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"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit.* lib. i. cap. 1. Not.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

——"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

CONTENTS OF VOL. XXXV.

(SIXTH SERIES).

NUMBER CCV.—JANUARY 1918.

NUMBER CCVI.—FEBRUARY.

NUMBER CCVII.—MARCH.

NUMBER CCVIII.—APRIL.

NUMBER CCIX.—MAY.

NUMBER CCX.—JUNE.

PLATES.

I. & II. Illustrative of Prof. Barton and Miss Browning's Paper on Variably-Coupled Vibrations: II. Unequal Masses or Periods.

III. Illustrative of Mr. N. Basu's Paper on the Diffraction of Light by Cylinders of Large Radius.

IV. Illustrative of Prof. S. Banerji's Paper on Aerial Waves generated by Impact.

V. Illustrative of Mr. S. K. Mitra's Paper on the Asymmetry of the Illumination-Curves in Oblique Diffraction.

VI.–VIII. Illustrative of Prof. R. W. Wood's Paper on Resonance Spectra of Iodine.

IX. Illustrative of Dr. G. W. Todd's Paper on General Curves for the Velocity of Complete Homogeneous Reactions between Two Substances at Constant Volume.

X. Illustrative of Mr. G. W. Walker's Paper on Relativity and Electrodynamics.

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[SIXTH SERIES.]

JANUARY 1918.

I. *Notes on the Theory of Lubrication.*

By Lord Rayleigh, *O.M.*, *F.R.S.*[*]

MODERN views respecting mechanical lubrication are founded mainly on the experiments of B. Tower[†], conducted upon journal bearings. He insisted upon the importance of a complete film of oil between the opposed solid surfaces, and he showed how in this case the maintenance of the film may be attained by the dragging action of the surfaces themselves, playing the part of a pump. To this end it is "necessary that the layer should be thicker on the ingoing than on the outgoing side"[‡], which involves a slight displacement of the centre of the journal from that of the bearing. The theory was afterwards developed by O. Reynolds, whose important memoir[§] includes most of what is now known upon the subject. In a later paper Sommerfeld has improved considerably upon the mathematics, especially in the case where the bearing completely envelops the journal, and his exposition[‖] is much to be recommended to those who wish to follow the details of the investigation. Reference may also be made to Harrison[¶], who includes the consideration of compressible lubricants (air).

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[*] Communicated by the Author.

[†] Proc. Inst. Mech. Eng. 1883, 1884.

[‡] British Association Address at Montreal, 1884; Rayleigh's Scientific Papers, vol. ii. p. 344.

[§] Phil. Trans. vol. 177. p. 157 (1886).

[‖] *Zeitschr. f. Math.* t. 50. p. 97 (1904).

[¶] Camb. Trans. vol. xxii. p. 39 (1913).

In all these investigations the question is treated as two-dimensional. For instance, in the case of the journal the width—axial dimension—of the bearing must be large in comparison with the arc of contact, a condition not usually fulfilled in practice. But Michell* has succeeded in solving the problem for a plane rectangular block, moving at a slight inclination over another plane surface, free from this limitation, and he has developed a system of pivoted bearings with valuable practical results.

It is of interest to consider more generally than hitherto the case of two dimensions. In the present paper attention is given more especially to the case where one of the opposed surfaces is plane, but the second not necessarily so. As an alternative to an inclined plane surface, consideration is given to a broken surface consisting of two parts, each of which is parallel to the first plane surface but at a different distance from it. It appears that this is the form which must be approached if we wish the total pressure supported to be a maximum, when the length of the bearing and the closest approach are prescribed. In these questions we may anticipate that our calculations correspond pretty closely with what actually happens,—more than can be said of some branches of hydrodynamics.

In forming the necessary equation it is best, following Sommerfeld, to begin with the simplest possible case. The layer of fluid is contained between two parallel planes at $y=0$ and at $y=h$. The motion is everywhere parallel to x, so that the velocity-component u alone occurs, v and w being everywhere zero. Moreover u is a function of y only. The tangential traction acting across an element of area represented by dx is $\mu(du/dy)dx$, where μ is the viscosity, so that the element of volume $(dx\,dy)$ is subject to the force $\mu(d^2u/dy^2)\,dx\,dy$. Since there is no acceleration, this force is balanced by that due to the pressure, viz. $-(dp/dx)\,dx\,dy$, and thus

$$\frac{dp}{dx}=\mu\frac{d^2u}{dy^2}. \qquad (1)$$

In this equation p is independent of y, since there is in this direction neither motion nor components of traction, and (1), which may also be derived directly from the general hydrodynamical equations, is immediately integrable. We have

$$u=\frac{1}{2\mu}\frac{dp}{dx}y^2+\mathrm{A}+\mathrm{B}y, \qquad (2)$$

where A and B are constants of integration. We now

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* *Zeitschr. f. Math.* t. 52. p. 123 (1905).

suppose that when $y=0$, $u=-\mathrm{U}$, and that when $y=h$, $u=0$. Thus

$$u=\frac{y^2-hy}{2\mu}\frac{dp}{dx}-\left(1-\frac{y}{h}\right)\mathrm{U}. \qquad (3)$$

The whole flow of liquid, regarded as incompressible, between 0 and h is

$$\int_0^h u\,dy=-\frac{h^3}{12\mu}\frac{dp}{dx}-\frac{h\mathrm{U}}{2}=-\mathrm{Q},$$

where Q is a constant, so that

$$\frac{dp}{dx}=-\frac{6\mu\mathrm{U}}{h^3}\left(h-\frac{2\mathrm{Q}}{\mathrm{U}}\right). \qquad (4)$$

If we suppose the passage to be absolutely blocked at a place where x is negatively great, we are to make $\mathrm{Q}=0$ and (4) gives the rise of pressure as x decreases algebraically. But for the present purpose Q is to be taken finite. Denoting $2\mathrm{Q}/\mathrm{U}$ by H, we write (4)

$$\frac{dp}{dx}=-\frac{6\mu\mathrm{U}}{h^3}(h-\mathrm{H}). \qquad (5)$$

When $y=0$, we get from (3) and (5)

$$\mu\frac{du}{dy}=\mu\mathrm{U}\frac{4h-3\mathrm{H}}{h^2}, \qquad (6)$$

which represents the tangential traction exercised by the liquid upon the moving plane.

It may be remarked that in the case of a simple shearing motion $\mathrm{Q}=\frac{1}{2}h\mathrm{U}$, making $\mathrm{H}=h$, and accordingly

$$dp/dx=0, \qquad du/dy=\mathrm{U}/h.$$

Our equations allow for a different value of Q and a pressure variable with x.

So far we have regarded h as absolutely constant. But it is evident that Reynolds' equation (5) remains approximately applicable to the lubrication problem in two dimensions even when h is variable, though always very small, provided that the changes are not too sudden, x being measured circumferentially and y normally to the opposed surfaces. If the whole changes of direction are large, as in the journal-bearing with a large arc of contact, complication arises in the reckoning of the resultant forces operative upon the solid parts concerned; but this does not interfere with the applicability of (5) when h is suitably expressed as a function of x. In the present paper we confine ourselves to the case where one surface (at $y=0$) may be treated as absolutely plane. The second surface is supposed to be limited at $x=a$ and at $x=b$, where h is equal to h_1 and h_2 respectively, and the pressure at both these places is taken to be zero.

For the total pressure, or load, (P) we have

$$\mathrm{P}=\int_a^b p\,dx=-\int_a^b x\frac{dp}{dx}dx,$$

on integration by parts with regard to the evanescence of p at both limits. Hence by (5)

$$\frac{\mathrm{P}}{6\mu\mathrm{U}}=\int_a^b\frac{x\,dx}{h^2}-\mathrm{H}\int_a^b\frac{x\,dx}{h^3}. \quad . \quad . \quad . \quad . \quad (7)$$

Again, by direct integration of (5),

$$0=\int_a^b\frac{dx}{h^2}-\mathrm{H}\int_a^b\frac{dx}{h^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

by which H is determined. It is the thickness of the layer at the place, or places, where p is a maximum or a minimum. A change in the sign of U reverses also that of P.

Again, if $\bar{x}$ be the value of x which gives the point of application of the resultant force,

$$\bar{x}\,.\,\mathrm{P}=\int_a^b p\,x\,dx=\tfrac{1}{2}\int_a^b x^2\frac{dp}{dx}dx,$$

so that

$$\frac{\bar{x}\,.\,\mathrm{P}}{3\mu\mathrm{U}}=\int_a^b\frac{x^2dx}{h^2}-\mathrm{H}\int_a^b\frac{x^2dx}{h^3}. \quad , \quad . \quad . \quad . \quad (9)$$

By (7), (8), (9) $\bar{x}$ is determined.

As regards the total friction (F), we have by (6)

$$\frac{\mathrm{F}}{\mu\mathrm{U}}=4\int_a^b\frac{dx}{h}-3\mathrm{H}\int_a^b\frac{dx}{h^2}. \quad . \quad . \quad . \quad . \quad (10)$$

Comparing (7) and (10), we see that the ratio of the tota friction to the total load is *independent of* μ *and of* U. And, since the right-hand members of (7) and (10) are dimensionless, the ratio is also independent of the linear scale. But if the scale of h only be altered, F/P varies as h.

We may now consider particular cases, of which the simplest and the most important is when the second surface also is flat, but inclined at a very small angle to the first surface. We take

$$h=mx, \quad \ldots \ldots \ldots \ldots \quad (11)$$

and we write for convenience

$$b-a=c, \qquad h_2/h_1=b/a=k, \quad \ldots \quad (12)$$

so that

$$m=(k-1)h_1/c. \quad \ldots \ldots \ldots \quad (13)$$

We find in terms of c, k, and h_1

$$\mathrm{H}=\frac{2kh_1}{k+1} \quad \ldots \ldots \ldots \ldots \quad (14)$$

$$\frac{\mathrm{P}}{6\mu\mathrm{U}}=\frac{c^2}{(k-1)^2h_1{}^2}\left\{\log_e k-\frac{2(k-1)}{k+1}\right\} \quad \ldots \quad (15)$$

$$\frac{\bar{x}}{\frac{1}{2}c}=\frac{k^2-1-2k\log k}{(k^2-1)\log k-2(k-1)^2} \quad \ldots \ldots \quad (16)$$

$$\frac{\mathrm{F}}{\mathrm{P}}=\frac{h_1}{c}\,\frac{2(k^2-1)\log k-3(k-1)^2}{3(k+1)\log k-6(k-1)}. \quad \ldots \quad (17)$$

U being positive, the sign of P is that of

$$\log k-\frac{2(k-1)}{k+1}.$$

If $k>1$, that is when $h_2>h_1$, this quantity is positive. For its derivative is positive, as is also the initial value when k exceeds unity but slightly. In order that a load may be sustained, the layer must be thicker where the liquid enters.

In the above formulæ we have taken as data the length of the bearing c and the minimum distance h_1 between the surfaces. So far k, giving the maximum distance, is open. It may be determined by various considerations. Reynolds examines for what value P, as expressed in (15), is a maximum, and he gives (in a different notation) $k=2{\cdot}2$. For values of k equal to 2·0, 2·1, 2·2, 2·3 I find for the coefficient of $c^2/h_1{}^2$ on the right of (14) respectively

·02648, ·02665, ·02670, ·02663.

In agreement with Reynolds the maximum occurs when $k=2{\cdot}2$ nearly, and the maximum value is

$$\mathrm{P}=0{\cdot}1602\frac{\mu\mathrm{U}c^2}{h_1{}^2}. \quad \ldots \ldots \ldots \quad (18)$$

It should be observed—and it is true whatever value be taken for k—that P varies as the *square* of c/h_1.

With the above value of k, viz. 2·2,

$$H=1·27h_1, \quad \ldots\ldots \quad (19)$$

fixing the place of maximum pressure.

Again, from (16) with the same value of k,

$$\bar{x}-a=0·4231\,c, \quad \ldots\ldots \quad (20)$$

which gives the distance of the centre of pressure from the trailing edge.

And, again with the same value of k, by (17)

$$F/P=4·70\,h_1/c. \quad \ldots\ldots \quad (21)$$

Since h_1 may be very small, it would seem that F may be reduced to insignificance.

In (18) .... (21) the choice of k has been such as to make P a maximum. An alternative would be to make F/P a minimum. But it does not appear that this would make much practical difference. In Michell's bearings it is the position of the centre of pressure which determines the value of k by (16). If we use (20), k will be 2·2, or thereabouts, as above.

When in (16) k is very large, the right-hand member tends to zero, as also does a/c, so that $\bar{x}-a$ tends to vanish, c being given. As might be expected, the centre of pressure is then close to the trailing edge. On the other hand, when k exceeds unity but little, the right-hand member of (16) assumes an indeterminate form. When we evaluate it, we find

$$\bar{x}-a=\tfrac{1}{2}c.$$

For all values of $k\,(>1)$ the centre of pressure lies nearer the narrower end of the layer of fluid.

The above calculations suppose that the second surface is *plane*. The question suggests itself whether any advantage would arise from another choice of form. The integrations are scarcely more complicated if we take

$$h=mx^n. \quad \ldots\ldots \quad (22)$$

We denote, as before, the ratio of the extreme thicknesses (h_2/h_1) by k, and c still denotes $b-a$. For the total pressure we get from (15)

$$\frac{P}{6\mu U}=\frac{c^2}{(k^{1/n}-1)^2h_1{}^2}\left\{\frac{3n-1}{(2n-1)(3n-2)}\,\frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{k^{-3+1/n}}\right.$$
$$\left.-\frac{k^{-2+2/n}-1}{2n-2}\right\}, \quad \ldots \quad (23)$$

from which we may fall back on (15) by making $n=1$.

For example, if $n=2$, so that the curve of the second surface is part of a common parabola, P is a maximum at

$$P=0{\cdot}163\frac{\mu U c^2}{h_1{}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

when $k=2{\cdot}3$. The departure from (18) with $k=2{\cdot}2$ is but small. In order to estimate the curvature involved we may compare $\frac{1}{2}(h_1+h_2)$ with the middle ordinate of the curve, viz.

$$\tfrac{1}{4}m(a+b)^2=\tfrac{1}{4}\{\sqrt{h_1}+\sqrt{(2{\cdot}3h_1)}\}^2=1{\cdot}58\,h_1,$$

which is but little less than

$$\tfrac{1}{2}(h_1+h_2)=\tfrac{1}{2}h_1(1+2{\cdot}3)=1{\cdot}65\,h_1.$$

It appears that curvature following the parabolic law is of small advantage.

I have also examined the case of $n=\infty$. It is perhaps simpler and comes to the same to assume

$$h=e^{\beta x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The integrals required in (7), (8) are easily evaluated. Thus

$$\int\frac{dx}{h^2}=\frac{e^{-2\beta a}-e^{-2\beta b}}{2\beta}=\frac{k^2-1}{2\beta k^2 h_1{}^2},$$

$$\int\frac{dx}{h^3}=\frac{e^{-3\beta a}-e^{-3\beta b}}{3\beta}=\frac{k^3-1}{3\beta k^3 h_1{}^3},$$

making

$$H=\frac{3kh_1(k^2-1)}{2(k^3-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In like manner

$$\int\frac{x\,dx}{h^2}=\frac{k^2(1+2\beta a)-1-2\beta b}{4\beta^2k^2h_1{}^2},$$

$$\int\frac{x\,dx}{h^3}=\frac{k^3(1+3\beta a)-1-3\beta b}{9\beta^2k^3h_1{}^3}.$$

Using these in (7), we get on reduction

$$P=\frac{3\mu U}{\beta^2k^2h_1{}^2}\left\{\frac{k^2-1}{6}+\frac{\beta(k^2-k^3)(b-a)}{k^3-1}\right\},$$

or, since $\beta c=\log k$,

$$P=\frac{3\mu U\,.\,c^2}{k^2(\log k)^2h_1{}^2}\left\{\frac{k^2-1}{6}-\frac{k^2(k-1)\log k}{k^3-1}\right\}. \quad . \quad (27)$$

If we introduce the value of β, the equation of the curve may be written

$$h = k^{x/c}, \quad \ldots \ldots \quad (28)$$

When we determine k so as to make P a maximum, we get $k = 2{\cdot}3$, and

$$P = 0{\cdot}165 \frac{\mu U c^2}{h_1^2}, \quad \ldots \ldots \quad (29)$$

again with an advantage which is but small.

In all the cases so far considered the thickness h increases all the way along the length, and the resultant pressure is proportional to the square of this length (c). In view of some suggestions which have been made, it is of interest to inquire what is the effect of (say) r repetitions of the same curve, as, for instance, a succession of inclined lines ABCDEF (fig. 1). It appears from (8) that H has the same value for the aggregate as for each member singly, and from (5) that the increment of p in passing along the series is r times the increment due to one member. Since the former increment is zero, it follows that the pressure is zero at the beginning and end of each member. The circumstances are thus precisely the same for each member, and the total pressure is r times that due to the first, supposed to be isolated. But if we imagine the curve spread once over the entire length by merely increasing the scale of x, we see that the resultant pressure would be increased r^2 times, instead of merely r times. Accordingly a repetition of a curve is very unfavourable. But at this point it is well to recall that we are limiting ourselves to the case of two dimensions. An extension in the third dimension, which would suffice for a particular length, might be inadequate when this length is multiplied r times.

Fig. 1.

The forms of curve hitherto examined have been chosen with regard to practical or mathematical convenience, and it remains open to find the form which according to (5) makes P a maximum, subject to the conditions of a given length and a given minimum thickness (h_1) of the layer of liquid. If we suppose that h becomes $h+\delta h$, where δ is the symbol of the calculus of variations, (8) gives,

$$2\int\frac{\delta h}{h^3}dx - 3\mathrm{H}\int\frac{dh}{h^4}dx + \delta\mathrm{H}\int\frac{dx}{h^3} = 0, \quad . \quad . \quad (30)$$

and from (7)

$$\frac{\delta \mathrm{P}}{6\mu \mathrm{U}} = \int\frac{\delta h(-2h+3\mathrm{H})x\,dx}{h^4} - \delta\mathrm{H}\int\frac{x\,dx}{h^3}, \quad . \quad . \quad (31)$$

the integrations being always over the length. Eliminating $\delta\mathrm{H}$, we get

$$\frac{\delta \mathrm{P}}{12\mu \mathrm{U}} = -\int\frac{\delta h}{h^4}\left\{x - \frac{\int h^{-3}x\,dx}{\int h^{-3}dx}\right\}\left\{h-\tfrac{3}{2}\mathrm{H}\right\}x\,dx. \quad . \quad (32)$$

The evanescence of $\delta \mathrm{P}$ for all possible variations δh would demand that over the whole range either

$$x = \frac{\int h^{-3}x\,dx}{\int h^{-3}dx}, \qquad \text{or} \qquad h = \tfrac{3}{2}\mathrm{H}. \quad . \quad . \quad . \quad (33)$$

But this is not the requirement postulated. It suffices that the coefficient of δh on the right of (32) vanish over that part of the range where $h > h_1$, and that it be negative when $h = h_1$, so that a *positive* δh in this region involves a decrease in P, a negative δh here being excluded *a priori*. These conditions may be satisfied if we make $h = h_1$ from $x = 0$ at the edge where the layer is thin to $x = c_1$, where c_1 is finite, and $h = \tfrac{3}{2}\mathrm{H}$ over the remainder of the range from c_1 to $c_1 + c_2$, where $c_1 + c_2 = c$, the whole length concerned (fig. 2). For the moment we regard c_1 and c_2 as prescribed.

Fig. 2.

For the first condition we have by (8)

$$\tfrac{2}{3}h_2 = \mathrm{H} = \frac{c_1/h_1^2 + c_2/h_2^2}{c_1/h_1^3 + c_2/h_2^3},$$

so that

$$c_2/c_1 = k^2(2k-3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

determining k, where as before $k = h_2/h_1$. The fulfilment of (34) secures that $h=\frac{3}{2}\mathrm{H}$ over that part of the range where $h=h_2$. When $h=h_1$, $h-\frac{3}{2}\mathrm{H}$ is negative; and the second condition requires that over the range from 0 to c_1

$$\frac{\int h^{-3}x\,dx}{\int h^{-3}\,dx}-x$$

be positive, or since c_1 is the greatest value of x involved, that

$$\int h^{-3}x\,dx-c_1\int h^{-3}dx=+. \quad . \quad . \quad . \quad (35)$$

The integrals can be written down at once, and the condition becomes

$$k^3<c_2^2/c_1^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

whence on substitution of the value of c_2/c_1 from (34),

$$k(2k-3)^2>1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

If k be such as to satisfy (37) and c_2/c_1 be then chosen in accordance with (34) and regarded as fixed, every admissible variation of h diminishes P. But the ratio c_2/c_1 is still at disposal within certain limits, while c_1+c_2 $(=c)$ is prescribed.

In terms of k and c by (34)

$$c_1=\frac{c}{1+2k^3-3k^2}, \quad c_2=\frac{c(2k^3-3k^2)}{1+2k^3-3k^2}, \quad . \quad . \quad (38)$$

and by (7)

$$\frac{\mathrm{P}}{\mu\mathrm{U}}=\frac{1}{h_1^2}\left\{c_1^2(3-2k)+\frac{2c_1c_2+c_2^2}{k^2}\right\}=\frac{c^2}{h_1^2}\frac{2k-3}{1+2k^3-3k^2}=\frac{c^2}{h_1^2}f(k), \quad . \quad . \quad . \quad (39)$$

The maximum of $f(k)$ is 0·20626, and it occurs when $k=1·87$. The following shows also the neighbouring values:

k.	$f(k)$.	$k(2k-3)^2$.
1·86	0·20624	0·964
1·87	0·20626	1·024
1·88	0·20617	1·086

It will seen that while $k=1·86$ is inadmissible as not satisfying (37), $k=1·87$ is admissible and makes

$$\mathrm{P}=·20626\frac{\mu\mathrm{U}c^2}{h_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

no great increase on (18). It may be repeated that k is the ratio of the two thicknesses of the layer (h_2/h_1), and that by (34)

$$c_2/c_1 = 2{\cdot}588. \qquad (41)$$

This defines the form of the upper surface which gives the maximum total pressure when the minimum thickness and the total length are given, and it is the solution of the problem as proposed. But it must not be overlooked that it violates the supposition upon which the original equation (5) was founded. The solution of an accurate equation would probably involve some rounding off of the sharp corners, not greatly affecting the numerical results.

The distance $\bar{x}$ of the centre of pressure from the narrow end is given by

$$\bar{x} = 0{\cdot}4262c, \qquad (42)$$

differing very little from the value found in (20). From (10) with use of (38) we get

$$\frac{F}{\mu U} = \frac{4c(k-1)^2}{h_1(1+2k^3-3k^2)} = \frac{4c}{(2k+1)h_1}, \qquad (43)$$

and

$$\frac{F}{P} = \frac{4h_1(k-1)^2}{c(2k-3)}. \qquad (44)$$

If $k = 1{\cdot}87$,

$$F/P = 4{\cdot}091h_1/c, \qquad (45)$$

a little less than was found in (21). The maximum total pressure and the corresponding ratio F/P are both rather more advantageous in the arrangement now under discussion than for the simply inclined line. But the choice would doubtless depend upon other considerations.

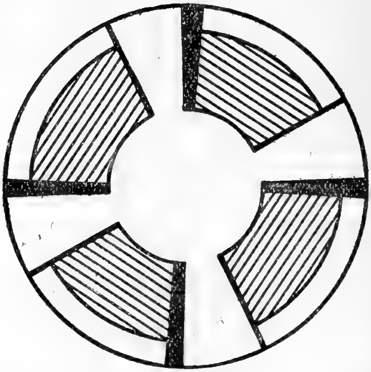
The particular case treated above is that which makes P a maximum. We might inquire as to the form of the curve for which F/P is a minimum, for a given length and closest approach to the axis of x. In the expression corresponding with (32), instead of a product of two linear factors, the coefficient of δh will involve a quadratic factor of the form

$$Bxh + Ch^2 + Dx + Eh + F, \qquad (46)$$

so that the curve is again hyperbolic in the general sense. But its precise determination would be troublesome and probably only to be effected by trial and error. It is unlikely that any great reduction in the value of F/P would ensue.

Fig. 3 is a sketch of a suggested arrangement for a footstep. The white parts are portions of an original plane surface. The 4 black radii represent grooves for the easy passage of lubricant. The shaded parts are slight depressions of uniform depth, such as might be obtained by etching with acid. It is understood that the opposed surface is plane throughout.

Fig. 3.



P.S. Dec. 13.—In a small model the opposed pieces were two pennies ground with carborundum to a fit. One of them—the stationary one—was afterwards grooved by the file and etched with dilute nitric acid according to fig. **3**, sealing-wax, applied to the hot metal, being used as a "resist." They were mounted in a small cell of tin plate, the upper one carrying an inertia bar. With oil as a lubricant the contrast between the two directions of rotation was very marked.

Opportunity has not yet been found for trying polished glass plates, such as are used in optical observations on "interference." In this case the etching would be by hydrofluoric acid *, and air should suffice as a lubricant.

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* Compare 'Nature,' vol. lxiv. p. 385 (1901); Scientific Papers, vol. iv. p. 546.

II. *On the Calculation of the Maximum Force between Two Coaxial Circular Currents.* By H. NAGAOKA, *Professor of Physics, Imperial University, Tokyo* [*].

THE problem of calculating the maximum force between two coaxial circular currents originated in the absolute measurement of electric current by means of a balance. The formula for calculating the force was developed by Lord Rayleigh [†] in his investigation on the electro-chemical equivalent of silver. Recently a similar method was used by Rosa, Dorsey, and Miller [‡], in the determination of the international ampere. The interesting question as to the position of the coils and the maximum force acting between them was taken up by F. W. Grover [§], who expressed the said quantities by means of Jacobi's q-series. In a note on the potential and the lines of force of a circular current [‖], I have shown how the expansion in q-series of ϑ-functions converges very rapidly in calculations of like nature. The expression for the mutual inductance between two coaxial coils and the force between the currents passing through them can be conveniently expressed in terms of q. Grover extended the expression for the force to terms involving q^{16} and q_1^{8} in the power series, thus increasing the accuracy of the expression to decimal places scarcely needed in practical measurements. From the integral expression for the force, we can by differentiation arrive at an expression giving the condition of maximum force. This method was followed by Grover, who obtained an expression for calculating the maximum force that can be applied for given coils in finding the distance between them. The expression in its final form is sufficiently convergent to be of practical value, but the approximation leading to the value of q which corresponds to the required maximum seems to offer another solution. Obviously the reduction of the integral involves the use of elliptic functions, which can be expressed in terms of ϑ-functions; there is, in addition, a factor containing the distance between the coils. This factor gives rise to a very convenient formula for finding the required distance when once the value of q is known. Thus the first step is essentially the evaluation of q. According to the method followed

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[*] Communicated by the Author.

[†] Rayleigh, B. A. R. p. 445 (1882); Phil. Trans. clxxv. pp. 411–460 (1884); Scientific Papers, ii. p. 278.

[‡] Rosa, Dorsey, & Miller, Bull. Bureau Stand. viii. pp. 269–393 (1911).

[§] Grover, Bull. Bureau Stand. xii. pp. 317–374 (1916).

[‖] Nagaoka, Journ. Coll. Sci., Tokyo, xvi. Art. 15 (1903); Phil. Mag. vi. p. 19 (1903); Proc. Math. Phys. Soc. vi. p. 156 (1911).

by Grover, this difficulty is overcome by finding the approximate value of the distance by means of Rayleigh's formula, and by using a relation which is rather empirical, and the ultimate result is arrived at after a number of successive approximations.

The process which I am now going to develop is similar to that already used in my former papers and is characterized by giving the value of q corresponding to the maximum force by a simple relation; it does not necessitate the knowledge of the approximate value of the required distance, but by two or three processes of approximation in finding the value of q, it leads to results which can be used in measurements of great accuracy even in the most unfavourable case.

Denote the radii of the coils by a and A, the distance between them by z, then the mutual inductance is given by

$$\mathrm{M}=4\pi\mathrm{A}a\int_0^{\pi}\frac{\cos\theta\,d\theta}{\sqrt{\mathrm{A}^2+a^2+z^2-2\mathrm{A}a\cos\theta}}, \quad . \quad . \quad (1)$$

whence the force between the unit currents passing through them is given by

$$\frac{\partial\mathrm{M}}{\partial z}=4\pi\mathrm{A}az\int_0^{\pi}\frac{\cos\theta\,d\theta}{(\mathrm{A}^2+a^2+z^2-2\mathrm{A}a\cos\theta)^{\frac{3}{2}}}. \quad . \quad (2)$$

For evaluating (1) and (2), we have to put, as usual,

$$4(s-e_1)(s-e_2)(s-e_3)=4s^3-g_2s-g_3=\mathrm{S},$$

where

$$e_1=\frac{2\beta}{\gamma}, \quad e_2=\frac{1-\beta}{\gamma}, \quad e_3=-\frac{1+\beta}{\gamma},$$

$$\beta=\frac{\mathrm{A}^2+a^2+z^2}{6\mathrm{A}a}, \quad \gamma=\left(\frac{2}{\mathrm{A}a}\right)^{\frac{1}{3}},$$

and $$u=\int_s^{\infty}\frac{ds}{\sqrt{\mathrm{S}}} \quad \text{or} \quad s=\wp(u).$$

We easily find that (2) is given by

$$\begin{aligned}\frac{\partial\mathrm{M}}{\partial z}&=-\pi\mathrm{A}a\gamma^4z\int_{\omega_3}^{\omega_2}\frac{\gamma\wp(u)+\beta}{\gamma\wp(u)-2\beta}du\\&=-\pi\mathrm{A}a\gamma^4z\left\{\omega_1-\frac{3e_1}{2(e_1-e_2)(e_1-e_3)}(\eta_1+e_1\omega_1)\right\}\\&=-\frac{\pi^4z}{\sqrt{\mathrm{A}a}}\left\{\vartheta_2{}^2(0)+\frac{1}{2\pi^2}\left(\frac{1}{\vartheta_0{}^4(0)}+\frac{1}{\vartheta_3{}^4(0)}\right)\vartheta_2{}''(0)\vartheta_2(0)\right\}\end{aligned}$$

$$. \quad . \quad . \quad (3)$$

Expressing the quantity under the parenthesis in terms of q, we obtain

$$F=\frac{\partial M}{\partial z}=\frac{192\pi^2 z}{\sqrt{Aa}}q^{\frac{5}{2}}\{1+20q^2+225q^4+1840q^6+12120q^8$$
$$+68052q^{10}+337465q^{12}+1513740q^{14}$$
$$+6247665q^{16}+\ldots\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

The expression in terms of q_1 is

$$F=\frac{\pi z}{16\sqrt{Aaq_1}}\Big\{(1+12q_1-192q_1^2+1232q_1^3-5634q_1^4$$
$$+21648q_1^5-73600q_1^6+226944q_1^7-648189q_1^8+\ldots)$$
$$-12q_1 \operatorname{logn}\frac{1}{q_1}.(1-10q_1+60q_1^2-300q_1^3+1300q_1^4$$
$$-4884q_1^5+16320q_1^6-49920q_1^7+142500q_1^8+\ldots)\Big\}.$$
$$. \quad . \quad . \quad (I'.)$$

These formulæ include many terms extended by Grover.

The condition required for the maximum force is simply given by

$$\frac{\partial F}{\partial z}=0 \quad \text{or} \quad \frac{\partial^2 M}{\partial z^2}=0.$$

This evidently leads to the following relation between two integrals

$$3z^2\int_0^{\pi}\frac{\cos\theta\, d\theta}{(A^2+a^2+z^2-2Aa\cos\theta)^{\frac{5}{2}}}$$
$$+\int_0^{\pi}\frac{\cos\theta\, d\theta}{(A^2+a^2+z^2-2Aa\cos\theta)^{\frac{3}{2}}}=0. \quad . \quad (4')$$

This equation was utilized by Grover for finding the distance between the coils when the said condition is satisfied. The evaluation of the integrals is not an easy process, and for finding q from (4′), we have to assume an approximate value of z given by Rayleigh's formula, and arrive at the final result by successive steps. The solution of the problem can, however, be obtained in another form by finding the value of q directly from the known value of a/A.

Reverting to the expression (I.) for F, we see that (4) is equivalent to

$$\frac{\partial \mathrm{F}}{\partial q} = 0,$$

provided z can be expressed by means of ϑ-functions. For this purpose we take advantage of the relation

$$\frac{e_1 - e_2}{e_2 - e_3} = \frac{(\mathrm{A}-a)^2 + z^2}{4\mathrm{A}a},$$

$$\frac{e_1 - e_3}{e_2 - e_3} = \frac{(\mathrm{A}+a)^2 + z^2}{4\mathrm{A}a}.$$

Expressed by means of ϑ-functions

$$\frac{z^2}{\mathrm{A}a} = 4\frac{\vartheta_0^4(0)}{\vartheta_2^4(0)} - \frac{(\mathrm{A}-a)^2}{\mathrm{A}a}$$

and

$$\frac{z^2}{\mathrm{A}a} = 4\frac{\vartheta_3^4(0)}{\vartheta_2^4(0)} - \frac{(\mathrm{A}+a)^2}{\mathrm{A}a}.$$

Adding them, we obtain, by utilizing the relation

$$\vartheta_3^4(0) - \vartheta_0^4(0) = \vartheta_2^4(0),$$

$$\frac{z^2}{\mathrm{A}a} = 2\left(\frac{\vartheta_3^4(0) + \vartheta_0^4(0)}{\vartheta_3^4(0) - \vartheta_0^4(0)}\right) - \left(\frac{\mathrm{A}}{a} + \frac{a}{\mathrm{A}}\right), \quad . \quad . \quad (5)$$

which enables us to expand z in terms of q.

For expansion, it is convenient to use the product series of ϑ's: thus

$$\vartheta_3(0) = \Pi(1-q^{2n})(1+q^{2n-1})^2,$$

$$\vartheta_0(0) = \Pi(1-q^{2n})(1-q^{2n-1})^2,$$

by which $(1-q^{2n})$ being common to the numerator and denominator is eliminated. On evaluation we obtain

$$\frac{z^2}{\mathrm{A}a}$$

$$= \frac{1 + 20q^2 - 62q^4 + 216q^6 - 641q^8 + 1636q^{10} - 3778q^{12} + 8248q^{14}}{4q} - r, \quad . \quad . \quad . \quad (6)$$

where $r = \frac{\mathrm{A}}{a} + \frac{a}{\mathrm{A}}$.

For expressing z in terms of q_1, we remark that

$$\frac{z^2}{\mathrm{A}a}=4\frac{\vartheta_2^4\left(0,-\frac{1}{\tau}\right)}{\vartheta_0^4\left(0,-\frac{1}{\tau}\right)}-\frac{(\mathrm{A}-a)^2}{\mathrm{A}a},$$

$$\frac{z^2}{\mathrm{A}a}=4\frac{\vartheta_3^4\left(0,-\frac{1}{\tau}\right)}{\vartheta_0^4\left(0,-\frac{1}{\tau}\right)}-\frac{(\mathrm{A}+a)^2}{\mathrm{A}a},$$

whence we obtain

$$\frac{z^2}{\mathrm{A}a}=2\left(\frac{\vartheta_2^4\left(0,-\frac{1}{\tau}\right)+\vartheta_3^4\left(0,-\frac{1}{\tau}\right)}{\vartheta_3^4\left(0,-\frac{1}{\tau}\right)-\vartheta_2^4\left(0,-\frac{1}{\tau}\right)}\right)-\left(\frac{\mathrm{A}}{a}+\frac{a}{\mathrm{A}}\right). \quad . \quad (5')$$

A similar remark as for (6) applies to the calculation of (7) in terms of q_1, resulting in the expression

$$\frac{z^2}{\mathrm{A}a}=2+64q_1(1+8q_1+44q_1^2+192q_1^3+718q_1^4+2400q_1^5$$
$$+7352q_1^6+20992q_1^7+\ldots.)-r. \quad . \quad . \quad . \quad . \quad (6')$$

It is generally sufficient to retain q_1^8, the remaining terms being negligible even in the most accurate work that can at present be attempted.

Squaring (I.) and substituting for $\frac{z^2}{\mathrm{A}a}$ in (6) and (6′), we find the required condition to be equivalent to

$$\frac{\partial \mathrm{F}^2}{\partial q}=0.$$

All the rest is a simple mechanical operation; the final equations for obtaining q or q_1 from the known ratio of the dimensions of the coils $r=\frac{\mathrm{A}}{a}+\frac{a}{\mathrm{A}}$ $\left(=\alpha+\frac{1}{\alpha}\right.$ according to Grover's notation) may be presented under the following form

$$r=\frac{1}{5q}+6{\cdot}8\,q-51{\cdot}6\,q^3+614{\cdot}4\,q^5-7934{\cdot}6\,q^7$$
$$+103683{\cdot}6\,q^9-1353676{\cdot}4\,q^{11}+17649300{\cdot}6\,q^{13}$$
$$-230{\cdot}1\times10^6\,q^{15}+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II.})$$

The coefficients are exact up to that of q^{13}; the rest of higher terms are insignificant and may be neglected even in the most unfavourable cases. For practical calculation, the above equation is applicable to values of r ranging from $r=\infty$ to $r=2·5$, the latter corresponding to the case of $a/\mathrm{A}=0·5$. From this value down to $r=2$, which is the smallest occurring in practice for the case of coils of equal radii, we have to use an expression in terms of q_1, which can be expressed by substituting (5) in (I.) and making $\frac{\partial \mathrm{F}^2}{\partial q}=0$. The formula is as follows :—

$$\begin{aligned} r\{&1-16q_1+376q_1^2-4672q_1^3+38948q_1^4-252192q_1^5 \\ &+1365888q_1^6-6463360q_1^7+27500946q_1^8-\ldots. \\ &-120q_1^2 \operatorname{logn}\frac{1}{q_1}.(1-16q_1+154q_1^2-1120q_1^3 \\ &+6680q_1^4-34272q_1^5+156268q_1^6-\ldots.\} \\ =2\{&1-264q_1^2+4096q_1^3-36828q_1^4+245760q_1^5 \\ &-860712q_1^6+6414336q_1^7-27377262q_1^8+\ldots. \\ &+24q_1^2 \operatorname{logn}\frac{1}{q_1}.(3-64q_1+702q_1^2-5376q_1^3 \\ &+32712q_1^4-169374q_1^5+775908q_1^6-\ldots.)\}. \quad \text{(II}'.) \end{aligned}$$

It is worthy of remark that (II′.) is to be used from $a/\mathrm{A}=0·5$ to 1, the value of q_1 ranging from $q_1=0·0113$ to $q_1=0$. The series in the numerator and denominator converge very rapidly, and we can sometimes utilize the formula for a somewhat larger value of q_1; the only tedious process of calculation is finding $\operatorname{logn}\frac{1}{q_1}$.

When once the value of either q or q_1 is found, we can calculate z by (6) or (6′); and then F by (I.) or (I′.).

As an example of practical calculation, let us take the case $a/\mathrm{A}=0·5$; i. e. $r=2·5$. From the first three terms in (II.), we find by inspection that q is nearly $0·11$; putting this value in (II.) and calculating to q^7, we find that the right-hand side is about $2·5059$, giving $\Delta r=-0·0059$, and hence $\Delta q=0·00054$; next putting $q=0·11054$, we find

$$\Delta r=-0·00003476$$

by taking all the terms into account; thus the final value of q corresponding to the required maximum is

$$q=0·110543224, \quad \log q=\bar{1}·04353213.$$

The number given by Grover is $\log q=\bar{1}{\cdot}4035322$; the value for z/A is 0·38353439, while Grover gives for it 0·3835341, agreeing to within a ten-millionth part.

For the convenience of practical calculation, the following tables of Δr and r are given for different values of q and q_1 respectively, calculated for me by Mr. Shobei Shimizu; Δr in the first table are residuals

$$614{\cdot}4\,q^5-7934{\cdot}6\,q^7+\ldots\ldots-230{\cdot}1\times10^6q^{15}$$

in (II.), while r in the second table refers to those calculated according to (II′.).

q.	Δr.	Δ_1.
0·00	0·0000000	
		1
0·01	0·0000001	
		19
0·02	0·0000020	
		128
0·03	0·0000148	
		469
0·04	0·0000616	
		1244
0·05	0·0001860	
		2705
0·06	0·0004565	
		5147
0·07	0·0009712	
		8885
0·08	0·0018597	
		14251
0·09	0·0032848	
		21574
0·10	0·0054422	
		31175
0·11	0·0085597	

q.	r.	Δ_1.
0·000	2·0000000	
		339002
0·001	2·0339002	
		366859
0·002	2·0705861	
		389351
0·003	2·1095212	
		409255
0·004	2·1504467	
		427502
0·005	2·1931969	
		444584
0·006	2·2376553	
		460811
0·007	2·2837364	
		476382
0·008	2·3313746	
		491447
0·009	2·3805193	
		506120
0·010	2·4311313	

From the table, it will be noticed that the residuals apparently cover a wide range in the calculation with q; this is at once evident from the fact that the range covered by the q-series is very great compared with that in which the q_1-series are applicable. This is in no way an impediment in practical calculations, as the formula in terms of q is simpler than that in q_1; especially in numerical evaluation of q from known values of r, the formula (II.) is characterized by the great facility with which all the terms can be calculated and the required approximations brought to test. As all the rest of the calculation depends on the value of q, when once this is accurately known, the distance between the coils and the force exerted at that distance can be found by the formulæ (1) and (6).

As regards the utility of the solution of the present problem, it would be unnecessary to spend words on its bearing in the construction of the current-balance for the absolute measurement of electric current. The numerical data calculated by Grover are of great value in researches of this kind. The solution which I have here given may be of service in the direct calculation when the dimensions of the coils are given. How far the accuracy of the instrument can be relied upon is of great interest to me, as I believe that the instrument can be used for a purpose totally different from the usual measurement of the electric current, and which seems not yet to have been well noticed.

The most exact method of measuring relative values of gravity is that of comparing the periods of invariable pendulums at the place of observation with those at the standard station. The great inconvenience and difficulty accompanying the method of observation lie in the extremely accurate measurement of time; the rate of the clock must be known to 1/60th part of a second per day, if the period is to be exact to one part in five million. For this we have to take a transit instrument of fairly large aperture, and when obstructed by bad weather we have to wait for days. In addition to this, the occasional change of the clock-rate necessitates the unintermittent continuation of observation which imposes a great burden on the observer. This tedious and unwelcome obstruction to the usual method of gravity determination may, to a great extent, be overcome by using a current-balance instead of invariable pendulums. The strength of the current is to be evaluated by means of known resistance of the circuit and the terminal potential difference, for which the electromotive force of the cadmium cell must be relied upon. It is a question if we can bring the constancy of the cadmium cells and of the coils to the same order as that of the pendulum and clock. The weight counterbalancing the attraction of the coils is an immediate measure of the force of gravity at the place of observation. For this purpose it is perhaps necessary to design an instrument anew in a transportable form, and construct the coils such that the attraction is of sufficient amount to give the desired accuracy. It must however be well noticed that the method of current-balance is not free from objections, as the current is liable to fluctuations and the coils are heated in course of measurement and give rise to convection current; moreover, the correction to be applied to such disturbances

18200

is difficult to calculate and almost impossible to estimate exactly. There may be other methods of dispensing with astronomical observations in gravity measurements, but I believe that the method of current-balance is one of the most accurate that can be easily brought into practice without sacrificing to any great extent the degree of precision usually attainable with invariable pendulums.

In the theory of atomic constitution, it is generally assumed that there are rings of electrons in rapid rotation: these are no doubt equivalent to currents and must exert mutual influence upon each other. When such atoms combine into a molecule and are in such a position that the planes of the rings are parallel to each other, then the position of maximum force between two circular rings here discussed will be of some significance in the atomic configuration of molecules.

III. *On the Nodal-Slide Method of Focometry.* *By* J. A. TOMKINS, *A.R.C.S.* (*Lond.*), *Lecturer in Physics, Technical College, Bradford* *.

PROF. A. ANDERSON has recently described (Phil. Mag. Jan. 1917, p. 157) an elegant method of determining the focal length, and other constants, of a lens system based upon a general theorem of which the ordinary nodal-slide method is but a particular application. This theorem is that for any lens system there is always one, and only one, point on the optic axis such that a small rotation of the system about a perpendicular axis through it will cause no lateral displacement of the image of an object in a given position. This point was shown by Prof. Anderson to divide the distance between the object and image, and also that between the nodal points, externally in a ratio equal to the value of the magnification.

A numerical example of the determination of the focal length of a diverging combination by this method was given, but it was pointed out by Mr. R. E. Baynes (Phil. Mag. April 1917, p. 357) that these data yielded very different results for the focal length and the positions of the nodal points when calculated in different ways. These discrepancies were explained by Prof. Anderson (Phil. Mag. July 1917, p. 76), who discussed the effect of various errors and showed that, while the method gave quite satisfactory values for the focal length, it failed to do so for the distance between

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* Communicated by the Author.

the nodal points, because small errors are multiplied by the numerical value of *d*, the distance through which the combination is moved, which may be large. He also described another method which gave satisfactory values for this distance. In a third paper (Phil. Mag. Sept. 1917, p. 174) he gave some further properties of this point, which he terms the *nul* point. There are, however, two possible sources of error mentioned by Prof. Anderson, viz. (1) want of precision in determining whether there is any displacement of the image, and (2) error in determining its position, which seem to call for further consideration.

With reference to the first it is to be noted that in the ordinary nodal-slide method there is one, and only one, possible axis of rotation of the lens system, viz. that passing through the second nodal point, whereas in the general method described by Prof. Anderson there is an infinite, or doubly infinite, number of possible axes. The object of this communication is to investigate the best position, if any, for the *nul* point, and to compare the results with those obtained by the ordinary nodal-slide method.

For the purpose of observing the displacement of the image the best position will be that for which a given small displacement of the axis from the *nul* position will, for a given small rotation of the lens system, produce the greatest displacement of the image.

To determine this it is necessary first to find an expression for the displacement of the image due to a small rotation about any axis.

Fig. 1.

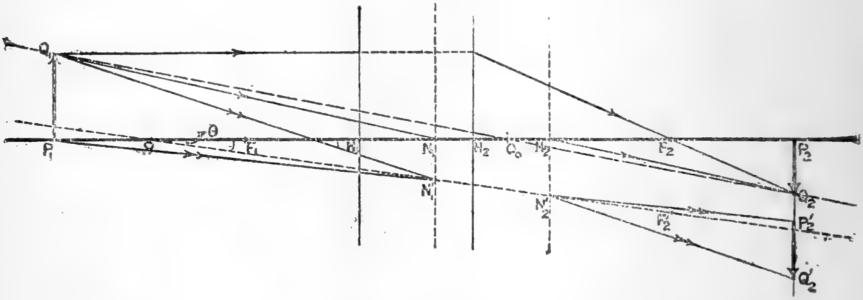

Fig. 1 shows the displacement produced by a convergent combination in the general case in which the first and last media are different, and in which, therefore, the principal and nodal points are not coincident. P_1Q_1 and P_2Q_2 are the object and image respectively, H_1 and H_2 the principal points, N_1 and N_2 the nodal points, F_1 and F_2 the principal foci. Suppose the system to be rotated about O through a small angle θ so that the principal axis moves into the position indicated by the dotted line. Then, to a first approximation, the nodal points N_1 and N_2 will move into the positions N_1' and N_2' and the image P_2Q_2 will move in the same plane into the position $P_2'Q_2'$ obtained by drawing $N_2'P_2'$ and $N_2'Q_2$, parallel to P_1N_1' and Q_1N_1' respectively.

Let $N_1O=l, \quad N_2N_1=a, \quad N_1P_1=u, \quad N_2P_2=v.$

Then $N_1N_1'=l\theta$ and $N_2N_2'=(a+l)\theta.$

Hence the displacement of the image is given by

$$\begin{aligned} s=Q_2Q_2'&=N_2N_2'+\frac{P_2N_2}{N_1P_1}\cdot N_1N_1' \\ &=(a+l)\theta-\frac{v}{u}\cdot l\theta \\ &=\{a+l(1-m)\}\theta, \quad \ldots \ldots \quad (1) \end{aligned}$$

where $m=\frac{v}{u}$, the magnification.

In order that the displacement of the image may be zero for a given value of θ we must have

$$s=a+l(1-m)=0,$$

or

$$m=\frac{a+l}{l}. \quad \ldots \ldots \ldots \quad (2)$$

There is thus one, and only one, position of the axis of rotation for which there will be no displacement of the image, viz. that which divides the distance between the nodal points externally in a ratio equal to the value of the magnification—a result obtained in another way by Prof. Anderson. The best position for the axis of rotation will, as already pointed out, be that for which $\frac{ds}{dl}$ is a maximum subject to the condition given by equation (2).

Differentiating (1) and substituting from (2), we get

$$\frac{ds}{dl}=(1-m)\theta=-\frac{a\theta}{l}. \quad \ldots \ldots \quad (3)$$

The rate of change of the displacement thus varies directly as a, the distance between the nodal points and inversely as l, the distance of the *nul* point from the first nodal point.

It is greatest when $l=0$, *i. e.* when $m=\infty$, and the *nul* point coincides with N_1. It thus appears that the best position is attained when the axis of rotation passes through the first nodal point, in which case the object, real or virtual, will be situated at the first principal focus.

The light will then emerge as a parallel pencil, and the image can be viewed through a telescope focussed for parallel rays, as in one of the well-known methods of determining the focal length of a thin lens. A further advantage of this position is that the nodal points are determined directly as in the ordinary nodal-slide method.

We will now apply these formulæ to the example given in Prof. Anderson's second paper (Phil. Mag. July 1917, p. 76), where

$$OP_1 = x_1 = 142 \text{ cm.}; \quad OP_2 = y_1 = 9{\cdot}4 \text{ cm.}; \text{ and } m_1 = \frac{9{\cdot}4}{142} = 0{\cdot}0662.$$

$$OP_1' = x_2 = 29{\cdot}1 \text{ cm.}; \quad OP_2' = y_2 = 8{\cdot}3 \text{ cm.}; \text{ and } m_2 = \frac{8{\cdot}3}{29{\cdot}1} = 0{\cdot}285.$$

$$d = 113{\cdot}8 \text{ cm.}; \quad H_2H_1 = a = 2{\cdot}43 \text{ cm.}$$

The distance H_1O in Prof. Anderson's figure (Phil. Mag. Jan. 1917, p. 158), in which the principal and nodal points are coincident, is given by

$$H_1O = -\frac{OP_1}{OP_1 - OP_2} \cdot H_2H_1 = -\frac{xa}{x-y} = -\frac{a}{1-m}, \quad . \quad . \quad (4)$$

which is but a particular case of the general expression obtained by Prof. Anderson.

Hence in the first position $H_1O = -\dfrac{a}{1-m_1} = -2{\cdot}60$ cm.

and in the second position $H_1O = -\dfrac{a}{1-m_2} = -3{\cdot}40$ cm.

These positions in relation to the nodal points are shown to scale in fig. 2.

Fig. 2.

Suppose now that $\theta = 5° = 0{\cdot}0873$ radian and that the axis of rotation is moved 1 mm. from the *nul* position towards the object.

Then, in the first case, $l_1 = -2{\cdot}60 + 0{\cdot}1 = -2{\cdot}5$ cm.; $m_1 = 0{\cdot}0662$; and the displacement of the image calculated by equation (1) is 0·0083 cm. In the second case we have $l_2 = -3{\cdot}40 + 0{\cdot}1 = -3{\cdot}3$ cm.; $m_2 = 0{\cdot}285$ and the displacement is 0·0062 cm. If, however, the ordinary nodal-slide method had been employed, in which case O would coincide with H_2, then $l=-2\cdot33$, $m=0$, and the displacement under the same conditions would have been 0·0087 cm.

These small displacements may also be calculated more simply by using the approximate formula

$$\delta s=\frac{ds}{dl}\,.\,\delta l=-\frac{a\theta}{l}\,.\,\delta l. \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

In the first case $l_1=-2\cdot6$ and $\delta s=0\cdot0082$, in the second case $l_2=-3\cdot4$ and $\delta s=0\cdot0062$; while in the ordinary nodal-slide method $l=-2\cdot43$ and $\delta s=0\cdot0087$. These results agree closely with those obtained by the more exact formula. Or, to put the matter rather differently, if we suppose the smallest observable displacement to be 0·01 cm., then the distances through which the axis would have to be moved from the *nul* position to produce this displacement would be 1·20, 1·61, and 1·15 mm. in the three cases respectively. These examples show that in this particular case the *nul* point can be determined with greater accuracy by the ordinary nodal-slide method than by the general method.

If, however, a convergent combination forming a real image, as in fig. 1, had been employed, the *nul* point would lie between N_1 and N_2, and its position could be determined with greater accuracy by the general method than by the ordinary nodal-slide method. In this case the *nul* point is indicated by O_0, which is the point of intersection of Q_1Q_2 with the optic axis.

Again, if, with the same convergent combination, a virtual image were formed, a case not likely to arise in the determination of focal length, the magnification would be positive and greater than unity. The *nul* point would therefore lie on the side of N_1 remote from N_2, and its position could be determined by the general method with greater or less accuracy than by the ordinary nodal-slide method according as it is at a distance from N_1 less or greater than a respectively.

If the nodal points are determined by either of the two direct nodal-slide methods, which will here be distinguished as the *first* and *second nul methods* respectively, the principal foci and focal lengths may be most readily and directly found by measuring the distances from the *nul* point of the object in the first method and of the image in the second.

Now in obtaining the position of the image there will be a certain range along the axis within which the object may lie and yet produce a distinct image on a fixed screen. And, conversely, there will be a certain range within which the screen may be moved and yet give a distinct image of an object in a given position.

The former range is called the *depth of focus* of the instrument, and may be shown (Heath's 'Optics,' pp. 269, 270) to be given by the expression

$$\Delta\xi = \pm \frac{\epsilon}{m \tan \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where ξ is the distance of the object from the entrance pupil, ϵ the maximum value of the circle of indistinctness, m the magnifying power, and α the angular aperture of the instrument.

In the second case the range through which the screen may be moved may be similarly shown to be given by the equation

$$\Delta\xi' = \pm \frac{m\epsilon}{\tan \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where ξ' is the distance of the image from the exit pupil.

In order to determine with precision the positions of the object and image the property required will be the inverse of these. In the first case, it will vary directly as the magnifying power and the angular aperture, and, in the second case, it will vary inversely as the magnifying power and directly as the angular aperture.

By comparing these magnitudes in the two cases just referred to it will be seen that the principal foci can be determined with greater accuracy by the first than by the second nul method.

Finally, the similarity of object and image will depend on the resolving power of the system, which increases with the angular aperture, and this again will be greater in the first nul method than in the second.

The conclusions arrived at in the foregoing were tested by the following experiments:—

Two thin plano-convex lenses, each having a focal length of 25·8 cm., were mounted with their convex surfaces inwards and at a distance of 12·9 cm. apart. The calculated focal length of the combination was 17·2 cm., and the distances of the first and second nodal points were 8·6 cm. measured inwards from the first and second lenses respectively, so that the distance between the nodal points was 4·3 cm.

The positions of the nodal points were then determined by the first and second nul methods.

In the first method it was found that the lens system could be moved through a total distance 1·7 mm. before any sensible displacement of the image was caused by a small rotation, whereas in the second method a motion through 5 mm. was required to produce the same result.

The principal foci of the combination were also found, a needle being used as object in the first method and to locate the image in the second method. The distances through which the needle could be moved before any sensible parallax was observed were 2 mm. and 4 mm. respectively.

This particular combination was chosen in order to get a fairly large distance between the nodal points, but as the lenses were uncorrected quite a small rotation caused the image to become confused owing to oblique aberration. A further test was therefore made with a 15 in. (38·1 cm.) Ross photographic lens in which the distance between the nodal points was only 1·6 cm. The following results were obtained for the ranges of adjustment in the two cases:—

First Method. For nodal points 0·2 mm.; for principal foci 3·5 mm.

Second Method. For nodal points 1·5 mm.; for principal foci 11 mm.

These figures must be taken as indicating the relative rather than the absolute accuracies of the two methods, since the nodal-slide employed, though not of the roughest, had no fine adjustments. They appear, however, to show that the first nul method is in every respect superior to the second.

In conclusion, I wish to thank my colleague, Mr. J. E. Rycroft, for his kindness in making the diagrams.

IV. *On the Value of the Mechanical Equivalent of Heat.*

By T. CARLTON SUTTON, *B.Sc.*[*]

THE following values of the Mechanical Equivalent of Heat (for references see Kaye and Laby's 'Tables' and Griffiths' 'Thermal Measurement of Energy') have been reduced to joules per mean calorie :—

1843	Joule	4·173
1878	Rowland..................	4·184
1893	Griffiths	4·188
1894	Schuster and Gannon	4·185
1897	Reynolds and Moorby	4·184
1899	Callendar and Barnes	4·184
1900	Griffiths (deduced)	4·184
1906	Jager and Steinwehr	4·188
1908	Crémieu and Rispail........	4·189
1909	Barnes (deduced)	4·185
1911	Bousfield..................	4·179
1914–5	Sutton-Henning (see below)..	4·185

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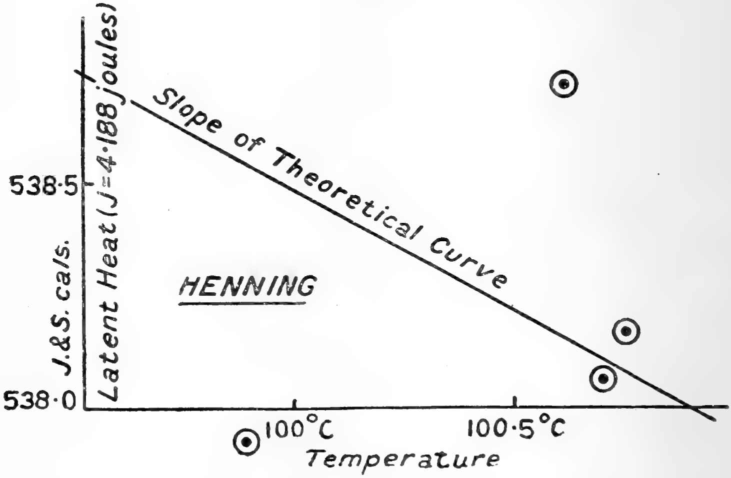
[*] Communicated by Principal E. H. Griffiths.

The results obtained since the year 1905 are not as concordant as might have been expected. It may be of interest therefore to compare the value of the heat of vaporization of water at 100° C. obtained electrically at the Reichsanstalt by Henning (*Ann. d. Phys.* 1906–9) with that obtained by the author in terms of the mean calorie directly (Proc. Roy. Soc. April 1917).

Henning's results are shown in the following graph, which indicates that the value at 100° C. is 538·5 ± 0·3. Probably the constant errors are even less than 0·3.

Henning's work may also be checked by comparing it with that of Griffiths' at 30° C. The values for the heat of vaporization at that temperature are in good agreement, 579·9 and 579·3 mean calories respectively.

Fig. 1.

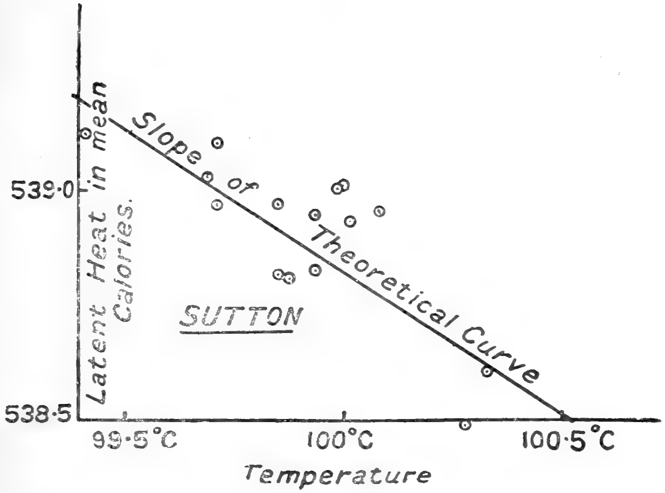


The corresponding values obtained by the author (Proc. Roy. Soc. April 1917) give 538·8 ± 0·1 mean calories at 100° C.

If, then, Henning's mean value is correct to 1 part in 2000, the deduced value of the Mechanical Equivalent should be correct to within two units in the third decimal figure.

Henning uses the Jager and Steinwehr value of the 15° calorie to convert joules to calories, and takes the Reichsanstalt value of the e.m.f. of the Clark Cell. What he actually determines as the Energy of Vaporization at 100° C. is 538·5 × 4·188 Reichsanstalt joules.

Fig. 2.



The theoretical curve is obtained from Clapeyron's relation

$$L=(V-v)T\frac{\partial p}{\partial T},$$

to show the change of latent heat with temperature. (Since the values of V and $\frac{\partial p}{\partial T}$ are not known with the same accuracy as L, the slope of the curve only has been taken from this relation. This is sufficient for the purpose of reducing the results.)

It will be noticed that the more accurate methods of measurement give values which show clearly the agreement with the thermodynamical result.

On comparing this with the author's results, which are measured directly in mean calories, it follows that

$$538{\cdot}5\times 4{\cdot}188=538{\cdot}88\ \mathrm{J},$$

i. e., $J=4{\cdot}185\pm 0{\cdot}002$ joules per mean calorie.

It will be noticed that this is in exact agreement with the values obtained by the earlier experimenters—Schuster and Gannon, Reynolds and Moorby, Callendar and Barnes,—and with the values deduced on separate occasions (from all data then available) by Griffiths and by Barnes; on the other hand, this value and those obtained by the more recent experimenters are certainly too discordant to be reconciled.

This agreement between the values given by the classical experiments and the value given by a method so different as the present substantially increases the probability that the value of the Mechanical Equivalent lies between 4·184 and 4·185 joules per mean calorie.

V. *Light Distribution round the Focus of a Lens, at various Apertures. By* L. SILBERSTEIN, *Ph.D., Lecturer in Natural Philosophy at the University of Rome* *.

Bibliographic and Introductory.

THE distribution of the intensity of light in the neighbourhood of a caustic has been studied by Sir G. Airy as early as in 1838 (Camb. Phil. Trans. vol. vi.), for the case, however, of an unlimited beam only. Some of the effects of spherical aberration of limited beams upon the central intensity and the definition of the image have been investigated by Lord Rayleigh in 1879 (Phil. Mag. vol. viii. pp. 403–411). The chief problem considered by him relates to a beam of *cylindrical* waves of rectangular section, their aberration being assumed proportional to the cube of the lateral coordinate x. The solution is reduced to the evaluation of an integral of the form $\int \cos(ax+bx^3)dx$. Availing himself of the numerical results of the mechanical quadratures recorded in Airy's paper, Lord Rayleigh calculates and draws three intensity curves for the focal plane (*loc. cit.* p. 406), corresponding to the case of no aberration ($b=0$), and to those in which the marginal aberrations amount to $\frac{1}{4}$ and $\frac{1}{2}$ period. The practically important consequence drawn from the aspect of these curves is that "aberration begins to be distinctly mischievous when it amounts to about a quarter-period." In the next case studied, that of symmetrical aberration proportional to x^4, Lord Rayleigh calculates, by the aid of a series, the intensity at the central point only. Passing, finally, to beams of circular section, he limits himself again to the calculation of the *central intensity*, viz. in the case of axially symmetric aberration proportional to the fourth power of the distance, and finds that, as in the preceding cases, aberration begins to be prejudicial when it mounts up to a quarter of a period. This result has since become widely known, having been incorporated into the *Enc. Brit.*† and several English text-books. In 1884 Lommel investigated the distribution of illumination in the diffraction image of a point given by a circular aperture ‡. The problem in this case reduces to the evaluation of integrals of the form

$$\int J_0(ax)\cos(bx^2)x\,dx,\quad \int J_0(ax)\sin(bx^2)x\,dx$$

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* Communicated by the Author.

† *Cf.* Lord Rayleigh's article on "Diffraction of Light" in the 11th ed. of *Enc. Brit.* vol. viii. pp. 238–255.

‡ E. Lommel, *Bayer. Akad. d. Wiss.* vol. xv.

which are developed into power series of the upper limit or of its reciprocal. Lommel's results have been adapted, in 1891, to the problem of pin-hole photography by Lord Rayleigh*, whose paper, besides a theoretical and experimental discussion of the subject, gives also five curves exhibiting the distribution of light round the centre of the image corresponding to different apertures. As will be seen later on, the typical phase aberrations of wave-surfaces emerging from lenses differ in kind from those involved in the pin-hole problem. Investigations aiming directly at a diffractional treatment of the images produced by lenses were undertaken, in 1893, by R. Straubel, whose papers are quoted in Winkelmann's 'Handbuch' (1906, vol. vi. p. 106), but unfortunately are not accessible to the writer, and a little later by K. Strehl in a very attractive book entitled 'Theorie des Fernrohrs auf Grund der Beugung des Lichts' (Barth: Leipzig, 1894)†. The earlier part of the work being dedicated to preparatory matter, Strehl investigates in Chap. V. and VI. the intensity along the optical axis and in the focal plane of an *aplanatic* object-glass, and since this is materially the same problem as that of a circular aperture treated by Lommel (*loc. cit.*), Strehl bases himself upon Lommel's results, as had already been done by Lord Rayleigh, and enunciates a number of theorems on the general features of the light distribution for the case in question. The effects of "*spherical aberration*" are treated in Chap. VII., where the intensity formula is developed for the case in which the emergent wave is an ellipsoid of revolution; the series developments (pp. 62–63) are very complicated, and it does not appear that they could conveniently be applied to concrete numerical calculations. They enable Strehl, however, to enunciate some general theorems about the symmetry relations of the diffraction effects associated with spherical aberration of the said kind, and an important conclusion on the true measure of the mischievous effect of spherical aberration (p. 65). In a sense, Strehl is right in declaring that by his investigation "the problem of spherical aberration is completely solved." So

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* Phil. Mag. vol. xxxi. (1891) pp. 87–99.

† After that Strehl has published several papers in *Zeitschr. f. Instrumentenkunde* for 1895–98 which I have not been able to consult. However, to judge from Winkelmann's quotation, the ground covered by these papers is essentially that of Strehl's book. Winkelmann (*loc. cit.* p. 403) quotes also, in connexion with the diffractional theory of the telescope, Ch. André's "Etude de la diffraction dans les instruments d'optique" (Paris, 1876), without, however, describing the contents of this paper, which was published in *Ann. sc. de l'école norm. supérieure*, vol. v.

fact it is, in its essence. None the less it seems desirable to treat problems relating to concrete lenses with all numerical (or graphical) details, and to express the results in terms of the attributes of the given lens or lens system.

It is precisely the object of the present paper to give a fully worked out example of this kind, as a part of investigations undertaken at the instance of Messrs. Adam Hilger in connexion with their Lens Interferometer which exhibits *ad oculos*, through its "contour map," the phase retardation of all the elements of an originally plane wave produced by the passage through a given lens. The example selected for the present purpose relates to the simplest possible lens, viz. the plano-convex lens, traversed by a beam of finite circular section along the optical axis. It has seemed that, owing to its extreme simplicity, it may be the best to show the reader a practicable and easy way of dealing with more complicated telescopic objectives.

To complete the above bibliographic sketch we have still to mention that the remaining chapters of Strehl's work are dedicated to the diffractional aspect of astigmatism and coma which are treated on similar lines as spherical aberration, to cylindrical waves, etc. These subjects, however, are beyond the scope of the present communication. Lastly, we have to mention a more recent paper by James Walker (Proc. Phys. Soc. London, vol. xxiv. 1912, pp. 160–164) in which the subject of Strehl's Chapter VII., viz. the intensity due to a rotationally ellipsoidal wave, is again taken up. Here the expression for the intensity is developed into a complicated double series (*cf.* last line of the paper quoted) which, although mathematically unobjectionable, does not seem convenient for actual calculation *.

It must be kept in mind that for physical applications hardly more than two significant figures in the final light intensity are required. Under these circumstances the method of mechanical quadratures or a graphic method, analogous to that of the Cornu spiral, seems by far the most convenient. Although laborious for very accurate work, it certainly becomes very handy when only the said degree of precision is aimed at. It will be explained and applied in what follows.

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* It has occurred to me that some of Walker's intermediate formulæ, as for instance that on top of p. 163, would easily yield a more "commodious" expression than is the final one.

General Formulæ.

Let the portion s of the surface of a fixed sphere, of centre O and radius R, which we will take as our reference sphere, be the seat of monochromatic luminous oscillations of constant amplitude a, but of different phases η,

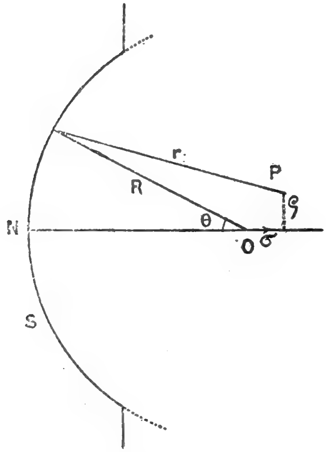
$$a \cos\left(\frac{2\pi t}{T} - \eta\right).$$

Given the distribution of η over s, find the intensity of light at the centre O and at points P near O. Let ds be an element of the reference surface, r its distance from the point P in question, c the light velocity and $\lambda = cT$ the wave-length, in vacuo. Then the usual way of applying Huyghens' principle gives, for the luminous vibration at P,

$$-\frac{a}{\lambda}\int \frac{1}{r} \sin\left(nt - \frac{2\pi}{\lambda} r - \eta\right) ds, \quad . \quad . \quad . \quad (1)$$

where $n = 2\pi/T$. Let N (fig. 1) be the pole of angular coordinates, *i. e.* ON the axis, θ and ϕ the pole distance and the longitude of an element ds. Let ϕ_P be the longitude

Fig. 1.



of the point P, further ρ its distance from the axis and σ its axial distance from O, away from N. Then, neglecting the squares of ρ/R, σ/R,

$$r = R\left[1 - \frac{\rho}{R} \sin\theta \,.\, \cos(\phi - \phi_P) + \frac{\sigma}{R} \cos\theta\right].$$

Introduce this into (1), writing in the denominator, $r = R$, and develop, remembering that $n\left(t-\frac{R}{c}\right)$ is constant all over s. Then the integral will split into two others multiplied by sin, and by $\cos n\left(t-\frac{R}{c}\right)$. Thus, taking the intensity at the sphere s as our unit, *i. e.* putting $\frac{1}{2}a^2=1$, the intensity of light at P will be

$$\mathrm{I} = \frac{1}{\lambda^2 R^2}(C^2+S^2), \quad \ldots \ldots \quad (2)$$

where

$$C, S = \int \cos, \sin \left[\eta - \frac{2\pi\rho}{\lambda} \sin\theta \,.\, \cos(\phi-\phi_{\mathrm{P}}) + \frac{2\pi\sigma}{\lambda}\cos\theta\right].\,ds. \quad (3)$$

These formulæ are valid for any distribution of phase, $\eta=\eta(\theta, \phi)$, and for any form of the edge of s (diaphragm).

If, as corresponds to the subject of the present paper, η is a function of θ alone[*], and if the edge is itself a circle of latitude, $\theta=\theta_1=$const., then we have *axial symmetry* round ON, so that C, S become independent of the longitude of the point P, and we can put $\phi_{\mathrm{P}}=0$. The most convenient integration variables being now θ and ϕ themselves, take

$$ds = R^2 \sin\theta \,.\, d\theta \,.\, d\phi$$

and integrate over $\phi=0$ to 2π and over $\theta=0$ to θ_1. Develop (3) and introduce the abbreviations

$$\alpha = \frac{2\pi\rho}{\lambda}, \quad \beta = \frac{2\pi\sigma}{\lambda}. \quad \ldots \ldots \quad (4)$$

Then, after some easy transformations, the expression for the light intensity at the point $\mathrm{P}(\rho, \sigma)$ will be

$$\mathrm{I} = \left(\frac{2\pi R}{\lambda}\right)^2 |w|^2, \quad \ldots \ldots \quad (5)$$

where $|w|$ is the absolute value of the complex integral

$$w = \int_0^{\theta_1} e^{i(\eta+\beta\cos\theta)} \mathrm{J}_0(\alpha\sin\theta) \,.\, \sin\theta \, d\theta. \quad \ldots \quad (6)$$

For the focal plane, as we shall henceforth call the plane

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[*] That is, if the "contour lines" exhibited by the Lens Interferometer are circles of latitude.

$\sigma=0$, or $\beta=0$, for cophasal vibrations ($\eta=0$), and for small θ_1, the integral (6) reduces to the familiar form

$$w = \int J_0(\alpha\theta)\theta\, d\theta = \frac{\theta_1}{\alpha} J_1(\alpha\theta_1),$$

giving for the intensity the well-known expression

$$I = \left(\frac{R\theta_1}{\rho}\right)^2 J_1^2\left(\frac{2\pi\rho}{\lambda}\theta_1\right).$$

Formulæ (5) and (6) are valid for any angular aperture $\theta_1 < \frac{\pi}{2}$, and for any given axially symmetrical phase distribution $\eta=\eta(\theta)$. The axial displacement β (of P) from the centre enters only through the factor $e^{i\beta\cos\theta}$; the transversal displacement α enters through the zeroth Bessel function, and the phase heterogeneity through $e^{i\eta}$. Notice in passing that, by (6), a phase distribution of the type $\eta=g\,.\cos\theta$ * is equivalent to a *rigid shift* of the whole image (luminous region) along the axis by $-\frac{\lambda g}{2\pi}$ and is, therefore, unessential. In the case of a wave issuing from a lens, with the centre O placed in its focus, and NO along its optical axis, the series development of η does not at all contain such a term, *i. e.*, practically, no term in θ^2. In fact, it will be seen that, with the above choice of the reference sphere, the series for η starts, for any "uncorrected" lens (such as the simple plano-convex lens), with θ^4, the next term being in θ^6.

In all practical cases, connected with lenses, the angular semiaperture θ_1 hardly exceeds 4° or 5°. Under these circumstances we can write in (6), both in the factor of J_0 and in J_0 itself, $\sin\theta \doteq \theta$, and in the exponential, $\beta\cos\theta \doteq \beta-\frac{1}{2}\beta\theta^2$. The first term, β, giving only the factor $e^{i\beta}$ outside the integral, does not influence the value of $|w|^2$ and can, therefore, be rejected.

Thus, introducing the new variable

$$u = \theta^2,$$

the formula for small θ_1 will be

$$2w = \int_0^u e^{i(\eta-\beta u)} J_0(\alpha\sqrt{u})\,.\,du, \quad . \quad . \quad . \quad . \quad (7)$$

where η is a given function of u. The corresponding intensity at P will be determined by (5), α and β being

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* Which in the case of a small θ_1 becomes $\eta=-\frac{1}{2}g\theta^2$, the additive constant term g being irrelevant.

the coordinates of P (with $\lambda/2\pi$ as unit length). The upper limit of the integral stands for θ_1^2, the suffix having been dropped.

For certain forms of η the integral (7) can be developed into more or less complicated series, as has been done by various authors, especially for the focal plane ($\beta=0$). In general, however, whether the phase distribution η be given graphically or analytically, the gaol will be reached much more easily and quickly by a number of comparatively small steps Δu or "du," starting from 0 and leading to the required $u=\theta_1^2$, either by mechanical quadratures or by a graphical construction analogous to that of the famous Cornu spiral. The latter method can now and then be checked by the former, which is particularly advisable for the first stages of the procedure. A curve drawn in this manner (for any fixed α, β) has also the advantage of exhibiting the local intensity as a function of the aperture; the process corresponds, in fact, to a gradual opening of (the pupil of) a lens, from no to the full required aperture.

Consider the plane of the complex variable

$$2w = x+iy = z,$$

so that, L being the distance of the point z from the origin, the corresponding intensity will be

$$\mathrm{I} = (\pi \mathrm{R} L/\lambda)^2.$$

To every fixed point $\mathrm{P}(\alpha, \beta)$ of the luminous region belongs, in the z-plane, a curve whose element is fully given by

$$dz = e^{i(\eta-\beta u)}\mathrm{J}_0(\alpha\sqrt{u})\,.\,du.$$

Thus the sloping angle ϵ at any point of the P-curve will be

$$\epsilon = \eta-\beta u, \quad \ldots\ldots\ldots \quad (8)$$

the length of an arc element

$$dl = |\,\mathrm{J}_0(\alpha\sqrt{u})\,|\,.\,du, \quad \ldots\ldots \quad (9)$$

and, therefore, the curvature

$$k = \frac{d\epsilon}{dl} = \frac{1}{|\mathrm{J}_0|}\left(\frac{d\eta}{du}-\beta\right). \quad \ldots\ldots \quad (10)$$

By means of these formulæ any P-curve can easily be drawn step by step, much in the same way as the Cornu spiral. For any point P of the focal plane ($\beta=0$) the angle ϵ is simply equal to the phase excess η, and for points outside the focal plane it is smaller by βu. The arc elements dl corresponding to equal steps du, for axial points $P(\alpha=0)$, are all equal, as in the case of the Cornu spiral. Outside the optical axis, however, the steps dl become smaller and smaller as we approach the first zero of J_0, which happens the sooner the larger α. At the same time the radius of curvature $\frac{1}{k}$ dwindles to nothing (unless $\frac{d\eta}{du}=\beta$); at the apertures corresponding to the zeros of $J_0(\alpha\sqrt{u})$ the P-curves have cusps, from which they emerge with increasing steps to be lessened again when the next zero is approached, and so on. For $u=0, \eta=0$ and, therefore, $\epsilon=0$; thus, all P-curves start from the origin tangentially to the x-axis. Again, since, in all cases of actual interest, $\frac{d\eta}{du}=0$ for $u=0$, the initial curvature of all curves belonging to the focal plane is nil; and the initial curvature for any other point P is $k=-\beta=-\frac{2\pi\sigma}{\lambda}$. But for any not highly corrected lens the term $\frac{d\eta}{du}$ in (10) soon becomes the more important one, unless σ mounts to many wave-lengths. Thus, with the exception of the first steps, the sloping angle is given primarily by η, and the curvature by $\frac{d\eta}{du}:|J_0|$, the modifications due to an axial displacement being comparatively small.

In most cases, therefore, it will be found that it is sufficient to draw in detail the P-curves for the *focal plane* only, when the construction data become

$$\epsilon=\eta, \quad dl=|J_0|\,du, \quad k=\frac{1}{|J_0|}\frac{d\eta}{du}.$$

If $d\eta/du$ preserves its sign, the sense of the windings of a P-curve remains throughout the same (say, anticlockwise), even in passing through a cusp. If η passes through a maximum or minimum (and $J_0\neq 0$), the curve becomes flat and inflected. If a P-curve, no matter after how many windings, happens to pass again through the origin, the light at P[*] is extinguished, and while the curve passes on (increasing aperture), the light will reappear there, and so on. A good check at any stage of the curve construction may be to measure the whole length l of the path already covered and to compare it with its correct length, which can

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[*] *i. e.*, along the circle through P, centred upon and normal to the optical axis.

at once be derived from (9). Up to the first cusp, *i. e.*, as long as the first root of J_0 is not exceeded, we have, in virtue of that formula,

$$l = \frac{2\sqrt{u}}{\alpha} J_1(\alpha\sqrt{u}), \quad \alpha\sqrt{u} \leqq 2{\cdot}4048. \quad . \quad . \qquad (10')$$

If $\alpha\sqrt{u}$ is contained between the first two roots x_1, x_2 of $J_0(x)=0$, then

$$l = \frac{2}{\alpha^2}\int_0^{x_1} J_0(x) x\,dx - \frac{2}{\alpha^2}\int_{x_1}^{x} J_0(x) x\,dx = \frac{4x_1}{\alpha^2} J_1(x_1) - \frac{2x}{\alpha^2} J_1(x),$$

$$\text{i. e.} \quad l = \frac{2{\cdot}4969}{\alpha^2} - \frac{2\sqrt{u}}{\alpha} J_1(\alpha\sqrt{u}), \quad x_1 \leqq \alpha\sqrt{u} \leqq x_2, \quad (10\,a)$$

and so on.

The arc length (10′) has, as well as the chord L, a noteworthy physical meaning. In fact, it represents the value of $2w$, for any point of the focal plane, for a perfect wave $(\eta=0)$, so that the *normal* intensity, due to such a wave, can be written, by (5),

$$N = \left(\frac{\pi R}{\lambda}\right)^2 l^2, \quad \alpha\sqrt{u} \leqq 2{\cdot}4048. \quad . \quad . \quad . \qquad (11)$$

On the other hand, the intensity due to the defective wave has been

$$I = \left(\frac{\pi R}{\lambda}\right)^2 L^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

More generally, for a ring-shaped aperture contained between θ_a and θ_b, the chord L in (12) is to be replaced by the chord ab joining the corresponding pair of points u_a, u_b of the curve. Thus, *up to the first cusp*, the ratio of the two intensities, at the same point of the focal plane, is

$$I : N = L^2 : l^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

In words, the defective intensity is to the normal intensity as the squared *chord* to the squared *arc* of the P-curve, from the origin to the point u in question. The latter will obviously be the longer of the two, the more so the greater the value attained by η at the aperture u. More generally, for a ring-shaped aperture θ_a, θ_b, the ratio of the two intensities will be equal to the square of the ratio of the chord L_{ab} to the arc l_{ab} of the curve. The simple relation (13) will enable us to see at a glance on the P-curves the relative value of the intensity due to the defective wave, for various apertures. The "definition" of the image will be exhibited by the mutual position of curves corresponding to various points of the field.

When the first cusp is exceeded, the arc length l loses its simple physical meaning *. But this will occur the farther away from the focus, the smaller the angular semi-aperture $\sqrt{u}$.

If the wave-surface which we are projecting, so to speak, on the sphere s, is produced by a centred lens or lens system from, say, a plane, axially incident, perfect wave, and if the centre O is the focus of that lens, then, as has already been mentioned, η will be of the form $b\theta^4 + b'\theta^6 + \ldots$, or in terms of u,

$$\eta = bu^2 + b'u^3 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

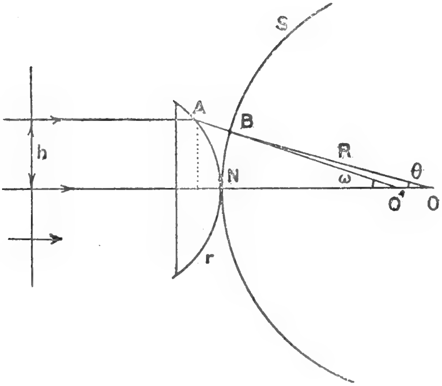
The constant coefficients b, b', etc. will depend on the individual properties of the lens.

The above generalities will be made plain presently by working out in detail the case of the most simple lens. Other cases can be treated on similar lines.

The Plano-Convex Lens.

Let a beam of parallel, cophasal, rays of monochromatic light of wave-length λ impinge upon the flat face of a plano-convex lens of refractive index n. Let r be the radius of curvature of the convex (spherical) face. Let our centre O be the focus, and $R = \overline{\mathrm{ON}}$ its shortest distance from the convex face of the lens. As our reference surface s (hitherto so called) let us take the sphere of radius R and centre O (fig. 2). A ray incident at height h will be refracted at A,

Fig. 2.



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* If $x_1 < \alpha\sqrt{u} < x_2$, and if l_1 be the arc from the origin to the first cusp, then l, in (11) and (13), is to be replaced by l_1 *minus* the arc length from the first cusp to the point u; and so on.

and will continue its path in air along BO′, piercing the sphere s at B. Write $\overline{\mathrm{AB}}=m$. Then, taking $\eta=0$ at N, the phase at B, corresponding to the angle $\mathrm{NOB}=\theta$, will be

$$\eta=\frac{2\pi}{\lambda}\left\{m-nr\left(1-\sqrt{1-\frac{h^2}{r^2}}\right)\right\}. \quad . \quad . \quad (15)$$

The required length m will be determined, for every h [*], by the following trigonometric set :

$$\sin i=\frac{h}{r}, \quad \sin i'=\frac{nh}{r}; \quad \omega=i'-i,$$

$$s'=\mathrm{O'N}=r\left(\frac{\sin i'}{\sin\omega}-1\right); \quad \Delta s'=s'-s_0'=s'-\mathrm{R},$$

$$\theta=\omega-i_2, \text{ where } \sin i_2=\frac{1}{\mathrm{R}}\,|\Delta s'|\sin\omega,$$

and, finally,

$$m=\frac{r\sin i-\mathrm{R}\sin\theta}{\sin\omega}.$$

By successive series developments I find, up to $\left(\frac{h}{r}\right)^6$ inclusively, the spherical aberration [†]

$$\Delta s'=-\frac{r}{2}\frac{n^2}{n-1}\left(\frac{h}{r}\right)^2\left\{1+\frac{n^2-n+1}{4}\left(\frac{h}{r}\right)^2+\frac{n(n^2+1)(n-1)+1}{8}\left(\frac{h}{r}\right)^4\right\}, \quad (16)$$

and $R=s_0'=\frac{r}{n-1}$, of course ; next, for the angle θ,

$$\sin\theta=(n-1)\frac{h}{r}\left\{1-\frac{n(n-1)}{2}\left(\frac{h}{r}\right)^2-\frac{n(n-1)(1+3n-n^2)}{8}\left(\frac{h}{r}\right)^4\right\}, \quad (17)$$

and ultimately, for the phase, as defined by (15),

$$\eta=-\frac{2\pi r}{\lambda}\cdot\frac{n^2(n-1)}{8}\left(\frac{h}{r}\right)^4\left\{1-\frac{n^2-n-1}{2}\left(\frac{h}{r}\right)^2\right\}, \quad (18)$$

a series starting with the *fourth* power of the relative aperture, as has been expected [the coefficient of the second power in the series of m is $\frac{1}{2}nr$, and this is exactly cancelled by the

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[*] Subject to the obvious condition $h<\frac{r}{n}$, beyond which there is total reflexion.

[†] Which, although not needed ultimately, may be of some interest.

corresponding coefficient in the second term of (15)]. If, for instance, $n=1\cdot5$, then

$$\eta = -\frac{2\pi r}{\lambda}\cdot\frac{9}{64}\left(\frac{h}{r}\right)^4\cdot\left(1+\frac{1}{8}\frac{h^2}{r^2}\right),$$

and
$$\sin\theta = \frac{h}{2r}\left(1-\frac{3}{8}\frac{h^2}{r^2}-\frac{39}{128}\frac{h^4}{r^4}\right).$$

Passing to the application of formula (18) to our diffractional problem, we can readily limit ourselves to values of $\frac{h}{r}$ less than or at any rate not considerably exceeding $\frac{1}{10}$. Under such circumstances, (17) can be written, with sufficient accuracy,

$$\sin\theta \doteqdot \theta = (n-1)\frac{h}{r} = \frac{h}{R},$$

and, therefore,
$$\eta = -\frac{\pi r}{\lambda}\frac{n^2}{4(n-1)^3}\theta^4.$$

The negative sign can be dropped, which makes no difference for points of the focal plane, and, for all other points, requires only a sign reversal of their abscissæ β. Thus, in our previous notation,

$$\eta = bu^2, \quad b = \frac{\pi r n^2}{4\lambda(n-1)^3}. \quad . \quad . \quad . \quad . \qquad (19)$$

It will be convenient to introduce as the integration variable, instead of u,

$$v = u\sqrt{\frac{2b}{\pi}} = u\sqrt{\frac{rn^2}{2\lambda(n-1)^3}}, \quad . \quad . \quad . \quad . \qquad (20)$$

so that

$$\eta = \frac{\pi v^2}{2}, \quad \frac{d\eta}{du} = v\sqrt{2\pi b}, \quad \text{and} \quad \alpha\sqrt{u} = \alpha'\sqrt{v},$$

where
$$\alpha' = \alpha\left(\frac{\pi}{2b}\right)^{1/4} = \frac{2\pi\rho}{\lambda}\left(\frac{\pi}{2b}\right)^{1/4}. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Thus the sloping angle, the arc element, and the curvature of any P-curve will be, by (8), (9), (10), remembering the sign reversal of β,

$$\left.\begin{aligned} \epsilon = \frac{\pi v^2}{2}+\beta\left(\frac{\pi}{2b}\right)^{1/2}v, \quad dl = \left(\frac{\pi}{2b}\right)^{1/2}|J_0(\alpha'\sqrt{v})|\,dv, \\ k = \frac{1}{|J_0|}\left(\sqrt{2\pi b}\,v+\beta\right). \end{aligned}\right\} \qquad (22)$$

It will be enough to consider in detail the intensity * at points P of the *focal plane* only. Then the required data are

$$\epsilon = \frac{\pi v^2}{2}, \quad dl = \left(\frac{\pi}{2b}\right)^{1/2} |J_0(\alpha' \sqrt{v})|\, dv, \quad k = \frac{\sqrt{2\pi b}\, v}{|J_0|}, \quad (22\,a)$$

where now α' defines the distance of P from the optical axis, and v the aperture at any stage. If L be the chord from the origin to the point v of the P-curve, then the intensity at P is, as in (12),

$$I = \left(\frac{\pi R}{\lambda}\right)^2 L^2 = \frac{\pi^2 r^2}{\lambda^2 (n-1)^2} L^2. \quad . \quad . \quad . \quad (23)$$

This is the graphical equivalent of the explicit expression

$$I = \left(\frac{\pi R}{\lambda}\right)^2 \frac{\pi}{2b} (A^2 + B^2), \quad . \quad . \quad . \quad (23\,a)$$

where

$$A, B = \int_0^v \cos, \sin \left(\frac{\pi v^2}{2}\right) \cdot J_0(\alpha' \sqrt{v})\, dv, \quad . \quad (24)$$

i. e. $\sqrt{\frac{\pi}{2b}}$ times the previous x and y, respectively.

The angle ϵ is independent of α', i.e. of ρ. Thus the tangents to the various curves at corresponding points (same v, i. e. same aperture) are parallel to one another, while their curvature is inversely proportional to $|J_0|$. For $v=1$, $\sqrt{2}$, $\sqrt{3}$, etc., ϵ becomes equal to one, two, three, etc., right angles. For the central point or focus, A, B become identical with the usual Fresnelian integrals,

$$A_0 = C(v) = \int_0^v \cos \frac{\pi v^2}{2}\, dv, \quad B_0 = S(v) = \int_0^v \sin \frac{\pi v^2}{2}\, dv;$$

the corresponding curve is the Cornu spiral †, well-known in connexion with the straight-edge problem. With increasing distance ρ from the focus, *i. e.* with increasing α', the curves change rapidly, and soon lose their simple spiral character, owing to the Bessel factor of the integrand. Starting from $v=0$ and ascending by steps "dv"$=0{\cdot}1$, the desired P-curves, or α'-curves, are at first more conveniently drawn by quadratures, the rectangular coordinates A, B being calculated by (24) as far as $v=1{\cdot}5$ or $2{\cdot}0$; after that

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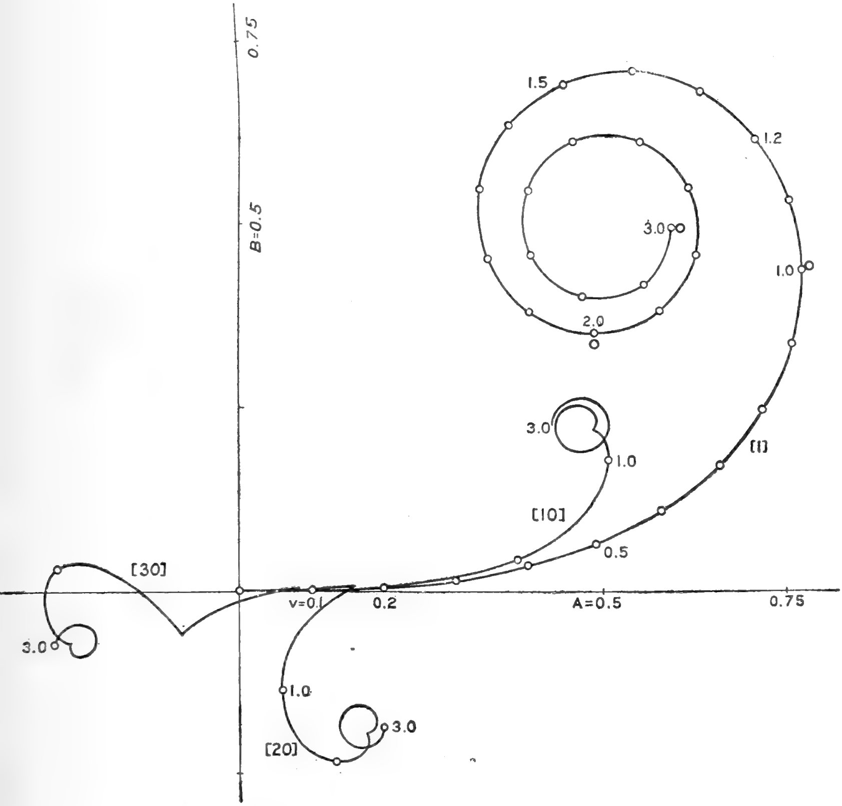
* In applying the diffraction formula to the present case we disregard, of course, the inequalities of amplitude at points of the sphere s due to the slightly different incident angles i and glass thicknesses traversed by the rays.

† In the present case, only one of its two branches comes into play.

the curves are more conveniently drawn by the help of (22 a). Formulæ (10′), or (10 a), etc. may, now and then, be used for a control of the length l of arc attained.

In this manner four curves have been drawn on millimetrated paper and are reproduced in the annexed fig. 3.

Fig. 3.



The curves marked [1], [10], [20], and [30], correspond to

$$\alpha' . \frac{10^{3/2}}{2\pi} = 1, \ 10, \ 20, \text{ and } 30.$$

If, for example, $n=1{\cdot}5$, and $r=\frac{1}{9}10^6\lambda$ (say, $\lambda=\frac{1}{2}$ micron, and $r=5{\cdot}55\ldots$ cm.), then these curves will correspond, by (21) and (19), to points of the focal plane whose distances from the focus are $\rho=\lambda$, 10λ, 20λ, and 30λ, respectively. Similarly for any other plano-convex lens. The circlets on the curves mark, by their centres, the apertures corresponding to $v=0{\cdot}1$, $0{\cdot}2$, and so on, up to $v=3{\cdot}0$. The solitary circlets near $v=1{\cdot}0$, $2{\cdot}0$, $3{\cdot}0$ belong to $\rho=0$, *i. e.* to the central or focal curve which is the usual Cornu spiral. It will be seen that the curve [1] deviates but little from the central one. The intensity, proportional to the square of distance from the origin, increases with v, or with the square of the aperture, first rapidly and then more and more slowly, and reaches its *maximum* a little beyond $v=1{\cdot}2$. Then it falls to a minimum which is considerably smaller than that maximum, and so on. The spiral character of the curve [1] would go on for over 3000 windings (v over 140) which, however, for any reasonable lens, would lie much beyond the limit-aperture ($h=r/n$) ; the spiral would reach its cusp, or first zero of J_0, much beyond that limit. The next curve, [10], is very soon deprived of its simple spiral character ; the Bessel function vanishes already at $v=1{\cdot}465$, where $k=\infty$, and the curve has a *cusp* in the neighbourhood of which the stations $v=1{\cdot}1$, $1{\cdot}2$, etc. are more and more crowded. Then the curve emerges from the singular point in an elegant fashion and follows on a nearly circular path ; it is drawn up to $v=3{\cdot}0$ only, but even beyond that it would never go far away from the spot. The intensity, for [10], reaches its maximum a little beyond $v=2{\cdot}7$; the value of this maximum is only about $\frac{1}{3}$ of that of the curve [1]. The curve [20] has one cusp at $v=0{\cdot}3662$, and another at $v=1{\cdot}929$. It also is drawn up to $v=3{\cdot}0$. Finally, the curve [30] has in the interval studied as many as three cusps, one between $v=0{\cdot}1$ and $0{\cdot}2$, another between $0{\cdot}8$ and $0{\cdot}9$, and yet another between $2{\cdot}1$ and $2{\cdot}2$. Neither the [30]-, nor the [20]-curve goes far away from the origin. The illumination is here very scanty, especially at $v \doteqdot 1{\cdot}2$ which is the most favourable aperture for the centre, and also for [1]. Other details can be seen from the drawing itself. Here but two more remarks: First, that it seems very doubtful whether there is at all a curve repassing exactly through the origin, *i. e.* whether there is at all a rigorously dark ring round the focus of the lens (physically it is enough that the light becomes very weak already for [20]). And, second, that the best illumination at the focus and, at the same time, the best definition is reached a little beyond $v=1{\cdot}2$. Opening the plano-convex lens so far as $v=2{\cdot}0$, for instance, would not only darken the focus, but make the intensity at [10] nearly as great as at the focus.

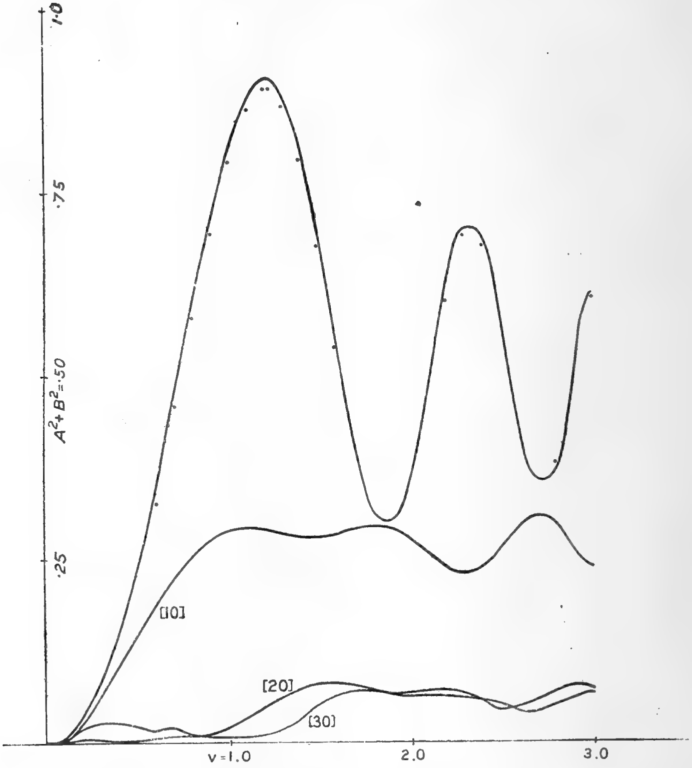
In the process of constructing the curves of fig. 3 many of the values of A, B needed have been calculated by quadratures ; to make the set complete all others have also been calculated. These values of A, B, as defined by (24), are not by themselves interesting, and will not be given here; the square sums A^2+B^2, however, proportional to the intensity, (23 *a*), may be useful. They are collected, together with those for $\alpha'=0$, in the following Table, which may be a good supplement to the P-curves themselves. The second column contains the sum of the squares of the Fresnelian integrals, the third etc. the values of A^2+B^2, for $v=0{\cdot}1$, $0{\cdot}2$ up to $v=3{\cdot}0$, corresponding to $\alpha'=2\pi\,.\,10^{-3/2}$ etc., that is, to the above curves [1], and so on.

$$A^2+B^2=\left|\int_0^v e^{i\frac{\pi v^2}{2}} J_0(\alpha'\sqrt{v})\,dv\right|^2.$$

v.	$\alpha'=0$.	$\alpha'\,.\,\frac{10^{3/2}}{2\pi}=1$.	10.	20.	30.
0·0	0·0000	0·0000	0·0000	0·0000	0·0000
0·1	·0099$_8$	·0099$_5$	·0096	·0068	·0042
0·2	·0399	·0399	·0327	·0174	·0050
0·3	·0898	·0895	·0663	·0243	·0025
0·4	·1591	·1583	·1058	·0253	·0003
0·5	·2466	·2450	·1474	·0218	·0010
0·6	·3499	·3471	·1879	·0162	·0042
0·7	·4648	·4604	·2243	·0202	·0078
0·8	·5849	·5781	·2542	·0099	·0101
0·9	·7004	·6913	·2758	·0131	·0108
1·0	·8004	·7886	·2886	·0213	·0101
1·1	0·8728	0·8571	0·2932	0·0337	0·0103
1·2	·9004	·8847	·2918	·0481	·0117
1·3	·8788	·8668	·2875	·0618	·0183
1·4	·8040	·7952	·2838	·0722	·0318
1·5	·6848	·6792	·2833	·0774	·0494
1·6	·5410	·5408	·2868	·0777	·0643
1·7	·4065	·4106	·2923	·0747	·0715
1·8	·3164	·3228	·2957	·0711	·0712
1·9	·3029	·3044	·2923	·0691	·0673
2·0	·3564	·3630	·2795	·0693	·0636
2·1	0·4781	0·4786	0·2599	0·0701	0·0611
2·2	·6123	·6049	·2411	·0687	·0614
2·3	·6981	·6852	·2332	·0634	·0617
2·4	·6921	·6786	·2434	·0554	·0588
2·5	·5926	·5860	·2693	·0494	·0521
2·6	·4537	·4578	·2973	·0504	·0449
2·7	·3593	·3712	·3094	·0595	·0430
2·8	·3718	·3820	·2957	·0714	·0486
2·9	·4846	·4832	·2636	·0771	·0589
3·0	0·6132	0·5987	0·2368	0·0715	0·0659

The curves in fig. 4 (which could also be taken from those of fig. 3) are drawn * directly according to this table, with v as abscissæ and A^2+B^2 as ordinates, and thus represent the intensity at selected places of the focal plane

Fig. 4.



as a function of the aperture of the lens. It will be remembered that v is proportional to the square of the aperture, viz., by (20),

$$v=\sqrt{\frac{rn^2}{2\lambda(n-1)^3}}\cdot\left(\frac{h}{R}\right)^2, \quad R=\frac{r}{n-1}. \quad . \quad . \qquad (25)$$

Thus, for example, if $n = 1·5$ and $r=\frac{1}{9}10^6\lambda$, then $\left(v=1000\frac{h}{2r}\right)^2$. The uppermost curve in fig. 4 corresponds

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* My thanks are due to Mr. W. Widigjör for the execution of both sets of curves.

to the centre or the focus, and to the usual Cornu spiral; that curve is well-known from the distribution of light outside the shadow of a semiplane; here its physical meaning is different: it represents the central intensity as function of (the area of) the aperture. It has its first, and highest, maximum at about $v=1{\cdot}23$, as already mentioned. A number of isolated points below and near this curve represent the intensity at the circle $\alpha'=\frac{2\pi}{\sqrt{1000}}$ (i.e. $\rho=\lambda$, for the said example). These points are too near the first curve to be joined into a continuous line. The next curve, [10], corresponds to $\alpha'=\frac{20\pi}{\sqrt{1000}}$ (i.e. $\rho=10\lambda$ for the above example); its first, as well as its second, maximum is very flat. Its interesting feature is that it nearly reaches the focal curve at about $v=1{\cdot}9$. The moral is obvious: at $v=1{\cdot}23$ the definition as well as the light intensity are excellent, at $v=1{\cdot}9$ very bad; at about $v=2{\cdot}3$ much better again. The next curve, [20], corresponds to $\alpha'=\frac{40\pi}{\sqrt{1000}}$, and the lowest, [30], to $\alpha'=\frac{60\pi}{\sqrt{1000}}$.

To form an opinion about the intensity as function of distance from the focus, at any fixed aperture $v\leqq 3{\cdot}0$, it is enough to read the above numerical table in horizontal rows (instead of columns). The corresponding curves would again corroborate the above remark, viz. that the greatest central intensity and the best definition of the image produced by a plano-convex lens, of refractive index n and curvature radius r, are obtained at an aperture corresponding to about $v=1{\cdot}23$, and to be determined for any concrete case by (25). If, for instance, $n=1{\cdot}5$ and $\lambda=\frac{1}{2}$ micron, then the best relative aperture is, in round figures,

for r =	5·56 cm.	2·78 cm.	6·9 mm.	1 mm.
h/r =	0·035	·041	·057	·090

The last of these would be a very minute lens indeed. Practically, not going much below $r=3$ cm., the best relative aperture will not exceed $\frac{1}{25}$. Opening the lens up to $v=1{\cdot}9$ would spoil the central intensity and the definition considerably. The next best aperture, after $v=1{\cdot}23$, would correspond to about $v=2{\cdot}3$; the next favourable opportunity will lie a little beyond $v=3$. The marginal phase retardation is, for the best aperture $v=1{\cdot}23$, $\eta=1{\cdot}51\frac{\pi}{2}$ or a little over 3/8 of a period.

Returning once more to the table on p. 45, notice that the v-values arrayed in the first column have themselves a simple meaning, their squares being proportional to the "normal" central intensity N_0 which would correspond to a perfect wave. Thus the relative central intensity, which is the squared ratio of chord and arc of the Cornu spiral, is given by

$$I_0 : N_0 = (A_0^2 + B_0^2) : v^2,$$

i. e. simply by the figures of the second divided by the squares of those of the first column. Thus, for $v = 0{\cdot}5$, $I_0 : N_0 = \frac{247}{250}$, and for $v = 1$, or η equal to a quarter-period, $I_0 : N_0 = {\cdot}8004$, agreeing with Lord Rayleigh's result of 1879 *, and so on. The relative intensity at the focus decreases steadily, of course; the absolute intensity, due to the defective wave, increases only up to a certain maximum and oscillates in decreasing amplitudes round a limit corresponding to the asymptotic value $A_0^2 + B_0^2 = \frac{1}{2}$. The central intensity by itself cannot, of course, inform us about the "definition." It so happens however that, in the case of the plano-convex lens at least, the best aperture, $v \doteq 1{\cdot}23$, for the focal absolute intensity is also sensibly the best for the definition of the image.

So much about the distribution of intensity in the focal plane of the lens. The intensity at points outside the focal plane will require but a few remarks, owing to the relative smallness of the axial intensity gradient (for small θ) already hinted at. Consider, for instance, the points of the optical *axis* itself, that is, $\alpha' = \alpha = 0$. Then, according to (22), the arc element is $dl = \left(\frac{\pi}{2b}\right)^{1/2} dv$, precisely as for the focal or Cornu-spiral itself, and, the sloping angle and the curvature of a β-curve,

$$\epsilon = \frac{\pi v^2}{2} + \beta\left(\frac{\pi}{2b}\right)^{1/2} v, \quad k = v\sqrt{2\pi b} + \beta, \quad . \quad . \quad (26)$$

where $\beta = 2\pi\sigma/\lambda$ measures the axial coordinate of the point in question. Take, for instance, our previous example $n = 1{\cdot}5$, $r/\lambda = \frac{1}{9}10^6$. Then, by (19),

$$\epsilon = \frac{\pi v^2}{2} + \frac{\beta}{1000} v, \quad k = 1000\pi v + \beta.$$

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* Phil. Mag. viii. p. 409. Rayleigh's figure, 0·8003, differs but insignificantly from the above one.

The β-curve even for $\sigma=10\lambda$ will, therefore, deviate but very little from the central or Cornu-spiral. In other words, the *axial* intensity-gradient, for appreciable v, will be very small (as compared with the transversal one, $dI/d\rho$). Both curves will start from the origin horizontally. The Cornu spiral is initially flat while the β-curve has the initial curvature $\beta=2\pi\sigma/\lambda$, and, if $\sigma<0$, an inflexion point for $v=\frac{|\sigma|}{500\lambda}$, *i. e.*, even for $|\sigma|=50\lambda$, very near the origin. With the exception of such details the β-curves deviate but insignificantly from the focal Cornu spiral, and it would therefore be hardly worth the trouble to draw them accurately.

London, October 1917,
Research Dept., Adam Hilger, Ltd.

VI. *Interferometers for the Experimental Study of Optical Systems from the point of view of the Wave Theory.* By F. Twyman[*].

CONTENTS.

1. *Description of the Interferometers.*

THESE instruments in their simplest form resemble the well-known Michelson interferometer, the main essential optical difference being that the two interfering beams of light are brought to a focus at the eye of the observer.

Optical elements or combinations suitable for examination by means of these instruments may almost all be classed in two categories. Into the one category fall those combinations which are required to receive a beam of light which has a plane wave-front and deliver it again after transmission with a plane wave-front; and into the other fall those the object of which is to impart spherical wave-fronts to beams which are incident on them with plane wave-fronts. The two corresponding arrangements will be referred to as the prism interferometer and the lens interferometer respectively.

