination.	Result of the Expansion. Pressure-fall = 18.5 cms.
0	Good shower (= Wilson effect).
About $\frac{1}{4}$ second.	Dense shower.
1 second.	Cloud.
2 seconds.	Cloud (same as above).

Thus the nuclei grow under the influence of the light and attain a maximum size in less than one second.

Table IV. shows that the nuclei disappear very quickly after their formation. The bulb was illuminated in each case for the same period and then kept in the dark for various intervals before expansion.

TABLE IV.

Period for which the Nuclei were kept in the dark before expansion.	Result of the Expansion. (Pressure-fall = 18.5 cms.)
3 minutes.	Good shower.
1 minute.	Good shower.
30 seconds.	Very dense shower.
10 seconds.	Cloud.
0	Cloud.

Thus nearly all the nuclei live for ten seconds, but all have disappeared in one minute. This may be due to the nuclei diffusing to the walls of the vessel, or to their actual break up. If the latter is the correct explanation, evidently the substance forming the nuclei is very unstable.

*Influence of the Gas in the Expansion Apparatus.*

Air, hydrogen, carbon dioxide, coal-gas, and oxygen were tried in turn in the expansion apparatus. In each case the apparatus was first calibrated before introducing any iodine in order to study the normal Wilson effects in these gases. With hydrogen,  $\text{CO}_2$ , and coal-gas, the introduction of the iodine produced absolutely no change. On the other hand, in the case of oxygen, the iodine gave rise to clouds similar to (possibly a little denser than) those obtained with air. Thus the presence of oxygen is necessary to the formation of the nuclei.

*Alcohol in the Expansion Apparatus.*

Some experiments were tried in which the water in the expansion apparatus and in the cloud-chamber was replaced by alcohol. With this liquid the normal Wilson effects begin at a pressure-fall of 10 cms., and fogs are obtained for a pressure-fall of about 12 cms. No increased effect could, however, be detected on admitting iodine. In fact, when the iodine had been in the apparatus for a day or two so that the alcohol had developed a bright yellow colour by the solution in it of some iodine vapour, the Wilson effects at any given expansion were then distinctly smaller than they were initially.

*Behaviour of the Nuclei in an Electric Field.*

In order to ascertain if the nuclei are charged, a cloud-chamber containing a horizontal platinum disk was constructed, the distance between the disk and the water surface being about 1.5 cms. A potential difference of 230 volts could be established when desired between the disk and the water. We could, however, detect no evidence of any motion of the nuclei under the electric force, for their rate of disappearance in the dark was the same with the field on as without. It may be concluded, therefore, that the nuclei do not carry an electric charge.

*Diminution of the Effect with Time.*

We noticed early on in the course of the experiments that the coloured clouds obtained with the iodine in the apparatus become less and less dense as time goes on. By the second day the result of an expansion of 18.5 is only a "dense shower." This decay continues from day to day, until by the fourth or fifth day the effect obtained is actually smaller than the normal Wilson effect for the same expansion. It was found, in addition, that the effect of the

iodine decreases much more rapidly when the apparatus is kept unshielded in bright diffused daylight than when kept in the dark. If the apparatus be taken down when in this non-sensitive state and thoroughly rinsed out with distilled water, the large effects already described once more make their appearance when iodine is admitted.

The following possible causes of the disappearance of the effect naturally suggested themselves and were investigated in turn :—

- (1) That some change took place in the properties of the iodine itself, say through its becoming damp. Introducing fresh iodine, however, failed to bring back the original clouds.
- (2) Thinking that possibly the nuclei might be due to an action of the iodine vapour on some impurity brought into the apparatus when it was filled with air, and that the disappearance of the effect in the course of a few days was due to this impurity being all used up, we tried admitting a fresh supply of air. On two or three occasions fresh air did *partly* bring back the clouds, but *generally* this was not the case, both air from boiling liquid air and dusty laboratory air being equally ineffective.
- (3) Again, it was thought possible that the original clouds were due to an action of the iodine on vapours evolved by the vaseline lubricating the taps. But we found that introducing fresh vaseline into the apparatus produced no change. Various other possible sources of impurity, such as tap-grease, indiarubber, red wax, cotton-wool, were placed in the apparatus. In every case there was no increased effect.
- (4) Again, we thought that the film of iodine which naturally forms in time on the sides of the cloud-chamber might possibly cut off the effective part of the light entering the bulb. But driving off this film by gently heating the glass failed to bring back the original clouds.
- (5) Again, it is well known that the value of the expansion required to catch nuclei of a given size depends upon the nature and condition of the liquid in the cloud-chamber. Now, after the iodine has been in the apparatus for two or three days, the water in the cloud-chamber develops a bright yellow colour owing to the solution in it of some of the iodine vapour. That the disappearance of the clouds is not due to this change in the water was shown in two ways :—
  - (a) The coloured water in the cloud-chamber was drained off through the tap E and fresh distilled water admitted through D. This process, however, had no effect.

- (b) A small quantity of radium was placed near the cloud-chamber and the minimum expansion required to catch the ions so produced carefully determined, first before any iodine has been introduced, and afterwards when the water in the apparatus had become strongly coloured by the iodine. The condensation on the ions was found to start at *exactly the same expansion* in both cases\*. Hence the vapour-pressure of the water is not appreciably altered by the iodine.
- (6) It has already been mentioned that the clouds after their disappearance are brought back by rinsing the apparatus out afresh with distilled water. We have also stated that admitting fresh water into the cloud-chamber through the tap D (fig. 2) is without effect. The two statements may appear contradictory. But an inspection of fig. 2 shows that there is a difference between the two operations. As may be seen from the figure, the water enters the apparatus through a nozzle F projecting into the bulb, and consequently settles in the bulb without flowing down the walls. With an earlier form of cloud-chamber, in which the nozzle F was absent, the water on admission ran down the sides of the bulb, and in this case the clouds were found to have been partly brought back. But the experiments with the cloud-chamber of fig. 2 show that this increase in the effect was due not so much to the changing of the water in the apparatus as to the rinsing of the glass walls by the water as it flowed into the bulb.

\* Taking as the measure of the expansion the ratio of the final to the initial volume of the gas, we found that in our apparatus the minimum expansion required to catch the ions produced by radium was 1.22. C. T. R. Wilson (Camb. Phil. Soc. Proc. vol. ix.) gives the same value for the same form of apparatus. It has already been mentioned that the ordinary Wilson effect was observed in our experiment to be smaller when the apparatus had had iodine in it for some days than it is for the same expansions in an apparatus free from iodine. And yet the experiment with the radium shows that the ionic condensation *commences* at the same point in the two cases. This fact suggests that the *spontaneous ionization* in a closed space is reduced by saturating the space with iodine vapour. A similar effect was noticed in the experiments with alcohol. We propose investigating this point further. It is possible that there may be a connexion between this decrease in the Wilson effects and the result obtained by Henry (Proc. Camb. Phil. Soc. 1897) in his experiments on the "Effect of Ultra-Violet Light on the Conductivity of Iodine Vapour." Henry found that the discharge of ions from a metal plate when illuminated by ultra-violet light was greatly reduced by admitting iodine vapour into the ionization-chamber, but he was uncertain as to whether this was a real effect or a spurious one due to the weakening of the light in its passage through the vapour.

From a consideration of the above we are led to regard as follows the disappearance of the clouds and their reappearance on washing the apparatus. As will be seen later on, the nuclei probably result from the production of some substance which is deposited on the walls of the apparatus, and it is also likely that the action ceases when a certain amount of this substance has been formed. The flow of water over the glass would remove this deposit and allow the action to proceed as before.

The view that the glass plays a part in the production of the clouds is supported by the results of experiments made with glass-wool. A passing reference to these experiments has already been made. We shall now consider them in greater detail.

#### *Experiments with Glass-wool.*

It has already been stated that when the iodine has been in the apparatus for a few days the coloured clouds obtained for an expansion corresponding to a pressure-drop of 18.5 cms. entirely disappear, and that the introduction of a number of foreign substances into the apparatus failed to bring the clouds back. We found, however, that the introduction of a *plug of glass-wool* gave rise to a cloud at a pressure-fall of 18.5. But after the wool had been in the apparatus for some twenty-four hours, the effect had again disappeared. Evidently, then, something was introduced with the glass-wool which for a time facilitated the production of nuclei. In order to ascertain whether this was accidental or whether it was a general property of glass-wool, we obtained through Messrs. J. H. and S. Johnson, of Liverpool, three different lots of pure glass-wool, guaranteed to have been supplied to them by three different makers. We found, however, that plugs of each specimen when placed in the apparatus brought back the clouds. Clearly, then, the property is general.

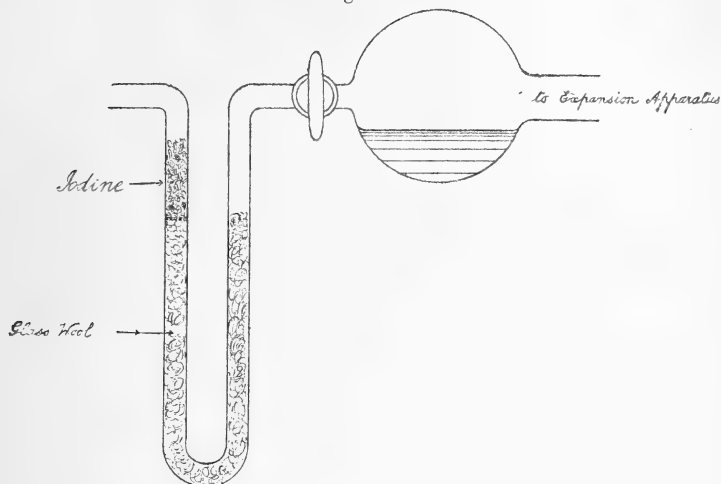
But the most effective way of making evident this action of the glass-wool is shown in fig. 3.

If a slow stream of air be drawn into the cloud-chamber through this arrangement, a *dense fog* is obtained for a pressure-drop of 18.5. Further, when the glass-wool has become coloured by the iodine along its whole length, clouds are obtained for much smaller expansions. Referring to Table III., it is seen, too, that illumination for so short an interval as one second is sufficient to produce the nuclei caught with a pressure-drop of 18.5. On the other hand, we found that to produce the large nuclei caught by small



expansion, the light had to be kept on for half a minute or more. This shows that the larger nuclei result from the growth of the smaller nuclei under the influence of the light.

Fig. 3.



The clouds obtained at large expansions become less dense as time goes on, showing that the glass-wool gradually loses its efficiency. Further, the decrease in the density of the clouds at large expansions is accompanied by the total disappearance of the clouds of small expansions. This shows that the nuclei only succeed in attaining a considerable size when their number is very large.

#### *Discussion of the Results.*

These experiments with the glass-wool suggest that the production of the nuclei when the iodine is put directly into the cloud-chamber (fig. 2) is influenced by the condition of the glass walls of the apparatus.

It now remains to consider the source of this influence. In the first place, it is not probable that the clouds obtained when the iodine is placed in the cloud-chamber are due to the presence of organic impurities on the glass, for they were invariably produced when the apparatus had previously been carefully washed with nitric acid and distilled water. And if this were the case it is difficult to see how the process of rinsing the apparatus with distilled water could bring back the clouds after they had once disappeared. On the other hand, the glass-wool itself is rendered less efficient as an aid

to the formation of the nuclei by soaking it in strong nitric acid and then washing it with distilled water. A small plug of glass-wool so treated did not cause a cloud when placed (in a dry condition) in the cloud-chamber. This same treatment, however, did not prevent the process of sweeping the iodine vapour through a long plug of it (as in fig. 3) from greatly increasing the density of the cloud at 18.5 pressure-fall, although it entirely stopped the formation of the large nuclei caught at small expansions.

If the nuclei originate as a surface action at the glass surface, this effect of the nitric acid on the glass-wool may be regarded in two ways:—(a) That the acid alters the catalytic properties of the surface of the glass-wool, or (b) that it partly removes from it some substance the presence of which aids the formation of the nuclei. One would not expect glass-wool to contain an "impurity" in the ordinary sense of the word. It is well known, however, that there is an action between glass and pure water resulting in the formation of minute traces of NaOH at the surface. Supposing for a moment that the production of the nuclei depends upon this action, it is not unreasonable to suppose that artificially increasing the alkalinity at the glass surface would increase the number of nuclei produced. To test this point the apparatus was rinsed out with a weak solution (5 per cent.) of NaOH and then worked with this same solution in the bulb and expansion cylinder instead of distilled water, as in the previous experiments. Under these circumstances we found the iodine to be entirely without effect, for the nucleation was identical with the Wilson effects which we studied before introducing the iodine. Thus the nuclei are not due to the minute traces of NaOH usually present on moist glass; or, at all events, the degree of extra alkalinity that we happened to try, instead of facilitating the production of nuclei, prevented it completely. We also found that treating the apparatus in the same way with a dilute solution of  $H_2SO_4$  prevented the production of the nuclei.

We thus find it difficult to say definitely what the nuclei are. They are evidently minute particles of some unstable compound requiring the co-presence of iodine-vapour, oxygen, and water-vapour for its production.

The effect is clearly photochemical in character, and in all probability belongs to that class of phenomena studied by Tyndall\*, Aitken†, and C. T. R. Wilson‡. Tyndall found

\* Tyndall, Phil. Trans. vol. clx. p. 333 (1870).

† Aitken, Proc. Roy. Soc. Edin. vol. xxxix. i. p. 15 (1897).

‡ Wilson, Phil. Trans. vol. cxcii. p. 403 (1899).

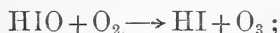
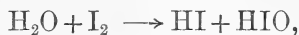
that ultra-violet light caused clouds to form without expansion in air containing traces of amyl nitrite, iodide of allyl, and other vapours.

Aitken found that clouds are produced in  $\text{NH}_3$ ,  $\text{H}_2\text{SO}_3$ ,  $\text{H}_2\text{S}$ ,  $\text{HCl}$ , and  $\text{Cl}_2$  when expanded after exposure to sunlight.

C. T. R. Wilson showed that the action of ultra-violet light on pure moist air or oxygen produced condensation nuclei which grew under the influence of the light, and with very intense light he obtained clouds without any expansion at all. Wilson suggested that the clouds obtained by him are due to the formation of  $\text{H}_2\text{O}_2$ , which, by dissolving in the drops as they form, lowers the vapour-pressure and thus makes it possible for the drops to grow where drops of pure water would evaporate.

Bevan\* again, in his experiments on the "Combination of Hydrogen and Chlorine under the Influence of Light," found that some substance is produced by the action of the light on the chlorine and water-vapour which acts as condensation nuclei, and that the formation of this intermediate substance is necessary for the production of the  $\text{HCl}$ .

With regard to the nuclei described in the present paper, it is possible that the reactions going on under the influence of the light are as follows:—



or it may be a reaction in which the iodine is oxidized directly. Moreover, it is difficult to say which of the products of such reactions would actually form the nuclei. Ozone, of course, is known to give rise to clouds, but, according to Meissner †, only as a consequence of reactions by which some of the ozone is destroyed.

The above reactions, being reversible, would cease when a certain amount of the products had been formed. This fact would explain why no clouds are obtained when the iodine has been for some days in the apparatus. It would also account for the reappearance of the clouds on washing the apparatus, for such treatment would remove the products of the reaction and allow it to go on once more. From this point of view the effect of introducing a plug of glass-wool into the cloud-chamber is readily understood, for the fibres

\* Bevan, *Phil. Trans.* vol. ccii. A. p. 347 (1903).

† Quoted by C. T. R. Wilson, *Phil. Trans.* vol. excii. p. 428 (1899).

will present a considerable area of fresh glass surface at which the action may proceed. On the other hand, the part played by the glass-wool in the experiments where the iodine vapour is swept through it before admission into the cloud-chamber is not quite so clear. It either means that the glass fibres by some preparatory catalytic action on the iodine-laden air facilitate the subsequent formation of the nuclei, or it means that there is some unexpected "impurity" in the glass-wool, which, reacting with the iodine, produces the results described. Since clean glass is known to affect certain chemical reactions, we are inclined to adopt the former view.

#### *Summary of the Main Results.*

In contradistinction to the results previously obtained with solid  $\text{CO}_2$ , we find that

(1) Camphor, naphthalene, benzoic acid, and iodine do not sublime in the form of particles sufficiently large to act as condensation nuclei for water-vapour. But

(2) When moist air (or oxygen) containing *iodine vapour* is *illuminated*, nuclei are produced, possessing the following properties :—

The nuclei are very unstable, disappearing in a few seconds in the dark. They do not carry an electric charge. They are not obtained except in the presence of oxygen and water-vapour. They grow under the action of the light, but, generally, do not attain a size greater than that requiring a pressure-fall of 18.5 cms. in order to catch them. The light required for their production need not be very intense nor of a high degree of refrangibility.

(3) No nuclei are produced after the iodine has been in the apparatus for some days. The cessation of the action is probably due to a state of chemical equilibrium having been attained. The equilibrium is destroyed by rinsing the glass walls of the apparatus with water.

(4) Glass-wool possesses the peculiar property of facilitating the formation of the nuclei, the number produced when iodine-laden air is admitted into the apparatus through a plug of glass-wool being much greater than the number obtained on placing iodine directly in the cloud-chamber. This property becomes less and less marked as the wool gets more and more saturated with the iodine. The action of the glass-wool is regarded either as being of a catalytic nature or as evidence of the wool being an unexpected source of

contamination for the iodine vapour during its passage through it.

In conclusion, we have pleasure in expressing our best thanks to our colleague, Dr. H. Bassett, for much valuable information on points pertaining to the chemistry of the subject.

Holt Physics Laboratory,  
Liverpool University.  
Feb. 8th, 1911.

LIV. Investigation of the Potentials required to produce Discharges in Gases at Low Pressures. By A. B. MESERVEY, New College, Oxford\*.

*Sparking Potentials between Concentric Cylinders.*

THE theory of sparking has been completely worked out for uniform fields only, but it would appear that some additional light might be thrown on the processes of ionization that take place at very low pressures from experiments on sparking in a non-uniform field. These experiments were undertaken with this end in view, and the field between concentric cylinders was selected as the simplest form of a field that is not uniform. Some preliminary experiments †, which were made in this laboratory, with concentric cylinders, showed, as might have been expected, that the sparking potential varied with the direction of the field, and the curves obtained in the two cases crossed at a point near the minimum sparking potential.

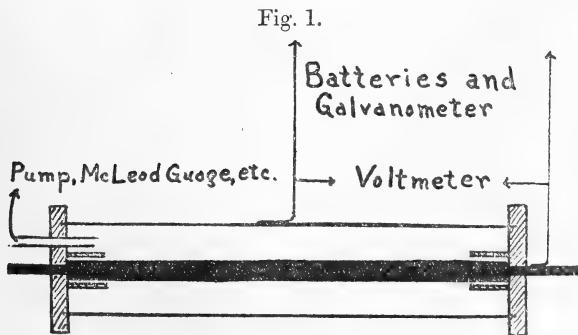


Fig. 1.

The apparatus used in the present experiments is represented in fig. 1. A brass cylinder, about 20 cm. in length

\* Communicated by Prof. J. S. Townsend, F.R.S.

† 'The Theory of Ionization of Gases by Collision,' John S. Townsend, p. 54.

and 4 cm. in diameter, was closed at the ends with ebonite disks, through which a concentric wire or tube of a given material and a given diameter was passed. The apparatus could be easily taken apart, and another cylinder substituted. As a precaution against leakage and irregular discharge, the inner cylinder was surrounded at the ends by short pieces of glass tubing, leaving about 16 cm. clear for discharge at the middle of the apparatus. The passage of a current was detected by a D'Arsonval galvanometer of 92 ohms resistance, giving a deflexion of 1 mm. on a scale 1 m. distant for a current of  $4.2 \times 10^{-8}$  ampere. The difference of potential between the cylinders was established by a battery of small storage-cells, and was measured by a Kelvin electrostatic voltmeter reading to 600 volts, connecting directly to the electrodes. The voltmeter had been accurately standardized, and during the course of the experiments was compared with a standard electromagnetic instrument. The gas was permitted to stand for a day or more in a drying-tube before being admitted to the apparatus, and there was another drying-tube in close proximity to the spark chamber.

When it was desired to take a set of readings, the connexions were made so that the difference of potential would be less than the sparking potential; then the high potential wire was moved up, a cell at a time, till a galvanometer deflexion indicated a discharge between the cylinders. The pressure was determined by the McLeod gauge, the connexions reversed, and another reading taken after a short interval. A small quantity of gas was then pumped out or admitted as the case might be, and, after time enough had elapsed to permit the pressure to become steady, the whole process was repeated at the new pressure. Readings were usually started at comparatively high pressure, and the latter was then decreased, a pair of readings being taken after each stroke of the pump, till the lowest desired pressure was reached. Small quantities of gas were then admitted, and the sparking potentials found in each case. The steps were more numerous during this process than during the other, as smaller changes of pressure were possible.

The readings were often somewhat less regular when the pressure was being changed by admitting fresh supplies of gas than when it was changed by the use of the pump, but the mean position of the curve through the points was not materially altered. It was found that when the apparatus had been unused for several hours, the potential required for the first spark was distinctly higher than that required

for subsequent sparks at the same pressure, and some experiments were made to determine the effect of previous discharge on the sparking potential. Different lengths of time were allowed to elapse, varying from a few seconds to two hours, and slight variations in the sparking potential were observed; but within the limits of time actually used in the main experiments, no regular variation of the potential was observed.

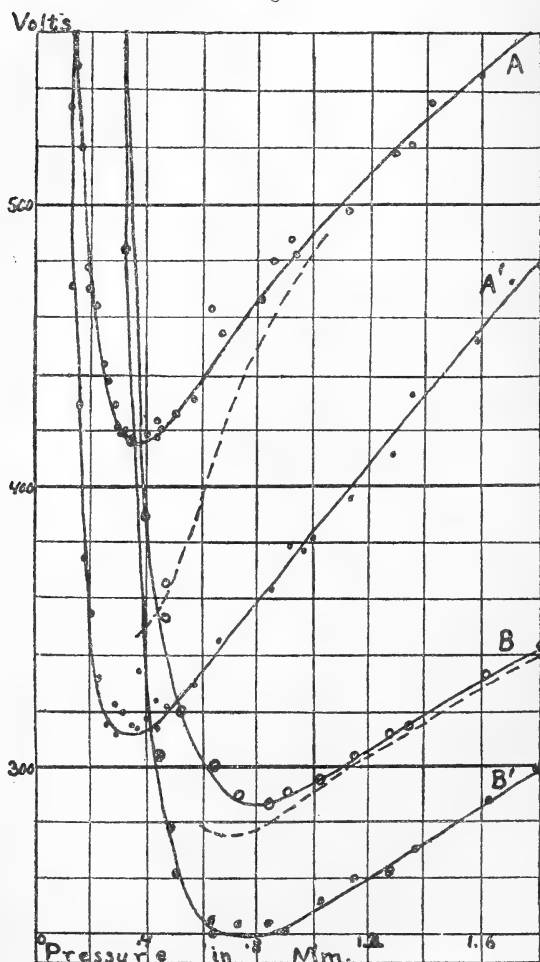
The results of the experiments showed that the sparking potential assumed quite different values according as the inner cylinder was positive or negative. In every case two distinct curves were obtained which, in all but the largest diameters, crossed near the critical pressure. For air the critical pressure varied from  $\cdot 1$  to  $\cdot 7$  mm., according to the diameter of the inner cylinder, and for hydrogen it varied from  $\cdot 3$  to  $1\cdot 1$  mm.; while the point of crossing of the curves varied only from  $\cdot 1$  to  $\cdot 2$  mm. and from  $\cdot 3$  to  $\cdot 5$  mm. for air and hydrogen respectively. As the diameter of the inner cylinder increased, the position of the positive and negative curves shifted so that the point of crossing of the two would be relatively nearer to that part of the curve which corresponded to the lowest values of the pressure, the curves gradually approached along their whole length, and, for the largest diameters employed, the two curves seemed to practically coincide as the pressure receded from the critical value. It is of course to be expected that the two curves would approach as the diameter increased, since for parallel plates they must completely coincide.

The minimum sparking potential was always lower when the inner cylinder was negative than when it was positive, though the difference was less for the large diameters. When the inner cylinder was positive, the minimum sparking potential was higher than for parallel plates; and when the inner cylinder was negative, the minimum sparking potential was in most cases lower than for parallel plates. The highest values obtained were 463 volts for air and 335 volts for hydrogen; the lowest were 311 volts for air and 240 volts for hydrogen.

Fig. 2 shows two typical pairs of curves in detail, together with curves calculated for parallel plate electrodes at the given distances apart in air and in hydrogen. The curves AA' are for air, with a brass rod 3.23 mm. in diameter as the inner electrode. For curves BB' the gas was hydrogen, and the inner electrode was a brass tube 8 mm. in diameter. The inner electrode was positive for A and B, negative for

A' and B'. The dotted lines show the corresponding parallel plate curves, calculated in accordance with Paschen's law

Fig. 2.



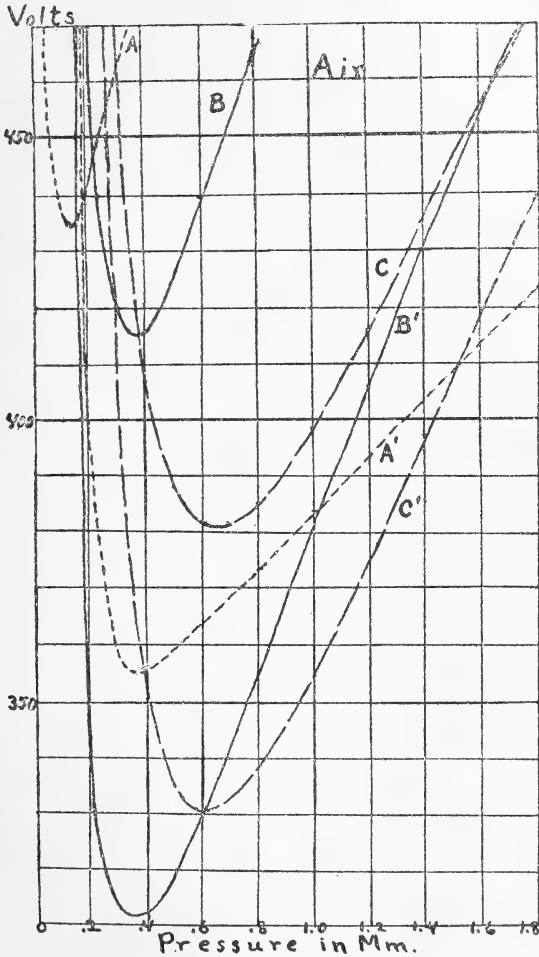
from the experimental curves of Prof. Townsend\*. Fig. 3 shows the curves obtained for three of the seven diameters used in air ; fig. 4 those for three of the five diameters used in

\* 'The Theory of the Ionization of Gases by Collision,' p. 60.



hydrogen. The others have been omitted in order to avoid making the diagram too complicated. For all the curves, the inner diameter of the outer cylinder was 39.6 mm.

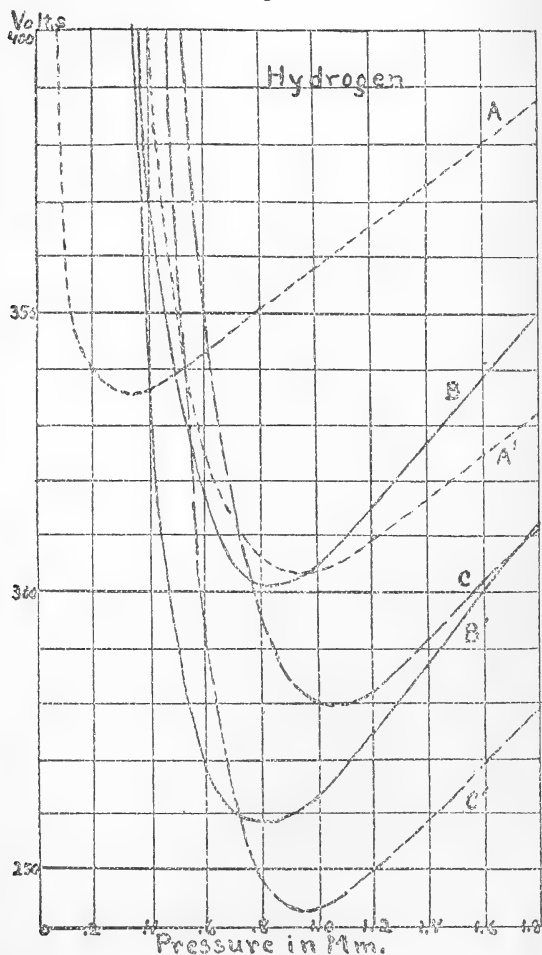
Fig. 3.



In fig. 3, curves	A and A'	diameter of inner cylinder was	.25 mm.
" "	" B and B'	" " " "	3.23 "
" "	" C and C'	" " " "	15.95 "
In fig. 4, curves	A and A'	" " " "	.25 "
" "	" B and B'	" " " "	5.10 "
" "	" C and C'	" " " "	15.95 "

A few sets of readings were taken with aluminium and platinum electrodes inside, but they showed no consistent differences from those obtained with brass. It seemed as if there might be some effect due to difference of material, but if so, it was different in air from what it was in hydrogen.

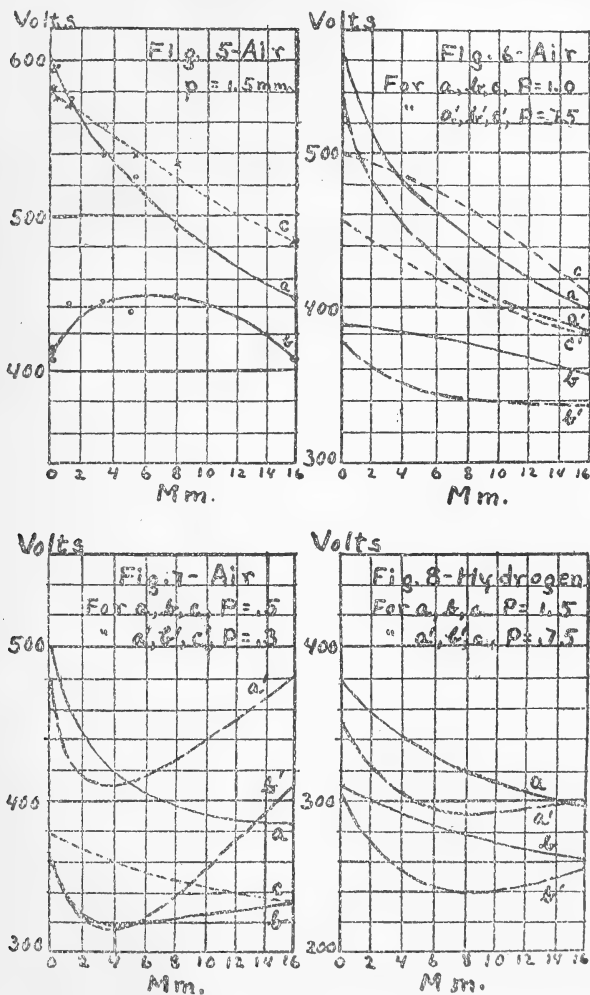
Fig. 4.



In fig. 5 the sparking potential is plotted against the diameter of the inner cylinder for a constant pressure of 1.5 mm. of mercury. The sparking potentials for parallel plates at distances equal to those between the cylinders for

the different diameters of the inner cylinder, have also been plotted (curve c). The sparking potential with the inner

Figs. 5-8.

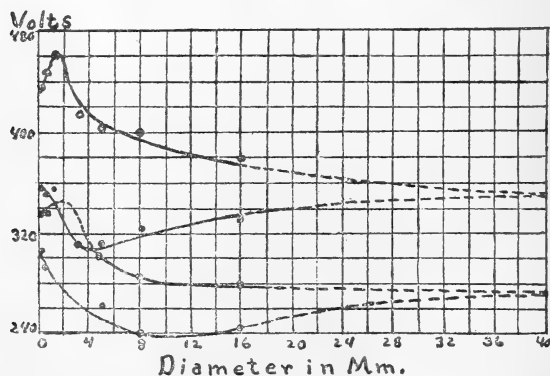


electrode positive (curve a) is below the parallel plate value, for the largest diameter, but becomes relatively higher as the diameter decreases ; with the inner electrode negative (curve b), the sparking potential for the largest diameter is lower than with either parallel plates or positive inner cylinder, and becomes relatively lower yet for the smaller

diameters. The positive and negative curves therefore make larger and larger angles with each other and with the parallel plate curve as the diameter decreases. Figs. 6 and 7 show similar curves (*a* and *a'* for positive, *b* and *b'* for negative inner electrodes, *c* and *c'* for parallel plates) for several different pressures in air, and exhibit a progressive change in the forms and relative positions of the different curves. As the pressure becomes less, the positive curve lies higher as a whole with reference to the parallel plate curve; while in the negative curve the divergence from the positive becomes less, and finally, instead of bending away from the positive, the negative actually rises and approaches it. The rise in the sparking potential for large diameters with negative inner electrodes at pressures of  $\cdot 5$  and  $\cdot 3$  mm., is due to the fact that for these diameters the pressure is below the critical value, so that the pressure-sparking potential curve is rising very rapidly. The curves for hydrogen, of which two pairs are shown in fig. 8, exhibit similar general characteristics, but are somewhat more confusing owing to the fact that the critical pressure is higher in hydrogen than in air, and the rise for the large diameters is much more marked for certain pressures.

In fig. 9 the diameters of the inner cylinder are taken as

Fig. 9.



abscissæ, and the corresponding values of the minimum sparking potential as ordinates. For the largest diameter employed the minimum sparking potential in air is 380 volts with the inner cylinder positive, or about 30 volts above the parallel plate value; when the inner cylinder is negative, the minimum sparking potential is 330 volts, or about 20 volts below the parallel plate value. As the diameter is

decreased, the minimum sparking potential rises for the first direction of the field, and becomes still lower for the second, the change becoming more and more rapid as we approach the small diameters. But finally a critical diameter is reached for the negative and a different one for the positive, and when the diameter is decreased below these critical values the effect on the minimum sparking potential is reversed, so that a sharp drop appears in the positive and a rise in the negative curve.

Fig. 9 also shows curves for hydrogen which are somewhat similar to those for air, with a maximum in the positive and a minimum in the negative. The number of points for the hydrogen curves is not such that we can consider the latter as definitely established, but they are probably approximately correct. If the curves in fig. 9 are produced, as seen by the dotted lines, they come to horizontal positions at approximately parallel plate values of the minimum sparking potential, and at the diameter of the outer cylinder. Additional readings might cause the curves to be drawn somewhat differently, but this ought to hold true in any case, since the field between concentric cylinders approaches that between parallel plates as a limit, and the minimum sparking potential for parallel plates is constant.

The phenomena of the discharge between concentric cylinders may be explained along lines suggested by Prof. Townsend in connexion with the collision theory\*. Above the critical pressure, all the main effects may be explained on the supposition that the ions are produced by the collision of positive and negative ions with the molecules of the gas. But below, and possibly in the region of the critical pressure, some other process of ionization in addition to ionization by collision evidently comes into play, and Prof. Townsend suggests that some form of non-penetrating radiation, due to the impact of negative ions against the positive inner electrode, may be the cause of the effects obtained. Such radiation, dependent upon the velocity of the negative ions, would affect only the positive curves in figs. 3 and 4, since only when the inner cylinder was positive would the negative ions approach the metal under a large force and impinge on the electrode with a high velocity. The radiation would naturally be absorbed at the higher pressures, but when the pressure first reached the point at which radiation would be present, the curve for positive inner electrode would begin to fall below the position which

\* 'The Theory of Ionization of Gases by Collision,' pp. 67 *et seq.* and 76.

it would occupy if this effect were not present, and would fall farther below the position as the pressure continued to diminish. If the effect were sufficiently great, it would eventually balance the effect of the non-uniformity of the field upon the ionization by collision, the two curves would cross, and below that point the radiation effect would keep the positive curve below the negative. This process would correspond exactly with what takes place in every case in the present experiments, for the small diameters; for the larger diameters the tendency is the same, though the actual point of crossing does not come within the limit of available potential difference as the field becomes more and more nearly uniform and the curves approach that for parallel plates.

The fact that the curves cross in different instances at very nearly the same pressure might also be explained on the radiation hypothesis. As the diameter of the inner cylinder decreased, the drop of potential near the positive inner electrode would become more abrupt, so that on the average the negative ions would strike the metal with greater velocity. The amount of the radiation effect tending to lower the positive curve would therefore be greater. On the other hand, the initial difference between the curves, due to the regular process of ionization by collision, would be greater in the less uniform field. The relative increase in the radiation effect with the diminution of pressure would probably remain the same, so that the larger amount of radiation might balance the larger initial difference at the same pressure at which the smaller amount of radiation would balance the smaller initial difference.

In addition to the fact that the radiation hypothesis fits in with the characteristics exhibited by these curves, it may be mentioned that the preliminary readings of some experiments now being carried on indicate the presence of some kind of radiation, at the pressures under consideration, the amount of which varies directly with the force and inversely with the pressure.

The curves in fig. 5 are just what the collision theory would lead us to expect for pressures somewhat above the critical pressure. Owing to the concentration of force near the inner electrode, the sparking potential when that electrode is negative should be lower than when it is positive and lower than the parallel plate value. Decreasing the diameter of the inner cylinder produces a still greater concentration of force near that electrode, and a corresponding weakening of the rest of the field. Such a decrease, therefore, should

tend to increase the sparking potential, as compared with the parallel plate value, if the inner electrode is positive, and to decrease it if that electrode is negative. That is, the positive curve should rise, with reference to the parallel plate curve, as the diameter decreases, and the negative should drop, the positive and negative therefore diverging toward the smaller diameters. This corresponds exactly with what takes place in fig. 5. The positions of all the curves may be somewhat affected by the fact that decreasing the diameter of the inner cylinder increases the distance between the electrodes, but their relative positions probably depend mainly upon the distribution of force in the field.

The curves in figs. 6 and 7 show the same characteristics in a general way, but there are certain steadily progressive changes which have been already pointed out. As the pressure diminishes, the position of the positive curve as a whole becomes higher with reference to the parallel plate curve, and the curvature increases; while the curvature of the negative is at first reduced and finally reversed as the pressure diminishes. The fact that these changes are continuous points to some continuously changing factor or factors in the determination of the sparking potential as the cause. Such continuously changing factors are to be found in  $\alpha$  and  $\beta$ , the coefficients of ionization for negative and positive ions respectively. Since  $\alpha$  and  $\beta$  depend upon both  $X$  and  $P$ , where  $X$  is the electric force and  $P$  the pressure, a change in  $P$  will necessitate a change in  $X$  in order to satisfy the conditions for sparking, and therefore in general a change in the total potential difference  $\int X dr$ . The change of  $\alpha$  and  $\beta$  with  $P$  may therefore very possibly be the cause of the change in the relative positions of the curves in figs. 5 to 7.

In fig. 9, where the minimum sparking potentials are taken as ordinates, and the diameters of the inner cylinder as abscissæ, the negative curve is very much like the curve in fig. 7 for a pressure of .5 mm. In fact, for the two larger diameters the critical pressure is near enough to .5 so that the potentials for these diameters lie practically on the .5 curve, while for all the other diameters the critical pressure is just over .3 and the curve for .3 very nearly coincides with that for .5 at all these diameters. The negative curve for minimum sparking potentials is therefore practically a constant pressure curve like those in figs. 5 to 8, and is probably to be explained in the same way.

For the positive curves the critical pressure does not remain so nearly constant, and the positive curve in fig. 9

does not follow any one curve of figs. 5 to 7. But it is of the same general form as all of them except for the drop at the two smallest diameters, and undoubtedly the change in distribution of force in the field would have the same kind of effect on the minimum sparking potential as on the others. Judging from the relative positions of the curves in figs. 5 to 7, the decrease in the critical pressure with the diameter should make the positive curve in fig. 9 less steep than the other positive curves, as it is, but this does not seem to account for the drop which occurs at the smallest diameters. But these two diameters are the only ones for which the critical pressure falls below the pressure at which the radiation effect seems to become large, that is, below the point at which the curves cross in fig. 3. The presence of the radiation would explain the lowering of the minimum sparking potential at these diameters for positive inner electrode, and therefore the sudden change in the curve in fig. 9. If, then, we consider the continuous change in the relative positions of the positive and negative curves in figs. 5 to 7 as due to continuous changes in  $a$  and  $\beta$ , we can explain all the curves obtained from these experiments by the collision theory and the presence of radiation at pressures in the vicinity of the critical pressure.

The results of these experiments may be summarized as follows :—

1. For every value of the diameter of the inner cylinder, two different sparking potential curves are obtained, corresponding to the two directions of the field. For smaller diameters, the two curves cross near the critical pressure, and that for a negative inner electrode is higher on the side of the lower pressures, but they tend to coincide as the field approaches that between parallel plates.

2. The minimum sparking potential depends on the diameter of the inner electrode, and is always lower when that electrode is negative.

3. The minimum sparking potential is always higher than that for parallel plates when the inner electrode is positive, while the lower values when the inner electrode is negative are below that for parallel plates. The lowest values obtained were 311 volts for air and 240 volts for hydrogen.

4. Curves drawn to show the relation between diameter and sparking potential at constant pressure exhibit a rise as compared with curves drawn for parallel plates if the inner electrode is positive. The negative curve is lower, and at the higher pressures diverges sharply from the positive for the smaller diameters, but the divergence lessens with the pressure.



5. Curves drawn to show the relation between diameter and *minimum* sparking potential are of the same nature, but the positive drops sharply for the two smallest diameters.

6. The results obtained at pressures down nearly to the critical pressure may be explained on the theory of ionization by collision. Below that, the results are entirely in harmony with the supposition that at these pressures some kind of radiation is produced by the impact of negative ions on the electrode under the action of a fairly strong force.

*The Effect of a Continuous Discharge upon Sparking Potentials at Low Pressures.*

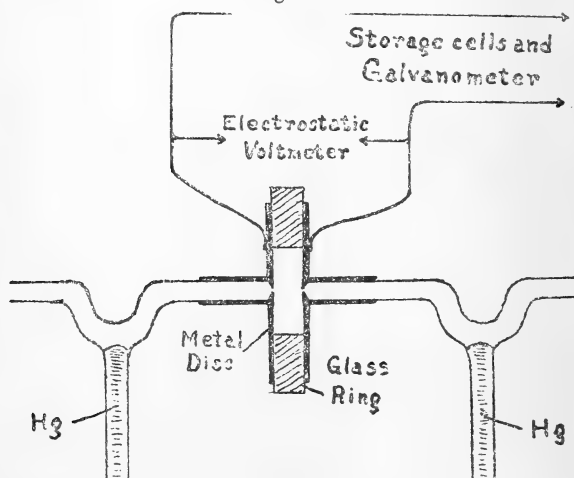
Some time ago a series of experiments was carried out by J. A. Brown to determine the relation between the sparking potential and the potential required to maintain a current in a gas\*. These experiments show that above the critical pressure, the potential required to maintain a current is less than the sparking potential, the difference increasing with the current, while below the critical pressure the effect is the reverse. The collision theory, which explains perfectly so many phenomena above the critical pressure, indicates that the potential required to maintain a current should be less than the sparking potential, regardless of the pressure, except for very small currents. Some other factor evidently comes into prominence at low pressures, in addition to the ordinary process of ionization by collision, and it is suggested, as an explanation of the relatively greater rise of the maintenance potential below the critical pressure, that the gas is heated by the discharge and part of it driven out from between the electrodes. The effect of such a diminution in the quantity of gas between the electrodes would be to lower the maintenance potential above the critical pressure, and raise it below the critical pressure. In the former case, the difference which should theoretically exist between the curves would be increased; but in the latter case the effect would be exactly the opposite, and if large enough would more than balance the difference demanded by the collision theory, so that the maintenance curve would be above that for the sparking potential. The present experiments were undertaken in order to test directly for the existence of any such expulsion effect.

R. F. Earhart has since published a paper embodying the results of experiments concerning the effect of temperature upon the potential required to produce a luminous discharge,

\* Philosophical Magazine, September 1906.

and upon that necessary to maintain it\*. These experiments indicate that, at least for the potential required to produce the luminous discharge, the effect at ordinary temperatures is similar to the heating effect mentioned above, that is, the curve for a higher temperature is higher below the critical pressure and lower above it than that for a lower temperature. In Brown's experiments, all the points on a sparking potential curve would be found at approximately the same temperature, while those on the corresponding maintenance curve might be at higher temperatures owing to the passage of the current. If the effect of temperature change on the maintenance curve were like that on the other curve, and sufficiently great, it might cause the maintenance curve to cross the other and run above it when the pressure was below the critical value. But the effect of temperature on the maintenance curve, in Earhart's experiments, does not seem to be very determinate, and at any rate is not large, so it would hardly seem that we could explain the peculiar relation between the two curves on the ground of temperature effects. However, the experiments were not undertaken for the purpose of settling this point, and did not give definite information on it, so it was decided to continue a method of direct investigation.

Fig. 10.



In the present experiments the spark chamber used (see fig. 10) was one of those used by Brown, and the rest of the apparatus was the same as his, with a few additions. The spark chamber consisted of two aluminium disks, set into

\* Physical Review, September 1909.

thicker supporting disks of brass, and separated by a ring of glass one centimetre thick, an arrangement which eliminated discharge from the backs and edges of the plates. Small holes in the centres of the disks enabled the gas to be pumped through the apparatus. A battery of small storage-cells provided the necessary difference of potential, and the passage of the current was detected by a galvanometer of the D'Arsonval type. The difference of potential was measured by a Kelvin multicellular electrostatic voltmeter reading to 600 volts. In place of the straight glass tubes leading from the spark chamber in Brown's apparatus, capillary tubes were inserted, with bends in them which could be filled with mercury and used as valves. The gas used was hydrogen. It was allowed to stand for at least a day in a drying-tube before being admitted to the apparatus, and there was another drying-tube in the apparatus near the spark chamber.

The sparking potential was found for a given pressure by increasing the potential difference until the galvanometer gave a deflexion. The current was then run for a few minutes, with the mercury in the position shown in the figure, so that gas heated between the electrodes would have an opportunity to expand along the tubes into the rest of the apparatus. The mercury was then raised far enough to block the bends in the tubes, and the current instantly shut off. After an interval a second determination of the potential was made. And finally, the valves were opened, the pressure permitted to adjust itself, and the sparking potential again found, the mean of the first and third values being taken as the value under normal conditions.

The results of this method are shown in Table I.  $V_1, V_2,$

TABLE I.

$V_1$ .	$V_2$ .	$V_3$ .	R.	P.
388	403	380	19	3.92
487	507	487	20	6.08
487	508	485	22 (?)	6.09
306	325	308	18	.61
333	338	334	5	.60
325	361	325	36 (?)	.60
411	446	410	35	4.55
425	453	426	27	4.58
269	280	271	10	1.08
319	342	320	22 (?)	.63
320	342	321	21	.65
321	342	323	20	.63
354	366	350	14	.58
350	362	348	13	.58
558	582	564	21	8.25

$V_3$ , are respectively the first, second, and third values of the sparking potential;  $R$  is the difference between  $V_2$  and the mean of  $V_1$  and  $V_3$ , reckoned positive when  $V_2$  is the greater:  $P$  is the pressure in millimetres of mercury. There was usually more or less variation in the value of the sparking potential, so several readings were generally taken for each of the quantities  $V_1$ ,  $V_2$ , and  $V_3$ . The values given in the table are in each case those that seemed most steady and reliable. A table was also made out in which the first reading obtained in each case was used, another in which the second was used, and a third in which the last reading taken was used. In these tables the values of  $R$  were sometimes quantitatively different from those given in Table I., but qualitatively they were the same, and in many cases differed by only a volt or two. A few sets of readings were rejected because "normal" conditions were not steady, apparently, as there were large differences between  $V_1$  and  $V_3$ . Three sets of readings are marked doubtful because of failure to record at the time the point in the experiment at which the spark chamber was opened, so that it was necessary to depend on memory for this. The doubt, however, as to the correctness of these readings is very slight indeed. The experiments show that  $R$  was positive in all cases, that is, that there was a rise in the sparking potential at all pressures, both above and below the critical pressure, with an average value of about 20 volts. The critical pressure was about 1 mm.

A further test was made by running the current while the spark chamber was closed. The initial sparking potential was found either before or after closing the valves, experiment having shown that the closing of the valves made no difference in it. The current was then run as before, but with the valves closed, the gas was given an opportunity to cool, and the sparking potential was again determined. Finally, the valves were opened and a third determination made, for comparison with the first. The results are shown in Table II., in which the same notation is employed as in Table I. In one case a small decrease in sparking potential is recorded, in all the others an increase. In no case does the sparking potential remain the same for  $V_2$  as for  $V_1$  and  $V_3$ .

Running the current with the valves open all the time did not seem to have any appreciable effect on the sparking potential. Two or three sets of readings were obtained which showed an increase of two or four volts from the average value before the current passed, to that afterward;

but this is very small compared with the differences shown for the same pressures in Table I., and is no greater than the variation in the values without regard to current.

TABLE II.

V <sub>1</sub> .	V <sub>2</sub> .	V <sub>3</sub> .	R.	P.
481	488	478	8	5.69
480	493	480	13	5.69
470	488	478	14	6.0
480	495	479	15	6.0
572	588	...	(16)	8.25
330	320	323	-7	.60

There seems to be no definite relation between R and the pressure in these experiments, nor much uniformity in the values of R themselves; but this is not surprising, as the conditions were not the same in all cases. The length of time the current was run was different in different instances, the currents used were of different strengths, and the discharge between the plates, though steady, was not always uniform over the surface of the plates. The experiments are of a qualitative rather than a quantitative nature, but qualitatively the results are the same, regardless of variations in the current.

It is evident from these experiments that the fact that a higher potential is required to maintain a current than to produce a discharge below the critical pressure, is not due to the expulsion of gas from between the electrodes. If in the first method (Table I.) some of the gas is driven out from the spark chamber by the passage of the current, the second value of the sparking potential should be lower than the normal above the critical pressure, and higher below it, since a smaller amount of gas is then occupying the same space that was previously occupied by a larger amount at the same temperature. For the gas cools to its original temperature very rapidly in contact with so much cooling surface, and the diminution of volume due to the rise of the mercury in the bend of the tube is entirely negligible. Table I. shows that, although the results obtained below the critical pressure are in harmony with the supposition that gas is driven from between the electrodes, those obtained

above the critical pressure are the direct opposite of what would result from the expulsion of gas. In the second method (Table II.), so far as expansion is concerned, the gas should be in exactly the same condition at all three times. The valves are closed before the current is started, and are kept closed, so that no gas can escape; and after the current has stopped, and sufficient time elapsed to allow the temperature to regain its original value, the gas should be exactly the same, with regard to temperature, pressure, and actual amount present, as it is at the start. And since it makes no difference at the start whether the sparking potential is found before or after the valves are closed, so at the end it should make no difference whether it is found before or after the valves are opened. That is, all the values should be the same, so far as any expansion of the gas originally between the plates is concerned, and the value of  $R$  should be zero. But in Table II., which gives the results of this method,  $R$  is never equal to zero. Table II., therefore, like Table I., shows plainly that the effects obtained in these experiments are due to some other cause than the expulsion of gas from between the electrodes.

In considering what may be the real cause of the differences in sparking potentials observed in these experiments, the first thing to be noticed is that, whatever its nature, it is confined to the gas. If the valves are left open during and after the passage of the current, a proceeding which leaves the gas free to circulate but which cannot have any effect upon the electrodes, running the current does not affect the sparking potential. This fact, and the fact that as soon as the gas is allowed to pass, by the opening of the valves, in the methods used in obtaining Tables I. and II., the potential drops to practically its original value, show that the cause of the effect is to be found in the gas.

The first thing which suggests itself as a cause is the driving out of gas from the electrodes themselves by the passage of the current. The gas which is most frequently met with in such cases is hydrogen, but the addition of a little hydrogen to that already present does not suffice to explain the results obtained. With the spark chamber closed, an increase in the amount of hydrogen should increase the sparking potential above the critical pressure, that is, the value of  $R$  should be positive; while below the critical pressure,  $R$  should be negative, and numerically greater because of the steepness of the sparking-potential curve in this region. Table II., taken by itself, is more or less in accord with this supposition, since the value of  $R$  is positive in each case

above the critical pressure, while in the single reading we have below the critical pressure it is negative. It may be noted, however, that the negative value of  $R$  is numerically smaller instead of larger than the positive values, and if compared with  $V_3$  instead of with the average of  $V_1$  and  $V_3$ , is only three volts. Furthermore, this negative value was obtained immediately after the set of readings in Table I., which gave the unusually small value of 5 volts for  $R$ , and it is possible that the conditions were in some way abnormal for both. Evidence of a much more definite nature is found upon examining the results obtained when the spark chamber is not closed till it is time to stop the current. In this case, the pressure has a chance to adjust itself all the time the current is running, so that when the valves are finally closed and the current stopped, the pressure and the kind of gas are the same inside the spark-chamber as outside, even if some hydrogen has been driven from the electrodes. Thus opening the valves should make no difference, or at most a small difference, if the pressure has not become quite equalized, and this small difference should be of opposite sign above and below the critical pressure. Table I. shows that the results are the exact opposite of these. Instead of a zero value of  $R$ , or small values which change sign at the critical pressure, we have much larger values of  $R$  on the average than with the spark-chamber closed all the time, and the sign of  $R$  does not change at any point. From these considerations it appears that we cannot explain the results by an increase in the amount of hydrogen between the plates any better than by a decrease.

If an explanation is sought in the presence of some other gas, it must be one whose sparking-potential curve lies above that of hydrogen at all points, as  $R$  does not change sign at any pressure, above or below the critical pressure. Since air fulfils this condition, it was thought possible that there was a slight leak in the black-wax joints of the spark-chamber, and that the air thus admitted, though not enough to change the pressure appreciably in the whole apparatus, might be sufficient to have an effect in the small space of the spark-chamber. Such an effect, however, would be independent of the presence of the current, and the fact that the sparking potential is sensibly the same whether the chamber is open or closed, so long as the current is not run, but changes after the passage of the current, disposes of this possibility.

If the effect were due to some gas of generally higher sparking potential than hydrogen, such as air, driven from the electrodes, one would naturally expect that the effect

would be much greater when the valves were closed all the time than when they were open during the passage of the current, as in the former case there would be no opportunity for diffusion. But as a matter of fact the readings indicate the opposite. As has already been pointed out, the readings are not of great value quantitatively, but one might reasonably expect that on the whole those taken with the valves closed would average higher than those taken with them open, while actually they are considerably lower.

Another possibility is the existence of some sort of electric fatigue in the gas after the passage of the current. At first thought it would seem that the existence of such an effect could not have escaped the observation of so many previous observers. But the conditions of previous observations have been such that the gas between the electrodes has been in free communication with that in the rest of the apparatus at all times, and under those conditions in the present experiments no change in the sparking potential is observed. On this supposition, however, the effect should be at least as great when the valves are closed during the time when the current is run as when they are closed only at the end of that time, though not necessarily greater, since the effect might be produced as rapidly as fresh gas could diffuse into the chamber. This supposition, therefore, is open to the same objection on the ground of quantitative results as is that mentioned in the last paragraph, but to a less extent.

It would have been well if a few more readings had been taken at certain points, but the apparatus was taken down immediately after the present readings had been completed, in order to make room for other apparatus, and the results were not considered in detail until later, so that the desirability of further readings was not at the time observed. As it is, the conclusions arrived at are:—

1. Running the current under the conditions of the experiments produces a rise in the sparking potential, and the cause of the change lies in the gas.

2. As to the main object of the experiments, the investigation of the suggested heating of the gas and expulsion from between the electrodes, with the resulting effect on the sparking potential, the effects obtained are distinctly inconsistent with this hypothesis. The explanation of the relation between the maintenance and sparking potential curves below the critical pressure is evidently to be sought on different grounds.

3. One possible explanation of the effects obtained is that air is driven out from the electrodes by the passage of the



current; or conceivably the current has some small temporary effect on the gas itself, as is the case with oxygen when ozone is formed. The quantitative evidence is contrary to both of these methods of explanation, though less strong against the second. Quantitatively, however, the readings are not very reliable, and further investigation may show that one of these explanations is correct.

In conclusion I wish to express my gratitude to Professor Townsend for the kindness he has shown in giving assistance and advice in the course of these experiments.

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LVI. *On the Pressure Displacement of Spectral Lines.*

By R. ROSSI, M.Sc.\*

SINCE the discovery of the pressure displacement of spectral lines several theories have been put forward by different authors to explain this phenomenon.

Schuster † first suggested that it ought to be ascertained whether the displacement of the lines is due to pressure only, *i. e.* to molecular impacts, or due to the proximity of molecules vibrating with equal periods.

FitzGerald ‡ and Larmor §, treating the atom as an electromagnetic oscillator, found that an increase of specific inductive capacity (due to an increase of pressure) would cause a displacement of the spectral lines; while Richardson ||, considering the forced vibrations set up in an atom by the surrounding atoms, arrived at the same conclusion.

Humphreys ¶, on the other hand, does not consider a change in the specific inductive capacity of a gas to be the main cause of pressure displacement.

If the pressure displacement is a linear function of the specific inductive capacity of the radiating vapour, the experimental methods so far available are not accurate enough to prove it on account of the small differences of the specific inductive capacities of different gases; but if it is proportional to  $(\mu^2 - 1)$ , as claimed by some authors, ( $\mu$  being the refractive index of the medium surrounding the vibrating atom), considerable changes in the displacements ought to be detected by surrounding the source of light with different gases.

\* Communicated by Prof. E. Rutherford.

† *Astrophysical Journal*, vol. iii. p. 292.

‡ *Astrophysical Journal*, vol. v. p. 210.

§ *Astrophysical Journal*, vol. xxvi. p. 120.

|| *Phil. Mag.* [6] vol. xiv. p. 557 (1907).

¶ *Astrophysical Journal*, vol. xxvi. p. 18.

Passing from air to carbon dioxide, for instance, the displacements ought to be very nearly in the ratio 2 to 3.

Some work on spark spectra in different gases under pressure has already been done by Hale and Kent\* and by Anderson †. The latter found larger displacements in carbon dioxide than in air; but, as pointed out by Humphreys ‡, his results were not conclusive as to the effect of specific inductive capacity on the pressure displacement.

In the following work the displacement of some lines of the spectrum of an iron arc burning in air and carbon dioxide under pressure are compared. Hydrogen was also tried; but the arc burns so poorly in that gas under pressure, that too long exposures of the photographic plate would have been needed, and the results would have been spoiled by leakages of the apparatus and changes of temperature of the room §.

Photographs were taken at 15, 30, and 50 atmospheres with the  $21\frac{1}{2}$  feet concave grating spectrograph of this laboratory. A small region of the spectrum containing some sharp lines was chosen, thus enabling the measurements to be made more accurately. The arc was found to burn in carbon dioxide just as well as in air.

The accompanying table gives the displacements at the various pressures. Figures in brackets denote doubtful readings, the letter R indicates that the line was found to be reversed at that pressure. It can be seen that, with a few exceptions, the displacements are the same in the two gases within the limits of experimental error. The mean displacement per atmosphere of all the 23 lines studied is found to be .00411 and .00401 Angström unit for the arc burning in air and carbon dioxide respectively.

There also are no very noticeable differences between the appearances of the spectra in the two gases. There are a few more reversals of lines when the arc burns in carbon dioxide than when it burns in air; and there seems to be a slight tendency of the lines to be broader and more diffuse in carbon dioxide than in air. The relative intensity remains the same.

So far then as this evidence from only two gases is worth, it points to the fact that the specific inductive capacity is of but secondary importance in the cause of the displacement

\* Publications of Yerkes Observatory, vol. iii. Part II. (1907).

† Astrophysical Journal, vol. xxiv. p. 221.

‡ Astrophysical Journal, vol. xxvi. p. 18.

§ At 15 atmospheres the exposure necessary when the arc was burning in air or  $\text{CO}_2$  was on the average 1 or 2 minutes, while it was estimated that for the same arc burning in hydrogen an exposure of 3 hours would have been needed.

Wave-length.	Displacements in thousandths of an Ångström Unit at					
	15 atms. in		30 atms. in		50 atms. in	
	Air.	CO <sub>2</sub> .	Air.	CO <sub>2</sub> .	Air.	CO <sub>2</sub> .
4422.67	40	39 R	69	67	89	97
27.44	68	65	142	108	190	196
42.46	90	91	138	105	195	200
43.30	62	73	108	90	100	124
47.85	67	80	145	120	193	173
54.50	46	56	86	77	108	113
59.24	64	68	147	126	245	240
61.75	40	53 R	69	78	103	169
66.70	40	45 R	84	64	108	104
76.20	41	38	79	76	116	112
82.35	66	70	120	119	225	237
94.67	82	63	147	146	250	250
4528.78	(71) R	72 R	(170)	142	260	255
31.25	45	47	88	86	126	118
47.95	45	39	90	83	124	148
92.75	50	39	84	94	130	142
4603.03	44	39	94	89	117	133
47.54	52	40	96	83	110	114
54.70	25	22	58	44	85	81
91.52	68	66	104	94	129	137
4707.45	191	180	315	310	(538)	(520)
10.37	68	69	104	112	127	129
36.91	179	189 R	319	302	(570)	(544)
Mean .....	67.5	67.1	124.2	113.7	184.3	188.6
Mean displacement per atmosphere.	} 4.5	4.5	4.1	3.8	3.7	3.8

of spectral lines. This is also in accordance with experiments made by using pure metals and alloys of those metals or carbon poles with metallic impurities, when the same pressure displacements were practically found in each case\*. All these facts seem to show that the pressure displacement is not a density effect, but is due to the total pressure of the radiating vapour, *i. e.* to the compactness in number of the atoms, irrespective of their kind.

In conclusion I wish to thank Professor Rutherford and Professor Schuster for the interest they have taken in this research.

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Manchester University.

\* Humphreys, *Astrophysical Journal*, vol. xxvi. p. 18. Duffield, *Phil. Trans. Roy. Soc. A.* vol. ccix. p. 218.

LVII. *The Common Sense of Relativity.*  
By NORMAN CAMPBELL\*.

THE Principle of Relativity has been discussed so often and in so many ways that it is perhaps presumptuous to attempt to add anything to the discussion except by offering original developments. But it appears to me that the needs of "the man in the laboratory"—to paraphrase a convenient modern expression—have been insufficiently considered by expositors. He has been offered profound mathematical investigations, which are intensely important and interesting, but tend to obscure the fundamental points at issue in the mind of one who thinks physically rather than mathematically. And on the other hand he has been offered collections of apparently paradoxical conclusions deduced from the Principle, which are sometimes elegant and entertaining, but more often fallacious. As a natural result he is inclined to think that this new development of science, the most important, in my opinion, since the days of Newton, is extremely abstruse and incomprehensible. In the following pages I desire to attempt to remove this misconception and to show that the view of the relations of moving systems adopted by the Principle of Relativity is very much simpler than that which it displaces, and that all its apparent difficulties are due to confusions of thought and misapprehensions. A few of the observations offered may be of interest to those who have studied the matter deeply, but it is to those whose knowledge is superficial that these remarks are primarily addressed.

As a basis of the discussion the admirable authoritative summary given by the original propounder of the Principle himself will be used (Einstein, *Jahrbuch der Radioaktivität*, Bd. iv. pp. 411 &c., 1907). The notation used there will be adopted without explanation.

I. *The Nature of the Principle.*

2. Let us first inquire exactly what the Principle of Relativity is and what it asserts.

The Principle is what is more often termed a "theory"—that is to say, it is a set of propositions from which experimental laws may be logically deduced. It can be proved to be true or false in a manner convincing to everybody only by comparing the laws so deduced with those found experimentally; but a theory which never conflicted

\* Communicated by the Author.

with experiment might yet (as I hold) be judged objectionable on other grounds, and, conversely, a theory which was not in complete accord with experiment might yet be judged satisfactory.

The special laws which it is the business of the Principle of Relativity to explain (that is, those which it is specially important to be able to deduce from the theory) are those which are met with in the study of the optical and electrical \* properties of systems in relative motion, but in this case, as in most cases, it turns out that laws other than those contemplated originally are deducible from the theory. It is important to notice that there is another theory, that of Lorentz †, which explains completely all the electrical laws of relatively moving systems, that the deductions from the Principle of Relativity are identical with those from the Lorentzian theory, and that both sets of deductions agree completely with all experiments that have been performed ‡. If, then, anyone prefers one theory to the other it must be either on the ground of differences in the laws not contemplated originally which are predicted respectively by the two theories, or because of some general grounds independent of experimental considerations.

3. The fundamental propositions of which the theory consists will now be enumerated and a few remarks made upon each. For the sake of brevity, I shall call a system the parts of which are all relatively at rest a "quiet" system, and one of which the parts are in relative motion a "disturbed" system. The terms are convenient to distinguish quiet and disturbed systems from those which are moving as wholes relatively to each other. Two quiet systems may be in relative motion as wholes.

(A.) The first assertion of the Principle of Relativity concerns quiet systems only. It asserts that any law and, consequently, any theory from which laws can be deduced, which has been found to hold for one quiet system which includes *all* the particles of which mention is made in the

\* "Optical" and "electrical" will be employed throughout as equivalent terms, one or the other being used according to the context.

† As given in the *Encyclo. d. Mathemat. Wissen.*

‡ This statement is only true if quantities of a higher order than the second in  $v/c$  are left out of account. Since such terms cannot be detected experimentally the conclusion given is not affected, and the terms will be neglected throughout our argument. In saying that both theories agree completely with experiment, I do not wish to offer any opinion as to the value of the experiments of Bucherer and others which have been subjected to criticism: I merely assume that the results announced in them will be accepted ultimately, because, if they are not, the Principle of Relativity would seem to be unworthy of further discussion.

laws, will hold for all quiet systems which are not accelerated relatively to that quiet system\*. In particular the theory expressed by the fundamental equations of the electron theory has proved perfectly satisfactory so long as it is applied to a quiet system. So long, that is, as all distances are measured relatively to axes fixed in the earth, all times are measured on clocks fixed in the earth, and the phenomena considered are those of charged bodies fixed in the earth in magnetic fields produced by instruments fixed in the earth, or those of sources of light fixed in the earth observed by instruments fixed in the earth, no conclusions have ever been obtained which are not consistent with that theory. The Principle asserts that, if the whole system of axes, scales, clocks, charges, magnets, light-sources, telescopes, and observers were placed on a ship moving uniformly relative to the earth and the experiments repeated, exactly the same relation would be found between the quantities measured.

This proposition is known as the First Postulate of Relativity. The justification for it is that it is in accordance with all known experimental facts: it is, moreover, directly implicated in the usual formulation of the theory of dynamics.

Now it is the main object of the Principle of Relativity to establish a connexion between the laws of a quiet system and those of a disturbed system. In order to establish such a connexion Einstein shows that it is only necessary to introduce a small number of additional fundamental propositions into his theory, and he shows also—it is this which makes his work so brilliantly ingenious—that the additional propositions which must be introduced do not concern the laws of any quiet system. Consequently, if the theory is true, it will tell

\* A difficulty arises from the fact that no system where anything "happens" can possibly be quiet, for almost all the changes which physics investigates involve relative motion of the parts of the system investigated. The logical questions raised by this difficulty will be discussed more fully in another place: for the present we may regard "laws for a quiet system" as propositions toward which the laws of actual systems tend as the relative velocity of the parts of those systems tends to zero. Thus the fundamental equations of the electron theory, involving the terms  $\rho v$ , cannot be applied to quiet systems, since they consider the motion of an electron (which is part of the system) relative to the rest of the system: they may be regarded as the limiting form of the equations valid for disturbed systems, as the velocity  $v$  tends to zero. There is no practical difficulty about this extrapolation, for the form of laws is not found to change with the relative velocity of the parts of disturbed systems over a large range limited, on the one hand, by a relative velocity of about  $10^6$  cm./sec., and, on the other, by the smallest velocity which can be detected experimentally. The laws for a quiet system are, then, the laws which hold within this range of velocities.

us how to change from the form suitable for a quiet system to that suitable for a disturbed system not only the laws which it is the special business of the theory to investigate, but any laws whatsoever. Instead of working out, as heretofore, the transformation necessary for each special law, we shall arrive at a transformation which is valid for all laws.

The three chief \* propositions necessary for this purpose are as follows :—

(B.) “Space is homogeneous and three-dimensional: time is homogeneous and one-dimensional.” Mathematically this means that the transformation of the space and time coordinates is to be linear. It would take us too far afield to inquire what it means in terms of observations, and since no difficulty appears to have been felt in connexion with it, it is unimportant for our present purpose.

(C.) “If the velocity of a system  $S'$  relative to  $S$  is determined by an observer on  $S$  to be  $v$ , then the velocity of  $S$  relative to  $S'$  determined by an observer on  $S'$  is  $v$ ” †. The necessity for introducing this proposition is generally overlooked. But it is a proposition which can be reasonably doubted and of which the truth can be tested by experiment only. Of course, all the evidence that there is is favourable: if it were not, we should not speak of “relative velocity.”

(D.) “The velocity of light determined by all observers who are not accelerated relatively to each other is the same, whatever may be the relative velocities of the observers.” This proposition is known as the Second Postulate of Relativity and more will be said about it hereafter.

4. The result which represents the attainment of the primary object of the Principle of Relativity is deduced from these fundamental propositions by purely mathematical argument. It may be stated as follows :—

Suppose that the disturbed system consists of two parts, A and B, each of which, regarded as a complete system, is quiet: let A be the part which contains the observer and his instruments for measuring  $x, y, z, t$ , and let the relative velocity of A and B be  $v$ . Then if A and B formed together

\* The other propositions are those which are implied in all physical measurement and in all theories. The propositions given are not those given by Einstein explicitly, but those which seem to be implied by his argument.

† The sign attributed to  $v$  is a matter of pure convention; so long as each observer adheres to his own convention throughout, it does not signify whether the same or contrary signs are attributed to the relative velocity by the two observers.

a quiet system, the known laws for quiet systems would state a relation between the time and the coordinates of the various parts of the system on the one hand, and some quantities P, Q, R, representing the physical state of the system on the other (forces, for example). Let this relation be represented by

$$f(x, y, z, t, P, Q, R \dots) = 0.$$

It is shown, as a consequence of the Principle of Relativity, that the analogous relation for the disturbed system is obtained by substituting for each set of coordinates  $x, y, z, t$ , belonging to a particle of B, the quantities  $x', y', z', t'$ , where

$$(x', y', z', t') = \left( \beta(x - vt), y, z, \beta\left(t - \frac{vx}{c^2}\right) \right).$$

It must be noted that the quantities P, Q, R, . . . will often involve implicitly the coordinates and the time, that is to say the values of P, Q, R, . . ., will be determined by certain measurements of  $x, y, z, t$  for certain identified particles. In fact there will be a relation of the form

$$\phi(P, Q, R \dots, x, y, z, t) = 0.$$

In this case the substitution of  $x', y', z', t'$  for  $x, y, z, t$  must be carried out consistently, and for P, Q, R, . . . must be substituted P', Q', R', . . ., where these latter quantities are given by

$$\phi(P', Q', R' \dots, x', y', z', t') \equiv \phi(P, Q, R \dots, x, y, z, t).$$

In this manner the relation is obtained

$$f(x', y', z', t', P', Q', R' \dots) = 0,$$

giving directly the relationship which holds for the disturbed system between  $x, y, z, t$  on the one hand, and P, Q, R, . . . on the other, all measurements being still made by the observer on A with the measuring instruments which form part of his quiet system. This relation is that which we set out to seek, the law for the disturbed system as observed by an observer who forms part of it\*.

So far surely nobody can find any difficulty: anything more beautifully straightforward it would be hard to conceive. Not only is the result magnificently simple, but it furnishes us with a mathematical instrument of extraordinary power. In place of the elaborate calculations which have hitherto been necessary in dealing with moving systems, all that we

\* The best examples of the process are, of course, those worked out by Einstein in the paper referred to.



have to do now is to solve the problem under consideration for the limiting case of infinitesimal velocity, and then effect a mere algebraical transformation. The only objection that seems likely to be raised is that the Principle proves too much, that it appears impossible that such far-reaching conclusions can be drawn from such simple assumptions: the only difficulty, in fact, is that the thing is too easy.

5. That the arguments by which the conclusion is attained are valid can, of course, only be proved by examining them, but I think a few remarks of a general nature may remove one cause of uneasiness. It is felt that the universal importance attributed to the velocity of light is strange, when it is proposed to apply the principle to laws which have nothing to do with optics. Why, it may be questioned, do we drag in the velocity of light rather than that of sound or of the trains on the twopenny tube? Some part of this uneasiness may arise from the unfortunate way in which Einstein introduces the Second Postulate in his paper: he seems almost to try to deduce it from the First Postulate. In describing the First Postulate he says:—"In particular the same number must be found for the velocity of light *in vacuo* for both reference systems." It is very pertinent to ask here why, then, the velocity of light rather than that of sound.

Of course the Second Postulate cannot really be deduced from the First. What the First Postulate asserts is that all laws must be the same for all quiet systems having no relative acceleration: in particular, for all such systems, the velocity of light or the velocity of sound determined from a source which forms part of the system to a receiver which forms part of the system must be the same. But the proposition which is necessary for the argument is quite different from this. It is that the velocity of light from some source common to two systems will be found to be the same by observers on both systems, even if those systems are in relative unaccelerated motion. Since the source is common to the two systems in relative motion, it is clear that both systems, if they both include the source, cannot be quiet, and, therefore, that the First Postulate, which refers only to quiet systems, can have nothing to do with the matter.

The Second Postulate is really made up of three distinct propositions\*. The first is that there is *some* velocity which

\* The complexity of the Second Postulate appears very clearly in Minkowski's 'Raum und Zeit.' Minkowski's treatment is somewhat different from that of Einstein and involves an entire rejection of the conceptions of space and time.

is found to be the same by all relatively unaccelerated observers; the second is that this velocity has been measured; the third is that it is the velocity of light. Only the first proposition is implicated in the result stated in the last paragraph: the quantity  $c$  is this universal velocity, whatever it may turn out to be. The second and third propositions are not introduced until the result is applied to the deduction of the optical and electrical laws for a disturbed system from those for a quiet system.

The first proposition is that which is really characteristic of the Principle of Relativity, and is the feature which distinguishes it from all other theories. It seems at first sight rather startling, but perhaps it may be made to appear more plausible, if it is pointed out that it means some velocity must be "physically infinite"—that is to say, it must be such that the addition to it or subtraction from it of finite velocities do not change its magnitude. If its magnitude had turned out to be represented on the scale of measurement of velocities ordinarily adopted by a mathematically infinite number, no difficulty would have been felt with regard to it: it is the fact that the second part of the Second Postulate proposes to represent the physically infinite velocity by a mathematically finite number which causes surprise. There is, however, nothing more difficult in such a representation than there is in the representation of the physically infinite low temperature by the mathematically infinitesimal number zero; both representations are merely consequences of the definitions of velocity and temperature adopted, and physical and mathematical infinity could be easily brought into agreement by a change of definition\*.

But if the first and second parts of the Second Postulate be accepted, there can be no doubt about the third, for if we are going to identify the physically infinite velocity with any velocity which has ever been observed, there is, on general grounds, no doubt as to its identification with the velocity of light. For this velocity clearly cannot be less than any velocity which has been measured: to suppose that it could would be self-contradictory. Hence if we are to identify the physically infinite velocity with any velocity which has been measured, it must be with the greatest velocity which has been measured, the velocity of light *in vacuo*. The agreement of the propositions deduced from the Principle of Relativity with the aid of this identification with the experimental work of Bucherer is strong evidence for the second part of the postulate—that is, for the view that

\* This line of thought will be developed in a later paper.

the physically infinite velocity has actually been measured. The first and third parts of the postulate are scarcely dubitable.

But perhaps such arguments are unconvincing to the physical instinct, so I proceed to considerations which should overcome any difficulty which is felt in connexion with the great importance attributed to the velocity of light when dealing with phenomena which appear to have nothing to do with light. These considerations are based on the fact, obvious when it is pointed out, that such velocities as distinguish practically a disturbed from a quiet system, velocities, that is to say, which are not physically infinitesimal, can only be measured by optical or electrical means. When we hear of the "velocity of the earth relatively to the sun" or the "velocity of a  $\beta$ -particle relatively to its source," association leads us first to think of a quantity measured by the distance travelled relatively to a metre scale during the passage of the hand of a chronometer over a certain part of its face. But a little reflection will show that this is not what we mean by these velocities: we have never held a metre rod up against the sun or an electron and observed the change in relative position. It would lead us too far to inquire here what exactly we do mean by such an expression as the "velocity of a  $\beta$ -particle," but it could be shown quite easily that that expression has no meaning whatever, unless we assume the truth of the fundamental equations of the electron theory\*. And since those equations involve the velocity of light, it is not surprising that that quantity enters when we are considering the velocity of an electron. "The velocity of a  $\beta$ -particle" is called a "velocity" because, within a certain range of values, the number representing it is the same as the number representing a velocity measured by a scale and clock (as is shown by the Rowland experiment), but, outside the range within which the scale-and-clock measurements of velocity are applicable, "the velocity of an electron" is dependent for its meaning on certain theories. To inquire whether, outside this range, this velocity would agree with that determined by the scale and clock is as absurd as to inquire whether, if all triangles had four sides, all circles would be square.

\* Some remarks on this subject are to be found in a paper on "The Principles of Dynamics," *Phil. Mag.* xix. p. 168.

II. *The Consequences of the Principle.*

6. But the chief objections which are raised against the Principle of Relativity are urged, not so much against the foundations of the Principle, as against its consequences. Two consequences seem to cause especial difficulty, and these will be considered.

The first difficulty concerns the "composition of velocities." The Principle of Relativity leads to the conclusion that, if an observer on a quiet system S measures the velocity of a quiet system S' relative to him and finds it  $u$ , and if an observer on S' finds the velocity relative to him of a third quiet system S'' to be  $v$ , then the observer on S will find the velocity of S'' relative to him to be

$$w = \frac{u + v}{1 + \frac{uv}{c^2}},$$

and not, as experience with small velocities might lead us to expect,  $u + v$ . ( $u$  and  $v$  are taken in the same direction.)

This conclusion seems absurd to many people. Let us inquire into the consequences of rejecting it and substituting the law  $w = u + v$ . We must then, of course, reject one of the fundamental propositions (B), (C), or (D); the rejection of (A) would not help us, because this proposition is not implied in the conclusion. Now, if an objector proposed to reject (B) or (C) I should have no argument to use against him, for the experimental evidence for these propositions is just as strong and just as weak as that for the proposition  $w = u + v$ . All these propositions, as well as (A), can be tested only by comparing the experiences of different observers, who have been moving relatively to each other with high velocities, when they meet again on a quiet system. Now since no two human beings have ever, within historic times, moved relatively to each other with a uniform velocity of  $10^4$  cm./sec. and subsequently compared their experiences, and since, on the other hand, we do not expect to detect divergencies from the laws of a quiet system or from the laws predicted by the Principle of Relativity until the relative velocity reaches at least  $10^6$  cm./sec., the evidence for all these propositions is extremely precarious. Nor does it seem in the least likely to become less precarious: so far as I know, nobody has made the faintest suggestion as to how a relative velocity of more than  $10^4$  between two human beings might be attained in such a way that they could perform delicate measurements. The one proposition among those which are fundamental to the Principle of Relativity which there appears to be some hope

of establishing definitely is (D) : we have the source of light relative to which we are moving with a velocity of  $3 \times 10^6$  always available in the stars, and it is not too much to hope that some day experimental ingenuity will succeed in measuring the velocity of the light from it with an accuracy of one part in ten thousand. It seems to me incredible that anyone, who understands what he is doing, will really propose to reject definitely a proposition which he may hope to prove in the near future in favour of one for which there is never likely to be the smallest direct experimental evidence.

But I think these people do not understand what they are doing : they have been confused by the most fruitful cause of confusion, the habit of using one word to denote two quite different ideas. "Velocity" is commonly used to mean either "mathematical velocity" or "physical velocity." Mathematical velocity is defined as the ratio of a certain variable  $x$  to a certain variable  $t$ . From the definition of a variable and a ratio, it follows that

$$\frac{x_1}{t} + \frac{x_2}{t} = \frac{x_1 + x_2}{t} :$$

this is a perfectly purely logical conclusion, and to deny it would be absurd. On the other hand, "physical velocity" in its simplest meaning is a number equal to the ratio of two numbers—one representing the groups of metre rods that have to be placed together in order that their ends may coincide with certain points, and the other expressing the occurrence of certain events in an instrument called a clock. From the definition nothing whatsoever can be predicted as to the relations of  $u$ ,  $v$ , and  $w$ , but experiment shows us that, for all values of  $u$  and  $v$  which can be attained practically in this way, if  $u = \frac{x_1}{t}$ ,  $v = \frac{x_2}{t}$ , then  $w = \frac{x_1 + x_2}{t}$ . This experimental proposition has become so familiar, and the association of the experimental  $u$  and  $v$  with the mathematical  $\frac{x_1}{t}$  and  $\frac{x_2}{t}$  so habitual, that people who do not think very deeply about these things have come to believe that  $u$  means the same thing as  $\frac{x_1}{t}$ ; and that, therefore, since it would be absurd to deny that  $\frac{x_1}{t} + \frac{x_2}{t} = \frac{x_1 + x_2}{t}$ , it is absurd to deny that  $u + v = w$ . There is no more absurdity in being forced to deny this assertion in the face of fresh evidence than there was in the necessity for Mill's Central African philosopher

having to deny in the face of fresh evidence his previously undoubted proposition that "all men are black."

7. But the greatest difficulties in connexion with the Principle of Relativity appear to concern certain propositions about length and time. In what follows I shall, for brevity, discuss only time: everything I say will apply, *mutatis mutandis*, to length.

The Principle of Relativity leads to the following conclusion. Suppose I examine a number of clocks which, with me and my instruments, form a quiet system, and I find that they all go  $n$  times as fast as my standard clock. That is to say, for the quiet system, the "law" of these clocks is that they return to some standard state when  $t = \frac{P}{n}$ , where  $P$  is any integer. Now one of these clocks is transferred to a system moving relatively to me with a velocity  $v$ . Let us suppose that, at the moment when  $t=0$ , this clock is just passing me, so that  $x=0$ . Then the Principle of Relativity states that the "law" for the disturbed system of myself and the clock is that the clock returns to a standard position when  $t' = \frac{P}{n}$ ,

or when 
$$\beta\left(t - \frac{vx}{c^2}\right) = \frac{P}{n},$$

or, since  $x=vt$ , when 
$$P = \beta \frac{P}{n}.$$

That is to say, the clock now agrees, not with the clocks with which it formerly agreed on the quiet system, but with one on the quiet which goes  $\frac{1}{\beta}$  as fast as those clocks.

There is nothing new in the form of this conclusion. The crudest arguments based on the oldest theory of light lead to the conclusion that the rate of a clock as observed by a certain observer must change with the relative motion of clock and observer. For, it will be argued, the observer does not see the clock "as it really is at the moment," but "as it was a time  $T$  earlier, where  $T$  is the time taken for light to reach the observer." And on these lines it is easy to show that the apparent rate of a clock moving away from the observer with a velocity  $v$  is  $\left(1 - \frac{v}{c}\right)$  times the rate of the same clocks observed at rest. It is only the magnitude of the change concerning which the two theories differ.

"Yes," says our objector, "that is all very well: of course the apparent rate of the clock changes with motion, but does

the real rate change?" We immediately inquire what the "real rate" means. He is at first inclined to assert that it is the rate observed by an observer travelling with the clock, but when we inquire relative to what clock that observer is to measure the rate he becomes uneasy. He cannot compare another clock travelling with him, for if the "real rate" of one clock has changed, so has the "real rate" of the other; and he cannot use a clock which is not travelling with him, because he admits that he does not see such a clock "as it really is."

Pressing our inquiries, I think we shall get an answer of this nature. "If I take a pendulum clock to some place where gravity is different, the rate of the clock will change. It is a change of this nature which I call a change in the 'real rate,' and I want to know whether there is any change of that kind, on the theory of Relativity, when the clock is set in motion." Now why does our objector call a change of the first kind a change in the "real rate"? The reply is to be found in the history of the word "real." The word is intimately associated with the philosophic doctrine of realism, which holds that the most important thing that we can know about any body is not what we observe about it, but its "real nature," which is something that is independent of observation. Now, of course, a quantity which is wholly independent of observation cannot play any part in an experimental science, but there are quantities which are independent of observation in the more limited sense that they are observed to be the same by whatever observer the observation is made. The term "real" has come to be transferred from the philosophical conception to such quantities. The "real rate" of the clock is said to change when it is transferred to a place where gravitation is different, because all observers agree that the rate of the clock which has been moved has undergone an alteration relatively to that which has not been moved.

Now in the conditions which we are considering the observers do not agree. If A and B, each carrying a clock with him, are moving relatively to each other, they will not agree as to the rate of either of their clocks relative to A's standard or to B's standard or to any other standard. The conditions which, in the case of the alteration of gravitation, gave rise to the conception of a "real rate" are not present: in this case there is no "real rate," and it is as absurd to ask whether it has changed as it would be to ask a question about the properties of round square. However, some people, who in their eagerness to escape the reproach of being metaphysicians have adopted without inquiry the

oldest and least satisfactory metaphysical doctrines, are so enamoured of the conception of "reality" that they refuse to give it up. Finding that the observations of different observers do not agree, they define a new function of those observations, such that it is the same for all observers, and proceed to call this the "real rate." This function, according to the Principle of Relativity, is  $\beta n'$ , where  $n'$  is the rate of the clock as seen by an observer relative to whom it is travelling with the velocity  $v$ : according to that Principle, if we substitute in that function the appropriate values for any one observer, the resulting number will always be the same.

So far no overwhelming objection can be raised. The function is important in the theory, and, if care is taken to note the precise meaning now attributed to the word "real," there is no harm in calling it by that name. But now certain writers commit an extraordinary series of blunders. They not only inquire whether the real rate changes with the velocity, a question which, as the real rate is defined as that function which does not change with the velocity, is utterly trivial, but they actually give a negative answer. They see that the expression for the real rate contains  $v$  explicitly and rush to the absurd conclusion that the real rate changes with the velocity. No wonder that they soon involve themselves in a hopeless maze of paradox.

As a matter of fact the "crude argument" given above shows that the second definition of "real" had been introduced before the Principle of Relativity. It had been recognised already that observers would not agree as to the rate of a clock: the conception of the clock "as it really is," introduced in that argument, means (if it means anything) that function of the observed rate of the clock and its velocity relative to the observer which is the same for all observers. But the logical order of the argument is reversed. Instead of proving from the "real rate" of the clock, which we do not know, the observed rate, which we do know, we should say that the observed rate of the clock is  $n$ , and that our theory of light leads to the conclusion that  $n / \left(1 - \frac{v}{c}\right)$  will be the same for all observers. Whether that conclusion or the conclusion reached by the Principle of Relativity is correct can only be determined by experiment, and the experiment has not yet been tried.

It is the great merit of the Principle of Relativity that it forces on our attention the true nature of the concepts of



“real time” and “real space” which have caused such endless confusion. If we mean by them quantities which are directly observed to be the same by all observers, there simply is no real space and real time. If we mean by them, as apparently we do mean nowadays, functions of the directly observed quantities which are the same for all observers, then they are derivative conceptions which depend for their meaning on the acceptance of some theory as to how the directly observed quantities will vary with the motion, position, etc. of the observers. “Real” quantities can never be the starting point of a scientific argument; by their very nature they are not quantities which can be determined by a single observation: the term “real” has always kept its original meaning of some property of a body which is not observed simply.

All the difficulties and apparent paradoxes of the Principle of Relativity will vanish if the attention is kept rigidly fixed upon the quantities which are actually observed. If anyone thinks he discovers that that Principle predicts some experimental result which is incomprehensible, let him dismiss utterly from his mind the conception of reality. Let him imagine himself in the laboratory actually performing the experiment: let him consider the numbers which he will record in his note-book and the subsequent calculation which he will make. He may then find that the result is somewhat unexpected—to meet with unexpected results is the usual end of performing experiments,—but he will not find any contradiction or any conclusion which is not quite as simple as that which he expected.

8. There is one further point sometimes raised in connexion with the Principle on which a few words may be said.

It is sometimes objected that the Principle “has no physical meaning,” that it destroys utterly the old theory of light based on an elastic æther and puts nothing in its place, that, in fact, it sacrifices the needs of the physical to the needs of the mathematical instinct. That the statement is true there can be no doubt, but the absence of any substitute for the elastic æther theory of light may simply be due to the fact that the Principle has been developed so far chiefly by people who are primarily mathematicians. It is well to ask, can any physical theory of light be produced which is consistent with the Principle?

The answer depends on what is meant by a “physical theory.” Hitherto the term has always meant a “mechanical theory,” a theory of which the fundamental propositions are

statements about particles moving according to the Newtonian dynamical formulæ. In this sense a physical theory is impossible if the Principle of Relativity be accepted, for the same reason that a corpuscular theory of light is impossible, if the undulatory theory of light be accepted. Newtonian dynamics and the Principle of Relativity are two theories which deal in part with the same range of facts; they both pretend to be able to predict how the properties of observed systems will be altered by movement. If they are not logically equivalent they must be contradictory: in either case an "explanation" of one in terms of the other is impossible.

It can be easily shown that they are contradictory: if the Principle of Relativity is true, Newtonian dynamics must be abandoned\*. I shall deal with this point rather fully in a later paper; here it will suffice to point out that Einstein has been forced in his development of the subject to deny Newtonian dynamics at an early stage. He states that the fundamental equations of his electron theory are

$$m\ddot{x} = \epsilon X, \text{ etc.},$$

and then puts  $\dot{x} = v$ , where  $v$  is the velocity of the electron relative to the instrument exerting the force  $\epsilon X$ . But, if Newtonian dynamics are true,  $\dot{x}$  is not this relative velocity, but the velocity of the electron relative to the centre of mass of the electron and the instrument. Since the mass of the electron can conceivably become infinite, the distinction, negligible in practice, is of great importance theoretically.

On the other hand, if a "physical theory" of light means, as I think it means, a theory which draws an analogy between light propagation and the propagation of a disturbance through some mechanism, composed of rods and strings and fluids and such things, then there is no reason apparent why a physical theory of light should not be constructed which is consistent with the Principle of Relativity. But, of course, the laws according to which rods and strings and so on are supposed to act, must be changed from those predicted by Newtonian dynamics to some laws predicted by a mechanical theory consistent with the Principle. This development also is left for future discussion.

\* This conclusion is reached by Sommerfeld in a recent paper, *Ann. d. Phys.* xxxiii. p. 684, &c. (1910).

*Summary.*

1-5. The assumptions made by the Principle of Relativity are stated and an attempt made to render some of them more plausible at first sight.

6. A difficulty connected with the composition of velocities is examined and found to be due to verbal confusion.

7. The confusions introduced by the word "real" are discussed.

8. The relation between dynamics and relativity is considered briefly.

Leeds, November 1910.

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LVIII. *On Apparatus for the Production of Circularly Polarized Light.* By A. E. OXLEY, M.Sc., Scholar of Trinity College, Cambridge\*.

THE relative merits and defects of the quarter-wave plate and Fresnel's rhomb as used in the production of circularly polarized light are well known. For clearness it may be well to state them here. When we are dealing with monochromatic light, the quarter-wave plate has the great advantage that as its axes are rotated the transmitted circularly polarized beam is not displaced laterally as the plate is rotated. Such a lateral motion is experienced with a Fresnel's rhomb. On the other hand, when we are dealing with white light, the use of a quarter-wave plate is impossible since the phase-difference introduced between the components transmitted, differs considerably for different wavelengths. In such a case the Fresnel rhomb must be used to convert plane polarized light into circularly polarized light, and *vice versa*.

The present investigation was undertaken with the idea of combining as far as possible the advantages of the Fresnel rhomb and the quarter-wave plate in one piece of apparatus; the end being to produce an apparatus for which the emergent beam should be in line with the incident beam, and the various angles of incidence being so arranged as to produce a phase-difference of  $\frac{\pi}{2}$  for as large a range of wavelength possible.

Let the incident vibration upon a refracting medium be represented by

$$y = a \cdot e^{nt};$$

\* Communicated by the Author, having been read before the British Association at the Meeting at Sheffield, 1910.

then supposing a phase-change to be produced on reflexion, the reflected amplitude may be represented by the complex quantity  $\alpha + \beta i$ , and the reflected vibration may be written

$$y = (\alpha + \beta i) \cdot e^{mt}$$

where  $\alpha$  and  $\beta$  are real.

This may be written

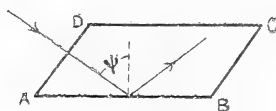
$$y = r \cdot e^{\theta i} \cdot e^{mt} = r \cdot e^{(nt + \theta) \cdot i},$$

where  $r = \sqrt{\alpha^2 + \beta^2}$ , and  $\theta = \text{arc. tan } \frac{\beta}{\alpha}$ .

Hence we see that a phase-change is produced on reflexion amounting to  $\theta$ .

Let  $\psi$  be the angle of incidence of a beam of plane-polarized light incident on the surface AB of the rhomb ABCD (fig. 1).

Fig. 1.



If  $\psi'$  and  $\mu$  are respectively the corresponding angle of refraction and index of refraction, then we have

$$\sin \psi = \mu \cdot \sin \psi'.$$

For total reflexion to take place,  $\psi'$  will be complex. Writing

$$\psi' = \xi + \eta i,$$

then

$$\begin{aligned} \sin \psi &= \mu \cdot \sin (\xi + \eta i) \\ &= \mu \cdot \sin \xi \cdot \cosh \eta + \mu \cdot i \cdot \cos \xi \cdot \sinh \eta. \end{aligned}$$

Equating real and imaginary parts,

$$\mu \cdot i \cdot \cos \xi \cdot \sinh \eta = 0,$$

$$\therefore \cos \xi = 0 \text{ or } \sinh \eta = 0,$$

whence

$$\xi = \frac{\pi}{2} \text{ or } \eta = 0.$$

$\eta = 0$  gives the ordinary case of refraction.

Taking  $\xi = \frac{\pi}{2}$ , we obtain the following law of refraction,—

$$\sin \psi = \mu \cdot \cosh \eta . . . . . (1)$$

Also 
$$\psi' = \frac{\pi}{2} + \eta t.$$

For the vibration whose plane of polarization is in the plane of incidence \*, Fresnel's sine law becomes with the law of refraction (1),

$$b = a \cdot \frac{\sin (\psi - \frac{\pi}{2} - \eta t)}{\sin (\psi + \frac{\pi}{2} + \eta t)}$$

“*b*” being the amplitude of the reflected, and “*a*” that of the incident light.

Writing  $b = r \cdot e^{\theta t}$ , we have

$$r \cdot e^{\theta t} = -a \cdot \frac{\cos (\psi - \eta t)}{\cos (\psi + \eta t)}$$

$$\therefore r \cdot e^{-\theta t} = -a \cdot \frac{\cos (\psi + \eta t)}{\cos (\psi - \eta t)}$$

Hence  $r^2 = a^2$ ,  $r = a$ ,

and 
$$e^{\theta t} = -\frac{\cos (\psi - \eta t)}{\cos (\psi + \eta t)} . . . . . (2)$$

For the component polarized perpendicular to the plane of incidence, the tangent law becomes

$$c = a \cdot \frac{\tan (\psi - \frac{\pi}{2} - \eta t)}{\tan (\psi + \frac{\pi}{2} + \eta t)}$$

or writing  $c = r' \cdot e^{\theta' t}$ ,

$$r' \cdot e^{\theta' t} = a \cdot \frac{\cot (\psi - \eta t)}{\cot (\psi + \eta t)}$$

$$\therefore r' \cdot e^{-\theta' t} = a \cdot \frac{\cot (\psi + \eta t)}{\cot (\psi - \eta t)}$$

and as before we find  $r' = a$ , and

$$e^{\theta' t} = \frac{\cot (\psi - \eta t)}{\cot (\psi + \eta t)} . . . . . (3)$$

\* Fresnel's theory is assumed.

Using equations (2) and (3) we see that the phase-difference produced at a single reflexion between the components polarized parallel and perpendicular to the plane of incidence, is given by  $\theta - \theta'$ , where

$$\begin{aligned} e^{(\theta - \theta')i} &= -\frac{\sin(\psi - \eta i)}{\sin(\psi + \eta i)} \\ &= -\frac{2 \cdot \sin^2(\psi - \eta i)}{2 \sin(\psi - \eta i) \sin(\psi + \eta i)} \\ &= \frac{\cos 2\psi \cdot \cosh 2\eta + i \cdot \sin 2\psi \cdot \sinh 2\eta - 1}{\cosh 2\eta - \cos 2\psi} \\ &= \cos \overline{\theta - \theta'} + i \sin \overline{\theta - \theta'}. \end{aligned}$$

Writing  $\theta - \theta' = \Delta$ ,

$$\cos \Delta = \cos \overline{\theta - \theta'} = \frac{\cos 2\psi \cdot \cosh 2\eta - 1}{\cosh 2\eta - \cos 2\psi} \dots \dots (4)$$

Since from (1)  $\sin \psi = \mu \cdot \cosh \eta$ , we have, using (4),

$$\frac{\cos 2\psi \left( \frac{2 \sin^2 \psi}{\mu^2} - 1 \right)}{\frac{2 \sin^2 \psi}{\mu^2} - 1 - \cos 2\psi} = \cos \Delta \dots \dots (4')$$

Writing  $x = \cos 2\psi$ ,

$$\therefore x(1 - \mu^2 - x) - \mu^2 = (1 - x - 1 + x \cdot \mu^2) \cdot \cos \Delta,$$

or

$$x^2 - x\{1 - \mu^2 + (1 + \mu^2) \cdot \cos \Delta\} + \cos \Delta + \mu^2(1 - \cos \Delta) = 0 \dots \dots (5)$$

If the beam of plane-polarized light incident normally upon the surface of the rhomb, be vibrating in a plane inclined at  $45^\circ$  to a principal section of the rhomb, a phase-difference amounting to  $\Delta$  will be produced between the components, and their amplitudes will be equal.

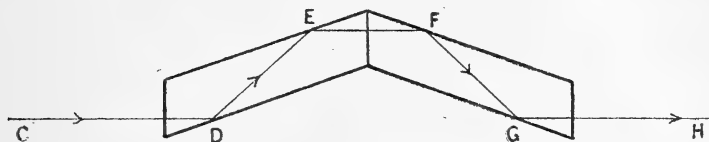
Now taking  $\mu = \frac{1}{1.51}$  (glass to air) and  $\Delta = \frac{7\pi}{8}^*$ , we find on substitution that  $x^2$  and therefore  $\cos^2 2\psi$  is real. Moreover, both values of  $\psi$  are real, one being about  $75^\circ$

\*  $\Delta$  may be of the form  $\frac{7\pi}{8} \pm 2n\pi + m\frac{\pi}{4}$  where  $n$  is any integer and  $m$  is either 1 or 0.

while the other is near the critical angle. The larger angle is chosen for reasons given below.

Now let there be two rhombs placed as shown in fig. 2,

Fig. 2.



which is a medial section. A beam of plane-polarized light enters along CD, is reflected at D, E, F, and G, and emerges along G, H, so that C, D, G, H, are collinear. There will be a retardation of one component on the other by  $\Delta$ , and

$$\Delta = \frac{7\pi}{8} \text{ if the angle of the rhomb is } 75^\circ \text{ for ordinary glass}$$

$$\left(\mu = \frac{1}{1.51}\right).$$

Hence for such a rhomb the total gain of one component on the other will be  $4 \times \frac{7\pi}{8} = 4\pi - \frac{\pi}{2}$ , and therefore the emergent light will be circularly polarized\*.

Further, since C, D, G, H, are collinear, on rotating the system there will be no lateral motion of the circularly polarized beam, and no readjustment of successive pieces of apparatus is needed.

The fact that the relative retardation of one component on the other has to be of the form  $n\pi \pm \gamma$ , where  $\gamma$  is an acute angle and  $n$  odd, in order that  $\psi$  should come out real is in accordance with the result first found by Lord Kelvin, that the phase-difference between the components resolved in and perpendicular to a principal section (for substances at our disposal) is oblique †.

In total reflexion the phase-difference introduced between the vibrations executed in and perpendicular to the plane of incidence, is not entirely independent of the wave-length, although nearly so, and it is best to choose the angle of incidence so that the dependence of  $\Delta$  on  $\lambda$ , or the "colour effect," as we may call it, is as small as possible.

\* According to Lord Kelvin the component polarized parallel to the plane of incidence is accelerated on the perpendicular component. See Baltimore Lectures, p. 400.

† *Loc. cit.*

From equation (4') we have

$$\left(\frac{2 \sin^2 \psi}{\mu^2} - 1 - \cos 2\psi\right) \cdot \cos \Delta = \cos 2\psi \left(\frac{2 \sin^2 \psi}{\mu^2} - 1\right) - 1,$$

and differentiating logarithmically,

$$\begin{aligned} \tan \Delta \cdot \frac{d\Delta}{d\mu} &= \frac{4 \sin^2 \psi}{\mu^2} \left\{ \frac{\cos 2\psi}{\cos^2 \psi (2 \sin^2 \psi - \mu^2) - \mu^2} - \frac{1}{2 \sin^2 \psi - \mu^2 (1 + \cos 2\psi)} \right\} \\ &= \frac{2 \sin^2 \psi}{\mu} \left\{ \frac{1}{\sin^2 \psi - \frac{\mu^2 \cos^2 \psi}{\cos 2\psi}} - \frac{1}{\sin^2 \psi - \mu^2 \cdot \cos^2 \psi} \right\} \dots \dots \dots \end{aligned}$$

Hence for given  $\Delta$  (say  $\Delta = \frac{7\pi}{8}$ ), and for all values of  $\mu$ ,

as  $\psi$  varies from 0 to  $\frac{\pi}{4}$ ,  $\frac{d\Delta}{d\mu}$  varies from 0 to

$$-\frac{1}{\mu} \cdot \frac{1}{\left(\frac{1}{2} - \frac{\mu^2}{2}\right)} \cdot \frac{1}{\tan \frac{7\pi}{8}} = \frac{1}{\mu} (1 - \mu^2) \cot \frac{\pi}{8}.$$

Here  $\mu < 1$ , if  $\nu = \frac{1}{\mu}$ , then

$$\frac{d\Delta}{d\nu} = -\frac{2\nu}{(\nu^2 - 1)} \cdot \cot \frac{\pi}{8}.$$

As  $\psi$  increases from  $\frac{\pi}{4}$  to  $\frac{\pi}{2}$ ,  $\frac{d\Delta}{d\mu} \rightarrow 0$ . Now for total reflexion,

$\psi$  must be greater than the critical angle. For a variation in  $\lambda$  corresponding to an increase in  $\nu$  by  $\delta\nu$ , the variation in phase  $\delta\Delta$  is greatest in the neighbourhood of the critical angle; while  $\delta\Delta \rightarrow 0$  for such a variation in  $\Delta$ , as  $\psi \rightarrow \frac{\pi}{2}$ .

Hence in so far as this effect is concerned large angles of incidence are desirable. In this respect the new arrangement has a further advantage over the Fresnel rhomb, for although the two reflexions in the latter are replaced by four reflexions in the former, yet the total colour effect, if we choose the larger value of  $\psi$ , is only half that of a Fresnel rhomb. The colour effect is examined below.

Uviol glass was chosen as the most suitable material for the Bi-rhomb (as this form of the apparatus is called) since



the loss of light by absorption is small and the dispersion low. Equation (5) becomes

$$x^2 - x \left\{ 1 - \mu^2 + (1 + \mu^2) \cdot \cos \frac{7\pi}{8} \right\} + \cos \frac{7\pi}{8} + \mu^2 \left( 1 - \cos \frac{7\pi}{8} \right) = 0,$$

where  $\mu = \frac{1}{1.5035}$  for the  $D_1$  line,

and  $x = \cos 2\psi$ , where  $\psi$  is the angle of the rhomb.

Each of the values of  $\psi$  is above the critical angle, which is  $41^\circ 42'$  for uviol glass. Both at  $\psi = 74^\circ 38'.2$  and  $\psi = 42^\circ 34'.8$ , we get relative retardation amounting to  $\frac{7\pi}{8}$  for each reflexion, and the larger value is chosen since then  $\Delta$  is practically independent of variations in  $\lambda$ . For fused quartz, of refractive index 1.5533 for the  $D_1$  line, the larger value of  $\psi$  is  $75^\circ 4'.5$ .

If we consider an interval comprising a range of  $\lambda$  equal to that between the C and F lines\*, and call the increment or decrement in  $\mu$  corresponding to it  $\pm d\mu$  ( $\mu < 1$ ) we can make an estimate of the variation in  $\Delta$  for any  $\lambda$  from the value  $\frac{7\pi}{8}$  which obtains for the  $D_1$  line. We have from (6),

$$d\Delta = \frac{2 \sin^2 \psi}{\mu \tan \frac{7\pi}{8}} \cdot \left\{ \frac{1}{\sin^2 \psi - \frac{\mu^2 \cdot \cos^2 \psi}{\cos 2\psi}} - \frac{1}{\sin^2 \psi - \mu^2 \cos^2 \psi} \right\} \mu^2 \cdot d\mu. \quad (7)$$

Here  $\psi = 74^\circ 38'$ ,

$$\left. \begin{aligned} \mu &= \frac{1}{1.5035} \text{ for } D_1 \text{ line} \\ d\mu_{C-F} &= \pm 0.00781 \end{aligned} \right\} \text{ for uviol glass.}$$

Substituting we find  $d\Delta_{C-F} = \pm 0^\circ.052$ .

Therefore for four reflexions, the extreme phase-difference for the C-F interval considered above will be

$$4 \cdot d\Delta_{C-F} = 0^\circ.208.$$

\* The total interval for which  $\delta\Delta$  is calculated comprises an interval  $\lambda_C - \lambda_F$  on each side of the  $D_1$  line, *i. e.* an interval of approximately 1700 A.U. on each side.

If  $\psi = 54^\circ 37'$ , which is the angle of Fresnel's rhomb ( $\mu$  being 1/1.51) equation (6) gives

$$\frac{d\Delta}{C-F} = \pm 0^\circ.203,$$

and there being two reflexions the extreme phase-difference for the C-F interval amounts to  $\pm 0^\circ.406$ . Hence, the variation of  $\Delta$  with  $\mu$  is only half as great in the Bi-rhomb as it is in Fresnel's rhomb, for a given wave-length.

*The dimensions of the Bi-rhomb.*—Owing to the large value of  $\psi$  the Bi-rhomb for a given aperture is rather inconveniently long. If  $S$  denote the aperture, and  $\chi$  the angle of the rhomb, the length in the direction of the incident light when the full aperture is utilized will be

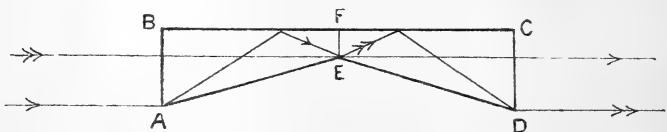
$$S \left\{ \tan \chi - \frac{1}{\tan 2\chi + \cot \chi} \right\},$$

and if  $A$  be 1.1 cms., and  $\psi = \chi = 74^\circ 38'$ , the length is 7.4 cms. This has to be doubled, and the total length of the Bi-rhomb of aperture 1.1 cms. is approximately 15 cms. Since, however, the value of  $\frac{d\Delta}{C-F}$  for  $\psi = 42^\circ 34'.8$  for the

four reflexions amounts to  $\pm 3^\circ.6$ , it is better to sacrifice compactness for efficiency.

In many experiments the length of the Bi-rhomb may be no serious objection to its use, but by the following method involving three reflexions an apparatus has been devised which combines practically all the advantages of the form just considered with compactness. ABCDE in fig. 3, shows

Fig. 3.



the bi-trapezoidal form of the section. The faint lines show the trace of a beam of light through the instrument, the beam passing through symmetrically with respect to the line EF. Thus the emergent beam, although it has suffered lateral inversion, is not displaced as a whole from the incident beam.

If  $\phi$  be the angle of incidence on the face AE,  $\phi$  is the angle of reflexion from the face ED. Let  $\phi'$  be the angle of incidence on the face BC. Clearly  $\phi'$  must be greater than

the critical angle from glass to air, and since we must have from the geometry of the figure, the relation

$$\phi = \frac{\phi'}{2} + \frac{\pi}{4}$$

satisfied, we must have  $\phi > 65^\circ 51'$ , taking  $\phi' = 41^\circ 42'$  as the critical angle.

Consider the component of the incident vibration which is polarized perpendicular to the plane of incidence. Let  $\phi$  be the angle of incidence,  $\theta$  the absolute phase-difference produced on the component for this angle of incidence,  $\theta'$  the corresponding phase-difference for angle of incidence  $\phi'$ .

Then, with the usual notation, we have

$$e^{\theta i} = -\frac{\cos \overline{\phi - \eta i}}{\cos \overline{\phi + \eta i}},$$

and for the three reflexions indicated in fig. 3 :

$$e^{(2\theta + \theta')i} = \left[ \frac{\cos \overline{\phi - \eta i}}{\cos \overline{\phi + \eta i}} \right]^2 \cdot \frac{\cos \overline{\phi' - \eta' i}}{\cos \overline{\phi' + \eta' i}}.$$

For the component polarized parallel to the plane of incidence, let  $\delta$  and  $\delta'$  correspond to  $\theta$  and  $\theta'$  respectively. Then as before

$$e^{(2\delta + \delta')i} = \left[ \frac{\cot \overline{\phi - \eta i}}{\cot \overline{\phi + \eta i}} \right]^2 \cdot \frac{\cot \overline{\phi' - \eta' i}}{\cot \overline{\phi' + \eta' i}}.$$

If  $\Delta'$  be the relative retardation, then will

$$\cos \Delta' = R * \left[ \frac{\sin \overline{\phi - \eta i}}{\sin \overline{\phi + \eta i}} \right]^2 \cdot \frac{\sin \overline{\phi' - \eta' i}}{\sin \overline{\phi' + \eta' i}}, \quad \dots \quad (8)$$

where

$$\cosh \eta = \frac{\sin \phi}{\mu}, \quad \cosh \eta' = \frac{\sin \phi'}{\mu},$$

and

$$\phi = \frac{\phi'}{2} + \frac{\pi}{4}.$$

From equation (8)

$$\begin{aligned} \cos \Delta' &= R \left[ \frac{\sin \phi \cdot \cosh \eta - i \cos \phi \cdot \sinh \eta}{\sin \phi \cdot \cosh \eta + i \cos \phi \cdot \sinh \eta} \right]^2 \times \\ &\quad \left[ \frac{\sin \phi' \cdot \cosh \eta' - i \cos \phi' \cdot \sinh \eta'}{\sin \phi' \cdot \cosh \eta' + i \cos \phi' \cdot \sinh \eta'} \right] \\ &= R \frac{(\sin \phi \cdot \cosh \eta - i \cos \phi \cdot \sinh \eta)^4}{(\sin^2 \phi \cdot \cosh^2 \eta + \cos^2 \phi \cdot \sinh^2 \eta)} \times \\ &\quad \frac{(\sin \phi' \cdot \cosh \eta' - i \cos \phi' \cdot \sinh \eta')^2}{(\sin^2 \phi' \cdot \cosh^2 \eta' + \cos^2 \phi' \cdot \sinh^2 \eta')} \end{aligned}$$

\* R is used to denote the real part of the function which follows it.

If now  $\Delta'$  is to be  $\frac{n\pi}{2}$ , where  $n$  is an odd integer, then the real part of the expression on the R.H.S. will be zero. This gives the following equation for  $\phi$  :

$$(\sin^4 \phi \cdot \cosh^4 \eta - 6 \sin^2 \phi \cdot \cosh^2 \eta \cdot \cos^2 \phi \cdot \sinh^2 \eta + \cos^4 \phi \cdot \sinh^4 \eta) \times \\ (\sin^2 \phi' \cdot \cosh^2 \eta' - \cos^2 \phi' \cdot \sinh^2 \eta') + 8 \cdot \sin \phi \cdot \cosh \eta \cdot \cos \phi \cdot \sinh \eta \cdot \sin \phi' \times \\ \cosh \eta' \cdot \cos \phi' \cdot \sinh \eta' (\cos^2 \phi \cdot \sinh^2 \eta - \sin^2 \phi \cdot \cosh^2 \eta) = 0, \quad \dots \dots (9)$$

where  $\cosh \eta = \frac{\sin \phi}{\mu}, \quad \cosh \eta' = \frac{\sin \phi'}{\mu},$

and  $\phi = \frac{\phi'}{2} + \frac{\pi}{4}.$

This is an equation of the sixteenth degree in  $\sin \phi$ , *i. e.* of the eighth degree in  $x$ . An approximate solution has been found by trial and error, and taking  $\mu_D = 1.5035$ , the value of  $\phi$  is approximately  $73^\circ 48'$ . From the relation

$$\phi' = 2\phi - \frac{\pi}{2}$$

we get  $\phi' = 57^\circ 36'.$

As a test of the accuracy of these values of  $\phi$  and  $\phi'$ , the equation (4') was used to find  $\Delta'$  for angles of incidence  $\phi$  and  $\phi'$ . If we call the relative phase-differences  $\theta - \delta$  and  $\theta' - \delta'$  in accordance with the notation on p. 525, then from the equation

$$\cos \Delta = \frac{x(1 - \mu^2) - \mu^2 - x^2}{1 - \mu^2 - (1 + \mu^2) \cdot x}$$

we find,  $\left. \begin{aligned} \theta - \delta &= \pi - 23^\circ 40' \cdot 2 \\ \theta' - \delta' &= \pi - 42^\circ 42' \cdot 2 \end{aligned} \right\}$

which gives

$$2(\theta - \delta) + (\theta' - \delta') = 2\pi + 89^\circ 57' \cdot 4.$$

A still more accurate value of  $\phi$  was found as follows.

Taking  $\phi = 73^\circ 49'$ , the total phase-difference for the three reflexions calculated as above is

$$2(\theta - \delta) + (\theta' - \delta') = 2\pi + 90^\circ 2' \cdot 2.$$

Taking  $\phi = 73^\circ 48'$ , it may be shown that

$$2(\theta - \delta) + (\theta' - \delta') = 2\pi + 89^\circ 57' \cdot 4.$$

On a simple interpolation we obtain  $\phi = 73^\circ 48'6$ . Therefore  $\phi' = 2\phi - \frac{\pi}{2} = 57^\circ 37'2$ .

These values give a phase-difference between the components on emergence equivalent to  $\frac{\pi}{2}$ , for the  $D_1$  line, and the emergent beam will be circularly polarized.

Since  $\phi = 73^\circ 48'6$ , the colour effect as calculated from equation (6) for a single reflexion will be nearly the same as for a single reflexion in the Bi-rhomb, for which  $\phi = 74^\circ 38'2$ . There are two such reflexions, and the colour effect is  $2 \cdot d\Delta = \pm 0.102$ , so far as they are concerned. For the

C-F reflexion at angle of incidence  $\phi'$ , the value of  $d\Delta = \pm 0.18$ .

Hence the total colour effect is a little less than that for an ordinary Fresnel's rhomb.

For an aperture of  $S$  cms. ( $S$  is the breadth of the beam of light) the length of the Bi-trapezoid will be  $2 \cdot S \cdot \tan \chi$ , where  $\chi$  is the acute angle of the trapezoid. Also the greatest breadth of the trapezoid is  $AB =$

$$S (\cot \chi - \tan 2\chi).$$

If  $\chi = 75^\circ$ ,  $S = 1$  cm., the length in the direction of the incident light is 7.5 cms., approx. *i. e.*, half the length of the previous form.  $CD$  is then about 1.7 cms.

#### *The Azimuth of the Incident Vibration.*

With  $\phi = 73^\circ 48'6$  the phase-difference for the three reflexions, between the components polarized parallel and perpendicular to the plane of incidence, is exactly  $\frac{\pi}{2}$  for sodium light,  $\mu$  being 1.5035.

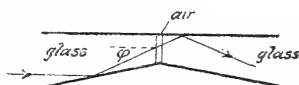
Since, however, the component polarized parallel to the plane of incidence is more copiously reflected on crossing the joint than the perpendicular component is, the usual azimuth (*viz.*  $45^\circ$ ) would produce elliptically polarized light, the major axis of the ellipse being parallel to  $CD$  (fig. 3). By adjusting the azimuth, the transmitted beam can be made circularly polarized. The new azimuth can be found in two ways.

(1) *Experimentally.*—Using a Babinet's compensator or the Bi-rhomb (which produces circularly polarized light, since here the beam crosses the joint normally) we can analyse the light transmitted by the Bi-trapezoid, and adjust

the azimuth of the incident vibration with respect to the latter so that the beam of plane-polarized light after passing through it would be circularly polarized.

(2) *By Calculation.*—The new azimuth can be determined more accurately by calculation. Consider the layer of air forming the joint. Let  $\phi$  be the angle of incidence in the

Fig. 4.



glass (fig. 4),  $\phi'$  the angle of refraction. The ratio of the refracted amplitudes in the air-gap is

$$\frac{2 \sin \phi' \cdot \cos \phi}{\sin (\phi + \phi')} \cdot \frac{\sin (\phi + \phi') \cos (\phi - \phi')}{2 \sin \phi' \cos \phi} = \cos (\phi - \phi'),$$

the azimuth of the incident vibration being  $45^\circ$ . After the second refraction this ratio is

$$\cos^2 (\phi - \phi').$$

If we make the azimuth of the incident vibration  $\Omega$  where

$$\tan \Omega = \frac{1}{\cos^2 \left\{ \phi - \arcsin (\mu \sin \phi) \right\}},$$

$\Omega$  being measured from the edge AB (fig. 3), the transmitted amplitudes will be equal, and since their phase-difference is still  $\frac{\pi}{2}$ , the emergent beam of light will be circularly polarized. Using the known values of  $\phi$  and  $\mu$ , we find

$$\Omega = 49^\circ 2'.$$

An estimation of the ellipticity of the orbit corresponding to any value of  $d\Delta = \zeta$  (say), can be obtained as follows. Let the transmitted components be represented by the equations

$$x = r \cdot \cos wt, \quad y = r \sin (wt \pm \zeta),$$

The equation of the orbit is

$$\frac{x^2(1 + \zeta^2)}{r^2} \pm 2xy \cdot \frac{\zeta}{r^2} + \frac{y^2}{r^2} = 1,$$

and defining the ellipticity ( $\epsilon$ ) as the ratio of the difference of axes to major axis, we have

$$\epsilon = 1 - \frac{1-\zeta}{1+\zeta} = \frac{2\zeta}{1+\zeta}.$$

Now taking an interval equal to that between the C and F lines, as before, the value of  $\zeta$  for the Bi-rhomb is  $0^{\circ} \cdot 208$ , and therefore

$$\epsilon = \frac{0 \cdot 208 \times \pi}{90} = \cdot 007.$$

For the Bi-trapezoid and the same limits of wave-length, we find  $\zeta = 0^{\circ} \cdot 28$ ,

$$\epsilon = \frac{0 \cdot 28 \times \pi}{90} = \cdot 0096,$$

while the corresponding ellipticity for the Fresnel rhomb is

$$\epsilon = \frac{0 \cdot 407 \times \pi}{90} = \cdot 014.$$

The ellipticities are practically the same for fused quartz as for uviol glass.

Table I. shows the value of  $\zeta$  for different wave-lengths for the Bi-rhomb (*i. e.* for angle of incidence  $74^{\circ} 38' \cdot 2$ ) and the ellipticity of the corresponding orbit as calculated from  $\epsilon = \frac{2\zeta}{1+\zeta}$ . Table II. shows the corresponding values of  $\lambda$ ,  $\zeta$ ,  $\epsilon$  for the Bi-trapezoid; Table III. is for the Fresnel rhomb.

TABLE I.

Line.	$\lambda$ (A.U.).	$\zeta$ (rad.).	$\zeta$ (deg.).	$\epsilon$ .
	9296·0	·0073	0·417	·014
K red .....	7668·5	·0038	0·215	·0076
	7596·0	·0036	0·208	·007
Li red .....	6708·2	·0017	0·099	·0034
Na (D <sub>1</sub> ) .....	5896·2	·0	0·0	·0
Tl green.....	5350·7	-·0012	-0·069	-·0024
H violet .....	4340·7	-·0033	-0·189	-·0066
	4196·0	-·0036	-0·208	-·007
	2496·0	-·0073	-0·417	-·014

TABLE II.

Line.	$\lambda$ (A.U.).	$\zeta$ (rad.).	$\zeta$ (deg.).	$\epsilon$ .
K red .....	9296.0	.0095	0.54	.019
	7668.5	.0057	0.32	.010
	7596.0	.0048	0.28	.0096
Li red .....	6708.2	.0023	0.13	.0046
Na (D <sub>1</sub> ) .....	5896.2	.0	0.0	.0
Tl green .....	5350.7	-.0015	-0.086	-.003
H violet .....	4340.7	-.0044	-0.25	-.009
	4196.0	-.0048	-0.275	-.0096
	2496.0	-.0095	-0.54	-.019

TABLE III.

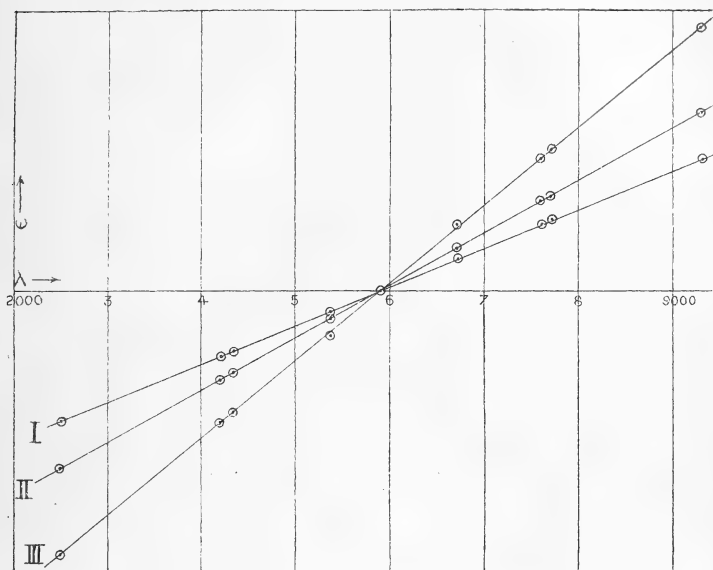
Line.	$\lambda$ (A.U.).	$\zeta$ (rad.).	$\zeta$ (deg.).	$\epsilon$ .
K red .....	9296.0	.0142	0.802	.028
	7668.5	.0074	0.424	.015
	7596.0	.0071	0.407	.014
Li red .....	6708.2	.0034	0.195	.007
Na (D <sub>1</sub> ) .....	5896.2	.0	0.0	.0
Tl green .....	5350.7	-.0023	-0.132	-.0046
H violet .....	4340.7	-.0064	-0.367	-.013
	4196.0	-.0071	-0.407	-.014
	2496.0	-.0142	-0.802	-.028

In fig. 5 the values of  $\epsilon$  given in the above tables are plotted against  $\lambda$ , and a comparison of ordinates for given wave-lengths shows the relative efficiency of the three forms for light of that wave-length. The efficiency is unity for the D<sub>1</sub> line. Curves I, II, III refer to Tables I, II, III. respectively. When  $\lambda$  is  $> 5896$  A.U.,  $\epsilon$  is regarded as positive, and after passing through  $\epsilon=0$  at  $\lambda=5896$ ,  $\epsilon$  is regarded as negative. Comparing each of the new forms with the Fresnel rhomb, it will be seen that the Bi-rhomb is about twice as efficient as the Fresnel rhomb for any given wave-length, while the Bi-trapezoid has an intermediate efficiency about one and a half times that of a Fresnel rhomb. But the Bi-rhomb has the disadvantage of being inconveniently long for a given aperture. This adds to the amount of absorption which, however, is reduced to a minimum by the use of uviol glass or fused quartz. In those experiments where length is not an objectionable factor, this form may prove useful. In the Bi-trapezoid, in which three reflexions are used, the efficiency is a little higher than that



of a Fresnel's rhomb, while the length is not much different. Hence in these respects the qualities of the new forms compare favourably with those of a Fresnel rhomb, but in

Fig. 5.



addition each has the advantage that the emergent beam of circularly polarized light is always in the same straight line as the incident beam whatever the orientation of the polarizer.

In the British Association Report for 1851, Prof. G. Stokes described a new form of elliptic analyser consisting of a plate of selenite which retarded waves of mean refrangibility by about a quarter wave-length. This, mounted in the manner described, was found to yield very accurate results, but a little experience is needed in experimenting with it as the tint produced perplexes the operator. It seems probable that the second form of polarizer described above would prove useful in the analysis of elliptic vibrations, as it could be accurately set with respect to the analysing Nicol's prism, an adjustment it is difficult to make with a quarter-wave plate, for in this case the positions of the axes are not accurately known. In the present form of analyser the edges of the glass (these should be accurately worked) forming the aperture take the places of the principal directions of the crystalline plate. Moreover, the tint would be absent. This

apparatus could be used for examining white light which had a definite vibration form.

The polarizers were tested with monochromatic light, and using a quarter-wave plate as analyser, the transmitted light was found to be circularly polarized. Removing the quarter-wave plate and using white light, the intensity of the field of view remained constant however the analysing Nicol's prism and polarizer were orientated, providing the azimuth of the incident vibration with respect to the polarizer was kept constant and at the requisite amount. Further, using the Bi-trapezoid polarizer to examine an elliptically polarized beam, produced by a Fresnel's rhomb with white light, no trace of tint could be discerned.

Each form of polarizer is mounted in a metal cylinder which is provided with central square apertures.

In conclusion I wish to tender my best thanks to Prof. W. M. Hicks and to Prof. Sir J. J. Thomson for the great interest they have taken in this work.

Trinity College, Cambridge,  
October, 1910.

*Note added March 16, 1911.*

In the Bi-trapezoid polarizer, since the azimuth  $\Omega$  of the incident vibration depends upon the refractive index, the efficiency of this form is slightly reduced, and becomes very nearly equal to that of Fresnel's rhomb.

LIX. *The Effect of Temperature on the Absorption Coefficient of Iron for  $\gamma$  Rays.* By W. WILSON, M.Sc.\*

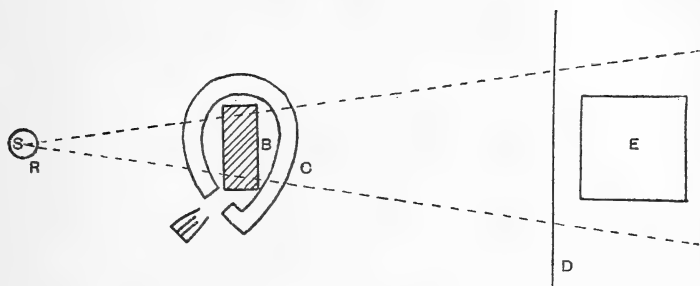
THE following experiment was made in order to determine if the absorption coefficient of iron for  $\gamma$  rays varies with the temperature. A negative result was obtained.

The arrangement of apparatus is shown in fig. 1.  $\gamma$  rays from a tube S containing about 30 m.g. radium bromide passed through an iron block B 7 cms. in thickness. The block was contained in a muffle-furnace C, and could be heated by means of a blowpipe. The intensity of the rays after passing through the furnace and iron was measured by an electroscope E. The distance from the furnace to the electroscope was 150 cms., and an asbestos sheet D was placed between them as shown. Under these circumstances, the heating of the furnace was found to have no direct influence on the electroscope.

\* Communicated by Prof. E. Rutherford, F.R.S.

In order to prevent a redistribution of the emanation in the tube containing the radium owing to heating effects, it was enclosed in a water-bath R and shielded from the furnace by a sheet of asbestos.

Fig. 1.



Experiments were made both while the iron was being heated and while it was cooling, the temperature being measured by means of a platinum-iridium thermo-couple. It was found that the readings were not so consistent while the block was being heated as while it was cooling. This is due to the fact that the temperature of the room rose considerably when heating was taking place, while when the heating was stopped the hot air soon dispersed, and the temperature did not vary while the iron was cooling.

Results obtained in three experiments are given in Table I.

TABLE I.

I.		II.		III.	
Temp. ° C.	Ionization.	Temp. ° C.	Ionization.	Temp. ° C.	Ionization.
630	3.55	678	3.57	570	3.38
595	3.60	631	3.48	480	3.36
524	3.52	594	3.47	410	3.35
490	3.50	525	3.48	210	3.36
447	3.50	470	3.48	90	3.36
411	3.50	411	3.49	15	3.33
389	3.50	351	3.48		
349	3.50	323	3.40		
290	3.50	307	3.47		
		178	3.47		
		147	3.47		
		134	3.45		

An experiment was made heating the furnace alone, and the ionization was the same at high and low temperatures.

It will be seen that the ionization in the electroscope is slightly greater at high than at low temperatures. The mean of a number of experiments showed the rate of increase of ionization to be .0020 per cent. per degree centigrade. This can be shown to be due to the change in density of the iron as follows.

Let  $I_0$  be the intensity of the rays falling on the block. The intensity,  $I$ , of the emergent rays is given by  $I = I_0 e^{-\lambda l}$ , where  $l$  is the thickness of the block and  $\lambda$  the coefficient of absorption.

Now  $\lambda$  is proportional to the density of the block, and therefore at any temperature  $\theta$  degrees higher is given by

$\frac{\lambda}{1 + 3\alpha\theta}$ , where  $\alpha$  is the coefficient of linear expansion of iron.

Similarly the thickness of the block at this temperature is  $(1 + \alpha\theta)l$ . We therefore have for small values of  $\alpha\theta$

$$I = I_0 e^{-\lambda l(1 - 2\alpha\theta)}$$

$$\frac{\partial I}{\partial \theta} = 2\alpha\lambda l I.$$

The percentage rise in ionization per degree centigrade is therefore  $200\alpha\lambda l$ .

In an actual experiment  $I_0$  was found to be 8.11 div. per min. and  $I$  3.55 div. per min.

$$\therefore \lambda l = \log I_0 - \log I = .825.$$

Taking  $\alpha$  as .0000117 we obtain an increase in ionization of .00193 per cent. per degree, which is in good agreement with the value .0020 obtained experimentally.

Thus the temperature of the iron within the limits examined affects the absorption coefficient by an amount to be expected from considerations of the change of density of the absorbing substance. That is, the absorption of  $\gamma$  rays by iron depends only on the actual number of atoms traversed by the rays without regard to the temperature of the body which they compose.

These experiments were carried out in the Physical Laboratories of the University of Manchester, and I wish to express my best thanks to Prof. Rutherford for the kind interest he took in them.

Gonville and Caius College Cambridge,  
February 1911

LX. *The Heat of Mixture of Substances and the Relative Distribution of the Molecules in the Mixture.* By R. D. KLEEMAN, D.Sc., B.A., Mackinnon Student of the Royal Society, and Clerk Maxwell Student of the University of Cambridge\*.

WHEN two quantities of different substances are mixed and no new molecules are formed, we will assume in this paper that the heat of mixture is the change in the total potential energy of the matter due to the "chemical" attraction between the molecules. We made a similar assumption in a previous paper in the case of the internal heat of evaporation of a pure liquid, which led to results agreeing well with the facts. But even if a change in the intra-molecular potential energy of a number of molecules with a change in their state of aggregation takes place, it could only be small in comparison with the change in the potential energy due to the attraction between the molecules, since the change in the internal latent heat of evaporation of a liquid with temperature is large in comparison with the corresponding changes in the intra-molecular energies of the vapour and liquid, that is, the specific heat at constant volume of the vapour and liquid.

We shall deduce formulæ on the above assumption in this paper applying to various cases of mixture of two or more substances, using the law of attraction between molecules which was deduced† from the internal latent heat and surface-tension of liquids. The law of attraction between

two molecules of the same kind is  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) \cdot \frac{(\Sigma\sqrt{m_1})^2}{z^5}$ , where

$T$  is the temperature of the molecules and  $z$  their distance of separation,  $x_c$  denotes the distance of separation of the molecules in the liquid state at the critical temperature  $T_c$ ,  $\Sigma\sqrt{m_1}$  denotes the sum of the square roots of the atomic weights of

the atoms of a molecule, and  $\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right)$  is an arbitrary function of the ratios  $\frac{z}{x_c}, \frac{T}{T_c}$ . As the exact form of the

function  $\phi_2$  cannot be deduced from latent heat or surface-tension data there is little other data relating to physical or chemical properties of substances from which any information on the exact form of  $\phi_2$  can be obtained. We have

\* Communicated by the Author.

† Phil. Mag. May 1910, p. 783.

shown, however, that  $\phi_2$  must be a function of both  $z$  and  $T^*$ , and that it probably varies very little with  $T$  or  $z$  †. Since the attraction between two molecules is equal to the product of their powers of attraction the function  $\phi_2$  must consist of two factors of the same form, that is

$$\phi_2\left(\frac{z}{x_c}, \frac{T}{T_c}\right) = \phi_a\left(\frac{z}{x_c}, \frac{T}{T_c}\right), \phi_a\left(\frac{z}{x_c}, \frac{T}{T_c}\right).$$

When the molecules are not of the same kind the attraction would be given by the expression

$$\phi_a\left(\frac{z}{x_c'}, \frac{T}{T_c'}\right) \phi_a\left(\frac{z}{x_c}, \frac{T}{T_c}\right) \Sigma\sqrt{m_1} \Sigma\sqrt{m_1'}.$$

Although the form of the function  $\phi_2$  is not yet exactly known, yet it will be of interest to deduce the general formulæ for the heat of mixture of substances. It will be found that in special cases useful information can be obtained from experimental data without an exact knowledge of the form of  $\phi_2$  being necessary.

We are quite justified in using the above law of attraction, as the distances of separation of the molecules in a mixture are not of a smaller order than those of the molecules in a pure liquid. When a new kind of molecule is formed by the combination of two molecules, these are probably after combination separated by a much smaller distance than the average distance of separation of the molecules of the mixture, but the above law of attraction probably holds down to distances of this magnitude. But other changes then happen, thus the total kinetic energy is not the same after as before combination. The heat of mixture might thus in some cases be appropriately divided into two parts, one due to the change of potential energy of the molecules that combine to form new molecules, supposing the molecules to be in the gaseous state at low pressure during the process, and the other part due to the change in potential energy in rendering into the gaseous state the molecules that combine to form new molecules, and then forming a mixture of these molecules and the remaining parts of the liquids.

#### *Relative Distribution of the Molecules in a Mixture.*

In order to be able to calculate the change in the potential energy of two quantities of matter on being mixed, we must know the relative distribution of the molecules in the mixture.

\* Phil. Mag. Jan. 1911, pp. 91-94.

† Phil. Mag. May 1910, pp. 795-798; Oct. 1910, pp. 667-670.

It is necessary, therefore, to investigate first of all this point.

The molecules in a liquid or mixture of liquids are undoubtedly in rapid motion, and the distance between any pair of molecules therefore continually changing. The term relative distribution of molecules has therefore at first sight no definite meaning. But consider the work at successive small intervals of time required to move a molecule in the centre of a mass of liquid to the surface and then to infinity. It is obvious that the average work approximates to some constant as the number of intervals is increased. We may therefore suppose, when we are concerned with the potential energy of the molecules, that they are not in motion but fixed in space in such a way that the work required to bring each molecule to infinity is equal to the above defined average work which corresponds to the molecule. This distribution in space is obtained when it satisfies the following conditions:—

(1) The forces on a molecule by the surrounding molecules must neutralize one another.

(This follows directly from the fact that the distribution is to represent an average effect.)

(2) The distribution must be such that keeping the density of the matter constant the potential energy due to molecular attraction is a minimum.

(3) In the case of a mixture of substances it is further necessary that conditions of symmetry are satisfied; in other words, the different molecules must be distributed as nearly as possible evenly throughout the volume of the mixture.

It will be easily seen that knowing the law of attraction between molecules and given the density of a mixture of substances and the numbers of different kinds of molecules it contains per c.c., it should be possible to calculate the relative distribution of the molecules subject to the above conditions. But the calculations in the majority of cases would be exceedingly difficult.

There are a few cases of mixture in which the relative distribution of the molecules, supposing no new molecules are formed, is at once obtained. The simplest case is that in which equal numbers of two different kinds of molecules, or masses proportional to their gram-molecules, are mixed. The distribution is in that case as follows. Suppose the mixture cut into a large number of equal cubes by three systems of parallel equidistant planes which are at right angles to one another. And suppose that the molecules of the mixture are situated at the various points of intersection of three planes, one at each point, but in such a way that each edge

of each cube has two different molecules adjacent to it. It will be easily seen on considering this distribution that it satisfies each of the above conditions.

It will also at once be recognized that if the relative concentration remains the same in a mixture when its density changes, the relative distribution of the molecules is unaltered, all the distances of separation of the molecules being increased in the same proportion by a change in density.

*Heat of Mixture of two or more different Liquids.*

Let us first obtain a formula for the heat of mixture of  $n_1$  grams of the liquid 1 and  $n_2$  grams of the liquid 2, on the supposition that no new molecules are formed in the mixture.

The mixing may be supposed to take place in the following way. Suppose the two liquids to evaporate into an infinitely large chamber, and let the work done against the attraction of the molecules in the process be denoted by  $L_1$  and  $L_2$  respectively. Let the mixture of vapours be now compressed till it is converted into liquid, and let the heat given out due to the attraction between the molecules be  $L_{n_1n_2}$ . Then on the supposition that the change in internal energy in the process is equal to the change in the potential energy due to the molecular attraction, we have that the heat of mixture  $H$  is equal to  $(L_{n_1n_2} - n_1L_1 - n_2L_2)$ .

Now according to the law of attraction given at the beginning of the paper

$$L_1 = \frac{A_1}{M_1} \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

and

$$L_2 = \frac{A_2}{M_2} \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2,$$

where

$$A_1 = \phi_1 \left( \frac{x_{a_1}}{x_{c_1}}, \frac{T}{T_{c_1}} \right) \quad \text{and} \quad A_2 = \phi_2 \left( \frac{x_{a_2}}{x_{c_2}}, \frac{T}{T_{c_2}} \right),$$

and  $x_a$  denotes the distance of separation of the molecules in the liquid,  $M$  the molecular weight, and  $\rho$  the density of the liquid\*. The form of  $A_1$  and  $A_2$ , it should be noticed, depends only on the form of the function  $\phi_2 \left( \frac{x_c}{x_c}, \frac{T}{T_c} \right)$ .

The work done in removing a molecule 1 from the mixture

\* Phil. Mag. May 1910, pp. 793-794.



to infinity may be divided into two parts—the work done against the attraction of the same kind of molecules in the mixture, and that done against the attraction of the other kind of molecules. The first part may be written

$$B_1 \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

where

$$B_1 = \phi_4 \left( \frac{x_{a_1}}{x_{c_1}}, \frac{T}{T_{c_1}} \right),$$

if the relative distribution of the molecules 1 in the mixture is expressed in terms of the distance of separation of the molecules in the pure liquid 1. The way the functions  $B_1$ ,  $B_2$ , and others of the same nature are formed will appear at once from a study of the way the functions  $A_1$  and  $A_2$  are formed which is given in the paper quoted above. The other part is obtained by substituting the law of attraction between two different molecules given at the beginning of the paper for  $\phi(z)(\Sigma \sqrt{m_1})^2$  in the equation for the internal latent heat of evaporation given (pp. 793–802) in a previous paper quoted above. If the relative distribution of the molecule 1 with respect to the molecules 2 is expressed in terms of the distance of separation of the molecules 1 in the pure liquid, this gives for the work done

$$C_1 \left( \frac{\rho_1}{M_1} \right)^{4/3} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2},$$

where

$$C_1 = \phi_5 \left( \frac{x_{a_1}}{x_{c_1}}, \frac{x_{a_1}}{x_{c_2}}, \frac{T}{T_{c_1}}, \frac{T}{T_{c_2}} \right).$$

In a similar manner it can be shown that the work done in removing a molecule 2 from the liquid to infinity is

$$B_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2 + C_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2},$$

where

$$B_2 = \phi_6 \left( \frac{x_{a_2}}{x_{c_2}}, \frac{T}{T_{c_2}} \right),$$

and

$$C_2 = \phi_7 \left( \frac{x_{a_2}}{x_{c_2}}, \frac{x_{a_2}}{x_{c_1}}, \frac{T}{T_{c_1}}, \frac{T}{T_{c_2}} \right).$$

The heat of mixture of  $n_1$  grams of liquid 1 with  $n_2$  grams of liquid 2 is therefore

$$\begin{aligned} & (B_1 - A_1) \frac{n_1}{M_1} \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2 \\ & + \left\{ C_1 \frac{n_1}{M_1} \left( \frac{\rho_1}{M_1} \right)^{4/3} + C_2 \frac{n_2}{M_2} \left( \frac{\rho_2}{M_2} \right)^{4/3} \right\} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} \\ & + (B_2 - A_2) \frac{n_2}{M_2} \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2, \end{aligned}$$

where  $B_1, A_1, B_2, A_2, C_1, C_2$ , are functions of the ratios

$$\frac{T}{T_{c_1}}, \frac{T}{T_{c_2}}, \frac{\rho_2}{\rho_{c_2}}, \frac{\rho_1}{\rho_{c_1}}, \frac{\rho_2}{\rho_{c_1}}, \frac{\rho_1}{\rho_{c_2}},$$

since

$$x_{\sigma_1} = \left( \frac{\rho_1}{M_1} \right)^{1/3}, \quad x_{\sigma_2} = \left( \frac{\rho_2}{M_2} \right)^{1/3}, \quad x_{c_1} = \left( \frac{\rho_{c_1}}{M_2} \right)^{1/3}, \quad \text{and} \quad x_{c_2} = \left( \frac{\rho_{c_2}}{M_2} \right)^{1/3}.$$

The form of the functions  $B_1, \dots, C_2$ , as will be seen, depends on the form of the function  $\phi_2 \left( \frac{z}{x_c}, \frac{T}{T_c} \right)$  in the law of attraction between molecules and the relative distribution of the molecules in the mixture. As the form of the function  $\phi_2$  is not yet exactly known, we are not yet able to evaluate these functions in any given case.

The expression for the value of  $L_{n_1, n_2}$  can be obtained in several different forms, depending on the quantity in terms of which the relative distribution of the molecules in the mixture is expressed. For example, if the relative distribution is expressed in terms of the mean distance of separation of the molecules in the mixture, *i. e.* in terms of

$$\left\{ \frac{\rho_3}{n_1 + n_2} \left( \frac{n_1}{M_1} + \frac{n_2}{M_2} \right) \right\}^{-1/3},$$

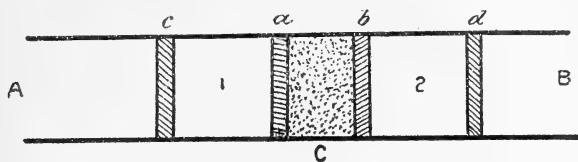
where  $\rho_3$  denotes the density of the mixture,  $L_{n_1, n_2}$  becomes

$$\begin{aligned} & \left\{ \frac{n_1 D}{M_1} (\Sigma \sqrt{m_1})^2 + \left( \frac{n_1 E_1}{M_1} + \frac{n_2 E_2}{M_2} \right) \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} \right. \\ & \left. + \frac{n_2 F}{M_2} (\Sigma \sqrt{m_2})^2 \right\} \left\{ \frac{\rho_3}{n_1 + n_2} \left( \frac{n_1}{M_1} + \frac{n_2}{M_2} \right) \right\}^{4/3}, \end{aligned}$$

where  $D, E_1, E_2$ , and  $F$  are functions of the ratios

$$\frac{\rho_3}{\rho_{c_1}}, \frac{\rho_3}{\rho_c}, \frac{T}{T_c}, \frac{T}{T_{c_2}}.$$

It is possible to develop formulæ for the heat of mixture of liquids which do not involve any knowledge of the relative distribution of the molecules in the mixture. Let AB in the figure be a cylinder containing four pistons  $c$ ,  $a$ ,  $b$ ,  $d$ .



Suppose the space C between the pistons  $a$  and  $b$  filled with a mixture of molecules 1 and 2 in the proportion of  $n_1$  to  $n_2$ . Suppose the piston  $a$  pervious to molecules 1 but not to molecules 2, and the piston  $b$  pervious to molecules 2 but not to molecules 1, so that the space between the pistons  $c$  and  $a$  is filled with saturated vapour of molecules 1, and the space between the pistons  $b$  and  $d$  filled with saturated vapour of molecules 2. Suppose the pistons  $c$  and  $d$  be moved towards the ends of the cylinder till the increase in the masses of the vapours 1 and 2 be equal to  $n_1$  and  $n_2$  respectively, and suppose that at the same time the pistons  $a$  and  $b$  are moved so as to remain in contact with the mixture. During the process the piston  $a$  has displaced a volume

$\rho_3 \frac{\delta V}{\delta c_1} \frac{n_1}{M_1}$  of the mixture, where  $V$  is the volume of a gram of the mixture and  $\rho_3$  its density, and  $c_1$  is the concentration of the molecules 1. The quantity  $\rho_3 \frac{\delta V}{\delta c_1}$  or  $\frac{1}{V} \frac{\delta V}{\delta c_1}$  may be called the volume a molecule 1 occupies in the mixture; it is a quantity which is of interest and importance and will be discussed in the next section. Similarly, the piston  $b$  displaces a volume of the mixture equal to  $\rho_3 \frac{\delta V}{\delta c_2} \cdot \frac{n_2}{M_2}$ . Let  $p_1$  denote the pressure per cm.<sup>2</sup> on the piston  $c$  and  $p_2$  that on  $d$ , and let the corresponding pressures on the pistons  $a$  and  $b$  be  $p_2'$  and  $p_1'$  respectively. Now according to Clapeyron's thermodynamical equation we have

$$L_{n_1} + p_1 v_1 - \rho_3 \frac{\delta V}{\delta c_1} \frac{n_1}{M_1} p_2' = n_1 T \frac{dp_1}{dT} - \rho_3 \frac{\delta V}{\delta c_1} \frac{n_1 T}{M_1} \frac{dp_2'}{dT},$$

$$L_{n_2} + p_2 v_2 - \rho_3 \frac{\delta V}{\delta c_2} \frac{n_2}{M_2} p_1' = n_2 T \frac{dp_2}{dT} - \rho_3 \frac{\delta V}{\delta c_2} \frac{n_2 T}{M_2} \frac{dp_1'}{dT},$$

where  $L_{n_1}$  and  $L_{n_2}$  denote the internal heats of evaporation of  $n_1$  grms. of molecules 1 and  $n_2$  grms. of molecules 2 respectively.

and  $v_1$  the volume displaced by the piston  $c$  and  $v_2$  that displaced by the piston  $d$ .  $v_1$  and  $v_2$  are given by the equations

$$\left(v_1 + \rho_3 \frac{\delta V}{\delta c_1} \frac{n_1}{M_1}\right) \rho_4 = n_1,$$

$$\left(v_2 + \rho_3 \frac{\delta V}{\delta c_2} \frac{n_2}{M_2}\right) \rho_5 = n_2,$$

where  $\rho_4$  and  $\rho_5$  denote the densities of the saturated vapours of molecules 1 and 2 respectively. Now  $L_{n_1} = L_{n_1}' - L_{n_1}''$ , where  $L_{n_1}'$  denotes the internal heat of evaporation of  $n_1$  grms. of molecules 1 from the mixture into a vacuum and  $L_{n_1}''$  the heat of evaporation of  $n_1$  grms. of the saturated vapour of molecules 1 into a vacuum. Similarly,

$$L_{n_2} = L_{n_2}' - L_{n_2}''.$$

Now we have seen that expressions for  $L_{n_1}''$  and  $L_{n_2}''$  are at once obtainable from the law of attraction between molecules. The values are

$$\frac{n_1 H_1}{M_1} \left(\frac{\rho_4}{M_1}\right)^{4/3} (\Sigma \sqrt{m_1})^2$$

and

$$\frac{n_2 H_2}{M_2} \left(\frac{\rho_5}{M_2}\right)^{4/3} (\Sigma \sqrt{m_2})^2,$$

where

$$H_1 = \phi_3 \left(\frac{x_{b_1}}{x_{c_1}}, \frac{T}{T_c}\right)$$

and

$$H_2 = \phi_3 \left(\frac{x_{b_2}}{x_{c_2}}, \frac{T}{T_c}\right),$$

$x_{b_1}$  and  $x_{b_2}$  denoting respectively the distances of separation of the molecules in the vapours of molecules 1 and 2. Since  $L_{n_1, n_2} = L_{n_1}' + L_{n_2}'$  we have for the heat of mixture by means of the above equations the expression

$$\left\{ \rho_3 \frac{\delta V}{\delta c_1} \frac{n_2}{M_1} \right\} \left[ p_2' - T \frac{dp_2'}{dT} \right] + \left\{ \frac{n_1}{\rho_4} - \rho_3 \frac{\delta V}{\delta c_1} \frac{n_1}{M_1} \right\} \left[ T \frac{dp_1}{dT} - p_1 \right]$$

$$+ \left\{ \rho_3 \frac{\delta V}{\delta c_2} \frac{n_1}{M_2} \right\} \left[ p_1' - T \frac{dp_1'}{dT} \right] + \left\{ \frac{n_2}{\rho_5} - \rho_3 \frac{\delta V}{\delta c_2} \frac{n_2}{M_2} \right\} \left[ T \frac{dp_2}{dT} - p_2 \right]$$

$$- \left\{ -H_1 \rho_4^{4/3} + A_1 \rho_1^{4/3} \right\} \frac{n_1}{M_1^{7/3}} (\Sigma \sqrt{m_1})^2$$

$$- \left\{ -H_2 \rho_5^{4/3} + A_2 \rho_2^{4/3} \right\} \frac{n_2}{M_2^{7/3}} (\Sigma \sqrt{m_2})^2.$$

It should be observed that the functions  $H_1, A_1, H_2, A_2$ , have all the same form, namely  $\phi_3$ . The form of  $\phi_3$  depends on that of  $\phi_2$  in the law of attraction between molecules, but this is not exactly known. There is, however, some evidence that  $\phi_3$  varies only slightly with the temperature and distance of the molecules\*. In the absence of further information we may therefore take  $A_1, A_2, H_1, H_2$  equal to the same constant for a given temperature. If the values of  $A_1$  and  $A_2$  are available, it is best to take the mean of  $A_1$  and  $A_2$  to represent the constants  $A_1, A_2, H_1, H_2$ , as they have the same value at corresponding temperatures. Otherwise the value of this constant can be obtained from Table V. p. 796, Phil. Mag. May 1910; at a temperature of 20° C. in the case of ether it is equal to 3742. This will probably give fairly accurate results.

If the density of the mixture of saturated vapours is so small that the constituent vapours obey Dalton's law of partial pressures, we have  $p_1 = p_1'$  and  $p_2 = p_2'$ . The formula for the heat of mixture then becomes

$$\left\{ \frac{n_1}{\rho_4} - \left( \frac{n_1 + n_2}{\rho_3} \right) \right\} \left( T \frac{dp_1}{dT} - p_1 \right) + \left\{ \frac{n_2}{\rho_5} - \left( \frac{n_1 + n_2}{\rho_3} \right) \right\} \left( T \frac{dp_2}{dT} - p_2 \right) \\ - A_1 \rho_1^{4/3} \frac{n_1}{M_1^{7/3}} (\Sigma \sqrt{m_1})^2 - A_2 \rho_2^{4/3} \frac{n_2}{M_2^{7/3}} (\Sigma \sqrt{m_2})^2,$$

since

$$\rho_3 \frac{\delta V}{\delta c_1} \frac{n_1}{M_1} + \rho_3 \frac{\delta V}{\delta c_1} \frac{n_2}{M_2} = \frac{n_1 + n_2}{\rho_3}.$$

When this is not the case there would be some difficulty in practice in obtaining the values of the quantities  $p, p_1', p_2, p_2'$ .

A special case of mixture which is of interest is a saturated solution of a salt in a liquid. To obtain a formula for the heat of mixture let the piston  $a$  in the figure be removed and the semi-permeable piston  $b$  replaced by a solid one. Let the piston  $c$  be moved away from the mixture till  $n_1$  grms. of liquid 1 have evaporated;  $n_2$  grms. of salt will then be deposited. From Clapeyron's equation we then have

$$L n_1' - L n_1'' + p_1 \left[ \frac{n_1}{\rho_4} - \left( \frac{n_1 + n_2}{\rho_3} - \frac{n_2}{\rho_2} \right) \right] \\ = T \left[ \frac{n_1}{\rho_4} - \left( \frac{n_1 + n_2}{\rho_3} - \frac{n_2}{\rho_2} \right) \right] \frac{dp_1}{dT},$$

\* *Loc. cit.*

where

$$\left( \frac{(n_1 + n_2)}{\rho_3} - \frac{n_2}{\rho_2} \right)$$

is the change in volume of the liquid and salt. The heat of mixture then is

$$\left[ \frac{n_1}{\rho_4} - \left( \frac{(n_1 + n_2)}{\rho_3} - \frac{n_2}{\rho_2} \right) \right] \left( T \frac{dp_1}{dT} - l_1 \right) + \frac{n_1 H_1}{M_1} \left( \frac{\rho_4}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2 \\ - \frac{n_1 A_1}{M_1} \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2.$$

It should be observed that the latent heat of evaporation  $L_1'$  of a grm. of liquid 1 into a vacuum, which was expressed by

$$\frac{A_1}{M_1} \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

may be obtained in a different form from Clapeyron's equation

$$L_1' - L_1'' + p \left( \frac{1}{\rho} - \frac{1}{\rho_1} \right) = T \left( \frac{1}{\rho} - \frac{1}{\rho_1} \right) \frac{dp}{dT},$$

where  $\rho$  is the density of the saturated vapour and  $p$  its pressure. The value of  $L_1''$  in the equation may be expressed in the form

$$A_1' \left( \frac{\rho}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2.$$

Another case of mixture of substances which is of interest is a saturated solution of a liquid—say of molecules 1—in a liquid of molecules 2. To obtain a formula for the heat of mixture in this case, let the piston  $b$  in the figure be replaced by one which is solid. Let the pistons  $c$  and  $a$  be moved in the same way as before till  $n_1$  grams of molecules 1 have evaporated from the mixture: the corresponding  $n_2$  grams of molecules 2 will form with a part of the mixture a saturated solution of molecules 2 in the liquid 1. Let  $\rho_3'$  denote the density of this solution, and suppose the ingredients are in the proportion of  $n_1'$  to  $n_2'$ . If  $x$  denotes the number of grams of the mixture which combine with the  $n_2$  grms. of molecules 2, we have

$$\frac{x n_1 / n_1 + n_2}{n_2 + x n_2 / n_2 + n_2} = \frac{n_1'}{n_2'}.$$

The change in volume of the liquid phase during the evaporation is therefore

$$(n_1 + n_2 + x) \frac{1}{\rho_3} - (v + n_2) \frac{1}{\rho_3'} = v \text{ say.}$$

From Clapeyron's equation we then have

$$L_{n_1}' - L_{n_1}'' + p_1 \left( \frac{n_1}{\rho_4} - v \right) - p_2' v = T \left\{ \left( \frac{n_1}{\rho_4} - v \right) \frac{d\rho_1}{dT} - v \frac{d\rho_2'}{dT} \right\},$$

where  $L_{n_1}''$ , as before, is given by

$$\frac{n_1 H_1}{M_1} \left( \frac{\rho_4}{M_1} \right)^{4/3} (\Sigma \sqrt{m_2})^2.$$

If the heat of mixture of a saturated solution of molecules 1 in 2 per gram of mixture be denoted by  $H_t$ , and that of a saturated solution of molecules 2 in 1 per gram of mixture by  $H_t'$ , we have

$$H_t (n_1 + n_2 + x) = L_{n_1}' - \frac{n_1}{M_1} A_1 \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2 + H_t' (n_2 + x),$$

where  $x$  and  $L_{n_1}'$  are given by the above equations. This equation expresses the relation between the quantities  $H_t$  and  $H_t'$ . If the molecules 2 instead of 1 be allowed to evaporate the above equations apply if the suffixes 1 and 2 and the symbols  $H_t$  and  $H_t'$  are interchanged, and  $\rho_5$  is written for  $\rho_4$ . The last equation thus becomes

$$H_t' (n_1 + n_2 + x) = L_{n_2}' - \frac{n_2}{M_2} A_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2 + H_t (n_1 + x).$$

The elimination of  $H_t'$  from the above two equations gives an expression for the value of  $H_t$ . The value of either  $H_t$  or  $H_t'$  may also be obtained by means of one of the formulæ given previously.

It will be observed that some of the formulæ for the heat of mixture obtained apply whatever the changes are that take place when the substances are mixed, such as the formation of new molecules, &c. Both types of formulæ obtained are perfectly general, and therefore apply when one or both of the substances are in the gaseous state.

It is unnecessary to develop formulæ for the mixture of three or more liquids, as these can now be developed without difficulty along the lines indicated.

*Volume of Occupation of a Molecule in a Mixture.*

The decrease in volume of a large mass of a mixture of substances on removing from it a single molecule, may be called the volume of occupation of the molecule in the mixture. This quantity it appears has not yet been definitely defined and discussed; it seems of importance, and a consistent study of it should lead to interesting results. It is intimately connected with the relative distribution of the molecules in a mixture. Using the same notation as before, and denoting the volume of occupation of a molecule 1 by  $\mathfrak{S}_1$  we have  $\mathfrak{S}_1 = \rho_3 \frac{\delta V}{\delta c_1}$ . We have further in the case of a mixture of molecules 1 and 2 in the proportion of  $n_1$  to  $n_2$  that

$$\mathfrak{S}_1 \frac{n_1}{M_1} + \mathfrak{S}_2 \frac{n_2}{M_2} = \frac{n_1 + n_2}{\rho_3}, \text{ or } \frac{\mathfrak{S}_1}{M_1} + \frac{\mathfrak{S}_2 \alpha}{M_2} = \frac{1 + \alpha}{\rho_3},$$

where  $\alpha$  denotes the ratio of the concentration of the molecules 2 to that of 1.

We have seen that the decrease in volume of a mixture when a molecule 1 is removed is  $\mathfrak{S}_1$ , therefore if  $P_{1,2}$  denote the intrinsic pressure of a liquid, *i. e.* the pressure due to the attraction between the molecules,  $P_{1,2} \mathfrak{S}_1$  is approximately the work done on the molecule by the mixture during its removal. If the temperature of the mixture is kept constant during the process, an equivalent amount of energy in the form of heat has to be supplied to the mixture. This amount of work is equal to the internal heat of evaporation  $L_1'$  of a molecule 1 into a vacuum, or equal to the ordinary heat of evaporation if the density of the saturated vapour is very small in comparison with that of the liquid. In the case of a mixture we have therefore at low temperatures  $L_1' = P_{1,2} \mathfrak{S}_1$  and  $L_2' = P_{1,2} \mathfrak{S}_2$ . From these two equations we have

$$L_1'/L_2' = \mathfrak{S}_1/\mathfrak{S}_2 = \frac{\delta V}{\delta c_1} / \frac{\delta V}{\delta c_2},$$

or the internal heats of evaporation of two molecules 1 and 2 are to one another as their volumes of occupation in the mixture.

When the density of the saturated vapour is not small in comparison with that of the liquid the value of  $L_1''$ , the heat of evaporation of a molecule 1 of the vapour into a vacuum, is not negligible in comparison with that of  $L_1'$ , and we have for the ordinary heat of evaporation  $L_1' - L_1'' = P_{1,2} \mathfrak{S}_1 - P'_{1,2} \mathfrak{S}'_1$ , where  $L_1'' = P'_{1,2} \mathfrak{S}'_1$ ,  $P'_{1,2}$  denoting the intrinsic



pressure of the mixture of saturated vapours and  $\mathfrak{S}_1'$  the volume of occupation of a molecule 1 in the vapour. Similarly we have for the heat of evaporation of a molecule 2,  $L_2' - L_2'' = P_{1,2} \mathfrak{S}_2 - P'_{1,2} \mathfrak{S}_2'$ , where  $L_2'' = P'_{1,2} \mathfrak{S}_2'$ .

If the temperature of a mixture is gradually increased, a temperature will ultimately be reached when  $L_1' - L_1'' = 0$ , in which case  $P_{1,2} \mathfrak{S}_1 = P'_{1,2} \mathfrak{S}_1'$ . Similarly,  $L_2' - L_2''$  will pass through zero for some temperature, when we have  $P_{1,2} \mathfrak{S}_2 = P'_{1,2} \mathfrak{S}_2'$ . If both the expressions pass through zero

at the same temperature we have  $\frac{\mathfrak{S}_1}{\mathfrak{S}_2} = \frac{\mathfrak{S}_1'}{\mathfrak{S}_2'}$ . Now this

equation would also apply when the relative concentrations of the different molecules in the liquid mixture and the saturated vapour are the same, since, as we have already remarked, the relative distribution of the molecules should then be the same. It appears, therefore, that when the internal heats of evaporation of the molecules of a mixture pass through zero at the same temperature the relative concentration of the different molecules in the vapour and the liquid must be the same. Further, since we then have  $L_1' = L_1''$  and  $L_2' = L_2''$  the density of the vapour and liquid must be the same.

A saturated solution of molecules 1 in a liquid of molecules 2 has the same partial vapour pressures as a saturated solution of molecules 2 in a liquid of molecules 1, since they remain in equilibrium in contact with one another. It follows, therefore, from Clapeyron's equation that at low temperatures, when the volume of the vapour of a grm. of molecules of the same kind is large in comparison with the corresponding volume of the mixture, the heat of evaporation of a molecule of the same kind is the same for each mixture. Therefore, if  $P'$  is the intrinsic pressure of one of the mixtures and  $V_1'$  and  $V_2'$  the volumes of occupation of the molecules 1 and 2, and the corresponding quantities for the other mixture are  $P''$ ,  $V_1''$ ,  $V_2''$ , we have  $P'V_1' = P''V_1''$ , and  $P'V_2' = P''V_2''$ . From these equations we have  $\frac{V_1'}{V_2'} = \frac{V_1''}{V_2''}$ ,

or the ratio of the volumes of occupation of the two kinds of molecules is the same in each mixture; further, we have  $\frac{P'}{P''} = \frac{V_1''}{V_1'}$ , or the intrinsic pressure of each of the mixtures is

inversely proportional to the volumes of occupation of one of the molecules.

*Test for the Formation of New Compounds in a Mixture.*

If two quantities of different liquids containing an equal number of molecules be mixed and no new molecules are formed, the relative distribution of the molecules in the mixture can be deduced as we have seen. The functions  $B_1$ ,  $B_2$ ,  $C_1$ , and  $C_2$ , in the formula for the heat of mixture given, can then at once be formed if the form of the function  $\phi_2$  in the law of attraction is known. It will then be most convenient to express the relative distribution of the molecules in terms of that of the molecules of one kind in the mixture.

We have pointed out that there is some evidence that the function  $\phi_3$  does not depend very much on the magnitude of the variables it contains. If we assume it approximately a constant an approximate formula for the heat of mixture can be obtained. Let us first find the value of  $L_{n_1 n_2}$  on this supposition. When a molecule 1 is removed from the mixture the work done against the attraction of the remaining molecules 1 is

$$u_1 \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2, \quad \text{where} \quad u_1 = \phi_3 \left( \frac{x_e}{x_{c_1}}, \frac{T}{T_{c_1}} \right),$$

$\rho_1$  denotes the density of the molecules 1 in the mixture, and  $x_e$  denotes their distance of separation. This is at once obtained by supposing the molecules 2 absent, when we are dealing with a liquid in which the molecules are evenly distributed as in a pure liquid, and a corresponding formula applies. The work done against the attraction of the molecules 2 in removing the molecule 1 is obtained as follows. Suppose the molecules 2 replaced by molecules 1. The work done against the attraction of all the molecules is then

$$W_1 \left( \frac{\rho_3 2}{M_1 + M_2} \right)^{4/3} (\Sigma \sqrt{m_1})^2, \quad \text{where} \quad W_1 = \phi_3 \left( \frac{x_{r_1}}{x_{c_1}}, \frac{T}{T_{c_1}} \right),$$

and  $x_{r_1}$  denotes the mean distance of separation of the molecules. The term which is raised to the 4/3 power is equal to  $\frac{1}{x_{r_1}}$ ; we have expressed this quantity in terms of quantities relating to the mixture. The work done against the molecules 1 which replaced the molecules 2 is therefore

$$W_1 \left( \frac{\rho_3 2}{M_1 + M_2} \right)^{4/3} (\Sigma \sqrt{m_1})^2 - u_1 \left( \frac{\rho_1}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

as is at once evident from the nature of the distribution of the molecules. The work done against the molecules 2 if they

are again put into their place is the above expression multiplied by  $\frac{\Sigma \sqrt{m_1} \Sigma \sqrt{m_2}}{(\Sigma \sqrt{m_1})^2}$ , and substituting  $x_{c_2}$  and  $T_{c_2}$  for certain of the quantities  $x_{c_1}$  and  $T_{c_1}$  occurring in  $W_1$  and  $u_1$ . This will at once be evident if we consider that this expression is supposed to be derived by a direct application of the law of attraction between two molecules 1 and 2, viz.

$$\phi_a\left(\frac{z}{x_{c_1}}, \frac{T}{T_{c_1}}\right) \cdot \phi_a\left(\frac{z}{x_{c_2}}, \frac{T}{T_{c_2}}\right) \cdot \Sigma \sqrt{m_1} \Sigma \sqrt{m_2}.$$

But since  $u_1$  and  $W_1$  are each of the form  $\phi_3$ , and therefore do not depend much on the magnitude of the variables they contain, we may suppose both these quantities equal to a constant. The work done when a molecule 1 is removed from a mixture against the attraction of the remaining molecules 1 and 2 is therefore

$$\left\{ W_1 \left( \frac{\rho_3^2}{M_1 + M_2} \right)^{4/3} - u_1 \left( \frac{\rho_7}{M_1} \right)^{4/3} \right\} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} + u_1 \left( \frac{\rho_7}{M_1} \right)^{4/3} (\Sigma \sqrt{m_1})^2,$$

where  $W_1 = u_1 = \text{constant}$  approximately. If the density of the molecules 1 in the mixture is expressed in terms of that of the mixture we have  $\rho_7 = \frac{\rho_3 M_1}{M_1 + M_2}$ . Similarly the work done when a molecule 2 is removed is

$$\left\{ W_2 \left( \frac{\rho_3^2}{M_1 + M_2} \right)^{4/3} - u_2 \left( \frac{\rho_8}{M_2} \right)^{4/3} \right\} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} + u_2 \left( \frac{\rho_8}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2,$$

where  $W_2 = u_2 = W_1 = u_1 = \text{a constant}$  approximately, and  $\rho_8 = \frac{\rho_3 M_2}{M_1 + M_2}$ .

This constant at 20° C. is equal to 3742, and the heat of mixture of  $n_1$  gram of molecules 1 with  $n_2$  grams of molecules 2 so that  $\frac{n_1}{M_1} = \frac{n_2}{M_2}$ , is at that temperature therefore given by

$$3742 \left\{ \left( \frac{n_1}{M_1} + \frac{n_2}{M_2} \right) \left( \frac{\rho_3}{M_1 + M_2} \right)^{4/3} 1.53 \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} \right. \\ \left. + \left( \left( \frac{\rho_3}{M_1 + M_2} \right)^{4/3} - \left( \frac{\rho_1}{M_1} \right)^{4/3} \right) \frac{n_1}{M_1} (\Sigma \sqrt{m_1})^2 \right. \\ \left. + \left( \left( \frac{\rho_3}{M_1 + M_2} \right)^{4/3} - \left( \frac{\rho_2}{M_2} \right)^{4/3} \right) \frac{n_2}{M_2} (\Sigma \sqrt{m_2})^2 \right\},$$

the molecular weights of  $M_1$  and  $M_2$  being expressed relative to that of hydrogen.

As a test for the formation of new molecules in a mixture of two substances we have then that if it is found that the heat of mixture of the substances in masses proportional to their molecular weights differs considerably from that calculated by the above or previous formula 2, we may conclude that new molecules are formed. The extent of the formation of new molecules in a mixture of two liquids in any given proportion will probably depend, however, on the proportion between the constituents.

The various formulæ for the heat of mixture of liquids all labour under the unavoidable disadvantage of appearing as the difference between quantities which are usually very large in comparison with this difference, which has the effect of magnifying all errors, and thus usually preventing a good agreement with the facts being obtained. If one of the substances is in the form of a vapour the formulæ suffer less from this defect. This is a point to be borne in mind when the formulæ for the heat of mixtures are used.

A simpler test than the above for the formation of new molecules in a mixture of equal numbers of different molecules is the following. It will be evident, from the relative distribution of the molecules in such a case (which we have discussed), that the volume of occupation is the same for each kind of molecule. It follows, therefore, that if  $nc$  new

molecules are formed we must have  $\frac{\delta V}{\delta c_1} = \frac{\delta V}{\delta c_2}$ . This formula

involves quantities which can be easily measured. In using this formula in practice, however, we are hampered by the difficulty—which occurs also in connexion with the previous formulæ—of not being sure of the molecular concentration of the liquids used, as these may be polymerized to some extent. Water is an example: it has been shown to be polymerized from surface-tension and other considerations, and consists probably of a mixture of molecules polymerized in different degrees. We are therefore not able to use the extensive data on the solution of substances in water in connexion with the above formula, till we possess more reliable information on the size of a water molecule. On the other hand, the data relating to other solvents is not sufficiently extensive to be of any use. But experiments having the object of testing the formula could be easily and rapidly carried out.