The Prism Interferometer.

The prism interferometer is shown in diagram (fig. 1) as arranged for the correction of a 60° prism, such as is used for spectroscopy.

The light used must consist of monochromatic rays. Such

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[*] Communicated by the Author.

a light may be obtained from a Cooper-Hewett Mercury-Vapour Lamp, combined with a suitable filter.

The light from the source is reflected by the adjustable mirror A through the condensing-lens B, by means of which it is condensed on the aperture of the diaphragm C.

The diverging beam of light is collimated by a lens D, and falls as a parallel beam on a plane parallel plate K, the second surface of which is silvered lightly so that a part of the light is transmitted and part reflected. The major part should be reflected. One part passes through the prism L in the same way as in actual use, and being reflected by the mirror F passes back through the prism to the plate K. The other part of the light is reflected to the mirror G and back again to the plate K. Here the separated beams recombine, and passing through the lens E each forms on the eye placed somewhat beyond the aperture in the diaphragm P an image of the hole in the diaphragm C. One of the reflecting mirrors in its mount is shown in fig. 2.

Fig. 1.

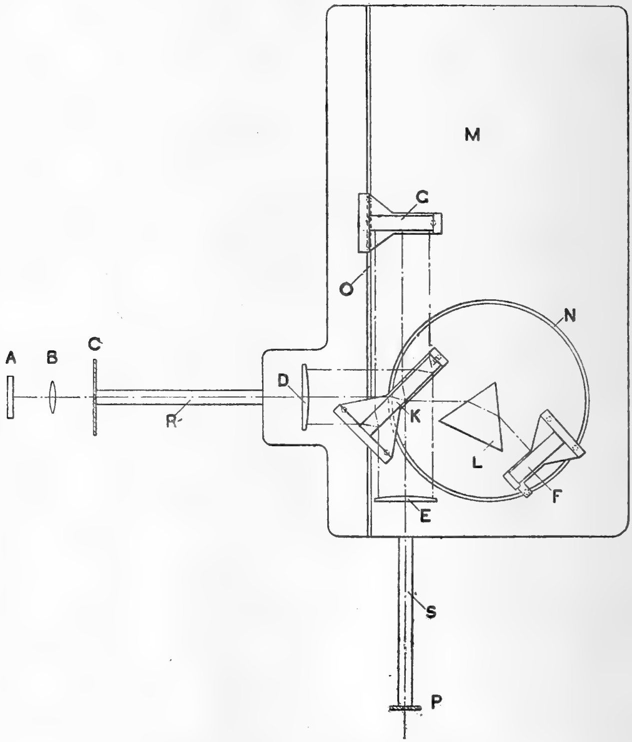
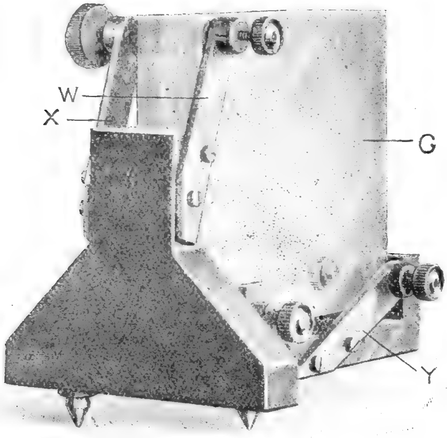


Diagram of Prism Interferometer.

When the mirrors are adjusted, interference-bands are seen which form a contour map to a scale of wave-lengths of the extent to which passage of the beam twice through the prism has distorted the wave-front. This distortion can be corrected by removing from each point of one prism face, by local polishing, an amount of glass proportional to the distortion of the wave-front at that point; hence it follows that the bands also form a contour map of the glass requiring to be removed in order to make the optical performance of the prism perfect.

Fig. 2.



Mirror in Mount.

Fig. 3.

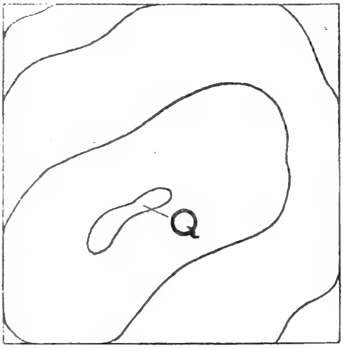


Fig. 3 represents in diagram a typical map, where Q represents the highest point of a "hill." The procedure in such a case is to mark out the contour lines on the surface of the prism with a paint-brush dipped in rouge, and then to polish first on the region Q, subsequently extending the area of polishing first partly, then wholly, to the next contour line; and so on. The marking out of the prism surface can be done while observing the bands.

Fig. 4.

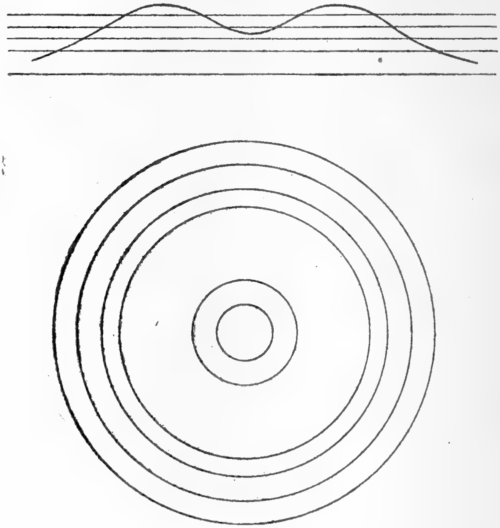
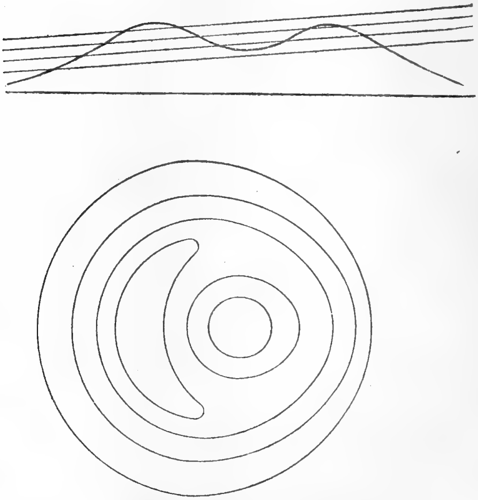


Fig. 5.



It should be noted that variations in the contour lines are obtained by a tilt of the plane of reference. Thus a slight adjustment of mirror F (fig. 1) might change a contour map from that shown in fig. 4 to that shown in fig. 5. The form of surface is in each case the same (see the sectional diagrams at the top of the figures), but correction can be carried out according to whichever plane of reference is the most favourable from the point of view of the operator. In order to find whether Q (fig. 3) is a hill or a valley, the cast-iron table M (fig. 1) can be bent with the fingers so as to tilt the mirror F in such a way as to lengthen the ray-path. If the contour line at Q expands to enclose a larger area, a hill is indicated, and *vice versa*. Although the words "hill" and "valley" are convenient to use, it must not be supposed that the imperfections necessarily result from want of flatness either of one or of both surfaces of the prism. The contour map gives the total effect on the wave-front produced by double passage through the prism, and shows in wave-lengths the departure from planeness of the resulting wave-surface.

The Lens Interferometer.

(See figs. 6, 7, and 8.)

Fig. 6.

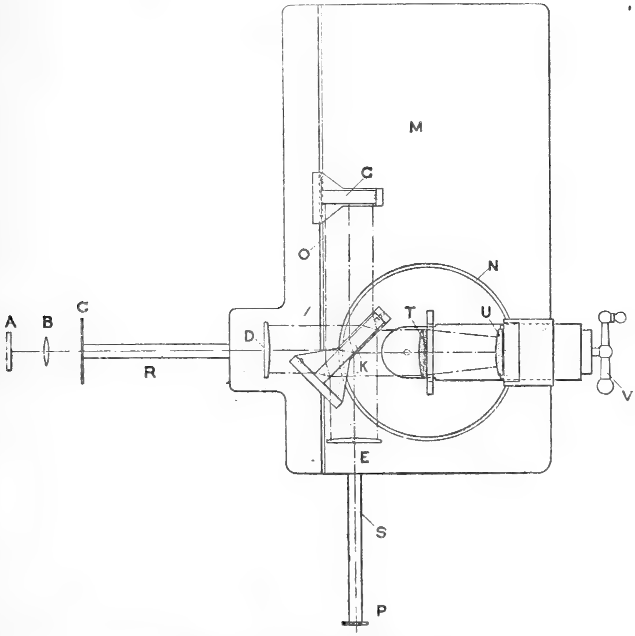
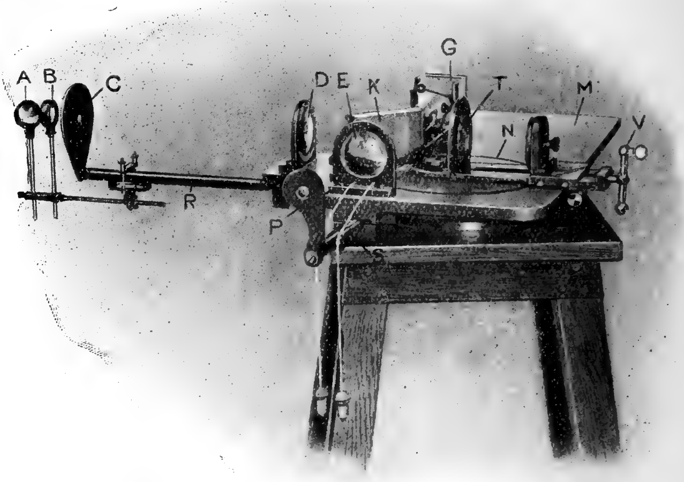


Diagram of Lens Interferometer.

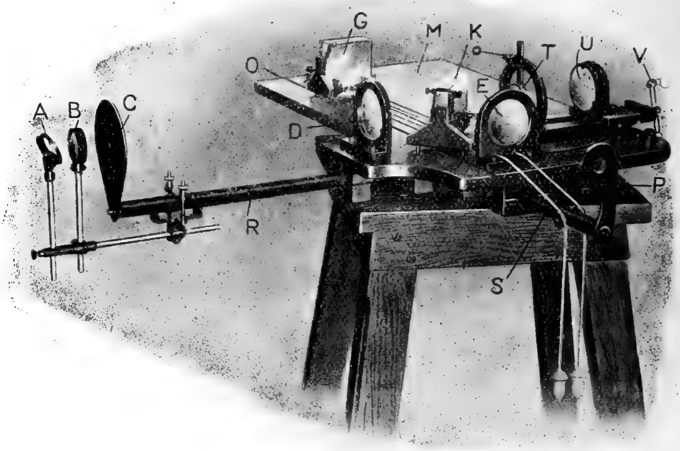
In the lens interferometer all parts are left as in the prism arrangement except that the mirror F is removed and replaced

Fig. 7.



Lens Interferometer.

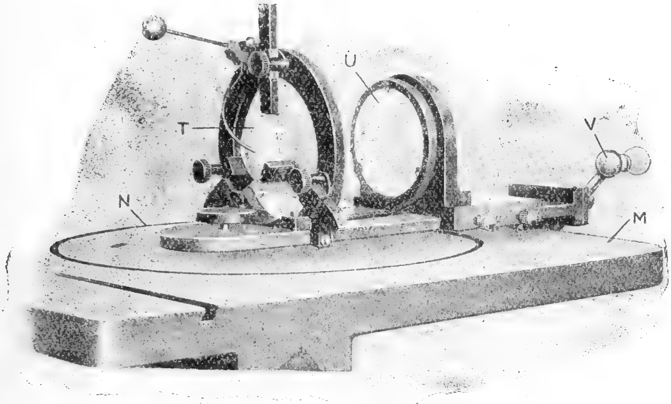
Fig. 8.



Lens Interferometer.

by the lens and mirror mount shown in fig. 9. T represents the lens under test, U a convex mirror in such a position that it reflects back along their own paths the rays received from T. The mirror U can be moved by a screw motion actuated by the handle V, so that its distance from T can be varied at will. It will be seen that when the adjustment of this part of the apparatus is correct, the whole lens addition will, if the lens T be perfect, receive the beam of plane wave-front and deliver it back again with a plane wave-front. If it does not do so the departures from planeness of the so delivered wave-front will form a contour map of the corrections which have to be applied to the lens in order to make its performance, when in actual use, perfect.

Fig. 9.



Lens and Mirror Mount.

The general procedure is the same as in the case of prisms, namely, to choose such an adjustment as presents a favourable aspect for working and to polish off those portions of the surface corresponding with the parts of the contour map which represent hills.

2. *Various Uses of the Interferometers.*

(a) *Control of "retouching."*

The instruments were primarily designed for the correction by retouching of object-glasses and prisms.

The process of retouching (viz. local polishing away of the glass) appears to have been adopted by all the great constructors of large astronomical lenses*. Although the necessity for this device has not generally been referred by them to the want of homogeneity of the glass, yet the almost invariable presence of such heterogeneity would render retouching essential even were there no other causes of optical imperfection. The method of carrying out the process with the aid of this apparatus has been sufficiently described above. It has been found both speedy and effective in the manufacture of prisms and lenses where the highest accuracy is desired.

Many other methods of testing telescope or camera objectives have been devised with a view to the control of retouching. With each of the well-known methods results have been attained of an excellence commensurate with the reputation for optical work of high quality which has distinguished the individual exponents; but none of the methods appear to give indications upon which the optician could take action without a more or less complex process of reasoning.

The apparatus here described, on the other hand, produces, as has been shown above, a system of interference-bands which may be regarded as a "contour map" of the imperfections. This contour map can for practical purposes be considered as located at any of the optical surfaces involved; and in the case of the control of retouching, the observer may, if he likes, draw this map upon the surface under treatmeat. He is then in a position without further preliminary to remove the superfluous material from the prominences by polishing with pads of suitable size and shape, the "contour map" giving all that is necessary for him to know both as to the location and the magnitude of the sources of the imperfections.

(b) *The testing of lens systems.*

The apparatus obviously affords, when applied to any optical system, all the data necessary for a complete and precise statement of the degree of optical perfection of that system. As regards small aberrations it is sufficient to quote the well-known and valuable generalization :—

"An obvious inference from the necessary imperfection of optical images is the uselessness of attempting anything like

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* The opinions of Schroeder, Grubb, Czapski, and Alvan Clark on this subject are cited in a summary by H. Fassbender of the then known methods of testing object-glasses: "Ältere und neuere Methoden zur Prüfung von Objektiven," *Deutsche Mechaniker Zeitung*, July 1913, pp. 133–138 & 149–155. This report should be read by all interested in the subject. It omits, however, an ingenious method due to Dr. Chalmers, see Proc. of the Optical Convention, vol. ii. p. 56 (1912).

an absolute destruction of aberration. In an instrument free from aberration the waves arrive at the focal point in the same phase. It will suffice for practical purposes if the error of phase nowhere exceeds $\frac{1}{4}\lambda$. This corresponds to an error of $\frac{1}{8}\lambda$ in a reflecting and $\frac{1}{2}\lambda$ in a (glass) refracting surface, the incidence in both cases being perpendicular "*.

In the case of larger aberrations (implying by that word the effect on the "image" of the deviation from sphericity of the wave-front) the writer is firmly of the opinion that the departure (expressed in wave-lengths) of the wave surface as it leaves the dioptric element from a spherical wave surface, should form the basis of all statements of imperfections of definition, and believes that such a procedure, besides being most rational according to our present knowledge, is at the same time very convenient in application by the manufacturer.

The co-ordination of the phenomena observed on this interferometer with the image-forming properties of the optical system under test is one of the objects of Dr. L. Silberstein's investigations, part of which appear in another paper in this Magazine.

(c) *The use of aspherical surfaces.*

In spite of the great advances in the technique of optical manufacture during the past thirty years, the definition of actual optical systems still leaves ample room for improvement. Even if we confine our remarks to the best makers, it is only in certain of the smaller optical instruments that the imperfections of definition are inconsiderable, and even then only on the best part of the field of view. This is in great part due to the difficulty with a limited choice of suitable dioptric materials (for, although extensive, the choice is not as varied as the lens designer could wish) of obtaining the results desired by calculations based on the utilization of spherical surfaces alone.

If we look for the reason why opticians have so far almost entirely limited themselves to the production of dioptric elements bounded by spherical surfaces, we find it at once in the comparative ease with which such surfaces can be generated with precision. But, given a sufficient incentive, it cannot be doubted that surfaces other than spherical could be produced. Indeed this has already been done with an

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* Scientific Papers of John William Strutt, Baron Rayleigh, vol. ii. Article on Optics, *Encyclopædia Britannica*, xvii. 1884.

accuracy suitable for the purpose in the case of a few exceptional types of lenses, for instance the special spectacle lenses designed by Von Rohr for those whose eyes have been operated on for cataract* ; and with considerable accuracy (though with a smaller departure from true sphericity) in the "figured" lenses or mirrors for large telescopes and for other special purposes.

Now, experience has shown that with the aid of this apparatus, and even with the ordinary means of retouching, the optician can face without dismay the task of making with precision quite considerable departures from the sphericity of his surfaces. It appears to the writer certain that by modifying from true sphericity the surfaces of systems which have been suitably computed for the purpose, results will be attained more perfect than has hitherto been possible in, for instance, camera lenses. It is even likely that valuable results may be attained at some future time by using apparatus of this kind to assist the correction to a pitch *of high accuracy* of lens systems wherein definitely aspherical surfaces have been generated by processes essentially different from those ordinarily used by opticians.

(d) *The experimental study of lens systems.*

There are probably few interested in optical systems who have not felt disappointment at the comparative lack of success which has attended attempts to deal with the passage of light through such systems by means of the Wave Theory. The difficulties are mathematical, and not of course to be directly relieved by such instrumental aids as those now described.

But it is perhaps not too much to hope that the existence of means of direct demonstration of the effect on wave surfaces of passage through optical systems may attract the attention of mathematicians to this aspect of the subject of dioptrics, with good result.

Research Department,
Adam Hilger Ltd.

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* See Proc. of the Optical Convention, 1912, p. 118.

VII. *On a New Secondary Radiation of Positive Rays. By* M. Wolfke, *Dr. phil., Lecturer at the Federal Technical High School and at the University of Zurich, Switzerland*[*].

UP to now, two secondary radiations of positive rays have been known: the slow electron rays[†] and the very soft X-rays recently discovered by J. J. Thomson[‡], which are probably caused by a retardation of the ions when encountering a rigid body.

Some years ago Chadwick[§] and Russell[||] showed that α-rays are capable of exciting the γ-radiation of heavy elements.

So far, however, it has not been known whether positive rays were capable of producing a similar effect. Now this question is of very vital interest, for if it can be proved that positive rays are able to excite characteristic rays, it then becomes possible to obtain a better insight into the connexion between the process of excitation and the chemical nature and charge of the exciting particles. And such an investigation might throw new light on the questions relating to the mechanism of excitation and emission of X-spectra.

These considerations have prompted me to investigate the question as to whether positive rays are capable of exciting the characteristic X-radiation.

The experimental method was based upon similar principles to that employed by Chadwick[¶] in his investigation of the excitation of the characteristic γ-radiation of gold by α-rays.

Through a channel of circular section 10 mm. wide a pencil of positive rays was let fall upon a circular opening provided in a brass box. This opening is divided into two halves, and each of these is covered up by a double foil made up of one foil of a heavy metal, say tin or lead, and of a second foil of a light metal, say aluminium, laid over the first one. These foils are so placed that over one half of the opening the heavy-metal foil is on the outside with the aluminium foil turned towards the inside of the box, while the second similar double foil is so arranged over the other half of the

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[*] Communicated by the Author.

[†] J. J. Thomson, Proc. of Cambr. Phil. Soc. xiii. p. 212 (1905); Ch. Füchtbauer, *Phys. Z. S.* vii. p. 153 (1906); L. W. Austin, Phys. Rev, xxii. p. 312 (1906).

[‡] J. J. Thomson, Phil. Mag. [6] xxviii. p. 620 (1914).

[§] J. Chadwick, Phil. Mag. [6] xxiv. p. 594 (1912); xxv. p. 193 (1913).

[||] J. Chadwick & A. S. Russell, Proc. Roy. Soc. A. lxxxviii. p. 217 (1913); A. S. Russell & J. Chadwick, Phil. Mag. [6] xxvii. p. 112 (1914).

[¶] J. Chadwick, Phil. Mag. [6] xxv. p. 193 (1913).

opening as to have its aluminium side outwards. Behind these foils the photographic plate is placed.

Thus on one half of the opening the positive rays fall upon a heavy-metal surface, and on the other half upon an aluminium surface. The characteristic radiation of the heavy metal being more intense and harder than that of aluminium, it reaches the photographic plate with an intensity not perceptibly diminished. On the other hand, the characteristic radiation of aluminium is weak and soft and is absorbed to a large extent by the layer of heavy metal through which it has to pass. Therefore, if it is true that the characteristic radiation of the heavy metal is excited by positive rays, then the impression produced on the photographic plate must be stronger beneath that half of the opening where the positive rays fall upon heavy metal, and less strong beneath the other half, where they encounter an aluminium foil. The higher the intensity of excitation of the characteristic rays, the more pronounced will be this difference in the strength of the impression obtained.

Such secondary cathode rays as might have been produced by the positive rays and the X-rays in the channel itself were deflected behind the channel, and prevented from entering the opening in the box by means of a field of sufficient power created between channel and box.

In order to eliminate the effect that might have been produced by any irregularity of thickness in the two foils, every test made was checked by a second exposure with the same foils as in the first case, but turned about so as to have their relative position reversed.

In order to avoid too strong heating of the foils by these powerful positive rays, the exposures were intermittent so that short exposures alternated with longer breaks.

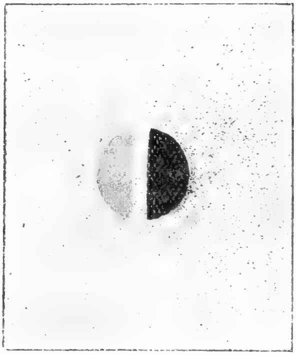
Two heavy metals, tin and lead, were treated. The foils used were ·016 mm. thick in the case of tin and ·028 mm. in the case of lead; the thickness of the aluminium foil was ·007 mm.

The experiments have shown that when acted upon by positive rays, either of these metals emitted a penetrating radiation of fair intensity which is probably its characteristic radiation.

When *tin* was tested, all photographs without exception showed a very marked contrast—that is, where the positive rays fell upon the tin surface, the darkening of the plate was strongly pronounced, while the other half of the circular imprint showed but a faint darkening. The photograph obtained is seen in the annexed figure, and this result was in accordance with anticipation.

Several similar exposures were made with an induction-coil and others with an influence-machine. The duration of exposure was varied between 2·5 and 22 minutes, the potential between 25 and 40 mm. spark-gap, and the pressure between ·0007 and ·0037 mm. mercury.

The annexed photograph was obtained with an influence-machine and an exposure of 22 minutes. The spark-gap was 25–30 mm. and the pressure ·0037 mm. mercury.



When photographs were taken with *lead* foil, with potentials ranging between 25 mm. and 40 mm. spark-gap, the darkening of the plate was only faint, and no difference was visible in intensity of the impression on either half of the image. As soon, however, as the spark-gap was increased to 45 mm. a very distinct contrast became visible in the darkening of the two halves.

The intensity of the photograph was greatest on that half where the positive rays fell upon the lead surface, and there only on that part where the intensity of the positive rays would be a maximum.

From this it would seem that the energy required for the excitation of characteristic X-rays has a lower limit, just as Duane, Hunt, Hull, Webster and others have observed in the case of cathode rays in a Coolidge tube.

For cathode rays the relation $\epsilon > h\nu$ has been established, where ϵ is the minimum energy of the electron necessary for exciting the K-series, and ν the maximum frequency of the line $K\beta_2$ that corresponds to this series. Supposing this relation to hold also for the excitation of characteristic rays by positive particles, then to excite the $K\beta_1$ line of tin ($\lambda = \cdot 432 . 10^{-8}$ cm.) the voltage required would have to exceed 57 KV. In the described experiments, however, the maximum width of spark-gap used was 45 mm., the spheres being 31 mm. in diameter; so that according to tests made by C. Müller* the voltage applied could not have exceeded 50 KV. The K-series could not then have been excited, and it is probable that the wave-length excited belonged to the L-series of either metal.

I propose to investigate further the wave-lengths and other properties of these radiations. A report of such studies will be published shortly.

Summary of Results.

1. For the first time the excitation of a *penetrating* radiation by positive rays was observed. This effect was retained on photographic plates in the case of tin and lead, and it is surmised that it is the characteristic X-radiation of these elements.
2. A lower *limit* was found to exist for the voltage necessary for excitation.
3. Einstein's quantum condition leads to the supposition that the new effect that has been observed is excitation of the L-characteristic rays of either element.

The Physical Laboratory,
Technical High School of Zurich.
August 1917.

VIII. *Variably-Coupled Vibrations:* II. *Unequal Masses or Periods. By* Edwin H. Barton, *D.Sc., F.R.S., Professor of Physics, and* H. Mary Browning, *B.Sc., Lecturer and Demonstrator in Physics, University College, Nottingham*†.

[Plates I. & II.]

Contents.

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* C. Müller, *Ann. d. Phys.* [4] xxviii. p. 585 (1910).

† Communicated by the Authors.

I. Introduction.

In a recent paper[*] two types of coupled pendulums were experimented with, their lengths and the masses of their bobs being in each case equal. The present paper, the second of the series, deals with the double-cord pendulum only, but in cases where either the masses of the bobs are unequal or else the lengths of their suspensions are unequal.

These mechanical cases may be regarded as somewhat analogous to the electrical cases of inductively-coupled circuits with unequal inductances or unequal periods respectively.

With unequal masses and equal lengths it is noticeable that with small couplings a great increase in the amplitude of vibration of the small bob entailed very little loss in that of the large bob. Indeed, for masses as 20 : 1 we almost realised the case of forced vibrations.

The funnel of the light bob was here of cardboard and so had an appreciable damping. This rendered it necessary to make corresponding modifications in the theory.

With unequal lengths and equal masses the response showed a great diminution for small couplings, whereas for larger couplings the mistuning seemed without appreciable effect.

The paper includes twenty-seven photographic reproductions of double sand traces obtained simultaneously one from each bob of the coupled pendulum.

II. Theory for Unequal Masses.

Equations of Motion and Coupling.—Throughout the work described in the present paper the double-cord pendulum was used. This was shown in figs. 1, 2, and 4 of the first paper. The equations of motion and coupling were given as (27)–(29) and may now be rewritten here as follows :—

$$\left.\begin{aligned} P\frac{d^2y}{dt^2}+\frac{P+Q+\beta Q}{(1+\beta)(P+Q)}P\frac{g}{l}y &= \frac{\beta}{(1+\beta)}\cdot\frac{PQ}{(P+Q)}\frac{g}{l}z, \\ Q\frac{dz^2}{dt^2}+\frac{P+\beta P+Q}{(1+\beta)(P+Q)}Q\frac{g}{l}z &= \frac{\beta}{(1+\beta)}\cdot\frac{PQ}{(P+Q)}\frac{g}{l}y \end{aligned}\right\} . \quad (1)$$

and

$$\gamma^2=\frac{\beta^2PQ}{(P+Q+\beta Q)(P+\beta P+Q)}. \quad . \quad . \quad . \quad (2)$$

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[*] Phil. Mag. (6) vol. xxxiv. no. 202, pp. 246–270 (Oct. 1917).

Let us now write in the above

$$\frac{\mathrm{Q}}{\mathrm{P}}=\rho, \quad \frac{g}{l}=m^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also divide the two equations (1) by P and Q respectively, and insert the frictional term $2k\,dy/dt$ in the first of them. We then obtain

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\frac{1+\rho+\beta\rho}{(1+\beta)(1+\rho)}m^2y=\frac{\beta}{(1+\beta)}\cdot\frac{\rho}{(1+\rho)}m^2z, \quad (4)$$

$$\frac{d^2z}{dt^2}+\frac{1+\rho+\beta}{(1+\beta)(1+\rho)}m^2z=\frac{\beta}{(1+\beta)}\cdot\frac{1}{(1+\rho)}m^2y. \quad (5)$$

These may be written

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+ay=\rho bz, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\frac{d^2z}{dt^2}+cz=by, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\left.\begin{aligned} a&=\frac{1+\rho+\beta\rho}{(1+\beta)(1+\rho)}m^2, \quad b=\frac{\beta m^2}{(1+\beta)(1+\rho)}, \\ \text{and} \quad c&=\frac{1+\rho+\beta}{(1+\beta)(1+\rho)}m^2 \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

Solution and Frequencies.—To solve (6) and (7) let us write

$$\left.\begin{aligned} &z=e^{xt}, \\ &\text{and, on inserting in (7), we have} \\ &y=\left(\frac{x^2+c}{b}\right)e^{xt}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Then (9) substituted in (6) gives

$$\left(\frac{x^2+c}{b}\right)(x^2+2kx+a)=\rho b,$$

or

$$x^4+2kx^3+(c+a)x^2+2kcx+ca-\rho b^2=0, \quad . \quad (10)$$

which is the auxiliary biquadratic in x. Though this equation has the form of the general biquadratic, an approximate solution, presenting all the accuracy needed for our purpose, may be easily obtained by noting that k is small compared with the other constants. For, as appears from the experiments, k is of the order one-thousandth of the coefficient of x^2 and of the constant term.

Then we may write for the roots of x in the biquadratic (10) the values

$$-r\pm ip \quad \text{and} \quad -s\pm iq, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where i denotes $\surd(-1)$, and r and s (being comparable to k) are to be treated as small quantities whose squares or products are negligible in comparison with p and q which depend upon the larger constants of the equation.

Thus, with the roots from (11) we may write instead of (10) the equivalent equation

$$(x+r-ip)(x+r+ip)(x+s-iq)(x+s+iq)=0,$$

or
$$x^4+2(r+s)x^3+(p^2+q^2+r^2+s^2+4rs)x^2$$
$$+2(p^2s+q^2r+r^2s+rs^2)x+(p^2+r^2)(q^2+s^2)=0. \quad (12)$$

This, on omitting the negligible quantities, becomes the approximate equation sufficiently accurate for our purpose,

$$x^4+2(r+s)x^3+(p^2+q^2)x^2+2(p^2s+q^2r)x+p^2q^2=0. \quad (13)$$

The comparison of coefficients in (10) and (13) yields

$$r+s=k, \quad \ldots \quad (14)$$

$$p^2+q^2=c+a, \quad \ldots \quad (15)$$

$$p^2s+q^2r=ck, \quad \ldots \quad (16)$$

$$p^2q^2=ca-\rho b^2. \quad \ldots \quad (17)$$

From (15) and (17) we may eliminate q^2 and obtain a quadratic in p^2 whose roots may be called p^2 and q^2. We thus find

$$\left.\begin{aligned} 2p^2&=c+a+\surd\{(a-c)^2+4\rho b^2\}, \\ \text{and} \quad 2q^2&=c+a-\surd\{(a-c)^2+4\rho b^2\}. \end{aligned}\right\} \quad \ldots \quad (18)$$

Again, from (14) and (16) we obtain

$$r=\frac{p^2-c}{p^2-q^2}k \text{ and } s=\frac{c-q^2}{p^2-q^2}k.$$

And by use of (18) these become

$$r=\frac{a-c+\surd\{(a-c)^2+4\rho b^2\}}{2\surd\{(a-c)^2+4\rho b^2\}}k. \quad \ldots \quad (19)$$

and

$$s=\frac{c-a+\surd\{(a-c)^2+4\rho b^2\}}{2\surd\{(a-c)^2+4\rho b^2\}}k. \quad \ldots \quad (20)$$

Then, inserting the values of a, b, and c from (8) in (18), (19), and (20), we obtain

$$p=m, \quad q=\frac{m}{\surd(1+\beta)}, \quad \ldots \quad (21)$$

$$\frac{p}{q}=\surd(1+\beta), \quad \ldots \quad (22)$$

$$r=\frac{\rho k}{1+\rho}, \quad \ldots \quad (23)$$

$$s=\frac{k}{1+\rho}. \quad \ldots \quad (24)$$

Thus, using (11) in (9) and introducing the usual constants, the general solution may be written in the form

$$z=e^{-rt}(\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+e^{-st}(\mathrm{C}e^{qit}+\mathrm{D}e^{-qit}), \quad . \quad (25)$$

and, omitting r^2 and s^2,

$$y=\frac{(-p^2+c)}{b}e^{-rt}(\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+\frac{(-q^2+c)}{b}e^{-st}(\mathrm{C}e^{qit}+\mathrm{D}e^{-qit})$$
$$+\frac{2pri}{b}e^{-rt}(-\mathrm{A}e^{pit}+\mathrm{B}e^{-pit})+\frac{2qsi}{b}e^{-st}(-\mathrm{C}e^{qit}+\mathrm{D}e^{-qit}). \quad . \quad (26)$$

Or, by transformation of (25) and (26) and use of (21)–(24), we may write the general solution in the form

$$z=\mathrm{E}e^{-pst}\sin(mt+\epsilon)+\mathrm{F}e^{-st}\sin\left(\frac{mt}{\sqrt{(1+\beta)}}+\phi\right), \quad (27)$$

and

$$y=-\rho\mathrm{E}'e^{-pst}\sin(mt+\epsilon')+\mathrm{F}'e^{-st}\sin\left(\frac{mt}{\sqrt{(1+\beta)}}+\phi'\right), \quad (28)$$

where

$$\left.\begin{aligned}(\mathrm{E}')^2&=\mathrm{E}^2\frac{\beta^2m^2+4(1+\beta)^2k^2}{\beta^2m^2},\\ \text{and}\quad \tan(\epsilon'-\epsilon)&=\frac{2(1+\beta)k}{\beta m};\end{aligned}\right\} \quad . \quad . \quad . \quad (29)$$

also

$$\left.\begin{aligned}(\mathrm{F}')^2&=\mathrm{F}^2\frac{\beta^2m^2+4(1+\beta)k^2}{\beta^2m^2},\\ \text{and}\quad \tan(\phi'-\phi)&=\frac{-2\sqrt{(1+\beta)}k}{\beta m};\end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

the exponential coefficient s being given by (24), and E, ϵ, F, and ϕ being the arbitrary constants dependent on the initial conditions. In many of the experimental cases E′ may be assimilated to E and F′ to F without appreciable error. The changes $(\epsilon'-\epsilon)$ and $(\phi'-\phi)$ of the phase angles may be distinctly appreciable for very small values of β. But in these cases the vibrations show a slow waxing and waning of amplitude and the phase is of very little importance. On the other hand, for β equal to unity, we have

$$\tan(\epsilon'-\epsilon)=4k/m \quad \text{and} \quad \tan(\phi'-\phi)=-2\sqrt{2}k/m.$$

And the numerical values of these are of the order 0·020 and 0·014, hence $\epsilon'-\epsilon=1^\circ\ 10'$ and $\phi'-\phi=0^\circ\ 48'$ nearly. Hence for all our present experimental cases, we may drop the four accents in equation (28).

Initial Conditions. *Case I.*—Suppose the heavy bob of mass Q (which $=\rho P$) is pulled aside and that the light one of mass P is allowed to hang at rest in its more or less displaced position according to the coupling in use. Then we may write:

$$\left.\begin{array}{l} \text{For } t=0 \text{ let } z=f, \\ \text{then it follows statically that } y=\dfrac{\beta\rho f}{1+\rho+\beta\rho}; \\ \text{also put} \qquad \dfrac{dz}{dt}=0 \quad\text{and}\quad \dfrac{dy}{dt}=0. \end{array}\right\} \quad . \quad . \quad . \quad (31)$$

Differentiating with respect to time (27) and (28) without its accents, and writing in the latter n for $m/\surd(1+\beta)$, we find

$$\frac{dz}{dt}=\mathrm{E}e^{-\rho st}[m\cos(mt+\epsilon)-\rho s\sin(mt+\epsilon)]$$
$$+\mathrm{F}e^{-st}[n\cos(nt+\phi)-s\sin(nt+\phi)], \quad . \quad (32)$$
$$\frac{dy}{dt}=-\rho\mathrm{E}e^{-\rho st}[m\cos(mt+\epsilon)-\rho s\sin(mt+\epsilon)]$$
$$+\mathrm{F}e^{-st}[n\cos(nt+\phi)-s\sin(nt+\phi)]. \quad . \quad (33)$$

The conditions (31) introduced in equations (27), (28), (32), and (33) give

$$f=\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \quad . \quad . \quad . \quad (34)$$
$$\frac{\beta\rho f}{1+\rho+\beta\rho}=-\rho\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \quad . \quad . \quad . \quad (35)$$
$$0=\mathrm{E}(m\cos\epsilon-\rho s\sin\epsilon)+\mathrm{F}(n\cos\phi-s\sin\phi), \quad . \quad (36)$$
$$0=-\rho\mathrm{E}(m\cos\epsilon-\rho s\sin\epsilon)+\mathrm{F}(n\cos\phi-s\sin\phi). \quad (37)$$

But, by reason of the smallness of ρs in comparison with m (of the order 0·01) and of s in comparison with n (still less), we may write instead of (36) and (37) the following:

$$0=\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi, \quad . \quad . \quad . \quad . \quad (38)$$

and

$$0=-\rho\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi. \quad . \quad . \quad . \quad (39)$$

These are satisfied by

$$\epsilon=\frac{\pi}{2} \quad\text{and}\quad \phi=\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

These values inserted in (34) and (35) give

$$f=\mathrm{E}+\mathrm{F},$$

and

$$\frac{\beta\rho f}{1+\rho+\beta\rho}=-\rho\mathrm{E}+\mathrm{F};$$

whence

$$\mathrm{E}=\frac{f}{1+\rho+\beta\rho} \quad\text{and}\quad \mathrm{F}=\frac{(1+\rho)\beta f}{1+\rho+\beta\rho}. \quad . \quad (41)$$

Hence, for the special solution with these initial conditions, we have

$$z=\frac{f}{1+\rho+\beta\rho}e^{-\rho st}\cos mt+\frac{(1+\rho)\beta f}{1+\rho+\beta\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \quad (42)$$

$$y=-\frac{\rho f}{1+\rho+\beta\rho}e^{-\rho st}\cos mt+\frac{(1+\rho)\beta f}{1+\rho+\beta\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \quad (43)$$

where $s=\frac{k}{1+\rho}$.

Thus the ratios of the amplitudes of the quick and slow components in the y and z vibrations are respectively given by

$$\frac{-\rho}{(1+\rho)\beta}e^{-(\rho-1)st} \text{ and } \frac{1}{(1+\rho)\beta}e^{-(\rho-1)st}. \quad . \quad (43\,a)$$

Case II.—Suppose now that the heavy bob (of mass $Q=\rho P$) is pulled aside while the light one (of mass P) is held undisplaced. Then we have:

$$\text{For} \quad \left.\begin{array}{l} t=0, \quad z=f, \quad y=0, \\ \frac{dy}{dt}=0 \text{ and } \frac{dz}{dt}=0. \end{array}\right\} \quad . \quad . \quad . \quad (44)$$

Putting (44) in (27), (28) without accents, (32) and (33), and omitting small quantities as before, we find

$$\left.\begin{array}{l} f=\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi, \\ 0=-\rho\mathrm{E}\sin\epsilon+\mathrm{F}\sin\phi; \end{array}\right\} \quad . \quad . \quad (45)$$

$$\left.\begin{array}{l} 0=\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi, \\ 0=-\rho\mathrm{E}m\cos\epsilon+\mathrm{F}n\cos\phi. \end{array}\right\} \quad . \quad . \quad . \quad (46)$$

Then (46) is satisfied by

$$\epsilon=\frac{\pi}{2} \quad \text{and} \quad \phi=\frac{\pi}{2},$$

and putting these values in (45), we obtain

$$\mathrm{E}=\frac{f}{1+\rho} \quad \text{and} \quad \mathrm{F}=\frac{\rho f}{1+\rho}.$$

Hence, for the special solution with these initial conditions, we have

$$y=-\frac{\rho f}{1+\rho}e^{-\rho st}\cos mt+\frac{\rho f}{1+\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}, \quad (47)$$

$$z=\frac{f}{1+\rho}e^{-\rho st}\cos mt+\frac{\rho f}{1+\rho}e^{-st}\cos\frac{mt}{\sqrt{(1+\beta)}}. \quad . \quad (48)$$

Accordingly the ratios of the amplitudes of the quick and slow vibrations in the y and z traces are respectively

$$-e^{-(\rho-1)st} \quad \text{and} \quad \frac{e^{-(\rho-1)st}}{\rho}. \quad . \quad . \quad . \quad (49)$$

Relation of Dampings in the Vibrations separate and coupled.—The vibrations of a separate damped pendulum of length l are derived from the equation of motion

$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+m^2y=0, \quad . \quad . \quad . \quad . \quad (50)$$

where $m^2=g/l$.

The solution of this involves simple harmonic vibrations of approximate period

$$\tau=2\pi/m,$$

and of damping factor

$$e^{-kt}.$$

Thus the ratio of successive amplitudes is

$$e^{k\tau/2}=e^{k\pi/m}$$

nearly.

But the logarithmic decrement λ (per half wave) is defined as the logarithm to base e of this ratio.

Hence

$$\lambda=\frac{k\pi}{m} \quad \text{or} \quad k=\frac{m\lambda}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

which gives the relation between damping coefficient and logarithmic decrement for a separate pendulum.

We have now to express in terms of λ the two damping coefficients r and s which apply to the superposed vibrations when the pendulums are coupled. Thus, combining (23) and (24) with (51), we find

$$r=\frac{\rho}{1+\rho}\cdot\frac{m\lambda}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and

$$s=\frac{1}{1+\rho}\cdot\frac{m\lambda}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

III. THEORY FOR UNEQUAL PERIODS.

Equations of Motion and Coupling.—Still using the double-cord pendulum, as shown in figs. 1, 2, and 4 of the first paper, we now make the masses of the bobs equal, but the lengths of the suspensions unequal. (The droops of the two bridles always remain equal.) In other words, Q=P or $\rho=1$, while the lengths of the suspensions for the

y and z vibrations are now denoted by ηl and l respectively, the droop of each bridle being βl as before.

Then the equations of motion of the pendulums may be written at first in the form :

$$\mathrm{P}\frac{d^2y}{dt^2}+\mathrm{P}g\theta=0, \quad \ldots \ldots \quad (54)$$

$$\mathrm{Q}\frac{d^2z}{dt^2}+\mathrm{Q}g\psi=0, \quad \ldots \ldots \quad (55)$$

where θ and ψ are the inclinations of the suspensions to the vertical.

But we have also

$$\theta=\frac{y-\beta l\omega}{\eta l} \text{ and } \psi=\frac{z-\beta l\omega}{l}, \quad \ldots \quad (56)$$

where ω is the inclination to the vertical of the planes of the bridles.

Neglecting masses of bridles, connector, and suspensions, ω must satisfy

$$\mathrm{Q}g(\psi-\omega)=\mathrm{P}g(\omega-\theta)=\mathrm{P}g(\psi-\omega). \quad \ldots \quad (57)$$

Then (56) in (57) gives

$$l\omega=\frac{y+\eta z}{\beta+\beta\eta+2\eta}. \quad \ldots \ldots \quad (58)$$

And (58) in (56) yields

$$\theta=\frac{(2+\beta)y-\beta z}{l(\beta+\beta\eta+2\eta)} \text{ and } \psi=\frac{(\beta+2\eta)z-\beta y}{l(\beta+\beta\eta+2\eta)}. \quad . \quad (59)$$

Then by (59), equations (54) and (55) become

$$\frac{d^2y}{dt^2}+\frac{2+\beta}{\beta+\beta\eta+2\eta}m^2y=\frac{\beta m^2}{\beta+\beta\eta+2\eta}z, \quad \ldots \quad (60)$$

$$\frac{d^2z}{dt^2}+\frac{\beta+2\eta}{\beta+\beta\eta+2\eta}m^2z=\frac{\beta m^2}{\beta+\beta\eta+2\eta}y, \quad \ldots \quad (61)$$

where m^2 is written for g/l.

So, for the coupling γ, we have

$$\gamma^2=\frac{\beta^2}{(2+\beta)(\beta+2\eta)}. \quad \ldots \ldots \quad (62)$$

Hence, for $\eta=1$, we recover the original relation

$$\gamma=\frac{\beta}{2+\beta}, \quad \ldots \ldots \ldots \quad (63)$$

which agrees with (32) of the first paper.

Solution and Frequencies.—In equation (61) try

then we have

$$\left.\begin{aligned} z&=e^{xt} \\ y&=\frac{x^2(\beta+\beta\eta+2\eta)+(\beta+2\eta)m^2}{\beta m^2}e^{xt} \end{aligned}\right\} \quad . \quad . \quad (64)$$

And, by (64) in (60), we obtain

$$\{x^2(\beta+\beta\eta+2\eta)+(\beta+2\eta)m^2\}\{x^2(\beta+\beta\eta+2\eta) \\ +(2+\beta)m^2\}=\beta^2m^4.$$

This reduces to the auxiliary biquadratic in x,

$$x^4(\beta+\beta\eta+2\eta)+2(1+\beta+\eta)m^2x^2+2m^4=0. \quad . \quad (65)$$

Solving this as a quadratic in x^2, we have

$$x^2=-m^2\frac{1+\beta+\eta\pm\sqrt{\{(1-\eta)^2+\beta^2\}}}{\beta+\beta\eta+2\eta}. \quad . \quad . \quad . \quad (66)$$

Or, let us write

$$x=\pm pi \text{ or } \pm qi. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Then, for the sake of brevity putting Δ^2 for $(1-\eta)^2+\beta^2$, we have

and

$$\left.\begin{aligned} p^2&=\frac{1+\beta+\eta+\Delta}{\beta+\beta\eta+2\eta}m^2, \\ q^2&=\frac{1+\beta+\eta-\Delta}{\beta+\beta\eta+2\eta}m^2, \\ \frac{p}{q}&=\left\{\frac{1+\beta+\eta+\Delta}{1+\beta+\eta-\Delta}\right\}^{1/2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (68)$$

Thus, using (67) in (64) and introducing the usual constants, we obtain

$$z=\mathrm{E}\sin(pt+\epsilon)+\mathrm{F}\sin(qt+\phi), \quad . \quad . \quad . \quad (69)$$

and

$$y=-\frac{1-\eta+\Delta}{\beta}\mathrm{E}\sin(pt+\epsilon)+\frac{\Delta-(1-\eta)}{\beta}\mathrm{F}\sin(qt+\phi), \quad (70)$$

p and q being defined by (68).

Initial Conditions.—Consider the case of pulling aside the bob Q of the pendulum of length l whose vibrations are denoted by z, the other bob hanging at rest in a more or less displaced position according to the magnitude of the coupling.

Thus, we may write :

$$\left.\begin{array}{l}\text{For} \quad t=0, \quad z=f, \\ \text{when it follows statically that, } y=\dfrac{\beta f}{2+\beta}, \\ \text{and we have also} \quad \dfrac{dy}{dt}=0, \quad \dfrac{dz}{dt}=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (71)$$

Differentiating (69) and (70) with respect to the time, and introducing (71) gives equations which are satisfied by

$$\epsilon=\frac{\pi}{2} \text{ and } \phi=\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Then, introducing (71) and (72) in (69) and (70) we find

$$\left.\begin{array}{l}\mathrm{E}=\dfrac{(2+\beta)(-1+\eta+\Delta)-\beta^2}{2(2+\beta)\Delta}f, \\ \mathrm{F}=\dfrac{(2+\beta)(1-\eta+\Delta)+\beta^2}{2(2+\beta)\Delta}f.\end{array}\right\} \quad . \quad . \quad . \quad (73)$$

Finally, (72) and (73) in (69) and (70) give as the required special solution

$$z=\frac{(2+\beta)(-1+\eta+\Delta)-\beta^2}{2(2+\beta)\Delta}f\cos pt$$

$$+\frac{(2+\beta)(1-\eta+\Delta)+\beta^2}{2(2+\beta)\Delta}f\cos qt, \quad . \quad (74)$$

and

$$y=-\frac{1+\beta+\eta-\Delta}{2(2+\beta)\Delta}\beta f\cos pt$$

$$+\frac{1+\beta+\eta+\Delta}{2(2+\beta)\Delta}\beta f\cos qt. \quad . \quad . \quad . \quad . \quad (75)$$

IV. Relations among Variables.

It is instructive to plot graphs with the values of the coupling γ as ordinates, the abscissæ being the corresponding values of β (ratio of droop of bridle to pendulum length). A different graph is needed for each value of η and ρ (which are respectively the ratios of pendulum lengths and masses of bobs).

The data for these graphs are derived from the equations and are given in Tables I., II., and III. (and Table I. p. 265 of October paper).

Table I.—Masses 20 : 1 and lengths equal

Coupling γ.	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}}=\beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q=\sqrt{(1+\beta)}$.
Per cent.		cm.	
0	0·0	0·0	1·00
1	0·05	10·9	1·025
2	0·10	20·8	1·05
3	0·151	29·9	1·07
4	0·207	39·7	1·10
5	0·265	48·1	1·12
10	0·60	85·9	1·27
20	1·53	138·6	1·59
30·4	3·00	171·8	2·00

Table II.—Masses 5 : 1 and lengths equal.

Coupling γ.	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}}=\beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q=\sqrt{(1+\beta)}$.
Per cent.		cm.	
0	0	0	1·00
1	0·027	6·0	1·014
2	0·058	13·3	1·029
5·2	0·141	28·2	1·07
9·4	0·259	47·1	1·12
25·5	1	114·5	1·414
39·5	2	152·7	1·732
48·8	3	171·8	2

Table III.—Masses equal and lengths 3 : 4.

Coupling γ.	$\frac{\text{Bridle Droop}}{\text{Pendulum length.}}=\beta$	Actual Droop for total length 229 cm.	Frequency Ratio $p:q$.
Per cent.		cm.	
0	0	0	1·154
5·5	0·1	20·8	1·16
10·3	0·2	38·2	1·175
22·4	0·5	76·3	1·29
36·5	1	114·5	1·48
59·9	2·6	165·4	2·00
63·2	3	171·8	2·11

The graphs referred to are given in fig. 1.

We may now, from the data in the same tables, plot graphs with the values of the frequency ratios $p:q$ as ordinates, the abscissæ being the corresponding values of the coupling γ.

These are shown in fig. 2, separate graphs being plotted for mass ratios 1, 5, and 20 and lengths equal, and also for lengths 3 : 4 and masses equal.

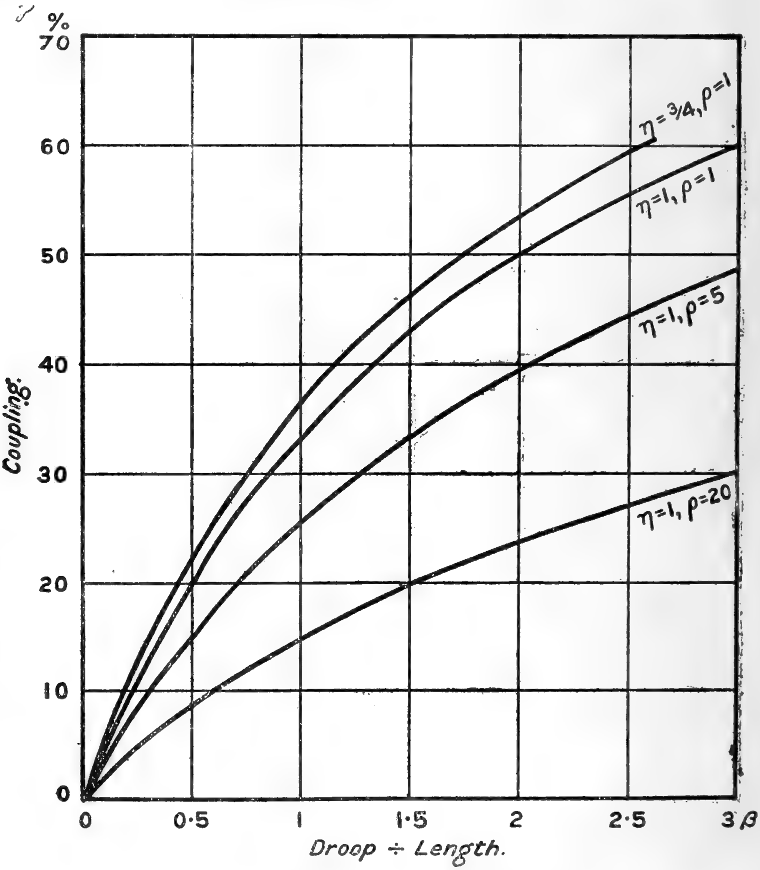

Fig. 1.—Couplings and Droop.

With the separate frequencies equal and a given coupling, it may be noted that the greater the inequality of the masses the greater is the inequality of the frequencies of the resulting superposed vibrations of the coupled system.

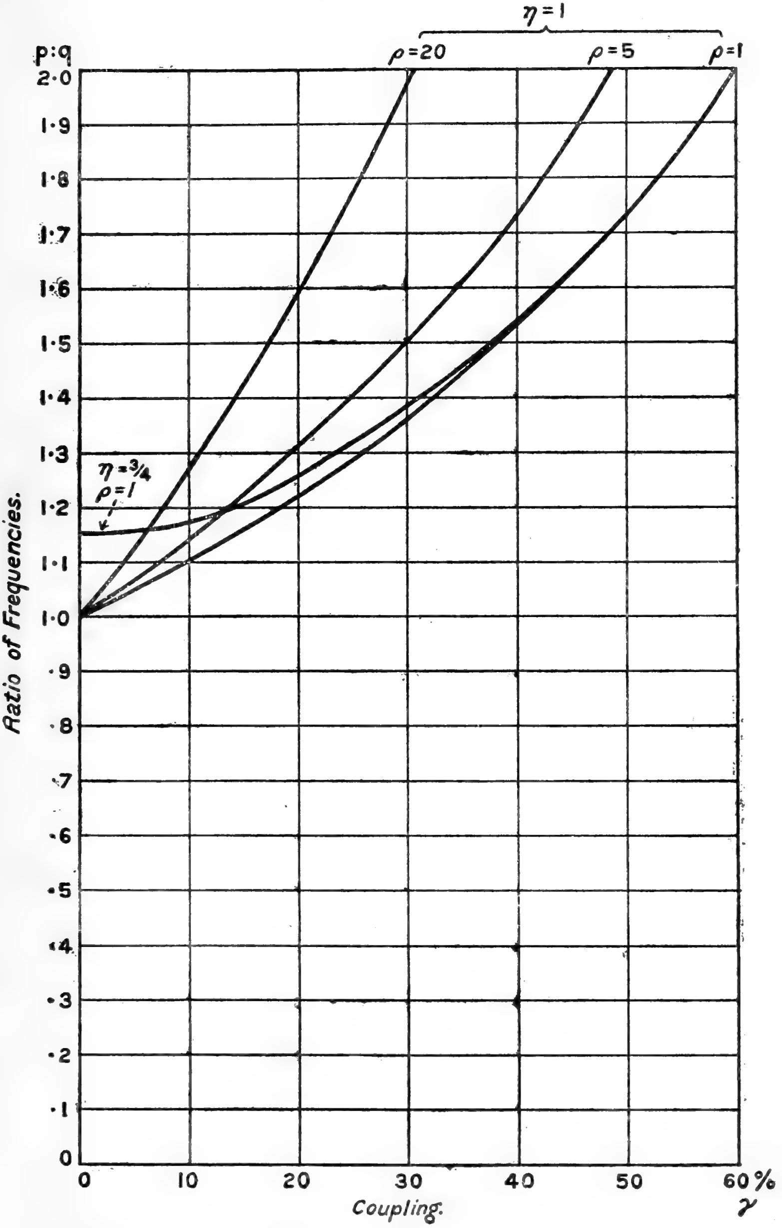

Fig. 2.—Frequency Ratios and Coupling.

When the coupling vanishes the frequencies of the separate vibrations are of course undisturbed. Thus for equal lengths, but any ratio of masses, we have for $\gamma=0$, $p:q$ equals unity. But for different separate frequencies (*i. e.*, η not equal to unity) we have for $\gamma=0$, $p:q$ greater than unity. But with large couplings the effect of unequal separate frequencies gradually disappears.

V. EXPERIMENTAL RESULTS.

Masses 20 : 1.—The bobs used in these experiments were of the order 1000 gms. and 50 gms. respectively. Figs. 1–11 of Plate I. give photographic reproductions of the double sand traces simultaneously obtained when the masses of the bobs Q and P were as 20 : 1, *i. e.*, $\rho=20$.

The couplings vary from 1 per cent. in the first to a little over 30 per cent. in the last, and are shown as percentages on every figure.

Figs. 1–8 were obtained by drawing the heavy bob aside horizontally, the light bob being allowed to hang at rest in its more or less displaced position according as the coupling was tight or loose. In figs. 9–11, while the heavy bob was pulled aside, the light one was held in its undisplaced position. Figs. 1–6 show a very marked effect due to the inequality of the masses. For, as the resultant vibrations of the light bob wax and wane in amplitude, those of the heavy bob scarcely change. Thus showing that with masses 20 : 1 we have in this respect almost reached the limiting case of forced vibrations in which the reaction of the driven on the driver is negligible. The frequencies, however, are still appreciably affected. The contrast with the case of equal masses may be seen by referring to figs. 1–5 in Plate V. of the October paper, where the waxings and wanings occur equally and alternately in both traces. Figs. 1–8 show that as the coupling increases the inequality of the frequencies of the superposed vibrations increases also. Hence there are fewer vibrations in the beat cycle and this fulfils the theory.

In fig. 9 the initial displacement of the heavy bob was so great that a collision occurred between the two as indicated. But its effect passed away after a few vibrations. This may be seen by fig. 10, in which with a slightly smaller displacement the collision was avoided.

Fig. 11 shows appreciable damping of the vibration of the light bob which was held undisplaced while the heavy one was drawn aside, whereas that of the heavy bob is not appreciably damped. This is exactly what might be expected from general considerations. But it seems at first sight in direct contradiction to the theory which shows that the y and z vibrations for the light and heavy bobs respectively involve the selfsame damping factors. But by equations (23) and (24) we see that one damping coefficient is ρ-times the other. Again, by equation (48) the amplitude of the slow vibrations of the heavy bob is ρ-times that of its quick ones. In the present experimental case ρ equals 20, hence almost all the vibration visible is the slow one with the negligibly small damping coefficient. On the other hand, by equation (47) we see that the amplitudes of the slow and quick vibrations of the light bob are numerically equal. Consequently the large damping coefficient, which is 20 times the small one, affects at least half of the amplitude visible.

Logarithmic decrements.—The lower trace on fig. 11 just dealt with, led to the theoretical introduction of the damping of the light bob as expressed by the constant k in equation (4). It also became necessary to estimate the experimental value of k. To do this one pendulum with a light bob was allowed to oscillate alone, the other being meanwhile disconnected. The traces for the lighter bobs P were taken when their masses were respectively as used in the experiments, so as to be one-twentieth and one-fifth of those of the corresponding heavy ones. The results are given in fig. 12. From the upper trace with the very light bob consisting simply of a cardboard funnel, a few weights and sand (total mass about 50 gms.), we find that the logarithmic decrement is of the order $\lambda = 0{\cdot}017$.

Then by (57) we have

$$k = \frac{m\lambda}{\pi} = (0{\cdot}005)m. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (76)$$

The lower trace with bob about 120 gms. shows considerably less damping and the decrement need not be evaluated.

Masses 5 : 1 —The masses of the bobs used in these experiments were of the order 600 gms. and 120 gms. respectively.

Figs. 13–19 in Plate II. show double traces obtained with this arrangement. In figs. 13–16 we see very plainly the beat effects on the lower trace which is left by the lighter bob. The traces of the heavier bob also show distinct but much slighter fluctuations of amplitude. In this respect they are seen to present an intermediate state between the cases of equal masses and masses as 20 : 1. And this is just what we should naturally expect. Further, the beat cycles contain fewer and fewer vibrations as the coupling increases.

This again is in accord with theory, for the frequencies of the superposed vibrations are then more unequal and therefore gain more quickly on each other.

Lengths 3 : 4.—Figs. 20–28 show double traces simultaneously obtained with the masses of the bobs equal, but the lengths of the suspensions as 3 : 4. The lower trace on each figure is that made by the shorter pendulum. In the case of fig. 20, the short pendulum was pulled aside, the long one hanging still in its slightly displaced position. In the cases of figs. 21–25, the long one was drawn aside while the short one hung at rest in its more or less displaced position. In figs. 26–28 the long pendulum was pulled aside while the short one was held in its zero position, as this favoured the exhibition of the compound harmonic trace which it was then sought to obtain.

The couplings in this set vary from about 5 per cent. to over 60 per cent. In the 5 and 10 per cent. couplings the response of the second pendulum is feeble and the beat cycles contain very few vibrations. These are the effects of the inequality of lengths. But as the coupling is further increased these effects of the inequality of the separate frequencies are seen to be overpowered. This is exactly in accord with the theory as exhibited in the graphs on fig. 2.

Fig. 26 shows an accidental collision of the lighter bob with the releasing apparatus. But the effect of the blow is seen to pass away after a few vibrations, as shown by comparison with fig. 27, which is a repetition of the conditions first intended. Figs. 26 and 27 are seen to present almost the appearance of the compound harmonic motion of a tone and its octave, the latter being too sharp. Fig. 28 shows the coupling reduced to 60 per cent., and this gives the relation of frequencies almost exactly 2 : 1.

The pair of simultaneous traces in fig. 28 is almost identical in type with those in fig. 11 of Plate V. in the October paper, in which latter case the lengths were equal. It may well seem surprising that the effect of the present mistuning (in which the frequency ratio exceeds 8 : 7) should be so completely obliterated by this coupling. But experiment and theory agree that it should be so.

VI. SUMMARY.

1. This second paper describes further experiments with the double-cord pendulum, but with the masses unequal as 20 : 1 and as 5 : 1, or the lengths unequal as 3 : 4. These

are somewhat analogous to coupled electrical circuits with different inductances or different periods.

2. The case of masses 20 : 1 is seen to be very nearly that of forced vibrations in which the light bob is driven by receiving energy from the heavy bob or driver, while the latter's loss, though equal in energy, entails only a very small decrease of amplitude. The case of masses as 5 : 1 is about midway in character between that of 20 : 1 and equal masses. Eighteen photographic reproductions of double traces are given for unequal masses.

3. It was noticeable on one of the traces that the light bob showed diminution of amplitude as the trace proceeded. This led to taking resistance into account in the equation of motion. It was also necessary to determine experimentally the actual damping of the light bob when vibrating separately. The theory thus developed and numerically applied fitted the observed facts.

4. In the case of unequal lengths but equal masses, a feebler response and a shorter beat cycle may naturally be expected than if mistuning were absent. Both these effects are quite striking with loose couplings. But with the tighter couplings the effect of mistuning is practically unnoticeable. The theory agrees with this experimental result. Nine sets of double traces are given for the unequal periods.

5. It is hoped that these methods may be shortly applied to the illustration of important phenomena in other branches of Physics.

Nottingham,
Nov. 19, 1917.

IX. *On the Diffraction of Light by Cylinders of Large Radius.*

By Nalinimohan Basu, *M.Sc., Sir Rashbehari Ghosh Research Scholar in the University of Calcutta* [*].

[Plate III.]

Introduction.

1. C. F. Brush has recently published a paper containing some interesting observations on the diffraction of light by the edge of a cylindrical obstacle [†]. Brush worked with

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[*] Communicated by Prof. C. V. Raman.

[†] "Some Diffraction Phenomena: Superposed Fringes," by C. F. Brush. Proceedings of the American Philosophical Society, 1913, pp. 276–282. See also 'Science Abstracts,' No. 1810 (1913).

cylinders of various radii (the finer ones being screened on one side so as to confine diffraction to the other side only), and observing the fringes formed within a few millimetres of the diffracting edge through a microscope, found that they appeared brighter and sharper with every increase in the radius of the cylinder. The fringes obtained with a smooth rod of one or two centimetres radius differed very markedly from those formed by a sharp edge or by a cylinder of small radius. They were brighter, more numerous, showed greater contrast between the maxima and minima of illumination, and their spacing was different from that given by the usual Fresnel formulæ. Brush also observed that when the radius of the cylinder was a millimetre or more, the fringes did not vanish when the focal plane of the microscope was put forward so as to coincide with the edge of the cylinder. Sharp narrow fringes were observed with the focal plane in this position, becoming broader and more numerous as the radius of the cylinder was increased.

2. To account for these phenomena Brush has suggested an explanation, the nature of which is indicated by the title of his paper. The diffraction-pattern formed by the cylinder is, according to Brush, the result of the superposition of a number of diffraction-patterns which are almost, but not quite, in register. He regards the cylindrical diffracting surface as consisting of a great many parallel elements, each of which acts as a diffracting edge and produces its own fringe-pattern, which is superposed on those of the other elements. Brush has made no attempt to arrive, mathematically or empirically, at any quantitative laws of the phenomena described in his paper. A careful examination of the subject shows that the view put forward by him presents serious difficulties, and is open to objection. One of the defects of the treatment suggested by Brush is that it entirely ignores the part played by the light regularly reflected from the surface of the obstacle at oblique or nearly grazing incidences. I propose in the present paper (*a*) to describe the observed effects in some detail, drawing attention to some interesting features overlooked by Brush; (*b*) to show that they can be interpreted in a manner entirely different from that suggested by him; and (*c*) to give a mathematical theory together with the results of a quantitative experimental test.

3. Reference should be made here to the problem of the diffraction of plane electromagnetic waves by a cylinder with its axis parallel to the incident waves. The solution of this problem for a perfectly conducting cylinder has been

given by J. J. Thomson [*], and for a dielectric cylinder by Lord Rayleigh [†]. These solutions are, however, suitable for numerical computation only when the radius of the cylinder is comparable with the wave-length. A treatment of the problem in the case of a cylinder of *any* radius has been recently given by Debye [‡]. He considers the electromagnetic field round a perfectly reflecting cylinder, whose axis is taken for axis of z, with polar co-ordinates r, ϕ, and waves in the plane xy polarized in the direction of z, the electric component in z being e^{ikx}. Expressing the disturbance-field in the form

$$Z = -\Sigma e^{in\frac{\pi}{2}} \frac{J_n(\kappa a)}{H_n(\kappa a)} \cdot H_n(\kappa r) \cos n\phi$$

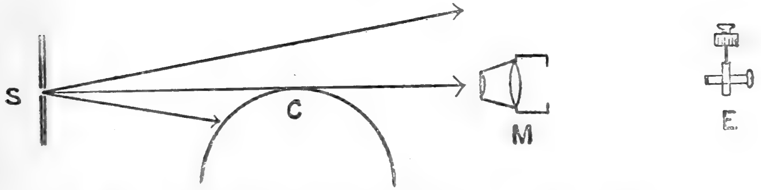
(in which J_n is the usual Bessel function, H_n is Hankel's second cylindrical function, and $\kappa = 2\pi/\lambda$), Debye transforms the solution into the simple form

$$Z = -\sqrt{\frac{a \cos\frac{\phi}{2}}{2r}} \cdot e^{-ik\left(r - 2a\cos\frac{\phi}{2}\right)}.$$

Debye's work is of considerable significance, but his final solution is valid only for points at a great distance from the surface of the cylinder, whereas the phenomena considered in the present paper are those observed in its immediate neighbourhood. No complete mathematical treatment of the subject now dealt with appears to have been given so far.

General Description of the Phenomena.

Fig. 1.



4. The experimental arrangements are those shown in the diagram (fig. 1). Light from a slit S falls on a polished

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[*] 'Recent Researches in Electricity and Magnetism,' p. 428.

[†] Phil. Mag. 1881. 'Scientific Works,' vol. i. p. 534.

[‡] P. Debye, "On the Electromagnetic Field surrounding a Cylinder and the Theory of the Rainbow," *Phys. Zeitschr.* ix. pp. 775–778, Nov. 1908. Also *Deutsch. Phys. Gesell. Verh.* 10, 20, pp. 741–749, Oct. 1908; and 'Science Abstracts,' No. 258 (1909).

cylinder of metal or glass and passes it tangentially at C *. The axis of the cylinder is parallel to the slit. A collimating lens may, if necessary, be interposed between the slit and the cylinder. The fringes bordering the shadow of the edge C are observed through the microscope-objective M and the micrometer eyepiece E. The latter may be placed at any convenient distance from the objective so as to give the necessary magnification. The effects are best seen with monochromatic light obtained by focussing the spectrum of the electric arc on the slit with a small direct-vision prism. For photographic work, the eyepiece E is removed and replaced by a long light-tight box in front of which the objective M is fixed, and at the other end of which the photographic plate is exposed. Sufficient illumination for photographing the fringes may be secured by using the arc and illuminating the slit by the greenish-yellow light transmitted by a mixture of solutions of copper sulphate and potassium bichromate.

5. The phenomena observed depend on the position of the focal plane of the objective with reference to the diffracting edge of the cylinder, and an interesting sequence of changes is observed as the focal plane of the objective is gradually moved, towards the light, up to and beyond the edge C (fig. 1) at which the incident light grazes the cylinder. Some idea of these changes will be obtained on a reference to Plate III., figs. I. to VIII., in which the fringes photographed with a cylinder of radius 1·54 cm. are reproduced. (A Zeiss objective of focal length 1·7 cm. was used, and the magnification on the original negative was 135 diameters.)

6. To interpret the phenomena it is convenient to compare them with those obtained by a sharp diffracting edge in the same position. Using the cylinder, it is found that when the focal plane is between the objective and the cylinder, but several centimetres distant from the latter, the fringes are practically of the same type as those due to a sharp diffracting edge. They are diffuse, few in number (not more than seven or eight being visible even in monochromatic light), and the first bright band is considerably broader and more luminous than the rest. The fringes become narrower (retaining their characteristics) as the focal plane is brought nearer the cylinder till the distance between the two is about two centimetres. At this stage some new features appear; the

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* A glass cylinder may be used without inconvenience as the light transmitted through the cylinder is refracted out to one side, and does not enter into the field under observation. Very little light is, in fact, transmitted through the cylinder at oblique incidences.

contrast between the minima and maxima of illumination becomes greater than in the fringes of the usual Fresnel type, and the number that can be seen and counted in monochromatic light increases considerably. These features become more and more marked as the focal plane approaches the cylinder, and the dark bands then become almost perfectly black. The difference between the intensity of the first maximum and of those following it also becomes less conspicuous. Figs. I., II., and III. in the Plate represent these stages. A considerable brightening-up of the whole field is also noticed as the focal plane approaches the cylinder, but this is not shown in the photographs, as the exposures obtained with the light of the arc were very variable. When the focal plane is within a millimetre or two of the edge at which the incident light grazes the cylinder, a change in the law of spacing of the fringes also becomes evident, the widths of the successive bright bands decreasing less rapidly than in the fringes of the Fresnel type. Fig. IV. in the plate illustrates this feature, which is most marked when the focal plane coincides with the edge of the cylinder. At this stage, of course, the fringes due to a sharp diffracting edge would vanish altogether.

7. When the focal plane is gradually moved further in, so that it lies between the cylinder and the source of light, some very interesting effects are observed. The fringes contract a little, and the first band, instead of remaining in the fixed position defined by the geometrical edge, moves into the region of the shadow, and is followed by a new system of fringes, characterized by intensely dark minima, that appears to emerge from the field occupied by the fringes seen in the previous stages. (See figs. V. and VI.) The first band of this new system is considerably more brilliant than those that follow it. It is evident on careful inspection that the fringes that move into the shadow form an independent system. For it is found that the part of the field from which the new system has separated out appears greatly reduced in intensity in comparison with the part on which it is still superimposed. When the separation of the field into two parts is complete, a few diffraction-fringes of the usual Fresnel type are observed at the geometrical edge of the shadow of the cylinder. (See figs. VII. and VIII. in the Plate, in which this position is indicated by an arrow.)

8. A comparison of the effects described in the preceding paragraph and of those obtained with a sharp diffracting edge in the same position, furnishes the clue to the correct explanation of the phenomena observed and dealt with in the

present paper. With a sharp edge, the fringes of the Fresnel type disappear when the focal plane coincides with it, and reappear without alteration of type when the focal plane is between the edge and the source of light. As mentioned above and shown in figs. VII. and VIII. of the Plate, fringes of this type may also be observed with the cylinder when the focal plane is in this position, and in addition we have, inside the shadow, an entirely separate system of fringes characterized by perfectly black minima and a series of maxima with intensities converging to zero. This latter system has nothing in common with the diffraction phenomena of the Fresnel class, and has obviously an entirely different origin. That it is formed exclusively by the light reflected from the surface of the cylinder is proved by the fact that it may be cut off without affecting the rest of the field by screening the surface. It is accordingly clear that the light reflected from the surface of the cylinder plays a most important part in the explanation of the phenomena, and that the edge of the cylinder grazed by the incident rays alone acts as a diffracting edge in the usual way, and not all the elements of the surface as supposed by Brush. We shall accordingly proceed on this basis to consider the theory of the fringes observed in various positions of the focal plane of the objective.

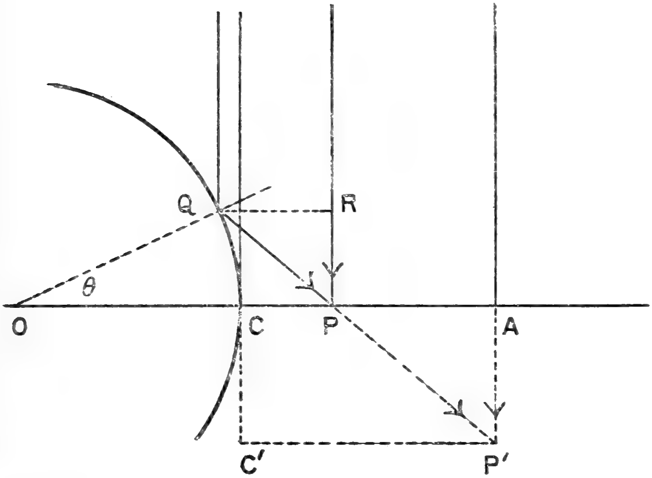
Theory of the Fringes at the edge of the cylinder.

9. When the focal plane coincides with the edge at which the incident light grazes the cylinder, it is permissible to regard the fringes seen as formed by simple interference between the light that passes the cylinder unobstructed and the light that suffers reflexion at the surface of the cylinder at various incidences; for, if a sharp diffracting edge be put in the focal plane in the same position, no diffraction-fringes would be visible. The positions of the minima of illumination in the field may be readily calculated.

In fig. 2, let O be the centre of the cross-section of the cylinder in the plane of incidence, and let C be the point at which the light grazes the cylinder. It is sufficient for practical purposes to consider the incident beam as a parallel pencil of rays. The ray meeting the cylinder at the point Q is reflected in the direction QP. Let $\angle \mathrm{QOA}=\theta$, so that $\angle \mathrm{OQP}=\frac{\pi}{2}+\theta$, and $\angle \mathrm{OPQ}=\frac{\pi}{2}-2\theta$. Let a be the radius of the cylinder and $\mathrm{CP}=x$. The difference of path, δ, between the direct ray and the reflected ray reaching the point P is evidently equal to QP−RP, which can be easily shown to be given by

$$\delta = a \sin\theta(\sec 2\theta - 1).$$

Fig. 2.



Similarly, we shall have

$$x = a \sec 2\theta(\cos\theta - \cos 2\theta).$$

Therefore, neglecting 4th and higher powers of θ, we have

$$\delta = 2a\theta^3, \quad \text{and} \quad x = 3a\theta^2/2,$$

so that

$$\delta = 2a\left(\frac{2x}{3a}\right)^{\frac{3}{2}}.$$

Since by reflexion the rays suffer a phase change of half a wave-length, the edge C will form the centre of a dark band, and the successive minima are therefore given by

$$x = \left(\frac{n\lambda}{2a}\right)^{\frac{2}{3}} \cdot \frac{3a}{2} = \frac{3}{4} \cdot (2a)^{\frac{1}{3}} \cdot (n\lambda)^{\frac{2}{3}},$$

where $n = 1, 2, 3,$ &c. The results calculated according to the above theory and those found in experiment are given in Table I.

TABLE I.

$a = 1{\cdot}54$ cm. $\lambda = 6562 \times 10^{-8}$ cm.

n.	Calculated width of Band.	Observed width of Band.
1.	·001775 cm.	·00174 cm.
2.	·001019 "	·00102 "
3.	·000875 "	·00086 "
4.	·000781 "	·00076 "
5.	·000717 "	·00069 "
6.	·000671 "	·00068 "

The discrepancies are within the limits of experimental error. When making these measurements, the focal plane was, in the first instance, set in approximate coincidence with the edge of the cylinder by noting the stage at which a further movement of the focal plane towards the light results in a movement of the fringes into the region of the shadow. There was, however, a slight uncertainty in regard to this adjustment, and the best position of the focal plane was finally ascertained by actual trial.

10. The ratio between the maxima and minima of illumination in the fringes at the edge may readily be calculated. Dividing up the pencil of rays incident on the cylinder into elements of width $a \sin\theta\, d\theta$ or $a\theta\, d\theta$ approximately, the width of the corresponding elements of the reflected pencil in the plane of the edge is dx, that is, $3a\theta\, d\theta$. The amplitude of the disturbance at any point in this plane due to the reflected rays is thus only $1/\sqrt{3}$ of that due to the direct rays, multiplied by the reflecting power of the surface. If the reflecting power be unity (as is practically the case at such oblique incidences), the ratio of the intensities of the maxima and minima is $(1+1/\sqrt{3})^2/(1-1/\sqrt{3})^2$, that is approximately 14 : 1. The dark bands are thus nearly, but not quite, perfectly black.

Theory of the Fringes at the edge of the shadow.

11. If the fringes be observed in a plane (such as C′P′ in fig. 2) which is farther from the source of light than the edge of the cylinder, the diffraction and mutual interference of the direct and the reflected rays have both to be taken into account. Since the reflected rays form a divergent

pencil while the incident rays are parallel, the effect of the former at any point sufficiently removed from the cylinder would be negligible in comparison with the effect of the latter. If d, the distance of the plane of observation from the edge of the cylinder, be sufficiently large, the problem thus practically reduces to one of simple diffraction of the incident waves by the straight edge C. The positions of the minima of illumination with reference to the geometrical edge of the shadow would then be given approximately by the simple formula

$$x' = \sqrt{2nd\lambda} = \sqrt{4n}\sqrt{d\lambda/2},$$

where $x' = \mathrm{C'P'}$ and $d = \mathrm{CC'}$,

or with great accuracy by Schuster's formula,

$$x' = \sqrt{(8n-1)d\lambda/4} = \sqrt{(8n-1)/2}\sqrt{d\lambda/2}.$$

The two formulæ give results which do not differ materially except in regard to the first two or three bands, as can be seen from Table II.

TABLE II.

1. n.	2. $\sqrt{4n}$.	3. $\sqrt{(8n-1)/2}$.	4. Proportionate widths of bands as per column 2.	5. Proportionate widths of bands as per column 3.
1.	2·000	1·871	2·000	1·871
2.	2·828	2·739	0·828	0·868
3.	3·464	3·391	0·636	0·652
4.	4·000	3·937	0·536	0·546
5.	4·472	4·416	0·472	0·479
6.	4·899	4·848	0·427	0·432
7.	5·292	5·244	0·393	0·396

12. If d be not large, the intensity of the reflected rays is not negligible. The following considerations enable us to find a simple formula for the positions of the minima of illumination which takes both diffraction and interference into account. We may, to begin with, find the positions of the minima assuming the case to be one of simple interference between the direct and the reflected rays. The expression for the path difference, δ', of the rays arriving at the point P′ is readily seen from fig. 2 to be given by the formula

$$\delta' = (d + a\sin\theta)(\sec 2\theta - 1).$$

Also, $x' = d \tan 2\theta + a(\cos\theta \sec 2\theta - 1)$.

These two relations may, to a close approximation, be written in the form

$$\delta' = 2d\theta^2 + 2a\theta^3,$$

and $$x' = 2d\theta + 3a\theta^2/2.$$

Putting $d=0$, we get the formula already deduced (see paragraph 9 above) for the fringes at the edge of the cylinder. On the other hand, if d be greater than a, we may, to a sufficient approximation, write

$$\delta' = 2d\theta^2,$$

and $$x' = 2d\theta,$$

and the positions of the points at which the direct and the reflected rays are in opposite phases are given by the formula

$$x' = \sqrt{2nd\lambda}.$$

13. But, as remarked above, the simple formula $x' = \sqrt{2nd\lambda}$ also gives the approximate positions of the minima in the diffraction-fringes at a considerable distance from the cylinder, where the effect of the reflected rays is negligible. It is thus seen that the formulæ

$$\left.\begin{aligned} n\lambda &= 2d\theta^2 + 2a\theta^3, \\ \text{and} \qquad x' &= 2d\theta + 3a\theta^2/2, \end{aligned}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (A)$$

suffice to give the approximate positions of the minima of illumination at the edge of the cylinder (at which point the fringes are due to simple interference of the direct and the reflected rays) and also at a considerable distance from it (in which case they are due only to the diffraction of the incident light). *A priori*, therefore, it would seem probable that the formulæ would hold good also at intermediate points, that is for all values of d. That this is the result actually to be expected may be shown by considering the effect due to the reflected rays at various points in the plane of observation. The reflected wave-front is the involute of the virtual caustic (see fig. 3 below). At the edge C, the radius of curvature of the wave-front is zero, and increases rapidly as we move outwards from the edge of the cylinder. The reflected rays accordingly suffer the most rapid attenuation due to divergence in the direction of the incident rays, and less rapid attenuation in other directions. In any plane C′P′, therefore, the effect of the reflected light is negligible in the immediate neighbourhood of the point C′, and would be most perceptible at points farthest removed from C′*, On the other hand, the fluctuations of intensity due to the diffraction of the direct rays are most marked in the neighbourhood of C′, that is, for the smallest values of θ. We should accordingly expect to find that when d is not zero, the first few bands are practically identical in position with those due to simple diffraction, and those following are due to simple interference between the direct and the reflected rays. The formulæ given above satisfy both of these requirements. For it is obvious from the manner in which they have been deduced that they satisfy the second requirement. The first requirement is also satisfied, as, by putting θ small, the formulæ reduce to $n\lambda=2d\theta^2$ and $x'=2d\theta$; or, in other words, $x'=\sqrt{2nd\lambda}$, for the minima of illumination, which is also the usual approximate diffraction formula. Accordingly, the complete formulæ $n\lambda=2d\theta^2+2a\theta^3$ and $x'=2d\theta+3a\theta^2/2$ would (on eliminating θ) give the positions of the minima over the entire field with considerable accuracy.

14. The statements made in the preceding paragraph are, however, subject to an important qualification. The validity of the formula obtained rests on the basis that, for large values of d, the positions of the minima of illumination are given by the simple relation $x'=\sqrt{2nd\lambda}$. This, however, is only an approximation, as the accurate values are to be found from Schuster's formula (see Table II., above), when the effect of the reflected light is negligible. When d is so large that the formulæ $n\lambda=2d\theta^2+2a\theta^3$ and $x'=2d\theta+3a\theta^2/2$ give nearly the same positions for the minima as the simple relation $x'=\sqrt{2nd\lambda}$, they should therefore cease to be strictly valid. The actual positions of the minima for such values of d should agree more closely with those given by Schuster's formula, and should, when d is very large, agree absolutely with the same. This qualification is, however, of importance only with reference to the first two or three bands obtained for fairly large values of d. The differences in respect of the other bands would be negligibly small.

15. To test the foregoing results, measurements were made of the widths of the bright bands for a series of values of d

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* Debye's formula (*loc. cit.*) shows that the intensity of the reflected light becomes very small as ϕ approaches π.

up to 2 cm. Table III. shows the observed values, the values calculated from my formulæ, and the values according to Schuster's formula (which would be valid for a sharp diffracting edge in the same position). To calculate the positions of the minima given by the relations $n\lambda = 2d\theta^2 + 2a\theta^3$, and $x' = 2d\theta + 3a\theta^2/2$, the first equation was solved for θ by Horner's method, and the resulting values substituted in the second equation. The measurements of the width of the first band were rather rough on account of the indefiniteness of its outer edge. The agreement between the observed widths and the widths calculated from my formulæ is seen to be fairly satisfactory for values of d up to 3 mm. For larger values of d the observed widths agree more closely with those calculated from Schuster's formula, as explained in paragraphs 11 and 14 above.

Theory of the Fringes between the edge and the source of light.

16. As already remarked in paragraph 7, the direct and the reflected pencils tend to separate into distinct parts of the field when the focal plane of the observing microscope is put forward so as to lie between the edge of the cylinder and the source of light. Why this is so will be readily understood on a reference to fig. 3. The rays reflected

Fig. 3.

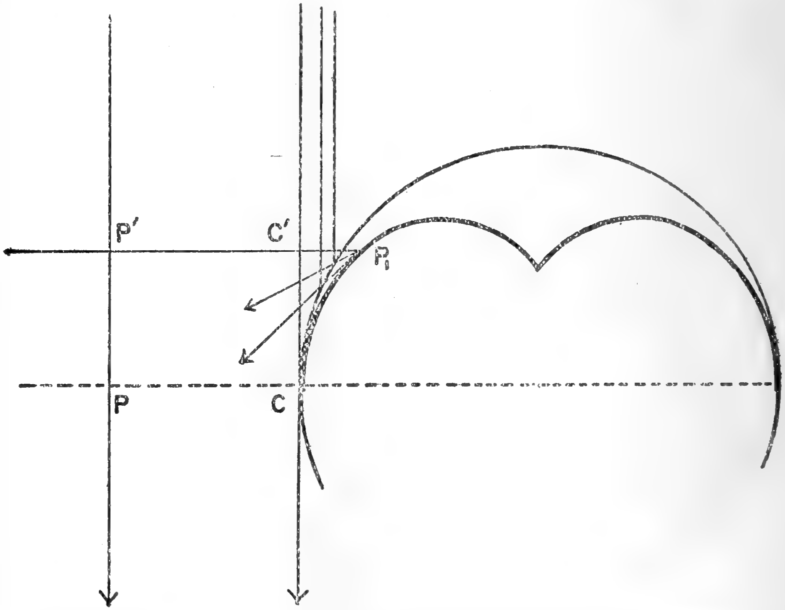

TABLE III.—Widths of Bright Bands in cm. × 10^{-5}.

$a=1\cdot 54$ cm. $\lambda=6562\times 10^{-8}$ cm.

n.	Observed widths.	Calculated [Formula (A)].	Calculated [Schuster's Formula].	Observed widths.	Calculated [Formula (A)].	Calculated [Schuster's Formula].	Observed widths.	Calculated [Formula (A)].	Calculated [Schuster's Formula].
		$d=\cdot 5$ mm.			$d=1$ mm.			$d=1\cdot 5$ mm.	
1......	277	294	239	397	385	339	471	458	415
2......	133	140	111	173	170	157	198	199	193
3......	107	111	83	134	134	118	154	155	145
4......	95	98	70	119	116	99	133	133	121
5......	86	89	61	100	103	87	114	119	106
6......	80	81	55	93	97	78	106	110	96
		$d=2$ mm.			$d=2\cdot 5$ mm.			$d=3$ mm.	
1......	527	522	479	594	578	536	606	636	587
2......	224	225	222	256	247	249	274	273	272
3......	173	175	167	194	197	187	211	201	202
4......	141	146	140	161	160	156	179	178	174
5......	131	133	123	138	145	137	153	154	150
6......	108 ?	121	115	129	129	124	141	141	136
		$d=5$ mm.			$d=10$ mm.			$d=20$ mm.	
2......	349	340	351	486	477	497	699	672	703
3......	267	263	264	375	366	373	514	516	528
4......	223	223	221	315	308	313	442	435	442

from the surface when produced backwards would touch the enveloping surface which lies within the cylinder. This surface, which is virtually the caustic of the reflected rays, terminates at the edge C of the cylinder, and when the focal plane of the observing microscope is moved forward from CP to a position $P'C'P_1$ in front of the edge, the boundary of the field on the right-hand side would shift into the region of the shadow, and would, in fact, lie on the surface of the caustic at the point P_1. If the focal plane $P'C'P_1$ is considerably forward of PC, the field is seen divided into two parts. The first part $P'C'$ consists of the direct rays alone (the reflected rays meeting $P'C'$ being too oblique to enter into the field of the microscope), and should obviously be bounded at C' by a few diffraction-fringes of the ordinary Fresnel type. The second part of the field P_1C' is due to the reflected rays alone, and requires separate consideration.

17. In the case considered above, that is, when the focal plane is considerably in advance of the edge, the fringe-system within the shadow due to the reflected light is of the same type as that found by Airy in his well-known investigation on the intensity of light in the neighbourhood of a caustic. For the elementary pencils into which the reflected rays may be divided up diverge from points lying along the caustic, and if the point P_1 at which the focal plane intersects the caustic is sufficiently removed from the edge C at which the latter terminates, Airy's investigation becomes fully applicable, but not otherwise. The rays emerging from the point P_1 after passage through the objective of the microscope become a parallel pencil, while pencils emerging from points on either side of P_1 become convergent and divergent respectively. The reflected wave-front after passage through the objective has thus a point of inflexion on either side of which it may be taken to extend indefinitely, provided the arc CP_1 be long enough. Assuming the focal length to be f and the equation of the wave-front to be $\xi = A\eta^3$, the value of A may be readily found. The equation of the caustic is

$$(4x^2+4y^2-a^2)^3-27a^4x^2=0.$$

From this, or directly by an approximate treatment, it may readily be shown that the radius of curvature of the caustic at the point C is $\frac{3}{4}$ the radius of the cylinder. For our present purpose, it is thus sufficient to treat the caustic as equivalent to a cylinder of radius $3a/4$ touching the reflecting surface at C. We have

$$A=\frac{1}{6}\frac{d^3\xi}{d\eta^3}=\frac{1}{6}\frac{d}{d\eta}\left(\frac{d^2\xi}{d\eta^2}\right),$$

where $\left(\frac{d^2\xi}{d\eta^2}\right)$ is the measure of the convergence or divergence of the normals to the wave-front in the neighbourhood of the point of inflexion. Substituting the values obtained from the formulæ of geometrical optics, it is found that

$$A=\frac{1}{6}\cdot\frac{3a}{4}\cdot\frac{1}{f^3}=\frac{a}{8f^3}.$$

The equation of the wave-front is accordingly

$$\xi=a\eta^3/8f^3.$$

The illumination in the fringe-system alongside the caustic is then given by Airy's formula

$$I=4\left[\int_0^\infty \cos\frac{\pi}{2}\left(w^3+mw\right)dw\right]^2,$$

where $m=4\,.\,2^{\frac{1}{3}}\,.\,a^{-\frac{1}{3}}\,.\,\lambda^{-\frac{2}{3}}\,.\,x_1,$

x_1 being the distance of any point in the focal plane measured from the point of intersection with the caustic. The integral gives a series of maxima of which the first is the largest, and the rest gradually converge to zero. The minima of illumination are zeroes*. As the focal plane is moved further and further towards the source of light, the fringe-system moves inwards along the caustic, but remains otherwise unaltered.

18. The foregoing treatment of the reflected fringe-system in terms of Airy's theory ceases to be valid when the focal plane is not sufficiently in advance of the edge, and the arc CP_1 of the caustic is therefore not large enough. For the reflected wave-front on one side of the point of inflexion then becomes limited in extent, and its equation cannot with sufficient accuracy be assumed to be of the simple form $\xi=A\eta^3$, extending to infinity in either direction. In fact, when the focal plane is at the edge of the cylinder and CP_1 is zero, the point of inflexion coincides with the extreme edge of the reflected wave-front. At this stage, of course, the

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* Graphs of Airy's integral and references to the literature will be found in an interesting paper by Aichi and Tanakadate (Journal of the College of Science, Tokyo, vol. xxi. Art. 3).

fringes seen in the field are due only to the interference of the direct and reflected wave-trains. The phenomena noticed as the focal plane is advanced towards the source of light, represent a gradual transition from this stage to one in which Airy's theory becomes fully applicable. In the transition-stages the field of illumination is a continuous whole, of which, however, the different parts present distinct characteristics. First, within the geometrical edge of the shadow, we have a finite number of fringes (one, two, or more according to the position of the plane of observation, but not an indefinitely large number as contemplated by Airy's theory); these may be regarded as the interference-fringes in the neighbourhood of the caustic due to the reflected light alone. Following these we have a long train of fringes due to the interference of the direct and the reflected pencils. The first few of these should evidently be modified by the diffraction which the direct rays suffer at the edge C before they reach the observing microscope. Finally, we may also have a part of the field in which the illumination is due only to the direct pencil, the reflected rays not entering the objective of the microscope owing to their obliquity. This part of the field should appear less brightly illuminated than the rest.

19. A complete theoretical treatment of the transition-stages described in the preceding paragraph is somewhat difficult, and has to be deferred to some future occasion. There is no difficulty, however, in calculating the positions of the fringes due to the interference of the direct and the reflected pencils when the focal plane is in advance of the edge, provided the diffraction-effect due to the edge is neglected. It is easily shown that the path-difference between the direct and reflected rays at a point x' is given by

$$\left.\begin{aligned}\delta' &= 2a\theta^3 - 2d\theta^2,\\ x' &= 3a\theta^2/2 - 2d\theta,\end{aligned}\right\} \quad , \quad . \quad . \quad . \quad . \qquad \text{(B)}$$

where x' is measured from C′ and $d = \mathrm{CC}'$. By putting $\delta' = n\lambda$ and eliminating θ, the positions of the minima of illumination may be calculated. A complete agreement of the results thus obtained with those found in experiment cannot, however, be expected, as the fringes are narrow and the modifications due to diffraction are not negligible. As regards the fringes alongside the caustic due to the reflected rays, we cannot expect to find a complete agreement between their widths and those found from Airy's theory, so long as the latter is not fully applicable. The divergence, if any, should be most marked when the region of the caustic under observation is nearest the edge of the cylinder, and for the fringes which are farthest from the caustic.

20. The foregoing conclusions have been tested by a series of measurements made with the focal plane in various positions in advance of the edge. To prove that the boundary of the field within the shadow is the caustic and not the surface of the cylinder, measurements were made of the length $C'P_1$, the rays incident on the cylinder being a parallel pencil.

	Observed value of $C'P_1$.	Calculated value.
$d=1$ mm.......	·00454 cm.	·00433 cm.
$d=1·3$ mm....	·00750 cm.	·00733 cm.

The following shows the widths of the fringes observed in the neighbourhood of the caustic when the focal plane was 1·6 mm. in advance of the edge, and the widths calculated from Airy's theory.

Width of fringes in cm. $\times 10^{-5}$.

Observed ...	159,	69,	56,	51,	45,	43
Calculated... (Airy's theory.)	155,	70,	57,	50,	46,	43

The agreement in both cases is satisfactory.

21. Table IV. shows the results of measurements made of the fringes in the transition-stages when the focal plane was only a little in advance of the edge, and Airy's theory is not fully applicable. The observed results are in general agreement with the indications of theory set out in paragraph 19. It will be seen that the fringes farthest within the region of the shadow show a fair agreement with Airy's theory, and the others are more nearly in agreement with the widths calculated from formula (B).

Summary and Conclusion.

22. C. F. Brush has recently published some observations of considerable interest on the diffraction of light by cylindrical edges. The views put forward by him to explain the phenomena, however, present serious difficulties and are open to objection. My attention was drawn to this subject by Prof. C. V. Raman, at whose suggestion the present work was undertaken by me in order to find the true explanation of the effects, and to develop a mathematical theory which would stand a quantitative test in experiment. This has now

TABLE IV.

Widths of Bright Bands in cm.$\times 10^{-6}$.

$a=1{\cdot}54$ cm. $\lambda=6562\times10^{-8}$ cm.

Observed widths.	Calculated [Airy's Formula].	Calculated [Formula (B)].	Observed widths.	Calculated [Airy's Formula].	Calculated [Formula (B)].
$d={\cdot}2$ mm.			$d={\cdot}4$ mm.		
1546	1550	......	1555	1550	......
852	696	894	777	696	754
722	570	777	628	570	674
672	504	705	590	504	636
622	461	637	566	461	594
590	429	602	500	429	543
$d={\cdot}6$ mm.			$d={\cdot}8$ mm.		
1513	1550	......	1466	1550	......
700	696	......	706	696	......
626	570	586	563	570	......
536	504	549	503	504	508
493	461	520	475	461	483
446	429	497	448	429	463

been done, and in the course of the investigation various features of importance overlooked by Brush have come to light. The following are the principal conclusions arrived at: (*a*) The fringes seen in the plane at which the incident light grazes the cylinder are due to the simple interference of the direct and the reflected rays, the positions of the dark bands being given by the formula $x=\frac{3}{4}\,.\,(2a)^{\frac{1}{3}}\,.\,(n\lambda)^{\frac{2}{3}}$; (*b*) the fringes in a plane further removed from the source of light than the cylinder are due to diffraction at the edge grazed by the incident rays but modified by interference with the light reflected from the surface of the cylinder. The positions of the dark bands in these fringes are (to a close approximation) given by the formulæ * $x=2d\theta+3a\theta^2/2$, and $n\lambda=2d\theta^2+2a\theta^3$, from which θ is to be eliminated; (*c*) when the focal plane of the observing microscope is on the side of the cylinder towards the light, the direct and reflected rays do not both cover exactly the same part of the field, and by putting the focal plane sufficiently forward towards the light,

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* This formula is subject to a small correction which is of importance only when d is large.

they may be entirely separated. When this is the case, the fringes of the ordinary Fresnel type due to the edge of the cylinder may be observed, and inside the shadow we have also an entirely separate system of fringes due to the reflected rays, the first and principal maximum of which lies alongside the virtual caustic formed by oblique reflexion; the distribution of intensity in this system can be found from the well-known integral due to Airy; (*d*) but when the focal plane is only a little in advance of the edge, the caustic and the reflecting surface are nearly in contact, and Airy's investigation of the intensity in the neighbourhood of a caustic requires modification. It is then found that only a finite number of bands (one, two, three, or more according to the position of the plane of observation) is formed within the limits of the shadow, and not an indefinitely large number as contemplated by Airy's theory. The rest of the fringes seen in the field are due to the interference of the direct and reflected rays, but modified by diffraction at the edge of the cylinder.

The Indian Association
for the Cultivation of Science,
Calcutta, 8th May, 1917.

X. *On Aerial Waves generated by Impact.* Part II. *By* SUDHANSUKUMAR BANERJI, *M.Sc., Assistant Professor of Applied Mathematics, University of Calcutta* [*].

[Plate IV.]

1. *Introduction.*

THE origin and characteristics of the sound produced by the collision of two solid spheres were discussed by me at some length in the first paper under the same title that was published in the Philosophical Magazine for July, 1916. It was shown in that paper that the sound is not due to the vibrations set up in the spheres, which in any ordinary material are both too high in pitch to be audible and too faint in intensity, but to aerial waves set up by the reversal of the motion of the spheres as a whole. The intensity of the sound in different directions for the case in which the two spheres were of the same material and diameter, was investigated by the aid of a new instrument which will be referred to as

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[*] Communicated by Prof. C. V. Raman.

"the ballistic phonometer [*]." The intensity was found to be a maximum along the line of collision, falling off gradually in other directions to a value which is practically zero on the surface of a cone of semi-vertical angle 67°, and rising again to a second but feebler maximum in a plane at right angles to the line of collision.

In view of the interesting results obtained for the case of two equal spheres, it was arranged to continue the investigation and to measure the distribution of intensity when the colliding spheres were not both of the same radius or material. A mathematical investigation of the nature of the results to be expected in these cases was also undertaken. In order to exhibit the results of the measurements and of the theoretical calculation, a plan has now been adopted which is much more suitable than the one used in the first paper. This will be best understood by reference to fig. 1 (Pl. IV.), which refers to the case of two spheres of the same material and diameter. The figure has been drawn by taking the point at which the spheres impinge as origin, and the line of collision as the axis of x, and setting off the indications of the ballistic phonometer as radii vectores at the respective angles which the directions in which the sound is measured make with the line of collision. The curve thus represents the distribution of intensity round the colliding spheres in polar coordinates, the points at which the intensity of the sound is measured being assumed to be all at the same distance from the spheres. The results are brought much more vividly before the eye by a diagram of this kind than by plotting the results on squared paper.

2. *Case of two spheres of the same material but of different diameters.*

Fig. 2, which shows the observed distribution of intensity when two spheres of wood of diameters 3 inches and $2\frac{1}{4}$ inches collide with each other, is typical of the results obtained when the impinging spheres are nearly of the same density and are of different diameters. There is a distinct asymmetry about a plane perpendicular to the line of impact. In addition to the maxima of intensity in the two directions of the line of collision, we have the maxima in lateral directions, which are not at right angles to this line. The

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[*] This name was suggested by Prof. E. H. Barton, D.Sc., F.R.S., writing in the 'Science Abstracts,' p. 399, Sept. 1916.

directions in which the intensity is a minimum are also asymmetrically situated.

For the explanation of these and other results, we have naturally to turn to the mathematical theory which rests upon the fact that the sound is due to the wave-motion set up in the fluid by the sudden reversal of the motion of the spheres. Let a and b be the radii of the two spheres and ρ_a and ρ_b be their densities. Then the masses of the spheres are $\frac{4}{3}\pi\rho_a a^3$ and $\frac{4}{3}\pi\rho_b b^3$ respectively. Denoting the changes in velocity which the balls undergo as a result of the impact by U_a and U_b respectively, by the principle of constant momentum we have $U_a/U_b=\rho_b b^3/\rho_a a^3$. The ratio $\frac{U_a}{U_b}$ thus depends only on the diameters and the densities of the spheres, while, of course, the actual values of U_a and U_b would depend on the relative velocity before impact and the coefficient of restitution. It is obvious that if we leave out of account the duration of impact, that is, regard the changes in velocity of the spheres as taking place practically instantaneously, the character and the ratio of the intensities of the sound produced in different directions would be completely determined by the sizes of the spheres and the ratio of their changes of velocity, that is, *by their diameters and their masses*; when the spheres are of the same material, the nature of the motion in the fluid set up by the impact would depend only on the radii of the spheres.

The complete mathematical problem of finding the nature of the fluid motion set up by the reversal of the motion of the spheres, taking the finite duration of impact into account, would appear to be of great difficulty. In my first paper, I have shown that when a single sphere of radius a undergoes an instantaneous change of velocity U, the wave-motion produced is given by the expression

$$\psi=-\frac{\sqrt{2}Ua^3}{4}\frac{\partial}{\partial r}\left[\frac{e^{-\left(\frac{ct+a-r}{a}\right)}}{r}\cos\left(\frac{ct+a-r}{a}-\frac{1}{4}\pi\right)\right]\cos\theta, \quad (1)$$

which indicates that it is of the damped harmonic type, confined to a small region near the front of the advancing wave. The wave-motion set up in the case of two spheres in contact assumed to undergo instantaneous changes of velocity would be of a more complicated type. In order to obtain a general idea of the results to be expected, particularly as to the intensity and character of the sound in different directions, we may consider the analogous acoustical problem of two rigid spheres nearly in contact, which execute small oscillations to and fro on the line of their centres. This problem may be mathematically formulated and approximately solved in the following manner :—

Given prescribed vibrations

$$U_a \cos\theta_1 \, . \, e^{ikct} \text{ and } U_b \cos\theta_2 \, . \, e^{ikct}$$

on the surfaces of two spheres of radii a and b nearly in contact, it is required to determine the velocity potential of the wave-motion started and the distribution of intensities round the spheres, where θ_1 and θ_2 are the angles measured at the centres A and B of the two spheres in opposite senses from the line joining the centres.

Supposing, now, that an imaginary sphere is constructed which is of just sufficient radius to envelop the two actual spheres (touching them externally), it is possible from a consideration of the nature of the motion that takes place in the immediate neighbourhood of the two spheres, to determine the aerial vibration on the surface of the imaginary sphere which would produce on the external atmosphere the same effect as the vibrations on the surfaces of the real spheres A and B. When the equivalent vibration on the surface of the enveloping sphere has been obtained, we can, by the use of the well-known solution for a single sphere, at once determine the wave-motion at any external point.

The radius of the enveloping sphere is evidently $a+b$, and its centre is at a point C, such that BC$=a$ and CA$=b$.

If the point C be taken as origin, and if the equivalent vibration on the surface of the enveloping sphere be expressed by the series

$$\Sigma A_n P_n (\cos\theta) \, e^{ikct}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A_n's are known constants, the velocity potential of the wave-motion is given by

$$\psi = -\frac{(a+b)^2}{r} e^{ik(ct-r+a+b)} \Sigma \frac{A_n P_n(\cos\theta)}{F_n(ik\,.\,\overline{a+b})} f_n(ikr), \quad . \quad (3)$$

where

$$f_n(ikr) = 1 + \frac{n(n+1)}{2\,.\,ikr} + \frac{(n-1)n(n+1)(n+2)}{2\,.\,4\,.\,(ikr)^2} + \dots$$

$$\dots + \frac{1\,.\,2\,.\,3 \dots 2n}{2\,.\,4\,.\,6 \dots 2n\,.\,(ikr)^n},$$

$$F_n(ikr) = (1+ikr) f_n(ikr) - ikr \, f_n'(ikr). \quad . \quad . \quad (4)$$

To obtain the equivalent vibrations on the surface of the imaginary enveloping sphere, we shall regard the small quantity of fluid enclosed by this sphere as practically incompressible, and use the well-known solution by the method of successive images for two spheres in an incompressible fluid.

We know that the velocity potential due to such a system of two spheres in an incompressible fluid can be expressed in the form

$$U_a\phi + U_b\phi', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where ϕ and ϕ' are to be determined by the conditions

$$\nabla^2\phi = 0, \qquad \nabla^2\phi' = 0,$$

$$\frac{\partial\phi}{\partial r_1} = -\cos\theta_1, \quad \text{and} \quad \frac{\partial\phi'}{\partial r_1} = 0, \quad \text{when } r_1 = a,$$

$$\frac{\partial\phi'}{\partial r_2} = -\cos\theta_2, \quad \text{and} \quad \frac{\partial\phi}{\partial r_2} = 0, \quad \text{when } r_2 = b,$$

r_1, r_2 being radii vectores drawn from A and B.

When ϕ and ϕ' have been determined so as to satisfy these conditions, the equivalent vibrations on the surface of the imaginary sphere can be taken to be very approximately given by the expression

$$-\left[U_a\frac{\partial\phi}{\partial r} + U_b\frac{\partial\phi'}{\partial r}\right]e^{ikct}_{r=a+b} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The functions ϕ and ϕ', as is well known, can be determined by the method of successive images, and if the expressions for the velocity potential due to these images be all transferred to the coordinates r, θ referred to the centre C of the enveloping sphere, we easily obtain

$$2\phi = a^3\left[1 - \frac{b^3}{(a+b)^3} + \frac{b^3}{(a+2b)^3} - \frac{b^3}{(2a+2b)^3} + \frac{b^3}{(2a+3b)^3} - \ldots\right]\frac{P_1(\cos\theta)}{r^2}$$

$$+ 2a^3\left[b - \frac{b^3(a^2+ab-b^2)}{(a+b)^4} + \frac{b^3(2b^2-a^2)}{(a+2b)^4} - \frac{b^3(2a^2+ab-2b^2)}{(2a+2b)^4} + \frac{b^3(3b^2-2a^2)}{(2a+3b)^4} - \ldots\right]\frac{P_2(\cos\theta)}{r^3}$$

$$+3a^3\left[b^2-\frac{b^3(a^2+ab-b^2)^2}{(a+b)^5}+\frac{b^3(2b^2-a^2)^2}{(a+2b)^5}-\frac{b^3(2a^2+ab-2b^2)^2}{(2a+2b)^5}\right.$$
$$\left.+\frac{b^3(3b^2-2a^2)^2}{(2a+3b)^5}-\dots\right]\frac{P_3(\cos\theta)}{r^4}$$
$$+4a^3\left[b^3-\frac{b^3(a^2+ab-b^2)^3}{(a+b)^6}+\frac{b^3(2b^2-a^2)^3}{(a+2b)^6}-\frac{b^3(2a^2+ab-2b^2)^3}{(2a+2b)^6}\right.$$
$$\left.+\frac{b^3(3b^2-2a^2)^3}{(2a+3b)^6}-\dots\right]\frac{P_4(\cos\theta)}{r^5}$$
$$+\ \&c., \quad \dots\dots\dots \quad (7)$$

and

$$2\phi'=-b^3\left[1-\frac{a^3}{(b+a)^3}+\frac{a^3}{(b+2a)^3}-\frac{a^3}{(2b+2a)^3}\right.$$
$$\left.+\frac{a^3}{(2b+3a)^3}-\dots\right]\frac{P_1(\cos\theta)}{r^2}$$
$$+2b^3\left[a-\frac{a^3(b^2+ab-a^2)}{(b+a)^4}+\frac{a^3(2a^2-b^2)}{(b+2a)^4}-\frac{a^3(2b^2+ab-2a^2)}{(2b+2a)^4}\right.$$
$$\left.+\frac{a^3(3a^2-2b^2)}{(2b+3a)^4}-\dots\right]\frac{P_2(\cos\theta)}{r^3}$$
$$-3b^3\left[a^2-\frac{a^3(b^2+ab-a^2)^2}{(b+a)^5}+\frac{a^3(2a^2-b^2)^2}{(b+2a)^5}-\frac{a^3(2b^2+ab-2a^2)^2}{(2b+2a)^5}\right.$$
$$\left.+\frac{a^3(3a^2-2b^2)^2}{(2b+3a)^5}-\dots\right]\frac{P_3(\cos\theta)}{r^4}$$
$$+4b^3\left[a^3-\frac{a^3(b^2+ab-a^2)^3}{(b+a)^6}+\frac{a^3(2a^2-b^2)^3}{(b+2a)^6}-\frac{a^3(2b^2+ab-2a^2)^3}{(2b+2a)^6}\right.$$
$$\left.+\frac{a^3(3a^2-2b^2)^3}{(2b+3a)^6}-\dots\right]\frac{P_4(\cos\theta)}{r^5}$$
$$-\ \&c., \quad \dots\dots\dots \quad (8)$$

the law of formation of the series within the brackets being obvious.

Coming now to the present problem of two unequal spheres of the same material, let us take

$$a=2 \text{ inches and } b=1 \text{ inch.}$$

Since the changes of velocities of the two spheres are inversely proportional to their masses, we must have

$$U_b=8U_a.$$

Substituting the values for a and b, we easily find that

$$
\begin{aligned}
2\phi = 2^3 \Bigg[& \left(1 + \frac{1}{4^3} + \frac{1}{7^3} + \frac{1}{10^3} + \dots\right) \\
& \qquad - \left(\frac{1}{3^3} + \frac{1}{6^3} + \frac{1}{9^3} + \dots\right)\Bigg] \frac{\mathrm{P}_1 (\cos\theta)}{r^2} \\
+ 2 \,.\, 2^3 \Bigg[& 1 - \left(\frac{5}{3^4} + \frac{8}{6^4} + \frac{11}{9^4} + \frac{14}{12^4} + \dots\right) \\
& \qquad - \left(\frac{2}{4^4} + \frac{5}{7^4} + \frac{8}{10^4} + \dots\right)\Bigg] \frac{\mathrm{P}_2 (\cos\theta)}{r^3} \\
+ 3 \,.\, 2^3 \Bigg[& \left(1 + \frac{2^2}{4^5} + \frac{5^2}{7^5} + \frac{8^2}{10^5} + \dots\right) \\
& \qquad - \left(\frac{5^2}{3^5} + \frac{8^2}{6^5} + \frac{11^2}{9^5} + \dots\right)\Bigg] \frac{\mathrm{P}_3 (\cos\theta)}{r^4} \\
+ 4 \,.\, 2^3 \Bigg[& 1 - \left(\frac{5^3}{3^6} + \frac{8^3}{6^6} + \frac{11^3}{9^6} + \frac{14^3}{12^6} + \dots\right) \\
& \qquad - \left(\frac{2^3}{4^6} + \frac{5^3}{7^6} + \frac{8^3}{10^6} + \dots\right)\Bigg] \frac{\mathrm{P}_4 (\cos\theta)}{r^5} \\
& \qquad + \text{\&c.}, \quad \dots \dots \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
2\phi' = -2^3 \Bigg[& \left(\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \frac{1}{11^3} + \dots\right) \\
& \qquad - \left(\frac{1}{3^3} + \frac{1}{6^3} + \frac{1}{9^3} + \dots\right)\Bigg] \frac{\mathrm{P}_1 (\cos\theta)}{r^2} \\
+ 2 \,.\, 2^3 \Bigg[& \left(\frac{1}{3^4} + \frac{4}{6^4} + \frac{7}{9^4} + \dots\right) \\
& \qquad + \left(\frac{4}{2^4} + \frac{7}{5^4} + \frac{10}{8^4} + \frac{13}{11^4} + \dots\right)\Bigg] \frac{\mathrm{P}_2 (\cos\theta)}{r^3} \\
- 3 \,.\, 2^3 \Bigg[& \left(\frac{4^2}{2^5} + \frac{7^2}{5^5} + \frac{10^2}{8^5} + \frac{13^2}{11^5} + \dots\right) \\
& \qquad - \left(\frac{1^2}{3^5} + \frac{4^2}{6^5} + \frac{7^2}{9^5} + \dots\right)\Bigg] \frac{\mathrm{P}_3 (\cos\theta)}{r^4} \\
+ 4 \,.\, 2^3 \Bigg[& \left(\frac{1^3}{3^6} + \frac{4^3}{6^6} + \frac{7^3}{9^6} + \dots\right) \\
& \qquad + \left(\frac{4^3}{2^6} + \frac{7^3}{5^6} + \frac{10^3}{8^6} + \frac{13^3}{11^6} + \dots\right)\Bigg] \frac{\mathrm{P}_4 (\cos\theta)}{r^5} \\
& \qquad - \text{\&c.} \quad \dots \dots \qquad (10)
\end{aligned}
$$

Summing the series, we easily find that the vibration on the surface of the enveloping sphere

$$= \quad \mathrm{U}_a\left[\frac{\partial\phi}{\partial r}+8\frac{\partial\phi'}{\partial r}\right]e^{ikct}_{r=3 \text{ inches}}$$

$$=-\frac{4}{27}\mathrm{U}_a\Big[\cdot 496\,\mathrm{P}_1(\cos\theta)+3\cdot 180\,\mathrm{P}_2(\cos\theta)$$

$$-1\cdot 708\,\mathrm{P}_3(\cos\theta)+2\cdot 600\,\mathrm{P}_4(\cos\theta)+\ldots\Big]e^{ikct} \quad (11)$$

We have seen that when the vibration on the surface of the enveloping sphere is

$$\Sigma\,\mathrm{A}_n\mathrm{P}_n(\cos\theta)\,.\,e^{ikct},$$

the velocity potential of the wave-disturbance is

$$\psi=-\frac{(a+b)^2}{r}e^{ik(ct-r+a+b)}\Sigma\frac{\mathrm{A}_n\mathrm{P}_n(\cos\theta)}{\mathrm{F}_n(ik\,.\,\overline{a+b})}f_n(ikr).$$

Now when r is large, $f_n(ikr)=1$, so that the factor on which the relative intensities in various directions depend is

$$\Sigma\,\mathrm{A}_n\frac{\mathrm{P}_n(\cos\theta)}{\mathrm{F}_n(ik\,.\,\overline{a+b})}.$$

Thus if we put this quantity $=\mathrm{F}+i\mathrm{G}$, the intensity of the vibrations in various directions is measured by $\mathrm{F}^2+\mathrm{G}^2$.

The distribution of intensities in different directions round the spheres will be influenced to a considerable extent by the value of the wave-length chosen. If we take $k(a+b)=2$, the wave-length is 3π inches, and if we take $k(a+b)=3$, the wave-length is 2π inches. From the expression (1) for the wave-motion produced by a single sphere undergoing an instantaneous change of velocity, it is seen that the wave-length to be chosen is of the same order as the circumference of the sphere. From this, it appears that for a system of two spheres whose radii are 1 inch and 2 inches respectively, the wave-length to be chosen should be some value intermediate between 2π and 4π, probably nearer 2π than 4π; for, in the actual case of impact, the smaller ball which would undergo by far the greater change in velocity would probably influence the character of the motion to a greater extent than the larger sphere. At the same time, it must not be forgotten that the analogy between the cases of impact and of periodic motion cannot be pushed very far, inasmuch as the fluid motion due to impact is undoubtedly of different character in different directions, and not all throughout the same as in the periodic case.

Now taking $k(a+b)=2$, we find (neglecting a constant factor)

$$F={\cdot}0992\,P_1(\cos\theta)+{\cdot}2840\,P_2(\cos\theta)-{\cdot}0354\,P_3(\cos\theta)$$
$$-{\cdot}0146\,P_4(\cos\theta)+\&c.,$$
$$G={\cdot}0496\,P_1(\cos\theta)-{\cdot}4040\,P_2(\cos\theta)-{\cdot}0177\,P_3(\cos\theta)$$
$$+{\cdot}0315\,P_4(\cos\theta)+\&c. \quad . \quad . \quad . \quad (12)$$

The values of F and G for different directions have been calculated and are shown in the following table :—

TABLE I.

Angles (in degrees).	F × const.	G × const.	(F^2+G^2) × const.
0	+329000	−338000	223144
10	+325908	−326525	212552
20	+297435	−273545	162738
30	+251537	−214337	109300
40	+189095	−114659	48946
50	+114215	− 24167	13572
60	+ 33341	+ 74407	6565
70	− 43647	+155306	25961
80	−107073	+205212	53574
90	−147250	+213625	67405
100	−158815	+178920	57322
110	−140329	+106238	30836
120	− 95149	+ 8675	9106
130	− 34099	− 99267	10957
140	+ 35677	−202159	42100
150	+102819	−289237	94130
160	+157869	−353605	150280
170	+192648	−392229	190913
180	+201000	−404000	203617

Now taking $k(a+b)=3$, we find (neglecting a constant factor)

$$F=\ {\cdot}105\,P_1(\cos\theta)+1{\cdot}060\,P_2(\cos\theta)+{\cdot}016\,P_3(\cos\theta)$$
$$-\ {\cdot}281\,P_4(\cos\theta)-\&c.$$
$$G=-{\cdot}122\,P_1(\cos\theta)+\ {\cdot}186\,P_3(\cos\theta)-{\cdot}024\,P_4(\cos\theta)$$
$$-\&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The values in different directions have been calculated from these expressions and are shown in the following table :—

TABLE II.

Angles (in degrees).	F × const.	G × const.	(F^2+G^2) × const.
0	+900000	+ 38000	811444
10	+890608	+ 28804	794265
20	+849305	− 2390	720805
30	+752167	− 45754	567620
40	+572469	− 90446	335284
50	+309902	−123998	111476
60	− 5783	−135346	18261
70	−313014	−118446	111893
80	−542728	− 73554	300178
90	−635375	− 9000	403306
100	−571364	+ 60786	329762
110	−371618	+118638	154545
120	− 96799	+149218	31610
130	+184472	+144494	54592
140	+412409	+105758	180980
150	+559907	+ 44650	315625
160	+630625	− 21410	398602
170	+654606	− 69748	433925
180	+658000	− 88000	440708

The values of F^2+G^2 shown in Tables I. and II., have been plotted in polar coordinates in figs. 3 and 4 (Pl. IV.). It is seen that in both cases the intensity in the direction of the larger ball is greater than in the direction of the smaller ball. The asymmetry is more marked when $k(a+b)$ has the larger value.

The intensity of the sound in different directions due to the impact of two spheres of wood of diameters 3 inches and $1\frac{1}{2}$ inches respectively has been measured with the ballistic phonometer and is shown in fig. 5. It is seen that this curve is intermediate in form between those shown in fig. 3 and fig. 4, exactly as anticipated. The agreement between theory and experiment is thus very striking in this case.

3. *Two spheres of the same diameter but of different materials.*

We have seen in the preceding section that in the expressions for F and G for two spheres of the same material but of unequal diameters, the terms containing the zonal harmonic of the second order $P_2(\cos\theta)$ usually preponderate, and that the intensity diagram is, accordingly, a curve which consists of four loops. A different result is obtained in the case of two spheres of the same diameter but of markedly unequal densities. The zonal harmonic of the first order preponderates in this case, and the intensity diagram is a curve consisting of only two loops. To obtain this result theoretically, we have to proceed on exactly the same lines as in the preceding pages.

Taking $a=1$ inch and $b=1$ inch, we easily find from the expressions (7) and (8) that

$$\begin{aligned}2\phi= &\left[1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\dots\right]\frac{P_1(\cos\theta)}{r^2}\\ &+2\left[1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\dots\right]\frac{P_2(\cos\theta)}{r^3}\\ &+3\left[1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\dots\right]\frac{P_3(\cos\theta)}{r^4}\\ &+4\left[1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\dots\right]\frac{P_4(\cos\theta)}{r^5}\\ &+\text{\&c.}, \qquad \dots\dots \qquad (14)\end{aligned}$$

and

$$\begin{aligned}2\phi'= &-\left[1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\dots\right]\frac{P_1(\cos\theta)}{r^2}\\ &+\left[1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\dots\right]\frac{P_2(\cos\theta)}{r^3}\\ &-\left[1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\dots\right]\frac{P_3(\cos\theta)}{r^4}\\ &+\left[1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\dots\right]\frac{P_4(\cos\theta)}{r^5}\\ &-\text{\&c.} \qquad \dots\dots \qquad (15)\end{aligned}$$

Summing the series, we find that the vibrations on the surface of the enveloping sphere, namely

$$\left[\mathrm{U}_a\frac{\partial\phi}{\partial r}+\mathrm{U}_b\frac{\partial\phi'}{\partial r}\right]e^{ikct}_{r=2\text{ inches}}$$

can be expressed in the form

$$\tfrac{1}{2}\big[(\mathrm{U}_a-\mathrm{U}_b)\times{\cdot}2254\,\mathrm{P}_1(\cos\theta)+(\mathrm{U}_a+\mathrm{U}_b)\times{\cdot}3550\,\mathrm{P}_2(\cos\theta)$$

$$+(\mathrm{U}_a-\mathrm{U}_b)\times{\cdot}3645\,\mathrm{P}_3(\cos\theta)+(\mathrm{U}_a+\mathrm{U}_b)\times{\cdot}3080\,\mathrm{P}_4(\cos\theta)$$

$$+(\mathrm{U}_a-\mathrm{U}_b)\times{\cdot}2325\,\mathrm{P}_5(\cos\theta)+\&\mathrm{c.}\big]e^{ikct}\quad.\;.\;.\;.\;.\;(16)$$

If the ball of radius b is four times heavier than the one of radius a, we have

$$\mathrm{U}_a=4\mathrm{U}_b.$$

So that the vibration on the surface of the enveloping sphere is proportional to the expression

$${\cdot}6762\,\mathrm{P}_1(\cos\theta)+1{\cdot}7750\,\mathrm{P}_2(\cos\theta)+1{\cdot}0935\,\mathrm{P}_3(\cos\theta)$$
$$+1{\cdot}5400\,\mathrm{P}_4(\cos\theta)+{\cdot}6975\,\mathrm{P}_5(\cos\theta)+\&\mathrm{c.}$$

Now taking $k(a+b)=1$, which will give a wave-length equal to the circumference of the enveloping sphere, we get (neglecting a constant factor)

$$\mathrm{F}=\ {\cdot}13524\,\mathrm{P}_1(\cos\theta)-{\cdot}04987\,\mathrm{P}_2(\cos\theta)-{\cdot}0074\,\mathrm{P}_3(\cos\theta)$$
$$+{\cdot}0007\,\mathrm{P}_4(\cos\theta)+\&\mathrm{c.}$$

$$\mathrm{G}=-\ {\cdot}0676\,\mathrm{P}_1(\cos\theta)-{\cdot}0798\,\mathrm{P}_2(\cos\theta)+{\cdot}0047\,\mathrm{P}_3(\cos\theta)$$
$$+{\cdot}0012\,\mathrm{P}_4(\cos\theta)-\&\mathrm{c.}\quad.\;.\;(17)$$

The values of F and G, and of $\mathrm{F}^2+\mathrm{G}^2$ in different directions obtained from the preceding expressions are shown in Table III.

The values of $(\mathrm{F}^2+\mathrm{G}^2)$ shown in Table III. have been plotted in polar coordinates and are shown in fig. 6. It is seen that the maximum intensity in the direction of the heavier ball is greater than that in the direction of the lighter one.

The experimental curve of intensity of sound due to impact of a sphere of wood, diameter $2\frac{1}{4}$ inches, with a billiard ball of nearly the same size is shown in fig. 7. It is found that the directions of minimum intensity are not quite in the plane perpendicular to the line of impact, being nearer the side of the lighter ball.

TABLE III.

Angles (in degrees).	F × const.	G × const.	(F^2+G^2) × const.
0	+ 786000	−1415000	2620021
10	+ 793732	−1374897	2521061
20	+ 813819	−1256037	2240132
30	+ 835068	−1068615	1839986
40	+ 845629	− 826059	1397992
50	+ 828667	− 549652	989741
60	+ 768690	− 262257	660005
70	+ 654594	+ 7901	427780
80	+ 482433	+ 237049	288493
90	+ 252125	+ 403500	225913
100	+ 228907	+ 495515	297466
110	− 331298	+ 509107	368642
120	− 647986	+ 454821	626929
130	− 954405	+ 347884	1031220
140	−1229335	+ 211923	1555385
150	−1458496	+ 71667	2130948
160	−1629521	− 47667	2655850
170	−1734880	− 128811	3026866
180	−1770000	− 157000	3157549

A result of some importance indicated by theory is that when one of the spheres is much heavier than the other, replacing the former by a still heavier sphere of the same diameter should not result in any important alteration in the distribution of the intensity of sound in different directions due to impact. This is clear from expression (16). For when U_a is much larger than U_b, any diminution in the value of U_b should not appreciably affect the value of the expression. This indication of theory is in agreement with experiment. Several series of measurements have been made with various pairs of balls of the same size but of different densities, *e. g.*, wood and marble, wood and iron, billiard ball and iron ball, and so forth. Generally, similar results are obtained in all cases. It was noticed also that the form of the intensity distribution as shown by the ballistic phonometer was not altogether independent of the thickness of the mica disk used in the instrument. This is not surprising, as the behaviour of the mica disk before the pointer attached to it ceases to touch the mirror of the indicator would no doubt depend, to some extent, on the relation between its natural frequency and the frequency of the sound-waves set up by the impact. The best results were obtained with a disk neither so thick as to be relatively insensitive nor so thin as to remain with its pointer in contact with the indicator longer than absolutely necessary.

4. *The general case of spheres of any diameter and density.*

When the impinging spheres are both of different diameter and of different density, the result generally obtained is that the sound is a maximum on the line of impact in either direction, and a minimum which approaches zero in directions asymmetrically situated with reference thereto. Generally speaking, no maxima in lateral directions are noticed, that is, the curve consists of two nearly closed loops. The difference of the intensity of the sound in the two directions of the line of impact may sometimes be very considerable. As a typical case, the results obtained by the impact of a sphere of wood 3 inches in diameter with a brass sphere only $1\frac{1}{8}$ inch in diameter are shown in fig. 8. It is observed that the sound due to impact is actually of greater intensity on the side of the small brass ball. As a matter of fact, the result generally obtained is that the intensity is greater on the side of the ball of the denser material even if its diameter be the smaller.

The mathematical treatment of the general case is precisely on the same lines as in the two preceding sections. It is found in agreement with the experimental result that in practically all cases in which both the densities and the diameters are different, the zonal harmonic of the first order is of importance and that the intensity curve consists of two nearly closed loops, as in the case of two spheres of the same diameter but of different density.

5. *Summary and Conclusion.*

The investigation of the origin and characteristics of the sound due to the direct impact of two similar solid spheres which was described in the Phil. Mag. for July, 1916, has been extended in the present paper to the cases in which the impinging spheres are not both of the same diameter or material. The relative intensities of the sound in different directions have been measured by the aid of the ballistic phonometer, and in order to exhibit the results in an effective manner, they have been plotted in polar coordinates, the point at which the spheres impinge being taken as the origin, and the line of collision as the axis of x. As might be expected, the curves thus drawn show marked asymmetry in respect of the plane perpendicular to the line of impact.

A detailed mathematical discussion of the nature of the results to be expected is possible by considering the analogous case of two rigid spheres nearly in contact which vibrate bodily along their line of centres. By choosing an appropriate wave-length for the resulting motion, intensity curves similar to those found experimentally for the case of impact are arrived at. A further confirmation is thus obtained of the hypothesis regarding the origin of the sound suggested by the work of Hertz and of Lord Rayleigh on the theory of elastic impact.

When the impinging spheres, though not equal in size, are of the same or nearly the same density, the intensity-curve drawn for the plane of observation shows the sound to be a maximum along the line of impact in either direction, and also along two directions making equal acute angles with this line. The sound is a minimum along four directions in the plane. In practically all other cases, that is when the spheres differ considerably either in density alone, or both in diameter and density, the intensity is found to be a maximum along the line of impact in either direction, and to be a minimum along directions which are nearly but not quite perpendicular to the line of impact. The form of the intensity curve is practically determined by the diameters and the masses of the spheres.

The investigation was carried out in the Physical Laboratory of the Indian Association for the Cultivation of Science. It is hoped when a suitable opportunity arises to study also the case of oblique impact. The writer has much pleasure in acknowledging the helpful interest taken by Prof. C. V. Raman in the progress of the work described in the present paper.

Calcutta,
15th June, 1917.

XI. *On the Asymmetry of the Illumination-Curves in Oblique Diffraction.* *By* Sisir Kumar Mitra, *M.Sc., Sir Rashbehary Ghosh Research Scholar in the University of Calcutta* *.

[Plate V.]

Introduction.

In the Phil. Mag. for May 1911, C. V. Raman has given the results of a photometric study of the unsymmetrical diffraction-bands due to an obliquely held rectangular reflecting surface previously observed by him †. The measurements showed a very marked asymmetry in the distribution of intensity in the diffraction pattern, the theoretical explanation of which is discussed in the papers quoted. The following were the principal conclusions arrived at by Raman as the result of the quantitative experimental study of the case :—

(*a*) The illumination at the points of minimum intensity in the diffraction pattern is zero at all angles of incidence, and the positions of the minima are accurately given by the formula

$$\delta = \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \&c.,$$

where $\delta = \frac{\pi a}{\lambda}(\sin i - \sin\theta)$, a being the width of the aperture, λ the wave-length, and i, θ the angles of incidence and diffraction respectively ; the fringes are wider on the side on which $\theta > i$ and their number is limited on that side, as θ cannot be greater than $\frac{\pi}{2}$.

(*b*) The formula of the usual type ($I = \sin^2\delta/\delta^2$) for the illumination in the pattern fails to represent the observed intensity-curves at oblique incidences except in regard to the position of the minima ($\delta = \pm\pi$, $\pm 2\pi$, &c.). The intensities at corresponding points on either side of the central fringe for which the values of δ are numerically the same are *not* equal.

(*c*) The observed distribution of intensity was found to fit in with the theoretical formula, if the latter is multiplied by a factor proportional to the square of the cosine of the

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* Communicated by Prof. C. V. Raman.

† C. V. Raman, M.A., "On the Unsymmetrical Diffraction Bands due to a Rectangular Aperture," Phil. Mag. Nov. 1906. See also Phil. Mag. Jan. 1909.

obliquity, which, of course, is not the same at all points in the diffraction pattern. In other words, the ordinates of the illumination curve were found to be proportional to the expression $\cos^2\theta\ \sin^2\delta/\delta^2$.

The question arises whether these results, practically those indicated in (*b*) and (*c*) above, are peculiar to the case of a surface of rectangular form, or whether similar phenomena might be expected with other forms of surface as well. The cases which it seemed of particular interest to examine are those in which the reflecting surface is not a single individual area but consists of two, three, or more parallel elements lying in the same plane. A satisfactory surface of this kind which can be used at very oblique incidences may be prepared by etching out deep grooves on the optically plane surface of a thick plate of glass with hydrofluoric acid, the edges of the reflecting strips left on the surface being subsequently ground so as to be sharp, straight, and parallel. I have prepared several such surfaces containing two and three equidistant reflecting strips respectively. By placing one of these on the table of a spectrometer, the diffraction pattern produced by reflexion at very oblique incidences may be readily observed through the telescope of the instrument. The present paper describes the results of the quantitative study of the phenomena thus obtained. Incidentally the opportunity has also been taken of testing the results obtained by Raman for the case of a single aperture using improved optical and photographic appliances. The experiments and determinations have throughout been made using monochromatic light. This was secured by illuminating the slit of the spectrometer with light of a definite wave-length isolated by a monochromator from sunlight or arc light.

Unsymmetrical Interference-fringes due to two parallel apertures.

Fig. I (Pl. V.) reproduces a photograph of the diffraction pattern due to a surface containing two reflecting elements each of width 0·48 cm., and 3·60 cm. apart. The direct image of the slit of the spectrometer also appears in the figures to the right of the diffraction-pattern. The photograph is reproduced from a dense negative taken to show the perfect blackness of the minima of illumination, and the progressive increase (from left to right) in the width of the interference-fringes of the light diffracted by the two reflecting elements. It was obtained by replacing the telescope of the spectrometer by a camera with a lens of long focus (176 cm.). Figs. II, III, and IV reproduce three photographs taken at three different angles of incidence, the reflecting strips in this case being 0·754 cm. wide and 1·446 cm. apart. In all the figures, the central fringe of the pattern is indicated by a small cross ×. The asymmetry of the luminosity curve will be evident on comparing the brightness of the corresponding bands on either side of the central fringe; for instance, the second band on the right and the second band on the left in figs. II and III, or the first band on the right and the first band on the left in fig. IV.

The positions of the interference minima in the pattern are given by the formula of the usual type

$$\delta = \pm\frac{\pi}{2}, \quad \pm\frac{3\pi}{2}, \quad \pm\frac{5\pi}{2}, \text{ \&c.}$$

where $\delta = \pi(a+b)$ $(\sin i - \sin \theta)/\lambda$, a being the width of each of the apertures, b their distance apart, and i, θ, λ having their usual significance. To test whether the formula holds good at the oblique incidences used, the negatives were measured under a travelling microscope. In photograph I, the distances between the successive interference minima were determined to find whether the relations

$$\sin\theta_1 - \sin\theta_2 = \sin\theta_2 - \sin\theta_3 = \sin\theta_3 - \sin\theta_4, \text{ \&c.},$$

indicated by the formula were valid. The results are shown in Table I.

TABLE I.

$a = 0{\cdot}48$ cm. $b = 3{\cdot}60$ cm.

Minima on the right of the centra	Observed value of $\sin\theta_{n+1} - \sin\theta_n$ × constant.	Minima on the left of the central fringe.	Observed value of $\sin\theta_n - \sin\theta_{n+1}$ × constant.
$\sin\theta_5 - \sin\theta_4$	0·712	$\sin\theta_1 - \sin\theta_2$	0·705
$\sin\theta_4 - \sin\theta_3$	0·703	$\sin\theta_2 - \sin\theta_3$	0·710
$\sin\theta_3 - \sin\theta_2$	0·711	$\sin\theta_3 - \sin\theta_4$	0·711
$\sin\theta_2 - \sin\theta_1$	0·711	$\sin\theta_4 - \sin\theta_5$	0·710
......	......	$\sin\theta_5 - \sin\theta_6$	0·713

For photographs II, III, and IV, the actual values of θ for the interference minima were calculated from the known constants a, b, i, λ and compared with the observed values. These are shown in Tables II., III., and IV.

Table II.

a=0·754 cm. b=1·446 cm. λ=·0000435 cm.
i=89° 15′·27.

Interference Minima.	Calculated diffraction angle, 90° − θ (in minutes).	Observed diffraction angle, 90° − θ (in minutes).
2nd on the right.	35′·98	35′·85
1st ,, ,, ,,	41′ 97	41′·83
1st ,, ,, left.	47′·21	46′·95
2nd ,, ,, ,,	51′·92	52′·05
5th ,, ,, ,,	64′·02	63′·95

Table III.

a=0·754 cm. b=1·446 cm. λ=·0000435 cm.
i=89° 22′·75.

Interference Minima.	Calculated 90°−θ.	Observed 90°−θ.
2nd on the right.	26′·15	26′·12
1st ,, ,, ,,	33′·88	33′·85
1st ,, ,, left.	40′·25	40′·09
2nd ,, ,, ,,	45′·69	45′·70
5th ,, ,, ,,	58′·90	58′·75

TABLE IV.

$a = \cdot 754$ cm. $b = 1\cdot 446$ cm. $\lambda = \cdot 0000435$ cm.
$i = 89^\circ\ 29'\cdot 90$.

Interference Minima.	Calculated $90^\circ - \theta$.	Observed $90^\circ - \theta$.
1st on the right.	$25'\cdot 72$	$25'\cdot 62$
1st on the left.	$33'\cdot 75$	$33'\cdot 45$
2nd ,, ,, ,,	$40'\cdot 08$	$39'\cdot 94$
5th ,, ,, ,,	$54'\cdot 94$	$54'\cdot 96$

The Asymmetry of the Illumination-Curves.

As remarked above, there is a very marked difference in the luminosity of the corresponding bands on either side of the central fringe of the pattern due to the reflecting surface of two elements. Similar effects are also noticeable when the reflecting surface consists of three elements. Figs. V and VI in the Plate reproduce two photographs obtained with a reflecting surface consisting of three elements. The difference between the intensities of the 2nd principal maximum on either side of the central one is very evident in the reproductions and might be made out even in respect of the secondary maxima on either side. This asymmetry demands an explanation. As is shown by the measurements given in Tables I. to IV., the positions of the minima of illumination are in good agreement with those calculated from the formula of the usual type, which are obtained on the assumption that each of the elements into which the reflecting surface may be divided diffracts light strictly in proportion to its area, and that the phase and intensity of the disturbance incident on the surface are the same as when the waves travel undisturbed. Further, the intensities at the points of minimum illumination are shown by observation and by the photographs to be zero, in agreement with the results indicated by these formulæ. On the other hand, the difference in the intensity at corresponding points of the pattern on either side of the central fringe remains unexplained according to such formulæ unless regarded as an *obliquity effect*.

A series of comparisons of the intensities of corresponding bands on the two sides of the pattern has been made for the cases in which the reflecting surface consists of one, two, and three reflecting elements respectively, for various angles of incidence. For this purpose, I have used a rotating-sector photometer of the Abney type supplied by Messrs. Adam Hilger, in which the free disk, which can be adjusted by handle while in rotation, is smaller in radius than the fixed disk. The sectors when in rotation thus present two annuli of different intensities, the ratio of which can be adjusted at pleasure by moving the handle of the instrument. The disk of the photometer is placed at the focal plane of the observing telescope, so that the diffraction pattern can be seen through it with an eyepiece, the fringes on the brighter side being observed through the inner annulus of the disk, and those on the fainter side through the outer annulus. To enable the intensities at corresponding points on the two sides of the pattern to be compared, a screen with two vertical slits is interposed immediately in front of the photometric disk so as to cut off everything except the regions under observation, which are then adjusted to equality of brightness by moving the handle of the photometer. Several readings can be taken in succession and their average struck. The diffraction angles θ and θ' of the two bands under comparison may then be measured under a micrometer eyepiece. Tables V., VI., and VII. show the observed ratios of the illumination and those calculated on the assumption that the formula for illumination includes a factor proportional to the square of the cosine of the obliquity. It is seen that the agreement is good except when the ratio is so large that it cannot be measured accurately, owing to the near approach of the fainter band towards the direct image of the slit.

Table V.

Single Reflecting Surface, width 0·90 cm.
Ratio of intensity of the first band on the right and the first band on the left.

Angle of Incidence.	Observed ratio of illumination.	Calculated ratio $\cos^2\theta/\cos^2\theta'$.
88° 42′	1·80	1·78
88° 53′	2·31	2·46
88° 56′	2·81	2·89
88° 4′	4·09	4·21

TABLE VI.

Reflexion grating of two elements.
$a=0{\cdot}754$ cm. $b=1{\cdot}446$ cm.

Angle of Incidence.	Ratio of the 1st maxima on the right and left. Observed.	Ratio of the 1st maxima on the right and left. Calculated.	Ratio of the 2nd maxima on the right and left. Observed.	Ratio of the 2nd maxima on the right and left. Calculated.
89° 6′	1·43	1·40	2·09	2·01
89° 23	2·25	2·01	3·91	4·88
89° 28′	2·37	2·44	7·50	11·06

TABLE VII.

Reflexion grating of three elements.
$a=0{\cdot}440$ cm. $b=0{\cdot}741$ cm.

Angle of Incidence.	Ratio of the 1st two Principal Maxima on either side of the central one. Observed.	Ratio of the 1st two Principal Maxima on either side of the central one. Calculated.	Ratio of the 2nd two Principal Maxima on either side of the central one. Observed.	Ratio of the 2nd two Principal Maxima on either side of the central one. Calculated.
88° 49′	1·48	1·43	1·83	1·70
89° 5′	1·63	1·71	2·90	3·15
89° 27′	2·65	2·87	×	more than 100 times.

Summary and Conclusion.

1. The unsymmetrical interference fringes of the light obliquely diffracted by two parallel reflecting surfaces in the same plane have been observed and photographed.

2. The illumination curve in the diffraction pattern (of the Fraunhofer class) due to an obliquely-held reflecting surface (which may consist of two or more separate parts in the same plane), is found to be markedly asymmetrical, corresponding points on either side of the central fringe being of very different intensities. As the positions of the points of minimum (*i. e.* zero) illumination are found to be in close agreement with those given by the formula of the usual type, the asymmetry of the illumination curve may be explained as due to the varying obliquity at different points in the diffraction pattern. Measurements of the ratio of the

intensities at corresponding points have been made with reflecting surfaces of rectangular form or consisting of either two or three elements in the same plane for various angles of incidence; the results show that the expression for the illumination at any point of the diffraction pattern contains a factor proportional to the square of the cosine of the obliquity at such point.

The experiments and observations described in the note were carried out in the Palit Laboratory of Physics. The writer hopes to carry out further work on the subject of oblique diffraction by various forms of aperture, and particularly in regard to the positions of the points of maximum intensity in the pattern, which would no doubt differ from those given by the usual formulæ owing to the asymmetry of the illumination curves.

Calcutta,
8th June, 1917.

XII. *On the Two-Dimensional Motion of Infinite Liquid produced by the Translation or Rotation of a Contained Solid.* *By* J. G. LEATHEM, *M.A., D.Sc., Fellow of St. John's College, Cambridge* [*].

1. PERIODIC conformal transformations—that is, transformations by which doubly connected regions in the plane of a variable $z=x+iy$, externally unbounded and bounded internally by a closed polygon or curve, may be represented conformally and repeatedly upon successive semi-infinite strips of width λ in the half-plane $\eta>0$ of a variable $\zeta=\xi+i\eta$—have been studied by the present writer in a previous paper [†]. It has there been shown how the knowledge of such a transformation for any particular curve makes possible the specification of a field of circulatory liquid flow (with or without logarithmic singularities) round a fixed solid body bounded by the curve.

It is now proposed to show that such knowledge makes possible also the determination of the field of irrotational motion due to any translation or rotation of the same solid in surrounding infinite liquid.

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[*] Communicated by the Author.

[†] J. G. Leathem, "On Periodic Conformal Curve-Factors and Corner-Factors," Proc. Royal Irish Academy, vol. xxxiii. Sec. A, August 1916.

2. The geometrical, or (z, ζ), relation may be either in the form

$$z = f(\zeta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

or in the differential form

$$dz = \mathscr{E}(\zeta) d\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where $\mathscr{E}(\zeta)$ is a periodic curve-factor of linear period λ and angular period 2π. With this angular period it is necessary * that both $\mathscr{E}(\zeta)$ and $f(\zeta)$ should, for η great and positive, tend to infinity like $\exp(2\pi\eta/\lambda)$. In fact, $\mathscr{E}(\zeta)$ is expansible in the form

$$\mathscr{E}(\zeta) = \exp(-2\pi i\zeta/\lambda) \,.\, \Sigma c_s \exp(2\pi s i\zeta/\lambda), \quad . \qquad (3)$$

where $s=0, 2, 3, 4 \ldots$, and the coefficients may be complex. The periodicity of z makes it necessary † that $c_1=0$.

From integration of (3) it follows that

$$z=f(\zeta)=c+(i\lambda/2\pi)\exp(-2\pi i\zeta/\lambda) \,.\, \Sigma\{c_s/(1-s)\} \exp(2\pi s i\zeta/\lambda), \; (4)$$

where c is another complex constant ‡.

It is also to be noticed that, if $|\mathscr{E}(\zeta)| = h$, and if $c_s=\kappa_s \exp(i\gamma_s)$,

$$h^2=\exp\left(\frac{4\pi\eta}{\lambda}\right)\left[\kappa_0^2+\exp\left(-\frac{4\pi\eta}{\lambda}\right)2\kappa_0\kappa_2\cos\left(\frac{4\pi\xi}{\lambda}+\gamma_2-\gamma_0\right)\right.$$
$$+\exp\left(-\frac{6\pi\eta}{\lambda}\right)2\kappa_0\kappa_3\cos\left(\frac{6\pi\xi}{\lambda}+\gamma_3-\gamma_0\right)$$
$$\left.+\exp\left(-\frac{8\pi\eta}{\lambda}\right)\left\{2\kappa_0\kappa_4\cos\left(\frac{8\pi\xi}{\lambda}+\gamma_4-\gamma_0\right)+\kappa_2^2\right\}\ldots\right], \quad . \qquad (5)$$

the terms containing ascending integral powers of

$$\exp(-2\pi\eta/\lambda).$$

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* *L. c.* § 5.

† *L. c.* § 4.

‡ The problem of obtaining a transformation of the type of formula (4), so that a given closed curve shall correspond to $\eta=0$, is the same as that of the parametric representation of the given curve by a formula of the type

$$x+iy=m\exp(-i\phi)+\mu_0+\Sigma\,\mu\,\exp(is\phi),$$

where ϕ is a real parameter, m and the μ's are complex constants, and s takes positive integral values.

It is to be noted that a formula of this type need not represent a curve free from nodes unless the constants are suitably restricted.

It is understood that the boundary in the z plane is the locus corresponding to $\eta=0$.

3. *Field of flow due to translation of the boundary.*—If the boundary have a velocity V in a direction making an angle μ with the axis of x, the superposition on the whole system of such uniform velocity as brings the boundary to rest gives an irrotational fluid motion which has zero normal velocity at the boundary and tends, for z infinite, to flow V in the direction $\mu+\pi$. If this motion have velocity potential ϕ and stream-function ψ, so defined that the velocity is the upward gradient of ϕ, and if $w\equiv\phi+i\psi$, w must tend, for z infinite, to the form

$$-\mathrm{V}z\exp(-i\mu)+\text{const.}, \quad . \quad . \quad . \quad . \quad (6)$$

or, in terms of ζ,

$$-\mathrm{V}\kappa_0(i\lambda/2\pi)\exp i(\gamma_0-\mu-2\pi\zeta/\lambda). \quad . \quad . \quad (7)$$

Now if

$$w=-(\mathrm{V}\kappa_0\lambda/\pi)\sin(2\pi\zeta/\lambda-\gamma_0+\mu), \quad . \quad . \quad (8)$$

this tends, for $\eta\longrightarrow+\infty$, to the form (7); and as

$$\psi=-(\mathrm{V}\kappa_0\lambda/\pi)\cos(2\pi\xi/\lambda-\gamma_0+\mu)\sinh(2\pi\eta/\lambda),$$

it is clear that ψ is zero along the boundary $\eta=0$. The corresponding form of ϕ shows that there is no circulation round the boundary; and w is free from infinities in the relevant region.

Hence formula (8) specifies that irrotational motion past the fixed boundary whose limit form, at indefinitely great distance, is the assigned uniform flow.

4. *The impulse of the motion due to translation.*—Though modern speculation tends to regard wave-motion as the preponderating factor in suction and other inertia phenomena of floating bodies, it can hardly be doubted that, in the case of a submarine at least, the ordinary inertia coefficients measure approximately the resistance to quick changes of velocity. Thus the evaluation of the impulse (X, Y) of the combined motion of solid and fluid, when the solid has translatory motion, is of interest.

If an approximation to w for $|z|$ great, closer than that afforded by formula (6), be

$$w=-\mathrm{V}z\exp(-i\mu)+\mathrm{C}+\mathrm{D}/z, \quad . \quad . \quad . \quad (6\,a)$$

where C and D are complex constants, it is known * that

$$X+iY=-2\pi\rho D,$$

where ρ is the density of the liquid, and it is supposed that the mean density of the solid is also ρ.

If formula ($6\,a$) be expressed in terms of ζ by means of formula (4), it yields the approximation

$$w=-V\kappa_0\left(\frac{i\lambda}{2\pi}\right)\exp i\left(\gamma_0-\mu-\frac{2\pi\zeta}{\lambda}\right)+C'$$

$$+\left\{V\kappa_2\left(\frac{i\lambda}{2\pi}\right)\exp i(\gamma_2-\mu)+D\left(\frac{2\pi}{i\lambda\kappa_0}\right)\exp(-i\gamma_0)\right\}\exp\left(\frac{2\pi i\zeta}{\lambda}\right.$$

where C′ is a constant. On comparison of this with formula (8), it appears that the coefficient of $\exp(2\pi i\zeta/\lambda)$ must equal $V\kappa_0(i\lambda/2\pi)\exp i(\mu-\gamma_0)$. Hence

$$D=-(V\lambda^2/4\pi^2)\{\kappa_0{}^2\exp(i\mu)-\kappa_0\kappa_2\exp i(\gamma_0+\gamma_2-\mu)\},$$

and therefore

$$X+iY=(\rho V\lambda^2/2\pi)\{\,|c_0|^2\exp(i\mu)-c_0c_2\exp(-i\mu)\}. \quad . \quad (9)$$

5. *Field of flow due to rotation of the boundary.*—When the boundary has a motion of rotation, the specification of the liquid motion presents greater difficulty; the outline of the procedure is as follows.

One motion is known which satisfies the proper condition at the moving boundary—namely, a rotation of the whole liquid, as if rigid, with the same angular velocity as the boundary. This may be called the *first motion.* It is not the required motion because it is rotational, and because it has infinite velocity at infinity.

Another motion, which will be called the *second motion*, can be specified. This also is rotational, having the same vorticity at every point as the first motion; but at the boundary its normal velocity is zero. It has infinite velocity at infinity.

If the second motion be subtracted from the first motion, the result is an irrotational motion whose normal velocity at the boundary is the same as that of the boundary itself. This may be called the *difference motion.* If the velocities of the first and second motions tend to equality at infinity in such manner that the difference motion tends to zero at infinity and has no circulation round the solid, the difference

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* J. G. Leathem, "Some Applications of Conformal Transformation to Problems in Hydrodynamics," Phil. Trans. Roy. Soc., A, vol. ccxv. 1915, § 17.

motion satisfies all the requirements of the problem, and is the motion due to the rotation of the solid.

6. *The first motion.*—The first motion will be taken to be a rotation, as if rigid, with angular velocity ω, about the point $z=\kappa \exp(i\gamma)$. There is no loss of generality in this choice, as the substitution of another centre of rotation can always be effected by superposing a motion due to translation of the boundary as explained in article 3.

The first motion may be specified by its stream function ψ_1, namely

$$\psi_1=-\tfrac{1}{2}\omega \mid z-\kappa \exp(i\gamma) \mid^2, \quad . \quad . \quad . \quad (10)$$

which is also expressible as a function of ξ and η. Another specification is by u_1, v_1, where

$$u_1=\partial\psi_1/\partial\eta, \qquad v_1=-\partial\psi_1/\partial\xi, \quad . \quad . \quad (11)$$

these also being regarded as functions of ξ and η. It is to be noted that u_1 and v_1 are not velocity components, but that the velocity components in the directions corresponding respectively to ξ and η increasing are u_1/h and v_1/h.

From formula (4) it follows that

$$x = \kappa \cos\gamma+\frac{\lambda}{2\pi}\left\{ \kappa_0 e^{2\pi\eta/\lambda} \sin\left(\frac{2\pi}{\lambda}\xi-\gamma_0\right)\right.$$
$$\left. +\sum_2 \frac{\kappa_s}{s-1} e^{-2\pi(s-1)\eta/\lambda} \sin\left(\frac{2\pi(s-1)\xi}{\lambda}+\gamma_s\right)\right\},$$

$$y = \kappa \sin\gamma+\frac{\lambda}{2\pi}\left\{ \kappa_0 \epsilon^{2\pi\eta/\lambda} \cos\left(\frac{2\pi}{\lambda}\xi-\gamma_0\right)\right.$$
$$\left. -\sum_2 \frac{\kappa_s}{s-1} e^{-2\pi(s-1)\eta/\lambda} \cos\left(\frac{2\pi(s-1)\xi}{\lambda}+\gamma_s\right)\right\},$$

so that, when η is great,

$$\psi_1=\frac{-\omega\lambda^2}{8\pi^2}\left\{ \kappa_0^2 e^{4\pi\eta/\lambda}-2\kappa_0\kappa_2 \cos\left(\frac{4\pi}{\lambda}\xi+\gamma_2-\gamma_0\right)\right.$$
$$\left. -\kappa_0\kappa_3 e^{-2\pi\eta/\lambda} \cos\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\dots\right\}, \quad (12)$$

and

$$u_1= -\frac{\omega\lambda}{2\pi}\left\{ \kappa_0^2 e^{4\pi\eta/\lambda}+\frac{1}{2}\kappa_0\kappa_3 e^{-2\pi\eta/\lambda} \cos\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\dots\right\},$$
$$. \quad . \quad . \quad . \quad . \quad (13)$$

$$v_1 = \frac{\omega\lambda}{2\pi}\left\{ 2\kappa_0\kappa_2 \sin\left(\frac{4\pi}{\lambda}\xi+\gamma_2-\gamma_0\right)\right.$$
$$\left. +\frac{3}{2}\kappa_0\kappa_3 e^{-2\pi\eta/\lambda} \sin\left(\frac{6\pi}{\lambda}\xi+\gamma_3-\gamma_0\right)+\dots\right\}, \quad (14)$$

the terms being arranged in descending order of importance.

7. *The second motion.*—The second motion, a rotational motion having zero normal velocity at the boundary, is got by an imaging of the vortex distribution; and the utility of the periodic conformal transformation consists in the fact that it makes this imaging process possible.

The specification of the motion may be by a stream-function ψ_2, or by functions (u_2, v_2) equal to $\partial\psi_2/d\eta$, $-\partial\psi_2/\partial\xi$, the corresponding velocity components in the z plane being u_2/h, v_2/h.

It is convenient, for a moment, to think of ψ_2 as the stream-function of a motion in the ζ plane, of which (u_2, v_2) would be the velocity. If such a motion, unhampered by any rigid boundary, were due to a line-vortex representing a circulation m, situated at $\zeta=\zeta'$, and periodically repeated at $\zeta'\pm r\lambda$, $(r=1, 2, 3, \ldots)$, it is known * that the corresponding stream-function would be

$$-\frac{m}{2\pi}\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\right|.$$

When the line $\eta=0$ is a rigid boundary an image must be introduced at the point $\zeta=\zeta''$, where ζ'' is the complex conjugate to ζ', and the stream-function is

$$-\frac{m}{2\pi}\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\Big/\sin\frac{\pi}{\lambda}(\zeta-\zeta'')\right|.$$

Instead of single vortices a continuous periodic distribution of vorticity may be postulated over the whole area between $\eta=0$ and $\eta=t$, the former line being still a rigid boundary; and if m be replaced by $\sigma(\zeta')\,d\mathrm{S}'$, where $d\mathrm{S}'$ is an element of area and σ a density of distribution, the stream-function is

$$-\frac{1}{2\pi}\int\sigma(\zeta')\log\left|\sin\frac{\pi}{\lambda}(\zeta-\zeta')\Big/\sin\frac{\pi}{\lambda}(\zeta-\zeta'')\right|d\mathrm{S}', \qquad (15)$$

the area integral being taken over a rectangle of length t and breadth λ.

If $d\mathrm{S}$ in the z plane and $d\mathrm{S}'$ in the ζ plane be corresponding elements of area,

$$d\mathrm{S} = \{h(\zeta')\}^2 d\mathrm{S}';$$

so, if the circulation round the contour of any area in the ζ plane is to be equal to 2ω times the corresponding area in the z plane, it is necessary that

$$\sigma(\zeta') = 2\omega\{h(\zeta')\}^2. \quad . \quad . \quad . \quad . \quad (16)$$

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* Proc. Royal Irish Academy, *l. c.* § 13.

The stream-function is then

$$\psi_t = -\frac{\omega}{\pi}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \log\left|\frac{\sin\frac{\pi}{\lambda}(\zeta-\zeta')}{\sin\frac{\pi}{\lambda}(\zeta-\zeta'')}\right| d\xi'\,d\eta', \tag{17}$$

and this specifies a motion in the z plane which has vorticity ω at all points for which $t > \eta > 0$, and has the curve corresponding to $\eta = 0$ as a fixed boundary.

The corresponding u, v functions are

$$u_t = \frac{\partial\psi_t}{\partial\eta} = -\frac{\omega}{\lambda}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \left\{\frac{\sinh\frac{2\pi}{\lambda}(\eta-\eta')}{\cosh\frac{2\pi}{\lambda}(\eta-\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} - \frac{\sinh\frac{2\pi}{\lambda}(\eta+\eta')}{\cosh\frac{2\pi}{\lambda}(\eta+\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')}\right\} d\xi'\,d\eta', \tag{18}$$

$$v_t = -\frac{\partial\psi_t}{\partial\xi} = \frac{\omega}{\lambda}\int_0^t\int_0^\lambda \{h(\zeta')\}^2 \left\{\frac{\sinh\frac{2\pi}{\lambda}(\xi-\xi')}{\cosh\frac{2\pi}{\lambda}(\eta-\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')} - \frac{\sin\frac{2\pi}{\lambda}(\xi-\xi')}{\cosh\frac{2\pi}{\lambda}(\eta+\eta')-\cos\frac{2\pi}{\lambda}(\xi-\xi')}\right\} d\xi'\,d\eta'. \tag{19}$$

It is to be noticed that if ζ is inside the area of integration the subjects of integration in (17), (18), and (19) have infinities at $\zeta' = \zeta$; these infinities, however, are not sufficiently powerful to make the integrals divergent.

8. These formulæ may be checked by noting directly what conditions u_t, v_t must satisfy if they are to represent the kind of motion in the z plane which has been described in the previous article.

ξ and η are curvilinear coordinates in the z plane, and the corresponding velocity components are $U = u_t/h$ and $V = v_t/h$. The boundary condition $V = 0$ or $v_t = 0$, when $\eta = 0$, is clearly satisfied. The equations of continuity and of vorticity are respectively

$$\frac{\partial}{\partial\xi}(\mathrm{U}h)+\frac{\partial}{\partial\eta}(\mathrm{V}h)=0,\quad \frac{\partial}{\partial\xi}(\mathrm{V}h)-\frac{}{\partial\eta}(\mathrm{U}h)=2\omega h^2,$$

or

$$\frac{\partial u_t}{\partial\xi}+\frac{\partial v_t}{\partial\eta}=0,\quad \frac{\partial v_t}{\partial\xi}-\frac{\partial u_t}{\partial\eta}=2\omega\{h(\zeta)\}^2, \quad . \quad (20)$$

when ζ is inside the area of integration.

The testing of these equalities involves the differentiation of the integrals of formulæ (18) and (19), and this cannot be done by the ordinary rule of differentiation under the sign of integration, since that would yield semi-convergent integrals. It is, however, easy to apply the method of differentiation explained in the Cambridge Tract on 'Volume and Surface Integrals used in Physics,' articles 21 and 23, and it is then readily verified that u_t and v_t satisfy both conditions.

A single compact formula giving both u_t and v_t is

$$v_t+iu_t=\frac{\omega}{\lambda}\int_0^t\int_0^\lambda\{h(\zeta')\}^2\left[\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')\right]d\xi'\,dn'. \quad (21)$$

The integral on the right-hand side has the appearance of being a function of the complex variable ζ; but this appearance is deceptive, for if v and u were conjugate functions there would be no vorticity *.

9. The next step that suggests itself is a passage to limits for t infinitely great. This is feasible in the case of v_t, but the integral representing u_t proves to be divergent.

It will be shown that this can be remedied by adding to u_t, before passage to limit, a suitable function of t which does not involve ξ or η. An addition to u_t means simply the superposition of an irrotational motion with circulation round the fixed boundary. This will serve to cancel an undesired circulation at infinity in the motion defined by u_t and v_t.

No corresponding addition to v_t need be or could be made.

It being necessary to consider not only the convergence of the u and v integrals but also the forms to which these, regarded as functions of ξ and η, tend for η very great and positive, it is important to notice two expansions of the function which appears in square brackets in formula (21).

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* On this point compare the writer's note "On Functionality of a Complex Variable" in the 'Mathematical Gazette,' early in 1918.

If $\eta' > \eta > 0$

$$\cot\frac{\pi}{\lambda}(\zeta-\zeta') - \cot\frac{\pi}{\lambda}(\zeta-\zeta'')$$
$$= 4i\left[\tfrac{1}{2} + \Sigma \exp\left(\frac{-2s\pi\eta'}{\lambda}\right)\cosh\frac{2s\pi}{\lambda}\{\eta - i(\xi-\xi')\}\right], \quad (22)$$

and if $\eta > \eta' > 0$

$$\cot\frac{\pi}{\lambda}(\zeta-\zeta') - \cot\frac{\pi}{\lambda}(\zeta-\zeta'')$$
$$= -4i\,\Sigma \sinh\left(\frac{2s\pi\eta'}{\lambda}\right)\exp\frac{2s\pi}{\lambda}\{i(\xi-\xi')-\eta\}, \quad (23)$$

where $s=1, 2, 3, \ldots$. The formulæ can be verified by noticing that each side of each equality is equivalent to

$$2i\left[\frac{1}{1-\exp\frac{2\pi}{\lambda}\{\eta-\eta'-i(\xi-\xi')\}} - \frac{1}{1-\exp\frac{2\pi}{\lambda}\{\eta+\eta'-i(\xi-\xi')\}}\right].$$

For η' great the most important terms of $\{h(\zeta')\}^2$ are

$$\{h(\zeta')\}^2 = \kappa_0^2 \exp\left(\frac{4\pi\eta'}{\lambda}\right) + 2\kappa_0\kappa_2 \cos\left(\frac{4\pi\xi'}{\lambda}+\gamma_2-\gamma_0\right)$$
$$+ 2\kappa_0\kappa_3 \exp\left(-\frac{2\pi\eta'}{\lambda}\right)\cos\left(\frac{6\pi\xi'}{\lambda}+\gamma_3-\gamma_0\right)+\ldots, \quad (24)$$

the terms decreasing by successive negative powers of $\exp(2\pi\eta'/\lambda)$; it is to be noted that the functions of ξ' which multiply these exponentials are sums which may contain constant as well as harmonic terms.

In studying the form of the subject of integration in formula (21), for great values of η', with a view to examining the divergence for $t \to \infty$, the product of the series (22) and (24) may be used. For this purpose a term of the product may be ignored if its integral with respect to ξ' through a range λ is zero, or if it contains as factor the exponential of a negative multiple of η'/λ. By one or other of these tests every term is negligible except one, namely,

$$2i\kappa_0^2 \exp(4\pi\eta'/\lambda),$$

which contributes to the integral, at its upper limit (after integration with respect to ξ'),

$$(i\kappa_0^2\lambda^2/2\pi)\exp(4\pi t/\lambda).$$

Hence the subtraction of $(i\kappa_0{}^2\omega\lambda/2\pi)\exp(4\pi t/\lambda)$ from the right-hand side of formula (21) gives an expression which has a definite limit for $t\to\infty$. This justifies the definition

$$v_2+iu_2=\frac{\omega}{\lambda}\operatorname*{Lim}_{t\to\infty}\left[-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}\right.$$
$$\left.+\int_0^t\int_0^\lambda\{h(\zeta')\}^2\left\{\cot\frac{\pi}{\lambda}(\zeta-\zeta')-\cot\frac{\pi}{\lambda}(\zeta-\zeta'')\right\}d\xi'\,d\eta'\right]. \quad (25)$$

10. *Limiting form of the second motion at infinity.*—The formula (25) defines a motion which has all the characteristics required for the second motion, with the, as yet, possible exception of tending to the proper form at infinity. It is now necessary to inquire what are the limiting forms to which u_2 and v_2, functions of ξ and η, tend with indefinite increase of η.

For this purpose the series-expansions of formulæ (22) and (23) may be used, each within the appropriate range of η'. If [22], [23] be used as abbreviations for the expressions on the right-hand side of these formulæ,

$$v_2+iu_2=\frac{\omega}{\lambda}\operatorname*{Lim}_{t\to\infty}\left[-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}+\int_\eta^t\int_0^\lambda\{h(\zeta')\}^2[22]\,d\xi'\,d\eta'\right.$$
$$\left.+\int_0^\eta\int_0^\lambda\{h(\zeta')\}^2[23]\,d\xi'\,d\eta'\right]. \quad (26)$$

As the integral of formula (25) is absolutely convergent in respect of the infinity of the subject of integration at $\zeta'=\zeta$, it is safe to use the series [22] and [23] right up to the critical value $\eta'=\eta$ which separates the ranges within which they are respectively valid. For $\{h(\zeta')\}^2$ the series of formula (24) is again employed.

In taking the term-by-term products of the two series which are multiplied under the sign of integration, any resulting term may be passed over whose integral with respect to ξ' over a range λ is zero. Thus a term of the the type

$$\cos\{(2\pi m/\lambda)(\xi'+\alpha)\}\cos\{(2\pi n/\lambda)(\xi'+\beta)\}$$

need not be considered unless $m=n$. Further, when only an approximation for η great is desired, an estimate of the importance of an exponential in η' and η is to be made on the hypothesis that η is very great but that η' is of a higher order of greatness at one of the limits of integration, while $\eta'=\eta$ at another of the limits. These considerations reduce the important terms in the equivalent of the square bracket in formula (26) to

$$-\frac{i\kappa_0{}^2\lambda^2}{2\pi}e^{4\pi t/\lambda}+4i\int_\eta^t\int_0^\lambda\Big[\frac{1}{2}\kappa_0{}^2e^{4\pi\eta'/\lambda}$$

$$+2e^{-4\pi\eta'/\lambda}\kappa_0\kappa_2\cosh\frac{4\pi}{\lambda}\{\eta-i(\xi-\xi')\}\cos\left(\frac{4\pi}{\lambda}\xi'+\gamma_2-\gamma_0\right)\Big]d\xi'd\eta'$$

$$-4i\int_0^\eta\int_0^\lambda 2\kappa_0\kappa_2\sinh\left(\frac{4\pi\eta'}{\lambda}\right)\cos\left(\frac{4\pi}{\lambda}\xi'+\gamma_2-\gamma_0\right)$$

$$\times\exp\frac{4\pi}{\lambda}\{i(\xi-\xi')-\eta\}d\xi'd\eta',$$

which reduces, after omission of some negligible elements, to a form whose limit, for $t\rightarrow\infty$, on substitution in (26) yields the formula

$$v_2+iu_2\backsim\frac{\omega\lambda}{2\pi}\left[-i\kappa_0{}^2\exp\left(\frac{4\pi\eta}{\lambda}\right)+2\kappa_0\kappa_2\sin\left(\frac{4\pi\xi}{\lambda}+\gamma_2-\gamma_0\right)\right]. \quad \ldots \quad (27)$$

11. *The difference motion.*—If formula (27) be compared with formulæ (13) and (14) it is seen that

$$v_2+iu_2-(v_1+iu_1)\rightarrow 0. \quad \ldots \quad (28)$$

Hence if (u, v) specify the difference motion, so that

$$u=u_1-u_2, \quad v=v_1-v_2,$$

u and v tend to zero at infinity.

Thus the difference motion is an irrotational motion, vanishing at infinity and having at the boundary a normal velocity corresponding to rotation about the point $\kappa\exp(i\gamma)$. It is free from circulation, as a circulation would involve, for η infinite, a definite limit value of u different from zero. It therefore constitutes the solution of the problem of motion due to the rotation of the boundary.

12. *Forms of boundary to which the method applies.*—The applicability of this method to solving the problems of motion due to translation and rotation depends upon the knowledge of a periodic conformal transformation which will make any particular form of boundary correspond to the real axis in the ζ plane. That a considerable variety of such transformations and their corresponding boundaries is

available is demonstrated in the writer's paper on the subject referred to above. In particular, mention may be made of polygonal boundaries (*l. c.* § 8); and it may be noticed that for a regular polygon of n sides the transformation is

$$\frac{dz}{d\zeta} = \mathrm{K}\left[\prod_{s=0}^{s=n-1} \sin\frac{\pi}{\lambda}\left(\zeta - \frac{s\lambda}{n}\right)\right]^{\frac{2}{n}} = \frac{\mathrm{K}}{4}\left\{-2\sin\frac{n\pi\zeta}{\lambda}\right\}^{\frac{2}{n}}, \quad (29)$$

so that

$$h^2 = \frac{\mathrm{K}^2}{16}\left[2\left\{\cosh\frac{2n\pi\eta}{\lambda} - \cos\frac{2n\pi\xi}{\lambda}\right\}\right]^{\frac{2}{n}}, \quad . \quad . \quad (30)$$

K being a constant; the latter expression, with accented letters, would be the first factor under the sign of integration in formula (25).

In all cases where the periodic transformation is known the solution of the hydrodynamical problems is reduced to quadratures.

In certain cases the integrations can be completed; this is noticeably the case when $f(\zeta)$, and therefore also h^2, is the sum of a finite number of terms harmonic in ζ. The integration may be accurately effected by the method used for approximation in article 10 above. Of the terms arising from the multiplication of h^2 into the series [22] and [23] there are only a finite number which do not yield zero result when integrated with respect to ξ' through a range λ, and each of these can be integrated separately with respect to η'.

The simplest example is the ellipse, for which the transformation is

$$z = c\cosh\{\alpha - (2\pi i/\lambda)\zeta\}, \quad . \quad . \quad . \quad . \quad (31)$$

so that

$$h^2 = \frac{2\pi^2c^2}{\lambda^2}\left\{\cosh 2\left(\alpha + \frac{2\pi}{\lambda}\eta\right) - \cos\left(\frac{4\pi}{\lambda}\xi\right)\right\}. \quad . \quad (32)$$

The working out of this case may be used to test the method, as the results are otherwise known.

Another simple integrable case corresponds to a boundary whose polar equation is

$$r = a + 2b\cos 2\theta, \quad (a > 2b). \quad . \quad . \quad . \quad (33)$$

The transformation is

$$z = b\exp(-2\pi i\zeta/\lambda) + a\exp(2\pi i\zeta/\lambda) + b\exp(6\pi i\zeta/\lambda). \quad (34)$$

12th November, 1917.

XIII. *On the Relation of the Audibility Factor of a Shunted Telephone to the Antenna Current as used in the Reception of Wireless Signals. By* Prof. G. W. O. Howe, *D.Sc., M.I.E.E.*[*]

[See paper with the same title by M. van der Pol, vol. xxxiv. p. 184.]

THE audibility factor of a radio-telegraph signal is defined as the ratio of the actual sound-producing current in the telephone-receiver to the minimum value to which this current could be reduced for the signals to remain just readable. It is assumed that the wave-form of the telephone current, and therefore also the character of the sound, remain the same in the two cases. This ratio is usually determined by shunting the telephone-receivers with a non-inductive resistance until the signals are only just readable. If there is any possibility of the total rectified current being affected by the decreased resistance of the detector circuit due to the addition of the shunt, a resistance should be inserted in series with the shunted receiver to maintain the total resistance of the detector circuit approximately constant. From the value of the shunt it is then necessary to calculate the ratio of the total or joint current to that through the receiver.

It is not clear from Mr. van der Pol's paper how he determined the resistance of the receiver which he gives as 1240 ohms; but since nothing is stated to the contrary, it would appear that he has treated the receiver as a non-inductive resistance equal in value to the actual resistance of the receiver to continuous current. If so, the results obtained will be in error for two reasons: firstly, because the effective resistance of a telephone-receiver at the frequency employed, viz. 467, is considerably greater than its resistance to continuous current; and secondly, because an alternating current divides between two alternative paths in a manner depending on the impedances and not on the resistances.

As an example of the magnitude of the error thus introduced, the following figures may be quoted: a 3200-ohm receiver had an effective resistance R_f at 750 cycles per second of 6200 ohms and an impedance Z of 9320 ohms, whilst at a frequency of 1000 these values were increased to 7250 and 11,200 ohms respectively. Thus $Z_{750}=2{\cdot}9R_0$ and

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[*] Communicated by the Author.

$Z_{1000} = 3{\cdot}15 R_0$. In the case of a 60-ohm receiver, it was found that $Z_{637} = 4{\cdot}27 R_0$ and $Z_{1115} = 6{\cdot}35 R_0$. As a rule at such frequencies the reactance is of the same order as the effective resistance, so that the current lags about 45° behind the terminal P.D.; this is, of course, merely a rough approximation.

There may be some doubt as to the correctness of treating the pulsating telephone current as a simple alternating current; but in the opinion of the writer, the pulsating current of audible frequency produced by the detector as the result of the successive wave-trains may be regarded as a steady current with a fundamental alternating current and a number of harmonics superposed upon it, the fundamental giving the pitch, and the harmonics the character of the sound heard in the receiver. If the character of the note remain constant, it would appear sufficient to consider the amplitude of the fundamental, and to assume that this sinusoidal current divides between the receiver and the shunt in accordance with the ordinary laws of alternating-current circuits.

The writer is well aware that references can be given to papers in which the ordinary continuous current-resistance of the receiver was apparently used in calculating the audibility factor, but in a recent paper Austin, who has done much experimental work on this subject, is careful to point out that the effective resistance of the receiver must be determined for the given frequency and telephone pulse form *.

Since Mr. van der Pol refers to papers by Hogan and Love, both of whom refer to the impedance and not the resistance of the telephone-receiver, it is possible that he has also used the impedance, notwithstanding the statement in his paper. If so, the paper would be of greater value and interest were this definitely stated.

If Mr. van der Pol did not take the precaution to keep the resistance of the detector circuit approximately constant, as mentioned above, the correctness of his experimental results is open to some doubt.

In order to see in what direction his results would be modified by employing the impedance of the receiver instead of the resistance, it has been assumed in the following table that the impedance Z is equal to four times the resistance R_0, and that the telephone current lags 45 degrees behind the P.D.

The values obtained from the simple vector diagram are as follows:—

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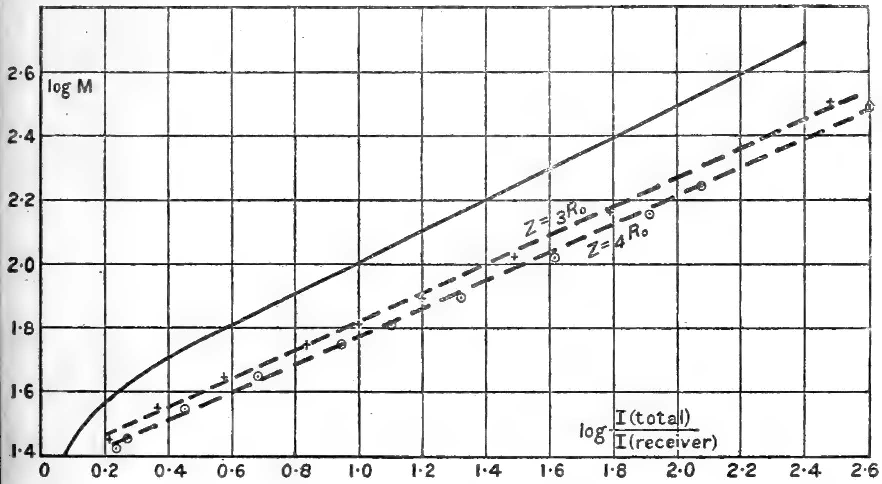
* Proc. Inst. Radio-Engineers, 1917, v. p. 239.

R_0/S.	$\frac{R_0+S}{S}$	$\frac{Z+S}{S}$	$\frac{\text{I (total)}}{\text{I (receiver)}}$	$\log\frac{R_0+S}{S}$	$\log\frac{I}{I_r}$.
0·25	1·25	2	1·85	0·097	0·267
0·5	1·5	3	2·8	0·176	0·447
1	2	5	4·77	0·301	0·679
3	4	13	12·72	0·602	1·104
10	11	41	40·71	1·041	1·61
30	31	121	120·7	1·491	2·081
100	101	401	400·7	2·004	2·602

In view of the unavoidable lack of precision in all audibility tests it is obviously sufficiently accurate to neglect the phase of the telephone current and take $\frac{Z+S}{S}$ as the audibility factor.

The figure is a reproduction of fig. 2 in Mr. van der Pol's paper, with the addition of two dotted lines which represent log M plotted against log $\frac{I}{I_r}$ instead of log $\frac{R_0+S}{S}$. For the upper dotted line it has been assumed that $Z=3R_0$, whilst for the lower $Z=4R_0$ as in the table above. It is seen that if the impedance of the receivers is three or four times their continuous current resistance, the resultant curve is considerably modified, and with it the conclusions based thereon. If further tests show that the results are not modified when the resistance of the detector circuit is kept constant, it would appear from the slope of the dotted lines that the audibility factor varies as a higher power than the square of the radio-frequency current. This appears improbable.



NOTE BY MR. VAN DER POL, JUNR.

The well-known facts respecting the division of current between a telephone and its shunt circuit are correctly stated by Prof. Howe, but he has rather lost sight of the motive of my paper and of certain experimental difficulties. The principal object in view was to test whether the audibility factor could be considered to vary as suggested by Prof. Love with the received antenna-current.

In discussing some experiments by Dr. Austin, Prof. Love makes use of the audibility factor defined as $R+S/S$, where S is the resistance of the shunt and R the telephone *resistance*[*]. In order to test experimentally his suggestions as to the proportionality of the so-defined audibility factor to the first power or square of the antenna-current I had to use the same constants. My experimental results appeared to be in close agreement with Prof. Love's suggestions.

The same definition (with the aid of the telephone resistance) of the audibility factor is used by several other writers. Prof. Howe refers in his paper to a very recent publication by Austin which was published after my paper had been sent to the Philosophical Magazine. Here Austin refers to the telephone-impedance, but on the other hand, in a former paper by the same experimentalist, he defines the audibility factor using the telephone *resistance* instead of the impedance[†].

It is by no means clear whether Austin or Hogan employed the true impedance of their telephones in the audibility factor, as in their papers cited no references at all are given how they determined these impedances.

Further, it is a matter of considerable difficulty to measure the true impedance of a telephone when used as in Wireless Telegraphy in series with a crystal detector, and therefore traversed by an intermittent or pulsatory current, *the waveform of which is not known.* From the pronounced variation in character of the tone in the telephone-receiver with different couplings it may further be concluded that, probably as a consequence of the irregular shape of the characteristic of most crystal detectors, the telephone current, while varying in intensity also (opposite to the suggestion of Prof. Howe) varies in wave-form, so that it is doubtful if the ordinary well-known theory of sine-form currents may be applied to the shunted telephone method.

Moreover, the current in the telephone circuit at the

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[*] Phil. Trans. Roy. Soc. Lond. ccxv. A. p. 128 (1915).

[†] Bull. Bureau of Standards, vi. no. 4, p. 531 (1910). See also J. Erskine Murray, 'A Handbook of Wireless Telegraphy' (1914), p. 349.

moment when the measurement is made is extremely small and quite beyond reach of any thermo-electric ammeter. The writer is therefore of opinion that an exact experimental determination of the telephone impedance under actual working conditions is a matter of higher order of difficulty than the measurement of received antenna-current itself.

It must further be borne in mind that in any case the shunt value which quenches the telephone sound is difficult to determine in practice with any but a rough approximation. In a very quiet room it may perhaps be determined to within 5 or 10 per cent., but in a wireless station or on board ship perhaps not within 30 or 40 per cent.

No assumptions as that made by Prof. Howe that the true impedance of the telephone under actual working conditions is equal to four times the steady resistance has been justified by any experiments. Hence, to avoid suppositions not based on experiment, the value taken for the calculation of the audibility factor in the case of my experiments was the steady resistance, although I was perfectly well aware that this was not identical with the true impedance for the waveform and frequencies used.

Having regard to the uncertainty attending the constants employed by Austin and Hogan, and the difficulty of determining exact values, it seemed better to base the reduction of the observations on known measurements rather than on assumptions as to the ratio of impedance to resistance.

XIV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xxxiv. p. 528.]

June 20th, 1917.—Dr. Alfred Harker, F.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Pre-Cambrian and Associated Rocks of the District of Mozambique.' By Arthur Holmes, A.R.C.S., D.I.C., B.Sc., F.G.S.

Beyond the coastal and volcanic beds of Mozambique (described in a previous contribution—Abs. Proc. Geol. Soc. 1916, No. 994, p. 72) the country assumes the form of a gently undulating plateau, gradually rising towards the west and diversified by innumerable inselberg peaks and abruptly-rising clusters of hills. The dominant rock throughout is a grey biotite-gneiss. Interfoliated with this are occasional lenticular masses of hornblende-gneiss and amphibolite, and within these smaller bands of crystalline limestone are sometimes preserved. In many places the gneisses become garnetiferous, while eclogites and basic granulites also occur. Schists—referable to arenaceous sediments—are found only near the coast, where they are interbanded with gneisses; and, as the latter are mainly of igneous origin, they are thought to be intrusive into, and therefore younger than, the schists. As a general rule, the foliation and the banding of the gneisses are well defined in parallel uncontorted planes, the strike being commonly along, or somewhat north of, a north-east to south-west direction. In certain inselberg peaks, the strike sweeps round the contours, while the foliation-surfaces dip quaquaversally from the summits. Into the gneisses later granites, belonging to at least two different periods, have penetrated, riddling them with enormous numbers of small intrusions, lit-par-lit injections, tongues, and apophyses. Rocks of later age are rarely met with; but, in a few places, dykes of picrite and pyroxenite have been found cutting the youngest pegmatites.

The succession of rocks in eight of the better-known districts is described, and the following general classification is based on the details thus provided:—

Ultrabasic Dykes.	Pyroxenites. Picrite.	Age unknown.
	Intrusive Contact.	
Granites and Coarse Pegmatites.	Biotite-Granites. Graphic Granite and other Pegmatites. Quartz-Veins.	Pre-Cretaceous and Post-Middle Pre-Cambrian.
	Intrusive Contact.	
Granulitic Granites, Pegmatites, and Associated Rocks.	Biotite-Granites (including porphyritic varieties). Pegmatites and Aplites. Pyroxene-Granite and Pyroxene Quartz-Diorite Series. Pyroxene-Granulites ?	Middle Pre-Cambrian. Pb/U=0·14 to 0·17.
	Intrusive Contact.	
Gneisses and Associated Rocks.	Biotite-Gneisses and Gneissose Granites. Hornblende-Gneisses. Amphibolites. Garnetiferous Gneisses, Granulites, and Eclogites.	Pb/U=0·21. Lower Pre-Cambrian.
	Intrusive Contact.	
Crystalline Schists and Limestones.	Quartz-Mica-Schists. Quartz-Magnetite-Schists. Hornblende-Garnet-Schists. Hæmatite-Schists. Forsterite-Marbles, and other Crystalline Limestones.	

The above correlations of certain groups of rocks with the Lower and Middle Pre-Cambrian of other regions are based on the determination of lead-uranium ratios of zircons derived from the gneisses and granulitic granites respectively, the zircons having been obtained by crushing and panning the rocks in the field. The gneisses give a ratio of 0·21, comparable with a ratio of 0·24 obtained for Canadian zircons of Laurentian age. The granulitic granites give ratios of 0·14 to 0·17, comparable to those of radioactive minerals of late Archæan: that is, late Middle Pre-Cambrian, age in Scandinavia (Moss 0·12 to 0·15, Arendal 0·16 to 0·18, and Ytterby 0·15 to 0·17), Canada (Villeneuve, Quebec, 0·17), and India (Singar 0·14).

The rocks are described in detail, with tables giving the quantitative mineral composition and the specific gravities and radium contents. Numerous examples of contact-phenomena between crystalline limestones and various types of igneous rock are recorded: pyroxene, amphibole, sphene, and soda-lime felspar being the new minerals chiefly developed between granite and limestone, with garnet and scapolite also in special cases.

With reference to the origin of the crystalline limestones and gneisses, the following conclusions are arrived at:—

(*a*) The crystalline schists and limestones are interpreted as arenaceous and calcareous facies respectively of an ancient sedimentary series, their argillaceous complements being unrepresented unless they enter into the composition of the biotite-gneisses.

(*b*) The limestones have controlled the formation of hornblende-gneiss and amphibolite by their interaction with a granitic magma that elsewhere is represented by biotite-gneisses. The cores of the limestones have been enabled to resist further silicification by being thus enclosed within a blanket of rocks impoverished in silica.

(*c*) If the ancient sedimentary series included argillaceous formations, it is thought probable that the gneisses are composite rocks produced by the concordant injection of granitic magma into such formations. This view, although not proved, is supported by mineralogical and radioactive evidence, and by the fact that in certain inselberg peaks the banding of the gneisses gradually dies away as the slopes are ascended, the rocks passing into granulitic granite nearly free from biotite and showing few traces of foliation. These peaks are interpreted as the irruptive foci of granulitic magmas which fed the lateral intrusions represented by the surrounding gneisses.

It is shown that there are at least three types of inselberg peaks that owe their survival to peculiarities of structure and composition. The first type is that just mentioned, in which the foliation is less marked and the biotite-content appreciably lower than in the surrounding gneisses. In the second, the peaks are mainly composed of granulitic granite (again poor in biotite compared with the gneisses), and in the third type the peaks are riddled with tongues and apophyses of pegmatite and aplite. In each case the greater resistance offered to denudation is related to the presence of less foliated and more felsic rocks than are found in the adjacent plains. There remains a fourth type—perhaps the most abundant—in which no differences have been recognized. Many of these seem to be isolated relics of gneissic escarpments; and it is suggested that desert erosion, involving the attack of slopes at their base by arid weathering, and the removal of disintegrated material by wind, is the most favourable condition for the development and maintenance of an inselberg landscape. Existing conditions of denudation are considered to be unfavourable to inselberg survival; for the peaks appear to be worn down by the removal of superficial layers by exfoliation more rapidly than the surface of the plateau is lowered.

2. 'The Inferior Oolite and Contiguous Deposits of the Crewkerne District (Somerset).' By Linsdall Richardson, F.R.S.E., F.G.S.

November 7th.—Dr. Alfred Harker, F.R.S., President, in the Chair.

A Lecture on 'The Nimrud Crater in Turkish Armenia' was delivered by **Felix Oswald**, B.A., D.Sc., F.G.S.

The Nimrud volcano, one of the largest volcanic craters in the world, is situated on the western shore of Lake Van, and was surveyed and investigated geologically for the first time by the speaker in 1898. The western half of the crater is occupied by a deep lake of fresh water, while the eastern half is composed of recent augite-rhyolites, partly cloaked in white volcanic ash. The crater-wall is highest on the north (9903 feet), rising in abrupt precipices over 2000 feet above the lake (7653 feet). The southern wall is also precipitous, but only reaches the height of 9434 feet (the south-eastern part). A large slice of the crater-wall has slipped down on the south-west, so as to form a narrow shelf, 800 feet above the lake. The crater is nearly circular, 8405 yards from west-south-west to east-north-east, while the transverse axis is 7905 yards. The lowest points lie on the long axis, reaching only 8139 feet on the western, and 8148 feet on the eastern rim.

The crater-wall has an external slope of 33° on the south and east, where it consists exclusively of overlapping lenticular flows of augite-rhyolite and obsidian. On the south-west, west, north-west, and north these are capped by thin sheets of cindery basalt which must have possessed great fluidity, extending for many miles to form wide plains of gentle slope and great fertility down to Lake Van on the east and into the Plain of Mush on the west. These basalt-flows dammed up the north-east to south-west valley between the Bendimahi and Bitlis rivers, and thus brought Lake Van into being.

The history of the Nimrud volcano may be summarized as follows from the speaker's observations:—

1. Its forerunner was the Kerkur Dagh on its southern flank,—a denuded mass of grey augite-trachyte, rising to 9000 feet, and crowned by many peaks. It was probably erupted in the Pliocene Period, subsequently to the folding of the Armenian area, in which the latest folded rocks are of Miocene (Helvetian–Tortonian) age, occurring north of the Nimrud Dagh and consisting of limestones with corals (*Cladocora articulata, Orbicella defrancei*, &c.), *Lithothamnion*, Foraminifera (Lepidocycline *Orbitoides*, *Amphistegina*, &c.), beds of *Pecten* (*P. urmiensis*, &c.) and of oysters (*Alectryonia virleti*). Nimrud and the other numerous volcanoes of Armenia came into existence at a period when the sedimentary rocks could no longer be folded, but were fractured along definite lines, and Nimrud is situated on the great fracture transverse to the Armenian folds at the apex of their bending round from the Antitauric (west-south-west to east-north-east) to the Persian (north-west to south-east) direction, and it also marks the point of intersection of this fracture with a great north-east to north-west fracture (Caucasian direction), which delimits on the south Lake Van and the faulted depression of the Plain of Mush, abruptly cutting off the Tauric horst of pre-Devonian marbles and mica-schists.

2. Numerous flows of augite-rhyolite built up the vast cone of the Nimrud Dagh, and the increasing pressure on the central vent became relieved by extrusions of augite-trachyte along radial fissures, forming the present promontories of Kizvag, Zighag, and Karmuch.

3. A presumably long period of inactivity was followed by violent explosions destroying the summit of the cone, and from this crater (smaller than the present one) vast lava-flows of a very fluid basalt (crowded with phenocrysts of labradorite, pale-green augite, and some olivine) flooded the country and filled up the Bitlis and Akhlat valleys, which have since then been eroded a little below their former depth. The Sheikh Ora crater of basic tuff (now breached by Lake Van) probably belongs to this period.

4. Further explosions widened the crater in which a large lake was formed, while the eastern half of the crater became filled by a succession of outflows of augite-rhyolite, in which numerous blow-holes were drilled, bringing to the surface large blocks of basaltic agglomerate and also affording sections showing the transition downwards from obsidian, spherulitic obsidian, and spherulitic rhyolite to banded augite-rhyolite (with sanidine and green augite in a micropœcilitic ground-mass).

5. The last eruption was recorded in 1441 by a contemporary Armenian chronicler, and resulted in the extrusion of a very viscous augite-rhyolite along a north-to-south zone of weakness, both inside

the Nimrud crater where it separated off part of the large lake to form the shallow, so-called 'hot lake,' and also to the north of Nimrud, where it rose up fissures and in a small crater.

6. A violent earthquake in 1881 which destroyed the village of Teghurt, at the eastern base of the crater-wall, was the last sign of activity; but earthquakes are still frequent in the Plain of Mush at the western foot of the Nimrud Dagh, and recent fault-scarps are clearly visible along the borders of this faulted depression.

The speaker mentioned that he had presented his model of the crater to the Museum of Practical Geology (Jermyn St.) and the rocks and slides to the British Museum (Natural History), where his fossils from Armenia are already preserved.

XV. *Intelligence and Miscellaneous Articles.*

COUPLED CIRCUITS AND MECHANICAL ANALOGIES,

Phil. Mag. Dec. 1917.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

CENTENARIAN PERIGAL is clean forgotten today, and his valuable kinematic work on his lathe. His method should be revived of drawing the ellipse or other Lissajous figures of combined vibration, as on p. 515, fig. 2.

The enveloping rectangle is divided up into elementary rectangles by lines spaced, not equidistant, but in equal time of simple vibration.

Perigal does this by describing a semicircle on each side of the rectangle, and then produces the ordinates of points at equal angular interval round the circumference.

Starting at any point of crossing and tacking across the diagonal of an elementary rectangle, a succession of points is made on an ellipse inscribed in the rectangle, and the points are close enough to be joined up in a continuous curve, such as Perigal could cut in his lathe.

If m and n steps are taken for a diagonal, the Lissajous curve appears for a combination of two vibrations of m and n fold frequency, and the phase difference of lead or lag is settled by the position of the starting point.

Yours sincerely,
G. GREENHILL.

Dec. 13, 1917.
1 Staple Inn, W.C. 1.

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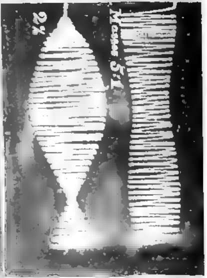

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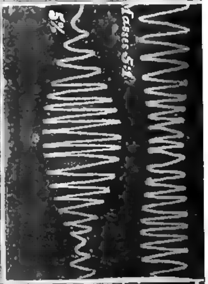

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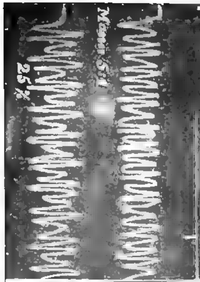

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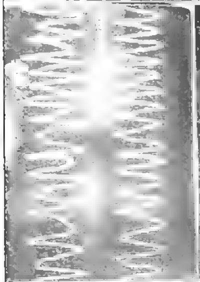

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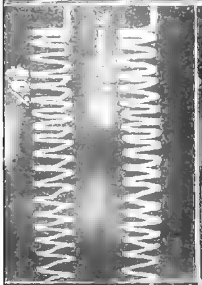

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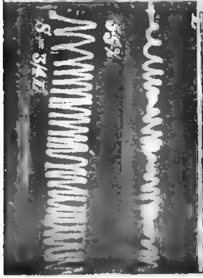

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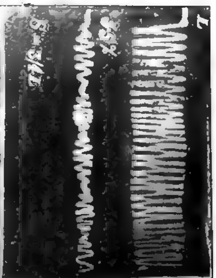

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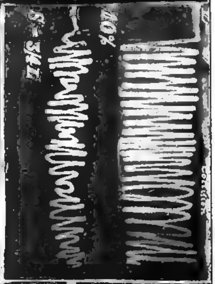

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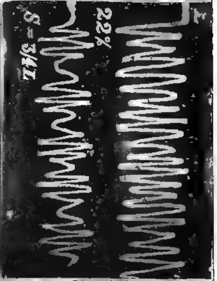

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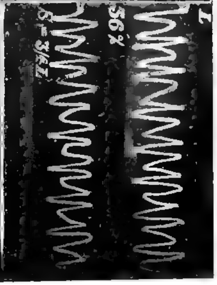

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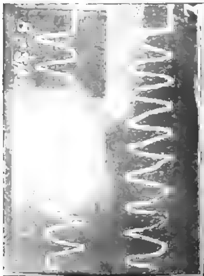

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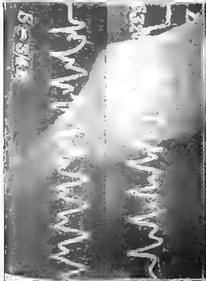

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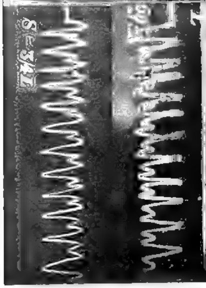

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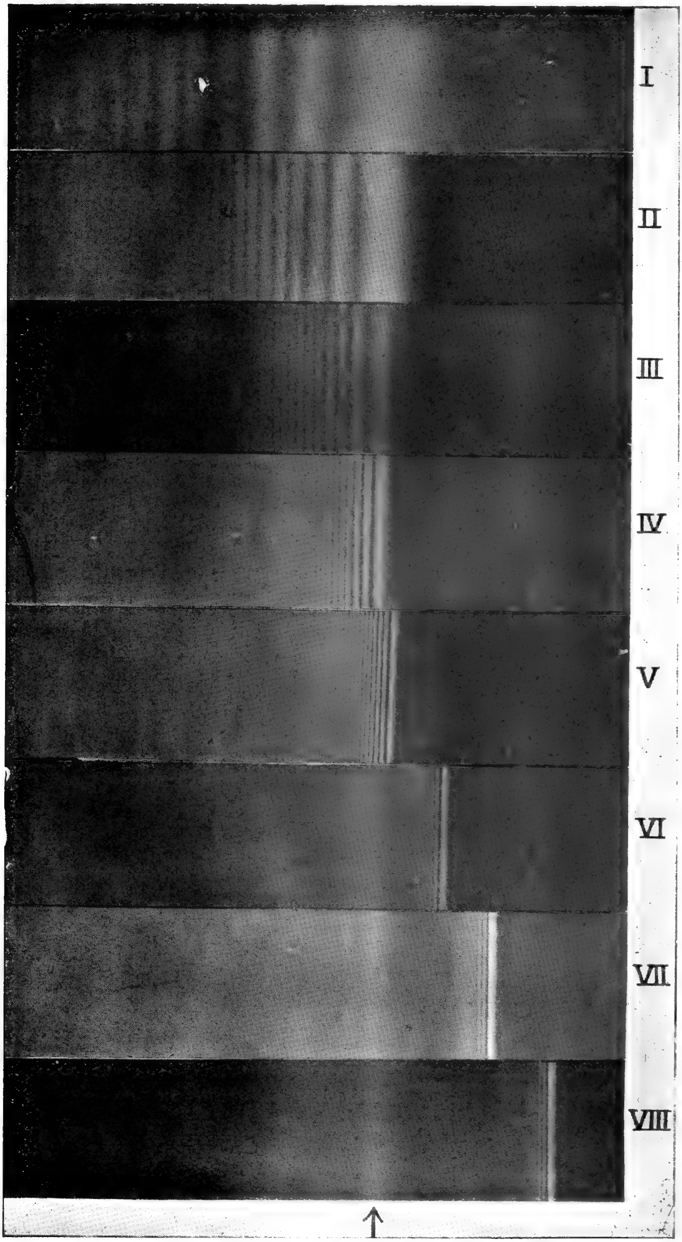

The Diffraction of Light by a Cylinder of radius 1·54 cm.

Barton & Browning.

Phil. Mag. Ser. 6. Vol. 35. Pl. I.

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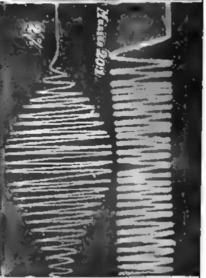

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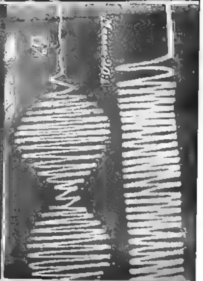

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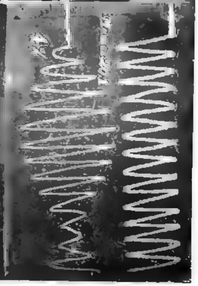

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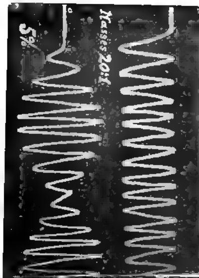

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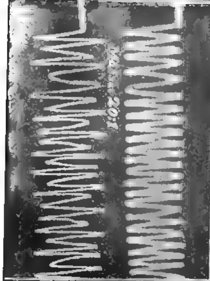

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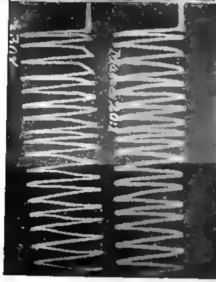

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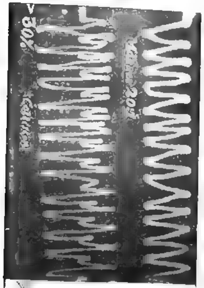

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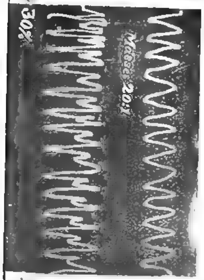

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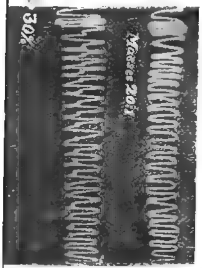

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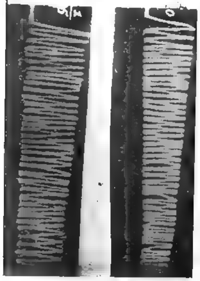

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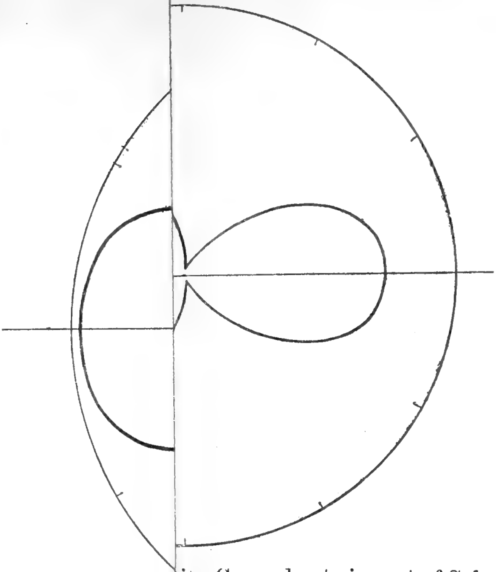
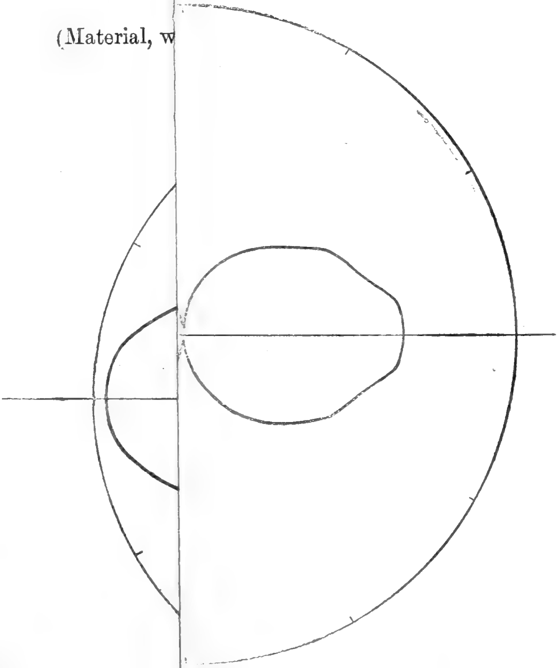


sity Curve due to impact of Spheres
rs but different materials.
ball; Diameter, $2\frac{1}{8}$ inches; Mass,
aterial, wood; Diameter, $2\frac{1}{4}$ inches;

Fig. 5.—C

(Material, w



nsity Curve due to impact of two
ameters and densities.

Fig. 6.—Ca

iameter, $1\frac{1}{8}$ inches; Mass, 118 gms.
Diameter, 3 inches; Mass, 158 gms.

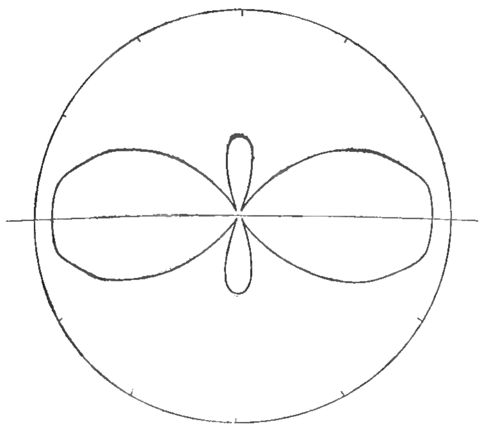


FIG. 1.—Observed distribution of sound intensity around two equal colliding spheres.

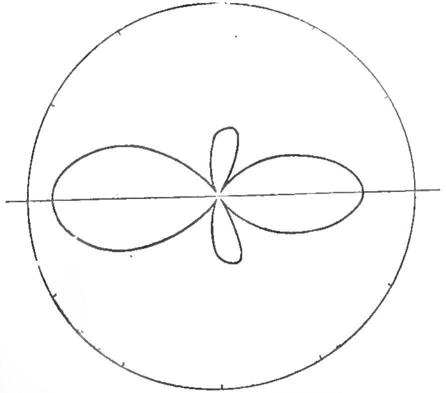


FIG. 2.—Observed distribution of intensity due to impact of two unequal spheres of wood.

Sphere on left; 3 inches diameter.

Sphere on right; 2¼ inches diameter.

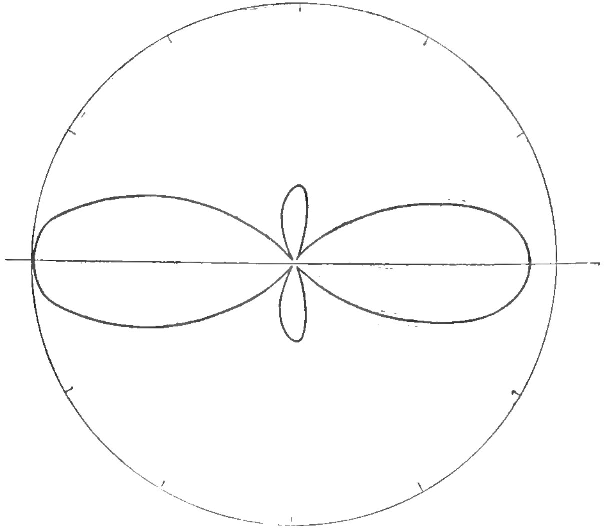


FIG. 3.—Calculated Form of Intensity Curve due to two Spheres of diameters 2 : 1. [$k(a+b)=2.$]

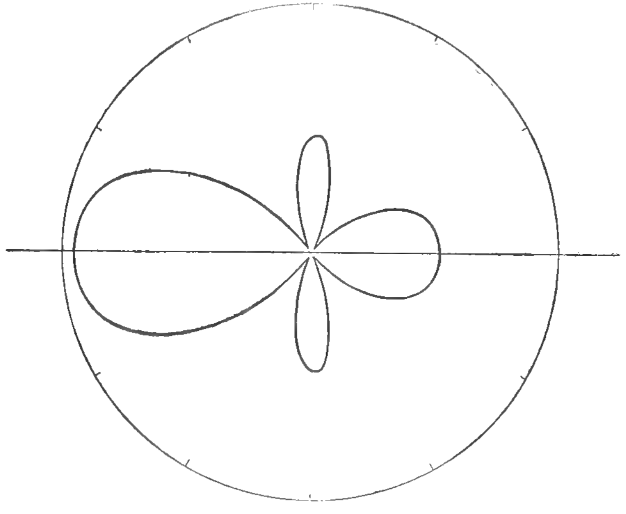


FIG. 4.—Same as fig. 3, but with $k(a+b)=3$.

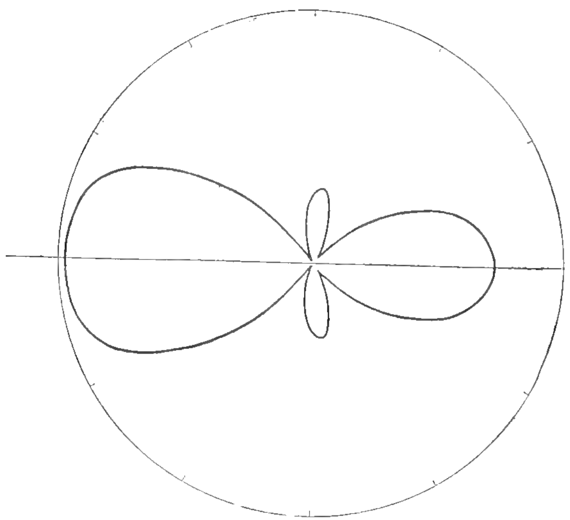


FIG. 5.—Observed Form of Intensity Curve due to impact of spheres of diameters 2 : 1.
(Material, wood; diameters 3 inches and 1½ inches respectively.)

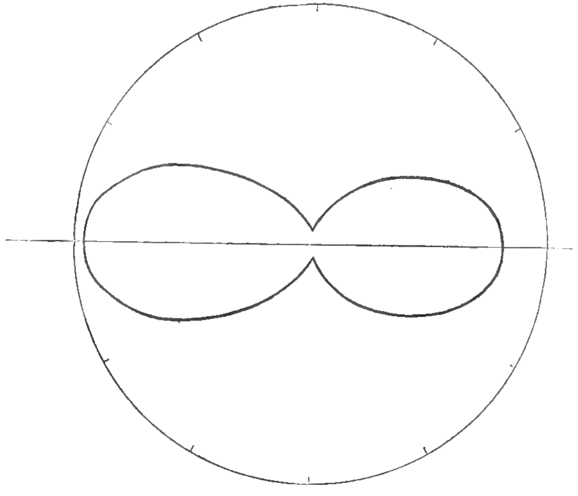


FIG. 6.—Calculated Form of Intensity Curve due to two equal Spheres of densities 4 : 1.

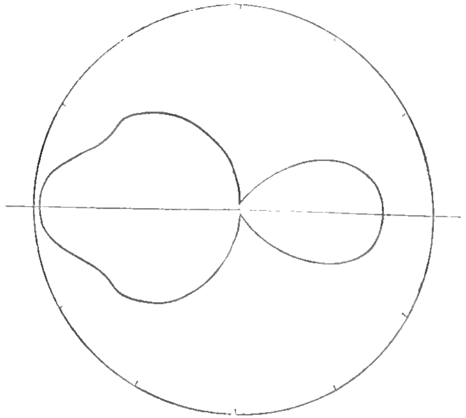


FIG. 7.—Observed Form of Intensity Curve due to impact of Spheres of nearly equal diameters but different materials.

Sphere on left: Material, billiard ball; Diameter, 2⅜ inches; Mass, 150 gms. Sphere on right: Material, wood; Diameter, 2¼ inches; Mass, 66 gms.

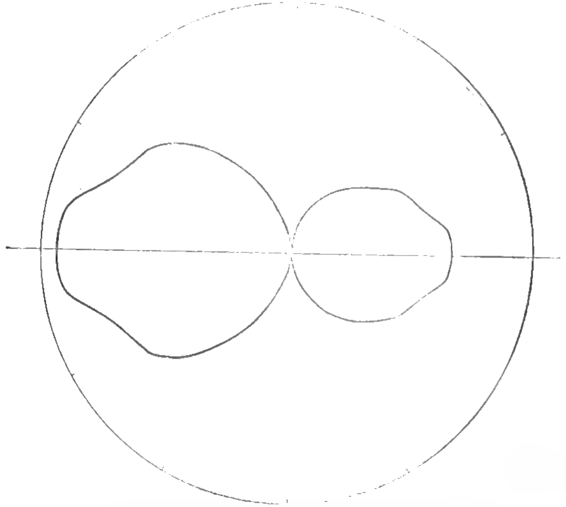
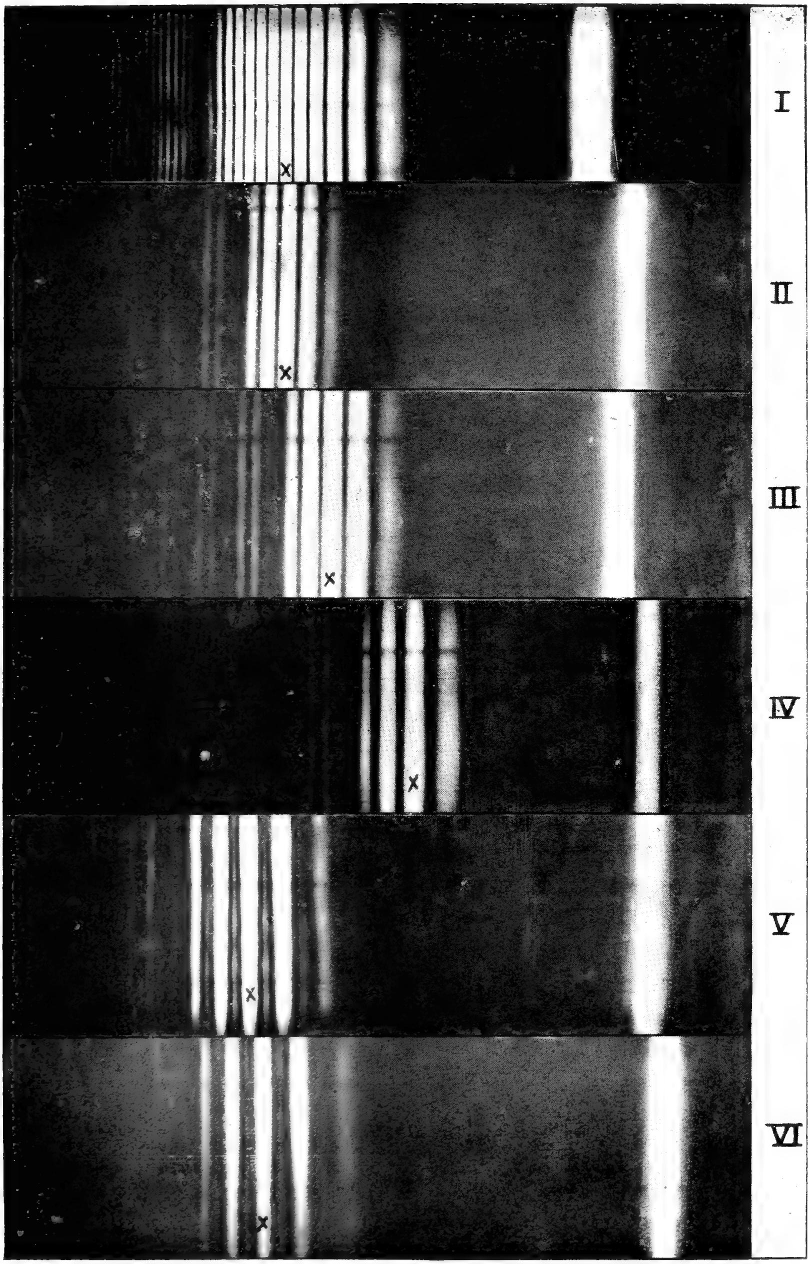


FIG. 8.—Observed Form of Intensity Curve due to impact of two Spheres of different diameters and densities.

Sphere on left: Material, brass; Diameter, 1⅛ inches; Mass, 118 gms.
Sphere on right: Material, wood; Diameter, 3 inches; Mass, 158 gms.



Illustrating the asymmetry of the fringes and of the illumination-curves in oblique diffraction by a reflecting surface consisting of two parts (Figs. I to IV) or three parts (Figs. V and VI) in the same plane.

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XVI. *Continued Discussion of the Astronomical and Gravitational Bearings of the Electrical Theory of Matter. By* Sir OLIVER LODGE *.

PART I.

MY short summary in the December number of the Phil. Mag., page 519, put prominently forward the idea that the expected effect required that the additional inertia due to motion should be independent of gravitative influence; for the conclusion seemed obvious that if weight and mass varied together there would be no change in acceleration, and that in that case it did not matter how much the mass of a revolving body varied. But I soon perceived that this was only attending to the transverse acceleration and neglecting the longitudinal, which is taken into account in Professor Eddington's completer theory in the October number of the Phil. Mag., page 322. He there re-determines the fundamental equation of particle dynamics, with momentum a function of speed, and shows that not the ratio F/m, but the product Fm, enters into the absolute term of that equation, so that it becomes $\frac{d^2u}{d\theta^2}+u=\frac{F/m_0}{h^2u^2}\cdot\frac{m}{m_0}$.

I take up the thread again here, and point out that that being so, the unexpected result follows, that if the additional inertia is acted on by gravity, in accordance with the ordinary

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* Communicated by the Author.

Newtonian law $F=\gamma mm'/r^2$, the varying factor m will enter twice into the equation of motion, and the whole perturbation will be increased instead of being annihilated. In other words, if the gravitative pull on the planet increases in the same way as the inertia increases, the effect is not to cancel, but to double the perturbing effect. On the other hand, if the extra inertia is not affected by gravity the perturbing effect is as already calculated. Consequently from this point of view some perturbation seems inevitable;—either the value reckoned by me in August, with the extra inertia independent of gravity, or else double that value, if the extra inertia is fully subject to the Newtonian law of attraction.