The above considerations suggest a slightly different way of testing for the formation of new molecules. Since the internal heat of evaporation of the molecules 1 and 2 when

the vapour pressure is small may be written  $P_{1,2} \cdot \frac{\delta V}{\delta c_1} \rho_3$  and  $P_{1,2} \cdot \frac{\delta V}{\delta c_2} \rho_3$  respectively, and we have  $\frac{\delta V}{\delta c_1} = \frac{\delta V}{\delta c_2}$  when no new molecules are formed, the internal heats of evaporation are equal to one another in such a case. This can only be approximately true, however (we should have pointed out before), since the intrinsic pressure per cm.<sup>2</sup> when we consider its effect upon such a small surface as that presented by the volume of occupation of a molecule, must depend somewhat on the area of this surface. The heats of evaporation could be obtained from the vapour-pressure curves of the mixture by means of Clapeyron's equation and compared with one another, and information thus obtained as to whether or no new molecules are formed in the mixture.

*Test whether in a dilute solution of molecules 1 in a liquid of molecules 2, a molecule 1 simply replaces and occupies the same position as a molecule 2.*

Suppose a mass of gas of molecules 1 which is at low pressure dissolves in a liquid of molecules 2 forming a dilute solution. Suppose that a dissolved molecule 1 simply replaces a molecule 2 and occupies its position. Now the heat of evaporation of a molecule 2 of the pure liquid into a vacuum we have seen is

$$A_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2, \quad \text{where} \quad A_2 = \phi_3 \left( \frac{x_{c_2}}{T_{c_2}}, \frac{T}{T_{c_2}} \right).$$

The heat of evaporation of a molecule 1 is therefore the above expression multiplied by  $\frac{\Sigma \sqrt{m_1} \Sigma \sqrt{m_2}}{(\Sigma \sqrt{m_1})^2}$ , and substituting

$x_{c_1}$  and  $T_{c_1}$  for certain of the quantities  $x_{c_2}$  and  $T_{c_2}$  occurring in  $A_2$ . But since  $A_2$  has the form  $\phi_3$  it does not vary much with the magnitude of the variables it contains, and we may therefore as before take it approximately equal to a constant. If the values of  $A_1$  and  $A_2$  in the expressions for the internal heat of evaporation of liquids 1 and 2 are available, this constant is best taken equal to  $\frac{A_1 + A_2}{2}$ . If these values are not

available the constant may be obtained from some single liquid, as explained before. At a temperature of 20° C. the heat of evaporation of a grm.-mol. of molecules 1 would be approximately equal to

$$\underline{3742} \left( \frac{\rho_2}{M_2} \right)^{4/3} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2} \text{ calories,}$$

and this is approximately equal to the heat of solution at that temperature. If it is found that the heat of solution is equal to this value, we may conclude that probably every molecule 1 replaces a molecule 2 in the mixture.

Let us apply these results to the heats of solution of a number of gases in large quantities of water. The internal heat of evaporation  $L_2$  of a grm.-mol. of water into a vacuum is given by

$$A_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2,$$

if the molecular weight of water is given by the chemical formula  $H_2O$ . But since the water molecules are polymerized we must multiply  $M_2$  and  $\Sigma \sqrt{m_2}$  each by some appropriate constant which expresses the degree of polymerization. The formula may therefore be written

$$L_2 = \mu A_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} (\Sigma \sqrt{m_2})^2,$$

where  $\mu$  is a constant. The heat of evaporation  $L_1$  of a grm.-mol. of molecules 1 from the solution is therefore

$$L_1 = \mu A_2 \left( \frac{\rho_2}{M_2} \right)^{4/3} \Sigma \sqrt{m_1} \Sigma \sqrt{m_2}.$$

From these two equations we have

$$L_1 = L_2 \frac{\Sigma \sqrt{m_1}}{\Sigma \sqrt{m_2}},$$

where  $L_1$  may now be taken as the heat of solution.

The following table contains the heats of solution per grm.-mol. of a number of gases in a large quantity of water at a temperature of  $18^{\circ}.2$  C. They were taken from Nernst's 'Theoretical Chemistry,' 4th edit. p. 599.

Gas.	Heat of Sol. per grm.-mol.	$L_2 \frac{\Sigma \sqrt{m_1}}{\Sigma \sqrt{m_2}}$	Gas.	Heat of Sol. per grm.-mol.	$L_2 \frac{\Sigma \sqrt{m_1}}{\Sigma \sqrt{m_2}}$
NH <sub>3</sub> ...	8,430	11,370	Cl <sub>2</sub> .....	4,870	20,060
HF .....	11,800	9,033	HCl ...	17,310	11,730
HBr ...	19,940	17,140	CO <sub>2</sub> ...	5,880	19,310
HI .....	19,210	20,650			

The third and sixth columns of the table contain the corresponding values of  $L_1$  calculated by means of the above equation. The internal heat of evaporation of a grm. of water at  $20^\circ$  C. is 561.5 cal., and the value of  $L_2$  therefore  $561.5 \times 18 = 10,110$ . It will be seen that there is a rough agreement between the two sets of values for the gases  $\text{NH}_3$ ,  $\text{HF}$ ,  $\text{HBr}$ ,  $\text{HI}$ . It seems, therefore, that a molecule of these gases replaces a water molecule on solution. This result is, however, not quite conclusive, as it may happen that hydrates are formed and the resultant molecules occupy such positions relative to the molecules of the solute that the total change in potential energy is the same as if no new molecules were formed and each dissolved molecule replaced a molecule of the solute.

The calculated heats of solution do not agree, however, with the calculated values in the case of  $\text{CO}_2$ ,  $\text{HCl}$ , and  $\text{Cl}_2$ . It is usually assumed that  $\text{CO}_2$  molecules do not change on solution in water, since the concentration obeys Henry's law. The nature of the disagreement with  $\text{CO}_2$  would then indicate that a  $\text{CO}_2$  molecule is on the average further away from the surrounding water molecules than a water molecule in pure water. But it also follows (referring to what has gone before) that the volume of occupation of a  $\text{CO}_2$  molecule is less than that of a water molecule. These two conclusions do not at first sight agree with one another. But it must be remembered that the volume of occupation of a molecule is the total change in the volume of the liquid when a molecule is removed, and it may therefore happen that when a molecule is removed from a mixture a change in the volumes of occupation of the neighbouring molecules takes place, most probably an increase, which changes the magnitude of the volume of occupation of the removed molecule considerably. These results combined, therefore, seem to indicate that a molecule of  $\text{CO}_2$  in water is surrounded by two shells of water molecules, the outside one being more dense and the inside one less dense than pure water. In the case of the other two gases  $\text{Cl}_2$  and  $\text{HCl}$ , all we can say is that a dissolved molecule either does not occupy a similar position as a molecule of water or a hydrate is formed.

Cambridge, Feb. 1, 1911.

LXI. *On Metallic Colouring in Birds and Insects.*  
*By A. A. MICHELSON\*.*

[Plate IV.]

WITH the exception of bodies which shine by their own light, the appearance of colour in natural objects is due to some modification which they impart to the light which illuminates them. In the great majority of cases this modification is caused by the absorption of part of the light which falls on the object, and which, penetrating to a greater or smaller depth beneath the surface, is reflected, and finally reaches the eye. If the proportion of the various colours constituting white light which is absorbed by the illuminated body is the same for each, the light which reaches the eye has the same composition as before, and we say the body itself is white; but if this proportion be different, the resulting light is coloured, and the colour of the body itself corresponds to that colour or colour combination which is least absorbed; it is complementary to the colours which are most strongly absorbed.

Thus the light from a green leaf in the sunshine, after penetrating a short distance in the substance of the leaf, is either transmitted or reflected to the eye. In its passage through the substance of the leaf it has lost a considerable part of the red light it originally contained, and the resulting combination of the remaining colours produces the effect of the complementary colour or green, as can readily be shown by analysing the light into its elementary colours by a prism, and comparing the resulting spectrum before and after the reflexion from the leaf.

The same explanation holds for all the paints, dyes, and pigments in common use.

These colour effects occur in such an immensely greater proportion than all others combined, that the occasional appearance of an exception is all the more striking. The rainbow and the various forms of halo are almost the only instances of prismatic colours which occur in nature.

There remain only two other possible methods of producing colour.

- A. Interference, including Diffraction.
- B. "Metallic" Reflexion.

It has been abundantly proved that the usual "flat," "dead," "uniform" colouring, brilliant as this sometimes

\* Communicated by the Author.



can be, *e. g.*, in birds, butterflies, and flowers, finds its simple explanation in the existence of pigment cells; so that the same cause (doubtless with many modifications) is here effective as in the great majority of cases previously considered. But the lively, variable "metallic" glitter of burnished copper or gold; the reflexion from certain aniline dyes; the colours of certain pigeons, peacocks, humming-birds, as well as a number of butterflies, beetles, and other insects, requires another explanation.

While cases under A occur occasionally in nature—for example, in the colours of thin films, in the iridescence of mother of pearl, and (as an accessory) in the colours of the rainbow and of certain halos—they are so rare and so readily distinguished from the true metallic colours that they may be most conveniently treated as exceptions after the subject of surface-colour has been considered.

The designation "metallic" at once suggests that there may be some common property of all these colours which is typified by the metals themselves. But, as is well known, the principal characteristic which distinguishes the metals from all other substances in regard to their action on light, is their extraordinary opacity.

A very important consequence of such great opacity is that light is practically prevented from entering the substance at all, but is thrown back, thus giving the brilliant metallic reflexion so characteristic of silver, gold, copper, &c. In fact, the distance to which light can penetrate in most metals is only a small fraction of a light-wave; so that a wave-motion such as constitutes light, strictly speaking, cannot be propagated at all. Again, as this opacity may be different for different colours, some would be transmitted more freely than others, so that the resulting transmitted light would be coloured; and the reflected light would be approximately complementary to the transmitted colour.

For most metals the difference is not very great; so that the reflected light, except in the case of gold and copper and a few alloys, is nearly white. In the case of the aniline dyes, however, there is a marked difference, as is clearly shown by their absorption spectrum. In transmitted light, even a very small thickness of fuchsine shows no yellow, green, or blue, and gives as a resultant of the remaining colours a beautiful crimson. The light which it reflects, however, is just this yellow and green which it refuses to transmit, and it accordingly shimmers with a metallic golden green colour, which changes when the surface is inclined, becoming full

green, or even bluish green when the illumination is sufficiently oblique\*.

The chief characteristics by which "metallic" reflexion may be distinguished may be summarized as follows:—

1. The brightness of the reflected light is always a large fraction of the incident light, varying from 50 per cent. to nearly 100 per cent.

2. The absorption is so intense that metal films are quite opaque even when their thickness is less than a thousandth of a millimetre.

3. If the absorption varies with colour, that colour which is most copiously transmitted will be the part of the incident white light which is least reflected—so that the transmitted light is complementary to the reflected.

4. The change of colour of the reflected light with changing incidence has already been mentioned. It follows the invariable rule that the colour always approaches the violet end of the spectrum as the incidence increases. If the colour of the normal reflexion is violet the light vanishes (changing to ultra-violet), and if the normal radiation be infra-red it passes through red, orange, and yellow as the incidence increases.

While the criteria just considered are the simplest and most convenient for general observation, it is to the more rigorous results of more refined optical methods that we must look for the final test of the quality of reflexion in any given case; to determine whether or not a colour phenomenon may be due to "metallic" reflexion or to one of the other general causes.

Such optical tests are furnished by the effect of reflexion upon polarized light. The elements of the resulting elliptic vibrations may be expressed in terms of the amplitude ratio  $R$  of the components, and of the phase difference  $P$  corresponding to the angle of incidence  $I$ , as in the following tables for silver and for glass.

The very marked difference in the run of the numbers in these tables may be rendered still more striking by plotting the values as ordinates of the curves shown in Pl. IV. fig. 1, which gives at a glance the form of the "phase" curve (full line) and the "amplitude" curve (dotted line) for silver, steel, graphite, selenium, flint glass, crown glass, and quartz. It is evident

\* The change in colour is very much more marked when the light is polarized perpendicularly to the plane of incidence. As the angle of incidence approaches the angle corresponding to the "polarizing angle" the colour is a deep blue or even purple.

Silver.			Glass.		
I.	P.	R.	I.	P.	R.
0°	·00	45	0°	·00	45
10°	·01	45	10°	·00	40
20°	·02	44	20°	·00	30
30°	·03	43	30°	·00	25
40°	·04	41	40°	·00	20
50°	·05	40	50°	·00	12
60°	·10	38	60°	·50	7
70°	·20	37	70°	·50	25
80°	·30	40	80°	·50	40
90°	·50	45	90°	·50	45

that metals have a smoother phase curve than semi-metallic substances like graphite and selenium, and these show less abrupt changes than do transparent substances such as glass and quartz.

In fact we may take the steepness of the curve where it is steepest (better where the phase difference is  $1/4$ ) as a measure of the transparency of the substance; and theory shows that this *steepness* is in fact inversely proportional to the *absorbing power* of the substance.

Starting with the formula of Cauchy

$$\tan \Delta = \frac{2k \sin^2 i \cos i}{(\nu^2 \cos^2 r + k^2) \cos^2 i - \sin^4 i},$$

differentiating for  $i$  and putting in the resulting expression  $\Delta = \pi/2$ , and  $I$  for the corresponding angle of incidence, we have

$$\frac{d\Delta}{di} = \frac{\sin I(\tan^2 I + 2)}{k} \quad (k = \text{coeff. of absorption}).$$

On the other hand, if the phase change be the result of a *surface film*, and we start with the corresponding approximate formula

$$\tan \Delta_1 = \epsilon \tan (i + r)$$

we find

$$\frac{d\Delta_1}{di} = \frac{1}{\epsilon \sin^2 I} \quad (\epsilon = \text{coeff. of ellipticity}).$$

In this case the steepness is inversely proportional to what Jamín has termed the "ellipticity."

In point of fact both causes are effective; and for semi-transparent substances it is impossible to obtain results which agree even approximately with experiment by either formula. But the rigorous expression of Cauchy, which contains both  $k$  and  $i$ , is so unwieldy to be practically useless.

The difficulty may be obviated by making use of the empirical relation  $E = E_e + E_k$ , where  $E = \frac{1}{d\Delta/di}$ ; which may be translated to mean that the actual "ellipticity" is made up of two parts which combine additively; one due to the surface film and the other resulting from absorption.

If the medium under observation be transparent  $E_k = 0$ , hence  $E_e = E$ . If it be opaque  $E_e$  is small compared with  $E_k$ , so that approximately  $E_k = E$ .

For semitransparent media it will be necessary to determine the absorption,  $k$ , by direct measurement, from which  $E_k$  may be calculated by means of the formula

$$E_k = \frac{k}{\sin I(\tan^2 I + 2)},$$

and  $E_e$  may then be determined by

$$E_e = E - E_k.$$

In the case of substances like fuchsine and diamond green, in which the medium is almost perfectly transparent for certain colours, we may find  $E_e$  for this colour; and if it be correct to assume that  $E_e$  does not vary with the colour, the value of  $E_k = E - E_e$  may be accurately determined for the semi-transparent and the opaque portions of the spectrum.

A fairly good test is that furnished by selenium. The incidence  $I$  corresponding to  $\Delta = 90^\circ$  is nearly independent of the colour, being about  $71^\circ$ .

The value of  $\frac{d\Delta}{di}$  calculated by the preceding formula is, very nearly,

$$\frac{d\Delta}{di} = \frac{1}{E_k} \cdot \frac{10}{k}.$$

Following are the values of  $\frac{d\Delta}{di}$ ,  $E$ ,  $E_k$ , and  $k^*$ .

\* These last are taken from the results of Professor Wood, *Phil. Mag.* 1902, vol. iii.

Selenium.

$\lambda$ .	$\frac{d\Delta}{di}$ .	E.	$E_k$ .	$10E_k$ .	$k$ .
6950	23	·043	·000	·000	·05
6410	21	·048	·005	·065	·09
6075	18	·056	·013	·13	·13
5745	16	·062	·019	·19	·20
5410	13	·077	·034	·34	·28
5070	11	·091	·048	·48	·40
4740	9	·111	·068	·68	·53
4405	8	·125	·082	·82	·61

Following are similar series of observations for fuchsine and for diamond green.

Fuchsine.

$\lambda$ .	$\frac{d\Delta}{di}$ .	E.	$10E_k$ .	$\log \frac{1}{R}$ .
670	50	·020	·00	·04
640	40	·025	·05	·05
615	18	·055	·35	·38
590	0	·11	·90	·75
560	7	·14	1·2	1·4
525	6	·16	1·4	1·7
500	4·5	·22	2·0	1·5
470	4·5	·22	2·0	1·0

Diamond Green.

$\lambda$ .	$\frac{d\Delta}{di}$ .	E.	$10E_k$ .	$\text{Log} \frac{1}{R}$ .
700	12	·08	0·7	0·8
680	8	·12	1·1	1·1
660	7	·14	1·3	1·3
640	5	·20	1·9	1·5
620	4	·25	2·4	1·5
600	4	·25	2·4	1·2
580	4	·25	2·4	1·0
560	6	·16	1·5	·65
540	25	·04	0·3	·23
520	60	·016	0·06	·08
500	80	·012	0·09	·02
480	70	·014	0·04	·01
460	45	·022	0·12	·25
440	15	·007	0·57	·40

$$R = \frac{I_0}{I} = \frac{\text{incident light}}{\text{transmitted light}} *$$

\* No account was taken of the loss by reflexion.

The quantities in the last column are proportional to  $k$ ; but the actual values of  $k$  thus deduced from observations of transmitted light are considerably less (about  $1/3$  of the value given), possibly because of the unevenness of the film which makes the measurement of the actual thickness (of the order of one thousandth of a millimetre) uncertain.

The agreement in the last two columns of the tables, while somewhat imperfect, is still enough to show that the results are of the right order of magnitude—and if it be remembered that the properties of the specimens vary considerably with the method of preparation, it is probable that the outstanding differences may be thus accounted for.

In any case the agreement is much better than it would be, had the ellipticity been attributed to absorption alone.

In the aniline colours the absorption varies enormously with the colour, and we have all the gradations from metallic reflexion to almost perfect transparence combined in a single specimen. The measurement of the phase-change and the amplitude-ratio for these substances show changes in the form of the curves almost identical with those given in the preceding figure.

Pl. IV. fig. 2 shows the curves obtained for fuchsine and for diamond green. It may be noted that in both these figures the "phase" curve is much more characteristic in its changes than the "amplitude" curve.

These specimens are prepared by dissolving the aniline colour in hot alcohol, filtering hot, and covering a hot glass surface with the solution. The alcohol evaporates rapidly, leaving a mirror surface of thickness of the order of a thousandth of a millimetre.

A quite remarkable alteration occurs in the phase curves when the solution is diluted. The film deposited is then very much thinner than before (from one-tenth to one one-hundredth of the former thickness) and, for some colours, the thickness is so small that considerable light is reflected from the surface of the glass. The resulting phase curve may then be *negative*, as shown in Pl. IV. fig. 3, for the colours red, orange, and yellow\*.

Such a result has been predicted from theoretical considerations†, but so far as I am aware, no attempt has been made to show that this depends on the colour of the incident

\* The lower curves show more clearly how the maximum value of  $k$  varies with the colour.

† Drude, 'Theory of Optics,' p. 294.

light. This, however, follows, if we consider that the condition for such a negative phase curve is that the transition layer have an index of refraction greater than that of the second medium; and as the refractive index for magenta is low at the blue end of the spectrum and high at the red end, the inversion of sign is strictly in accord with the theory, of which indeed it furnishes a striking confirmation.

On applying the simpler general tests of metallic reflexion to the case of iridescent plumage of birds, scales of butterflies, and wing-cases of beetles, one is at once struck with the close resemblance these bear to the aniline colours, in every particular: for

1st. The intensity of the reflected light is much greater than for the "non-metallic" plumage, &c., in some cases approaching the value of the reflexion factor of the metals themselves.

2nd. The reflected light is always coloured, showing either a rapid change of index of refraction, or of coefficient of absorption with the wave-length or colour; and, indeed, it may perhaps be objected that these colours are far more vivid than any of the reflexion hues of the aniline dyes, or of any other case of "surface colour" hitherto observed.

3rd. In the cases which could be investigated for this relation (unfortunately rather few) the transmitted light is approximately complementary to that which is reflected.

4th. The change of colour with changing incidence strictly follows the law already mentioned—the colour always changing towards the blue end of the spectrum as the incidence increases.

This remarkable agreement has been pointed out by Dr. B. Walter in an admirable essay, "Die Oberflächen- oder Schiller-Farben," and it is shown that none of the other causes of colour phenomena (in particular interference and diffraction) can be effective; the laws which govern these last being totally different. It is, therefore, somewhat surprising to find that the contrary view is still held by many eminent naturalists, and it is hoped that the further evidence here presented may serve to emphasize the distinction between "metallic" or "surface" colour and the remaining classes of colour (due to pigments, interference, and diffraction).

In attempting to apply the more rigorous optical test of the measurement of the phase-difference and amplitude-ratios, one is met at the outset with the serious difficulty of the absence of true "optical" surface. In fact, the materials we

have to deal with (feathers, butterfly scales, beetle wing-cases) are so irregular that the quantity of "regularly" reflected light which is brought to a focus by the observing telescope is insignificant, and is often masked by the light diffusely reflected. But by the simple device of replacing the objective of the collimator and of the observing telescope by low-power microscope objectives of small aperture, these difficulties are so far removed that it has been possible to obtain results which compare favourably with those obtained with the aniline films. In some of the measurements it has been found possible to deal with a single butterfly scale; and in these the irregularities of the surface were often insignificant, or of such nature that they could be taken into account.

Following is a diagram showing the results of a set of measurements on a beetle having a lustre resembling burnished copper. Beside it is a duplicate of the preceding observations on a *thin* film of magenta (Pl. IV. fig. 4).

The correspondence between the two sets of curves is so remarkable that it leaves no room to doubt that in this case the metallic coppery colour of the wing-case is due to an extremely thin film of some substance closely analogous in its optical qualities to the corresponding aniline dye\*. The thickness of the magenta film was not very accurately determined, but from the fact that it was deposited from a solution of 1/20 of the concentration of that which produced the corresponding *thick* film (whose thickness is about 0.005 mm.), it is estimated that the thickness of the *thin* film is of the order of 0.00025 mm. It is, doubtless, unsafe from this to draw any more definite conclusion regarding the film of the wing-case, than to say it is probably of the same order.

An attempt was made to check this estimate by the following simple device.

A portion of the ellipsoidal wing-case of mean radius  $R$  was removed by passing over it very lightly a piece of the finest emery-paper fastened to a flat piece of wood. This left a clean elliptical hole of mean radius  $r$  showing the edges of the "metallic" film, whose width,  $h$ , could not be appreciated in a microscope with a half-inch objective. If this be estimated at less than 0.001 mm. the relation  $\frac{t}{h} = \frac{r}{R} = \frac{0.5 \text{ mm.}}{5 \text{ mm.}}$  gives  $t$ , the thickness of the film, less than a ten-thousandth of a millimetre.

A second specimen of the same general coppery lustre,

\* The character of the curves for the organic film is considerably more "metallic" than the corresponding curves for magenta.



gave a set of curves (Pl. IV. fig. 5) which showed a double reversal; the phase-curve being positive for crimson and red, negative for orange and orange yellow, and positive again for the yellow, green, and blue.

A series of curves for a very thin film of magenta (estimated thickness 0.00005 mm.) gave results surprisingly resembling those of the beetle. The second point of inversion being, however, in the green instead of the yellow, and the "metallic" character of the film being much less marked than in the beetle wing-case. The resemblance in the lower curves, showing the variation of maximum steepness with the colour, is even more striking. It can scarcely be doubted, therefore, that here again the "metallic" colour is produced in a film whose thickness is of the order of a ten thousandth of a millimetre or less.

A third example (Pl. IV. fig. 6) is added, in which the correspondence is less marked, for the purpose of illustrating the general character of the curves for a case of green metallic lustre. There is in fact no aniline colour which shows an accurate correspondence, but the same magenta curves may be referred to for comparison.

The beetle wing-cases furnish in many cases a fairly smooth surface, and the difficulties in obtaining the necessary measurements are far less than when working with feathers of birds or with butterfly scales. Nevertheless, as Pl. IV. fig. 7 shows, the same general characteristics obtain in these, in both the phase-curves and the amplitude-ratios. It may be noted that the two curves do not always correspond\*, but it is probable that the difference may be explained by the difficulty in obtaining accurate results with surfaces so irregular.

It is worthy of note that in all of these curves (except that furnished by a red humming-bird feather) the curves are negative; from which it is fair to conclude that the film which produces the surface colour is very thin.

The total number of specimens which have been examined is perhaps not so large as it should be to draw general conclusions, and it is clearly desirable that it be extended; but so far the evidence for *surface film*, as the effective source

\* If we take the approximate formula  $k = \tan 2\psi$ , it is at once apparent that the dotted curves in our diagrams should have the highest minimum value for all the cases of great opacity. Thus the opacity may be inferred from the dotted curve for  $\psi$  as well as from the full curve for  $d\Delta/di$ , and in general the indications in the two cases show a rough agreement, the steeper full curves corresponding to the lower dotted curves, and *vice versa*.

of the metallic colours in birds and insects, is entirely conclusive.

It is clear that in all of these curves the descriptive colour corresponds in general to that colour for which the full curve is least steep, and for which the dotted curve is highest; and is complementary to the colour for which the full curve is steepest and the dotted curve is lowest, as we should expect; since the former corresponds to high reflective power, while the latter is characteristic of transparent substances with but moderate reflecting power.

#### EXCEPTIONS.

##### *Morpho alga.*

The measurement of the phase-difference in the light reflected from the blue-winged butterfly (*Morpho alga*), instead of being zero at normal incidence, had values which ranged from 0.15 to -0.15, and which were found to vary with the orientation of the specimen. There were also corresponding changes in the general character of the phase and amplitude curves, all of which showed clearly that the whole phenomenon is considerably complicated by a structure of the scales.

An examination under the microscope revealed the presence of exceedingly fine hairs (which can only be seen in reflected light) arranged without much regularity with their length parallel with that of the scale\*.

It was at first natural to attribute the blue colour to the light diffracted from these hairs; and it is not impossible that some of the silky sheen which these butterflies exhibit, is at least in part due to these hairs, whose diameter is much less than a light-wave, and which are therefore in the same relation to the light-waves as the small particles which cause the blue colour of the sky. But the changes in colour with varying incidence, so characteristic of true "surface colours," were precisely the same in this specimen, and were practically independent of the orientation; whereas the changes with the angle of incidence, which should result on the hypothesis that the colour is due to diffraction, should follow an entirely different law.

Another species of butterfly (*Papilio Ulysses*) was also examined and found to yield normal surface-colour curves, as

\* There are three varieties of scales, of different shapes. These are arranged in overlapping layers, the outer layer being quite transparent and the lower one opaque. The middle layer is the one showing blue by reflexion and brownish-yellow in transmitted light.

shown in Pl. IV. fig. 7 (No. 6). There is in this case no such minute linear structure as in the case of *Morpho alga*; and as here the phenomenon is clearly a case of "surface-colour," so it is highly probable that the same cause is effective in the case of *Morpho*.

Many other specimens were subsequently examined, but all fell into one or other of the two classes typified by these two.

#### *Diamond Beetle.*

If a specimen of the beetle popularly known as the Diamond Beetle be examined with a low power under the microscope, the bright green dots on the wing-case are seen to consist of depressions from which spring brilliant and exquisitely coloured scales; the colours varying throughout the range of the spectrum (green, however, predominating).

The colours exhibited by these scales are so vivid and varied, and the changes so rapid with varying incidence, that it was at once evident that the effect must be due to diffraction from regular striations, which were accordingly looked for under a magnification of about 1000 diameters. There were occasionally faint indications of striated structure, but so uncertain that if other indications had been less decided it might have been concluded that some other cause must have been effective. But on putting the microscope out of focus a moderately pure *spectrum* was observed, and by measuring the angles of incidence and diffraction of the various colours, the "grating" space could be determined, and was found to be of the order of a thousandth to a two thousandth of a millimetre.

The specimen was next examined by reflected light\* and the striations at once appeared, the count of the striations giving numbers agreeing very well with the calculated values. Frequently a single scale showed two or even three series of striations, giving corresponding spectra in three different directions. Another important feature of these "gratings" was shown in the fact that the light is all concentrated in a single spectrum, showing that the striations must have an unsymmetrical saw-tooth shape †.

\* The observation is somewhat difficult on account of the very small working space when using high powers.

† It may be noted that the objection that the colours of birds and insects cannot be due to diffraction on account of the equalizing effects of the varying angles of incidence and diffraction, would not apply if the striations are so fine as to give practically a single spectrum extending over a range of 45°.

On immersing the specimen in oil or other liquid little or no change is observed, except in those specimens in which a small communicating aperture exists in the neck (point of support) of the scale. The oil can be seen to gradually fill the interior, and simultaneously all trace of colour vanishes\*.

It appears, then, that the colour in this case is due to fine striations on the interior surface of the scale.

*Plustiotis resplendens.*

This is a beetle whose whole covering appears as if coated with an electrolytic deposit of metal, with a lustre resembling brass. Indeed, it would be difficult for even an experienced observer to distinguish between the metal and the specimen.

On examination with the Babinet compensator it was found that the reflected light was *circularly polarized* even at normal incidence, whether the incident light was polarized or natural. The proportion of circularly polarized light is greatest in the blue, diminishing gradually in the yellow portion of the spectrum and vanishing in the orange-yellow—for which colour the light appears to be completely depolarized. On progressing towards the red end of the spectrum traces of circular polarization *in the opposite sense* appear, the proportion increasing until the circular polarization is nearly complete in the extreme red.

It was at first suspected that the phase difference (not always as great as one quarter, but varying between  $\cdot 15$  and  $\cdot 25$ ) was due to linear structure, as in the case of *Morpho alga*; but on rotating the specimen about the normal no change resulted. The effect must therefore be due to a "screw structure" of ultra microscopic, probably of molecular dimensions. Such a structure would cause a separation of natural incident light into two circularly polarized pencils travelling with different speeds, and having different coefficients of absorption.

Such cases have been observed in some absorbing crystals; but whereas in these the difference in absorption between the two circularly polarized pencils is quite small compared with the total absorption—here one of the two is almost totally reflected, while there is scarcely a trace of the other.

If this hypothesis be correct, however, the selective absorption (or reflexion) being reversed at the other end of the

\* Sometimes a faint indication of colour remains (usually greenish) which shows the characteristics of surface colour. It is probable that this surface colour acts conjointly with the effect of diffraction, and indeed the character of the spectrum indicates an excess of green which may be thus accounted for.

spectrum—then for the orange-yellow the resulting light should be compounded of these two; and the resulting light should be plane-polarized, not depolarized.

The depolarization is in fact only apparent; for on using a moderately high power objective it is at once evident that there is a structure in the wing-case which causes a difference of phase between the components varying very rapidly from point to point; and the resulting plane of the plane-polarized light varies with corresponding rapidity, leaving no trace of polarization when the observation is made with a telescope.

The absorption coefficient for this specimen is quite of the order of that of the metals; and the thickness of the "metallic" film is of the order of a ten thousandth of a millimetre.

I take this opportunity of expressing my appreciation of the courtesy of Messrs. R. M. Strong, V. E. Shelford, W. L. W. Field, and H. B. Ward, to which I am indebted for bringing the literature of the subject to my notice, and for the specimens on which these observations are based.

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LXII. *On a Physical Interpretation of Schlömilch's Theorem in Bessel's Functions.* By Lord RAYLEIGH, O.M., F.R.S.\*

**T**HIS theorem teaches that any function  $f(r)$  which is finite and continuous for real values of  $r$  between the limits  $r=0$  and  $r=\pi$ , both inclusive, may be expanded in the form

$$f(r) = a_0 + a_1 J_0(r) + a_2 J_0(2r) + a_3 J_0(3r) + \dots \quad (1)$$

$J_0$  being the Bessel's function usually so denoted; and Schlömilch's demonstration has been reproduced with slight variations in several text-books †. So far as I have observed, it has been treated as a purely analytical development. From this point of view it presents rather an accidental appearance; and I have thought that a physical interpretation, which is not without interest in itself, may help to elucidate its origin and meaning.

The application that I have in mind is to the theory of

\* Communicated by the Author.

† See, for example, Gray & Mathews' 'Bessel's Functions,' p. 30; Whittaker's 'Modern Analysis,' §165.

aerial vibrations. Let us consider the most general vibrations in one dimension  $\xi$  which are periodic in time  $2\pi$  and are also symmetrical with respect to the origins of  $\xi$  and  $t$ . The condensation  $s$ , for example, may be expressed

$$s = b_0 + b_1 \cos \xi \cos t + b_2 \cos 2\xi \cos 2t + \dots, \dots \quad (2)$$

where the coefficients  $b_0, b_1, \&c.$  are arbitrary. (For simplicity it is supposed that the velocity of propagation is unity.) When  $t=0$ , (2) becomes a function of  $\xi$  only, and we write

$$F(\xi) = b_0 + b_1 \cos \xi + b_2 \cos 2\xi + \dots, \dots \quad (3)$$

in which  $F(\xi)$  may be considered to be an arbitrary function of  $\xi$  from 0 to  $\pi$ . Outside these limits  $F$  is determined by the equations

$$F(-\xi) = F(\xi + 2\pi) = F(\xi). \dots \dots \quad (4)$$

We now superpose an infinite number of components, analogous to (2) with the same origins of space and time, and differing from one another only in the direction of  $\xi$ , these directions being limited to the plane  $xy$ , and in this plane distributed uniformly. The resultant is a function of  $t$  and  $r$  only, where  $r = \sqrt{(x^2 + y^2)}$ , independent of the third coordinate  $z$ , and therefore (as is known) takes the form

$$s = a_0 + a_1 J_0(r) \cos t + a_2 J_0(2r) \cos 2t + a_3 J_0(3r) \cos 3t + \dots, \quad (5)$$

reducing to (1) when  $t=0$  \*. The expansion of a function in the series (1) is thus definitely suggested as probable in all cases and certainly possible in an immense variety. And it will be observed that no value of  $\xi$  greater than  $\pi$  contributes anything to the resultant, so long as  $r < \pi$ .

The relation here implied between  $F$  and  $f$  is of course identical with that used in the purely analytical investigation. If  $\phi$  be the angle between  $\xi$  and any radius vector  $r$  to a point where the value of  $f$  is required,  $\xi = r \cos \phi$ , and the mean of all the components  $F(\xi)$  is expressed by

$$f(r) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} F(r \cos \phi) d\phi. \dots \dots \quad (6)$$

The solution of the problem of expressing  $F$  by means of  $f$  is obtained analytically with the aid of Abel's theorem. And here again a physical, or rather geometrical, interpretation throws light upon the process.

\* It will appear later that the  $a$ 's and  $b$ 's are equal.

Equation (6) is the result of averaging  $F(\xi)$  over all directions indifferently in the  $xy$  plane. Let us abandon this restriction and take the average when  $\xi$  is indifferently distributed in all directions whatever. The result now becomes a function only of  $R$ , the radius vector in space. If  $\theta$  be the angle between  $R$  and one direction of  $\xi$ ,  $\xi = R \cos \theta$ , and we obtain as the mean

$$\int_0^{\frac{1}{2}\pi} F(R \cos \theta) \sin \theta \, d\theta = \frac{1}{R} \{F_1(R) - F_1(0)\}, \quad (7)$$

where  $F_1' = F$ .

This result is obtained by a direct integration of  $F(\xi)$  over all directions in space. It may also be arrived at indirectly from (6). In the latter  $f(r)$  represents the averaging of  $F(\xi)$  for all directions in a certain plane, the result being independent of the coordinate perpendicular to the plane. If we take the average again for all possible positions of this plane, we must recover (7). Now if  $\theta$  be the angle between the normal to this plane and the radius vector  $R$ ,  $r = R \sin \theta$ , and the mean is

$$\int_0^{\frac{1}{2}\pi} f(R \sin \theta) \sin \theta \, d\theta. \quad (8)$$

We conclude that

$$R \int_0^{\frac{1}{2}\pi} f(R \sin \theta) \sin \theta \, d\theta = F_1(R) - F_1(0), \quad (9)$$

which may be considered as expressing  $F$  in terms of  $f$ .

If in (6), (9) we take  $F(R) = \cos R$ , we find \*

$$\int_0^{\frac{1}{2}\pi} J_0(R \sin \theta) \sin \theta \, d\theta = R^{-1} \sin R.$$

Differentiating (9), we get

$$F(R) = \int_0^{\frac{1}{2}\pi} f(R \sin \theta) \sin \theta \, d\theta + R \int_0^{\frac{1}{2}\pi} f'(R \sin \theta) (1 - \cos^2 \theta) \, d\theta. \quad (10)$$

Now

$$\begin{aligned} R \int_0^{\frac{1}{2}\pi} \cos^2 \theta f'(R \sin \theta) \, d\theta &= \int \cos \theta \cdot df(R \sin \theta) \\ &= -f(0) + \int_0^{\frac{1}{2}\pi} f(R \sin \theta) \sin \theta \, d\theta. \end{aligned}$$

\* *Enc. Brit.* Art. Wave Theory, 1888; Scientific Papers, vol. iii. p. 98. *Phil. Mag.* S. 6. Vol. 21. No. 124. April 1911. 2 P

Accordingly

$$F(R) = f(0) + R \int_0^{\frac{1}{2}\pi} f'(R \sin \theta) d\theta. \quad (11)$$

That  $f(r)$  in (1) may be arbitrary from 0 to  $\pi$  is now evident. By (3) and (6)

$$\begin{aligned} f(r) &= \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} d\phi \{ b_0 + b_1 \cos(r \cos \phi) + b_2 \cos(2r \cos \phi) + \dots \} \\ &= b_0 + b_1 J_0(r) + b_2 J_0(2r) + \dots, \dots \dots \dots (12) \end{aligned}$$

where

$$b_0 = \frac{1}{\pi} \int_0^\pi F(\xi) d\xi, \quad b_n = \frac{2}{\pi} \int_0^\pi \cos n\xi F(\xi) d\xi. \dots (13)$$

Further, with use of (11)

$$b_0 = f(0) + \frac{1}{\pi} \int_0^\pi d\xi \cdot \xi \cdot \int_0^{\frac{1}{2}\pi} f'(\xi \sin \theta) d\theta, \dots (14)$$

$$b_n = \frac{2}{\pi} \int_0^\pi d\xi \cdot \xi \cos n\xi \cdot \int_0^{\frac{1}{2}\pi} f'(\xi \sin \theta) d\theta, \dots (15)$$

by which the coefficients in (12) are completely expressed when  $f$  is given between 0 and  $\pi$ .

The physical interpretation of Schlömilch's theorem in respect of two-dimensional aerial vibrations is as follows:— Within the cylinder  $r=\pi$  it is possible by suitable movements at the boundary to maintain a symmetrical motion which shall be strictly periodic in period  $2\pi$ , and which at times  $t=0, t=2\pi$ , &c. (when there is no velocity), shall give a condensation which is arbitrary over the whole of the radius. And this motion will maintain itself without external aid if outside  $r=\pi$  the initial condition is chosen in accordance with (6),  $F(\xi)$  for values of  $\xi$  greater than  $\pi$  being determined by (4). A similar statement applies of course to the vibrations of a stretched membrane, the transverse displacement  $w$  replacing  $s$  in (5).

Reference may be made to a simple example quoted by Whittaker. Initially let  $f(r)=r$ , so that from 0 to  $\pi$  the form of the membrane is conical. Then from (12), (14), (15)

$$b_0 = \frac{\pi^2}{4}, \quad b_n = 0 \text{ (} n \text{ even)}, \quad b_n = -\frac{2}{n^2} \text{ (} n \text{ odd)};$$



and thus

$$f(r) = \frac{\pi^2}{4} - 2\{J_0(r) + \frac{1}{9}J_0(3r) + \frac{1}{25}J_0(5r) + \dots\}, \quad (16)$$

the right-hand member being equal to  $r$  from  $r=0$  to  $r=\pi$ .

The corresponding vibration is of course expressed by (16) if we multiply each function  $J_0(nr)$  by the time-factor  $\cos nt$ .

If this periodic vibration is to be maintained without external force, the initial condition must be such that it is represented by (16) for all values of  $r$ , and not merely for those less than  $\pi$ . By (11) from 0 to  $\pi$ ,  $F(\xi) = \frac{1}{2}\pi\xi$ , from which again by (4) the value of  $F$  for higher values of  $\xi$  follows. Thus from  $\pi$  to  $2\pi$ ,  $F(\xi) = \frac{1}{2}\pi(2\pi - \xi)$ ; from  $2\pi$  to  $3\pi$ ,  $F(\xi) = \frac{1}{2}\pi(\xi - 2\pi)$ ; and so on. From these  $f$  is to be found by means of (6). For example, from  $\pi$  to  $2\pi$ ,

$$\begin{aligned} f(r) &= r \int_0^{\sin \theta = \pi/r} \sin \theta \, d\theta + \int_{\sin \theta = \pi/r}^{\sin \theta = 1} (2\pi - r \sin \theta) \, d\theta \\ &= r - 2\sqrt{(r^2 - \pi^2)} + 2\pi \cos^{-1}(\pi/r), \dots \dots \dots (17) \end{aligned}$$

where  $\cos^{-1}(\pi/r)$  is to be taken in the first quadrant.

It is hardly necessary to add that a theorem similar to that proved above holds for aerial vibrations which are symmetrical in all directions about a centre. Thus within the sphere of radius  $\pi$  it is possible to have a motion which shall be strictly periodic and is such that the condensation is initially arbitrary at all points along the radius.

LXIII. *On the Ionization of Different Gases by the Alpha Particles from Polonium and the Relative Amounts of Energy required to produce an Ion.* By T. S. TAYLOR\*.

*Introduction.*

IN previous papers† the writer has shown that the air-equivalents‡ of metal foils decrease with the speed of the alpha particles entering the foils. For sheets of different metals of equal air-equivalents, the rates of decrease are

\* Communicated by the Author.

† Amer. Journ. Sci. vol. xxvi. pp. 169-179, Sept. 1908; *ibid.* vol. xxviii. pp. 357-372, Oct. 1909; Phil. Mag. vol. xviii. p. 604, Oct. 1909.

‡ By air-equivalent is meant the amount by which the range of the alpha particle is cut down by its passage through the foil.

approximately proportional to the square roots of the respective atomic weights. On the contrary, the air-equivalents of hydrogen sheets increase while the hydrogen-equivalents of air sheets decrease with the speed of the entering alpha particles, and at such a rate as to be in agreement with the square root law observed for the decrease of the air-equivalents of the metal sheets.

A comparison of the Bragg ionization curves, obtained in atmospheres of air and hydrogen, when the pressure of the air was so reduced that the range of the alpha particles from polonium was the same in air as it was in hydrogen at atmospheric pressure, showed differences which are sufficient to account for the variations in the air-equivalents of the hydrogen sheets with the speed of the alpha particles. These differences between the Bragg ionization curves in air and hydrogen suggested that some such differences might be found between the ionization curves obtained in other gases, and it was for the purpose of making a detailed comparison of the ionization curves obtained in different gases that the present experiments were begun.

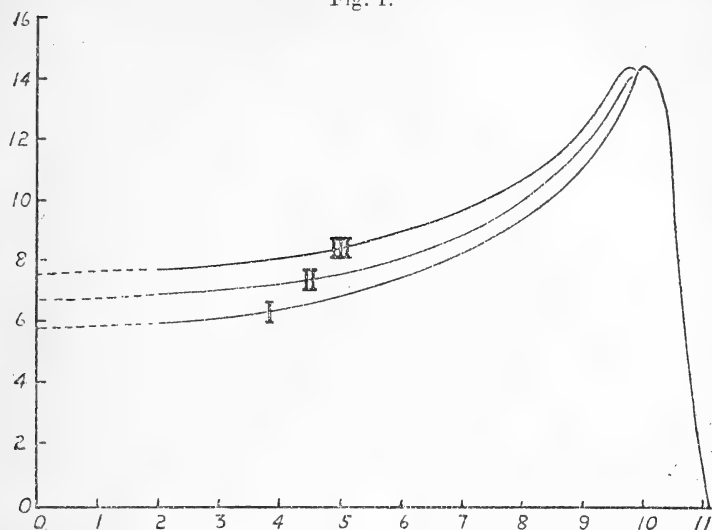
#### *Continuation of Experiments.*

The apparatus used was the same as had been used in the previous experiments\*. The sheet-iron case, enclosing the apparatus proper, was replaced by a solid iron case which could be readily exhausted. Polonium was used as the source of rays, and was placed in a brass cylinder of such dimensions that the rays emerging from the cylinder fell well within the limits of the ionization chamber for all available distances of the source of rays from the ionization chamber.

In the determination of the ionization curve in any gas, the vessel enclosing the apparatus was first evacuated and then the gas admitted very slowly till the pressure it exerted was such that the range of the alpha particles was exactly 11.1 centimetres, which was the maximum range available with the apparatus. The Bragg ionization curve was then obtained in the usual manner by observing the deflexion of the needle of the Dolezalek electrometer in scale-divisions per second for various distances of the source of rays from the ionization chamber. In this manner, the Bragg ionization curves were obtained in the gases and vapours given in Table I. The curves in figs. 1 and 2, and the dotted ones in fig. 3 represent the ionization curves obtained in the above manner in the gases as indicated below the figures,

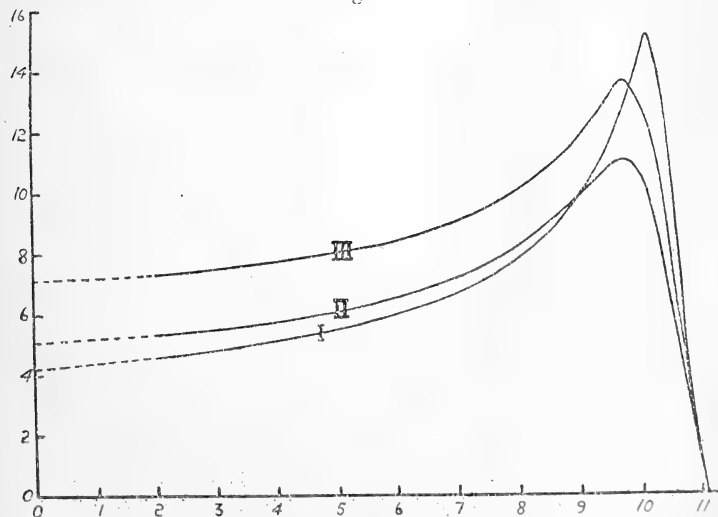
\* *Loc. cit.*

Fig. 1.



The ordinates are the deflexions in millimetres of the electrometer needle per second. The abscissæ are the distances in centimetres of the polonium from the ionization chamber. Curves I, II, and III were obtained when the maximum range of the alpha particle was exactly 11.1 centimetres in hydrogen, air, and methyl iodide, respectively.

Fig. 2.



The ordinates are the deflexions in millimetres of the electrometer-needle per second. The abscissæ are the distances in centimetres of the polonium from the ionization chamber. Curves I, II, and III were obtained when the maximum range of the alpha particle was exactly 11.1 centimetres in methane, ethyl chloride, and carbon disulphide, respectively.

respectively. The dotted portion of each curve in figs. 1 and 2 is assumed to be the form it would take were it possible to move the polonium entirely up to the ionization chamber. At any rate, such assumed portions of the curves can differ but little from the actual curves. It is to be noted, that the ionization curves shown in figs. 1 and 2 are plotted differently from the regular Bragg ionization curve in that the values of ionization are taken as ordinates and distances of the source of rays from the chamber as abscissæ, instead of *vice versa* as is usually done.

Although the curves in figs. 1, 2, and 3 represent some differences from one another in regard to the relative amounts of ionization for corresponding distances of the source of rays from the ionization chamber, all of them are of the same general form. From a re-determination of the velocity of the alpha particle at different points in its path, and the assumption that the ionization produced at any point in the path of the particle is proportional to the energy consumed, Geiger\* has shown that the ionization  $I$  at any point in the path is given by the relation

$$I = \frac{c}{(r-x)^{1/3}},$$

where  $c$  and  $r$  are constants and  $x$  is the distance from the source of rays. By comparing this theoretical ionization curve with the experimental curve obtained in hydrogen for a pencil of rays, Geiger found the two to agree very closely.

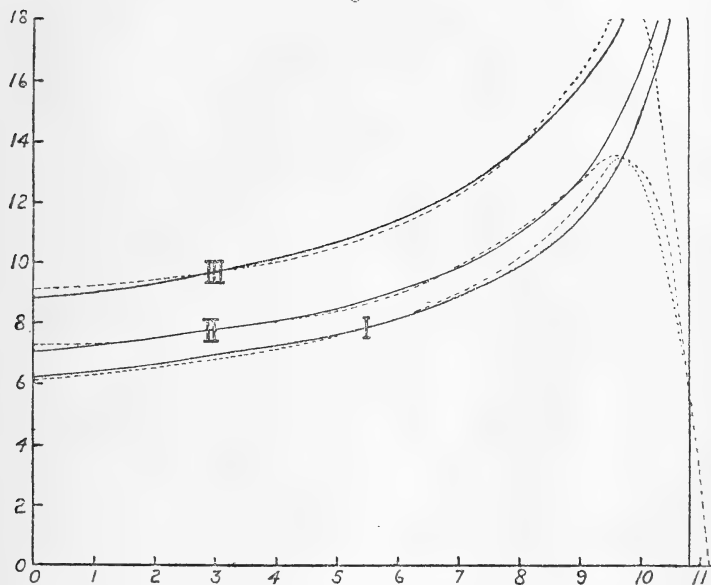
This theoretical curve has been compared with the experimental curves obtained in each of the gases and vapours given in Table I., and a very close agreement between theoretical and experimental curves was found for each gas. To make this comparison, it was necessary to determine the constants  $r$  and  $c$  for each gas. For the value of  $r$ , Geiger used the average range of the alpha particles in the pencil of rays. Since the maximum range of the alpha particles in the cone of rays used in the present experiments was always 11.1 centimetres, the average range of the alpha particles in this cone of rays emerging from the cylinder containing the polonium was slightly less than 11.1 centimetres. Consequently 10.8 centimetres was taken as the value of the average range of the alpha particle, that is 10.8 centimetres is supposed to represent the average distance the alpha particles travelled in each gas before losing

\* Proc. Royal Society, Series A, vol. lxxxiii, no. A 365, p. 505.

their power of producing ions. In order to determine  $c$  for any one gas, the ionization (ordinate of the ionization curve, figs. 1, 2, and 3) and the corresponding distance  $x$  of the source of rays from the ionization chamber (abscissa of curve) were substituted in the equation

$$I = \frac{c}{(10.8 - x)^{1/3}},$$

Fig. 3.



The full line curves I, II, and III are the theoretical ionization curves for nitrogen, sulphur dioxide, and ether, respectively as obtained by substituting the corresponding values of  $c$  given in column 2, Table I., in the equation

$$I = \frac{c}{(r-x)^{1/3}}, \quad \text{where } r=10.8.$$

The dotted curves I, II, and III are the experimental ionization curves for nitrogen, sulphur dioxide, and ether, respectively, and are plotted similarly to the curves in figs. 1 and 2.

and the equation solved for  $c$ . Separate values of  $c$  were thus obtained for various distances of the source of rays from the ionization chamber between  $x=0$  and 9.5 centimetres, and the mean value of these separate determinations found for each gas. The mean values of  $c$  as found in the

above manner for all the gases and vapours used are recorded in column 2, Table I.

TABLE I.

Gas or Vapour.	$c$ or area under theoretical curve divided by 7.33.	Area under experimental curve as measured with planimeter.	Ratio of area under experimental curve to $c$ .	Ratio of the total ionization in the gas to that in air.		Relative energy required to produce an ion.
				Taylor.	Bragg.	
Air .....	11.24	980	87	...	...	1.00
H <sub>2</sub> .....	10.00	966	96	0.99	1.00	1.01
CH <sub>3</sub> I .....	14.73	1301	88	1.33	1.33	0.75
CH <sub>4</sub> .....	12.65	1156	91	1.18	...	0.85
C <sub>2</sub> H <sub>5</sub> Cl ...	14.05	1251	89	1.29	1.32	0.77
CS <sub>2</sub> .....	15.60	1355	87	1.38	1.37	0.73
Air .....	14.64	1249	85	...	...	1.00
N <sub>2</sub> .....	13.81	1206	87	0.96	0.96	1.04
CO <sub>2</sub> .....	15.01	1262	84	1.01	1.08	0.99
O <sub>2</sub> .....	16.72	1415	85	1.13	1.09	0.88
C <sub>4</sub> H <sub>10</sub> O ...	19.42	1702	88	1.36	1.33	0.74
Air .....	13.27	1182	89	...	...	1.00
SO <sub>2</sub> .....	15.30	1223	80	1.03	...	0.97
HCl .....	17.70	1530	86	1.29	...	0.77
HBr .....	18.32	1527	83	1.29	...	0.77
Air .....	13.36	1190	89	...	...	1.00
HI .....	17.68	1535	87	1.29	...	0.77

The full line curves I, II, and III in fig. 3 represent the theoretical curves for nitrogen, sulphur dioxide, and ether, respectively, as obtained by using the values of  $c$  as recorded in column 2, Table I. for the respective gases. The dotted curves are the corresponding experimental curves and, as can be seen, agree very well with the theoretical curves. The agreement between the theoretical and the experimental curves for the other gases was equally as good as it is for those given in fig. 3. In some cases the agreement was much closer. This agreement between theoretical and experimental curves confirms the assumption that the energy consumed is proportional to the ionization produced.

The ionization at any point of the path of the particle being given by the relation

$$I = \frac{c}{(r-x)^{1.3}}$$

the total area under this theoretical curve is a measure of the total ionization produced by the alpha particle in the gas. If  $A_t$  represents the area under the theoretical curve, then

$$\begin{aligned} A_t &= \int_0^r I dx = \int_0^r \frac{c dx}{(r-x)^{1.3}} \\ &= 3/2 c(r)^{2/3} = 7.33 c \end{aligned}$$

( $r$  being equal to 10.8 centimetres).

Hence  $c$  is 3/22 of the area under the theoretical curve when the average range of the alpha particle is 10.8 centimetres in any gas whatever. The values of  $c$  recorded in column 2 of Table I. are then 3/22 of the area under the theoretical ionization curves in the respective gases.

The areas under the ionization curves being proportional to the energies consumed in the production of ions in the respective gases, the value of  $c$  in any one gas depends upon the total ionization produced in the gas, and consequently upon the energy required to produce an ion in the gas. Then the ratio of the area under the experimental curve to  $c$  should be a constant. By dividing the areas under the experimental curves as measured with a planimeter and recorded in column 3, Table I., by the values of  $c$  for the corresponding gases, the values recorded in column 4 were obtained and, as can be seen, are approximately constant.

The areas under the ionization curves being the measures of the relative ionizations produced in the gases, the ratios of the total ionization produced in the gases to that produced in air were determined by finding the ratio under each curve to the area under the corresponding comparison air curve. After the determination of the ionization curve in each gas the ionization curve was always obtained in air to be used as a basis of comparison. The ratios of the ionizations produced in the different gases to that produced in air are recorded in column 5 of Table I. Bragg\*, by a less direct process, determined the ratio of the total ionizations in gases to that in air and his values are recorded in column 6. There is a fairly good agreement between the values as found by Bragg and those found by a more direct process of measurement of the area enclosed by the axes of references and the ionization curve for each gas.

Since the energy of the alpha particle is entirely consumed before it ceases to produce ions, the energy required to produce an ion in any given substance will vary inversely as the ratio of the total ionization in the substance to the total

\* Bragg, Phil. Mag. vol. xiii. pp. 333-357, March 1907.

ionization in air if the energy required to produce an ion in air is always taken as the basis of comparison. The values of column 5 of the Table are the ratios of the total ionizations produced in the gases as compared with the total ionization produced in air. Consequently the reciprocals of these ratios are the relative amounts of energy required to produce an ion in the substance as compared with the energy required to produce an ion in air. The values recorded in column 7 are these reciprocals of the values in column 5, and hence are the relative amounts of energy required to produce an ion in the gases as compared with that required to produce an ion in air. These values indicate a considerable variation of the energy required to produce an ion. The heavier and more complex molecules are apparently more readily ionized than the lighter and less complex ones. This is probably due to the electrons in the heavier and more complex molecules being in a less stable arrangement than they are in the lighter and less complex molecules, and hence more readily drawn out.

In conclusion I wish to express my thanks to Professor Bumstead for his valuable suggestions in connexion with the work and for loaning me the apparatus. I am also indebted to Professor Boltwood for furnishing me the preparation of polonium.

#### *Results.*

1. The ionization curve obtained in various gases and vapours with polonium as the source of rays is of the general form

$$I = \frac{c}{(r-x)^{1/3}},$$

where  $I$  is the ionization;  $c$  is a constant for any one gas depending upon the total ionization produced, and consequently upon the energy required to produce an ion in the given gas;  $r$  is the average range of the alpha particles in the cone of rays; and  $x$  is the distance from the source of rays.

2. The agreement between the theoretical and the experimental curves confirms the assumption made in previous papers by the writer\* and by Geiger†, that the ionization produced by the alpha particle is proportional to the energy consumed.

3. The values of the ratio of the total ionization produced by the alpha particle in different gases to the total ionization

\* *Loc. cit.*

† *Loc. cit.*



produced in air as found by Bragg have been confirmed by a more direct process.

4. The energy of the alpha particle consumed in the production of an ion depends upon the nature of the molecule ionized. It apparently requires less energy to produce an ion in the gases or vapours which have heavy or relatively complex molecules than it does in those gases of lighter or less complex molecules.

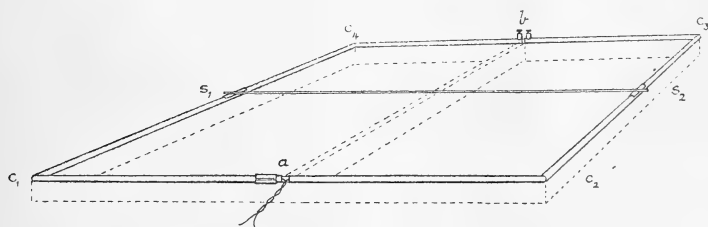
Laboratory of Physics, University of Illinois,  
Urbana, Illinois, January 28, 1911.

LXIV. *A new Form of Earth Inductor.* By JAMES E. IVES, *Ph.D.*, Associate Professor of Physics, and S. J. MAUCHLY, of the University of Cincinnati\*.

THE instrument described in this paper is the result of a desire of the authors to construct an Earth Inductor which could be used to measure the vertical component of the Earth's magnetic field *directly*, without the use of the magnetometer and the ordinary rotating Earth Inductor. In the form described, it was only used to measure the vertical component, but it could easily be adapted to determine, *directly*, all of the elements of the Earth's field.

The essential parts of the device were a square of brass tubing supported in a horizontal plane upon a wooden frame,  $c_1 c_2 c_3 c_4$  in the perspective drawing of fig. 1; a sliding conductor,  $s_1 s_2$ , connecting two sides of the square, which

Fig. 1.



could be moved parallel to itself; and an insulated wire enclosed within the tubing. In the figure, the brass tubing is shown in solid, and the wooden frame in dotted lines. The tubing had outer and inner diameters of 1.12 and .97 cm. respectively. The square was 100 cms. long on a side, from centre to centre of the tubing. It was cut in two places, shown at  $a$  and  $b$  in fig. 1, on opposite sides. At  $a$  the gap was 1.2 cm. long, and at  $b$  .3 cm. These gaps were filled in

\* Communicated by the Authors.

with closely fitting, split, hard rubber tubes, having shoulders on them to keep the two parts of the brass tube the proper distance apart. At  $b$ , two binding posts were soldered to the ends of the brass tubes. The tubes forming the square were carefully soldered together at the corners,  $c_1, c_2, c_3, c_4$ . The insulated copper wire within the tubing had a diameter of 0.1007 cm. The ends of this wire were brought out at  $a$ .

The sliding conductor,  $s_1 s_2$ , was attached securely to the lower side of a wooden bar, and consisted of a stout brass wire, .318 cm. in diameter, soldered at its ends to square brass plates, 3.25 cm. on a side, to which were attached half-cylinders of brass, 3.20 cm. long, engaging with the brass tubing. It could be moved parallel to itself through any desired distance, between wooden stops attached to the wooden bar  $ab$ .

At  $a$  was a closely-fitting split brass sleeve, about 3 cms. long, sliding upon the brass tube, which could be slid over the gap at  $a$ , forming a conducting bridge between the two tubes  $c_1 a$  and  $c_2 a$ , when desired.

The binding posts at  $b$  were connected to a two-coil Du Bois-Rubens armoured galvanometer made by Siemens and Halske, having a resistance, with the two coils in parallel, of about 2.5 ohms, a period of about 4 seconds, and a sensitiveness, as adjusted for this experiment, of about  $2 \times 10^{-8}$  ampere for one millimetre at a distance of one metre. This galvanometer is described in the *Zeitschrift für Instrumentenkunde*, Jahrg. 1900, p. 65.

The arrangement of the circuits is shown in fig. 2, where

$s_1 s_2$  is the slider ;

G, the galvanometer ;

$R_1$ , a resistance in the tube circuit to give a convenient deflexion ;

$c_1 c_2 c_3 c_4$ , the square of brass tubing ;

B, battery in the wire circuit ;

K, key " "

$A_1$ , ammeter " " and

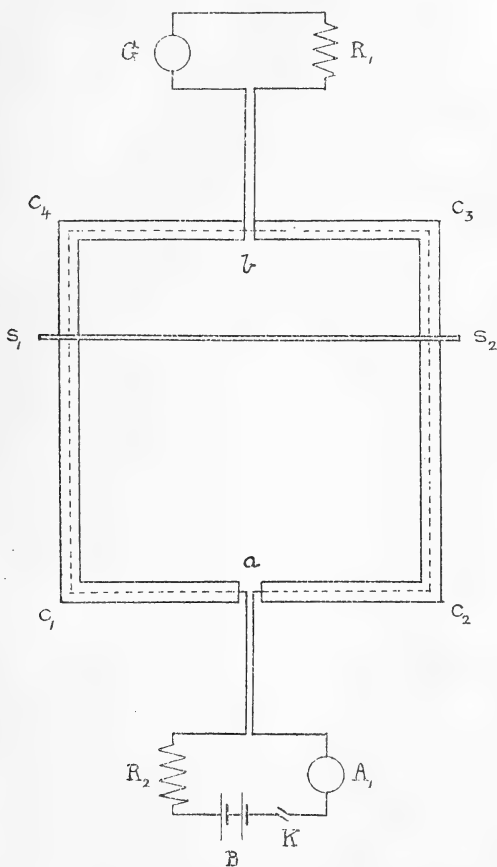
$R_2$ , a rheostat " " to adjust the current to a suitable value.

The experiment consisted in varying the flux, due to the earth's field, through the rectangle  $s_1 s_2 c_3 c_4$  by moving the slider  $s_1 s_2$ , and then calibrating the galvanometer by making or breaking a current in the wire circuit, and noting the deflexion produced in the tube circuit.

The procedure was as follows:—With the tube circuit open at  $a$ , and the wire circuit open at K, the slider was moved through a suitable distance between stops, and the

ballistic deflexion,  $\delta_z$ , on the galvanometer noted. The actual distance moved through was 15.1 cm. The deflexion was proportional to the flux cut, which was equal to  $Z$ , the vertical component of the Earth's field, multiplied by the area,  $A$ , described by the slider.

Fig. 2.



To determine to what change of flux this deflexion of the galvanometer corresponded, the slider  $s_1 s_2$  was removed, the tube circuit closed at  $a$ , and the ballistic deflexion  $\delta_i$  of the galvanometer observed when a current of two amperes was made or broken in the wire circuit. The value of the current in the wire circuit was given by the ammeter  $A_1$ , which had been previously calibrated, and by means of which its value could be determined to within one per cent. If the self-inductance,  $L$ , of the square of brass tubing is known, the

flux,  $\Phi$ , producing this deflexion can be calculated, for  $\Phi = LI$ , where  $I$  is the current made or broken in the wire circuit. This follows from the fact that since there is no magnetic field within a metal tube due to a current flowing uniformly along it, the *Mutual Inductance* of the two circuits is equal to the *Self Inductance* of the square of tubing. We then have :

$$Z = \frac{\delta_2}{\delta_1} \times \frac{LI}{A} \dots \dots \dots (1)$$

$L$  can be calculated approximately by taking the mean radius,  $\rho$ , of the tube, and assuming that its walls are vanishingly thin. In this case, the current lies entirely on the surface, and

$$L = 8l \left[ \log_e \frac{l}{\rho} + \frac{\rho}{l} - 0.774 \right], \dots \dots \dots (2)$$

where  $l$  is the length of one side of the square\*.

$L$  can be determined to a greater degree of accuracy by substituting for  $\rho$ , in formula (2), the geometric mean distance,  $a$ , of the cross-section of the tube.  $a$  is given by

$$\log_e a = \log_e a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{3a_2^2 - a_1^2}{a_1^2 - a_2^2}, \dots \dots (3)$$

where  $a_1$  and  $a_2$  are the outer and inner radii of the tube †.

The mean radius of the tube was .523 cm., and the inductance of the square circuit of brass tubing, 100 cm. on a side, from (2), was therefore equal, approximately, to 3587 cm.

The geometric mean distance of the cross-section of the tube by (3) was .536 cm. Using this instead of the mean radius in (2) we get, to a greater degree of approximation,

$$L = 3568 \text{ cm.}$$

The mean throw of the galvanometer when the slider was moved through 15.1 cms. was 7.06 cms. The mean throw when a current of 2 amperes was made or broken in the wire circuit was 6.19 cms. We therefore have

$$Z = \frac{7.06}{6.19} \times \frac{3568 \times 2}{1510} = 0.539 \text{ c.g.s. unit.}$$

An *indirect* determination of  $Z$ , made in the same place and under the same conditions with a magnetometer and Edelmann Earth Inductor, gave .548 c.g.s. unit for the value of  $Z$ .

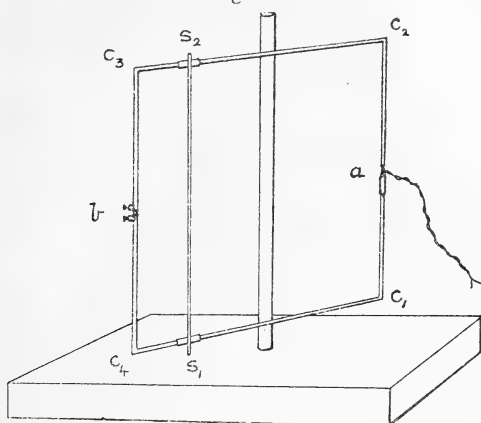
Our inductor was only used to determine  $Z$ , but it could easily be adapted to determine the horizontal intensity,  $H$ ,

\* See Fleming, 'Electric Wave Telegraphy,' pp. 98-100.

† Maxwell, 'Electricity and Magnetism,' vol. ii, § 692; Rosa & Cohen, Bull. Bur. Standards, 5, 1908, pp. 50-52.

and the magnetic declination by supporting it in a vertical plane, in such a manner that it could be rotated about a vertical axis, as shown in fig. 3. Its plane could be rotated

Fig. 3.



until no deflexion of the galvanometer was obtained when moving the slider. This would give the *direction* of  $H$ . It could then be turned at right angles to this position, and the magnitude of  $H$  determined in the same manner as has already been described for  $Z$ .

Our inductor was 100 cms. on a side. It could, however, without any serious loss of efficiency be reduced in size to 50 cms. on a side or less.

An advantage of this instrument, when suitably constructed, would be that it would be universal, as it could be used to determine all the elements of the Earth's field. The method is also very direct, involving only the determination of a current, and the calculation of a self-inductance.

University of Cincinnati,  
June 1910.

### LXV. Notices respecting New Books.

*Physics.* By C. RIBORG MANN and G. RANSOM TWISS. Revised Edition. Chicago and New York: Scott Foresman & Company. Pp. 424. With Illustrations.

TO say that this book will be received with very different feelings by different readers is only to say that it possesses individuality. It is an attempt to replace the usual elementary course of physics by one in which the problems are more likely to be significant to the average high-school boy or girl just beginning the study of physics. Instead of the "Physics which every Physicist must know" they give what they consider to be the "Physics every child should know." The authors are to be commended on the

choice of their material and in general upon the way they present it. At least this is so if it be clearly understood that the book is to be treated as an introductory account intended to rouse the interest of the pupil in scientific things. At the same time lovers of the logical presentation of ideas will continue to wonder whether it is really necessary to sacrifice so much in order to rouse the pupils' interest. Our own opinion is that the *via media* is best. The pupils' interest must, of course, be aroused; but unless, at the same time, an endeavour be made to logically develop the subject, they go without one of the main advantages of a scientific training.

Regarded as an introduction to physics, however, the work of the authors has been well done. Although terms such as force, energy, and work are introduced with the slightest definition only, yet we do not think the pupil will have anything serious to unlearn at a later date. The book covers the ground of mechanics, heat, light, sound, electricity, and magnetism, the illustrative cases being selected from the technical side (electric bells, telephones, &c.). We observe only one mistake. Radium has not the heaviest atom known. This honour is possessed by Uranium, with Thorium as a good second.

*Crystalline Structure and Chemical Constitution.* By Dr. A. E. H. TUTTON, F.R.S. London: Macmillan & Co. 1910.

THIS is an account of the work of the author during the last twenty years in connexion with the properties of certain crystals. The detailed accounts of this work are only to be found in scattered journals and proceedings of scientific societies, the result being that it is perhaps not so well known as it ought to be. At any rate, by bringing the various parts of it together into a short monograph, the author has succeeded in presenting a picture of the researches on which he has expended so much time and ingenuity, which will go far to bring him the greater credit he deserves. There are few cases which form a better illustration of the enormous amount of scientific advance which can be effected by the continuous and thoroughgoing application of an investigation to the examination of one small department of knowledge.

Dr. Tutton's work has consisted in the measurement, with the highest possible degree of precision, of some of the chief physical properties of selected groups of crystals, with the object of ascertaining, without any of the uncertainty existing at the time when he began his researches, whether any small differences exist in these properties and whether these differences present any correlation with the position of the metallic bases of the crystal in Mendelejeff's series.

For the results obtained reference must be made to this book itself, where a sufficiently full summary is given of them and of the apparatus by which the results were obtained. But it may be said here that Dr. Tutton has succeeded in his aim to set free from uncertainty some of the disputed questions of crystallography.

The book is written by an enthusiast; and if sometimes his enthusiasm drives him into digressions which do not seem to belong to the main theme, we do not doubt that this small defect will be forgiven him.

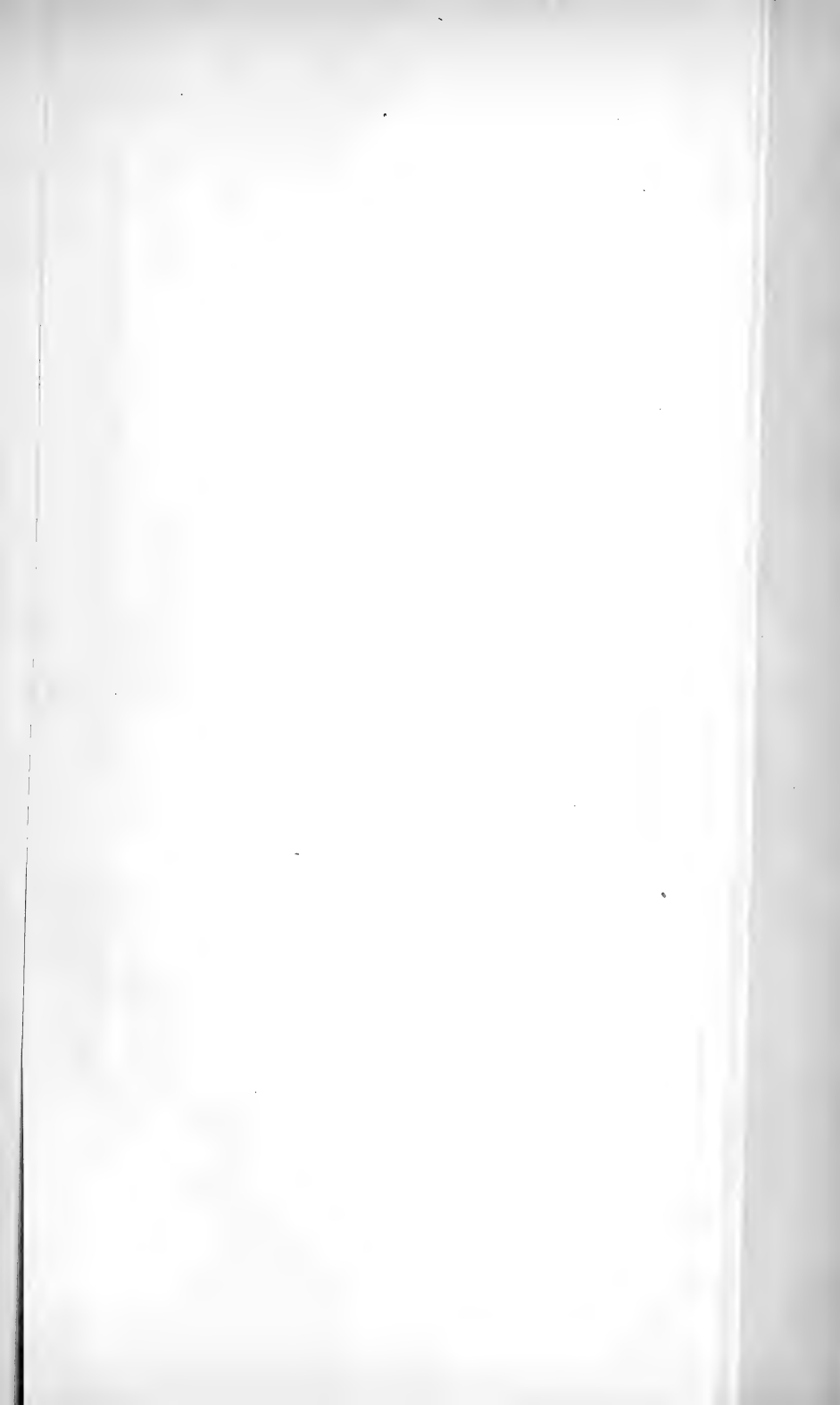






FIG. 1.

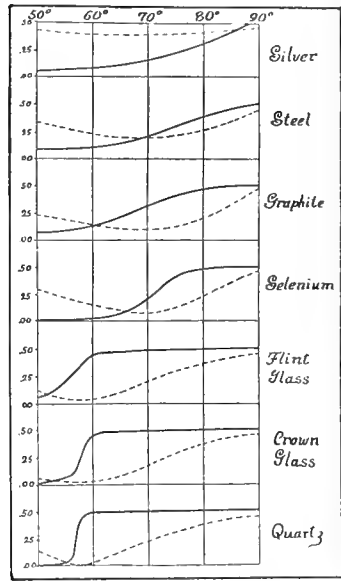


FIG. 2.

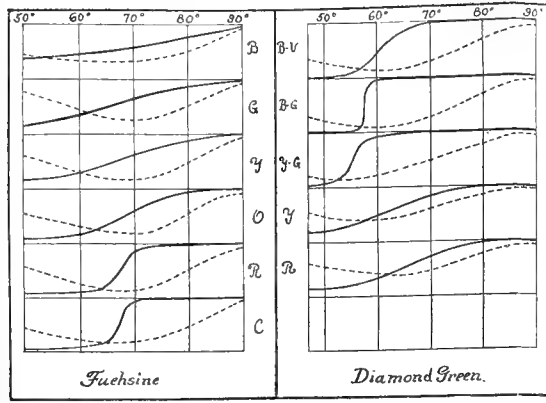


FIG. 3.

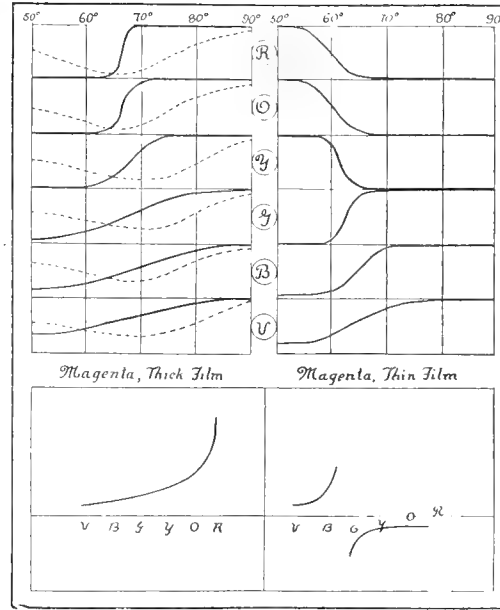


FIG. 7.

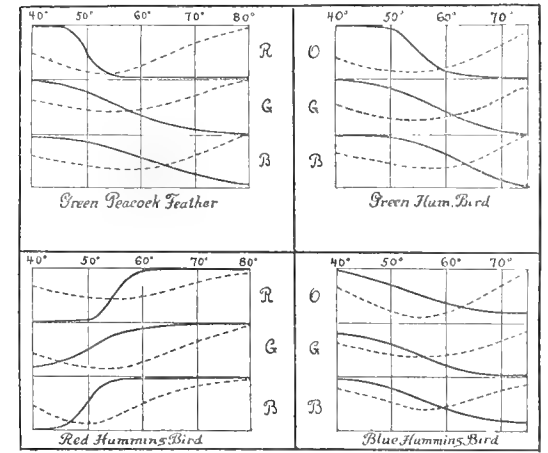


FIG. 4.

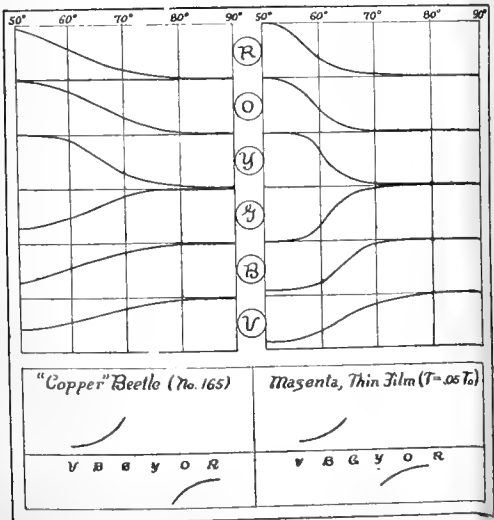


FIG. 5.

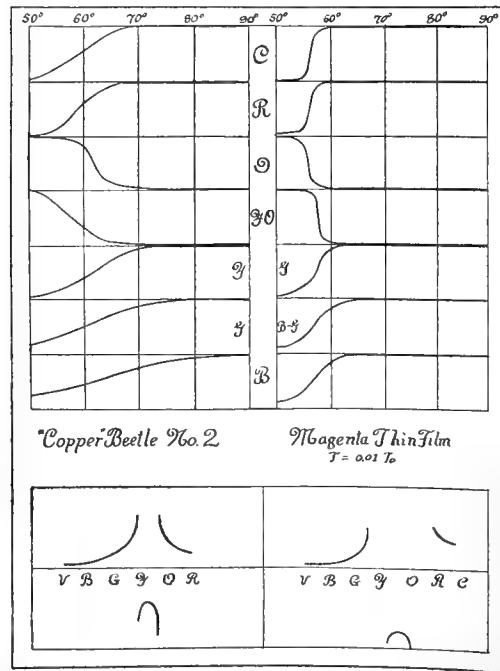
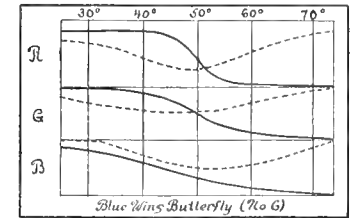
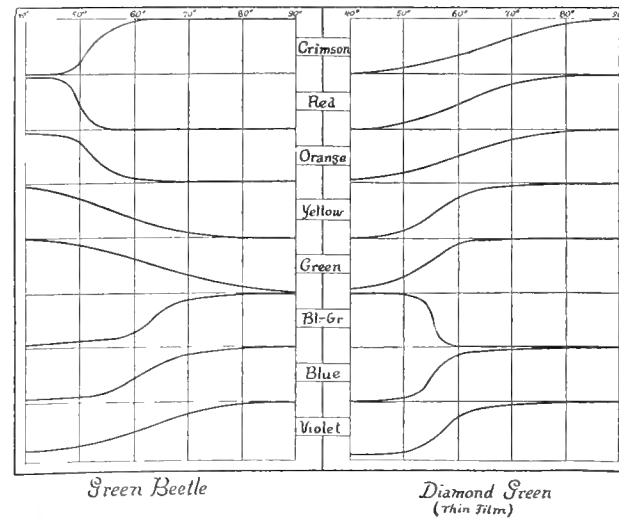
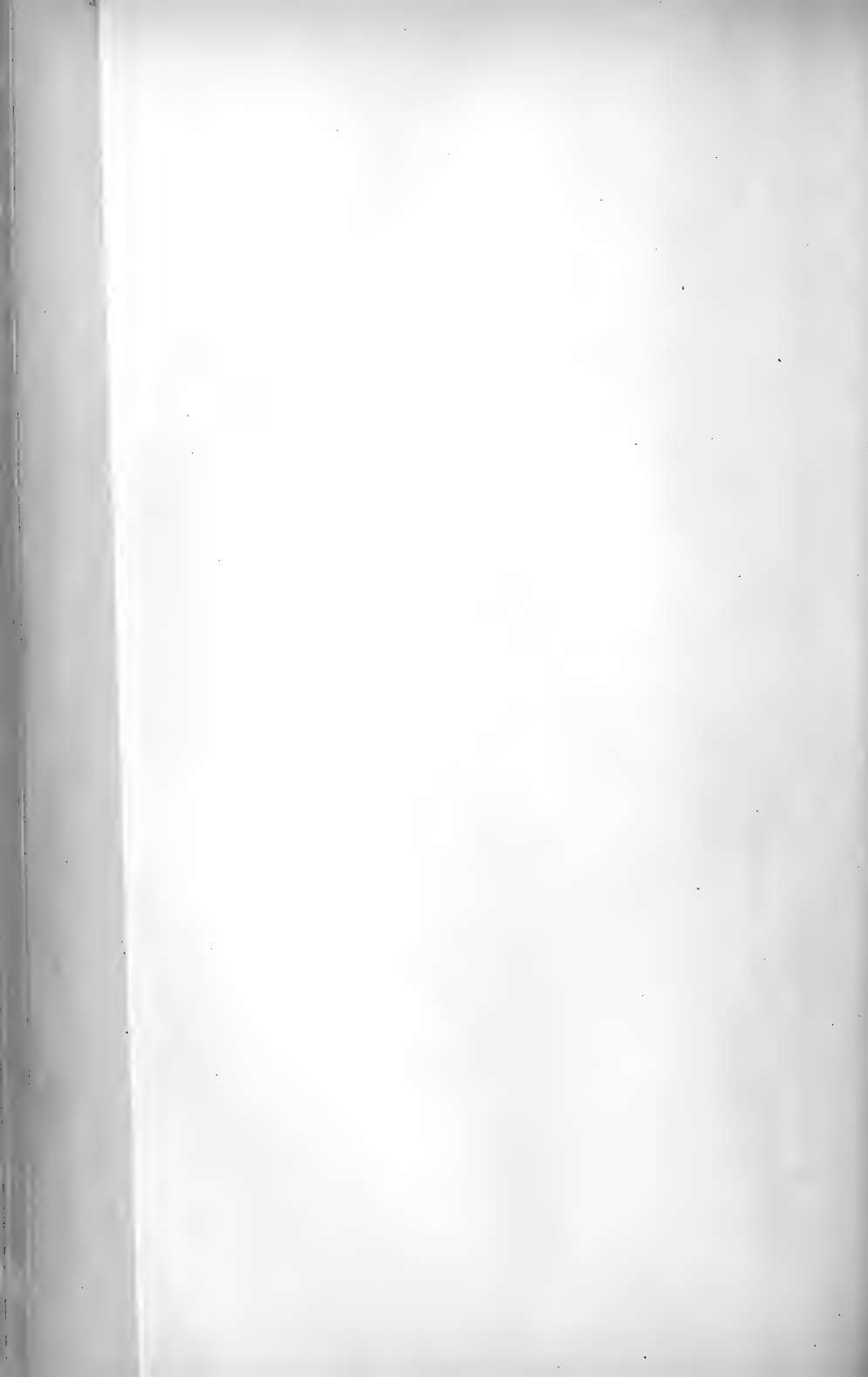


FIG. 6.





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[SIXTH SERIES.]

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LXVI. *On the Discharge from an Electrified Point.* By  
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in Physics in the University of Bristol\*.

[Plate V.]