The question arises therefore, rather pressingly, how much dependence can be placed on the theory? It will be granted I think that the fact that a correct value for Mercurial apsidal progress can be deduced from the electrical theory of matter by a reasonable assumption of solar drift is not a negligible fact. For if the theory were completely inapplicable the value of drift required might have been of an altogether unreasonable order of magnitude. The fact that the same drift gave a Martial apsidal progress also of the right magnitude (see August Phil. Mag. pp. 91 & 92) seemed to me at the time very confirmatory. But I admit that the changes in excentricity are not thus accounted for correctly, and that the calculated perturbations for Earth and Venus exceed any probable value for those planets.

I perceived in my August paper that a difficulty of this kind would arise, but thought that it might be got over by choosing a particular longitude for the projected component of the solar drift which should almost nullify the result for those two planets; and so I chose the longitude 294° as being half-way between the perihelia of Mercury and Mars, and inclined to their major axes at a reasonable angle, while at the same time it happens to be practically half-way between the perihelia of Earth and Venus, though inclined to their major axes at a much smaller angle, and so being less effective. I hoped therefore that it might be possible to contrive to get rid of the calculated too great progress of perihelion for Earth and Venus, especially as the roundness of their orbits must make the exact position of perihelion difficult to determine.

Professor Eddington, however, countered all this contrivance, in September, by working out the theoretical changes of excentricity more thoroughly than I had done, and showed that it was highly improbable that the calculated perturbations could be admitted for some and evaded for others of the four inner planets, by any judiciously selected direction for solar drift.

If therefore the theory fails to give all the known perturbations correctly, something must be wrong; and by finding out what is wrong, we may perhaps discover something instructive. At first I thought that it would suffice to say that the extra inertia must after all be fully subject to gravity, and that therefore the acceleration was unaltered. But in the light of Eddington's improved theory, that loop-hole is closed; for the ratio F/m turns out not to be really involved. So the alternative that next occurred to me was to suppose that the gravitative constant γ may likewise be a function of velocity, and that its changes act in a compensating manner; that is to say, that the γ for bodies in motion through the æther differs from the value appropriate for bodies at rest in one or other of the two following ways:—

$$\gamma=\gamma_0\sqrt{\left(1-\frac{v^2}{c^2}\right)}$$

if extra inertia is independent of gravity,

or
$$\gamma=\gamma_0\left(1-\frac{v^2}{c^2}\right)$$

if the extra inertia has its full complement of weight. But then, so far as I know, there has never been any reason for suspecting such a variability in the Newtonian gravitative constant.

If there should really turn out to be such a change in the value of the gravitative attraction between bodies in rapid motion, it would be a very remarkable and noteworthy fact, and one that would serve to strengthen belief in some form of the Principle of Relativity. Hitherto the experimental foundation of that Principle has been the fact that whether luminous or telegraphic signals were used, or whether the interval of time between reception and transmission of signals was made to depend on cohesion or some other property of solid matter, there was always a disconcertingly complete compensation, so that no motion of matter as a whole through the æther could be demonstrated. I had hoped that by aid of the astronomical perturbations calculable from the electrical theory of matter this compensating influence might be overcome, and that the force of gravity would not, so to speak, join the conspiracy to defeat our object. We must, however, now face the possibility that gravitation too obeys a compensating law, and declines to enable us to receive information about absolute motion of matter through æther.

We must face the possibility: I do not say we must accept it; but it becomes necessary to consider what other loop-holes there are out of the conclusion which has thus suggested itself.

Is it possible that there is anything wrong about the theory that the inertia of matter is increased as a certain function of the speed? There can hardly be any doubt about it for an isolated charge such as an electron, inasmuch as the calculation based upon the behaviour of its lines of force is very straightforward, and because in the case of certain extra-high-speed ejections from radium the extra inertia has been observed and measured and found to correspond with theory.

But it may be argued that when electrons are packed together into an atom,—positive and negative together, so as to be on the whole neutral,—this effect, due to their individual electrostatic lines of force, is masked, and that the whole neutral atom ought not to show any but perhaps a residual effect of that kind.

Against this argument I urge several considerations:—

First, that if the inertia of the negative electron is wholly electrical, it becomes exceedingly probable that all inertia is of that kind; and if so, the abolition of electrical inertia would mean the abolition of all inertia.

Secondly, although we may speak of electrons as "packed together" into an atom, the packing is very far from being close; for to all intents and purposes they are well separated from each other and almost isolated, even though several are contained in a sphere the ten-millionth of a millimetre in diameter. It will be remembered that inertia is due to the *concentration* of the electric charge, and this concentration is only marked within a minute distance from a sphere so small as an electron. Taking its linear dimensions as 10^{-13}, all but one-hundred-thousandth of the inertia lies within a range smaller than 10^{-8}. Regarding the electron of a hydrogen atom as a sphere an inch in radius, the adjacent opposite charge is something like a mile away. Moreover, the fields of opposite charges are subject to the law of simple superposition. And again, the inertia in superposed fields of force does not depend on the sign of each component, but is proportional to the square of each component. Consequently there seems no adequate reason against merely adding their inertias in simple arithmetical fashion, even for atoms much more closely packed than hydrogen.

It may be said that little is known about the positive nucleus, wherein most of the mass of an atom resides according to the now prevalent view. That is true; but then, according to analogy, it would appear likely that the additional mass is due to still further concentration, and that the size of the positive unit, at least in a hydrogen atom, must be a thousand or say seventeen-hundred times smaller lineally than even the electron; in which case the argument for its practical isolation, and for its effective inertia being within an exceedingly small distance of its surface, is intensified.

On the whole then, I think that the weight of argument is strongly in favour of the full applicability of the theory of electrical inertia to every kind of atom, and to all masses of matter; though it is just possible that for atoms of high atomic weight some modification may have to be made owing to their presumably more complicated structure, especially the more complicated structure of their nuclei.

If this loop-hole is to be considered closed, what other is there? Yet in view of the importance of the threatening deduction that the gravitative constant is a function of velocity, we must seek every way out of the negative conclusion that the perturbations predicted by the electrical theory of matter do not in the case of the two planets Earth and Venus really occur; or do not occur to anything like the extent required by the quantitative explanation of the perturbations of Mercury by the same theory. If the theory fails to account correctly for the outstanding perturbations of the four inner planets—especially if it makes those perturbations too great,—it seems at present as if a variation of the gravitative constant for bodies in motion is proved: a result too important to be lightly regarded.

It seems therefore worth while to expend some labour in calculating what those perturbations would be for the four inner planets, given some favourable value of the solar drift.

PART II.

It may be remembered that in the October Philosophical Magazine Professor Eddington tentatively adopted but partly modified my theory, and also introduced terms depending on the excentricity, getting as his final result two equations representing the main perturbations to be expected from varying inertia, correct up to the first power of e. These equations, for convenience, I will here quote from page 326, vol. xxxiv. as they stand :—

$$\left.\begin{aligned} ed\varpi &= -\frac{u_0 V\theta}{2c^2}\sin\varpi + \frac{u_0{}^2\theta}{2c^2}e + \frac{V^2\theta}{8c^2}e\cos 2\varpi, \\ de &= \frac{u_0 V\theta}{2c^2}\cos\varpi + \frac{V^2\theta}{8c^2}e\sin 2\varpi, \end{aligned}\right\} \quad . \quad (1)$$

where V is the component of true solar drift projected on plane of orbit,

ϖ is longitude of planet's perihelion reckoned from the direction of V as zero,

u_0 is that constant component of the velocity of a planet which is normal to its radius vector,

θ is the angle turned through by radius vector per century,

and c is the velocity of light.

I proceed to apply this improvement on what I published in August, so as to ascertain whether or not the theory can be made to work.

Let the solar drift be k times the planet's velocity as specified, say $V = ku_0$; and introduce an aberration angle $\alpha = u_0/c$; then we can write the above equations thus :—

$$\left.\begin{aligned} ed\varpi &= \tfrac{1}{2}\alpha^2\theta(-k\sin\varpi + \tfrac{1}{4}k^2 e\cos 2\varpi + e), \\ de &= \tfrac{1}{2}\alpha^2\theta(k\cos\varpi + \tfrac{1}{4}k^2 e\sin 2\varpi). \end{aligned}\right\} \quad . \quad . \quad (2)$$

The common factor outside the brackets, $\frac{1}{2}\alpha^2\theta$, is independent of solar drift and cannot be evaded. It varies as the reciprocal of the 5/3rd power of the periodic time for different planets, or as the $-2{\cdot}5$th power of their distances; as can be seen thus :—

The large angle θ is $2\pi n$, so it is inversely as T; α is proportional to u_0, which is practically the same in magnitude as the average orbital velocity ; so α varies inversely as $\sqrt{r}$ or as $T^{-\frac{1}{3}}$ by Kepler's third law.

Hence $\frac{1}{2}\alpha^2\theta$ is proportional to $T^{-\frac{5}{3}}$ or $n^{\frac{5}{3}}$ or $r^{-\frac{5}{2}}$. It becomes small, therefore, for the outer planets. It is also plain that the values of $\frac{1}{2}\alpha^2\theta \times k$ for different planets vary inversely as the square of their distances from the sun.

The value of $\frac{1}{2}\alpha^2\theta$ for the Earth is $100\pi \times 10^{-8} = 0''{\cdot}648$, and from this it can be reckoned for the other planets by dividing by $r^{5/2}$. The result is tabulated here, along with other fixed planetary data for convenience of reference.

I. *Fixed Planetary Data.*

	Excentricity of orbit. e.	Longitude of perihelion.	Distance from sun. r.	Perturbation constant, $\frac{1}{2}\alpha^2\theta$ ($=0''\cdot648\, r^{-5/2}$).
Mercury	·2056	75°	0·387	6″·95
Venus	·0068	129	0·723	1·46
Earth	·0167	100	1·000	0·648
Mars	·0933	333	1·524	0·227
Jupiter	·0483	12	5·203	0·0105
Saturn	·0559	90	9·539	0·0023
Uranus	·0463	168	19·18	0·0004
Neptune	·0090	47	30·04	0·00013

Now consider the bracketed factors of (2). We see that the dominating part of both of these factors is *k*, and that whatever longitude is chosen for ϖ neither factor can exceed $\pm k$ to any considerable extent; they will, in fact, usually be both smaller than *k*. By suitable choice of ϖ either of the factors may be made small or zero; but if so, the other will thereby usually tend to be big.

To make this more obtrusively clear we might write them respectively

$$\left.\begin{aligned} &-k\sin\varpi(1+\tfrac{1}{2}ke\sin\varpi)+e(1+\tfrac{1}{4}k^2), \\ &\ \ k\cos\varpi(1+\tfrac{1}{2}ke\sin\varpi). \end{aligned}\right\} \quad . \quad . \quad (3)$$

If *k* is zero or small, *i. e.* if the solar system is nearly at rest in the æther, the *de* perturbation vanishes, but not the $d\varpi$. It is rather remarkable that there should be any residual perturbation due to fluctuating mass in a stationary solar system. But of course the velocity in an orbit with any excentricity is not quite constant, and the equations show that when *k* is 0, whatever the value of *e*, there will still be a cumulative $d\varpi$ (progress of perihelion) equal to $\frac{1}{2}\alpha^2\theta$; that is, $\frac{1}{2}\alpha^2$ times the angle turned through by the radius vector in any given time. This is a curious and interesting result, since it is independent of V and of e and of ϖ. The fact may require attention in another branch of physics later on (see concluding remarks in Part III.).

Over the main factor, $\frac{1}{2}\alpha^2\theta$, we have no control; but we can partly determine the bracketed factors of (2) by judicious selection of ϖ and k for any particular planet; though, having chosen for one, the others all follow.

The observational values to be accommodated by theory are stated by Mr. Harold Jeffreys in the 'Monthly Notices' of the Royal Astronomical Society for December 1916, whence I get the permissible range:—

II. *Unexplained Secular Variations per century, as observed.*

	$e\,d\varpi$.	$e\,d\varpi$. Permissible range.	de.	de. Permissible range.
Mercury	$8''{\cdot}48 \pm 0''{\cdot}43$	+8·91 to +8·05	−0·88±0·50	−1·38 to −0·38
Venus	−0·05±0·25	−0·30 to +0·20	0·21±0·31	+0·52 to −0·10
Earth	0·10±0·13	+0·23 to −0·03	0·02±0·10	+0·12 to −0·08
Mars............	0·75±0·34	+1·09 to +0·41	0·29±0·27	+0·56 to +0·02

Hence to get results for Mercury and Mars from the above equations, such as will correspond in sign with the entries in the above table, a solar drift must be chosen so as to make $\sin\varpi$ negative for both planets, with $\cos\varpi$ negative for Mercury and positive for Mars. This suggests an obtuse negative angle between Mercury's perihelion and the solar drift, and an acute negative angle for that of Mars; but this can be seen to be impossible, though the converse would be easy. A troublesome accommodation difficulty lies in the fact that the major axes of these two planets happen to cross nearly at right-angles, so that what suits one is hardly likely thoroughly to suit the other.

The above table shows moreover that both the observed perturbations for Earth and Venus are small, and might even be zero. But looking at the equations (3), we see that in general both the theoretical perturbations cannot vanish or be small together for any reasonable value of k. (There is no sense in a negative value for k such as $-2/e$.)

Alternatives.

Now let us consider the numerical values of the theoretical perturbations for different planets, with specially selected values for the solar drift. Write them

$$\left.\begin{aligned} e\,d\varpi &= \tfrac{1}{2}\alpha^2\theta\,.\,k\,.\,(-\sin\varpi + \tfrac{1}{4}ke\cos 2\varpi + e/k), \\ de &= \tfrac{1}{2}\alpha^2\theta\,.\,k\,.\,(\cos\varpi + \tfrac{1}{4}ke\sin 2\varpi), \end{aligned}\right\} \quad . \quad (4)$$

so as to show in each case

(1) A purely planetary factor $\frac{1}{2}\alpha^2\theta$, concerning which we have no choice ;

(2) A numerical factor k, depending on the magnitude of assumed solar drift and varying directly as $\sqrt{r}$ for different planets ;

(3) A trigonometrical factor, depending mainly on the direction of the assumed solar drift.

For moderate values of k, and for round orbits, the first term of each direction factor (viz. $-\sin\varpi$ for the one, $\cos\varpi$ for the other) is by far the most important.

Large values of k require delicate and practically impossible adjustments, so that if for any planet both perturbations are wanted small (as they are), it is essential to keep k down to the smallest value which will give anything like the desired result for some one definite planet.

Let us then choose k or V/u_0 as small as possible. We might even try it no greater, for the Earth, than suits the known solar motion towards Vega (viz. 19·5 $\frac{\text{km.}}{\text{sec.}}$ directed towards R.A. $18^{\text{h}}\ 2^{\text{m}}$ or longitude 270° 46′, and declination 34° or latitude 57° 30′), of which the projected component will be about half of two-thirds of the earth's orbital speed, or $\frac{1}{3}\times 10^{-4}c$, which makes k for the earth $=\frac{1}{3}$. Then for the other planets we shall have k varying as $1/u_0$ or $\sqrt{r}$, and so get the column k_1 in Table III. below.

But the resulting value of k for Mercury, viz. ·21, is far too small to explain the progress of Mercury's perihelion: for that purpose Mercury's k must be comparable to unity, in order to give a reasonable factor with which to multiply $\frac{1}{2}\alpha^2\theta$: but what its necessary value is will depend on ϖ, *i. e.* on the direction chosen for the solar drift with reference to the planet's major axis. Consider then what is the best direction to choose.

The idea on which we started (see Phil. Mag. for last

August) was to explain simply the outstanding perihelion progression of Mercury, 43″ per century or $ed\varpi = 8''\cdot6$; so it is natural to select values of k and ϖ which shall give an approximation to this value in the case of Mercury and yet keep k small enough not to spoil agreement for the other planets. The most favourable possible direction is 90° beyond the perihelion longitude of Mercury, which is 75°. For this drift longitude of 165° will make $\varpi = -90$ and $\sin\varpi = -1$. The drift-factor for $ed\varpi$ then attains its maximum, viz.

$$k + e(1 - \tfrac{1}{4}k^2),$$

while the factor for de is zero. In that case the value of k for Mercury may be as low as ·9; for, since $e = \cdot2$, this will give a factor, 1·1, sufficient to convert the 7″ belonging to $\frac{1}{2}\alpha^2\theta$ for Mercury into very near the desired 8″. Corresponding values of k for other planets are tabulated in the column k_2 below.

But then the perihelion longitude for Mars happens to be 333°, so that with drift longitude 165° the ϖ for Mars would be $+168°$ or $180 - 12°$; and the value of $\sin\varpi$ will be small and positive, and of $\cos\varpi$ big and negative—an arrangement which does not suit Mars at all; hence a compromise is necessary.

If we choose the compromise which I suggested in the August Phil. Mag., page 91, viz. 294°, or rather that direction reversed, viz. 114°, we have got a direction equally inclined to the major axes of Mercury and Mars, and also, as it happens, practically equally inclined to the major axes of Earth and Venus, which are another pair that may be considered together. The angle ϖ for Mercury in this case is $-39°$, and for Mars is $180 + 39°$; so in both cases $\sin\varpi$ is negative, as wanted, though the less said about $\cos\varpi$ the better. The corresponding values of k for all the planets, including that needed to suit Mercury's perihelion for this drift direction 114°, namely 1·5, are tabulated below as k_3.

The table shows some corresponding values of k and ϖ for all the planets, with the corresponding assumed longitude for solar drift. This is here called l; and it must be remembered that this solar drift direction, and not the first point of Aries, is the artificial zero from which to reckon ϖ. The ϖ are got by subtracting l from the perihelion longitude, as cited. The k vary as $\sqrt{r}$.

III. *Alternative values of k and ϖ.*

Longitude of perihelion. $\varpi+l$.	Distance from sun. r.		$l_1=271°$.		$l_2=165°$.		$l_3=114°$.		$l_4=173°$.	
			ϖ_1.	k_1.	ϖ_2.	k_2.	ϖ_3.	k_3.	ϖ_4.	k_4.
75	·4	Mercury.	164	·21	270	·9	321	1·5	262	1·056
129	·7	Venus.	218	·28	324	1·2	15	2·0	316	1·44
100	1·0	Earth.	189	·33	295	1·4	346	2·4	287	1·7
333	1·5	Mars.	62	·41	168	1·7	219	2·9	160	2·1
12	5·2	Jupiter.	101	·76	207	3·2	258	5·5	199	—
90	9·5	Saturn.	179	1·03	285	4·4	336	7·4	277	—
168	19	Uranus.	257	1·45	3	6·1	54	10·5	355	—
47	30	Neptune.	136	1·82	242	7·5	293	13·1	234	—

The columns with suffix 1 represent the known solar motion with reference to the stars; but for the present purpose this has no necessary importance.

The columns with suffix 2 represent the most favourable direction for explaining Mercury's perihelion-progress alone.

The columns with suffix 3 represent a compromise intended to suit the perihelion-progress of the four inner planets so far as possible.

The columns with suffix 4 represent the values got by satisfying *both* the observed perturbations for Mercury, and letting other planets look after themselves; *i. e.* by solving the equations,

$$e\,d\varpi \div \tfrac{1}{2}\alpha^2\theta = -k\sin\varpi + \tfrac{1}{4}k^2 e\cos 2\varpi + e$$

$$= 8''\cdot38 \div 6''\cdot95 = 1\cdot206,$$

$$de \div \tfrac{1}{2}\alpha^2\theta = k\cos\varpi + \tfrac{1}{4}k^2 e\sin 2\varpi$$

$$= -0''\cdot90 \div 6''\cdot95 = -\cdot13\,;$$

or, since $e = \cdot 206$, or say $\cdot 2$ for correction terms,

$$\left.\begin{array}{l} \dfrac{k^2}{20}\cos 2\varpi - k\sin\varpi = 1, \\ \dfrac{k^2}{20}\sin 2\varpi + k\cos\varpi = -\cdot 13. \end{array}\right\} \quad \cdots\cdots \quad (5)$$

A solution of these equations * is very nearly

$$\varpi = 262^\circ = -98^\circ,$$
$$k = 1\cdot 056.$$

And since the perihelion position for Mercury is

$$75^\circ = \varpi + l,$$

we get as the longitude of solar drift proper to account for *both* perturbations of Mercury

$$l = 75^\circ - 262^\circ = -187^\circ = 173^\circ.$$

We attend to this case further in Table V. below.

Numerical calculation for special cases.

In taking out the trigonometrical functions so as to get the proper factors in each case we have to pay particular

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* I solved these by successive approximation, the upper equation mainly determining k, while the lower mainly determines ϖ; general consideration, about signs, etc., showing that ϖ must be something big in the third quadrant, *i. e.* not far from 270°; but my brother has now solved them in much neater fashion, as thus:—

$$\left.\begin{array}{l} x^2\cos 2\theta - 20x\sin\theta = 20, \\ x^2\sin 2\theta + 20x\cos\theta = -2\cdot 6, \end{array}\right\}$$

$$x^2(\cos 2\theta + i\sin 2\theta) + 20\,ix(\cos\theta + i\sin\theta) = 20 - 2\cdot 6i,$$
$$(xe^{i\theta})^2 + 20i(xe^{i\theta}) + (10i)^2 = -80 - 2\cdot 6i,$$
$$(xe^{i\theta} + 10i)^2 = (a + ib)^2,$$

where $\qquad a^2 - b^2 = -80 \quad \text{and} \quad ab = -1\cdot 3.$

$$\therefore \quad x\cos\theta = a, \quad x\sin\theta = b - 10, \quad \text{and } x^2 = a^2 + b^2 + 100 - 20b.$$

The value of $a^2 + b^2$ is $80\cdot 04224$ nearly, and

$$a = \pm\cdot 1453252, \qquad b = \mp 8\cdot 94545.$$

$$\therefore \quad \left.\begin{array}{l} x\cos\theta = +\cdot 145325 \\ x\sin\theta = -18\cdot 94545 \end{array}\right\} \text{ or } \left.\begin{array}{l} x\cos\theta = -\cdot 145325, \\ x\sin\theta = -1\cdot 05455; \end{array}\right\}$$

$$\left.\begin{array}{l} x = 18\cdot 9460 \\ \theta = 270^\circ\ 26\tfrac{1}{2}', \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l} x = 1\cdot 0640, \\ \theta = 262^\circ\ 9'. \end{array}\right\}$$

attention to sign. It may be worth while to give details just for one case, and we may as well choose the one (No. 3) in Table III. when solar drift has longitude 114°.

The perturbations are

$$\left.\begin{aligned} e\,d\varpi &= \tfrac{1}{2}\alpha^2\theta \,.\, k \,.\, \mathrm{A} \\ de &= \tfrac{1}{2}\alpha^2\theta \,.\, k \,.\, \mathrm{B} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where
$$\mathrm{A} = -\sin\varpi + \tfrac{1}{4}ke\cos 2\varpi + e/k,$$

and
$$\mathrm{B} = \cos\varpi + \tfrac{1}{4}ke\sin 2\varpi.$$

They consist therefore of a purely planetary factor depending only on u_0 and T; a solar drift magnitude factor $k = \mathrm{V}/u_0$; and a relative direction factor A and B respectively.

We have now to reckon the direction factors A and B for the particular case marked above with the suffix 3.

For the values of $\frac{1}{2}\alpha^2\theta$ and e see Table I.

IV. *Calculated Perturbations for the Case of* $l = 114°$ *and* $\mathrm{V} = 2{\cdot}4 \times 10^{-4}\, c$.

	ϖ.	sin ϖ.	cos ϖ.	sin 2ϖ.	cos 2ϖ.	k.	A.	B.	$e\,d\varpi$.	de.
									″	″
Mercury	−39	−·63	+·777	−·98	+·21	1·5	·78	·70	+8·1	+7·2
Venus...	+15	+·26	+·97	+·50	+·87	2·0	−·26	·97	−0·76	+2·8
Earth...	−14	−·24	+·97	−·47	+·88	2·4	·24	·96	+0·37	+1·5
Mars ...	180+39	−·63	−·777	+·98	+·21	3·0	·675	−·71	+0·46	−·048
Jupiter.	180+78	−·978	−·208	+·41	−·91	5·5	·93	−·18	+·054	−·01
Saturn .	−24	−·407	+·91	−·74	+·67	7·4	·48	·83	+·0083	+·014
Uranus.	+54	+·809	+·59	+·95	−·31	10·5	−·84	·71	−·0035	+·003
Neptune	−67	−·92	+·39	−·72	−·70	13·1	·92	·37	+·0016	+·0007

The last two columns of this Table show that the superior planets will give us no trouble, whatever their aspect, and in spite of their comparatively large value of k; the smallness of their own factor $\frac{1}{2}\alpha^2\theta$ reduces all perturbations due to varying inertia to practically nil. Even if V were as high as 10 times the earth's orbital velocity, or say 200 miles a second, the superior planets would still give no trouble.

Of the four inner planets the entries for $e\,d\varpi$, compared with those in Table II., are not bad; but the de are deplorable. And no choice of direction or of drift which satisfies one set of perturbations seems likely to satisfy all.

Perhaps the simplest plan of calculating perturbations for a given V, is to reckon $\frac{1}{2}\alpha^2\theta k$ for one planet, say the Earth; divide this by relative r^2 for all the others; and then multiply the numbers so obtained by the respective direction factors A and B; which consist of $-\sin\varpi$ and $\cos\varpi$ primarily, corrected by addition or subtraction of small functions of k and e and $\sin 2\varpi$ and $\cos 2\varpi$. Thus a first approximation is

$$-\frac{{\cdot}648k_0}{r^2}\,.\,\sin\varpi, \quad \text{for } e\,d\varpi,$$

and

$$\frac{{\cdot}648k_0}{r^2}\,.\,\cos\varpi, \quad \text{for } de,$$

where k_0 refers to the Earth, being $\mathrm{V}/10^{-4}\,c$.

The next approximation can be taken from equations (3) above, viz. the factor $(1+\frac{1}{2}ke\sin\varpi)$ for both, with a supplementary term for one.

The only other case which I will work in detail is the case when a direction is chosen to satisfy both $e\,d\varpi$ and de for Mercury. We can then see what happens to the others. This is the case which we previously tabulated as k_4 and ϖ_4 in Table III., and which depends on the solution of equations (5).

V. *Calculated Perturbations for the Case of* $l=173°$ *and* $\mathrm{V}=1{\cdot}7\times10^{-4}\,c$.

	ϖ.	$\sin\varpi$.	$\cos\varpi$.	k.	A.	B.	$e\,d\varpi$.	de.
							″	″
Mercury...	180+82	−·990	−·139	1·056	1·14	−·124	+8·34	−0·91
Venus......	−44	−·695	+·719	1·44	·70	·72	+1·46	+1·52
Earth......	−73	−·956	+·292	1·7	·96	·29	+1·04	+0·32
Mars	180−20	+·342	−·940	2·1	−·26	−·97	−0·12	−0·46

On the whole perhaps these perturbations agree rather better with observation than do those in Table IV., though now Mars is disappointing. If they are in any degree tolerable, one may note that the drift, of magnitude 1·7 in direction 173°, thus postulated, can be considered as the resultant of a component of the known solar motion with reference to the stars

$$\left(\frac{19{\cdot}5}{29{\cdot}76} \times \cos 57\tfrac{1}{2}^{\circ} = {\cdot}352, \times 10^{-4}c, \text{ in direction } 270^{\circ}{\cdot}46\right)$$

compounded with a true drift in plane of ecliptic

$$1{\cdot}78 \times 10^{-4}\, c \text{ in direction } 162^{\circ}.$$

I really do not know whether astronomers could pass, as at all possible, outstanding perturbations such as those last tabulated. Comparing Tables V. and II. they are clearly not of orthodox size ; they are too big for Earth and Venus, and too small for Mars. But I suppose that the recognized values are in reality dissected out from a group system of small discrepancies of which the total is more certain than the precise distribution among individual members.

I submit also that even the forced agreement for Mercury is not to be wholly set aside as mere algebra ; for the postulated solar drift is of a not unreasonable order of magnitude, and the figures are got from it absolutely by a theory which if not in the least degree valid might have given them millions of times wrong. So the fact that absolute values not quite hopelessly discordant with observation can be thus reckoned ought to be taken into account.

Assuming that astronomers will not pass them, however, we must face the question why not. Full gravitational influence on the extra inertia might halve the necessary values of k, but would not otherwise improve things. Total absence of solar drift is unlikely ; so in order to explain the hypothetical absence of perturbations which ought to occur but do not, we may be driven to conclude that the gravitation-constant itself is a function of the speed of the attracting masses, in some such way as that suggested in Part I. above : thus adding to the evidence for an uncompromising Principle of Relativity.

PART III.—*Suggested Possibilities.*

In support of the idea that gravitative attraction may be a function of speed, I may point out that if the attraction were of an electrical order, such dependence on speed would be reasonable, and even the amount of the dependence would be appropriate; for the attraction between two charges moving together in parallel lines is $\frac{e^2}{\mathrm{K}r^2}\left(1-\frac{v^2}{c^2}\right)$; while if one charge is revolving round another, the attraction between them presumably has for its main term

$$\mathrm{F}=\frac{ee'}{\mathrm{K}r^2}\left(1-\frac{\mathrm{V}u\sin\theta}{c^2}\right).$$

(See J. J. Thomson's "Report on Electrical Theories," British Association volume for Aberdeen, p. 110 (1885).)

Hence if

$$m=m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}=m_0\left(1-\frac{\mathrm{V}^2+u^2+2\mathrm{V}u\sin\theta}{c^2}\right)^{-\frac{1}{2}}$$

then, u being in this case far larger than any probable V,

$$\mathrm{F}m=\mathrm{F}_0m_0\left(1-\frac{\mathrm{V}u\sin\theta}{c^2}+\frac{2\mathrm{V}u\sin\theta}{2c^2}+\frac{\mathrm{V}^2+u^2}{2c^2}\right),$$

and the terms involving $\sin\theta$, which are responsible for the cumulative terms in the solution of the differential equation quoted in paragraph 1, cancel.

Were it not so, some curious consequences could be deduced for an electron revolving at immense speed inside an atom round a nucleus under the inverse-square law, especially when such an atom is shot away at high speed; for the angle θ or $2\pi n$ is enormous.

Assuming it *possible*, then, that a quantitatively similar law holds in the case of gravity, the force of attraction $\mathrm{F}=\gamma m\mathrm{M}/r^2$ will diminish as m increases (M the central body, moving steadily at speed V, will not *change* its value whatever it is), and accordingly the product Fm (involving γm^2) will remain constant at whatever varying speed m moves through the æther: the variation of the gravity-constant γ just compensating for the double variation of mass m.

But it will be very remarkable if such compensation really occurs; and if such a fact is established it may begin to throw some light on the family relationship of the force of gravity.

XVII. *On the Lubricating and other Properties of Thin Oily Films.* *By* Lord RAYLEIGH, *O.M., F.R.S.*[*]

THE experiments about to be described were undertaken to examine more particularly a fact well known in most households. A cup of tea, standing in a dry saucer, is apt to slip about in an awkward manner, for which a remedy is found in the introduction of a few drops of water, or tea, wetting the parts in contact. The explanation is not obvious, and I remember discussing the question with Kelvin many years ago, with but little progress.

It is true that a drop of liquid between two curved surfaces draws them together and so may increase the friction. If d be the distance between the plates at the edge of the film, T the capillary tension, and α the angle of contact, the whole force is[†]

$$\frac{2\mathrm{AT}\cos\alpha}{d}+\mathrm{BT}\sin\alpha,$$

A being the area of the film between the plates and B its circumference. If the fluid wets the plate, $\alpha=0$ and we have simply $2\mathrm{AT}/d$. For example, if $d=6\times10^{-5}$ cm., equal to a wave-length of ordinary light, and T (as for water) be 74 dynes per cm., the force per sq. cm. is 25×10^5 dynes, a suction of $2\frac{1}{2}$ atmospheres. For the present purpose we may express d in terms of the radius of curvature (ρ) of one of the surfaces, the other being supposed flat, and the distance (x) from the centre to the edge of the film. In two dimensions $d=x^2/2\rho$, and A (per unit of length in the third dimension) $=2x$, so that the force per unit of length is $8\rho\mathrm{T}/x$, inversely as x. On the other hand, in the more important case of symmetry round the common normal $\mathrm{A}=\pi x^2$, and the whole force is $4\pi\rho\mathrm{T}$, *independent of x*, but increasing with the radius of curvature. For example, if $\mathrm{T}=74$ dynes per cm., and $\rho=100$ cm., the force is 925 dynes, or the weight of about 1 gram. The radius of curvature (ρ) might of course be much greater. There are circumstances where this force is of importance; but, as we shall see presently, it does not avail to explain the effects now under consideration.

My first experiments were very simple ones, with a slab of

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[*] Communicated by the Author.

[†] See for example Maxwell on Capillarity. Collected Papers, vol. ii. p. 571.

thick plate glass and a small glass bottle weighing about 4 oz. The diameter of the bottle is $4\frac{1}{2}$ cm., and the bottom is concave, bounded by a rim which is not ground but makes a fairly good fit with the plate. The slab is placed upon a slope, and the subject of observation is the slipping of the bottle upon it. If we begin with surfaces washed and well rubbed with an ordinary cloth, or gone over with a recently wiped hand, we find that at a suitable inclination the conditions are uniform, the bottle starting slowly and moving freely from every position. If now we breathe upon the slab, maintained in a fixed position, or upon the bottle, or upon both, we find that the bottle *sticks* and requires very sensible forces to make it move down. A like result ensues when the contacts are thoroughly wetted with water instead of being merely damped. When, after damping with the breath, evaporation removes the moisture, almost complete recovery of the original slipperiness recurs.

In the slippery condition the surfaces, though apparently clean, are undoubtedly coated with an invisible greasy layer. If, after a thorough washing and rubbing under the tap, the surfaces are dried by evaporation after shaking off as much of the water as possible, they are found to be sticky as compared with the condition after wiping. A better experiment was made with substitution of a strip of thinner glass about 5 cm. wide for the thick slab. This was heated strongly by an alcohol flame, preferably with use of a blowpipe. At a certain angle of inclination the bottle was held everywhere, but on going over the surface with the fingers, not purposely greased, free movement ensued. As might have been expected, the clean surface is sticky as compared with one slightly greased; the difficulty so far is to explain the effect of moisture upon a surface already slightly greased. It was not surprising that the effect of alcohol was similar to that of water.

At this stage it was important to make sure that the stickiness due to water was not connected with the minuteness of the quantity in operation. Accordingly a glass plate was mounted at a suitable angle in a dish filled with water. Upon this fully drowned surface the bottle stuck, the inclination being such that on the slightest greasing the motion became free. In another experiment the water in the dish was replaced by paraffin oil. There was decided stickiness as compared with surfaces slightly greasy.

The better to guard against the ordinary operation of surface tension, the weight of the bottle was increased by inclusion of mercury until it reached 20 oz., but without material modification of the effects observed. The moisture of the breath, or drowning in water whether clean or soapy, developed the same stickiness as before.

The next series of experiments was a little more elaborate. In order to obtain measures more readily, and to facilitate drowning of the contacts, the slab was used in the horizontal position and the movable piece was pulled by a thread which started horizontally, and passing over a pulley carried a small pan into which weights could be placed. The pan itself weighed 1 oz. (28 grams). Another change was the substitution for the bottle of a small carriage standing on glass legs terminating in three feet of hemispherical form and 5 mm. in diameter. The whole weight of the carriage, as loaded, was 7¾ oz. The object of the substitution was to eliminate any effects which might arise from the comparatively large area of approximate contact presented by the rim of the bottle, although in that case also the actual contacts would doubtless be only three in number and of very small area.

With ½ oz. in pan and surfaces treated with the hand, the carriage would move within a second or two after being placed in position, but after four or five seconds' contact would stick. After a few minutes' contact it may require 1½ oz. in pan to start it. When the slab is breathed upon it requires, even at first, 3½ oz. in the pan to start the motion. As soon as the breath has evaporated, ½ oz. in pan again suffices. When the weight of the pan is included, the forces are seen to be as 1 : 3. When the feet stand in a pool of water the stickiness is nearly the same as with the breath, and the substitution of soapy for clean water makes little difference.

In another day's experiment paraffin (lamp) oil was used. After handling, there was free motion with 1 oz. in pan. When the feet stood in the oil, from 2¾ to 3 oz. were needed in the pan. Most of the oil was next removed by rubbing with blotting-paper until the slab looked clean. At this stage ¾ oz. in pan sufficed to start the motion. On again wetting with oil 2 oz. sufficed instead of the 2¾ oz. required before. After another cleaning with blotting-paper ½ oz. in pan sufficed. From these results it appears that the friction is greater with a large dose than with a minute quantity of the *same* oil, and this what is hard to explain. When olive oil was substituted for the paraffin oil, the results were less strongly marked.

Similar experiments with a carriage standing on *brass* feet of about the same size and shape as the glass ones gave different results. It should, however, be noticed that the brass feet, though fairly polished, could not have been so smooth as the fire surfaces of the glass. The present carriage weighed (with its load) 6½ oz., and on the well-handled glass slide moved with ¼ oz. in pan. When the slide was breathed upon, the motion was as free as, perhaps more free than, before. And when the feet stood in a pool of water, there was equal freedom. A repetition gave confirmatory results. On another day paraffin oil was tried. At the beginning ¼ oz. in pan sufficed on the handled slab. With a pool of oil the carriage still moved with ¼ oz. in pan, but perhaps not quite so certainly. As the oil was removed with blotting-paper the motion became freer, and when the oil-film had visibly disappeared the ¼ oz. in pan could about be dispensed with. Doubtless a trace of oil remained. The blotting-paper was of course applied to the feet and legs of the carriage, as well as to the slab.

In attempting to interpret these results, it is desirable to know what sort of thickness to attribute to the greasy films on handled surfaces. But this not so easy a matter as when films are spread upon water. In an experiment made some years ago * I found that the mean thickness of the layer on a glass plate, heavily greased with fingers which had touched the hair, was about $\frac{1}{5}$ of the wave-length of visible light, viz. about 10^{-4} mm. The thickness of the layer necessary to induce slipperiness must be a small fraction of this, possibly $\frac{1}{10}$, but perhaps much less. We may compare this with the thickness of olive oil required to stop the camphor-movements on water, which I found † to be about 2×10^{-6} mm. It may well be that there is little difference in the quantities required for the two effects.

In view of the above estimate and of the probability that the point at which surface-tension begins to fall corresponds to a thickness of a single layer of moleules ‡, we see that the phenomena here in question probably lie outside the field of the usual theory of lubrication, where the layer of lubricant is assumed to be at least many molecules thick. We are rather in the region of incipient *seizing*, as is perhaps not surprising when we consider the smallness of the surfaces

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* Phil. Mag. vol. xix. p. 96 (1910); Scientific Papers, vol. v. p. 538.

† Proc. Roy. Soc. vol. xlvii. p. 364 (1890); Scientific Papers, vol. iii. p. 349.

‡ Phil. Mag. vol. xlviii. p. 321 (1899); Scientific Papers, vol. iv. p. 430.

actually in contact. And as regards seizing, there is difficulty in understanding why, when it actually occurs, rupture should ensue at another place rather than at the recently engaged surfaces.

It may perhaps be doubted whether the time is yet ripe for a full discussion of the behaviour of the thinnest films, but I will take this opportunity to put forward a few remarks. Two recent French writers, Devaux * and Marcelin †, who have made interesting contributions to the subject, accept my suggestion that the drop of tension in contaminated surfaces commences when the layer is one molecule thick; but Hardy ‡ points out a difficulty in the case of pure oleic acid, where it appears that the drop commences at a thickness of $1{\cdot}3 \times 10^{-6}$ mm., while the thickness of a molecule should be decidedly less. Many of Devaux' observations relate to the case where the quantity of oil exceeds that required for the formation of the mono-molecular layer, and he formulates a conclusion, not accepted by Marcelin, that the thickness of the layer depends upon the existence and dimensions of the globules into which most of the superfluous oil is collected, inasmuch as experiment proves that when a layer with fine globules exists beside a layer with large globules, the former always contracts at the expense of the latter. As to this, it may be worth notice that the tension T of the contaminated surface could not be expressed as a function merely of the volume of the drop and of the two other tensions, viz. T_1 the tension of an air-oil surface and T_2 that of a water-oil surface. It would be necessary to introduce other quantities, such as gravity, or molecular dimensions. I am still of the opinion formerly expressed that these complications are the result of *impurity* in the oil. If the oil were really homogeneous, Devaux' views would lead one to regard the continued existence of two sizes of globules on the same surface as impossible. What would there be to hinder the rapid growth of the smaller at the expense of the greater until equality was established? On the other hand, an impurity, present only in small proportion, would naturally experience more difficulty in finding its way about.

The importance of impurities in influencing the transformations of oil-films was insisted on long ago by Tomlinson §;

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* A summary of Devaux' work, dating from 1903 onwards, will be found in the *Revue Gén. d. Sciences* for Feb. 28, 1913.

† *Annales d. Physique*, t. i. p. 19 (1914).

‡ Proc. Roy. Soc. A, vol. lxxxviii. p. 319 (1913).

§ Phil. Mag. vol. xxvi. p. 187 (1863).

and as regards olive oil, Miss Pockels showed that the behaviour of purified oil is quite different from that of the common oil. She quotes Richter (Nature, vol. xlix. p. 488) as expressing the opinion that the tendency of oil to spread itself on water is only due to the free oleic acid contained in it, and that if it were possible to completely purify the oil from oleic acid, it would not spread at all*. Some confusion arises from the different meanings attached to the word "spreading." I suppose no one disputes the rapid spreading upon a clean surface which results in the formation of the invisible mono-molecular layer. Miss Pockels calls this a solution current—a rather misleading term, which has tended to obscure the meaning of her really valuable work. It is the second kind of spreading in a thicker layer, resulting in more or less rapid subsequent transformations, which is attributed to the presence of oleic acid. Miss Pockels says:—"The Provence oil used in my experiment was shaken up twice with pure alcohol, and the rest (residue) of the latter being carefully removed, a drop of the oil was placed upon the freshly formed water-surface in a small dish by means of a brass wire previously cleaned by ignition. The oil did not really spread, but after a momentary centrifugal movement, during which several small drops were separated from it, it contracted itself in the middle of the surface, and a second drop deposited on the same vessel remained absolutely motionless." I have repeated this experiment, using oil which is believed to have come direct from Italy. A drop of this placed upon a clean water-surface at once drives dust to the boundary in forming the mono-molecular layer, and in addition flattens itself out into a disk of considerable size, which rapidly undergoes the transformations well described and figured by Devaux. The same oil, purified by means of alcohol on Miss Pockels' plan, behaves quite differently. The first spreading, driving dust to the boundary, takes place entirely as before. But the drop remains upon the water as a lens, and flattens itself out, if at all, only very slowly. Small admixtures of the original oil with the purified oil behave in an intermediate manner, flattening out slowly and allowing the beautiful transformations which follow to be observed at leisure.

Another point of importance does not appear to have been noticed. Water-surfaces on which purified olive oil stands in drops *still allow the camphor movements.* Very small fragments spin merrily, while larger ones by their slower

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* 'Nature,' vol. l. p. 223 (1894).

movements testify to the presence of the oil. Perhaps this was the reason why in my experiments of 1890 I found the approximate, rather than the absolute, stoppage of the movements to give the sharpest results. The absolute stoppage, dependent upon the presence of impurity, might well be less defined.

If, after the deposition of a drop of purified oil, the surface be again dusted over with sulphur or talc and then touched with a very small quantity of the original oil, the dust is driven away a second time and camphor-movements cease.

The manner in which impurity operates in these phenomena merits close attention. It seems pretty clear that from pure oil water will only take a layer one molecule thick. But when oleic acid is available, a further drop of tension ensues. The question arises how does this oleic acid distribute itself? Is it in substitution for the molecules of oil, or an addition to them constituting a second layer? The latter seems the more probable. Again, how does the impurity act when it leads the general mass into the unstable flattened-out form? In considering such questions Laplace's theory is of little service, its fundamental postulate of forces operating over distances large in comparison with molecular dimensions being plainly violated.

Terling Place, Witham,
Dec. 31, 1917.

XVIII. *On the Second Postulate of the Theory of Relativity: Experimental Demonstration of the Constancy of Velocity of the Light reflected from a Moving Mirror. By* Q. MAJORANA, *Professor of Physics at the Polytechnic School of Turin**.

THE Theory of Relativity is based upon two well-known fundamental postulates. The first affirms the impossibility of discovering the movement of a system without referring this system to other systems; that is to say, it denies the physical reality of *absolute motion.* The second postulate affirms that the velocity c of propagation of light *in vacuo* is a universal constant. Both these postulates are generalizations of facts or principles already admitted by physicists.

In fact, we may regard the first as the extension to optical or electrical phenomena of a classical principle of mechanics,

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* Communicated by the Author.

an extension justified by the negative results of certain experiments (Michelson and Morley, Trouton and Noble) by which it was sought to discover the absolute motion of the earth, or the *æthereal wind* which must traverse all terrestrial objects. The second postulate is the generalization of a fundamental principle in the theory of æthereal or electromagnetic undulations.

But if these two principles, derived from quite different chapters of physics, have been fully accepted severally by modern physicists, their origin has been almost forgotten; an ingenious structure arose upon their union: the theory of relativity. This theory, while repudiating according to Einstein and others a theoretical conception which had given occasion for the formulation of the second postulate (the æther), serves well to explain the insuccess of the above-cited experiments.

Now our imagination, accustomed, as W. Ritz has said, to "substantialize" physical phenomena, if it easily grasps the essence of the first postulate, does not do so in the case of the second; and the more so since, as has already been said, some adherents to relativistic theories do not retain as necessary the existence of a medium of transmission (the æther) in order to explain the constancy of c. Moreover on the second postulate, or, more precisely, on a certain portion of this, depend the conclusions which appear artificial or extraordinary in the whole relativistic theory *. The second postulate must be understood in the sense that an observer who measures the velocity of light finds always the same value if both he and the source be at rest, relative or (if the possibility be admitted) absolute, or if the source or the observer, or both, have a uniform motion of translation. In short, the second postulate affirms the absolute independence of c of any contingency whatever of uniform motion of translation of the source or of the observer.

It is known that an hypothesis of a mechanical character (emissive or ballistic), according to which to the ordinary velocity of light must be added that of the source, can explain, like the theory of relativity, the failure of the above-quoted experiments. But this hypothesis is radically in contrast with the electromagnetic theory, and consequently is not much favoured †. But in any case laboratory experiments can be conceived which should decide between the

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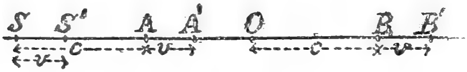
* Carmichael, Phys. Rev. xxxv. p. 168 (1912).

† In this connexion should be recalled the important critical work of W. Ritz (*Œuvres*, p. 317) which perhaps has not been taken into sufficient consideration by physicists.

above-mentioned hypothesis or mechanical theory and the relativistic one. It is indeed possible to see that some method, one moreover already in use, adopted for the verification of Doppler's principle may serve for the solution of the above-quoted problem.

In order to see this, let us consider a luminous source S which emits waves of length λ and of frequency n moving towards the observer fixed at O (fig. 1). If we suppose that the waves are transmitted through a stationary æther, the n waves emitted in a second by S will be distributed over the segment $S'A=c-v$. In the same time all the n waves distributed in the segment $OB=c$ will have passed through O; we have therefore

Fig. 1.



$$\frac{c-v}{n}=\frac{c}{n'}, \text{ or } n'=n\frac{c}{c-v}.$$

If we put $v/c=\beta$ and neglect terms of higher order than the first in β we have

$$n'=n(1+\beta).$$

The new wave-length is obtained by the relations $c=n\lambda=n'\lambda'$:

$$\lambda'=\lambda(1-\beta).$$

If now instead of the hypothesis of a stationary medium we adopt the ballistic or emissive hypothesis of which we have spoken above, we shall find that in one second the n waves emitted by S will be distributed over the segment $S'A'=c$. In the same time there will pass through O, n' waves which will be distributed over the segment $OB'=c+v$. We have, therefore,

$$\frac{c}{n}=\frac{c+v}{n'}, \text{ or } n'=n(1+\beta).$$

And since $c=n\lambda$ and $c+v=n'\lambda'$ we see that, in this case, $\lambda'=\lambda$.

As regards the frequency we arrive, therefore, at the same conclusions (with the exception of the terms in β^2) whether we adopt the æthereal or the ballistic hypothesis; but for the wave-length we obtain different values from the two hypotheses, and these values differ by a term of the first order in β. If then the Doppler effect is measured, by observation of the wave-length, different results should be obtained according as the one or the other hypothesis is adopted [*]. Now observations of the Doppler effect have already been made by measuring the displacement of the spectral lines, employing either prisms or diffraction-gratings. In the case of prisms it may be observed that all the theories of dispersion hitherto admitted lead to the supposition that this phenomenon can only depend on the frequency of the incident luminous vibrations. Consequently the displacement of the spectral lines may be caused by the simple variation in frequency due to the Doppler effect, and this whether, for the light, the hypothesis of a stationary æther is adopted, or a ballistic or emissive theory. From this point of view, therefore, the question whether the velocity of propagation of the light emitted by a source does or does not change with the velocity of the latter remains unanswered.

But the Doppler effect has been established with diffraction-gratings as well as with prisms, and for astronomical as well as terrestrial sources [†]. Now the function of a grating, from the geometrical point of view, may be regarded as depending exclusively on the values of the incident wave-lengths ; the positions of the successive spectral lines remain exactly determined by those values. But since, according to the ballistic or emissive hypothesis, the value of λ does not vary with the velocity of the source, we see that the grating should not give an appreciable result in the study of the Doppler effect, and this, as is known, is not in agreement with experience. We may then conclude from observations of the Doppler phenomenon in the stars and the limb of the sun with moving mirrors (Galitzin & Wilip), or again in the canal rays (Stark, Paschen), that the velocity of light is absolutely constant and independent of the movement of the source ; this is equivalent to the rejection of the ballistic or emissive theory. This is Tolman's opinion [‡], in contradiction to that of Stewart [§]. Indeed, it should be borne in mind that the ordinary grating theory [‖] may not apply exactly in the case of a mechanical (ballistic or emissive) theory of

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[*] These conclusions are identical with those already published by other authors ; see, *e. g.*, Tolman, Phys. Rev. xxi. p. 26 (1910).

[†] Galitzin & Wilip, *Communications Acc. Russe*, 1907, p. 213 ; Stark, *Ann. d. Phys.* xxviii. p. 974 (1909).

[‡] Phys. Rev. xxxv. p. 136 (1912).

[§] Phys. Rev. xxxii. p. 418 (1911).

[‖] La Rosa, *Nuovo Cimento*, iii. p. 356 (1912).

light. In any case it should be remarked that astronomical observations of the Doppler effect are not always made with an *à priori* knowledge of the relative velocities of source and observer. In the case of the solar limb it is necessary, moreover, to be cautious in establishing a relation between the measure of displacement of the lines and the velocity of the limb established by observation of the solar spots; in fact, the light from the limb may be strongly refracted by the perispherical incandescent gases, and consequently the value of the Doppler effect may vary considerably *. So far as terrestrial observations are concerned, and those on the canal rays (Stark, Paschen), they give measures of the phenomenon of only small precision, and it is impossible to foresee by another method the exact velocity of the luminous particles ; finally, observations made with moving mirrors bear no relation to those with moving sources, and these may produce different consequences †.

From all this we may conclude that up to the present time we do not possess any quite certain evidence of the immutability of c with variable velocity of the source if, be it understood, we are not willing to admit as conclusive the simple electromagnetic theory or that of bodies in motion according to Lorentz or else Einstein's theory of relativity. This conclusion is confirmed by the study of the works of the chief supporters of the last theory, and, implicitly, of the second postulate. In these works we frequently find expressed the desire to discover further facts which will definitely confirm the said theory : this desire corresponds with the crisis of the latter years of the said theory.

But on the other hand, as Levi-Civita observes, after the latest researches of Einstein, which collect in an admirably comprehensive synthesis all the physical phenomena (gravitation included), it is difficult to avoid the impression that we are, as regards the theory of relativity, face to face with some definite acquisition. But, while taking account of this, it is not expedient to neglect any attempt at a definite confirmation, from an experimental point of view, of a theory which has subverted to so large an extent our simplest physical notions. This confirmation may follow from a precise study of the velocity of propagation of light emitted from a moving source, or, which is equivalent, of the wavelength of this light.

To realize this study we must devise an arrangement

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* Michelson, Astrophys. Journ. xiii. p. 192 (1901) ; Harnack, *Ann. d. Phys.* xlvi. p. 558 (1915).

† See the theory proposed by Ritz, *Œuvres*, pp. 321, 371, 444.

which will permit us to identify the structure of the luminous wave, freed from all external action, in its free propagation (or transmission) when the velocity of translation of the source can be varied at will. But, apart from the fact that we must inevitably experiment under the eventual action of our earth [*], two serious and almost insurmountable difficulties oppose themselves to the realization of such a programme. In the first place, it is not easy artificially to endow a luminous source with rapid movement [†], especially if this source (as is necessary in some interference methods) has to be very rigorously monochromatic; moreover, I shall publish an account in a forthcoming paper of a disposition of this nature with which I am about to experiment. Secondly, in order to be able to examine the structure of the light emitted by a moving source, with whatever disposition, the light itself has to be subjected to reflexions, refractions, &c., sometimes fairly numerous; that is to say, the luminous ray must encounter ponderable matter after leaving the source. It does not follow, therefore, that even if c in a vacuum varies with the particular velocity of the source, this quantity does not return to the same fixed value after the said phenomena of reflexion, refraction, &c. It will be well, therefore, to endeavour to eliminate as far as possible, in an experiment of this nature, all causes tending to complicate the phenomenon, and in every case to consider its results carefully.

Meanwhile, to begin with a relatively simple experiment, we may undertake the study of the wave-length of a ray of light reflected by a moving mirror [‡]. This may correspond with the experiment already realized, some years ago, by Belopolski, and afterwards repeated by Galitzin & Wilip; but if the first of these authors employed prisms for the observation of the Doppler effect (and consequently the question of the eventual variation of λ remained unsolved), the other two made use of a diffraction-grating, by which the controversy spoken of above arises. It would be better

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[*] I cannot succeed in imagining an interplanetary experiment of the nature of that proposed (in jest) by Rose-Innes; see Phil. Mag. xxvii. p. 150 (1914).

[†] I understand by this a velocity higher than some hundreds of metres per second; this value may perhaps be attained, but it is difficult to conceive a practical disposition for a higher velocity. Naturally I leave out of account the employment of canal rays, which do not give simple and well-known velocities.

[‡] While this article was in the press, M. Michelson has called my attention to his paper on the same subject, published in the 'Astrophysical Journal,' April 1913, the conclusions of which agree with those that I am stating.

therefore to examine the ray reflected from a moving mirror by an interference method simpler than that on which the action of the diffraction depends, as has been said above.

Before expounding this method it is well to recall that many theoretical researches have been made on the influence of the motion of the mirror upon the reflected luminous wave, amongst them those of Abraham, Brown, Edser, Harnack, Larmor, Planck. These researches make of the problem either a simply geometrical investigation, or an application of the electromagnetic theory of light. But without discussing the result of these researches we may accept the conclusion of Harnack * respecting the frequency of the vibrations reflected by a mirror in uniform motion. Let v be the velocity of the latter, normal to its plane, reckoned as positive towards the source; c the velocity of the luminous ray *in vacuo* which makes the angle of incidence θ with the mirror; n, n' the frequencies of the ray before and after reflexion, the source and observer being at rest. If we put $\beta=v/c$ we shall have

$$n'=n\frac{1+2\beta\cos\theta+\beta^2}{1-\beta^2},$$

which, neglecting the terms in β^2, reduces to

$$n'=n(1+2\beta\cos\theta).$$

This relation is the same as that of Ketteler †, which was employed by Belopolski ‡ in his investigation of the Doppler effect, and follows simply from the consideration that the image of the source moves with the velocity $2v$ in the direction of the normal to the mirror and, consequently, the component of this velocity in the direction of the reflected ray is $2v\cos\theta$.

If now we suppose that the ray is, by suitable arrangements, reflected with the incidence θ, k times from several mirrors in motion with the velocity v, we shall have

$$n'=n(1+2k\beta\cos\theta).$$

Therefore, according to the hypothesis of constant velocity of light, neglecting the terms in β^2 we shall have

$$\lambda'=\lambda(1-2k\beta\cos\theta).$$

If, on the contrary, we suppose that the velocity of the

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* *Ann. d. Phys.* xxxix. p. 1053 (1912) and xlv. p. 547 (1915).

† *Astronomische Undulationstheorie.*

‡ *Communications Acc. Russe,* xiii. p. 451 (1900).

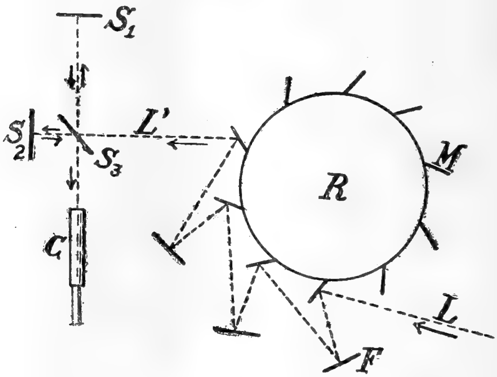
reflected light is variable (and equal to the sum of $c=3.10^{10}$ cm. and the component of the velocity of the image in the direction of the ray) we shall have

$$c'=c+2kv\cos\theta.$$

And since $c'=n'\lambda'$ and $c=n\lambda$ we have $\lambda'=\lambda$. It remains then to see by experiment whether or not we can observe, in addition to the Doppler effect, a variation in the value of λ; from this we can ascertain whether c remains constant or not on reflexion from a moving mirror. I have not proceeded to the observation of the Doppler effect in these researches since there is no doubt about its existence, already proved experimentally by the authors quoted ; I have rather sought to find out whether and in what way λ varies when the velocity of the moving mirrors changes.

Belopolski's arrangement for the study of the Doppler effect was inconvenient on account of the excessive subtility of the luminous ray necessary to obtain multiple reflexions from the same mirrors ; for this reason the author mentioned was unable to observe the displacement of the rays except by photography. I prefer to adopt the arrangement shown diagrammatically in fig. 2. A horizontal brass wheel R, 35 cm. in diameter (6 mm. thick), which can be made to revolve with a maximum velocity of 80 turns per second, bears on its periphery ten mirrors similar to M, rectangular, plane, vertical, of glass silvered at the back. The velocity of the centres of the mirrors, corresponding to the greatest velocity of rotation, amounts therefore to more than 100 metres per

Fig. 2.



second. The number of revolutions of the wheel was determined acoustically in each experiment. The mirrors, at equal intervals on the periphery of the wheel, are inclined to the radius from R passing through the centre of each of them at an angle $\alpha = 29°$. They are fixed solidly to R by screw movements capable of permitting a rigorous adjustment. The support for the bearings of the axle of R carries also fixed mirrors F, vertical like M, of which the number in the figure is three ; but this number may, at will, be reduced, or increased up to nine. The position of the F's and M's is such that a parallel beam of light L, after a certain number of reflexions from the F's and M's (seven in the figure), may be received at L′ when R has determinate angular positions. Naturally the intensity of L′ is much weaker than that of L, and this enfeeblement is much more marked if R is in rotation, because in this case the light arrives at L′ only during certain very short instants (ten times per revolution). I have observed in practice, however, that the four moving and three fixed reflexions of the figure allow of experimenting with light sufficiently intense at L′ even if R is in motion : that is to say, that direct observation (without photography) suffices to establish the luminous phenomenon of which we have spoken above.

To study the value of λ the light L′ was examined with the well-known interferometer of Michelson, shown diagrammatically in the figure. It is known that if the distances S_1S_3 and S_2S_3 are exactly equal fringes are observed with the telescope C even if the light is not monochromatic ; these fringes then have the coloration of Newton's rings. As soon as a difference of path occurs (even if only of a few microns) observation with white light is no longer possible. Monochromatic light must then be used, and the order of the interference fringes increases with this difference. Their visibility is greater, the simpler the luminous vibrations. From the researches of Michelson * it is known that from this point of view the line that gives the greatest visibility of the fringes with the greatest difference in path is the green one of mercury ($\lambda = 546\mu\mu$). In this case numberless circular fringes are visible even for a difference of path $l = 2(S_1S_3 - S_2S_3) = 40$ cm. I have therefore employed as source L a mercury arc *in vacuo* the light of which is conveniently filtered by solutions of chromate of potassium and chloride of nickel to absorb the violet and yellow rays ;

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* *Travaux et Mémoires*, Bur. Int. de poids et mésures, xi. p. 146 (1895).

in this manner I have been able to observe with the telescope C, with sufficient clearness, countless circular fringes, even for $l=32$ cm. But for these researches I have limited the difference of path to $l=13$ cm., or still less.

The disposition described above is particularly suitable for detecting very small differences in the value of the incident wave-length; in fact, the value of l being large a very great number of wave-lengths is comprised in this length (*e. g.*, 200,000 if $\lambda=0{\cdot}5\mu$, and $l=10$ cm.), and correspondingly for the same variations very sensible displacements can be observed in the position of a fringe.

With the apparatus disposed as above, let us note with the micrometer wire of the telescope the position of a fringe, for instance the first central bright one, when R is in the position shown in the figure, or, still better, when it revolves with a negligible velocity (one turn per second). If, now, this velocity be increased to sixty turns per second *a displacement of the fringe under observation is distinctly visible*; if the mirrors are moving against the incident ray this displacement indicates a diminution of λ, and it changes sign when the direction of rotation of the wheel is reversed, and this indicates an increase of λ. In order to define the sense of the displacement, I will say that on examining the system of circular fringes with the telescope focussed for infinite distance the diameter of each of these increases when the mirrors move against the incident ray, and the fringes themselves crowd together as those of large diameter are very little displaced; at the same time some new fringes come out from the centre of the system. On the other hand, when the mirrors are moving in the sense of propagation of the incident light the diameter of each fringe diminishes; they become more widely separated, and some of them are as it were swallowed up by the centre.

Before stating the measure of the displacement observed we will see what it should amount to, making the hypothesis that the velocity of the light reflected from a mirror is the same as that of the incident light. Let g be the number of revolutions of R per second and d its diameter, reckoned between the centres of two opposite mirrors M, then πdg will be the instantaneous velocity of translation of the latter. Since the mirrors are inclined at an angle α to the radius of the wheel passing through each of them, the component of the given velocity in the direction normal to the plane of each mirror will be

$$v=\pi dg\cos\alpha.$$

We have, therefore,

$$n' = n\left(1 + \frac{2k\pi dg \cos\alpha \cos\theta}{c}\right);$$

and, by the hypothesis of the invariability of c,

$$\lambda' = \lambda\left(1 - \frac{2k\pi dg \cos\alpha \cos\theta}{c}\right).$$

If l is the difference of path of two interfering rays in Michelson's apparatus, the number of fringes which are seen to cross the micrometer thread of the telescope when λ becomes λ' (that is to say when the velocity of rotation varies between zero and g turns per second) is

$$f = \frac{l}{\lambda} \frac{2k\pi dg \cos\alpha \cos\theta}{c}.$$

If the observation is made by noting the position of the fringes when the wheel turns in one sense with the velocity g, and that corresponding to an equal and contrary velocity, the number of fringes crossing the micrometer thread will be $2f$.

Now, in my apparatus $d=38$ cm., $\alpha=29°$, $\theta=27°$, $k=4$ (as in the figure); if λ is put equal to $0{\cdot}546\mu$ (green mercury line), $l=13$ cm., $c=3\,.\,10^{10}$ cm., and $g=60$ (turns of R per second in one sense and afterwards in the other), we may expect, according to the preceding formula, a fringe displacement $2f=0{\cdot}71$.

Experiment gives, for the case mentioned, a displacement of between 0·7 and 0·8 fringes; and it is not possible, for reasons of visibility, to carry the precision of the observations further. But, as is seen, the agreement between the predicted result and observation is sufficient; this agreement is confirmed by observations made by choosing other convenient values of l and g, of which for brevity's sake I shall not speak here.

Experiment, therefore, authorizes the conclusion that *reflexion of light by a moving metallic mirror does not alter the velocity of propagation of the light itself, in air, and consequently, with great probability, also in vacuo*; at least, in the conditions of the experiment above described. This experimental result, as to which no doubt can be entertained, is contrary to the hypothesis of some physicists who, like Stewart *, basing themselves upon the electromagnetic emission theory of Thomson, maintain the possibility that

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* Phys. Rev. xxxii. p. 418 (1911).

light, after reflexion, is propagated with the velocity $c+v$, where v is the component of the velocity of the image in the direction of the reflected ray.

To complete these researches I intend, as I have said above, to investigate further with the same interferential arrangements, the velocity of propagation of light from a source set in motion artificially; but of these, and of the general conclusions to be drawn from these investigations, I reserve mention for a future occasion.

XIX. *The Visibility of Radiation.*

By PRENTICE REEVES *.

THE theory of this subject has been given previously by Nutting † and Ives ‡, and in those papers may be found a thorough treatment of the early literature. In this paper the writer wishes to present further data obtained by a method similar to that employed by the above writers but using a different apparatus. The writer has data from thirteen subjects, five of whom were also used as observers by Nutting in his list of twenty-one subjects. The values for the spectral energy distribution of acetylene were those offered by Nutting, and were obtained by weighting the data accessible up to that time as well as his own results in this laboratory. By using these values the writer was able to directly compare results with those of the other writers, and by using the values offered by Coblentz § and revised by Coblentz and Emerson ‖, we can see the effect of various values for the spectral energy distribution of acetylene. The variations in the acetylene values are probably due to the different kinds of burners used, as Coblentz has shown that the spectral energy distribution in the longer wave-lengths is affected by the thickness of the radiating layer of incandescent particles in the flame.

The apparatus represented in fig. 1 is a modification of the Nutting monochromatic colorimeter ¶ as manufactured

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* Communicated by Dr. C. E. Kennett Mees, being communication No. 55 from the Research Laboratory of the Eastman Kodak Company.

† P. G. Nutting, Phil. Mag. xxix. p. 301 (1915); Trans. Illum. Eng. Soc. ix. p. 633 (1914).

‡ H. E. Ives, Phil. Mag. xxiv. p. 149 (1912).

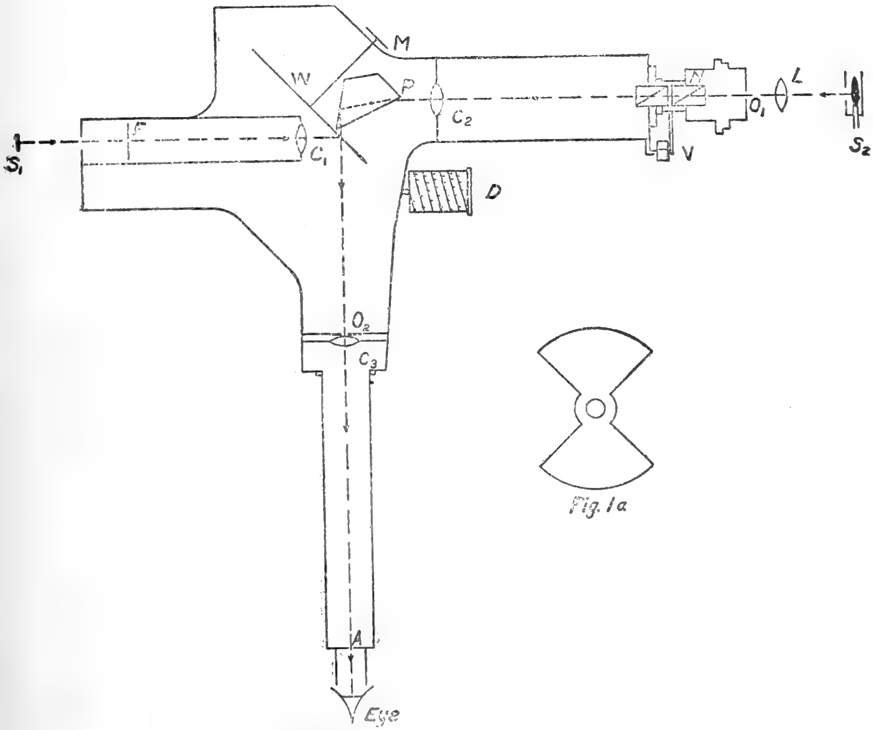
§ W. W. Coblentz, Bull. Bur. Stds. vii. p. 243 (1911); reprinted, ix. p. 109 (1912).

‖ W. W. Coblentz and W. B. Emerson, Bull. Bur. Stds. xiii. p. 1 (1916).

¶ P. G. Nutting, Bull. Bur. Stds. ix. p. 1 (1913); *Zsch. f. Instrumentenkund.*, xxxiii. p. 20 (1913).

by Adam Hilger of London. This type of apparatus has been described by Jones *, but the modifications made for its use in this experiment warrant a separate description in this paper. The light from an acetylene burner † S_2 is focussed on the slit O_1 by the lens L. The pressure of the gas was kept constant at 9 cm. as indicated on a water manometer, and the width of O_1 was determined by a series of preliminary experiments. A pair of nicol prisms N controlled the intensity of light from S_2, and by means of the vernier V and a graduated quadrant attached to the movable nicol we are able to determine the ratio of the incident to the transmitted intensities. C_2 is a collimating lens, P a constant deviation dispersing prism operated by a screw carrying a wave-length drum D, which indicates directly

Fig. 1.



Modification of Nutting Colorimeter.

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* L. A. Jones, Phys. Rev. iv. p. 454 (1914); Trans. I. E. S. ix. p. 687 (1914).

† L. A. Jones, Trans. I. E. S. ix. p. 716 (1914).

the quality of the light through this part of the system. S_1 is a gas-filled tungsten lamp the light from which passes through a daylight filter, F, and by means of the collimating lens C_1 strikes the matte surface on the Whitman disk W. This disk, shown in fig. 1 *a*, is rotated by a motor belted to M and turns so that at one instant a reflecting quadrant sends a beam of standard white light from S_1 to the eye, and the next instant a blank quadrant allows the coloured light through P to reach the eye. The eye sees the light image at O_2, which is screened down so as to restrict vision to the fovea, and an artificial pupil A was used.

The white light source S_1 was regulated so as to give an illumination of 13 foot candles at W, and was kept constant at this intensity by means of connexions through a Wolff potentiometer and a sensitive galvanometer. With both light sources constant and three independent series taken on different days, we may safely assume the resultant average curves to be representative of the observers.

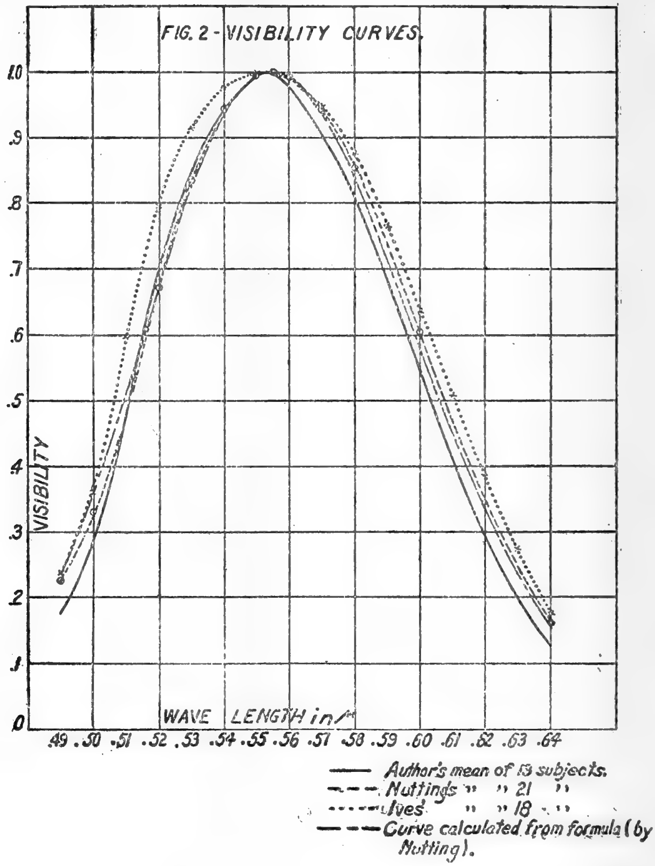
A constant width of slit was used throughout. In order to determine the necessary slit-width correction, the wave-length interval corresponding to the width of the slit, as one edge of the image is moved across the field, was determined throughout the spectrum by sighting on certain lines of mercury, hydrogen, and helium. The relative slit-width for any wave-length thus determined multiplied by the corresponding acetylene emission gave the value of the relative energy.

When taking a series of observations the necessary preliminary adjustments were made, the movable nicol N (fig. 1) was set for maximum intensity and the balance was made by shifting the wave-length drum. For observations between wave-lengths 500 and 680 the balance was made by setting the drum and moving the nicol. An electric tachometer was belted to the motor and the speed could be changed by variable resistance. As all observers were familiar with the theory of the so-called "critical frequency" in flicker balances, each observer regulated the speed, but no record was kept of these values. The relative energy for a given wave-length and the sine square of the angle read at V gives us the relative energy for equivalent luminosity. The values for the three independent series are averaged and plotted for each individual's curve. From this average curve the visibility curve for that observer was obtained, and the visibility curves were reduced to equal areas by weighting the ordinates according to height in order to compare separate curves and obtain the average curve of the group. These results are shown in Table I., and fig. 2 shows the

TABLE I.—Individual Visibility Results.

Subject.	λ·49	·50	·51	·52	·53	·54	·55	·56	·57	·58	·59	·60	·61	·62	·63	·64	λ max.
J.B.	·173	·253	·512	·740	·875	1·006	1·036	·986	·894	·803	·666	·536	·406	·269	·156	·110	·550
F.A.E.	·189	·382	·553	·712	·799	·880	·946	·955	·898	·799	·662	·558	·395	·287	·196	·116	·555
M.F.F.	·199	·303	·471	·706	·907	·996	1·018	1·005	·907	·778	·626	·502	·360	·261	·152	·101	·552
M.B.H.	·151	·212	·381	·674	·840	·936	·990	1·003	·891	·786	·681	·577	·464	·354	·225	·159	·556
K.H.	·153	·303	·466	·671	·808	·920	·990	·985	·941	·862	·722	·551	·395	·284	·200	·135	·554
L.A.J.	·206	·330	·491	·749	·864	·965	1·001	·982	·882	·764	·622	·518	·391	·276	·188	·105	·554
C.E.K.M.	·145	·268	·474	·773	·913	1·021	1·038	·962	·846	·723	·607	·505	·388	·283	·192	·117	·547
P.G.N.	·236	·318	·467	·716	·861	·984	·996	·986	·891	·785	·654	·519	·388	·282	·190	·120	·555
P.R.	·118	·209	·392	·594	·748	·867	936	·936	·890	·841	·775	·667	·494	·354	·242	·155	·555
F.E.R.	·216	·334	·503	·726	·862	·950	1·011	1·007	·931	·884	·652	·496	·361	·243	·141	·095	·553
R.G.S.	·153	·254	·406	·616	·749	·875	·935	·958	·935	·869	·766	·626	·474	·343	·239	·161	·560
A.F.W.	·187	·321	·580	·750	·857	·970	·994	·970	·882	·783	·650	·501	·374	·271	·193	·136	·552
R.B.W.	·151	·274	·485	·701	·865	·976	·980	·957	·887	·813	·700	·571	·427	·306	·211	·138	·548
Average	·175	·289	·475	·702	·842	·950	990	·977	·898	·807	·676	·548	·409	·293	·194	·127	·553

mean curve as compared with the curves of other writers. In Table II. are shown the results obtained by using the different values for the acetylene emission and the mean



from Nutting's and Ives's results, as well as the results computed from the formula offered by Nutting. This formula was found to represent visibility very closely between the wave-lengths of ·48 and ·67 and is of the form

$$V = Vm\,R^{a}e^{a(1-R)}$$

where $R=\lambda$ max./λ and $a=181$. When the writer used Coblentz's revised data for acetylene the results agreed remarkably well with these computed results.

In Table III. the acetylene values used by the writer and

Nutting are given, as well as the two sets of values published by Coblentz. The greatest differences in these values occur in the region of the longer wave-lengths and, as has been said, are probably due to the different types of burners used.

TABLE II.—Comparative Visibility Results.

Wave-Length.	Mean V from Table I.	Coblentz data publ. 1911.	Coblentz data publ. 1916.	Nutting's Mean.	Ives's Mean.	Computed from formula.
·49	·175	·172	·172	227	·235	·232
·50	·289	·283	·275	·330	·363	·358
·51	·475	·471	·474	·477	·596	·514
·52	·702	·705	·686	·671	·794	·675
·53	·842	·851	·841	·835	·912	·824
·54	·950	·947	·935	·944	·977	·933
·55	·990	·988	·993	·995	1·000	·994
·56	·977	·982	·985	·993	·990	·993
·57	·898	·926	·935	·944	·948	·939
·58	·807	·825	·836	·851	·875	·839
·59	·676	·693	·710	·735	·763	·717
·60	·548	·552	·580	·605	·635	·585
·61	·409	·417	·446	·468	·509	·456
·62	·293	·294	·319	·342	·387	·343
·63	·194	·185	·214	·247	·272	·235
·64	·127	·125	·140	·163	·175	·158

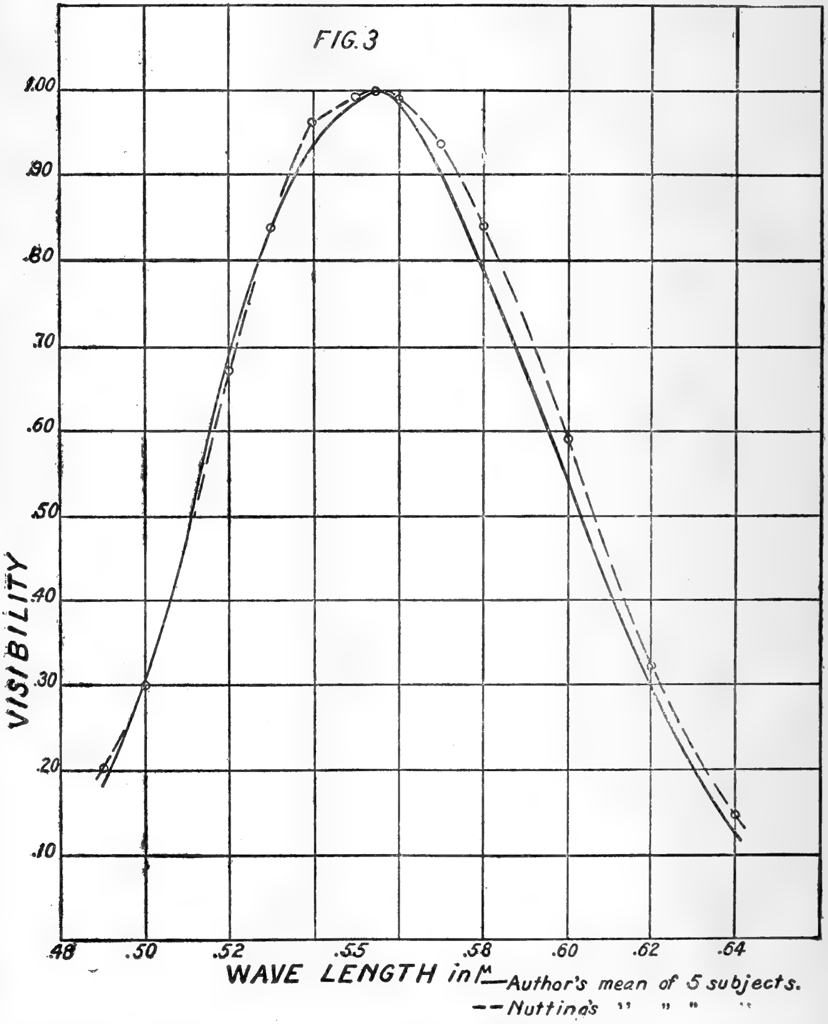
TABLE III.—Spectral Energy Distribution of an Acetylene Flame.

Wave-length in μ.	Nutting data.	Coblentz old data.	Coblentz new data.
·48	11·1	16·5	17·0
·50	14·4	21·7	21·9
·52	18·4	27·6	27·9
·54	23·2	34·8	35·0
·55	26·1	...	38·9
·56	29·1	43·7	42·9
·58	36·2	54·0	52·2
·60	44·2	66·3	62·1
·62	53·7	80·5	73·0
·64	63·8	96·5	84·7
·66	74·6	112·8	97·4
·68	86·1	130·1	110·9
·70	98·2	147·0	124·6

The spectral energy distribution of acetylene is probably better known than that of any other light source, and the burner used in this experiment gives an extremely constant quality of light *.

In comparison with the other curves shown in fig. 2, the writer's visibility curve for the thirteen subjects is slightly more contracted. The maximum visibility occurs at wave-length ·553 in agreement with Ives's curve, as against ·555 for Nutting's curve.

FIG. 3



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* Standardized burners may be obtained from the Research Laboratory, Eastman Kodak Company, Rochester, N.Y.

Fig. 3 shows the mean of the writer's results and Nutting's results on the five subjects who served as observers in both experiments. The average maximum visibility found by the writer is ·555, and by Nutting ·554.

This work was carried out at the suggestion of Dr. P. G. Nutting, and the writer wishes to thank him and the other members of the laboratory for their assistance.

Author's Note.—This paper had been completed before a paper by Coblentz and Emerson* appeared, so that their latest data have not been included.

XX. *On the Hodographic Treatment and the Energetics of undisturbed Planetary Motion. By* Professor ANDREW GRAY, *F.R.S.*†

1. THE following bit of Newtonian dynamics was suggested by some passages in a recent discussion in the Philosophical Magazine. It sets forth a mode of dealing with the elementary theory of planetary motion which may have some novelty, and states certain results, which I think are interesting, with regard to the energetics of such motion. Into problems of relativity I do not at present enter.

As a matter of scientific history there would seem to be no doubt that the first idea of the hodograph is due to August Ferdinand Möbius, the inventor of the barycentric calculus. In § 22 of his book, *Die Elemente der Mechanik des Himmels,* which was published in 1843, Möbius specifies a point H which moves so that its distance OH from a fixed point O is continually equal in length to, and in the same direction as, the line which represents the velocity of a particle moving in a given path. He derives the result that the acceleration of the particle in the path is represented in magnitude and direction by the velocity of the point H. This of course is the whole idea of the hodograph very clearly stated. So far as I have been able to see, Möbius does not make any particular use of the idea; and he does not seem to be aware that the curve described by the point H is, for a planet moving in a conic section, a circle, with O as an eccentric point within it.

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* Coblentz, W. W., Emerson, W. B. "Relative Sensibility of the Average Eye to Light of Different Colors and some Practical Applications to Radiation Problems." Scientific Paper 303, Bureau of Standards, issued Sept. 12, 1917.

† Communicated by the Author.

2. Apparently Hamilton was ignorant of the work of Möbius when he published his paper on "The Law of the Circular Hodograph" in the Proceedings of the Royal Irish Academy in 1847. More than ten years after, in a letter of date March 3, 1858, to De Morgan, he says:—"Do you know much about Möbius and his works? . . . He wrote to me a couple of years ago . . . that he had been lecturing on my theory of the circular hodograph, to which he might, very plausibly, have put in a sort of claim, or at least a claim to the general conception." [Graves, 'Life of Hamilton,' vol. iii. p. 543.]

That the hodograph for the motion of an undisturbed planet is a circle, was of course the great discovery, and without doubt this discovery was Hamilton's, and Hamilton's alone. That a name was given to the curve may have had something to do with the progress of discovery. When the curve had thus been endowed with a kind of personality, such questions naturally presented themselves as: What is the nature of the hodograph of a planet? What is the hodograph of an unresisted projectile? and so on. The names given to electric and magnetic quantities have certainly helped to distinguish sharply between one idea and another closely related to it, and led to the evaluation of the individualities which the names emphasized.

3. I was surprised to see Professor Eddington's statement in the Philosophical Magazine for October that the theorem of the resolution of the orbital velocity of a planet into two components of constant amount, one of them also in a constant direction, seemed to be overlooked in dynamical textbooks. It was given in Frost's 'Newton' in 1854, and is stated on p. 147 of the fourth edition. It is to be found in Routh's 'Dynamics of a Particle,' § 397, and in a book on Dynamics by Dr. J. G. Gray and myself, § 134. In the last mentioned work there is a rather full treatment of the subject referred to recently in the Phil. Mag.—the motion of bodies of varying mass.

The existence of these two components of velocity, one of constant amount v_1, say, and constant direction, the other of constant amount v_2, directed always forward at right angles to the radius vector, affords the most elegant mode of passing from the circular hodograph to the orbit. I shall show first that the hodograph is a circle, and then pass from the hodograph to the orbit.

Let the planet, denoted by P and supposed to be of unit mass, be acted on by a force directed towards a fixed point S, and varying inversely as the square of the distance SP. If $\dot{\theta}$

be the angular speed with which SP is turning about S, and $r = \text{SP}$, we have $r^2\dot{\theta} = h$, where h is a constant. Thus $\dot{\theta} = h/r^2$. From a centre C let a circle be described with some radius $2a$ in the plane of the path or in a parallel plane. Let a radius CH of the circle be always parallel to SP. The velocity of H is $2a\dot{\theta}$, and is therefore proportional to $1/r^2$. If the force per unit mass on the planet at distance r be μ/r^2, the acceleration is $\mu\dot{\theta}/h$, and so

$$\text{velocity of H} = \frac{2ah}{\mu} \cdot \frac{\mu}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Thus, to the factor $2ah/\mu$, the velocity of H represents the magnitude of the acceleration of the planet. Its direction is at right angles to SP, and represents the direction that PS would have if SP were turned 90° about P in the opposite direction to that in which SP turns, as P moves in the orbit.

Drawing the chord H_0H, which on the scale adopted represents the total change of velocity, between a previous point H_0, corresponding to a radius vector SP_0, and H, we see that, if we choose the proper point O in the plane of the circle, OH_0 and OH will represent the velocities at the beginning and end of the time t. It is clear that O must lie within the circle, for the vector OH, which represents the velocity, must turn through an angle 2π while the planet traverses the orbit once, which would not be the case if O were outside the circle.

The position of O must be independent of the value of t, otherwise the speed for the radius vector SP would depend on the choice of the initial radius vector SP_0. Thus O is a definite point. This result, taken along with the representation of the acceleration by the motion of a point in the circle, shows that the hodograph of the planet is a circle. The velocities are represented in magnitude and direction by the radii vectores drawn to the circle from the eccentric point O.

OH_0 is perpendicular to the direction of motion at P_0, and OH to the direction of motion at P. The members of the family of lines drawn from O to the sequence of points H are perpendiculars to the corresponding tangents to the path.

4. It is obvious that OH may be resolved into the two components OC, CH, that is into two components of fixed amounts, v_1, v_2, at right angles respectively to the line OC

and to the radius vector SP. We can now instantaneously find the orbit.

Let θ be the angle which SP makes with the direction OC. The angular momentum of the planet about S is $r(v_2+v_1\cos\theta)$, which has a constant value h. Thus if we write e for v_1/v_2, we get

$$r(1+e\cos\theta)=\frac{h}{v_2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

that is the path is a conic section of which S is a focus.

The length of the major axis is the sum of the lengths r_1, r_2 of r obtained by putting $\cos\theta=1$, $\cos\theta=-1$, respectively. If it is denoted by $2a$ we have

$$2a=\frac{h}{v_2}\left(\frac{1}{1+e}+\frac{1}{1-e}\right)=\frac{2h}{v_2(1-e^2)}. \quad . \quad . \quad . \quad (3)$$

Thus the perihelion and aphelion distances are $a(1-e)$ and $a(1+e)$.

5. It is convenient to make the radius of the hodograph equal to the $2a$ just found. The length of the line OH, representing the velocity, is then equal to twice the length of the perpendicular let fall from O on a line which is parallel to the tangent to the orbit at the corresponding position of the planet. In fact a circle of radius $2a$ described from S as centre serves very conveniently as hodograph. The hodographic origin O is then coincident with the empty focus, as will easily be seen from the fact that the perpendicular from the empty focus to any tangent of the ellipse at a point P intersects the radius vector SP on the circle.

6. As has been seen in (2) the two components v_1, v_2 of velocity give an instantaneous integration of the differential equations of the orbit. The radial differential equation

$$\ddot{r}-r\dot{\theta}^2=-\frac{\mu}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

shows this more clearly, and incidentally gives the value of h in terms of v_2. Since the component v_1 is at right angles to the major axis, and is directed towards the side of that axis to which the planet passes at perihelion, we have

$$\dot{r}=v_1\sin\theta, \quad \dot{\theta}=\frac{v_2+v_1\cos\theta}{r}, \quad . \quad . \quad . \quad . \quad (5)$$

where θ is the angle traversed by the radius vector from perihelion to the position considered. Hence

$$\ddot{r}=v_1\cos\theta\,.\,\dot{\theta}=v_1\cos\theta(v_2+v_1\cos\theta)/r,$$

and the equation of motion is

$$v_2(v_2+v_1\cos\theta)=\frac{\mu}{r},$$

or

$$r=\frac{\dfrac{\mu}{{v_2}^2}}{1+e\cos\theta}, \quad \ldots\ldots\ldots \quad (6)$$

the equation again of the path. But we have already obtained by considerations of the angular momentum the equation

$$r=\frac{\dfrac{h}{v_2}}{1+e\cos\theta}, \quad \ldots\ldots\ldots \quad (7)$$

and therefore we obtain the very simple and remarkable relations

$$h=\frac{\mu}{v_2}=\frac{e\mu}{v_1}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

that is the double rate of description of area by the radius vector is equal to the ratio of the "intensity of the centre" to the constant component of velocity at right angles to the radius vector.

The time t occupied in describing any part of an orbit is thus equal to $2\mathrm{A}v_2/\mu$, where A is the corresponding area swept over by the radius vector. This, expressed in terms of the focal radii and the chord of the arc described and the axes, is Lambert's theorem.

A steamer rounding a buoy, in a uniform tidal stream, with constant speed v_2 always directed at right angles to a line joining it with the buoy, describes a conic section with reference to the land. The curve is a hyperbola if v_2 is less than the speed v_1 of the stream, and an ellipse in the contrary case. The focal axis is at right angles to the stream, and $e=v_1/v_2$. [I believe that this illustration is originally due to Greenhill.]

When $\theta=0$, (7) becomes

$$a(1-e)(1+e)=h/v_2,$$

or (with $v_1/v_2=e$)

$$\left.\begin{aligned} v_2&=\frac{h}{a(1-e^2)}=\left\{\frac{\mu}{a(1-e^2)}\right\}^{\frac{1}{2}}=\left(\frac{\mu a}{b^2}\right)^{\frac{1}{2}},\\ v_1&=e\left\{\frac{\mu}{a(1-e^2)}\right\}^{\frac{1}{2}}.\end{aligned}\right\}\ldots \quad (9)$$

7. We deduce the energy equation as follows. The resultant speed v is given in terms of v_1, v_2 by

$$v^2 = v_1{}^2 + v_2{}^2 + 2v_1v_2 \cos\theta, \quad . \quad . \quad . \quad . \quad (10)$$

which by (6) and (9) can be written

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{2a}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The term μ/r represents the potential energy exhausted by the passage of the planet under the central attraction from infinity to the distance r. Writing the equation in the form

$$\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and taking $-\mu/r$ as the potential energy, we see that the kinetic and potential energies have a constant negative sum, $-\mu/2a$. From this equation we shall draw some conclusions which appear interesting. [It is to be understood that when $e > 1$, that is when the orbit is a hyperbola, the sum of the energies $\frac{1}{2}v^2$ and $-\mu/r$ is $+\mu/2a$.]

The constant angular momentum is $\mu/v_2 = e\mu/v_1 = b(\mu/a)^{\frac{1}{2}}$. Hence the period of revolution is

$$\mathrm{T} = 2\pi\frac{abv_2}{\mu} = 2\pi\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and is therefore independent of the eccentricity.

8. This gives an interesting instantaneous solution of the elementary problem of polar dynamics. *The orbital motion of a planet is annulled when the distance from the sun is* d; *find the time which the planet will take to fall to the centre of force.* Since the period is independent of the eccentricity, let e be less than but very nearly equal to 1. The foci are practically at the ends of the major axis. When the planet has just passed (not rounded) the aphelion end, the speed along the major axis is zero. The time taken to reach the other focus is half the period, and so the time taken by the planet to fall into the sun is $\pi(\frac{1}{8}d^3/\mu)^{\frac{1}{2}}$, as may be verified at once by direct integration.

The earth in the circumstances stated would fall into the sun in $\frac{1}{2}\mathrm{T}/2\sqrt{2}$, that is in about 65 days.

[It may be noticed incidentally that if we use the value of v^2 given on the right of (11), with the values of v_1, v_2 given in (9) and (10), and the values of the action and the period just found, we obtain after a little reduction the integral

$$\int_0^{2\pi} \frac{\cos\theta \, d\theta}{(1 + e\cos\theta)^2} = -2\pi\frac{e}{(1-e^2)^{\frac{3}{2}}}.]$$

Some years ago I came upon the examination question: *Prove that the action for a complete revolution of a planet in its orbit is independent of the eccentricity of the orbit.* The following proof of this interesting proposition presented itself: it is simple and I think elegant. The action, A, is given by the equation

$$A = \int v^2 dt = \int v\,ds, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where the space integration is taken round the orbit and the time integration for a complete period. Let p, p' be the lengths of the perpendiculars let fall from S and the empty focus on the tangent at P to the orbit. We have

$$pv = h = \mu/v_2, \quad \text{or} \quad v = (\mu/v_2 b^2)p' = (\mu/ab^2)^{\frac{1}{2}}p', \text{ by (9),}$$

Hence

$$A = \int v\,ds = \left(\frac{\mu}{ab^2}\right)^{\frac{1}{2}} \int p'ds = 2\pi(\mu a)^{\frac{1}{2}}, \quad . \quad (15)$$

since $\int p'ds$ is $2\pi ab$, twice the area of the (elliptic) orbit. The action is thus independent of the eccentricity, and there is no variation of the total action from one orbit to another, provided both possess the same major axis. The action may also be written as $2\pi b v_2$. [There is no difficulty in writing down an expression for the action in any finite part of the orbit.]

If T be the period we get by (13)

$$A = T\frac{\mu}{a} = 2T\frac{\mu}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This is the time integral of v^2. The time average of the kinetic energy in the orbit is $\mu/2a$.

From this proof * it appeared that the action is proportional to the area swept over in any time by the radius vector from the empty focus to the planet. Thus while the radius vector from the sun to the planet is the timekeeper, measuring as it does time in the orbit by the area it sweeps over, the other radius vector "keeps" the *action*. This proposition I found had already been stated by Professor Tait.

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* Another proof naturally occurs in which the integration is effected with the aid of the eccentric angle: but it is long and unsuggestive. Since this paper was sent in I have found a memoir by Grinwis, *Akad. van Wetens., Amsterdam,* ix. 1891–2, in which the proposition regarding the action is given. Probably this was the origin of the examination question.

9. The term μ/r in the energy equation is the potential energy exhausted when the planet is brought by the sun's attraction from infinity to the distance r. Thus we obtain by (11) the curious theorem that the time-average value of this term for a complete revolution of the planet is twice the time-average of the kinetic energy in the orbit, that in fact the time-average of this exhaustion of potential energy is equal to the action. This is also, of course, independent of the eccentricity.

As a particular case of this the kinetic energy of a planet in a circular orbit is half the potential energy exhausted in the journey from infinity. Hence also, if the planet were transferred from one circular orbit to one of (say) smaller radius, the increase of kinetic energy would be only one half of the additional potential energy exhausted in the passage.

This result for the *circular* orbit is of course well known; the corresponding relation which holds for the mean kinetic energy in an elliptic orbit was, I believe, first stated by myself in a letter in 'Nature,' August 7, 1913. The fixed ratio ($\frac{1}{2}$) of the mean orbital kinetic energy to the mean potential energy, exhausted to the different points of the orbit, is curious; but there is always a fixed ratio of the energy dissipated in the interactions of bodies to the whole available energy. For example, a body of mass m moving with speed v collides inelastically with a body of mass m' at rest; and the kinetic energy dissipated bears to the original kinetic energy the ratio $m'/(m+m')$, which is quite independent of the details of the action between the bodies.

The theorem for passage from one elliptic orbit to another is exactly parallel to that stated above for a circular orbit.

In the planetary case then, the bodies are only left moving in elliptic orbits, when the proper adjustment has taken place; others if left moving too slowly will fall into the central body, or if moving too quickly may recede from the central body to undergo further energy modifications by collision or otherwise.

Now let us compare a hyperbolic orbit with an elliptic orbit as regards this affair of energy. We have the theorem that the kinetic energy at distance r in a hyperbolic orbit exceeds, and in an elliptic orbit falls short of, the potential energy exhausted from infinity to this distance by the mean value of the kinetic energy in the orbit. This result has not been formally proved for the hyperbolic orbit, but has been inferred by analogy from the result for the ellipse. There is no difficulty in framing a formal proof, but the following may be sufficient.

At a considerable distance from the centre of force the kinetic energy in the hyperbolic orbit will have become practically constant and equal to $\mu/2a$. For r has become very great and μ/r very small, so that $\frac{1}{2}v^2=\mu/2a$. The planet therefore moves along the curve, ultimately along the asymptote, with more and more exactly constant speed, $(\mu/a)^{\frac{1}{2}}$, and as it continues at this speed, in its coming and its going, for an infinite time, the time-average of the kinetic energy is $\mu/2a$, as in the elliptic orbit.

It may seem a hard saying that the time-average of the kinetic energy in a parabolic orbit is zero, but in this case the energy equation is $\frac{1}{2}v^2=\mu/r$, and so v is very small when r is very large. Thus during an infinite time the value of $\frac{1}{2}v^2$ is the evanescent quantity μ/r, and thus the time-average of the kinetic energy is evanescent.

This theory leads to the result that the exhaustion of gravitational potential energy alone cannot have led to motion of a planet or comet in a hyperbolic orbit. It would seem that the necessary excess of mean kinetic energy must have been produced by some cataclysm within a body, from which the planet was thrown off after the sun had done the work of bringing the body within a finite distance r of the centre of force. Only by exhaustion of internal energy does it seem possible to make up the necessary additional energy.

The question of equipartition of energy between the different stars has been a good deal discussed. If a system of stars has its origin in the exhaustion of potential energy by the attraction of some great central system, the energy relations here discussed would appear to negative equipartition. If there is an approach to equipartition of kinetic energy of translational motion, and there is evidence apparently that the more massive stars move the slower, such an origin becomes on one more ground improbable.

The University, Glasgow,
Dec. 31, 1917.

XXI. *A Criticism of Wien's Distribution Law.*

By FRANK EDWIN WOOD [*].

1. *Introduction.*

THE purpose of this article is to criticise Wien's distribution law from a mathematical point of view :—To show (1) that although the derivation of Wien's distribution law is generally made by steps which are not mathematically justified, or for which no rigorous justification is given, still, by using Wien's assumptions [†], a rigorous derivation is possible; (2) that the law obtained by Wien is inconsistent with other results which follow from the same assumptions; and (3) that this inconsistency is eliminated and a new law [‡] obtained if Wien's implicit assumption be replaced by a simpler and more probable one.

Incidentally several theorems, new so far as the author knows, in the kinetic theory of gases will be obtained from the Maxwell law for the distribution, with respect to their velocities, of the molecules in a gas; a simple proof of the Wien displacement law will be obtained from these theorems and the Wien assumptions, and two interesting relations regarding the dependence of the radiation of a molecule upon its velocity will be given. Also there will be found some criticisms of the treatment of the distribution law and of the displacement law as given in the standard treatises.

Mendenhall and Saunders [§], Waidner and Burgess [‖], Rayleigh [¶] and others have criticised from a physical point of view the assumptions used by Wien to prove his distribution law, while Lummer and Pringsheim [**], Paschen [††], and others have considered the agreement of this law with experimental results. So far as the author knows, no criticism from a mathematical point of view has been published.

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[*] Communicated by the Author.

[†] It will be necessary to include under Wien's assumptions one which he has made implicitly, but not explicitly; or else to assume that he has made a fundamental error.

[‡] I am indebted to Professor Lunn, of the University of Chicago, for this new formula, and for the observation that Wien either made a mistake or an unstated assumption.

[§] Astrophysical Journal, xiii. p. 25 (1901).

[‖] Bull. Bur. of Standards, i. p. 189 (1904).

[¶] Phil. Mag. xlix. p. 539 (1900).

[**] *Verh. d. Deutsch. Phys. Ges.* i. p. 1 (1900).

[††] Astrophysical Journal, x. p. 40 (1899); xi. p. 288 (1900).

Wien's distribution law is

$$\phi(\lambda, \theta) = \frac{Ce^{c/\lambda\theta}}{\theta^5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\phi(\lambda, \theta)d\lambda$ represents the intensity of radiation of a black body at a temperature θ produced by waves whose lengths lie between λ and $\lambda + d\lambda$, and where C and c are constants [*]. M. Planck [†] obtained this same formula (and also another [‡]) but from considerations entirely different from those used by Wien. The criticism of this article does not apply to Planck's work; however, it has been suggested that a similar criticism might apply to Planck's derivation of this same law. Other formulas for $\phi(\lambda, \theta)$ have been obtained by Callendar [§] and Rayleigh [‖]; although these formulas may be in closer accord with experimental results, still the Wien formula has considerable importance due to its use by Drude and many other investigators.

2. *A derivation of the Distribution Law along the lines proposed by Wien.*

Wien takes a gas as the black body, and uses Maxwell's law that the number of molecules whose velocities lie between v and $v + dv$ is proportional to

$$v^2 e^{-v^2/\alpha^2} dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\alpha^2 = \frac{2}{3}\bar{v}^2$, and $\bar{v}$ is the root-mean-square velocity; α^2 is proportional to θ, the absolute temperature of the gas.

Wien makes the hypotheses:

(*a*) That the length of the wave sent out by a molecule depends only upon the velocity of that molecule:—then v is a function of λ only.

(*b*) That the intensity of the radiation for wave-lengths between λ and $\lambda + d\lambda$ is proportional to the number of molecules, as given by Maxwell's law (2), which send out waves with lengths between λ and $\lambda + d\lambda$.

Wien states that it follows from these two hypotheses that

$$\phi(\lambda, \theta) = F(\lambda) e^{-\frac{f(\lambda)}{\theta}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $F(\lambda)$ and $f(\lambda)$ are two unknown functions [¶].

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[*] Wied. *Ann.* lviii. p. 662 (1896).

[†] Wied. *Ann.* i. pp. 69, 719 (1900).

[‡] *Verh. d. Deutsch. Phys. Ges.* ii. p. 202 (1900).

[§] Phil. Mag. xxvi. pp. 787 (1913); xxvii. p. 870 (1914).

[‖] Phil. Mag. xlix. p. 539 (1900).

[¶] In our development of Wien's distribution law, it is assumed that $F(\lambda)$ and $f(\lambda)$ are continuous functions. Just what physical significance these assumptions have can be seen from § 6 of this article.

He then states: "Now the variation of the radiation with the temperature according to the law given by Boltzmann and myself consists of an increase of the total energy in proportion to the fourth power of the absolute temperature, and a variation of the length of the waves associated with an energy quantum and lying between λ and $\lambda+d\lambda$ in such a way that the corresponding wave-lengths are inversely proportional to the temperature. So if one plots for any one temperature the energy as a function of the wave-length, then for any other temperature this curve will be the same if the scale units of the graph are so varied that the ordinates are made smaller in the ratio $\frac{1}{\theta^4}$ * and the abscissæ are made larger in the ratio θ. This latter is possible for our value of $\phi(\lambda, \theta)$ only when λ and θ appear in the exponential as a product $\lambda\theta$. Then

$$\frac{f(\lambda)}{\theta}=\frac{c}{\lambda\theta}, \quad \dots\dots \quad (4)$$

where c denotes a constant."

Since I have found no simple proof of (4) from the above statements, I propose the following derivation of the form of $F(\lambda)$ and $f(\lambda)$ based entirely upon Wien's statements. Let C_1 be the curve obtained by plotting λ as abscissa and $y=\phi(\lambda, \theta)$ as ordinate for an arbitrary temperature θ_1; then the equation of C_1 will be

$$y=F(\lambda)e^{-\frac{f(\lambda)}{\theta_1}}.$$

Let C_2 be a corresponding curve for a temperature θ_2; then the equation of C_2 will be

$$y=F(\lambda)e^{-\frac{f(\lambda)}{\theta_2}}.$$

Now Wien's statement, as corrected, is that the transformation

$$y=\left(\frac{\theta_2}{\theta_1}\right)^5 y', \quad \dots\dots \quad (5)$$

$$\lambda=\frac{\theta_1}{\theta_2}\lambda', \quad \dots\dots \quad (5')$$

will transform C_2 into a curve congruent to C_1. This transformation gives as the equation of the transform of C_2

$$y'=\left(\frac{\theta_1}{\theta_2}\right)^5 F\left(\frac{\theta_1\lambda'}{\theta_2}\right)e^{\frac{-f\left(\frac{\theta_1\lambda'}{\theta_2}\right)}{\theta_2}} \quad \dots\dots \quad (6)$$

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* This ratio should be $\frac{1}{\theta^5}$; see Wien, *Berlin Sitzungsberichte*, vi. p. 55 1893); Lorentz, 'The Theory of Electrons,' p. 74.

Now if we make the transformation $y'=y$, $\lambda'=\lambda$, this curve with equation (6) coincides with C_1, and so

$$F(\lambda)e^{-\frac{f(\lambda)}{\theta_1}} \equiv \left(\frac{\theta_1}{\theta_2}\right)^5 F\left(\frac{\theta_1\lambda}{\theta_2}\right)e^{-\frac{1}{\theta_2}f\left(\frac{\theta_1\lambda}{\theta_2}\right)} \quad . \; . \; . \quad (7)$$

This is an identity for all values of θ_1, θ_2, and λ.

Now (7) can be written in the form

$$\left(\frac{\theta_1}{\theta_2}\right)^5 \frac{F\left(\frac{\theta_1\lambda}{\theta_2}\right)}{F(\lambda)} \equiv e^{\left\{\frac{1}{\theta_2}f\left(\frac{\theta_1\lambda}{\theta_2}\right)-\frac{1}{\theta_1}f(\lambda)\right\}} \quad . \; . \; . \quad (8)$$

If θ_1 and θ_2 be replaced by $k\theta_1$ and $k\theta_2$, where k is an arbitrary constant different from zero, the value of the left-hand member of (8) is unchanged, and therefore the value of the right member is also unchanged. This gives

$$e^{\left\{\frac{1}{\theta_2}f\left(\frac{\theta_1\lambda}{\theta_2}\right)-\frac{1}{\theta_1}f(\lambda)\right\}} \equiv e^{\frac{1}{k}\left\{\frac{1}{\theta_2}f\left(\frac{\theta_1\lambda}{\theta_2}\right)-\frac{1}{\theta_1}f(\lambda)\right\}},$$

which is an identity for all values of $k\neq 0$. Therefore

$$\frac{1}{\theta_2}f\left(\frac{\theta_1\lambda}{\theta_2}\right)-\frac{1}{\theta_1}f(\lambda)=0, \quad . \; . \; . \; . \; . \quad (9)$$

or

$$\frac{\left(\frac{\theta_1}{\theta_2}\right)f\left(\frac{\theta_1}{\theta_2}\lambda\right)}{f(\lambda)}=1. \quad . \; . \; . \; . \; . \; . \quad (10)$$

Substituting (9) in (8) gives

$$\frac{\left(\frac{\theta_1}{\theta_2}\right)^5 F\left(\frac{\theta_1}{\theta_2}\lambda\right)}{F(\lambda)}=1. \quad . \; . \; . \; . \; . \; . \quad (11)$$

In order to obtain the form of $F(\lambda)$ and $f(\lambda)$ we will prove the *Lemma*: *The most general solution of the functional equation*

$$\frac{k^a\phi(kx)}{\phi(x)}=1 \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

is $\phi(x)=\frac{C}{x^a}$, *where* k $(|k|\neq 0)$, C, *and* a *are constants. and* ϕ *is a continuous function of* x.

Let

$$A(x)=x^a\phi(x); \quad . \; . \; . \; . \; . \; . \; . \quad (13)$$

then $A(x)$ is a continuous function of x. From (12) and (13) it follows that

$$A(kx)=A(x). \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

Suppose first $|k|<1$; then the A function has the same value at the points $x, kx, \ldots\ldots k^n x, \ldots$ This set of points converges to the point $x=0$, and therefore $A(x)=\lim_{x=0} A(x)$; but since $A(x)$ is continuous

$$\lim_{x=0} A(x)=A(0), \text{ and } A(x)=A(0).$$

This is true for every value of x, and therefore $A(x)$ is a constant.

Now suppose $|k|>1$; make the transformation $kx=y$ in (14), then (14) becomes $A(y)=A\left(\frac{y}{k}\right)$, and therefore the function has the same value at the set of points $y, \frac{y}{k}, \ldots\ldots \frac{y}{k^n}, \ldots$ which converges to the point $x=0$, and therefore, by the same reasoning as for $|k|<1$, $A(x)$ is a constant.

Therefore $A(x)=C$ in every case, and by (13) $\phi(x)=\frac{C}{x^a}$, which proves the lemma.

Equations (10) and (11) are of the form (12), where $k=\frac{\theta_1}{\theta_2}$ and therefore

$$f(\lambda)=\frac{c}{\lambda}; \quad \ldots\ldots \quad (15)$$

$$F(\lambda)=\frac{C}{\lambda^5}. \quad \ldots\ldots\ldots \quad (15')$$

These values, substituted in (3), give Wien's distribution law.

3. *A criticism of Drude's proof of the Wien Displacement* and Distribution Laws.*

Equation (5) is the Wien displacement law and is generally derived from the Stefan-Boltzmann law, which states that

$$\int_0^\infty \frac{\phi(\lambda, \theta)}{\theta^4} d\lambda = \text{a const.}, \quad \ldots\ldots \quad (16)$$

or that the total radiation varies directly as the fourth power

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* The relation $\lambda\theta=$a const. is sometimes known as the Wien displacement law, and sometimes as a part of that law, but in this paper will be regarded as a distinct law. The equation $\lambda\theta=$a const. of itself means nothing, since λ and θ are independent. The equation is a shorthand way of stating that the radiation for $\lambda=\lambda_1$, $\theta=\theta_1$ will become the radiation for $\lambda=\lambda_2$ when $\theta=\theta_2$, where $\lambda_1\theta_1=\lambda_2\theta_2$. Wien obtains this relation by actually determining the change of wave-length, the temperature being increased by an adiabatic compression of the gas. No criticism of the derivation of this relation is intended in this article.

of the absolute temperature. Another form of the Stefan-Boltzmann law is that $\psi(\theta)=C\theta^4$, where $\psi(\theta)$ is the total radiation, and C is a constant.

Drude * attempts to prove a statement that is equivalent to (5), namely that

$$\frac{\phi(\lambda,\theta)}{\theta^5}=f(\lambda\theta), \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where f is an unknown function of $\lambda\theta$ alone. He states that (17) follows from

$$\frac{\psi(\theta)}{\theta^4}=\int_0^\infty \frac{\phi(\lambda,\theta)}{\theta^5}d(\lambda\theta), \quad . \quad . \quad . \quad . \quad (18)$$

an equation which is an immediate consequence of the definition of $\psi(\theta)$, viz.

$$\psi(\theta)=\int_0^\infty \phi(\lambda,\theta)d\lambda.$$

Now (17) cannot follow from (18) unless $\frac{\psi(\theta)}{\theta^4}=$ a const.; since this is true, by the Stefan-Boltzmann law, Drude's conclusion is not impossible. Suppose then that

$$\int_0^\infty \frac{\phi(\lambda,\theta)}{\theta^5}d(\lambda\theta)=\text{a const.} \quad . \quad . \quad . \quad (19)$$

Now (17) does not follow from (19) alone, as the following example will show. Let $\phi(\lambda,\theta)=\frac{\theta^4}{1+\lambda^2}$; then $\frac{\phi(\lambda,\theta)}{\theta^5}$ is not a function of $\lambda\theta$ alone; however,

$$\int_0^\infty \frac{\frac{\theta^4}{1+\lambda^2}}{\theta^5}d(\lambda\theta)=\int_0^\infty \frac{\theta}{\theta^3+\lambda^2\theta^2}d(\lambda\theta)=\frac{\pi}{2}\dagger.$$

Even if we put $\phi(\lambda,\theta)=F(\lambda)e^{-\frac{f(\lambda)}{\theta}}$, it is still possible to choose $F(\lambda)$ and $f(\lambda)$ so that (19) is satisfied, while (17) is not.

For if $F(\lambda)=\lambda^3$, $f(\lambda)=\lambda$, then $\frac{\phi(\lambda,\theta)}{\theta^5}$ is not a function of $\lambda\theta$ while

$$\int_0^\infty \frac{\lambda^3}{\theta^5}e^{-\frac{\lambda}{\theta}}d(\lambda\theta)=6\dagger.$$

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* *Lehrbuch der Optik,* p. 480 (1900). This treatment is repeated in the second edition.

† Pierce, 'A Short Table of Integrals,' formulas 480 and 493.

Therefore, Drude has not proved the displacement law; if, however, (17) be regarded as proved, then the remainder of Drude's argument for proving the distribution law can be made rigorous*.

Let
$$\frac{F(\lambda)}{\theta^5} e^{-\frac{f(\lambda)}{\theta}} \equiv \Phi(\lambda\theta). \,, \quad \ldots \ldots \quad (20)$$

where $F(\lambda), f(\lambda)$, and $\Phi(\lambda\theta)$ are unknown functions. Put
$$F(\lambda) = \frac{A(\lambda)}{\lambda^5}, \; f(\lambda) = -\frac{B(\lambda)}{\lambda};$$

then (20) becomes
$$A(\lambda) e^{\frac{B(\lambda)}{\lambda\theta}} \equiv \frac{1}{\lambda\theta} \Phi(\lambda\theta). \quad \ldots \ldots \quad (21)$$

Now if λ be replaced by $k\lambda$ and θ by $\frac{\theta}{k}$, the right-hand member of (21) is unchanged, and therefore
$$A(\lambda) e^{\frac{B(\lambda)}{\lambda\theta}} \equiv A(k\lambda) e^{\frac{B(k\lambda)}{\lambda\theta}}. \quad \ldots \ldots \quad (22)$$

If the expressions $e^{\frac{B(\lambda)}{\lambda\theta}}$, $e^{\frac{B(k\lambda)}{\lambda\theta}}$ be expanded, (22) becomes
$$A(\lambda) \left[1 + \frac{B(\lambda)}{\lambda\theta} + \frac{1}{2\,!} \frac{B^2(\lambda)}{(\lambda\theta)^2} + \ldots\right]$$
$$\equiv A(k\lambda) \left[1 + \frac{B(k\lambda)}{\lambda\theta} + \frac{1}{2\,!} \frac{B^2(k\lambda)}{\lambda\theta} + \ldots\right],$$

or
$$[A(\lambda) - A(k\lambda)] + \frac{1}{\lambda\theta}[A(\lambda)B(\lambda) - A(k\lambda)B(k\lambda)] + \ldots = 0. \quad (23)$$

Since (23) is an identity in θ,
$$A(\lambda) = A(k\lambda); \quad \ldots \ldots \quad (24)$$

and since (24) is an identity in k, $A(\lambda) = C$ where C is a constant. Also from (23)
$$A(\lambda)B(\lambda) - A(k\lambda)B(k\lambda) = 0,$$

from which $B(\lambda) = c$, where c is a constant; so
$$F(\lambda) = \frac{C}{\lambda^5}, \quad f(\lambda) = \frac{c}{\lambda}.$$

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* This derivation of the form of $F(\lambda)$ and $f(\lambda)$ is due to Professor Moulton, of Northwestern University.

4. *A proof of some auxiliary theorems in the Kinetic theory of Gases.*

Before giving our derivation of Wien's displacement law, it will be necessary to obtain some laws in the kinetic theory of gases. Let us consider the distribution of the molecules of a given homogeneous gas at a temperature θ_1 with regard to their velocities, v; if we plot v^2 as abscissa, and y, the corresponding number of molecules, as ordinate [*], then the curve C_1, which gives the distribution, will by Maxwell's [†] law have the equation $y=\frac{kv^2}{\theta_1^{3/2}}e^{-\frac{lv^2}{\theta_1}}$, where k and l are independent of v and θ_1. The equation of the corresponding curve C_2 for the temperature $\theta_2>\theta_1$ (this restriction is convenient, but unnecessary) upon the (V^2, Y) plane will be $Y=\frac{kV^2}{\theta_2^{3/2}}e^{-\frac{lV^2}{\theta_2}}$.

The transformation $V^2=\frac{\theta_2}{\theta_1}v^2$, $Y=\sqrt{\frac{\theta_1}{\theta_2}}y$ takes C_2 into C_1. Let v_1 and $\sqrt{\frac{\theta_2}{\theta_1}}v_1$ be called *corresponding velocities for the temperatures* θ_1 *and* θ_2 *respectively*; and the intervals $v_1\leq v\leq v_2$ and $\sqrt{\frac{\theta_1}{\theta_2}}v_1\leq v\leq\sqrt{\frac{\theta_2}{\theta_1}}v_2$ be called *corresponding velocity intervals for the temperatures* θ_1 *and* θ_2 *respectively.* Now

$$\int_{v_1^2}^{v_2^2}\frac{kv^2}{\theta_1^{3/2}}e^{-\frac{lv^2}{\theta_1}}d(v^2)=\sqrt{\frac{\theta_1}{\theta_2}}\int_{\frac{\theta_2v_1^2}{\theta_1}}^{\frac{\theta_2v_2^2}{\theta_1}}\frac{kV^2}{\theta_2^{3/2}}e^{-\frac{lV^2}{\theta_2}}d(V^2), \quad (25)$$

and (25) holds for all finite values of v_1 and v_2 and also for $v_1=0$, $v_2=\infty$, *i. e.*

$$\int_0^{\infty}\frac{kv^2}{\theta_1^{3/2}}e^{-\frac{lv^2}{\theta_1}}d(v^2)=\sqrt{\frac{\theta_1}{\theta_2}}\int_0^{\infty}\frac{kV^2}{\theta_2^{3/2}}e^{-\frac{lV^2}{\theta_2}}d(V^2). \quad (26)$$

It follows from (26) that the total area under the curve C_1 does not equal the total area under the curve C_2, as one might expect from the fact that in each case the area is proportional to the total number of molecules in the gas, a number which is unchanged by the rise in temperature.

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[*] Strictly speaking, there will be in general no molecule of the gas having a given velocity v at a given instant; however, if N molecules have their velocities within a very small interval δv about v, where the width of δv is preassigned, then we shall say for brevity that N molecules have the velocity v.

[†] Scientific Papers of James Clerk Maxwell, i. p. 381.

However, the areas associated with corresponding velocity intervals have the same ratio, by (25), as the ratio of the total areas, and so we have

Theorem I. The number of molecules in a gas at temperature θ_1 which have velocities in a given interval is equal to the number of molecules of this gas at temperature θ_2 having velocities in the corresponding velocity interval.

From this theorem we have the

Corollary. The number of molecules in a gas at temperature θ_1 which have velocities less than or equal to an arbitrary velocity v_1 is equal to the number of molecules in the same gas at temperature θ_2 which have velocities less than or equal to the corresponding velocity $\sqrt{\frac{\theta_2}{\theta_1}}v_1$ *.

The velocity of any particular molecule of this gas at temperature θ_1 is constantly changing, owing to impacts, etc.; so it is difficult to consider the change of velocity due to a rise of temperature alone. Let us consider an ideal gas in which the velocities of the various molecules are such that (1) the distribution of velocities obeys Maxwell's law for any temperature, and (2) the velocity of each molecule does not change while the temperature remains constant. We assume that the results obtained from a study of this ideal gas will hold also for an actual gas. By this device, we can consider a single molecule, or a group of molecules, at a given velocity instead of the succession of molecules, or groups, which have that velocity at various moments of time. Moreover, if the temperature of this gas be raised to θ_2, each molecule, in general, will have a new velocity, since the mean-square velocity has been changed, and the new distribution of velocities will also obey Maxwell's law. The new velocity of a molecule for given values of θ_1 and θ_2 will be a continuous function of the former velocity; also molecules whose velocities for $\theta=\theta_1$ are zero, will again have zero (or nearly zero) velocities for $\theta=\theta_2$. Then the molecules having velocities in the interval $0\leq v\leq v_1$ for $\theta=\theta_1$ must have their velocities in some interval $0\leq v\leq v_2$ for $\theta=\theta_2$. From

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* If the velocity at which the maximum number of molecules move be called the *mode*, then it follows from this theorem and corollary, (1) that the mode varies directly as the square root of the absolute temperature, (2) that when the temperature is increased, the number of molecules whose velocity is at the new mode is less than the number whose velocity was at the former mode, varying inversely as the temperature, and (3) that the fraction of the total number of molecules which have velocities less than or equal to the mode is independent of the temperature.

this and the Corollary to Theorem I. we have

Theorem II. The molecules having velocities in a given interval for $\theta=\theta_1$ will have their new velocities in the corresponding velocity interval for $\theta=\theta_2$.

If the given velocity interval be taken sufficiently small, we have

Theorem III. The velocities v_1 and v_2 of a molecule of a gas at temperatures θ_1 and θ_2 respectively, satisfy the relation

$$\frac{v_1}{v_2}=\frac{\sqrt{\theta_1}}{\sqrt{\theta_2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Theorem III. may be restated in this way:—If with certain units $v_1{}^2=\kappa\theta_1$, where v_1 is the velocity of a molecule of a gas at temperature θ_1 and κ is independent of θ_1 and v_1, then the velocity of this molecule when the temperature of the gas is θ_2 will be v_2 where $v_2{}^2=\kappa\theta_2$, and this relation holds for any velocity v_1 and any two temperatures θ_1 and θ_2. Consider two molecules m_1 and m_2 of a gas at temperature θ_1 having velocities v_1 and v_2 respectively, and producing waves of length λ_1 and λ_2 respectively; if the temperature be increased so that the velocity of m_1 is v_2, the temperature now being θ_2, then by Wien's first hypothesis the length, λ_2', of the waves produced by m_1 will be the same as the length, λ_2, of the waves produced by m_2 when the temperature was θ_1. Now, from (27) and the law $\lambda\theta=$a const., and since $\lambda_2=\lambda_2'$,

$$\frac{v_1}{v_2}=\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}}.$$

This proves

Theorem IV. The length of the waves produced by a molecule in a gas at temperature θ varies inversely as the square of the velocity; i. e.

$$v^2=\frac{b}{\lambda}, \text{ where } b \text{ is a constant.} \quad . \quad . \quad . \quad (28)$$

5. *A derivation of the Displacement Law from Wien's hypotheses.*

Let us consider the distribution of ϕ, the intensity of radiation from a gas at temperature θ_1 with regard to λ, the wave-length. Denote by K_1 the curve obtained by taking λ as the abscissa and ϕ as the ordinate. We will fix our attention upon the radiation associated with values of λ in the interval, $\lambda_1\leq\lambda\leq\lambda_1+\delta\lambda_1$, where λ_1 and $\delta\lambda_1$ are arbitrarily chosen (except that $\delta\lambda_1$ is very small). Suppose that the temperature be raised to θ_2 and that the corresponding curve be plotted. From the above theorems and the relation $\lambda\theta=\text{const.}$, it follows that the molecules which produced waves of length λ_1 will now produce waves of length $\frac{\theta_1}{\theta_2}\lambda_1$, and those which produced waves of length $\lambda_1+\delta\lambda_1$ will now produce waves of length $\frac{\theta_1}{\theta_2}(\lambda_1+\delta\lambda_1)$. Therefore the number of molecules producing waves with lengths in an interval with width $\delta\lambda_1$ about λ_1 in a gas at a temperature θ_1 is equal to the number producing waves with lengths in an interval with width $\frac{\theta_1}{\theta_2}\delta\lambda_1$ about $\frac{\theta_1}{\theta_2}\lambda_1$ when the temperature is θ_2.

Now $\phi(\lambda_1, \theta_1)\delta\lambda_1$ is the intensity of radiation carried by waves with lengths in the interval $\lambda_1 \leq \lambda \leq \lambda_1+\delta\lambda_1$ for $\theta=\theta_1$, and $\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)\frac{\theta_1}{\theta_2}\delta\lambda_1$ is the intensity of radiation carried by the waves with lengths in the interval $\frac{\theta_1}{\theta_2}\lambda_1 \leq \lambda \leq (\lambda_1+\delta\lambda_1)\frac{\theta_1}{\theta_2}$. When θ_1 and θ_2 are given, $\phi(\lambda_1, \theta_1)\delta\lambda_1$ and $\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)\frac{\theta_1}{\theta_2}\delta\lambda_1$ will be called *corresponding radiation elements*. In general, two corresponding radiation elements are not equal, since the total radiation increases when the temperature is increased (Stefan-Boltzmann law).

The intensity of radiation produced by a molecule is a function of the velocity of that molecule; then

$$\epsilon=g(v^2), \quad \ldots\ldots \quad (29)$$

where ϵ is the intensity of radiation produced by a molecule with velocity v and g is an unknown function of v^2 alone.

Therefore, using Wien's second hypothesis,

$$\frac{\phi(\lambda_1, \theta_1)\delta\lambda_1}{\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)\frac{\theta_1}{\theta_2}\delta\lambda_1}=\frac{\mathrm{N}g(v_1^2)}{\mathrm{N}g(v_2^2)}=\frac{g(v_1^2)}{g(v_2^2)}, \quad \ldots \quad (30)$$

where N is the number of molecules producing waves with lengths in the interval $\lambda_1 \leq \lambda \leq \lambda_1+\delta\lambda_1$, and v_1 and v_2 are the velocities of these molecules when the temperature is θ_1 and θ_2 respectively.

It follows from (30) and Theorem III. that

$$\frac{\phi(\lambda_1, \theta_1)\delta\lambda_1}{\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)\frac{\theta_1}{\theta_2}\delta\lambda_1}=\frac{g(\kappa\theta_1)}{g(\kappa\theta_2)} \quad \ldots \quad (31)$$

Therefore, *the ratio of corresponding radiation elements is a function of θ_1 and θ_2 respectively; in particular this ratio does not depend upon the value of λ about which one element is chosen.*

Since the area of each radiation element under K_2 is equal to a constant times the area of the corresponding radiation element under K_1, then the total area under K_2 is equal to the total area under K_1 multiplied by this constant; but by the Stefan-Boltzmann law this constant is $\left(\frac{\theta_1}{\theta_2}\right)^4$. Therefore

$$\frac{\phi(\lambda_1, \theta_1)\delta\lambda_1}{\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)\frac{\theta_1}{\theta_2}\delta\lambda_1} = \frac{\theta_1^4}{\theta_2^4},^* \quad . \quad . \quad . \quad . \quad (32)$$

and therefore

$$\frac{\phi(\lambda_1, \theta_1)}{\theta_1^5} = \frac{\phi\left(\frac{\theta_1}{\theta_2}\lambda_1, \theta_2\right)}{\theta_2^5}.$$

This is another form of equation (5), and our derivation of the displacement law is complete.

From (31) and (32) it follows that

$$\frac{g(\kappa\theta_1)}{g(\kappa\theta_2)} = \frac{\theta_1^4}{\theta_2^4},$$

from which

$$\frac{g(\kappa\theta_1)}{\theta_1^4} = \frac{g(\kappa\theta_2)}{\theta_2^4} = c,$$

where c is a constant. Then

$$g(\kappa\theta) = c\theta^4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and from (29) it follows that $\epsilon = c^2v^8$; i. e. *the intensity of radiation given off by a molecule varies directly as the eighth power of its velocity.*

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[*] In Wien's first article upon the displacement law he assumed that (32) followed immediately from the Stefan-Boltzmann law; in his second article the error is corrected, but the derivation given is difficult, even though rigorous, so most authors have followed Wien's first method and have not remedied the errors: *cf.* Wood, Physical Optics, p. 620; Sachur, *Thermochemie und Thermodynamik*, p. 302; Winkelmann, *Handbuch der Physik*, iii. pp. 379–80.

6. *The inconsistency in the Wien Distribution Law, and the revised form.*

Let us consider the method by which Wien obtained (**3**) from his two hypotheses. From the first hypothesis one can write

$$v=m(\lambda), \quad \ldots\ldots \quad (34)$$

where m is an unknown function of λ *.

If $$\epsilon=\rho(\lambda) \quad \ldots\ldots \quad (35)$$

denote the intensity of radiation produced by a molecule giving off waves of length λ, then by using (34), (35), and Wien's second hypothesis

$$\phi(\lambda,\theta)=k_1\rho(\lambda)m^2(\lambda)m'(\lambda)e^{-\frac{k_2m^2(\lambda)}{\theta}},$$

where k_1 and k_2 are constants and $m'(\lambda)=\frac{d}{d\lambda}m(\lambda)$. So Wien has written for brevity

$$F(\lambda)=k_1\rho(\lambda)m^2(\lambda)m'(\lambda); \quad \ldots \quad (36)$$

$$f(\lambda)=-k_2m^2(\lambda). \quad \ldots\ldots \quad (37)$$

Now, from (37) and (15)

$$m(\lambda)=\sqrt{\frac{-c}{k_2\lambda}}; \quad \ldots\ldots \quad (38)$$

and from (28), (29), (33), and (35)

$$\rho(\lambda)=\frac{a}{\lambda^4}, \quad \ldots\ldots \quad (39)$$

where a is a constant. Substituting (38) and (39) in (36) gives

$$F(\lambda)=\frac{A}{\lambda^{13/2}}, \quad \ldots\ldots \quad (40)$$

where A is a constant depending upon the constants a, c, k_1, and k_2. But (40) is inconsistent with (15′). Therefore, *the Wien formula is inconsistent with other results obtained from the same hypotheses.*

Now by the Maxwell law †, the number of molecules having velocities between v and $v+dv$ is

$$\frac{4}{\sqrt{\pi}}N\frac{v^2}{\alpha^3}e^{-\frac{v^2}{\alpha^2}}dv, \quad \ldots\ldots \quad (41)$$

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* However, the form of $m(\lambda)$ has already been obtained in this article from the relation $\lambda\theta$=a const., and certain theorems in the kinetic theory of gases; see Theorem IV. That this form agrees with the form obtained by Wien's method is an agreeable fact.

† Scientific Papers of James Clerk Maxwell, i. p. 381.

where N is the total number of molecules in the gas, and α^2 is proportional to the absolute temperature θ—see § 2. If one assumes that N does not change with θ [*], then (3) does not follow from Wien's assumptions, but instead

$$\phi(\lambda, \theta) = \frac{F(\lambda)}{\theta^{3/2}} e^{-\frac{f(\lambda)}{\theta}} . \quad . \quad . \quad . \quad . \quad (42)$$

If (3) be replaced by (42) and the argument given in § 1 be repeated, it will be found that

$$F(\lambda) = \frac{A}{\lambda^{13/2}} ; \quad f(\lambda) = \frac{c}{\lambda} ; \quad . \quad . \quad . \quad . \quad (43)$$

where A and c are constants.

Moreover (43) is consistent with (40) and (27), and so *Wien's assumptions lead to the law*

$$\phi(\lambda, \theta) = \frac{C}{\lambda^{13/2}} e^{-\frac{c}{\lambda\theta}} . \quad . \quad . \quad . \quad . \quad , \quad (44)$$

Evanston, Illinois,
July 5, 1917.

XXII. *Coupled Circuits and Mechanical Analogies. By* E. H. Barton, *D.Sc., F.R.S., and* H. M. Browning, *B.Sc.*[†]

1. As the valuable "Note on the Action of Coupled Circuits and their Mechanical Analogies" by Prof. H. C. Plummer[‡] seems in places to imply a slight misunderstanding of a previous one dealing with the same subject, a brief reply appears desirable.

Prof. Plummer regards the matter chiefly from the mathematical standpoint, whereas the authors of the October paper[§] were concerned chiefly with the physical phenomena and their visible representation to average electrical students (see pp. 246–7). And in some colleges not one per cent. of these are masters of the problem mathematically.

2. Prof. Plummer expresses "a doubt whether a simple electrical problem really is made easier for the average student by a complicated mechanical analogy" (p. 510). It was never intended that the *mathematics* of the mechanical

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[*] If $N = \sigma(\theta)$, where σ is an undetermined function, and if Theorem IV. § 4 be regarded as valid, then it can be proved that $\sigma(\theta) =$ a const. Wien's implicit assumption is that $N = c'\theta^{3/2}$, where c' is a constant.

[†] Communicated by the Authors.

[‡] Phil. Mag. pp. 510–517, vol. xxxiv., Dec. 1917.

[§] Phil. Mag. pp. 246–270, vol. xxxiv., Oct. 1917.

analogy should explain or render easier that of the electrical case. In its mathematical aspect the electrical case is probably simpler than that of any mechanical model yet put forward.

The utility of the mechanical analogies given in the October paper lies in their power to give with such simple apparatus actual traces of the vibrations in question, and that under the most various conditions as to masses of bobs, lengths of the separate pendulums, and looseness or closeness of their coupling. And these advantages can be reaped by students who are too weak to assimilate the mathematics either of the original problem or its analogies, though for completeness' sake the equations of both were naturally included in the paper.

There is also another aspect of the matter. For it may be hoped that the study of these mechanical models may throw some light on the hidden mechanism of the electromagnetic phenomena.

3. Equations (27) and (28) of the October paper for the double-cord pendulum were in the form

$$\left.\begin{aligned} \mathrm{P}\frac{d^2y}{dt^2}+\frac{\mathrm{P}+\mathrm{Q}+\beta\mathrm{Q}}{(1+\beta)(\mathrm{P}+\mathrm{Q})}\mathrm{P}\frac{g}{l}y &= \frac{\beta}{1+\beta}\cdot\frac{\mathrm{PQ}}{\mathrm{P}+\mathrm{Q}}\cdot\frac{g}{l}\cdot z,\\ \mathrm{Q}\frac{d^2z}{dt^2}+\frac{\mathrm{P}+\beta\mathrm{P}+\mathrm{Q}}{(1+\beta)(\mathrm{P}+\mathrm{Q})}\mathrm{Q}\frac{g}{l}z &= \frac{\beta}{1+\beta}\cdot\frac{\mathrm{PQ}}{\mathrm{P}+\mathrm{Q}}\cdot\frac{g}{l}\cdot y.\end{aligned}\right\}\quad .\ .\quad (1)$$

Prof. Plummer prefers the form

$$\left.\begin{aligned} (\mathrm{P}+\mathrm{Q}+\beta\mathrm{P})\frac{d^2y}{dt^2}+(\mathrm{P}+\mathrm{Q})gl^{-1}y &= -\beta\mathrm{Q}\frac{d^2z}{dt^2},\\ (\mathrm{P}+\mathrm{Q}+\beta\mathrm{Q})\frac{d^2z}{dt^2}+(\mathrm{P}+\mathrm{Q})gl^{-1}z &= -\beta\mathrm{P}\frac{d^2y}{dt^2};\end{aligned}\right\}\quad .\ .\ .\quad (2)$$

and adds "the analogy is now exact." Is not this too much to claim? As to the occurrence of the variables and their derivatives the equations (2) do indeed present a formal agreement with those for the electrical circuits, viz.,

$$\left.\begin{aligned} \mathrm{L}\frac{d^2y}{dt^2}+\frac{y}{\mathrm{R}} &= \mathrm{M}\frac{d^2z}{dt^2},\\ \mathrm{N}\frac{d^2z}{dt^2}+\frac{z}{\mathrm{S}} &= \mathrm{M}\frac{d^2y}{dt^2}.\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\quad (3)$$

Hence in a certain restricted mathematical sense the analogy is exact. But this exactness of analogy does not extend to the coefficients and their physical significance. Thus in (3) for the electrical case it will be seen that the same coefficient occurs on the right side of each equation. This is M the coefficient of mutual induction. Also the first coefficients at the left of each equation respectively are L and N the separate self-inductions. Now this correspondence of coefficients does *not* hold between (3) and (2) but *does* hold between (3) and (1). And for this reason the form (1) is probably preferable to the experimental physicist, though the other form (2) is distinctly illuminating and may be preferred by the mathematician.

4. That the pendulums represented by equations (1) and (2) are not in the complete sense an exact analogy to the electrical case of (3) may also be seen from the relations of the frequencies of the coupled vibrations in the two cases. Suppose the two separate vibrations for pendulums or those for the electrical circuits to be equal and denote them by cos mt. Let the superposed coupled vibrations for each system be denoted by cos pt and cos qt. Then, for the electrical case, we have

$$p > m > q.$$

Whereas, for the mechanical analogy, we have

$$p = m, \text{ and } m > q.$$

5. What would seem to the writers to be an exact mechanical analogy to the electrical case would be one capable of representation as follows:

$$\left.\begin{aligned} \mathrm{P}\frac{d^2y}{dt^2} + \mathrm{P}m^2y &= \mathrm{J}\,\frac{d^2z}{dt^2}, \\ \mathrm{Q}\frac{d^2z}{dt^2} + \mathrm{Q}n^2z &= \mathrm{J}\,\frac{d^2y}{dt^2}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In these P and Q denote the masses which vibrate in the two systems. Their separate vibrations are to be obtained by writing $\mathrm{J}=0$. Thus giving as their separate vibrations cos mt and cos nt respectively. Further, it should be noted that in the above we are supposing that the introduction of the cross-connexion terms on the right has not modified the coefficients on the left. A model fulfilling these conditions seems to be still a desideratum. In these equations it may be seen by the theory of dimensions that J must be a mass like P and Q.

6. Prof. Plummer appears to consider it a mistake to regard as a mass the mutual induction M of the electrical case.

But does not the current view regard the mutual induction as an inertia factor of some sort? Thus in Sir J. J. Thomson's model referred to in the October paper (p. 251) both self and mutual inductions are represented by masses. In Prof. J. A. Fleming's 'Alternate Current Transformer' (vol. i. pp. 97–98, 1889), the electrical energy $\frac{1}{2}Li^2$ is likened to the mechanical energy of rotation $\frac{1}{2}I\omega^2$. Again, in Sir Oliver Lodge's 'Modern Views of Electricity' (p. 496, 1907), coefficient of induction (self or mutual) is given as inertia per unit area.

It is true that the coupling in the electrical case is made by a change of configuration which fixes the value of M. But this does not prevent M from being a mass (*i. e.*, an inertia) like the inductances L and N which are also dependent solely on configurations, provided no iron or other magnetizable substances are present. As to whether the coefficient M in the electrical equations is to be represented by a mass in any one mechanical model incompletely analogous to it, is another matter.

Nottingham,
Dec. 17, 1917.

XXIII. *The Radioactivity of Archæan Rocks from the Mysore State, South India.* *By* W. F. Smeeth, *D.Sc., A.R.S.M.*, *and* H. E. Watson, *D.Sc., A.I.C.*[*]

Preliminary Investigation.

THIS investigation was started some years ago on a number of samples of the hornblendic schists of the Kolar Gold Field, selected by Mr. H. M. A. Cooke, Superintendent of the Ooregum Gold Mining Company.

The samples were taken from the Kolar mines at different depths, with a view to ascertaining whether the radium content varied with the depth from the surface in rock of fairly uniform character and composition. These hornblendic schists and epidiorites are all ancient lava flows, or sills, of fairly uniform composition, notwithstanding petrological distinctions in texture and structure.

[*] Communicated by the Authors.

An account of the method used and results obtained was published in the Philosophical Magazine (6) xxviii. p. 44, 1914, and the results are repeated in Table I., Nos. 1 to 15. It will be seen that the radium content is very low, remarkably constant, and that there is no variation in depth down to a vertical depth of some 3500 feet from the surface.

Two of the samples—Nos. 12 and 15—gave results considerably higher than the others, but microscopic examination showed that these samples did not represent normal types of the hornblendic schists or "country" of the mines, but had undergone considerable mineral alteration, such as is common in the immediate vicinity of the quartz veins or other acid intrusives, and there is no doubt that the higher values are due to the intrusion of acid material of higher radium content.

Further Investigation.

It was then decided to obtain a number of representative samples from the various components of the Archæan complex of Mysore, in order to ascertain how far the various formations or groups might be distinguishable from one another by their radioactivity, and what variations existed amongst the members of each group as a result of magmatic segregation. The experimental procedure was the same as before, viz., 10 gms. of the finely powdered rock were fused with potassium hydroxide, under reduced pressure, and the resulting gases led to an electroscope after removal of the hydrogen and drying. Towards the middle of the experiments the leaf system in the electroscope broke down, and was replaced by a smaller and more sensitive one, which was subsequently carefully standardized. With this a leak of 1 scale division an hour corresponded to $1{\cdot}67 \times 10^{-13}$ gm. of radium. Control experiments showed no discontinuity between the two series of values obtained. All experimental details have already been given (*loc. cit.*), and will not be repeated. Altogether, fifty samples have been selected from specimens in the Department of Mines and Geology of Mysore and the radium determined, but each group is itself so complex and variable that a much larger number would be required before fair averages or estimates could be obtained. In spite of this, certain interesting variations appear to be indicated, and the results obtained have been grouped, in Table I., under the various formations taken in order of age from the oldest to the youngest.

The following classification is inserted for convenience of reference, and shows the order of succession and relationship of the various formations in Mysore as at present adopted by the Mysore Geological Survey*.

Classification of Mysore Rocks.

1. Recent soils and gravels.
Possibly Tertiary. 2. Laterite. Horizontal sheet capping Archæans.
Pre-Cambrian (Animikean) } 3. Basic dykes. Chiefly various dolerites.

Great Eparchæan Interval.

Archæan. {
4. Felsite and porphyry dykes.
5. Closepet granite and other massifs of corresponding age.
6. Charnockite, norite, and pyroxenite dykes.
7. Charnockite massifs. (Complex.)
8. Various hornblendic and pyroxene-granulite dykes.
9. Peninsular gneiss. (Granite and gneissic complex.)
10. Champion gneiss. (Granite porphyry, micaceous gneisses, felsites and quartz porphyries; usually containing opalescent quartz and frequently associated with autoclastic conglomerates.)

Eruptive Unconformity.

Dharwar System (probably Keewatin). {
11. Upper (chloritic) division (greenstones and chloritic schists).
12. Lower (hornblendic) division (epidiorites and hornblendic schists).
} { Including also:—
Amphibolites, peridotites, &c., mostly intrusive.
Conglomerates (autoclastic).
Banded ferruginous quartzites, origin doubtful, possibly igneous.
Quartzites and quartz-schists, mostly intrusive.
Limestones, probably secondary.
Mica schists; metamorphic igneous.
Intrusive masses of dioritic and diabasic character.

Unknown.

We may now very briefly consider the groups of figures presented in Table I., and the following summary of the results will help to bring out such points of similarity or distinction as exist amongst them, although the observations are too few in number to permit of final conclusions being drawn.

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* 'Outline of the Geological History of Mysore,' by W. F. Smeeth, Bulletin No. 6—Department of Mines and Geology, Mysore State.

(Numbers in brackets refer to the classification given above.)

	Group.	Radium, gm. per 10^{12} gm. of rock. Limits.
Dharwar	Hornblendic series (12)	0·14 to 0·25
	,, ,, altered types	0·34 ,, 0·96
	Chloritic series (11)	0·20 ,, 0·54
	Basic intrusives (11) and (12)	0·05 ,, 0·16
	Champion gneiss (10)	0·85 to 1·45
	,, ,, auriferous quartz	1·28
	Peninsular gneiss (9)	0·40 to 1·50
	,, ,, pegmatites	1·44 to 6·90
	Charnockites (7)	0·04 to 0·12
	,, hypersthenite (6)	0·06
	,, quartz-magnetite ore	0·08
	Closepet granite (5)	0·27 to 2·14
	,, ,, porphyries (4)	1·37 ,, 2·42
	Dolerite dykes (Post-Archæan) (3)	0·45

The hornblendic rocks of the Dharwar System (Nos. 1–18) are low in radium and exhibit no great variation from the mean, though many petrological types are included, such as hornblende schists, hornblende diabases, amphibolite, and hornblende granulite. When, however, these rocks are altered in contact with intrusions of the Champion gneiss and of the related quartz veins of the Kolar Field (Nos. 26–30), all of which contain much more radium than the normal schists, the radioactivity of the altered types is considerably increased (Nos. 12, 15, and 16) to a point intermediate between the radioactivities of the two reacting masses.

The rocks of the Chloritic series (Nos. 19–21) do not appear to differ much in radium from those of the hornblendic series. The higher value in No. 21 may possibly be the result of alteration due to a neighbouring granitic intrusion.

The basic intrusives of Dharwar age—that is to say, intrusives into the general body of the Dharwar schists prior to the period of the Peninsular gneiss—contain much less radium than even the schists themselves. This is particularly noticeable in the Bellara trap (No. 23) and the Grey trap of Chitaldrug (No. 24), and these rocks afford an interesting example of the possible use of such determinations in the correlation of these very old and much altered Archæan types. Some years ago the Grey trap was considered to be a modification of the less altered Bellara trap, but subsequent work showed that the Santaveri trap of the Kadur District possessed a considerable resemblance to the Grey trap of Chitaldrug, so much so that it had almost been decided to class these two formations together and separate them from the Bellara trap. The radium determinations, however, show that the Grey trap and the Bellara trap have practically the same radioactivity (*vide* Nos. 23 and 24), which is about one third of that of any of the schists proper, while the Santaveri trap (No. 19) contains about three times as much radium and falls within the limits so far ascertained for the schists. This appears to confirm the original classification, which correlated the Grey trap with the Bellara trap, and which, consequently, has been allowed to stand. The case affords an illustration of the possible use of radium determinations as an aid to correlation of the highly metamorphosed members of the Archæan complex, amongst which it may often happen that the chemical and mineral composition, and the field relationships, do not afford sufficiently definite points of similarity or distinction. The value of such determinations will depend on the possibility of ascertaining fairly definite limits for the radium contents of the various rock groups or of the various members of such groups.

Amongst the various gneisses and granites, which have been divided into four great groups of distinctly different ages, it will be noted that the Charnockites stand apart from the others in virtue of their excessively low radioactivity, which is much lower even than that of the Dharwar schists.

The Champion gneiss, Peninsular gneiss and the Closepet granite—which last is also a variable complex—all contain from 12 to 15 times as much radium as the Charnockites and four to five times as much as the Dharwar schists. The Charnockites have been shown by Holland to form a distinct petrographical province amongst the gneisses of Southern India and vary, as the result of magmatic segregation, from highly acid granites to norites and hypersthenites—all characterized by the presence of hypersthene and certain physical features—and the radium determinations fully confirm the distinct individuality of the parent magma. The varieties of Charnockite which have been examined show an increase of radium with increasing basicity, but the hypersthenite and quartz-magnetite ore—which are considered to be end products of the segregative process—show a relapse towards the mean value.

The other gneisses and granites are very complex, and the determinations are not sufficiently numerous to permit of very definite conclusions. It may be noted that the members of the Champion gneiss series show least variation, those of the Peninsular gneiss rather more, and those of the Closepet granite the greatest variation.

The auriferous quartz of the Kolar Field (No. 29) is interesting as falling into line with the members of the Champion gneiss series, with which it has been correlated on other grounds, and, as already pointed out, the altered schists (lode matter) in contact with, or in continuation of, the auriferous quartz veins show values intermediate between those of the normal schists and of the quartz.

It is interesting also to note that the matrix of the Conglomerate (No. 27) has the same radioactivity as the clearly intrusive granite (No. 28), and this is in accordance with the view that the conglomerate is autoclastic and due to crushing of portions of the Champion gneiss series.

The pegmatite cross-course (No. 39) is remarkable as yielding the highest result so far obtained. The pegmatite contains a large amount of tourmaline, and it was thought that this mineral might account for the high value. A small quantity of the tourmaline was separated and, though a definite determination was not made, the test was sufficient to show that it was not abnormally high, and that the high result of the rock as a whole was not due to this mineral.

A single determination (No. 50) has been made of one of the very numerous dolerite dykes which are considered to be of Pre-Cambrian age but subsequent to the formation and folding of the Archæan complex. This rock is very similar in composition to the old hornblendic schists, which probably were originally diabasic flows and sills of a much earlier period. The result shows that the later rock contains more than twice as much radium as the earlier type, but no further inference can be drawn from a single observation.

Summary.

1. These very ancient rocks, all of which are considered to be of igneous origin, contain remarkably little radium.

2. Amongst the various groups which have been differentiated on geological grounds, there are some striking differences in the radium contents of some of them.

3. In the case of a fairly uniform group of rocks (viz., the hornblendic schists of the Kolar Field) the radium content does not appear to vary with the depth from the surface.

4. Different igneous magmas appear to contain very different amounts of radium, and the latter, or the minerals

which carry it, is subject to magmatic segregation. The amount of radium in the segregated portions of a magma sometimes increases and sometimes decreases with increase of basicity.

5. Amongst magmas, the more basic appear to be lower in radium than the more acid, and, in the products of granitic magmas, the pegmatites appear to carry more radium than the corresponding granites. The Charnockite magma, which was probably of intermediate composition, forms a striking exception, and is notable for its extremely low radioactivity.

6. In the case of rocks of somewhat similar character and composition, and in which other means of distinction or identification are lacking, a marked difference in the radium contents may afford a means of correlating them with known groups or formations for which the radium limits have been sufficiently determined.

Bangalore, March 1917.

Table I.

Radioactivity of Rocks from the Mysore State.

(The rock groups are arranged in order of age from the oldest to the youngest. The radium is given in units of 10^{-12} gramme per gramme of rock.)

Serial No.	Registered No.	Description.	Radium.	Remarks.
		Group 1. Dharwar System—Lower (hornblendic) series.		
		ft.		
1.		Balaghat Mine — depth 1000	0·21	(The first 15 samples are all hornblende schists and epidiorites from the Kolar Gold Field. The depths are measured on the dip of the lode.)
2.		,, ,, ,, 2000	0·14	
3.		,, ,, ,, 3000	0·21	
4.		Nundydroog Mine ,, 1000	0·17	
5.		,, ,, ,, 3000	0·14	
6.		Ooregum Mine ,, 1000	0·18	
7.		,, ,, ,, 2000	0·17	
8.		,, ,, ,, 4000	0·21	
9.		Champion Reef Mine ,, 2000	0·24	
10.		,, ,, ,, ,, 3000	0·18	
11.		,, ,, ,, ,, 4000	0·14	
12.		Mysore Mine ,, 1000	0·34	Altered schist.
13.		,, ,, ,, 2000	0·21	
14.		,, ,, ,, 3000	0·23	
15.		,, ,, ,, 4000	0·96	Much altered schist, part of the lode formation; contact alteration close to quartz of lode.
16.	S2/377	Amphibolite, footwall of lode, Ooregum.	0·82	Contact alteration.
17.	Z3/510	Fine hornblende schist, Bababudan Hills, Kadur District	0·25	
18.	J1/535	Hornblende granulite; Kolar schists.	0·19	

Serial No.	Registered No.	Description.	Radium.	Remarks.
Group 2. Dharwar System—Upper (Chloritic) series.				
19.	Z3/258	Santaveri trap, Bababudan Hills.	0·20	Chloritic trap, with some hornblende.
20.	Z4/757	Banded chlorite schist, Sacrebail, Shimoga.	0·27	
21.	Z4/703	Calc-chlorite trap, Tarikere series.	0·54	Probably altered hornblendic trap.
Group 3. Intrusives of Dharwar age—subsequent to the schists.				
22.	S2/943	Hornblende diabase, Bababudan Hills.	0·16	
23.	W3/69	Bellara trap, Tumkur District.	0·05	Altered diabase intrusive into chloritic series.
24.	Z4/95	Grey trap, Chitaldrug District.	0·07	Chloritic trap, probably related to Bellara trap.
25.	J4/117	Titaniferous iron ore, Ubrani, Shimoga.	0·05	Associated with ultrabasic intrusives.
Group 4. Champion Gneiss Series—intrusive into Dharwars.				
26.	J1/335	Grey micro granite, Kolar.	0.85	
27.	J1/156	Fine micaceous granite-matrix of crush conglomerate, Kolar Gold Field.	1·45	
28.	S3/313	Fine mica granite, Mysore Mine at depth of 3000 ft.	1·34	
29.		Auriferous quartz from the Mysore Mine.	1·28	These are considered to be products of the Champion gneiss magma.
30.	A/624	Quartz-felspar porphyry.	1·05	
31.	S/402	Massive phyrrhotite from Champion Reef Mine	0·07	From Gifford's shaft.
Group 5. Peninsular Gneiss Series—later than Champion Gneiss.				
32.	J3/238	Dark porphyritic granite, Kolar.	0·91	These are in order of intrusion, beginning with the oldest.
33.	J3/278	Dark grey granite, Kolar.	0·99	
34.	J3/212	Fine dark grey granite, Kolar.	1·28	
35.	J3/276	Fine light grey granite, Kolar.	0·47	
36.	J3/63	Light crushed granite, east of Kolar Field.	0·40	
37.	J3/122	Uniform granite mass, Patna, Kolar.	1·50	
38.	S/326	Pegmatite cross-course, Balaghat Mine.	1·44	These probably belong to the Peninsular gneiss. In No. 39 the tourmaline itself is not abnormally high in radium.
39.	S/354	Pegmatite cross-course, with tourmaline, Ooregum Mine.	6·90	
Group 6. Charnockite Series—later than the Peninsular Gneiss.				
40.	J2/824	Acid Charnockite, Chamrajnagar, Mysore District.	0·04	Order of intrusion doubtful.
41.	J2/822	Intermediate Charnockite, Chamrajnagar, Mysore District.	0·10	
42.	J2/722	Basic Charnockite, Heggaddevankotte, Mysore District.	0·12	
43.	J3/631	Hypersthenite, Nanjangud.	0·06	These are considered to be derived from the Charnockite magma.
44.	J3/713	Quartz-magnetite ore, Heggaddevankotte.	0·08	

Serial No.	Registered No.	Description.	Radium.	Remarks.
		Group 7. Closepet Granite—later than Charnockite.		
45.	J3/434	Coarse grey porphyritic granite, Closepet, Bangalore District.	0·27	In order of intrusion, beginning with the oldest.
46.	J3/430	Grey granite, Closepet.	0·63	
47.	J3/429	Red granite, Closepet.	2·14	
48.	Z2/621	Dark quartz-felspar porphyry, Yelwal, Mysore District.	2·42	These porphyries are subsequent to the Closepet granite and may belong to the same magma.
49.	Z2/648	Pink felspar-porphyry, Kirangur, Mysore District.	1·37	
		Group 8. Post-Archæan dykes.		
50.	J1/333	Normal dolerite dyke, Kolar Gold Field.	0·45	Typical of a large series of Post-Archæan dykes which may be of Cuddapah age (Animikean).

XXIV. *The Effect of Interionic Force in Electrolytes.* *By* S. R. Milner, *D.Sc.*[*]

Part I.

Do the electrical forces which exist between the ions in an electrolytic solution produce an effect on the ionic mobilities? This question has often been asked but has never received a satisfactory answer, although it is of fundamental importance in the theory of electrolytic dissociation. In Part II. of the following paper an attempt is made to provide an answer by establishing on the principles of the kinetic theory a general proposition on the effect of interionic forces. Briefly stated this is that whatever effect such forces produce on the osmotic pressure of the free ions in an electrolyte in reducing it below what it would be were the forces nonexistent, they will produce the same reduction in the average velocity with which the ions move in carrying a current. The bearing of this result on the theory of the extent and character of the dissociation of strong electrolytes is then considered. In this part it is proposed to consider in this connexion the well-known difficulty in the theory of Arrhenius connected with the failure of the law of mass action for strong electrolytes. The failure can, I think, be shown to be of such a kind as to form an objection apparently insuperable, not only to the original

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[*] Communicated by the Author.

theory, but to *any* theory of dissociation in which the reduction of the molecular conductivity is ascribed solely to a reduction in the number of the free ions, instead of to a direct effect produced by the interionic forces on their mobilities. In considering it we will confine attention to dilute solutions of strong binary electrolytes in water, where the essential facts to be explained by any theory are these:—

(1) The molecular lowering of the freezing-point τ is found to be nearly, but not quite, twice as great as the theoretical value τ_0 valid for a non-electrolyte. Let us express this fact thus:

$$\frac{\tau}{\tau_0}=2-\beta_1. \quad \ldots \ldots \quad (1)$$

β_1 is in the first place a purely experimental quantity, but by strict thermodynamic reasoning it can be identified with the reduction in the value of $\frac{\mathrm{PV}}{\mathrm{RT}}$ for the electrolyte below the value which would apply to an electrolyte completely dissociated and obeying the gas law. In the theory of Arrhenius it is further identified with the fraction of the whole number of ions which, as the concentration is increased from zero, have associated into molecules. Equation (1) is usually written with the symbol γ for the fraction of molecules *dis*sociated, but for the present purpose the use of β_1 $(=1-\gamma)$ in the equation is more convenient. We are dealing with strong electrolytes where at the most there are only small departures from complete dissociation, and of these according to the theory β_1 forms the direct measure.

(2) The molecular conductivity λ diminishes with increase in the concentration, or we may write, if λ_0 is its extrapolated value at zero concentration,

$$\frac{\lambda}{\lambda_0}=1-\beta_2. \quad \ldots \ldots \quad (2)$$

β_2 can not be identified thermodynamically with anything, but on the theory it also represents the fraction of the ions associated.

(3) $$\beta_1=\beta_2 \text{ approximately.} \quad \ldots \ldots \quad (3)$$

Much work has been done in testing the extent of the agreement, the general result of which seems to be that the equality is practically complete but the limit of experimental error is still somewhat wide *.

(4) β_1 and β_2 both vary with the concentration in a way which requires to be accounted for.

The original theory of Arrhenius of course explains perfectly the equality $\beta_1=\beta_2$ by identifying each of them with β the fraction of the ions associated. The variation of β with the concentration C which it requires, namely

$$\frac{(1-\beta)^2C}{\beta}=K, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

is, however, quite inconsistent with the experimental variation of β_1 and β_2. Numerous attempts have been made to get over the difficulty by modifications in the mass action equation (4), but this procedure, whether the modifications have a theoretical basis or are purely empirical, will be found to increase rather than diminish the difficulty in which the theory is involved. For the law (4) is a thermodynamical result the truth of which is independent of any theory of the mechanism, and rests solely on the assumption that the osmotic pressure of the ions and the molecules obeys the law for perfect gases. If the phenomenon is one of pure dissociation (and no successful explanations of the discrepancy have been reached on other lines), the fact that (4) is not obeyed is conclusive evidence that either the ions or the molecules do not obey the gas law. But when this is the case, the experimental quantity β_1 in (1) is no longer the same as β the true association. β_1 will in fact be equal to β *plus* the additional reduction in the value of PV/RT for the electrolyte due to the non-obedience of the osmotic pressure to the gas law, and calculation shows that, as far as can be estimated from the experimental curves, the last term is many times greater than the first. Comparison of β_1 with a modified mass-action law is thus invalid.

An equally great objection applies to any attempt to represent the β_2 of the conductivity variation by means of a modification of the mass-action law. For if β_2 is actually the same as β the true association it must be different from β_1, which is not the same as β except in the single case when (4) is obeyed. This is, however, in conflict with the experimental result (3). It may be thought that the differences between β_1 and β_2 thus necessitated are small second order ones concerning which experiment is inconclusive, but this is not the case. They are of the same order

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* *Cf.* A. A. Noyes and K. G. Falk, Amer. Chem. Soc. Journ. xxxiv. p. 485 (1912), where a systematic comparison of all the best data is given.

of magnitude as those which occur in the comparison of β_2 with the unmodified mass-action law. The point may be illustrated by a brief consideration of one of the most recent positions reached in the development of the original theory.

Kraus and Bray * have found as a result of a detailed examination of a great many solutions in various solvents, that the variation of λ can be closely represented by the empirical formula

$$\frac{\gamma^2 C}{1-\gamma} = K + D(C\gamma)^m,$$

where $\gamma = \frac{\lambda}{\lambda_0}$ $(=1-\beta_2)$, and K and D are constants. According to this formula the law of mass action (4) is obeyed when the concentration is sufficiently small, the term $D(C\gamma)^m$ being ultimately negligible. As the solution becomes more concentrated the mass-action "constant" K becomes increased by a term depending on the concentration of the ions. They conclude from their examination that in all cases the conductivity ratio λ/λ_0 is a true measure of the ionization, and that the simple law of mass action applies if the solution is made sufficiently dilute. This view has been mentioned with approval by Arrhenius †, who, however, lays stress on the fact that the fundamental difficulty of the failure of the strict mass-action law is still unremoved by the use of an empirical equation. Curve I., fig. 1, shows the agreement of the equation

$$\frac{\gamma^2 C}{1-\gamma} = \cdot 080 + 2\cdot 707 (C\gamma)^{\cdot 673}, \quad \gamma = \frac{\lambda}{128\cdot 3},$$

with experiment in aqueous solutions of KCl (the constants are given by Kraus and Bray, and the experimental numbers by Noyes and Falk, *loc. cit.*). The curve represents well the conductivities over a wide range except at the lowest concentrations. Now assuming that $\frac{\lambda}{\lambda_0}$ represents the true ionization, it is a simple matter to apply the general differential equation of mass action which is applicable in all circumstances, to determine the variation of the molecular freezing-point lowering with the concentration which is thermodynamically necessitated by this assumption ‡. The

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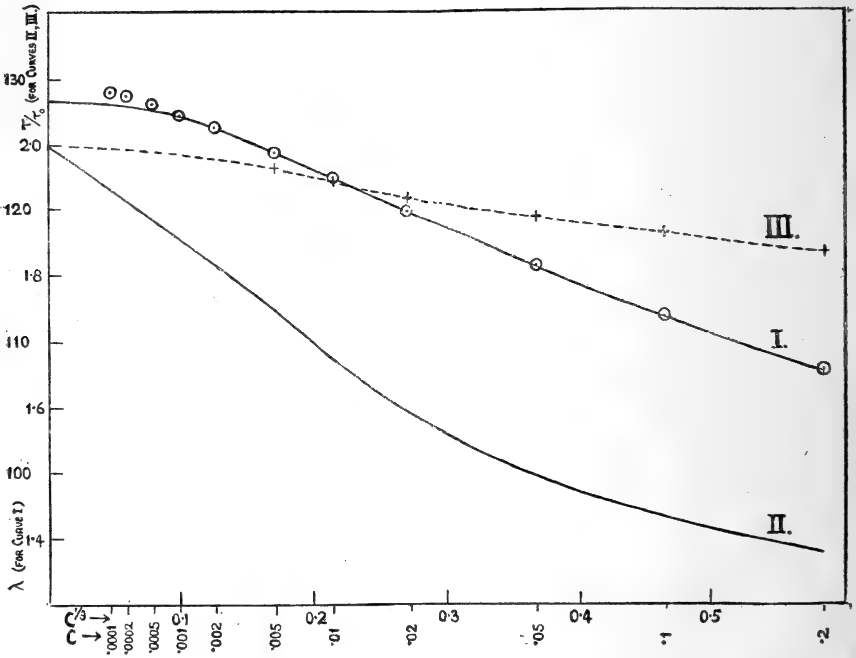
* Amer. Chem. Soc. Journ. xxxv. p. 1315 (1913).

† Chem. Soc. Journ. cv. p. 1414 (1914).

‡ See Appendix.

result is shown in Curve II. along with the experimental variation of τ/τ_0 (Curve III.). The difference between the curves is several times as great as the amount (β_1) by which the experimental curve differs from the theoretical value 2.

Fig. 1.



This type of difficulty is inherent in any theory in which β_2 is identified with β, *i. e.*, in which the reduction in the conductivity with increasing concentration is ascribed solely to a reduction in the number of the free ions, their mobilities remaining unchanged. We may say in general that with a suitable assumption as to the osmotic pressure of the ions it is always possible to derive a mass-action law which will express the experimental results either for β_1 or for β_2 whatever they may be, but only at the cost of entailing a theoretical difference between these two quantities which does not actually exist. The view that the electrical forces between the ions affect their mobilities (to a sufficient extent to account for the greater part of the variation of λ in strong electrolytes) will of course dispose of this difficulty; and so far as I can see it is the only view that will do so.

APPENDIX.

The problem to be solved is as follows :—
Being given

$$(1) \qquad \frac{\gamma^2 C}{1-\gamma} = K + D(C\gamma)^m$$

an empirical equation for an experimental fact when γ stands for $\frac{\lambda}{\lambda_0}$,

$$(2) \qquad C = c + c', \quad P = p + p',$$

where the capitals, undashed, and dashed letters mark the total, molecular, and ionic, concentrations and pressures.

$$(3) \qquad \gamma = \frac{c'}{C},$$

an assumption which the result of the calculation shows to be erroneous,

$$(4) \qquad \frac{dp}{c} = \frac{dp'}{c'} = \frac{dP}{C},$$

a thermodynamic result indisputable in all cases if the phenomenon is to be ascribed to dissociation,

$$(5) \qquad p = RTc, \quad \text{but} \quad p' \text{ is not} = RTc',$$

an admission that the failure of the strict mass-action law is connected with the non-obedience of p' to the gas law (this has to be made for either p or p' if (4) is to hold),

$$(6) \qquad \frac{\tau}{\tau_0} = \frac{P}{RTC},$$

a thermodynamic approximate equation indisputable in dilute solutions—express $\frac{\tau}{\tau_0}$ as a function of the measured quantities C and $\frac{\lambda}{\lambda_0}$.

We have from (1), (2), and (3),

$$(7) \qquad \frac{c'^2}{c} = K + Dc'^m,$$

and from (4) and (5),

$$dp' = \frac{c'}{c} RTdc.$$

Substituting the values of c and dc in terms of c' and dc' obtained from (7) we get

$$dp' = \mathrm{RT}\left(2 - \frac{m\mathrm{D}c'^m}{\mathrm{K}+\mathrm{D}c'^m}\right)dc'.$$

Integrating this equation we get p' as a function of c', *i. e.*, of $\gamma\mathrm{C}$; substituting this in (2) we get P and from (6) $\frac{\tau}{\tau_0}$ in the form

$$\frac{\tau}{\tau_0} = 1 + \gamma - \frac{m}{\mathrm{C}}\int_0^{\mathrm{C}\gamma} \frac{\mathrm{D}(\mathrm{C}\gamma)^m}{\mathrm{K}+\mathrm{D}(\mathrm{C}\gamma)^m}\, d(\mathrm{C}\gamma).$$

In order to draw the curve of fig. 1 the integral was evaluated graphically using the constants given by Kraus and Bray.

XXV. *Notices respecting New Books.*

Researches of the Department of Terrestrial Magnetism. Vol. III. *Ocean Magnetic Observations* 1905–1916, *and Reports on Special Researches.* By L. A. Bauer, Director, Washington D.C. Published by the Carnegie Institution of Washington, 1917. Quarto, pp. v+447, with 25 plates and 35 figures in the text.

THIS volume is issued also in three parts, dealing respectively with the earlier magnetic observations taken at sea by the 'Galilee' 1905-1908, the later sea magnetic observations by the 'Carnegie' 1909–1916, and the results of the observations on Atmospheric Electricity taken on both vessels. There is a very full account of the instruments, the methods of observation and the reduction formulæ, and elaborate tables of results and particulars of the errors in existing charts—American, British, and German. The vessels and the instruments are illustrated in the plates from a variety of points of view. The observing vessel 'Carnegie' was specially built for magnetic work and is almost free from magnetic material. This has proved a great simplification in the reduction of the observations.

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XXVI. *Rain, Wind, and Cyclones.*

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DURING recent years our conceptions respecting the conditions obtaining in cyclones have undergone very considerable alterations. The old idea, which made a cyclone consist of a lower spirally inflowing current of air directed towards the centre of an area of low barometric pressure, an internal rising current of warm air, and an upper stream of air flowing outwards from the same area, requires very considerable modification in view of modern discoveries.

That the cause of rain is due to the upward flow of masses of air, which being cooled below the dew-point by expansion, condense out some of their moisture, still holds true; but the distribution of temperature and the actual nature of the air circulation prove to be very different from what was once thought to be the case.

In all scientific work accurate observation and correct records are essential. Here, however, meteorological records fail us in a very important particular. The wind directions shown on our charts are not always accurate. In Great Britain, for example, wind directions are given to the nearest of sixteen points of the compass. It thus comes about that winds which are observed to differ in direction only *one* or *two* degrees from each other, may be charted as differing by as much as *twenty* degrees.

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[*] Communicated by the Author.

Owing to the direct flow of the wind being prevented by trees, buildings, hills, &c., it is by no means easy to observe the correct direction of movement. It often happens that a wind blowing up a river will follow closely the river windings; passing clouds, especially rain-clouds, also produce, for short intervals, marked changes both in the direction and force of the wind; but the difficulties that are met with in observing the general direction should be overcome by the exercise of extreme care so as to obtain a result as accurate as possible. The observed results should then be recorded in degrees from the true north, working clockwise round the compass.

To render the points I wish to draw attention to clear, a number of charts or diagrams of cyclonic disturbances &c. have been taken from Shaw and Lempfert's paper on "The Life History of Surface Air Currents"*. The wind directions are sometimes shown by continuous lines instead of arrows, and the isobars by dotted lines. In the original charts the arrows often show many irregularities in the wind directions of the same wind province; but these are due to local causes such as hills, falling rain, &c. It is near the centres of cyclones that the errors arising from the methods of observation now adopted appear to be most marked. Indeed, until strict accuracy is arrived at in recording and plotting observations of wind directions, many features of atmospheric circulation will continue to be obscure.

Shaw and Lempfert, in the paper already referred to, show that the wind directions in cyclones, if they do not always close in as regular spirals towards the cyclonic centre, do travel in such a way that the air suffers horizontal contraction and rises. That this is the case is rendered evident by the consideration of the air trajectories they have worked out.

In the case of quite a deep cyclone the difference of pressure between the centre and the margin seldom amounts to more than one inch of mercury, and this is about the change of pressure due to a rise of 400 feet. To give rise to heavy rain, masses of land over which winds are blowing and rising as they advance must be much higher than this. In a cyclone the rain-producing effect resulting from the fall of the barometer is very small indeed. Rain when it does fall nearly always results from the rise of air in the cyclone or the mounting of one air current over another, the quantity of rain depending upon the humidity of the rising air and the height to which it is lifted.

According to the old view, the air rises spirally in the

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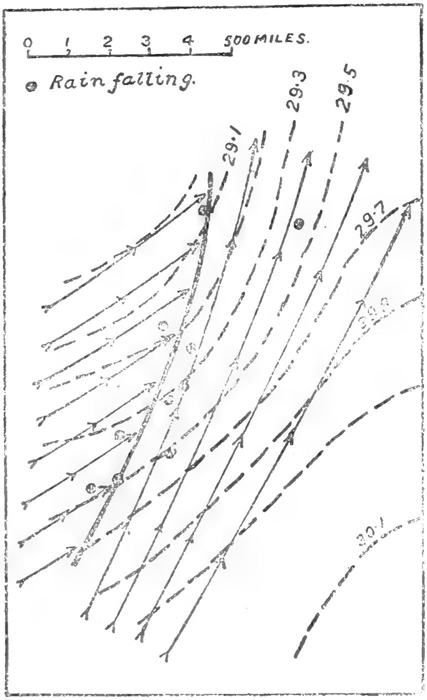
* Meteorological Office Publication No. 174.

central portion of a depression. If such were the case, the greatest rainfall would occur around the centre. But it does not often do so. The greatest rainfall is generally on one of its sides, and considered with regard to the direction in which the cyclone is moving, not always on the same side. However, we may safely conclude that the area over which rain is falling is that above which the air is rising, and this fact, considered in the light of the directions of the wind on the earth's surface at the time, throws light upon some points in the construction of the upper portion of the cyclone which might otherwise be obscure.

Fig. 1.



A very obvious case of a rising wind producing rain, without any marked barometric changes, is described by Shaw and Lempfert*. The wind directions are shown in fig. 1. Here we have a wind from the south-west blowing against one coming from a little west of south; the line

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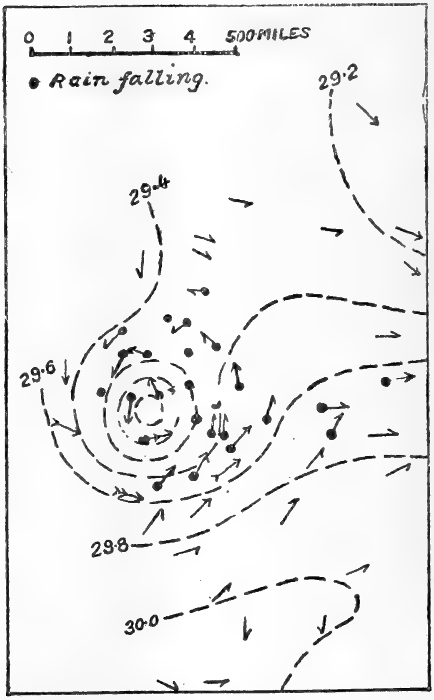
* Meteorological Office Publication No. 174, p. 62.

separating the two wind provinces was moving westwards at about 29 miles per hour, whereas the wind following it up was only moving at 20 miles per hour. Rain was falling near to and on both sides the dividing line between the winds. The upper portion of the westerly wind appears to have been travelling more rapidly than the portion near the ground, and was descending and forcing itself beneath the more southerly wind. The latter being forced to rise precipitated some of its moisture through the lower south-westerly wind as well as along its westerly margin.

Fig. 2.



All the weather charts figured are for the area of the British Isles and portions of western Europe. The points where rain was falling were generally on land, observations out at sea not being available in many instances.

Occasionally cyclones, accompanied by rain, do occur which show, even as far as their lower levels are concerned, a striking resemblance to the old theoretical cyclone*. Fig. 2 is such a one. Its centre lay over the sea between Wales

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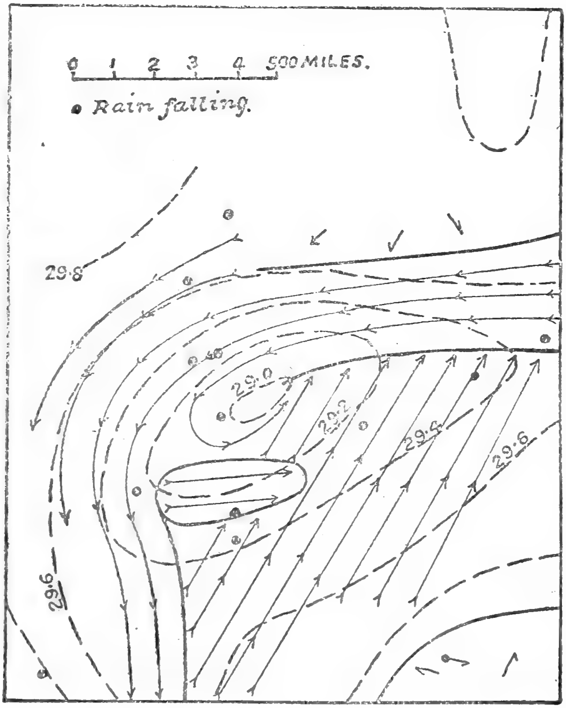
* Met. Office Pub. 174, Plate IV. fig. 3.

and Ireland. The wind directions show that the air was circulating around the centre of the depression, and was rising over the whole of it, with the exception, perhaps, of a small area to the south-west. It was travelling from west to east at a velocity of about 35 miles per hour, and the wind velocities were high. The irregular nature of the wind directions shown by the arrows is most probably due to the flow of air from areas of heavy rainfall, resulting from the mechanical effect of the falling rain. To the east of the storm centre, in the direction of Holland, a south-westerly wind

Fig. 3.



and westerly wind were in conflict, and rain was the result there as well as near the centre.

With the exception, perhaps, of the portions near their centres, the cyclones now to be illustrated do not show a regular spiral flow of the lower air into the cyclone; but are areas where several winds are approaching or receding from an area of low pressure. These lower winds often pass over or beneath each other, and cause rain.

Fig. 3 shows the conditions in a cyclone which passed over

England on Nov. 13, 1901, based upon a chart given by Shaw and Lempfert*. The wind over France, Germany, and the south-east of England blew from the south-west. An easterly wind from the Baltic Sea cut off abruptly the south-westerly wind, and backing as it travelled round the western side of the depression, finally became a northerly wind. The conditions obtaining in this depression are well described by Shaw and Lempfert. Its average rate of motion from west to east was only about 17 miles per hour, whereas in one locality the velocity of the wind was 59 miles per hour. The rainfall was exceptionally heavy, four inches being recorded at several stations in Ireland. The area of precipitation was a broad band stretching from west to east along the line separating the south-westerly wind from the easterly Baltic wind. Shaw and Lempfert remark, "As might be expected, the air from these two sources was at decidedly different temperatures, and . . . . suggest that the process going on in the depression consisted in the warm air from the south rising up over the top of the cold air from the north-east." The fact that rain was falling in Scotland when the centre of the depression was well over the centre of the North Sea, shows that the south-westerly wind, after rising over the easterly wind, curved round and passed to the north and west of the cyclonic centre.

A peculiar feature of the above described depression was the area of westerly wind that prevailed over the Channel. This small wind province was to the south-west of the cyclonic centre and travelled with it to the east. Over this area the south-westerly winds were interrupted.

Another cyclone is shown in fig. 4†, which reproduces many of the features of the one already described. It was of the slow travelling type, moving from west to east at about 11·5 miles per hour. It originated over the west of the British Isles and grew deeper and deeper as it moved in an easterly direction. At the time of its formation, October 7, 1903, there were south-westerly winds over the area and easy pressure gradients. At 8 A.M. on October 8 there was a well-marked but shallow depression over the south-west of England. An inflow set in from the Baltic area, the southerly wind commenced to rise over it, and rain fell in the northern counties of England. These features had become well marked at 8 A.M. on Oct. 9, and the depression amounted to 0·6 inch of mercury. Fig. 4 shows the conditions obtaining at this

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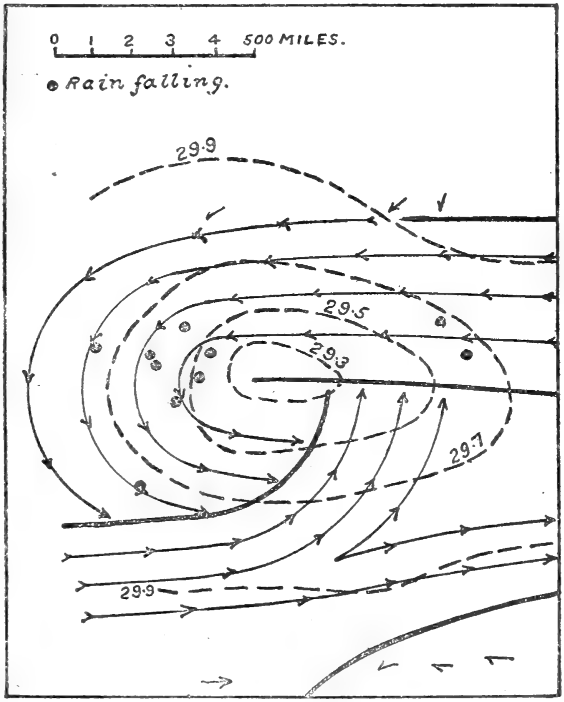
* Met. Office Pub. 174, Plate VII. fig. 14.

† Ibid. Plate IX. fig. 12.

time. The growth of the depression appears to have been from above downwards.

Another condition of wind and pressure which often gives rise to heavy rain is known as a V-shaped depression. Such a one formed over England during the interval from Jan. 6 to 8 in 1900. There was a low-pressure area over Iceland, and a tongue of it was thrust in between a high-pressure area over Scandinavia, and another in the region of the Bay of Biscay. Fig. 5 shows the condition of affairs at 6 P.M. on Jan. 6th. The depression subsequently extended in a south-westerly direction, low pressure still holding to the north-west. It shows a strip of southerly wind, with westerly winds to the west of it, and easterly winds to the east. The southerly wind is moving at a low angle across the isobars, from high to lower pressures. The depression is thus a marginal feature of the great Icelandic depression. The rain is considered by

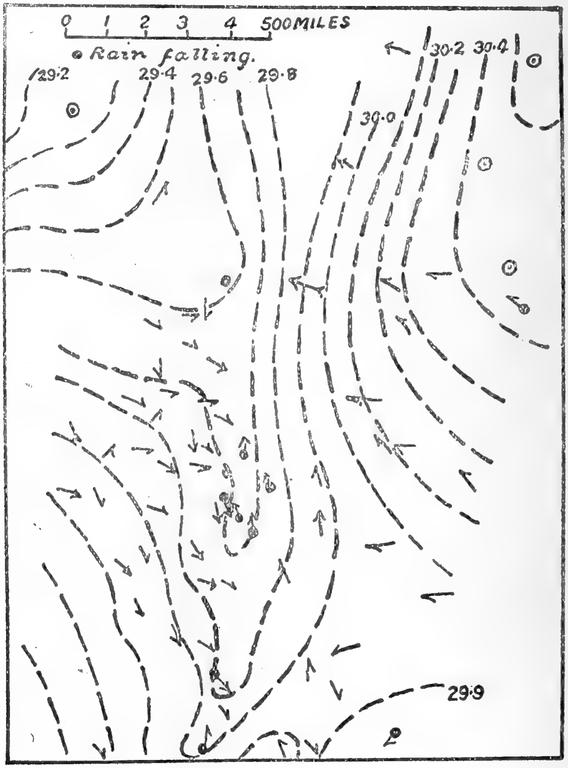
Fig. 4.



Shaw and Lempfert to have been due to the mounting of the southerly current over the northerly and westerly wind on its westerly margin.

But there are other phenomena, of which rain and wind are features, which are not the result of great differences of barometric pressure, but arise from the variable heating and cooling of the land and water areas by the sun. In the Troposphere we have the atmosphere in a condition approaching convective equilibrium; due to the upward rise of heated air, such air cooling by expansion as it rises. There is a generally held opinion that the full moon dissipates the clouds. The fact that clouds often melt away after sunset is probably the cause of this view. During the day in quiet weather, the air over the land is more heated than is the case over the surrounding seas, and it rises, thereby producing

Fig. 5.



clouds; but during the night the land cools more than the seas, the air descends and the clouds melt away. Thunderstorms originate in this way, the air rising locally to such great heights that heavy rain and hail result.

The conditions shown on the diagrams that have been considered are such as occur near the earth's surface. At low levels the winds often approach and recede from the low-pressure area much as comets approach and recede from the sun. But the wind near the earth's surface encounters considerable frictional resistance, and some of it is drawn in and rises in the cyclone. Indeed, if it were not for the friction of the air and the ground, a cyclone once started should persist by reason of its own momentum.

It must be remembered that the water content of the air rapidly decreases with elevation, owing to the decrease of temperature. Above 3000 metres the moisture content of the atmosphere is very small owing to the low temperatures prevailing at that level. Indeed, this lower region of the atmosphere has been called the storm layer. In it the temperature gradients are very irregular, for all upward currents throw down moisture as they rise, and are prevented from falling in temperature as much as they otherwise would do by the liberation of the latent heat of condensation, whereas descending currents undergo simple adiabatic changes of temperature. In very cold weather the temperature gradient is often reversed by reason of the cooling of the air in contact with the ground.

The diagrams of cyclones shown in the figures that have been given make it clear that the winds near the earth's surface very often cross the gradients at high angles. Gold * has carefully considered the relationship of the isobars and winds at heights of about 3000 feet. Assuming that the isobars near the earth's surface hold true for greater heights, it is found that the strength of the upper winds closely approximates, by calculation, to what would be expected from the gradients at the earth's surface. Gold remarks "It is to be noted, however, that on the average, even for anticyclones, the tendency is for the wind at 1000 metres altitude to blow slightly across the isobars from high pressure to low." In a preface to the paper referred to, Shaw says: "The general result of the investigation is, in my opinion, to confirm the suggestion that the adjustment of wind velocity to gradient is an automatic

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* "Barometric Gradient and Wind Force," Meteorological Office Publication 190, p. 9.

process which may be looked upon as a primary meteorological law, the results of which are more and more apparent as the conditions are more and more free from disturbing causes, mechanical or meteorological."

It was previously suggested that the surface winds often followed directions resembling the eccentric paths of comets. Their courses, however, at higher levels more closely resemble planetary orbits. In the last paragraph of the preface to Gold's paper Shaw remarks: "The whole question of the cause and meaning of the discrepancies between the gradient wind and the actual wind is, of course, bound up with the origin of pressure differences. To put the point in a crude form, I do not know whether, in practice, the winds have to adjust themselves to the pressure conditions, or the pressure distribution is the *result* of the motion of the air."

Perhaps the most promising way of ascertaining the cause of the circulation of the winds is to be found by studying the distribution of atmospheric temperatures. In this direction a great deal has been done in the exploration of the upper atmosphere by means of pilot balloons. In this connexion we cannot do better than consult a paper by Shaw and Dines* on "The Free Atmosphere in the Region of the British Isles."

Fig. 6 is a diagram showing the distribution of temperature within a cyclone 2000 miles in diameter. It will be noticed that the temperatures below about 10 kilometres are lower in the centre than in the margin of the depression. It is clear, therefore, that the temperature distribution below this level is such as would cause a descent of air at the cyclonic centre. The old theory that a cyclone results from the ascent of warm air from the earth's surface must be abandoned.

The dotted line B B shows the dividing line between the Stratosphere and Troposphere. In the stratosphere the temperature not only rises as the height increases, but over the cyclone there is a local mass of heated air. I would suggest that this heated air extends to the confines of the atmosphere, and there is there formed a gravity gradient for the flow of air outwards. The displacement of this air reduces the weight of the air column, and the result is felt at the earth's surface.

Instead of the air moving into and filling up the low-pressure area at once, it circulates round it, in a manner depending upon the direction of the air-currents in the atmosphere at

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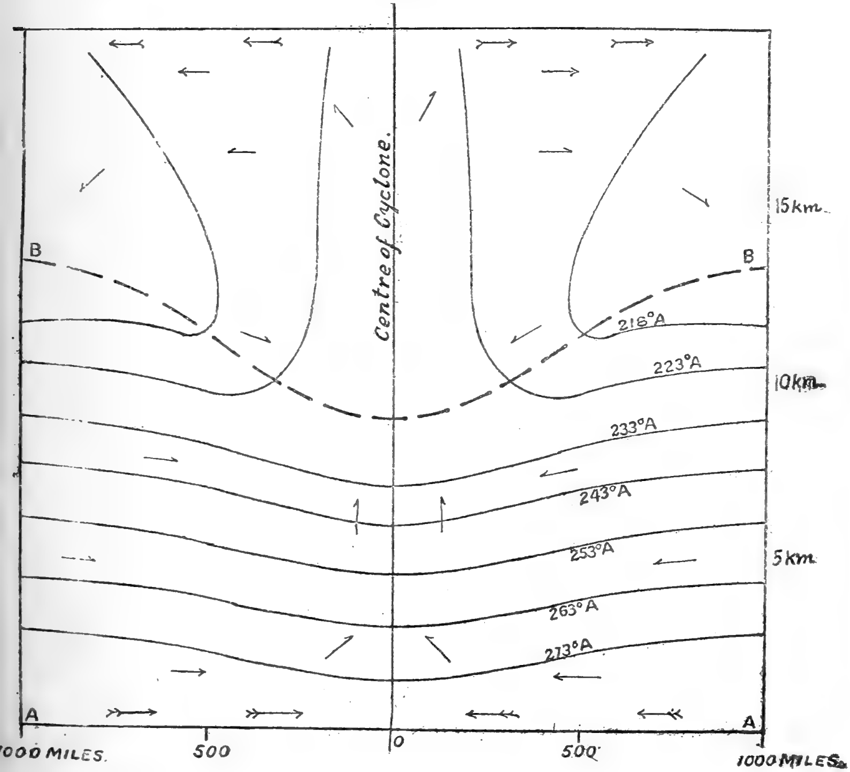
* Meteorological Office Report, No. 210 *b*.

the time of its formation. Dines* remarks: "The inference drawn is that a cyclone is produced by the withdrawal laterally of the air at a height of from 8 to 10 kilometres; for if we choose this height the observed and the theoretical

Fig. 6.



variations of pressure and temperature agree, whereas they would not do so if any other height were chosen for the outflow of the air which undoubtedly flows in along the earth's surface." My suggestion is that the air does not flow outwards from the cyclone in any volume anywhere except where a corresponding volume enters somewhere else; there being, as regards horizontal flow to or from the centre, above 2000 or 3000 feet—except at very great heights,—a balance maintained between incoming and outgoing air.

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* Met. Office Pub. 210 *b*, p. 50.

The low pressure and inward flow of surface currents are maintained by the buoyancy of the warm air of the stratosphere, in which there is also a cyclonic circulation.

According to this conception a cyclone is the result of the outward flow of a volume of heated air in the upper portion of the stratosphere, an inflow near the earth's surface, and a slight bodily lifting of the mass of air between.

A section of the atmosphere between the equator and the pole shows the same temperature distributions as does a cyclone. Fig. 7 is a generalized section from the pole to the equator showing what would appear to be the temperature distribution in the atmosphere, as revealed by observations made with self-registering pilot balloons. The dotted isotherms are purely theoretical. A A is the earth's surface and the dotted line B B the lower surface of the stratosphere. Fig. 7, it will be noticed, closely resembles the left side of the cyclone shown in fig. 6. Over both low-pressure areas there are masses of warm air in the stratosphere.

It is only possible to obtain an idea of the temperatures of the upper portions of the stratosphere by a study of the atmospheric pressure on the earth's surface and the temperature of the troposphere. The troposphere has been sounded in many places, and its temperature is fairly well known at several latitudes. In constructing the diagram fig. 7 the temperatures of the troposphere, as ascertained for Batavia, Milan, Pavia, England, and Pavlovsk, have been plotted, and the isotherms drawn in. Comparing the variations in weight of the air columns at these places resulting from their different temperatures with the actual barometric readings, it is clear that a ridge of cold air in the stratosphere must encircle the earth at latitude 30° north and south of the equator, and there is a trough of heated air in the upper atmosphere over the equatorial regions, and a basin of hot air over each pole. Isotherms have been drawn in the stratosphere to illustrate this; but to obtain an accurate representation of the variation of temperature further observations are necessary.

The high-pressure ridge is rather a string of anticyclones and the north polar basin has two minima, one over the North Atlantic and the other over the Behring Sea. The South Polar Continent is the centre of one great cyclone due to the heated basin of air in the stratosphere. Were it not for the powerful action of this basin, anticyclonic conditions would prevail there.

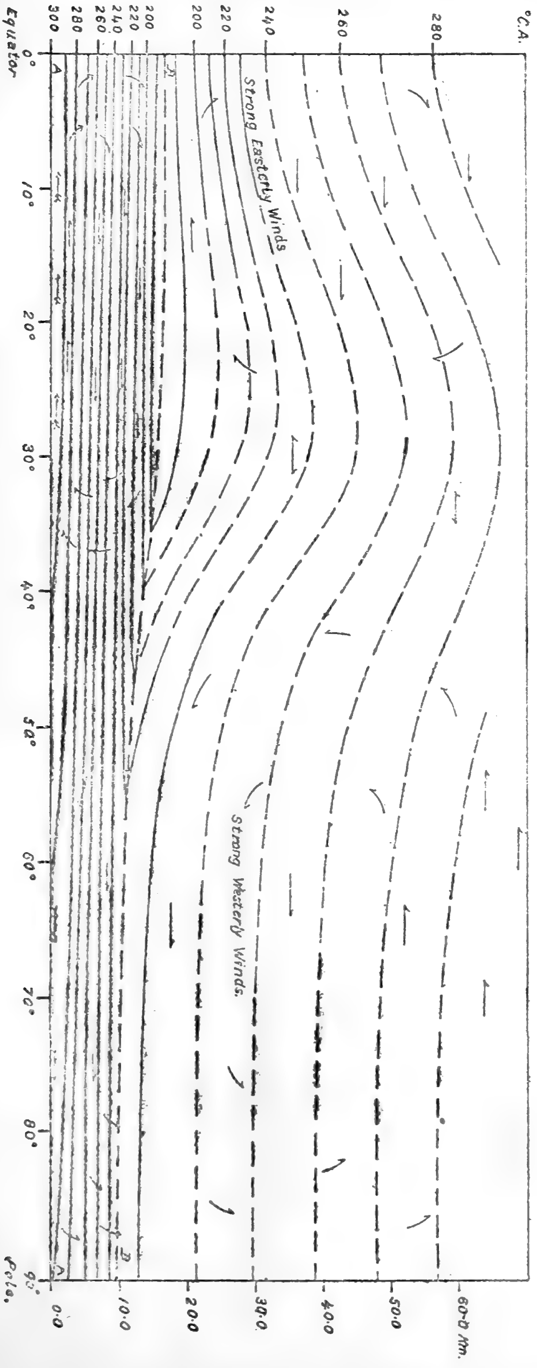
°C.A.
300
280
260
240
220
200
200
220
240
260
280
Equator
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
Pole.
0.0
10.0
20.0
30.0
40.0
50.0
60.0 Km.
A
B
Strong Easterly Winds
Strong Westerly Winds.

Fig. 7.

The general surface winds of the globe, like those of cyclones, generally cross the isobars at angles more or less acute. It is probable, however, that all the winds of the world above 1000 metres, or even lower, follow the isobars very closely, and that the energy of cyclones both large and small is dissipated mainly by friction at or near the earth's surface. Although the energy represented by such movements is very large, it would be easy to overrate the strength of the forces required to keep them going or to start them. We must regard the circulation of the atmosphere much as we do the movement of the planets, moons, comets, &c. of the solar system; but the retarding frictional forces are greater in the case of the atmosphere than they are in the case of the solar system.

I have ventured to show what I regard as the general directions of the slow movement of the winds across the isobars in figs. 6 & 7.

In the case of fig. 6, the arrows show the assumed very slow movement to or from the centre of the cyclone during its growth. The velocity of the wind along the isobars may be high. In fig. 7 the arrows also show the assumed steady movement to and from the centre of the cyclone. The velocity along the isobars may also be great.

The temperature distribution in the atmosphere appears to be the result of heating mainly in two ways. At the upper surface there may be arrested many kinds of radiation (undulatory and material) and also cosmic matter moving at high velocities. It is thus heated at its upper surface, and this heat passes downwards and gives us the conditions of temperature found in the stratosphere. Light and heat rays pass through the atmosphere until they reach clouds in the lower atmosphere or the earth's surface. From here the heat rises and the temperature conditions of convective equilibrium are established in the lower atmosphere.

I have suggested * that the great heating of the upper surface of the atmosphere over the poles is primarily due to the electrons shot out by the sun, which, being caught by the earth's magnetic field, are directed towards the poles, the air in the neighbourhood of which they heat and probably ionize. But we have to account for the local heating of the upper surface of the atmosphere required to produce cyclones. It may be due to pencils of high-velocity cosmic matter; for the optical properties of the upper atmosphere, as indicated by

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* Phil. Trans. vol. xxxi., April 1916.

the varying sunrise and sunset effects, are otherwise difficult to explain.

The energy of impact of the cosmic dust need not be as great as the energy of the cyclone produced; for the winds and isobars of a cyclone are to a large extent a modified arrangement of existing isobars and winds.

Cyclones according to this theory must travel with the winds of the upper atmosphere. That cyclones originate at high levels and extend downwards would seem to be implied by the following remark of Shaw and Lempfert*: "This disturbance moved slowly in a north-westerly direction and finally passed away to the North Sea. In the early stages small 'secondary' minima of pressure developed near the primary minimum, and the process of travel appears to consist in the formation of a 'secondary' in front of the storm, and the filling up of the original minimum."

The local heating of the upper portion of the stratosphere which is considered to result in the formation of travelling cyclones, is regarded as being produced rapidly and as dying away slowly. In the case of the polar fixed cyclones the heating of the upper atmosphere must be a continuous process or nearly so; the heat is always passing downwards as rapidly as the air moving into the lower portion of the cyclone raises the upper surface of the stratosphere, and the vertical distribution of temperature in the atmosphere remains nearly constant.

One would expect the weather conditions to be much less variable if they depended wholly upon the physical features of the earth's surface and the radiations received from the sun. Indeed the Trade Winds, and even the general circulation of the atmosphere, are fairly regular. It is to wandering cyclones that our short-period weather variability is due, and the want of regularity in the manner of their occurrence would be only what might be expected if they were caused by irregular streams of cosmic matter. Indeed, our atmosphere probably protects us from a bombardment from space, not only of matter but of many undesirable radiations. However, the energy received from the sun is probably somewhat irregular in its amount owing to the formation and disappearance of spots on the sun, and some relationship undoubtedly exists between weather condition variations and sun-spot periods.

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* Met. Office Pub. 174, p. 45.

If the driving force of cyclones originates in the upper portion of the stratosphere where it is locally heated, then this heated air must be carried along with the stratosphere wind, and the course of the cyclone on the earth's surface should indicate the direction of flow of the stratosphere wind above. The generally easterly movement of cyclones favours this assumption; but the track of cyclones from south to north in North America seems to require explanation. In this connexion it is interesting to note that the dust of the Krakatoa eruption adhered closely to the area of the equatorial trade winds, except over the Atlantic, where the dust was carried to the north apparently along the American Cyclone track.

XXVII. *Resonance Spectra of Iodine. By* R. W. WOOD, *Professor of Experimental Physics, Johns Hopkins University*[*].

[Plates VI.-VIII.]

SINCE the appearance of the last paper on this subject (Phil. Mag. ser. 6. vol. xxvi.), the study of these interesting spectra has been continued without interruption, and some new and very important relations have been brought to light. As has been shown in previous communications, the vapour of iodine in vacuo, when excited to luminosity by the light of the Cooper-Hewitt mercury lamp (glass), emits a spectrum consisting of a series of doublets, with a separation of about 1·5 ÅU., very regularly spaced along the spectrum and separated by intervals of about 70 ÅU. These intervals increase gradually, however, as we pass away from the green mercury line, at which point the doublet series has its origin, until, in the extreme red, the distance between the last two doublets observed is about 102 ÅU., and the separation of the components of the doublet has increased to 2·8 ÅU. By the use of dicyanine plates the series has been followed to its termination at wave-length 7685 and the wave-lengths of the seven new doublets accurately measured. The doublets are not all of uniform intensity, and some are missing entirely, and it is the connexion between this circumstance and the way in which the doublet series is related to the band absorption spectrum, that is the most interesting point brought out by the recent investigations. By varying the conditions of the experiment it has been found possible to excite by the green mercury line not only the doublet series, but a simplified

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[*] Communicated by the Author.

system of fluted bands, few in number and regularly spaced if the iodine is in vacuo, increasing in number and complexity if a gas of the helium group is mixed with the iodine, or if more than a single iodine absorption line is excited by the mercury lamp. It is probable that the lines forming the doublets are themselves constituents of the fluted bands, and the transfer of energy from one part of the vibrating system to another, as a result of collisions between iodine and helium molecules, enables us to build up, so to speak, the complicated system of fluted bands shown in the absorption spectrum, out of a number of simpler systems which can be excited separately. This constitutes a very great advance in the analysis of band spectra, and brings us a step nearer to the point at which we can picture some idea of the vibrating mechanism.

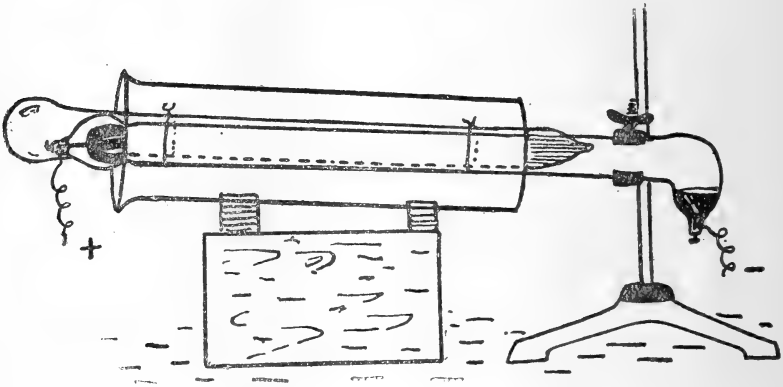
In the more recent work, a method of illumination has been employed which is distinctly superior to any previously used, and as it is well adapted to purposes of demonstration I shall describe it in some detail. The iodine tubes which I now employ are of soft glass, about 40 cm. long and 3 cm. in diameter. One end is blown out into a thin bulb, taking care to avoid having the thick drop near the centre of the bulb. This is best accomplished by drawing off the tube in an oblique direction, which brings the drop—formed by the melting down of the pointed end—well to one side. If this is not done the drop is apt to form a small lens on the surface of the bulb exactly on the axis of the tube.

The other end is drawn down, and a few flakes of iodine introduced into the tube. It is a good plan to provide the tube with a lateral branch, by which the density of the vapour can be controlled, though this is not necessary for demonstration purposes. The iodine flakes are now brought into the bulb, or to the bottom of the lateral tube, and the tube joined to a Gaede pump, interposing a U-tube immersed in liquid air or solid CO_2, or a tube filled with caustic potash, to keep the iodine out of the pump. During the exhaustion it is a good plan to heat the walls with a bunsen flame, except where the iodine is located. Then allow the tube to cool down to the temperature of the room, and heat the portion where the iodine is located. The flakes will sublime rapidly and crystallize on the cooler portions of the wall. The tube is now sealed off from the pump and the drawn-down end painted black for a distance of a few centimetres. For the illumination I used a very simple modification of the "light furnace" described in the earlier paper.

The iodine tube is fastened alongside of and in contact with a small Cooper-Hewitt mercury lamp (glass, not quartz). The bulb should project a centimetre or two beyond the cap on the positive electrode, and the drawn-down end should reach not quite down to the negative electrode bulb. Two small pads of thick asbestos paper should be placed between the two tubes, which are then securely fastened together with copper wire.

The Cooper-Hewitt lamp is supported in a clamp fastened close to the negative bulb, just beyond the end of the iodine tube, as shown in fig. 1.

Fig. 1.



A cylindrical reflector is now prepared by cutting off the bottom of a beaker glass measuring about 12 × 25 cm., and silvering the outside with Brashear's solution. This can be done with a minimum amount of solution by rotating the beaker slowly in a glass or porcelain tray, tipped slightly on its longer side. A preliminary trial with water shows at once the minimum amount that can be used. It is sufficient if the solution wets one side of the beaker from one end to the other. After the silvering the inside of the beaker is cleaned with a cloth dampened with dilute nitric acid, and the hollow reflecting cylinder slipped over the iodine tube and mercury lamp. The lamp is clamped at a suitable angle for operating, say 5° from the horizontal, and started by tipping the clamp-stand. The reflector should be supported so that the tubes are centrally placed. The luminous iodine vapour is viewed "end-on" through the bulb. If a prism of about 8°, such as is used for mounting Lippman photographs, is placed in front of and close to the bulb, it has the effect of bringing the tube into the horizontal position, which is advantageous if an image is to be projected on the slit of a spectroscope.

This is the arrangement which I have used in all of the recent work, and besides having a very high efficiency, it is easy to construct.

The fluorescence of the iodine is so intense that the doublets excited by the green mercury line can be seen widely separated in the first order spectrum of a large grating with a telescope of three metres focus. In a small prism spectroscope the complete resonance spectrum is extremely brilliant.

The excitation of the iodine vapour results from the circumstance that the green mercury line and the two yellow lines coincide with absorption lines of the iodine, of which, as I have previously shown, there are between forty and fifty thousand in the visible spectrum.

We will consider, first, the resonance spectrum excited by the green Hg line. To obtain this pure, it is necessary to screen off the light of the two yellow lines. This can be done with a solution of neodymium chloride, or the double salt of neodymium and ammonium, and as the use of a fluid screen is impractical in the case of the method of illumination just described, it is necessary in this case to illuminate the tube with a large condensing lens. As a matter of fact, however, the yellow lines are comparatively feeble in the case of the Cooper-Hewitt glass lamp, and the doublets excited by the green line are so intense that, in the greater part of the work, no screen has been employed.

The more recent investigations have brought some extremely interesting phenomena to light, especially with respect to the transfer of energy from the doublet series to the band spectra, as a result of the admixture of helium or other rare gases with the iodine.

On account of the complexity of the subject, it will be necessary to touch briefly on some of the relations which have been discussed in the earlier papers.

The band absorption spectrum of iodine covers the spectrum range comprised between wave-lengths 5100 and 7700. It is made up of exceedingly fine lines averaging 20 to the Ångström unit in the green and yellow regions, or some 50,000 in all making the estimate on the above average. It is covered on the short wave-length side by a band of continuous absorption in the blue-green region, which makes the exact determination of its end impossible. In the red it has been followed by means of dicyanine plates sensitive to $\lambda = 9000$, and its termination discovered at about $\lambda = 7700$.

A portion of this spectrum, in the vicinity of the green mercury line, reproduced from an earlier paper, is shown by fig. *a*, Plate VI. The entire spectrum, reproduced on the same scale, would be about 80 metres in length. In the yellow, orange, and red regions the lines form fluted bands, or rather series of overlapping bands. In the green region there appears to be so much superposition of bands that all appearance of regularity vanishes. A good idea of the general appearance of this spectrum in the region 5460—5700 is given by fig. *d*, Plate VI. This is in reality the emission spectrum of iodine in a vacuum-tube, reproduced as a negative, and with the dispersion employed could scarcely be distinguished from the absorption spectrum, except for the strong dark lines, which are iodine emission lines not belonging to the band emission.

With this as an introduction, we will now take up the remarkable spectrum emitted by the iodine when illuminated by the green line of the Cooper-Hewitt lamp. This line is shown in coincidence with the absorption spectrum in fig. *a*, Plate VI. As will be seen, the main line falls nearly midway between two of the iodine absorption lines. It is probable that only the left-hand absorption line is stimulated, as the width of the mercury line is not quite sufficient to enable it to reach the other. The short wave-length satellite is also in coincidence with an absorption line, but, for the present, we shall neglect the effect due to this. The resonance spectrum excited by the stimulation of this absorption line consists of a series of close doublets (doublet separation about 1·50 ÅU.) very regularly spaced along the spectrum. For convenience we will designate, as before, the one in coincidence with the exciting line as the doublet of 0 order, those lying on the long wave-length side as +1, +2, +3, &c., orders, and those on the other side as −1, −2, &c.

The doublet of 0 order is indicated on fig. *a* (Plate VI.) immediately above the absorption spectrum. One line (5460·74) is in coincidence with the iodine absorption line covered by the mercury line, the other (5462·25) lies 1·5 ÅU. to the right. The former is in reality a re-emission of the absorbed energy without change of wave-length (Resonance radiation), and I have accordingly named this line the R.R. line. The other line we may designate the companion line. On the scale of fig. 1 the next doublet (+1 order) would lie on the right at a distance of nearly two metres.

By means of plates sensitized with dicyanine, which were prepared for me by Mr. Meggers, I have succeeded in photographing the doublets as far as the 27th order, with a large plane grating and a Cooke lens of one metre focus. This permits of wave-length determinations correct to about 0·1 ÅU.

This marks the end of the resonance spectrum, I believe, as the wave-length of the last doublet recorded on the plate was 7685, and the plates are highly sensitive to well beyond 8500. Moreover, the absorption spectrum terminates at about this point.

Photographs of the doublets are reproduced on Plate VI. Fig. *e* shows the orders 0, +1, and +3. The doublet of order +2 is missing, though a pair of faint lines appear nearly in the position in which it should be found.

Fig. *b* shows the doublets +6 to +13 inclusive, and fig. *c* +15 to +22 inclusive; orders 9, 14, 16, 19, and 21 are also missing. The variable intensity of the doublets is also to be noted.

Comparison Spectrum Neon Short Line.

The law governing the spacing of these doublets will be discussed in a separate communication immediately following the present one, in which only the general nature of the phenomena will be treated.

Fig. *d* shows the doublets of order 0, +1, and +3 taken with a large plane grating and an objective of 3 metres focus (exposure 15 minutes), in superposition with the emission band spectrum of iodine electrically excited in a vacuum-tube. All of the photographs, with the exception of fig. *a*, are reproduced as negatives. The resolving power employed in the case of *d* was, of course, quite insufficient to completely resolve the band spectrum, as can be seen by comparing the width of the doublets with the width indicated in fig. *a*. It nevertheless gives an idea of the relation of the doublets to the band absorption spectrum.

If we give a longer exposure, we find that the doublets are accompanied by faint companion lines. These appear in fig. *e*, which was exposed for an hour and a quarter. Some of these lines are due to the excitation of other iodine absorption lines by the satellites of the green mercury line, but others, I feel sure, result from the stimulation of the absorption line covered by the main line. The former come out strong when the iodine is excited by the quartz mercury arc, in which case the green line can be broadened until it covers all of the seven absorption lines between the two arrows in fig. *a*.

If now we give a greatly prolonged exposure, we find that a band spectrum also appears. Fig. *f* is a 20-hour exposure for the same region of the spectrum. The doublets have fused to a wide band, owing to over exposure. The companion lines, above referred to, come out strong, and in addition there is a fluted band to the right of the doublets of order +1 and +3.

It will be observed that these doublets lie just within the heads of the fluted bands, a circumstance which is better shown by fig. *h*, in which the heads of the bands are indicated by arrows. In the case of fig. *g* the iodine tube, instead of being highly exhausted, contained xenon at a pressure of 1·5 mm. As is apparent, the effect of the xenon is to reduce tremendously the intensity of the doublets, and bring out strongly a number of fluted bands between the doublets, of which scarcely a trace can be seen in the case of iodine in vacuo. In the case of fig. *h*, we have the iodine in helium at 4 mm. The doublets are still further reduced in intensity, the bands are stronger, and a new band appears at the centre, no trace of which can be seen in fig. *g*. The heads of the bands are not resolved, though on the original plate a number of the component lines can be seen to the left of the doublets. The doublet of the second order, which is missing, would fall at a considerable distance from the head of the band. There is in fact a group of lines at this point in fig. *f*, but it is my opinion that they result from excitation of the vapour by some of the satellite lines; at all events, none of them fits into the series of doublets excited by the main line.

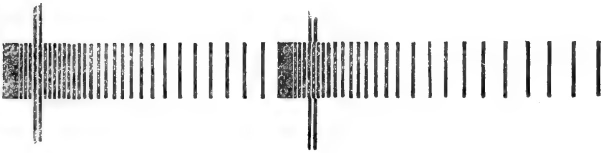
If we compare fig. *h* with fig. *d*, we see at once that the band spectrum emitted by iodine in helium with monochromatic excitation is much simpler than the complete band spectrum. For example, there is in fig. *d* a strong band-head at A, of which no trace appears in fig. *h*. Moreover, fewer of the bands appear in the case of iodine in vacuo than in the case of iodine in helium.

If the excitation is by the quartz mercury arc the bands become more complicated, and in place of the doublets we have groups of lines, which will be discussed more in detail presently.

Relation between the Doublets and the Band Spectrum.

The absorption spectrum of iodine is made up of more or less regular fluted bands, resolvable under high dispersion into fine lines. The heads of these bands lie towards the region of shorter wave-lengths, and there is considerable overlapping which gives rise to considerable irregularity in appearance, especially in the green region. The emission spectrum of iodine, electrically excited in a vacuum-tube, closely resembles the absorption spectrum, though they are not exactly complementary, as has been shown recently (Wood and Kimura, Astrophysical Journal, Sept. 1917). Now the green line of mercury, which excites the series of doublets, lies just within the head of a well-marked band in the emission spectrum, and it will be observed that the doublets of order +1 and +3 are similarly located. This was ascertained by superposing the resonance spectrum on a band emission spectrum. It is less well shown, except for the doublet of the +3 order, by fig. *d*, Plate VI., which was taken under conditions not well suited to emphasize the heads of the bands, the line spectrum being too prominent. The three bands above specified appear as emission bands accompanying the doublets when the iodine is excited in vacuo, as shown diagrammatically by fig. 2, in which the doublets have been drawn a little longer than the lines forming the bands. The band accompanying the doublet of 0 order is not as strongly developed as the other two, and only its head shows in fig. *f*, Plate VI.

Fig. 2.



By comparing the plates of the resonance spectrum with those of the band spectrum, it has been found that the doublet of the fourth order also lies just within the head of a band. Above this point the relations have not yet been exactly determined, for the band spectrum accompanying the resonance doublets has not yet been photographed in the red. Though the fourth order doublet, which is faint, lies near the head of a band shown on the plate made of the electrically excited vapour, it does not occupy a corresponding position with respect to the band which forms a member of the simpler system shown in fig. *h*, the spacing of which is two-fifths of the distance between the doublets, *i. e.*, there are five bands between the doublets of first and third order.

It will be necessary to trace this simpler band spectrum throughout the orange and red region, before we can be sure that all of the strong doublets are located near the heads of the bands, and the missing ones near the tails.

The doublet of the sixth order is very strong, and it lies just within the head of a strong band shown by electrical excitation, and the same thing appears to be true of the eighth and tenth order doublets. The interesting point, however, is that a simple system of fluted bands, spaced apparently according to a law similar to that which governs the spacing of the doublets, is excited by the stimulation of a single absorption line.

Multiplex Excitation.

If, instead of the glass Cooper-Hewitt lamp, we employ a quartz mercury arc (Westinghouse, Cooper-Hewitt) for the excitation of the iodine vapour, we find complicated groups of lines in place of the simple doublets. This is due to the fact that the green mercury line has broadened to such a degree that it covers a number of the iodine absorption lines. This we may call multiplex excitation.

The first point of interest which we should note is that the intensity distribution among the groups is practically the same as for the doublets, *i. e.*, groups of strong lines are built up around the strong doublets, weak groups around weak doublets, and only a few very faint lines at the points where the doublets are missing. This means that the dynamics of the vibrating system excited is very much the same in the case of the several absorption lines covered by the broadened mercury line.

The complexity of the groups depends upon the width of the green line which increases with the potential drop across the terminals of the quartz arc, as has been shown in previous communications.

If sufficient resistance is put in circuit with the arc to keep the potential down to 35 volts, the iodine emits the doublets only, fig. *j*, Plate VII. With the potential at 60 volts we have two new lines to the left of the doublets, as shown by fig. *k*, Plate VII., while with a potential difference of 110 volts we have the complicated groups shown by figs. *l* and *m*, the latter showing the group of −1 order. These groups are so similar in appearance that, until very recently, I have considered that the lines corresponded to each other, that is to say, that the fourth line from the left in each group was excited by the same absorption line. I now feel certain, however, that we must be a little careful about accepting this conclusion, for reasons which will appear presently.

In discussing the manner in which the groups are formed by multiplex excitation, we must recall that in the case of strictly monochromatic excitation, where a single absorption line only is stimulated, we have a series of doublets, the shorter wave-length component of the first doublet coinciding with the absorption line.

It has been found that the doublets conform very nearly to the following formula, in which $1/\lambda$ represents the frequency of the left-hand component of the doublet of order m,

$$\frac{1}{\lambda}=183075-2132m+\frac{m(m-1)}{2}13,$$

or, putting it in words, that (approximately) the distance between the doublets increases by a constant amount as we pass from each one to the one of next higher order. The degree of accuracy with which this formula is followed will be discussed in the communication following this one. The circumstance that we have a group of lines formed around the (unresolved) absorption lines which are excited by the broadened mercury line, furnishes us with the clue as to how the groups originate.

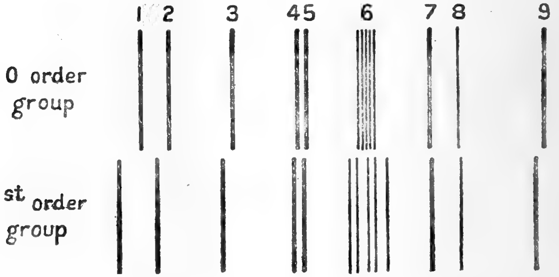
These groups originate in the following way: The seven absorption lines which are covered by the broadened green mercury line are simultaneously excited, and the vapour emits these seven wave-lengths without change. These lines we may call the R.R. lines (resonance radiation). Each one of these is moreover the first member of a series such as is expressed by the formula previously given. The R.R. lines are not resolved by the spectrograph employed in photographing the resonance spectra and consequently appear superposed. But each one is accompanied by one or more companion lines, lying to the right or left, and it is these companion lines which form the group of 0 order.