SOME experiments, chiefly on (1) the pressure of the Electric Wind †, and (2) the field in the neighbourhood of a discharging point ‡, led Prof. A. P. Chattock and the author to adopt certain theories as to the nature of point discharge which at the time were capable of explaining most of the results obtained. The author has since extended the experiments, and has obtained results which are to a certain extent at variance with these theories. It seems desirable, therefore, to review the situation in the light of the more recent experimental facts.

The theories of point discharge originally suggested may be briefly summarized thus:—

Suppose that a point P is gradually charged with electricity; the field near its surface is at first unable to do more than clear away the few initially present ions in its neighbourhood as fast as they are formed. But as soon as it is strong enough to impart to the positives among them sufficient energy to enable these to ionize fresh molecules in their turn,

\* Communicated by the Author.

† Chattock and Tyndall, *Phil. Mag.* [6] xix. p. 449 (1910).

‡ Chattock and Tyndall, *Phil. Mag.* [6] xx. p. 277 (1910).

ordinary discharge with glow sets in. For both signs of discharge the supply of positive ions is kept up by ionization due to negative ions, these having been produced by previously formed positive ions and so on. Both signs of ion have, therefore, to be able to ionize as each produces the other. Since positive ions require a stronger field for this than negative, it is always the field required by the positive ions which has to occur at the point P.

Suppose now that ions of opposite sign to the charge on P are supplied to it from an external point N in its neighbourhood. When P is negative the field necessary for glow at its surface is unaltered.

When P is positive, the supply of negative ions, which is in other cases kept up by ionization near P by positive ions, is now continuously supplied from without. Glow therefore appears at P when the field near it has reached that in which negative ions will ionize. This field ( $f^-$ ) is less than that ( $f^+$ ) in which positive ions will ionize.

Experiments on the pressure of the electric wind further suggested that the ions streaming away from a glowing point are not at first fully formed, and travel a distance from the point varying from 4 centimetres in the case of negative ions in pure hydrogen to 3 or 4 millimetres in air, before they completely cluster.

Now unclustered ions ionize in a lower field than ions fully grown, so that, as N is made to approach P, the field for glow at the latter begins to drop when the distance between N and P is decreased to values less than this critical distance. Thus when P is positive and N very close to it, P is subjected to a corpuscular bombardment and its field sinks to that ( $f_c$ ) necessary for corpuscles to ionize.

Lastly, when the current is small, ions of one sign only traverse the greater part of the distance between point and plate, except in negative discharge in pure hydrogen.

In the light of the experiments described below it appears that certain parts of this theory must now be abandoned. Thus it will be shown that except in the case of negative ions in certain oxygen-free gases the growing ion theory presents very serious difficulties, and that effects originally attributed to unclustered ions are better explained on the view that, accompanying the process of ionization by positive ions, there is an emission of rapidly moving neutral bodies which have a range of a centimetre or two at atmospheric pressure.

The statement that in air for small currents, ions of one sign only are present between the glow and the point and the plate is probably also not correct. It seems likely that

at all currents there is a back discharge from the plate increasing with current and becoming very intensified at particular points on the plate under certain special conditions.

### EXPERIMENTAL RESULTS.

The experimental results may be divided into two groups : (1) Experiments on the strength of field at the surface of a point ; and (2) Experiments on the pressure of the Electric Wind.

#### (1) *Experiments on the strength of field at the surface of a point.*

Most of the original theories were apparently verified by some experiments, in which the field at a hemispherically ended point P was determined in terms of the mechanical pull on its surface.

If  $r$  is the radius of the point,  $f$  the field at the centre of its surface, and  $P$  the mechanical pull due to the lines of force which start from the end of P, then

$$f = \frac{\sqrt{P}}{r} \times \text{constant.}$$

The value of the constant is 2.83 for a positive point and about 3.07 for a negative point. The particular value of  $f$  at which under given conditions glow first appears at P is referred to below as  $f_0$ .

The supply of "external ions" was obtained by spraying ions on to the point P from a point N in its neighbourhood. N was a point of smaller radius than P, so that N discharged more readily than P. The tendency of N to start first was further increased by causing P to project through a plate, the plane of which was only a few millimetres behind P. P and N were both horizontal and in the same vertical plane, and they were so arranged that the vertical component  $x$  and the horizontal component  $y$  of the distance between them could be varied at will.

When P was positive and external ions were supplied to it from N placed opposite to it, and at a distance  $y$  greater than about 2 cms.,  $f_0$  was reduced to about 0.6 of the value it had when N was replaced by a plate. On the above theories this field when N was present was  $f-$ , and when N was absent  $f+$ .  $f-$  was about 0.8  $f+$ .

As  $y$  was decreased below 2 cms.  $f_0$  rapidly decreased, and apparently the curve of  $f_0$  and  $y$  cut the axis of  $f_0$  at a positive

value of  $f_0$ , assumed to be the field  $f_c$  in which corpuscles ionize.  $f_c$  was about  $0.14 f_+$ .

The author has extended the work on these smaller values of  $y$ , and has found that a variation in the radius of a positively charged P, which causes a marked variation in the value of the field at P, does not appreciably affect the value of the critical distance (referred to in what follows as  $y_0$ ) at which the drop in field sets in.

Thus in Curves 1 (Pl. V.) the values of  $f_0$  are plotted as ordinates with the corresponding values of  $y$  as abscissæ. Two points were experimented upon, one of radius 0.062 cm. (crosses) and the other of radius 0.018 cm. (circles); the vertical scales of the curves are arbitrary, and are adjusted to make the two curves identical at large values of  $y$ .

As a result it appears that the two curves are then coincident throughout; this points to the conclusion that the cause of the drop in  $f_0$  does not lie with P but with N. On the other hand, except when the two points are a millimetre or two apart, the values of these fields are independent of the size of N, provided that the latter is discharging vigorously before P starts. Under these conditions  $y_0$ , therefore, has a value which is independent of the sizes of P and N; it is about 2 centimetres.

Similar experiments have been carried out with a negative P. These were not previously tried because P and N always sparked at distances under several centimetres. Assuming, however, that the field just when the spark passes is the true glow field  $f_0$ , it has since been found that, in this case also,  $f_0$  drops rapidly as  $y$  decreases. This is shown in Curves 2, where the upper curve is the  $f_0$ - $y$  curve and the lower the current- $y$  curve. P was of radius 0.026 cm. and N was a very fine wire.

Whether it is permissible or not to take the field just when the spark passes as the true ionizing field, the drop in field at short distances is shown to be real by the current- $y$  curve, which is similar in form to that obtained when P is positive and is giving true point discharge; this drop in current at short distances implies a falling field.

Also by removing the plate behind P, true point discharge was obtained for smaller values of  $y$ , and the same drop in  $f_0$  was observed. The observed currents were in this case too great, because although the lines of force ending on the sides of the point do not contribute to the pull, an appreciable amount of current is supplied to them when the plate has been removed.

The reason why the two points sparked when P was backed

by a plate can be explained as follows: N is giving positive ions to P at the ordinary high values of field at N. P starts discharging: this means that a sudden supply of negative ions enters the high field at N, the value of the field there being considerably greater than that necessary to cause the arriving ions to ionize. When the flow from P to N is concentrated by a plate behind P, this may mean a very sudden increase in current from N to P which may possibly result in a spark. Throughout the work with a negative P the point was very unsteady, no doubt owing to this tendency to spark, and attempts to find the effect of varying  $r$  have given exceedingly irregular results. With each point investigated, however, a similar drop in  $f_0$  has been observed for small values of  $y$ , and it appears not unlikely that, as in the case of a positive P,  $y_0$  is also independent of the size of P. For negative discharge  $y_0$  is about 1.5 cms.

Sometimes the discharge takes the "streamer" form, in which there is a continuous glow stretching across between the two points even when they are a centimetre or two apart. The field at P is in this case greatly reduced, especially when P is large. An isolated case of this effect, which has since been frequently observed, was given in a previous paper (*loc. cit.*).

When P is positive, sparks also pass between P and N at short distances, but it has been found that the value of  $y$  at which this occurs becomes smaller as the radius of N is decreased. In fact the results obtained when N was a piece of the finest platinum wire cut obliquely with scissors show that the values of the fields  $f_0$  given in the previous work are not correct. These were obtained by extrapolating the  $f_0$ - $y$  curve to  $y=0$ , N being throughout a sewing-needle. The dotted curve in fig. 1 shows similar results which were obtained when P was of radius .062 cm. and N .039 cm. But using the above very fine N the values shown in full lines in Curves 1 were obtained, and it appears not unlikely from these that  $f$  and  $y$  become zero almost simultaneously. The same applies to the  $f_0$ - $y$  curve for a negative P with a very fine N in Curves 2.

The reason for this sparking when P is positive is also clear. When  $y$  is quite small the positive ions from P at first find a field at N which is larger than necessary to give ionization there, so that a sudden increase in current results. The phenomenon is, therefore, similar to that occurring at a negative P, except that at a positive P it only occurs at small values of  $y$ . For a very sharp point, the lines of force spread rapidly and the distance  $y$  must be decreased, so that the field

along the path of the discharge may be uniform enough to cause the sudden current to take the spark form.

Some interesting results were obtained when N was made larger than P. P was a positively charged point 0.0115 cm. in radius. N was 0.039 cm. in radius, and was backed by a plate which was adjusted at such a distance that though N was larger than P, it started to glow either before or after P according to the distance  $y$  between them. In Curves 3 the full line shows the glow-fields at P plotted with  $y$  under these conditions. The dotted curve was obtained with N a piece of the finest platinum wire, and hence always first to start discharging. In the region BC of the full curve N started glowing first; near A, P started first, and between A and B there was a transition stage in which, as far as could be ascertained by eye, the two points started together. Curves 4a give a few of the field-current curves obtained with the large N point at varying values of  $y$ . The circle on each of the upper curves gives the current at which, as far as one could tell by eye, the point N started to glow. It is probable that the currents thus circled are too great because N could not be viewed in darkness owing to the proximity of a glowing P.

It will be seen that in the BC region the current-field curves are similar to those obtained in previous work with a fine N point (Phil. Mag. xx. 1910). In the region AB, where P either started first or simultaneously with N, it will be seen that there was no sudden drop in field at P when N started. It would appear, therefore, that though N was glowing, the negative ions supplied by it to P were not numerous enough to bring the field at P down to that in which negative ions will ionize. As the distance  $y$  was increased, the proportion of current supplied by N increased and the field at P decreased until the conditions were reversed and N supplied practically all the current. The field at P was then that in which negative ions will ionize.

These experiments also throw light upon the effect which was obtained when N was large but not so large as P. It was found in this case that at very short distances the field at P tended to become constant, as is shown in the dotted line of Curves 1 discussed above. Now at these distances, which were only slightly greater than those at which sparks occurred, N seemed to discharge almost simultaneously with P. Consequently the number of ions supplied to P was small, and hence the field at P was higher than that when N was a fine point discharging vigorously.



When P was negative the presence of N, whether discharging or not, made practically no difference to the field at P, as shown in the field-current curves 4b. This is also what one would expect on the above theories, if N at short distances is only supplying a small proportion of the current between the points. At greater distances, whatever the size of N, a large supply of positive ions from it to P has no effect on the field at P because that must still be the field in which positive ions ionize.

Lastly, an interesting effect was obtained when P was made a white hot platinum loop. When P was positively charged  $f_0$  was greatly reduced, being from 40 to 70 per cent. lower than that when P was cold and other conditions remained the same. The greatest reduction occurred when  $y$  was small. A much smaller reduction of from 5 to 10 per cent. occurred when P was negative. This is what might be expected from the well-known fact that the mobility of a negative ion increases much more rapidly with temperature than that of a positive ion. The fields, therefore, in which these ionize will be correspondingly lower, and the decrease in  $f_0$  will be most marked when N is supplying negative ions.

## (2) *Experiments on the pressure of the Electric Wind.*

It has been shown by Chattock\* for various gases, and later by Chattock and the author (*loc. cit.*), for mixtures of hydrogen and oxygen, that if a point is placed opposite a plate at a distance  $z$  from it, the average pressure on the plate  $p$  for a given current density is a linear function of  $z$  when the values of the latter are of the order of a few centimetres. If, however, these values of  $p$  and  $z$  are plotted and the curve is produced to  $p=0$ , it cuts the axis of  $z$  at some positive value, referred to in what follows as  $z_0$ .

For negative discharge in "pure" hydrogen  $z_0$  was as much as 4 centimetres, but it rapidly decreased to 3 or 4 millimetres on the addition of small percentages of oxygen or air. For positive discharge under the same conditions it was 3 or 4 millimetres throughout. In air  $z_0$  appeared to have about the same value, 3 or 4 millimetres, for both signs of discharge.

It was suggested that  $z_0$  was the average distance that an ion travelled before clustering. Support for this view was obtained from some experiments of Franck † when discharge

\* Chattock, *Phil. Mag.* [5] xlviii. p. 401 (1899).

† J. Franck, *Ann. d. Phys.* [4] Bd. xxi. p. 984.

occurred in air from the surface of a fine wire, his value of 7 mms. instead of 3-4 mms. being attributed at the time to the fact that the average field within 7 mms. of a straight wire was greater than the average field within 3 mms. of a fine point.

The possible effects of field and current upon  $z_0$  seemed interesting, and the wind pressure apparatus was set up again by the author, to investigate the question.

The apparatus was that previously used for the work on hydrogen, the experiments now, however, being confined to dry dust-free air.

Values of  $p$  and  $z$  were obtained for various values of current densities  $C$ . If it be assumed that in a distance  $dz$  ions of one sign only are present, then the specific velocity  $v$  of the ions may be calculated from the expression

$$v = C / \frac{dp}{dz}.$$

The typical result is given in Curves 5 for a current of 12.2 microamperes, with the values of  $p$  as ordinates and those of  $z$  as abscissæ.

It will be seen that the curve is made up of three parts, of which the centre part AB is straight; it is from this part AB that the values of  $v$  in past work have been calculated.

Omitting for the present the ends OA and BC from the discussion, it is clear that if the above expression holds the slope of AB should be proportional to  $C$ . The author has found, however, that this is not the case. Curves 6 show the relation between  $\frac{dp}{dz}$  and  $C$  for positive and negative discharge,  $C$  being measured in microamperes per sq. centimetre.

The full curves are the experimental curves and the dotted lines are calculated from the above expressions, assuming that  $v$  has constant values 1.32 and 1.84 cms. per sec. per volt cm. for positive and negative discharge respectively.

It will be seen that as the current increases  $\frac{dp}{dz}$  falls short of the value necessary for proportionality with  $C$ ; consequently the calculated values of the velocities of the ions increase.

The readings were all very irregular, and the points given on the curves are actually means of a number of readings differing from one another sometimes by nearly 20 per cent.

The author found that these irregularities were due, at any rate in part, to the presence of ozone in the discharge-tube, because (1) they increased with the length of the experiment and diminished again with a long wait, and (2) they were reduced by placing a tray of powdered manganese dioxide in the discharge vessel, to decompose the ozone which was formed during discharge. Effects due to ozone could not be completely eliminated in this way, because it is generated in the path of the discharge itself, but the oxide no doubt decreased the amount of it which was present at any given moment.

The effect on  $z_0$  of altering the size of the point and the current was then determined. Three points were used, all of platinum: two, A and B, had hemispherical ends and were of diameter 0.078 cm. and 0.0043 cm. respectively. The third, C, was a piece of the finest platinum wire cut obliquely with scissors. Now the field at A was nearly four times as great as that at B\*; that at C probably many times greater still. The values of  $z_0$  obtained were, however, nearly the same for all.

Thus the values of  $z_0$  for A were 0.43 cm. in positive discharge, and 0.40 cm. in negative discharge. The corresponding values for B were 0.44 and 0.41, and those for C were 0.39 and 0.36.

Those values were also independent of the current, as will be seen from the following table, if experimental discrepancies are allowed for.

$C$ (microamperes per sq. cm.).	$z_0+$ .	$z_0-$ .
0.05	.45	.40
0.14	.47	.38
0.14	.38	.28
0.31	.43	.49
0.31	.42	.40
0.69	.45	.35
1.17	.43	.43
	.44	.40

\* Obtained from the expression  $f^{0.45} = \text{constant}$ , Phil. Mag. xx. p. 270 (1910).

## DISCUSSION OF RESULTS.

It would seem at first sight that the drop in field at short distances when  $P$  is negative may be explained, as when  $P$  is positive, by the theory that at these distances  $P$  enters the unclustered region of the discharging  $N$ . That is to say,  $z_0$  and  $y_0$  would receive a common explanation which would be that they are indications of a region near a discharging point in which some of the ions are unclustered. As a detector of this region the field method would be more sensitive than the wind pressure method because the presence of a very few unclustered ions has a much greater effect on the value of the ionizing field than on the value of the wind pressure.

There are, however, great difficulties in the way of the acceptance of this theory, and of these the following are the chief:—

It is not easy to explain on the above view why  $z_0$  is constant for different points. Thus one may imagine that an ion tends to cluster when its velocity has decreased to some critical value; if the velocity of an ion close to the point is proportional to the field, this implies that growth occurs when an ion enters a certain critical field. Now assuming that the inverse square law of field holds near a point, the values of the fields at say 4 millimetres from the two points  $A$  and  $B$  may be calculated. Their ratio is about 65 to 1. It is of course probable that the inverse square law does not hold as far from these points as this, but the difference between the two fields is too great to be accounted for in this way.

Moreover, the following deductions on the growing ion view do not fit the experimental facts.

When no external ions are supplied to  $P$ , glow discharge starts when the positive ions in its neighbourhood ionize. These having just been produced by negative ions are new and consequently unclustered. On this view then, when discharge has once started, the field at such a point is that in which unclustered positive ions ionize. Now when  $P$  is negative and external ions are supplied from  $N$  at very short distances, the positive ions which start the discharge are again unclustered. The fields in the two cases should therefore be the same. It is found experimentally, however, that the field at a point opposite a plate is at least six times greater than that at the same point supplied with external ions at small values of  $y$ .

Again, assume that of these two fields the one at a point

opposite a plate is the true ionizing field for unclustered positives, and that the effects of external ions at short distances must be explained in some other way. Then when N supplies negative ions from a distance greater than  $y_0$ , the field at P is  $f-$ . One would expect, however (though not with certainty), that  $f-$  would be greater than the ionizing field for unclustered positives, whereas experiment shows it to be considerably less.

Suppose, on the other hand, that the positive ion at birth is already of molecular magnitude, and that the negative ion is the only one to go through a clustering stage. This would explain the effects at a positive P for all values of  $y$ . In fact, in the previous paper (*Phil. Mag.* xx. 1910) the possibility of a positive clustering ion was ignored from lack of experimental data. The theory, however, would not explain (1) why the field at a negative P decreases as N is brought within  $y_0$ , and (2) why there is a  $z_0$  wind effect in positive discharge.

On these grounds the author considers that, at any rate for air, the above growing ion theory must be abandoned. If in air the negative ion is at birth a corpuscle, it must immediately cluster by taking on an oxygen molecule or molecules. Further evidence for this view is given below. With regard to possible alternatives the following considerations present themselves:—

1. A  $z_0$  effect will be present if there is a large amount of ionization occurring within a few millimetres of a discharging point. Such ionization may occur if the point is emitting radiations with powerful ionizing properties, the resultant positive and negative ions giving mutually neutralizing wind pressures. On this view, it is not clear why the field at P should fall when brought close to a discharging N, because although ions are being produced close to P by the radiations from N, the field at P before it can discharge must be such as to make these ions produce more. It is true that radiations falling on the surface of a negatively charged P may give photoelectric effects, and a supply of negative ions may thus be obtained at a low field, but to fit the facts there must be similar effects at P when it is positively charged. Also it is probable that any such photoelectric currents will be negligibly small compared with the currents used in the above experiments.

2. The author tentatively suggests the following hypothesis as one which seems to fit most of the experimental facts.

It is well known that electrically neutral bodies—possibly doublets formed by the union of positive and negative ions—are present in discharge-tubes at low pressures; evidence

has also been obtained by Lonsdale\* for their existence at atmospheric pressure. It has further been suggested by Sir J. J. Thomson† that the emission of these bodies accompanies the process of ionization, so that it is not unreasonable to suppose that such are emitted from the glow region near a point. If their initial velocity is high, they may travel some distance from the point before their kinetic energy falls to that of the molecules surrounding them.

Now suppose that they are emitted from N and that the distance  $y_0$  is a measure of their range: then they will have no effect on a point P when it is placed at a distance  $y$  greater than  $y_0$ . But when P is brought within the distance  $y_0$ , doublets possessing considerable energy will strike its surface. By this bombardment energy will be communicated to P in an amount which will increase as  $y$  is made smaller, and this supply of energy may be available to aid the process of ionization at P, so that the nearer that P approaches N, the smaller will be the field necessary to start the discharge at P.

The supply of doublets, however, must be kept up by N, hence the size of P will not affect the value of  $y_0$ . Also the range of the doublets will be independent of N if it is assumed that the initial velocity of expulsion from N is constant. Some of the doublets in the distance  $y_0$  will either ionize the gas or will themselves break up, so that  $z_0$  receives an explanation on the lines suggested in (1). It is consistent with this view that  $z_0$  is independent of the size of the discharging point, and is practically the same for positive and negative discharge in air.

Now  $z_0$  is much less than  $y_0$ . It is, however, really the value of  $z$  from 0 to A in Curve 5 that should be compared with  $y_0$ : this value is about 0.8 cm. in air. Also it is possible that  $z_0$  may be too low because the wind pressure method may not be sensitive enough to detect the small number of ions occurring at greater distances.

In negative discharge in very pure hydrogen  $z_0$  is many times greater than in air, but for various reasons, detailed below, the growing ion view gives in this particular case a better explanation.

It seems necessary to suppose that these doublets are produced only during the ionization of molecules by means of positive ions. If, for instance, doublets were produced during ionization by negatives they would take the place of the positives in being a fresh source of negative ions, and the

\* Lonsdale, *Phil. Mag.* [6] xx. p. 464 (1910).

† Thomson, *Phil. Mag.* [6] xviii. p. 821 (1909).

field, therefore, at a point would be that in which negative ions will ionize, and this does not seem to be the case.

This doublet theory is thus consistent with all the experimental facts above stated.

#### *Changes of Wind Pressure with Current.*

There is still the fact to be explained that in the wind-pressure work the values of  $\frac{dp}{dz}$  per unit current decrease with increasing values of current.

The fact that  $\frac{dp}{dz}$  is constant for values of  $z$  included in the centre portion of Curve 5 implies that the carriers of the current are not generated in the main body of the gas, but travel a distance which does not differ very appreciably from the whole distance  $z$  between the point and the plate.

The observed fall in  $\frac{dp}{dz}$  per unit current might occur if the current is carried by two kinds of ions, clustered and unclustered, both travelling from point to plate and varying in relative numbers with the current. Although, as shown below, such a view is possible for negative discharge in hydrogen containing traces of oxygen, it does not seem probable in air\*.

The simplest explanation is that the current between the point and the plate is not wholly carried by ions of one sign only, but that there is a certain amount of back discharge from the plate which at very small currents is negligible in amount, but which increases as the current and the field at the plate increase.

That such a back discharge can exist in air, even at low currents, is shown by the part BC of Curve 5. In this region there is a rapid fall in  $\frac{dp}{dz}$ . At a current of 1 microampere the value of  $z$  at which the curve leaves the straight is about 6.0 centimetres, and at 10 microamperes it is about

\* It is true that in a recent paper (Phil. Mag. Feb. 1911) Sir J. J. Thomson has shown that positive ions  $O_3$  and  $O_6$  are present in the positive rays at low pressures. It is, therefore, conceivable that these and similar clusters with negative charges may be present at atmospheric pressure in varying amounts depending on the value of the current flowing from the point; if they carry an appreciable percentage of that current, the observed changes in  $\frac{dp}{dz}$  may thus be produced.

2.8 centimetres. This sudden change in  $\frac{d\rho}{dz}$  was always accompanied by a speck of light on the plate; this is no doubt the source of a back discharge which effectually reduces the value of  $p$ . The effect was observed in both positive and negative discharge. Such a phenomenon will occur, if at any region on the plate the back discharge accidentally increases; the lines of force from the point will then converge towards that region and concentrate the current there, thus tending to increase the back discharge still further, and so intensify the concentration of the lines. In the earlier work on hydrogen when, in negative discharge, this back discharge was known to be present, very marked effects of this kind were observed (see *Phil. Mag.* xix. p. 455).

This instability apparently sets in when the field at the plate reaches some critical value, since the necessary distance between point and plate decreases as the current increases. For values of  $z$  less than that from O to B, it may be assumed that the back discharge is general over the surface of the plate, but increases with increasing field and current.

#### *Discharge in Hydrogen.*

Franck\* has suggested that gases may be arranged in the following order according to the magnitude of the affinity which their molecules possess for negative electrons:—chlorine, nitric oxide, oxygen, hydrogen, nitrogen, argon, and helium. According to him chlorine and oxygen molecules, for instance, have a strong affinity for negative ions, but the gases at the other end of the series have comparatively little. Thus he has shown that when argon and nitrogen have had all electropositive impurities such as oxygen and chlorine eliminated from them, the velocities of the negative ions rise to very high values, even at atmospheric pressure. For instance, in pure argon the velocity of a negative ion was 206.3 cms. per volt cm., and in pure nitrogen 80–145 cms.; but traces of oxygen reduced these to normal values 1.70 and 1.84 respectively. The velocity of the positive ion was normal throughout; thus in pure nitrogen it was 1.27 and in impure nitrogen 1.30.

Previous to this Prof. A. P. Chattock and the author (*Phil. Mag.* xix. 1910) found effects in the wind-pressure work in hydrogen which may be similarly explained. These experiments, however, were complicated by the presence of an unknown amount of back discharge from the plate, so that, in negative discharge in pure hydrogen, the direction of the

\* *Verh. d. D. Phys. Gesell.* xii. pp. 291 & 613 (1910).



electric wind was sometimes even reversed. It is, therefore, impossible to say what the actual velocity of the negative ions was, but in the light of Franck's work it is almost certain to have been high. It was found that the wind pressure rapidly rose with the addition of slight traces of air, that is to say, the calculated velocity of the negative ions rapidly fell, as was found by Franck in argon and nitrogen. On Franck's view this is due to the tendency possessed by the negative ions, when oxygen is present, to attach themselves to molecules and become clustered; in oxygen-free inert gases they remain for a longer time in a corpuscular state.

This is supported by the results obtained for  $z_0$  in pure and impure hydrogen. In pure hydrogen for negative discharge,  $z_0$  was about 4 centimetres, but with the addition of 2 per cent. of oxygen it fell to about 0.3 cm. If it is assumed that 4 centimetres is the distance that negative ions travel in pure hydrogen before clustering, one would expect a rapid decrease in this distance when small percentages of oxygen are added; the negative ion in pure hydrogen would thus cluster within a distance depending on the amount of oxygen present. The growing ion view as an explanation of  $z_0$  may, therefore, be retained for the particular case of hydrogen, either pure or containing not more than a few traces of oxygen.

It is not so easy to explain the form of the  $p$ - $z$  curves for different percentages of oxygen. If for negative discharge in hydrogen, curves of the type of curve 5 are taken, it is found that, in general, whatever the percentage of oxygen present the part AB is straight but that its slope for a given current rapidly increases with the addition of oxygen. Thus at 0.15 per cent. oxygen  $z_0$  was found to be 2.4 centimetres, but the slope of the curve was constant and about 1/6th of that in ordinary impure hydrogen. The fact that at a given percentage  $\frac{dp}{dz}$  is constant, implies that outside the distance  $z_0$  no further clustering takes place, and yet the ions appear to move much faster than in impure hydrogen. At the time the facts were explained by the theory, that in pure hydrogen there was a considerable amount of back discharge from the positive plate, which gradually ceased with the addition of oxygen. Now the slightest reversal of the electric wind in pure hydrogen shows that back discharge was undoubtedly present; but if the negative ions are corpuscular in nature they will not contribute any appreciable wind, and the amount of back discharge which is necessary to give the observed reversal need, therefore, only be very slight.

A bright glow was also observed on the plate in pure hydrogen. This might be explained either on the view that the plate was bombarded by corpuscles or that the plate was a source of back discharge: a corpuscular bombardment may of course itself be a cause of back discharge. At any rate, the glow was connected with the above phenomena in that it gradually disappeared as oxygen was introduced.

The following argument reconciles the results with Franck's theory:—

In some work on the combination produced by point discharge in hydrogen containing traces of oxygen\*, evidence was obtained for the theory that, at low percentages of oxygen much of the oxygen was concentrated in films at the electrode surfaces. Owing to the suggested affinity between oxygen molecules and negative electricity, the density of this film at the cathode will be far greater than at the anode, and will increase as the percentage of oxygen in the gas increases. Now in negative discharge the negative ions which take part in the discharge will be produced at the point, and if they are ejected from the point surface they will pass through this concentrated layer of oxygen. A certain number of them will immediately take on oxygen and others will escape uninfluenced. The percentage of the clustered to the unclustered will depend on the density of the film, that is to say, on the percentage of oxygen present. Once free of this film they will continue their path without further change, since it may be shown that for the percentages of oxygen considered the chances of further collisions between ions and oxygen molecules in the main body of the gas are very remote.

Since the clustered ions are wind producing and the unclustered are comparatively not, the value of  $\frac{dp}{dz}$  will increase with increasing percentages of oxygen. Also, since  $z_0$  for these clustered ions is on this view very small, the resultant  $z_0$  for the whole discharge will also decrease with increasing percentages. To explain the residual  $z_0$  of about 3 millimetres in very impure hydrogen, when all the ions are clustered, some such theory as the doublet theory suggested above is necessary. The unclustered ion is, therefore, a special case for negative discharge in oxygen-free gases only.

Franck's theory thus offers a very satisfactory explanation of all the wind-pressure results obtained in hydrogen containing traces of oxygen.

\* Chattock and Tyndall, *Phil. Mag.* [6] xvi. p. 24 (1908).

*Changes of Pressure accompanying Point Discharge  
in Hydrogen.*

This view, that the negative ions in pure hydrogen are corpuscular in nature, also throws light on the net changes of pressure which result in closed vessels from point discharge in hydrogen.

If there is any oxygen present water is formed, and there is in consequence a small decrease in pressure. It was found, however, that even if oxygen is very carefully excluded, a very small contraction still occurs (*loc. cit.* Phil. Mag. 1908). This effect was attributed to an absorption of ions at the surfaces of the metal electrodes. With a copper plate this contraction amounted to 2 atoms per ion—the expression “per ion” being defined as “per hydrogen atom set free in a water voltameter placed in series with the discharge vessel.” At the time it was generally supposed that an ion was a charged molecular cluster. From the results of the work of Wellisch and others, on the mobilities of the ions, there is now however considerable evidence for the theory that an ion is either molecular or atomic in size at atmospheric pressure. Now, if in hydrogen the negative ion is a corpuscle, there can only be absorption of gas at one electrode, the plate in positive discharge and the point in negative discharge. The two atoms absorbed, “per ion,” must thus be carried in the gas by a single positive hydrogen ion. This is, therefore, evidence for the view that the positive ion in pure hydrogen at atmospheric pressure is a charged molecule.

But if there was an evolution of gas from the cathode and an absorption of gas at the anode, as has been found by Skinner\* to occur in glow-discharge in hydrogen at low pressures, the observed fall in pressure might have been merely a small difference effect. The author rejects this view for the following reason. The point used was a piece of the finest platinum wire and therefore of very small surface. By making it the cathode it could be easily denuded of surface gases by discharge—at any rate temporarily. The same electrodes were used for months without change, and for the greater part of that time the point was the cathode. If the Skinner effect were responsible for the result, one would expect that after a long negative discharge from the point, the contractions per coulomb in following discharges would differ widely according to whether the discharge was positive or negative—that is to say, according to whether the plate or the denuded point was the cathode.

\* Skinner, Phil. Mag. xii. p. 481; Phys. Rev. xxi. pp. 1 & 169.

It was found, however, that the contractions on the average were the same in both cases. It seems, therefore, that in the purest possible hydrogen, with point and plate no longer "fresh," emissions from the cathode are only corpuscular in nature.

Again it may be argued that this contraction was due to residual impurity in the gas. The reasons for not accepting this view are set forth in the above paper.

There is no doubt that, in negative discharge in pure hydrogen, a little back discharge occurs. If the ions in this are generated in the gas at the surface of the plate, the above conclusions will not be affected. But if they are emitted from the metal itself, this is not the case. However, it is shown above that it is very probable that the amount of back discharge was quite small, so that its effect on the result may be neglected.

The theory of Franck will also explain the amount of combination which is produced in negative discharge between hydrogen and oxygen, when the latter is present in small quantities. In the above paper it was shown that when oxygen is present in hydrogen, combination occurs between them, and as far as one could tell between them only, even when nitrogen is present in large excess.

The results receive a simple explanation on the view that to make oxygen and hydrogen combine, they must be suitably presented to one another together with an electric charge. The efficiency of negative discharge is greater because the affinity of negative ions for oxygen is great. Since their affinity for nitrogen is, according to Franck, even less than that for hydrogen, the tendency to unite nitrogen and hydrogen will be small, at any rate with small currents. One may assume that having caused a certain amount of combination, a negative ion may either be retained by the products or in some way may be rendered no longer efficient as a combining agent. Other corpuscles will attract oxygen, but may not be suitably presented to hydrogen before the time of entry into a molecular life.

Now if, as suggested, some of this oxygen in negative discharge is concentrated as a film on the point, then, as the percentage of oxygen increases, the density of this film increases. At first the chances of free corpuscles producing combination in the neighbourhood of the point will greatly increase with this increasing density, and the combination per coulomb will rapidly increase. With higher percentages of oxygen, however, oxygen will predominate over hydrogen in the surface film, and the free corpuscles will be decreased in

number by clustering with oxygen before hydrogen can be suitably encountered. The curve of combination per coulomb and percentage of oxygen will thus pass through a maximum value; this was found to occur experimentally, the combination per coulomb being greatest at 0.003 per cent. oxygen, and being then three times greater than at 14 per cent. oxygen.

There was apparently a much smaller maximum in positive discharge; this also is reasonable because the main oxygen layer is then at the negative plate, where the number of free corpuscles is very much less, if not *nil*.

#### SUMMARY.

1. Further measurements of the field at the surface of a discharging point have been made, and it is suggested that some of the observed effects, originally attributed to the action of unclustered ions, are due to an emission of uncharged doublets during ionization by positive ions.
2. When extrapolated, the curve of wind pressure on a plate ( $p$ ) and the distance between point and plate ( $z$ ) cuts the axis of  $z$  at a positive value  $z_0$ .  $z_0$  is independent of the field at the point surface and of the current. It is in general about 4 millimetres in length, but is much greater in negative discharge in pure hydrogen. The effects in air are explained by the doublet theory, and in hydrogen by the view that the negative ions in that gas are, in the main, corpuscular in nature.
3. The apparent velocities of the ions in air, as measured by the wind-pressure method, become greater as the current between point and plate increases. This is explained by the presence of a back discharge from the plate, which prevents the method from being applied with accuracy except when the current is small.
4. Evidence is adduced in support of the view that the positive ion in pure hydrogen is a charged molecule.
5. The amount of combination which point discharge produces in hydrogen containing different percentages of oxygen and nitrogen, and the form of the wind-pressure curves in these mixtures are discussed and explanations are offered.

LXVII. *A New Method of Measuring the Luminosity of the Spectrum.* By FRANK ALLEN, M.A., Ph.D., Professor of Physics, University of Manitoba, Winnipeg\*.

WHEN a ray of light entering the eye is periodically interrupted by a rotating sectored disk, a sensation of flickering is produced until the interruptions reach a certain critical frequency at which the impressions of the separate flashes of light become fused into one continuous sensation. This peculiarity of vision, in one form or another, was observed and commented on by philosophers in ancient times, but was first quantitatively investigated by the Chevalier D'Arcy, who in 1765 measured the least time a revolving glowing coal required to trace an apparently continuous circle of light. In more recent times this subject has been studied by numerous investigators, and many phenomena of interest and importance elucidated.

It was discovered by Ferry †, and subsequently, in another manner, by Porter ‡, that the duration of the sensation of undiminished brightness of a flash of light, at the critical frequency of interruption, depended only on the luminosity of the light and not in any way on the colour. The duration of the impression was found by both investigators to be inversely proportional to the logarithm of the luminosity of the light.

In the course of some investigations on colour vision in which constant use was made of the measurement of the critical frequency of flicker, it became desirable to measure the luminosity of the spectrum in some direct manner. As apparatus for the more commonly used methods was not available, a method based on the above principle of Ferry and Porter was devised which is believed to possess some new features.

The arrangement of apparatus is shown in fig. 1. Light from an acetylene flame (A), after concentration by a lens (B), passed through an open sector of the disk (D), which was rotated by an electric motor, then through two Nicol prisms (E and F) arranged with their principal sections horizontal, thence through the spectrometer (G), and was finally viewed in a Hilger eyepiece (H) in which all the light of the spectrum, except a narrow central band of any desired colour, was cut off by means of adjustable shutters.

In the path of the light at C a small mirror was set so as

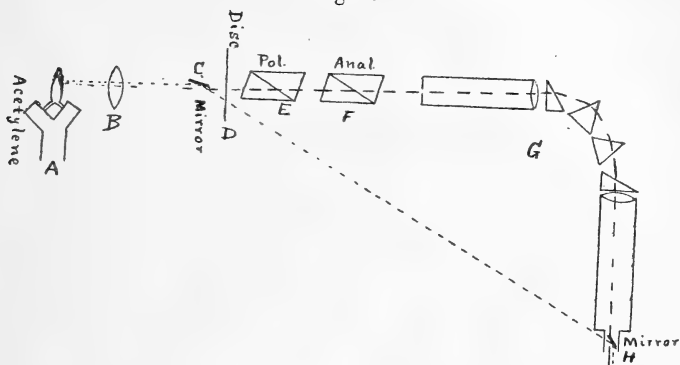
\* Communicated by the Author.

† E. S. Ferry, Am. Journ. Sci. vol. xlv., 1892.

‡ T. C. Porter, Proc. Roy. Soc. vol. lxiii., 1898; vol. lxx., 1902.

to reflect white light to a similar mirror mounted on the eyepiece H, which reflected it down through a hole in the eyepiece upon the polished sloping top of a steel pointer,

Fig. 1.



with which the instrument was provided instead of cross-hairs, which finally reflected the light to the eye. The observer, therefore, looking into the eyepiece could see a small patch of white light, and immediately above it a patch of suitable size of any desired colour of the spectrum. The white light could be reduced in intensity to any desired amount by changing the inclination of the mirrors at C or H, while the intensity of the spectrum was controlled by rotating the polarizer (E). When the disk (D) rotated both patches of light flickered necessarily at the same rate.

The spectrometer used was of the Hilger automatic type, which gave a dispersion slightly in excess of twelve degrees.

In measuring the luminosity of the spectrum the method of procedure was, first, to lower the intensity of the patch of white light until it was of the same luminosity as a patch of violet ( $\lambda = .414 \mu$ ) of undiminished brightness, the principal sections of the nicols being parallel, as near the end of the spectrum as it was possible to make exact measurements upon. The intensities of the white and violet lights were considered equal when the critical frequency of flicker was the same for both. The white light now became the standard of comparison, and was maintained continuously at this low intensity through all the observations. Each selected part of the spectrum was in turn brought into view and reduced to the luminosity of the white patch by rotating the polarizer an amount depending on the brightness of the part of the spectrum under observation. In every case the same critical

frequency of flicker of both white and coloured lights was taken to mean equality of brightness. The brighter the spectral light the greater was the rotation of the polarizer required, since a smaller portion of light was sufficient in that case to equal the luminosity of the standard of comparison. The luminosity of each part of the spectrum is inversely proportional to the intensity of the portion of light passing through the nicols, which is proportional to the square of the cosine of the angle between the principal sections of the prisms.

Observations were made upon nineteen portions of the spectrum. These are given in Table I. The results are shown graphically in the luminosity curve in fig. 2.

TABLE I.

$\lambda$ .	Angle between planes of nicols = $\alpha$ .	$\cos^2 \alpha$ .	$\frac{1}{\cos^2 \alpha}$ .	Reduced to max. = 100.	Remarks.	
·414 $\mu$	0°	1	1	0·16	Standard of comparison.	
·442	9°	·975	1·02	0·16		
·460	35°	·671	1·49	0·24		
·480	63°	·206	4·85	0·78		
·491	70°	·117	8·55	1·42		
·500	76° 45'	·052	19 05	3·10		
·515	83°	·0148	67·34	11·1		
·530	84° 54'	·0079	126·53	20·6		
·564	87°	·0033	365·4	59·4		
·593	87° 40'	·0016	603·5	98·0		
(·598)	.....	.....	(615)	(100)		Not observed.
·601	87° 40'	·0016	603·5	98		
·621	87° 15'	·0023	434·8	70·6		
·648	86°	·0048	205·5	33·4		
·663	84°	·0109	91·57	15·2		
·677	79°	·0364	27·47	4·47		
·695	73° 24'	·0816	12·25	1·99		
·712	55° 24'	·322	3·1	0·50		
·733	37° 30'	·629	1·59	0·25		
·748	0°	1	1	0·16	{ Equal to standard of comparison.	

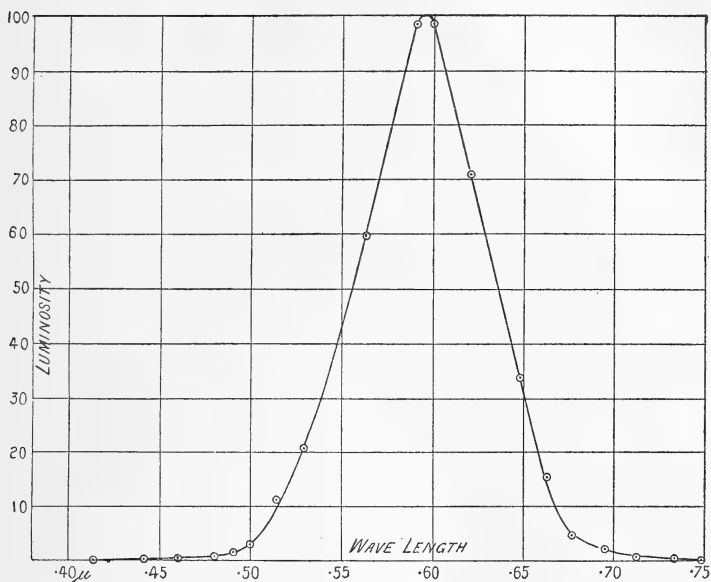
In plotting the curve the maximum ordinate is given an arbitrary value of 100, and the others are reduced proportionately. At least three independent observations were made on each colour, and these always agreed with each other very closely. In thirteen of the nineteen cases the settings of the polarizer differed from the mean value by less than one per cent. ; in the remaining six the differences were greater.

The ordinates of the curve do not represent absolute luminosities of the spectrum, but only values relative to the



chosen standard, which was ultimately the violet colour of wave-length  $0.414 \mu$ . Since, however, this standard was of

Fig. 2.



very low intensity, it need not be considered except in the ends of the spectrum where the luminosity is small. The accuracy of the measurements is influenced, especially in the extreme red and violet, by stray white light, which was unavoidably present in sufficient amount to influence the measurements of the brightness of the spectrum in those feebly luminous regions.

The method as originally devised was to adjust the speed of the sectored disk until the critical frequency of flicker of the standard violet was reached. The disk was then to be maintained constantly at this rate of rotation, and each colour in succession reduced in intensity by rotating the polarizing prism until critical frequency was reached. The luminosity could then be determined as before. By this method the auxiliary white standard could be dispensed with, and the luminosity of the spectrum obtained directly. It was found, however, impossible to maintain the motor at a sufficiently constant speed, and the method modified as described in this paper was substituted.

LXVIII. *On the Comparison of Two Self-Inductions.*

By Professor A. ANDERSON\*.

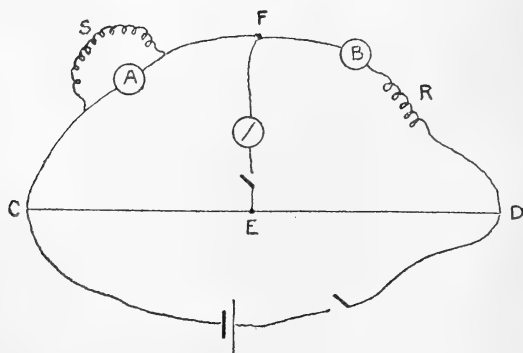
**T**WO conductors connected in parallel, whose resistances are  $P$  and  $Q$ , coefficients of self-induction  $L$  and  $N$ , and coefficient of mutual induction  $M$ , are equivalent to a single conductor of resistance  $\frac{PQ}{P+Q}$ , and coefficient of self-induction  $\frac{2MPQ + LQ^2 + NP^2}{(P+Q)^2}$ , in cases where the current is not oscillatory. It follows that a system consisting of a coil of resistance  $P$  and coefficient of self-induction  $L$  connected in parallel with a non-inductive resistance  $S$  has an inductance equal to  $\frac{LS^2}{(P+S)^2}$ .

It is thus possible, by shunting a coil with a non-inductive resistance, to reduce its effective self-inductance, and to make the latter equal to that of another coil whose coefficient of self-induction is less.

A very easy and, possibly, useful method of comparing the coefficients of self-induction of two coils is readily deduced; and, though it involves a double balance, there is little more experimental difficulty in it than in the measurement of a resistance.

Referring to fig. 1, the coils  $A$  and  $B$ , of which  $A$  has the higher self-induction, are placed in the two arms of

Fig. 1.



a Wheatstone bridge,  $B$  in series with a variable resistance  $R$ , and  $A$  shunted by a variable shunt of resistance  $S$ . The

\* Communicated by the Author.

resistance CE is equal to the resistance ED. If the resistance in the arm DF is equal to that in CF there will be no permanent current in the galvanometer, and if the self-induction in DF is equal to that in CF, there will be no transient current when the battery key is put down after the galvanometer key. The method consists in varying S and R till both these conditions are fulfilled.

Denoting the resistances of A and B by X and Y, and their coefficients of self-induction by L and N, we have, then,

$$\frac{L}{N} = \left(1 + \frac{X}{S}\right)^2, \quad \text{and} \quad \frac{XS}{X+S} = Y + R.$$

It is possible that Y may be greater than  $\frac{XS}{X+S}$ ; and, in that case, the resistance R should be in CF in series with the system consisting of A and S. Any inconvenience of moving the variable resistance R from DF to CE will be avoided by having variable resistances in both arms.

The following experiment will illustrate the ease with which the adjustment can be made.

Resistance of A = 109.3 ohms.

Resistance of B = 14.5 ohms.

R in ohms.	S in ohms.	Kick on taking out Battery Key.
0	16.7	Right.
10	31.5	Right.
20	50	Right.
30	75	Right.
40	109	Right.
50	157	Right.
60	233	Right.
70	371	Right, but small.
80	697	Left.
72	414	Right, one division.
73	434	No kick.
74	465	Left, one division.

N the coefficient of self-induction of B is therefore equal to

$$\frac{434^2 L}{(434 + 109.3)^2}, \quad \text{or} \quad \frac{L}{N} = 1.56^*.$$

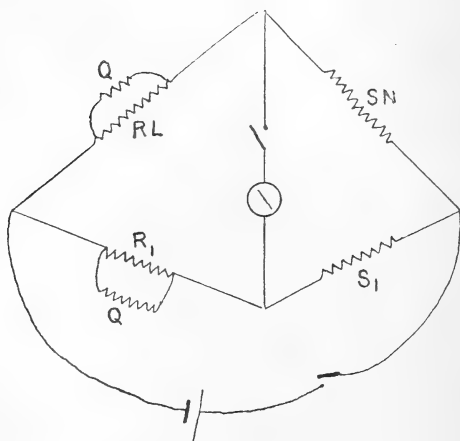
\* Very careful measurements by Mr. W. G. Griffith of the self-inductances of these coils, in which Lord Rayleigh's method was used, gave, for A, 0.236 henry, and, for B, 0.151 henry.

When the kick is large it is not necessary to have an accurate balance for permanent currents; a rough balance will suffice. But when the transient currents are nearly zero, the permanent balance must be as good as possible.

In the above experiment an ordinary mirror galvanometer of the Thomson type was used.

The following is, perhaps, the easiest way of applying the principle of the method:—

Fig. 2.



The self-inductions,  $L$  and  $N$ , of the coils whose resistances are  $R$  and  $S$  are to be compared.  $R_1$  and  $S_1$  are non-inductive resistances equal, respectively, to  $R$  and  $S$ . There is thus a balance for steady currents. The  $Q$ 's are equal non-inductive shunts which are varied till there is no transient current.

We have then  $N = \frac{LQS}{R(R+Q)}$ . Thus  $N$  must not be greater than  $\frac{LS}{R}$ . Otherwise, the shunts must be applied to  $S$  and  $S_1$ .

University College, Galway.  
March 13, 1911.

LXIX. *Notes on the Electrification of the Air near the Zambesi Falls.* By W. A. DOUGLAS RUDGE, M.A., Professor of Physics, University College, Bloemfontein\*.

THE electrification of the atmosphere is always very marked near a waterfall, and it seemed of sufficient interest to take a series of observations in the neighbourhood of what is probably the largest fall in the world. It is hardly necessary to describe the structure of the fall, but it may be noted that it possesses the peculiar feature of falling over the side of the gorge rather than over one end. The Zambesi just above the fall is nearly a mile and a half wide, but at the fall it narrows to about 1900 yards. The width of the chasm is about 350 ft., the depth 420 ft. The river below the fall pursues a course at right angles to its original direction, and a very energetic churning of the water ensues, an enormous cloud of spray being formed, which may be seen for many miles. The actual height of the cloud above the gorge varies considerably, the maximum height noted by the observer being about 600 ft. in the early morning, falling to about one-third of this height at midday, increasing again towards the evening.

The instrument used in taking observations was an electroscope of the Exner type attached to a telescopic stand, so that its height above the ground could be varied, a wire tipped with radium serving as the collector. Observations were taken at a fixed station and also at different distances from the falls.

Owing to the peculiar formation of the river-bed just above the fall, it is possible to take observations at the very edge of the fall itself, and also at various points in the river above the fall, as well as from the side of the gorge opposite; but the results obtained in the immediate neighbourhood are not of much value, as the charges obtained were so great that the instrument used was incapable of measuring them. A wire poorly insulated, and stretched halfway across the gorge close to the bridge, about half-a-mile from the fall, became so strongly charged that sparks 2 or 3 mm. in length were easily obtained from it.

*Observations at the Fixed Station.*

These consisted in taking the potential gradient at frequent intervals.

\* Communicated by the Author.

The potential varied enormously but to some extent uniformly with the time, and changed sign during the morning. Observations were begun before sunrise 6.30 to 7, and were continued at intervals up to 10 P.M. The position chosen was such that the cloud from the falls intervened between the observer and the rising sun. In the early morning at a distance of about one mile from the fall the charge was negative. The amount of electrification depended upon the joint effect of the sun and the cloud, a large increase being seen when the rays of the sun were able to break through the cloud. The maximum was reached when the position of the sun allowed it to shine above the cloud. During the whole period of observations, the sky was quite free from ordinary clouds. The maximum electrification occurred at about 8 A.M., though no quite precise time can be stated, for, as might be expected, the wind had a very appreciable influence. From 8 o'clock the potential fell and reached a minimum value about 10.30 at the station where the observations were made, but when the electroscope was carried nearer to the falls, the potential rose to a negative value; and on taking the instrument some distance further from the falls, a slight positive electrification was obtained. From this period on to 2 P.M. the electrification was nearly always zero, but afterwards a positive value was developed.

As the country round the falls is thickly wooded, observations could only be taken satisfactorily where free from trees. The fixed station was at a height of about 100 ft. above the falls, and no high trees intervened, or, rather, the line drawn from the station to the falls' cloud passed well over the intervening trees. At 6 P.M. the positive electrification was well-marked, and it increased in value up to 7 o'clock, after which it fluctuated, falling once or twice to zero and changing sign. At 9.50 it was positive.

In the evening when the charge was positive at the fixed station, it was still as strongly negative near the falls; the sign changed at about 2000 paces from the falls.

Observations were taken in a canoe at points on the river above the falls. At a distance of a mile above, the value had fallen to zero, at which it remained for a considerable distance. At  $7\frac{1}{2}$  miles above, a landing was made upon a rock in the rapids, and here the electrification was decidedly positive but comparatively small in amount—100 volts per metre. The potential falls very rapidly above the falls. Livingstone Island is perched at the very edge of the fall, and is about 200 yards in length measured in the direction

TABLE I.  
Potential Gradient at Fixed Station.

Time.	Height of Electroscope above ground.	Charge.	Potential difference in volts.	P.D. per metre.
6.38 A.M. ....	160 cms.	—	75	47
7.0 .....	"	—	75	47
7.17 .....	"	—	175	110
7.20 .....	"	—	200	125
7.23 .....	"	—	250	156
7.25 .....	"	—	375	237
7.27 .....	"	—	500	314
7.30 (1) .....	"	—	250	156
7.50 .....	"	—	510	317
8.0 .....	"	—	525	330
8.30 .....	"	—	275	171
9.00 .....	"	—	175	110
10.30 .....	"	—	0	0
↓				
2.0 P.M. ....	"	+	25	16
2.30 .....	"	+	25	16
2.45 .....	"	+	25	16
3.30 .....	"	+	40	25
4.0 .....	"	+	50	31
4.30 .....	"	+	50	31
5.00 .....	"	+	100	62.5
6.0 .....	"	+	120	75
6.30 (2) .....	"	+	0?	0
6.35 .....	"	+	150	93
6.45 (3) .....	"	+	150	93
7.00 .....	"	+	125	78
8.40 .....	"	+	140	88
9.50 .....	"	+	100	62.5
10.30 .....	"	+	100	62.5

Between 10.30 and 2 the charge was very small.

of the river. At the end nearer to the falls, the electroscope was at once charged to the maximum, while at the further end it had a value of only 600 to 800 volts per metre.

A set of observations was next taken at different distances from the falls, starting from the fixed station at a time when the electrification at that place was at a minimum. Distances were measured by pacing, and detours had to be made in order to obtain spaces free from trees; therefore the results are only approximate. The total distance between the fixed point and the falls was about 2,500 paces, and observations were made at intervals of 100 paces, with the result shown in the table. After 1000 paces had been made, the charge was too great to be measured directly by the instrument, so

that the comparative values had to be obtained by noting the time required by the leaves to reach the maximum.

At 1000 paces	the time was	50 secs.
„ 1100	„ „	36 „
„ 1200	„ „	30 „
„ 1300	„ „	25 „

At this point the indication became rather erratic owing to the spray being carried by the wind towards the instrument. When the spray was encountered on the confines of the "Rain Forest," the electroscope was charged to a maximum in a second or two, in some cases in even less, so that the leaves were continuously moving up and down.

TABLE II.  
Potential Gradient at Different Distances from Fall.

Distance from Falls in paces.	Height of Electroscope above ground.	Charge.	Potential Gradient. Volts per metre.
500 .....	20 cms.	—	5,000
600 .....	20 „	—	*
700 .....	20 „	—	3,500
800 .....	50 „	—	*
900 .....	50 „	—	2,400
1,000 .....	50 „	—	1,700
1,100 .....	50 „	—	1,390
1,200 .....	50 „	—	1,000
1,300 .....	50 „	—	*
1,400 .....	50 „	—	*
1,500 .....	190 „	—	500
1,600 .....	160 „	—	425
1,700 .....	160 „	—	300
1,800 .....	160 „	—	250
1,900 .....	160 „	—	200
2,000 .....	160 „	—	100
2,100 .....	160 „	—	75
2,200 .....	160 „	—	75
2,300 .....	160 „	—	50
2,400 .....	160 „	—	50
2,500 .....	160 „	0	0

\* Values uncertain.

At the bottom of the gorge, but screened from the falls by a bend, the electroscope diverged to a maximum in 15 seconds, but if the spray got carried by the wind towards the instrument, it was at once charged to a maximum. At Livingstone, eight miles from the falls, the potential had fallen to a low value, and always, during the short time spent there, was positive. On the way to Livingstone some



observations were made from the moving train by projecting a radium-tipped wire connected with the electroscope out of the carriage window. When the steam from the engine passed over the carriage, a very large positive charge was shown by the electroscope. The potential gradient appeared to be of the order of more than 1000 volts per metre, due to the steam, as when that was blown away from the carriage, the potential fell to practically zero.

On the southward journey, observations were taken at most of the stopping-places, including the summit of the Matoppos Hills. The electrification was always positive, no abnormal values being observed.

At the edge of the falls the potential gradient if it increased at the same rate would have been enormous, but no doubt in the cloud itself there would be some conduction going on (certainly convection). From the rate at which the electroscope leaves diverged to a maximum, the charge at the falls would be twenty-five times as great as that at a distance of 1100 paces, viz. : 25,000 volts per metre.

The curve shows the rate up to within 500 paces.

This of course is one day's observation; no doubt the potential gradient would vary from time to time.

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LXX. *Photographs of Vibration Curves.*  
By C. V. RAMAN, M.A.\*

[Plate VI.]

THE photographs of vibration-curves forwarded with this note possess certain features of interest which seem to justify their publication. Experimental work on vibration-curves relating to the sonometer, violin, and pianoforte that has been published in recent issues of this Journal, considerably interested me and induced me to undertake some work in the same direction.

The photographs (figs. 1 to 9, Pl. VI.) were obtained, working with an apparatus a description of which has already been published elsewhere †.

The idea of the construction of this apparatus was suggested to me in 1908 by the problem of the motion of the bridge of the violin. I recognized that the bridge is subject to a normal forcing of double the frequency of the oscillation

\* Communicated by the Author.

† See 'Nature,' Dec. 9, 1909, on "The Maintenance of Forced Oscillations of a New Type," and 'The Journal of the Indian Mathematical Club,' October 1909.

of the string, and might, under suitable circumstances, be expected to oscillate with the double frequency. To verify this point, I did, at that time, think only of direct aural observation on a specially constructed model. This idea was worked out by me immediately and with success. To hear a note of the double frequency it was essential that all sounding parts that would emit the fundamental should be abolished. In other words, the model sonometer (for so it was) had only, so to speak, a very much magnified bridge, *i. e.*, only a sounding-board *normal* to the wire, instead of, as usual, one parallel to it. This was arranged without difficulty. The sounding-board was fixed in a rigid frame; one end of the stretched wire was attached to the frame, and the other end normally to the centre of the sounding-board. It was verified by comparison with a sonometer of the ordinary type that the note emitted by the instrument had double the frequency of the vibrations of the wire, in whatever way the latter was set in vibration. A better musical effect was obtained when the sounding-board was replaced by a membrane stretched on a circular ring.

The frequency-relation was also verified by the following (it is believed novel) method. A point on the vibrating wire quite close to the sounding-board has only a microscopic motion. This has, however, two components: one transverse to the wire having the same frequency as its oscillation; the other normal to the sounding-board, *i. e.*, lengthwise of the wire, having double the frequency of the vibrations of the latter. The path described is the characteristic Lissajous's figure, *i. e.* a parabolic arc. This was verified by observation.

For photographing the vibration-curves of the wire and the sounding-board in my apparatus, a three-legged optical lever was as usual employed. The amplitude of oscillation of the sounding-board was not, however, so small as to require very considerable magnification. One leg rested upon the edge of the sounding-board, the other two in a hole and a slot cut in a brass plate kept fixed nearly flush with the sounding-board. The lever had only a plane mirror attached to it. One source of light was a horizontal slit, and the other was a vertical slit placed immediately behind the oscillating wire. Both were illuminated by sunlight and had collimating lenses in front of them. The light from the former after reflexion at the oscillating mirror, and the light from the latter after reflexion at a fixed mirror, fell upon the lens (having an aperture of about 2 inches diameter) of a roughly constructed camera. In the focal plane of the latter was placed a brass plate with a

vertical slit cut in it. The images of the horizontal and vertical slits fell, one immediately above the other, on the slit in the plate. Only a very small length of the former, *i. e.*, practically only a point of light, was allowed to fall upon the photographic plate. The dark slide which held this was moved quickly by hand in horizontal grooves behind the slit in the focal plane of the camera. No arrangements were made to obtain a strictly uniform velocity of the plate, as the main point of interest in the photographs was not interfered with by want of it.

Figs. 3 to 9 all show that the motion of the sounding-board had a frequency strictly double that of the vibration of the wire. From the vibration-curves of the wire in those figures, it will be seen that the string was excited (by plucking with the fingers) in various ways. In fact, subject to the remarks made below, it may be stated that it was a matter of indifference how the wire was excited: the frequency of the sounding-board was always twice that of the vibration of the wire.

Fig. 1, however, distinctly shows the presence of an oscillation of the same frequency as the wire, superposed upon an oscillation of the double frequency. This is also perceptible in fig. 2. The cause of the appearance of the "fundamental" in these cases was successfully investigated. It will be seen that the oscillation of the wire recorded on the plate is quite large in fig. 1, and fairly large in fig. 2. This does not mean that the oscillation of the wire had a smaller amplitude in the other cases, but only that the vertical oscillation of the wire (which alone was recorded upon the plate) was comparatively larger when figs. 1 and 2 were photographed. As a matter of fact, the wire was plucked almost horizontally when all the photographs except figs. 1 and 2 were taken. In photographing fig. 2 it was the intention to pluck horizontally, but it appears to have been carelessly done. In the case of fig. 1 the wire was plucked vertically. Now it was found that there was an important difference between the oscillation of the wire in a horizontal and in a vertical plane. The equilibrium position of the wire is not quite a straight line, but a catenary of small curvature. The oscillation of the sounding-board depends upon the second-order differences in the length of the displaced and equilibrium positions of the wire. When the oscillation is in a vertical plane, the fact that the equilibrium position is a catenary becomes of importance. The changes in length about the equilibrium position become unsymmetrical, and this introduces a component of the "fundamental" frequency into the oscillation

of the sounding-board which would otherwise be of double frequency.

While the work of setting up the apparatus for photographing the vibration-curves was in progress, Barton and Ebbelwhite's paper in the *Phil. Mag.* for September 1910 appeared, in which it was shown that the motion of the bridge of the violin in some cases had a frequency double that of the string. This effect seems to be clearly analogous to that shown in my photographs, though the effect is not obtainable in all cases with the bridge of the violin.

Fig. 10 illustrates still another method, which was devised, of demonstrating the 1 : 2 frequency relation. This was to compound the motion of the sounding-board and the wire *optically*. An illuminated horizontal slit is placed just below a point on the wire, transversely to it. The light issuing therefrom after reflexion at a mirror fixed over it and then at the oscillating mirror of the optical lever is focussed by the lens of a camera. The two motions of the shadow of the wire on the slit are at right angles to each other, and its path is the appropriate Lissajous's figure. It is seen in the photograph that this is approximately a parabolic arc.

Nagpur, C. P. India,  
10th November, 1910.

LXXI. *The Photometric Measurement of the Obliquity Factor of Diffraction* \*. By C. V. RAMAN, M.A. †

[Plate VII.]

**I**N an earlier paper on "The Experimental Study of Huygens's Secondary Waves," published in this Journal (*Phil. Mag.* Jan. 1909), I showed that the study of diffraction-patterns formed at oblique incidences is of particular interest, as the phenomena observed throw a fresh light on Huygens's principle and on the theory of secondary waves. Quantitative measurements of the effects described in that paper have since been made with the rectangular aperture, this being the one which lends itself most readily to the work. The present paper deals with the detailed and quantitative verification of the explanation suggested by me to account for the effects observed: its net result is to show that it is possible actually to observe and measure the

\* A preliminary note on this subject was published in 'Nature,' dated the 18th of November, 1909.

† Communicated by the Author.

variation of the amplitude of vibration on a Huygens's secondary wave.

My first observations on oblique diffraction by a rectangular aperture or reflecting surface were published in a note in the *Phil. Mag.* for Nov. 1906. The principal points noted were that the diffraction-pattern was unsymmetrical in character, the width of the bands instead of being the same throughout, as in the case of normal incidence, increasing continuously from one side of the pattern to the other: again, that the number of bands on one side of the pattern was limited. These results were shown to be simple consequences of the formula giving the positions of the minima of illumination in the pattern, *i. e.*

$$\sin i - \sin \theta = \pm \frac{n\lambda}{a}.$$

Later observations, in which the unsymmetrical character of the distribution of *intensity* in the pattern was noted, were described in the second paper quoted above. These observations were inexplicable on the ordinary (non-analytical) theory of diffraction. It was found that the broader bands on one side of the pattern were considerably feebler in intensity than the corresponding bands on the other side; whereas the ordinary theory required that they should be of equal intensity. The differences of intensity became very large as grazing incidence was approached. It was also observed that for any particular angle of incidence, the illumination in the pattern died away as the plane which sets the limit to the number of bands (on the side of the pattern at which these were broader) was approached. According to the ordinary theory, on the other hand, the intensity should have remained finite right up to this plane (which is the plane of the reflecting surface or aperture) and fallen discontinuously to zero at this point. It was this last, rather anomalous consequence of the ordinary theory which first drew my attention to the fact that the illumination in the diffraction-pattern was actually unsymmetrical in character.

The only explanation for these effects that I could find was that they were due to the varying obliquities at different points of the pattern; in other words, that the effects were due to the obliquity-factor of diffraction, of which no account is taken in the expression for the intensity derived from the ordinary theory. This explanation was developed in the paper quoted above, and was shown to be capable of accounting for the unsymmetrical character of the intensity-distribution.

One method of verifying the theory suggested would probably be to show that no other explanation of the observed effects is feasible. Any suggestion about defects in the experimental arrangements was found to be inadmissible. The focussing of the bands was or could be rendered perfect. The reflecting surface used was the face of a first-class prism of the kind used for accurate spectrometric work, and was absolutely fresh and clean: in fact, no perceptible effect was found to be produced by any moderate amount of dust or grease, the reason apparently being that the incidence was very oblique. The effect of scattered light, if any, in the field was in the opposite direction, and it could almost entirely be excluded by special precautions (to be described below in connexion with the photometric work). In fact, all these factors were found to be quite incapable of accounting for the observed effects.

Even assuming that there was at work in the experiments some unknown factor (other than an obliquity-factor), it is difficult to see how it could have altered the relative intensity of the bands on the two sides of the pattern to a considerable degree without either (1) shifting the positions of the minima of illumination from those given by the ordinary theory of diffraction; or (2) without causing a finite sensible intensity to exist at the minima, which according to the ordinary theory are absolute zeroes of illumination. Both of these points were carefully tested by direct observation and by photography. With a sufficiently homogeneous but powerful source of light, there is no difficulty in observing perfectly black bands alternating with bright bands of great intensity. To secure a satisfactory negative showing clear minima, it is necessary first carefully to abolish halation by backing the photographic plates before use. If care is taken to avoid chemical fogging of the plates, it is possible to get photographs of the diffraction-pattern showing great density at the maxima with practically clear glass at the minima. An enlargement from such a negative is shown as fig. 1 in Plate VII. In the diffraction-pattern photographed the theoretical ratios of illumination, calculated by a method to be explained below, should have been as follows:—

The 1st narrow bright band 1·81 times as intense as the 1st broad band.

2nd	”	”	2·92	”	”	2nd	”	”
3rd	”	”	5·68	”	”	3rd	”	”
4th	”	”	about 20	”	”	4th	”	”

which is too faint to be visible in the negative.

From an inspection of the original negative (or even of the enlargement, due regard being paid to the reduction of contrasts), it could be seen that the calculated values were of the right order.

The photographs were measured under a travelling microscope. The instrument was one which had been constructed by Messrs. Hilger for spectrum-photograph measurements. The following tables show how nearly the observed positions of the *dark* bands agree with the calculated values.

TABLE I.

Positions on the plate of

	Observed.	Calculated.
The Direct image .....	7.5833	7.5833
1st Broad Dark Band .....	7.2550	7.2551
1st Narrow " " .....	7.1671	7.1667
2nd " " " .....	7.1335	7.1335
3rd " " " .....	7.1035	7.1041

TABLE II.

Positions on the plate of

	Observed.	Calculated.
The Direct image .....	8.3626	8.3626
1st Narrow Dark Band ...	8.6579	8.6579
2nd " " " ...	8.6997	8.6997
3rd " " " ...	8.7339	8.7347
4th " " " ...	8.7656	8.7653

TABLE III.  
Positions on the plate of

	Observed.	Calculated.
The Direct image .....	8·4837	8·4837
2nd Broad Dark Band ...	8·8330	8·8332
1st „ „ „ ...	8·8861	8·8857
1st Narrow „ „ ...	8·9610	8·9602
2nd „ „ „ ...	8·9898	8·9903
3rd „ „ „ ...	9·0166	9·0175

TABLE IV.  
Positions on the plate of

	Observed.	Calculated.
The Direct image .....	7·0966	7·0966
3rd Broad Dark Band .....	8·2190	8·2191
2nd „ „ „ .....	8·3186	8·3187
1st „ „ „ .....	8·4010	8·4025
1st Narrow „ „ .....	8·5424	8·5427
2nd „ „ „ .....	8·6028	8·6038
3rd „ „ „ .....	8·6622	8·6607
4th „ „ „ .....	8·7164	8·7141

The measurements do not reveal any decided departure from theory, of the positions of the minima of illumination.

The considerations advanced above seem to me to be sufficient proof of the general correctness of the explanation suggested by me. I realized, however, that actual measurements of the obliquity-effects would form a more convincing proof, and at the same time would have intrinsic experimental interest and serve also to test the correctness of the mathematical work.

There are *two* classes of obliquity-effects, which in practice



are capable of experimental determination. The variation of the obliquity-factor over the diffraction-pattern profoundly modifies the distribution of illumination. In the first place the illuminations in corresponding bands on either side of the central band, instead of being equal, become widely different. The ratio of the illumination in two corresponding bands is capable of photometrical determination and furnishes a method of measuring the obliquity-factor. Then, again, the curve of distribution of illumination in any particular band of the pattern, *i. e.* from one minimum to the next, is modified, and the position of the point of *maximum* illumination is shifted. This shift furnishes a second method. Both these lines of investigation have been worked on. In the present paper only the first method will be dealt with, the second being reserved for a separate communication.

The ratio of the illuminations at corresponding points (on opposite sides of the central band) is, as shown in the paper quoted above (Phil. Mag. Jan. 1909, page 214), equal to  $\cos^2 \theta_1 / \cos^2 \theta_2$ , where  $\theta_1$  and  $\theta_2$  are the angles made by the diffracted rays with the normal to the plane of the reflecting surface or transmitting aperture. Putting

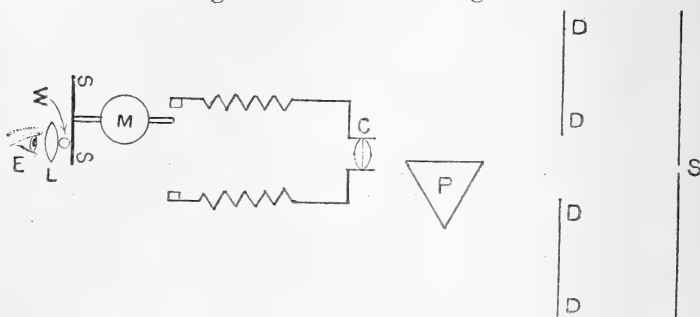
$$\theta_1 = \left( \frac{\pi}{2} - \delta_1 \right) \quad \text{and} \quad \theta_2 = \left( \frac{\pi}{2} - \delta_2 \right),$$

the ratio of illumination may be put equal to  $\delta_1^2 / \delta_2^2$ , provided  $\delta_1$  and  $\delta_2$  are fairly small.

The ratio was determined experimentally by the method of revolving sectors. Plate VII. fig. 2 shows the photometric arrangement adopted. Two disks of blackened cardboard about 24 cms. and 21 cms. in diameter respectively, are mounted concentrically, just touching each other on the shaft of a small electric motor. The outermost annulus of the first disk having a breadth of 6 cms. is divided into 12 equal sectors of  $30^\circ$  each and alternate sectors are removed. The 2nd disk is similarly treated for a breadth of 3 cms. from its circumference inward. It is arranged so that the two disks can be clamped on the shaft in any desired position relative to each other. On working the motor, the two annuli (having a breadth of 3 cms. each) are seen to transmit unequal intensities of light. The photometric disk lies and rotates in a vertical plane. The diffraction-pattern is focussed on the photometric disk, so that the central bright band (which, as also the other feebler ones on either side of it, is vertical) lies on the edge between the two annuli at right angles to the horizontal diameter of the disk. The broad

bands on one side of the pattern lie on the brighter annulus the narrow ones on the other side in the feebler annulus.

The general disposition of the apparatus when at work is shown in the diagram below. The light from the slit S



passes directly and partly also after reflexion at the surface of the prism P to the lens of a camera. The image of the slit and the diffraction-pattern are focussed by the lens on to the plane of the photometric disk, the adjustments being as described above. DD is a screen placed in front of the prism to shut off superfluous light. To further reduce the amount of scattered light, all the faces of the prism except the one used are completely blackened with a coat of varnish. The pattern is observed with the aid of a lens L placed behind the photometric disk. The central bright band of the pattern which would otherwise dazzle and confuse the eye, is cut off by a vertical wire W placed in front of the eyepiece.

The relative intensity of the first bands on either side of the central band is observed. Their brightnesses are adjusted to equality by sliding the disks one over the other, and thereby altering the angle of the smaller sectors. To secure accuracy, it was found essential to have the bands well-illuminated by using a brilliant source (in practice sunlight) to light up the slit. The adjustment to equality of brightness was found to be greatly facilitated if the upper or lower edge of the bands stood out sharply against a perfectly dark background.

When the adjustment is made, the lengths of the chords of the six larger and the six smaller sectors are read off with a scale. A divided glass scale is then placed in the focal plane of the camera so that the diffraction-pattern falls upon it, and the positions of (*a*) the direct image of the slit, (*b*) the maximum intensity in the central band, (*c*) and (*d*) the maximum intensities in the bands compared, are read off and noted. The theoretical ratio of the illuminations can be computed from these measurements and compared with the value

determined from the measurements of the sectors. Table V. illustrates the method.

TABLE V.

Chords of small Sectors.	Chords of large Sectors.	Scale-Readings.
1·60	4·10	(a) = 7·55
1·69	4·10	(b) = 4·85
1·67	4·10	(c) = 5·15
1·55	4·15	(d) = 4·57
1·62	4·10	(a+b) / 2 = 6·20
1·63	4·14	(a+b) / 2 - c = 1·05
<u>Mean</u> 1·63	<u>Mean</u> 4·12	(a+b) / 2 - d = <u>1·63</u>

$$\text{Observed illumination ratio} = \frac{4.12}{1.63} = \underline{2.53.}$$

$$\text{Calculated illumination ratio} = \left(\frac{1.63}{1.05}\right)^2 = \underline{2.41.}$$

The results of measurements made on this plan over a pretty wide range of incidences are shown in Table VI.

TABLE VI.

Ratio of Illumination according to the ordinary theory.	Ratio of Illumination actually determined.	Ratio of Illumination calculated from obliquity.
1·00	1·48	1·44
1·00	1·72	1·81
1·00	2·44	2·22
1·00	2·53	2·41
1·00	3·01	3·18
1·00	3·89	4·00
1·00	4·88	5·10
Total .....	19·95	20·16

The totals at the foot of the table have no significance except that they serve to show by their agreement that the differences between the calculated and observed values in individual experiments lie pretty evenly on either side. The measurements entirely confirm the theory.

*Summary.*—General considerations lead to the conclusion that the unsymmetrical character of the distribution of illumination in diffraction-patterns observed at oblique incidences, must be an obliquity effect. This is entirely verified by photometric work on the diffraction-pattern. The measurements show that the obliquity-factor follows the cosine-law, which may be stated thus :—“In the hemispherical wavelets emitted by each element of a transmitting aperture or reflecting surface upon which waves are incident at any angle, the amplitude of the light vector is, at any point in the plane of incidence, proportionate to the cosine of the angle made by the line joining that point and the element, with the normal to the plane of the element.”

This investigation was commenced at the Physical Laboratory of the Indian Association for the Cultivation of Science, Calcutta. My removal from that station necessitated its being completed elsewhere.

## LXXII. *Relativity and the Conservation of Momentum.*

By NORMAN CAMPBELL \*.

1. **I**N the March number of the ‘Philosophical Magazine’ Dr. Tolman attempts to show that the number of independent propositions necessary to formulate the theory of electromagnetic action may be reduced if the “law” of the conservation of momentum be assumed. This conclusion appears to me fallacious, and the fallacy to be one which deserves notice. The error is introduced, in my opinion, by a false application and by a misunderstanding of a result proved by Prof. Lewis and Dr. Tolman in a previous paper (*Phil. Mag.* [6] xviii. p. 510, 1909). This result consisted in a demonstration that the formula for the variation of the mass of a body with its velocity relative to the observer could be deduced by assuming only the “law” of the conservation of momentum and the fundamental principles of relativity. It will be convenient to examine this conclusion first.

2. The argument as it is stated appears to me to contain many of the errors and confusions noted in a recent paper†.

\* Communicated by the Author.

† “The Common Sense of Relativity,” *Phil. Mag.* April 1911.

“The observer A, considering himself at rest, concludes that the real change in velocity . . .” (p. 518). If A considers himself to be at rest, while he admits the principle of relativity, which states that he can have no rational ground for such a belief, his pronouncements on any subject must be received with considerable scepticism. When he proceeds to calculate the “real change” without explaining what he means by that term, I have a suspicion that he is using words to which he cannot attach any significance whatsoever. However, the argument can be expressed in a more satisfactory form as follows.

A and B, two observers furnished with instruments which, when compared at relative rest, are similar, are moving past each other with a relative velocity  $\phi$ . Each projects a body,  $a$  or  $b$ , appearing to him of mass  $m$ , towards the other. The bodies collide and each observer determines the velocity of each body relatively to his measuring instruments before and after the collision. Let the components of these velocities as determined by A be  $u_a, v_a, w_a, u'_a, v'_a, w'_a, u_b, v_b, w_b, u'_b, v'_b, w'_b$ : and let those determined by B be  $U_a, \dots, W_b'$ . Then Einstein's formulæ for the composition of velocities give relations between the  $u$ 's and the  $U$ 's, etc. Thus, if the  $u$  and  $U$  axis coincides in direction with  $\phi$ ,

$$u_b = \frac{\phi + U_b}{1 + \frac{\phi U_b}{c^2}}, \quad v_b = \frac{(1 - \frac{\phi^2}{c^2})^{\frac{1}{2}} V_b}{1 + \frac{\phi V_b}{c^2}}, \quad \text{etc.}$$

If the “law” of the conservation of momentum is true, and  $m'$  is the mass of  $b$  as it appears to A, we have

$$m(u_a - u'_a) + m'(u_b - u'_b) = 0, \text{ etc. . . . . (A)}$$

But, by the first postulate of relativity,  $u_a - u'_a = U_b - U'_b$ , etc. Hence, if all the  $u$ 's and  $U$ 's are small compared to  $c$ , we have, if the bodies move in the direction  $v$  or  $w$ ,

$$\frac{m'}{m} = \left(1 - \frac{\phi^2}{c^2}\right)^{-\frac{1}{2}}, \quad \dots \dots \dots (1)$$

or, if they move in the direction  $u$ ,

$$\frac{m'}{m} = 1. \quad \dots \dots \dots (2)$$

On the other hand, if the  $u_a$  etc., or  $U_a$  etc., are not small, the value of  $m'/m$  is a function of the  $u_a$  etc., and  $U_a$  etc.

3. (1) represents Prof. Lewis and Dr. Tolman's result. It appears of very much less generality than they imagined:

even the argument which they used would seem to show that it is not valid for collisions in the line of relative motion. That they omitted to note that it is only valid for small values of  $u$  seems to be due to a neglect to observe that, when a body appears to B to be moving in a direction making an angle  $\theta$  with the direction of relative motion, the same angle does not in general appear as  $\theta$  to A.

On these grounds, then, the application of the result which Dr. Tolman has made in his last paper appears quite unjustifiable. But the error which I wish to discuss is much more interesting and deep-seated.

4. It must be remembered that, if (A) is to state the foundation of Newtonian mechanics, the quantities  $u_a$ ,  $u_b$ , etc. must be measured relative to axes which have no absolute acceleration. Now I have pointed out in another paper \* that no amount of experiment can determine whether a body is or is not absolutely accelerated: any conclusion which we reach on the point is determined wholly by the assumptions which we make as to the nature of the circumstances which determine acceleration. We can take any body we please as absolutely unaccelerated, so long as the further assumptions we thus bind ourselves to make are accepted as soon as the necessity for them arises. Prof. Lewis and Dr. Tolman in their argument are dealing with a state of affairs of which we have no experience whatsoever; we have not observed, and are not likely to observe, a collision between two bodies projected from two systems, moving with a relative velocity approaching that of light, and each containing an observer who can afterwards tell us the value he found for  $m$ . Accordingly they can make any assumptions they like, happy in the confidence that they will never have to make any more, unless they want to do so, because their theory cannot possibly come within the range of experiment. In general, then, there is not the smallest objection to their taking the systems A and B to be absolutely unaccelerated, and applying (A) to values of the velocities determined relative to those systems.

5. But there is one system which cannot be taken as absolutely unaccelerated and on which the axes to which the velocities mentioned in (A) are referred must not be placed. This system is either of those which are taking part in the reaction. For (A) involves the change of velocity for each of the bodies: if velocities are measured relative to one of the bodies, change of velocity relative to that body means the change of velocity of a body relative to itself, an expression

\* Phil. Mag. [6] xix. p. 168 (1910).

which is absolutely self-contradictory and meaningless. This mistake of placing the axes of reference on one of the reacting bodies has been made twice by Dr. Tolman.

The first case \* occurs in an attempt to prove by a different method the same proposition, given previously by himself and Prof. Lewis, which has been discussed above. There are only two bodies mentioned, and the observer is placed on one of them. They collide, and in discussing the collision the author speaks of the velocity of the body on which the observer is placed. The conclusion is then attained that the variation of the mass of a body with its velocity can be ascertained by one of the observers if he remembers that, while he cannot determine his absolute velocity, he can determine that velocity of his which occurs in (A). Of course this velocity is precisely that absolute velocity which he cannot, *ex hypothesi*, determine. If there are only two bodies present, the only velocity which can possibly be determined is their relative velocity ; it cannot be split into one part which is the velocity of one body and another which is the velocity of the other, unless there is present also a body absolutely unaccelerated.

6. The second case occurs in the attempt made in the March number of this Magazine to apply the result to electro-dynamics. A charged body reacts with the instruments exciting electric and magnetic fields, which are measured by an observer at rest relative to those instruments. That is to say, that the axes of reference are placed on one of the reacting bodies, and (1), which only applies when the axes of reference are placed on an absolutely unaccelerated body, is utterly irrelevant.

7. Of course the same logical error is committed every day in the laboratory : it is committed, for instance, every time that  $g$  is determined by pendulum observations. We then measure displacements relative to the earth and identify them with absolute displacements in equations based upon Newton's principles, which deny that the earth, when reacting with a pendulum, can be absolutely unaccelerated. The justification in this case is that the method turns out to yield consistent and satisfactory results. But practical truths can be used legitimately only in practical circumstances : Dr. Tolman is attempting to prove a proposition of the utmost generality, and he is using in his attempt the consideration of circumstances which can never be approached practically. His results may be true : indeed his results are much more

\* Phys. Rev. xxxi. p. 26 (1910).

certain already than his assumptions, for we have far more evidence concerning the variation of the mass of an electron with its velocity relative to the measuring instruments than we have of the truth of the "law" of the conservation of momentum as applied to the interaction of two systems moving relative to each other with velocities approaching that of light. But his logical methods are not justifiable. A great part of the confusion into which the subject of the motion of electric charges was plunged before the principle of relativity came to enlighten us was due to a confusion between logical proof and the mere extension of experimental results to regions where there was no experimental evidence. It is desirable to avoid such confusions for the future.

### *Summary.*

The paper contains an examination of certain conclusions reached by Prof. Lewis and Dr. Tolman, and by the latter author alone. It is concluded (1) that the result attained by the joint authors is of much less generality than is claimed by them, and (2) that the application of the result to prove other propositions is unjustifiable, both on this ground and on the ground of a general confusion between absolute and relative velocity.

Leeds, March 12, 1911.

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LXXIII. *On Thunderbolts.* By W. M. THORNTON, D.Sc.,  
D.Eng., Professor of Electrical Engineering in Armstrong  
College, Newcastle-upon-Tyne\*.

THE term thunderbolt is given in common use both to the rare phenomenon of ball lightning and to meteoric stones. In the latter case it only has meaning, in so far as their luminous path resembles lightning or that they cause great atmospheric disturbance, and it is here used to describe the former. The singularity of ball lightning lies in the complete isolation of a gaseous sphere having no envelope, yet within which there is energy stored by previous electrical action. This is in the end liberated with explosive violence.

From the scattered records of its appearance the following facts may be regarded as established. It is observed as a luminous blue ball, occurring after lightning flashes of great intensity, and either falling slowly from clouds or moving

\* Communicated by the Author.



horizontally some feet above the earth's surface. It is seen more often at sea than on land, and both vertical and horizontal movements are recorded in each case. One of the most interesting records of its appearance is given in an account of a storm at sea in Hakluyt's *Voyages* by Pedro Fernandez de Quiros, three falling in one day. Ball lightning appears to move under gravitative action on a mass somewhat denser than air, or horizontally in a feeble air-current or electric field of force. It has been observed to follow the course of a conductor such as a water-main, and in most cases to burst on reaching water. It has also been seen to burst in mid-air. That it has some elastic cohesion is shown by its spherical shape and by its rebounding from the earth—in one case at least—after falling vertically. The features of its end are significant; the ball simply ceases to be and an explosion wave travels outwards from the spot. In all cases its disappearance is followed by a strong smell of ozone.

There are records of its curious selective behaviour in the neighbourhood of conductors. Thus a fireball came down a chimney, approached a person in the room (who slowly avoided it), retired up an old flue papered over, breaking through the paper, and finally burst with great violence on reaching the chimney-top, doing considerable damage. It may be inferred from this that its undoubted immense energy is not in the form of any surface charge which would have had many opportunities of dissipation in such a journey.

From the circumstances of its origin, it is clear that there can be nothing present in it but the gases of the atmosphere. That their molecular condition is abnormal is shown by the light which permeates the whole, and the only possible inference from this is that there is atomic rearrangement proceeding actively within the mass. This blue colour is characteristic of a state of air in which there is proceeding intense electric dissociation, as for example in the immediate neighbourhood of a highly charged needle-point. The chief product of molecular change under electric stress in air is ozone. This is shown by the fact that at a charged point ozone is given off, freely at the negative, and to a much less extent at the positive pole. Nitric oxide is not produced in this case, and it appears to be necessary to have streams of sparks to give rise to the formation of nitrogen compounds in air. The absence of nitrogen compounds is shown by the action of the electric wind from charged points on paper dipped in a solution in alcohol of tetramethyl p. p. diamido-diphenyl-methane, which in the presence of ozone turns violet-blue and with nitrogen

compounds yellow \*. In the electric wind no yellow coloration is to be seen. If a stream of ozone produced electrically in a Siemens tube from oxygen is passed over a metal plate attached to an electroscope charged with positive electrification, the leaves collapse, and the rate of decay is proportional to the speed at which the gas is passed through the ozonizer. The discharge is accelerated by the influence of a negatively charged plate, showing clearly that the fresh ozone carries a negative charge.

This suggests an explanation of the origin of the energy in ball lightning. On the occurrence of a flash of lightning from a charged cloud there is an immediate readjustment of the surface electrical conditions, and in certain cases there is the so-called return flash, closely following the first, caused by such a readjustment of the distribution of charge on cloud and earth. If at any projecting part of a negatively charged cloud the stress is nearly but not quite sufficient for a second flash, there will be for a time ionization on a great scale with the formation of ozone which, when sufficiently local in production, gathers into a ball, is repelled and falls. The volume produced depends on the energy immediately available. The process is of the same nature as the point discharge; but whereas under the stress possible in a laboratory the space in which the glow occurs has a radius of about half a millimetre, under the colossal stress in thunder-clouds it may quite well occur simultaneously in a space a yard in diameter, that of the largest fireball known. At the mast-heads and yards of ships at sea in tropical thunderstorms a blue light is frequently seen—St. Elmo's fire—a foot or more in radius.

All records agree that a thunderbolt is somewhat heavier than air. Nitrogen is lighter than air, and no allotropic form of it is known, though oxides of nitrogen are produced under the influence of streams of electric sparks. Oxygen is slightly heavier than air, ozone is nearly 70 per cent. heavier. The gravitating force on a sphere of ozone a metre diameter in air is 430 grammes—nearly a pound weight. Such a sphere would descend at a rate quick enough to be called a fall. On one of half this size the force would be 54 grammes. For such quantities not to fall but to travel horizontally there must be electrostatic repulsion from the earth requiring, since ozone carries a negative charge, a similar charge on its surface, which is known to generally exist. It is improbable that the cloud and earth below it

\* Fischer & Braemer, *Ber.* vol. xxxviii. No. 3, p. 2633 (1905), "On the Production of Ozone by Ultra-violet Light."

should both be negatively charged at the same instant. It would be the normal thing for the ball to approach the earth with considerable velocity, as is recorded in well authenticated cases. The fact that sometimes it turns off parallel to the earth's surface indicates that if, at the moment of the last discharge, the earth was locally positive, its sign has changed by reason of the electrical discharge, and is as usual negative.

The reason why the gas gathers into a sphere is that since the energy of ozone is for a given mass greater than that of oxygen, whilst the volume is less, the force between molecules of oxygen and ozone is an attraction, which decreases in the aggregate as recombination proceeds until it has the same value as the repulsion due to the usual molecular bombardment in gases. The temperature rising on account of the heat set free equilibrium would be quickly reached, the cooling of the ball by radiation and its motion through the air giving it stability. On reaching water, for which ozone has a strong affinity, or anything which causes ozone to decompose with great rapidity, the ball explodes.

The most conclusive evidence for any suggested constituent is whether this contains energy in such a form that it can be quickly liberated. It is well known that ozone reverts to oxygen, and it remains to see whether the energy liberated by this change is sufficient to account for the effects observed on its sudden occurrence.

In the conversion of a gramme of oxygen into ozone 29.6 kilogramme-degree-centigrade units of heat are absorbed. A sphere of 50 cm. diameter contains 62.5 litres. For the complete change of oxygen at 1.45 grammes per litre to this volume of ozone at 2.14 grammes per litre, there would be required 2615 of the above heat units. A clearer view of what this means is obtained by expressing it in mechanical units. Each kilogramme-degree unit is equivalent to 3094 foot-pounds; the total energy of transition is therefore 8 million foot-pounds.

It is unlikely, since there is recombination proceeding by diffusion within the ball as it falls, that the whole mass could be pure ozone on reaching the earth, but the dissipation of one tenth of the above energy explosively in the tenth of a second is at the rate of 15,000 horse-power.

The energy of the explosion-wave is well accounted for by this. There is in addition the sudden expansion when ozone is changed into oxygen, in this case 20 litres for a sphere of 50 cm. diameter of ozone.

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The facts which may then be stated in favour of thunderbolts consisting mostly of ozone in active recombination are :—

1. Ozone is said to be observed on their dissipation.
2. The gas of which they are composed is heavier than air. Ozone is the only gas denser than air produced in quantity under electric stress in air, as distinct from streaming spark-discharge.
3. On reaching the earth thunderbolts are frequently deflected and travel horizontally as if repelled. The earth's surface and ozone are both in general negatively charged.
4. The energy liberated on the transition of ozone to oxygen in the volume of a fireball is sufficient to account for the explosive violence with which it bursts.
5. The blue colour usually observed with it is associated with the sparkless electrical discharge in air which causes the production of ozone. It is also observed when oxygen and hydrogen combine explosively; when nitrogen is present the colour of the explosion flame is yellow.

These considerations lead one to suggest that the principal though not perhaps the only constituent of thunderbolts is an aggregation of ozone and partially dissociated oxygen, thrown off from a negatively charged cloud by an electric surge after a heavy lightning discharge.

LXXIV. *The Discharge of Positive Electricity from Hot Bodies.* By WILLIAM WILSON, Ph.D., Assistant Lecturer in Physics, University of London, King's College\*.

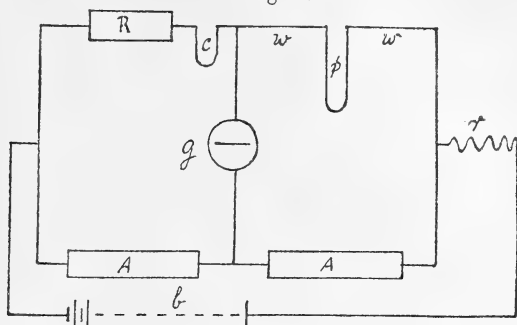
**I**N a recent paper on the positive electrification due to heated aluminium phosphate (Phil. Mag. Oct. 1910), A. E. Garrett has described the effect of the presence of water in the salt in temporarily increasing the positive electrification produced. As I have observed a similar phenomenon, I propose to publish a preliminary report of research which is still in progress on the discharge of electricity from hot bodies.

*Experimental Arrangement.*—The platinum wire, *p* (fig. 1), formed one arm of a Wheatstone bridge arrangement, and was 13 cm. in length and 0.2 mm. in diameter. The adjacent arm contained a known adjustable resistance, *R*, of thick eureka wire (immersed in paraffin oil), and a shorter

\* Communicated by the Author.

piece,  $c$  (3 cm. in length), of platinum wire exactly similar to  $p$ . This shorter piece of wire had also leads like those

Fig. 1.

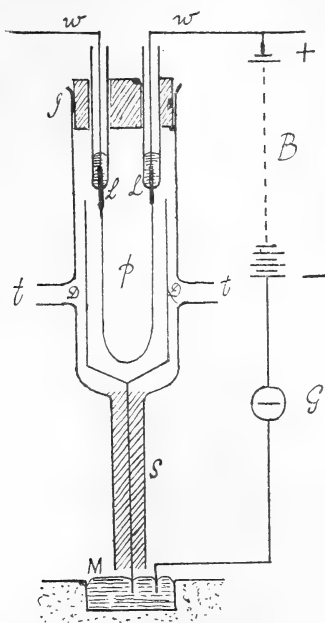


of  $p$ , and served to compensate for their resistance. The wire was of pure platinum supplied by Messrs. Johnson & Matthey. The other arms of the bridge consisted of two resistances,  $A A$ , each of 5000 ohms. A battery,  $b$ , of 10 to 12 accumulators supplied the current, which could be regulated by means of a variable resistance,  $r$ . Before a measurement of the leak from the platinum was carried out, the adjustable resistance,  $R$ , was given a suitable value. The current was then started and the resistance,  $r$ , reduced till the galvanometer,  $g$ , indicated no current. In this way the wire,  $p$ , was given a definite temperature, namely that at which the resistance of a definite portion of it was equal to  $R$ . The arrangement thus served to heat the wire,  $p$ , and to determine its temperature.

In fig. 2 (p. 636) is shown the way in which the platinum wires,  $p$  and  $c$ , were connected to the thick copper wires,  $w w$ , conveying the current, and the arrangement for measuring the "thermionic" current. The wire,  $p$ , was connected to thick platinum leads fused through the ends of glass tubes,  $L L$ . The wires,  $w w$ , dipped into mercury which covered the platinum leads. The whole was fixed by means of a tightly fitting indiarubber stopper,  $g$ , in a test-tube shaped vessel provided with side tubes,  $t t$ , by means of which dry air could be supplied to it. Surrounding the platinum wire was an aluminium cylinder,  $D$ , 2 cm. in diameter, supported at the end of a straight copper wire which was insulated from the rest of the vessel by means of sulphur,  $S$ , and the other end of which dipped into mercury contained in a cup,  $M$ , in a paraffin block. One terminal of a delicate moving coil-galvanometer,  $G$  (1 scale-division =  $6.32 \times 10^{-10}$  ampere) was connected to the negative pole of a battery,  $B$ , of 100 volts

whose positive pole was connected through  $w$  to  $p$ . When  $p$  had been raised to the desired temperature in the manner

Fig. 2.



already described, the other terminal of the galvanometer was connected with the mercury in  $M$  and the ensuing current from  $p$  to  $D$  measured. The experiments were conducted in air, at atmospheric pressure, which had been freed from dust and moisture by passing through tubes containing cotton-wool, calcium chloride, caustic potash, and phosphorus pentoxide.

*Diminution of Positive Leak from Hot Platinum not due to Heating only.*—It is a well-known fact that the positive leak from hot platinum diminishes with continued heating. My experiments point to the conclusion that this phenomenon is not due to the temperature alone, but is rather a consequence of the discharge of positive electricity. The following observations illustrate this:—

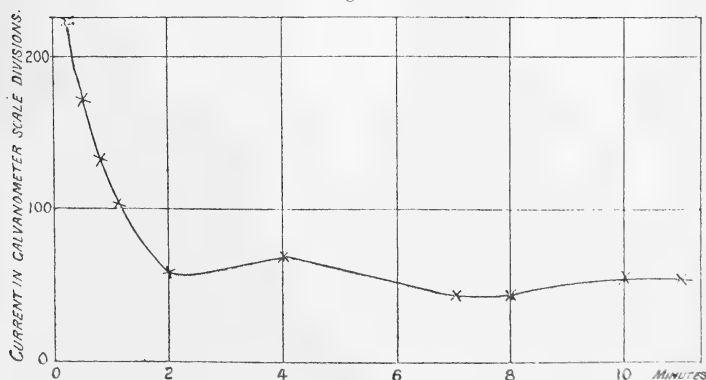
In the initial observation the current was represented by 186 scale-divisions, the potential-difference between the wire,  $p$ , and the cylinder,  $D$  (fig. 2), being 100 volts. The temperature of the wire was about  $1100^{\circ}\text{C}$ . After this measurement the cylinder,  $D$ , was completely insulated by disconnecting at  $M$ , the wire being maintained at *exactly the same temperature as before*. At the end of 10 minutes the connexion

at M was re-established and the current again measured. It was now found to be 176 scale-divisions—that is to say, not very much smaller than the initial leak. The potential-difference of 100 volts was maintained for 10 minutes (with the same temperature as before) and the current noted at the end of this period. This time it was found to be only 111 scale-divisions. The discharge from the hot wire was thus reduced to a much greater extent during the second than during the first period. Finally the cylinder was again insulated for a further period of 10 minutes, the temperature of the wire not being allowed to vary. When the current was again measured it was found to be 111 scale-divisions.

These observations show that the diminution in the positive current from the platinum is not an ordinary fatigue effect, but is due to the decrease in the quantity of ionizable matter which the platinum contains.

*Influence of Water on the Positive Leak from Platinum.*—One of the most remarkable phenomena in connexion with the emission of positive electricity by hot platinum is the increase in the discharge caused by the presence of water. I observed this effect by accident during experiments on potassium sulphate. This salt when heated on platinum wire apparently discharged large quantities of positive electricity. It was put on the wire by dipping the latter in a concentrated aqueous solution of the salt. At a temperature of  $1100^{\circ}$  C. and with a potential-difference of 100 volts the leak was initially far beyond the range of measurement of the galvanometer. This leak decayed with great rapidity at first and more slowly and somewhat irregularly afterwards. The

Fig. 3.



way in which the current varies with the time is shown in fig. 3. As it was found that the same effect could be produced

by simply dipping the platinum wire in distilled water, the discharge in the case of potassium sulphate was due mainly, if not entirely, to water. The similar effect observed by A. E. Garrett, *l. c.*, in his experiments on hot aluminium phosphate is possibly an increased activity of the platinum on which he heated his salt, and occasioned by the comparatively large quantity of water which the salt contains.

To obtain this increase in the positive discharge from platinum, it is not necessary that there should be water vapour present in the atmosphere surrounding the wire *during the measurement of the discharge*. This is shown by the following experiment:—The platinum wire was placed in an atmosphere saturated with water vapour and heated for some minutes by a current, the temperature being raised as far as was possible without risking fusion of the wire. The latter, after being allowed to cool in this atmosphere, was then placed in the measuring apparatus (fig. 2), the air in which was perfectly dry, and with the usual potential-difference and temperature the leak was found to be greatly increased. It would appear therefore that water causes or accelerates the production *in the platinum* of something which can emerge at sufficiently high temperatures in the form of positive ions. In all cases the activity induced by water decreased with great rapidity, which indicates that it is confined to the outer portion of the wire. It should be mentioned that a much greater increase in activity was produced by actually dipping the wire into water and introducing the wet wire directly into the measuring apparatus.

Another experiment, which showed the effect of water in a very striking way, consisted in dipping the wire into a solution of calcium nitrate. It was then heated in the measuring apparatus (fig. 2), till the salt was reduced to calcium oxide and the discharge was reduced to small dimensions. So long as the wire with its coating of oxide remained in the dry atmosphere of the apparatus, the leak continued to be almost inappreciable. A 5 minutes' exposure of the coated wire to the air of the room, however, was sufficient to induce a leak of 50 to 60 scale-divisions. Again the leak decayed with great rapidity.

*Nature of the Ions in the Induced Discharge.*—The carriers of the positive electricity in the case of the normal leak from hot platinum consist probably to a large extent of carbon monoxide. The value of  $\frac{e}{m}$  obtained by Richardson for the positive ions from platinum, and Horton's spectroscopic work in the case of aluminium phosphate heated on platinum



(Proc. Roy. Soc. Dec. 1910), support this view. Further, the value of  $\frac{e}{m}$  found by Garrett, *l. c.*, for some of the ions emitted when aluminium phosphate is heated on platinum is consistent with the view that hydrogen ions are also emitted from hot platinum.

Now if we suppose—as I think we are justified in doing—that even the purest platinum contains traces of carbon, the water effect can easily be explained. When the platinum is heated in the presence of water vapour, the water will be decomposed with the formation of carbon monoxide and the liberation of hydrogen. This may occur even at temperatures considerably below that at which the positive leak begins to be appreciable. These gases will naturally diffuse into the platinum and re-emerge under suitable conditions in the form of positive ions. There is also the further possibility that water may accelerate catalytically the production of carbon monoxide when platinum is heated in air.

*Positive Leak from Hot Aluminium Phosphate.*—Both Horton and Garrett heated the aluminium phosphate on platinum. The salt was made into a paste with water, and moreover contains in any case a large amount of water. As this water will affect, temporarily at any rate, the activity of the platinum, the question naturally arises: What part of the discharge from the hot phosphate are we entitled to ascribe to the salt itself? Even when it has been heated so long that the effect due to the presence of water has died down, it is still conceivable that the salt may facilitate in some way the escape of ions from the platinum. I failed to observe any positive discharge from aluminium phosphate when the latter was heated on a Nernst filament, the temperature of which was very much higher than that employed in the experiments with a platinum wire. The experiment was carried out both on a Nernst filament provided with the usual heating arrangement and on one not so provided. In the former case the galvanometer showed an appreciable deflexion (20 to 30 scale-divisions) *while the heater was in operation*. This deflexion diminished to zero immediately the heater was cut out. It was therefore due to the platinum wire of the heater. While the Nernst filament was glowing no deflexion of the galvanometer could be observed. The experiment was done under conditions as closely resembling those in which a platinum wire was used as possible.

It seems quite likely that aluminium phosphate, at any rate when prepared from aluminium acetate, may contain traces of carbon, and therefore some part of the positive

discharge observed when the salt is heated on platinum may have its origin in this carbon. The experiments described above, however, suggest that the platinum plays an important rôle in the *ionization* of the carbon monoxide and other products formed by heating platinum or platinum coated with aluminium phosphate.

*Summary.*—(a) The activity of platinum is not reduced by continued heating merely, but only under conditions which admit of a positive discharge from the metal. The loss of activity is therefore due to the diminution of the quantity of matter—carbon, carbon monoxide, or whatever it may be—which emerges from the platinum in the form of positive ions at sufficiently high temperatures. (b) The activity is greatly increased by heating the platinum in the presence of water. This effect is possibly due to the production of carbon monoxide and hydrogen in the platinum or at its surface, and can be observed even when the platinum is heated in a dry atmosphere, provided it has been previously heated and allowed to cool in an atmosphere saturated with aqueous vapour.

(c) There is apparently no positive leak (or only a very small one) when aluminium phosphate is heated on a Nernst filament, and therefore the leak observed when the salt is heated on platinum is either mainly a leak from the platinum itself, or the latter plays an important rôle in its production.

Further research on the subject is being carried out, of which a full account will be published later.

Wheatstone Laboratory,  
University of London, King's College.  
February 1911.

LXXV. *The Initial Accelerated Motion of a Perfectly Conducting Electrified Sphere.* By G. H. LIVENS, B.A.,  
*Lecturer in Mathematics, Sheffield University* \*.

SEVERAL papers have recently been published dealing with this subject; those particularly under review here are by G. W. Walker (Proc. R. S. vol. lxxvii. and Phil. Trans. 1910).

In the present paper the same subject is dealt with in a manner similar to that given by Walker, but the complications of considering any material mass that the sphere may possess are entirely avoided, the subject being discussed purely from the electromagnetic standpoint. The use of

\* Communicated by the Author.

spherical polar coordinates is also adopted, following a very kind suggestion from Dr. Bromwich.

The general method consists in imparting to the sphere, in a manner which will hereafter appear, a uniform acceleration and deducing the initial field purely from geometrical considerations. The effective force on the sphere is then calculated, and the coefficient of the acceleration in the expression for this force is taken as representing the electromagnetic mass of the sphere.

I. *When the sphere starts from rest.*

The sphere is perfectly conducting of radius  $a$ , with a total charge  $e$ , and the acceleration is  $s$ , applied in a direction which is taken as the polar axis of the coordinates, the centre of the sphere coinciding initially with the origin.

The acceleration is considered so small that the displacement of the sphere in the time taken by radiation to travel across the sphere is small compared with the radius;  $\frac{sa^2}{c^2}$  is small compared with  $a$ .

We have obviously only to deal with a case of symmetry about the polar axis. The Maxwell equations for the field outside the sphere can then be written

$$\frac{1}{c^2} \frac{d}{dt} (X, Y) = \left( -\frac{1}{r^2} \frac{\partial \psi}{\partial \mu}, \quad -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \right)$$

$$-\frac{dy}{dt} = \frac{1}{r} \left( \frac{\partial}{\partial r} (rY) - \frac{\partial X}{\partial \theta} \right),$$

( $X, Y, Z$ ;  $\alpha, \beta, \gamma$ ) being the usual components of the electric and magnetic vectors along the polar directions, and  $\mu = \cos \theta$  and  $\psi = r\gamma \sin \theta$ . This leads as usual to the equation for  $\psi$ ,

$$\frac{1}{c^2} \frac{d^2 \psi}{dt^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1 - \mu^2}{r^2} \frac{\partial^2 \psi}{\partial r^2},$$

of which the known general solution is

$$\psi = r^{n+1} \left( -\frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{f(ct-r)}{r} \right] (1 - \mu^2) \frac{\partial P_n}{\partial \mu}.$$

The solution being restricted to the only necessary case of expanding waves.

We now attempt to find the field outside the sphere at the end of a time  $t$ , to the first order in the acceleration. The small displacement of the centre of the sphere is  $\xi = \frac{st^2}{2}$ , and the equation to its surface is

$$r = a + \xi \cos \theta.$$

We try a solution involving the first order harmonics only. We add on to the initial field the first harmonic solution of the general equations, and attempt to satisfy the boundary conditions. We take

$$X = \frac{e}{r^2} + \frac{2 \cos \theta}{r^3} (rf' + f), \quad Y = \frac{\sin \theta}{r^3} (r^2 f'' + rf' + f),$$

$$c\gamma = \frac{\sin \theta}{r^2} (rf'' + f''),$$

$f$  is now interpreted as a function of  $(ct - r + a)$ .

As Prof. Love points out, there are two conditions to be satisfied, one at the front of the advancing wave boundary, which started out from the sphere at the initial instant, and the other one at the surface of the sphere itself. The conditions at the wave front can easily be seen to be

$$X = X_0, \quad Y - c\gamma = Y_0 \quad \text{at} \quad r = a + ct,$$

the initial field  $(X_0, Y_0, Z_0, \alpha_0, \beta_0, \gamma_0)$  existing undisturbed outside this boundary. These give

$$f'(0) = 0, \quad f(0) = 0.$$

The condition at the surface of the sphere is that the tangential electrodynamic force is zero. In the case under discussion this is the same as that the tangential electric force should be zero, the magnetic part of the former, containing the product of two small quantities, being zero to the first order.

This leads to the condition that

$$Y - \frac{\xi \sin \theta X}{a} = 0,$$

account being taken of the fact that the centre of the sphere is not at the origin of the coordinates; this is equivalent to

$$a^2 f''(ct) + af'(ct) + f(ct) - e\xi = 0.$$

We now use  $x \equiv ct$  and  $A \equiv \frac{es}{2c^2}$ , and then the equation for  $f$  is

$$a^2 f''(x) + af'(x) + f(x) - Ax^2 = 0.$$

The solution subject to the wave-boundary conditions is

$$f = \frac{4Aa^2}{\sqrt{3}} e^{-\frac{x}{2a}} \sin \frac{x\sqrt{3}}{2a} + Ax^2 - 2Aax.$$

With this form of  $f$  ( $x$  being now interpreted generally as  $ct - r + a$ ) we have fulfilled all the conditions, satisfied by the field, to the first order. The field is, therefore, completely determined under the restrictions imposed.

The density of the charge on the sphere is given to the first order by

$$4\pi\sigma = X \quad \text{at} \quad r = a + \xi \cos \theta$$

$$= \frac{e}{a^2} + \frac{2 \cos \theta}{a^3} (af'(ct) + f(ct) - e\xi);$$

or considering the equation satisfied by  $f$ ,

$$4\pi\sigma = \frac{e}{a^2} - \frac{2f''}{a} \cos \theta.$$

We now find the resultant force on the sphere, which is obviously along the polar axes. The component of the electrodynamic force in this direction is

$$P' = X \cos \theta - Y \sin \theta,$$

and at the surface of the sphere this reduces to

$$P' = \left( \frac{e}{a^2} - \frac{2f'' \cos \theta}{a} \right) (\cos \theta).$$

Thus the total force on the sphere in the direction of motion is

$$P = \frac{1}{2} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P' \sigma a^2 \sin \theta \, d\theta \, d\psi$$

$$= \frac{2}{3} \frac{ef''}{a}.$$

This is the effective electromagnetic force on the sphere. Now

$$f'' = 2A \left( 1 - e^{-\frac{x}{2a}} \cos \frac{x\sqrt{3}}{2a} \right) - \frac{2A e^{-\frac{x}{2a}}}{\sqrt{3}} \sin \frac{x\sqrt{3}}{2a},$$

and since

$$A = \frac{1}{2} \frac{se}{c^2}$$

we have

$$P = \frac{2}{3} \frac{e^2 s}{ac^2} \left[ \left( 1 - e^{-\frac{ct}{2a}} \cos \frac{ct\sqrt{3}}{2a} \right) - \frac{e^{-\frac{ct}{2a}}}{\sqrt{3}} \sin \frac{ct\sqrt{3}}{2a} \right].$$

The electromagnetic mass of the sphere, defined as the ratio  $P/s$ , is therefore

$$m = \frac{2}{3} \frac{e^2}{ac^2} \left( 1 - e^{-\frac{ct}{2a}} \cos \frac{ct\sqrt{3}}{2a} \right) - \frac{2}{3} \frac{e^2}{ac^2} \frac{e^{-\frac{ct}{2a}}}{\sqrt{3}} \sin \frac{ct\sqrt{3}}{2a}.$$

The value of  $m$  is initially zero, but rapidly approaches the ordinarily assigned value  $\frac{2}{3} \frac{e^2}{ac^2}$ . The sphere therefore starts off without offering any electromagnetic inertia to its motion initially. The force and mass, however, both differ from zero at any finite time, however small, after the initial instant. This fact is explained quite easily from general principles. The system to be moved is specified by a certain state in the æther around the sphere. Now the æther at any place is not affected by any motion of the sphere until after the time that radiation, leaving the sphere at the initial instant of its disturbance, takes to reach that place; and it cannot, therefore, offer any reaction to the motion of the sphere until after that time.

Other conclusions, similar to those deduced by Walker, can be deduced; the only distinction being that none of the results here given involve the "material" mass of the sphere. The production of a small uniform acceleration causes a readjustment of the charge distribution. The readjustment of the charge, however, involves an oscillation which sends out a damped periodic wave train into the æther. The oscillations and wave-motion are, however, soon damped out of the system, and a sort of steady state is reached in which

$$4\pi\sigma = \frac{e}{a^2} \frac{2se}{ac^2} \cos \theta$$

$$P = \frac{2}{3} \frac{e^2}{ac^2} s.$$

## II. *The sphere accelerated from a uniform motion with velocity v.*

The velocity is supposed to have been uniform for an indefinite time before the initial instant considered.

Now, according to Larmor\*, if we refer the phenomena to a set of axes moving with a velocity  $v$ , the fundamental equations of the theory, the Maxwell equations referred to moving axes, assume exactly the same form as they had originally referred to fixed axes.

\* See 'Æther and Matter,' pp. 173-175.

In fact, if the motion of the sphere is along the axis of  $x$ , and if  $(x_1 y_1 z_1 t_1)$  be the coordinates of space and time in the moving axes, connected with  $(x, y, z, t)$  those in the fixed axes by the relations

$$\begin{aligned} x_1 &= \epsilon^{\frac{1}{2}} x' = \epsilon^{\frac{1}{2}} (x - vt), \\ y_1 &= y' = y, \\ z_1 &= z' = z, \\ t_1 &= \epsilon^{-\frac{1}{2}} \left( t - \frac{v\epsilon}{c} x' \right); \end{aligned} \quad \epsilon = \frac{1}{1 - \frac{v^2}{c^2}},$$

then Maxwell's equations referred to moving axes assume the form

$$\begin{aligned} 4\pi \frac{d}{dt_1} (f_1, g_1, h_1) &= \text{Curl}_1 (a_1, b_1, c_1) \\ - \frac{1}{4\pi c^2} \frac{d}{dt_1} (a_1, b_1, c_1) &= \text{Curl}_1 (f_1, g_1, h_1), \end{aligned}$$

where  $(f_1, g_1, h_1; a_1, b_1, c_1)$  are related to the actual vectors in the field by the equations

$$\begin{aligned} (f_1, g_1, h_1) &= \epsilon^{\frac{1}{2}} \left( \epsilon^{-\frac{1}{2}} f, g - \frac{vc}{4\pi c^2}, h + \frac{v}{4\pi c^2} b \right), \\ (a_1, b_1, c_1) &= \epsilon^{\frac{1}{2}} (\epsilon^{-\frac{1}{2}} a, b + 4\pi v h, c - 4\pi v g). \end{aligned}$$

Thus if the values  $(f_1, g_1, h_1; a_1, b_1, c_1)$  given as functions of  $(x_1, y_1, z_1, t_1)$  express the course of change of the æthereal vectors of any electrical system referred to the axes  $(x_1, y_1, z_1, t_1)$  at rest in the æther, then

$$\epsilon^{\frac{1}{2}} \left( \epsilon^{-\frac{1}{2}} f, g - \frac{v}{4\pi c^2} c, h + \frac{v}{4\pi c^2} b \right), \epsilon^{\frac{1}{2}} (\epsilon^{-\frac{1}{2}} a, b + 4\pi v h, c - 4\pi v g)$$

expressed by the same functions of the variables

$$\epsilon^{\frac{1}{2}} x', y', z', \quad \epsilon^{-\frac{1}{2}} t' - \frac{v}{c^2} \epsilon^{\frac{1}{2}} x',$$

will represent the course of change of the æthereal vectors  $(f, g, h; a, b, c)$  of a correlated system of moving charges referred to axes  $(x', y', z')$  moving through the æther with uniform translatory velocity  $(v, 0, 0)$ . Moreover, in this correlation between the courses of change, in the two systems, elements of charge occupy corresponding positions in the two systems and are of equal strengths. However, electrodynamic forces per unit charge are not the same in the two systems. They are, however, related in an obvious manner.

In fact, if the force per unit charge on any element executing any motion in the system at rest be determined as  $(P_1 Q_1 R_1)$ , then the force per unit charge on the element executing the corresponding motion relative to the moving axes is determined by  $(P, Q, R)$ , where

$$(P_1, Q_1, R_1) = \epsilon^{\frac{1}{2}} (\epsilon^{-\frac{1}{2}} P, Q, R).$$

Again, the accelerations of any point determined as  $(s_1, s_1', s_1'')$ , when referred to the axes at rest in the æther, become when they refer to the point executing a corresponding motion in the correlated system  $(s, s', s'')$ , where

$$(s_1, s_1', s_1'') = \epsilon^{\frac{1}{2}} (\epsilon s, \epsilon^{\frac{1}{2}} s', \epsilon^{\frac{1}{2}} s''),$$

if the point is always near the origin of the coordinates.

Thus if in the case of a system referred to fixed axes we have determined the equations of linear motion, considered merely in its electrodynamic aspects, in the form

$$(P_1 Q_1 R_1) = (m_1 s_1, m_1' s_1', m_1'' s_1''),$$

$m_1 m_1' m_1''$  being determined as the electromagnetic masses for the system in motion, as a whole, along the three axes of coordinates; then the equations of motion for the same system executing a corresponding motion referred to axes in uniform motion along the  $x$ -axes will be

$$(P, Q, R) = (m_1 \epsilon^{3/2} s, m_1' \epsilon^{\frac{3}{2}} s', m_1'' \epsilon^{\frac{3}{2}} s'').$$

The electromagnetic masses for the new system, which has the additional motion with velocity  $v$  along the  $x_1$  axis, are therefore

$$(m, m', m'') = (m_1 \epsilon^{3/2}, m_1' \epsilon^{\frac{3}{2}}, m_1'' \epsilon^{\frac{3}{2}}).$$

From these preliminary remarks it will be at once seen that the solution which we have already obtained for the initial accelerated motion of a charged perfectly conducting sphere can be transferred to the case for the sphere initially in uniform motion with any velocity  $v$ , if the vectors are interpreted properly. One other condition has to be satisfied. The equation of the sphere has to remain of the form

$$x^2 + y^2 + z^2 = a^2,$$

and to ensure this we must accept the Lorentz contraction hypothesis, which I propose to do, and have already tacitly done in the previous general discussion. In the solution transferred to the case of initial uniform motion the general electrodynamic vectors  $(f_1 g_1 h_1)(a_1 b_1 c_1)$  determined in terms



of the actual field vectors by relations already given, correspond to the values already obtained for the electric and magnetic vectors, when these are expressed as functions of the coordinates  $(x, y, z, t')$  referred to the moving axes.

Thus if  $(X_1 Y_1 Z_1; \alpha_1 \beta_1 \gamma_1)$  are the components of the electrodynamic vectors referred to spherical polar coordinates moving with the cartesian axes, but whose polar axis is in the direction of the applied acceleration, which need not necessarily be the direction of the uniform motion, then the field referred to moving coordinates is determined by relations similar to those already obtained.

$$\begin{aligned} X_1 &= \frac{e}{r_1^2} + \frac{2 \cos \theta_1}{r_1^3} (r_1 f_1' + f_1), \\ Y_1 &= \frac{\sin \theta_1}{r_1^3} (r_1^2 f_1'' + r_1 f_1' + f_1), \\ c\gamma_1 &= \frac{\sin \theta_1}{r_1^3} (r_1^2 f_1'' + r_1 f_1'), \end{aligned}$$

where

$$f_1 = \frac{4A_1 a^2}{\sqrt{3}} e^{-\frac{x_1}{2a}} \frac{\sin x_1 \sqrt{3}}{2a} + A_1 x_1^2 - 2A_1 a x_1,$$

and

$$x_1 = ct_1 - \gamma_1 + a, \quad A_1 = \frac{e s_1}{2c^2}.$$

The density of the charge on the sphere is given by

$$4\pi\sigma_1 = \frac{e}{a^2} - \frac{2f_1'' \cos \theta}{a},$$

and the force on the sphere in the direction of the acceleration is

$$P_1 = \frac{2}{3} \frac{e^3 s_1}{ac^2} \left( 1 - e^{-\frac{ct_1}{2a}} \cos \frac{ct_1 \sqrt{3}}{2a} \right) - \frac{2}{3} \frac{e^2 s_1}{ac^2} \frac{e^{-\frac{ct_1}{2a}}}{\sqrt{3}} \sin \frac{\sqrt{3} ct_1}{2a}.$$

Whence we deduce that the electromagnetic masses of the sphere are

$$(\epsilon^{3/2}, \epsilon^{1/2}, \epsilon^{1/2}) \left[ \frac{2}{3} \frac{e^2}{ac^2} \left( 1 - e^{-\frac{ct}{2a_1}} \cos \frac{ct_1 \sqrt{3}}{2a} \right) - \frac{2}{3} \frac{e^2}{ac^2} \frac{e^{-\frac{ct}{2a}}}{\sqrt{3}} \sin \frac{\sqrt{3} ct_1}{2a} \right].$$

They are all three initially zero, but tend rapidly in an oscillating manner to the values usually obtained from quasi-stationary principles

$$\frac{2}{3} \frac{e^2}{ac^2} (\epsilon^{3/2}, \epsilon^{1/2}, \epsilon^{1/2}).$$

Conclusions can be drawn similar to those indicated in the previous case considered. The introduction of acceleration into the motion of the electrified sphere, which was previously uniform, results in a small disturbance of the initial uniform distribution. The disturbance and rearrangement of the charge give rise to its oscillation which sends out the damped wave-train into the æther, but the oscillation and æther disturbance soon die away, and the system settles down into a steady state of motion with a uniform acceleration. The electrical reaction to the starting of the accelerated motion is initially zero, but rapidly assumes its steady value.

The case of a rigidly charged dielectric sphere possesses some additional characteristics, and I shall reserve the discussion of it for a future paper.

Sheffield, February 1911.

LXXVI. *Note on the Energy of Scattered X-radiation.*

By CHARLES G. BARKLA\*.

IN 1904 the writer made an experimental determination of the energy of Röntgen radiation scattered by light elements (Phil. Mag. May 1904), and applied the result to calculate the number of scattering electrons in a known quantity of matter, on the theory of scattering given by Sir J. J. Thomson †. With the data for  $\frac{e}{m}$  and  $e$  of an electron at that time available, the number of electrons in a cubic centimetre of air under normal conditions of pressure and temperature was found to be  $6 \times 10^{21}$ , or between 100 and 200 per molecule of air. Using the more recently determined values of  $\frac{e}{m}$ ,  $e$ , and  $n$  (the number of molecules per cubic centimetre of gas) ‡, the calculation gives the number of scattering electrons per atom as about half the atomic weight of the element.

In a recent paper, however (Proc. Roy. Soc. A. lxxxv. pp. 29-44), Mr. Crowther has determined the energy of the scattered radiation from experiments on aluminium. He

\* Communicated by the Author.

† The theory was first given, together with the expression for the energy of the scattered radiation, in the First Edition of 'Conduction of Electricity through Gases.'

‡  $\frac{e}{m} = 1.73 \times 10^7$  e.m.u. (Bucherer);  $e = 1.55 \times 10^{-20}$  e.m.u. (Rutherford & Geiger);  $n = 2.8 \times 10^{19}$  (Rutherford).

estimates the energy of radiation from a given mass to be six times that given by the writer, and concludes that the number of scattering electrons per atom is three times the atomic weight.

Now it was early shown by the writer\* that elements of low atomic weight, up to and including sulphur, scatter to the same extent mass for mass, and that the scattering is independent of the penetrating power of the Röntgen radiation used. Mr. Crowther later verified both these results †. We should therefore expect to find the same amount of energy scattered in the two cases from equal masses of air and aluminium. The experimental values are evidently in conflict, and it becomes a matter of interest to examine them, not merely for the sake of any evidence the result may afford as to atomic constitution, but in order to explain certain phenomena of absorption.

From the results of experiments by the writer it was concluded that a layer of atmospheric air of 1 centimetre thickness scatters about 00024 of the energy of Röntgen radiation passing through it. Thus if  $-dI_s$  represents the diminution due to scattering in the intensity of a beam during transmission through a layer of air of thickness  $dx$ , then  $dI_s = -00024 I dx$ . Calling the quantity 00024 the coefficient of scattering  $s$ , we get  $\frac{s}{\rho} = 2$  approximately, where  $\rho$  is the density of air or any substance of low atomic weight. Mr. Crowther's value in the case of aluminium is 1.18.

We will briefly consider these results in the light of other experiments.

(1) As the radiation considered is scattered radiation, it involves a corresponding diminution in the intensity of the primary beam. That is if the intensity of a beam proceeding in a given direction be expressed by the equation  $I = I_0 e^{-\lambda x}$ ,  $\lambda$  cannot be less than  $s$ , or the total diminution of intensity of the beam proceeding in the original direction of propagation cannot be less than the loss due to scattering alone ‡. Yet the total

\* Phil. Mag. June 1903, pp. 685-698; May 1904, pp. 543-560; June 1906, pp. 812-828.

† Phil. Mag. Nov. 1907, pp. 653-675. (Hydrogen is a possible though not certain exception to the first law.)

‡ For want of a better term  $\lambda$  will throughout be termed the absorption coefficient, though some of the energy is merely scattered and some is re-emitted in a different form. Correctly it is the rate of diminution with distance of the primary beam, as a *primary beam*, or it is the rate of diminution of intensity of an infinitely narrow pencil of radiation during transmission through matter. In experiments on absorption care has to be taken to get the primary beam after transmission practically free from the scattered and re-scattered radiations, as well as from the fluorescent X-radiation and the corpuscular radiation, or at any rate to arrange for the effects to be small and to correct for them.

mass absorption coefficient  $\left(\frac{\lambda}{\rho}\right)$  in carbon of a certain X-radiation of quite ordinary penetrating power, such as certainly obeys the laws of scattering, was shown by Barkla and Sadler\* to be only about  $\cdot 41$ . It is difficult to reconcile this with Mr. Crowther's estimate of scattering, *i. e.*  $\frac{s}{\rho} = 1\cdot 18$ .

(2) Much more penetrating beams of homogeneous X-radiation have since been found by the writer, and the mass-absorption-coefficient for these in carbon has been found as low as about  $\cdot 25$ , or about  $\frac{1}{2}$  of Mr. Crowther's scattering coefficient. Thus either for these rays the scattering is much less than for the more absorbable rays, or there is a considerable discrepancy between these results. Neither theory nor experiment indicates that for these very penetrating rays there is any diminution in scattering; both in fact indicate that the scattered radiation carries away the same fraction of the energy of primary radiations differing widely in penetrating power.

(3) In aluminium itself, the mass-absorption-coefficient  $\left(\frac{\lambda}{\rho}\right)$  of certain penetrating rays is much less than  $1\cdot 18$ . The lowest value experimentally found is about  $\cdot 6$  for a fluorescent X-radiation characteristic of cerium.

(4) The simple laws of absorption found by Barkla and Sadler point to the conclusion that the mass scattering coefficient is of the order of magnitude of  $\cdot 2$ . The absorption of Röntgen radiation in a substance A bears an approximately constant ratio to the absorption in a substance B, through long ranges of penetrating power. The limits to this law are that the radiation used must not extend in penetrating power beyond that of any radiation characteristic of either A or B, and must not be near one of these characteristic radiations on its more penetrating side. This proportionality is evidently true only when the total absorption is great compared with the portion of it due to scattering, for as has been pointed out, the scattering is independent of the penetrating power of the radiation as well as the particular light element which is producing the scattering. There is thus a constant term in the variable mass-absorption-coefficient  $\frac{\lambda}{\rho}$ . If then we subtract a constant quantity  $\frac{s}{\rho}$  due to scattering from this, we expect the law to hold even for small values of  $\frac{\lambda}{\rho}$ , for there is no obvious reason why the law should be departed from just when scattering becomes an appreciable

\* Phil. Mag. May 1909, pp. 739-760.

fraction of the whole absorption. Now the quantity  $x$  which the writer has found from the equation

$$\frac{\left(\frac{\lambda}{\rho}\right)_C - x}{\left(\frac{\lambda}{\rho}\right)_{Al} - x} = \text{constant}$$

[suffixes denoting the absorbing substance] varies in different cases from .16 to .25.

It is very probable that with care the absorption coefficients will be determined with greater accuracy, and consequently  $x$  brought within narrower limits. The results are, however, sufficiently accurate to give the order of magnitude. This agrees very well with the value directly determined by the writer. It should also be pointed out that a value for  $x$  (i. e.  $\frac{s}{\rho}$ ) as high as 1.18 would involve complete violation in place of otherwise close agreement with simple laws of absorption.

(5) The penetrating power of the fluorescent radiations from various elements shows no tendency with increasing atomic weight of the radiator to approximate to the limit suggested by a scattering coefficient ( $\frac{s}{\rho}$ ) as large as 1.18. On the other hand, there is such an approximation in the absorption in carbon to an inferior limit for ( $\frac{\lambda}{\rho}$ ) equal to something of the order of .2.

There thus appears abundant, consistent, and apparently conclusive evidence that the intensity of Röntgen radiation scattered by light elements is of the order of magnitude found directly by the writer in early experiments, and that Mr. Crowther's result is several times too great.

The theory of scattering as given by Sir J. J. Thomson leads to the conclusion that the number of scattering electrons per atom is about half the atomic weight in the case of light atoms\*.

\* This applies to atomic weights not greater than 32, with the possible exception of hydrogen. Accurate results have not been obtained for the intensity of radiation scattered from heavier elements owing to the difficulty in many cases of getting rid of the fluorescent X-radiations superposed on the scattered radiation. Barkla & Sadler estimated the intensity of the radiation scattered from silver to be about 6 times that from an equal mass of the light elements. There is also indirect, but by no means conclusive evidence that still heavier atoms scatter to a greater extent. Measurements might easily be made in a number of cases with a fair degree of accuracy. The subject is worth further investigation.

Evidence is also given of a limit to the penetrating power of Röntgen radiation. Unless the laws of scattering somewhere break down, the lowest possible value for  $\frac{\lambda}{\rho}$  is about .2. This has been approached in the case of absorption by carbon.

LXXVII. *The Ratio between Uranium and Radium in Minerals.* II. By RUTH PIRRET, *B.Sc.*, and FREDERICK SODDY, *M.A., F.R.S.\**

IN a previous paper on this subject (*Phil. Mag.* 1910 [6] xx. p. 345), a short account was given of the determination of the ratio of radium to uranium in Ceylon thorianite and a specimen of Portuguese autunite. The preliminary results went to confirm those of Mlle. Gleditsch (*Compt. Rend.* 1909, cxlviii. p. 1451; cxlix. p. 267) in that the ratio in autunite was found to be considerably lower than in pitchblende; but the results with thorianite were not equally conclusive. Only one specimen of thorianite was compared with the old pitchblende standards prepared some years ago. Further investigations with several different specimens of thorianite, pitchblende, and autunite were therefore carried out on the same lines.

*Uranium Analysis.*—The methods of estimating the uranium in thorianite and autunite have already been described. In the case of the pitchblendes the mineral was first dissolved in nitric acid, the solution diluted, filtered, evaporated to dryness, the residue dissolved in hydrochloric acid, treated with sulphuretted hydrogen and filtered. The filtrate, after heating and oxidizing, was poured into a mixture of ammonium hydrate, sulphide, and carbonate, corked, and left over night. The filtrate from this precipitate was heated, acidified by nitric acid, and the uranium precipitated by microcosmic salt and sodium thiosulphate in presence of acetic acid. The precipitate was ignited in a porcelain crucible and weighed in the form of a green compound of constant composition. It was then converted, by means of a few drops of strong nitric acid, into uranium pyrophosphate and weighed again after ignition at a dull red heat (Brearley's 'Analytical Chemistry of Uranium,' p. 7). In some cases Patera's Method (Fresenius, 'Quantitative Analysis,' vol. ii. p. 310) was employed, or a modification of it in which, instead of the uranium being weighed as sodium uranate, it was, after separation by

\* Communicated by the Authors.

means of sodium carbonate, estimated as pyrophosphate as before.

Table I. contains the results of the estimation of uranium in the various minerals. The thorianites were all specimens of Ceylon thorianite. "Th I, Th I<sub>a</sub>, Th I<sub>b</sub>" were from the same sample of the variety richest in uranium and containing a small residue (3 per cent.) insoluble in acid. "Th X<sub>2</sub>" was a poor specimen of the mineral, and contained nearly 24 per cent. of insoluble material. "Th CCC" and "Th E" were specimens of mixtures of the two varieties (which differ chiefly in their respective high or low percentage of uranium). The pitchblende "P I" was a specimen from Joachimsthal which contained comparatively little uranium and proved troublesome in the analysis and unsatisfactory in the results. The other ("J.P.A" and "J.P.B") was a picked

TABLE I.

		Percentage of Uranium.					
		(1)	(2)	(3)	(4)	(5)	Mean.
Ceylon Thorianites.	{ Th I, Th I <sub>a</sub> , Th I <sub>b</sub> ...	20.80 <i>a</i>	20.11 <i>c</i>	19.86 <i>c</i>	19.32 <i>c</i>	19.80 <i>c</i>	} 20.06
	Th X <sub>2</sub> 1 & 2 .....	20.56 <i>b</i>	20.28 <i>d</i>	19.99 <i>d</i>	19.74 <i>d</i>	20.15 <i>d</i>	
	Th CCC .....	9.49 <i>a</i>	10.71 <i>d</i>	10.40 <i>d</i>	.....	.....	} 10.00
	Th E .....	9.39 <i>b</i>	.....	.....	.....	.....	
	Th E .....	13.02 <i>a</i>	14.04 <i>a</i>	13.44 <i>a</i>	.....	.....	} 13.61
Th E .....	13.20 <i>b</i>	14.14 <i>b</i>	13.82 <i>b</i>	.....	.....		
Joachimsthal Pitchblendes.	{ P I .....	17.21 <i>a</i>	17.42 <i>a</i>	.....	.....	.....	} 17.62
	P I .....	17.93 <i>b</i>	17.95 <i>b</i>	.....	.....	.....	
	{ J.P.A, J.P.B .....	33.35 <i>c</i>	28.06 <i>c</i>	30.81 <i>d</i>	34.99 <i>a</i>	31.20 <i>e</i>	} 31.86
	J.P.A, J.P.B .....	33.57 <i>d</i>	28.22 <i>d</i>	.....	34.71 <i>b</i>	.....	
	G.E.A.P. ....	62.37 <i>a</i>	60.73 <i>a</i>	61.77 <i>c</i>	61.34 <i>a</i>	.....	} 61.23
G.E.A.P. ....	61.39 <i>b</i>	59.98 <i>b</i>	.....	61.01 <i>b</i>	.....		
Portuguese* Autunites.	{ German East African pitchblende.	72.30 <i>c</i>	.....	.....	.....	.....	} 71.3
	German East African pitchblende.	70.31 <i>d</i>	.....	.....	.....	.....	
	{ A I, A II .....	20.04 <i>a</i>	19.90 <i>a</i>	19.54 <i>d</i>	21.56 <i>d</i>	19.95 <i>a</i>	} 20.10
	A I, A II .....	.....	.....	.....	.....	19.65 <i>b</i>	
	{ S I .....	47.82 <i>a</i>	47.85 <i>a</i>	.....	.....	.....	} 48.44
S I .....	49.30 <i>b</i>	48.79 <i>b</i>	.....	.....	.....		
Pilbarite .....	20.45 <i>a</i>	21.16 <i>a</i>	.....	.....	.....	} 21.28	
	20.85 <i>b</i>	21.72 <i>b</i>	.....	.....	.....		

\* The other autunites referred to in Tables II. and III. were used in connexion with other work (*Le Radium*, 1910, vii. p. 295), and details of the uranium analysis need not be given.

*a* denotes that the uranium was weighed as the green compound before conversion into pyrophosphate.

*b* " " " " as uranium pyrophosphate.

*c* " " " " as U<sub>3</sub>O<sub>8</sub>.

*d* " " " " as UO<sub>2</sub>.

*e* " " " " as sodium uranate.

specimen of Joachimsthal pitchblende containing a much larger proportion of uranium. "G.E.A.P." was a German East-African pitchblende. The autunites were all specimens of Portuguese autunite. A specimen of the new mineral pilbarite was obtained from Mr. Simpson of the West Australian Government Survey. It is described in the 'Australian Mining Standard,' 7/9/10 (Chem. News, 1910, cii. p. 283).

*Method of Estimation of Radium.*—The radium was estimated in the solution of the mineral by the emanation method. The apparatus used was at first (series A) that described by one of us (Phil. Mag, 1909 [6] xviii. p. 846). Later a new apparatus was employed with a microscope of about four times less magnifying power. With this instrument the leaf was charged positively instead of negatively, and was not kept charged during the period of three hours preceding the measurements (series B and C). A period of about three months elapsed between series B and C. In all twenty-seven radium preparations were used including the six old pitchblende standards already described (*loc. cit.*), representing twelve different minerals. These were prepared by dissolving quantities of minerals which could be accurately weighed, and taking a known fraction by weight of the solution.

Table II. shows the results. Column (1) gives the designation of the preparation, those bracketed being of the same specimen of the same mineral. Column (2) gives the number of milligrams of uranium in the preparation. Columns 3, 4, 5 give the number of divisions of leak of the electro-scope-leaf per minute in the three respective sets of observations, and 6, 7, 8 the same per milligram of uranium present.

The old standards I. to VI. had existed for nearly four years, and comparing the results in Table II. with the determinations previously published (*loc. cit.*) made at the time of preparation, the variations among the standards are rather larger than they were initially, but do not show any certain influence of age. "R I" and "R II" were made up on June 16, 1910, from a solution of radium bromide obtained from Professor Rutherford, and described by him as containing  $1570 \times 10^{-12}$  gram of radium. The Rutherford-Boltwood ratio is assumed in stating the quantity of uranium in these ( $3.4 \times 10^{-7}$  gram of radium per gram of uranium).



TABLE II.

Specimen.	Uranium (milligrams).	Leak (divisions per 1 min.).			Leak per 1 min. per 1 milli-gram Uranium.			
		A.	B.	C.	A.	B.	C.	
Pitchblendes.	{ St. I. ....	.308	20.28	.....	3.99	65.83	.....	[12.98]
	{ St. II. ....	1.49	.....	19.72	20.88	.....	13.23	14.23
	{ St. III. ....	.696	.....	9.56	9.83	.....	13.74	14.14
	{ St. IV. ....	6.66	.....	86.80	.....	.....	13.02	.....
	{ St. V. ....	.149	9.70	.....	1.80	65.10	.....	[12.10]
	{ St. VI. ....	.74	.....	9.75	10.01	.....	13.23	13.53
	{ P. I. ....	.395	23.55	4.91	5.02	59.62	12.43	12.71
	{ J.P.A. ....	.171	.....	2.25	.....	.....	13.16	.....
	{ J.P.B. ....	.3539	.....	4.66	5.00	.....	13.17	14.13
{ G.E.A.P. ...	.6534	49.64	10.39	10.56	75.97	15.90	16.16	
R I .....	.2077	14.80	2.61	2.71	71.25	[12.54]	[13.07]	
R II .....	1.0066	72.20	14.25	14.85	71.78	14.17	14.77	
Ceylon Thorionites.	{ Th I .....	.6103	41.16	8.81	8.93	67.44	14.43	14.62
	{ Th Ia .....	3.027	.....	43.65	43.64	.....	14.42	14.41
	{ Th Ib .....	1.926	.....	.....	27.88	.....	.....	14.47
	{ Th X <sub>2</sub> (1)...	8.4103	.....	111.5	.....	.....	13.25	.....
	{ Th X <sub>2</sub> (2)...	1.03	.....	.....	13.91	.....	.....	13.51
	{ Th CCC ...	.3475	.....	4.56	4.68	.....	13.14	13.46
	{ Th E .....	3.168	.....	.....	41.64	.....	.....	13.14
Portuguese Autunites.	{ A I .....	.834	24.28	5.12	.....	29.1	6.14	...
	{ A II .....	.92	26.22	4.42	.....	28.5	4.80	...
	{ A A I .....	.568	26.18	5.33	.....	46.1	9.39	...
	{ A A II .....	1.69	.....	16.29	.....	.....	9.63	...
	{ A B I .....	.562	16.24	3.29	.....	28.9	5.85	...
	{ S I .....	2.531	.....	.....	8.43	.....	.....	3.33
	{ A D .....	1.265	.....	.....	12.92	.....	.....	10.22
Pilbarite ...	1.359	.....	.....	12.10	.....	.....	8.90	

*Discussion of Results.*—In interpreting the results of Table II. it must be remembered that the determinations have not all equal weight. The specimens on which the A series of measurements was performed contained as a rule too small quantities of radium to give the best results with the less sensitive electroscopes and method employed in the B and C series. The measurements of Table II. enclosed in square brackets may be at once eliminated from further consideration on this account. In addition many of the solutions had been prepared a considerable time, and may have undergone changes by keeping. Three months elapsed between the measurements in the B and C series, and it is evident from the Table that the sensitiveness of the instrument had somewhat increased in the interval, as has been observed before with the old instrument.

The first thing is to reduce the three series of measurements given in the last three columns of Table II. to a common standard. This has been done by taking first the mean value in each of the three series for each of the minerals, except the autunites, and then taking the mean of these means. The result is, in divisions per milligram of uranium :—

A 68·00,            B 13·79,            C 14·24.

Hence to reduce all the determinations to the same standard as the C series, the A series must be multiplied by the factor 0·212, and the B series by 1·033. This has been done in Table III.

TABLE III.

	Divisions per milligram of Uranium (reduced values).			
	A (cor.).	B (cor.).	C.	Means.
St. I. ....	13·95	.....	.....	} 13·85
St. II. ....	.....	13·66	14·23	
St. III. ....	.....	14·18	14·14	
St. IV. ....	.....	13·44	.....	
St. V. ....	13·82	.....	.....	} 12·72
St. VI. ....	.....	13·66	13·53	
P I. ....	12·64	12·82	12·71	
J.P.A. ....	.....	13·58	.....	
J.P.B. ....	.....	13·59	14·13	} 13·77
G.E.A.P. ....	16·10	16·41	16·16	16·22
R I. ....	15·10	.....	.....	} 14·90
R II. ....	15·20	14·62	14·77	
Th I. ....	14·32	14·89	14·62	} 14·60
Th Ia. ....	.....	14·88	14·41	
Th Ib. ....	.....	.....	14·47	
Th X <sub>1</sub> (1) ....	.....	13·68	.....	} 13·55
Th X <sub>2</sub> (2) ....	.....	.....	13·51	
Th CCC. ....	.....	13·56	13·46	
Th E. ....	.....	.....	13·14	
A I. ....	6·20	6·32	.....	} 7·15
A II. ....	6·05	4·96	.....	
AA I. ....	9·80	9·68	.....	
AA II. ....	.....	9·94	.....	
AB I. ....	6·12	6·03	.....	
AS I. ....	.....	.....	3·33	
AD. ....	.....	.....	10·22	
Pilbarite ....	.....	.....	8·90	

Dealing first with the pitchblendes, the mean results for the old standards are :—

A 13·83,            B 13·74,            C 13·97,

which gives a mean result for this mineral of 13·85. The mean of the three determinations for the mineral "J. P. A" and "J. P. B" is 13·77. So that from these two results the mean value 13·8 may be taken with considerable confidence as representing Joachimsthal pitchblende. The specimen P I, giving a mean result of 12·72, must be rejected, for, as will be seen from Table I., its uranium analysis is far from satisfactory. Indeed this mineral proved most troublesome to analyse, the uranium content being low and the proportion of foreign constituents high. The mean of the four determinations with Rutherford's radium standard is 14·9; so that, if this is taken as the primary standard, the ratio of radium to uranium in Joachimsthal pitchblende is  $3\cdot15(\times 10^{-7})$ . The original value given by Rutherford and Boltwood was 3·8, which was lowered subsequently to 3·4 owing to an error in the uranium analysis (Boltwood, *Am. Journ. Sci.* 1911, xxv. p. 296). The radium solution provided by Professor Rutherford was part of the original employed by these investigators; so that, assuming the solution has not changed since its preparation, our results indicate that the corrected value is still somewhat high. The value we have arrived at,  $3\cdot15(\times 10^{-7})$ , is in good agreement with the following results of Mlle. Gleditsch, obtained with Mme. Curie's standards of radium (Mme. Curie, *Radioactivité*, ii. p. 441):—St. Joachimsthal pitchblende 3·21, Norwegian Clèveite 3·23, Brögerite 3·22, Portuguese chalcocite 3·24.

Dealing now with the thorianites, the Table shows that although the first specimen investigated, "Th I," as recorded in the last paper, gives an undoubtedly higher value than Joachimsthal pitchblende, it is alone of those examined in this respect. Five different uranium analyses and six estimations of the radium in three solutions of this mineral give the mean result 14·6, which is about 6 per cent. higher than that for Joachimsthal pitchblende. Even if the highest uranium result and the lowest radium result are compared, the value arrived at would still be as high as the mean for Joachimsthal pitchblende. But the other thorianites examined give an entirely different result. The mean five determinations on the three minerals is 13·47. Omitting "Th E," for which only one determination has so far been done, the mean is 13·55. Hence the later results have not confirmed

these of Mlle. Gleditsch that the ratio for thorianite is 20 per cent. higher than for pitchblende. It is interesting to notice that investigations published subsequently to Mlle. Gleditsch's work (Marckwald, *Ber. Chem. Ges.* 1910, xliii. p. 3420; Soddy, *Trans. Chem. Soc.* 1911, xcix. p. 72) show that in separating the radium she must also have separated the mesothorium quantitatively from the thorianite, though this does not account for her results.

We are inclined to ascribe the undoubtedly high value obtained for "Th I" to contamination with radium before it came into our hands. All the other specimens were samples of large quantities purchased direct from the importers, but "Th I" was obtained from a retail dealer who handles radium preparations and sells spinthariscopes. We attach no importance to the high result for the single specimen of German East-African pitchblende, as the same doubt arises. It came to us through Mr. Russell from Prof. Marckwald's laboratory, where chemical work on radium has long been carried on.

With regard to the autunites, all of which came from various mines in Portugal, the values range from that of "A D," which has 74 per cent. of the equilibrium amount, to that of "A S," which has only 24 per cent. and is the lowest yet recorded, that described by Mr. Russell (*Nature*, Aug. 25, 1910) from Autun, France, having 27 per cent. It is interesting to note that both these specimens "A D" and "A S" were from the same property (compare *Ann. Reports, Chem. Soc.* 1910, vii. p. 264). The pilbarite is also low (64 per cent.), but the mineral, described as probably a hydrous pseudomorph of an anhydrous parent mineral, is evidently much altered.

The main point at issue, that possibly the life-period of ionium is sufficiently extended to cause the equilibrium ratio of radium to uranium to be less in a geologically recent mineral like Joachimsthal pitchblende than in an ancient mineral like thorianite, although no doubt still an open one, certainly receives no support from these measurements.

Physical Chemical Laboratory,  
University of Glasgow.  
March 22nd, 1911.

LXXVIII. *An Apparent Softening of Röntgen Rays in Transmission through Matter.* By CHARLES A. SADLER, D.Sc., Oliver Lodge Fellow, and ALFRED I. STEVEN, M.A., B.Sc., Lecturer in Physics, University of Liverpool\*.

A LARGE proportion of the rays emitted by the anticathode of an ordinary X-ray tube must of necessity be absorbed in their passage through the glass walls, and consequently the nature of the emergent radiation is somewhat modified. A recognition of this fact has led various investigators to place a thin aluminium window in the walls of the tube, and to examine the nature of the radiation proceeding through it. In particular, Kaye† found that, with such a bulb, the amount of the radiation emitted for a given potential difference varied with the nature of the anticathode used, and also gave data which indicated that in some cases a fairly homogeneous beam was emitted. Subsequently it was pointed out that these beams were largely composed of the homogeneous radiation characteristic of the particular metal used as anticathode, but at the same time there was also present a certain amount of scattered radiation.

If a piece of glass, 1 mm. thick, were placed in the path of such a beam, it would practically absorb the whole of the homogeneous radiation together with the softer constituents of the scattered, the remainder being similar in character to the radiation from an ordinary bulb. Should the characteristic radiation, however, be very easily absorbed, and of no great intensity, a comparatively thin layer of glass or other substance would suffice to cut out this portion, but the beam would still contain components which are much softer than those present under ordinary circumstances.

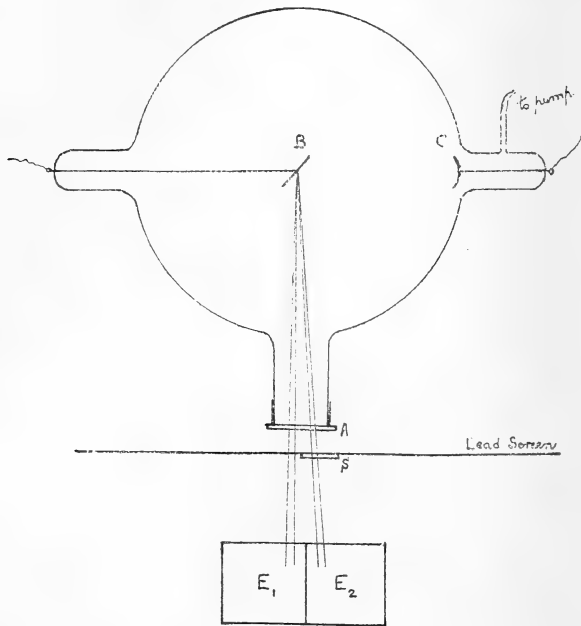
If these conditions can be experimentally realized, it is obvious that the issuing beam approximates much more closely to the scattered radiation as it leaves the anticathode itself. Now, as the characteristic radiation of aluminium is known to be very soft and of feeble intensity compared with the scattered radiation, it seemed desirable to investigate the nature of the radiation proceeding from a bulb having a thin aluminium window and fitted with an aluminium anticathode. The scope of the inquiry was limited to a measurement of the penetrating power and heterogeneity of the rays emitted at different stages of exhaustion, but here an unexpected difficulty arose. When the beam had reached a certain penetrating power, it appeared to become *softer* on cutting

\* Communicated by the Authors. The expenses of this research have been partially defrayed by a Government grant through the Royal Society.

† Phil. Trans. A. ccix. pp. 123-151.

off part of the radiation by sheets of different substances. As the explanation of this softening was not manifest, it was decided to make it the subject of further study.

Fig. 1.



The arrangement of the apparatus employed is shown in fig. 1, drawn to scale. The bulb of the usual spherical form had sealed into it three side-tubes all in one plane, two of which were at opposite ends of a diameter, while the third was along a radius perpendicular to the others. One of these tubes had a fixed aluminium cathode C, curved so as to concentrate the cathode rays on the anticathode B placed at the centre of the bulb. A thin sheet of aluminium ( $\cdot 00367$  cm.) was fixed between two similar brass disks. One of these was soldered to a tube which slid on the glass tube facing the anticathode. The radiation under examination was limited to two beams, which passed through two holes ( $\cdot 4$  cm. diameter) in the brass disk situated symmetrically with respect to the axis of the tube, the distance between their centres being  $1\cdot 5$  cm.

A lead screen prevented any stray radiation from the bulb from entering the two electroscopes (of the usual Wilson type),  $E_1$ ,  $E_2$ , which were used for testing the radiation. In the lead screen were two holes ( $\cdot 5$  cm. diameter) corresponding exactly to the two holes in the brass plate. The

radiation entered the two electroscopes by precisely similar holes. A guide was placed at S, so that absorbers could be placed and replaced in exactly the same position as required.

Between the bulb and the pump a tube containing charcoal was connected, and by surrounding this with liquid air the exhaustion was facilitated. When the liquid air had been in position for some time, the bulb reached a steady state.

The nature of the radiation from the bulb was then tested by placing sheets of different substances at S, and the amount by which the rays were absorbed could be deduced from the readings of the two electroscopes.

When the radiation was cut down by sheets of aluminium and the absorbability of the remainder tested by a thin sheet of aluminium (.00305), the beam appeared very heterogeneous as shown by the following table:—

TABLE I.

Previous per cent. absorption by Aluminium sheets.	Subsequent per cent. absorption by Aluminium test-piece.
0	34.3
37.4	26.9
56.5	23.4
75.5	17.8
80.6	12.9
90.0	10.4

When, however, the beam was cut down by sheets of copper and the remainder still tested with the same aluminium sheet, it now appeared fairly homogeneous, as shown in Table II.

TABLE II.

Previous per cent. absorption by Copper sheets.	Subsequent per cent. absorption by Aluminium test-piece.
0	31.6
48.3	31.7
73	30.0
83	30.1

In general, it was found that when the beam was cut down by a substance which, under the stimulus of a suitable Röntgen radiation, emits a characteristic homogeneous radiation considerably in excess of that which it scatters, *e. g.* Ni, Fe, &c.,

results similar to that for copper were obtained. On the other hand, if cut down by substances of low atomic weight, which chiefly scatter incident radiation, *e. g.* C, Al, &c., or by substances of higher atomic weight, *e. g.* Au, in which the characteristic radiation is not excited except by very penetrating beams, the results were similar to those given in Table I.

It might be supposed that these effects were due to secondary radiation superposed on the primary beam, but direct experiment showed that when the beam was cut down by copper or iron at S, the amount of secondary radiation entering either of the electroscopes  $E_1$  or  $E_2$  would not account for 1 per cent. of the observed ionization.

The use of the charcoal cooled by liquid air as an aid to exhaustion had however its disadvantages. After the bulb had been running for a considerable time the discharge tended to become intermittent, and the readings obtained were very irregular. The charcoal was therefore dispensed with, and the pump alone depended on for exhaustion. Under these new conditions, the discharge at any stage appeared much more regular, and there was the added advantage of being able to investigate the rays at an earlier stage in the exhaustion. By continuing the pumping, higher stages of exhaustion could be reached, and the process, although slower, proved more reliable.

The beam was now tested by a thin sheet of aluminium (.00305 cm.), and the absorption was found both before and after cutting down the beam by iron (.00124 cm.). With a moderately penetrating beam the absorption by the aluminium was considerably greater after transmission through the iron than before transmission, and this apparent softening was greater as the initial primary beam became still harder. This is clearly shown in the following table:—

TABLE III.

Per cent. absorption by Al (.00305) of initial beam.	Per cent. absorption by Al (.00305) after cutting down primary beam by iron.	Per cent. absorbed by the iron sheet (.00124 cm.).	Per cent. increase in absorption by the aluminium.
48.8	55.7	65	14.2
45.5	57.2	67	25.8
39.4	50.7	67.6	28.7
30.4	46.0	63.8	51.3
21.4	39.6	54.5	85.0



Similar results were found when the beam was cut down by nickel and copper, aluminium still being used as test-substance.

When the beam was cut down by aluminium and tested by aluminium, the absorption decreased slightly at first, but more rapidly as successive sheets were placed in the primary beam.

TABLE IV.

Per cent. previously absorbed by Al sheets.	Per cent. subsequently absorbed by Al (·00305).
0	27·2
30·7	26·8
67·5	20·1
88·0	12·8

If the beam was cut down by aluminium and tested by nickel, the same apparent softening occurred.

TABLE V.

Per cent. previously absorbed by Al sheets.	Absorption by a Nickel sheet (·00095 cm.).
0	45·7
35	55·4
0	47·9

As this softening had not been observed when the charcoal tube cooled by liquid air was used, it was again resorted to in order to bring the bulb to the same stage of hardness. The observations showed that the phenomena were similar but not so pronounced.

*Discussion of Results.*

When a primary beam passes through a thin sheet of an element, there is a loss of energy (measured by the decrease in the ionization it is able to produce in a given volume of air) which may be due to :—

1. A scattering of a portion of the incident energy ;
2. A transformation of energy into the production of a homogeneous radiation characteristic of the element ;
3. A production of a corpuscular radiation accompanying both the scattered and the homogeneous radiation.

It has been shown\* that this characteristic radiation is only produced by a more penetrating radiation, and the increase in absorption accompanying its production is a maximum for an exciting beam only slightly more penetrating. In Table VI.\* are given the absorption coefficients in different substances of the characteristic radiations of the elements Cr ..... Ag. From columns II. and III. it will be seen that these radiations are in increasing order of penetrating power when tested by elements in which they excite little or no characteristic radiation; and from column IV. that the absorption coefficient in iron is a maximum for nickel radiation.

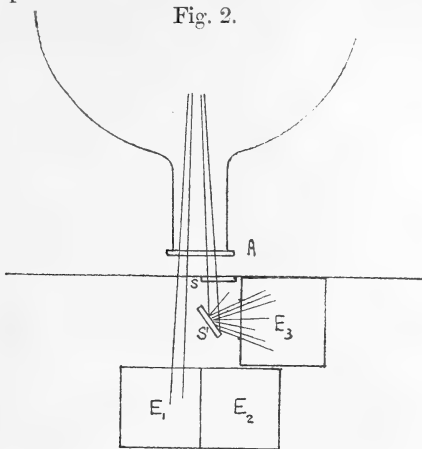
TABLE VI.  
Mass Absorption Coefficients ( $\lambda/\rho$ ).

RADIATOR.	ABSORBER.					
	C.	Al.	Fe.	Ni.	Cu.	Zn.
Cr .....	15.3	136.0	103.8	129	143	170.5
Fe .....	10.1	88.5	66.1	83.8	95.1	112.5
Co .....	7.96	71.6	67.2	67.2	75.3	91.5
Ni .....	6.58	59.1	31.4	56.3	61.8	74.4
Cu .....	5.22	47.7	268	62.7	53.0	60.9
Zn .....	4.26	39.4	221	265	55.5	50.1
As .....	2.49	22.5	134	166	176	203.5
Se .....	2.04	18.9	116.3	141.3	149.8	174.6
Ag .....	.41	2.5	17.4	22.7	24.3	27.1

We should expect that if we placed a sheet of iron in our heterogeneous primary beam, it would specially absorb the constituents of about the penetrating power of nickel radiation, and more penetrating constituents to a lesser extent. The portion of the beam too soft to excite iron radiation would be absorbed only to a limited extent. For example, it will be seen that a constituent of the hardness of chromium radiation, of which the absorption coefficient in aluminium is 136, would be absorbed to a less extent by iron than that of the radiation from selenium, the absorption coefficient of which in aluminium is only 18.9. On the whole, then, the beam after passing through iron would be richer proportionately in these softer rays than before, but these latter are much more easily absorbed by aluminium than those specially absorbed by iron, and therefore we would expect the beam to be softer to aluminium after passing through iron than before.

\* Barkla & Sadler, Phil. Mag. May 1909, pp. 739-76

This explanation of the phenomena is borne out by the following experiments.



An electroscope  $E_3$  (fig. 2) was placed with its aperture parallel to the primary beam. A guide  $S'$  was arranged so that different substances could be placed in the path of the beam entering  $E_2$ . The secondary radiations excited in these could be measured by  $E_3$ . The ionization in electroscopes  $E_1$  serving as a standard.

Strips of Ti, Cr, Fe, Cu, Zn were used as radiators. The amounts by which the ionization in  $E_3$  was diminished when sheets of iron and aluminium respectively (absorbing the same percentage of the primary) were put at  $S$ , were noted. These diminutions in the secondary radiation indicated which constituents of the primary were cut off by the iron and aluminium respectively. The results are tabulated below.

TABLE VII.

Secondary Radiator.	$\lambda/\rho$ for secondary radiation in Al.	Per cent. by which Al in primary (cutting off 63 per cent.) cuts down secondary radiation.	Per cent. by which Fe in primary (cutting off 62.8 per cent.) cuts down secondary radiation.	$\lambda/\rho$ for secondary radiation in Iron.
Ti .....	230	68.1	54.2	173
Cr .....	136	63.7	62	104
Fe .....	88.5	53.5	<u>69.6</u>	66.2
Cu .....	47.7	45.3	55	268
Zn .....	39.4	42.3	46.2	221

It will be seen from column IV. that the iron specially cuts down the secondary radiation from iron as predicted, *i. e.*, the iron selectively absorbs those constituents of the primary capable of exciting radiations in itself. It will be seen also that the softer radiations, too soft to excite iron radiation but capable of producing considerable radiation from Ti and Cr, are cut down by the iron to a lesser extent than the average. These latter, on the other hand, are cut down more readily by the aluminium than those which specially excite on iron (see column III.).

It is interesting in this connexion to show the subsequent absorption by aluminium of the primary beam after being cut down by iron and aluminium respectively to the same extent (63 per cent.).

Of the initial beam		aluminium '00305 absorbs 28.9%
” ” ” cut down by Fe	”	” ” ” 41.4%
” ” ” ” Al	”	” ” ” 22.2%

The softening of the primary beam when cut down by various substances and tested by a thin sheet of a substance other than aluminium, has also been observed. The following two examples indicate, however, that the occurrence of the effect depends on certain relations existing between the radiations characteristic of the absorber and of the test substance.

#### I. Cutting down the primary beam 60.3 per cent. by iron:

Test Substance.	Per cent. absorbed by test substance :		Per cent. of secondary from test substance cut down by Iron.
	<i>Before cutting down by Iron.</i>	<i>After cutting down by Iron.</i>	
Ni .....	49	56.6	54.9
Fe .....	60	52.6	65

#### II. Cutting down the primary beam 50.7 per cent. by nickel :

Test Substance.	Per cent. absorbed by test substance :		Per cent. of secondary from test substance cut down by Nickel
	<i>Before cutting down by Nickel.</i>	<i>After cutting down by Nickel.</i>	
Ni .....	47.7	46.2	62.7
Fe .....	60.3	67.9	52

From what has been said in a previous portion of the

paper, it will be gathered that a primary beam, after passing through a thin sheet of iron, is deficient in those constituents which are especially capable of exciting iron radiation. If, then, the issuing beam is absorbed by a further sheet of iron of the same thickness, we should expect the second sheet to absorb much less. This is well shown in the first example above, for the iron sheet which previously absorbed 60 per cent. now only absorbs 52·6 per cent.

A similar line of reasoning explains why the beam is apparently harder to nickel after passing through nickel.

An examination of Table VI. shows clearly that nickel absorbs those constituents which especially excite the characteristic radiation of iron (*e. g.* those of the hardness of nickel and copper radiations) to a much less extent than it absorbs more penetrating radiations (*e. g.* those of the hardness of Zn, As, Se, &c.). But Table VII. column IV. clearly shows that the beam after passing through iron is richer in those constituents which can be specially absorbed by nickel, for these are cut down to a lesser extent than the average.

As a further test of this point, the secondary radiations from iron and nickel respectively were measured before and after cutting down the primary beam by iron. A reference to the last column of the first example shows that while the whole beam is reduced by 60 per cent., the constituents specially capable of exciting secondary homogeneous radiation in nickel are only reduced by 54·9 per cent.; it also well illustrates the fact that iron selectively absorbs those constituents specially exciting radiation in itself—this absorption 65 per cent. being above the average.

The results given in example II. equally confirm these views.

The phenomena so far described can be readily duplicated with a beam composed of suitable proportions of various homogeneous radiations. For instance, a beam composed of the homogeneous radiations from iron, nickel, zinc, and arsenic each equally contributing to the ionization produced, shows the following properties when tested by the sheets used in these experiments:—

A sheet of nickel absorbs of the composite beam 63·3 per cent., but after transmission, the absorption by a similar sheet of nickel *falls* to 52 per cent.; on the other hand, if the beam had previously passed through iron (cutting off 76·2 per cent.), the subsequent absorption by the same sheet of nickel *rises* to 78·6 per cent.

A sheet of aluminium absorbing 33·7 per cent. of the

composite beam, absorbs 38·6 per cent. after cutting down by iron, and 38·2 per cent. after cutting down by nickel.

Other experiments have been carried out on similar lines, using anticathodes of different metals. In general, the phenomena observed are of the same kind as those already described. There are one or two outstanding features which are being further investigated. These include:—1. An apparent considerable softening of a primary beam from certain anticathodes when tested by aluminium after having been cut down by aluminium. 2. A variation in the components of primary beams of the same average hardness produced under different conditions.

In conclusion, we wish to point out the importance of using as a test-substance, when comparing the penetrating powers of different beams, one in which the fraction of the total absorption due to the emission of secondary characteristic radiation is small, *e.g.* Al, C, &c.

Attention may also be directed to the use of the distinguishing properties of the characteristic radiations of various elements as affording an effective means for the analysis of heterogeneous beams.

#### *Summary.*

An apparent softening of a heterogeneous primary beam in the process of transmission through matter has been observed.

This effect is shown to be connected with the selective absorption by a substance of those constituents of the beam, which can readily excite its characteristic homogeneous radiation.

Confirmation of this view has been obtained by an analysis of the primary beam.

We wish to place on record our appreciation of the kindly interest Professor Wilberforce has shown and the encouragement we have received from him throughout the course of these experiments.

George Holt Physics Laboratory,  
University of Liverpool.  
March 27th, 1911.

LXXIX. *The Scattering of  $\alpha$  and  $\beta$  Particles by Matter and the Structure of the Atom.* By Professor E. RUTHERFORD, F.R.S., University of Manchester\*.

§ 1. IT is well known that the  $\alpha$  and  $\beta$  particles suffer deflexions from their rectilinear paths by encounters with atoms of matter. This scattering is far more marked for the  $\beta$  than for the  $\alpha$  particle on account of the much smaller momentum and energy of the former particle. There seems to be no doubt that such swiftly moving particles pass through the atoms in their path, and that the deflexions observed are due to the strong electric field traversed within the atomic system. It has generally been supposed that the scattering of a pencil of  $\alpha$  or  $\beta$  rays in passing through a thin plate of matter is the result of a multitude of small scatterings by the atoms of matter traversed. The observations, however, of Geiger and Marsden † on the scattering of  $\alpha$  rays indicate that some of the  $\alpha$  particles must suffer a deflexion of more than a right angle at a single encounter. They found, for example, that a small fraction of the incident  $\alpha$  particles, about 1 in 20,000, were turned through an average angle of  $90^\circ$  in passing through a layer of gold-foil about  $\cdot 00004$  cm. thick, which was equivalent in stopping-power of the  $\alpha$  particle to 1.6 millimetres of air. Geiger ‡ showed later that the most probable angle of deflexion for a pencil of  $\alpha$  particles traversing a gold-foil of this thickness was about  $0^\circ\cdot 87$ . A simple calculation based on the theory of probability shows that the chance of an  $\alpha$  particle being deflected through  $90^\circ$  is vanishingly small. In addition, it will be seen later that the distribution of the  $\alpha$  particles for various angles of large deflexion does not follow the probability law to be expected if such large deflexions are made up of a large number of small deviations. It seems reasonable to suppose that the deflexion through a large angle is due to a single atomic encounter, for the chance of a second encounter of a kind to produce a large deflexion must in most cases be exceedingly small. A simple calculation shows that the atom must be a seat of an intense electric field in order to produce such a large deflexion at a single encounter.

Recently Sir J. J. Thomson § has put forward a theory to

\* Communicated by the Author. A brief account of this paper was communicated to the Manchester Literary and Philosophical Society in February, 1911.

† Proc. Roy. Soc. lxxxii. p. 495 (1909).

‡ Proc. Roy. Soc. lxxxiii. p. 492 (1910).

§ Camb. Lit. & Phil. Soc. xv. pt. 5 (1910).

explain the scattering of electrified particles in passing through small thicknesses of matter. The atom is supposed to consist of a number  $N$  of negatively charged corpuscles, accompanied by an equal quantity of positive electricity uniformly distributed throughout a sphere. The deflexion of a negatively electrified particle in passing through the atom is ascribed to two causes—(1) the repulsion of the corpuscles distributed through the atom, and (2) the attraction of the positive electricity in the atom. The deflexion of the particle in passing through the atom is supposed to be small, while the average deflexion after a large number  $m$  of encounters was taken as  $\sqrt{m} \cdot \theta$ , where  $\theta$  is the average deflexion due to a single atom. It was shown that the number  $N$  of the electrons within the atom could be deduced from observations of the scattering of electrified particles. The accuracy of this theory of compound scattering was examined experimentally by Crowther\* in a later paper. His results apparently confirmed the main conclusions of the theory, and he deduced, on the assumption that the positive electricity was continuous, that the number of electrons in an atom was about three times its atomic weight.

The theory of Sir J. J. Thomson is based on the assumption that the scattering due to a single atomic encounter is small, and the particular structure assumed for the atom does not admit of a very large deflexion of an  $\alpha$  particle in traversing a single atom, unless it be supposed that the diameter of the sphere of positive electricity is minute compared with the diameter of the sphere of influence of the atom.

Since the  $\alpha$  and  $\beta$  particles traverse the atom, it should be possible from a close study of the nature of the deflexion to form some idea of the constitution of the atom to produce the effects observed. In fact, the scattering of high-speed charged particles by the atoms of matter is one of the most promising methods of attack of this problem. The development of the scintillation method of counting single  $\alpha$  particles affords unusual advantages of investigation, and the researches of H. Geiger by this method have already added much to our knowledge of the scattering of  $\alpha$  rays by matter.

§ 2. We shall first examine theoretically the single encounters † with an atom of simple structure, which is able to

\* Crowther, Proc. Roy. Soc. lxxxiv. p. 226 (1910).

† The deviation of a particle throughout a considerable angle from an encounter with a single atom will in this paper be called "single" scattering. The deviation of a particle resulting from a multitude of small deviations will be termed "compound" scattering.



produce large deflexions of an  $\alpha$  particle, and then compare the deductions from the theory with the experimental data available.