The actual width of the group of seven R.R. lines is only about 1/30 of the width of the group formed by the companion lines.

Let us now see *how* the groups of higher order are built up. Suppose each of the seven R.R. lines to be the first member of a series such as was represented by our formula, and suppose that for each one we have the same values of the constants. Suppose moreover that each member of any given series is accompanied by a companion line. In this case the group of 0 order will be exactly duplicated at intervals along the spectrum. The centre of each group will be composed of seven superposed lines (in reality separated by the same small intervals as the R.R. lines) each one of which is accompanied by a companion line to the right or left as the case may be. In an earlier paper I spoke of the seven superposed lines as the "core" of the group. As a matter of fact, the spacing is not exactly the same for the seven series of main lines, consequently, as we ascend to higher group orders they begin to separate, even with the resolving power employed in photographing the resonance spectra. This accounts for the fact that the groups of higher order differ in appearance from those of lower.

I have photographed the groups of 0 and +1 order with the 7-inch grating and 3-metre objective in the fourth order spectrum with an exposure of 48 hours. The lines were very faint but perfectly sharp. The appearance of the two groups is shown in fig. 3. The resolving power in this case was but little less than that required to separate the iodine absorption lines, and we find that the centre of the group of 0 order is a narrow band (line No. 6 in the figure) made up of five barely resolved lines. It was possible to count the lines by holding the plate somewhat foreshortened under a magnifying-glass. These are the R.R. lines. The other lines which form the group are the companion lines, and the fact that there are more of them than R.R. lines suggests that probably some of the R.R. lines have two companions instead of one.

Fig. 3.



Passing now to the first order group, we find that the main lines which form the core (each one of which belongs to a series of which a R.R. line is the first member) are more widely separated than in the 0 order group, the spectral range having about doubled. This is due to the fact that the value of the constant in the second term of our formula is not the same for each series.

We will now consider the subject of the companion lines.

In the case of the doublets excited by the Cooper-Hewitt lamp the companion lines lie on the long wave-length side of the main lines, at distances which gradually increase with increasing group order. The widths of the doublets in the various orders are given in the following table:—

Order.	Width.	Order.	Width.
0	1·48	15	2·04
1	1·54	17	2·19
3	1·64	18	2·25
5	1·72	20	2·43
8	1·76	22	2·45
10	1·85	23	2·53
11	1·90	25	2·50
12	1·95	27	2·80
13	1·96		

The increment is not quite regular, and it is my hope that a new set of plates made with a more powerful spectrograph will show no discrepancies. It is pretty clearly established, however, that the distance of the companion line from the main line increases progressively. If this is true of the companion lines of the other main lines, this circumstance, combined with the fact that the group of main lines widens as we pass to groups of higher orders, explains fully the fact that the groups gradually change in appearance as we ascend the series.

Groups 4, 5, 6, 7, and 8 are shown by figs. *n* and *o* of Plate VII., the former excited by the Cooper-Hewitt lamp, the latter by the quartz arc at 115 volts. In the case of fig. *n* we have faint series excited by the two yellow mercury lines, one of which (5790) lies within the fifth order group excited by the green line. The lines marked by small crosses are the ghosts of the yellow mercury lines, and should not be confused with the resonance lines.

In fig. *o* it will be observed that the doublets shown in fig. *n* have become relatively weak, and that we have a new series of strong doublets displaced towards the left with respect to the old ones. This is due to the fact that, in the case of the quartz arc operating at 115 volts, the green mercury line is strongly reversed and the excitation of absorption line No. 4 (Plate VI. fig. *a*) becomes relatively weak, as it coincides with the reversed core of the mercury line. In the seventh order of fig. *n* there are four lines. Two of these, marked by dots, form the doublet excited by the green mercury line; the other two belong to a series of doublets excited by the yellow line 5790. The same condition is found in the fourth order group, the dotted doublet in this case lying to the right of the doublet of order -1

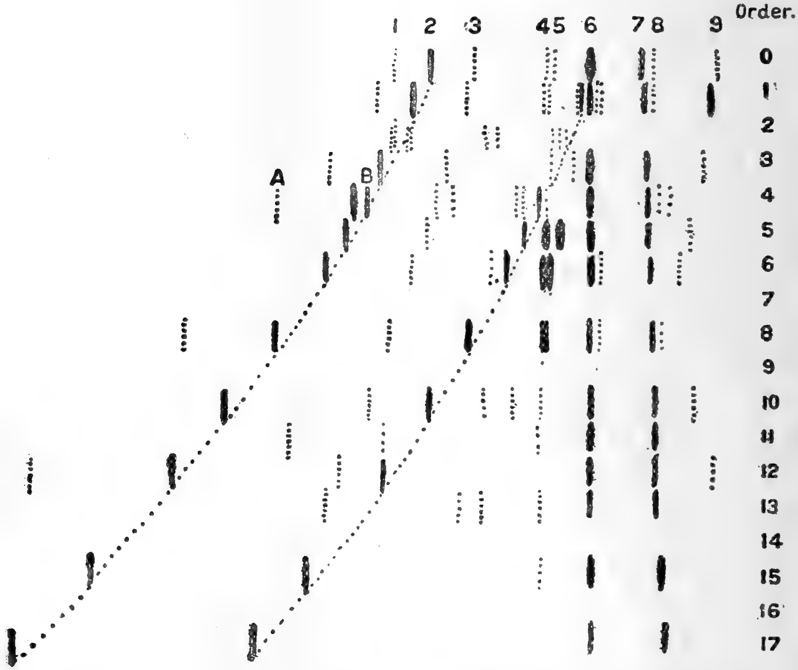
excited by 5790. Similar complications, of course, occur at other higher orders.

If we could excite the iodine absorption lines one at a time there would be no difficulty in finding out how the groups are built up, but this is impossible with present facilities.

By varying the voltage at which the mercury lamp operates, and by filtering the light through bromine vapour, some clues have been obtained regarding the relations existing between the absorption lines and the lines forming the groups, but a complete analysis has not yet been made.

In fig. 4 I have given a diagram of the groups up to the

Fig. 4.



seventeenth order excited by the green mercury line of the quartz lamp operating at 115 volts. The doublets (lines 6 and 7) excited by the Cooper-Hewitt lamp appear in all of these groups, though they are relatively faint owing to the reversal of the exciting line, and these doublets are brought into coincidence in the diagram.

When the iodine is excited by the lamp operating at 60 volts, lines 2, 4, 5, 6, and 7 appear in the group of

0 order, line 6 being of course the unresolved complex of emission lines corresponding to the absorption lines covered by the green mercury line. Line 7 is the companion line which, together with the "R.R." line corresponding to absorption line 3, forms the doublet of 0 order. The doublets of higher order lie immediately below, the increasing distance between the components being very apparent.

Now line 2 is a companion line to the R.R. line corresponding to absorption line 4, indicated also by line 6 in the diagram. These two lines form another doublet of zero order. The higher orders do not lie immediately below, but drift to the left, as indicated by the dotted lines. This is due to the fact that the constant in the second term of the formula is a little less than in the case of the first series of doublets considered, in other words the doublets are closer together.

In the group of the first order the main line of this series of doublets can be separated from the main line of the other series only in the fourth order spectrum of the grating. In the third order group it is so far detached, that it was confused for a long time with line 5 of the first order group. If we compare the orders 0 and 6 we shall see another case of this kind: If it were not for this diagram arrangement of the groups, we should probably assume that the first line to the left of group 6 corresponded to line 1 in group 0, whereas the diagram shows clearly that it corresponds to line **2**. Moreover, it appears in the 60-volt excitation, which does not bring out lines 1 and 3.

In the construction of the diagram it is, of course, necessary to leave blank spaces for the missing orders, otherwise the corresponding lines will not lie on a smooth curve.

It is a little difficult to explain in words just how this diagram is to be interpreted, though it is clear enough if the theory of the group formation which I have given is understood. All of the lines with the exception of 6 in the 0 order group must be companion lines, line 6 being made up of the unresolved R.R. lines. In the case of the doublets, the superposition of which forms the other groups, we must distinguish between what I have called the main line and the companion. As we run up the diagram the main lines should lie on curves intersecting line 6, for example, the dotted curve shown which belongs to the 2, 6 doublet.

I have not yet been able to identify certainly any other main lines, though I suspect that the one corresponding to companion line 9 descends from line 6 on a curve sloping to the left at a lesser angle than the dotted curves, *i. e.*, at about the angle taken by companion line 3.

Various modifications in the conditions of excitation have been made with a view of establishing which absorption lines are responsible for the various doublets.

For example, it was found that the lateral emission and the end-on emission of a Cooper-Hewitt lamp showed a very different intensity distribution in the green mercury line, as shown by figs. *r* and *s*, Plate VIII., which were made with a very fine plane grating by Dr. Anderson. If the iodine vapour is excited by the lateral emission of the lamp, as with the "light-furnace" companion line No. 1 appears in addition to the strong doublets. See 0 and +1 orders of fig. *j*, Plate VII. After several failures I succeeded in obtaining a record of the iodine resonance excited by the end-on emission, and in this spectrum companion line No. 2 appeared also. Now companion line No. 1 does not appear in the case of excitation by the quartz arc operating at 35 volts, and the short wave-length satellite of the green line is weaker, with respect to the main line, in this case, than in the case of the Cooper-Hewitt lamp, as is shown by figs. *t* and *u*, Plate VIII. (*t* being the Cooper-Hewitt line and *u* the quartz arc). This makes it appear probable that companion line No. 1 arises from the excitation of the absorption line which is in coincidence with the short wave-length satellite.

Companion line No. 2 is probably due to the excitation of absorption line No. 4. It comes out with excitation by the "end-on" emission of the Cooper-Hewitt lamp owing to the broadening of the main line which occurs under this condition, and for the same reason it is the first line to appear when the terminal voltage of the quartz arc is increased. No very definite conclusions have been drawn from the numerous experiments which have been made with the exciting light filtered through bromine vapour and nitrogen tetroxide. With a potential of 90 volts on the quartz arc companion lines 4 and 5 appear. If the exciting light is filtered through bromine vapour contained in an exhausted bulb about 30 cm. in diameter, line No. 5 disappears in the groups of order 0 and + 1. In the third order group line No. 5 is much stronger than 4 and bromine filtration of the exciting light equalizes the intensity. Line No. 4 must therefore be due to the excitation of an absorption line which is not in coincidence with a bromine line, and which is first covered by the mercury line when the lamp operates at 90 volts. This seems to be absorption line No. 5, while the other component, which is removed by filtration of the exciting light through bromine, is probably due to absorption line 6.

With a potential of 110 volts on the lamp, companion line No. 3 appears, and this also is removed by the bromine filtration of the exciting light, as is shown by figs. *p* and *q*, Plate VIII., in which *q* is the resonance spectrum obtained when the exciting light is filtered through bromine. It appears to be due to the stimulation of absorption line 7 which is in coincidence with a bromine line.

The difficulty in interpreting the results obtained is due to the fact that the mercury line widens both to the right and left as the voltage increases, so that two absorption lines may be attacked simultaneously. If this happens, we can differentiate between them only if one of them is in coincidence with a bromine line and the other not. What is most needed just now is one or more other filters similar to bromine vapour, but I have not been able to find anything with sufficiently narrow lines, though I have tried a number of vapours which looked promising. What would be still better would be to alter the wave-length of a narrow exciting line so as to cause it to pass by degrees from one absorption line to the next.

Excitation by the Yellow Lines.

The resonance spectra excited by the two yellow lines have not been completely investigated as yet, though a large number of photographs have been made. Each yellow line excites a series of nearly equidistant groups which resemble roughly the groups excited by the green line. Six pairs of these groups, from -1 order to $+4$ order, photographed with rather low dispersion are shown by fig. *i*, Plate VI. In this case the excitation was by the quartz mercury arc operating at 140 volts, the green line having been cut off by means of a glass trough filled with a solution of eosine. Some difficulty was found in securing the spectrum excited by the Cooper-Hewitt arc, as the yellow lines are comparatively weak in this case, but satisfactory results were finally obtained with the light furnace, the iodine tube being wrapped around with a sheet of gelatine stained to a deep orange-yellow.

In this case each yellow line excited a series of doublets, but both series were much more irregular than the series excited by the green line.

The separation of the components of the doublets excited by the 5790·7 line varied in an irregular manner from 2·1 to 5·6 ÅU. In the case of the excitation by the 5769·6 line we have also a series of doublets, though the companion line is missing at the zero order, in other words the R.R. line has no companion. The separation of the components of the doublets is less irregular in this case, varying from 4·8 to 5·4 ÅU. The table of wave-lengths will be given in the communication following this one.

XXVIII. *The Series Law of Resonance Spectra. By* Prof. R. W. Wood, *Johns Hopkins University, and* Prof. M. Kimura, *University of Kyoto*[*].

IN the previous communication a general account of the results which have been obtained, up to the present time, on the resonance spectra of iodine has been given.

The present paper will deal with the measurements of wave-length of the lines in the groups, and the subject of the series law which governs their spacing.

The wave-lengths in the lines in the groups of 0 and +1 order were determined from plates made in the fourth order spectrum of a large plane grating with a telescope of 3 metres focus. They are correct probably to 0·01 ÅU. The groups +2, +3, and +4 were made in the second order spectrum, and the higher order groups in the first order spectrum.

The series which has been most definitely determined, and to which the greatest amount of study has been given, is the series of strong doublets excited by the Cooper-Hewitt lamp.

The two components of each doublet appear to be of equal intensity, although, in the case of two or three, a different ratio appears in the photograph as a result of absorption. It was found, as has been stated in earlier papers, that the first order group, which is usually recorded with the component of shorter wave-length three or four times as intense as the other, comes out with its lines of nearly equal intensity if the lateral branch of the iodine tube is cooled to zero, while the right hand component disappears entirely if the light from the tube is passed through a large glass bulb containing iodine vapour before it enters the spectroscope.

In studying the series law it has been found necessary to

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[*] Communicated by the Authors.

reduce all wave-lengths to vacuum, and convert them into frequencies.

We will take up first the study of the doublets, the wave-lengths of which and their reciprocals are given in the following table, on the International Scale and reduced to vacuum.

Doublets excited by Green Line of Cooper-Hewitt Lamp.

Group Order.		$\frac{1}{\lambda}$ (Obs.)	Freq. Dif.	$\frac{1}{\lambda}$ (Cal.)	Difference between Obs. & Cal.
0......	5462·23 (Hg or R.R. Line) 5463·74	183075 183025	50	183075	0
1......	5526·55 5528·10	180945 180894	51	180942	+3
2......	Missing				
3......	5658·71 5660·38	176719 176667	52	176715	+4
4......	5726·59 5728·25	174624 174573	51	174621	+3
5......	5795·79 5797·51	172539 172488	51	172539	0
6......	5866·14 5867·85	170469 170420	49	170470	−1
7......	Missing				
8......	6010·66 6012·50	166371 166320	51	166370	+1
9......	Missing				
10......	6160·63 6162·48	162321 162272	49	166322	−1
11......	6237·68 6239·56	160316 160268	48	160316	0
12......	6216·16 6218·14	158324 158270	50	158324	0
3......	6396·08 6398·05	156346 156297	49	156344	+2
14......	Missing				
15......	6560·56 6562·68	152426 152377	49	152423	+3
16......	6645·0 6647·0	150489 150443	46	150481	+8
17......	6731·12 6733·28	148564 148516	48	148552	+12

From this point on values determined from plates made with telescope of 1 metre focus. They are correct only to about 0·1 ÅU.

18......	6618·63 6820·01	146657 146608	49	146636	+21
19......	Faint and masked	by mercury line			
20......	6998·96 7001·39	142878 142828	50	142842	+36
21......	Missing				
22......	7186·23 7188·68	139155 139107	48	139099	+56
23......	7282·39 7284·92	137318 137270	48	137247	+71
24......	Missing				
25......	7480·4 7482·9	133682 133638	44	133580	+102
26......	Missing				
27......	7685·7 7688·5	130110 130060	50	129964	+146

The first point established by this table is that, while the separation of the components of the doublets increases progressively from 1·51 ÅU. at 0 order to 2·5 ÅU. at the twenty-seventh order, THE FREQUENCY DIFFERENCE BETWEEN THE COMPONENTS IS A CONSTANT, 50. The extreme low values 46 and 44 found in the sixteenth and twenty-fifth orders are undoubtedly due to the fact that the lines were extremely faint, and the wave-lengths could not be very accurately determined. The last doublet (the twenty-seventh order) was fairly strong, and the frequency difference in this case is exactly the same as in the case of the 0 order.

We will now consider the law governing the spacing of the doublets along the spectrum, applying the calculations to the first member of each doublet (shorter λ component). If we confine our attention to the first few orders, it seems as if the distance between the doublets increased by a constant small increment. This would mean a constant second difference of wave-lengths. It was found, however, that this condition held only for the first few orders. The reciprocals of the wave-lengths were next examined, and it was found that a constant second difference existed, at least over a considerable range of the spectrum.

If this condition held rigorously the series would be represented by the formula

$$\frac{1}{\lambda_m} = 183075 - 2132\,m + 13\frac{m\,(m-1)}{2},$$

in which λ_m is the wave-length of the doublet of the *m*th order, 2132 is the frequency difference between orders 0 and +1, 13 the constant second difference of frequency, and *m* the order of the doublet. The most accurate value of the second constant would be obtained by calculating it from a doublet of high order, as a small error would be enormously magnified by the term $\frac{m\,(m-1)}{2}$, the value of which is 351 for the twenty-seventh order.

Calculating the constants 2130 and 12·2 from orders 0 and 5 gave calculated values of $1/\lambda$ which differed from the observed by the following amounts:—

Doublet Order.	Difference.	Doublet Order.	Difference.
1	0	7	+ 2
3	+1	10	+ 4
4	−1	13	+ 14
5	−1	18	+ 65
5	0	23	+166
6	−1	27	+284

The large discrepancies in the higher orders are due to incorrect determination of the constants. In spite of this, though, the series is well represented up to the seventh order. The following formula gave the best results over the entire range:—

$$\frac{1}{\lambda_m} = 183075 - 2131{\cdot}414 = m - 12{\cdot}734\frac{m\,(m-1)}{2}.$$

The values given in the table were calculated by this formula and, as will be seen, the agreement is good up to the doublet of the fifteenth order. There is a small discrepancy in the orders 1, 3, and 4, which appears to be inevitable if the constants are so chosen as to make the formula cover a wide range. Of course, the formula is not correct for the entire series, and though we have tried formulæ involving higher powers of *m* than the square, we have been unable to develop anything superior to the one given. The discovery of the fact that the frequency difference between the components of the doublets is a constant, has been of assistance in picking out other series of doublets in the series of complicated groups excited by the quartz arc.

For example, we may take the wider doublets shown united by dotted lines in the previous paper (fig. 4).

The frequency differences for these doublets are given in the following table:—

Order.	Freq. Dif.
0	161
1	159
3	159
5	158
6	157
8	158
10	157
12	157
15	152
17	156

It will be remembered that the frequency difference of the first series of doublets considered was 50.

The spacing of this series along the spectrum is only fairly well represented by the formula

$$\frac{1}{\lambda_m} = 183075 - 2119\,m + 13\frac{m\,(m-1)}{2}.$$

The observed and calculated values of $1/\lambda$ for the components of longer wave-length (the companion line is to the left in this case) are given in the following table. It will be observed that the doublets are missing in the fourth, eleventh, and thirteenth orders, as well as in the orders in which the doublets of the first series failed to appear.

Order.	$\frac{1}{\lambda}$ (Obs.)	$\frac{1}{\lambda}$ (Cal.)	Dif.
0	183075	183075	0
1	180956	180956	0
3	176754	176757	3
5	172599	172600	1
6	170543	170556	13
8	166470	166487	17
10	162448	162470	22
12	158479	158505	26
15	152621	152655	34
17	148786	148820	34

In the following table are given the wave-lengths and their reciprocals, on the International Scale and reduced to vacuum, of all of the lines in the groups between 0 and 17, in the case of iodine vapour excited by the quartz mercury arc operating at 115 volts.

The doublets excited by the Cooper-Hewitt lamp are marked thus * and the other doublets which we have studied thus †. This table corresponds to the diagram in the previous paper. In the fourth order group lines A and B were added from an old series of measurements. The line between them is the only one which appears on our recent plates, and this line only is given in the table.

0 Order.

No. of Line.	λ.	$\frac{1}{\lambda}$
1	5	
†2	5457·43	183236
3	5458·33	183189
4	5460·88	183121
5	5461·07	183114
†*6	5462·23	183075
*7	5463·74	183025
8	5464·05	183014
9	5466·04	182948

First Order.

No. of Line.	λ.	$\frac{1}{\lambda}$
1	5520·30	181149
2	†5521·33	181115
3	5522·92	181064
4	5525·17	180989
5	5525·38	180983
6	†5526·20	180956
	·22	955
	·47	47
	*5526·55	45
	·71	39
	·80	36
7	*5528·10	180894
8	5528·39	180884
9	5530·06	180830

Second Order.

No. of Line.	λ.	$\frac{1}{\lambda}$
	5585·08	179048
	86·33	179008
	86·81	178992
	89·05	921
	89·42	909
	91·02	858
	91·17	853
	91·38	847
	93·33	784
	93·77	770

Third Order.

No. of Line.	λ.	$\frac{1}{\lambda}$
1	5651·06	176958
2	†5652·49	176913
3	5654·55	176849
	†5657·57	176754
6	*5658·71	176719
7	*5660·38	176667
9	5661·97	176617
	5663·15	176580

and, in addition, faint lines as follows:

5656·87
57·17
57·35
57·98
58·24
58·96
59·27
59·50
60·70
61·02

Fourth Order.

No. of Line.	λ.	$\frac{1}{\lambda}$
	†?5719·62	174837
	5722·05	174763
	22·55	4747
	24·47	4688
	24·72	4681
	†5725·13	4668
	25·35	4661
	*5726·59	4624
	26·84	4616
	*5728·25	4573
	28·56	4564
	28·95	4552

Fifth Order.

λ.	$\frac{1}{\lambda}$
†5788·45	172757
91·03	2681
†5793·77	2599
94·48	2578
94·88	2566
*5795·79	2539
*5797·51	2488
98·70	2452

Sixth Order.

λ.	$\frac{1}{\lambda}$
†5858·23	170700
60·8	625
63·2	555
†5863·6	543
64·75	510
64·9	505
*5866·14	469
*5867·85	420
68·1	413
68·1	395

Eighth Order.

λ.	$\frac{1}{\lambda}$
5998·6	166704
†6001·38	628
04·88	531
†6007·07	471
09·32	411
09·40	406
*6010·66	371
10·91	364
*6012·50	320
12·80	310

Tenth Order.

λ.	$\frac{1}{\lambda}$
†6149·87	162605
54·07	494
†6155·81	448
57·52	403
58·31	382
59·14	360
*6160·63	321
*6162·48	272
63·65	241

Eleventh Order.

λ.	$\frac{1}{\lambda}$
6228·79	160545
31·44	477
36·12	356
*6237·68	316
*6239·56	268
42·0	205

Twelfth Order.

λ.	$\frac{1}{\lambda}$
6294·87	158859
99·40	745
†6303·71	636
08·57	514
†6309·99	479
*6316·16	324
*6318·14	274
19·6	237

Thirteenth Order.

λ.	$\frac{1}{\lambda}$
6388·24	156538
92·14	442
92·82	425
94·54	383
*6396·08	346
*6398·05	297

Fifteenth Order.

λ.	$\frac{1}{\lambda}$
†6545·65	152773
†6552·19	621
59·05	461
*6560·56	426
*6562·64	378

Seventeenth Order.

λ.	$\frac{1}{\lambda}$
†6714·00	148923
†6721·03	787
*6731·09	564
*6733·27	516

Excitation by the Yellow Lines.

We have measured the wave-lengths of the lines in the resonance spectrum excited by the yellow lines of the Cooper-Hewitt arc, and the quartz arc operating at 115 volts. The values given in the following table are on the International Scale and reduced to vacuum. They were determined from plates made with the plane grating and Cooke lens of 1 metre focus, and can be considered correct only to about 0·1 ÅU. We have, however, made some measurements of the doublets photographed with the 3-metre lens, which are correct probably to 0·02 ÅU., and as the same irregularities were found in the spacing, we have not thought it worth while to measure the complete spectrum to the highest degree of accuracy.

The wave-lengths are given in the following table. The lines or doublets excited by the Cooper-Hewitt lamp are marked thus *.

The series excited by the Cooper-Hewitt lamp in the case of the 5769 line differs from that excited by the green line in a number of respects.

In the first place, at the point of excitation we have only the R.R. line with no companion. At orders 1, 2, 3, 6, and 9 we have doublets, 4 and 8 are missing, and at 5, 7, and 10 we have single lines.

Excitation by Hg 5769·6 (5771·2 *Reduced to Vacuum*).

−1 Order.

	$\frac{1}{\lambda}$	
5701·8	175383	
5705·2	278	

0 Order.

5766·0	173430	
67·8	376	
*5771·2	274	
75·2	154	

+1 Order.

5834·4	171397	
38·5	277	
*5842·5	159	137
*5847·2	022	

Second Order.

5911·2	169170	
14·7	070	
*5915·8	039	143
*5920·2	168896	

Third Order.

5985·2	167078	
88·3	166992	
*90·0	945	142
*95·1	803	

Fourth Order.

6060·4	165006	
64·6	164891	
69·1	164769	

Fifth Order.

	$\frac{1}{\lambda}$	
6139·0	162893	
41·0	839	
*43·1	784	

Sixth Order.

6215·1	160898	
17·0	849	
18·9	800	
*22·1	717	129
*27·1	588	

Seventh Order.

6294·9	158858	
99·6	740	
*6303·5	641	

Eighth Order.

6374·3	156879	

Ninth Order.

6459·7	154806	
63·7	710	
*67·9	609	126
*73·2	483	

Tenth Order.

6544·1	152809	
48·7	702	
*53·0	602	

The $1/\lambda$ difference, in the case of the components of the doublets, is not constant, as in the previous case, but varies from 143 to 126.

As to the spacing of the doublets along the spectrum, we find that in this case the $1/\lambda$ difference is very nearly *constant*, as is shown by the following table:—

Order.	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$ Dif.	Order.	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$ Dif.
−1	175383	2109	5	162784	2069
0	173274	2015	6	160715	2074
1	171159	2120	7	158641	
2	169039	2094	8		2014
3	166945	2080	9	154612	2010
4			10	152602	

The variation is irregular, and it is obvious that the series is of a different type from the one excited by the green line, a portion of which at least was well represented by a formula.

In the case of the excitation by the 5790·7 line, we obtained different values of λ in the case of the Cooper-Hewitt lamp, consequently these values only are given in the table. It is probable that in the case of the quartz lamp at 115 volts reversal of the line causes the disappearance of the doublets excited by the lamp running at a lower temperature :—

Excitation by Cooper-Hewitt 5790·7 (5792·3 *Reduced to Vacuum.)*

λ.	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$ Dif.	λ.	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$ Dif.
	−2 Order.			Fourth Order.	
5658·6	176722	53	6084·3	164357	108
5660·3	669		88·3	249	
	−1 Order.			Fifth Order.	
5722·1	174761	34	6163·9	162235	58
5723·2	727		66·1	177	
	0 Order.			Sixth Order.	
5792·3	172644	168	6242·3	160197	69
5797·9	476		45·0	128	
	First Order.			Seventh Order.	
5871·3	170320	61	6325·8	158083	86
5873·4	259		29·2	157997	
	Second Order.			Eighth Order.	
5936·8	168441	61	6404·0	156152	66
38·9	380		06·7	086	
	Third Order.				
6010·8	166367	69			
13·3	298				

In the case of the series excited by this line the doublets are present in all orders, but the 1/λ difference between their components varies in a very irregular manner from a minimum value of 34 to a maximum value of 168.

The spacing of the series along the spectrum is more regular, however, the 1/λ differences being as follows (the last significant figure is omitted):

196
211
232
188
207
203
209

205
211
193
212
200
209

It does not appear to be worth while at this stage of the investigation to give the wave-lengths of the lines in the more complicated groups excited by the quartz arc operating at various voltages, as the simpler series excited by the Cooper-Hewitt lamp does not appear at the present time to conform to any law. The cause of this may appear when the relation of this spectrum to the band spectrum developed when the iodine is in helium has been studied. This will require exposures of many days, however.

XXIX. *On the Pressure Effect in Corona Discharge.*

By A. M. TYNDALL, *D.Sc., and* Miss N. S. SEARLE, *B.Sc.*[*]

SEVERAL papers have been published within the past few years on what has been termed "the ionization pressure in corona discharge." When a glow or "corona" discharge starts between a cylinder and an axial wire in a closed tube at atmospheric pressure, a sudden rise in pressure is observed. Farwell[†] and Kunz[‡] have contended that this pressure effect is quite distinct from that produced by the heating effect of the discharge, and have suggested that it is due to the increase in the number of gas particles resulting from ionization in the tube.

Kunz also has deduced the following formula connecting the rise in pressure "p_1-p_0" with the current "i," potential difference "e," and the volume of the tube "v_0":—

$$i=\frac{v_0}{e}(p_1-p_0).$$

In support of this formula Warner[§] has shown experimentally in a number of gases that, if the potential difference is constant, "i" is proportional to "p_1-p_0." Arnold[‖], however, has suggested that the pressure effect can be accounted for by thermal considerations alone, and has pointed out that the small magnitude of the current excluded the possibility of any appreciable contribution to pressure by ionization. Recently Warner[¶] has replied to this criticism. He has described a number of experiments on the pressure

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[*] Communicated by the Authors.

[†] Farwell, Proc. A. M. I. E. E. vol. xxxiii. p. 1717 (1914).

[‡] Kunz, Phys. Rev. vol. viii. p. 28 (1916).

[§] Warner, Phys. Rev. vol. viii. p. 285 (1916).

[‖] Arnold, Phys. Rev. vol. ix. p. 93 (1917).

[¶] Warner, Phys. Rev. vol. x. p. 483 (1917).

effect (1) due to discharge, and (2) due to a heating current passed through the wire, and has re-affirmed the view that the two effects are quite distinct.

Now in the past there has been general agreement among physicists that in an ordinary glow discharge the fraction of molecules which are ionized is a negligible quantity. It seems therefore desirable to settle conclusively what is the origin of the pressure rise.

The object of the present paper is to point out several fallacies in the arguments of the advocates of the ionization theory, and to present a quantitative verification of the statement that this sudden rise of pressure accompanying the start of corona discharge is purely a thermal effect.

It must first be pointed out that the formula given above, upon which the advocates of the ionization theory base much of their work, is necessarily incorrect. This may readily be seen by applying the test of dimensions to the terms of the equation. Before the dimensions of the two terms can be equated, the left-hand term must be multiplied by a time "t." The equation then takes the form it would have if the pressure effect was due entirely to heat generated in a vessel of constant volume, from which the radiation losses were always a constant fraction of the heat supplied. If reference be made to the proof of this formula in Kunz's paper, it will be seen that in steps 1 and 4 a time factor has been erroneously omitted.

But this does not dispose of a further argument which has been brought forward: namely, that the pressure effect at the start of the discharge is far too sudden to be accounted for by heat generated during discharge, and that the rise in pressure due to the latter is only appreciable after the discharge has been passing for some time.

Thus Warner has published a number of curves showing that the "corona pressure" reached its full value within 3 seconds of the start of discharge, whereas in the corresponding effect produced by heating the wire this did not occur until about 15 seconds after the heating current was switched on. Moreover, the rise in pressure produced by the dissipation of a given amount of energy in the wire was much greater than that accompanying the same dissipation of energy in the glow discharge.

But the argument that the pressure effects in the two cases must therefore be different in origin entirely breaks down if one takes into account the potent influence of the electric wind in distributing the heat generated in the gas and conveying it to the walls of the surrounding tube. It may be presumed that the following process is in operation. As in all cases of discharge from wires or points, the outgoing ions set up movements of the gas to which the name electric wind has been given. Heat generated in the gas by the corona discharge causes a rise in pressure; but since the wind distributes this heat throughout the gas, the rise in pressure is quickly checked by the large cooling effect of the metal wall of the tube to which the heat is being rapidly conveyed. With a large cooling surface and a small rate of dissipation of electrical energy, the heat may be removed by the walls as fast as it is generated, in which case the pressure, after a small initial sudden rise, remains constant. If the rate of generation is, however, too great for this to be the case, a subsequent gradual rise in pressure will occur as the temperature of the gas and of the wall of the tube rises. This is well seen in figures 3 and 4 in Warner's second paper.

In the case of a supply of heat by a current in the wire, no electric wind is present and the cooling effect of the walls can only operate through the ordinary convection currents, which are far less effective. Hence, for the same energy supply the rise in temperature of the gas, and consequently its pressure, is considerably greater and reaches its full value later in time than when the electric wind is present. This is in agreement with experimental results.

Again, Warner states that the return to normal pressure after a heating current in the wire is cut off is less rapid than it is after the cessation of corona discharge; the numbers quoted for the two cases under similar conditions being 25 and 18 seconds respectively. But this also receives a simple explanation in terms of the electric wind, since the movement of air does not instantaneously cease when the discharge current is cut off.

It follows that computations in which conduction and convection losses are neglected break down entirely when the electric wind is present, however admissible they may be in its absence.

Exception must also be taken to some other opinions expressed in support of the ionization theory.

First, the statement that near the wire every molecule may be ionized and still the resultant current may be very small is incompatible with the statement that the potential gradient near the wire is very high. The latter is, no doubt, correct, and the result is that the ions near the wire have a high ionic velocity. But if this is coupled with a huge ionic density the currents would greatly exceed those which are observed.

Secondly, Warner attempts to obtain support for the theory from the variation of discharge current with the initial pressure "p_0" for a constant potential difference between the wire and the tube. Since, however, the pressure "p_1" is not a constant and its variation with "p_0" is unknown, it is impossible that the curves he gives can furnish us with any information on the subject.

Perhaps it may be well to show that the ordinary pressure of the electric wind makes no appreciable contribution to the effect. In the experiments cited above, the manometer communicated directly with a hole in the wall of the metal tube receiving the current. Now the discharge sets up a radial gradient of pressure between the wire and the tube, the pressure at the former being below, and at the latter above, atmospheric pressure. The following argument gives the maximum value of excess pressure which the wind can set up at the tube-wall. Circulation of air would decrease this value.

Let the radii of the wire and of the metal tube be "a" and "A" respectively. Consider a thin cylindrical layer of gas, of thickness δr and radius "r," coaxial with the wire. Let a discharge current of "i," per unit length of the wire, pass uniformly through the layer. This will give rise to the transfer of a small quantity of gas through the layer, and to a difference in pressure, $\delta \mathrm{P}$, per unit area, between the two sides of the layer.

Now
$$\delta \mathrm{P} = \frac{i\delta r}{2\pi r k}\text{*},$$

where k is the specific ionic velocity.

This difference in pressure *e* may be expressed as a rise of pressure δp per unit area outside the layer, and a fall $\delta p'$ inside it, so that

$$\delta \mathrm{P} = \delta p + \delta p'.$$

Then
$$\delta p(\mathrm{A}^2 - r^2) = \delta p'(r^2 - a^2);$$

$$\therefore \quad \delta p = \frac{i\delta r(r^2 - a^2)}{2\pi r k(\mathrm{A}^2 - a^2)}.$$

* Chattock, Phil. Mag. [5] vol. xlviii. p. 401.

The excess pressure "p" at the wall of the metal tube is given by

$$p=\int_{a}^{\mathrm{A}} dp=\frac{i}{4\pi k}-\frac{a^2}{2\pi k}\frac{\log_e \mathrm{A}/a}{(\mathrm{A}^2-a^2)}.$$

When "A" is large compared with "a," this approximately $=\dfrac{i}{4\pi k}$.

Putting $\mathrm{K}=1{\cdot}4$ cm./sec., volt/cm.,

and $i=5\times 10^{-6}$ amp. per cm.,

p approximately $=0{\cdot}003$ cm. of water.

This is about 0·5 per cent. of the pressure actually observed by the authors under those conditions, and is therefore a quantity which may be neglected.

The authors have been able to verify quantitatively that the corona pressure effect is solely due to heat generated in the discharge, by supplying heat *simultaneously* from a current in the wire and from corona discharge, instead of experimenting separately with each source of supply. The electric wind thus operates in rapidly conveying heat to the wall of the tube from *both* sources of supply. Let it be assumed that the heat from corona discharge is mainly evolved close to the wire; then the wind will be almost equally effective in distributing heat, whether it comes from discharge or from a current in the wire. In other words, if the observed pressure rise is due to heat, it should be approximately independent of the source of the heat when wind is present. Experiments confirming this conclusion are herewith described.

Experiments.

The discharge vessel consisted of a platinum wire of diameter ·006 mm. placed along the axis of a horizontal brass tube of diameter 2·21 cm. and length 28 cm. To a side tube in the latter, one limb of an oil manometer was attached. The brass tube was closed by ebonite stoppers, through which the axial wire passed. The wire was attached to one pole of a high-potential dynamo giving voltages up to 5500 volts. The current between wire and tube was measured by a Paul microammeter, shunted when necessary. The potential difference was measured by a Braun electrometer previously calibrated. One terminal of the galvanometer, the case of the electrometer, and one pole of the dynamo were earthed. The wire also formed part of a separate heating circuit; the heat generated in the wire was measured by a voltmeter and a milliammeter. The electrical arrangements included a key so arranged that the discharge and heating currents could be switched on simultaneously or separately, as desired.

It may be noted in passing that the tube for which Farwell gives dimensions was 4·45 cm., and the wire ·019 mm. in diameter. This necessitated the use of higher potentials, but these were not so convenient in the present experiments.

In Table I. some results for positive and negative discharge are given; "p_1-p_0" is the initial corona pressure effect due to a current "i" and a potential difference "e." The last column gives the ratio of "p_1-p_0" to "ie," the watts dissipated in the tube.

TABLE I.

Sign of Discharge.	p_1-p_0.	i 2×10^{-3} amp.	e volts.	$\frac{p_1-p_0}{ie}$.
+	·08 (?)	6·2	4450	2·96 (?)
	·13	16·5	4620	1·78
	·16	22·6	4600	1·58
	·20	27·4	4780	1·50
	·24	34·1	4780	1·51
	·35	55·5	5050	1·26
	·46	76·0	5200	1·17
	·52	87·0	5230	1·13
	·66	110·0	5350	1·11
	·71	120·0	5500	1·08
−	·09	21·7	4700	·90
	·16	35·4	4820	·91
	·20	44·5	4770	·93
	·29	64·2	5030	·91
	·37	81·0	5150	·89
	·42	94·3	5200	·86
	·77	169·0	5400	·84
	1·01	220·0	5480	·84

It will be noticed that in positive discharge the ratio shows a marked decrease in value as the current increases, though it appears to approach some limiting value at still higher currents. This decrease may be attributed to the growing strength of the wind as the current increases. Though a decrease is also observable in negative discharge, it is much smaller.

In positive discharge the value of the ratio is consistently higher than that in negative. This may be simply explained in the following way. If a circulation of air is to be set up, air dragged from the neighbourhood of the wire by discharge must be returned to it by other paths. The greater the uniformity of discharge along the length of the tube the less easily can this occur. Now the uniform nature of the positive corona implies that the discharge in this case is sensibly uniform along the whole length of the tube; whereas in the negative corona the glow is concentrated in a number of small beads on the wire. Consequently, the circulation is more restricted in positive discharge and the cooling action of the wall of the tube less effective than in negative. This is less noticeable at higher currents when the wind pressure gradient causing circulation is steeper; hence the decrease in the value of the ratio under these conditions.

The degree of circulation, other conditions being the same, will no doubt depend upon the size of the tube, and this will have its effect upon the value of the ratio obtained and its dependence upon discharge current.

The results of discharging from the wire, and heating it simultaneously, are shown in Table II. (*a*) for positive and Table II. (*b*) for negative discharge. "w_1" and "w_2" are the watts dissipated in discharge and heating current respectively; "W" is the total watts. As before, "p_1-p_0" is the initial rise of pressure.

TABLE II.

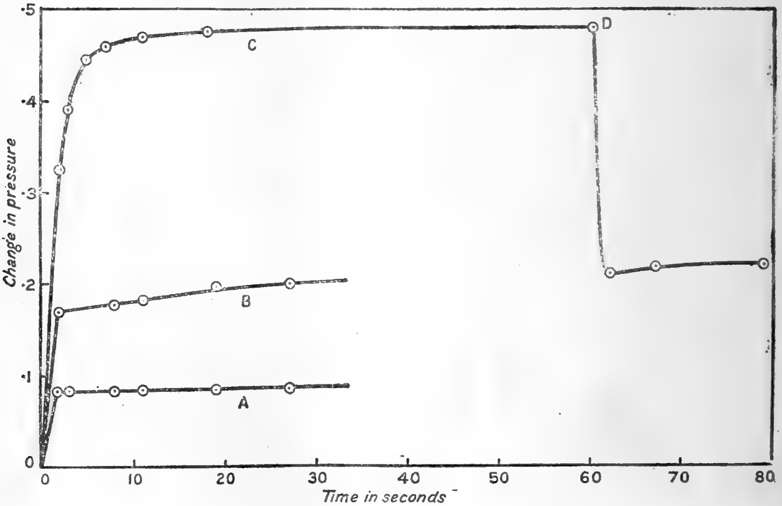
...					$(p_1-p_0)/\mathrm{W}$.	
	p_1-p_9.	w_1.	w_2.	W.	Observed.	Calculated from Table I.
(*a*)	·18	·071	·031	·102	1·75	1·66
	·26	·103	·084	·178	1·46	1·44
	·38	·158	·137	·295	1·28	1·25
	·43	·204	·213	·417	1·15	1·16
	·56	·263	·256	·519	1·08	1·12
(*b*)	·17	·091	·093	·184	·95	·92
	·28	·157	·138	·295	·94	·90
	·52	·240	·338	·578	·92	·86
	·78	·242	·573	·815	·96	·84
(*c*)	·24	......	·045	·045	5·5	
	·48	......	·094	·094	5·1	
	·66	......	·137	·137	4·8	
	1·19	......	·330	·330	3·6	
	1·98	......	·66	·66	3·0	

The values of the ratio of "p_1-p_0" to "W" are given in a separate column, and may be compared with the values of the ratio for the same dissipation of watts with discharge operating alone. The latter have been calculated from a smooth graph constructed from the values in Table I.

The agreement between corresponding numbers in the two columns is striking. In fact such a close agreement was somewhat unexpected, since the strength of the wind is not the same in the two cases which are compared. Such differences as exist between the observed and calculated values are in general in the direction which neglect of wind variation would cause. These results show conclusively that the pressure effect is purely a heating effect.

Table II. (*c*) shows values obtained for the rise in pressure produced by heat alone. It will be seen that owing to the absence of the electric wind the values of the ratio are much higher.

This is also shown in curves A, B, and C, which are typical of a number which have been obtained. Curve A is the pressure time curve when 0·091 watt was supplied by discharge alone. Curve B, that for combined supply of 0·094 by discharge and 0·091 by a heating current in the wire. Curve C, up to the point D, for 0·094 watt supplied by current alone.



At the time marked by the point D, in curve C, a discharge current supplying ·094 watt was switched on. The cooling effect of the wind is well shown by the drop in pressure, the final value of which was practically that which would have been reached after combined action from zero time.

Lastly, mention must be made of an experiment by Warner, by which he claims to have shown that instead of a heating effect at the instant of starting of the discharge from an unheated wire, the gas near the wire actually cools. This he deduced from the movement of the spot of a galvanometer connected to a thermo-junction, placed in the tube at a distance of about 4 mm. from the axial wire.

The authors repeated this experiment, and found that the deflexion observed is not due to a thermo-electric current, but to an electrostatic effect when the thermo-junction takes up the potential of the air. Thus this deflexion is also observed when the galvanometer is not included in the circuit containing the thermo-couple, but is merely in electrical contact with it. The cooling of a red-hot wire which Warner quotes as occurring when discharge starts from it, is undoubtedly caused by the electric wind which is thus set up.

Summary.

When glow discharge starts between a cylinder and an axial wire, in a closed tube at atmospheric pressure, a sudden rise of pressure is observed. It has been argued by some that this cannot be due to heat generated in the discharge, and an alternative theory, that it is due to ionization, has been advanced.

In the above paper the authors criticise the argument for the ionization theory, and the interpretation of the experimental results upon which it is based. They also verify quantitatively that the effect is purely thermal in origin.

Physics Department,
University of Bristol.
Jan. 7, 1918.

XXX. *Some Problems of Evaporation.* *By* HAROLD JEFFREYS, *M.A., D.Sc., Fellow of St. John's College, Cambridge**.

THE problem of evaporation is practically one of gaseous diffusion: that is to say, if V denote the fraction of the density of the air at any point that is due to water vapour, V will vary from time to time and place to place according to the equation

$$\frac{dV}{dt}=\frac{\partial}{\partial x}\left(k\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial V}{\partial z}\right), \quad . \quad (1)$$

where k is the effective coefficient of diffusion, t the time, and d/dt denotes the total differential following a particle of the fluid. In general V, as above defined, will be referred to as the concentration.

Then if u, v, w be the components of velocity and V be supposed expressed as a function of x, y, z, and t,

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z}. \quad . \quad . \quad . \quad (2)$$

The boundary conditions are that the air in contact with a liquid surface is saturated, so that V is there equal to the concentration in saturated air at the temperature of the liquid, and that at a great distance from any liquid V tends to a finite value.

The equation (1) is identical in form with those that determine the transference of heat and momentum; and when the transference is due entirely to turbulence the quantity k has the same value in all three cases †. In contact with a solid or liquid surface, on the other hand, the velocity of the medium is zero, and k diminishes to the value it has when there is no turbulence: that is to say, in the evaporation problem k is equal to the coefficient of diffusion of the vapour; in the thermal problem it is the thermometric conductivity of air; and in the equations of motion it is the kinematic viscosity of air. These three quantities are of the same order of magnitude, but are not equal. When the air is at rest k is a constant in all cases and the problem is simply that of solving the equation

$$\frac{dV}{dt}=k\nabla^2 V, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ∇^2 denotes the Laplacian operator.

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* Communicated by the Author.

† G. I. Taylor, "Eddy Motion in the Atmosphere," Phil. Trans. 215 A. (1915).

When the air is in motion, k at a distance from a boundary is almost wholly due to turbulence, and is practically independent of position. The velocity near a boundary rapidly increases from zero to about half its amount at a considerable distance; this transition is accomplished in a thin layer of shearing, whose thickness in centimetres is estimated at $40/\mathrm{U}$, where U is the velocity in centimetres per second at a considerable distance *. In outdoor problems it is therefore usually of the order of a millimetre or smaller. The concentration and temperature also change rapidly within this layer, and in most cases it will be justifiable to assume that at the outer boundary of it each is constant and midway between the values at the surface and at a great distance. Outside this layer transference of heat and vapour will take place according to equation (3), where k is now put equal to the eddy viscosity.

If the dimensions of the liquid surface are of order l, and the time needed for any considerable change of condition over it is of order τ, we see that the terms like $\partial \mathrm{V}/\partial t$, $u\partial \mathrm{V}/\partial x$, and $k\partial^2 \mathrm{V}/\partial x^2$ are relatively of orders

$$1/\tau, \quad \mathrm{U}/l, \quad \text{and} \quad k/l^2.$$

Taking k to be of the order of 10^3 cm.^{2}/sec., which is a somewhat small estimate for outdoor problems, $\tau=1$ second, and $\mathrm{U}=400$ cm./sec., we see that the first term is small compared with the last provided l is less than 10 cm., and for slower variations this term will be small for still larger values of l. Again, we see that the second term is small compared with the last, provided l is less than 2 cm.; otherwise it must be taken into account. In indoor problems U is probably not far from zero, $k=0{\cdot}24$ cm.^{2}/sec., and the whole of d/dt can therefore be neglected for a surface 1 cm. across provided the saturation near it does not change considerably in 8 seconds.

When l is sufficiently small to satisfy both these conditions the equation of transference reduces to $\nabla^2\mathrm{V}=0$, subject to the same boundary conditions as before. Without loss of generality we can take the concentration at a great distance to be zero. For if it is actually V_d, $\mathrm{V}-\mathrm{V}_d$ will satisfy the same differential equation and will be constant over all the wetted surfaces, and hence if a problem is solved for $\mathrm{V}_d=0$ the solution for any other value of V_d can at once be found by merely writing $\mathrm{V}-\mathrm{V}_d$ for V throughout.

Now, $\nabla^2\mathrm{V}=0$ is the equation of steady diffusion and

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* Private communication from Major G. I. Taylor.

is also satisfied by the potential in electrostatic problems; it follows that, provided the initial change is not too abrupt and the dimensions are not too great, the value of V at any point is the same as the electric potential at that point when the wetted surface is regarded as a conductor charged to potential V_0, where V_0 is the concentration at the edge of the layer of shearing when the air is moving, and the saturation concentration at the boundary when it is at rest. The charge needed for this is CV_0, where C is the electrostatic capacity of the conductor. Now the rate of transference outwards from the boundary is

$$-\iint k\rho \frac{\partial V}{\partial \nu} dS,$$

where $\partial \nu$ denotes the element of the outward normal, ρ is the density of air, and the integral is taken over the boundary.

But in the electrostatic problem, if σ denotes the density of the electric charge on the surface,

$$\frac{\partial V}{\partial \nu} = -4\pi\sigma,$$

and hence

$$-\iint \frac{\partial V}{\partial \nu} dS = 4\pi \iint \sigma dS = 4\pi CV_0.$$

Hence the rate of transference outwards is

$$4\pi k\rho CV_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This determines the rate of evaporation, which is shown, other things being equal, to be proportional to the linear dimensions, since the electrostatic capacity varies in this way for bodies of the same shape.

The above is practically Stefan's * solution, which has been experimentally proved to be correct subject to the conditions stated †. When the velocity is large enough to need to be taken into account, a general solution is no longer possible, but for a large wetted surface the curvature may be neglected, and the problem reduces to that of a wind blowing over a flat surface. This is treated in the next section.

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* *Wien. Akad. Ber.* lxxxiii. Abteil 2, p. 613 (1881).

† H. T. Brown & F. Escombe, Phil. Trans. 193 B. pp. 223–291.

Steady wind blowing over a flat surface of water.

Take the plane of the surface for that of $z=0$. Let u be the velocity of the wind in the direction of the x axis. There is no wind in the directions of the other two axes. Then outside of the layer of rapid shearing the equation of diffusion when a steady state has been attained is

$$u\frac{\partial \mathrm{V}}{\partial x}=k\frac{\partial^2 \mathrm{V}}{\partial z^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

provided the distance from the margin is great enough for $\frac{\partial^2 \mathrm{V}}{\partial x^2}$ and $\frac{\partial^2 \mathrm{V}}{\partial y^2}$ to be neglected in comparison with $\frac{\partial^2 \mathrm{V}}{\partial z^2}$. The velocity u is a function of z only, and k/u is supposed constant, equal to h^2, say. The value of V over the surface when x is positive is V_0. Then all the conditions are satisfied if

$$\mathrm{V}=\mathrm{V}_0\left(1-\operatorname{Erf}\frac{zx^{-\frac{1}{2}}}{2h}\right) \text{ when } x \text{ is positive.} \quad . \qquad (6)$$

$\mathrm{V}=0$ when x is negative.

This makes $\frac{\partial \mathrm{V}}{\partial z}=\frac{\mathrm{V}_0}{h\sqrt{(\pi x)}}$ over the wet surface.

Hence the rate of evaporation is

$$k\rho\frac{\partial \mathrm{V}}{\partial z}=\mathrm{V}_0\rho\sqrt{\frac{ku}{\pi x}} \text{ per unit area.} \quad . \quad . \quad . \qquad (7)$$

Then the amount evaporated between 0 and x over a strip dy in diameter is by integration

$$2\rho\mathrm{V}_0(kux/\pi)^{\frac{1}{2}}dy.$$

Finally, if the length of the strip from one margin to the other be l, and parts near enough to the edges for the end corrections at both ends to be important be neglected, the amount evaporated can be found at once to be

$$2\rho\mathrm{V}_0\left(\frac{ku}{\pi}\right)^{\frac{1}{2}}\int l^{\frac{1}{2}}dy \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

taken over the whole area.

Hence for areas of the same shape and linear dimensions proportional to a the rate of evaporation would be proportional to $a^{1\cdot5}$. In particular, for a circular area of radius a it would be

$$3\cdot95\,\rho\mathrm{V}_0(kua^3)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

It has been assumed in this treatment that the linear dimensions are so large that $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^2 V}{\partial y^2}$ are negligible in comparison with $\frac{\partial^2 V}{\partial z^2}$.

Now the value of V in (6) makes the first two quantities of order V_0/a^2 and the last of order $V_0/h^2 a$. The condition required for the method to be justifiable is therefore that a shall be large compared with h^2; in other words, ua/k shall be large.

It has also been assumed that k/u is constant. This requires that the distance out to which the disturbance extends shall be sufficiently small for the change in k/u from its value just outside a thin surface-layer to be small compared with the whole. This distance is of order $2ha^{\frac{1}{2}}$. Thus $2a^{\frac{1}{2}}dh/dz$ must be small. Now $\frac{1}{h}\frac{dh}{dz}$ is probably of the same order as $\frac{1}{u}\frac{du}{dz}$. But $k\rho\frac{du}{dz}$ when $z=0$ is about $\cdot 003\rho u^2$, for each quantity is equal to the skin friction per unit area. Thus $\cdot 006(ua/k)^{\frac{1}{2}}$ must be small. This method is therefore correct provided ua/k lies between, say, 4 and 10^4.

Consider first an outdoor case, with $u=400$ cm./sec., $k=1000$ cm.^{2}/sec. Then a must be between 10 cms. and 250 metres. In a room with little draught, we may have $u=4$ cm./sec., $k=1$ cm.^{2}/sec., and a must be between 1 cm. and 25 metres.

When ua/k is much less than unity, the velocity of the wind may be neglected and the problem becomes one of steady diffusion; the rate of evaporation is then proportional to the linear dimensions. When ua/k is large compared with 10^4, the effect will extend upwards for such a distance that in most of the volume concerned u and k will be nearly constant and equal to their values at a considerable height. The air up to such a height will be practically saturated, and vapour will diffuse from the upper surface of the saturated air precisely as before; the law that the rate of evaporation is proportional to $a^{1\cdot 5}$ will therefore still hold, with different values of u and k.

The result that for bodies of medium dimensions the rate of evaporation is practically proportional to $a^{1\cdot 5}$ has been discovered experimentally by Thomas and Ferguson*, who also point out that this gives a fair representation of Renner's †

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* N. Thomas and A. Ferguson, Phil. Mag. xxxiv. pp. 308-321 (1917).

† O. Renner, *Flora*, 100. p. 474 (1910).

results. Brown and Escombe also found that the actual rate of evaporation varied more rapidly than the radius but not so rapidly as the area. Thomas and Ferguson find that when the surface of the water is below the rim of the containing vessel and the evaporation is assumed proportional to a^n, where n is a constant for a given depth, n tends to 2 when the depth is large.

The evaporation from a circular cylinder, wet at the bottom and open at the top, may be considered here. So long as the depth is either great or small compared with the radius, we can assume that within the cylinder the layers of equal concentration are parallel to the base. Let l be the depth, a the radius, and R the rate of evaporation. Let the concentration at the bottom be V_0, and at the top V_1. Then inside the cylinder the condition for steady flow makes

$$R=\pi a^2 k\rho \frac{\partial V}{\partial z}=\pi a^2 k\rho \frac{V_0-V_1}{l}.$$

Outside the cylinder, if the air is at rest, the vapour is practically diffusing from one side of a disk at concentration V_1, and therefore the rate of evaporation is $4k\rho a V_1$ * ; this also must be equal to R, else the concentration at the mouth would be changing. Eliminating V_1 we find at once

$$R=\pi a^2 k\rho V_0/(l+\tfrac{1}{4}\pi a),$$

agreeing with a result of Brown and Escombe (*loc. cit.* p. 258).

If a strong wind is blowing over the top, the conditions inside will be unaltered, but at the top we shall have

$$R=3{\cdot}95\,\rho V_1(kua^3)^{\frac{1}{2}},$$

giving, after elimination of V_1,

$$R=\rho V_0\div\left(\frac{l}{\pi a^2 k}+\frac{1}{3{\cdot}95\,(kua^3)^{\frac{1}{2}}}\right),$$

which varies with l in the way found by Thomas and Ferguson.

Evaporation from the surface of a leaf.

The surface of a leaf consists of an almost impermeable cuticle perforated by a large number of small holes, called stomata, through which respiration and absorption of carbon dioxide take place. It is a matter of some uncertainty

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* The capacity of a circular disk is $2a/\pi$, and therefore that of one side of it is a/π.

whether carbon dioxide can by simple diffusion enter these holes as fast as is indicated by the observed rate of carbon assimilation. If the rate of absorption were proportional to the area of the holes it could not be great enough, but if each hole absorbed at a rate proportional to its radius (this being the correct law for an isolated hole as small as a stoma) the total absorption would be much greater than is required. Some points in the theory still require examination; it is not obvious that the surrounding stomata will not interfere with the action of any individual to an important extent, and a wind blowing over the surface, though unimportant for a single stoma, may be important when there are thousands of them spread over a considerable area. The problem is mathematically the same as that of evaporation from the stomata, by which it will be replaced.

First, consider the leaf to be in a steady state and wind absent. Let the radius of a stoma be a, and the number per unit area n^2. Then the average distance between stomata is of order $1/n$, and is large compared with a. The value of V over the surface of any stoma is V_0. Then at a distance r from an isolated stoma V is of order V_0a/r. Now, if the stomata acted independently of one another, consider some particular stoma. By itself it would make $V=V_0$ over it; the others will together add to this an amount $\Sigma\frac{V_0a}{r}$, which is not very different from $\iint\frac{V_0a}{r}n^2dS$, taken over the whole surface of the leaf. This is of order $2\pi V_0an^2l$, where l is of the order of the dimensions of the leaf. Now Jörgensen and Stiles give* for a typical case $2a=0\cdot00107$ cm.; $n^2=33,000/\text{cm.}^2$ Thus the addition to V by the neighbouring stomata would be of the order of 300 V_0 for a leaf of radius 3 cm. This is of course impossible, for V cannot be greater than V_0. The meaning of the result is that the surroundings are enough to cause the air at any point to be practically saturated, and only a small portion of the vapour-pressure over any stoma is maintained by that stoma itself. The total evaporation from the surface of a leaf is therefore the same as would take place if V were equal to V_0 over the whole surface, and its amount is therefore $4\pi kCV_0$, where C is the electrostatic capacity of the whole surface of the leaf.

Looking at the matter in another way, the rate of evaporation from a single stoma uninfluenced by its surroundings

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* 'Carbon Assimilation,' p. 63.

would be $2\pi kcV_0$ [*], where c is the electrostatic capacity of a disk of the dimension of the stoma, and from all the stomata on a leaf of area A it would be $2\pi kn^2cV_0A$, if they did not influence one another. On the other hand, the evaporation in unit time from the whole surface of a leaf when completely wet is $4\pi kCV_0$. The ratio of these two rates is of order $n^2aA/2l$, or practically n^2al. Now, taking l to be 3 cm., and $a=0{\cdot}00053$ cm., it follows that n^2al is unity if n^2 is as large as 600. Hence, if there are more than 600 stomata per square centimetre, the rate of evaporation from them will be greater than that from the whole surface of a wet leaf, which is absurd. If follows that evaporation must be enormously restricted by the presence of other stomata. Brown and Escombe have stated that "the interference of the density shells of small holes set at 10 diameters or more apart is small, each hole beyond this limit acting almost independently according to the diameter law." The above result shows that this is erroneous [†]; in fact, the authors themselves imply its error in the diagram they give of the lines of flow through a multiperforate septum, when they make them become approximately parallel at a short distance from the septum.

Again, taking $n^2=33000/\text{cm.}^2$ and $l=3$ cm., we see that n^2al is greater than unity unless a is less than 10^{-5} cm. Hence we shall still have the result that the evaporation is practically independent of the opening of the stomata as long as a exceeds this limit; in other words, until the stomata contract to $\frac{1}{50}$ of their original diameter, the rate of evaporation will be practically independent of the diameter [‡]; when they have closed still further it will decrease, and will finally vanish when they are quite closed. This appears to contain the answer to Sir F. Darwin's criticism [§] of some results of Lloyd [‖]. Lloyd has stated that the regulatory function of stomata is almost *nil*, which Sir F. Darwin

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[*] We must have 2 instead of 4, as evaporation can only take place from one side of the stoma.

[†] In Brown and Escombe's experiments with multiperforate septa the radius a was 0·019 cm., n^2 was 100 to 2·77, and l about 2 cm., making n^2al equal to about 4 at the most, and in most cases much smaller. The results are therefore irrelevant to the case of a leaf.

[‡] Brown and Escombe, Phil. Trans. 193 B. p. 278 (1900), say that stomata can close to $\frac{1}{15}$ of their diameter without affecting assimilation.

[§] "The Relation between Transpiration and Stomatal Aperture," Phil. Trans. 207 B. p. 413 (1915-16).

[‖] F. E. Lloyd, "The Physiology of Stomata," Carnegie Institution, 1908.

regards as inconsistent with his further results that "complete closure . . . . . reduces transpiration to or nearly to cuticular rate," and "when the stomata are open to their utmost limit the highest rate of transpiration is the maximum of which the leaf is capable." A more satisfactory statement would be that until the stomatal aperture is reduced to a certain very small value the possible rate of transpiration is practically independent of the aperture, and nearly all of the reduction to zero when the stoma closes takes place in the last 2 per cent. of the reduction of aperture.

Next, consider the effect of wind. Suppose for simplicity that the stomata are arranged in straight rows, the distance between consecutive rows and between consecutive stomata on the same row being b. Then $b=1/n$. Consider a square column of air of side b. To pass over a stoma it would take a time b/u, and if it were unsaturated at the commencement it would therefore acquire a weight of vapour $2\pi k\rho c\mathrm{V}_0 b/u$, if there were no mutual influence between stomata. Now suppose the air to have moved forward a distance x, in time x/u. Then the vapour in it will have spread out by diffusion through a radius comparable with $2(kx/u)^{\frac{1}{2}}$; and if x/b is great diffusion parallel to the surface of the leaf and across the wind will have practically ceased, and thus the vapour will occupy half a flat cylinder of radius $2(kx/u)^{\frac{1}{2}}$ and thickness b, its centre being of course at the point x. Thus the concentration in it will be of order $c\mathrm{V}_0/x$. Further, the number of stomata much affected will be of order

$$4n^2b(kx/u)^{\frac{1}{2}}=4n(kx/u)^{\frac{1}{2}}.$$

Similarly, the number of stomata whose influence at this time will have affected the column of air when it has travelled a distance between $x-\frac{1}{2}b$ and $x+\frac{1}{2}b$ is $4n(kx/u)^{\frac{1}{2}}$, and therefore the total concentration produced by them is $4nc\mathrm{V}_0(k/ux)^{\frac{1}{2}}$. This is then the concentration acquired by a mass of air on account of what happened between times $(x\pm\frac{1}{2}b)/u$ previously, and the total produced by *all* times is to be found by summing the series for all such intervals. Put $x/b=r$. The total is then $4nc\mathrm{V}_0(k/ub)^{\frac{1}{2}}\Sigma r^{-\frac{1}{2}}$, the summation being from $r=1$ to $r=l/b$, where l is the distance of the mass from the stoma nearest the margin. When l is great this is of the order of $8n^2c\mathrm{V}_0(kl/u)^{\frac{1}{2}}$. Now b is about 0·05 mm., and thus is usually small compared with the thickness of the layer of rapid shearing [*]. *A fortiori* a, and hence c, are

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[*] See p. 271.

smaller. Thus u, being the velocity within the region to which the diffusion from a stoma extends, is much less than the velocity outside. Similarly k is the true coefficient of diffusion, about 0·24 cm.^{2}/sec. The thickness of the layer of shearing being 40/U, it follows that u is of order $U^2a/40$. Thus the quantity just obtained is of order $100n^2V_0(kla)^{\frac{1}{2}}/U$, which is about 100 V_0. It follows by argument similar to that used in the case of no wind that the earlier stomata saturate the air before the later ones are reached. Thus the total evaporation is not very different from that in the case where the whole surface of the leaf is wet *, and is therefore proportional to $l^{1\cdot5}$, where l is now proportional to the linear dimensions of the leaf.

This approximation will break down if n^2a is much smaller, for then the residual saturation from the earlier stomata may be small compared with V_0, and the rate of evaporation will then be the same as that obtained by summing the results for the individual stomata, each being supposed isolated; this sum is $2\pi n^2k\rho cV_0A$. A similar result may be obtained if u/k is much greater. It must always be noted, however, that this formula can be applied only when the result it gives is less than the rate of evaporation from a wet leaf with the wind blowing over it; otherwise we should again have the absurdity of the evaporation from a part being greater than that from the whole.

The best method of determining whether it is better to employ the sum of the possible evaporations from the individual stomata, or to regard the whole surface of the leaf as wet, is probably to calculate the rate of evaporation on both bases and take the smaller of the two results as supplying the correct upper limit to the amount of respiration the leaf can perform. Similar remarks will apply to the possible absorption of substances from the air.

It may be remarked that when the number of stomata is so large as to make the problem reduce to that of a wet leaf, the total evaporation is not a function of the number of stomata, but that from any single stoma is inversely proportional to the number. Thus increasing the number diminishes the work thrown on any individual, which may be of some physiological importance.

The above investigation concerns only the purely physical

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* O. Renner, *Flora*, vol. 100. pp. 451–547 (1910), states on p. 485 that the evaporation from a leaf is the same as that from a water surface

side of diffusion. It does not preclude the possibility that a reduction of the stomatal aperture may be associated with a reduction of the rate of evaporation; but it does show that in most cases the cause of such reduction is not the mere extra mechanical obstruction to the passage of water vapour, but must depend on the internal conditions. The importance of these is obvious. For instance, in the problem considered here the air has been supposed saturated when in contact with a stoma and perfectly dry at a great distance. Actually the concentration at a great distance has the finite value V_d; and that at a stomatal aperture is probably somewhat less than the saturation concentration. Let it be V_1. The latter question is further complicated by the facts that the dissolved substances within the cells must diminish the pressure of saturated vapour; that the leaf is normally at a somewhat higher temperature than its surroundings, so that the pressure of saturated vapour will on this account be greater than that at the temperature of the surroundings; and that owing to internal restrictions to the supply of water to the stoma the vapour-pressure may be reduced. The effect of these changes is that in all the formulæ we must substitute $V_1 - V_d$ for V_0.

Another complication arises from the fact that the stomata are not usually mere pores with saturated air in their planes; in most cases they are pits sunk in the leaf-surface. As long as their number is large this is not likely to produce any great effect on the rate of evaporation, for in exactly the same way as with flat stomata the earlier ones met by the air will partially saturate it, and the air when it meets the later ones will be nearly at the same saturation as that *inside* them. When the number is small, on the other hand, the formula for evaporation from depressed stomata must be used. For circular cylindrical stomata this gives for the rate of evaporation when the depth l is great compared with the radius

$$\frac{\pi n^2 k \rho a^2 (V_1 - V_d) A}{l + \frac{1}{4}\pi a}.$$

XXXI. *General Curves for the Velocity of Complete Homogeneous Reactions between Two Substances at Constant Volume.* By George W. Todd, *D.Sc.(Birm.), B.A. (Camb.)**.

[Plate IX.]

WHEN m molecules of a substance A react with n molecules of a substance B to give one or more resultants, there being no back reaction, the velocity of the reaction is given by

$$\frac{dx}{dt}=k\,.\,C_A^m\,.\,C_B^n,$$

where x is the change in the concentration C in time t and k is the velocity constant. If k is known, the changes in concentration for various initial concentrations of the reacting substances can be worked out by integrating the above equation, but the integration often absorbs valuable time. By choosing suitable quantities it is possible to plot curves which will apply generally to all reactions of a similar type. The author has worked out some of these, and puts them on record hoping that they may save much time and labour.

Bi-molecular Reaction.

If the reaction is bi-molecular of the type A+B→1 or more resultants, the reaction velocity is given by

$$\frac{dx}{dt}=k(a-x)(b-x),$$

where a, b are the initial concentrations of A, B respectively. The equation may be written

$$\frac{dx}{dt}=ka^2\left(1-\frac{x}{a}\right)\left(\frac{b}{a}-\frac{x}{a}\right).$$

Putting $\frac{x}{a}=X$, where X = fraction changed,

$$\frac{dX}{dt}=K(1-X)(p-X),$$

where $K=ka$ and $p=\frac{b}{a}$.

Take (i.) initial concentrations equal, *i. e.* $a=b$ or $p=1$, then

$$Kt=\int_0^X\frac{dX}{(1-X)^2}=\frac{X}{1-X}.$$

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* Communicated by the Author.

The maximum value of $X=1$. Giving X values up to 1 we get

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·111	·250	·428	·666	1·00	1·50	2·33	4·00	9·00	∞

These are plotted on fig. 1 ($p=1$), and the curve will apply to any bi-molecular reaction in which the initial concentrations are equal.

Take (ii.) one of the substances in excess, say $\frac{b}{a}=p$, where $p>1$. Then we have

$$Kt=\frac{1}{1-p}\left\{\log_e \frac{p(1-X)}{p-X}\right\}.$$

$p=1·5.$

X ...	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	·95	1·0
K*t* ...	0	·071	·159	·267	·401	·572	·808	1·15	1·69	2·77	8·66	∞

$p=2.$

X ...	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	·95	1·0
K*t* ...	0	·054	·117	·194	·288	·405	·559	·771	1·10	1·70	2·32	∞

$p=3.$

X ...	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	·94	·97	1·0
K*t* ...	0	·036	·077	·126	·183	·255	·344	·386	·549	·851	1·09	1·42	∞

These are plotted on fig. 1 (Pl. IX.), and the curves will apply to any bi-molecular reaction of the type $A+B\rightarrow 1$ or more resultants.

Ter-molecular Reaction.

Let the reaction be represented by $2A+B\rightarrow 1$ or more resultants, then the velocity of reaction is given by

$$\begin{aligned}\frac{dx}{dt}&=k(a-x)^2(b-x)\\ &=ka^3\left(1-\frac{x}{a}\right)^2\left(\frac{b}{a}-\frac{x}{a}\right).\end{aligned}$$

Bringing to the same notation as before gives

$$Kt=\int_0^X \frac{dX}{(1-X)^2(p-X)},$$

where $K=ka^2$.

Take (i.) equal initial concentrations, *i. e.* $p=1$, then

$$Kt=\int_0^X \frac{dX}{(1-X)^3}=\frac{1}{2}\,\frac{(2-X)X}{(1-X)^2},$$

which gives

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·117	·281	·520	·889	1·50	2·63	5·06	12·0	50	∞

These are plotted on figs. 2 and 3, and the curve applies to any ter-molecular reaction in which the initial concentrations are equal.

Take (ii.) B in excess, *i. e.* $p=\frac{b}{a}>1$. We have

$$\mathrm{K}t=\int_0^{\mathrm{X}}\frac{d\mathrm{X}}{(1-\mathrm{X})^2(p-\mathrm{X})}=\frac{1}{(1-p)^2}\left\{\log_e\frac{p(1-\mathrm{X})}{p-\mathrm{X}}-\frac{\mathrm{X}(1-p)}{1-\mathrm{X}}\right\},$$

$p=1·5$.

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·077	·180	·323	·440	·848	1·38	2·37	4·62	12·48	∞

$p=2$.

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·046	·133	·234	·379	·595	·941	1·56	2·90	7·5	∞

$p=3$.

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·038	·086	·151	·247	·373	·577	·932	1·68	4·01	∞

$p=4$.

X	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
K*t*......	0	·028	·064	·111	·177	·274	·416	·665	1·07	2·7	∞

These figures and the graphs plotted from them (fig. 2) will apply to any ter-molecular reaction of the type $2\mathrm{A}+\mathrm{B}\rightarrow 1$ or more resultants, B being in excess.

Take (iii.) A in excess, and let $\frac{a}{b}=p>1$, then we have

$$\mathrm{K}t=\int_0^{\mathrm{X}}\frac{d\mathrm{X}}{(p-\mathrm{X})^2(1-\mathrm{X})}=\frac{1}{(p-1)^2}\left\{\log_e\frac{p-\mathrm{X}}{p(1-\mathrm{X})}-\frac{\mathrm{X}(p-1)}{p(p-\mathrm{X})}\right\},$$

where $\mathrm{K}=kb^2$.

$p=1\frac{1}{4}$.

X	0	·4	·7	·8	·9	1·0
K*t*.........	0	·576	2·05	3·69	8·24	∞

$p=1\frac{1}{2}$.

X	0	·1	·2	·5	·7	·8	·9	·94	1·0
K*t*.........	0	·050	·114	·482	1·13	1·85	3·54	6·02	∞

$p=2$.

X	0	·1	·2	·3	·4	·5	·7	·9	·95	·97	1·0
K*t*......	0	·027	·062	·110	·162	·239	·503	1·29	1·87	2·37	∞

$p=3$.

X	0	·1	·5	·7	·9	·95	·98	1·0
K*t*	0	·012	·093	·224	·531	·575	·797	∞

These figures and the corresponding graphs (fig. 3) apply to any ter-molecular reaction of the type 2A + B→1 or more resultants, A being in excess.

Quadri-molecular Reaction.

Let the reaction be of the type 3A + B→1 or more resultants, and let the initial concentrations be equal. The general equation is

$$\frac{dx}{dt}=(a-x)^3(b-x)$$

$$=ka^4\left(1-\frac{x}{a}\right)^3\left(\frac{b}{a}-\frac{x}{a}\right),$$

when $\frac{b}{a}=1$

$$Kt=\int_0^X\frac{dX}{(1-X)^4}=\frac{1}{3}\left\{\frac{1}{(1-X)^3}-1\right\},$$

and $K=ka^3$.

X	0	·1	·2	·3	·4	·5	·6	·65	·7	1·0
K*t*	0	·123	·32	·64	1·21	2·33	4·87	7·43	12·0	∞

Now take B in excess, putting $\frac{b}{a}=p>1$. The equation becomes

$$Kt=\int_0^X\frac{dX}{(1-X)^3(p-X)}$$

$$=-A\log_e(1-X)+B\frac{X}{1-X}+\frac{C}{2}\left\{\frac{1}{(1-X)^2}-1\right\}+D\log_e\frac{p}{p-X},$$

where A, B, C, D are constants depending on the value of p.

$p=2$. Then $A=1$. $B=-1$. $C=1$. $D=-1$.

X	0	·2	·4	·6	·7	·8	1·0
K*t*	0	·149	·501	1·68	3·49	9·09	∞

$p=3$. Then $A=\frac{1}{8}$. $B=-\frac{1}{4}$. $C=\frac{1}{2}$. $D=-\frac{1}{8}$.

X	0	·3	·5	·7	·8	·9	1·0
K*t*	0	·184	·564	2·06	5·16	22·7	∞

$p=6$. Then $A=\frac{1}{125}$. $B=-\frac{1}{25}$. $C=\frac{1}{5}$. $D=-\frac{1}{125}$.

X	0	·5	·7	·8	·9	1·0
K*t*	0	·265	·925	2·25	9·52	∞

These figures give the graphs (fig. 4) for a quadri-molecular reaction of the type 3A + B→, the substance B being in excess.

Now let the substance A be in excess, so that $\frac{a}{b}=p>1$. We have

$$\mathrm{K}t=\int_0^{\mathrm{X}}\frac{d\mathrm{X}}{(p-\mathrm{X})^3(1-\mathrm{X})},$$

where $\mathrm{K}=kb^3$.

The solution is

$$\mathrm{K}t=-\mathrm{A}\log_e\frac{p-\mathrm{X}}{p}+\mathrm{B}\frac{\mathrm{X}}{p(p-\mathrm{X})}+\frac{\mathrm{C}}{2}\left\{\frac{1}{(p-\mathrm{X})^2}-\frac{1}{p^2}\right\}-\mathrm{D}\log_e(1-\mathrm{X}),$$

A, B, C, D being constants depending on p.

$p=1\frac{1}{4}$. Then $\mathrm{A}=-64$. $\mathrm{B}=-16$. $\mathrm{C}=-4$. $\mathrm{D}=64$.

X	0	·2	·4	·6	·7	·8	1·0
Kt	0	·15	·47	1·51	3·01	11·49	∞

$p=1\frac{1}{2}$. Then $\mathrm{A}=-8$. $\mathrm{B}=-4$. $\mathrm{C}=-2$. $\mathrm{D}=8$.

X	0	·2	·5	·6	·7	·8	·9	1·0
Kt	0	·082	·411	·602	1·15	2·15	4·76	∞

$p=2$. Then $\mathrm{A}=-1$. $\mathrm{B}=-1$. $\mathrm{C}=-1$. $\mathrm{D}=1$.

X	0	·3	·6	·8	·9	·95	·97	1·0
Kt	0	·065	·216	·575	1·01	1·58	2·02	∞

The graphs for these are shown in fig. 5.

There is yet the quadri-molecular reaction of the type $2\mathrm{A}+2\mathrm{B}\rightarrow 1$ or more resultants. In this case we have

$$\mathrm{K}t=\int_0^{\mathrm{X}}\frac{d\mathrm{X}}{(1-\mathrm{X})^2(p-\mathrm{X})^2},$$

where $\mathrm{K}=ka^3$ and $p=\frac{b}{a}>1$.

The solution is

$$\mathrm{K}t=\frac{2}{(p-1)^3}\log_e\frac{p(1-\mathrm{X})}{p-\mathrm{X}}+\frac{1}{(p-1)^2}\left\{\frac{p+1-2\mathrm{X}}{(1-\mathrm{X})(p-\mathrm{X})}-\frac{p+1}{p}\right\}.$$

$p=1\frac{1}{4}$.

X	0	·2	·4	·6	·7	·8	1·0
Kt.........	0	·21	·70	2·2	4·7	11·7	∞

$p=2$.

X.........	0	·1	·3	·5	·75	·85	·9	1·0
Kt	0	·03	·129	·36	1·465	3·33	6·00	∞

$p=3$.

X	0	·5	·8	·9	·95	1·0
Kt......	0	·141	·707	1·80	4·14	∞

The general curves for this type of quadri-molecular reaction are shown in fig. 6.

Application of the Curves.

(i.) To find the velocity coefficient k.

When the order of the reaction and the initial concentrations are known we have only to measure the fraction (X) changed in a given time t in order to find Kt from the curve and therefore k.

(ii.) To find the fraction (X) changed in a given time t.

This requires a knowledge of k, the order of the reaction, and the initial concentrations. Then X for a given t can be read straight from the curve.

(iii.) To find the Order of a reaction.

Two or more determinations of X and t are needed together with a knowledge of the initial concentrations. The particular curve on which the points (X, t) best lie, determines the order of the reaction.

London,
December 1917.

XXXII. *On the Coefficients of Potential of Two Conducting Spheres.* *By* Prof. A. Anderson [*].

It may be of interest to show how the coefficients of potential of two conducting spheres may be obtained directly without a previous determination of the coefficients of capacity and induction, and without making use of electric images. For this purpose the following elementary proposition, which is easily seen to be true, may be used.

If a conducting sphere whose radius is a have a charge E, and if other charged bodies be brought into the field, the potential V of the charge E at an external point P whose distance from the centre of the sphere is r will be given by the equation

$$rV = E + a(U - U'),$$

where U is the potential at the centre of the sphere of the introduced charges, and U' their potential at P', the inverse point of P in the sphere.

Let A and B be the centres of two spheres whose radii are a and b, the distance AB being c. Let I_1 be the inverse point of A in the sphere B, I_2 the inverse of I_1 in A, I_3 the inverse of I_2 in B, and so on. Also, let J_1 be the inverse of B in the sphere A, J_2 the inverse of J_1 in B, J_3 the inverse of J_2 in A, and so on.

Let U be the potential of the sphere A at B, U_1, U_2, U_3, &c. its potentials at I_1, I_2, I_3, &c., and U_1', U_2', U_3', &c. its potentials at J_1, J_2, J_3, &c. Also, let V be the potential of the sphere B at A, V_1, V_2, V_3, &c. its potentials at I_1, I_2, I_3,

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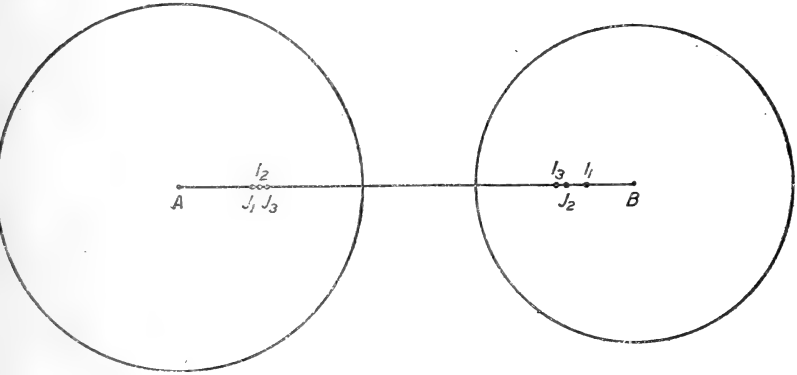
[*] Communicated by the Author.

&c., and V_1', V_2', V_3', &c. its potentials at J_1, J_2, J_3, &c. Let the charge on A be unity, and let B have no charge.

Fig. 1.



We have, then, the two sets of equations,—

$$\begin{aligned} cU &= 1+a(V-V_1'), \\ BJ\,.\,V_1' &= \quad b(U-U_2'). \\ AJ_2\,.\,U_2' &= 1+a(V-V_3'), \\ BJ_3\,.\,V_3' &= \quad b(U-U_4'), \\ AJ_4\,.\,U_4' &= 1+a(V-V_5'). \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

$$\begin{aligned} cV &= \quad b(U-U_1), \\ AI_1\,.\,U_1 &= 1+a(V-V_2), \\ BI_2\,.\,V_2 &= \quad b(U-U_3), \\ AI_3\,.\,U_3 &= 1+a(V-V_4), \\ BI_4\,.\,V_4 &= \quad b(U-U_5). \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}$$

The first set gives at once

$$\begin{aligned} cU = (1+aV)\Bigg[1 &+ \frac{ab}{BJ_1\,.\,AJ_2} + \frac{a^2b^2}{BJ_1\,.\,BJ_3\,.\,AJ_2\,.\,AJ_4} \\ &+ \frac{a^3b^3}{BJ_1\,.\,BJ_3\,.\,BJ_5\,.\,AJ_2\,.\,AJ_4\,.\,AJ_6} + \dots\Bigg] \\ -abU\Bigg[\frac{1}{BJ_1} &+ \frac{ab}{BJ_1\,.\,BJ_3\,.\,AJ_2} \\ &+ \frac{a^2b^2}{BJ_1\,.\,BJ_3\,.\,BJ_5\,.\,AJ_2\,.\,AJ_4} + \dots\Bigg] \end{aligned}$$

or $$cU = (1+aV)G_j - abUF_j,$$

where G_j and F_j are the sums of the two series, the subscript j denoting that J's enter into the expressions.

In like manner, the second set gives immediately

$$cV = bU\left[1+\frac{ab}{AI_1 \, . \, BI_2}+\frac{a^2b^2}{AI_1 \, . \, AI_3 \, . \, BI_2 \, . \, BI_4}+\dots\right]$$

$$-b(1+aV)\left[\frac{1}{AI_1}+\frac{ab}{AI_1 \, . \, AI_3 \, . \, BI_2}+\dots\right],$$

or
$$cV = bUG_i - b(1+aV)F_i.$$

Now
$$p_{11} = V + \frac{1}{a}, \text{ and } p_{12} = U,$$

hence
$$(c+abF_j)p_{12} = aG_jp_{11},$$

and
$$c\left(p_{11}-\frac{1}{a}\right)+abF_ip_{11} = bG_ip_{12}.$$

Hence

$$p_{11} = \frac{1}{a}\frac{1+\frac{ab}{c}F_j}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G_iG_j},$$

$$p_{12} = \frac{1}{c}\frac{G_j}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G_iG_j},$$

$$p_{22} = \frac{1}{b}\frac{1+\frac{ab}{c}F_i}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G_iG_j}.$$

Since, however, p_{12} must be equal to p_{21}, it follows that $G_i = G_j$, and, in fact, on examining the two series, it will be seen that they are identical if

$$BJ_1 \, . \, AJ_2 = AI_1 \, . \, BI_2,$$
$$BJ_3 \, . \, AJ_4 = AI_3 \, . \, BI_4,$$
$$\&c. \quad = \quad \&c.$$

These equalities, though they can be proved without difficulty, are not at all self-evident, and imply the further relations

$$AI_1 - BJ_1 = AI_3 - BJ_3 = \dots = AI_{2n+1} - BJ_{2n+1} = \dots = \frac{a^2-b^2}{c}.$$

Thus we have a case where theorems in pure geometry are suggested by purely electrical ones.

We have, then, writing G for G_i and G_j,

$$p_{11}=\frac{1}{a}\,\frac{1+\frac{ab}{c}F_j}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G^2},$$

$$p_{12}=\frac{1}{c}\,\frac{G}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G^2},$$

$$p_{22}=\frac{1}{b}\,\frac{1+\frac{ab}{c}F_i}{\left(1+\frac{ab}{c}F_i\right)\left(1+\frac{ab}{c}F_j\right)-\frac{ab}{c^2}G^2}.$$

It will be seen that

$$q_{11}=a\left(1+\frac{ab}{c}F_i\right),$$

$$q_{12}=-\frac{ab}{c}G,$$

$$q_{22}=b\left(1+\frac{ab}{c}F_j\right).$$

It has thus been shown that the values of the coefficients of induction, capacity, and potential depend on the sums of two series,

$$1+\frac{ab}{AI_1\,.\,BI_2}+\frac{a^2b^2}{AI_1\,.\,AI_3\,.\,BI_2\,.\,BI_4}+\frac{a^3b^3}{AI_1\,.\,AI_3\,.\,AI_5\,.\,BI_2\,.\,BI_4\,.\,BI_6}+\ldots$$

which we have denoted by G, and

$$\frac{1}{AI_1}+\frac{ab}{AI_1\,.\,BI_2\,.\,AI_3}+\frac{a^2b^2}{AI_1\,.\,AI_3\,.\,AI_5\,.\,BI_2\,.\,BI_4}+\ldots,$$

which we have denoted by F. Both of these series can be expressed in terms of a, b, c.

Taking the first series and denoting

$AI_1\,.\,BI_2$ by p_{12}, $AI_3\,.\,BI_4$ by p_{34}, $AI_5\,.\,BI_6$ by p_{56}, &c....,

it can be shown that

$$p_{34}=\frac{(c^2-a^2-b^2)\,p_{12}-a^2b^2}{p_{12}},$$

$$p_{56}=\frac{(c^2-a^2-b^2)\,p_{34}-a^2b^2}{p_{34}},$$

&c., &c.,

or, if we write for $c^2-a^2-b^2$, κ^2, and for ab, p^2,

$$p_{12}=\kappa^2,$$

$$p_{34}=\frac{\kappa^4-p^4}{\kappa^2},$$

$$p_{56}=\frac{\kappa^4-2p^4}{\kappa^2},$$

$$p_{78}=\frac{\kappa^4-3p^4}{\kappa^2},$$

&c....

The series is, therefore,

$$1+\frac{p^2}{\kappa^2}+\frac{p^4}{\kappa^4-p^4}+\frac{\kappa^2p^6}{(\kappa^4-p^4)(\kappa^4-2p^4)}$$
$$+\frac{\kappa^4p^8}{(\kappa^4-p^4)(\kappa^4-2p^4)(\kappa^4-3p^4)}+\dots$$

or, denoting $\frac{\kappa^2}{p^2}$, or $\frac{c^2-a^2-b^2}{ab}$, by α,

$$1+\frac{1}{\alpha}+\frac{1}{\alpha^2-1}+\frac{\alpha}{(\alpha^2-1)(\alpha^2-2)}+\frac{\alpha^2}{(\alpha^2-1)(\alpha^2-2)(\alpha^2-3)}+\dots$$

Hence we have

$$q_{12}=-\frac{ab}{c}\left(1+\frac{1}{\alpha}+\frac{1}{\alpha^2-1}+\frac{\alpha}{(\alpha^2-1)(\alpha^2-2)}+\dots\right).$$

Similarly, for the second series, we can show that

$$p_{45}=\frac{p_{23}(c^2-a^2-b^2)-a^2b^2}{p_{23}},$$

$$=\frac{\kappa^2p_{23}-p^4}{p_{23}},$$

$$p_{67}=\frac{\kappa^2p_{45}-p^4}{p_{45}},$$

&c....,

also $\mathrm{AI}_1=\frac{c^2-b^2}{c}$, and $p_{23}=\frac{(c^2-b^2+ac)(c^2-b^2-ac)}{c^2-b^2}$.

Let p_{23} be denoted by h^2.

Then
$$p_{23}=h^2,$$
$$p_{45}=\frac{\kappa^2h^2-p^4}{h^2},$$
$$p_{67}=\frac{\kappa^4h^2-(\kappa^2+h^2)p^4}{\kappa^2h^2-p^4},$$
$$p_{89}=\frac{\kappa^6h^2-\kappa^2(\kappa^2+2h^2)p^4+p^8}{\kappa^4h^2-(\kappa^2+h^2)p^4}.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The series is therefore

$$\frac{c}{c^2-b^2}\left[1+\frac{ab}{h^2}+\frac{a^2b^2}{\kappa^2h^2-p^4}+\frac{a^3b^3}{\kappa^4h^2-(\kappa^2+h^2)p^4}\right.$$
$$\left.+\frac{a^4b^4}{\kappa^6h^2-\kappa^2(\kappa^2+2h^2)p^4+p^8}+\dots\right],$$

or

$$\frac{c}{c^2-b^2}\left[1+\kappa^2\left(\frac{1}{h^2\alpha}+\frac{1}{\alpha(h^2\alpha-1)}+\frac{1}{\alpha\left(h^2\alpha^2-\alpha^2-\frac{h^2}{\kappa^2}\alpha\right)}+\dots\right)\right].$$

Thus

$$q_{11}=a+\frac{a^2b}{c^2-b^2}\left(1+\frac{ab}{h^2}+\frac{a^2b^2}{\kappa^2h^2-p^4}+\frac{a^3b^3}{\kappa^4h^2-(\kappa^2+h^2)p^4}+\dots\right).$$

For q_{22} we must write b for a and a for b, and h^2 will now denote

$$\frac{(c^2-a^2+bc)(c^2-a^2-bc)}{c^2-a^2}.$$

XXXIII. *Notices respecting New Books.*

Centennial Celebration of the United States Coast and Geodetic Survey, April 5 *and* 6, 1916. Washington Government Printing Office, 1916. Large octavo, pp. 196.

THE Coast and Geodetic Survey, under its present Superintendent, Mr. E. Lester Jones, celebrated the 100th Anniversary of its institution in April 1916. Addresses dealing with various aspects of the scientific work of the Survey were delivered by fifteen eminent Americans, and further addresses, including one by the President of the United States, were delivered at a banquet, which completed the proceedings. The volume contains these addresses, and extracts from the principal Acts of Congress relating to the Survey. A series of plates contain illustrations of surveying vessels and apparatus of various kinds, and there are full page photographs of the eleven gentlemen who have successively held the post of superintendent. There is much in the volume that is of interest even to those who are not Americans.

XXXIV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 140.]

November 21st, 1917.—Dr. Alfred Harker, F.R.S., President, in the Chair.

THE following communication was read:—

'The Shap Minor Intrusions.' By James Morrison, B.A., B.Sc.

The paper deals with the minor igneous intrusions occurring in the triangular area between Shap, Windermere, and Sedbergh.

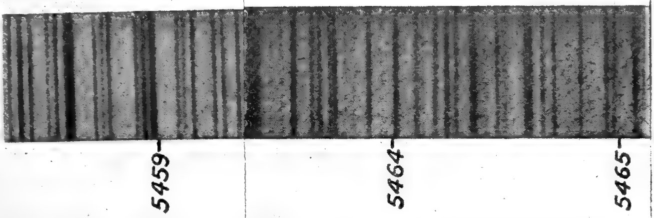
From their field relations and petrographic characters the intrusions are found to belong to one or the other of two well-marked groups, a division which is regarded as connoting also an age-classification.

The rocks of the earlier set, characterized by the presence of large orthoclase-felspars of the granitic type, are intimately associated with the granite, to the immediate neighbourhood of which they are practically confined. The rocks range from quartz-felsites to lamprophyres. Of considerable interest in this group is a series of hybrid intrusions, consisting essentially of rocks of a more or less basic magma enclosing xenocrysts of a more acid (but allied) magma obtained by settlement under intratelluric conditions. The constitution of any given member of the series is determined by two factors: the abundance of xenocrysts and the composition of the matrix, an increasing basicity in the latter (due to original magmatic differentiation) and a decrease in the former marking the successive stages. The more acid have affinities with the porphyrites, the more basic with the lamprophyres, the series ranging from modified biotite-porphyrites to modified pilitic lamprophyres.

The later intrusions are typically free from the large orthoclase-felspars, though quartz-grains may occur even in the basic members. Associated centrally with the earlier set they are distributed over a much wider area, overlapping the former in every direction. They are the result of a further differentiation, and are assigned to a later period when igneous activity was renewed on a more or less regional scale. The rocks include acid felsites and spessartites.

The rocks of the earlier set agree in general direction with the north-north-west fractures transverse to the strike of the country-rock, while the later intrusions trend generally east of north.

a.



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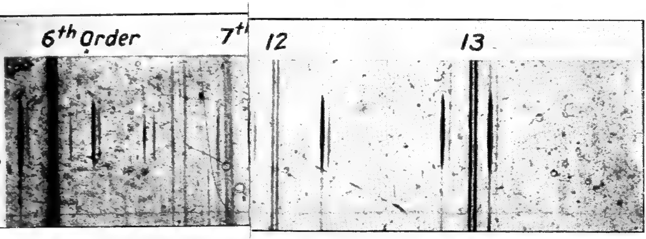

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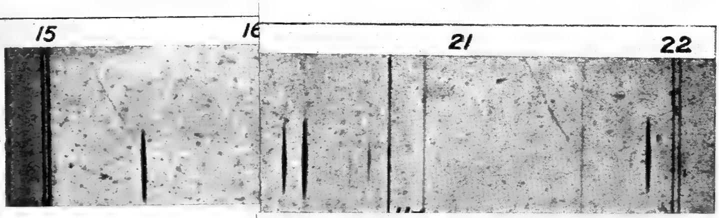

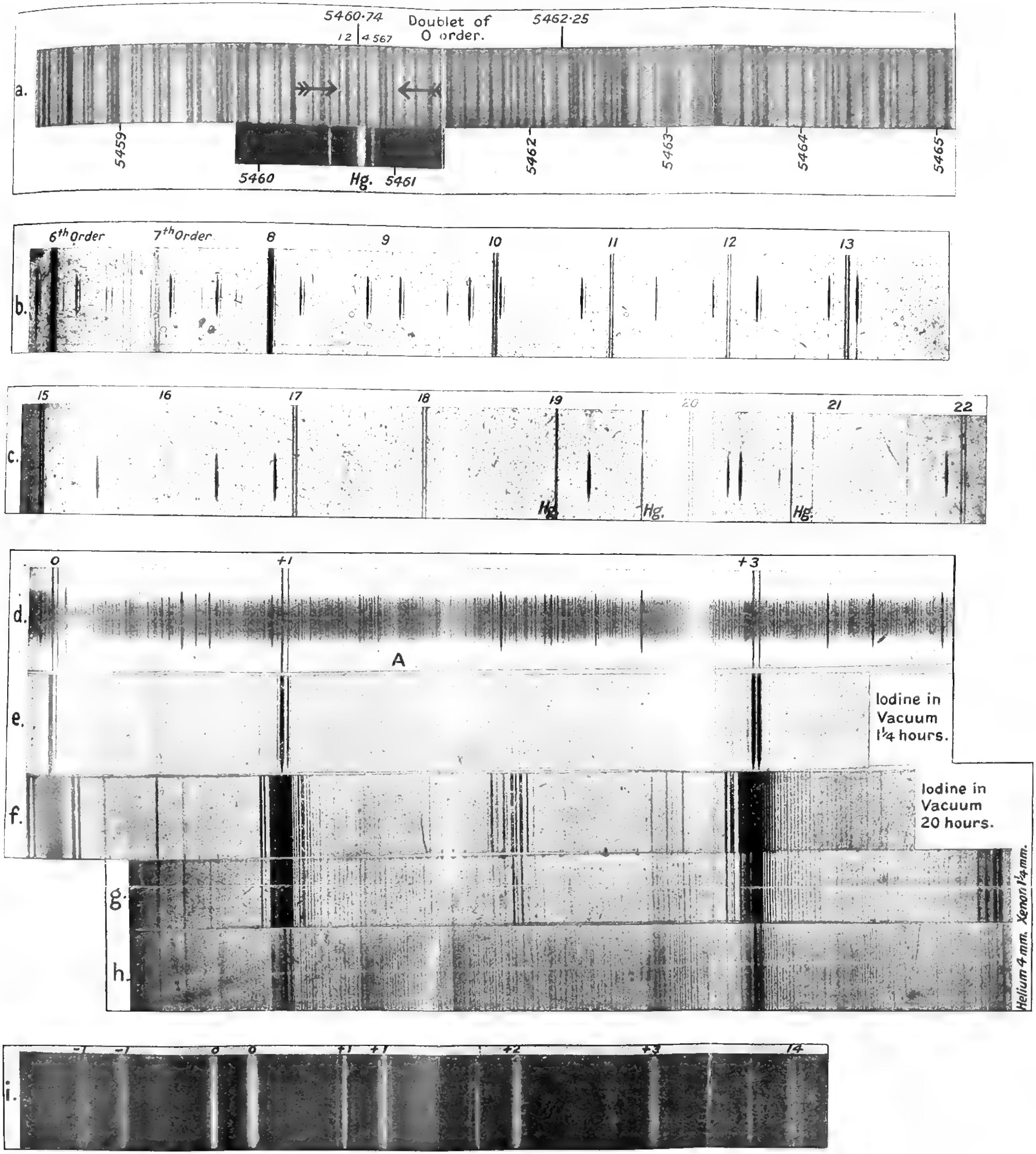
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Vacuum
1¼ hours.
f.
Iodine in
Vacuum
20 hours.
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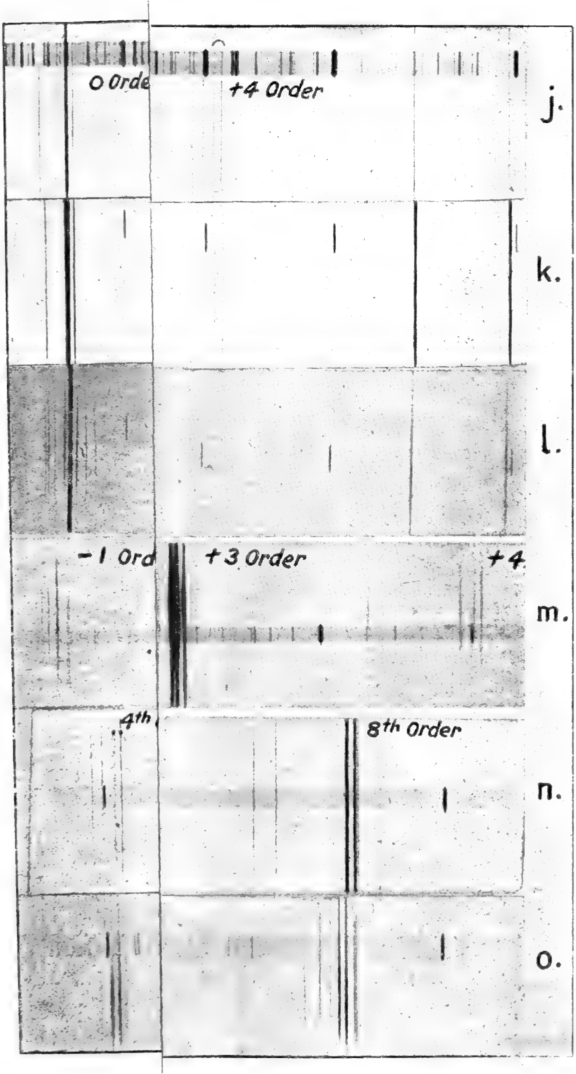
14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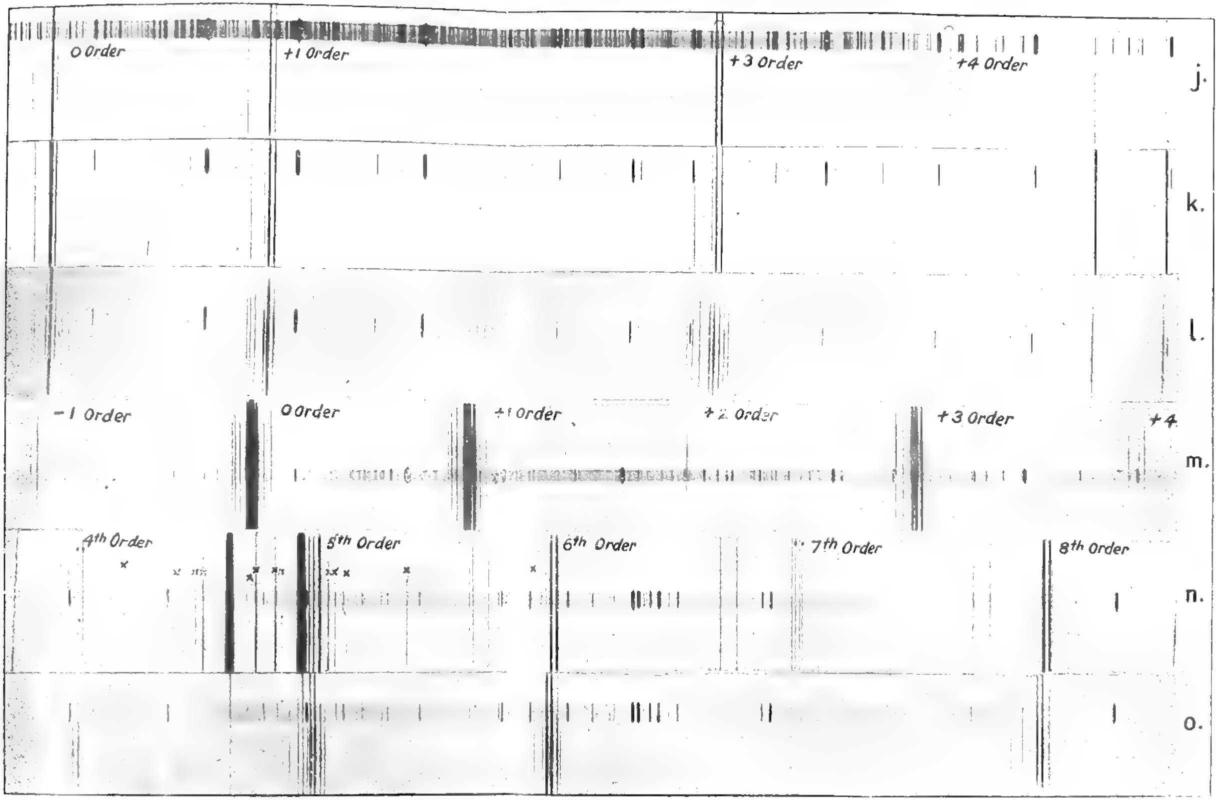
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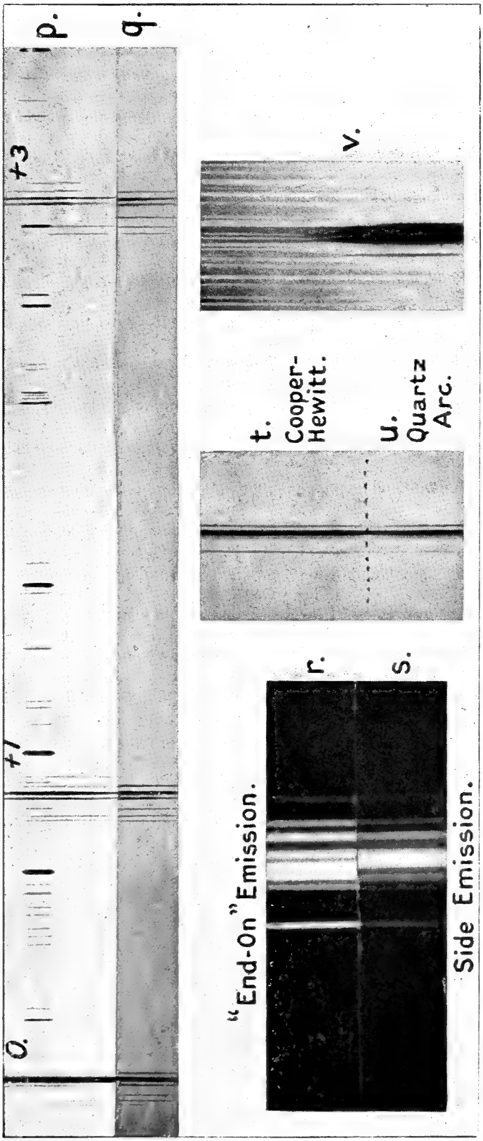
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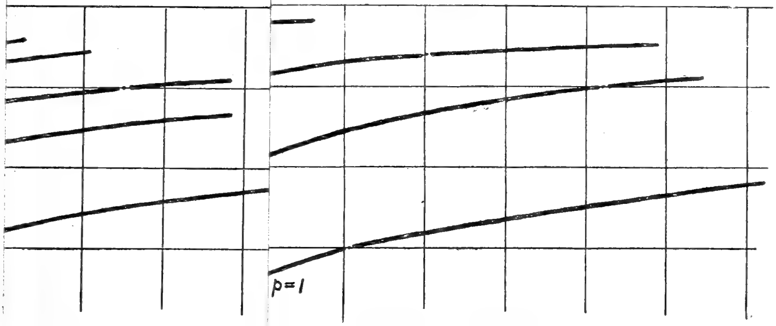


Reaction

resultants

cess

ntration of B

tration of A

Ter - molecular Reaction

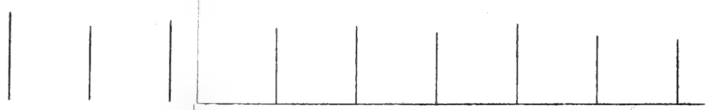
$2A+B \longrightarrow 1$ or more resultants

A is in excess

$$p = \frac{\text{initial concentration of } A}{\text{initial concentration of } B}$$

$K = k \cdot b^2$

Fig. 3



Phil. Mag. Ser. 6, Vol. 35, Pl. IX.

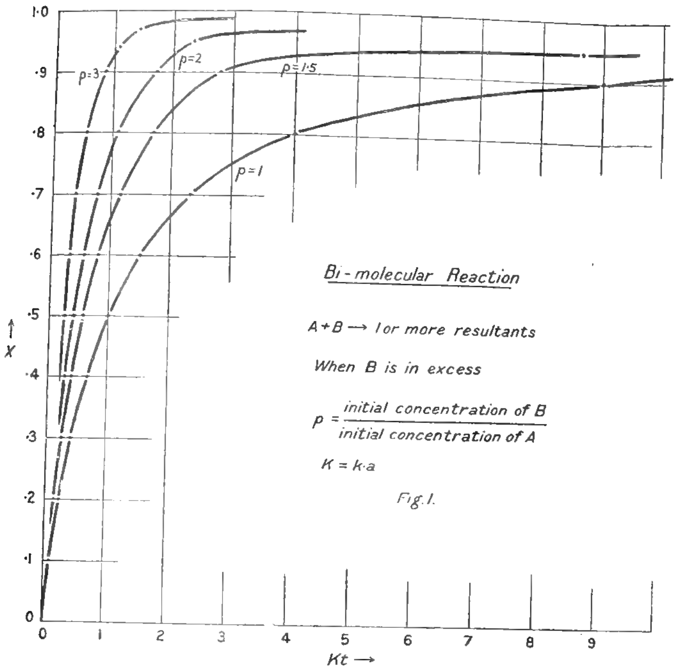


Fig. 1.