Consider an atom which contains a charge  $\pm Ne$  at its centre surrounded by a sphere of electrification containing a charge  $\mp Ne$  supposed uniformly distributed throughout a sphere of radius  $R$ .  $e$  is the fundamental unit of charge, which in this paper is taken as  $4.65 \times 10^{-10}$  E.S. unit. We shall suppose that for distances less than  $10^{-12}$  cm. the central charge and also the charge on the  $\alpha$  particle may be supposed to be concentrated at a point. It will be shown that the main deductions from the theory are independent of whether the central charge is supposed to be positive or negative. For convenience, the sign will be assumed to be positive. The question of the stability of the atom proposed need not be considered at this stage, for this will obviously depend upon the minute structure of the atom, and on the motion of the constituent charged parts.

In order to form some idea of the forces required to deflect an  $\alpha$  particle through a large angle, consider an atom containing a positive charge  $Ne$  at its centre, and surrounded by a distribution of negative electricity  $Ne$  uniformly distributed within a sphere of radius  $R$ . The electric force  $X$  and the potential  $V$  at a distance  $r$  from the centre of an atom for a point inside the atom, are given by

$$X = Ne \left( \frac{1}{r^2} - \frac{r}{R^3} \right)$$

$$V = Ne \left( \frac{1}{r} - \frac{3}{2R} + \frac{r^2}{2R^3} \right).$$

Suppose an  $\alpha$  particle of mass  $m$  and velocity  $u$  and charge  $E$  shot directly towards the centre of the atom. It will be brought to rest at a distance  $b$  from the centre given by

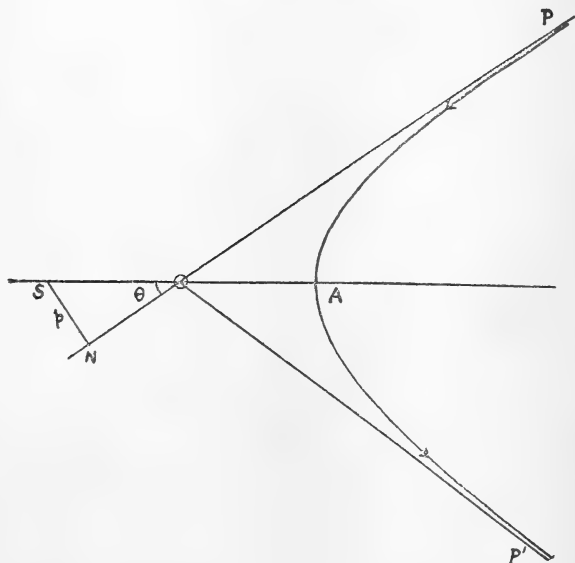
$$\frac{1}{2}mu^2 = NeE \left( \frac{1}{b} - \frac{3}{2R} + \frac{b^2}{2R^3} \right).$$

It will be seen that  $b$  is an important quantity in later calculations. Assuming that the central charge is  $100e$ , it can be calculated that the value of  $b$  for an  $\alpha$  particle of velocity  $2.09 \times 10^9$  cms. per second is about  $3.4 \times 10^{-12}$  cm. In this calculation  $b$  is supposed to be very small compared with  $R$ . Since  $R$  is supposed to be of the order of the radius of the atom, viz.  $10^{-8}$  cm., it is obvious that the  $\alpha$  particle before being turned back penetrates so close to

the central charge, that the field due to the uniform distribution of negative electricity may be neglected. In general, a simple calculation shows that for all deflexions greater than a degree, we may without sensible error suppose the deflexion due to the field of the central charge alone. Possible single deviations due to the negative electricity, if distributed in the form of corpuscles, are not taken into account at this stage of the theory. It will be shown later that its effect is in general small compared with that due to the central field.

Consider the passage of a positive electrified particle close to the centre of an atom. Supposing that the velocity of the particle is not appreciably changed by its passage through the atom, the path of the particle under the influence of a repulsive force varying inversely as the square of the distance will be an hyperbola with the centre of the atom  $S$  as the external focus. Suppose the particle to enter the atom in the direction  $PO$  (fig. 1), and that the direction of motion

Fig. 1.



on escaping the atom is  $OP'$ .  $OP$  and  $OP'$  make equal angles with the line  $SA$ , where  $A$  is the apse of the hyperbola.  $p = SN =$  perpendicular distance from centre on direction of initial motion of particle.

Let angle POA =  $\theta$ .

Let V = velocity of particle on entering the atom,  $v$  its velocity at A, then from consideration of angular momentum

$$pV = SA \cdot v.$$

From conservation of energy

$$\frac{1}{2}mV^2 = \frac{1}{2}mv^2 - \frac{NeE}{SA},$$

$$v^2 = V^2 \left(1 - \frac{b}{SA}\right).$$

Since the eccentricity is  $\sec \theta$ ,

$$SA = SO + OA = p \operatorname{cosec} \theta (1 + \cos \theta)$$

$$= p \cot \theta / 2,$$

$$p^2 = SA(SA - b) = p \cot \theta / 2 (p \cot \theta / 2 - b),$$

$$\therefore b = 2p \cot \theta.$$

The angle of deviation  $\phi$  of the particle is  $\pi - 2\theta$  and

$$\cot \phi / 2 = \frac{2p}{b} \dots \dots \dots (1)$$

This gives the angle of deviation of the particle in terms of  $b$ , and the perpendicular distance of the direction of projection from the centre of the atom.

For illustration, the angle of deviation  $\phi$  for different values of  $p/b$  are shown in the following table:—

$p/b \dots$	10	5	2	1	.5	.25	.125
$\phi \dots\dots$	5°·7	11°·4	28°	53°	90°	127°	152°

§ 3. Probability of single deflexion through any angle.

Suppose a pencil of electrified particles to fall normally on a thin screen of matter of thickness  $t$ . With the exception of the few particles which are scattered through a large angle, the particles are supposed to pass nearly normally through the plate with only a small change of velocity. Let  $n$  = number of atoms in unit volume of material. Then the number of collisions of the particle with the atom of radius  $R$  is  $\pi R^2 nt$  in the thickness  $t$ .

\* A simple consideration shows that the deflexion is unaltered if the forces are attractive instead of repulsive.

The probability  $m$  of entering an atom within a distance  $p$  of its centre is given by

$$m = \pi p^2 nt.$$

Chance  $dm$  of striking within radii  $p$  and  $p + dp$  is given by

$$dm = 2\pi pnt \cdot dp = \frac{\pi}{4} nt b^2 \cot \phi/2 \operatorname{cosec}^2 \phi/2 d\phi, \quad (2)$$

since  $\cot \phi/2 = 2p/b$ .

The value of  $dm$  gives the *fraction* of the total number of particles which are deviated between the angles  $\phi$  and  $\phi + d\phi$ .

The fraction  $\rho$  of the total number of particles which are deflected through an angle greater than  $\phi$  is given by

$$\rho = \frac{\pi}{4} nt b^2 \cot^2 \phi/2. \quad \dots \quad (3)$$

The fraction  $\rho$  which is deflected between the angles  $\phi_1$  and  $\phi_2$  is given by

$$\rho = \frac{\pi}{4} nt b^2 \left( \cot^2 \frac{\phi_1}{2} - \cot^2 \frac{\phi_2}{2} \right). \quad \dots \quad (4)$$

It is convenient to express the equation (2) in another form for comparison with experiment. In the case of the  $\alpha$  rays, the number of scintillations appearing on a *constant* area of a zinc sulphide screen are counted for different angles with the direction of incidence of the particles. Let  $r$  = distance from point of incidence of  $\alpha$  rays on scattering material, then if  $Q$  be the total number of particles falling on the scattering material, the number  $y$  of  $\alpha$  particles falling on unit area which are deflected through an angle  $\phi$  is given by

$$y = \frac{Q dm}{2\pi r^2 \sin \phi \cdot d\phi} = \frac{nt b^2 \cdot Q \cdot \operatorname{cosec}^4 \phi/2}{16r^2} \dots \quad (5)$$

Since  $b = \frac{2NeE}{mu^2}$ , we see from this equation that the number of  $\alpha$  particles (scintillations) per unit area of zinc sulphide screen at a given distance  $r$  from the point of

incidence of the rays is proportional to

- (1)  $\operatorname{cosec}^4 \phi/2$  or  $1/\phi^4$  if  $\phi$  be small ;
- (2) thickness of scattering material  $t$  provided this is small ;
- (3) magnitude of central charge  $Ne$  ;
- (4) and is inversely proportional to  $(mv^2)^2$ , or to the fourth power of the velocity if  $m$  be constant.

In these calculations, it is assumed that the  $\alpha$  particles scattered through a large angle suffer only one large deflexion. For this to hold, it is essential that the thickness of the scattering material should be so small that the chance of a second encounter involving another large deflexion is very small. If, for example, the probability of a single deflexion  $\phi$  in passing through a thickness  $t$  is  $1/1000$ , the probability of two successive deflexions each of value  $\phi$  is  $1/10^6$ , and is negligibly small.

The angular distribution of the  $\alpha$  particles scattered from a thin metal sheet affords one of the simplest methods of testing the general correctness of this theory of single scattering. This has been done recently for  $\alpha$  rays by Dr. Geiger \*, who found that the distribution for particles deflected between  $30^\circ$  and  $150^\circ$  from a thin gold-foil was in substantial agreement with the theory. A more detailed account of these and other experiments to test the validity of the theory will be published later.

#### § 4. *Alteration of velocity in an atomic encounter.*

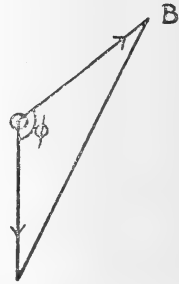
It has so far been assumed that an  $\alpha$  or  $\beta$  particle does not suffer an appreciable change of velocity as the result of a single atomic encounter resulting in a large deflexion of the particle. The effect of such an encounter in altering the velocity of the particle can be calculated on certain assumptions. It is supposed that only two systems are involved, viz., the swiftly moving particle and the atom which it traverses supposed initially at rest. It is supposed that the principle of conservation of momentum and of energy applies, and that there is no appreciable loss of energy or momentum by radiation.

\* Manch. Lit. & Phil. Soc. 1910.

- Let  $m$  be mass of the particle,  
 $v_1$  = velocity of approach,  
 $v_2$  = velocity of recession,  
 $M$  = mass of atom,  
 $V$  = velocity communicated to atom as result of encounter.

Let OA (fig. 2) represent in magnitude and direction the momentum  $mv_1$  of the entering particle, and OB the momentum of the receding particle which has been turned through an angle  $AOB = \phi$ . Then BA represents in magnitude and direction the momentum  $MV$  of the recoiling atom.

Fig. 2.



$$(MV)^2 = (mv_1)^2 + (mv_2)^2 - 2m^2v_1v_2 \cos \phi. \quad (1)$$

By the conservation of energy

$$MV^2 = mv_1^2 - mv_2^2. \quad (2)$$

Suppose  $M/m = K$  and  $v_2 = \rho v_1$ , where  $\rho$  is  $< 1$ .

From (1) and (2),

$$(K+1)\rho^2 - 2\rho \cos \phi = K-1,$$

$$\text{or} \quad \rho = \frac{\cos \phi}{K+1} + \frac{1}{K+1} \sqrt{K^2 - \sin^2 \phi}.$$

Consider the case of an  $\alpha$  particle of atomic weight 4, deflected through an angle of  $90^\circ$  by an encounter with an atom of gold of atomic weight 197.

Since  $K = 49$  nearly,

$$\rho = \sqrt{\frac{K-1}{K+1}} = .979,$$

or the velocity of the particle is reduced only about 2 per cent. by the encounter.

In the case of aluminium  $K = 27/4$  and for  $\phi = 90^\circ$   $\rho = .86$ .

It is seen that the reduction of velocity of the  $\alpha$  particle becomes marked on this theory for encounters with the lighter atoms. Since the range of an  $\alpha$  particle in air or other matter is approximately proportional to the cube of the velocity, it follows that an  $\alpha$  particle of range 7 cms. has its range reduced to 4.5 cms. after incurring a single

deviation of  $90^\circ$  in traversing an aluminium atom. This is of a magnitude to be easily detected experimentally. Since the value of  $K$  is very large for an encounter of a  $\beta$  particle with an atom, the reduction of velocity on this formula is very small.

Some very interesting cases of the theory arise in considering the changes of velocity and the distribution of scattered particles when the  $\alpha$  particle encounters a light atom, for example a hydrogen or helium atom. A discussion of these and similar cases is reserved until the question has been examined experimentally.

§ 5. *Comparison of single and compound scattering.*

Before comparing the results of theory with experiment, it is desirable to consider the relative importance of single and compound scattering in determining the distribution of the scattered particles. Since the atom is supposed to consist of a central charge surrounded by a uniform distribution of the opposite sign through a sphere of radius  $R$ , the chance of encounters with the atom involving small deflexions is very great compared with the chance of a single large deflexion.

This question of compound scattering has been examined by Sir J. J. Thomson in the paper previously discussed (§ 1). In the notation of this paper, the average deflexion  $\phi_1$  due to the field of the sphere of positive electricity of radius  $R$  and quantity  $Ne$  was found by him to be

$$\phi_1 = \frac{\pi}{4} \cdot \frac{NeE}{mu^2} \cdot \frac{1}{R}.$$

The average deflexion  $\phi_2$  due to the  $N$  negative corpuscles supposed distributed uniformly throughout the sphere was found to be

$$\phi_2 = \frac{16}{5} \frac{eE}{mu^2} \cdot \frac{1}{R} \sqrt{\frac{3N}{2}}.$$

The mean deflexion due to both positive and negative electricity was taken as

$$(\phi_1^2 + \phi_2^2)^{1/2}.$$

In a similar way, it is not difficult to calculate the average deflexion due to the atom with a central charge discussed in this paper.

Since the radial electric field  $X$  at any distance  $r$  from the

centre is given by

$$X = Ne \left( \frac{1}{r^2} - \frac{r}{R^3} \right),$$

it is not difficult to show that the deflexion (supposed small) of an electrified particle due to this field is given by

$$\theta = \frac{b}{p} \left( 1 - \frac{p^2}{R^2} \right)^{3/2},$$

where  $p$  is the perpendicular from the centre on the path of the particle and  $b$  has the same value as before. It is seen that the value of  $\theta$  increases with diminution of  $p$  and becomes great for small values of  $\phi$ .

Since we have already seen that the deflexions become very large for a particle passing near the centre of the atom, it is obviously not correct to find the average value by assuming  $\theta$  is small.

Taking  $R$  of the order  $10^{-8}$  cm., the value of  $p$  for a large deflexion is for  $\alpha$  and  $\beta$  particles of the order  $10^{-11}$  cm. Since the chance of an encounter involving a large deflexion is small compared with the chance of small deflexions, a simple consideration shows that the average small deflexion is practically unaltered if the large deflexions are omitted. This is equivalent to integrating over that part of the cross section of the atom where the deflexions are small and neglecting the small central area. It can in this way be simply shown that the average small deflexion is given by

$$\phi_1 = \frac{3\pi}{8} \frac{b}{R}.$$

This value of  $\phi_1$  for the atom with a concentrated central charge is three times the magnitude of the average deflexion for the same value of  $Ne$  in the type of atom examined by Sir J. J. Thomson. Combining the deflexions due to the electric field and to the corpuscles, the average deflexion is

$$(\phi_1^2 + \phi_2^2)^{1/2} \quad \text{or} \quad \frac{b}{2R} \left( 5.54 + \frac{15.4}{N} \right)^{1/2}.$$

It will be seen later that the value of  $N$  is nearly proportional to the atomic weight, and is about 100 for gold. The effect due to scattering of the individual corpuscles expressed by the second term of the equation is consequently small for heavy atoms compared with that due to the distributed electric field.



Neglecting the second term, the average deflexion per atom is  $\frac{3\pi b}{8R}$ . We are now in a position to consider the relative effects on the distribution of particles due to single and to compound scattering. Following J. J. Thomson's argument, the average deflexion  $\theta_t$  after passing through a thickness  $t$  of matter is proportional to the square root of the number of encounters and is given by

$$\theta_t = \frac{3\pi b}{8R} \sqrt{\pi R^2 \cdot n \cdot t} = \frac{3\pi b}{8} \sqrt{\pi n t},$$

where  $n$  as before is equal to the number of atoms per unit volume.

The probability  $p_1$  for compound scattering that the deflexion of the particle is greater than  $\phi$  is equal to  $e^{-\phi^2/\theta_t^2}$ .

Consequently 
$$\phi^2 = -\frac{9\pi^3}{64} b^2 n t \log p_1.$$

Next suppose that single scattering alone is operative. We have seen (§ 3) that the probability  $p_2$  of a deflexion greater than  $\phi$  is given by

$$p_2 = \frac{\pi}{4} b^2 \cdot n \cdot t \cot^2 \phi / 2.$$

By comparing these two equations

$$p_2 \log p_1 = -\cdot 181 \phi^2 \cot^2 \phi / 2,$$

$\phi$  is sufficiently small that

$$\tan \phi / 2 = \phi / 2,$$

$$p_2 \log p_1 = -\cdot 72.$$

If we suppose  $p_2 = \cdot 5$ , then  $p_1 = \cdot 24$ .

If  $p_2 = \cdot 1$ ,  $p_1 = \cdot 0004$ .

It is evident from this comparison, that the probability for any given deflexion is always greater for single than for compound scattering. The difference is especially marked when only a small fraction of the particles are scattered through any given angle. It follows from this result that the distribution of particles due to encounters with the atoms is for small thicknesses mainly governed by single scattering. No doubt compound scattering produces some effect in equalizing the distribution of the scattered particles; but its effect becomes relatively smaller, the smaller the fraction of the particles scattered through a given angle.

### § 6. Comparison of Theory with Experiments.

On the present theory, the value of the central charge  $Ne$  is an important constant, and it is desirable to determine its value for different atoms. This can be most simply done by determining the small fraction of  $\alpha$  or  $\beta$  particles of known velocity falling on a thin metal screen, which are scattered between  $\phi$  and  $\phi + d\phi$  where  $\phi$  is the angle of deflexion. The influence of compound scattering should be small when this fraction is small.

Experiments in these directions are in progress, but it is desirable at this stage to discuss in the light of the present theory the data already published on scattering of  $\alpha$  and  $\beta$  particles.

The following points will be discussed :—

- (a) The “diffuse reflexion” of  $\alpha$  particles, *i. e.* the scattering of  $\alpha$  particles through large angles (Geiger and Marsden).
- (b) The variation of diffuse reflexion with atomic weight of the radiator (Geiger and Marsden).
- (c) The average scattering of a pencil of  $\alpha$  rays transmitted through a thin metal plate (Geiger).
- (d) The experiments of Crowther on the scattering of  $\beta$  rays of different velocities by various metals.

(a) In the paper of Geiger and Marsden (*loc. cit.*) on the diffuse reflexion of  $\alpha$  particles falling on various substances it was shown that about 1/8000 of the  $\alpha$  particles from radium C falling on a thick plate of platinum are scattered back in the direction of the incidence. This fraction is deduced on the assumption that the  $\alpha$  particles are uniformly scattered in all directions, the observations being made for a deflexion of about  $90^\circ$ . The form of experiment is not very suited for accurate calculation, but from the data available it can be shown that the scattering observed is about that to be expected on the theory if the atom of platinum has a central charge of about  $100 e$ .

(b) In their experiments on this subject, Geiger and Marsden gave the relative number of  $\alpha$  particles diffusely reflected from thick layers of different metals, under similar conditions. The numbers obtained by them are given in the table below, where  $z$  represents the relative number of scattered particles, measured by the number of scintillations per minute on a zinc sulphide screen.

Metal.	Atomic weight.	$z$ .	$z/A^{3/2}$ .
Lead .....	207	62	208
Gold .....	197	67	242
Platinum .....	195	63	232
Tin .....	119	34	226
Silver .....	108	27	241
Copper .....	64	14.5	225
Iron .....	56	10.2	250
Aluminium ...	27	3.4	243
Average			233

On the theory of single scattering, the fraction of the total number of  $\alpha$  particles scattered through any given angle in passing through a thickness  $t$  is proportional to  $n \cdot A^2 t$ , assuming that the central charge is proportional to the atomic weight  $A$ . In the present case, the thickness of matter from which the scattered  $\alpha$  particles are able to emerge and affect the zinc sulphide screen depends on the metal. Since Bragg has shown that the stopping power of an atom for an  $\alpha$  particle is proportional to the square root of its atomic weight, the value of  $nt$  for different elements is proportional to  $1/\sqrt{A}$ . In this case  $t$  represents the greatest depth from which the scattered  $\alpha$  particles emerge. The number  $z$  of  $\alpha$  particles scattered back from a thick layer is consequently proportional to  $A^{3/2}$  or  $z/A^{3/2}$  should be a constant.

To compare this deduction with experiment, the relative values of the latter quotient are given in the last column. Considering the difficulty of the experiments, the agreement between theory and experiment is reasonably good\*.

The single large scattering of  $\alpha$  particles will obviously affect to some extent the shape of the Bragg ionization curve for a pencil of  $\alpha$  rays. This effect of large scattering should be marked when the  $\alpha$  rays have traversed screens of metals of high atomic weight, but should be small for atoms of light atomic weight.

(c) Geiger made a careful determination of the scattering of  $\alpha$  particles passing through thin metal foils, by the scintillation method, and deduced the most probable angle

\* The effect of change of velocity in an atomic encounter is neglected in this calculation.

through which the  $\alpha$  particles are deflected in passing through known thicknesses of different kinds of matter.

A narrow pencil of homogeneous  $\alpha$  rays was used as a source. After passing through the scattering foil, the total number of  $\alpha$  particles deflected through different angles was directly measured. The angle for which the number of scattered particles was a maximum was taken as the most probable angle. The variation of the most probable angle with thickness of matter was determined, but calculation from these data is somewhat complicated by the variation of velocity of the  $\alpha$  particles in their passage through the scattering material. A consideration of the curve of distribution of the  $\alpha$  particles given in the paper (*loc. cit.* p. 496) shows that the angle through which half the particles are scattered is about 20 per cent greater than the most probable angle.

We have already seen that compound scattering may become important when about half the particles are scattered through a given angle, and it is difficult to disentangle in such cases the relative effects due to the two kinds of scattering. An approximate estimate can be made in the following way:—From (§ 5) the relation between the probabilities  $p_1$  and  $p_2$  for compound and single scattering respectively is given by

$$p_2 \log p_1 = -\cdot721.$$

The probability  $q$  of the combined effects may as a first approximation be taken as

$$q = (p_1^2 + p_2^2)^{1/2}.$$

If  $q = \cdot5$ , it follows that

$$p_1 = \cdot2 \quad \text{and} \quad p_2 = \cdot46.$$

We have seen that the probability  $p_2$  of a single deflexion greater than  $\phi$  is given by

$$p_2 = \frac{\pi}{4} n \cdot t \cdot b^2 \cot^2 \phi / 2.$$

Since in the experiments considered  $\phi$  is comparatively small

$$\frac{\phi \sqrt{p_2}}{\sqrt{\pi n t}} = b = \frac{2NeE}{mu^2}.$$

Geiger found that the most probable angle of scattering of the  $\alpha$  rays in passing through a thickness of gold equivalent in stopping power to about  $\cdot76$  cm. of air was  $1^\circ 40'$ . The angle  $\phi$  through which half the  $\alpha$  particles are turned thus corresponds to  $2^\circ$  nearly.

$$t = \cdot00017 \text{ cm. ; } n = 6\cdot07 \times 10^{22} \text{ ;}$$

$$u \text{ (average value)} = 1\cdot8 \times 10^9.$$

$$E/m = 1\cdot5 \times 10^{14} \text{ . e.s. units ; } e = 4\cdot65 \times 10^{-10}.$$

Taking the probability of single scattering = .46 and substituting the above values in the formula, the value of N for gold comes out to be 97.

For a thickness of gold equivalent in stopping power to 2.12 cms. of air, Geiger found the most probable angle to be  $3^{\circ} 40'$ . In this case  $t = .00047$ ,  $\phi = 4^{\circ} 4'$ , and average  $u = 1.7 \times 10^9$ , and N comes out to be 114.

Geiger showed that the most probable angle of deflexion for an atom was nearly proportional to its atomic weight. It consequently follows that the value of N for different atoms should be nearly proportional to their atomic weights, at any rate for atomic weights between gold and aluminium.

Since the atomic weight of platinum is nearly equal to that of gold, it follows from these considerations that the magnitude of the diffuse reflexion of  $\alpha$  particles through more than  $90^{\circ}$  from gold and the magnitude of the average small angle scattering of a pencil of rays in passing through gold-foil are both explained on the hypothesis of single scattering by supposing the atom of gold has a central charge of about 100  $e$ .

(d) *Experiments of Crowther on scattering of  $\beta$  rays.*—

We shall now consider how far the experimental results of Crowther on scattering of  $\beta$  particles of different velocities by various materials can be explained on the general theory of single scattering. On this theory, the fraction of  $\beta$  particles  $p$  turned through an angle greater than  $\phi$  is given by

$$p = \frac{\pi}{4} n \cdot t \cdot b^2 \cot^2 \phi/2.$$

In most of Crowther's experiments  $\phi$  is sufficiently small that  $\tan \phi/2$  may be put equal to  $\phi/2$  without much error. Consequently

$$\phi^2 = 2\pi n \cdot t \cdot b^2 \quad \text{if } p = 1/2.$$

On the theory of compound scattering, we have already seen that the chance  $p_1$  that the deflexion of the particles is greater than  $\phi$  is given by

$$\phi^2 / \log p_1 = - \frac{9\pi^3}{64} n \cdot t \cdot b^2.$$

Since in the experiments of Crowther the thickness  $t$  of matter was determined for which  $p_1 = 1/2$ ,

$$\phi^2 = .96\pi n t b^2.$$

For a probability of 1/2, the theories of single and compound

scattering are thus identical in general form, but differ by a numerical constant. It is thus clear that the main relations on the theory of compound scattering of Sir J. J. Thomson, which were verified experimentally by Crowther, hold equally well on the theory of single scattering.

For example, if  $t_m$  be the thickness for which half the particles are scattered through an angle  $\phi$ , Crowther showed that  $\phi/\sqrt{t_m}$  and also  $\frac{mu^2}{E} \cdot \sqrt{t_m}$  were constants for a given material when  $\phi$  was fixed. These relations hold also on the theory of single scattering. Notwithstanding this apparent similarity in form, the two theories are fundamentally different. In one case, the effects observed are due to cumulative effects of small deflexions, while in the other the large deflexions are supposed to result from a single encounter. The distribution of scattered particles is entirely different on the two theories when the probability of deflexion greater than  $\phi$  is small.

We have already seen that the distribution of scattered  $\alpha$  particles at various angles has been found by Geiger to be in substantial agreement with the theory of single scattering, but cannot be explained on the theory of compound scattering alone. Since there is every reason to believe that the laws of scattering of  $\alpha$  and  $\beta$  particles are very similar, the law of distribution of scattered  $\beta$  particles should be the same as for  $\alpha$  particles for small thicknesses of matter. Since the value of  $mu^2/E$  for the  $\beta$  particles is in most cases much smaller than the corresponding value for the  $\alpha$  particles, the chance of large single deflexions for  $\beta$  particles in passing through a given thickness of matter is much greater than for  $\alpha$  particles. Since on the theory of single scattering the fraction of the number of particles which are deflected through a given angle is proportional to  $kt$ , where  $t$  is the thickness supposed small and  $k$  a constant, the number of particles which are undeflected through this angle is proportional to  $1-kt$ . From considerations based on the theory of compound scattering, Sir J. J. Thomson deduced that the probability of deflexion less than  $\phi$  is proportional to  $1-e^{-\mu/t}$  where  $\mu$  is a constant for any given value of  $\phi$ .

The correctness of this latter formula was tested by Crowther by measuring electrically the fraction  $I/I_0$  of the scattered  $\beta$  particles which passed through a circular opening subtending an angle of  $36^\circ$  with the scattering material. If

$$I/I_0 = 1 - e^{-\mu/t},$$

the value of  $I$  should decrease very slowly at first with

increase of  $t$ . Crowther, using aluminium as scattering material, states that the variation of  $I/I_0$  was in good accord with this theory for small values of  $t$ . On the other hand, if single scattering be present, as it undoubtedly is for  $\alpha$  rays, the curve showing the relation between  $I/I_0$  and  $t$  should be nearly linear in the initial stages. The experiments of Madsen\* on scattering of  $\beta$  rays, although not made with quite so small a thickness of aluminium as that used by Crowther, certainly support such a conclusion. Considering the importance of the point at issue, further experiments on this question are desirable.

From the table given by Crowther of the value  $\phi/\sqrt{t_m}$  for different elements for  $\beta$  rays of velocity  $2.68 \times 10^{10}$  cms. per second, the values of the central charge  $Ne$  can be calculated on the theory of single scattering. It is supposed, as in the case of the  $\alpha$  rays, that for the given value of  $\phi/\sqrt{t_m}$  the fraction of the  $\beta$  particles deflected by single scattering through an angle greater than  $\phi$  is .46 instead of .5.

The values of  $N$  calculated from Crowther's data are given below.

Element.	Atomic weight.	$\phi/\sqrt{t_m}$ .	N.
Aluminium .....	27	4.25	22
Copper .....	63.2	10.0	42
Silver .....	108	15.4	78
Platinum .....	194	29.0	138

It will be remembered that the values of  $N$  for gold deduced from scattering of the  $\alpha$  rays were in two calculations 97 and 114. These numbers are somewhat smaller than the values given above for platinum (viz. 138), whose atomic weight is not very different from gold. Taking into account the uncertainties involved in the calculation from the experimental data, the agreement is sufficiently close to indicate that the same general laws of scattering hold for the  $\alpha$  and  $\beta$  particles, notwithstanding the wide differences in the relative velocity and mass of these particles.

As in the case of the  $\alpha$  rays, the value of  $N$  should be most simply determined for any given element by measuring

\* Phil. Mag. xviii. p. 909 (1909).

the small fraction of the incident  $\beta$  particles scattered through a large angle. In this way, possible errors due to small scattering will be avoided.

The scattering data for the  $\beta$  rays, as well as for the  $\alpha$  rays, indicate that the central charge in an atom is approximately proportional to its atomic weight. This falls in with the experimental deductions of Schmidt\*. In his theory of absorption of  $\beta$  rays, he supposed that in traversing a thin sheet of matter, a small fraction  $\alpha$  of the particles are stopped, and a small fraction  $\beta$  are reflected or scattered back in the direction of incidence. From comparison of the absorption curves of different elements, he deduced that the value of the constant  $\beta$  for different elements is proportional to  $nA^2$  where  $n$  is the number of atoms per unit volume and  $A$  the atomic weight of the element. This is exactly the relation to be expected on the theory of single scattering if the central charge on an atom is proportional to its atomic weight.

#### § 7. *General Considerations.*

In comparing the theory outlined in this paper with the experimental results, it has been supposed that the atom consists of a central charge supposed concentrated at a point, and that the large single deflexions of the  $\alpha$  and  $\beta$  particles are mainly due to their passage through the strong central field. The effect of the equal and opposite compensating charge supposed distributed uniformly throughout a sphere has been neglected. Some of the evidence in support of these assumptions will now be briefly considered. For concreteness, consider the passage of a high speed  $\alpha$  particle through an atom having a positive central charge  $Ne$ , and surrounded by a compensating charge of  $N$  electrons. Remembering that the mass, momentum, and kinetic energy of the  $\alpha$  particle are very large compared with the corresponding values for an electron in rapid motion, it does not seem possible from dynamic considerations that an  $\alpha$  particle can be deflected through a large angle by a close approach to an electron, even if the latter be in rapid motion and constrained by strong electrical forces. It seems reasonable to suppose that the chance of single deflexions through a large angle due to this cause, if not zero, must be exceedingly small compared with that due to the central charge.

It is of interest to examine how far the experimental evidence throws light on the question of the extent of the

\* *Annal. d. Phys.* iv. 23. p. 671 (1907).



distribution of the central charge. Suppose, for example, the central charge to be composed of  $N$  unit charges distributed over such a volume that the large single deflexions are mainly due to the constituent charges and not to the external field produced by the distribution. It has been shown (§ 3) that the fraction of the  $\alpha$  particles scattered through a large angle is proportional to  $(NeE)^2$ , where  $Ne$  is the central charge concentrated at a point and  $E$  the charge on the deflected particle. If, however, this charge is distributed in single units, the fraction of the  $\alpha$  particles scattered through a given angle is proportional to  $Ne^2$  instead of  $N^2e^2$ . In this calculation, the influence of mass of the constituent particle has been neglected, and account has only been taken of its electric field. Since it has been shown that the value of the central point charge for gold must be about 100, the value of the distributed charge required to produce the same proportion of single deflexions through a large angle should be at least 10,000. Under these conditions the mass of the constituent particle would be small compared with that of the  $\alpha$  particle, and the difficulty arises of the production of large single deflexions at all. In addition, with such a large distributed charge, the effect of compound scattering is relatively more important than that of single scattering. For example, the probable small angle of deflexion of a pencil of  $\alpha$  particles passing through a thin gold foil would be much greater than that experimentally observed by Geiger (§ *b-c*). The large and small angle scattering could not then be explained by the assumption of a central charge of the same value. Considering the evidence as a whole, it seems simplest to suppose that the atom contains a central charge distributed through a very small volume, and that the large single deflexions are due to the central charge as a whole, and not to its constituents. At the same time, the experimental evidence is not precise enough to negative the possibility that a small fraction of the positive charge may be carried by satellites extending some distance from the centre. Evidence on this point could be obtained by examining whether the same central charge is required to explain the large single deflexions of  $\alpha$  and  $\beta$  particles; for the  $\alpha$  particle must approach much closer to the centre of the atom than the  $\beta$  particle of average speed to suffer the same large deflexion.

The general data available indicate that the value of this central charge for different atoms is approximately proportional to their atomic weights, at any rate for atoms heavier than aluminium. It will be of great interest to examine

experimentally whether such a simple relation holds also for the lighter atoms. In cases where the mass of the deflecting atom (for example, hydrogen, helium, lithium) is not very different from that of the  $\alpha$  particle, the general theory of single scattering will require modification, for it is necessary to take into account the movements of the atom itself (see § 4).

It is of interest to note that Nagaoka \* has mathematically considered the properties of a "Saturnian" atom which he supposed to consist of a central attracting mass surrounded by rings of rotating electrons. He showed that such a system was stable if the attractive force was large. From the point of view considered in this paper, the chance of large deflexion would practically be unaltered, whether the atom is considered to be a disk or a sphere. It may be remarked that the approximate value found for the central charge of the atom of gold ( $100 e$ ) is about that to be expected if the atom of gold consisted of 49 atoms of helium, each carrying a charge  $2 e$ . This may be only a coincidence, but it is certainly suggestive in view of the expulsion of helium atoms carrying two unit charges from radioactive matter.

The deductions from the theory so far considered are independent of the sign of the central charge, and it has not so far been found possible to obtain definite evidence to determine whether it be positive or negative. It may be possible to settle the question of sign by consideration of the difference of the laws of absorption of the  $\beta$  particle to be expected on the two hypotheses, for the effect of radiation in reducing the velocity of the  $\beta$  particle should be far more marked with a positive than with a negative centre. If the central charge be positive, it is easily seen that a positively charged mass if released from the centre of a heavy atom, would acquire a great velocity in moving through the electric field. It may be possible in this way to account for the high velocity of expulsion of  $\alpha$  particles without supposing that they are initially in rapid motion within the atom.

Further consideration of the application of this theory to these and other questions will be reserved for a later paper, when the main deductions of the theory have been tested experimentally. Experiments in this direction are already in progress by Geiger and Marsden.

University of Manchester,  
April 1911.

\* Nagaoka, *Phil. Mag.* vii. p. 445 (1904).

LXXX. *On Extremely Long Waves, emitted by the Quartz Mercury Lamp.* By H. RUBENS and O. VON BAEYER\*.

AN advance in the spectrum towards the side of the long waves is extremely difficult, while using pure thermoradiators. If the source of heat does not possess selective qualities, the intensity of radiation in the long-waved spectrum diminishes with the 4th power of the wave-length. It is true this intensity of radiation grows in proportion to the temperature of the source; but in a much higher degree (with the 4th power of the absolute temperature) the total energy of the radiant body augments, from which the desired part of radiation must be sorted out by certain processes. An increase of the temperature of the source of light will, therefore, in many cases scarcely involve an advantage for the present purpose. In the long-waved spectrum the Welsbach mantle has proved the most advantageous source of heat of purely thermoradiant character, because of its very favourable selective qualities. But even here, no rays of much greater wave-length than  $108\mu$  have been obtained.

This paper gives a description of experiments undertaken with a view to enlarge the knowledge of the infra-red spectral region, by employment of sources of light, from which the radiation is emitted by incandescent gases. As far as pure radiation of temperature is concerned, such light-sources are selective in the highest degree. Besides, the possibility of an existing long-waved infra-red luminescence radiation must here be considered.

Our arrangement of apparatus is identical with that used recently by R. W. Wood and one of us for the isolation of long-waved rays †. It is founded on the use of quartz-lenses, which, because of the extreme difference of the index of refraction for heat-rays on both sides of the region of absorption in quartz (1.50 to 2.14), can be so adjusted as to concentrate the emitted long-waved radiation on a given diaphragm, while the ordinary heat-waves are dispersed. Our method is further founded on the selective absorption of quartz and on the effect of certain central screens. For all details concerning the apparatus and the method reference must be made to the above-cited paper.

The first sources of light we now used were strong leyden-jar sparks between electrodes of zinc, cadmium, aluminium, iron, platinum, and bismuth; the sparks were produced by a

\* Communicated by the Authors.

† H. Rubens & R. W. Wood, *Phil. Mag.* Feb. 1911.

40 cm. inductor, using alternating current for the inner coil. We have, however, in no case succeeded in obtaining a perceptible radiation in the observed long-waved spectral region. As little success was gained when we used the electric arc with carbon electrodes, or with Bremer carbons and carbons with iron-salt filling, if the investigation was limited to the electric arc itself. It is true, in both these last-named cases our micro-radiometer always showed small irregular deviations, which undoubtedly were due to long-waved radiation; but it is not improbable that this radiation is emitted by solid particles in the electric arc. The observed effects were neither regular enough nor sufficiently strong to allow of a closer investigation.

A comparatively very strong long-waved radiation was, however, obtained with the quartz-mercury lamp, especially at higher consumption of energy\*. With a current of 4 amperes on 100 volts, the arc being about 80 mm. long, a deflexion of more than 50 mm. appeared in our micro-radiometer. When the lamp had burned some time this deviation proved so constant that it could easily be measured down to fractions of a per cent.

A few preliminary experiments showed us that the observed long-waved radiation of the mercury-lamp must possess a composition essentially different from that of the Welsbach mantle, the mean wave-length of which had, under the same conditions, amounted to 108  $\mu$ . We found, for instance, that

Substance.	<i>d.</i> mm.	D <sub>1</sub> . Per cent.	D <sub>2</sub> . Per cent.	D <sub>3</sub> . Per cent.	D <sub>4</sub> . Per cent.
Quartz .....	41·7	12·1	25·4	51·8	58·9
Amorphous quartz .	2·00	12·5	24·2	—	60·0
Fluorite .....	0·59	5·3	19·4	39·5	42·2
Rock-salt .....	1·29	0·5	5·7	16·5	22·5
Sylvine .....	2·10	0	3·6	11·7	16·7
Diamond .....	1·26	45·3	64·5	—	—
Selenium .....	0·52	6·8	12·9	24·2	—
Mica .....	0·055	16·6	38·8	51·5	55·0
Glass .....	0·18	2·1	9·0	21·1	25·9
Paraffin .....	3·03	57·0	72·3	82·5	85·5
Hard rubber .....	0·40	39·0	51·5	58·8	65·3
Auber .....	1·28	11·2	16·4	32·2	34·8
Wood .....	1·80	0·7	2·9	10·0	—
Black Paper .....	0·11	33·5	52·4	78·5	79·0
Black Cardboard ...	0·38	2·1	11·7	29·8	36·7
Celluloid .....	0·26	16·2	27·6	38·7	43·5
Water .....	{ 0·019	—	55·5	60·3	62·7
	{ 0·038	20	33·0	38·4	39·8

\* A mercury-amalgam lamp, containing 20 per cent. bismuth and 20 per cent. lead on 60 per cent. mercury gave nearly the same results.

a 14.66 mm. thick layer of quartz transmitted 46.6 per cent. of the isolated radiation, when the mercury-lamp served as source of light; and only 21.7 per cent. when the Welsbach mantle was adopted as radiation-source.

This table shows, for a large number of substances, the transmission of long-waved radiation isolated by means of quartz-lenses. Both sources of light were in use,  $D_1$  being the transmission observed with the Welsbach mantle,  $D_2$  with the mercury-lamp as radiator. We have, moreover (under  $D_3$ ), exhibited, for the same substances, the transmission of the radiation of the mercury-lamp filtered by a 2.0 mm. thick layer of amorphous quartz. It could be assumed from the beginning that the observed radiation of the mercury-lamp consisted of two parts, one of which emanated from the hot quartz-walls of the tube, the other from the mercury vapour itself. For the separation of the latter part we at first deemed a filter of melted quartz most suitable. Later on, we found that a ray-filter of black cardboard proved still more efficient for the isolation of the long-waved radiation emerging from the mercury vapour. In the last column of our table ( $D_4$ ) we have therefore exhibited the results of the measurements on transmission, obtained after substituting a filter of black cardboard, 0.38 mm. thick, for the amorphous quartz.

By reference to the table it will immediately be seen that for all substances the values  $D_1$ ,  $D_2$ ,  $D_3$ , and  $D_4$  form an ascending series. So far as substances are concerned whose region of absorption is known to be situated at shorter wave-lengths (as quartz, fluorite, rock-salt, and sylvine), this course indicates an increase of the mean wave-lengths of the corresponding radiations. We must therefore attribute a greater mean wave-length to this radiation of the mercury-lamp than to that emitted by the Welsbach mantle; we must further assume a greater mean wave-length for the radiation of the mercury-lamp filtered by black cardboard, than for that which passed through amorphous quartz. This assumption is in a still higher degree justified by the behaviour of black paper and black cardboard, because in such media, in which the principal loss of energy is due to diffuse dissipation, the transmission must strongly increase with growing wave-length. The rise of the mean wave-length, which the radiation of the mercury-lamp shows after the introduction of the radiation-filters employed, is according to our opinion due to the fact that the short-waved radiation of the quartz-tube (which is nearly of the same quality with that of the Welsbach mantle) is much more strongly absorbed by these filters than

the evidently much longer-waved radiation of the mercury vapour. The extremely high transmission of quartz is of particular interest for these kinds of radiation. On calculating the coefficient of absorption

$$q_r = \frac{1}{d} \log \text{nat.} \frac{100}{D_r'}$$

from the transmission for the 41.7 mm. thick quartz plate, cut perpendicularly to the axis ( $d$  being the thickness of the plate in mm.,  $D_r'$  the transmission, corrected on account of the loss by reflexion), we obtain for the here investigated radiations the following values of  $q_r$ :—

$$q_1 = 0.044 ; q_2 = 0.026 ; q_3 = 0.0089 ; q_4 = 0.0057.$$

It is evident that the rays of the mercury-lamp filtered by black cardboard must penetrate about 8 times as thick a quartz-layer as the rays emerging from the Welsbach mantle, before being attenuated to the same fraction of their primary intensity. Similar circumstances prevail with amorphous quartz, but here the absorption-power for the four investigated radiations is about 20 times as great as at the natural modification.

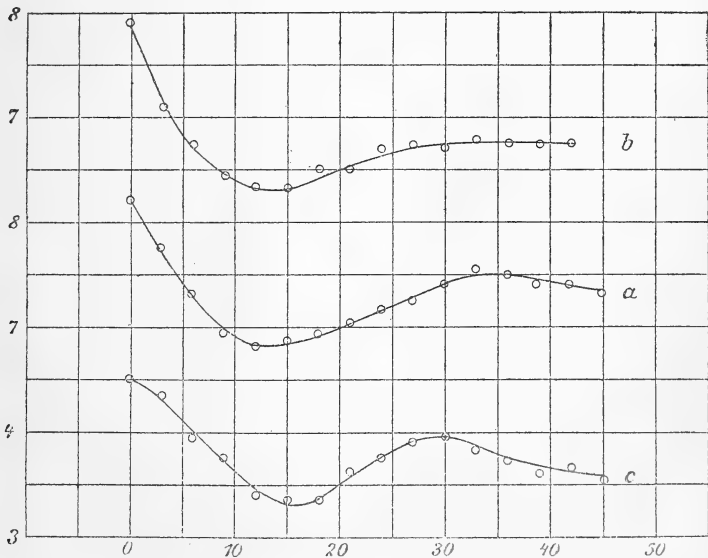
Glass and mica seem, like fluorite, rock-salt, and sylvine, to belong to the substances whose main region of absorption lies among the wave-lengths below 100  $\mu$ . The high transmission of paraffin, hard rubber, and amber, well-known as good isolators, is not surprising; neither is the small absorption of the elements diamond and selenium.

Water shows a far smaller absorption power for the radiation emitted by the mercury-lamp (particularly after its filtration through quartz or black cardboard) than for the rays emanating from the Welsbach mantle. The reflexion from the water surfaces can also not be considerable, as the values of the coefficient of absorption calculated from both layers of different thickness agree satisfactorily without consideration of the reflecting power. This would not be so in case of a considerable loss by reflexion. We may, therefore, assume that in these spectral regions water still possesses a refractive index of small magnitude, lying far closer to the value observed in the visible spectrum than to the square-root of the dielectric constant for slow vibrations.

As our measurements of absorption cannot give quantitative determinations as to the average wave-length of the investigated radiations, we have attempted to measure the wave-length by aid of the previously employed interferometer\*. The interference-curves obtained with the

\* H. Rubens & H. Hollnagel, *Phil. Mag.* [6] xix. p. 761 (1910).

quartz-mercury lamp—omitting the radiation filter—showed very irregular character. Nevertheless, it was evident that the main element of the investigated radiation was supplied by a radiation of about the same mean wave-length as that resulting from the Welsbach mantle with this arrangement. But as soon as a 15 mm. thick layer of quartz was inserted, the aspect was changed. The first minimum, which had been observed for unfiltered radiation at a thickness of the air-film of about 5 divisions of the drum\* ( $26\ \mu$ ), now did not appear before a thickness of the layer of air of 8 divisions ( $42\ \mu$ ). If the thickness of the inserted layer of quartz was increased to 42 mm., the first minimum appeared only at a distance of the interferometer plates of about 13 divisions ( $68\ \mu$ ). At the same time the interference curve showed a much smoother course. The originally observed irregular maxima and minima had nearly quite disappeared; and, besides the mentioned minimum at 13 divisions, in some series of observations a faintly marked maximum appeared in



the further course of the curve. Such an interferometer-curve is exhibited in the accompanying figure (curve *a*). Curve (*b*) of the same figure was observed in the same way with insertion of the 2 mm. thick plate of amorphous quartz; curve (*c*) with insertion of the black cardboard (0.4 mm. thick). Curve (*c*) shows the wave-character most distinctly. Here the minimum lies at 15 divisions ( $78.4\ \mu$ ) and the

\* One division of the drum to  $5.23\ \mu$ .

maximum at 30 is more conspicuous than in the other curves. But even in this curve an accurate determination of these points is very difficult. The assumption is certainly justifiable, that the radiation filtered through black cardboard contains a greater amount of this long-waved part than it does after purification by the quartz-filter. We had already deduced this fact from the results of the absorption-table.

It is still an open question, whether this long-waved radiation consists of several nearly homogeneous kinds of rays of different wave-length—as would be expected upon the assumption of a luminiferous radiation of mercury vapour—or whether it is a continuous radiation, covering a larger spectral region, such as thermo-radiators mostly possess. The results of the interferometer measurements are unable to warrant us in settling this question. But we can safely deduce from our observations, that a large part of this radiation emerging from the quartz-mercury lamp possesses a mean wave-length of about  $30 \times 2 \times 5.23 \mu = 314 \mu$  or nearly  $\frac{1}{3}$  mm.

With a view to confirming our supposition, that this extremely long-waved radiation originates in the mercury vapour itself and not in the hot quartz-tube of the lamp, we cite the following reflection:—As the intensity of the radiation from a black body diminishes with the 4th power in the region of great wave-lengths, amorphous quartz (which at  $\lambda = 100 \mu$  behaves nearly like a black body) might at the threefold wave-length send forth at most the 81st part of the energy it emits at  $100 \mu$ . But at the relatively low temperature of the quartz-mantle such a feeble radiation would not be discernible. We could, moreover, show by experiment that the observed long-waved radiation came from the mercury vapour itself. The intensity of radiation was measured shortly before and after the break of the current of the lamp. On introduction of the cardboard-filter the observed intensity of radiation fell, immediately after the interruption of the current, to about 30 per cent. of the initial value, and then slowly diminished more and more. The same experiment without cardboard-filter only produced a decrease of radiation of about 30 per cent. after the interruption of the lamp-current. We have, lastly, investigated the radiation we obtained by substituting for the quartz-mercury lamp a piece of amorphous quartz, heated by a Bunsen-flame. This radiation proved to be even of a somewhat smaller wave-length than that emitted by the Welsbach mantle under equal conditions. Less than two per cent. of it passed through black cardboard, and only ten per cent. through 2 mm. of amorphous quartz.



That the observed long-waved radiation is not emitted by the quartz walls is, therefore, an established fact, and it is highly probable that it originates in the luminous mercury vapour\*.

But the question is not solved, whether we are dealing with a radiation of temperature or of luminosity. According to measurements of Messrs. Küch and Retschinsky†, the mercury vapour of the quartz-mercury lamp possesses a temperature which amounts to many thousand degrees. In this case, the observation of such long-waved pure temperature radiation is not impossible, if the radiating mercury vapour possesses strongly defined selective absorption in that spectral region.

The main result of this investigation is the fact that heat rays of a wave-length of about 0.3 mm. may be extracted from the radiation of the mercury-lamp in sufficient force to permit an investigation of their qualities. The infra-red spectrum thereby sustains another enlargement of  $1\frac{1}{2}$  octaves.

### LXXXI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 392.]

December 21st, 1910.—Prof. W. W. Watts, Sc.D., M.Sc., F.R.S.,  
President, in the Chair.

THE following communication was read:—

‘The Keuper Marls around Charnwood Forest.’ By Thomas Owen Bosworth, B.A., B.Sc., F.G.S.

The area under consideration comprises some 300 square miles, including the towns of Leicester, Loughborough, Coalville, and Hinckley. As has been shown by Prof. Watts, the Charnian rocks project through a mantle of Triassic deposits which once completely covered them. In numerous quarry-sections the relation of the Keuper to the pre-Cambrian rocks is well exposed.

The quarries generally have been opened in the summits of the more or less completely buried hills. A quarry is so worked that its outline follows the contour of the buried hill: consequently, the section presents but a dwarfed impression of the irregularity of the rock-surface. Nevertheless, considerable undulations are observed,

\* It is, moreover, not quite out of the range of possibility that this long-waved radiation could consist of relatively very short Hertzian waves, which are produced by electric oscillations in small mercury drops. But it seems improbable that any condensation of mercury vapour would take place in the path of the current, *i. e.* in the hottest part of the tube.

† Küch & Retschinsky, *Ann. d. Phys.* xxii. p. 595 (1907).

and wherever there are any sections at right angles to the contours, the rock-slopes are seen to be remarkably steep. Contoured maps have been prepared, showing the features of some of these covered peaks.

On the buried slopes, and in the gullies, are scree and breccias: and bands of stones and grit are present in the adjacent beds of marl. All these stones, in every case, are derived only from the rock immediately at hand. They never resemble pebbles, but often are fretted into irregular shapes. Where exposed to the present climate, the Charnian igneous rocks are deeply weathered and disintegrated. But the same rocks beneath the Keuper are fresh right up to the top, as also are the rock-fragments in the marls.

The Keuper marls lie in a catenary manner across the gullies, and probably across the large valleys also; for they dip away steeply in all directions around each buried peak.

There has been almost no post-Triassic movement in Charnwood. Nevertheless, the beds must have been originally laid down horizontally, for they are in no way peculiar, and contain the normal seams of shallow-water sediment. All the points of contact of any one bed with the Charnian rocks lie on one horizontal plane. The inclination of the strata must, therefore, be due to subsequent sagging.

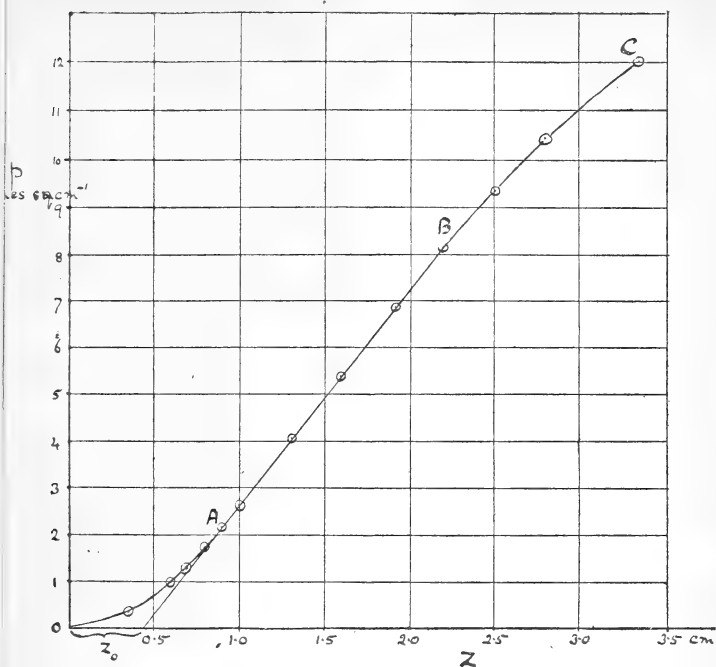
The Upper Keuper deposits accumulated in a desert basin, of which parts were dry and parts were occupied by ever-shifting salt-lakes and pools. In these waters the red marls were laid down. The red marls are of several different types, and are usually well-bedded. The principal ingredients are a certain aluminous mineral in very small particles and a much smaller proportion of very fine quartz-sand. There is generally 20 or 30 per cent. of dolomite present, in the form of minute rhombs.

The grey bands include various kinds of rock. Each band usually contains one or more seams of well-bedded sandstone or quartzose dolomite, and may safely be relied upon to indicate the bedding. The irregularities are due to irregularity in the bleaching above and below these porous seams.

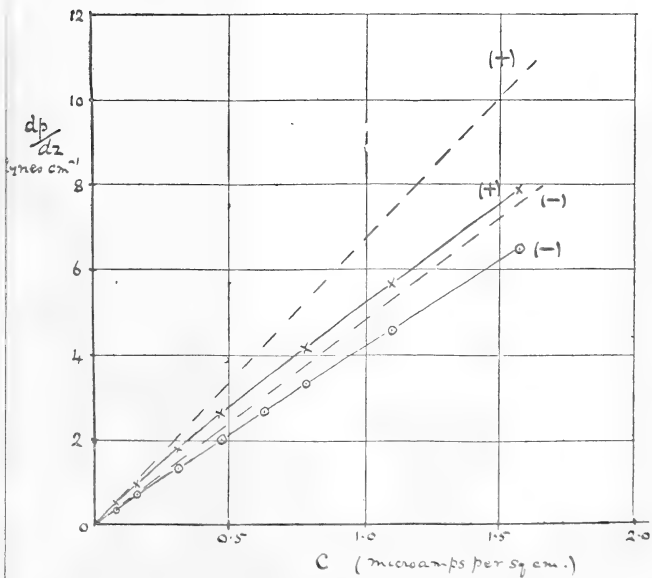
The abundant heavy minerals are garnet, zircon, tourmaline, staurolite, rutile, magnetite. These are found in every sediment—marls, sandstones, grits, breccias, etc. The grains are intensely worn.

The quartz-grains are sometimes evidently wind-worn. The sand in the grey bands is coarser and more abundant than that in the red marls. Each grey band marks the introduction of coarser sediment into the basin. The false bedding is mainly from the south-west. The bands are of wide extent and are due to inflows of fresh water from the surrounding hills, which from time to time spread themselves far and wide over the dry portions of the desert, and were often completely desiccated before reaching any pre-existing pool. Where these waters evaporated the quartzose-dolomite seams were formed, bearing ripple-marks and salt-pseudomorphs. The ripples indicate prevalent south-westerly winds.

CURVES 5.

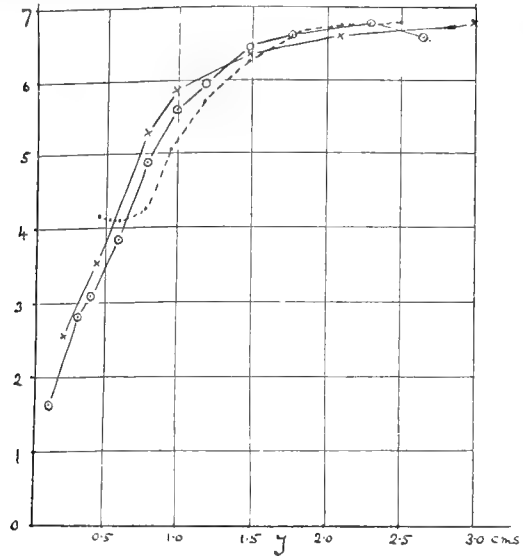


CURVES 6.

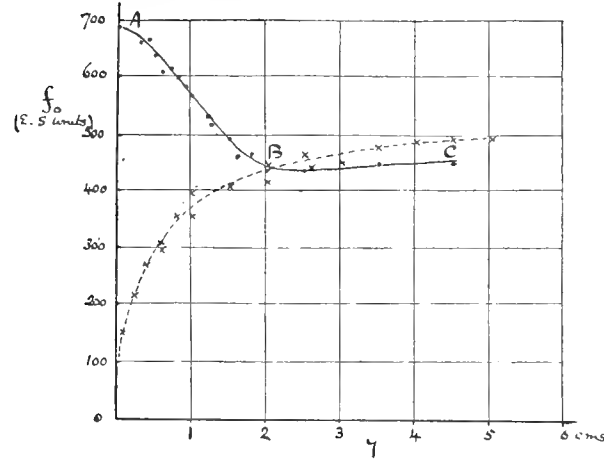




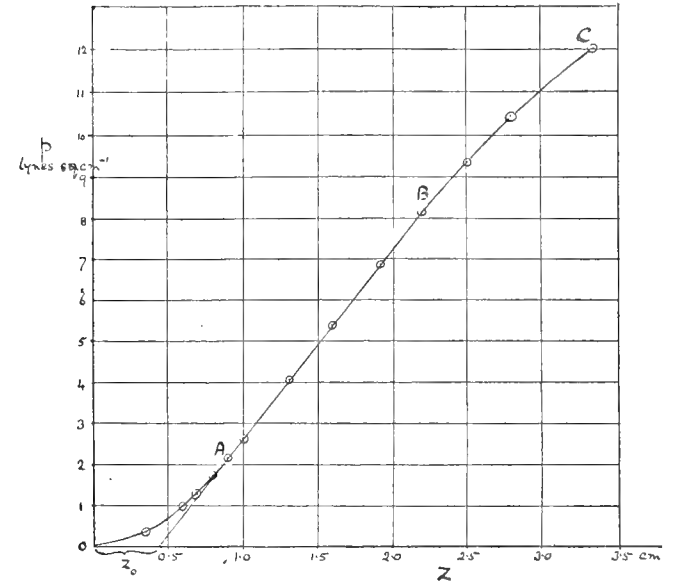
CURVES 1.



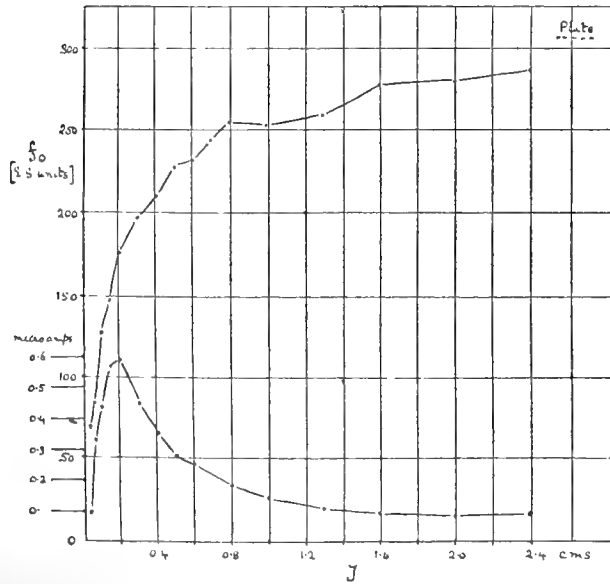
CURVES 3.



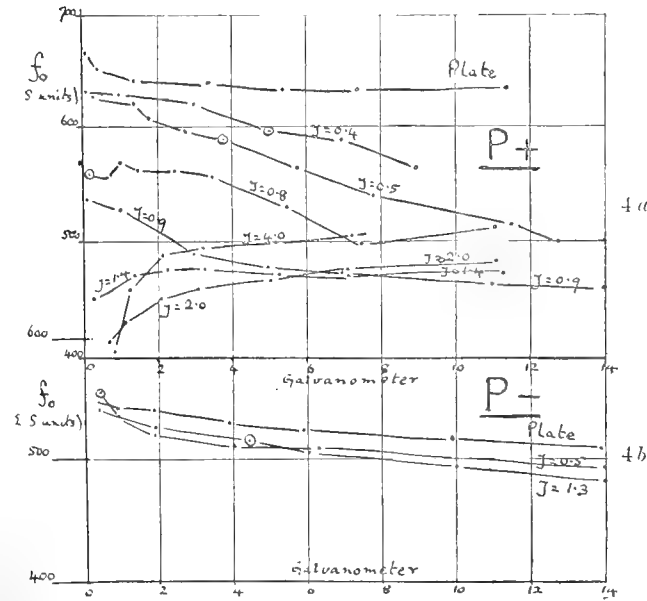
CURVES 5.



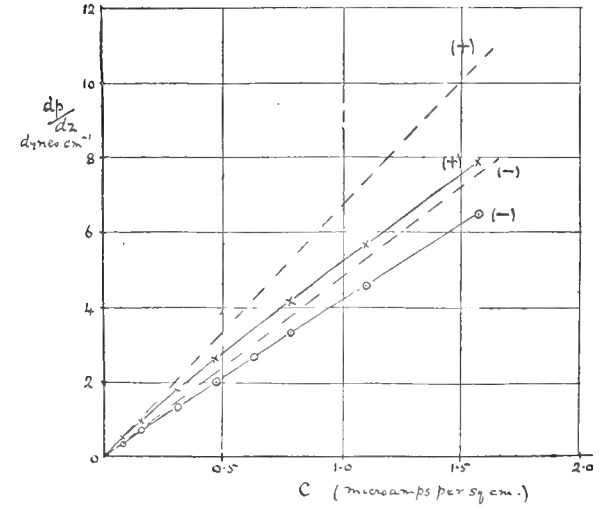
CURVES 2.

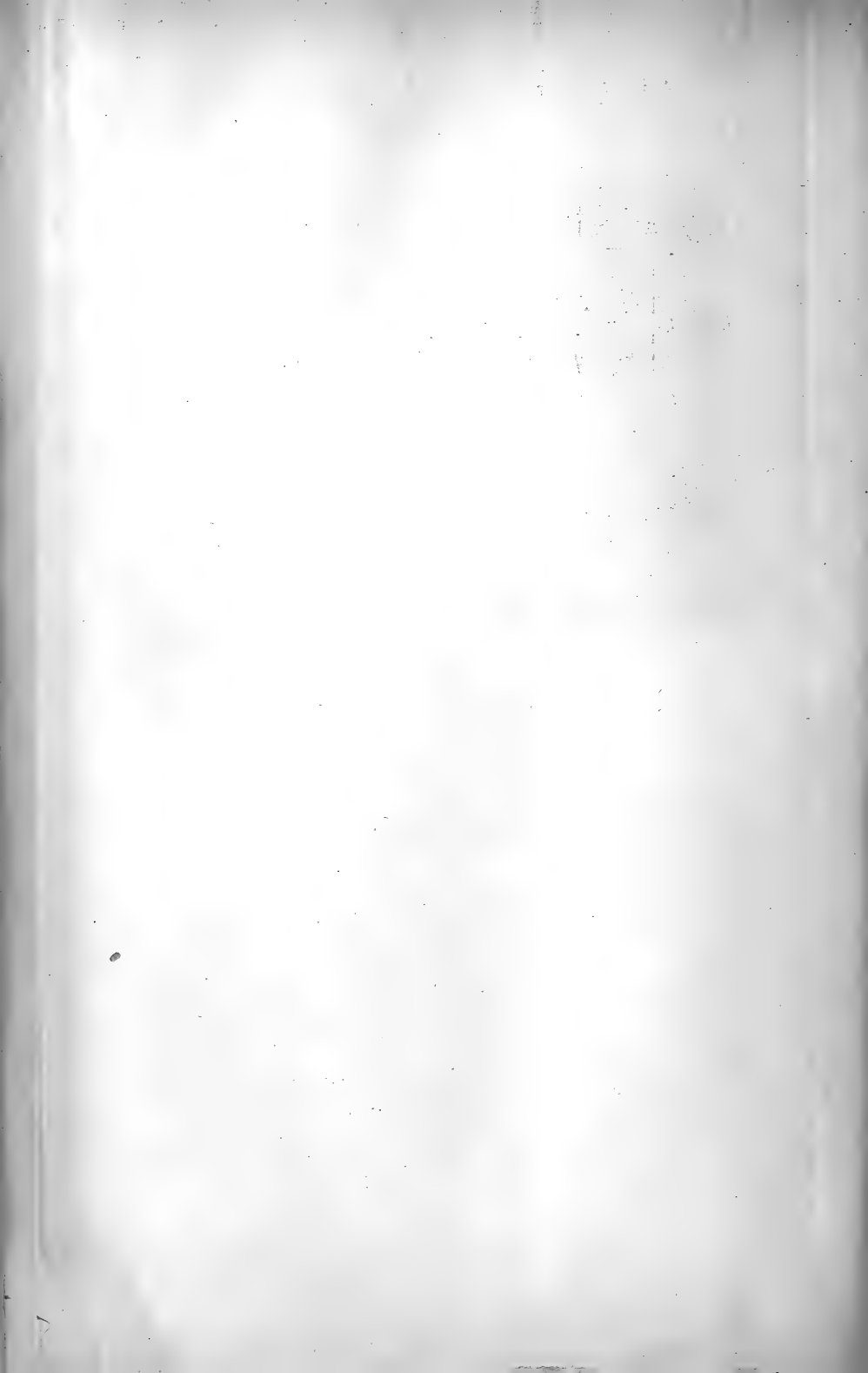


CURVES 4.



CURVES 6.





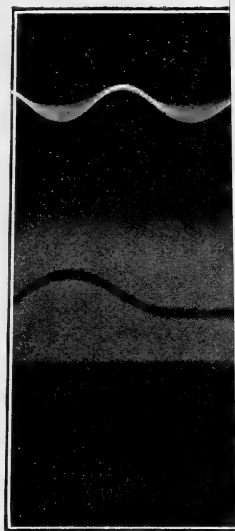






FIG. 1.

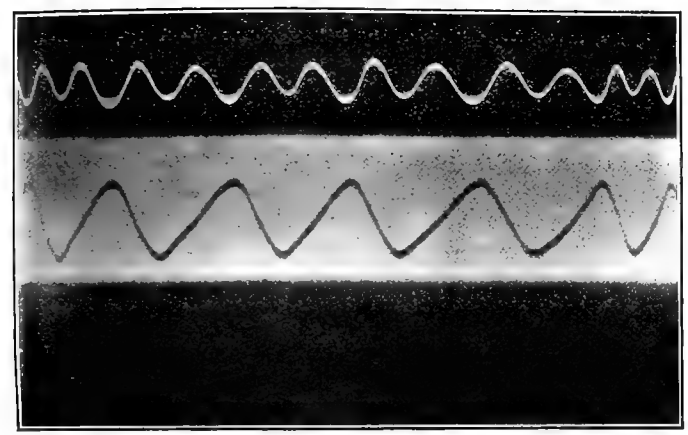


FIG. 2.

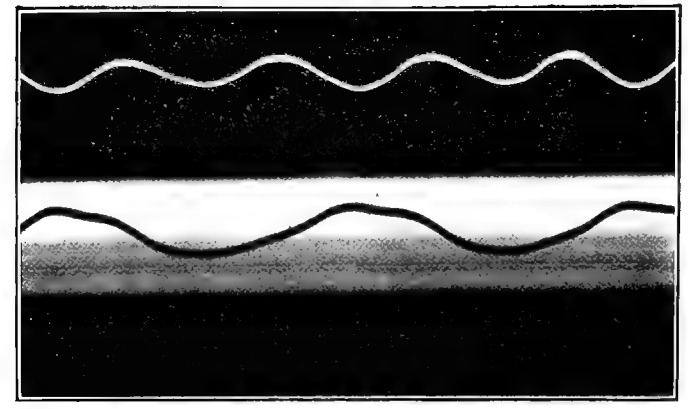


FIG. 3.

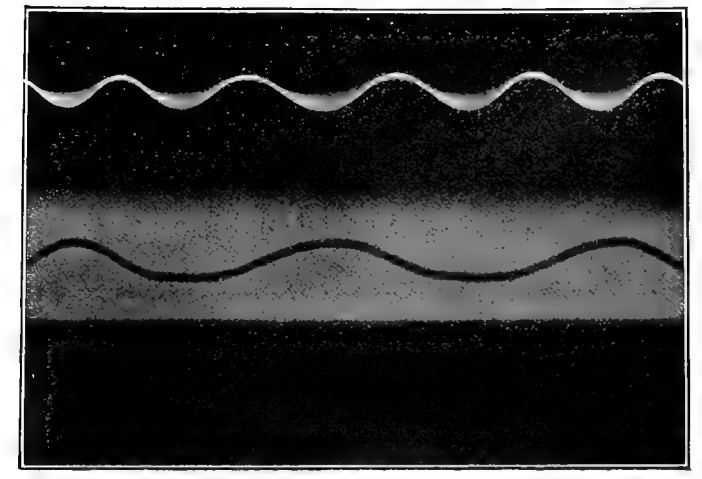


FIG. 4.

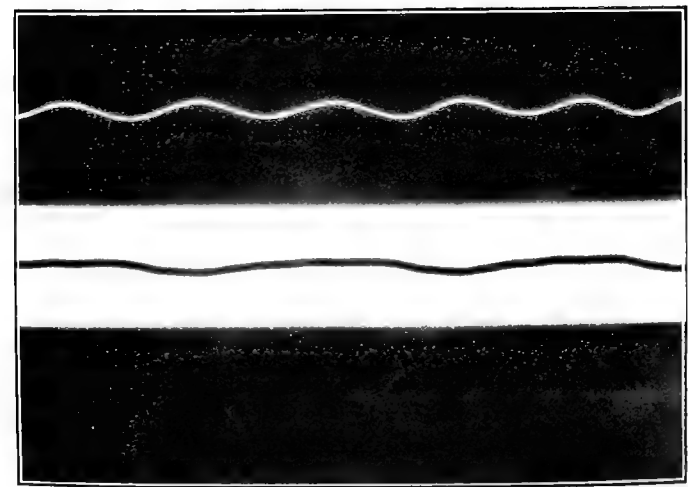


FIG. 5.

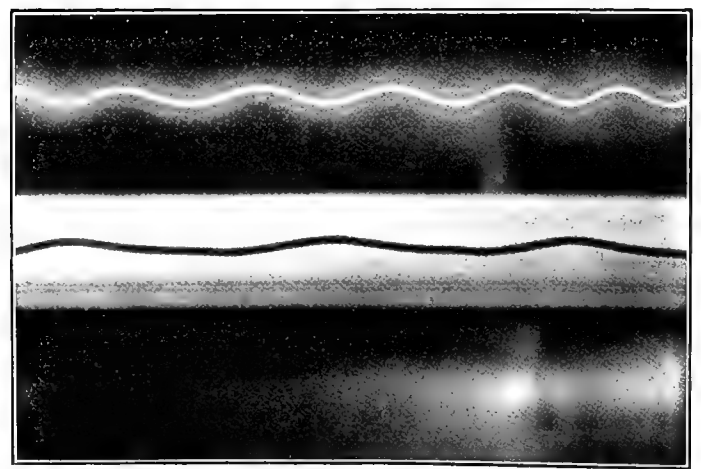


FIG. 6.

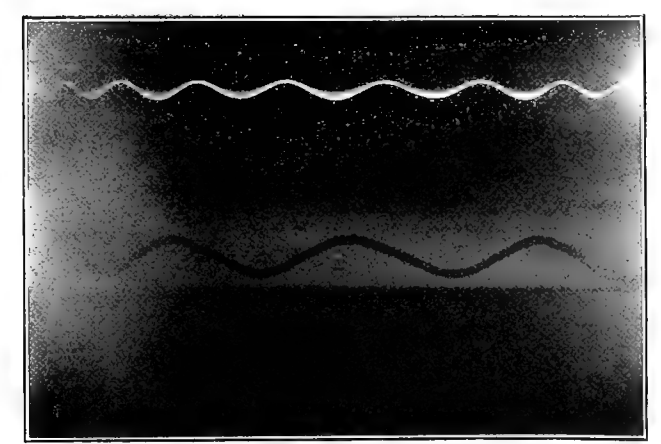


FIG. 7.

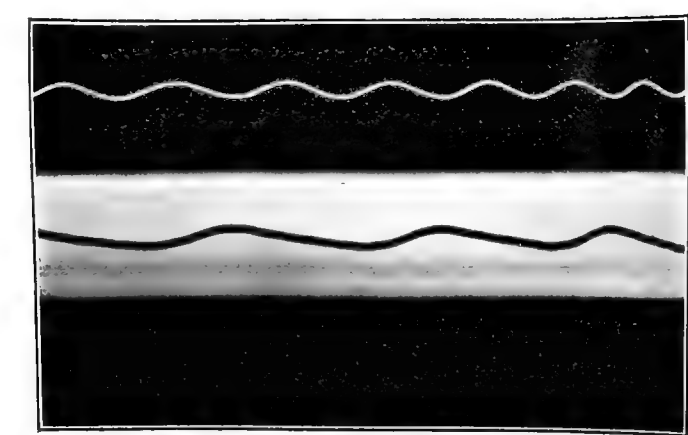


FIG. 8.

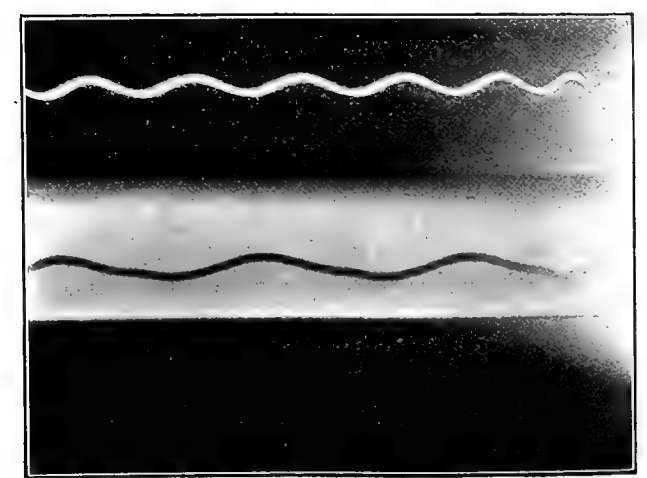


FIG. 9.

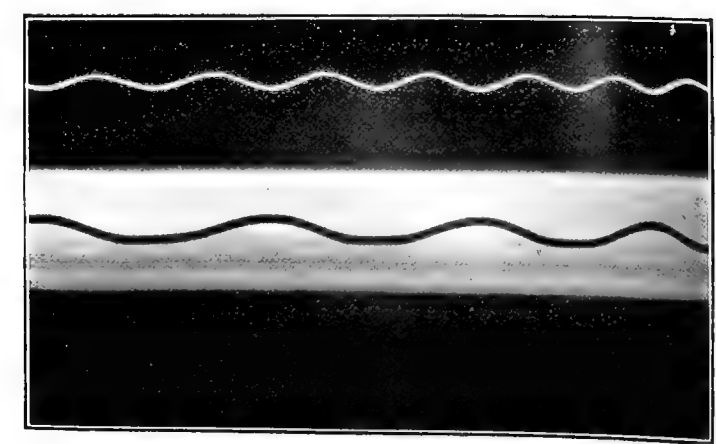


FIG. 10.

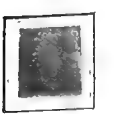
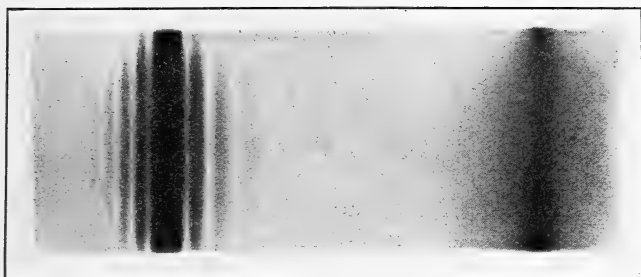


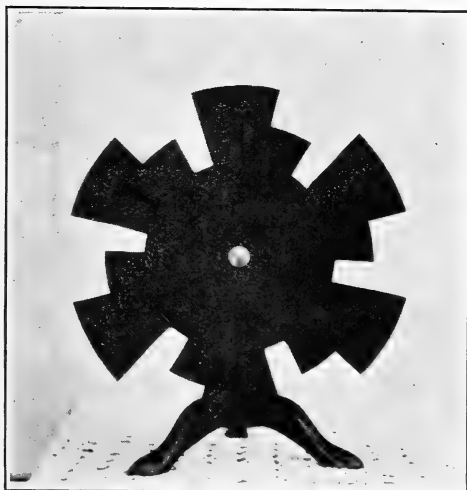


FIG. 1.



Enlarged photograph showing diffraction pattern.

FIG. 2.



Photograph of black photometric disk.



INDEX

THE  
LONDON, EDINBURGH, AND DUBLIN  
PHILOSOPHICAL MAGAZINE  
AND  
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1911.

LXXXII. *On the Motion of Solid Bodies through Viscous Liquid.* By Lord RAYLEIGH, O.M., F.R.S.\*

§ 1. **T**HE problem of the uniform and infinitely slow motion of a sphere, or cylinder, through an unlimited mass of incompressible viscous liquid otherwise at rest was fully treated by Stokes in his celebrated memoir on Pendulums†. The two cases mentioned stand in sharp contrast. In the first a relative steady motion of the fluid is easily determined, satisfying all the conditions both at the surface of the sphere and at infinity; and the force required to propel the sphere is found to be finite, being given by the formula (126)

$$-F = 6\pi\mu aV, \quad \dots \dots \dots (1)$$

where  $\mu$  is the viscosity,  $a$  the radius, and  $V$  the velocity of the sphere. On the other hand in the case of the cylinder, moving transversely, no such steady motion is possible. If we suppose the cylinder originally at rest to be started and afterwards maintained in uniform motion, finite effects are propagated to ever greater and greater distances, and the motion of the fluid approaches no limit. Stokes shows that more and more of the fluid tends to accompany the travelling cylinder, which thus experiences a continually decreasing resistance.

\* Communicated by the Author.  
† Camb. Phil. Trans. ix. 1850; Math. & Phys. Papers, vol. iii. p. 1.  
*Phil. Mag.* S. 6. Vol. 21. No. 126. June 1911. 2 Z

§ 2. In attempting to go further, one of the first questions to suggest itself is whether similar conclusions are applicable to bodies of other forms. The consideration of this subject is often facilitated by use of the well-known analogy between the motion of a viscous fluid, when the square of the motion is neglected, and the displacements of an elastic solid. Suppose that in the latter case the solid is bounded by two closed surfaces, one of which completely envelopes the other. Whatever displacements  $(\alpha, \gamma, \beta)$  be imposed at these two surfaces, there must be a corresponding configuration of equilibrium, satisfying certain differential equations. If the solid be incompressible, the otherwise arbitrary boundary displacements must be chosen subject to this condition. The same conclusion applies in two dimensions, where the bounding surfaces reduce to cylinders with parallel generating lines. For our present purpose we may suppose that at the outer surface the displacements are zero.

The contrast between the three-dimensional and two-dimensional cases arises when the outer surface is made to pass off to infinity. In the former case, where the inner surface is supposed to be limited in all directions, the displacements there imposed diminish, on receding from it, in such a manner that when the outer surface is removed to a sufficient distance no further sensible change occurs. In the two-dimensional case the inner surface extends to infinity, and the displacement affects sensibly points however distant, provided the outer surface be still further and sufficiently removed.

The nature of the distinction may be illustrated by a simple example relating to the conduction of heat through a uniform medium. If the temperature  $v$  be unity on the surface of the sphere  $r=a$ , and vanish when  $r=b$ , the steady state is expressed by

$$v = \frac{a}{b-a} \left( \frac{b}{r} - 1 \right). \quad \dots \dots \dots (2)$$

When  $b$  is made infinite,  $v$  assumes the limiting form  $a/r$ . In the corresponding problem for coaxial cylinders of radii  $a$  and  $b$  we have

$$v = \frac{\log b - \log r}{\log b - \log a} \dots \dots \dots (3)$$

But here there is no limiting form when  $b$  is made infinite. However great  $r$  may be,  $v$  is small when  $b$  exceeds  $r$  by

only a little; but when  $b$  is great enough  $v$  may acquire any value up to unity. And since the distinction depends upon what occurs at infinity, it may evidently be extended on the one side to oval surfaces of any shape, and on the other to cylinders with any form of cross-section.

In the analogy already referred to there is correspondence between the displacements ( $\alpha, \beta, \gamma$ ) in the first case and the velocities ( $u, v, w$ ) which express the motion of the viscous liquid in the second. There is also another analogy which is sometimes useful when the motion of the viscous liquid takes place in two dimensions. The *stream-function* ( $\psi$ ) for this motion satisfies the same differential equation as does the transverse displacement ( $w'$ ) of a plane elastic plate. And a surface on which the fluid remains at rest ( $\psi=0, d\psi/dn=0$ ) corresponds to a curve along which the elastic plate is clamped.

In the light of these analogies we may conclude that, provided the square of the motion is neglected absolutely, there exists always a unique steady motion of liquid past a solid obstacle of any form limited in all directions, which satisfies the necessary conditions both at the surface of the obstacle and at infinity, and further that the force required to hold the solid is finite. But if the obstacle be an infinite cylinder of any cross-section, no such steady motion is possible, and the force required to hold the cylinder in position continually diminishes as the motion continues.

§ 3. For further developments the simplest case is that of a material plane, coinciding with the coordinate plane  $x=0$  and moving parallel to  $y$  in a fluid originally at rest. The component velocities  $u, w$  are then zero; and the third velocity  $v$  satisfies (even though its square be not neglected) the general equation

$$\frac{dv}{dt} = \nu \frac{d^2v}{dx^2}, \quad \dots \dots \dots (4)$$

in which  $\nu$ , equal to  $\mu/\rho$ , represents the kinematic viscosity. In § 7 of his memoir Stokes considers periodic oscillations of the plane. Thus in (4) if  $v$  be proportional to  $e^{int}$ , we have on the positive side

$$v = A e^{int} e^{-x\sqrt{(in/\nu)}}. \quad \dots \dots \dots (5)$$

When  $x=0$ , (5) must coincide with the velocity ( $V$ ) of the plane. If this be  $V_n e^{int}$ , we have  $A=V_n$ ; so that in real

quantities

$$v = V_n e^{-x\sqrt{(n/2\nu)}} \cos \{nt - x\sqrt{(n/2\nu)}\} \dots (6)$$

corresponds with

$$V = V_n \cos nt \dots (7)$$

for the plane itself.

In order to find the tangential force ( $-T_3$ ) exercised upon the plane, we have from (5) when  $x=0$

$$\left(\frac{dv}{dx}\right)_0 = -V_n e^{int} \sqrt{(in/\nu)}, \dots (8)$$

and

$$\begin{aligned} T_3 &= -\mu(dv/dx)_0 = \rho V_n e^{int} \sqrt{(in\nu)} \\ &= \rho \sqrt{(\frac{1}{2}n\nu)} \cdot (1+i) V_n e^{int} \\ &= \rho \sqrt{(\frac{1}{2}n\nu)} \cdot \left(V + \frac{1}{n} \frac{dV}{dt}\right), \dots (9) \end{aligned}$$

giving the force per unit area due to the reaction of the fluid upon one side. "The force expressed by the first of these terms tends to diminish the amplitude of the oscillations of the plane. The force expressed by the second has the same effect as increasing the inertia of the plane." It will be observed that if  $V_n$  be given, the force diminishes without limit with  $n$ .