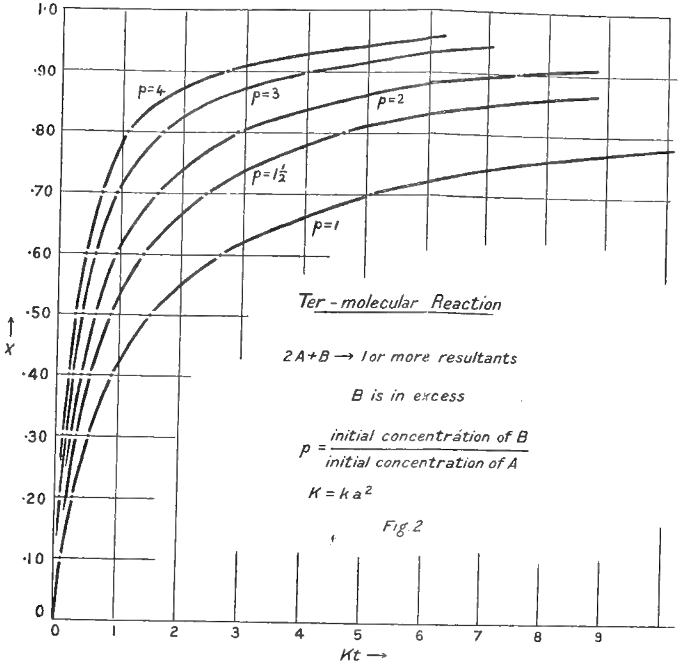


Fig. 2

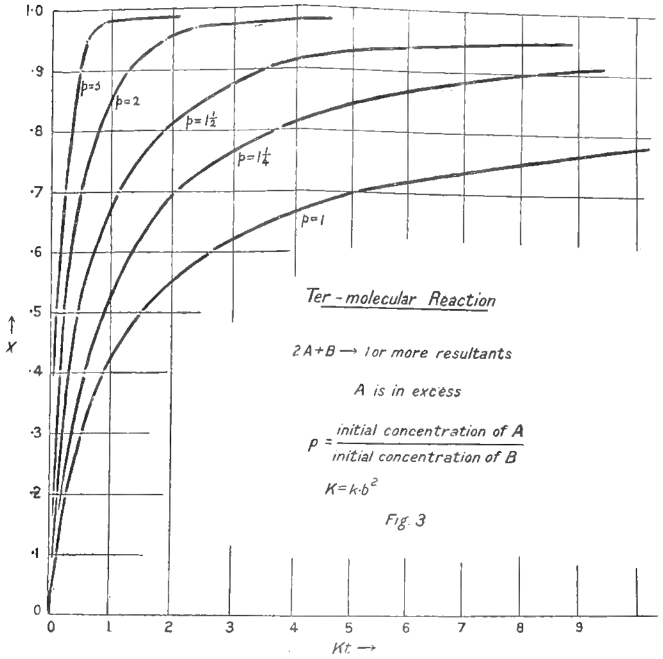


Fig. 3

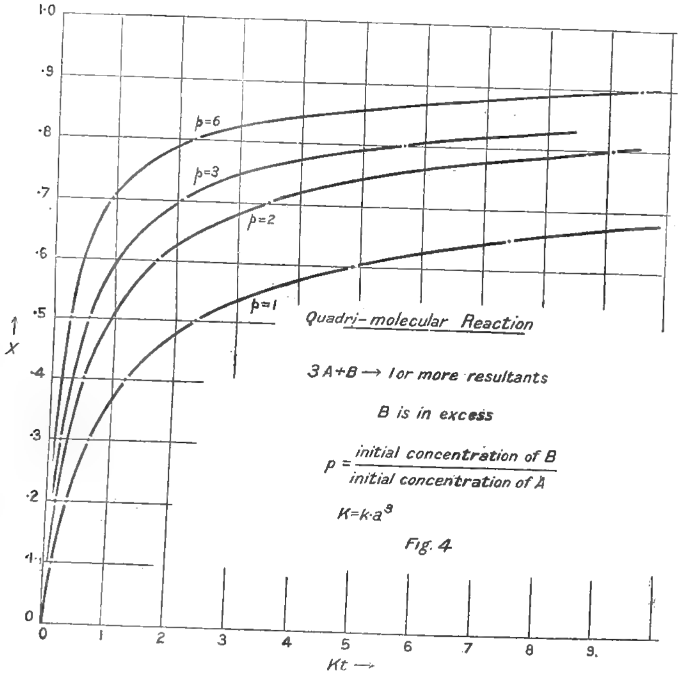


Fig. 4

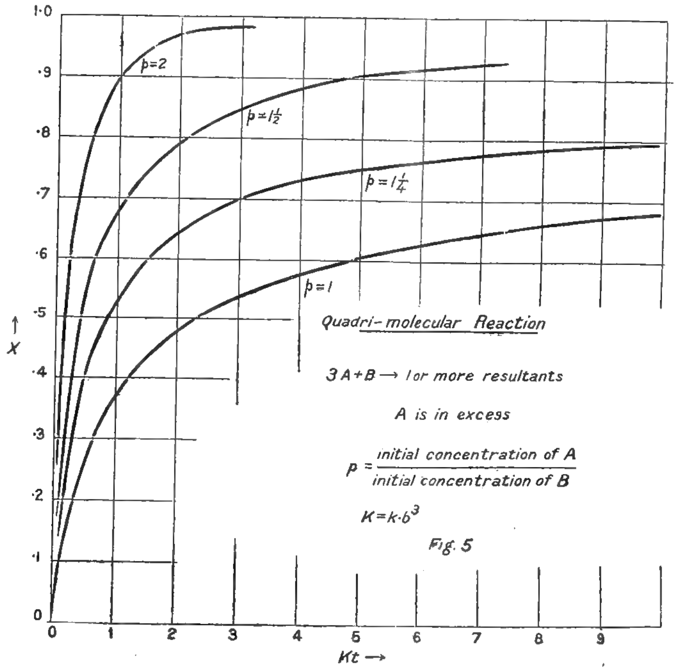


Fig. 5

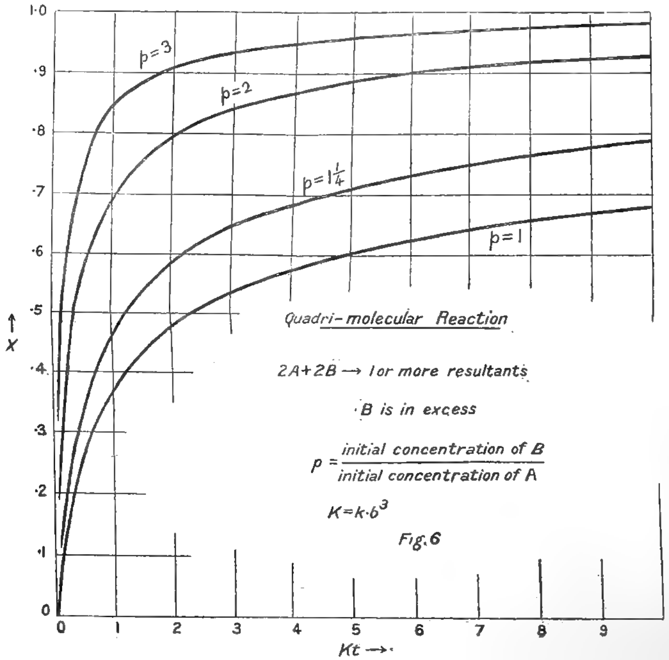


Fig. 6

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

APRIL 1918.

XXXV. *The X-Ray Spectra and the Constitution of the Atom.*

By L. VEGARD, *Dr. phil. University Christiania* *.

§ 1. ACCORDING to Rutherford † the atom consists of a positive nucleus and electrons circulating round it. A number of different methods have led to the result that the number of electrons in the neutral atom is equal to the atomic number N, and that consequently the charge of the nucleus is $+\mathrm{N}e$. The stability of the system is not secured by the usual electrodynamic forces, but Bohr ‡ by his ingenious explanation of the series spectra has been able to fix orbits of stability by means of the energy quanta introduced by Planck.

The fundamental assumptions underlying Bohr's theory of atoms and their light emission are the following :—

I. In the normal state of the atom the electrons are arranged in groups (rings) in such a way that for each electron the angular momentum is equal to $\frac{h}{2\pi}$ or

$$m\omega a^2 = \frac{h}{2\pi}, \quad \ldots \ldots \quad (1\,a)$$

where m is the mass, ω the angular velocity of the electron, a the radius of the circular orbit, and h Planck's constant.

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* Communicated by the Author. The results of the present investigation were communicated to the Kristiania Vid. Selsk. on October 23 and 30, and read before the Society November 23.

† E. Rutherford, Phil. Mag. xxi. p. 669 (1911), and xxviii. p. 488 (1914).

‡ N. Bohr, Phil. Mag. xxvi. pp. 1, 476, 857 (1913).

II. To produce the line of a series it is necessary that an electron should be removed in some way or other from one of the rings. The recombination of an electron towards the broken ring may take place in steps between stability orbits determined by the condition :

$$m\omega a^2 = \tau \frac{h}{2\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1\,b)$$

III. To each stability circle of the recombining electron corresponds a certain energy of the atomic system. When the electron in *one step* passes from a stability circle τ_2 to one τ_1 the energy must be removed from the system, and Bohr assumes it to be radiated out in *one* quantum homogeneous radiation, or

$$h\nu = W_2 - W_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ν is the frequency and W the total energy. The latter relation we may call Bohr's frequency law. The orbits corresponding to the normal state of the atom we shall call primary orbits, and the stability orbits which the electron may take up during its recombination we denote as secondary orbits.

By means of these assumptions Bohr was able to deduce the complete series spectrum of hydrogen and a spectrum of helium which was emitted when a single electron recombined to the isolated nucleus. He was also able to give an explanation of the appearance in the series formula of the universal frequency constant (R) of Rydberg, and his frequency law explained in a simple manner the combination principle of Ritz. But up to the present the atomic model based on the assumption I. has not been able to yield any theoretical deduction of the series spectra in general.

In the case of hydrogen and those spectra similar to that of hydrogen, the theory of Bohr has proved to be very successful indeed. Through the introduction of noncircular orbits already Bohr * was able to deduce a formula for the Stark-effect which gave the right order of magnitude and the right type of variation. Later on the question of noncircular orbits has been much more completely treated by Sommerfeld †, Schwarzschild ‡, and Epstein §.

Sommerfeld gave a generalization of the quant-conditions

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* N. Bohr, Phil. Mag. xxviii. p. 506 (1914).

† A. Sommerfeld, *Sitz. Ber. d. Münchener Akad. d. Wiss.* 1915; *Ann. d. Phys.* li. pp. 1 & 125 (1916).

‡ K. Schwarzschild, *Berliner Akad. d. Wiss.* p. 548 (1916).

§ P. Epstein, *Ann. d. Phys.* l. p. 489 (1916).

which proved to be of very great importance. He introduces generalized coordinates q_i (coordinates of position) and p_i (coordinates of momentum), and takes as the general quant-condition for stable orbits:

$$\int p_i dq_i = nh, \quad \ldots \ldots \quad (3)$$

where the integral is to be extended over one period.

This general quant-condition involves the new question as to the choice of coordinates.

Sommerfeld carries out the calculation in polar coordinates $(r\psi\vartheta)$ for a single electron moving round a positive charge. The three quant-conditions then take the form

$$\int_0^{2\pi} p_\psi d\psi = n_1 h, \quad \int p_r d_r = n'h, \quad \int p_\vartheta d\vartheta = n_2 h. \quad . \quad (4)$$

If we merely consider orbits in a plane, only the first two conditions will be wanted.

Carrying out the calculation in the case of hydrogen he gets:

$$\nu = \mathrm{R}\left\{\frac{1}{(n_1+n')^2} - \frac{1}{(m_1+m')^2}\right\}, \quad . \quad . \quad . \quad (5)$$

m_1, m' being the value of n_1, n' for the orbit from which the electrons recombine.

In order to give the ordinary Balmer series it is necessary that

$$\mathrm{N} = n_1 + n' = 2$$

and that $\mathrm{M} = m_1 + m'$ is an integer number > 2. Further, the number of combinations is limited by the conditions

$$m_1 \overset{=}{>} n_1,$$
$$m' \overset{=}{>} n'.$$

This new form of the frequency formula is so far identical with that of Bohr that it gives the same position of the lines, but it is different with regard to the way in which we may imagine the lines to be produced. Thus the H-line may be produced in four different ways, as, *e.g.*, by recombination from an elliptic orbit $(m_1=2,\ m'=1)$. The case of circular orbits treated by Bohr is only a special case of many, and corresponds to a recombination from $(m_1=3,\ m'=0)$ to $(n_1=2,\ n'=0)$.

The H-lines which are produced in these different ways are only with certainty identical provided we may treat the system as consisting of two constant masses attracting each other by a force inversely proportional to the square of the distance. When other forces come into play, or if the masses are not to be regarded as constant, the H-lines produced in the different possible ways will no longer be identical.

In fact, when the mass of the electron is supposed to vary with the velocity according to the law given by Lorentz or the principle of relativity, Sommerfeld finds that the lines are split up, and carrying out the calculation he has been able to explain even quantitatively the splitting up of the hydrogen lines and to give a general theory of the formation of multiple lines.

Further, we see that the frequency formula of Sommerfeld gives us a possibility of explaining the Stark-effect, for when a uniform electrostatic field is introduced into the system the various ways in which a certain line may be produced will no longer give the same frequency.

A complete determination of the Stark-effect was finally given by Epstein [*], by an ingenious method of selecting the generalized coordinates by means of the equations of Hamilton–Jacobi.

Application of Bohr's Conceptions to the High Frequency Spectra of the Elements.

§ 2. The law connecting the high-frequency spectra of the elements which was brought out through the beautiful experiments of Moseley [†] was simply explained by the atomic model of Rutherford [‡] and showed that the atomic number played a fundamental part in the constitution of atoms. In fact, we may say that the evidence gathered from various sources leaves no doubt as to the correctness of Rutherford's conception of the atom. I think we may safely take it as a fact that the normal atom has a positive nucleus of charge Ne, surrounded by N electrons.

The atomic problem is then resolved into the following two questions :—

1. The arrangement of the electrons which surround the nucleus and the laws that govern this arrangement.
2. The constitution of the nucleus.

The nucleus is the seat of gravitation and the radioactive transformations, and a possible theory of the nucleus would have to gather evidence from these two phenomena. The

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[*] P. Epstein, *l. c.*

[†] H. G. J. Moseley, Phil. Mag. xxvi. p. 1024 (1913), and xxvii. p. 703 (1914).

[‡] *Loc. cit.*

outer electronic system is responsible for the light emission, for the homogeneous X-rays, and for the chemical properties of the atom.

Bohr[*] was able to show that the K-spectrum could be approximately explained by assuming it to be produced by the removal and recombination of an electron next to the nucleus.

Moseley[†] found a better agreement by assuming four electrons in the system next to the nucleus; but the way in which he deduced his formula was open to criticism. His formula may therefore be regarded as empirical, although Nicholson[‡] points out that it can be deduced by a proper modification of Bohr's frequency law to *systems* of electrons.

Kossel[§] was the first to point out some very interesting relations between the lines of the K- and L-series. Denoting the frequency by ν, the following relation very nearly holds true:

$$\nu_{\mathrm{K}_\beta} - \nu_{\mathrm{K}_\alpha} = \nu_{\mathrm{L}_\alpha}. \qquad (6)$$

According to Bohr's frequency law (III.) the frequency is proportional to the differences of energy of the electron in the initial and final state, and, as pointed out by Bohr[‖], the above relations would naturally convey the following conception with regard to the formation of the high-frequency spectra.

The electrons may be supposed to be arranged in rings round the nucleus. When an electron is removed from the ring nearest the nucleus, an electron from the next ring may replace it and give rise to the emission of K_α. If the electron is taken from the third ring, we get K_β. When an electron of the second ring is removed and replaced by one from the third, we might get L_α, and in this way Kossel's frequency relations should be explained.

Sommerfeld[¶], following up this line of thought, has been able to express a number of lines of the X-ray spectra by introducing a number of "terms" peculiar to the various X-ray series. Thus, *e.g.*, he introduces a K-term $\mathrm{K} = \frac{(\mathrm{N} - 1\cdot 6)^2}{1^2}$ and a L-term $\mathrm{L} = \frac{(\mathrm{N} - 3\cdot 5)^2}{2^2}$ and finds

$$\nu_{\mathrm{K}} = \mathrm{K} - \mathrm{L}.$$

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[*] N. Bohr, Phil. Mag. (6) xxvi. p. 408 (1913).

[†] *Loc. cit.*

[‡] J. W. Nicholson, Phil. Mag. (6) xxviii. p. 562 (1914).

[§] W. Kossel, *Verh. d. D. Phys. Ges.* 1914.

[‖] N. Bohr, Phil. Mag. xxx. p. 394 (1915).

[¶] A. Sommerfeld, *Ann. d. Phys.* li. p. 125 (1916).

Further, he has been able to show that a number of lines in the X-ray spectra appear as doublets. Thus he can show that K_α and K_α' form a doublet, and the difference of frequency can be calculated from his theoretical formula.

The L-series consists of at least 13 lines. It is of importance to adopt definite names for these lines.

Arranging them in the order of increasing frequency, Siegbahn * and Sommerfeld † have the following somewhat different denotation :—

Siegbahn .... $l\,\alpha_2\,\alpha_1\,\eta\,\beta_4\,\beta_1\,\beta_2\,\beta_3\,\beta_5\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4$
Sommerfeld .... $\epsilon\,\alpha'\,\alpha\,\eta\,v\,\beta\,\gamma\,\phi\,\zeta\,\delta\,\vartheta\,\chi\,\psi$

Sommerfeld finds the following doublets :

$(\alpha'\beta)$ or perhaps $(\alpha\beta)$, $(\gamma\delta)$, $(\epsilon\eta)$, and $(\zeta\vartheta)$.

All four doublets give nearly the same value for the difference of frequency of its two components, thus :

$$\beta-\alpha'=\delta-\gamma=\eta-\epsilon=\vartheta-\zeta.$$

Recently Debye ‡ has attacked the problem in a somewhat different way. His method may be considered as an application of Bohr's frequency law to systems consisting of more than one electron.

Assuming that all atoms which give the K-radiation have a ring next to the nucleus of p electrons, he is able to deduce a theoretical formula for the K-line by putting $p=3$.

His deduction is based on the following assumptions :—

(*a*) The angular momentum of each electron in the normal ring is that supposed by Bohr for the normal state of the atom (equation 1).

(*b*) When an electron is removed from the ring the angular momentum of the remaining electrons is supposed to be preserved.

(*c*) A line belonging to the K-series is supposed to be formed when an electron recombines to the broken ring from a secondary stability state (circle) corresponding to an angular momentum $\tau\frac{h}{2\pi}$ where τ is a whole number greater than 1.

(*d*) When an electron recombines in one step from a

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* M. Siegbahn, "Bericht über die Röntgenspektren der chemischen Elemente," *Jahrbuch d. Radioakt. u. Elektronik*, xiii. p. 296 (1916).

† *Loc. cit.* p. 133.

‡ P. Debye, *Phys. Z.S.* xviii. p. 276 (1917).

secondary circle to re-establish a normal circuit, the difference of energy of the whole system (electron + circuit) before and after recombination is radiated in a single quantum.

Let the effective charge of the nucleus be $+Ne$ and the angular momentum of each electron $\tau\frac{h}{2\pi}$; then, as shown by Bohr and Debye, the total energy of a ring of p electrons will be

$$E = C - pRh\frac{(N-S_p)^2}{\tau^2}$$

where C is a constant, and as we have only to deal with differences we put

$$W = phR\frac{(N-S_p)^2}{\tau^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

and we take simply $-W$ as the total energy. R is Rydberg's universal frequency. Debye puts $R = 2{\cdot}7337 \,.\, 10^{15}\,\frac{1}{\text{sec.}}$, or in wave-number per cm.

$$R = 109740\,\frac{1}{\text{cm.}}.$$

Further,

$$S_p = \frac{1}{4}\sum_{i=1}^{i=p-1}\frac{1}{\sin i\frac{\pi}{p}}.$$

Let the total energy of the restored ring be $-W_1$, that of the broken ring $-W_2$, and that of the electron in the secondary circuit $-W_2'$, then Debye's application of Bohr's frequency law gives

$$h\nu = W_1 - (W_2 + W_2'). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Putting $p=3$ and $t=2$ he finds a frequency formula which gives very good agreement with observations for values of N smaller than 30. Above this value there is a considerable deviation, which he shows to be due to the fact that for higher atomic numbers the mass of the electron will increase on account of increase of velocity.

Adopting a similar way of procedure to that followed by Sommerfeld in his theory of doublets, Debye calculates the energy on the supposition that the motion takes place subject to the principle of relativity. The equations of motion under these conditions can be derived from the ordinary equations simply by putting

$$m = m_0 \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this case a fairly simple calculation gives for the total energy

$$W = \frac{2phR}{\rho}\left\{1-\sqrt{1-\frac{\rho}{\tau^2}(N-S_p)^2}\right\}, \quad . \quad (10)$$

where

$$\rho = \frac{2hR}{m_0 c^2} = 5{\cdot}30 \,.\, 10^{-5}. \quad . \quad . \quad . \quad . \quad (11)$$

Forming from equation (10) the expressions for the energies of the broken and unbroken system, the equation (8) gives the frequency of the radiation.

Letting the electron recombine from a state corresponding to $\tau=3$, Debye finds frequencies which are fairly near to those of K_β.

Ambiguities with regard to the Determination of the K-System.

§ 3. The question as to the system of electrons which produces the K-series is a most important one, being the first step towards a determination of the outer system of electrons in the atom, and it should not be out of place to see how far the most promising solution found by Debye is the only one which may give a satisfactory agreement with observations.

When we stick to the assumptions of Debye (*a, b, c, d*) and suppose the K-lines to be produced by the removal of one single electron, his result that there are three electrons in the inner ring is the only one possible.

We might, however, more generally assume that a group of q electrons were removed from the inner circuit of p electrons, and we could assume that the lines were formed when the system of q electrons reproduced the original system under the emission of q energy quanta.

From equation (7) we get for the energies :—

For the normal ring,

$$W_1 = phR(N-Sp)^2.$$

For the broken system,

$$W_2 = (p-q)hR(N-S_{p-q})^2 + q\frac{hR}{2^2}(N-p+q-Sq)^2.$$

Hence we get for the frequency:

$$\frac{\nu}{R}=\frac{p}{q}(N-S_p)^2-\left(\frac{p}{q}-1\right)(N-S_{p-q})^2-\frac{(N-p+q-S_q)^2}{2^2}$$

$$=3/4N^2+B_{pq}N+C_{pq},$$

where B_{pq} and C_{pq} are functions of p and q. Independent of p and q we get the undoubtedly right factor 3/4 of N^2.

The combination $p=3$ and $q=1$ represents the solution found by Debye.

If q is different from 1, there is one and only one other possible solution corresponding to

$$p=4 \quad \text{and} \quad q=4,$$

which give for the frequency :

$$\frac{\nu}{R}=3/4(N-0{\cdot}9571)^2, \quad . \quad . \quad . \quad . \quad (12\,a)$$

which is just the formula given by Moseley *.

When we take into account the variation of the mass according to equation (10), we easily get from equation (11) :

$$\frac{\nu}{R}=\frac{2}{\rho}\left\{\sqrt{1-\frac{\rho}{4}(N-S_4)^2}-\sqrt{1-\rho(N-S_4)^2}\right\}, \quad (12\,b)$$

or approximately

$$\nu/R=3/4(N-S_4)^2(1+5/16\rho(N-S_4)^2). \quad . \quad (12\,c)$$

The accuracy of the formula (12 *c*) will be seen from Table I., where the values of $\frac{\nu}{R}$ are calculated for a number of elements.

TABLE I.

N.	ν/R obs.	ν/R Debye.	Difference in Percent.	ν/R from 12 c.	Difference in Percent.
15 P	147·7	146·4	0·9	148·4	0·4
20 Ca	271·6	272·4	0·3	273·6	0·7
25 Mn	435·4	437·3	0·4	437·7	0·5
30 Zn	635·9	641·5	0·9	641·5	0·9
35 Br	880·4	885·6	0·6	885·9	0·6
40 Zr	1156·0	1171·0	1·3	1172·0	1·4
45 Rh............	1482·0	1499·0	1·1	1502·0	1·3
50 Sn	1871·0	1877·0	0·3	1876·0	0·3
55 Os	2290·0	2297·0	0·3	2296·0	0·3
60 Nd	2761·0	2777·0	0·6	2766·0	0·2

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* *Loc. cit.*

We see that the formula (12 *c*) satisfies the observations with the same accuracy as does Debye's formula. In fact the two formulas give almost identical values. If we assume the whole ring of four electrons to the normal state from a three-quantical secondary, we should get frequencies nearly equal to those of K. The accuracy would be about the same as that found by Debye.

If we adopt the frequency law in this more general sense there are two possible solutions. The one would give four, the other three electrons in the K-ring. Which of the two is the right solution cannot be determined from a mere numerical comparison with the observations, but we must have regard to the physical consequences to which they lead, and in this respect the solution of Debye has the advantage.

In the case of four electrons the whole ring of electrons would have to operate intact. We should have to assume that the system formed a kind of unity, in such a way that they tried to keep the same angular momentum. I think that there is reason to believe that as a component of the atomic system the electrons are not to be considered as independent unities; but that they are linked together by forces which are different from the ordinary central attraction and repulsion between the centres. The arrangement of electrons in conformity with certain quant-conditions is one aspect of these as yet unknown forces. But energy considerations seem to show that this mutual attachment cannot be so close as to prevent one single electron from leaving the system.

The experiments of Barkla and Sadler * have shown that in order to excite the K-radiation by means of Röntgen rays the hardness of the incident rays must just surpass that of the excited radiation. The frequency K_A of those rays which are just sufficient to produce the K-radiation is accurately determined by Wagner † and de Broglie ‡, and they also find K_A just a little greater than the frequency K_β.

This fact is simply explained by the quantum theory and Bohr's conception of the X-ray-and-light emission. An electron will be expelled from the ring when the energy quantum is equal to or greater than the energy required to remove it from the atom. Debye's formula involves the assumption that *one* quantum is sufficient to excite radiation. The assumption of four electrons would mean that four

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* C. G. Barkla and C. A. Sadler, Phil. Mag. xvi. p. 550 (1908).

† E. Wagner, *Ann. d. Phys.* xlvi. p. 868 (1915); *Phys. Z.S.* p. 432 (1917).

‡ M. de Broglie, *C. R.* clxiii. pp. 87, 354 (1916).

energy quanta $h\nu_{K_A}$ should be necessary to produce the K-radiation. If so, an atom should have the power of accumulating energy until it had taken up a sufficient number of quanta to break the whole ring. In trying to explain the mutual relation between γ- and β-rays Rutherford [*] has supposed that several *quanta* can accumulate to give energy to *one* β-particle; but the assumption of such an accumulation would, on the other hand, make it very difficult indeed to understand why a certain frequency at all should be necessary to produce the K-radiation.

Also the transformation of cathode-ray energy to that of Röntgen rays gives a similar result. The most important investigations of Whiddington [†] and the more recent measurements of Webster [‡] by means of the reflexion method have proved, that in order to excite the K-radiation the energy of *one* cathode ray must surpass the energy quantum $h\nu_{K_A}$. If we suppose that a single cathode particle can produce radiation, the assumption of four electrons is therefore not justified; but the assumption of Debye should be in accordance with physical facts. The fact that we can deduce two formulæ which both give a satisfactory agreement with observations is still of some importance, as it shows that we should be cautious in relying too much on a mere numerical agreement.

The Explanation of the L-Radiation.

§ 4. Although the agreement between observed and calculated values was less good in the case of K_β, there is probably no doubt that also this line is caused by a recombination to the inner circle, from a three-quantical secondary. Kossel's relation (6), if it indicates a physical connexion between the K- and the L-lines, would indicate that L_α should be produced by an electron recombining towards the K-circles, between the two secondaries corresponding to $\tau=3$ and $\tau=2$. This assumption would indeed give the right factor $\left(\frac{1}{2^2}-\frac{1}{3^2}\right)$ to N^2 in the formula for L_α.

This assumption, however, does not agree with the absorption phenomena. According to Barkla [§] and Sadler [‖] an L-absorption takes place when the hardness of the

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[*] E. Rutherford, Phil. Mag. (6) xxviii. p. 305 (1914).

[†] R. Whiddington, Proc. Roy. Soc. 1911.

[‡] D. L. Webster, Proc. Amer. Acad. vol. ii. p. 90 (1916).

[§] C. G. Barkla, Phil. Mag. xxiii. p. 987 (1912).

[‖] C. A. Sadler, Phil. Mag. xviii. p. 107 (1909).

incident rays surpasses that of the L-radiation of the substance; and Wagner * and de Broglie † have by means of the reflexion method observed two absorption bands in the L-region and accurately determined the sharp edges L_{A_1} and L_{A_2}, which have just the sufficient frequency to give absorption. The measurements of Wagner and de Broglie show that the frequency L_{A_1} is just a little greater than L_γ, and L_{A_2} just greater than L_δ.

If we stick to the assumption that one quantum $h\nu$ is sufficient to excite radiation, an energy quantum $h\nu_1$ or $h\nu_2$, cannot produce L_α if this line is emitted by an electron recombining to a broken K-ring. Now absorption takes place when the radiation falls upon atoms in the normal state, and we cannot assume that atoms are found in the normal state with a broken K-ring and the recombining electron in some secondary circuit.

We might then naturally try to follow up the idea suggested by Bohr to explain Kossel's frequency relations, and assume the electron removed from the K-ring to be replaced by electrons belonging to some outer system. If, however, we uphold Bohr's first assumption that each electron in the normal state has an angular momentum $\frac{h}{2\pi}$, there seems to be no possibility of explaining the high-frequency spectra by interchange of electrons from outer to inner systems.

Suppose we have a ring of p electrons, and inside this, systems which are made up of r electrons surrounding the nucleus. Outside the p-ring follows a ring of q electrons. According to Bohr ‡, when the diameters of the rings are not nearly equal, the forces due to the external systems can be neglected, and the effect of inner systems is approximately equivalent to a reduction of the charge of the nucleus equal to the charge of the inner electrons.

Hence the effective nucleus charge of the p-ring:

$$N_p = N - r;$$

of the q-ring:

$$N\ = N - (p + r).$$

We now regard the system when one electron is removed from the p-ring. The energy of the p- and q-ring is then:

$$W_2 = W_{p-1} + W_q = hR\{(p-1)(N_p - S_{p-1})^2 + q(N_q - S_q)^2\}.$$

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* *Loc. cit.*

† *Loc. cit.*

‡ N. Bohr, Phil. Mag. xxvi. p. 476 (1913).

The energy after an electron from the q-ring has regenerated the p-ring will be :

$$W_1 = hR\{p(N_p-S_p)^2+(q-1)(N_q-S_{q-1})^2\}.$$

Applying Bohr's emission law

$$h\nu = W_1-W_2$$

and inserting the values of W_1 and W_2, the frequency takes the form :

$$\frac{\nu}{R} = A_{pqr}N+B_{pqr}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where A_{pqr} and B_{pqr} are functions of p, q, and r. We see that the factor of N^2 vanishes for all values of (p, q, r) ; whence we conclude that the above equation cannot express any of the Röntgen-ray spectra, or the high-frequency radiation cannot be explained by interchange of electrons between rings of the normal atom, when each electron in this state has an angular momentum $\frac{h}{2\pi}$.

It can also be shown that from the assumptions of Bohr and Debye we cannot get the right formula for L_α by any recombination to a system, where each electron has an angular momentum of $\frac{h}{2\pi}$. On the other hand, the relation between the absorption and emission frequencies leads us to the assumption that the L-series is produced by *a recombination to some system which exists in the atom in its normal state.* If then we are to explain the L-radiation, we must in some way alter at least one of the assumptions at first made by Bohr.

In all cases where Bohr's principles have led to a complete or satisfactory determination of the spectrum, we have always dealt with systems next to the nucleus. It might then be natural to suppose that it is only for this inner ring that Bohr's assumption I. is fulfilled. The quant-condition to be satisfied by circular systems in the normal state of the atom might more generally be written

$$m\omega a^2 = n\frac{h}{2\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where n is an integer.

This would indeed seem to complicate matters, as we introduce a new parameter n into the conditions which secure the stability of the electronic systems. But still it seems the right procedure to begin work with n as an arbitrary whole number to be determined so as to fit in with observations, because there seems *à priori* no necessity for putting $n=1$ for all systems in the normal state of the atom.

Under these generalized conditions we shall calculate the frequencies which are produced when an electron is recombining to a normal system with q electrons which have lost one electron.

Let the number of electrons between the q-ring and the nucleus be p, then the effective nucleus charge is

$$N_q = N - p\,;$$

and forming the energies from equation (7) in the same way as before, we get:—

Energy of unbroken q-ring: $W_1 = \frac{q}{n^2}(N_q' - S_q)^2 hR$;

„ broken „ : $W_2' = \frac{q-1}{n^2}(N_q - S_{q-1})^2 hR$;

and energy of recombining electron in the secondary circle:

$$W_2'' = \frac{1}{\tau^2}(N_q - q + 1)^2 hR.$$

If we still stick to Bohr's frequency law, we get

$$h\nu = W_1 - (W_2' + W_2'').$$

Hence

$$\left.\begin{aligned}
\frac{\nu}{R} &= \left(\frac{1}{n^2} - \frac{1}{\tau^2}\right)N^2 + f_1(r, q, n, \tau)N + f_2(r, q, n, \tau),\\
f_1(pqn\tau) &= \frac{2}{\tau^2}(p+q-1) - \frac{2}{n^2}(p + S_{q-1}) - \frac{2q}{n^2}(S_q - S_{q-1}),\\
f_2(pqn\tau) &= \frac{q}{n^2}(2p + S_q + S_{q-1})(S_q - S_{q-1}) + \left(\frac{p - S_{q-1}}{n}\right)^2\\
&\qquad - \left(\frac{p+q-1}{\tau}\right)^2.
\end{aligned}\right\}\quad(15)$$

On the more general assumption (14) we might now also possibly explain the formation of lines by assuming that an electron removed from a ring is replaced by an electron coming from one of the other normal ring-systems.

Let, as before, an electron be removed from the q-ring, and let it be replaced by an electron from another ring with

q' electrons. Let the effective charge of the q'-ring be $N_{q'} = N - p'$.

In order to find expressions for the energy, we suppose that the removal of an electron from one ring does not change the energy of any of the other systems which keep their electron number unaltered. This will mean that the removal of an electron will change the radius of the external rings if we suppose the angular momenta to be kept unaltered.

Let the energy of the normal and broken q-ring be W_q and W_{q-1}.
" " " " q'-ring be $W_{q'}$ and $W_{q'-1}$.

Then

$$\nu h = W_q + W_{q'-1} - (W_{q'} + W_{q-1}).$$

From equation (7) we get:

$$\frac{\nu}{R} = \frac{1}{n^2}\{q(N_q - S_q)^2 - (q-1)(N_q - S_{q-1})^2\} \\ - \left\{\frac{1}{n_1^2} q'(N_{q'} - S_{q'})^2 - (q'-1)(N_{q'} - S_{q'-1})^2\right\},$$

n and n_1 being the quant numbers corresponding to the two circles q and q'. The formula may also take the form:

$$\left.\begin{aligned} \frac{\nu}{R} &= \left(\frac{1}{n^2} - \frac{1}{n_1^2}\right)N^2 - 2\left(\frac{\phi_1(pq)}{n^2} - \frac{\phi_1(p'q')}{n_1^2}\right)N \\ &\qquad + \frac{\phi_2(pq)}{n^2} - \frac{\phi_2(p'q')}{n_1^2}, \\ &\text{where} \\ \phi_1(pq) &= q(S_q - S_{q-1}) + S_{q-1} + p, \\ \phi_2(pq) &= q(2p + S_q + S_{q-1})(S_q - S_{q-1}) + (p + S_{q-1})^2. \end{aligned}\right\} . \quad (16)$$

Comparing the equations (15) and (16) we see that, if we assume the normal atom to have rings with increasing number of n as we pass from the centre, the assumption that the recombining electron comes from one other ring with a higher quant number leads to an equation of the right type. Whether we shall assume that the X-ray spectra are produced by recombination from a secondary or a primary circle is a question to be decided by the ability of the assumption to give a satisfactory agreement with observational data.

To test whether the second hypothesis (eq. 16) may be possible, let us try if the K-line can be explained by a recombination from an L-ring.

In this case we put

$$n = 1, \quad n_1 = 2, \quad p = 0, \quad p' = q,$$

and get:

$$\left.\begin{aligned} \frac{\nu}{\mathrm{R}} &= 3/4\mathrm{N}^2 - 2\left(\phi_1(oq) - \frac{\phi_1(qq')}{4}\right)\mathrm{N} + \phi_2(oq) - \frac{\phi_2(qq')}{4} \\ \text{or} \quad \frac{\nu}{\mathrm{R}} &= 3/4\mathrm{N}^2 + \mathrm{BN} + \mathrm{C}. \end{aligned}\right\} \quad (17)$$

The factor 3/4 is an accordance with observations. The essential further condition is that the factor B has the proper value, and in order to fit in with observations it should be nearly equal to $-1{\cdot}5$.

Putting $\phi(q) = q(\mathrm{S}_q - \mathrm{S}_{q-1}) + \mathrm{S}_{q-1}$,

we should have

$$4\phi(q) - q - 3 = \phi(q').$$

This equation is to be approximately fulfilled for whole numbers q and q'.

Values of $\phi(q)$ for varying q are given in the following Table:—

Table II.

q.	$\phi(q)$.	$4\phi(q)-q-3$.
1	0	−4·000
2	0·500	−3·000
3	*1·231*	−2·307
4	2·097	*1·388*
5	3·052	*4·208*
6	*4·082*	*7·328*
7	5·173	10·692
8	6·305	14·220
9	*7·512*	18·048

First we have to select nearly equal values in the second and third columns, and the corresponding values of q give the number of electrons in the K- and L-ring. As is seen, the following values of q and q' might be possible:

q	4	5	6
q'	3	6	9

If we now calculate the coefficients B and C of equation (17), we find the values set up in Table III.

TABLE III.

q_1.	q'.	B.	C.
4	3	−1·578	− 4·02
5	6	−1·563	−13·29
6	9	−1·408	−29·18

For the sake of comparison we can write down the corresponding formulæ of Moseley and Debye, which give very close agreement with observations for $N < 40$.

$$\frac{\nu}{R} = 3/4N^2 - 1{\cdot}5N + \tfrac{3}{4} \quad \text{(Moseley)},$$

$$\frac{\nu}{R} = 3/4N^2 - 1{\cdot}464N - 0{\cdot}125 \quad \text{(Debye)}.$$

The combination $q=4$ and $q'=3$ gives the best agreement and leads to the formula :

$$\frac{\nu}{R} = 3/4N^2 - 1{\cdot}578N - 4{\cdot}02. \quad . \quad . \quad . \quad . \quad (18)$$

For substances with fairly small atomic numbers the Debye formula gives a much closer agreement. But if we would correct for the variation of mass with velocity, the corrected formula (18) would probably give better agreement for the interval $35 < N < 55$, where the Debye formula gives too high values for $\frac{\nu}{R}$; but as a whole it must be said that the Debye formula gives by far the better agreement. But still the difference is so small, that we cannot decide from a mere numerical comparison which is the right formula so long as the calculations are not carried out with perfect exactness, and there may of course also be effects of unknown origin, which taken into account might put the assumption of recombination between primary systems in a more favourable position. But so far as the present investigation has been carried, it must be said that the assumption of Debye that the K-spectrum is produced by recombination from secondary circles has given the best agreement with observations; and we shall also see that the assumption of an inner ring of three electrons very well fits in with the chemical relations of the elements as they are expressed in the periodic system. It may also in this connexion be mentioned that Sommerfeld *, from his expresssions of the frequencies by means of his "terms," comes to the conclusion "that none of the L-lines can be derived from the K-lines by means of the principle of combination."

In our further investigation we shall then build on the assumption that the electron producing the X-radiation recombines from secondary systems in a way which is independent of the systems exterior to the primary ring to which recombination takes place.

On the Explanation of the L_α-Line.

§ 5. We now suppose the result of Debye as regards the K-radiation to be true, and we ask whether it is possible in equation (15) to give q, n, and τ such values that the observations are satisfied.

We put $p=3$, and from the formula of Moseley for the L_α-line,

$$\frac{\nu}{R}=\left(\frac{1}{2^2}-\frac{1}{3^2}\right)(N-7{\cdot}4)^2,$$

we see that we must put

$$n=2, \quad \tau=3.$$

If in (15) we further put

$$q=1,\ 2,\ 3\ldots\ldots,$$

we find that $q=7$ gives:

$$f_1(p,q,n,\tau)=-2{\cdot}0846,$$

$$f_2(p,q,n,\tau)=+5{\cdot}3\,;$$

and we get for the frequency:

$$\frac{\nu}{R}=\frac{5}{36}N^2-2{\cdot}0846N+5{\cdot}3. \quad . \quad . \quad . \quad (19\,a)$$

The formula of Moseley put into the same form gives:

$$\frac{\nu}{R}=\frac{5}{36}N^2-2{\cdot}0556N+7{\cdot}6. \quad . \quad . \quad . \quad (19\,b)$$

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* A. Sommerfeld, *Ann. d. Phys.* li. p. 155 (1916).

We see that the agreement between the two formulæ is remarkably good. Curiously enough, we can also in this case find a good agreement when we assume the whole L-ring to recombine from a secondary circle. If we now suppose 4 electrons in the K-ring and 9 electrons in the L-ring, we get

$$\frac{\nu}{\mathrm{R}}=\frac{5}{36}(\mathrm{N}-7{\cdot}328)^2,$$

in close agreement with Moseley's formula.

If we take into account the variation of the mass of the electron with its velocity, we can easily find the frequency by means of equations (2) and (10).

Putting
$$3+\mathrm{S}_7=5{\cdot}3048=a,$$
$$3+\mathrm{S}_6=4{\cdot}8274=b,$$

we obtain

$$\frac{\nu}{\mathrm{R}}=\frac{2}{\rho}\left\{6\sqrt{1-\frac{\rho}{4}(\mathrm{N}-b)^2}+\sqrt{1-\frac{\rho}{9}(\mathrm{N}-9)^2}-7\sqrt{1-\frac{\rho}{4}(\mathrm{N}-a)^2}\right\};\quad(20\,a)$$

and if expanded into series, we have approximately

$$\frac{\nu}{\mathrm{R}}=\frac{\nu_0}{\mathrm{R}}+\rho\left(\frac{7}{64}(\mathrm{N}-a)^4-\frac{6}{64}(\mathrm{N}-b)^4-\frac{1}{324}(\mathrm{N}-9)^4\right),\quad(20\,b)$$

ν_0 being the frequency calculated from the uncorrected formula (19 *a*).

Observed and calculated values are given in Table IV.

TABLE IV.

N.	ν/R obs.	$(\nu/\mathrm{R})_{\mathrm{M}}$.	$(\nu_0/\mathrm{R})_{\mathrm{V}}$.	$(\nu/\mathrm{R})_{\mathrm{V}}$.	P_{M}.	P_{V}.
30 Zn	73·8	71·0	67·8	67·9	−3·8	−8·0
35 Br	108·6	105·8	102·5	102·8	−2·6	−5·3
40 Zr	149·8	147·6	144·1	144·8	−1·5	−3·3
45 Rh.........	198·3	196·4	192·7	193·9	−1·0	−2·2
50 Sn	253·5	252·0	248·3	250·5	−0·6	−1·2
55 Cs	315·2	314·7	310·8	314·0	−0·2	−0·4
60 Nd	394·6	384·3	380·2	384·9	−0·1	0·0
65 Tb.........	461·8	460·8	456·6	463·4	−0·2	+0·3
70 Ad.........	545·6	544·3	539·9	549·5	−0·2	+0·7
80 Hg	734·9	732·1	727·4	745·5	−0·4	+1·4
90 Th.........	952·2	947·6	942·7	972·5	−0·5	+2·1

The column headed

$\left(\frac{\nu}{R}\right)_M$ contains values from the formula of Moseley.

$\left(\frac{\nu}{R}\right)$,, ,, calculated from (20 *b*).

$\left(\frac{\nu_0}{R}\right)_V$,, ,, ,, ,, (19 *a*).

P_M and P_V give the errors in percent. for the formula of Moseley and that of the writer.

The agreement between theory and experiment is seen to be remarkably good. The variation of P_V with the atomic number may be due to influences of external systems which we have not taken into account.

If our conception as to the production of the L_α-line is right, the relation of Kossel does not mean that L_α is produced by an electron recombining towards a broken K-ring between secondaries corresponding to $\tau=3$ and $\tau=2$. Nor does it involve that the lines are produced in the way suggested by Bohr by recombination between primaries. The relation is more or less accidental, and due to the fact that the first secondary K-circle has the same quant number as the primary L-ring.

On the Origin of the other L-Lines.

§ 6. As already mentioned, Sommerfeld found[*] that a number of lines in the L-series could be grouped into doublets.

To L_α corresponds L_β as a second component of the doublet, and if we stick to the explanation given by Sommerfeld, we should have to suppose that the normal L-system has two stationary states—one circular and one elliptic. Now the recombination to the elliptic state gives the higher frequency, and we should suppose, as we have already done, that L_α corresponds to the circular and L_β to the elliptic state. We are then led to the conception of one elliptic state common to a whole system of electrons, which would mean that the electrons forming one ring are in some way mutually connected.

Now Wagner has shown that the absorption edges A_1 and A_2 have the same difference of frequency as the doublet, or

$$\nu_{A_2}-\nu_{A_1}=\nu_{L_\beta}-\nu_{L_\alpha}.$$

This leads to the assumption that L_{A_1} and L_{A_2} are due to the transference of an electron from the circular and the

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[*] A. Sommerfeld, *Ann. d. Phys.* li. p. 125 (1916).

elliptic state respectively to the same final state, and that both states are to be found in the normal atom.

Absorption will set in when the quantum of energy $h\nu$ is great enough to overcome a certain amount of energy. Wagner has adopted the idea that absorption sets in when $h\nu$ is just sufficiently great to bring the electron from its place in the atom outside the sphere of influence of the atom; but in the case of the L-absorption the quanta $h\nu_{A_1}$ and $h\nu_{A_2}$ are probably too small to bring the electron outside the atom. Thus in the case of gold, $\lambda_{A_1}=1{\cdot}042$, $\lambda_{A_2}=0{\cdot}914$; but some of the lines belonging to the L-series have an even smaller wave-length, or an electron recombining to the L-circle should be able to radiate more energy than $h\nu_{A_2}$.

Neither does absorption set in when $h\nu$ is just sufficient to bring an electron to some secondary circle; for if so, the absorption edges should give the same wave-length as some of the emission lines.

Valuable information as to the relation between absorption and emission lines would be got by studying the relative intensity of the spectrum for velocities of the cathode rays varying from just below to just above the critical value.

With regard to the origin of the other lines of the L-spectrum, we might naturally assume the doublet (γ, δ) to be produced by recombination from secondaries corresponding to $\tau=4$ to the two primary states.

If in equation (15) we put

$$n = 2, \quad \tau = 4, \quad p = 3, \quad q = 7,$$

we get

$$\frac{\nu}{\mathrm{R}} = \frac{3}{16}\mathrm{N}^2 - 2{\cdot}9596\,\mathrm{N} + 9{\cdot}2 \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

Values of ν/R calculated from this formula and corrected for variation of mass are given in Table V., which also contains the corresponding values for some of the lines of the L-series.

TABLE V.

ν/R.

N.	Calc.	γ.	δ.	χ.
60	512	448	486	513
65	616	542	595	620
70	732	644	720	745
81	1021	906	1054	1085
90	1297	1143	1392	1435

It appears that the calculated frequencies come out considerably greater than those of L; at any rate, for lower atomic numbers they very nearly fall into the γ region of Siegbahn, being quite close to Sommerfeld's lines δ and χ. As to how this discrepancy is to be explained, I should not venture to express any definite idea. In any case it shows that equation (15) cannot be applied for values of τ higher than 3 without the introduction of certain correction terms,—or the secondary stability circles of higher quant numbers cannot be determined as if the electrons outside the normal L-ring were removed.

When proceeding to interpret the L-spectrum we should also be aware of the possibility that an L-radiation can be produced when an electron is recombining towards the K-ring—to a secondary corresponding to $\tau=2$. A recombination from a circle corresponding to $\tau=3$ would give

$$\frac{\nu}{R}=\frac{5}{36}(N-2)^2\left\{1+\frac{13\rho}{144}(N-2)^2\right\}. \quad . \quad . \quad (22)$$

This formula, which takes into account the variation of mass, would give frequencies in the β-region of Siegbahn.

Whether lines of the L-series are produced by a recombination towards a broken K-ring, can be tested experimentally by exciting the L-radiation with cathode rays which have not sufficient energy to excite the K-radiation.

The l-Series.

§ 7. The softest line given by the ring of 7 electrons should be the L_α-line; but now we have in the L-series the l-line discovered by Siegbahn, with a wave-length considerably greater than that of L_α; and to this line the lines η, ζ, ϑ are probably closely related. These lines we shall call the l-series.

The explanation suggests itself that the l-series is due to a second ring surrounding the L-ring and with the same quant number $n=2$. This l-ring would have a radius just a little greater than that of the L-ring, and the calculation of the frequencies due to the l-ring would be very complicated, as we can no longer suppose the effective atomic number to be $N-p$, where p is the number of electrons inside the ring.

If, however, we take $N-p$ to represent the effective atomic number, and assuming 8 electrons in the l-ring, we get:

$$\frac{\nu_0}{R}=\frac{5}{36}N^2-4{\cdot}37\,N+30{\cdot}8. \quad . \quad . \quad . \quad . \quad (23\,a)$$

Corrected for variation of mass with velocity, we get:

$$\frac{\nu}{\mathrm{R}}=\frac{\nu_0}{\mathrm{R}}+\rho\left\{\frac{8}{64}(\mathrm{N}-a')^4-\frac{7}{64}(\mathrm{N}-b')^4-\frac{1}{324}(\mathrm{N}-17)^4\right\}, \quad (23\,b)$$

where

$$a' = 10+\mathrm{S}_8 = 12{\cdot}805,$$

$$b' = 10+\mathrm{S}_7 = 12{\cdot}305.$$

The agreement between observed and calculated values will be seen from Table VI.

TABLE VI.

N.	ν/R obs.	ν/R calc.
70	482	411
74	545	475
78	608	544
82	676	618
92	855	826

The calculated frequencies are too small, especially for low atomic numbers, but, as already mentioned, we cannot claim any great accuracy.

This *l*-ring should probably be responsible for the doublets $(\epsilon\eta)$ and $(\zeta\vartheta)$.

We should also expect the *l*-ring to give absorption bands. An idea which might naturally suggest itself, is that the absorption bands A_2 and A_1 are due to the two rings respectively. Such a coordination is not impossible, but still the assumption that A_1 and A_2 are due to different states of the same ring seems, for the reasons previously mentioned, to be the more probable. As the *l*-lines are very weak, also the absorption ought to be weak and thus may have escaped detection.

The M-Series.

We have seen that the typical features of the L-radiation can be explained by the assumption of two rings with 7 and 8 electrons both with a quant number $n=2$. We might naturally try to explain the M-radiation by means of an M-ring with quant number 3.

The line M_α, say, can be approximately expressed by the empirical formula :

$$\frac{\nu}{R}=\frac{7}{144}N^2-2\cdot37\,N+40. \quad . \quad . \quad . \quad (24\,a)$$

If in equation (15) we would try to give p and q such values as to agree with this empirical equation, we should find the best possible agreement by putting

$$p=18 \quad \text{and} \quad q=9 \text{ or } 10.$$

Thus in order to explain the M-series we should have to assume a number of electrons inside the M-ring just equal to that found for the K- and L-rings. Thus the assumption of an outer l-ring of 8 electrons is also necessary to explain the M-radiation. 9 electrons in the M-ring give

$$\frac{\nu}{R}=\frac{7}{144}N^2-2\cdot45\,N+28\,; \quad . \quad . \quad . \quad (24\,b)$$

10 electrons give

$$\frac{\nu}{R}=\frac{7}{144}N^2-2\cdot55\,N+31. \quad . \quad . \quad . \quad (24\,c)$$

Although the agreement between observed and calculated values is not so good as in the case of L_α, the empirical and theoretical equations are of essentially the same type, and even the numerical agreement is surprisingly good.

The radius of the rings is given by the expression

$$a=\frac{e^2n^2}{2hR(N-p-S_q)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where p and q have the same meaning as in equation (15). To correct for variation of mass we should have to multiply with $\sqrt{1-\frac{\rho}{n^2}(N-p-S_q)^2}$.

The expression for the velocity is

$$v=\sqrt{\frac{2hR}{m}}\frac{N-p-S_q}{n}=2\,.\,10^8\frac{N-p-S_q}{n}, \quad . \quad (26)$$

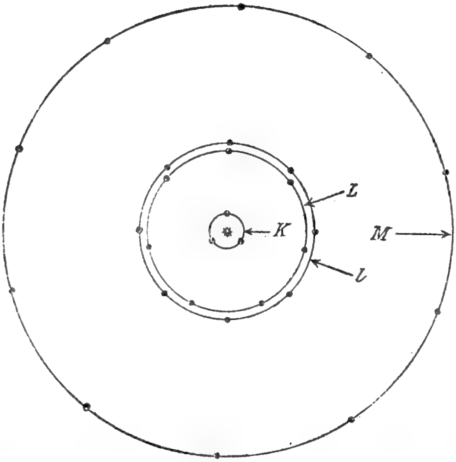
and is not changed when the variation of the mass is taken into account.

The values of a for the K-, L-, l-, and M-rings for a couple of elements are given in Table VII. and in fig. 1.

TABLE VII.

a.	Zr (N=40).	Hg (N=80).
K	$1{\cdot}34\times10^{-10}$ cm.	$0{\cdot}63\times10^{-10}$ cm.
L	6·08 ,,	2·82 ,,
l	7·76 ,,	3·18 ,,
M	25·4 ,,	8·10 ,,

Fig. 1.



I am far from claiming that the theoretical interpretation of the L- and M-series given in this paper can be considered as proved. As I have already stated, we must be cautious in building too much on a numerical agreement; but if at all we shall be able to proceed further in the direction pointed out by Bohr, I think we can hardly avoid the assumption that systems of electrons exist in the normal atom with quant numbers greater than 1.

So long as we do not know all the forces which are engaged in forming the stability of the atoms, we have more or less to grope in the darkness and feel our way forward. The test of the correctness of the previous theory will be whether it is in accordance with observation. At any rate, I hope the attempt made to elucidate the laws governing the X-ray spectra may prove to be of importance as a guidance in experimental research.

The Constitution of the Elements based on the X-ray Spectra.

The theory of the X-ray spectra involves the determination of the electronic systems next to the nucleus, and may give us valuable information as to the way in which the electrons round the nucleus are grouped together.

The previous investigation has shown that the K-series should be attached to a ring with quant number 1 next to the nucleus and containing 3 electrons. The L-series should be due to two rings with quant numbers 2 and consisting of 7 and 8 electrons. The M-series should probably have rings with a quant number 3 and 9–10 electrons.

If this theory is right, it would mean that if a ring is formed for lower atomic numbers the same ring is kept throughout the whole system of elements. Indeed I think that this is to be considered as a necessary consequence to be derived from the simple laws governing the X-ray spectra, and is independent of any special theory which we propose to explain the frequencies and the type of the spectra. For a change in the number of electrons in the K-ring, say, would necessarily involve a discontinuity in the formula expressing the relation between frequency and atomic number.

Now it might be legitimate as an hypothesis to take this rule as a fundamental property of the atomic structure, and quite general to assume that a system of electrons once formed is kept also for elements of higher atomic numbers. There is no reason why this rule should cease to hold because we pass to lower frequencies.

Now if we would further build on the result of our present theory as to the number of electrons in the K- and L-rings, we should get a quite definite system for the first eighteen elements, and from this start we should be able to see how the electrons are arranged in a series of elements forming one period in the periodic system. If we have proceeded so far, we can get further by the assumption that elements of the same family, such as Li, Na, and K say, must have the same number of electrons in the outer ring. I think this is an assumption which is very well founded, because the chemical properties must be mainly determined by the outer electrons, and the assumption is independent of any other special hypothesis with regard to the grouping of the electrons.

Proceeding in this way, we assume in accordance with Bohr that the strongest electropositive elements have

1 electron in the outer ring. Now the elements from Li to Fl are assumed to maintain an internal system of 2 electrons and add one in the external ring for each step in atomic number. This will make an external ring of 7 electrons. By Ne one electron is added which, however, goes to the central ring, and hence forward we get the K-ring. If so, the K-radiation should begin with Ne or Na; and in fact this result is in agreement with experiments, for Na is the first element for which the K-radiation has been observed. Now the ring of 7 electrons is kept to form the inner L-ring, and a new ring comes into existence for Na. From Ar we have both L-rings with 7 and 8 electrons formed, and the L-radiation might perhaps be expected to begin with potassium; perhaps some of the lines might be traced to Na.

Now we come to the long period from Ar to Kr. At first a ring of 10 electrons is formed, completed by the elements Fe, Co, and Ni with 8, 9, and 10 electrons in the external ring respectively; this should be the first M-ring with quant number 3. At Cu a new ring comes into existence, and we get a monovalent electropositive element. During the next long period from Kr to Xe the same process is repeated.

The next and longest of all periods which go from Xe to Ra Em is peculiar because it contains the rare earths. Now I think that the view here adopted with regard to the constitution of the electronic systems may afford a very simple and natural explanation of this peculiar group of elements.

When we pass from Xe, a new external ring is formed, with 1 electron for Cs, 2 for Ba, and so on until for Ce we get a ring of 4 electrons. Passing now to the next elements we assume the external ring to be kept, and that the new electrons are forming a new *internal* ring. From our point of view such an assumption is a quite legitimate one. It would only mean that the new electronic system had a smaller quant number than the external ring: for a smaller quant number will, according to equation (25), give a smaller radius of the ring. Thus the new electrons which are taken up in the series of rare earths when we pass to higher atomic numbers are, so to speak, soaked into the atom, and the surface systems mainly determining the chemical properties are kept unaltered. How these new internal electrons are arranged we do not know. In the graphical representation (fig. 2) I have assumed them to form one system inside the surface electrons.

When at last the atom has become saturated as it were, we pass from the rare earths ; new electrons are added as before to the surface system, and we get systems of the same type as those of the two long series.

The whole system here shortly sketched is graphically represented in fig. 2. Along the horizontal axis the elements are arranged in the order of increasing atomic numbers. The principle adopted, that an electronic system once formed is kept throughout the whole series of elements, makes it natural to represent an electron by a horizontal line. These lines are arranged into groups, and each group represents an electronic ring system. The arrangement of electrons for a certain element is got by drawing a vertical line from the place of the element on the horizontal axis. The points of intersection with the horizontal lines give the number of electrons and their arrangement into ring systems.

On the Electron Affinity of the Elements.

When we pass from elements that follow an inert gas, we begin with the strong electropositive elements, and as we pass on they become more electronegative. The transition from electronegative to electropositive elements may either take place by the passage through an inert gas or by passing the groups Fe Co Ni, Ru Rh Pd, and Os Ir Pt. In our system the strong electropositive elements set in with the formation of a new surface ring.

It might now be asked which quantity might rightly be selected to express the chemical electro-affinity.

The idea would naturally suggest itself that the electron affinity is measured by the energy necessary to remove an electron from the external ring. This, however, is identical with the energy necessary to ionize the atom and is proportional to the ionizing potential, which is no measure of the chemical electronegativity [*].

Nor can we take the energy which binds an additional electron ; for the experiments of J. J. Thomson [†] on positive rays have shown that the power of an atom to bind electrons does not follow the chemical electronegativity.

I think the explanation of these facts may be found in the following considerations. The electrons forming part of a normal atomic ring system are not to be considered as free electrons, but as linked together in some way, the nature of

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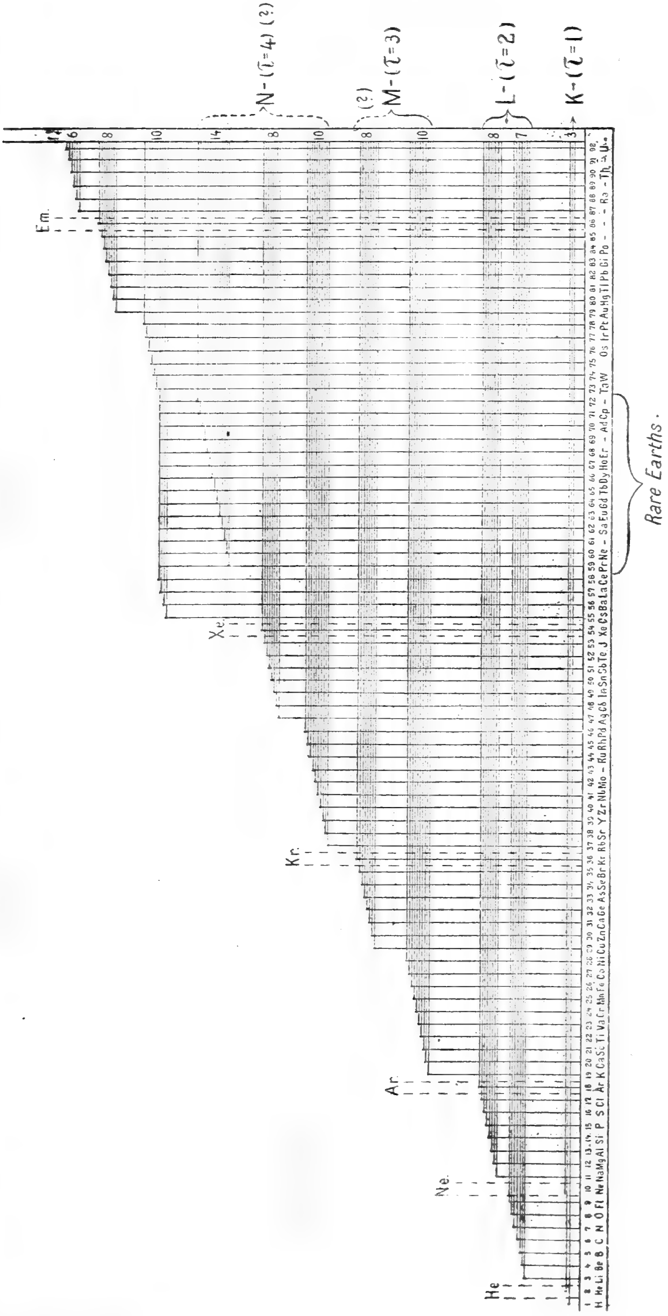
[*] See J. Stark, "Ionisierung der chemischen Elemente durch Elektronenstoss," *Jahrb. d. Rad. u. Elektronik*, xiii.

[†] Sir J. J. Thomson, 'Rays of Positive Electricity,' p. 40 (1913), p. 395 (1916).

Fig. 2.


N-(τ=4) (?)
(?) M-(τ=3)
L-(τ=2)
K-(τ=1)
14
6
8
10
14
8
10
8
10
8
7
3
Em.
Xe.
Kr.
Ar.
Ne.
He
Rare Earths.
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92
H He Li Be B C N O Fl Ne Na Mg Al Si P S Cl Ar K Ca Sc Ti Va Cr Mn Fe Co Ni Cu Zn Ga Ge As Se Br Kr Rb Sr Y Zr Nb Mo - Ru Rh Pd Ag Cd In Sn Sb Te J Xe Cs Ba La Ce Pr Ne - Sa Eu Gd Tb Dy Ho Er - Ad Cp - Ta W Os Ir Pt Au Hg Tl Pb Bi Po - - - Ra - Th - U.

which we do not know. Now the forces which are engaged in the chemical binding of elements do not act on a single electron as in the case of an ionizing agency, but much more on the ring as a unity. From this consideration it might be more natural to take the energy necessary to remove an electron when all the other electrons of the ring were removed simultaneously. According to Bohr this energy is equal to the kinetic energy of the electron, and thus elements with the more slowly moving electrons are the more electropositive. Equation (7) gives for this energy

$$w = h\mathrm{R}\frac{(\mathrm{N}' - \mathrm{S}p)^2}{n^2},$$

where approximately $\mathrm{N}' = p$ and p is the number of electrons in the surface ring; $h\mathrm{R}$ is the value of the energy w_{H} for hydrogen.

$$\frac{w}{w_{\mathrm{H}}} = \frac{(p - \mathrm{S}p)^2}{n^2} = \sigma. \quad . \quad . \quad . \quad . \quad (27)$$

Let us first consider the variation of σ for elements which have the same number of ring systems and only differ with regard to the number of electrons in the external ring. Suppose, *e. g.*, that we consider the elements from Na to Cl. For such a series the quant number n is constant and σ consequently proportional to $(p - \mathrm{S}_p)^2$, and we can easily see that σ increases with increasing values of p by forming

$$\sigma_{p+1} - \sigma_p = \{2p + 1 - (\mathrm{S}_{p+1} - \mathrm{S}_p)\}\{1 - (\mathrm{S}_{p+1} - \mathrm{S}_p)\}.$$

As both factors on the right side are positive, $\sigma_{p+1} > \sigma_p$.

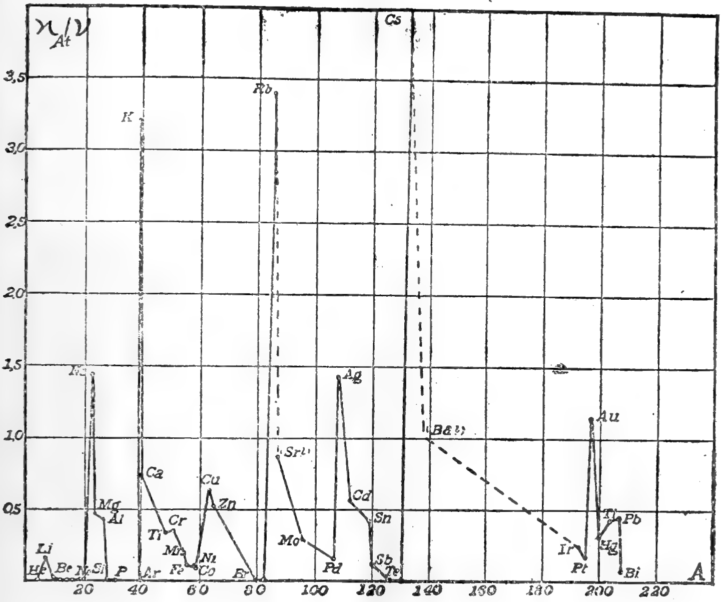
If σ could be taken to represent the electronegativity, the elements in each such group would pass from electropositive to more electronegative as we proceed towards increasing atomic numbers.

Let us next consider elements which are chemically related; such elements have the same value of p. As we pass from low to high atomic numbers, the quant number n will increase and the value of σ will diminish. Thus elements of the same chemical family should be more electropositive as we pass towards increasing atomic numbers, which is indeed a well-known property of the elements.

The Electric Conductivity.

There can probably be no doubt that the electric conductivity in some way or other is related to the energy which binds the electrons of the surface system. Introducing a quantity which he calls the atomic conductive capacity, Benedicks[*] has given a curve which most beautifully brings out the periodic character of the electric conductivity. This curve is shown in fig. 3.

Fig. 3.



Now it is to be expected that a great atomic conductive capacity corresponds to a small value of σ. For, taking the view of Benedicks, the atomic conductivity should be proportional to the frequency ν of the atom and equal to $c\nu$, where c is the conductive capacity. But c must be proportional to the probability that an oscillation shall result in a free electron. Now this probability, and thus c, ought to increase when σ diminishes, and thus we might expect the conductivity to show a similar variation to $1/\sigma$.

Fig. 4 gives the variation of σ and $1/\sigma$ when for a given value of n the number of electrons in the surface ring increase from $p=1$. From this curve we should expect the conductivity suddenly to take a high value each time a new surface ring is commenced. This is exactly the type of variation which is brought out in the curve of Benedicks.

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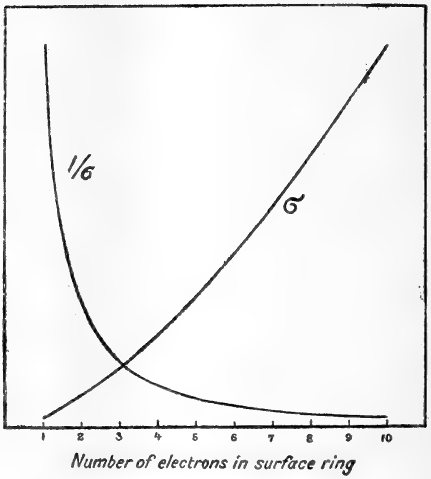
[*] C. Benedicks, *Jahrb. d. Rad. u. Elektronik,* xiii. p. 362 (1916).

For the same value of p the conductivity ought to increase with the atomic number on account of the increase of n. Also this variation is clearly exhibited by the curve of Benedicks. At any rate the rule seems to hold without exception for elements with atomic numbers smaller than those of the rare earths. Above these elements there is again a drop in the conductivity, as if there had been a drop in the quant numbers.

Fig. 4.



The Electron Systems and Radioactivity.

The constitution of atoms here proposed gives us also the simplest possible conception with regard to the changes of the electronic system which accompany a transformation process. The elements formed only have to add one or drop two electrons from the surface ring, according as the product is formed by a β-ray or an α-ray transformation, and the element comes into its right place in the periodic system. No further arrangement of the external system should be necessary.

According to the view put forward in this paper the quant number may vary as an integer from the value $n=1$ nearest to the nucleus. Now the idea suggests itself that passing inwards from the K-ring the electrons which partake in the constitution of the nucleus might have quant numbers smaller than unity.

As I have already shown in previous papers [*], an electron moving with an angular momentum which is a fraction of $h/2\pi$ would possess a very great kinetic energy and move in an orbit with very small diameter. Now according to Rutherford [†] and his collaborators the radioactive elements give γ-radiations which are very much more penetrating than those of the K-series, and we might ask whether these γ-radiations could be due to electronic systems moving inside the K-ring and with an angular momentum

$$\frac{n}{m}\frac{h}{2\pi},$$

where n and m are whole numbers and

$$m > n.$$

This assumption, however, would meet with the difficulty that for elements of high atomic numbers the velocity would soon exceed the velocity of light. The velocity of an electron moving inside the K-ring with an angular momentum $\frac{h}{2\pi m}$ would be

$$v = 2 \,.\, 10^8 (\mathrm{N}+1) m.$$

For uranium $\mathrm{N}=92$, and if v is to be smaller than the velocity of light, we must have

$$m < 1{\cdot}6.$$

It might, however, be argued that electrons bound up in the atom may acquire velocities greater than that of light, because they are not setting up any radiating electromagnetic field when moving in a stationary state. Of course, when an electron is set free in the form of a β-ray, the ordinary electromagnetic field would be active; the electron would meet a sudden retarding impulse which at once would reduce its velocity to less than that of light. Or we might say, at the very moment of release the electron moves according to the principle of relativity, which makes the light velocity an upper limit.

The line $\lambda = 0{\cdot}072 \,.\, 10^{-8}$ cm. observed by Rutherford and

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[*] L. Vegard, Phil. Mag. xxix. p. 651 (1915); *Ann. d. Phys.* liii. p. 27 (1917).

[†] E. Rutherford and E. N. da C. Andrade, Phil. Mag. xxviii. p. 263 (1914).

Andrade [*], which forms a mixture of RaB and RaC, might be accounted for by assuming an electron to recombine between circles with angular momenta $\frac{4}{6}\frac{h}{2\pi}$ and $\frac{3}{6}\frac{h}{2\pi}$.

Let more generally an electron pass from a circle with angular momentum $\frac{n_2}{m}\frac{h}{2\pi}$ to one with a momentum $\frac{n_1}{m}\frac{h}{2\pi}$. If we take into account the variation of mass with velocity, we get

$$\frac{\nu}{\mathrm{R}}=(\mathrm{N}+1)^2m^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)\left\{1+\frac{\rho}{4}(\mathrm{N}+1)\left(\frac{1}{n_1^2}+\frac{1}{n_2^2}\right)\right\}. \quad (28)$$

Putting $\mathrm{N}=82$, $m=6$, $n_1=3$, $n_2=4$, we get

$$\frac{\nu}{\mathrm{R}}=1{\cdot}23\,.\,10^4,$$

while the observed wave-length gives

$$\frac{\nu}{\mathrm{R}}=1{\cdot}26\,.\,10^4.$$

The numerical agreement is good enough, but I think we must be very careful in drawing conclusions from a single coincidence. I merely put it down as a suggestion which might be worth consideration.

Recent experiments of Barkla and Miss White have given indications of a homogeneous J-radiation more penetrating than the K-radiation, which Barkla calls the Y-series. From the absorption coefficient of the rays just hard enough to excite the J-radiation they find for Al a wave-length

$$\lambda=0{\cdot}37\,.\,10^{-8}\ \mathrm{cm}.$$

If this series really exists it can hardly be explained by electrons belonging to the external system, but should be produced by the electrons forming part of the nucleus. The equation (28) would give nearly the right frequency when we put

$$m=4, \quad n_1=1, \quad \text{and} \quad n_2=2.$$

Physical Institute, Christiania.
December 14, 1917.

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[*] E. Rutherford and E. N. da C. Andrade, Phil. Mag. xxviii. p. 263 (1914).

XXXVI. *Relativity and Electrodynamics.* By G. W. Walker, *M.A., F.R.S., A.R.C.Sc., formerly Fellow of Trinity College, Cambridge**.

[Plate X.]

SIR OLIVER LODGE'S recent papers in the Philosophical Magazine have brought into prominence once more the difference of attitude of the protagonists in "Relativity Doctrine" and "Newtonian Dynamics." That Sir Oliver's equation of motion for a moving planet requires some amplification in order to take full account of the special features of electrical inertia, will be recognized, and Prof. Eddington has suggested a method of dealing with the problem. Unfortunately, Eddington's method introduces an assumption which is frequently made by relativists in dealing with electrical inertia, and which in my opinion is inconsistent with the fundamental equations of electrodynamics. In former papers I have drawn attention to this assumption, which is closely linked with the "quasi-stationary principle," and I had not intended to raise the point again. But Sir Oliver has suggested to me that an exposition of my views as to the parting of the ways between the logical development of electrodynamics and the doctrine of relativity would be of value, and I have agreed to his request. My remarks must, however, be confined to electric inertia, and I do not propose to enter on the gravitational and astronomical developments of Einstein's hypothesis.

The *main* point at issue may, I think, be put very concisely. Relativists assume that "the kinetic energy of a moving electrical system is a function of the resultant speed only and is independent of the direction of motion."

My thesis is that this assumption is not consistent with the fundamental electromagnetic equations for the æther (supposed immobile), and that "the energy, or preferably the modified Lagrangean function, depends on the acceleration as well as on the speed of the system and involves also the relative direction of these."

While the above appears to me to be the main point, there is no doubt that subsidiary considerations arise. Theory and experiment have interacted in a curious way, and I think the discussion should proceed by taking notice of the historical development. Sir Joseph Thomson was the first to

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* Communicated by Sir Oliver Lodge.

prove theoretically that a moving electrified system would possess inertia, which Heaviside showed would depend on the speed with which the system moves. A later calculation by Thomson referred to a *particular form* of nucleus and to the momentum which it would carry with it in virtue of a *uniform* translation. It is extremely important to realize that the character of the nucleus determines the manner in which the speed enters in the expression for the momentum or for the energy. It is also vital to realize that while the momentum or the energy can be calculated for a particular form of nucleus moving with a *uniform* speed, it has not so far been found possible to give a complete solution when the speed is *variable*.

M. Abraham extended Thomson's calculations, and he assumed that while the nucleus was still a sphere it was a perfect conductor, and he consequently obtained a value for the momentum in a state of uniform translation which differed from that found by Thomson when squares of the speed were retained. He emphasized the distinction between the effective inertia for acceleration along and perpendicular to the direction of motion.

But finding that he could not obtain the exact solution for a variable speed, Abraham made use of what is called the "quasi-stationary principle," which amounts to saying that if we can calculate the momentum, or if we prefer it the Lagrangean function, for a uniform motion we can infer the equations of motion for a small departure from this state in the ordinary way. My contention is that we can no more do this logically for electromagnetic systems than we can for ordinary dynamical systems. We know quite well that we do not get the correct equations for small departures from a steady state, when the steady motion values are inserted in the Lagrangean function *before* the differential equations of motion are formed. The steady motion values may be inserted *after* the equations have been formed from the general Lagrangean function.

Abraham calculated expressions for longitudinal and transversal electric inertia by means of the quasi-stationary principle. Experiments on transverse inertia became possible with the discovery of the Becquerel rays, and of the minute negatively charged particles projected from radium with speeds only little short of that of light.

The matter was taken up first by W. Kaufmann, and I have a special personal interest in this since I was working side by side with him in the laboratory at Göttingen while his experiments were in progress.

Kaufmann deflected the particles by crossed electric and magnetic fields, by which the particles are sifted out according to their speed, so that the ends of their trajectories form a curve on a photographic plate.

Kaufmann considered that his measurements proved that Abraham's expression for transverse inertia was correct, and that the inertia of the particles was purely electromagnetic in origin.

We must now retrace our steps to consider the important contributions to the theory of moving systems made by Prof. H. A. Lorentz and Sir Joseph Larmor. They proved that a mathematical correlation held between an electrical system at rest and a certain system maintained in *uniform* translation. "If the moving system has a uniform speed kc (where c is the velocity of light) in the direction x, and the x linear extent of the moving system is $(1-k^2)^{\frac{1}{2}}$ of the linear extent of the fixed system, and the variables time t and distance x in the fixed system are transformed to t' and x' in the moving system by a certain linear transformation involving k, then the state of the fixed system in terms of t and x is the same as that of the moving system in terms of t' and x'."

It is reasonable to inquire if the contraction in the proportion $(1-k^2)^{\frac{1}{2}}$ actually takes place when a system at rest is put into uniform translation, for if so it provides an explanation of the Michelson-Morley experiment.

Now it is quite certain that the mathematical transformation is not true when k is variable, and therefore not true at any intermediate stage by which the system at rest might conceivably pass to the correlated system in uniform motion, if it ever does so at all. But relativists have assumed that the correlation proved by Lorentz and Larmor for a *uniform translation only* is true, and that the change actually takes place, when the speed is variable. It appears to me that if the primary equations are correct, the assumption is not merely not permissible, but is not true; and, on the contrary, if the assumption does represent actual truth, then the primary equations are wrong and must go. We await proof, which so far has not been offered.

The longitudinal and transverse inertia of a "contracted" electron have been calculated by the quasi-stationary method, so that there is a double source of error in the result.

Experiments by Kaufmann, Bestelmeyer, and others have been offered as experimental proof that the formula for transverse inertia of a contracted electron on relativity doctrine is correct. My contention is that while the experiments do not conflict with the relativity formula, the formula is inconsistent with the electrodynamic equations, and that several other formulæ correctly deduced from the primary equations agree with the experiments equally well.

I doubt if many people in this country realize the very meagre character of the experimental results, and I therefore give a full-sized reproduction (Pl. X.) of the photographic plate from which Kaufmann made his measurements. The electric deflexion is across the paper and the magnetic deflexion up the paper, and it may be pointed out that if the inertia of the particles were quite independent of speed, the small curved arcs would be parabolas, and that it is only in so far as these arcs differ from parabolas that any dependence of inertia on speed can be made out at all. Further, the highest speed particles are those for which the deflexion is least.

I now return to the theoretical treatment of electric inertia. In order to avoid the error of the quasi-stationary principle, I developed some time ago a method of obtaining the longitudinal and transversal inertia directly from the primary equations by Newtonian methods. The method is rather tedious, but its correctness has not been called in question. Its application is general, but to get definite results the character of the nucleus must be specified. Various systems may be examined provided they do not violate any fundamental restriction imposed by electrodynamic conditions. In this way I examined the nucleus assumed by Sir Joseph Thomson and was able to confirm his result for transverse inertia, but obtained a different result for longitudinal inertia. On the other hand, with the nucleus assumed by Abraham I was able to confirm his result for longitudinal inertia, but not that for transverse inertia.

Again, recently I examined the case of a contracted conducting spheroid which agreed in form with Lorentz's contracted electron for the uniform speed, but did not alter its form when acceleration was imposed *. The results for both longitudinal and transverse inertia differ from those adopted by relativists.

The differences that arise in these examples only become important when squares and higher terms in the speed are retained, and they arise from the fact that when acceleration is imposed, additional electric forces are set up which have to be allowed for in utilizing the boundary conditions at

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* The restriction is unnecessary, as I now find that my results are not altered when the surface deforms under acceleration as Lorentz assumes.

the surface of the electron. Thus, for instance, with a conductor there is a redistribution of the charge, which depends on the acceleration, the speed, and the direction of the acceleration relative to that of the speed. So the Lagrangean function must involve these things, and I am doubtful if it is the resultant speed that alone enters.

The results I have obtained for electric inertia by my direct, if pedestrian, method, can be shown to prove that the energy cannot be expressed as a function of resultant speed only. For if

$$\mathrm{T}=f(v^2)$$

we find that in the direction of motion, say along x, we get

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{x}}=\ddot{x}(2f'+4v^2f''),$$

and at right angles to this, say along y, we get

$$\frac{d}{dt}\frac{d\mathrm{T}}{d\dot{y}}=\ddot{y}2f\,,$$

therefore longitudinal inertia $=m_1=2f'+4v^2f''$,

and transversal inertia $=m_2=2f'$.

Hence, since

$$\frac{d}{dv}m_2v=\frac{d}{dv}2vf'=2f'+4v^2f'',$$

$$m_1=\frac{dm_2v}{dv}$$

must be satisfied. But it does not follow that $\mathrm{T}=f(v^2)$ if $m_1=\dfrac{dm_2v}{dv}$ is satisfied.

My results for the cases mentioned do not satisfy this condition, and unless it can be shown directly from the primary equations that arithmetical error has entered into my calculations, it follows that the kinetic energy of a system in variable motion is not expressible as a function of the resultant speed only.

The conclusion is that Eddington's proposed treatment of the astronomical problem is invalid, and I see no help for it but to start with the equations in the tangential form,

$$m_1v\frac{dv}{ds}=\mathrm{T},$$

$$m_2v^2/\rho=\mathrm{N},$$

where m_1 and m_2 are different functions of v^2, which can be calculated when the electrical system is fully specified.

Starting thus with longitudinal and transverse electric inertia given by m_1 and m_2 as functions of the resultant speed, and employing what I hold to be the correct procedure in forming the equations of motion, viz., resolving along and perpendicular to the resultant direction of motion, we can proceed as follows :—

Let the origin be the sun moving in space with components of velocity u, v, w, which are constant, and let x, y, z be the co-ordinates of a planet relative to the sun and referred to axes through the sun. The components of relative velocity of the planet are

$$\dot{x},\quad \dot{y},\quad \dot{z},$$

and of velocity in space $\dot{x}+u$, $\dot{y}+v$, $\dot{z}+w$, with resultant say V.

The components of acceleration are

$$\ddot{x},\quad \ddot{y},\quad \ddot{z}.$$

The acceleration along the resultant direction of velocity is

$$\{(\dot{x}+u)\ddot{x}+(\dot{y}+v)\ddot{y}+(\dot{z}+w)\ddot{z}\}/\mathrm{V},$$

and if the components of force are X, Y, Z, the component along the direction of V is

$$\{\mathrm{X}(\dot{x}+u)+\mathrm{Y}(\dot{y}+v)+\mathrm{Z}(\dot{z}+w)\}/\mathrm{V}.$$

Hence the equation

$$\begin{aligned} m_1\{(\dot{x}+u)\ddot{x}+(\dot{y}+v)\ddot{y}+(\dot{z}+w)\ddot{z}\} & \\ &= \mathrm{X}(\dot{x}+u)+\mathrm{Y}(\dot{y}+v)+\mathrm{Z}(\dot{z}+w) \\ &= \mathrm{S} \quad \text{say} \quad . \; . \; . \; . \; . \; . \; . \quad (1) \end{aligned}$$

Resolve along any direction λ, μ, ν, at right angles to the direction of V,

then $$m_2\{\lambda\ddot{x}+\mu\ddot{y}+\nu\ddot{z}\}=\mathrm{X}\lambda+\mathrm{Y}\mu+\mathrm{Z}\nu,$$

and $$(\dot{x}+u)\lambda+(\dot{y}+v)\mu+(\dot{z}+w)\nu=0.$$

Hence $$\begin{aligned} m_2\ddot{x} &= \mathrm{X}+(\dot{x}+u)k, \\ m_2\ddot{y} &= \mathrm{Y}+(\dot{y}+v)k, \\ m_2\ddot{z} &= \mathrm{Z}+(\dot{z}+w)k, \end{aligned}$$

where k is some quantity to be determined. Multiply these in order by $\frac{m_1}{m_2}(\dot{x}+u)$, etc., and add and use (1). Then we find

$$S=\frac{m_1}{m_2}S+\frac{m_1}{m_2}kV^2,$$

or

$$k=\frac{(m_2-m_1)}{m_1}S/V^2 \text{ where } V^2=(\dot{x}+u)^2+(\dot{y}+v)^2+(\dot{z}+w)^2.$$

Hence the Cartesian equations of motion are

$$m_2\ddot{x}=X+\frac{(m_2-m_1)}{m_1}\frac{(\dot{x}+u)}{V^2}\{X(\dot{x}+u)+Y(\dot{y}+v)+Z(\dot{z}+w)\},$$

$$m_2\ddot{y}=Y+\frac{(m_2-m_1)}{m_1}\frac{(\dot{y}+v)}{V^2}\{X(\dot{x}+u)+Y(\dot{y}+v)+Z(\dot{z}+w)\},$$

$$m_2\ddot{z}=Z+\frac{(m_2-m_1)}{m_1}\frac{(\dot{z}+w)}{V^2}\{X(\dot{x}+u)+Y(\dot{y}+v)+Z(\dot{z}+w)\}.$$

Also for a central attraction

$$X=-\mu x/r^3, \qquad Y=-\mu y/r^3, \qquad Z=-\mu z/r^3.$$

From these it appears that a planet's orbit cannot remain a plane orbit during transference through the æther, except in the special case when the direction of the sun's motion is in and remains in the plane of the orbit.

We can readily transform the equations to polars since the disturbance is along the direction of the resultant velocity in space.

For the example under recent discussion by Sir Oliver Lodge and as a first approximation, we might take

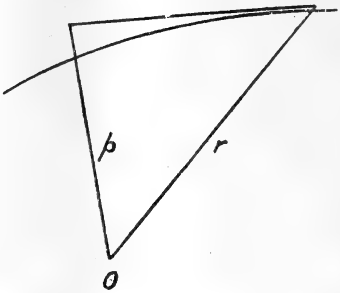
$$m_1=m_0(1+aV^2/c^2), \qquad m_2=m_0(1+bV^2/c^2),$$

and neglecting squares of V^2/c^2, treat the problem as one of a disturbed central orbit.

Astronomers doubtless know the best mode of dealing with this, and could obtain a solution without difficulty.

It is important to remember that these equations are first approximations only to the general equations connecting moving material systems with the state of the æther.

A special case amenable to elementary treatment is that of the orbit of a particle, having electric inertia, round a fixed centre which is at rest in space.



Let v = resultant velocity at any point of the orbit.
ρ = radius of curvature.
r = radius vector.
p = perpendicular on tangent.
μ/r^2 = attraction to the centre.
m_1 = longitudinal inertia $= m_0(1+k_1v^2/c^2)$.
m_2 = transverse inertia $= m_0(1+k_2v^2/c^2)$ neglecting squares of v^2/c^2.

The orbit is plane, and resolving along and perpendicular to the path we get

$$m_1 v\frac{dv}{ds} = -\frac{\mu}{r^2}\cdot\frac{dr}{ds}, \quad \ldots\ldots \quad (1)$$

and

$$m_2v^2/\rho = \frac{\mu}{r^2}\cdot\frac{p}{r}. \quad \ldots\ldots\ldots \quad (2)$$

Integrating (1) we get

$$m_0(v^2+\tfrac{1}{2}k_1v^4/c^2) = \frac{2\mu}{r} - \frac{\mu}{a},$$

where a is a constant.

From (2) $\quad m_0(v^2+k_2v^4/c^2) = \dfrac{\mu p}{r^2}\dfrac{dr}{dp}, \quad$ since $\rho = r\dfrac{dr}{dp}$.

Let $u=1/r$ and $u_0=1/a$,

$$\therefore\ m_0(v^2+\tfrac{1}{2}k_1v^4/c^2) = 2\mu u - \mu u_0,$$

and

$$m_0(v^2+k_2v^4/c^2) = -\mu p\frac{du}{dp},$$

$\therefore$ approximately $m_0 v^2 = 2\mu u - \mu u_0 - \frac{1}{2}\frac{k_1}{m_0 c^2}(2\mu u - \mu u_0)^2$,

and
$$(2\mu u - \mu u_0) + \frac{(k_2 - \frac{1}{2}k_1)}{m_0 c^2}(2\mu u - \mu u_0)^2 = -\mu p \frac{du}{dp},$$

or putting
$$x = 2\mu u - \mu u_0,$$

and
$$\lambda = (k_2 - \tfrac{1}{2}k_1)/m_0 c^2,$$

we get
$$x + \lambda x^2 = -\tfrac{1}{2}p\frac{dx}{dp},$$

or
$$-2\frac{dp}{p} = \delta x\left\{\frac{1}{x} - \frac{\lambda}{1+\lambda x}\right\},$$

$\therefore$ integrating we get

$$b^2/p^2 = \frac{x}{1+\lambda x} = x - \lambda x^2 \text{ neglecting squares of } \lambda,$$

where b is a constant of integration.

Now
$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

$$\therefore \quad b^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = (2\mu u - \mu u_0) - \lambda(2\mu u - \mu u_0)^2,$$

or
$$b^2\left(\frac{du}{d\theta}\right)^2 = (2\mu u - \mu u_0) - \lambda(2\mu u - \mu u_0)^2 - b^2 u^2,$$

so that the integral is of the form

$$lu = 1 + e\cos\left[\left\{1 + \frac{4\mu^2\lambda}{b^2}\right\}^{\frac{1}{2}}\theta - \eta\right],$$

where η is an arbitrary constant and l and e are determinate constants in μ, a, b, and λ. This solution implies an elliptic orbit slowly revolving in its own plane. The eccentricity does not change, but the apses will advance in the direction of description of the orbit by

$$\frac{2\pi}{\left\{1 + \frac{4\mu^2\lambda}{b^2}\right\}^{\frac{1}{2}}} - 2\pi \text{ for each description of the orbit,}$$

i. e. since λ is small,
by
$$-\frac{4\pi\mu^2\lambda}{b^2},$$

or
$$\frac{4\pi\mu^2}{m_0 b^2}(\tfrac{1}{2}k_1 - k_2)/c^2.$$

Expressed in terms of the semi-major axis R_0, eccentricity e and periodic time T, the progress per revolution is

$$\left\{\frac{2\pi R_0}{T(1-e^2)^{\frac{1}{2}}}\right\}^2 \frac{(\frac{1}{2}k_1-k_2)}{c^2} \,.\, 4\pi.$$

The apses therefore progress or regress according as $(\frac{1}{2}k_1-k_2)$ is positive or negative.

We have no knowledge as to the proper forms of m_1 and m_2 for matter in bulk, but the following are results for hypothetical single nuclei.

For the contracted electron using relativity methods,

$$m_1=m_0(1+\tfrac{3}{2}v^2/c^2), \qquad m_2=m_0(1+\tfrac{1}{2}v^2/c^2),$$

so that $\frac{1}{2}k_1-k_2=\frac{1}{4}$ (or progression).

From the primary electromagnetic equations my results* are :—

For the contracted conducting electron :

$$m_1=m_0(1+\frac{13}{10}v^2/c^2), \qquad m_2=m_0(1+\frac{41}{60}v^2/c^2),$$

so that $$\tfrac{1}{2}k_1-k_2=-\frac{1}{30} \text{ (or regression).}$$

For Thomson's electron with special surface condition :

$$m_1=m_0(1+\frac{7}{10}v^2/c^2), \qquad m_2=m_0(1+\frac{4}{10}v^2/c^2),$$

so that $$\tfrac{1}{2}k_1-k_2=-\frac{1}{20} \text{ (or regression).}$$

For spherical conductor which does not change in shape :

$$m_1=m_0(1+\tfrac{6}{5}v^2/c^2), \qquad m_1=m_0(1+\frac{19}{60}v^2/c^2),$$

so that $$\tfrac{1}{2}k_1-k_2=\frac{17}{60} \text{ (or progression).}$$

This last case is numerically almost the same as that for the contracted electron by relativity methods. This is important, because it shows that so far as inertia enters in the astronomical problem we can get practically the same result

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* Proved only for disturbance from a steady state.

as a logical sequence from the fundamental equations as has been obtained by relativity doctrine.

Our results for the apsidal progression per revolution are: Contracted electron by relativity method,

$$\left\{\frac{2\pi R_0}{T(1-e^2)^{\frac{1}{2}}}\right\}^2\frac{\pi}{c^2}.$$

Spherical electron by orthodox method,

$$\left\{\frac{2\pi R_0}{T(1-e^2)^{\frac{1}{2}}}\right\}^2\frac{17}{15}\frac{\pi}{c^2}.$$

Now Einstein obtained

$$\left\{\frac{2\pi R_0}{T(1-e^2)^{\frac{1}{2}}}\right\}^2\frac{6\pi}{c^2},$$

which numerically is in close agreement with observations on "Mercury."

This result is obtained by assuming that the attraction depends on the velocity. It is easily seen from our analysis that if

$$\mu=\mu_0(1+k_3v^2/c^2),$$

we get for the apsidal progress

$$\left\{\frac{2\pi R_0}{T(1-e^2)^{\frac{1}{2}}}\right\}^2\frac{4\pi}{c^2}\left(\tfrac{1}{2}k_1+\frac{k_3}{2}-k_2\right).$$

In order to get the observed value for "Mercury" k_3 would have to be 5/2 if $\frac{1}{2}k_1-k_2$ is 1/4.

It is important to recognize that it is only by introducing either explicitly or implicitly this comparatively large dependence on speed, of the attraction between bodies that Einstein can get the numerical agreement. Such dependence based on the known forces between electrical currents has been recognized before now in the theory of electrodynamics, but is hardly acceptable in gravitational theory. On these lines it appears that orthodox electrodynamics is quite as capable of providing an explanation of this astronomical feature as Einstein's theory. It is, however, important that endeavour should be made to determine, if possible, the numerical value of $\frac{1}{2}k_1+\frac{1}{2}k_3-k_2$ for matter in bulk.

There still remains the question of the effect of transference in space as suggested by Sir Oliver Lodge. Eddington's conclusions on this problem may be modified considerably when what I hold to be more correct equations of motion are used.

Note.—At Sir Oliver Lodge's request I have calculated m_1 and m_2 for Bucherer's electron which has the same form as Lorentz's electron but keeps its volume unchanged. My results are

$$m_1 = m_0\left(1+\frac{17}{15}v^2/c^2\right), \qquad m_2 = m_0\left(1+\frac{31}{60}v^2/c^2\right),$$

so that

$$\tfrac{1}{2}k_1 - k_2 = \frac{1}{20}.$$

XXXVII. *Molecular Frequency and Molecular Number.* By H. Stanley Allen, *M.A., D.Sc., University of London, King's College*[*].

Part I.

§ 1. *Molecular Number.*

The work of Moseley on the high-frequency spectra of the elements has established securely the importance of the "atomic number" of an element: that is, the number which determines the place in the periodic classification and fixes the charge carried by the central part of the atom. It is now certain that the atomic number is more fundamental than the atomic weight. Recent investigations of the atomic weight of lead of radioactive origin have shown that the value obtained for this quantity depends upon the source from which the material is derived. An interesting account of these researches has been given by Soddy[†], who points out that the atomic weight as ordinarily understood is not the unique quantity hitherto supposed. In the future increasing importance will be attached to the atomic number. It is the conviction of the present author that this will prove true not only in connexion with the properties of the chemical elements but also in dealing with compounds. In the latter case it is convenient to introduce the term "molecular number" to signify the sum of the positive charges carried by the atomic nuclei contained in the molecule. Thus when a molecule contains a atoms of an element A, b atoms of B, c atoms of C, so that its chemical formula is $A_aB_bC_c$, the molecular number $N = aN_a + bN_b + cN_c$, where N_a, N_b, N_c are the atomic numbers of the component elements. For

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[*] Communicated by the Author.

[†] Royal Institution Lecture, 'Nature,' vol. xcix. p. 414 (1917).

example, the molecular number of water (H_2O, hydrol) is 10[*], for the nuclear charge of hydrogen is 1, and of oxygen is 8.

It may be remarked that the molecular number is usually even. This arises from the fact that when the valency is odd, the atomic number is usually odd also. But in the case of an element such as copper, which may be either univalent or divalent, or in the case of some of the metals of the eighth group, the molecular number may be odd.

In former papers[†] it has been shown that simple relations exist between the atomic number of an element and the characteristic frequency deduced from observations of the specific heat in the solid state. In the present communication similar results are found in connexion with the molecular number of a compound and its characteristic frequency. So far as the writer is aware, this is the first attempt to establish a relationship involving molecular number, previous work in different branches of physics having been restricted to considerations of atomic number only.

§ 2. *Characteristic Molecular Frequency.*

At high temperatures the law as to the specific heat of compounds enunciated by Joule[‡] and verified by Kopp[§] shows that, as the specific heat is then mainly additive, the heat energy arises for the most part from the vibrations of the individual atoms[‖]. At sufficiently high temperatures the vibrational energy of each atom approaches the value 3RT. At low temperatures, on the other hand, Nernst[¶] supposes that the vibrations of the molecules play a more important part than the vibrations of the atoms in the molecule. In the case of regular monatomic solids Debye has deduced an

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[*] This fact is probably at the bottom of the remarkable numerical relations involving powers of 10, pointed out by the author in a paper read before the Physical Society of London (Proceedings, vol. xxvii. p. 425, 1915). It was shown that there must be a numerical connexion between the unit of length and the unit of mass in the C.G.S. system, "and there is no reason why it should not involve the number 10." This negative statement may now be changed to a positive one. There *is* a reason, in the constitution of water itself, why the number 10 should be introduced.

[†] H. S. Allen, Proc. Roy. Soc. vol. xciv. p. 100 (1917); Phil. Mag. vol. xxxiv. p. 478, p. 488 (1917).

[‡] Joule, Phil. Mag. [3] vol. xxv. p. 334 (1844).

[§] Kopp, *Lieb. Ann.* vol. iii. pp. 1 & 289 (1864).

[‖] *Cf.* Sutherland, Phil. Mag. [5] vol. xxxii. p. 550 (1891).

[¶] Nernst, *Vorträge über die Kinetische Theorie*, p. 79 (1914). 'The Theory of the Solid State,' p. 81 (1914).

expression for the specific heat, C_v, which is reduced, at sufficiently low temperatures, to a simple law of proportionality between C_v and T^3. That this is also true for certain regular polyatomic substances has been shown experimentally by Eucken and Schwers* in the case of fluorite, CaF_2, and pyrites, FeS_2. Thus it would appear that near the absolute zero the forces uniting the atoms in the molecule are sufficiently great, as compared with the forces uniting the molecules, to compel the individual atoms to follow the movements of the molecule of which each forms a part. At low temperatures the specific heat can be represented by Debye's formula assuming a single characteristic frequency. At higher temperatures Nernst introduces one or more Einstein terms, with appropriate characteristic frequencies, to include the vibrations of the atoms in the molecule. It will be shown that the characteristic frequency, ν, for the molecular movement conforms to the relation previously found to hold for the elements, viz.

$$N\nu = n\nu_A \quad \text{or} \quad N\nu = (n + \tfrac{1}{2})\nu_A,$$

where N is now the molecular instead of the atomic number, n is an integer, and ν_A is a fundamental frequency having a value very near to 21×10^{12} sec.$^{-1}$. The term "frequency number" is suggested to denote the numerical factor, n or $n + \frac{1}{2}$.

It would, of course, be possible to avoid the introduction of the fraction $\frac{1}{2}$ by introducing a fundamental frequency which is $\frac{1}{2}$ that just quoted, but as the number of cases requiring the fractional value is comparatively small, it seems better to retain for the present the larger value for ν_A.

§ 3. *Characteristic Frequency from Specific Heat.*

For a small number of compounds low-temperature measurements are available, and the characteristic frequency can be deduced from the specific heat. In 1912 Nernst and Lindemann † published observations on the specific heat of rock-salt and sylvin at temperatures down to 22° K. For NaCl the characteristic frequency, ν, was determined by the equation $\beta\nu = 287{\cdot}3$, whilst for KCl $\beta\nu = 217{\cdot}6$, where $\beta = 4{\cdot}78 \times 10^{-11}$. From these results we find the value of $N\nu$ for rock-salt to be $8 \times 21{\cdot}0 \times 10^{12}$, whilst for sylvin it is nearly the same, $8 \times 20{\cdot}5 \times 10^{12}$. In his address on the Kinetic Theory Nernst gives different values for the Debye

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* Eucken and Schwers, *Ber. deutsch. phys. Gesell.* vol. xv. p. 578 (1913).

† *Preuss. Akad. Berlin,* p. 1160 (1912).

term which is predominant at low temperatures. For NaCl he finds $\beta\nu=229$, which would require $N\nu=6\frac{1}{2}\times 20{\cdot}6\times 10^{12}$. As Lindemann's formula gives a smaller value (215) for $\beta\nu$, it may be suspected that the true value for $N\nu$ at very low temperatures is $6\times 21\times 10^{12}$. This is supported by the value for KCl, for which Nernst gives $\beta\nu=166$, so that $N\nu=6\times 20{\cdot}8\times 10^{12}$. The change in the frequency number from 8 to 6 must be attributed to the introduction of the Einstein term, and points to the relation $N\nu=n\nu_A$ being obeyed by the corresponding frequency. This is actually found to be the case, as is shown in a separate paper by the writer.

Nernst also gives values for the characteristic frequency of chloride of mercury and chloride of silver. For HgCl $\beta\nu=115$, from which we find $N\nu=11\times 21{\cdot}2\times 10^{12}$, and for AgCl $\beta\nu=102$, giving $N\nu=6\frac{1}{2}\times 21{\cdot}0\times 10^{12}$.

Experiments by Eucken and Schwers* which are believed to be very accurate give the characteristic frequency for two compounds containing three atoms in the molecule. For fluorite (CaF_2) $\beta\nu=474$, resulting in $N\nu=18\times 20{\cdot}9\times 10^{12}$. For pyrites ($FeS_2$) $\beta\nu=645$, and as $N=58$ the product $N\nu$ is so large ($782{\cdot}6\times 10^{12}$) that it is difficult to be certain of the value to be assigned to n. If $n=37$ the product $N\nu=37\times 21{\cdot}2\times 10^{12}$; if, as is more probable, $n=36$ the product $N\nu=36\times 21{\cdot}8\times 10^{12}$.

Eucken† found for carbon dioxide (CO_2) in the solid state $\beta\nu=119$, giving $N\nu=2\frac{1}{2}\times 21{\cdot}9\times 10^{12}$. This is of interest as indicating that the rule applies in the case of non-metallic compounds.

§ 4. *Characteristic Frequency from Lindemann's Formula.*

For the majority of compounds no measurements of the specific heat at low temperatures have been made, and in such cases it is necessary to have recourse to some more or less empirical formula such as that of Lindemann. This formula gives

$$\nu=3{\cdot}08\times 10^{12}\sqrt{(T_s/MV^{\frac{2}{3}})},$$

where T_s denotes the melting-point, M the molecular weight, and V the molecular volume.

It is easy to understand that the frequency calculated in this way may not be identical with the frequency determined from the specific heat at low temperatures, for the

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* Eucken & Schwers, *D. P. G. V.* vol. xv. p. 578 (1913).

† Eucken, *D. P. G. V.* vol. xviii. p. 4 (1916).

molecule of the solid at or near the melting-point may not have the same constitution as the molecule near the absolute zero of temperature. The frequency as given by Lindemann's formula must be taken to represent the characteristic frequency the substance would take at low temperatures, on the assumption that the molecular structure remained unchanged in cooling from the melting-point to the absolute zero. If either polymerization or dissociation occur in the process of cooling, then a change in the characteristic frequency is to be anticipated.

A definite decision as to whether the relation between N and ν is exact or only approximate cannot be reached until further accurate determinations of the specific heat of compounds at very low temperatures are available. Empirical formulæ, such as that of Lindemann, may be employed for the evaluation of ν, but it must be remembered that such formulæ generally give only approximations to the true value, and therefore cannot furnish a decisive test. It has been suggested that to render Lindemann's formula accurate an additional factor is required depending upon the relation between the molecular volume at absolute zero and that at the melting-point*. It may be anticipated that the formula in its present form should give comparable results for chemical compounds of similar constitution. This would be shown by agreement between the values of $N\nu$, or by concordant values of ν_A. It has, in fact, been shown previously that such agreement exists in the case of similar elements, and evidence is now put forward that relations of the same kind hold for similar compounds.

It must be borne in mind in considering the results not only that Lindemann's formula is merely an approximate one, but that the numerical factor is purely empirical. Further, it is to be noticed that the molecular volume is usually found from determinations of the density made at ordinary temperatures. It might be better to employ the density of the solid near the absolute zero of temperature or at the melting-point.

The data employed for the calculation of the characteristic frequencies in the following paragraphs have been taken mainly from Nostrand's 'Chemical Annual' for 1913, which gives Tables convenient for this purpose. Kaye and Laby's 'Physical and Chemical Constants' (1911), and the Smithsonian Physical Tables (1914) have also been made use of for certain compounds. In cases where a range of values

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* *Cf.* Sutherland, Phil. Mag. vol. xxx. p. 318 (1890); vol. xxxii. p. 524 (1891); and Grüneisen, *Ann. d. Physik*, vol. xxxix. p. 298 (1912).

is given for the melting-point, the highest value quoted has been used; similarly the largest value of the density has been taken in calculating the molecular volume.

§ 5. *Inorganic Compounds (Lindemann's Formula).*

It may be pointed out in the first place that the product $N\nu$ frequently has the same value for compounds which are similar in their chemical constitution and behaviour. Thus in the case of the alkali metals we find for the chlorides of sodium, potassium, and rubidium the values:—

$NaCl$, $123 \cdot 2 \times 10^{12}$; KCl, $125 \cdot 8 \times 10^{12}$; $RbCl$, $124 \cdot 7 \times 10^{12}$,

Similarly for the iodides of the same metals:—

NaI, $145 \cdot 2 \times 10^{12}$; KI, $143 \cdot 1 \times 10^{12}$; RbI, $145 \cdot 4 \times 10^{12}$.

Further it may be noted that the product for the iodide exceeds that for the chloride by an amount which is approximately constant, and equal to about 20×10^{12}. Similarly the difference between the product for sodium chloride ($123 \cdot 2 \times 10^{12}$) and that for lithium chloride ($101 \cdot 6 \times 10^{12}$) is $21 \cdot 6 \times 10^{12}$, whilst the difference between the product for sodium iodide ($145 \cdot 2 \times 10^{12}$) and that for lithium iodide ($124 \cdot 7 \times 10^{12}$) is $20 \cdot 5 \times 10^{12}$. All such relations, and their number is far too great for them to be fortuitous, may be included in the formulæ

$$N\nu = n\nu_A \quad \text{and} \quad N\nu = (n + \tfrac{1}{2})\nu_A,$$

where ν_A is approximately 21×10^{12} sec.$^{-1}$.