In note B Stokes resumes the problem of § 7: instead of the motion of the plane being periodic, he supposes that the plane and fluid are initially at rest, and that the plane is then ( $t=0$ ) moved with a constant velocity  $V$ . This problem depends upon one of Fourier's solutions which is easily verified\*. We have

$$\frac{dv}{dx} = -\frac{V}{\sqrt{(\pi\nu t)}} e^{-x^2/4\nu t}, \dots (10)$$

$$v = V - \frac{2V}{\sqrt{\pi}} \int_0^{x/2\sqrt{(\nu t)}} e^{-z^2} dz. \dots (11)$$

For the reaction on the plane we require only the value of  $dv/dx$  when  $x=0$ . And

$$\left(\frac{dv}{dx}\right)_0 = -\frac{V}{\sqrt{(\pi\nu t)}} \dots (12)$$

Stokes continues † "now suppose the plane to be moved

\* Compare Kelvin, Ed. Trans. 1862; Thomson & Tait, Appendix D.

† I have made some small changes of notation.



in any manner, so that its velocity at the end of the time  $t$  is  $V(t)$ . We may evidently obtain the result in this case by writing  $V'(\tau)d\tau$  for  $V$ , and  $t-\tau$  for  $t$  in [12], and integrating with respect to  $\tau$ . We thus get

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}} \int_{-\infty}^t \frac{V'(\tau) d\tau}{\sqrt{(t-\tau)}} = -\frac{1}{\sqrt{(\pi\nu)}} \int_0^\infty V'(t-t_1) \frac{dt_1}{\sqrt{t_1}}; \quad \dots (13)''$$

and since  $T_3 = -\mu dv/dx_0$ , these formulæ solve the problem of finding the reaction in the general case.

There is another method by which the present problem may be treated, and a comparison leads to a transformation which we shall find useful further on. Starting from the periodic solution (8), we may generalize it by Fourier's theorem. Thus

$$\left(\frac{dv}{dx}\right)_0 = -\int_0^\infty V_n e^{int} \sqrt{(in/\nu)} dn \quad \dots (14)$$

corresponds to

$$V(t) = \int_0^\infty V_n e^{int} dn, \quad \dots (15)$$

where  $V_n$  is an arbitrary function of  $n$ .

Comparing (13) and (14), we see that

$$\int_0^\infty V_n e^{int} n^{\frac{1}{2}} dn = \frac{1}{\sqrt{(i\pi)}} \int_{-\infty}^t \frac{V'(\tau) d\tau}{\sqrt{(t-\tau)}}. \quad \dots (16)$$

It is easy to verify (16). If we substitute on the right for  $V'(\tau)$  from (15), we get

$$\frac{1}{\sqrt{(i\pi)}} \int_{-\infty}^t \frac{d\tau}{\sqrt{(t-\tau)}} \int_0^\infty in V_n e^{in\tau} dn;$$

and taking first the integration with respect to  $\tau$ ,

$$\int_{-\infty}^t \frac{e^{in\tau} d\tau}{\sqrt{(t-\tau)}} = \int_0^\infty \frac{e^{in(t-t_1)}}{\sqrt{t_1}} dt_1 = \sqrt{\left(\frac{\pi}{in}\right)} \cdot e^{int},$$

whence (16) follows at once.

As a particular case of (13), let us suppose that the fluid is at rest and that the plane starts at  $t=0$  with a velocity which is uniformly accelerated for a time  $\tau_1$  and afterwards remains

constant. Thus from  $-\infty$  to 0,  $V(\tau)=0$ ; from 0 to  $\tau_1$ ,  $V(\tau)=h\tau$ ; from  $\tau_1$  to  $t$ , where  $t > \tau_1$ ,  $V(\tau)=h\tau_1$ . Thus ( $0 < t < \tau_1$ )

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}} \int_0^t \frac{h d\tau}{\sqrt{(t-\tau)}} = -\frac{2h\sqrt{t}}{\sqrt{(\pi\nu)}}; \quad (17)$$

and ( $t > \tau_1$ )

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{(\pi\nu)}} \int_0^{\tau_1} \frac{h d\tau}{\sqrt{(t-\tau)}} = -\frac{2h}{\sqrt{(\pi\nu)}} \{\sqrt{t}-\sqrt{(t-\tau_1)}\}. \quad (18)$$

Expressions (17), (18), taken negatively and multiplied by  $\mu$ , give the force per unit area required to propel the plane against the fluid forces acting upon *one* side. The force increases until  $t=\tau_1$ , that is so long as the acceleration continues. Afterwards it gradually diminishes to zero. For the differential coefficient of  $\sqrt{t}-\sqrt{(t-\tau_1)}$  is negative when  $t > \tau_1$ ; and when  $t$  is great,

$$\sqrt{t}-\sqrt{(t-\tau_1)} = \frac{1}{2}\tau_1 t^{-\frac{1}{2}} \text{ ultimately.}$$

§ 4. In like manner we may treat any problem in which the motion of the material plane is prescribed. A more difficult question arises when it is the *forces* propelling the plane that are given. Suppose, for example, that an infinitely thin vertical lamina of superficial density  $\sigma$  begins to fall from rest under the action of gravity when  $t=0$ , the fluid being also initially at rest. By (13) the equation of motion may be written

$$\frac{dV}{dt} + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}} \int_0^t \frac{V'(\tau)d\tau}{\sqrt{(t-\tau)}} = g, \quad (19)$$

the fluid being now supposed to act on *both* sides of the lamina.

By an ingenious application of Abel's theorem Boggio has succeeded in integrating equations which include (19)\*. The theorem is as follows:—If  $\psi(t)$  be defined by

$$\psi(t) = \int_0^t \frac{\phi'(\tau)d\tau}{(t-\tau)^{\frac{1}{2}}}, \quad (20)$$

then

$$\int_0^t \frac{\psi(\tau)d\tau}{(t-\tau)^{\frac{1}{2}}} = \pi \{\phi(t) - \phi(0)\}. \quad (21)$$

\* Boggio, *Rend. d. Accad. d. Lincei*, vol. xvi. pp. 613, 730 (1907); also Basset, *Quart. Journ. of Mathematics*, No. 164, 1910, from which I first became acquainted with Boggio's work.

For by (20), if  $(t-\tau)^{\frac{1}{2}}=y$ ,

$$\psi(t) = 2 \int_0^{\sqrt{t}} \phi'(t-y^2) dy;$$

so that

$$\begin{aligned} \int_0^t \frac{\psi(\tau) d\tau}{(t-\tau)^{\frac{1}{2}}} &= 2 \int_0^{\sqrt{t}} \psi(t-x^2) dx \\ &= 4 \int_0^{\sqrt{t}} dx \int_0^{\sqrt{t-x^2}} \phi'(t-x^2-y^2) dy \\ &= 2\pi \int_0^{\sqrt{t}} \phi'(t-r^2) r dr = \pi \{ \phi(t) - \phi(0) \}, \end{aligned}$$

where  $r^2 = x^2 + y^2$ .

Now, if  $t'$  be any time between 0 and  $t$ , we have, as in (19),

$$V'(t') + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}} \int_0^{t'} \frac{V'(\tau) d\tau}{\sqrt{(t'-\tau)}} = g.$$

Multiplying this by  $(t-t')^{-\frac{1}{2}} dt'$  and integrating between 0 and  $t$ , we get

$$\int_0^t \frac{V'(t') dt'}{(t-t')^{\frac{1}{2}}} + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}} \int_0^t \frac{dt'}{(t-t')^{\frac{1}{2}}} \int_0^{t'} \frac{V'(\tau) d\tau}{(t'-\tau)^{\frac{1}{2}}} = g \int_0^t \frac{dt'}{(t-t')^{\frac{1}{2}}}. \quad (22)$$

In (22) the first integral is the same as the integral in (19). By Abel's theorem the double integral in (22) is equal to  $\pi V(t)$ , since  $V(0) = 0$ . Thus

$$\int_0^t \frac{V'(\tau) d\tau}{\sqrt{(t-\tau)}} + \frac{2\rho\nu^{\frac{1}{2}}\pi^{\frac{1}{2}}}{\sigma} V(t) = 2g\sqrt{t}. \quad (23)$$

If we now eliminate the integral between (19) and (23), we obtain simply

$$\frac{dV}{dt} - \frac{4\rho^2\nu}{\sigma^2} V = g - \frac{4\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}} g\sqrt{t} \quad (24)$$

as the differential equation governing the motion of the lamina.

This is a linear equation of the first order. Since  $V$

vanishes with  $t$ , the integral may be written

$$\begin{aligned} \frac{4\rho^2\nu V}{g\sigma^2} &= e^{t'} \int_0^{t'} e^{-t} \left(1 - \frac{2\sqrt{t}}{\sqrt{\pi}}\right) dt \\ &= \frac{2\sqrt{t'}}{\sqrt{\pi}} - 1 + \frac{2}{\sqrt{\pi}} e^{t'} \int_{\sqrt{t'}}^{\infty} e^{-x^2} dx, \quad \dots \quad (25) \end{aligned}$$

in which  $t' = t \cdot 4\rho^2\nu/\sigma^2$ . When  $t$ , or  $t'$ , is great,

$$\int_{\sqrt{t'}}^{\infty} e^{-x^2} dx = \frac{e^{-t'}}{2\sqrt{t'}} \left(1 - \frac{1}{2t'} + \dots\right); \quad \dots \quad (26)$$

so that

$$\frac{4\rho^2\nu V}{g\sigma^2} = \frac{2\sqrt{t'}}{\sqrt{\pi}} - 1 + \frac{1}{\sqrt{(\pi t')}} \left(1 - \frac{1}{2t'} + \dots\right). \quad \dots \quad (27)$$

Ultimately, when  $t$  is very great,

$$V = \frac{g\sigma}{\rho} \sqrt{\left(\frac{t}{\pi\nu}\right)}. \quad \dots \quad (28)$$

§ 5. The problem of the sphere moving with arbitrary velocity through a viscous fluid is of course more difficult than the corresponding problem of the plane lamina, but it has been satisfactorily solved by Boussinesq\* and by Basset†. The easiest road to the result is by the application of Fourier's theorem to the periodic solution investigated by Stokes. If the velocity of the sphere at time  $t$  be  $V = \dot{V}_n e^{int}$ ,  $a$  the radius,  $M'$  the mass of the liquid displaced by the sphere, and  $s = \sqrt{(n/2\nu)}$ ,  $\nu$  being as before the kinematic viscosity, Stokes finds as the total force at time  $t$

$$F = -M' V_n n \left\{ \left( \frac{1}{2} + \frac{9}{4sa} \right) i + \frac{9}{4sa} \left( 1 + \frac{1}{sa} \right) \right\} e^{int}. \quad \dots \quad (29)$$

Thus, if

$$V = \int_0^{\infty} V_n e^{int} dn, \quad \dots \quad (30)$$

$$F = -M' \int_0^{\infty} V_n n e^{int} \left\{ \left( \frac{1}{2} + \frac{9}{4sa} \right) i + \frac{9}{4sa} \left( 1 + \frac{1}{sa} \right) \right\} dn, \quad \dots \quad (31)$$

\* C. R. t. 100. p. 935 (1885); *Theorie Analytique de la Chaleur*, t. ii. Paris, 1903.

† Phil. Trans. 1888; *Hydrodynamics*, II. ch. xxii. 1888.

Of the four integrals in (31),

$$\text{the first} = \frac{1}{2} \int_0^\infty in V_n e^{int} dn = \frac{1}{2} V';$$

$$\text{the fourth} = \frac{9\nu}{2a^2} \int_0^\infty V_n e^{int} dn = \frac{9\nu}{2a^2} V.$$

Also the second and third together give

$$\frac{9(1+i)\sqrt{(2\nu)}}{4a} \int_0^\infty V_n n^{\frac{3}{2}} e^{int} dn,$$

and this is the only part which could present any difficulty. We have, however, already considered this integral in connexion with the motion of a plane and its value is expressed by (16). Thus

$$F = -M' \left\{ \frac{1}{2} \frac{dV}{dt} + \frac{9\nu}{2a^2} V + \frac{9\nu^{\frac{1}{2}}}{2a\pi^{\frac{1}{2}}} \int_{-\infty}^t \frac{V'(\tau) d\tau}{(t-\tau)^{\frac{3}{2}}} \right\}. \quad (32)$$

The first term depends upon the inertia of the fluid, and is the same as would be obtained by ordinary hydrodynamics when  $\nu=0$ . If there is no acceleration at the moment, this term vanishes. If, further, there has been no acceleration for a long time, the third term also vanishes, and we obtain the result appropriate to a uniform motion

$$F = -\frac{9\nu M' V}{2a^2} = -6\pi a \rho \nu V = -6\pi \mu \alpha V,$$

as in (1). The general result (32) is that of Boussinesq and Basset.

As an example of (32), we may suppose (as formerly for the plane) that  $V(t)=0$  from  $-\infty$  to  $0$ ;  $V(t)=ht$  from  $0$  to  $\tau_1$ ;  $V(t)=h\tau_1$ , when  $t > \tau_1$ . Then if  $t < \tau_1$ ,

$$F = -hM' \left[ \frac{1}{2} + \frac{9\nu t}{2a^2} + \frac{9\nu^{\frac{1}{2}} t^{\frac{3}{2}}}{a\pi^{\frac{1}{2}}} \right]; \quad \dots \quad (33)$$

and when  $t > \tau_1$ ,

$$F = -hM' \left[ \frac{9\nu\tau_1}{2a^2} + \frac{9\nu^{\frac{1}{2}}}{a\pi^{\frac{1}{2}}} \left\{ \sqrt{t} - \sqrt{(t-\tau_1)} \right\} \right]. \quad (34)$$

When  $t$  is very great (34) reduces to its first term.

The more difficult problem of a sphere falling under the influence of gravity has been solved by Boggio (*loc. cit.*). In the case where the liquid and sphere are initially at rest, the solution is comparatively simple; but the analytical form

of the functions is found to depend upon the ratio of densities of the sphere and liquid. This may be rather unexpected; but I am unable to follow Mr. Basset in regarding it as an objection to the usual approximate equations of viscous motion.

§ 6. We will now endeavour to apply a similar method to Stokes's solution for a *cylinder* oscillating transversely in a viscous fluid. If the radius be  $a$  and the velocity  $V$  be expressed by  $V = V_n e^{int}$ , Stokes finds for the force

$$F = -M' in V_n e^{int} (k - ik'). \dots (35)$$

In (35)  $M'$  is the mass of the fluid displaced;  $k$  and  $k'$  are certain functions of  $m$ , where  $m = \frac{1}{2} a \sqrt{(n|\nu)}$ , which are tabulated in § 37. The cylinder is much less amenable to mathematical treatment than the sphere, and we shall limit ourselves to the case where, all being initially at rest, the cylinder is started with unit velocity which is afterwards steadily maintained.

The velocity  $V$  of the cylinder, which is to be zero when  $t$  is negative and unity when  $t$  is positive, may be expressed by

$$V = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin nt}{n} dn, \dots (36)$$

in which the second term may be regarded as the real part of

$$\frac{1}{\pi i} \int_0^\infty \frac{e^{int}}{n} dn. \dots (37)$$

We shall see further below, and may anticipate from Stokes's result relating to uniform motion of the cylinder, that the first term of (36) contributes nothing to  $F$ ; so that we may take

$$F = -\frac{M'}{\pi} \int_0^\infty e^{int} (k - ik') dn,$$

corresponding to (37). Discarding the imaginary part, we get, corresponding to (36),

$$F = -\frac{M'}{\pi} \int_0^\infty (k \cos nt + k' \sin nt) dn, \dots (38)$$

Since  $k, k'$  are known functions of  $m$ , or ( $a$  and  $\nu$  being given) of  $n$ , (38) may be calculated by quadratures for any prescribed value of  $t$ .

It appears from the tables that  $k, k'$  are positive throughout.

When  $m=0$ ,  $k$  and  $k'$  are infinite and continually diminish as  $m$  increases, until when  $m=\infty$ ,  $k=1$ ,  $k'=0$ . For small values of  $m$  the limiting forms for  $k$ ,  $k'$  are

$$k=1+\frac{\frac{1}{4}\pi}{m^2(\log m)^2}, \quad k'=-\frac{1}{m^2 \log m}; \dots \quad (39)$$

from which it appears that if we make  $n$  vanish in (35), while  $V_n$  is given,  $F$  comes to zero.

We now seek the limiting form when  $t$  is very great. The integrand in (38) is then rapidly oscillatory, and ultimately the integral comes to depend sensibly upon that part of the range where  $n$  is very small. And for this part we may use the approximate forms (39).

Consider, for example, the first integral in (38), from which we may omit the constant part of  $k$ . We have

$$\int_0^\infty k \cos nt \, dn = \frac{\pi}{4} \int_0^\infty \frac{\cos nt \, dn}{m^2(\log m)^2} = \frac{4\pi v}{a^2} \int_0^\infty \frac{\cos(4va^{-2}t.x)dx}{x(\log x)^2} \dots \quad (40)$$

Writing  $4vt/a^2=t'$ , we have to consider

$$\int_0^\infty \frac{\cos t'x \cdot dx}{x(\log x)^2} \dots \dots \dots \quad (41)$$

In this integral the integrand is positive from  $x=0$  to  $x=\pi/2t'$ , negative from  $\pi/2t'$  to  $3\pi/2t'$ , and so on. For the first part of the range if we omit the cosine,

$$\int_0^{\pi/2t'} \frac{dx}{x(\log x)^2} = \int \frac{d \log x}{(\log x)^2} = \frac{1}{\log(2t'/\pi)}; \dots \quad (42)$$

and since the cosine is less than unity, this is an over estimate. When  $t'$  is very great,  $\log(2t'/\pi)$  may be identified with  $\log t'$ , and to this order of approximation it appears that (41) may be represented by (42). Thus if quadratures be applied to (41), dividing the first quadrant into three parts, we have

$$\frac{\cos \frac{\pi}{12}}{\log \frac{\pi}{6t'}} + \cos \frac{3\pi}{12} \left[ \frac{1}{\log \frac{3t'}{\pi}} - \frac{1}{\log \frac{6t'}{\pi}} \right] + \cos \frac{5\pi}{12} \left[ \frac{1}{\log \frac{2t'}{\pi}} - \frac{1}{\log \frac{3t'}{\pi}} \right],$$

of which the second and third terms may ultimately be neglected in comparison with the first. For example, the coefficient of  $\cos(3\pi/12)$  is equal to

$$\log 2 \div \log \frac{3t'}{\pi} \cdot \log \frac{6t'}{\pi}.$$

Proceeding in this way we see that the cosine factor may properly be identified with unity, and that the value of the integral for the first quadrant may be equated to  $1/\log t'$ . And for a similar reason the quadrants after the first contribute nothing of this order of magnitude. Accordingly we may take

$$\int_0^\infty k \cos nt \, dn = \frac{4\pi\nu}{a^2 \log t'} \cdot \dots \quad (43)$$

For the other part of (38), we get in like manner

$$\begin{aligned} \int_0^\infty k' \sin nt \, dn &= -\frac{8\nu}{a^2} \int_0^\infty \frac{\sin t'x \cdot dx}{x \log x} \\ &= \frac{8\nu}{a^2} \int_0^\infty \frac{\sin x' dx'}{x' \log (t'/x')} \cdot \dots \quad (44) \end{aligned}$$

In the denominator of (44) it appears that ultimately we may replace  $\log (t'/x')$  by  $\log t'$  simply. Thus

$$\int_0^\infty k' \sin nt \, dn = \frac{4\pi\nu}{a^2 \log t'}, \dots \quad (45)$$

so that the two integrals (43), (45) are equal. We conclude that when  $t$  is great enough,

$$F = -\frac{8\nu M'}{a^2 \log t'} = -\frac{8\nu M'}{a^2 \log (4vt/a^2)} \cdot \dots \quad (46)$$

But a better discussion of these integrals is certainly a desideratum.

§ 7. Whatever interest the solution of the approximate equations may possess, we must never forget that the conditions under which they are applicable are very restricted, and as far as possible from being observed in many practical problems. Dynamical similarity in viscous motion requires that  $Va/\nu$  be unchanged,  $a$  being the linear dimension. Thus the general form for the resistance to the uniform motion of a sphere will be

$$F = \rho\nu Va \cdot f(Va/\nu), \dots \quad (47)$$

where  $f$  is an unknown function. In Stokes's solution (1)  $f$  is constant, and its validity requires that  $Va/\nu$  be small\*. When  $V$  is rather large, experiment shows that  $F$  is nearly proportional to  $V^2$ . In this case  $\nu$  disappears. "The second

\* Phil. Mag. xxxvi. p. 354 (1893); Scientific Papers, iv. p. 87.



power of the velocity and independence of viscosity are thus inseparably connected”\*.

The general investigation for the sphere moving in any manner (in a straight line) shows that the departure from Stokes's law when the velocity is not very small must be due to the operation of the neglected terms involving the squares of the velocities; but the manner in which these act has not yet been traced. Observation shows that an essential feature in rapid fluid motion past an obstacle is the formation of a *wake* in the rear of the obstacle; but of this the solutions of the approximate equations give no hint.

Hydrodynamical solutions involving surfaces of discontinuity of the kind investigated by Helmholtz and Kirchhoff provide indeed for a wake, but here again there are difficulties. Behind a blade immersed transversely in a stream a region of “dead water” is indicated. The conditions of steady motion are thus satisfied; but, as Helmholtz himself pointed out, the motion thus defined is unstable. Practically the dead and live water are continually mixing; and if there be viscosity, the layer of transition rapidly assumes a finite width independently of the instability. One important consequence is the development of a suction on the hind surface of the lamina which contributes in no insignificant degree to the total resistance. The amount of the suction does not appear to depend much on the degree of viscosity. When the latter is small, the dragging action of the live upon the dead water extends to a greater distance behind.

§ 8. If the blade, supposed infinitely thin, be moved edge-ways through the fluid, the case becomes one of “skin-friction.” Towards determining the law of resistance Mr. Lanchester has put forward an argument† which, even if not rigorous, at any rate throws an interesting light upon the question. Applied to the case of two dimensions in order to find the resistance  $F$  per unit length of blade, it is somewhat as follows. Considering two systems for which the velocity  $V$  of the blade is different, let  $n$  be the proportional width of corresponding strata of velocity. The momentum communicated to the wake per unit length of travel is as  $nV$ , and therefore on the whole as  $nV^2$  per unit of time. Thus  $F$  varies as  $nV^2$ . Again, having regard to the law of viscosity and considering the strata contiguous to the blade, we see that  $F$  varies as  $V/n$ . Hence,  $nV^2$  varies as  $V/n$ , or  $V$  varies as  $n^{-2}$ , from which it follows that  $F$  varies as  $V^{3.2}$ . If this

\* Phil. Mag. xxxiv. p. 59 (1892); Scientific Papers, iii. p. 576.

† Aerodynamics, London, 1907, § 35.

be admitted, the general law of dynamical similarity requires that for the whole resistance

$$F = c\rho v^{\frac{1}{2}} l b^{\frac{3}{2}} V^{\frac{3}{2}}, \dots \dots \dots (48)$$

where  $l$  is the length,  $b$  the width of the blade, and  $c$  a constant. Mr. Lanchester gives this in the form

$$F/\rho = cv^{\frac{1}{2}} A^{\frac{3}{2}} V^{\frac{3}{2}}, \dots \dots \dots (49)$$

where  $A$  is the area of the lamina, agreeing with (48) if  $l$  and  $b$  maintain a constant ratio.

The difficulty in the way of accepting the above argument as rigorous is that complete similarity cannot be secured so long as  $b$  is constant as has been supposed. If, as is necessary to this end, we take  $b$  proportional to  $n$ , it is  $bV/n$ , or  $V$  (and not  $V/n$ ), which varies as  $nV^2$ , or  $bV^2$ . The conclusion is then simply that  $bV$  must be constant ( $v$  being given). This is merely the usual condition of dynamical similarity, and no conclusion as to the law of velocity follows.

But a closer consideration will show, I think, that there is a substantial foundation for the idea at the basis of Lanchester's argument. If we suppose that the viscosity is so small that the layer of fluid affected by the passage of the blade is very small compared with the width ( $b$ ) of the latter, it will appear that the communication of motion at any stage takes place much as if the blade formed part of an infinite plane moving as a whole. We know that if such a plane starts from rest with a velocity  $V$  afterwards uniformly maintained, the force acting upon it at time  $t$  is per unit of area, see (12),

$$\rho V \sqrt{(v/\pi t)}. \dots \dots \dots (50)$$

The supposition now to be made is that we may apply this formula to the element of width  $dy$ , taking  $t$  equal to  $y/V$ , where  $y$  is the distance of the element from the leading edge. Thus

$$F = l\rho (v/\pi)^{\frac{1}{2}} V^{\frac{3}{2}} \int y^{-\frac{1}{2}} dy = 2l\rho (v/\pi)^{\frac{1}{2}} V^{\frac{3}{2}} b^{\frac{1}{2}}, \dots (51)$$

which agrees with (48) if we take in the latter  $c = 2/\sqrt{\pi}$ .

The formula (51) would seem to be justified when  $v$  is small enough, as representing a possible state of things; and, as will be seen, it affords an absolutely definite value for the resistance. There is no difficulty in extending it under similar restrictions to a lamina of any shape. If  $b$ ,

no longer constant, is the width of the lamina in the direction of motion at level  $z$ , we have

$$F = 2\rho(\nu/\pi)^{\frac{1}{2}}V^{\frac{3}{2}} \int b^{\frac{1}{2}} dz. \quad . \quad . \quad . \quad (52)$$

It will be seen that the result is not expressible in terms of the *area* of the lamina. In (49)  $e$  is not constant, unless the lamina remains always similar in shape.

The fundamental condition as to the smallness of  $\nu$  would seem to be realised in numerous practical cases; but any one who has looked over the side of a steamer will know that the motion is not usually of the kind supposed in the theory. It would appear that the theoretical motion is subject to instabilities which prevent the motion from maintaining its simply stratified character. The resistance is then doubtless more nearly as the square of the velocity and independent of the value of  $\nu$ .

When in the case of bodies moving through air or water we express  $V$ ,  $a$ , and  $\nu$  in a consistent system of units, we find that in all ordinary cases  $\nu/Va$  is so very small a quantity that it is reasonable to identify  $f(\nu/Va)$  with  $f(0)$ . The influence of linear scale upon the character of the motion then disappears. This seems to be the explanation of a difficulty raised by Mr. Lanchester (*loc. cit.* § 56).

LXXXIII. *The Velocity of the Ions of Alkali Salt Vapours in Flames.* By Prof. H. A. WILSON, F.R.S., F.R.S.C., McGill University, Montreal\*.

IT was shown by the writer in 1899 † that, in flames, all the alkali metals give positive ions which have equal velocities due to an electric field. This result has been confirmed by Marx and Moreau. The value of the velocity is about 70 cms. per sec. for one volt per cm.

The fact that the maximum quantity of electricity which can be carried by a definite amount of any alkali salt vapour is equal to that required to electrolyse the same amount in a solution ‡, shows that the product  $\bar{N}e$  has the same value in salt vapours as in solutions. Here  $\bar{N}$  is the number of positive ions formed from one gram molecule of the salt when completely ionized and  $e$  the charge carried by each ion. Since in solutions each atom of the alkali metal forms

\* Communicated by the Author.

† Phil. Trans. A. cccxxvii. (1899).

‡ H. A. Wilson, Phil. Trans. A. cccxvi. (1901).

one monovalent positive ion, the result just mentioned makes it probable that the same thing happens in the vapours.

Prof. O. W. Richardson \* has recently measured the ratio of the charge  $e$  to the mass  $m$  for the positive ions of vapours of the sulphates of all the alkali metals, and finds it equal to the value which obtains in solutions. This makes it very probable that the positive ions are metal atoms. The fact that the haloid salt vapours give ions having the same velocities as the oxy-salts in flames, shows that all salts of any one metal give ions identical in nature.

The equality of the velocities of a lithium ion and a caesium ion is difficult to explain on the view that they are simply single atoms, for we should expect the velocity to depend on the atomic weight. The main object of this paper is to point a way out of this difficulty.

In my experiments two electrodes were placed one above the other in a Bunsen flame, and the current between them was measured. If the upper electrode was positively charged and a bead of salt was placed just below it, it was found that the current was not appreciably increased by the salt unless the potential difference between the electrodes was greater than about 100 volts. This was taken to mean that 100 volts was just enough to make the positive ions move down the flame. The potential gradient in the flame is nearly uniform except near the electrodes, so that the current density ( $i$ ) is given by  $i = en(v_1 + v_2)$ , where  $n$  is the number of ions of either sign per c.c.,  $v_1$  and  $v_2$  the velocities of the positive and negative ions. If the gas is moving upwards with velocity  $u$  and  $X$  denotes the electric intensity, then we have

$$v_1 = k_1 X - u$$

$$v_2 = k_2 X + u,$$

so that the current is equal to  $enX(k_1 + k_2)$ , and is independent of  $u$ .

When the upper part of the flame is filled with salt vapour  $n$  will be much larger in that part than in the rest of the flame, so that for a given current  $X$  will be proportionally smaller. This diminution of  $X$ , however, does not lead to an appreciable increase in the current, when the upper electrode is positive, because nearly all the resistance to the passage of the current is close to the negative electrode, where the greater part of the fall of potential takes place.

\* Phil. Mag. Dec. 1910.

Thus if  $X'$  and  $n'$  refer to the upper part of the flame containing the salt, we have

$$i = X'n'e(k_1 + k_2) = Xne(k_1 + k_2),$$

which is independent of  $u$ , so that it does not appear at first sight why the salt should increase the current with any potential difference.

If, however,  $X$  is big enough, the metal ions will move down against the upward stream of gas and will be deposited on the negative electrode. The alkali metal will consequently accumulate at the lower electrode, and since it is strongly ionized it will diminish the resistance there and so increase the current. In the case of sodium salts this accumulation can be easily observed by the appearance of sodium light near the lower electrode\*.

It appears, therefore, that the increase in the current is not as was originally supposed, due merely to the current carried by positive ions coming down, for before the salt is put in there are already present far more than enough ions to carry the current allowed by the resistance at the negative electrode.

Suppose that an alkali metal atom in the flame is ionized for a fraction  $f$  of the time. Then its velocity due to an electric field will be  $f k_1 X$  instead of  $k_1 X$ . If, then,  $X_0$  denotes the least value of  $X$  for which the metal accumulates at the lower electrode, we have

$$f(k_1 X_0 - u) = (1 - f)u,$$

or

$$f k_1 = \frac{u}{X_0},$$

for during the fraction  $(1 - f)$  of the time the atom will be carried upwards with velocity  $u$ .

The quantity which was determined experimentally was therefore not  $k_1$  as was supposed, but  $f k_1$ . Now the conductivity imparted to a flame by equal numbers of molecules of different alkali metal salts increases rapidly with the atomic weight of the metal. This shows that a caesium atom is ionized for a much larger fraction of the time than a lithium atom. Hence, since both give the same value for  $f k_1$ , it follows that  $k_1$  for lithium must be really much greater than  $k_1$  for caesium.

In hot air at about 1000° C. the  $f$ 's will be much smaller

\* H. A. Wilson, Proc. R. I. 1909.

than in a Bunsen flame, so that the values found for  $fk_1$  should be much less, as was found to be the case\*.

The relative values of the fraction  $f$  for different salts can be deduced from the conductivities which they impart to the flame. In the determination of  $fk_1$  the concentration of the metallic atoms which move down the flame is extremely small, not enough to appreciably colour the flame except in the case of sodium. The equilibrium between the atoms and negative electrons will, therefore, be determined by the equation

$$q = \beta(N - n) = \alpha n m, \quad \dots \dots (1)$$

where  $q$  is the number of positive ions produced per c.c. per sec. by ionization of the metal atoms,  $N$  the total number of metal atoms present per c.c.,  $n$  the number of metal atoms which are ions per c.c.,  $m$  the number of negative electrons per c.c., and  $\beta$  and  $\alpha$  are constants.  $m$  will be large compared with  $n$  because the ions of the flame will be much more numerous than the ions from the metal vapour. Now  $f = n/N$ , so that equation (1) becomes

$$\beta(1 - f) = \alpha m f;$$

for another salt we have in the same way

$$\beta'(1 - f') = \alpha' m f'.$$

Hence

$$\frac{\beta \alpha'}{\alpha \beta'} = \frac{f}{f'} \cdot \frac{1 - f'}{1 - f} \dots \dots \dots (2)$$

When the conductivities imparted to the flame by different salts were compared, the salts were present in comparatively large concentrations, and the number of ions due to the salts was large compared with the number due to the flame gases. In this case, therefore,

$$q = \beta(N - n),$$

and for another salt having the same molecular concentration

$$q' = \beta'(N - n').$$

Here  $n$  will be small compared with  $N$ , so that approximately

$$\frac{q}{q'} = \frac{\beta}{\beta'} \dots \dots \dots (3)$$

Then (2) and (3) give

$$\frac{q \alpha'}{q' \alpha} = \frac{f}{f'} \cdot \frac{1 - f'}{1 - f} \dots \dots \dots (4)$$

\* H. A. Wilson, Phil. Trans., A. ccxxxvii. (1899).

It will be observed that in this equation the  $f$ 's apply to very small concentrations, while the  $q$ 's are for a comparatively large concentration.

Sir J. J. Thomson has given the theory\* of the relation between the potential difference (V) between two electrodes in a flame and the current ( $i$ ). He obtains the equation

$$V = \left(\frac{\alpha}{q}\right)^{\frac{3}{2}} \frac{1}{ek_2} \left( li + \frac{\alpha i^2}{4\pi e^2 q \sqrt{k_1 k_2}} \right).$$

When the electrodes are near together, as was the case in the measurements of the currents with different salts, the first term can be neglected, so that

$$V = \frac{i^2 \alpha^{\frac{3}{2}}}{4\pi e^3 k_2^{\frac{3}{2}} k_1^{\frac{1}{2}} q^{\frac{3}{2}}}.$$

If  $i'$  denotes the current obtained with another salt and the same V, then

$$\left(\frac{i}{i'}\right)^2 = \left(\frac{\alpha' q}{\alpha q}\right)^{\frac{3}{2}} \left(\frac{k_1}{k_1'}\right)^{\frac{1}{2}}.$$

The experiments show that  $f k_1 = f' k_1'$ . Hence, using (4),

$$\left(\frac{i}{i'}\right)^2 = \left(\frac{f}{f'} \cdot \frac{1-f'}{1-f}\right)^{\frac{3}{2}} \left(\frac{f'}{f}\right)^{\frac{1}{2}},$$

or 
$$\frac{i}{i'} = \left(\frac{f}{f'}\right)^{\frac{1}{2}} \left(\frac{1-f'}{1-f}\right)^{\frac{3}{4}} \dots \dots \dots (5)$$

Here  $i$  and  $i'$  are the currents given by the same P.D. with two different salt vapours of equal concentration. The ratio  $i/i'$  ought therefore to be independent of the concentration. When the concentration of the salt is not too large this is true, for then the current is proportional to a power of the concentration, which is about one-half for all salts.

The following table† gives the currents observed when a  $\frac{1}{10}$  normal solution of the chloride was sprayed into the flame

\* 'Conduction of Electricity through Gases,' 2nd ed.  
 † "The Electrical Conductivity and Luminosity of Flames containing Vaporized Salts," Smithells, Dawson, and Wilson: Phil. Trans. A. cexli. (1899).

using a P.D. of 5.60 volts. With this P.D., the first term in Sir J. J. Thomson's equation can be safely neglected.

Metal.	Current.
Cæsium . . .	123
Rubidium . . .	41.4
Potassium . . .	21
Sodium . . .	3.5
Lithium . . .	1.3

In order to use these values of the currents to calculate the  $f$ 's we require another relation. If the positive ions consist of single atoms, then their velocities ought to be approximately inversely proportional to the square roots of their atomic weights ( $M$ ), consequently  $f$  ought to be proportional to  $\sqrt{M}$ .

Instead of (5) we can write

$$i = \frac{A f^{\frac{3}{2}}}{(1-f)^{\frac{3}{4}}},$$

where  $A$  is a constant. Putting  $f = B \sqrt{M}$  this becomes

$$i = \frac{A B^{\frac{3}{2}} M^{\frac{3}{4}}}{(1 - B M^{\frac{1}{2}})^{\frac{3}{4}}}.$$

Two values of  $i$  and  $M$  then suffice to determine  $A$  and  $B$ . Using the values for cæsium and sodium gives  $B = 0.08594$  and  $A = 3.67$ .

With these values of  $A$  and  $B$  we get the values of  $f$  given in the second column of the following table:—

Metal.	$f$ .	$f'$ .	$k_1$ .	$k_1'$ .
Cæsium.....	0.99	0.99	71	71
Rubidium.....	0.79	0.96	89	73
Potassium.....	0.54	0.91	130	77
Sodium.....	0.41	0.41	170	170
Lithium.....	0.23	0.21	305	333

The column headed  $f'$  contains the values of  $f$  required by the observed currents. The differences between  $f$  and  $f'$  are not very great, except in the case of potassium. It seems, therefore, that the assumption that  $f$  varies as  $\sqrt{M}$  is roughly true. The column headed  $k_1$  contains the values of the velocities of the positive ions got by using the numbers for  $f$ , and that headed  $k_1'$  those corresponding to the numbers



under  $f'$ . The value of  $fk_1$  was taken to be 70 cms. per sec.

The value of  $k_1$  can be calculated roughly on the kinetic theory of gases, for in a flame at about 2000° C. the free path ( $\lambda$ ) of an atom is probably about  $10^{-4}$  cm.\* The well known formula  $k_1 = e\lambda/mV$  gives  $k_1 = 300$  cms. per sec. for an atom of hydrogen, or 120 for an atom of lithium. This is about one-third the value of  $k_1$  as estimated above, which is as near as could be expected.

I think, therefore, that the evidence provided by the measurements made with the object of finding the velocities of the positive ions in flames is not inconsistent with the view that these ions are single atoms of the alkali metal.

Measurements on the negative ions in flames have also been made by the writer and others †. The negative ions appear to be free electrons, so that their deposition on the positive electrode cannot be supposed to cause an increase in the current, as in the case of the positive ions at the negative electrode. It seems, therefore, that the supposed determinations of the velocity of the negative ions in flames, by finding the least P.D. required to make them move against or across the stream of gas, are based on a fallacy. When the salt is put in near the negative electrode the large resistance there is diminished, so that the current ought to be increased, whether the P.D. is big enough to make the negative ions move against the stream or not. The following table contains the currents observed taken from my paper ‡.

P.D. (Volts).	Current.	
	(Without salt.)	(With salt.)
0	-3	-13
0.25	-2	-10
0.5	0	- 7
0.75	+2	0
1.0	+3	+ 5
1.5	+3	+20
2.0	+3	+30
3.0	+3	+33

\* Sir J. J. Thomson, 'Conduction of Electricity through Gases,' 2nd ed.

† Marx, Moreau, and E. Gold.

‡ Phil. Trans. A. ccxxxvii. p. 517 (1899).

The increase in the current between one and two volts with salt below the upper (negative) electrode was supposed to show that above one volt the negative ions from the salt moved down the flame. The increase in the current below one volt is however quite marked. It was then supposed that the flame without salt was not strongly ionized, but now it is known that the small current without salt is due to the great resistance close to the negative electrode, and that the ions present are sufficient to carry a current of probably many amperes. Such experiments, therefore, do not give any information with regard to the velocity of the negative ions.

Under these circumstances it is necessary to fall back on indirect evidence. Measurements of the effect of a magnetic field on the conductivity of a Bunsen flame made by the writer\* indicated that the velocity of the negative ions was about 9000 cms. per sec., which is about the value to be expected for negative electrons.

We may therefore conclude that the positive ions of alkali salts in flames are probably single atoms of the metal, and that the negative ions are electrons.

In a recent paper Mr. Lusby † finds 290 cms. per sec. for the velocity of the positive ions of salt vapours in flames. In his experiments the electrodes were only 3 cms. apart, so that in the absence of salt the uniform gradient was not present because the negative drop extends more than 2 cms. from the lower electrode. On putting in the salt near the upper electrode he observed a very small uniform gradient which is evidently due to the high conductivity of the salt vapour. To calculate the velocity of the ions correctly the value of the uniform gradient just below the salt vapour is required, and this should be equal to the gradient in the absence of salt since the current is unchanged at the critical potential. I think therefore that Mr. Lusby's result is too high.

LXXXIV. *The Number of Electrons in the Atom.* By Prof. H. A. WILSON, F.R.S., F.R.S.C., McGill University, Montreal ‡.

ACCORDING to Sir J. J. Thomson's theory § atoms may be regarded as spheres of positive electricity containing negative electrons which can move about freely inside the positive charge. The total negative charge on the electrons

\* Proc. Roy. Soc. A., vol. lxxxii.

† Proc. Camb. Phil. Soc. vol. xvi. Pt. 1, 1911.

‡ Communicated by the Author.

§ 'The Corpuscular Theory of Matter,' 1907.

is equal to the positive charge on the sphere in a neutral atom.

The object of the present paper is to show how to obtain an approximate solution of the problem of the distribution of  $n$  electrons in a positive sphere and how to deduce the number of electrons in any atom from the atomic weights of the elements.

Consider an electron having a charge  $e$  inside a sphere of positive electricity of uniform density of charge  $\rho$  per c.c. Close to the electron the electric field is of strength  $\frac{e}{r^2}$ , where  $r$  is the distance from the electron, so that  $4\pi e$  tubes of electric force come out of the electron, if the number of tubes per sq. cm. is taken to be equal to the field strength. Consider one of these tubes of force and let  $ds$  be an element of its length and  $\alpha$  its cross-section at  $ds$ . The charge in the length  $ds$  is  $\rho\alpha ds$ , so that

$$-\frac{d}{ds}(F\alpha) = 4\pi\rho\alpha,$$

where  $F$  is the electric force along  $ds$ . Integrating along the tube this gives

$$F_1\alpha_1 - F\alpha = 4\pi\rho \int \alpha ds,$$

where  $F_1\alpha_1$  denotes the value of  $F\alpha$  at the surface of the electron. This shows that as we go along the tube  $F\alpha$  diminishes and when

$$F_1\alpha_1 = 4\pi\rho \int \alpha ds$$

it will be zero and the tube will end. Now  $F_1 = e/a^2$ , where  $a$  is the radius of the electron, and  $\alpha_1 = a^2/e$ , so that  $F_1\alpha_1 = 1$ , hence  $4\pi\rho \int \alpha ds$  from the surface of the electron to the end of the tube is equal to unity. Thus the volume of each tube is

$\frac{1}{4\pi\rho}$  and the volume of all the  $4\pi e$  tubes is therefore  $e/\rho$ .

Thus the tubes of force starting from the electron occupy a volume  $e/\rho$ , and this is true in any case whether other electrons are near or not. Also, since every tube of force must end on positive electricity, it is clear that the volume  $e/\rho$  can only contain the one electron from which the tubes start. Thus when any number of electrons are present each one will be surrounded by its own field which will occupy the volume  $e/\rho$ . The positive charge in the volume  $e/\rho$  is equal to  $e$ , so that if the sphere has a positive charge equal to the total negative charge on the  $n$  electrons in it, it will be divided up into  $n$  equal volumes each containing one electron.

The energy in an element of a tube of force is equal to

$F^2 \alpha ds / 8\pi$ , and if the tube is slightly distorted the volume of each element and the value of  $F\alpha$  remain unchanged, so that the change in the energy in the element will be due to the change in  $F$ . The energy will be a minimum when the tube is in equilibrium, so that  $F$  will be as small as possible and therefore  $\alpha$  as large as possible. This means that the tubes tend to become as short as possible, their volumes remaining constant. The effect of this will evidently be to make the field round each electron tend to become as nearly spherical as possible with the electron in the middle.

Consequently, to determine approximately the distribution of the  $n$  electrons in the positive sphere, it is sufficient to find how the sphere can be divided up into  $n$  equal volumes all as nearly spherical as possible and to put an electron at the centre of each of the  $n$  volumes. When  $n$  is large it is easy to see that this requires the electrons to be arranged like the centres of the shot in a pile of shot. Thus with thirteen electrons we should expect to have one in the middle and twelve arranged round it all at the same distance from it.

It is easy to see from considerations of symmetry that the electrons will arrange themselves on nearly spherical surfaces concentric with the surface of the positive sphere. The condition that the fields of the electrons shall be as nearly spherical as possible evidently requires the distances between the successive surfaces to be all equal. The fields of the electrons on the surface of a sphere will form a layer the cube of the thickness of which will be approximately equal to the volume of the field of one electron.

According to Sir J. J. Thomson's theory each element in a series of similar elements, such as fluorine, chlorine, bromine, iodine, is derived from the one before it in the series by the addition of a spherical layer of electrons together with the necessary amount of positive electricity to keep the atom neutral.

Let  $n_1, n_2, n_3, \&c.$ , denote the numbers of electrons in the atoms of a series of similar elements and let  $A_1, A_2, A_3, \&c.$  denote their atomic weights. Then, if we assume that the number of electrons in an atom is proportional to its atomic weight, we can write  $\beta A_1 = n_1, \beta A_2 = n_2, \&c.$  where  $\beta$  is a constant.

Let  $r_1, r_2, \&c.$  denote the radii of the positive spheres and let  $v = e/\rho$  be the volume of the field round each electron.

Then we have

$$\frac{4}{3} \pi r_m^3 = n_m v = \beta v A_m,$$

$$\frac{4}{3} \pi r_{m+1}^3 = n_{m+1} v = \beta v A_{m+1}.$$

Hence

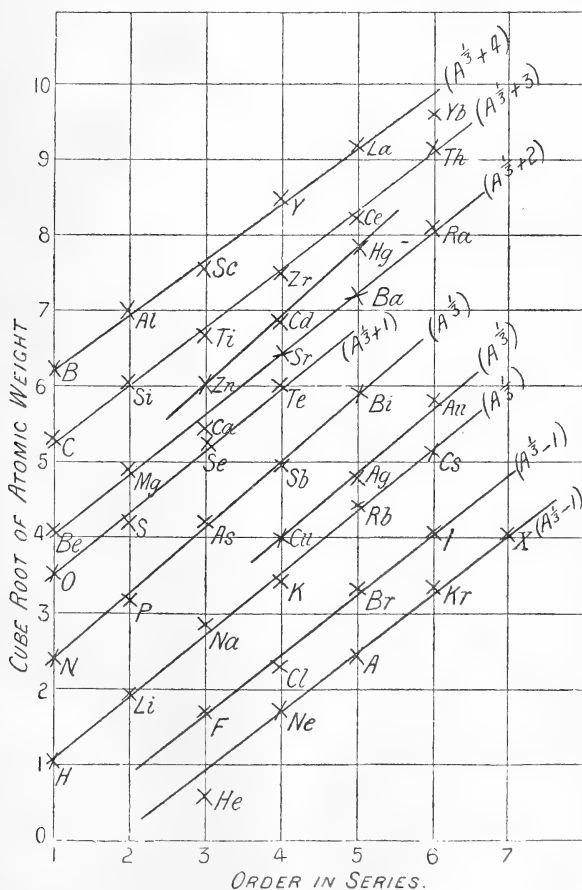
$$\left(\frac{4\pi}{3\beta v}\right)^{\frac{3}{2}}(r_{m+1}-r_m) = A_{m+1}^{\frac{3}{2}} - A_m^{\frac{3}{2}} = C,$$

where C is a constant which should be the same for all series of similar elements.

Also  $(r_{m+1}-r_m)^3 = v$  approximately so that

$$\beta = \frac{4\pi}{3C^3}.$$

According to the theory therefore we ought to be able to find the number of electrons per atom from the atomic weights.



In the figure the values of  $A^{3/2}$  for series of similar elements are plotted against the order of the elements in the series.

For some series a constant has been added to the values of  $A^{\frac{1}{3}}$  to prevent the different lines falling too close together. It will be seen that the values of  $A^{\frac{1}{3}}$  for each series fall nearly on straight lines and that the different lines are nearly parallel. This shows that  $A_{m+1}^{\frac{1}{3}} - A_m^{\frac{1}{3}} = C$  is nearly constant, as was to be expected from the theory. The mean value of  $C$  is about 0.81. Hence we get  $\beta = 8$ , so that the number of electrons per atom comes out about 8 times the atomic weight in all cases.

This estimate agrees as well as could be expected with recent estimates depending on the scattering of radiation by different elements.

Since  $n = \beta A$  we have

$$\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} = n_{m+1}^{\frac{1}{3}} - n_m^{\frac{1}{3}}.$$

By means of this equation it is easy to calculate the number of electrons in successive spherical layers. If we take  $n_1 = 8$  we get the following values of  $n_m$  :—

$m$ .	$n_m$ .	$\frac{n_m}{8}$ .	
1.....	8	1	H = 1
2.....	47	6	Li = 7
3.....	142	18	Na = 23
4.....	320	40	K = 39
5.....	600	75	Rb = 85
6.....	1020	128	Cs = 133

The last column contains the atomic weights of the alkali metals, which do not differ very much from the values of  $n/8$ . Since the calculations made are only approximate, the agreement is as good as could be expected.

LXXXV. *The Behaviour of Radium Emanation at Low Temperatures.* By R. W. BOYLE, M.Sc., Ph.D., 1851 Exhibition Science Scholar, McGill University\*.

THE researches of Rutherford † and of Gray and Ramsay ‡ have shown that at temperatures from  $-127^{\circ}$  C. to  $104^{\circ}$  C. the emanation of radium has definite and constant values of vapour pressure corresponding to every temperature. At temperatures below  $-127^{\circ}$  C. the only knowledge

\* Communicated by Prof. E. Rutherford, F.R.S.

† Phil. Mag. [6] xvii. p. 723 (1909).

‡ Journ. Chem. Soc. xcv. p. 1073 (1909).

we have concerning the process of volatilization of condensed emanation is that given by the flow method of experiment originally devised by Rutherford and Soddy \*. This method is best adapted, and so far has been used, for experiments with small quantities of emanation. Under the circumstances of its use condensation of the emanation can only result in a very sparse distribution of emanation molecules over a considerable area of cooled surface, so that the condensed "layer" will be of much less than molecular thickness. In these cases it is probable that the phenomenon is entirely one of surface adhesion or occlusion.

The object of the present paper is to describe briefly some experiments which were performed to seek further information on the process of volatilization at low temperatures. The conditions in the experiments were quite different from those in the flow method. The emanation was contained in sealed glass tubes which were as free as possible from all other gases; condensation and volatilization were confined to the point where the minute volume of condensed emanation was situated; and no current of air or other gas was required.

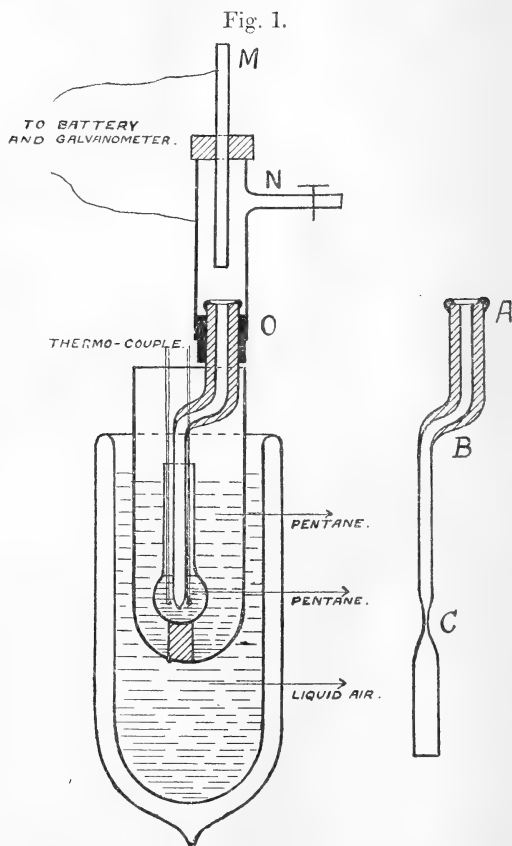
#### *Apparatus and Method of Experiment.*

The tubes containing the emanation were of the shape ABC shown in fig. 1 (p 724). The wall of the part AB was 2.5 mm. thick, and of the part BC 1 mm. thick. The bore of the tube was usually about 2.5 mm. The end of the tube at A was closed by a very thin sheet of mica, which was secured to the wall by a special kind of marine glue. The thickness of the mica was equivalent to 1.9 cm. of air in its stoppage of  $\alpha$ -particles; nevertheless the sheet was strong enough to support the full atmospheric pressure over the opening, and thus maintain a vacuum inside the tube.

For an experiment the glass tube was first evacuated to a charcoal vacuum. The required amount of purified emanation received from Prof. Rutherford was then introduced, and the tube was sealed at C. Four hours after admitting the emanation the amount present was determined by the  $\gamma$ -ray method. The tube was then fitted to a small ionization vessel MNO in the manner shown in the diagram. The vessel and fittings were made air-tight so that the pressure of air in the ionization chamber could be adjusted to any value. With such an arrangement the ionization was usually very intense, and it was often very difficult to obtain saturation.

\* Phil. Mag. [6] v. p. 561 (1903).

The method of performing an experiment was to condense the emanation at the extreme end C of the containing tube



by immersing this end in a bath of pentane cooled with liquid air. Condensation was maintained for four hours, during which time the active deposit about the tube practically all decayed, and the ionization gradually decreased to a small constant value. The double right-angled bend in the tube at B prevented the  $\alpha$ -rays from the condensed emanation and its active deposit at C having any ionizing effect in the vessel above.

After keeping the emanation condensed for four hours the temperature was allowed to rise slowly, and continuous observations of ionization were taken. Under these conditions, whenever any of the condensed emanation at C volatilized, the emanation vapour quickly distributed itself throughout the tube, and the portion of it going to the upper part AB



marked its presence there by causing an increase of ionization. In this way the ionization measurements indicated roughly the amount of emanation in the upper part of the tube corresponding to the various temperatures, and thus gave a means of following the progress of the volatilization.

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The condensing arrangement finally used is shown in fig. 1. The free end of the glass tube which contained the emanation was dipped into a very small glass bulb containing just enough pentane to cover the end of the tube and the junctions of the thermo-couples which were used for determining the temperatures. Surrounding this small bulb was an outer bath of pentane, which was itself surrounded by a bath of liquid air contained in a 6-inch, silvered, Dewar cylinder. The Dewar cylinder was kept filled to the top with liquid air until it was desired to allow the temperature to rise. The liquid air was then allowed to evaporate slowly, and the gradual lowering of the level caused a slow variation of temperature at the point where the emanation was condensed. The temperature usually rose at a rate of  $0^{\circ}.5$  C. per minute. Without the double bath arrangement it was found that the rate of temperature rise was not sufficiently uniform.

Some trial experiments with moderate quantities of emanation showed, after the baths were removed, the presence of a bright point of light at the extreme end of the glass tube, and a uniform fluorescence over the rest. This bright spot was due to the active matter which had been deposited by the condensed emanation. Its concentration at this one point showed that the emanation had condensed not over any considerable area but at the very tip of the tube. Consequently the junctions of the thermo-junction were placed in the inner pentane bath exactly at the tip of the glass tube which contained the emanation. The thermo-couple was a double, copper-constantan element of number 30 double-cotton covered wires. The warmer junctions were maintained at the temperature of melting ice.

It has already been mentioned that there was a difficulty in obtaining saturation in the ionization chamber. Since the ionization-temperature curves afterwards shown could not be used to determine the actual amount of emanation volatilized at a given temperature, complete saturation was not essential; nevertheless, in all experiments saturation was approximately attained. For this purpose the ionization vessels employed were made very small. The one mostly used was a brass cylinder, 1 cm. in diameter and 5 cms. long, fitted with the usual central electrode and ebonite insulation.

Since the quantities of emanation employed in different

experiments varied over a wide range, it was necessary in measuring the ionizations to use instruments varying widely in sensitiveness. With the larger amounts of emanation a Kelvin astatic galvanometer, of which the greatest sensitiveness was 1 scale-division (millimetre) for  $1.3 \times 10^{-10}$  ampere, could be used; with the smaller quantities a gold-leaf electroscope sufficed.

When using the galvanometer and large quantities of emanation, it was necessary to reduce the pressure in the ionization chamber in order to obtain saturation. But the pressure could not be too far reduced, for a diminution of the pressure caused a decrease in the current, and it was desirable to work with a fairly large deflexion of the galvanometer-needle. (Increased voltage sometimes helped to this end.) In general, approximate saturation with satisfactory deflexion was obtained by manipulating the pressure, the voltage, and the position of the control magnet of the galvanometer. The maximum deflexions in different experiments varied from 200 to 400 divisions at a scale-distance of 1.41 metres.

When using the gold-leaf electroscope the only possible adjustment to secure saturation was to lower the pressure of air in the ionization chamber.

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Some possible causes of error which may enter into the experiments should be mentioned. The emanation produces gases—mostly carbon dioxide—by its action on the marine glue with which the mica sheet was secured to the containing tube. If the volume of the tube were very small, as in the case of a capillary, these gases would have an effect in retarding by diffusion the passage of the volatilized emanation from the lower to the upper part of the tube. Again, with capillaries, another error due to viscosity would affect the readings in the same direction. The frictional resistance of the gas in the capillary would retard the passage of the volatilized emanation from the lower to the upper part of the tube. Some trial experiments showed the necessity of avoiding these troubles.

The best conditions of experiment were

- (1) to use well cleaned tubes of not too small a bore, and therefore of not too small a cubical capacity;
- (2) to perform the experiment as soon as possible after the admission of purified emanation into the tube;
- (3) to obtain as nearly as possible saturation of ionization.

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From a number of experiments in which these conditions were fulfilled the curves shown in fig. 2 are given as samples.

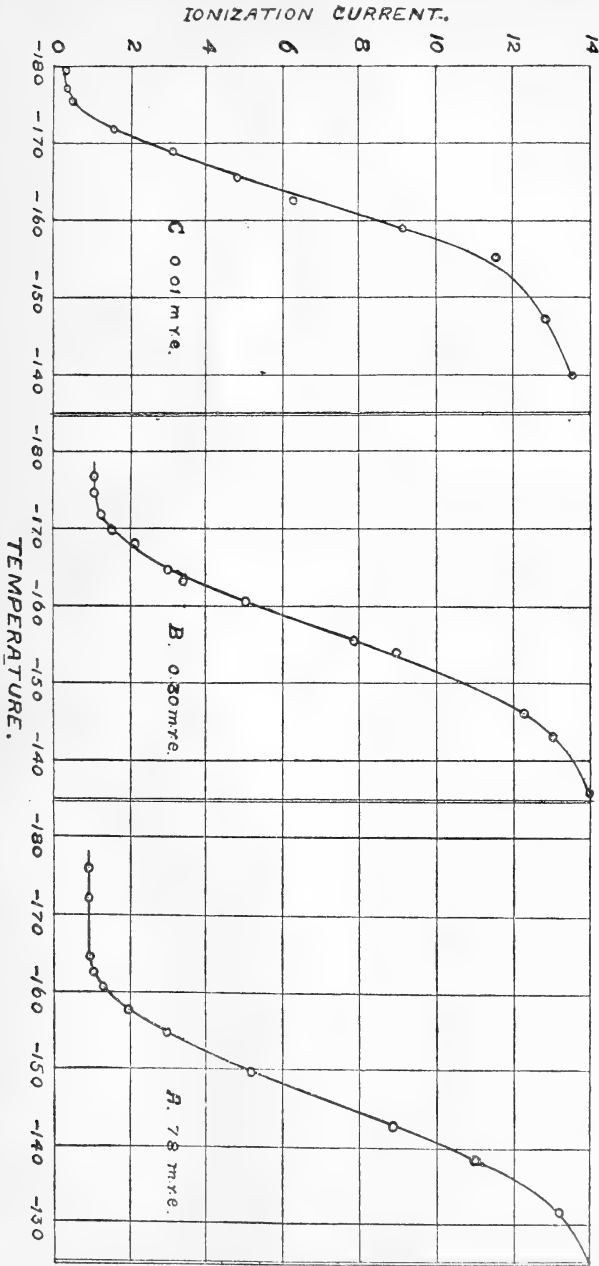


FIG. 2.

The ordinates represent ionization current and the abscissæ the corresponding temperatures. The three curves, A, B, and C, represent three widely different quantities of emanation, which correspondingly required three measuring devices differing widely in sensitiveness. The scales of ordinates are therefore very unequal: scale of C > scale of B > scale of A.

In Table I. are given the experimental details concerning the curves shown in fig. 2. The "Partial Pressures of Emanation" are calculated—assuming Boyle's law to hold approximately—from the fact that the equilibrium amount of emanation of 1 gm. of radium has a volume of 0.6 cub.mm. at N.T.P. These pressures are included merely to give an idea of the extreme tenuity of the emanation before or after condensation. (The abbreviation "m.r.e" means the amount of emanation in radioactive equilibrium with the stated number of milligrams of radium element.)

TABLE I.

(1) Curve.	(2) Quantity of Emanation.	(3) Vol. of Tube.	(4) Partial Pressure of Emanation.	(5) Measuring Device.
A .....	78 m.r.e.	0.6 c.c.	$6 \times 10^{-2}$ mm. Hg.	Galvanometer { 100 mm. Hg. pressure in ionization chamber.
B .....	0.30 "	1.0 "	$1.5 \times 10^{-3}$ "	Electroscope { Only a few mm. Hg. pressure in ionization chamber.
C .....	0.01 "	2.3 "	$2 \times 10^{-6}$ "	Electroscope { 760 mm. Hg. pressure in ionization chamber.

The general form of the curve consists of three parts. First there is an initial, flat, or nearly flat, portion; then a steep portion rising from the temperature axis; and then a bend towards the temperature axis, after which the curve continues to rise but much less markedly. The last portion extends much further than is shown in the diagram. As an example the following table shows the readings in an experiment where the galvanometer was used.

TABLE II.

Deflexion before condensation ..... 25.4 cm.  
 Deflexion 4 hours after applying liquid air 0.8 ,,

On allowing the temperature to rise :—

Temperature.	Deflexion.
-177	0.8 cm.
-172	0.8
-164.5	0.8
-160.5	1.15
-154.5	2.55
-142.5	7.65
-131.5	11.5
-118.5	13.6
-110.5	14.6

Followed by a long period of very slow rise of ionization.

The form of the curve can be very simply explained on a basis of a normal behaviour of the emanation.

Consider, first, curve A which corresponds to 78 m.r.e., and in which a galvanometer was used to measure the ionization. The galvanometer being a very insensitive instrument required an enormous ionization to affect it. Apparently there was no appreciable rise of ionization, and therefore no appreciable volatilization of emanation until the temperature approached  $-163^{\circ}\text{C}$ . To this is due the initial, flat, portion of the curve. About  $-163^{\circ}$  the emanation began to volatilize in larger quantity, the vaporized emanation distributed itself throughout the tube, and Ra A and Ra C began to grow. As the temperature increased all three products—the emanation, Ra A, and Ra C—increased more rapidly in the upper part of the tube. The continuously increasing number of  $\alpha$ -rays sent out by these products caused a corresponding increase of ionization in the chamber above, thus giving the steep, rising, portion of the curve. After the emanation has all volatilized the rate of increase of emanation vapour in the top part of the tube must fall off very greatly. On account of the low temperature at the bottom of the tube, the density of the emanation is greater at this part than in the upper part, although all the emanation has volatilized. But as the bottom temperature continues to rise slowly the density here decreases, and this causes a slow transference of emanation molecules from the lower to the

upper part of the tube. This slow increase of emanation in the upper part of the tube, with the consequent growth of Ra A and Ra C, causes the ionization to increase, but much less rapidly than before, and thus gives the final, long-continued, portion of the curve.

Within the range of temperature covered by the initial, flat, portion of the curve, the emanation was certainly volatilizing, though not in sufficient quantity to affect the galvanometer. But a more sensitive instrument should be affected within this range, and this is shown to be the case by curve B. This curve corresponds to a quantity of 0.30 m.r.e. of emanation and an electroscope with a pressure of a few mm. Hg in the ionization chamber for measuring the ionization. Instead of the flat portion extending from the lowest condensing temperature to  $-163^{\circ}$  C., as in the case of curve A, it extends here only to  $-171^{\circ}$  C., after which the curve rises, and then bends towards the temperature axis, as explained. The measuring device in this case was so much more sensitive than the galvanometer, that it could detect the changes in the vapour phase of the emanation at temperatures as low as  $-171^{\circ}$  C., whereas the galvanometer could only do this as low as  $-163^{\circ}$  C.

A still more sensitive device, viz. an electroscope with atmospheric pressure in the ionization chamber, could detect the changes in the vapour phase at a lower temperature still. This is the case of curve C, where 0.01 m.r.e. was employed. From the above it follows that if it were feasible to condense a very large quantity of emanation, and employ at the same time only one measuring instrument possessing the required ranges of sensibility, we could obtain a single ionization curve of the form already shown. But this curve would rise immediately from the lowest temperature of condensation, and would cover a wide range of temperature before bending towards the axis of temperature. In other words, the condensed emanation would begin to volatilize at the lowest temperature of condensation, and would continue to volatilize through a wide range of temperature until the emanation was entirely free from the condensing surface.

In the experiments we cannot be sure that the amount of emanation volatilized at any temperature was the exact amount required to saturate the space of the containing tube at that temperature, and from the curves given we cannot calculate the vapour pressures. The experiments were only qualitative, and the complications introduced by the rate of rise of temperature, the growth of Ra A and Ra C with their different ranges of  $\alpha$ -particles, effectively prevent our utilizing

the ionization measurements for quantitative calculations. The slight ionizations at the beginning of each of the curves shown were due to the  $\gamma$  and some  $\beta$  rays from the radioactive products of the condensed emanation.

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The experiments show that a vapour phase corresponding to condensed radium emanation can easily be traced to a temperature as low as  $-180^{\circ}$  C.

Gray and Ramsay\*, as the result of an experiment in which the opacity of the condensed emanation was the test of solidity, state that the emanation solidifies at  $-71^{\circ}$  C., the vapour pressure then being 500 mm. Hg. Under the infinitesimal partial pressures and low temperatures in the present experiments the state of the condensed emanation is not known, but whether it exists as solid, or as liquid, or as an adsorbed layer, we should expect on a basis of behaviour like ordinary gases under familiar conditions :

- (1) that at any temperature a vapour phase of the emanation would exist ;
- (2) that volatilization from the condensed to the vapour phase would set in as soon as the temperature commenced to rise ; and
- (3) that volatilization would proceed gradually, becoming more and more rapid as the temperature increased.

The experiments described bear out these expectations, and thus far the behaviour of the emanation may be said to be normal. It can be seen that the temperature of final volatilization from the condensing surface will depend on the quantity of emanation, and therefore it cannot be said, unless particular conditions are stated, that the emanation when condensed will volatilize at any particular temperature.

The experiments cannot tell us whether the phenomenon of volatilization under these conditions is affected by surface adsorption or adhesion. Such matters will be settled when it can be shown definitely that in equilibrium with these infinitesimal volumes of condensed emanation there is, or is not, at any fixed, low, temperature an invariable value of vapour pressure. Already we have from Russ and Makower† a few incidental observations which suggest that at the liquid air temperature the amount of emanation vapour in equilibrium with condensed emanation is not a fixed quantity but depends on the amount of emanation condensed.

Following the work already described in this paper, the

\* *Loc. cit.*

† *Le Radium*, vi. 1909, p. 182; *Proc. Roy. Soc. A*, lxxxii. p. 205.

writer made a number of experiments with the object of determining quantitatively the vapour pressures at temperatures upwards from  $-180^{\circ}\text{C}$ . The experiments were not successful, but they may be briefly referred to.

The emanation was condensed in a manner somewhat similar to the one described on p. 725, in the bottom of a narrow glass tube connected to the side of a larger vessel which terminated at the top in a series of small bulbs. The whole apparatus was exhausted to a charcoal vacuum before the emanation was introduced. After condensing the emanation and securing temperature conditions as steady as possible, the condensing tube was opened to the larger vessel for half an hour in order to ensure a constant distribution of the emanation. The connexion was then closed, and the emanation distributed in the larger volume was compressed over mercury into the topmost small bulb. This bulb was then sealed off, the mercury was lowered to its original position, and the same process was repeated at another temperature. After a set of experiments, the emanation contents of the different bulbs were measured, and from them could be calculated the vapour pressures corresponding to the different temperatures employed.

The result of these experiments supported the conclusions arrived at from the former experiments, and showed that the pressures were of such small orders as the figures of column 4, Table I., would suggest. But the numerical results were very irregular, and they could not be even approximately repeated under the same conditions. There was great difficulty in maintaining constant temperatures; but the chief cause of the failure was due to the action of the emanation in producing, in the course of an experiment, appreciable quantities of carbon dioxide and other gases from the impurities introduced into the apparatus by the mercury and the stop-cocks. The emanation behaved as if it condensed along with these gases, thereby becoming entrapped and not being able to escape until the gases escaped also. The experience showed that in this type of experiment special apparatus will have to be used to prevent any foreign gas entering the condensing chamber. It is hoped that the experiments may be taken up again in the near future.

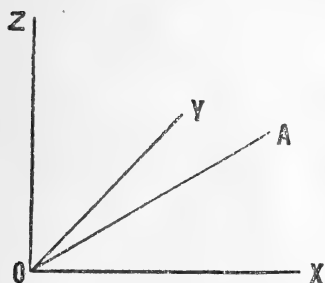
The writer is greatly indebted to Prof. Rutherford for the loan of apparatus and supplies of emanation, and also for his helpful suggestions and advice throughout the course of the experiments.



LXXXVI. *The Longitudinal and Transverse Mass of an Electron.* By W. F. G. SWANN, D.Sc., A.R.C.S., Assistant Lecturer in Physics at the University of Sheffield\*.

**I**N his paper on "Recent Theories of Electricity" (Phil. Mag. February 1911) Prof. L. T. More refers in a note, page 214, to the difficulty of realizing the existence of a transverse mass for an electron when the velocity is zero, in view of the fact that, as he remarks, transverse mass is defined as mass due to a change in direction only. I think that the following method of deducing the expressions for the masses, while of course it rests on the same fundamental bases as those hitherto employed in former investigations, has the advantage of bringing out more clearly the real meaning of the masses, and further it does not involve the consideration of a curvilinear motion at all †.

We first define force as equal to the rate of increase of momentum produced by it in the direction in which it acts.



Let us find the component accelerations  $\lambda, \mu, \nu$  produced by the unit forces in the three coordinate directions X, Y, Z at the instant when the electron is moving in any direction OA with velocity  $p, q, r$ . Let U, V, W be the components of the momentum of the electron expressed as functions of  $p, q, r$ . According to our definitions of the unit forces we have

$$1 = \frac{\partial U}{\partial t} = \frac{\partial U}{\partial p} \cdot \frac{\partial p}{\partial t} = \lambda \frac{\partial U}{\partial p},$$

with similar expressions involving  $\frac{\partial V}{\partial q}$  and  $\frac{\partial W}{\partial r}$ ,

so that

$$\lambda^{-1} = \frac{\partial U}{\partial p}, \quad \mu^{-1} = \frac{\partial V}{\partial q}, \quad \nu^{-1} = \frac{\partial W}{\partial r}.$$

\* Communicated by the Author.

† The point raised by Professor More may also be met, by observing that the expression deduced for the transverse mass by the method adopted by Abraham (see 'Ions, Electrons, and Corpuscles,' by Abraham & Langevin) is independent of the radius of curvature of the curve which the electron is supposed to describe, so that it holds for an infinite radius of curvature, *i. e.* for a rectilinear motion.

Now, confining ourselves to the case in which the electron is moving along the axis of X with velocity  $v$ , we have

$$\lambda^{-1} = \left( \frac{\partial U}{\partial p} \right)_{\substack{p=v \\ q=0 \\ r=0}} \quad \mu^{-1} = \left( \frac{\partial V}{\partial q} \right)_{\substack{p=v \\ q=0 \\ r=0}} \quad \nu^{-1} = \left( \frac{\partial W}{\partial r} \right)_{\substack{p=v \\ q=0 \\ r=0}}$$

These expressions are of course the values of the so-called longitudinal and transverse masses, the last two being the two transverse masses, which are of course equal. Now although when  $p = v$ ,  $q = 0$ ,  $r = 0$ ,  $V$  is zero, it does not follow that  $\frac{\partial V}{\partial q}$  is also zero. Again, though each of the quantities  $U, V, W$  is zero when  $p = q = r = 0$ , it does not follow that the derivatives are zero also. Of course from symmetry, when  $p = q = r = 0$ , all three masses are the same.

Let us now proceed to the deduction of the expressions for the masses: to do this it is necessary to find the general expression for the momentum of an electron moving along any line. Take axes of  $\xi, \eta, \zeta$ , not coincident with those of  $x, y, z$ , and let the electron move along the axis of  $\xi$  with velocity  $\omega$ . Let  $\alpha, \beta, \gamma$  be the magnetic vector due to the motion; then the kinetic energy per unit volume is

$$T = \frac{1}{8\pi} (\alpha^2 + \beta^2 + \gamma^2).$$

The resultant momentum of the electron per unit volume is

$$\frac{\partial T}{\partial \omega} = \frac{1}{4\pi} \left( \alpha \frac{\partial \alpha}{\partial \omega} + \beta \frac{\partial \beta}{\partial \omega} + \gamma \frac{\partial \gamma}{\partial \omega} \right).$$

Since if  $f, g, h$  are the components of the ætherial displacement

$$\alpha = 0, \quad \beta = -4\pi h\omega, \quad \gamma = 4\pi g\omega,$$

therefore

$$\frac{\partial \alpha}{\partial \omega} = 0, \quad \frac{\partial \beta}{\partial \omega} = -4\pi h, \quad \frac{\partial \gamma}{\partial \omega} = 4\pi g,$$

and

$$\frac{\partial T}{\partial \omega} = 4\pi(h^2 + g^2)\omega = 4\pi P^2\omega,$$

where  $P$  is the component of the ætherial displacement resolved perpendicular to the line of motion of the electron. The total momentum is  $4\pi \iiint P^2 \omega d\xi d\eta d\zeta$ , the integral being taken throughout all space. The rest of the analysis depends on the shape and nature of the electron. If we take the

ellipsoidal electron of Lorentz the value of our integral is (see Lorentz's 'Electrons,' p. 211)

$$\frac{2}{3} \cdot \frac{e^2}{ac^2} \omega \left(1 - \frac{\omega^2}{c^2}\right)^{-\frac{1}{2}*},$$

$e$  being the charge in electrostatic units,  $a$  the semi-major axis of the ellipsoid, and  $c$  the velocity of light. Returning to the axes of  $X$ ,  $Y$ ,  $Z$ , and putting  $\omega^2 = p^2 + q^2 + r^2$ , we at once obtain for the components  $U$ ,  $V$ ,  $W$  of the momentum resolved along the axes

$$U = \frac{2}{3} \frac{e^2}{ac^2} \left\{ 1 - \frac{p^2 + q^2 + r^2}{c^2} \right\}^{-\frac{1}{2}} p,$$

$$V = \frac{2}{3} \frac{e^2}{ac^2} \left\{ 1 - \frac{p^2 + q^2 + r^2}{c^2} \right\}^{-\frac{1}{2}} q,$$

$$W = \frac{2}{3} \frac{e^2}{ac^2} \left\{ 1 - \frac{p^2 + q^2 + r^2}{c^2} \right\}^{-\frac{1}{2}} r.$$

Differentiating these expressions with regard to  $p$ ,  $q$ , and  $r$  respectively, and afterwards putting  $p=v$ ,  $q=0$ ,  $r=0$ , we obtain for the three masses the expressions usually given †, the last two being of course identical.

$$m_1 = \left( \frac{\partial U}{\partial p} \right)_{\substack{p=v \\ q=0 \\ r=0}} = \frac{2e^2}{3ac^2} \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}},$$

$$m_2 = \left( \frac{\partial V}{\partial q} \right)_{\substack{p=v \\ q=0 \\ r=0}} = \frac{2e^2}{3ac^2} \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}},$$

$$m_3 = \left( \frac{\partial W}{\partial r} \right)_{\substack{p=v \\ q=0 \\ r=0}} = \frac{2e^2}{3ac^2} \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}}.$$

If, instead of the ellipsoidal electron, we take the conducting spherical electron we of course obtain the well known expressions corresponding to that case.

\* Lorentz uses the "Rational unit" of charge, which results in an expression slightly different from the above.

† In obtaining this expression the field of the electron at each point in space is taken as the field corresponding to the steady motion of the electron. All methods of determining the electromagnetic masses involve this assumption. It is a very legitimate assumption for the purpose in hand, because, as is easily shown, practically the whole momentum of the field of the electron is contained within a space of the same order of size as the electron itself, and consequently the field in this region follows the motion of the electron practically instantaneously. It may be noted that it is easy to show that even if the electron were not small, the assumption would be justified for the case of the motion of an electron starting from rest.

LXXXVII. *The Oscillations of Chains and their Relation to Bessel and Neumann Functions.* By JOHN R. AIREY, M.A., B.Sc., late Scholar of St. John's College, Cambridge\*.

THE oscillations of chains afford interesting examples of the practical applications of Bessel functions to physical problems. These functions, in fact, first presented themselves in connexion with the problem of the small oscillations of a uniform chain suspended by one end—Bernoulli's problem. The times of vibration in this case depend upon the roots of the equation  $J_0(z)=0$ . The more general function of the same kind but of higher order, viz.  $J_n(z)$ , appears in the expression for the time of vibration of a chain whose line-density varies as the  $n$ th power of the distance from the free end. When a uniform chain is loaded at the free end—a more general case than Bernoulli's,—the complete solution includes both kinds of Bessel functions, viz.  $J_n(z)$  and  $Y_n(z)$ , and their differential coefficients. The  $Y_n(z)$  functions are sometimes called Neumann functions†. The following experiments were carried out for the purpose of comparing the observed periods of oscillation of certain "chains" with those of "ideal chains" calculated from the expressions giving the periods in terms of these functions.

(A) *Oscillations of a uniform chain.*

The periodic times  $\tau$  of the small "normal" oscillations of a uniform chain of length  $l$ , suspended by one extremity and hanging under the action of gravity, are determined by the equation

$$\tau = (4\pi/\rho)(l/g)^{\frac{1}{2}},$$

where  $\rho$  is a root of the equation  $J_0(z)=0$ . The equation  $J_0(z)=0$  has an infinite number of real positive roots corresponding to the different modes of vibration of the chain. The first root  $\rho_1=2.405$  gives the period when the whole of the chain lies on the same side of its original vertical position; the second root  $\rho_2=5.520$  gives the period when the chain has one node; the third root  $\rho_3=8.654$  gives the period when the chain has two nodes and so on.

In order to compare the calculated results with those obtained by experiment, the times of oscillation of a long chain were observed. A bicycle chain was employed so that the vibrations might be restricted as far as possible to one vertical plane. The observation of the periods presented no

\* Communicated by the Author.

† Gray and Mathews, 'Treatise on Bessel Functions,' p. 14.

difficulty when the chain was vibrating in the first and second modes, but when the chain had three or four nodes, only a limited number of vibrations were executed without assistance, and it was necessary in these cases to maintain the motion by gentle pressure of the hand near the top of the chain. The error thus introduced is however quite small.

Twenty sets of 100 vibrations each were recorded for each mode of oscillation of the chain. The time was measured by means of an accurate stop-watch.

This experiment, which is quite easily performed, is perhaps the simplest example of a physical problem involving the use of Bessel functions.

Value of  $g$  at Morley, Yorks = 981.4 cms./sec.<sup>2</sup>  
 Length of chain = 219.9 cms.

Mode of vibration.	Time of vibration in secs. Calculated.	Time of vibration in secs. Observed.
First (no node).....	2.473	2.470
Second (one node) ...	1.077	1.075
Third (two nodes) ...	.687	.685
Fourth (three nodes).	.504	.504
Fifth (four nodes) ...	.398	.397

(B) *Oscillations of a heterogeneous chain whose line-density varies as the  $n$ th power of the distance from the free end.*

This extension of Bernoulli's problem is due to Prof. Sir Geo. Greenhill. The form of this chain, when executing its principal oscillations, is given by

$$y = Ax^{-\frac{n}{2}} J_n(2b\sqrt{x}) \sin(pct + k),$$

where  $4c^2 = g$ ,  $4b^2 = p^2(n+1)$ ;  $x$  is measured upwards from the free end, and  $y$  is measured horizontally. The fact that the upper end is fixed imposes the condition that

$$J_n(2b\sqrt{l}) = 0.$$

If  $\rho$  be one of the roots of this equation,  $l$  the length of the chain, and  $\tau$  the time of vibration, it is easily shown that

$$\tau = (4\pi/\rho) [(n+1)l/g]^{\frac{1}{2}}. \quad . . . . (1)$$

To realize the conditions of the problem practically, a number of "blinds" were constructed, each consisting of about fifty or sixty wood rods, with square cross-section, sides one cm. long and fixed .25 cm. from one another. The uppermost rod was generally about 40 cms. long. The shape of the "blind" was determined by the curves  $y = \pm cx^n$ , where  $n$  had the values  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{3}{4}$ , 1, &c., the first curve being a parabola, the fourth a triangle, &c. The rods were held in position by means of a string passing tightly into vertical saw cuts at their ends. The whole arrangement was then suspended from two loops on the string above the uppermost rod.

The following observations were made of the times of vibration in the different modes, ten sets of 100 vibrations each being recorded for each mode. For comparison, the values calculated from equation (1) have been added. The roots of the equation  $J_n(z) = 0$  are easily found from the formula given by Prof McMahon.

Mode of vibration.	Value of $n$ .	Length of "blind" in cms.	Time of vibration secs. Calculated.	Time of vibration secs. Observed.
1	$\frac{1}{2}$	107.1	1.618	1.629
2			.809	.817
3			.539	.539
4			.404	.402
1	$\frac{2}{3}$	101.6	1.547	1.552
2			.799	.798
3			.539	.536
4			.407	.404
1	$\frac{3}{4}$	87.9	1.425	1.437
2			.747	.745
3			.507	.504
4			.384	.375
1	1	107.4	1.535	1.538
2			.838	.835
3			.578	.571
4			.441	.437
5			.357	.359
1	$\frac{5}{4}$	86.4	1.342	1.352
2			.758	.758
3			.530	.524
4			.406	.404

(C) *Oscillations of a uniform chain loaded at the free end.*

The Bessel functions  $J_0(z)$ ,  $J_1(z)$ , and  $J_2(z)$ , and the Neumann functions  $Y_0(z)$ ,  $Y_1(z)$ , and  $Y_2(z)$  appear in the expression for the times of vibration of a loaded chain\*. If the load attached to the lowest point of the chain be  $n$  times the mass of the chain, the periods of oscillation in the different modes can be found from the roots of the equation

$$\frac{Y_0(\lambda z)}{J_0(\lambda z)} = \frac{zY_0(z) - 2Y_1(z)}{zJ_0(z) - 2J_1(z)} = \frac{Y_2(z)}{J_2(z)}, \quad \dots \quad (2)$$

where  $z = (4\pi/\tau_s)(nl/g)^{\frac{1}{2}}$ ,  $\lambda = [(n+1)/n]^{\frac{1}{2}}$ ,

and  $\tau_s$  = time of complete vibration in the  $s$ th mode.

A bicycle-chain about 150 cms. long was suspended by one extremity and a load was attached to the other. Through two openings in the lower end of the lowest link of the chain, a steel rod was passed which supported a number of perforated iron disks about  $\frac{1}{3}$  cm. thick. The radii of the disks varied from 1 cm. to 4 cms. The load could by this means be made any multiple or submultiple of the mass of the chain.

Load equal to or greater than the mass of the chain.

Mode of vibration.	Value of $n$ .	Length of chain. cms.	Time of vibration secs. Calculated.	Time of vibration secs. Observed.
1	1	219.9	2.812	2.809
2			.730	.725
1	1	171.4	2.484	2.472
2			.645	.638
1	2	148.8	2.370	2.361
2			.475	.472
1	5	148.8	2.410	2.410
2			.326	.325
1	10	148.8	2.425	2.425
2			.237	.236

\* Routh, 'Advanced Rigid Dynamics,' 1905, p. 405.

Load less than mass of chain.

Mode of vibration.	Value of $n$ .	Length of chain, cns.	Time of vibration. Calculated.	Time of vibration. Observed.
1 2	$\frac{4}{5}$	148.8	2.297 .639	2.274 .638
1 2	$\frac{1}{2}$	do.	2.255 .715	2.246 .715
1 2 3	$\frac{1}{3}$	do.	2.211 .772 .428	2.200 .766 .426
1 2 3	$\frac{1}{8}$	do.	2.128 .856 .506	2.114 .850 .500
1 2 3	$\frac{1}{15}$	do.	2.092 .883 .537	2.076 .877 .531
1 2 3 4	$\frac{1}{24}$	do.	2.072 .890 .557 .386	2.058 .882 .552 .382

(D) The general expression for the roots of the equation

$$\frac{Y_0(\lambda z)}{J_0(\lambda z)} = \frac{zY_0(z) - 2Y_1(z)}{zJ_0(z) - 2J_1(z)} = \frac{Y_2(z)}{J_2(z)},$$

where  $\lambda$  is greater than unity, can be obtained by following the method adopted by Prof. McMahon\* in finding the roots of Bessel and other related functions.

Subtract  $\log 2 - \gamma$  from each side †.

$$\text{Then} \quad \frac{N_0(\lambda z)}{J_0(\lambda z)} = \frac{N_2(z)}{J_2(z)}, \quad \dots \dots \dots (3)$$

where

$$\frac{\pi}{2} N_n(z) = Y_n(z) - (\log 2 - \gamma) J_n(z).$$

Substitute the semiconvergent series for  $J_0(\lambda z)$ ,  $N_0(\lambda z)$ , &c. in equation (3) and put

$$Q_0(\lambda z) = R \sin \theta, \quad P_0(\lambda z) = R \cos \theta \quad \dots \dots (4)$$

$$Q_2(z) = S \sin \eta, \quad P_2(z) = S \cos \eta \quad \dots \dots (5)$$

\* McMahon, *Annals of Mathematics*, 1895.

† Jahnke u. Emde, *Funktionentafeln*, 1907.



We find, after simplification, that

$$\tan(\lambda z - \frac{\pi}{4} + \theta) = \tan(z - \frac{\pi}{4} + \eta);$$

$$\therefore \lambda z - \frac{\pi}{4} + \theta = z - \frac{\pi}{4} + \eta + n\pi$$

$$\text{or } (\lambda - 1)z = n\pi + \eta - \theta. \quad [n = 1, 2, 3 \dots]$$

Writing  $y$  for  $\frac{1}{8z}$ , we get from (4)

$$\tan \theta = -\frac{y}{\lambda} + \frac{33y^3}{\lambda^3} - \frac{6834y^5}{\lambda^5} + \dots,$$

and making use of Gregory's series,

$$\theta = -\frac{1}{8\lambda z} + \frac{100}{3\lambda^3(8z)^3} - \frac{34336}{5\lambda^5(8z)^5} + \dots \quad (6)$$

Similarly 
$$\eta = \frac{15}{8z} - \frac{180}{(8z)^3} - \frac{47520}{(8z)^5} \dots \quad (7)$$

Since 
$$z = \frac{n\pi}{\lambda - 1} + \frac{1}{\lambda - 1}(\eta - \theta),$$

we find, on substituting the values of  $\eta$  and  $\theta$ , and writing  $\beta$  for  $\frac{n\pi}{\lambda - 1}$ ,

$$z = \beta + \frac{15\lambda + 1}{8\lambda(\lambda - 1)z} - \frac{540\lambda^3 + 100}{3\lambda^3 \cdot 8^3 \cdot (\lambda - 1)z^3} - \frac{237600\lambda^5 - 34336}{5\lambda^5 \cdot 8^5 (\lambda - 1)z^5} \dots,$$

an equation of the form

$$z = \beta + \frac{p}{z} + \frac{q}{z^3} + \frac{r}{z^5} + \dots,$$

where 
$$p = \frac{15\lambda + 1}{8\lambda(\lambda - 1)}, \quad q = -\frac{135\lambda^3 + 25}{384\lambda^3(\lambda - 1)},$$

and 
$$r = -\frac{7425\lambda^5 - 1073}{5120\lambda^5(\lambda - 1)}.$$

By Lagrange's theorem

$$z = \beta + \frac{p}{\beta} + \frac{q - p^2}{\beta^3} + \frac{r - 4pq + 2p^3}{\beta^5} \dots$$

Close approximations to the earlier roots are not given by

this series. In these cases, closer values are obtained by interpolation from tables of the  $J_0, J_1, Y_0$  and  $Y_1$  functions.

The following table gives the roots of equation (2) for different values of  $\lambda$ ; the earlier roots were found by interpolation, the higher roots from the general expression given above.

Table of roots of equation (2).

$$\frac{J_0(\lambda z)}{Y_0(\lambda z)} = \frac{J_2(z)}{Y_2(z)}, \lambda > 1.$$

Value of $\lambda$ .	First root.	Second.	Third.	Fourth.
$\sqrt{1.1}$	6.380	65.264	129.587	194.116
$\sqrt{1.2}$	4.540	33.536	66.145	98.956
$\sqrt{1.5}$	2.920	14.581	28.268	42.144
$\sqrt{2.0}$	2.114	8.147	15.441	22.908
1.5	1.905	6.852	12.870	19.055
$\sqrt{3.0}$	1.534	4.837	8.882	13.078
2.0	1.278	3.659	6.594	9.626
3.0	0.813	2.017	3.416	4.906
4.0	0.604	1.431	2.352	3.328
5.0	0.482	1.122	1.794	2.586

LXXXVIII. *An Approximate Theory of an Elastic String vibrating, in its fundamental mode, in a Viscous Medium.*

By JAMES E. IVES, Ph.D., Associate Professor of Physics in the University of Cincinnati\*.

**I**N the theory of the elastic string, usually given, the effect of the internal and external friction is assumed to be so small that it can be neglected. In certain cases, however, this is not permissible. For instance, if the string vibrates in a viscous medium the external friction can no longer be disregarded.

Since the displacement and velocity of any point on the string vary from point to point along it, the system is really one having an infinite number of degrees of freedom and is difficult to treat. An approximate solution may, however, be obtained by making use of its *mean* velocity, and regarding it as a system having only *one* degree of freedom. To do this, we must know the transverse displacement,  $q_x$ , of the

\* Communicated by the Author.

string at any point  $x$  as a function of  $x$ . We are, I think, justified in assuming that, for its fundamental vibration,

$$q_x = q \sin \frac{\pi x}{l},$$

where  $q$  is its displacement at its middle point and  $l$  is its length, since we know that this is true for an undamped string, and *observation* does not show any sensible variation from this form of displacement when the string is vibrating in a viscous medium.

The velocity,  $v_x$ , of any point, will then be given by

$$v_x = v \sin \frac{\pi x}{l},$$

where  $v$  is the velocity at the middle point. The *mean velocity* will therefore be equal to  $\frac{2}{\pi}v$ , and the equivalent momentum of the whole string to  $\frac{2}{\pi}Mv$ , where  $M$  is its mass.

In the same way, the *mean frictional force* will be given by  $\frac{2}{\pi}Rv$ , where  $R$  is the force which would be necessary to overcome the internal and external friction of the whole string if every point of it were moving with unit velocity.

The normal pressure on the string, at any point, tending to bring it back to its position of equilibrium, is equal to  $\frac{\tau}{\rho}$ , where  $\tau$  is the tension to which the string is subjected, and  $\rho$  the radius of curvature of the string at that point. For small curvature,

$$\frac{1}{\rho} = \frac{d^2 q_x}{dx^2} = -\frac{\pi^2}{l^2} q \sin \frac{\pi x}{l}.$$

Therefore, for any point,  $x$ , on the string, the normal pressure is given by

$$-\frac{\tau \pi^2}{l^2} q \sin \frac{\pi x}{l}.$$

The *total force* on the string tending to restore it to its position of equilibrium is equal to

$$-\int_{x=0}^{x=l} \frac{\tau \pi^2}{l^2} q \sin \frac{\pi x}{l} dx = -\frac{2\pi\tau}{l} q.$$

The equation of motion of the string, considered as a

system having only one degree of freedom, is then, for its fundamental vibration, given by

$$\frac{d\left(\frac{2}{\pi} Mv\right)}{dt} + \frac{2}{\pi} Rv + \frac{2\pi\tau}{l} q = 0,$$

or

$$M \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{\pi^2 \tau}{l} q = 0.$$

Its period will be given by

$$T = \frac{2}{\sqrt{\frac{\tau}{Ml} - \frac{R^2}{4\pi^2 M^2}}}$$

If  $R$  is small compared with  $M$ , this reduces to

$$T = 2\sqrt{\frac{Ml}{\tau}}.$$

Writing  $M = ml$ , where  $m$  is the mass of unit length, we have

$$T = 2l\sqrt{\frac{m}{\tau}},$$

which is the well-known formula for the period of an undamped string.

The motion of the middle point of the string will be given by

$$q = q_m e^{-\frac{R}{2M}t} \cos \frac{2\pi}{T}t,$$

where  $q_m$  is its maximum displacement.

The relaxation-time is equal to  $\frac{2M}{R}$ .

The string will cease to vibrate when

$$R = 2\pi\sqrt{\frac{M\tau}{l}} = 2\pi\sqrt{m\tau}.$$

This is the *critical frictional resistance* which will make its motion non-oscillatory. That the motion can be made non-oscillatory can easily be shown by immersing a string made of indiarubber, which vibrates freely in air, in glycerine. In such a viscous medium, when displaced from its position of equilibrium and then released, it no longer oscillates about this position, but simply returns to it.

University of Cincinnati,  
January 1911.

LXXXIX. *Some Problems in the Theory of Probability.*  
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1. **I**N a note at the end of a paper by Prof. Rutherford and Dr. Geiger †, I gave a method of finding the chance that exactly  $n$   $\alpha$ -particles should strike a screen in a given interval of time  $t$ , when the average number  $x$  of  $\alpha$ -particles which strike the screen in an interval of length  $t$  is already known. If the source of  $\alpha$ -particles is kept constant and the value of  $x$  is determined from a very large number of observations, the chance in question is found to be

$$\frac{x^n}{n!} e^{-x}. \quad \dots \quad (1)$$

This law of probability is not new but it is not very well known, and has sometimes been used in a slightly different form. In view of the recent interesting applications of the formula, it may be useful to add a few references to my former note.

In a recent article by R. Greiner, "Ueber das Fehlersystem der Kollektivmasslehre," *Zeitschrift für Mathematik und Physik*, vol. lvii. (1909) p. 150, it is stated that the formula is due to Poisson and is known as the law of probability for rare events. Greiner refers to a treatise by Borkiewitsch, "Ueber das Gesetz der kleinen Zahlen," and considers the question of the correlation of errors when the law is valid.

The formula is usually obtained by a limiting process.

In J. W. Mellor's 'Higher Mathematics for Students of Chemistry and Physics,' 3rd edition, p. 495, the theorem is stated in the following form:—

"If  $p$  denotes the very small probability that an event will happen on a single trial, the probability,  $P$ , that it will happen  $r$  times in a very great number,  $n$ , of trials is

$$P = \frac{(np)^r}{r!} e^{-np}.$$

"Thus if  $n$  grains of wheat are scattered haphazard over a surface  $s$  units of area, the probability that  $a$  units of area will contain  $r$  grains of wheat is

$$\frac{(an)^r}{r!} e^{-\frac{an}{s}}."$$

\* Communicated by the Author.

† *Phil. Mag.* October 1910.

The particular case in which  $r=0$  is well known in the Kinetic Theory of Gases. It was shown in fact by Clausius\* that the chance that a single molecule, moving in a swarm of molecules at rest, will traverse a distance  $x$  without collision, is

$$P=e^{-x/l},$$

where  $l$  denotes the mean free path or the probable length of the free path which the molecule can describe without a collision.

The average number of collisions which occur when a molecule describes a path of length  $x$  may be taken to be equal to  $\frac{x}{l}$ , and so by applying the general formula we find that the chance that the molecule experiences  $n$  collisions while describing a path of length  $x$  is

$$\frac{1}{n!} \left(\frac{x}{l}\right)^n e^{-x/l}.$$

The general formula may also be used in the Kinetic Theory of Gases in quite a different way, as M. von Smoluchowski has shown in an interesting paper published in 1904†.

Imagine a certain volume in a mass of gas to be geometrically but not mechanically bounded, and let the number of molecules which would be contained in this volume in a uniform distribution be  $\nu$ . In consequence of the molecular motion the number will sometimes be greater, sometimes less than this mean value. The chance that exactly  $n$  molecules are present in the volume at a given time is

$$\frac{\nu^n e^{-\nu}}{n!}.$$

The relative momentary deviation  $\delta$  from the mean value  $\nu$  being defined by the equation

$$\delta = \frac{n - \nu}{\nu}.$$

Smoluchowski determines the mean value of all these momentary positive and negative deviations. Assuming

\* Phil. Mag. [4] xvii. p. 81 (1859).

† "Über Unregelmässigkeiten in der Verteilung von Gasmolekülen und deren Einfluss auf Entropie und Zustandsgleichung." Boltzmann *Festschrift*, p. 626.

that the Boyle-Gay Lussac law holds, he finds that when  $\nu$  is large the mean value is given by the equation

$$\bar{\delta} = \sqrt{\frac{2}{\nu\pi}},$$

but that when  $\nu$  is not very large

$$\bar{\delta} = \frac{2\nu^k e^{-\nu}}{k!},$$

where  $k$  denotes the largest integer which is not greater than  $\nu$ .

If Boyle's law does not hold, then for large values of  $\nu$

$$\bar{\delta} = \sqrt{\frac{2}{\nu\pi}} \sqrt{\frac{\beta}{\beta_0}},$$

where  $\beta$  is the true compressibility and  $\beta_0$  the compressibility derived from Boyle's law.

These formulæ have been applied by The Svedberg\* to the study of colloidal solutions. He finds that in great dilution the distribution of particles corresponds very exactly to the theory and that Boyle's law is practically exact for dilute solutions.

2. Having indicated some of the known applications of the formula, we now proceed to a few developments which may perhaps be useful in the future. Consider first the case of a number of particles which carry either a positive or negative unit charge. If the average number of these particles which are present within the given volume is  $\nu$ , what is the chance that at any given time the volume contains a total charge  $r$  on account of the presence of particles of these types?

This problem is analogous to one considered by Whetham in an electrical theory of coagulation, 'Theory of Solution,' p. 396. In Whetham's problem, however, the electric charges are supposed to be all of one sign, and the probability is calculated from a different point of view with the result that

Poisson's law  $\frac{\nu^n}{n!} e^{-\nu}$  is replaced by the simpler law  $(A\nu)^n$

where  $A$  is a constant.

If we suppose that positive and negative charges are equally likely to be present, then the chance that a group of

\* "Eine neue Methode zur Prüfung der Gültigkeit des Boyle-Gay-Lussacschen Gesetz für kolloide Lösungen." *Zeitschr. für Phys. Chemie*, Bd. lxxiii. Heft 5, p. 547 (1910).

$n$  particles has a total charge  $r$  is zero if  $n-r$  is odd, and equal to

$$\frac{n!}{\left(\frac{n-r}{2}\right)! \left(\frac{n+r}{2}\right)!} \frac{1}{2^n}$$

if  $n-r$  is even. The chance is in fact the coefficient of  $t^r$  in

$$\left(\frac{t}{2} + \frac{1}{2t}\right)^n.$$

Now we have seen that if  $\nu$  is the probable number of particles in the given volume, the chance that at a given time there are exactly  $n$  particles is

$$\frac{\nu^n}{n!} e^{-\nu}.$$

Hence the chance that the volume contains a total charge equal to  $r$  units is the coefficient of  $t^r$  in the expansion

$$\sum_{n=0}^{\infty} \frac{\nu^n}{n!} \left(\frac{t}{2} + \frac{1}{2t}\right)^n e^{-\nu};$$

that is in the expansion of the function

$$e^{\frac{\nu}{2} \left(t + \frac{1}{t}\right) - \nu}.$$

Now if we use the notation employed by Basset\*, we may write

$$e^{\frac{\nu}{2} \left(t + \frac{1}{t}\right) - \nu} = \sum_{r=-\infty}^{\infty} t^r I_r(\nu). \quad \dots \quad (2)$$

Hence the chance of getting a total charge of  $r$  units is represented by

$$e^{-\nu} I_r(\nu). \quad \dots \quad (3)$$

The probable value of  $r$  is clearly zero, but we may find the probable value of  $r^2$  by summing the series

$$\sum_{r=-\infty}^{\infty} r^2 e^{-\nu} I_r(\nu).$$

\* 'Hydromechanics,' vol. ii.



To do this we differentiate equation (2), this gives

$$\frac{\nu}{2} \left(1 - \frac{1}{t^2}\right) e^{\frac{\nu}{2} \left(t + \frac{1}{t}\right)} = \sum_{r=-\infty}^{\infty} r t^{r-1} I_r(\nu).$$

Therefore

$$\frac{\nu}{2} \left(t - \frac{1}{t}\right) e^{\frac{\nu}{2} \left(t + \frac{1}{t}\right)} = \sum_{r=-\infty}^{\infty} r t^r I_r(\nu).$$

Differentiating again and putting  $t=1$ , we get

$$\nu e^{\nu} = \sum_{r=-\infty}^{\infty} r^2 I_r(\nu).$$

Hence the probable value of  $r^2$  is  $\nu$ .

A somewhat similar result is obtained in Rayleigh's 'Sound,' vol. i. p. 36, where it is shown that if  $n$  unit vectors whose signs can be either positive or negative, are combined so as to give a resultant of magnitude  $r$ , then the probable value of  $r^2$  is  $n$ .

3. To find the most probable value of  $r$  we take the recurrence formula

$$I_{n-1}(\nu) - I_{n+1}(\nu) = \frac{2n}{\nu} I_n(\nu).$$

Since  $I_n(\nu)$  is always positive, we have

$$I_{n-1} > I_{n+1}, \quad n > 0.$$

Also

$$I_n'(\nu) = \frac{1}{2} [I_{n-1} + I_{n+1}];$$

$$\therefore I_n'(\nu) < I_{n-1}(\nu) \quad \text{and} \quad I_n'(\nu) > I_{n+1}(\nu).$$

Consequently,

$$\frac{I_{n-1}(\nu)}{I_n(\nu)} > \frac{I_n'(\nu)}{I_n(\nu)} > \frac{I_{n+1}(\nu)}{I_n(\nu)}.$$

These inequalities show that if  $\frac{I_n'(\nu)}{I_n(\nu)} \geq 1$ ,

then

$$I_{n-1}(\nu) > I_n(\nu);$$

while if  $\frac{I_n'(\nu)}{I_n(\nu)} \leq 1$ , we have

$$1 > \frac{I_n'(\nu)}{I_n(\nu)} > \frac{I_{n+1}(\nu)}{I_n(\nu)};$$

and so

$$I_n(\nu) > I_{n+1}(\nu).$$

We may conclude from these inequalities that

$$I_1, I_2, I_3, \dots$$

form a decreasing set of quantities; we have finally to find whether

$$I_0(\nu) \gtrless I_1(\nu).$$

Now it follows at once from the equations \*

$$I_0(\nu) = \frac{1}{\pi} \int_0^\pi \cosh(\nu \cos \phi) d\phi,$$

$$I_1(\nu) = \frac{1}{\pi} \int_0^\pi \sinh(\nu \cos \phi) \cos \phi d\phi,$$

that

$$I_1(\nu) < I_0(\nu);$$

for clearly

$$\sinh(\nu \cos \phi) \cos \phi < \cosh(\nu \cos \phi)$$

for all values of  $\phi$ . Hence  $r=0$  is the most probable value of  $r$ .

4. It is also of some interest to find the value of  $\nu$  for which the chance

$$e^{-\nu} I_r(\nu)$$

is a maximum when  $r$  is given. To do this we have to solve the equation

$$I_r'(\nu) = I_r(\nu).$$

Putting  $X_n = \frac{I_n'(\nu)}{I_n(\nu)}$ , we easily find from the recurrence formulæ

$$I_n' + \frac{n}{\nu} I_n = I_{n-1},$$

$$I'_{n-1} - \frac{n-1}{\nu} I_{n-1} = I_n,$$

that

$$\left(X_n + \frac{n}{\nu}\right) \left(X_{n-1} - \frac{n-1}{\nu}\right) = 1.$$

\* Cf. Whittaker's 'Analysis,' p. 307.

Hence if  $X_n=1$ ,  $X_{n-1}<1$ , we have

$$1 - \frac{n-1}{\nu} > \frac{1}{1 + \frac{n}{\nu}};$$

$$\therefore \frac{1}{\nu} - \frac{n(n-1)}{\nu^2} > 0;$$

$$\therefore \nu > n(n-1).$$

On the other hand, if we assume

$$X_n=1, \quad X_{n+1} > 1,$$

the equation

$$\left(X_{n+1} + \frac{n+1}{\nu}\right) \left(X_n - \frac{n}{\nu}\right) = 1$$

gives

$$\nu < n(n+1).$$

This suggests that the value of  $\nu$  for which  $X_n(\nu)=1$  lies between  $n(n-1)$  and  $n(n+1)$ .

By using Lodge's tables\* for the functions  $I_r(\nu)$  I have been able to make a rough estimate of the position of the roots of  $X_1(\nu)=1$  and  $X_2(\nu)=1$ . The values found are

$$1.58 \text{ for } X_1(\nu)=1,$$

$$4.58 \text{ for } X_2(\nu)=1.$$

The ratio of the two numbers is thus about three.

It was thought that the ratio might turn out to be the ratio of the atomic weights of two elements such as hydrogen and helium, whose atoms sometimes carry charges of one and two units respectively. This idea was based on the supposition that the average number of particles required for the formation of a particular atom may be such as to make the chance of getting a given valency charge as large as possible †. The idea need not be altogether abandoned because we have left out of account the occurrence of neutral particles (*e. g.* doublets) and particles carrying more than one unit charge. This brings us to the consideration of a more general problem.

\* British Association Reports (1889). To solve  $X_r(\nu)=1$  we must find a value of  $\nu$  such that  $I_r = \frac{1}{2}(I_{r-1} + I_{r+1})$ . The tables do not go far enough to enable me to carry the calculations any further.

† It should be noticed, however, that the probable number of particles  $\nu$  is equal to the probable value of  $r^2$ .

5. We may show by a simple extension of the previous method that if  $z$  is the chance of a particle having no charge,  $x_p$  the chance that it carries a charge of  $p$  positive units,  $y_p$  the chance that it carries a charge of  $p$  negative units, then the chance of getting a total charge of  $r$  units within a certain volume at a given time is equal to the coefficient of  $t^r$  in the expansion of

$$e^{\nu[z-1+x_1t+y_1t^{-1}+x_2t^2+y_2t^{-2}+\dots]}.$$

where  $\nu$  is the average number of particles within the given volume. In the case of an atom or molecule the volume may be taken to be the probable sphere of influence of the atom or molecule.

The probable value of  $r$  is easily found to be

$$\nu[x_1 - y_1 + 2(x_2 - y_2) + 3(x_3 - y_3) + \dots],$$

and the probable value of  $r^2$  is

$$\begin{aligned} &\nu[x_1 + y_1 + 2^2(x_2 + y_2) + 3^2(x_3 + y_3) + \dots] \\ &+ \nu^2[(x_1 - y_1) + 2(x_2 - y_2) + \dots]^2. \end{aligned}$$

If we call  $W_r$  the chance of getting a total charge of  $r$  units, it is easy to see that  $W_r$  is a solution of partial differential equations of the type

$$\begin{aligned} \frac{\partial^2 V}{\partial x_1 \partial y_1} &= \frac{\partial^2 V}{\partial z^2}, \\ \frac{\partial^2 V}{\partial x_1^2} &= \frac{\partial^2 V}{\partial x_2 \partial z}, \quad \&c. \end{aligned}$$

In the simple case when  $x_2 = x_3 = \dots = y_2 = y_3 = \dots = 0$  the value of  $W_r$  is easily found to be

$$W_r = \left(\frac{y_1}{x_1}\right)^r e^{-\nu} I_r(2\nu\sqrt{x_1 y_1});$$

the probable value of  $r$  is  $\nu(x_1 - y_1)$ , and the probable value of  $r^2$  is

$$\nu(x_1 + y_1) + \nu^2(x_1 - y_1)^2.$$

If the probable value of  $r^2$  is known the value of  $\nu$  may be derived from this formula, or if  $\nu$  be also known and  $x_1 = y_1$ , the formula may be used to estimate the value of  $x_1$ .

XC. *On the Question of Valency in Gaseous Ionization.* By R. A. MILLIKAN, *Professor of Physics in the University of Chicago,* and HARVEY FLETCHER, *Fellow in Physics in the University of Chicago*.\*

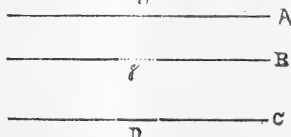
§ 1. *Historical Survey.*

UP to the year 1908 the only experiments which threw any light whatever upon the question of valency in gaseous ionization were those made by Townsend† in 1900. He found that the value of  $ne$  obtained from measurements on ionized gases was identical with the value of  $nE$  obtained from measurements on the deposition of the products of electrolysis. In the foregoing expressions  $n$  denotes the number of molecules in 1 c.c. of gas at  $15^{\circ}$  C. 76 cm. pressure,  $e$  the mean charge carried by an ion in an ionized gas, and  $E$  the charge carried by a univalent ion in electrolytic conduction. The mean value of this product came out in both cases  $1.23 \times 10^{10}$  electrostatic units, although in the measurements on gases the individual experiments gave differences from the mean as large as 30 per cent. The values of  $ne$  were obtained from the equation  $ne = \frac{K}{D} P$ , in which  $K$  denotes the "mobility" of the gaseous ion,  $D$  its coefficient of diffusion, and  $P$  normal atmospheric pressure. Townsend's work consisted simply in the measurement of  $D$ , the value of  $K$  having been taken from the work of previous observers.

In view of these results it has been customary for the past ten years to assume that all gaseous ions are univalent. In 1908, however, Townsend‡ devised a method of measuring directly the ratio  $\frac{K}{D}$  and revised his original conclusions.

His method consisted essentially in driving ions by means of an electric field from the region between two plates A and B (fig. 1), where they had been produced by the direct action

Fig. 1.



of X rays, through the gauze  $g$ , and observing what fraction of these ions was driven by a field established between the plates B and C to the central disk D, and what fraction drifted by virtue of diffusion to the guard-ring C.

\* Communicated by the Authors.

† J. S. Townsend, *Phil. Trans. A.* vol. cxiii. p. 129 (1900).

‡ J. S. Townsend, *Proc. Roy. Soc. A.* vol. lxxx. p. 207 (1908).

By this method Townsend found that  $ne$  for the negative ions was accurately  $1.23 \times 10^{10}$ , but for the positive ions it was  $2.41 \times 10^{10}$ . From these results the conclusion was drawn that in X-ray ionization all of the positive ions are bivalent, *i. e.*, presumably, that the act of ionization by X rays consists in the detachment from a neutral molecule of two elementary electrical charges.

Townsend accounted for the fact that his early experiments had not shown this high value of  $ne$  for the positive ions by the assumption that by the time the doubly charged positive ions in these experiments had reached the tubes in which D was measured, most of them had become singly charged through drawing to themselves the singly charged negative ions with which they were mixed. This hypothesis found some justification in the fact that in the early experiments the mean value of  $ne$  for the positive ions had indeed come out some 15 per cent. or 20 per cent. higher than  $1.23 \times 10^{10}$ —a discrepancy which had at first been regarded as attributable to experimental errors, and which in fact might well be attributed to such errors in view of the discordance between the observations on different gases.

Franck and Westphal\*, however, have recently redetermined  $ne$  by a slight modification of Townsend's original method, measuring both K and D independently, and have not only found exactly the same value of D when the positive and negative ions are separated by means of an electric field so as to render such recombination as Townsend suggested impossible as when they were not so separated, but they have also found  $ne$  for the positive ions produced by X rays but  $1.4 \times 10^{10}$  instead of  $2.41 \times 10^{10}$ . Since this is in fair agreement with Townsend's original mean, the authors conclude that only a small fraction—about 9 per cent.—of the positive ions formed by X rays are doubles, or other multiples, and the rest singles. The authors attempt to reconcile the discrepancy between their result and Townsend's later finding by pointing out that the singly charged positives formed between A and B (fig. 1) would have twice the diffusion coefficient of the doubly charged positives, and would therefore largely be lost to the wires of the gauze  $g$  in passing through it, while a much larger proportion of the slowly diffusing positives would pass through the meshes. In other words, the gauze would strain out the doubles from the mixture by the process of fractional diffusion. In support of this point of view they placed several very narrow-meshed pieces of gauze in the path of the ions which were moving

\* J. Franck and W. Westphal, *Verh. D. Phys. Ges.* März 5, 1909.

to the diffusion tubes, and found, indeed, that when the current of air was made exceeding slow, the diffusion coefficient of the ions which got through became smaller as the number of sheets of gauze was increased. Thus they found without the insertion of gauze  $D=0.029$ ; with one sheet of thick gauze  $D=0.020$ ; with three sheets  $D=0.0175$ .

It is to be noted, however, that these experiments actually show only that some of the ions in the ionized gas had smaller diffusion coefficients than others. They do not show directly that the cause of this difference in  $K$  is a difference in the charge carried. Indeed, in later experiments\* Franck and Westphal find that the diffusion coefficient of the negative as well as the positive ions produced by point discharge decreases to half value upon the reduction of the speed of the air current or the insertion of gauze, and in this case they do not draw the conclusion of double charges, but merely infer the presence of a certain number of Langevin's large, slowly-diffusing, ions †. It is obvious, then, that the assumption of doubly charged ions is not necessitated by any of Franck and Westphal's results. Indeed, it seems altogether likely, from Langevin's and Pollock's work ‡, that a certain number of large or "intermediate" ions may have been present in their experiments with X rays. In their experiments on the ionization produced by  $\alpha$  rays,  $\beta$  rays, and  $\gamma$  rays, they themselves find no evidence whatever for the existence of doubly charged ions.

These experiments of Franck and Westphal's, while in good agreement with Townsend's original work, do not seem to us to be easily reconcilable with his later experiments, since they certainly make untenable his explanation of the cause of the difference between his earlier and later findings, and yet do not seem to us to replace it by a thoroughly adequate one, for it does not appear likely that Townsend's one sheet of gauze could have produced such complete separation of the doubly and singly charged ions as he found. The two sets of experiments are also difficult to reconcile in that Franck and Westphal find that the hardness of the rays has no effect upon the proportion of doubles and singles, while Townsend § in a later paper finds that very soft X rays produce singles while hard ones produce only doubles.

This later paper deals with the action of secondary X rays

\* J. Franck and W. Westphal, *Verh. D. Phys. Ges.* Juli 2, 1909.

† Langevin, *C. R.* 1140. p. 232 (1905).

‡ Pollock, 'Science,' vol. xxix. p. 919 (1909).

§ T. Townsend, *Proc. Royal Soc.* vol. lxxxii. p. 464 (1908).

and reports experiments by his later method, in which, when the secondary radiator was a brass plate covered with a coat of vaseline, singles only were formed, while when the plate was freshly cleaned a considerable proportion of the ions were doubles. This is explained by the assumption that the vaseline absorbs the very soft secondaries which are alone responsible for the singles. According, however, to Hazelfoot's\* observation Townsend's later method does not appear to yield very consistent results for varying values of the field between B and C, the value of  $ne$  for the positives formed by radium coming out 1.37 for a 4 volt P.D. between B and C, and 1.24 for a 2 volt P.D.

In summarizing then the work of all preceding observers in this field it may be said that although both Townsend and Franck and Westphal draw the conclusion that doubly charged ions exist in gases ionized by X rays, there are such contradictions and uncertainties in their work as to leave the question still an open one. In gases ionized by other agencies than X rays no one has yet found any evidence for the existence of ions carrying more than a single charge.

§ 2. *A study of the act of ionization by primary X rays through the capture upon oil drops of the products of the ionization.*

The method of studying gaseous ionization recently devised by one of us† seemed capable of furnishing a direct and unmistakable answer to the question as to whether or not the phenomenon of valency exists at all in gaseous ionization, and if so to what extent. The conclusion to which we come as a result of a number of months of study of the subject, is that under no circumstances which we have as yet been able to investigate does the act of ionization of air consist in the detachment of more than one elementary charge from a neutral molecule. This conclusion is contrary to an opinion expressed by one of us in the first report made upon the catching of individual ions by oil drops, for although it was stated in this report that "the results already obtained demonstrated that the great majority of atmospheric ions of both positive and negative sign carried but one single elementary electrical charge," nevertheless the statement was added that strong evidence had been obtained that doubles and triples were in occasional instances formed. The evidence here referred to was found in the fact that although

\* C. E. Hazelfoot, Proc. Royal Soc. vol. lxxxii. p. 18 (1909).

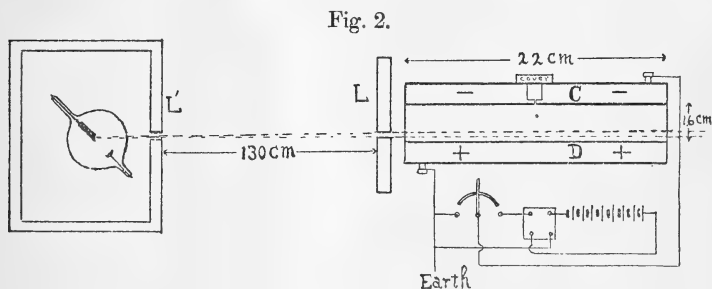
† Millikan, 'Science,' Sept. 30, 1910. See also Phys. Review, April 1911, for a more detailed account of this work.



the vast majority of all changes of charge which the oil drops experienced corresponded to the advent upon the drop under observation of a single elementary electrical charge, a few changes were observed, *even when the field was on*, which corresponded to the advent of two or three such charges.

It was of course recognized from the first that it is very difficult to distinguish between the practically simultaneous advent upon a drop of two or three separate ions and the advent of a doubly or trebly charged ion, but a consideration of the frequency with which ions were being formed in the experiments under consideration seemed to render it improbable that the few double or treble catches observed when the field was on could represent the simultaneous advent of separate ions. It was obvious, however, that the question could be conclusively settled by working with smaller and smaller drops; for the proportion of double to single catches made in a field of strength between say 1000 and 6000 volts per centimetre should be independent of the size of the drops if the doubles are due to the advent of doubly charged ions, while this proportion should decrease with the square of the radius of the drop if the doubles are due to the simultaneous capture of separate ions.

Accordingly, in the work herein described, we suspended by the method detailed in the papers referred to, a very small positively charged drop in the upper part of the field between C and D (fig. 2), adjusting either the charge upon



the drop or the field strength until the drop was nearly balanced. We then produced beneath the drop a sheet of X-ray ionization. With the arrangement shown in the figure, in which C and D are the plates of the condenser previously described, and L and L' are thick lead screens, the positive ions are thrown practically at the instant of formation to the upper plate. When one of them strikes the drop it increases the positive charge upon it, and the

amount of the charge added by the ion to the drop can be computed from the observed change in the speed of the drop.

For the sake of convenience in the measurement of successive speeds, a scale containing 70 equal divisions was placed in the eyepiece of the observing cathetometer telescope, which in these experiments produced a magnification of about 15 diameters. The method of procedure was in general to first get the drop nearly balanced by shaking off its initial charge by holding a little radium near the observing chamber, then, with a switch, to throw on the X rays until a sudden start in the drop revealed the fact that an ion had been caught, then to throw off the rays and take the time required for it to move over 10 divisions, then to throw on the rays until another sudden quickening in speed indicated the capture of another ion, then to measure this speed, and to proceed in this way without throwing off the field at all until the drop got too close to the upper plate, when the rays were thrown off and the drop allowed to fall under gravity to the desired distance from the upper plate. In order to remove the excess of positive charge which the drop now had because of its recent captures, some radium was brought near the chamber and the field thrown off for a small fraction of a second. As explained in the preceding papers, ions are caught by the drop many times more rapidly when the field is off than when it is on; also, as heretofore explained, negatives are caught more easily than positives. Hence it was in general an easy matter to bring the positively charged drop back to its balanced condition, or indeed to any one of the small number of working speeds which it was capable of having, and then to repeat the series of catches described above. In this way we kept the same drop under observation for hours at a time and in one instance we recorded 100 successive captures of ions by a given drop, and determined in each case whether the ion captured carried a single or a double charge.

The process of making this determination is exceedingly simple and very reliable. For, since electricity is atomic in structure, there are only, for example, three possible speeds which a drop can have when it carries 1, 2, or 3 elementary charges, and it is a perfectly simple matter to adjust conditions so that these speeds are of such different values that each one can be unfailingly recognized even without a stopwatch measurement. Indeed, the fact that electricity is atomic is in no way more beautifully shown than by the way in which in the following tables these relatively few possible

working speeds recur. After all the possible speeds have been located it is only necessary to see whether one of them is ever skipped in the capture of a new ion, in order to know whether or not that ion was a double. Table I. represents the results of experiments made with very hard X rays produced by means of a powerful 12-inch Scheidel coil, a mercury-jet interrupter, and a Scheidel tube whose equivalent spark-length was about 5 inches. The radius  $a$  of the drop is computed, as explained in previous papers, from the modified form of Stokes's law

$$v_1 = \frac{2}{9} \frac{ga^2(\sigma - \rho)}{\mu} \left(1 + A \frac{l}{a}\right).$$

The absolute values of the charges carried by the drops have been computed as in the preceding papers and agree in every case, within the limits of observational error (4 or 5 per cent.), with the value of  $E$  previously found, viz.  $4.891 \times 10^{-10}$ . However, no attempt has been made in these experiments to make precise determinations of speed since a high degree of accuracy of measurement was not necessary for the purpose for which the investigation was undertaken. We are here only concerned with the relative values of the charges carried by a given drop after the capture of one or more ions from the air. These relative values could be determined, if desirable, with a very high degree of accuracy. For the equation

$$e_n = \frac{mg}{Fv_1} (v_1 + v_2)^*$$

shows that for a given drop the charge is proportional simply to  $(v_1 + v_2)$ . Furthermore, these relative values are quite independent of errors due to convection currents, or other constant disturbing forces; for  $v_1$  is simply the velocity under the joint action of all the outside forces before the field is thrown on, and  $v_2$  is the velocity under the action of all these forces plus that of the field. Whatever then the size of the drop it is only necessary, for our present purpose, to determine the greatest common divisor of all the observed values of  $v_1 + v_2$ . The number of elementary charges upon the drop is then the value of  $(v_1 + v_2)$  under consideration divided by this greatest common divisor. With most of the drops actually worked with this greatest common divisor was one of the observed values of  $(v_1 + v_2)$ , so that the differences were so great that no high degree of accuracy was necessary in order to be very certain of each number of the multiple

\* *Loc. cit.*

series of speeds. For convenience of observation, therefore, the following readings were all made with a stop-watch rather than with a chronograph. The observational errors in the timings may be two or three tenths of a second. Table I. is a good illustration of the character of the observations. The time of fall under gravity recorded in the

TABLE I.

Plate Distance 1.6 cm.				Distance of Fall .0975 cm.			
Volts 1015. Temperature 23° C.				Radius of Drop .000063 cm.			
<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
19.0	100.0	1 P	1 P	20.0	10.0*	1 N	1 P
	16.0	2 P	1 P		20.0*	0	1 P
	8.0	3 P			100.0	1 P	1 N
20.0	16.0	2 P	1 P	20.0*	0	1 P	
	8.0	3 P		100.0	1 P	1 P	
				16.0	2 P	1 P	
				8.0	3 P	1 P	
	100.0	1 P	1 P				
	17.0	2 P	1 P	104.0	1 P		1 P
	8.2	3 P	1 P	15.0	2 P		1 P
	6.0	4 P	1 P	9.0	3 P		1 P
				6.0	4 P		
		7.0*	2 N	1 P			
	9.8*	1 N	1 N	6.5*	2 N		1 P
	7.0*	2 N		10.0*	1 N		1 P
21.0	20.0*	0	1 P	20.0*	0		1 P
	95.0	1 P	1 P	100.0	1 P		1 P
	16.5	2 P	1 P	15.5	2 P		1 P
	8.0	3 P	1 P	8.0	3 P		1 P
	6.0	4 P	1 P	6.0	4 P		1 P
	100.0	1 P	1 P	100.0	1 P		1 P
	16.0	2 P	1 P	16.5	2 P		
	8.4	3 P					
				20.0*	0		1 P
20.0	100.0	1 P	1 P	100.0	1 P		1 P
	16.0	2 P	1 P	16.6	2 P		1 P
	8.4	3 P	1 P	8.8	3 P		1 P
				5.7	4 P		1 P
	10.0*	1 N	1 P				
	20.0*	0	1 P	100.0	1 P		1 N
	100.0	1 P	1 P	20.0*	0		1 N
	16.0	2 P	1 P	10.0*	1 N		1 P
				20.0*	0		
100.0	1 P	1 P	44 catches, all singles.				
16.0	2 P	1 P					
8.0	3 P	1 P					

column headed "g" varies slightly both because of observational errors and because of Brownian movements\*. Under the column F are recorded the various observed values of the times of rise through 10 divisions of the scale in the eyepiece. A star after an observation in this column signifies that the drop was moving *with* gravity instead of *against* it. The procedure was in general to start with the drop either altogether neutral (so that it fell when the field was on with the same speed as when the field was off) or having one single positive charge, and then to throw on positive charges until its speed came to the 6.0 second value, then to make it neutral again with the aid of radium and to begin over.

It will be seen from Table I. that in 4 cases out of 44 we caught negatives, although it would appear from the arrangement shown in fig. 2 that we could catch only positives. These negatives are doubtless due to secondary X rays which radiate in all directions from the air molecules when these are subjected to the primary X ray radiation. The *smallness of the number of negatives so caught shows conclusively that the greater part of the ionization of a gas by X rays is due to the direct action of the primary rays.*

Towards the end of Table I. is an interesting series of catches. The drop was as at the beginning of this series charged with 2 negatives which produced a speed in the direction of gravity of 6.5 seconds. It caught in succession six single positives before the field was thrown off. The corresponding times were 6.5\*, 10\*, 20\*, 100, 15.5, 8.0, 6.0. The mean time during which the X rays had to be on in order to produce a "catch" was in these experiments about six seconds, though in some instances it was as much as a minute. The majority of the times recorded in column F were actually measured with a stop-watch as recorded, but since there could be no possibility of mistaking the 100 second speed it was observed only four or five times. It will be seen from Table I. that *out of 44 catches of ions produced by very hard X rays there is not a single double.*

Table II. is even more convincing than is Table I. This drop was held under observation for about 4 hours and 100 different catches were observed, every one of which was a single. The rays were somewhat softer than those used in obtaining Table I., and corresponded to a spark distance of about 2 inches.

\* Millikan and Fletcher, *Phys. Zeit.* xii. pp. 161-3 (1911).

TABLE II.

Plate Distance 1.6 cm.				Distance of Fall .0941 cm.			
Volts 1970. Temperature 22°·5 C.				Radius of Drop .00007 cm.			
<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
12.6	12.4*	0		12.0	12.4*	0	
12.0	25.5	1 P	1 P		26.0	1 P	1 P
	12.4*	0			12.4*	0	1 N
	25.5	1 P	1 P		5.0*	1 N	1 N
12.2	6.6	2 P	1 P		12.4*	0	1 P
	12.4*	0			26.5	1 P	1 P
	25.0	1 P	1 P		6.8	2 P	1 P
12.7	6.6	2 P	1 P		12.4*	0	1 P
	12.4*	0			25.0	1 P	1 P
	25.5	1 P	1 P		6.8	2 P	1 P
12.3	6.6	2 P	1 P		25.0	1 P	1 P
12.6	25.5	1 P	1 P		6.8	2 P	
	6.5	2 P	1 P		24.8	1 P	1 P
12.2	26.5	1 P	1 P	12.5	6.3	2 P	
12.8	6.5	2 P	1 P		25.8	1 P	1 N
12.2	12.4*	0			12.4*	0	1 P
12.8	25.3	1 P	1 P		25.0	1 P	1 P
13.0	6.4	2 P	1 P		6.4	2 P	1 P
	25.5	1 P	1 P		25.0	1 P	1 P
	6.4	2 P	1 P		6.7	2 P	
	26.5	1 P	1 P		25.0	1 P	1 P
	6.4	2 P	1 P		6.8	2 P	
12.3	12.4*	0			25.0	1 P	1 N
	25.0	1 P	1 P		12.4*	0	1 P
	6.5	2 P	1 P		25.0	1 P	1 N
	25.0	1 P	1 P		12.4*	0	1 P
	6.5	2 P	1 P		25.0	1 P	1 N
	12.4*	0			12.4*	0	1 P
	27.0	1 P	1 P		25.5	1 P	1 P
	6.7	2 P	1 P		6.6	2 P	
	3.8	3 P	1 P		12.4*	0	1 P
	12.4*	0			25.0	1 P	1 N
	25.0	1 P	1 P		12.4*	0	
	6.6	2 P	1 P		25.0	1 P	1 P
	3.7	3 P	1 P		6.6	2 P	
				12.4	25.0	1 P	1 P
					6.6	2 P	

TABLE II. (continued).

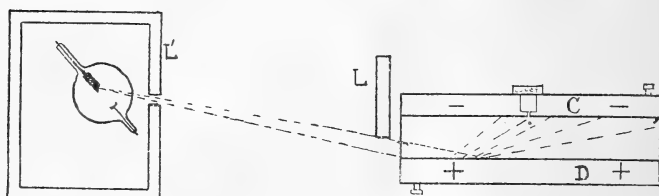
<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
12.3	25.0	1 P	1 P	12.0	12.4*	0	1 P
	6.6	2 P	1 P		25.0	1 P	1 P
	3.8	3 P		13.0	6.8	2 P	1 P
			4.0		3 P		
	25.0	1 P	1 P		12.4*	0	
	6.6	2 P		12.5	25.0	1 P	1 P
					6.3	2 P	1 P
12.4	12.4*	0	1 P		12.4*	0	
	25.0	1 P	1 N	12.7	27.0	1 P	1 P
	12.4*	0	1 P		6.2	2 P	1 P
	25.0	1 P	1 P				
6.3	2 P	1 P	12.4*		0		
	4.0	3 P		5.0*	1 N	1 N	
11.9	5.0*	1 N	1 P		12.4*	0	
	12.4*	0	1 P	12.7	25.0	1 P	1 P
	25.0	1 P	1 P		6.6	2 P	1 P
	6.8	2 P					
11.9	5.0*	1 N	1 P		25.0	1 P	1 P
	12.4*	0	1 P		6.6	2 P	
12.2	25.0	1 P	1 P		12.4*	0	
	6.6	2 P		12.4	25.0	1 P	1 P
					6.8	2 P	1 P
12.4	25.0	1 P	1 P		25.0	1 P	1 P
	6.6	2 P		12.0	6.7	2 P	1 P
					25.0	1 P	1 P
	12.4*	0	1 P		6.5	2 P	
	25.0	1 P	1 P		25.0	1 P	1 P
	6.6	2 P		12.8	6.4	2 P	1 P
	5.0*	1 N	1 P	12.4	25.0	1 P	1 P
	12.4*	0	1 P		6.4	2 P	1 P
	26.0	1 P	1 P		3.6	3 P	
12.0	6.7	2 P	1 P		5.0*	1 N	1 P
	3.4	3 P			2.4*	0	1 N
					5.0*	1 N	
	5.0*	1 N	1 P				
12.8	12.4*	0	1 P				
	27.0	1 P	1 P				
	6.7	2 P					
12.6	12.4*	0	1 P				
	27.0	1 P	1 P				
	6.7	2 P					
12.1	12.4*	0	1 P				
	27.0	1 P	1 P				
	6.8	2 P					
12.7	24.6	1 P	1 N				
	12.4*	0	1 P				
	25.0	1 P	1 P				
	6.6	2 P	1 P				
	3.7	3 P	1 P				

100 catches, all singles.

## § 3. Ionization by Secondary X rays.

Table III. represents observations taken on the ionizing effect of secondary X rays under conditions as nearly as possible like those described by Townsend when he found that the positives produced by secondaries were chiefly doubles. Fig. 3 shows the arrangement of the apparatus.

Fig. 3.



The plate D was covered by a thin coat of oil and only the left-hand portion of it could be hit by the primary X rays. Despite the fact that the secondary rays from the plate D traverse the air on all sides of the drop, we succeeded in catching chiefly positives by holding the drop always within 2 or 3 millimetres of the plate C. We had hoped to clear up the apparent contradictions in preceding work on this subject by finding that homogeneous secondary X rays do in fact produce double positives, but it will be seen from Table III. that out of 84 catches there were but 3 which could possibly correspond to multiples of any kind. Of these 3 catches the two followed by an interrogation mark unfortunately happened when Millikan, who was observing at this time, had glanced away to read the stop-watch, and hence may have corresponded to two or three separate catches following in rapid succession. The third catch marked "good" appeared to the observer (Fletcher) to correspond to a change in speed which happened all at once rather than in two successive steps. Nevertheless the changes were happening here, when the field was on, at about one or two second intervals, instead of six second intervals as in the work recorded in Tables I. and II. It is not at all permissible, therefore, to draw the conclusion that this catch corresponded to the advent of a double ion, rather than to the nearly simultaneous advent of two separate single ions. The legitimate conclusion which can be drawn from this table is that if doubles were formed at all by the secondary rays here used, their numbers were certainly exceedingly small in comparison with the number of singles formed.



TABLE III.

Plate Distance 1.6.				Distance of Fall .0744 cm.				
Volts 1750. Temperature 24° C.				Radius of Drop .000095 cm.				
<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	
7.5	bal.† 5.4	2 P 3 P	1 P	6.8	7.2* 18* bal.	0 1 P 2 P	1 P 1 P	
7.4	bal. 18* 5.5	2 P 1 P 4 P	1 N 3 P(?)		9.0 5.0	3 P 4 P	1 P	
	18* bal. 9.0	1 P 2 P 3 P	1 P 1 P		18* bal. 9.0 5.5	1 P 2 P 3 P 4 P	1 P 1 P 1 P	
	18* bal. 9.0 5.0	1 P 2 P 3 P 4 P	1 P 1 P 1 P	6.9	bal. 9.0	2 P 3 P	1 P	
	18* bal. 9.0	1 P 2 P 3 P	1 P 1 P	6.9	7.2* 18* bal. 9.0 5.0 9.0 5.0	0 1 P 2 P 3 P 4 P 3 P 4 P	1 P 1 P 1 P 1 N 1 P	
	18* bal. 9.0 5.2	1 P 2 P 3 P 4 P	1 P 1 P 1 P		bal. 9.2 5.0	2 P 3 P 4 P	1 P 1 P	
	18* bal. 9.2 5.5	1 P 2 P 3 P 4 P	1 P 1 P 1 P		bal. 9.2 5.0	2 P 3 P 4 P	1 P 1 P	
	17* bal. 5.0	1 P 2 P 4 P	1 P 2 P(?)	6.9	5.0* 7.0* 18.0* bal. 9.2 4.9 bal. 9.1 5.0	1 N 0 1 P 2 P 3 P 4 P 2 P 3 P	1 P 1 P 1 P 1 P 1 P 1 P 1 P	
	7.0	18* bal.	1 P 2 P	1 P				
	6.9	18* bal. 9.0	1 P 2 P 3 P	1 N 1 P 1 P				
		7.3* 18* bal. 9.0	0 1 P 2 P 3 P	1 P 1 P 1 P				

† This speed marked balanced was actually measured two or three times and corresponded to a time of about 48 seconds against *g.*

TABLE III. (continued).

<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
7.0	7.0*	0	1 P	7.5	6.9*	0	1 P
	18*	1 P	1 P		17*	1 P	1 P
	bal.	2 P	1 P		bal.	2 P	2 P good
	9.7	3 P	1 P		5.8	4 P	
	5.3	4 P	1 P				
	7.0*	0	1 P		6.9*	0	1 P
	18*	1 P	1 P		16*	1 P	
	bal.	2 P	1 P		6.9*	0	1 P
	9.0	3 P	1 P		16*	1 P	1 P
	5.5	4 P	1 P		bal.	2 P	1 P
	bal.	2 P	1 P		17.0*	1 P	1 P
	10.0	3 P			bal.=40	2 P	1 P
	bal.	2 P	1 N		9.0	3 P	1 P
	17*	1 P	1 P		5.5	4 P	
	bal.	2 P	1 P		18*	1 P	1 P
	10.0	3 P	1 P		bal.	2 P	1 P
bal.	2 P	1 P	9.0	3 P			
10.0	3 P		bal.	2 P	1 P		
18*	1 P	1 P	9.0	3 P	1 P		
bal.	2 P	1 P	5.5	4 P			
18*	1 P	1 P	bal.	2 P	1 P		
bal.	2 P	1 P	9.0	3 P			
10.0	3 P	1 P	bal.	2 P	1 P		
5.8	4 P	1 P	8.8	3 P			
3.4	5 P	1 P					
bal.	2 P	1 P					
10.0	3 P						

The above tables do not represent one half of the observations which we have made upon ionization produced by either primary or secondary X rays of varying hardness, and Table III. contains the best evidence which we have been able to obtain at all in support of the hypothesis that doubles are even in rare instances formed. Despite the fact that we began this investigation with the firm belief in the correctness of this hypothesis, we are forced to the conclusion that *we have now no reason whatever for assuming that the act of ionization of air by X rays ever consists in the detachment of more than one elementary charge from a neutral molecule.*

§ 4. Ionizing effect of  $\beta$  and  $\gamma$  rays of Radium.

Although there was no evidence from other quarters that ionizing agents other than X rays ever produce doubles in air, we have made observations similar to the above on the ionizing action of the  $\beta$  and  $\gamma$  rays of radium. No attempt was made to separate the effects of the  $\beta$  and  $\gamma$  rays, but we estimate from rough measurements that about three-fourths of the ions formed in these experiments were due to the  $\beta$  rays.

A sample of the results obtained with the use of very small drops is shown in Table IV. *Every one of the 20*

TABLE IV.

Plate Distance 1.6 cm.				Distance of Fall .0741 cm.			
Volts 1368. Temperature 25° C.				Radius of Drop .0000392 cm.			
<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
34.4	40*	0	1 P		40*	0	1 P
	6.0	1 P			6.1	1 P	1 P
30.6	40*	0	1 N	41.4	40*	0	1 P
30.6	6.0	1 N			5.8	1 P	
33.0	40*	0	1 P	41.7	40*	0	1 N
40.6	6.0	1 P			6.0	1 N	
41.0	40*	0	1 P		40*	0	1 P
44.0	5.8	1 P			5.8	1 P	
37.0	40*	0	1 P		40*	0	1 P
	6.0	1 P			6.0	1 P	
44.0	40*	0	1 P		40*	0	1 P
	5.5	1 P			6.4	1 P	
45.6	40*	0	1 P	46.0	40*	0	1 P
45.0	5.9	1 P			6.0	1 P	
	40*	0	1 P		40*	0	1 P
	5.9	1 N	1 N		5.9	1 P	1 P
39.0	40*	0	1 N		40*	0	1 P
	6.2	1 N		46.0	6.0	1 N	1 N
	40*	0	1 N	20 catches, all singles.			
	5.8	1 N					
	40*	0	1 P				
	6.1	1 P					

TABLE V.

Plate Distance 1.6 cm. Distance of Fall .0741 cm.  
Volts 3200. Temperature 25° C. Radius of Drop .000093 cm.

<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.	<i>g.</i>	F.	No. of charges on drop.	No. of charges on ion caught.
7.0	120	1 P	1 N	7.4	100	1 N	1 N
	7.0*	0			6.6	2 N	
	120	1 N	1 P		100	1 N	1 N
	7.0*	0			6.8	2 N	
	120	1 N	1 P		100	1 N	1 P 1 N 1 N
	7.0*	0			7.2*	0	
	120	1 P	1 N 1 P		100	1 N 1 N	
	7.0*	0			6.7	2 N	
	200	1 P	1 P		100	1 P	1 P
	7.0*	1 P			6.8	2 P	
	200	1 P	1 P		100	1 N	1 N
	6.6	2 P			6.4	2 N	
	200	1 N	1 P		100	1 N	1 N
	6.4	2 P			6.8	2 N	
200	1 N	1 P	100	1 N	1 N		
7.2*	0		6.7	2 N			
120	1 P	1 N 1 N 1 P 1 P	100	1 N	1 N		
7.2*	0		6.4	2 N			
120	1 N	1 N 1 P	90	1 P	1 P		
6.5	2 N		6.5	2 P			
120	1 N	1 P	125	1 P			
6.4	2 N		7.2*	0			
120	1 N	1 N	125	1 N	1 N		
7.8*	0		6.8	2 N			
120	1 N	2 N(?)	125	1 P	1 P		
6.8	2 N		6.8	2 P			
7.2	100	1 N	1 N	125	1 P	1 P	
	4.8	3 N		7.0	2 P		
	100	1 P	1 N	125	1 P	1 N	
	7.2*	0		7.0*	0		
				125	1 P	1 P	
				6.4	2 P		
			120	1 N	1 N		
			6.8	2 N			

*catches observed with this drop was a single.* When large drops were used the chance that the same  $\beta$  particle would ionize two adjacent molecules, and thus throw two separate ions simultaneously upon the drop, became larger, and yet, in general, we caught only singles even with drops which were so large that a  $\beta$  particle in going the length of a diameter of a drop would have to pass through at least 8 molecules. The drop shown in Table V. has a diameter which is 20 times the mean free path of an air molecule, and  $\frac{20}{5.7} = 3.5$  times the mean free path of an electron, if the latter be regarded as a point, yet out of 34 catches there is but one which could possibly be a double. This means of course that a  $\beta$  particle ionizes but a small fraction of the molecules through which it actually passes—a conclusion which can also be reached in other ways. With the use of drops whose diameters were ten times the mean free path of an electron we occasionally, as in our early experiments, observed double changes when the field was on, but since these were never observed with drops of one-tenth the size, the conclusion is inevitable that these apparent doubly charged ions were in fact two separate singly charged ions.

The above observations represent only a small portion of those which we have made within the past three months, but they present the best evidence which we have been able to find for the formation of doubly charged ions. We would, however, point out this difference between our experiments and those of Townsend and Franck and Westphal. While their measurements have to do with the charges carried by ions at a considerable time after the formation of these ions *our experiments have to do solely with the measurement of the charge which is freed from a neutral molecule in the act of ionization.* If two single positive charges attached themselves after formation to a minute dust particle or other molecular aggregate, and thus formed a doubly charged ion, our experiments would not reveal the fact. Or, again, if X rays were capable of detaching more readily an elementary charge from an already singly charged ion than from a neutral molecule, and thus forming double positives (a very improbable hypothesis), our method would not be able to detect such doubles.

We are extending investigations of the types herein described to other gases and to vapours.

### § 6. Conclusion.

Our conclusion may be stated as follows:—Although we entered upon this investigation with the expectation of proving the existence of valency in gaseous ionization, we have instead

obtained direct, unmistakable evidence that the act of ionization of air molecules by both primary and secondary X rays of widely varying degrees of hardness, as well as by  $\beta$  and  $\gamma$  rays, uniformly consists, under all conditions which we have been able to investigate, in the detachment from a neutral molecule of one single elementary electrical charge.

Ryerson Physical Laboratory,  
University of Chicago,  
February 14, 1911.

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XCI. *The Radioactivity of some Igneous Rocks from Antarctic Regions.* By ARNOLD L. FLETCHER, B.A.I., Research Assistant to the Professor of Geology in the University of Dublin\*.

THE question of the mean radioactivity of the Earth's surface materials is of sufficient importance, and at the same time so far from being definitely agreed upon, that experimental data tending to increase our knowledge of the subject seem deserving of record.

The accompanying analyses of rock-specimens from the Antarctic region of South Victoria were determined upon a series of rocks collected on the recent Expedition under Lieutenant Shackleton, and kindly obtained and lent to me by Professor Joly. They embrace rock-types of varied chemical and petrographical characters. The determinations were made by the solution method of Professor Strutt in the manner described in a former communication.

An examination of the constants of the two new electroscopes used was made by the standard uraninite solution used in former experiments, and by a standard radium solution sent by Prof. Rutherford. It was found that two experiments with Prof. Rutherford's solution on the constant of one of the electroscopes read 0.63 and 0.62, and with Prof. Joly's solution 0.61; and in the case of the other electroscope the constant read 0.83 and 0.80 respectively with the two solutions. The truth of the Uraninite Standard as used all along in the calibration of the electroscopes is therefore supported by these experiments.

It is not without interest to note that on first comparing these standards, that coming from Prof. Rutherford gave a higher constant for the electroscopes referred to above, *i. e.* 1.1 and 1.7 respectively. This was traced to insufficient acidification when diluting the standard solution sent. On adding an additional quantity of HCl and warming, the almost perfect agreement recorded above was obtained. The concealment of the emanation in the perfectly limpid solution

\* Communicated by the Author.

before sufficient acidification, suggests possibilities when traces of radium emanation are sought for in bulky and chemically rich solutions—especially those which are examined in the alkaline state.

The following results were obtained:—

No.	Rock.	Locality.	Radium gr. per gr.	Thorium gr. per gr.	$\frac{\text{Radium}}{\text{Thorium}}$
1.	Basalt .....	Mt. Erebus, McMurds Sound .....	$2.13 \times 10^{-12}$	$1.45 \times 10^{-5}$	$1.5 \times 10^{-7}$
*2.	Kenytt lava .....	Mt. Erebus, Ross Sound .....	2.17 „	1.45 „	1.5 „
3.	Alkali trachyte ...	Mt. Erebus (South cliffs) .....	2.16 „	1.30 „	1.6 „
4.	Trachytic doleritic kenytt } }	Cape Royds, Ross Sound .....	1.3 „	0.8 „	1.6 „
5.	Kenytt lava .....	Cape Royds, Ross Sound .....	1.46 „	0.7 „	2.0 „
6.	Kenytt (coloured red by geyser action.) } }	McMurds Sound, Ross Sound .....	4.02 „	1.84 „	2.2 „
7.	Vesicular basic lava } }	McMurds Sound, Ross Sound .....	0.58 „	0.47 „	1.2 „
8.	Olivine augite basalt. } }	McMurds Sound, Ross Sound .....	0.38 „ ?	0.48 „	0.8 „ ?
9.	Olivine felspar basalt. } }	Cape Royds, Ross Sound .....	0.00 „ ?	0.40 „	.....
10.	Hornblende basalt. } }	Cape Barn, Ross Sound .....	0.47 „ ?	1.00 „	0.5 „ ?
11.	Erratic granite block. } }	Cape Royds, Ross Sound .....	0.82 „ ?	0.53 „	1.5 „
12.	Erratic granite block. } }	McMurds Sound, Ross Sound .....	0.70 „	0.37 „	1.9 „
13.	Erratic granite block. Another variety. } }	“Lower ground” of S. Victoria land	0.20 „	0.14 „	1.4 „
			Mean {	excluding queried results }	1.7

\* For description of rock-type see Quart. Journ. Geol. Soc. Lond. 1900, vol. lvi. p. 209.

One of the most noticeable features of the above results is the remarkable constancy obtaining in the case of both radium and thorium among specimens from the same locality, and probably taking their origin from a common magma. This feature, which is as much marked in the case of the thorium as in that of the radium, seems to be independent of the basicity or acidity of the rock. Thus in the case of Nos. 1, 2, and 3 in the above table, the figures are, within the limits of experimental error, identical, yet the rocks, although all from Mt. Erebus, are by no means similar

chemically or petrographically. Nos. 4 and 5 in the table, probably taking their origin in the same lava-flow, similarly show a striking equality in their radium and thorium contents.

Some of the lowest results obtained were from erratic granites. These contained but a small quantity of biotite, and consisted almost entirely of orthoclase and quartz. With the single exception of No. 11, the granites showed a complete absence of precipitation, and are probably therefore approximately correct.

Speaking generally, the results do not bear out the conclusion of Farr and Florence in their research on the radium content of the Sub-Antarctic islands of New Zealand (Phil. Mag. Nov. 1909), that the radioactivity depends roughly upon the acid or basic character of the rock. The mean value, 2.15, obtained for the basic rocks of Mt. Erebus is in good agreement with the minimum value 2.38 found by them (*loc. cit.*) on a specimen of lava from the same locality.

The radioactivity of No. 6 in the table is of especial interest. This rock is stained red probably under the action of infiltrating geyser waters, and it seems probable that in this case the kenyt owes its relatively high radioactivity to this cause.

In addition to the constancy shown in the quantities of radium and thorium present in the different groups of rocks in the table, these quantities seem to show a decidedly definite relationship between themselves, from the highest to the lowest reliable values obtained; a rise in the radium present being almost invariably accompanied by a nearly corresponding rise in thorium present. This feature which, in the case of some weathered andesites examined, might find an explanation in the effects of the weathering action itself, cannot be similarly explained in the case of granite and the unweathered Antarctic rocks, where a similar relationship appears to hold.

In the case, however, of the kenyt No. 6 in the table, the same cause which probably gave rise to its relatively high radium content appears at the same time to have proportionately increased the thorium content, for the ratio of radium to thorium present fairly approximates to the general mean ratio. Hence in this case the radioactivity, if due chiefly to the external effects of infiltrating waters rather than to any peculiarity inherent in its original magma, involves a similar relationship between the radium and thorium in those waters.

In conclusion I desire to express my thanks to Professor Joly for his interest in the work.

Geological Laboratory,  
Trinity College, Dublin,  
March 17th, 1911.



NOTE.—A. Gockel (*Jahrbuch der Radioaktivität*, vii. p. 503, 1910), examined various minerals found by Prof. Strutt to be relatively rich in radium, and obtained radium contents far below his values, accompanied at the same time by unusually high thorium richness. Gockel suggests the possibility of a source of error from the active deposit of thorium, consequent upon the introduction of thorium emanation into the electroscope.

Some experiments were made to test the possibility of such an error affecting the estimation of radium in bodies rich in thorium.

A solution containing as much as 0.1 gram thorianite, having been first boiled to expel radium emanation, was then treated in the same manner as that employed in the estimation of radium. Four experiments were made using both slow and fast admission, in the latter case the gases being admitted into the electroscope as fast as the safety of the gold leaf would permit. The whole of the gases were in this case transferred to the electroscope within two minutes subsequent to being cut off from the parent solution. In no case was any certain increase noticed in the rate of collapse of the leaves of the electroscope in three hours after admission.

In view of the fact that no effect was observable in the case of a solution so rich in thorium, it is safe to infer that in no case could the leak of an electroscope be noticeably affected as read three hours after the slow admission of such quantities of thorium emanation as are associated with amounts of thorium of the order of magnitude dealt with in the estimation of rocks and minerals.

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XCII. *On Water Waves as Asymmetric Oscillations and on the Stability of Free Wave-Trains.* By ANDREW STEPHENSON\*.

1. **W**ATER waves furnish a complex example of asymmetric oscillations, and it is natural to inquire whether they exhibit the marked energy absorption under direct force of double frequency, which is characteristic of the asymmetric system with one degree of freedom†. The problem is most simply considered as one of steady motion. Direct force may be applied to a deep stream flowing uniformly by a stationary periodic variation of pressure along its surface. Such variation will produce standing waves of equal length. Is this state of motion stable when the wave-length is half that of the free standing waves?

For the purpose of testing the stability of a train of waves

\* Communicated by the Author.

† "On a Peculiar Property of the Asymmetric System," *Phil. Mag.* Jan. 1911.

for small disturbance of a given character, we seek the force system necessary to maintain the disturbance. If the force when acting alone tends continually to damp it, we conclude that the steady motion is unstable for such variation. Thus in the present problem we seek the conditions under which a small standing train of double wave-length, and therefore of nearly free type, may be maintained by pressure variation of the corresponding period in such phase that its action tends to damp out the train without producing other change.

As the discussion involves quantities of different orders of smallness, we shall discard the customary stream and velocity-potential functions in favour of more direct coordinates, thus obtaining equations which are readily applicable to all cases in which the magnitude of the amplitude is involved.

2. Let  $(x, y + \eta)$  be the coordinates of a particle the mean level of which is at distance  $y$ , positive upwards, from the undisturbed surface. Then the velocity in the stream-line of mean level  $y$  is  $c\sqrt{1 + \left(\frac{d\eta}{dx}\right)^2} / \left(1 + \frac{d\eta}{dy}\right)$ , where  $c$  is a function of  $y$ , and its horizontal and vertical components are  $c / \left(1 + \frac{d\eta}{dy}\right)$  and  $c \frac{d\eta}{dx} / \left(1 + \frac{d\eta}{dy}\right)$ .

A displacement  $\delta x$  horizontally is given by a displacement along the stream-line of horizontal component  $\delta x$ , and a vertical displacement  $-\frac{d\eta}{dx} \cdot \delta x$ . For irrotational motion therefore

$$\left(\frac{d}{dx} - \frac{\frac{d\eta}{dx}}{1 + \frac{d\eta}{dy}} \frac{d}{dy}\right) \frac{c \frac{d\eta}{dx}}{1 + \frac{d\eta}{dy}} - \frac{1}{1 + \frac{d\eta}{dy}} \frac{d}{dy} \frac{c}{1 + \frac{d\eta}{dy}} = 0;$$

$$\text{i. e.,} \quad \left(1 + \frac{d\eta}{dy}\right)^2 \frac{d^2\eta}{dx^2} + \left(1 + \left(\frac{d\eta}{dx}\right)^2\right) \frac{d^2\eta}{dy^2} - 2\left(1 + \frac{d\eta}{dy}\right) \frac{d\eta}{dx} \frac{d^2\eta}{dx dy} - \frac{1}{c} \frac{dc}{dy} \left(1 + \frac{d\eta}{dy}\right) \left(1 + \left(\frac{d\eta}{dx}\right)^2\right) = 0 \dots (1)$$

The pressure is given by

$$p/\rho = -g\eta - \frac{1}{2}c^2 \frac{1 + \left(\frac{d\eta}{dx}\right)^2}{\left(1 + \frac{d\eta}{dy}\right)^2} + \text{a constant.}$$

4. The forced motion due to a small simple pressure variation of wave-length  $\pi/k$  is

$$\eta = ae^{2ky} \cos 2kx,$$

and the equation of motion for any additional disturbance,  $\zeta$ , is therefore from (1)

$$\begin{aligned} \frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + 4ake^{2ky} \cos 2kx \frac{d^2\zeta}{dx^2} + 4ake^{2ky} \sin 2kx \cdot \frac{d^2\zeta}{dx dy} \\ - 8ak^2 e^{2ky} \cos 2kx \frac{d\zeta}{dy} + 8ak^2 \sin 2kx \frac{d\zeta}{dx} = 0. \quad \dots (2) \end{aligned}$$

Putting

$$\zeta = \alpha \cos kx + \beta \sin kx,$$

where  $\alpha$  and  $\beta$  are functions of  $y$ , we have from (2)

$$\begin{aligned} \alpha &= p(e^{ky} + \frac{3}{2}ak e^{3ky}), \\ \beta &= q(e^{ky} - \frac{3}{2}ak e^{3ky}). \end{aligned}$$

Now the pressure at the surface is proportional to

$$-\frac{g}{c^2} \zeta + \frac{d\zeta}{dy} - 6ak \cos 2kx \frac{d\zeta}{dy} + 2ak \sin 2kx \cdot \frac{d\zeta}{dx},$$

which is  $-p(ak - \sigma) \cos kx + q(ak + \sigma) \sin kx$ ,

where 
$$c^2 = \frac{g}{k} (1 + \sigma).$$

This pressure, being nearly of free period, tends to damp the disturbance if its phase is  $-1/4$ : that is, if

$$p^2(ak - \sigma) = q^2(ak + \sigma).$$

$|\sigma|$  must therefore be less than  $ak$ . Thus there is instability for any value of  $\sigma$  numerically less than  $ak$ , the standing disturbance of wave-length  $2\pi/\{k(1 + \sigma)\}$  being magnified in one phase  $-\tan^{-1} \sqrt{\frac{ak + \sigma}{ak - \sigma}}$ , and diminished in the numerically equal phase of opposite sign.

Interpreting the result in terms of progressive waves in still water, it is evident that if a periodic pressure variation moves uniformly over the surface, the forced train of equal wave-length constitutes an unstable state of motion if the ratio of the wave-length to that of the free wave of equal speed lies within a range  $2ak$  about the value  $1/2$ ; for a small disturbance will result in a series of waves of double wave-length, which is continually magnified through the periodic pressure until the amplitude is large compared with the original motion. The process consists essentially in the continual enlargement of an asymmetric oscillation of approximately free type by a direct force of frequency lying within a range about the double frequency of free oscillation.

For the purposes of experimental illustration it would be simplest to take the case of a stream flowing over a corrugated bed.

3. Since the free wave of finite amplitude is appreciably asymmetric, it would seem possible from the foregoing that the free train might tend to magnify other relatively stationary trains of nearly the same wave-lengths and relatively small amplitudes. This question is a partial test of the stability of a free train of waves. To determine the free wave of finite amplitude\*, putting

$$\eta = \alpha \cos kx + \beta \cos 2kx,$$

where  $\alpha$  and  $\beta$  are functions of  $y$ , we find

$$\alpha'' - k^2\alpha + \frac{1}{4}k^2\alpha^2\alpha'' - \frac{5}{4}k^2\alpha\alpha'^2 - 3k^2\alpha\beta' - 6k^2\alpha'\beta - \frac{1}{c}c'\alpha' = 0,$$

$$\beta'' - 4k^2\beta = 0,$$

$$\frac{1}{c}c' + 2k^2\alpha\alpha' = 0,$$

subject to the boundary conditions  $\alpha = \beta = 0$  when  $y = -\infty$ , and

$$\frac{g}{c^2}\alpha - 2\alpha' + 3\alpha'\beta' - 3\alpha'^3 + 2k^2\alpha\beta - \frac{1}{2}k^2\alpha^2\alpha' = 0,$$

$$\frac{g}{c^2}\beta - \beta' + \frac{3}{4}\alpha'^2 - \frac{1}{4}k^2\beta^2 = 0,$$

when  $y = 0$ .

$$\text{Hence} \quad \alpha = ae^{ky} + \frac{5}{8}a^3k^2e^{3ky},$$

$$\beta = \frac{1}{2}a^2ke^{2ky},$$

$$c^2 = \frac{g}{k} \{1 + k^2a^2(1 - 2e^{2ky})\}.$$

The equation for the disturbance,  $\zeta$ , is

$$\begin{aligned} \frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + 2ake^{ky} \left\{ \cos kx \left( \frac{d^2\zeta}{dx^2} - k \frac{d\zeta}{dy} \right) + \sin kx \left( \frac{d^2\zeta}{dxdy} + k \frac{d\zeta}{dx} \right) \right\} \\ + a^2k^2e^{2ky} \left\{ \frac{1}{2} \left( \frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} \right) + \frac{1}{2} \cos 2kx \left( 5 \frac{d^2\zeta}{dx^2} - \frac{d^2\zeta}{dy^2} - 8k \frac{d\zeta}{dy} \right) \right. \\ \left. + \sin 2kx \left( 3 \frac{d^2\zeta}{dxdy} + 4k \frac{d\zeta}{dx} \right) \right\} = 0. \end{aligned}$$

\* The method is evidently applicable when the depth is finite. In the case of long waves the process is necessarily different. We have then

$$\eta = \eta_0 + y \left( \frac{d\eta}{dy} \right)_0 + \frac{y^2}{2!} \left( \frac{d^2\eta}{dy^2} \right)_0 + \frac{y^3}{3!} \left( \frac{d^3\eta}{dy^3} \right)_0$$

when small quantities of order higher than the second are neglected. Hence, putting  $y = -h$ , and substituting for the  $y$  derivatives from (1) and the surface condition we obtain

$$\frac{d^2}{dx^2} \left( \frac{\eta}{h} \right) + \frac{3}{h^2} \left\{ \left( \frac{gh}{c^2} - 1 \right) \frac{\eta_0}{h} + \frac{3}{2} \left( \frac{\eta_0}{h} \right)^2 \right\} = 0,$$

the well-known equation for the contours.

The pressure at the surface is proportional to

$$-k(1 + a^2k^2)\zeta + (1 + \frac{7}{2}a^2k^2 - 3ak \cos kx - \frac{1}{2}a^2k^2 \cos 2kx) \frac{d\zeta}{dy} + ak \sin kx \frac{d\zeta}{dx}.$$

The solution

$$\zeta = (e^{(k+s)y} + \frac{7}{4}a^2k^2e^{3ky}) \cos (k+s)x + \frac{9}{8}a^2k^2e^{3ky} \cos (k-s)x \\ + ak e^{(2k+s)y} \cos (2k+s)x + ak e^{2ky} \cos sx - ak e^{1s1y} \cos sx$$

gives surface pressure

$$(s - a^2k^3) \cos (k+s)x - a^2k^3 \cos (k-s)x :$$

and

$$\zeta = \left( e^{(k+s)y} + \frac{7}{4}a^2k^2e^{3ky} \right) \sin (k+s)x - \frac{9}{8}a^2k^2e^{3ky} \sin (k-s)x \\ + ak e^{(2k+s)y} \sin (2k+s)x - ak e^{(2k+s)y} \sin sx - ak e^{1s1y} \sin sx$$

the pressure

$$(s - a^2k^3) \sin (k+s)x + a^2k^3 \sin (k-s)x.$$

Hence the disturbance

$$e^{(k+s)y} \{ A \cos (k+s)x + C \sin (k+s)x \} + e^{(k-s)y} \{ B \cos (k-s)x + D \sin (k-s)x \}$$

is maintained by the pressure

$$\{ (s - a^2k^3)A - a^2k^3B \} \cos (k+s)x + \{ (s - a^2k^3)C + a^2k^3D \} \sin (k+s)x \\ + \{ (-s - a^2k^3)B - a^2k^3A \} \cos (k-s)x + \{ (-s - a^2k^3)D + a^2k^3C \} \sin (k-s)x.$$

The pressure acting alone would tend simply to change the intensity of the disturbance if the two components are proportional to the amplitudes of the trains of corresponding wave-lengths, and if the phases differ by a quarter period from those of the trains; that is, if

$$\frac{(s - a^2k^3)A - a^2k^3B}{C} = - \frac{(s - a^2k^3)C + a^2k^3D}{A} \\ = \frac{(-s - a^2k^3)B - a^2k^3A}{D} = - \frac{(-s - a^2k^3)D + a^2k^3C}{B} = q, \text{ say.}$$

Hence  $q^2 = -s^2$ ;

$q$  is therefore always complex, and the free train has no tendency to develop a periodicity of amplitude.

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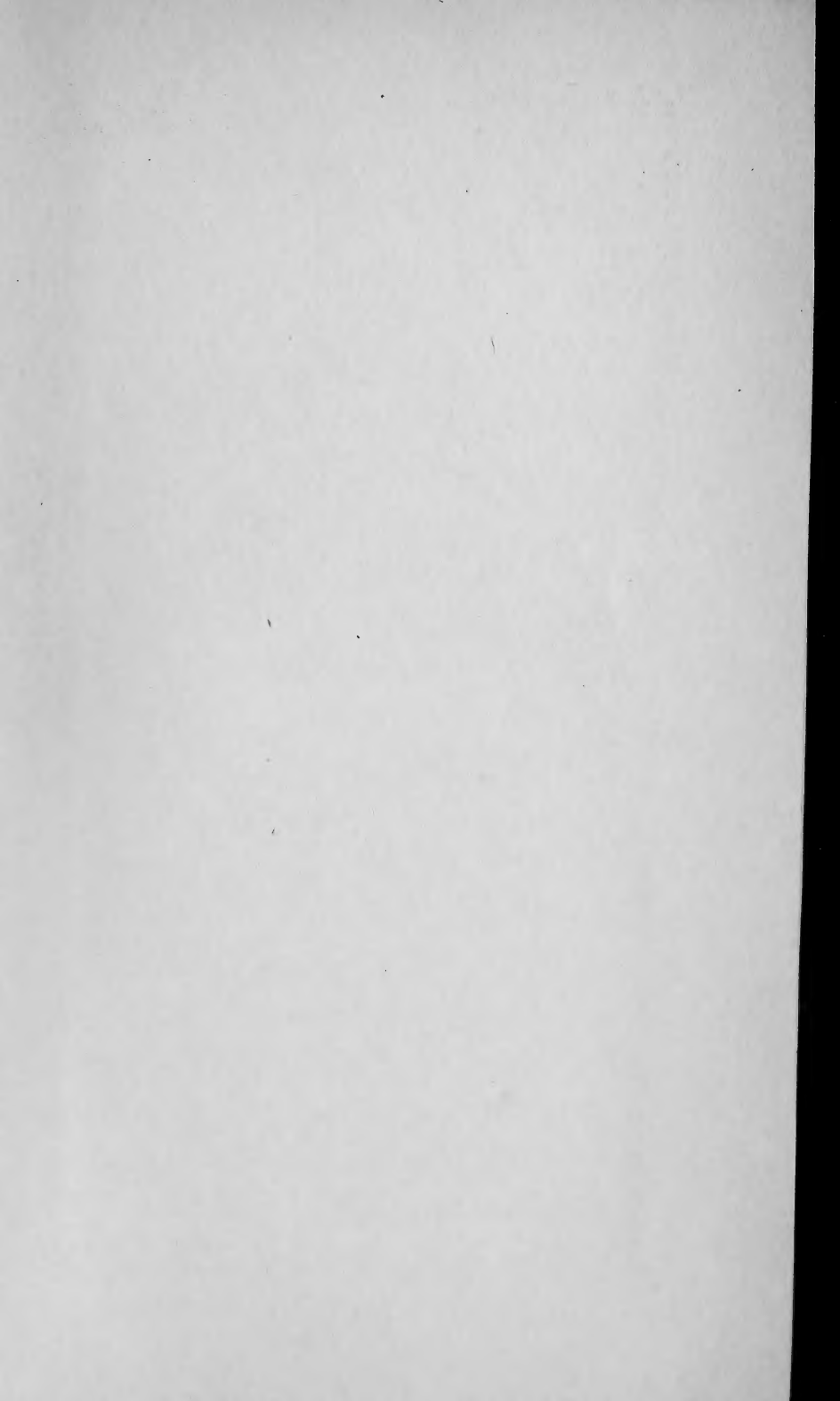
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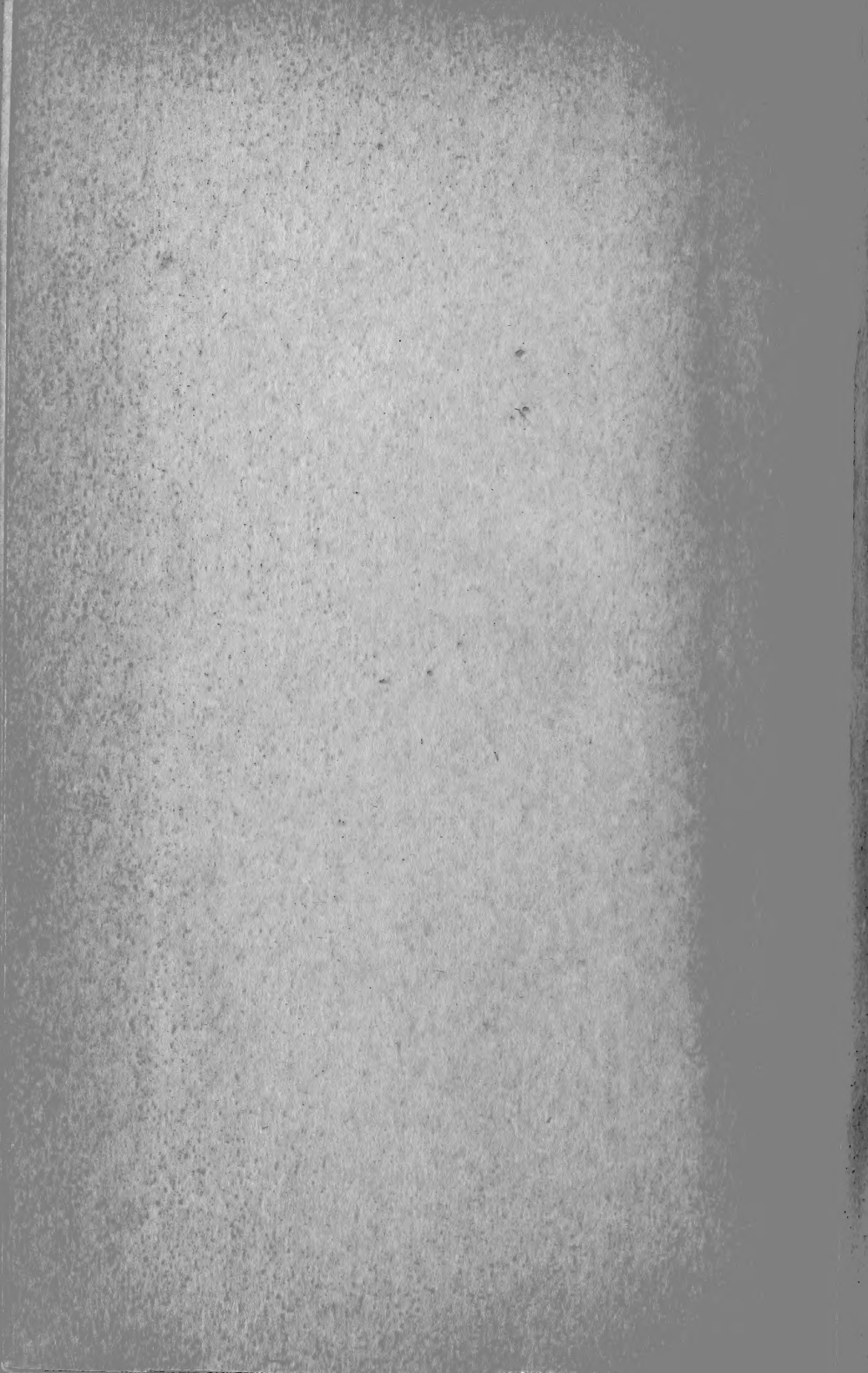
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