These formulæ have now been tested for those inorganic compounds for which the necessary data are recorded, and it has been found that the number of cases in which one or other of the formulæ cannot be applied is small. It is hoped to publish details of these results later; at present it will suffice to quote the figures for two series of compounds. In Table I. are given the results for all the lithium compounds for which data are available. This element has been selected on account of the small atomic number ($N = 3$) rendering the product $N\nu$ comparatively small; consequently it is not necessary to employ large values for n, and a more satisfactory test of the new relation can be obtained. The frequency number in the last column of the Table falls between $3\frac{1}{2}$ and 8, whilst the extreme values for ν_A are $20 \cdot 1 \times 10^{12}$ sec.$^{-1}$ and $21 \cdot 3 \times 10^{12}$ sec.$^{-1}$ Application of the theory of probability to the figures shows that there is only 1 chance in 282 that these results should occur by accident.

TABLE I.

Lithium Compounds.

Name.	Formula.	N.	$\nu \times 10^{-12}$.	$N\nu \times 10^{-12}$.
LITHIUM				
amide	$LiNH_2$	12	6·073	$3\frac{1}{2} \times 20{\cdot}8$
bromide	$LiBr$	38	3·234	$6 \times 20{\cdot}5$
carbonate	Li_2CO_3	36	3·435	$6 \times 20{\cdot}6$
chloride	$LiCl$	20	5·079	$5 \times 20{\cdot}3$
fluoride	LiF	12	9·207	$5\frac{1}{2} \times 20{\cdot}1$
iodide	LiI	56	2·227	$6 \times 20{\cdot}8$
nitrate.................	$LiNO_3$	34	2·821	$4\frac{1}{2} \times 21{\cdot}3$
perchlorate............	$LiClO_4$	52	1·742	$4\frac{1}{2} \times 20{\cdot}1$
phosphate	$Li_3PO_4 . H_2O$	66	2·345	$7\frac{1}{2} \times 20{\cdot}6$
silicate*	Li_2SiO_3	44	3·756	$8 \times 20{\cdot}8$
sulphate (acid)	$LiHSO_4$	52	1·636	$4 \times 21{\cdot}3$
sulphate (normal) ...	Li_2SO_4	54	2·680	$7 \times 20{\cdot}7$

Mean value of $\nu_A = 20{\cdot}66 \times 10^{12}$.

In Table II. are recorded the figures for the chlorides of the alkali metals, and the monochlorides of copper and silver. No data have been obtained for aurous chloride. The probability calculated in the same way as for Table I. is about 1/18. It will be noticed, however, that all the frequency numbers in Table II. are integers. The probability that this should be the case is about 1/2000.

TABLE II.

Monochlorides of the Metals of Group I.

Formula.	N.	$\nu \times 10^{-12}$.	$N\nu \times 10^{-12}$.
$LiCl$...............	20	5·079	$5 \times 20{\cdot}3$
$NaCl$	28	4·401	$6 \times 20{\cdot}5$
KCl...............	36	3·493	$6 \times 20{\cdot}9$
$RbCl$	54	2·310	$6 \times 20{\cdot}8$
$CsCl$	72	2·065	$7 \times 21{\cdot}2$
$CuCl$	46	2·726	$6 \times 20{\cdot}9$
$AgCl$	64	2·343	$7 \times 21{\cdot}4$

Mean value of $\nu_A = 20{\cdot}86 \times 10^{12}$.

In view of the fact that so many other inorganic compounds as well show fair agreement with the proposed relation, it is hardly possible to doubt that it must give at least a close approximation to the truth.

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* The melting-point of lithium silicate is given as a standard temperature (1201°) in the Smithsonian Physical Tables.

One point of interest may be mentioned. Inorganic compounds which contain water of crystallization conform to the general rule. In some cases the frequency number of the dehydrated salt is the same as that of the hydrated compound. Thus for sodium sulphate (Na_2SO_4) the value of $N\nu$ is $8\times 20{\cdot}4\times 10^{12}$, for the hydrated salt ($Na_2SO_4 . 10\,H_2O$) it is $8\times 20{\cdot}6\times 10^{12}$. In other cases there is a change in the frequency number. An example is afforded by calcium nitrate—the anhydrous salt ($Ca(NO_3)_2$) gives for the product $N\nu$ the value $6\frac{1}{2}\times 21{\cdot}3\times 10^{12}$, whilst the hydrated compound ($Ca(NO_3)_2 . 4\,H_2O$) gives $4\times 21{\cdot}5\times 10^{12}$.

A comparison between the results obtained from the specific heat at low temperatures and those found by calculation from Lindemann's formula is only possible in a few cases.

TABLE III.

Compound.	$N\nu\times 10^{-12}$.	
	Specific heat.	Lindemann's formula.
NaCl	$8\times 21{\cdot}0$ $6\frac{1}{2}\times 20{\cdot}6$	$6\times 20{\cdot}5$
KCl	$8\times 20{\cdot}5$ $6\times 20{\cdot}8$	$6\times 20{\cdot}9$
AgCl	$6\frac{1}{2}\times 21{\cdot}0$	$7\times 21{\cdot}4$
CaF_2	$18\times 20{\cdot}9$	$9\times 20{\cdot}1$
FeS_2	$36\times 21{\cdot}8$	$10\times 21{\cdot}7$

It is curious that there should be such a large difference between the two values of the frequency numbers for calcium fluoride and iron disulphide. Interpreted according to the theory of Nernst, this may indicate that for these compounds the contribution to the specific heat arising from the internal vibrations of the molecule forms an important part of the whole.

§ 6. *Organic Compounds (Lindemann's Formula).*

Chemists have not, as a rule, devoted great attention to the determination of the density of organic compounds in the solid state. On examining such a Table as that given in the 'Chemical Annual' it will be found in general that where the density of the solid is recorded, the melting-point is wanting and *vice versa*. Amongst the results available up to the present time are to be found many suggestive cross-relationships between the values of $N\nu\times 10^{-12}$, in which a number approximating to 21 or $\frac{1}{2}\times 21$ is of frequent occurrence. This is illustrated for some aliphatic derivatives in the following Table.

TABLE IV.

Name.	Formula.	N.	$N\nu \times 10^{-12}$.
Maleic anhydride ...	$<(CH.CO)_2>O$	50	60·17
Succinic anhydride...	$<(CH_2CO)_2>O$	52	70·56
Malic acid (*l*)	$CO_2H.CH_2.CHOH.CO_2H$	70	82·11
Oxalic acid	$CO_2H.CO_2H+2H_2O$	66	82·24
Maleic acid	$CO_2H.CH:CH.CO_2H$	60	82·42
Citric acid	$(CO_2H.CH_2)_2C(OH)CO_2H+H_2O$	110	93·71
Tartaric acid	$CO_2H[CH(OH)]_2CO_2H$	78	93·79

Here the characteristic difference is found between the two anhydrides; the three dibasic acids, malic, oxalic, and maleic, have a common value for the product but different values for N; and citric acid (a monohydroxy tribasic acid) and tartaric acid (a dihydroxy dibasic acid) have the same value for $N\nu \times 10^{-12}$, but a value exceeding the previous common value by the characteristic difference.

Data are available for a larger number of aromatic derivatives, and amongst these compounds many interesting correspondences occur. In Table V. are given the results for a number of hydroxyl derivatives containing the benzene ring, and in Table VI. some of the halogen derivatives of benzene in which the two substituted groups occupy the para position.

TABLE V.

Name.	Formula.	N.	$N\nu \times 10^{-12}$.
Phenol	C_6H_5OH	50	63·29
Cresole (*p*)	$CH_3.C_6H_4OH$	58	64·23
Xylenol	$(CH_3)_2C_6H_3OH$	66	72·84
Hydroquinone (*p*) ...	$C_6H_4(OH)_2$	58	82·07
Pyrogallol..............	$C_6H_3(OH)_3$	66	82·52

TABLE VI.

Name.	Formula.	N.	$N\nu \times 10^{-12}$.
Chlor phenol	ClC_6H_4OH	66	68·40
Dichlor benzene	$C_6H_4Cl_2$	74	69·63
Chlor nitrobenzene ...	$ClC_6H_4NO_2$	80	78·86
Brom phenol	BrC_6H_4OH	84	79·41
Dibrom benzene	$C_6H_4Br_2$	110	88·64
Brom nitrobenzene ...	$BrC_6H_4NO_2$	98	89·95

Data are available for a number of ketones of the benzene series.

TABLE VII.

Name.	Formula.	N.	$N\nu \times 10^{-12}$.
Methyl-phenyl ketone (aceto-phenone) ...	$CH_3 . CO . C_6H_5$	64	63·16
Ethyl-phenyl ketone ...	$C_2H_5 . CO . C_6H_5$	72	64·48
Methyl-benzyl ketone ...	$CH_3 . CO . CH_2 . C_6H_5$	72	65·03
Propyl-phenyi ketone ...	$C_3H_7 . CO . C_6H_5$	80	65·81

Here the values of $N\nu \times 10^{-12}$, though not differing greatly for the various compounds, tend to increase slightly with the complexity of the chemical molecule, indicating the presence of a constitutive influence.

An attentive examination of the results just recorded for organic compounds will have shown that in a large number of cases the value of the product $N\nu$ can be expressed in the form already employed for inorganic compounds. In other cases, however, the simple form of the relation cannot be applied successfully. The two Tables following (VIII. and IX.) contain the figures for a number of well-known aliphatic and aromatic compounds for which the relation is found to hold good.

It is noteworthy that the frequency number for organic compounds is usually small, varying from about 3 to about 5. This fact may be correlated with the low melting-point of these compounds and the small atomic numbers of the constituent elements.

TABLE VIII.

Aliphatic Compounds.

Name.	Formula.	N.	$\nu \times 10^{-12}$.	$N\nu \times 10^{-12}$.
Butyl carbinol (tert.) ...	$(CH_3)_3C . CH_2OH$	50	1·242	$3 \times 20·7$
Urethane	$NH_2CO_2 . C_2H_5$	48	1·307	$3 \times 20·9$
Urea	$CO(NH_2)_2$	32	2·243	$3\frac{1}{2} \times 20·5$
Ethyl urea	$C_2H_5NH . CONH_2$	48	1·500	$3\frac{1}{2} \times 20·6$
Ethylene iodide	$CH_2I . CH_2I$	122	0·672	$4 \times 20·5$
Iodoform	CHI_3	166	0·549	$4\frac{1}{2} \times 20·3$
Carbon tetrabromide ...	CBr_4	146	0·703	$5 \times 20·5$
Oxamide	$CONH_2 . CONH_2$	46	2·209	$5 \times 20·3$

Table IX.

Aromatic Compounds.

Name.	Formula.	N.	$\nu \times 10^{-12}$.	$N\nu \times 10^{-12}$.
Xylidene (1 : 4 : 2)	$(CH_3)_2C_6H_3 . NH_2$	66	0·954	$3 \times 21·0$
Chlorquinoline (py 2) ...	C_9H_6ClN	84	0·842	$3\frac{1}{3} \times 20·2$
Chlorquinoline (py 4)...	C_9H_6ClN	84	0·858	$3\frac{1}{2} \times 20·6$
Naphthalene..	$C_6H_4 . C_4H_4$	68	1·062	$3\frac{1}{2} \times 20·5$
Triphenyl phosphine ...	$(C_6H_5)_3P$	138	0·592	$4 \times 20·4$
Hydrazo benzene	$C_6H_5 . NH . NH . C_6H_5$	98	0·842	$4 \times 20·6$
Camphor	$C_{10}H_{16}O$	84	0·989	$4 \times 20·8$
Nitraniline (*p*)	$NO_2 . C_6H_4 . NH_2$	72	1·173	$4 \times 21·1$
Tetrachlor benzene (*s*)...	$C_6H_2Cl_4$	106	0·874	$4\frac{1}{2} \times 20·6$
Camphoric acid (*d*)	$C_8H_{14}(CO_2H)_2$	108	0·859	$4\frac{1}{2} \times 20·6$
Camphoric anhydride...	$C_{10}H_{14}O_3$	98	0·949	$4\frac{1}{2} \times 20·7$
Anthraquinone	$C_6H_4 : (CO)_2 : C_6H_4$	108	0·961	$5 \times 20·7$

It has been mentioned already that for a considerable number of organic compounds the proposed relation, in its simple form, does not hold. One possible suggestion as to the reason for this failure will be considered in the following section.

§ 7. *Molecular Association in Solids.*

A question of great importance, which can only be considered briefly in this paper, is the determination of the true value of the molecular weight of a solid compound. Nernst has shown how the constitution of the molecule may be inferred from the correspondence between the molecular heat at low temperatures and the value calculated by Debye's formula. For example, he concludes that the molecule of ice is not hydrol ($\beta\nu = 227$) but dihydrol ($\beta\nu = 155$). On the first supposition we find $N\nu = 47·5 \times 10^{12}$, which cannot be expressed as a multiple of ν_A, but the second supposition gives $N\nu = 64·8 \times 10^{12} = 3 \times 21·6 \times 10^{12}$, showing good agreement with the new relation.

If, in the next place, we apply the formula of Lindemann to ice ($T_s = 273°·1$ K, $\rho = 0·917$), we find that neither H_2O nor $(H_2O)_2$ gives satisfactory agreement. When, however, we assume that the ice molecule at the melting-point is trihydrol, $(H_2O)_3$, we find $N\nu = 2\frac{1}{2} \times 21·4 \times 10^{12}$. The method does not give a unique determination of the degree of association, for equally good concordance is obtained by assuming that the ice molecule contains 9 groups of H_2O, which makes $N\nu = 3 \times 21·4 \times 10^{12}$.

On general grounds it seems most probable that ice at the melting-point is pure trihydrol. This was the conclusion reached by Sutherland*, who emphasized the fact that water crystallizes in the hexagonal system, whilst trihydrol can be represented by three linked oxygen atoms at the corner of an equilateral triangle with the six hydrogen atoms arranged symmetrically round them.

It is easy to examine the effect of association on the frequency as determined by Lindemann's formula. If M denote the molecular weight of the simplest molecule, that of the associated molecule may be written xM. The molecular volume V will also be increased x times. Hence the frequency, determined by the equation

$$\nu = \frac{k}{V^{\frac{1}{3}}} \sqrt{\left(\frac{T}{M}\right)}$$

will be divided by $x^{\frac{1}{3}+\frac{1}{2}} = x^{\frac{5}{6}}$. The atomic number will be increased x times, and consequently the value of $N\nu$ will be multiplied by $x^{\frac{1}{6}}$. It is, then, a simple matter to find an integral value for x so as to satisfy the relation $N\nu = n\nu_A$. As in most cases there is no independent check on the degree of association of a solid, it does not seem desirable in the present state of our knowledge to attempt to apply this method in detail.

§ 8. *Conclusion.*

In this preliminary survey of the subject of the relation between molecular frequency, ν, and the molecular number, N, it has been proved that the product $N\nu$ frequently shows related values for analogous compounds. There is considerable evidence for the validity of a formula of the type $N\nu = n\nu_A$ in the case of most inorganic solid compounds and of a number of organic compounds. To what extent the formula is to be considered approximate can be decided only when further data as to the specific heat of solids at low temperatures are available. The physical significance of such a formula has been discussed in an earlier paper. Perhaps the simplest interpretation that can be suggested for the "frequency number," n, is that it is related to the number of valency electrons concerned in imparting to the solid its crystalline structure. Further investigation on these lines may serve to throw more light on the nature of the forces connecting the molecules, and the problem of molecular association, in crystalline solids.

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* Sutherland, Phil. Mag. vol. l. p. 460 (1900); Faraday Soc. vol. vi. (1910).

XXXVIII. *On Transpiration through Leaf-Stomata.*

By Sir JOSEPH LARMOR, *F.R.S.*[*]

THE acute and valuable paper by Dr. Harold Jeffreys (Phil. Mag. for March, pp. 270–280) on evaporation and diffusion shows, by inadvertence, less than due appreciation of the investigations of H. T. Brown and F. Escombe on transpiration through the stomata of leaves. The question is so important in plant-economy that misunderstandings should not be allowed to persist. The statement held to be erroneous on p. 277 (for which originally I had some degree of responsibility) still appears to me to be quite correct, in its proper context. The diffusional suction to or from each stomatal opening is local ; thus, when the openings are as much as ten diameters apart the interference between the adjacent shells of diffusion is surely very slight, as stated. But Dr. Jeffreys finds that this would make the transpiration of vapour from the stomata very many times greater than the evaporation from the entire leaf when wet. Whence, then, the discrepancy ? His calculation compares a system of actual stomata, each of diameter 10^{-3} cm., and at about $5 . 10^{-3}$ cm. apart, with a single giant stoma the size of the whole wet leaf and so of diameter 6 cm. But the calculation for the latter case implies that the air is absolutely still throughout the shells of diffusion, which then extend to several diameters from the leaf, so that most of this mass of air remains highly saturated with vapour. Under natural conditions, where the atmosphere and the leaf are not quite still, the evaporation will be at the very least hundreds of times greater.

The natural comparison is that discussed later by Dr. Jeffreys (p. 279), where he passes on to consider the effect of wind or movement of the air. But here again, by reasoning similar to that described above, he seems to arrive at misunderstanding of Sir F. Darwin, and holds that when the stomata become constricted "until the stomatal aperture is reduced to a certain very small value the possible rate of transpiration is practically independent of the aperture and nearly all of the reduction to zero when the stoma closes takes place on the last 2 per cent. of the reduction of aperture." The reasoning on which this statement is based should be capable of some less paradoxical form of conclusion. If I am not mistaken, the underlying idea may be developed in altered form as follows. We may imagine partitions erected perpendicular to the surface of the leaf, so that each stomatal

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[*] Communicated by the Author.

aperture is isolated from the others and the transpiration occurs within its own cylindrical tube of diffusion; this will not sensibly affect the course of the phenomena. The electric idea of conductance of the diffusion-current along this path is now the appropriate aid to discussion. We can imagine the tube prolonged on the other side beyond the stoma, and thus consider the parallel case of electric flow along a conductor having a sharp local constriction representing the stoma, whose area is much smaller than the cross-section of the conductor. The resistance of the whole tube is proportional to its length, provided the latter is increased by a constant correction in order to include the extra resistance arising at the constriction*. The methods by which this correction may be practically estimated were developed by Lord Rayleigh in 1870: *cf.* 'Theory of Sound,' ii. ch. xvi. The current being the same all along the tube, the resistance in any segment of it is proportional to the fall of head (of potential, or of density of diffusing substance) between its ends. When the constriction by a transverse barrier is, as here, to less than one-fifth of the radius of the tube, it will not be far wrong to estimate the fall of head across the constriction as if the enclosing tube were absent. This procedure leads, on the same lines as in Brown and Escombe quoted by Dr. Jeffreys (p. 275), for a circular constriction of radius a, to a correction to the length of each half l of the doubled tube, of amount equal to the area of the section of the tube divided by $4a$.

Now Dr. Jeffreys considers that under natural conditions a layer of air, as much as 1 mm. thick before the disturbed motion beyond is reached around the leaf, may be regarded as still. For the dimensions of stomata quoted by him, a would be $\frac{1}{2} . 10^{-3}$ cm. and the area of section $3 . 10^{-5}$ cm.^{2}, while l would be taken as 1 mm. The correction to l would then be $\frac{1}{7}$ mm. As this is a small fraction of l the main resistance to transpiration would arise in getting across this highly saturated layer of air, as much as 1 mm. thick around the leaf. The stomata would transpire into it, and the vapour would not get away as rapidly as it could be supplied; the case is analogous in a lesser degree to a leaf enclosed in a bottle with narrow, open neck, in which the air would soon become nearly saturated with vapour and the transpiration would be

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* For the problem in two dimensions of space the exact solution is known, and a diagram of flow is given by Prof. Lamb, 'Hydrodynamics,' § 306. In that case the correction to be added to the half length l (*infra*) in order to obtain the effective half length, when the resistance of the constrictions is included, proves to be $-\log_e \sin \frac{1}{2}\pi k$, where k is the ratio of the area of the straight stomatal strips to the whole area. But this result is hardly applicable even to illustrate the actual problem of local stomata.

reduced to a very small amount. Each stoma is fully efficient in proportion to its radius; but the output will be diminished for all sizes of stoma because it has to transpire into a nearly saturated space. When the stoma becomes constricted it operates more feebly, but under better conditions; if a circular stomatal aperture of dimensions as above is constricted to $\frac{1}{7}$ of its radius, its own resistance will be half the total resistance to transpiration instead of only $\frac{1}{8}$; if it is constricted to $\frac{1}{50}$, the resistance of the stoma itself will be $\frac{7}{8}$ of the whole, and the layer of moist air outside will hardly count. The question is whether we can assume a still layer of air anything like 1 mm. thick. As moist air is sensibly lighter than dry, such a layer could hardly be established if the leaf itself is not quite still, unless possibly to some degree on the lower surface of a horizontal leaf.

The problem whether a cause like this, which I take to be the essence of Dr. Jeffreys's important suggestion, has really intervened in observations such as those quoted by Darwin from F. E. Lloyd (p. 277) could only be probed by further experiment under suitable precautions.

The question whether the sap-current in trees is reduced on still days is much simpler, for the whole region of the tree-top may become nearly saturated.

These considerations, which apply to the transpiration of vapour from the stomata, are pertinent, of course, equally to the diffusion of carbon dioxide into them. They seem directly to confirm, from a different aspect, the conclusions of Brown and Escombe that the stomatal cavities are capable of much more absorption than they are called upon to perform.

Cambridge, March 9.

XXXIX. *The Effect of Interionic Force in Electrolytes.*

By S. R. Milner, *D.Sc.*[*]

Part II.

Ionic Mobility and Osmotic Pressure.

THE first attempt to determine the effect of interionic force on the ionic mobility is due to Sutherland[†], whose method is based on the following idea:—The effect of the forces will approximate to what would be obtained if the ions were regularly distributed throughout the liquid, say at the centres of equal cubes. When they are displaced

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[*] Communicated by the Author.

[†] Phil. Mag. xiv. p. 1 (1907).

from these positions by an applied electric field, the interionic forces act as restoring forces in a way which would give rise to a sort of rigidity of the ionic configuration were it not for the fact that the "actions which produced the original uniformity" (thermal motions?) will cause the rigidity to be continually breaking down. The process of breaking down originates a special type of viscosity, which acts in addition to ordinary viscosity when conduction is taking place. A second type of viscosity due to the polarization of the medium is also discussed, and the conclusion is reached that when these viscosities are taken into consideration the diminution of λ, with increase in the concentration, can be accounted for without any association of the ions into molecules taking place.

Sutherland's calculation does not bring the conductivity variation into any relation with that of the freezing-point; and it is based on several speculative hypotheses which are not always convincing. This is particularly the case in regard to the assumed configuration of the ions; the special type of regularity of this is a feature which is inconsistent with the general theory of the distribution of ions to which the kinetic theory leads.

Effect of permanence of the distribution on the mobility.—The method of calculation adopted here is based on the assumption that the distribution of the ions remains undisturbed in the interior of the electrolyte when a current is being carried. Suppose we have a mixture of positive and negative ions contained in a volume, and in the first place suppose that they are subject to no interionic forces. We must assume that they are distributed at random throughout the volume, for there are no data for assuming anything else. Now suppose that an external electric field is applied which gives each positive ion a velocity to the right, and each negative ion one to the left. In the interior of the volume the random distribution will not be disturbed, as is easily seen whether the velocities are all equal or whether they vary arbitrarily from ion to ion.

When interionic forces are present the distribution is no longer random. It becomes modified in such a way that the chance of a positive ion being found in a given position will depend on the mutual potential energy which it possesses in that position with the other ions. If we now imagine all the positive ions displaced to the right and the negative ones to the left—with the same or with arbitrary velocities—the distribution will be disturbed. It will, in fact, tend to be converted into a random distribution. Consequently we see that, if the distribution is to remain permanent when the electrolyte is carrying a current, the velocity with which each ion must be supposed to move under the influence of the applied electric field must be a function at each instant of the mutual potential energy which the ion possesses with the others.

The way in which the random distribution will be modified when the ions are subject to interionic force is given by a theorem due to Boltzmann. Let us suppose that we take a large number of instantaneous views of a certain region of the liquid, and that in each view we observe the positions and signs of all the ions which are present in it. We will confine our attention in the first place to those views alone, n in number, in which the region contains m ions, $A_1 \ldots A_m$, and no more, and we will suppose that these are all so far away from the ions outside the region that the forces between the ions inside and outside are negligible. This will simplify the statement of the argument without affecting the generality of it in any way, since the region may, if necessary, comprise the whole of the liquid. In a certain number, say ν, of the n views, the m ions will be found in small equal volumes $dv_1 \ldots dv_m$, situated at the points $P_1 \ldots P_m$. For shortness we will call this the P configuration, and speak of the ion A_1 as "occupying the position" P_1, the uniform size of the elementary volumes being understood. In another number ν' of views the ions will be found in positions $P_1' \ldots P_m'$ (P' configuration). On a purely random distribution we should have

$$\nu = \nu',$$

but in the modification caused by the presence of interionic force,

$$\nu = \nu' e^{-(\phi - \phi')/kT}, \quad . \quad . \quad . \quad . \qquad (5)$$

ν/ν' here stands for the probability of the P configuration relative to that of the P'. ϕ and ϕ' are the respective mutual potential energies of the ions in each configuration, *i. e.* the work done by the system when the ions are moved to infinite distances apart. When the forces are attractive ϕ and ϕ' are negative quantities. $kT = \frac{2}{3} \times$ average translatory kinetic energy of an ion.

Equation (5) can easily be transformed so as to represent the absolute probability of a given configuration (estimated under the special conditions attached to the total number of views n) by writing it in the form

$$\nu = Kne^{-\phi/kT} dv_1 \ldots dv_m. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

The essential feature of the ionic distribution represented by (6) is of course its permanence, that is, that it is undisturbed by the thermal motions of the ions. Let us suppose that the n views were taken at certain times $t_1 \ldots t_n$. If we were to take another set of views at times $t_1+\tau, \ldots t_n+\tau$, where τ is a very small time, we should find the same number of views in which the ions have the configuration P; the actual views will not be identical in the two cases, in some of the n views at t the ions will have left the volumes dv, but in an equal number of other cases ions will have come into them.

Now suppose that the electrolyte is under the influence of an external electric field when these two sets of views are being taken. In addition to the thermal displacement which each ion undergoes in each of the times τ, it will be dragged by the electric field a certain distance to the right or left. We can effect a considerable simplification by observing that, since the thermal displacements do not affect the distribution, we can imagine them to be non-existent without affecting in any way the result of reasoning concerning the effect on the distribution of the displacements due to the electric field. With this simplification the problem to be solved becomes this:—A set of views at times $t_1, \ldots t_n$, and another set at $t_1+\tau, \ldots t_n+\tau$, being taken of a system of ions existing in configurations the probability of which is given by (6) and at rest during the intervals τ so far as thermal motions are concerned, but in which each ion is dragged during the intervals by an external force to right or left according as it is positive or negative, what is the average velocity with which an ion such as A_1 in a given position must move in order that the distribution may not be disturbed? By the average velocity is meant the average for all the views which show the ion in the given position, but as it makes no difference whether it varies from view to view or is uniform, we may in the calculation treat it as uniform.

Let $u_1, \ldots u_m$ be the velocities of each ion $A_1, \ldots A_m$. Describe small cylinders of area a and of arbitrary lengths $dx_1, \ldots dx_m$ near each of the points $P_1, \ldots P_m$ to the right or left of the points according to the sign of the ion. The number of cases in which an ion is to be found in each of these cylinders will be the same in the views at $t+\tau$ as it is in the views at t. Consequently the number ν_1 of views in which an ion will enter each cylinder during the intervals τ is the same as the number ν_2 in which an ion will leave each cylinder. But ν_1 comprises all those views in which the m ions are situated in the infinitesimal volumes $au_1\tau, \ldots au_m\tau$ at $\mathrm{P}_1, \ldots \mathrm{P}_m$, or

$$\nu_1 = \mathrm{K}ne^{-\phi/k\mathrm{T}}au_1\tau\ldots au_m\tau.$$

A similar expression holds for ν_2 except that both ϕ and the u's are infinitesimally different :

$$\nu_2 = \mathrm{K}n\left\{e^{-\phi/k\mathrm{T}} + \frac{d}{dx_1}e^{-\phi/k\mathrm{T}}dx_1\ldots + \frac{d}{dx_m}e^{-\phi/k\mathrm{T}}dx_m\right\}$$
$$\times\, a\left(u_1 + \frac{du_1}{dx_1}dx_1\right)\tau\ldots a\left(u_m + \frac{du_m}{dx_m}dx_m\right)\tau.$$

Equating ν_1 and ν_2 we get

$$\frac{d}{dx_1}\left\{e^{-\phi/k\mathrm{T}}u_1\right\} = \ldots = \frac{d}{dx_m}\left\{e^{-\phi/k\mathrm{T}}u_m\right\} = 0. \qquad (7)$$

Now, if we write ϕ in the form

$$\phi = \phi_1 + \phi_1' = \phi_2 + \phi_2',$$

where ϕ_1 is the mutual energy of A_1 with all the rest, ϕ_1' the mutual energy of all the $m-1$ ions other than A_1, ϕ_2 that of A_2 with all the rest, &c., we see that the only part of ϕ which is affected by d/dx_1 is ϕ_1, and similarly for each of the other ions. Consequently, integrating (7), we see that the conditions which must be satisfied in order that the distribution may not be disturbed are

$$u_1e^{-\phi_1/k\mathrm{T}} = \text{const.},$$
$$u_2e^{-\phi_2/k\mathrm{T}} = \text{const., \&c.}$$

The constant is independent of x, that is, of ϕ_1, ϕ_2, &c., and is equal to the velocity with which an ion will move when it is so far away from the others that its mutual energy with them is zero. In these circumstances the velocity will be conditioned simply by the friction of the water, and it is clearly the same for all ions of the same sign. Calling it u_0 in unit field, we shall then have for the mobility u_1, which an ion must be reckoned to possess when it exists in a place where its mutual energy with other ions is ϕ_1,

$$u_1 = u_0e^{\phi_1/k\mathrm{T}}. \qquad (8)$$

Effect on Osmotic Pressure.—In the method of the kinetic theory of considering the pressure of a gas as the rate at which momentum is transferred through a unit plane within it, a careful distinction must be made between "internal" and "external" pressure. Consider a gas in which—say by impressed mechanical forces—the potential energy of a molecule when in a certain region R is ϕ (a negative quantity) and zero elsewhere. The distribution will be such that the chance of a molecule occupying a position inside R is to that of its occupying one outside as $e^{-\phi/k\mathrm{T}}:1$, and, in fact, the densities in the two parts will adjust themselves in this proportion. If we imagine a unit plane situated inside R, the momentum transferred through it per second will be the total pressure inside R, but it is only a certain fraction—$e^{+\phi/k\mathrm{T}}$—of the molecules passing through the plane which are capable of transferring their momentum outside the region. We can thus divide the pressure in R into two parts—the external pressure, which is due to momentum capable of being transferred outside it, and which is, in fact, in equilibrium with the pressure outside, and the internal pressure, which in this case will be exerted on the mechanical constraints which cause the increased density in R.

A similar state of things occurs when we deal with a group of ions existing momentarily in a liquid. The whole momentum passed per second by the ions of the group through a unit area drawn in the interior of the group will not be delivered to places of zero potential energy, and the fraction of it that is so delivered will be in statistical equilibrium with the pressure exerted by those ions which are in positions of zero energy. We may call this fraction the external pressure p of the ions in the group or the pressure of the "free" ions—understanding by free ions those which momentarily have no mutual energy with any others. Let us inquire how much of the momentum of an ion existing in a group such as that considered above would, on the average, be capable of being transferred to a place of zero potential.

Consider a single ion of mass m moving in a random direction with velocity v. The average rate at which it transfers, parallel to a given direction, the component in that direction of its momentum is $\frac{1}{3}mv^2$, or, if we take into the average all the possible velocities which it may have, $\frac{1}{3}m\overline{v^2}$ or $k\mathrm{T}$. The sum of this quantity for every ion in the mixture gives the total (*i. e.* internal + external) pressure × volume, PV, of the electrolyte. The contribution of each ion to the total PV is thus a scalar quantity $\frac{1}{3}m\overline{v^2}$ associated with the ion—these contributions will therefore obey the same law of distribution as do the ions themselves.

Consider now two configurations P and P′ of the group of m ions dealt with above. In P the ions occupy the positions $P_1 \dots P_m$, the mutual energy is $\phi = \phi_1 + \phi_1'$, and the number of views in which it is found is

$$\nu = \mathrm{K}n e^{-\phi_1'/k\mathrm{T}} e^{-\phi_1/k\mathrm{T}} dv_1 \dots dv_m. \quad . \quad . \quad . \quad . \quad (9)$$

In P′ let $A_2 \dots A_m$ be in the same positions as before, but let A_1 be in a place of zero mutual energy. The mutual energy of the group is now ϕ_1' and the number of views in which this configuration is found is

$$\nu' = \mathrm{K}n e^{-\phi_1'/k\mathrm{T}} dv_1 \dots dv_m.$$

In the series of n views taken one after the other at arbitrary times the configuration of the system is changed between one view and the next by complex thermal motions. Omitting all the other views, let us confine our attention to the ν views of the configuration P and the ν' views of P′. These are observed at certain successive times and are all the views of these configurations which are observed in the series. In the intervals between them the configurations change over one into the other as the result of thermal motions, but not indiscriminately. It is only in a fraction ν'/ν of the ν views of the P configuration that a change by thermal motions into the P′ configuration will occur. On a random distribution the fraction would be unity. The change in the configuration considered consists simply in the transference by thermal motion of the ion A_1 from a position of mutual energy ϕ_1 to one of zero mutual energy. We see that, given the ion in this position, the probability that such a transference will take place is not the same as it would be on a random distribution (*i. e.* in the absence of interionic forces), but ν'/ν or $e^{\phi_1/k\mathrm{T}}$ times as great *.

On a random distribution the whole of the scalar property $\frac{1}{3}m\overline{v^2}$ associated with each ion is capable of being transferred from one position to another. This gives a random, or on the large scale a uniform, distribution of the pressure throughout the volume. It follows from the preceding proposition that,

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* The proposition is so far only proved for the case in which the $m-1$ ions other than A_1 remain fixed during the transference of A_1. If these also undergo displacements we shall have a simultaneous alteration of ϕ_1 and ϕ_1'. In so far as these displacements affect ϕ_1 only, it is immaterial to the argument whether A_1 reaches a state of zero mutual energy by its own displacement or by suitable ones of the other ions, the expression for ν'/ν being the same in either case. If, on the other hand, they alter ϕ_1', the expression (9) shows that this is an event independent of the change in A_1's mutual energy, the probability of the simultaneous occurrence of the two events being the product of the two probabilities. The truth of the proposition is thus unaffected by all other ionic changes which may proceed simultaneously with the transference of the ion A_1 to a position of zero energy.

when interionic force is present, the average amount of the property which is capable of being transferred from a position in which the mutual energy of its ion is ϕ_1 to a position of zero mutual energy, is

$$\tfrac{1}{3}m\overline{v^2} \,.\, e^{\phi_1/kT}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

This quantity represents the contribution to the external pV made by the ion when it forms part of a group. The summation of it for each ion in the electrolyte would give the external pV or, strictly speaking, its instantaneous value in the view observed. The external pressure of a system of ions subject to interionic force determined in this way is a perfectly definite thing everywhere in equilibrium throughout the volume. It is identical with the pressure of the free ions as defined above. Superposed on it in the interior of groups is the internal pressure (got by summing $\tfrac{1}{3}m\overline{v^2}(1-e^{\phi_1/kT})$ for each ion), which is exerted against the mutual forces and is not effective on the walls.

A comparison of the result (10) for the contribution of an ion to the external pV when it is in a position of mutual energy with others with that (8) for the mobility of the ion in the same circumstances, shows that both are affected by the mutual energy in exactly the same way. Suppose now we follow in imagination the history of an individual ion in an electrolyte for a long time. We shall observe that its state as regards the mutual energy ϕ_1 which it possesses with other ions is continually varying. If we take the average value over a sufficiently long time of the quantities

$$u_0 e^{\phi_1/kT} \text{ and } kT \,.\, e^{\phi_1/kT},$$

we shall get in the first instance the average value of the mobility u_1 of the ion (which is of course the same as that u of any other ion of the same kind, but u_0 differs if the kind differs), and in the second instance a quantity which, when multiplied by the total number N of ions in the electrolyte (combined or not) gives the product pV of the pressure of the free ions into the volume. It is clear that, whatever the effect of ϕ_1 may be on the actual values, we shall always have

$$\frac{u}{pV} = \frac{u_0}{NkT} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Hence in an electrolyte, while alterations with the concentration of u and pV may, and indeed will, be produced as the result of the presence of interionic forces, the alterations will always be such that the ratio of u to pV remains unaffected.

Part III.

The theorem thus proved can be applied to throw some light on the nature of the interionic forces in strong electrolytes, but before applying it it is necessary to identify u_0 and p clearly with measured physical quantities. As regards u_0 there is no difficulty: it represents the mobility of an ion in a region where it is free from interionic force. u_0 is there conditioned only by the friction of the water, and can thus be identified with the experimentally determined mobility at zero concentration to which the same condition applies. The case is different for p the free ionic pressure. In what relation does this stand to the measured osmotic pressure? It will be useful to consider this point in connexion with three possible theories of the constitution of electrolytes.

(1) In the original theory of Arrhenius electric interionic forces are neglected, and an ion is assumed to be definitely either associated (when it contributes nothing to the free ionic pressure p) or free. Although p is not susceptible to direct measurement, a clear conception of it can be got.

The general theorem is of course independent of the law of force between the ions and applies to this theory equally with others. Interionic forces (of the kind referred to below as "chemical" forces) produce an increase in the frequency of occurrence of ions in an associated state with a consequent reduction in the pressure of the free ions in the ratio of $1-\beta:1$. The average mobility of an ion, taken over a period long enough to include its being combined as well as free, is reduced in the same ratio. This agrees with the experimental requirement of an equality in the freezing-point and conductivity variations, for in calculating the complete osmotic pressure P, allowance must be made for the molecular pressure of the fraction $\beta/2$ of associated ions. The reduction in P is thus in the ratio $\left(1-\beta+\frac{\beta}{2}\right):1$, and consequently

$$\frac{\tau}{\tau_0}=2-\beta, \quad \frac{\lambda}{\lambda_0}=1-\beta,$$

in agreement with the experimental results (*v.* Part I.).

The difficulty here, as already mentioned, is the failure of the mass action law (4) to represent correctly the variation of β. When we consider the dynamical assumptions on which (4) is based, we find it essential that the forces which tend to produce association of the ions must fall off very rapidly with the distance*. The law indeed forms a limiting case for which the field of force surrounding each ion is infinitely strong but confined to an infinitely thin shell. This means that the mass law (4) applies solely to association and dissociation which is the result of what we may call "chemical" forces, using this name to distinguish forces of this type from the electrical forces of the free ionic charges which fall off very slowly with the distance. If association is produced by these (4) will not apply to it.

(2) The preceding conclusion suggests that the failure of the mass law may be due to a mistaken view of the nature of the interionic forces which cause association. The apparent association may be partly—or wholly—due to the electric forces. In attempting to investigate this view the first difficulty is to know exactly what is the law of force between the ions. Are we to assume that every + ion attracts every − ion and repels every + ion in the liquid according to the inverse square law? If the mixture of ions were a gaseous one this assumption would presumably be a sound one, but its validity is more doubtful in the case of an electrolyte, where the forces between the ions are affected by the intervening water molecules. As the first step, however, it seems the most straightforward assumption to make, and in previous papers † I have worked out by what is, I think, a strict method, the approximate effect on the osmotic pressure of a mixture of ions in which interionic force of this character is assumed to exist. The calculation, as might be expected from the complexity of the forces, is lengthy and need not be further referred to. The net effect of the interionic forces was found to give a reduction of the osmotic pressure which appears to be in accurate agreement with the experimentally found results for dilute aqueous solutions of strong binary electrolytes. In these cases therefore there is ground for believing that "chemical association," if existent, is extremely small, and that the effects observed are due entirely to the electrical interionic forces. Before we can apply to this view the proposition of the present paper, it is necessary to settle the relation in which the free ionic pressure p stands to the measured osmotic pressure P. Unless they are different from each other, an *in*equality

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* This is the kinetic aspect of the thermodynamical stipulation that the osmotic pressures of the ions must obey the perfect gas law. The assumption is formally made in Boltzmann's original deduction, and it can easily be shown that any other will result in a law different from (4).

† Phil. Mag. xxiii. p. 551 (1912); xxv. p. 742 (1913).

between the freezing-point and conductivity variations will result from the proposition. For if β is the fractional alteration of pV, and therefore of u, and if further $p=\mathrm{P}$, we shall get

$$\frac{\tau}{\tau_0}=\frac{\mathrm{PV}}{\mathrm{RT}}=\frac{p\mathrm{V}}{\mathrm{RT}}=2(1-\beta)\;;\;\frac{\lambda}{\lambda_0}=1-\beta,$$

which is in conflict with the experimental result (3) (Part I.).

p and P, however, cannot be the same, as is evident from the following consideration :—Consider a pair of ions which happen to be fairly close together and under the influence of each other's attraction, and let a number of representative views be taken. In a certain fraction—$e^{\Phi_1/k\mathrm{T}}$—of the cases the ions contribute to the free ionic pV, in the remainder of the cases the ions act as though bound together and contribute nothing to pV, but in these cases the pair will make to the measured PV exactly the same contribution as if the ions formed an actual molecule. The effect of electrical forces is in this respect exactly similar to that produced by chemical association. The electrical bond it is true persists when the ions are well separated from each other, while the chemical bond acts only at very small distances, but this difference is immaterial in considering the molecular pressure which a pair of bound ions will exert. For the type of force considered, however, the bonds are not confined to single pairs of ions, but each pair must be considered as forming part of a large group, with the rest of the ions in which it possesses a certain mutual energy, and so is not free to exercise its full molecular pressure. While a difference between p and P may on these lines be inferred to exist, it is difficult to settle exactly what it is*. It seems doubtful that it would be such as to give the exact equality between β_1 and β_2 which experiment suggests unless the mutual energy between each pair and the rest of the group is negligible.

(3) A theory which is to some extent intermediate between (1) and (2) has much to recommend it. We must infer from the preceding comparisons that the interionic forces must extend over considerable distances, as it is only

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* The straightforward way to settle this point is to calculate p by a method in accordance with its definition on p. 357 and compare it with P. The calculation can be carried out strictly by exactly the same method as that by which P was originally determined (*loc. cit.*). It has been done, but unfortunately the numerical results in both cases can only be obtained in an approximate form which is not sufficiently accurate to determine definitely what is the difference between them.

in this way that a satisfactory explanation of the failure of the mass action law can be got. On the other hand, the idea of the molecular pressure of a pair of associated ions seems also necessary to obtain an accurate agreement with the experimental equality of the freezing-point and conductivity variations. Both conditions will be satisfied if the law of interionic force be such as practically to confine the electrical attraction to pairs of nearest ions. Now, the view that in an electrolyte each + ion attracts every − ion and repels every + ion is undoubtedly a highly artificial one. How artificial it is is made evident by observing that the mutual energy of an ion with others would have to be expressed as a sum of hundreds of terms before any close approximation to its value could be obtained. These represent the mutual energies with the nearest ion, the next nearest, the third nearest, &c., and form terms which partly cancel each other as the successive ions are + and −. It is unlikely that this state of things represents a physical reality. Indeed, the assumption on which it is based, that the action of the water molecules can be simulated by that of a continuous medium of S.I.C. the same as that of water in mass, is hardly likely to be true. It is probably a good deal nearer to the truth to imagine surrounding each ion a number of polarized water molecules which tend to form chains linking together pairs of temporarily nearest oppositely charged ions. Such an action would not be the same as that of a uniform medium; it would be more analogous to the action of iron filings in forming chains between two magnetic poles. The general effect would be to increase the attraction between an ion and the nearest one to it of unlike sign at the expense of the attraction of more distant ones, which latter might well be negligible in consequence.

The view of the constitution of an electrolyte which is thus attained will satisfy both the requirements mentioned above, which are essential to a satisfactory theory. On it we may imagine all the ions divided into pairs formed of ions which are temporarily nearest together, the individuals of each pair undergoing continual change. Between the ions of each pair electrical force exists, which in many ways is similar to a chemical bond, but is different in others. Thus, with chemical association, an ion is either free or combined, it cannot be both together, but here it possesses simultaneously characteristics of both conditions. The ions in each pair, for instance, will be separated spatially from each other nearly as widely as if they were quite free; they *are* free (*e. g.* to carry current or exert ionic pressure) in a fraction of the cases in which the pair is observed, while they act as combined (exert a molecular pressure) in the remainder. It seems to me that it is a theory on lines similar to these which will ultimately succeed in reconciling all the difficulties connected with strong electrolytes.

SUMMARY.

(1) A critical discussion of the way in which the law of mass action fails for strong electrolytes leads to the conclusion that the reduction in the molecular conductivity with increasing concentration must be ascribed mainly to a reduction in the mobilities of the ions, and not to a reduction in their number by association into molecules.

(2) A theoretical investigation of the effect of interionic force shows that identical variations with the concentration will be produced in the conductivity and in the osmotic pressure of the "free" ions (as defined on p. 357).

(3) The application of this result to strong electrolytes shows that the variation in the conductivity and the freezing-point can be best explained by a modification in the view we take of what constitutes association. According to this ions in strong electrolytes are not associated into *molecules*; they are neither completely associated nor completely free, but pairs of ions which are temporarily nearest together, in consequence of the electric forces between them, will, in a fraction of cases, act as if bound together, and in the remaining cases as if free.

The University, Sheffield,
December 1917.

XL. *Bessel Functions of Equal Order and Argument. By* G. N. WATSON, *M.A., D.Sc., Assistant Professor of Pure Mathematics at University College, London* *.

1. APPROXIMATE formulæ for the Bessel function and its derivate, $J_n(n)$ and $J_n'(n)$, (when n is large) have been discussed in numerous papers during the last few years †; several of these papers have appeared in this Magazine.

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* Communicated by the Author.

† Debye, Math. Ann. lxvii. pp. 535–558 (1909). Rayleigh, Phil. Mag. Dec. 1910. Nicholson, Phil. Mag. Dec. 1907, Aug. 1908, Feb. 1910. Watson, Proc. London Math. Soc. (2) xvi. pp. 150–174 (1917); Proc. Camb. Phil. Soc. xix. pp. 42–48 (1917). Various numerical results have also been given by Airey in a series of recent papers in the Phil. Mag.

The associated function

$$\int_0^1 \mathrm{J}_n(nx)dx,$$

which does not occur in many of the physical problems in which the other Bessel functions present themselves, appears to play a prominent part in connexion with various series arising in the theory of Electromagnetic Radiation, and consequently Professor Schott has asked me to determine whether there is any approximate formula analogous to the results

$$\mathrm{J}_n(n) \backsim \frac{\Gamma(\frac{1}{3})}{\pi 2^{\frac{2}{3}} 3^{\frac{1}{6}} n^{\frac{1}{3}}}, \quad \mathrm{J}_n'(n) \backsim \frac{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}{\pi 2^{\frac{1}{3}} n^{\frac{2}{3}}}.$$

This note, in which I prove the remarkably simple result that

$$\int_0^1 \mathrm{J}_n(nx)dx \backsim \frac{1}{3n}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the outcome of his inquiry. A closer approximation is given in § 4, but it involves the gamma function of 1/3; this more precise result is

$$\int_0^1 \mathrm{J}_n(nx)dx \backsim \frac{1}{3n} - \frac{2^{\frac{2}{3}}}{3^{\frac{1}{3}} 5 n^{\frac{2}{3}} \Gamma(\frac{1}{3})}.$$

In order to obtain this approximate formula I propose to employ not the elementary methods which I have used elsewhere* in connexion with $\mathrm{J}_n(n)$ and $\mathrm{J}_n'(n)$, but the methods which depend on the contour integrals of Debye; the latter methods yield the desired result with a much smaller expenditure of labour.

2. We take the well-known contour integral

$$\mathrm{J}_n(nx) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\frac{1}{2}nx(t-1/t)} t^{-n-1} dt,$$

(in which the contour starts from $-\infty$, encircles the origin once counter-clockwise, and then returns to $-\infty$), and on integrating under the integral sign we get

$$\begin{aligned}\int_0^1 \mathrm{J}_n(nx)dx &= \frac{1}{n\pi i}\int_{-\infty}^{(0+)} \left\{e^{\frac{1}{2}n(t-1/t)} - 1\right\} \frac{t^{-n}dt}{t^2-1} \\ &= \frac{1}{n\pi i}\int_{-\infty}^{(0+)} \left\{t^{-n} e^{\frac{1}{2}n(t-1/t)} - 1\right\} \frac{dt}{t^2-1} \\ &\quad + \frac{1}{n\pi i}\int_{-\infty}^{(0+)} \frac{1-t^{-n}}{t^2-1} dt.\end{aligned}$$

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* See the last of the papers cited.

In the last integral of all, we deform the contour into a circular arc of indefinitely great radius, starting and ending at $-\infty$. Since n is supposed to be positive the integrand is $O(t^{-2})$ on the deformed contour, and so the integral is zero. We thus obtain the formula

$$\int_0^1 J_n(nx)dx = \frac{1}{n\pi i}\int_{-\infty}^{(0+)} \left\{ t^{-n} e^{\frac{1}{2}n(t-1/t)} - 1 \right\} \frac{dt}{t^2-1}. \qquad (2)$$

Now that the large variable n only occurs in a single term of the integrand, we proceed to apply the methods of Debye by choosing a contour on which

$$\tfrac{1}{2}(t-1/t) - \log t$$

is purely real.

Writing $t=re^{i\theta}$, we see that the contour has to satisfy the condition

$$(r+1/r)\sin\theta - 2\theta = 0;$$

this equation is satisfied if $\theta=0$ or if

$$r = \theta \operatorname{cosec} \theta \{1 \pm \sqrt{}(1-\theta^{-2}\sin^2\theta\}.$$

Taking the upper sign, so that

$$r = \theta \operatorname{cosec} \theta \{1 + \sqrt{}(1-\theta^{-2}\sin^2\theta)\}, \qquad . \quad . \quad (3)$$

we obtain a contour of the required type if we take θ to vary from $-\pi$ to π.

The contour, which is symmetrical with respect to the real axis, passes through (1, 0) and has an abrupt change of direction at that point; its direction immediately above the real axis is inclined $\frac{1}{3}\pi$ to the positive direction of the real axis *.

We thus get

$$\int_0^1 J_n(nx)dx = \frac{1}{n\pi i}\int_{-\pi}^{\pi} \left\{ e^{-nF(\theta)} - 1 \right\} \frac{t}{t^2-1} \left\{ \frac{1}{r}\frac{dr}{d\theta} + i \right\} d\theta,$$

where r is given as a function of θ by equation (3), $t=re^{i\theta}$ and †

$$F(\theta) = \tfrac{1}{2}(r-1/r)\cos\theta - \log r.$$

Now

$$\frac{t}{t^2-1} = \frac{1}{(r-1/r)\cos\theta + i(r+1/r)\sin\theta}$$
$$= \frac{(r-1/r)\cos\theta - i(r+1/r)\sin\theta}{r^2+1/r^2-2\cos 2\theta}.$$

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* This is easily proved; it is suggested by the fact that $\frac{1}{2}(t-1/t)-\log t$ has a triple zero at $t=1$. Various properties of the contour are given in the first of my papers to which reference is made in § 1. The curve obtained by giving the lower sign to the radical is the inverse of the curve obtained by taking the upper sign.

† In my previous paper, it was convenient to call this function $F(\theta, 1)$.

Since r is an even function of θ, we easily deduce that

$$\int_0^1 J_n(nx)\,dx = \frac{2}{n\pi}\int_0^\pi \{1-e^{-nF(\theta)}\} \times \frac{(r+1/r)\sin\theta(dr/rd\theta)-(r-1/r)\cos\theta}{r^2+1/r^2-2\cos 2\theta}\,d\theta. \quad (4)$$

Now it is easy to show that

$$\begin{aligned}\frac{(r+1/r)\sin\theta(dr/rd\theta)-(r-1/r)\cos\theta}{r^2+1/r^2-2\cos 2\theta} &= \frac{1}{2}\frac{d}{d\theta}\tan^{-1}\frac{2r\sin\theta}{1-r^2} \\ &= \frac{1}{2}\frac{d}{d\theta}\left\{\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}+\tan^{-1}\frac{r\sin\theta}{1-r\cos\theta}\right\}.\end{aligned}$$

But the contour starts from $(1, 0)$ in a direction making an angle $\frac{2}{3}\pi$ with the line joining this point to the origin; and so $\tan^{-1}\{r\sin\theta/(1-r\cos\theta)\}$ decreases from $\frac{2}{3}\pi$ to 0 as θ increases from 0 to π, while $\tan^{-1}\{r\sin\theta/(1+r\cos\theta)$ increases from 0 to π in the same circumstances.

We therefore write equation (4) in the form

$$\begin{aligned}\int_0^1 J_n(nx)\,dx &= \frac{1}{n\pi}\int_0^\pi \left\{1-e^{-nF(\theta)}\right\}\frac{d}{d\theta}\left\{\tan^{-1}\frac{2r\sin\theta}{1-r^2}-\pi\right\}d\theta \\ &= \frac{1}{n\pi}\left[\left\{1-e^{-nF(\theta)}\right\}\left\{\tan^{-1}\frac{2r\sin\theta}{1-r^2}-\pi\right\}\right]_0^\pi \\ &\qquad + \frac{1}{\pi}\int_0^\pi e^{-nF(\theta)}\left\{\pi-\tan^{-1}\frac{2r\sin\theta}{1-r^2}\right\}F'(\theta)\,d\theta\end{aligned}$$

on integrating by parts; that is to say

$$\int_0^1 J_n(nx)\,dx = \frac{1}{\pi}\int_0^\pi e^{-nF(\theta)}\left\{\pi-\tan^{-1}\frac{2r\sin\theta}{1-r^2}\right\}F'(\theta)\,d\theta, \quad (5)$$

it being observed that $F(0)=0$, $F(\pi)=\infty$, so that the integrated part vanishes at each limit.

Now $F'(\theta)$ is positive* when $0 \leq \theta \leq \pi$, so that $F(\theta)$ is a steadily increasing function of θ. Moreover, we can show that $\pi-\tan^{-1}\{2r\sin\theta/(1-r^2)\}$ is a steadily decreasing function of θ. For we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d\theta}\left\{\pi-\tan^{-1}\frac{2r\sin\theta}{1-r^2}\right\} &= \frac{(r+1/r)\sin\theta(dr/rd\theta)-(r-1/r)\cos\theta}{r^2+1/r^2-2\cos 2\theta} \\ &= -\frac{\theta+\sin\theta\cos\theta-2\theta^2\cot\theta}{2(\theta^2-\sin^2\theta\cos^2\theta)\sqrt{(\theta^2-\sin^2\theta)}}\sin^2\theta. \quad (6)\end{aligned}$$

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* Proc. London Math. Soc. (2) xvi. p. 153.

Now $\theta + \sin\theta\cos\theta - 2\theta^2\cot\theta$ vanishes when $\theta = 0$ and has the positive derivate $2(\cos\theta - \theta\operatorname{cosec}\theta)^2$; hence it is positive when $0 < \theta < \pi$. The function

$$\pi - \tan^{-1}\{2r\sin\theta/(1-r^2)\}$$

therefore has a negative derivate, and the desired result is proved.

Taking $F(\theta)$ as a new variable, F, and writing $\phi(F)$ for the function $\pi - \tan^{-1}\{2r\sin\theta/(1-r^2)\}$, we have

$$\int_0^1 J_n(nx)dx = \frac{1}{\pi}\int_0^\infty \phi(F)e^{-nF}dF,$$

where $\phi(F)$ is a positive decreasing function of F such that $\phi(0) = \frac{1}{3}\pi$.

Hence, since the integral on the right is uniformly convergent for large values of n *,

$$\lim_{n\to\infty}\left[n\int_0^1 J_n(nx)dx\right] = \lim_{n\to\infty}\frac{1}{\pi}\int_0^\infty \phi(u/n)e^{-u}du$$
$$= \phi(0)/\pi = \tfrac{1}{3}.$$

That is to say,

$$\int_0^1 J_n(nx)dx \backsim \frac{1}{3n},$$

which is the result stated.

From the formula

$$n\int_0^1 J_n(nx)dx = \frac{1}{\pi}\int_0^\infty \phi(u/n)e^{-u}du$$

it is evident that the function on the left steadily increases (for *all* positive values of n) as n increases.

3. When n is an odd integer we can express the integral as a sum of Bessel functions, since we have

$$n\int_0^1 J_n(nx)dx = \int_0^n J_n(y)dy$$

and

$$J_m(y) = J_{m-2}(y) - 2J'_{m-1}(y),$$

whence we find

$$n\int_0^1 J_n(nx)dx$$
$$= \int_0^n \{J_1(y) - 2J_2'(y) - 2J_4'(y) - \ldots - 2J'_{n-1}(y)\}dy$$
$$= 1 - \{J_0(n) + 2J_2(n) + 2J_n(n) + \ldots + 2J_{n-1}(n)\}.$$

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* Bromwich, 'Infinite Series,' pp. 434, 436.

When n is an even integer, we find that

$$n\int_0^1 J_n(nx)dx = \int_0^n J_0(y)dy - 2\{J_1(n) + J_3(n) + \ldots + J_{n-1}(n)\}.$$

The integral on the right does not seem to be expressible by elementary functions, but we have

$$\int_0^n J_0(y)dy = 1 - \int_n^\infty J_0(y)dy$$
$$\backsim 1 - \frac{\sin(\frac{1}{4}\pi - n)}{\sqrt{(\frac{1}{2}\pi n)}}$$

by integrating the ordinary asymptotic expansion of $J_0(y)$.

The following table indicates the mode of increase of $n\int_0^1 J_n(nx)dx$ to its limit (namely $\frac{1}{3}$) as n increases through the odd integral values 1, 3, 5, 7, ...

n.	$n\int_0^1 J_n(nx)dx$.	n.	$n\int_0^1 J_n(nx)dx$.
1	0·2348023	13	0·3175245
3	0·2878694	15	0·3190390
5	0·3020013	17	0·3202397
7	0·3087657	19	0·3212187
9	0·3128134	21	0·3220347
11	0·3155424	23	0·3227271

4. We can obtain a closer approximation to the integral in the following manner :—

When θ is small we have

$$\theta + \sin\theta\cos\theta - 2\theta^2\cot\theta \backsim 8\theta^5/45,$$
$$\theta^2 - \sin^2\theta \backsim \theta^4/3,$$
$$\theta^2 - \sin^2\theta\cos^2\theta \backsim 4\theta^4/3,$$
$$F(\theta) \backsim 4\theta^3/(9\sqrt{3}).$$

Hence from (6) we find that $\pi - \tan^{-1}\{2r\sin\theta/(1-r^2)\}$ is a function of θ approximately equal to

$$\tfrac{1}{3}\pi - \theta^2/(5\sqrt{3}),$$

and the complete expansion of this function involves even powers of θ only.

Since $F(\theta)$ is an odd function of θ, it follows that $\phi(F)$ is expansible in a series of ascending powers of $F^{\frac{2}{3}}$, convergent when F is sufficiently small, in which the first two terms are given by the formula

$$\phi(F) \backsim \tfrac{1}{3}\pi - 2^{\frac{2}{3}}3^{\frac{7}{6}}F^{\frac{2}{3}}/20,$$

and hence the integral of $J_n(nx)$ possesses an asymptotic expansion in which the ratio of consecutive terms is of order $n^{\frac{2}{3}}$, the first two terms being given by the formula

$$\int_0^1 J_n(nx)dx \backsim \frac{1}{3n} - \frac{2^{\frac{2}{3}}3^{\frac{7}{6}}}{20\pi n^{\frac{5}{3}}}\Gamma(\tfrac{5}{3})$$

$$\backsim \frac{1}{3n} - \frac{2^{\frac{2}{3}}}{3^{\frac{1}{3}}5n^{\frac{5}{3}}\Gamma(\frac{1}{3})}.$$

This approximation gives the value of the integral correct to four places of decimals when $n=23$.

XLI. *Notices respecting New Books.*

The Electron. Its isolation and measurement and the determination of some of its properties. By ROBERT ANDREWS MILLIKAN. Pp. xii+268. The University of Chicago Press. Price 7*s.* net.

STARTING with a brief historical account of the rise of the electron theory, the author soon reaches the question of the determination of the electronic charge *e*, and, after describing briefly the early work of the Cavendish school, and the difficulties encountered, he devotes special attention to the experiments carried out by himself and his students in the Ryerson laboratory of the University of Chicago. In these experiments, in place of Wilson's cloud, a single oil drop was observed; and Professor Millikan gives a most interesting account of the method by which the capture of single electrons by a drop was observed, the corrections to Stokes' law necessitated by the small radii of some of the drops, and the final determination of *e* to be $(4{\cdot}774 \pm {\cdot}005) \times 10^{-10}$ electrostatic units, and N, Avogadro's constant, to be $(6{\cdot}062 \pm {\cdot}006) \times 10^{23}$. He also details experiments carried out on the Brownian movement in gases to determine N.

Not long before the war Ehrenhaft published an account of series of experiments which he considered to demonstrate the existence of a sub-electron, or charge very much smaller than the electron. Few English physicists found the work convincing, but, nevertheless, it aroused some attention. Professor Millikan devotes a chapter of his book to discussing this question of a sub-electron, and brings forward very strong arguments, based on his own experience, for supposing Ehrenhaft's results to be a development of experimental errors and uncertainties. There seems no doubt that, according to the best experimental evidence at present available, the electronic charge *e* is constant and indivisible, as was universally assumed.

The last two chapters are devoted to the structure of the atom, and the nature of radiant energy. The author gives a clear account of Moseley's famous experiments on the characteristic X rays, and of Bohr's very successful atom model of Rutherford type. In discussing the quantum theory he describes his experiments on the initial velocity of the photo-electrons, and the consequent determination of h. He concludes with a sketch of the still unexplained difficulties which beset the construction of a satisfactory theory of radiation.

The style of the book is very clear and pleasant, all but the very simplest mathematical considerations being dealt with in a little series of appendices. The Cavendish work, so familiar to English physicists, is dealt with briefly, while Professor Millikan's own work, which has been less described in text-books, is exposed at greater length. The short accounts of the most recent theories are excellent. The book will be read with pleasure and profit by the physicist, while at the same time its clarity and directness render it available to any man of general scientific training.

Napier Tercentenary Memorial Volume. Edited by C. G. KNOTT. Published for the Royal Society of Edinburgh by Longmans, Green and Co. Quarto. Price 21*s*. net.

THIS volume of Essays and Addresses, contributed to the Tercentenary Congress, forms a worthy memorial of the publication of Napier's wonderful discovery. It is appropriate, too, that on this occasion special attention should be drawn to the extensive logarithmic tables calculated by Edward Sang, which have too long remained unpublished. "What more fitting outcome of the Napier Tercentenary could there be than making accessible to the civilized world the fundamental part of these great tables, calculated in the very city where John Napier invented the logarithm and gained undying fame as a benefactor of his race?"

In his inaugural address, Lord Moulton traces the different stages in the development of the discovery of logarithms, whilst other papers deal with the life and work of Napier. The question of Napier's claim to priority of discovery is fully discussed, and valuable papers on logarithms and logarithmic computations form an important part of the volume. The Essays more directly concerned with mathematical tables will be of most interest to calculators, whose labours will be considerably reduced thereby. Prof. Andoyer contributes an interesting and suggestive paper on fundamental trigonometric and logarithmic tables. The arrangement of tables, the reduction of the number of entries and of the mean error, and a method of extending the accuracy of tables by improvement of differences, include only a few of the numerous and important essays contained in the volume. Mathematicians interested in history and methods of calculation are greatly indebted to the Editor, Dr. Knott, and to the experts who have made such valuable contributions to the study of this subject.

Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition. Edited by E. M. Horsburgh. Geo. Bell & Sons and the Royal Society of Edinburgh. Price 6*s.*

The Handbook is a mine of information on all questions relating to calculations, and the aim of the Editor and the Committee to make this volume useful to those engaged in computation has been fully realized. Special mention should be made of the list of mathematical tables, including tables of logarithms and other functions and of the chapters devoted to calculating machines and mathematical laboratory instruments. Logarithmic computation is being extensively supplemented by that of mechanical calculators, and the increasing number of those who use these machines will find the descriptive article on this subject of much interest. The chapter on instruments gives a detailed account of integrometers, planimeters, harmonic analyzers, and other mechanisms required for special purposes. Other chapters deal with slide-rules, ruled papers, and mathematical models. The Handbook forms a fitting companion to the Memorial volume, and will be a valuable addition to the library of every student of mathematics.

Elliptic Integrals (Mathematical Monographs, No. 18). By Professor Harris Hancock. Pp. 104. New York: John Wiley & Sons; London: Chapman & Hall. 1917. Price 6*s.* net.

This excellently produced volume is one of a series of mathematical monographs now appearing in America. It contains an account of the three elliptic integrals, the integral of the third kind, however, receiving only passing notice. Starting with the definition of elliptic integrals as the integrals of expressions when cubics and quartics occur under the root sign, the author gives the reduction to Legendre's normal form, and illustrates some of the more obvious properties of the functions with excellent graphs. After treating the *sn*, *cn*, *dn*, functions and the Gudermannian he deals in detail with the reduction of various types of integrals of the first kind to Legendre's form, and then passes on to the methods of numerical computation of the first and second kind of integral. This part of the subject is developed at some length, and illustrated with detailed numerical workings. After a few well-selected examples and exercises the book closes with five place tables of elliptic functions of the first and second kind, taken from Levy's well-known work.

The book is well-planned and clearly written, and can be understood by any student well-grounded in the elements of the calculus. It forms an excellent introduction, and will, we think, be welcomed by those who, without having much time to devote to the subject, wish to possess a concise and accurate account of the fundamental properties of elliptic integrals.

WALKER. Phil. Mag. Ser. 6, Vol. 35. Pl. X.

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AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1918.

XLII. *On the Scattering of Light by a Cloud of similar small Particles of any Shape and oriented at random.*

By Lord RAYLEIGH, *O.M., F.R.S.*[*]

FOR distinctness of conception the material of the particles may be supposed to be uniform and non-magnetic, but of dielectric capacity different from that of the surrounding medium; at the same time the results at which we shall arrive are doubtless more general. The smallness is, of course, to be understood as relative to the wave-length of the vibrations.

When the particles are spherical, the problem is simple, as their orientation does not then enter[†]. If the incident light be polarized, there is no scattered ray in the direction of primary electric vibration, or if the incident light be unpolarized there is complete polarization of the light scattered at right angles to the direction of primary propagation. The consideration of elongated particles shows at once that a want of symmetry must usually entail a departure from the above law of polarization and may be one of the causes, though probably not the most important, of the incomplete polarization of sky-light at 90° from the sun. My son's recent experiments upon light scattered by carefully filtered gases[‡] reveal a decided deficiency of polarization in the light emitted

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[*] Communicated by the Author.

[†] Phil. Mag. vol. xli. pp. 107, 274, 447 (1871), vol. xii. p. 81 (1881), vol. xlvii. p. 375 (1899); Scientific Papers, vol. i. pp. 87, 104, 518 vol. iv. p. 397.

[‡] Roy. Soc. Proc. Feb. 28, 1918.

perpendicularly, and seem to call for a calculation of what is to be expected from particles of arbitrary shape.

As a preliminary to a more complete treatment, it may be well to take first the case of particles symmetrical about an axis, or at any rate behaving as if they were such, for the calculation is then a good deal simpler. We may also limit ourselves to finding the *ratio* of intensities of the two polarized components in the light scattered at right angles, the principal component being that which vibrates parallel to the primary vibrations, and the subordinate component (vanishing for spherical particles) being that in which the vibrations are perpendicular to the primary vibrations. All that we are then concerned with are certain resolving factors, and the integration over angular space required to take account of the random orientations. In virtue of the postulated symmetry, a revolution of a particle about its own axis has no effect, so that in the integration we have to deal only with the direction of this axis. It is to be observed that the system of vibrations scattered by a particle depends upon the direction of primary *vibration* without regard to that of primary *propagation*. In the case of a spherical particle the system of scattered vibrations is symmetrical with respect to this direction and the amplitude of the scattered vibration is proportional to the cosine of the angle between the primary and secondary vibrations. When we pass to unsymmetrical particles, we have first to resolve the primary vibrations in directions corresponding to certain principal axes of the disturbing particle and to introduce separate coefficients of radiation for the different axes. Each of the three component radiations is symmetrical with respect to its own axis, and follows the same law as obtains for the sphere *.

In fig. 1 the various directions are represented by points on a spherical surface with centre O. Thus in the rectangular system XYZ, OZ is the direction of primary vibration, corresponding (we may suppose) to primary propagation parallel to OX. The rectangular system UVW represents in like manner the principal axes of a particle, so that UV, VW, WU are quadrants. Since symmetry of the particle round W has been postulated, there is no loss of generality in taking U upon the prolongation of ZW. As usual, we denote ZW by θ, and XZW by ϕ.

The first step is the resolving of the primary vibration Z in the directions U, V, W. We have

$$\cos ZU = -\sin\theta, \quad \cos ZV = 0, \quad \cos ZW = \cos\theta. \quad . \ (1)$$

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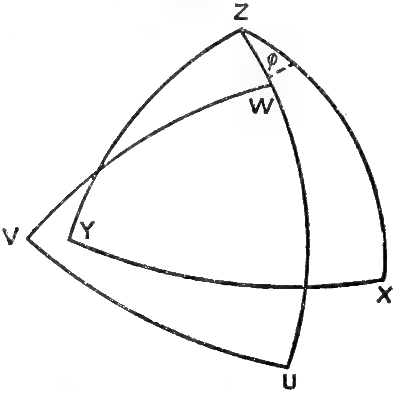
* Phil. Mag. vol. xliv. p. 28 (1897); Sci. Papers, vol. iv. p. 305.

The coefficients, dependent upon the character of the particle, corresponding to U, V, W may be denoted by A, A, C; and we seek the effect along the scattered ray OY, perpendicular to both primary vibrations and primary propagation. The ray scattered in this direction will not be completely polarized, and we consider separately vibrations parallel to Z and to X. As regards the former, we have the same set of factors over again, as in (1), so that the vibration is $A\sin^2\theta + C\cos^2\theta$, reducing to C simply, if $A=C$. This is the result for a single particle whose axis is at W. What we are aiming at is the aggregate intensity due to a large number of particles with their positions and their axes distributed at random. The *mean* intensity is

Fig. 1.



$$\int_0^{\frac{1}{2}\pi}\{A+(C-A)\cos^2\theta\}^2\sin\theta\,d\theta \div \int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta$$

$$=A^2+\frac{2(C-A)A}{3}+\frac{(C-A)^2}{5}=\tfrac{1}{15}(8A^2+3C^2+4AC). \quad (2)$$

This represents the intensity of that polarized component of the scattered light along OY whose vibrations are parallel to OZ.

For the vibrations parallel to OX the second set of resolving factors is cos UX, cos VX, cos WX. Now from the spherical triangle UZX,

$$\cos UX=\sin(90^\circ+\theta)\cos\phi=\cos\theta\cos\phi.$$

Also from the triangles VZX, WZX,

$$\cos VX=\cos VZW=\cos(90^\circ+\phi)=-\sin\phi,$$

$$\cos WX=\sin\theta\cos\phi.$$

The first set of factors remains as before. Taking both sets into account, we get for the vibration parallel to X

$$-\mathrm{A}\sin\theta\cos\theta\cos\phi+\mathrm{C}\cos\theta\sin\theta\cos\phi,$$

the square of which is

$$(\mathrm{C}-\mathrm{A})^2\sin^2\theta\cos^2\theta\cos^2\phi. \quad . \quad . \quad . \quad . \quad (3)$$

The mean value of $\cos^2\phi$ is $\frac{1}{2}$. That of $\cos^2\theta$ is $\frac{1}{3}$ and that of $\cos^4\theta$ is $\frac{1}{5}$, as above, so that corresponding to (2) we have for the mean intensity of the vibrations parallel to X

$$\tfrac{1}{2}(\mathrm{C}-\mathrm{A})^2(\tfrac{1}{3}-\tfrac{1}{5})=\tfrac{1}{15}(\mathrm{C}-\mathrm{A})^2. \quad . \quad . \quad . \quad (4)$$

The ratio of intensities of the two components is thus

$$\frac{(\mathrm{C}-\mathrm{A})^2}{8\mathrm{A}^2+3\mathrm{C}^2+4\mathrm{AC}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Two particular cases are worthy of notice. If A can be neglected in comparison with C, (5) becomes simply *one-third.* On the other hand, if A is predominant, (5) reduces to *one-eighth.*

The above expressions apply when the primary light, propagated parallel to X, is completely polarized with vibrations parallel to Z, the direction of the secondary ray being along OY. If the primary light be unpolarized, we have further to include the effect of the primary vibrations parallel to Y. The two polarized components scattered along OY, resulting therefrom, both vibrate in directions perpendicular to OY, and accordingly are both represented by (4). In the case of unpolarized primary light we have therefore to double (4) for the secondary vibrations parallel to X, and to add together (2) and (4) for the vibrations parallel to Z. The latter becomes

$$\tfrac{1}{15}(9\mathrm{A}^2+4\mathrm{C}^2+2\mathrm{AC}),$$

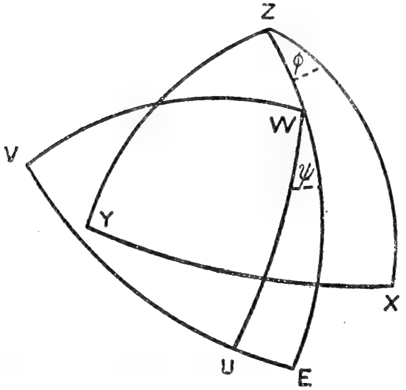
and for the ratio of intensities of the two components

$$\frac{2(\mathrm{C}-\mathrm{A})^2}{9\mathrm{A}^2+4\mathrm{C}^2+2\mathrm{AC}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $\mathbf{A}=0$, this ratio is *one-half.*

For a more general treatment, which shall include all forms of particle, we must introduce another angle ψ to represent the inclination of WU to ZW produced, fig. 2. The direction cosines of either set of axes with respect to

Fig. 2.



the other are given by the formulæ *

$$\left.\begin{aligned}\cos XU &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta\\ \cos YU &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta\\ \cos ZU &= -\sin\theta\cos\psi\end{aligned}\right\}, \quad . \quad (7)$$

$$\left.\begin{aligned}\cos XV &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta\\ \cos YV &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta\\ \cos ZV &= \sin\theta\sin\psi\end{aligned}\right\}, \quad . \quad (8)$$

$$\left.\begin{aligned}\cos XW &= \sin\theta\cos\phi\\ \cos YW &= \sin\theta\sin\phi\\ \cos ZW &= \cos\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Supposing, as before, that the primary vibration is parallel to Z, we have as the first set of factors

$$\left.\begin{aligned}\cos ZU &= -\sin\theta\cos\psi,\\ \cos ZV &= \sin\theta\sin\psi,\\ \cos ZW &= \cos\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

For the vibrations propagated along OY which are parallel to Z, we have the same factors over again with coefficients

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* See, for example, Routh's 'Rigid Dynamics,' Part I. § 258, 1897. ψ and ϕ are interchanged.

A, B, C as before, and the vibration is expressed by

$$A\sin^2\theta\cos^2\psi+B\sin^2\theta\sin^2\psi+C\cos^2\theta\,; \quad . \quad (11)$$

while for the intensity

$$\begin{aligned}I={}&A^2\sin^4\theta\cos^4\psi+B^2\sin^4\theta\sin^4\psi+C^2\cos^4\theta\\&+2AB\sin^4\theta\cos^2\psi\sin^2\psi+2BC\sin^2\theta\cos^2\theta\sin^2\psi\\&+2CA\sin^2\theta\cos^2\theta\cos^2\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)\end{aligned}$$

This is for a single particle, and we have now to take the mean for all orientations. The mean value of $\sin^4\psi$, or $\cos^4\psi$, is $\frac{3}{8}$; that of $\sin^2\psi\cos^2\psi$ is $\frac{1}{8}$; and that of $\sin^2\psi$ is $\frac{1}{2}$. The averaging with respect to ψ thus yields

$$\begin{aligned}I={}&\tfrac{3}{8}(A^2+B^2)\sin^4\theta+C^2\cos^4\theta+\tfrac{1}{4}AB\sin^4\theta\\&+(A+B)C\sin^2\theta\cos^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)\end{aligned}$$

Again, the mean value of $\sin^4\theta$ is $\frac{8}{15}$, that of $\cos^4\theta$ is $\frac{1}{5}$, and that of $\sin^2\theta\cos^2\theta$ is $\frac{2}{15}$. Thus, finally, the mean value of I over the sphere is given by

$$\text{mean } I=\tfrac{1}{15}\{3(A^2+B^2+C^2)+2(AB+BC+CA)\}. \quad . \quad (14)$$

This refers to the vibrations parallel to Z which are propagated along OY.

For the vibrations parallel to X, the second set of factors is cos XU, cos XV, cos XW, as given above, and the vibration is expressed by

$$\begin{aligned}&-A\sin\theta\cos\psi(-\sin\phi\sin\psi+\cos\phi\cos\psi\cos\theta)\\&+B\sin\theta\sin\psi(-\sin\phi\cos\psi-\cos\phi\sin\psi\cos\theta)\\&+C\cos\theta\sin\theta\cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)\end{aligned}$$

Accordingly for the intensity

$$\begin{aligned}I={}&A^2\sin^2\theta\cos^2\psi(\sin^2\phi\sin^2\psi+\cos^2\phi\cos^2\psi\cos^2\theta\\&\qquad\qquad-2\sin\phi\cos\phi\sin\psi\cos\psi\cos\theta)\\&+B^2\sin^2\theta\sin^2\psi(\sin^2\phi\cos^2\psi+\cos^2\phi\sin^2\psi\cos^2\theta\\&\qquad\qquad+2\sin\phi\cos\phi\sin\psi\cos\psi\cos\theta)\\&+C^2\sin^2\theta\cos^2\theta\cos^2\phi\\&-2AB\sin^2\theta\sin\psi\cos\psi(\sin^2\phi\sin\psi\cos\psi\\&\qquad-\cos^2\phi\sin\psi\cos\psi\cos^2\theta+\sin\phi\cos\phi\sin^2\psi\cos\theta\\&\qquad\qquad-\sin\phi\cos\phi\cos^2\psi\cos\theta)\\&+2BC\sin^2\theta\cos\theta\sin\psi\cos\phi(-\sin\phi\cos\psi\\&\qquad\qquad-\cos\phi\sin\psi\cos\theta)\\&-2CA\sin^2\theta\cos\theta\cos\psi\cos\phi(-\sin\phi\sin\psi\\&\qquad\qquad+\cos\phi\cos\psi\cos\theta). \quad . \quad (16)\end{aligned}$$

In taking the mean with respect to ϕ, the terms which are odd in $\sin\phi$, or $\cos\phi$, disappear, while the mean value of $\sin^2\phi$, or $\cos^2\phi$, is $\frac{1}{2}$. We get for the mean

$$\begin{aligned}
\mathrm{I} = {} & \tfrac{1}{2}\mathrm{A}^2 \sin^2\theta \cos^2\psi (\sin^2\psi + \cos^2\psi \cos^2\theta) \\
& + \tfrac{1}{2}\mathrm{B}^2 \sin^2\theta \sin^2\psi (\cos^2\psi + \sin^2\psi \cos^2\theta) \\
& + \tfrac{1}{2}\mathrm{C}^2 \sin^2\theta \cos^2\theta \\
& - \mathrm{AB} \sin^2\theta \sin\psi \cos\psi \,.\, \sin\psi \cos\psi \sin^2\theta \\
& - \mathrm{BC} \sin^2\theta \cos\theta \sin\psi \,.\, \sin\psi \cos\theta \\
& - \mathrm{CA} \sin^2\theta \cos\theta \cos\psi \,.\, \cos\psi \cos\theta . \quad . \quad . \quad . \qquad (17)
\end{aligned}$$

The averaging with respect to ψ now goes as before, and we obtain

$$\begin{aligned}
& \tfrac{1}{2}(\mathrm{A}^2 + \mathrm{B}^2) \sin^2\theta (\tfrac{1}{8} + \tfrac{3}{8}\cos^2\theta) + \tfrac{1}{2}\mathrm{C}^2 \sin^2\theta \cos^2\theta \\
& - \tfrac{1}{8}\mathrm{AB} \sin^4\theta - \tfrac{1}{2}(\mathrm{A}+\mathrm{B})\mathrm{C} \sin^2\theta \cos^2\theta \,; \quad . \quad . \quad . \quad . \qquad (18)
\end{aligned}$$

and, finally, the averaging with respect to θ gives

$$\begin{aligned}
\text{mean } \mathrm{I} & = \frac{\mathrm{A}^2 + \mathrm{B}^2}{16} (1 - \tfrac{1}{3} + \tfrac{6}{15}) + \frac{\mathrm{C}^2}{15} - \frac{\mathrm{AB}}{15} - \frac{(\mathrm{A}+\mathrm{B})\mathrm{C}}{15} \\
& = \tfrac{1}{15}\{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 - \mathrm{AB} - \mathrm{BC} - \mathrm{CA}\} . \quad . \quad . \quad . \qquad (19)
\end{aligned}$$

This represents the intensity of the vibrations parallel to X dispersed along OY, due to primary vibrations parallel to Z. It vanishes, of course, if $\mathrm{A}=\mathrm{B}=\mathrm{C}$; while, if $\mathrm{A}=\mathrm{B}$ merely, it reduces to (4).

The *ratio* of the two polarized components is

$$\frac{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 - \mathrm{AB} - \mathrm{BC} - \mathrm{CA}}{3(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2) + 2(\mathrm{AB} + \mathrm{BC} + \mathrm{CA})}, \quad . \quad . \qquad (20)$$

reducing to (5) when $\mathrm{B}=\mathrm{A}$.

If the primary light travelling in direction OX is unpolarized, we have also to include primary vibrations parallel to Y. The secondary vibrations scattered along OY are of the same intensity whether they are parallel to Z or to X. They are given by (19), where all that is essential is the perpendicularity of the primary and secondary vibrations. Thus, in order to obtain the effect along OY of unpolarized primary light travelling along OX, we have merely to add (19) to both components. The intensity of the component vibrating parallel to Z is thus

$$\begin{aligned}&\tfrac{1}{15}\{3(A^2+B^2+C^2)+2(AB+BC+CA)\}\\+&\tfrac{1}{15}\{A^2+B^2+C^2-AB-BC-CA\}\\&=\tfrac{1}{15}\{4(A^2+B^2+C^2)+AB+BC+CA\}\,; \quad (21)\end{aligned}$$

while that of the component vibrating parallel to X is simply

$$\tfrac{2}{15}\{A^2+B^2+C^2-AB-BC-CA). \quad (22)$$

The ratio of the two intensities is

$$\frac{2(A^2+B^2+C^2-AB-BC-CA)}{4(A^2+B^2+C^2)+AB+BC+CA}, \quad (23)$$

reducing to (6) when B=A.

It may be observed that, since (21)=(14)+(19), we obtain the same intensity whether we use a polarizer transmitting vibrations parallel to Z and no analyser, or whether we use an analyser transmitting vibrations parallel to Z and no polarizer.

If neither polarizing nor analysing apparatus is employed, we may add (21) and (22), thus obtaining

$$\tfrac{1}{15}[6(A^2+B^2+C^2)-AB-BC-CA]. \quad (24)$$

When the particles are supposed to be of uniform quality, with a specific inductive capacity K′ as compared with K for the undisturbed medium, and to be of *ellipsoidal* form with semi-axes a, b, c, we have

$$A^{-1}:B^{-1}:C^{-1}=1+\frac{K'-K}{4\pi K}L:1+\frac{K'-K}{4\pi K}M:1+\frac{K'-K}{4\pi K}N,$$

where $\quad (25)$

$$L=2\pi abc\int_0^\infty\frac{d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{1/2}(c^2+\lambda)^{1/2}}, \quad (26)$$

with similar expressions for M and N.

If the ellipsoid be of revolution the case is simplified *. For example, if it be of the elongated or ovary form with eccentricity e,

$$a=b=c\sqrt{(1-e^2)}\,; \quad (27)$$

$$L=M=2\pi\left\{\frac{1}{e^2}-\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad (28)$$

$$N=4\pi\left\{\frac{1}{e^2}-1\right\}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\}. \quad (29)$$

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* See the paper of 1897.

For the sphere ($e=0$)

$$L=M=N=\frac{4\pi}{3}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In the case of a very elongated ovoid, L and M approximate to the value 2π, while N approximates to the form

$$N=4\pi\frac{a^2}{c^2}\left(\log\frac{2c}{a}-1\right), \quad . \quad . \quad . \quad . \quad (31)$$

vanishing when $e=1$. It appears that, when K'/K is finite, mere elongation does not suffice to render A and B negligible in comparison with C. The limiting value of C : A is in fact $\frac{1}{2}(1+K'/K)$. If, however, as for a perfectly conducting body, $K'=\infty$, then C becomes paramount, and the simplified values already given for this case acquire validity*.

Another question which naturally presents itself is whether a want of equality among the coefficients A, B, C interferes with the relation between attenuation and refractive index, explained in my paper of 1899 †. The answer appears to be in the affirmative, since the attenuation depends upon $A^2+B^2+C^2$, while the refractive index depends upon $A+B+C$, so that no simple relation obtains in general. But it may well be that in cases of interest the disturbance thus arising is not great.

The problem of an ellipsoidal particle of uniform dielectric quality can be no more than illustrative of what happens in the case of a molecule; but we may anticipate that the general form with suitable values of A, B, C still applies, except it may be under special circumstances where resonance occurs and where the effective values of the coefficients may vary greatly with the wave-length of the light.

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* But the particle must still be small relatively to the wave-length *within the medium* of which it is composed.

† An equivalent formula was given by Lorenz in 1890, 'Œuvres Scientifiques,' t. i. p. 496, Copenhagen, 1898. See also Schuster's 'Theory of Optics,' 2nd ed. p. 326 (1909).

XLIII. *Note on Fox Talbot's Method of obtaining Coloured Flames of Great Intensity.* *By* G. A. HEMSALECH, *Honorary Research Fellow in the University of Manchester*[*].

LORD RAYLEIGH has very kindly called my attention to a short but interesting paper by Talbot, in which the latter describes an exceedingly simple device of producing intensely luminous metal vapours in the flame of a spirit-lamp[†]. In view of the historic interest attached to this paper as recording an example of the endeavours made by the early spectroscopists to improve upon the efficiency of the light sources at their disposal, and also in consideration of the fact that the method is not mentioned in modern works on spectroscopy, it may perhaps be of some use to recall Talbot's experiment, and to give the results of a few personal observations on the same subject.

In Talbot's own words the method is as follows:—"It is only requisite to place a lump of common salt upon the wick of a spirit-lamp, and to direct a stream of oxygen gas from a blowpipe upon the salt. The light emitted is quite homogeneous and of dazzling brightness. If instead of common salt we use the various salts of strontium, barytes, etc., we obtain the well-known coloured flames, which are characteristic of those substances, with far more brilliancy than by any other method with which I am acquainted." It is to be noted that Talbot precisely states that the stream of oxygen is directed upon the salt. No explanation is, however, given to account for the result achieved, and it is not certain whether the greater luminosity obtained in the manner set forth is due to an actual combustion of the salt or to a more effective dissociation of the latter, caused by a rise in temperature of the flame. The following series of experiments was made with the object of elucidating the mode of action of the oxygen in Talbot's experiment and of applying his method to the air-coal gas flame.

1. *Observations with a spirit flame.*

With a small piece of salt on the wick the flame appeared feebly coloured yellow and showed the D-lines only. A current of oxygen gas escaping from a glass nozzle was then directed towards the salt, and the latter immediately burst out in a brilliant light, emitting the D-lines most strongly,

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[*] Communicated by Sir E. Rutherford, F.R.S.

[†] H. F. Talbot, Philosophical Magazine, vol. iii. p. 35 (1833).

as also a fainter line on the more refrangible side of the former, probably the pair at λ 5685. The greater part of the flame, although more intensely coloured than without the oxygen, remained, however, relatively feeble as compared with the vivid brightness exhibited by the immediate vicinity of the salt. Thus the light distribution in the flame is not very uniform, there being a most pronounced maximum near the salt.

In addition to the sodium lines the bands of the Swan spectrum were likewise very prominent near the base of the flame where the oxygen passed through. In fact the path of the oxygen through the flame is rendered visible by the more vigorous combustion of unburnt hydrocarbons, which gives rise to the emission of a greenish light (more bluish and fainter when the oxygen passes through the upper parts of the flame, where less unburnt gas prevails).

When the oxygen was passed through the coloured region of the flame situated just above the salt there was hardly any increase in the intensity of the light emitted, as though the free radiating sodium atoms remained unaffected by a rise in temperature. (The fact that the temperature of the flame is appreciably raised by the oxygen will be demonstrated in § 2.)

Similar results were obtained with calcium chloride. But in this case there was a notable brightening of the spectrum bands when the oxygen was sent through the middle region of the flame well above the salt. This may be due to undissociated particles of the salt being carried upwards through the flame and broken up as they enter the presumably hotter zone of the oxygen path.

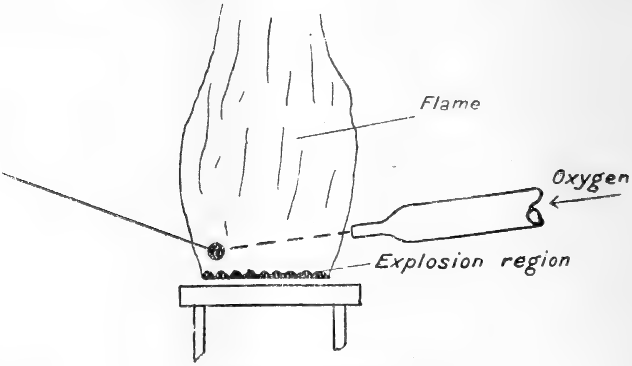
Thallium chloride showed no line when placed on the wick of the spirit-lamp. But as soon as oxygen was blown against it, the green line appeared not only in the vicinity of the salt but also in the upper part of the flame.

Thus it is evident that Talbot's method constituted an important improvement in the days when the spirit flame was practically the only means in general use of vaporizing substances for spectroscopic purposes.

2. *Observations with an air-coal gas flame.*

When oxygen is passed into the flame given by an ordinary Bunsen or Méker burner, there is a great tendency for the former to strike down the burner-tube, owing to the increased velocity of the explosion. This inconvenience was completely obviated by placing on the top of a Méker burner a perforated brass plate, $\frac{1}{8}$ inch thick, having 49 one-millimetre holes per square centimetre. Experiments were made with the salts of sodium and calcium. The greatest intensity of the metal vapour emission was also in this case obtained when the salts were held near the base of the flame or placed on the perforated plate. When the oxygen was passed through the luminous vapour in the region above the salt, about half way up the flame, the brightening along the oxygen path was relatively feeble; but when now the salt was raised and held in the path of the oxygen, the intensity again was fairly great, though much less so than when both the stream of oxygen and the salt were close to the explosion region. On the other hand, near the tip of the flame, where combustion of the gases is nearly complete, no luminous effect was observed when oxygen was blown against the salt. Thus the oxygen, in order to be effective by this method, must be directed upon the substance to be vaporized in the presence of unburnt gases (hydrocarbons and hydrogen). This fact is further illustrated by the following observations, which at the same time prove that the light effect obtained in Talbot's experiment is not due to mere combustion of the material placed on the wick: when the salt is held in the flame near the border, and the oxygen enters from the opposite side (fig. 1) in such a way that it traverses a long zone of flame containing unburnt gases before encountering the salt, the luminous effect is a maximum. If, however, the salt be held near to the point of the border at which the oxygen enters, no effect is observed (fig. 2).

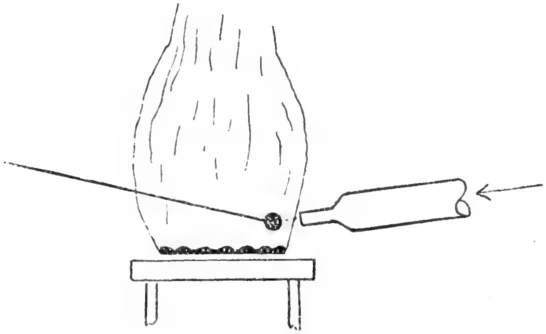
Fig. 1.



Maximum Effect.

That the temperature of the flame is raised when oxygen is passed through near the base is easily shown in the following way:—A piece of iron wire 1 millimetre thick is held in the flame about half way up from the burner-plate.

Fig. 2.



No Effect.

In this position it will be raised to a bright red heat. If now a stream of oxygen be directed through the flame just above the explosion region, but well below the wire, the latter immediately becomes very much brighter. The same heating effect is observed when the oxygen is blown against the wire within the flame.

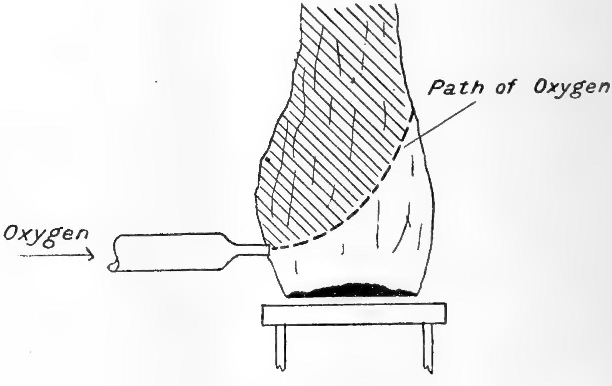
Hence it is evident that the increased intensity of the light emission in Talbot's experiment is of thermal origin, at least in so far as the rise in the temperature of the flame entails a more vigorous chemical dissociation of the salt. This conclusion was further corroborated by directing the high-temperature flame from an oxy-coal gas blowpipe upon the salt on the wick of the spirit-lamp. The salt in this case gave out an exceedingly brilliant light, in fact much more so than with the oxygen alone, and a large volume of luminous vapour rose into the spirit flame above.

3. *Application of Talbot's method to flames coloured by means of a sprayer.*

The foregoing experiments were made in accordance with Talbot's original procedure of blowing the oxygen against a lump of salt held in the flame. But the method is equally well applicable to flames into which the material to be vaporized is introduced in a very finely divided state, such as is provided by a sprayer. Also in this case it was found that the greatest effect is obtained when the oxygen enters the flame near its base, and the effect vanishes near the top. The reason for this is of course the same as before, namely, the accelerated combustion of the unburnt gases which predominate near the base. It is, however, well to remark that the intensity of the coloured flame produced in this way falls appreciably short of that obtained when the oxygen is thoroughly mixed with the air and coal gas before these gases reach the burner plate*.

In order to observe the effect of varying the relative proportions of the gases in the air-coal gas mixture, an electric sprayer was used in connexion with burner No. 1 described in a previous paper†. With this arrangement the fundamental condition for the successful working of Talbot's method, namely, the presence of unburnt gases, is most strikingly demonstrated by the following experiment:—The glass nozzle through which passes the oxygen is held close to the edge of the flame about ½ inch above the burner-plate. The velocity of the oxygen is such that its curved path can be distinctly followed to the opposite edge. When the air-coal gas mixture is so adjusted that the explosion region just begins to rise from the burner-plate, otherwise stated when there is an excess of coal gas in the mixture with consequent deficiency of combustion, then the portion of the coloured flame situated above the sharply defined oxygen path is most notably increased in luminosity, as indicated by the shaded portion of fig. 3. If now more air

Fig. 3.



Excess of Coal Gas.

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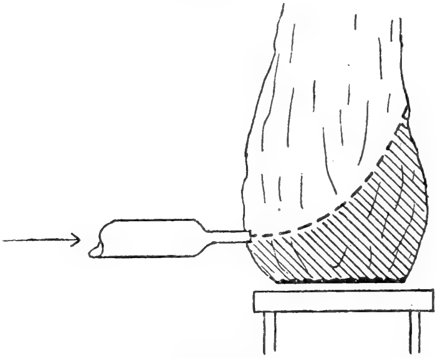
* Hemsalech, Philosophical Magazine, vol. xxxiv. p. 243 (1917).

† Hemsalech, ibid. vol. xxxiii. p. 6 (1917).

be admitted and the combustion of the coal gas thereby rendered more complete, the luminosity of the flame above the oxygen path gradually decreases and finally becomes less than that of the region below, as sketched in fig. 4. In fact, when viewed in the direction of the oxygen path, the luminosity along and above it is found to have been completely arrested, the reduced light as seen sideways being merely due to the thin borders of flame rising up on either side of the oxygen stream. That the extinction of the luminous vibrations in this case is caused principally by actual cooling of the vapour and not merely by oxidation, is shown by the fact that when a stream of nitrogen is passed into the flame the same extinction is observed. On the other hand, a stream of ammonia does not reduce the intensity of the light emission in the upper zone; the feebly luminous flame with which the ammonia burns as it passes through the air-coal gas flame seems to be just sufficient to keep up the temperature of the latter. Also when a high temperature oxy-coal gas flame is directed through the air-coal gas flame under these conditions, the portion of the latter situated above the path of the oxy-coal gas flame remains always a little brighter. All the above observations were made on the luminous vapours of sodium, strontium, lithium, calcium, and barium.

Fig. 4.



Excess of Air.

Summary.

The series of observations recorded in this note clearly shows that the rôle of the oxygen in Talbot's experiment is to increase the rate of combustion of the unburnt gases in the flame, thus causing a rise in temperature of the latter. The temperature is highest along the path of the oxygen, and when the latter flows in the direction of the substance to be vaporized, the full effect of the temperature is, as it were, concentrated upon it. The effect is greatest near the base of the flame, where combustion is only beginning, and least near the tip. It is consequently essential that the oxygen, prior to reaching the substance, should pass through a region of the flame containing unburnt gases.

In conclusion, I wish to thank Lord Rayleigh for having provided me with the opportunity of paying a modest tribute to the work of one who came so near to discovering spectrum analysis.

Manchester, Feb. 18, 1918.

XLIV. *On the Problem of Two and that of Three Electrified Spherical Conductors.* *By* Prof. A. ANDERSON, *M.A.*[*]

WHEN an insulated conducting sphere of radius a is charged to potential A, the potential V due to the charge at any external point, P, whose distance from the centre of the sphere is r, is given by

$$rV = aA.$$

If now, charged bodies are brought into the field, this equation no longer holds : we have, instead,

$$rV + aV' = aA,$$

where V' is the potential that the introduced bodies have at P' the inverse point or, as we may call it for shortness, the image of P in the sphere. A has, of course, altered in value and V is, as before, the potential due to the charge on the sphere.

This equation may be used to find the coefficients of capacity and induction of two conducting spheres.

Let the potentials of the spheres be A and B, their centres O_1 and O_2, their radii a and b, and the distance apart of their centres c. Also, in fig. 1, let I_1 be the image of O_1 in B, I_2 the image of I_1 in A, I_3 the image of I_2 in B, and so on, I_{n+1} in either sphere being the image of I_n in that sphere. We have thus a series of points I_1, I_3, I_5, I_7, &c. inside the

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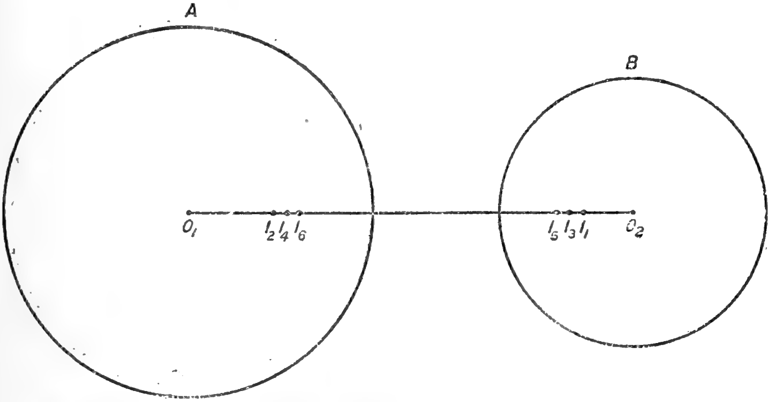
[*] Communicated by the Author.

sphere B, and a series I_2, I_4, I_6, I_8, &c. inside A. Let A_n, B_n denote the potentials due to the charges on the spheres A and B at the point I_n, and B_0 the potential due to the charge on B at the centre of A.

Fig. 1.



We have
$$cB_0+bA_1=bB,$$
$$O_1I_1 \cdot A_1+aB_2=aA.$$
$$\therefore \quad \frac{c}{b} \cdot B_0-\frac{a}{O_1I_1} \cdot B_2=B-\frac{aA}{O_1I_1},$$
and, likewise,
$$\frac{O_2I_2}{b} \cdot B_2-\frac{a}{O_1I_3} \cdot B_4=B-\frac{aA}{O_1I_3},$$
$$\frac{O_2I_4}{b} \cdot B_4-\frac{a}{O_1I_5} \cdot B_6=B-\frac{aA}{O_1I_5}.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{O_2I_{2n}}{b} \cdot B_{2n}-\frac{a}{O_1I_{2n+1}} \cdot B_{2n+2}=B-\frac{aA}{O_1I_{2n+1}}.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
Hence we have
$$\frac{c}{b} \cdot B_0=B-\frac{aA}{O_1I_1}+\left(B-\frac{aA}{O_1I_3}\right)\frac{ab}{O_1I_1 \cdot O_2I_2}$$
$$+\left(B-\frac{aA}{O_1I_5}\right)\frac{a^2b^2}{O_1I_1 \cdot O_1I_3 \cdot O_2I_2 \cdot O_2I_4}$$
$$+\left(B-\frac{aA}{O_1I_7}\right)\frac{a^3b^3}{O_1I_1 \cdot O_1I_3 \cdot O_1I_5 \cdot O_2I_2 \cdot O_2I_4 \cdot O_2I_6}$$
$$+\cdots\cdots$$
$$+\left(B-\frac{aA}{O_1I_{2n+1}}\right)\frac{a^nb^n}{O_1I_1\ldots\ldots O_1I_{2n-1} \cdot O_2I_2\ldots\ldots O_2I_{2n}}$$
$$+\cdots\cdots$$

But if E is the charge on A, $A = B_0 + \frac{E}{a}$, and therefore

$$E = aA - aB_0.$$

Thus

$$E = A\left\{a + \frac{a^2b}{c}\left(\frac{1}{O_1I_1} + \frac{ab}{O_1I_1 \,.\, O_1I_3 \,.\, O_2I_2} + \frac{a^2b^2}{O_1I_1 \,.\, O_1I_3 \,.\, O_1I_5 \,.\, O_2I_2 \,.\, O_2I_4} + \dots + \frac{a^nb^n}{O_1I_1 \dots O_1I_{2n+1} \,.\, O_2I_2 \dots O_2I_{2n}} + \dots\right)\right\},$$

$$-B\frac{ab}{c}\left\{1 + \frac{ab}{O_1I_1 \,.\, O_2I_2} + \frac{a^2b^2}{O_1I_1 \,.\, O_1I_3 \,.\, O_2I_2 \,.\, O_2I_4} + \dots + \frac{a^nb^n}{O_1I_1 \,.\, O_1I_3 \dots O_1I_{2n-1} \,.\, O_2I_2 \dots O_2I_{2n}} + \dots\right\}.$$

Thus q_{11} and q_{12} have been found, and, of course, also q_{22}, by a simple application of the above equation.

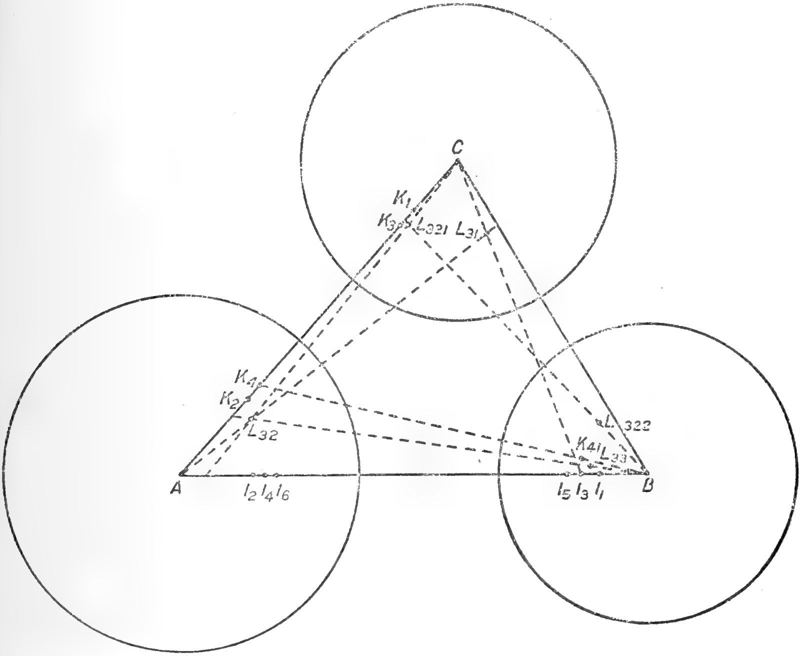
The same method is applicable to the case of several conducting spheres. For three spheres whose centres are at the corners of a triangle the work is necessarily much longer than that for two, but it is possible to find the values of the coefficients of induction and capacity to any degree of approximation.

We require for the solution of the problem a set of points I_1, I_3, I_5,... inside B, and a set I_2, I_4, I_6,... inside A, as in the problem for two spheres, and corresponding to these, for A and C, a set K_1, K_3, K_5,... inside C, and a set K_2, K_4, K_6,... inside A. But other points besides these are needed. Take one of the points, say I_3, inside B. We take its image in C and denote it by L_{31}, the image of this in A by L_{32}, the image of L_{32} in B by L_{33}, and so on, going round again and again in the positive direction.

For the I points inside A we do a similar thing, the image of I_4 in C being L_{41}, and that of this in B L_{42}, and so on, going round in the negative direction. We do the same thing for the K points. Thus the image of K_5 in B is M_{51}, and the image of this in A is M_{52}, and so on. But this does not exhaust all the points required. Starting from any L or M point, we reverse the direction and find an infinite series of points for it. Thus, taking the point M_{52}, its image in C is M_{53}, but we also take its image in B and call it M_{521}, the image of this in C, M_{522}, and so on. A few of the points are shown in fig. 2.

The centres of the spheres are the points A, B, C, and we shall use these letters to denote also their potentials; α, β, γ are the lengths of the radii of the spheres, and a, b, c the lengths of the sides of the triangle. The potential due to the sphere A at any point, say K_{41}, will be denoted by $A_{k_{41}}$, the potential due to B at L_{31} by $B_{l_{31}}$, and so on. The object of the problem is to find $B_a + C_a$.

Fig. 2.



We have

$$c \,.\, B_a + \beta(A_{i_1} + C_{i_1}) = \beta B,$$

$$b \,.\, C_a + \gamma(A_{k_1} + B_{k_1}) = \gamma C.$$

$$\therefore \quad B_a + C_a + \frac{\beta}{c} A_{i_1} + \frac{\gamma}{b} A_{k_1} + \frac{\beta}{c} C_{i_1} + \frac{\gamma}{b} B_{k_1} = \frac{\beta}{c} B + \frac{\gamma}{b} \,.\, C.$$

Thus the first two terms of the expression for $B_a + C_a$ are

$$\frac{\beta}{c} B + \frac{\gamma}{b} \,.\, C.$$

Again,
$$\mathrm{AI}_1 \,.\, \mathrm{A}_{i_1} + \alpha(\mathrm{B}_{i_2} + \mathrm{C}_{i_2}) = \alpha\mathrm{A},$$
$$\mathrm{AK}_1 \,.\, \mathrm{A}_{k_1} + \alpha(\mathrm{B}_{k_2} + \mathrm{C}_{k_2}) = \alpha\mathrm{A}\ ;$$

hence

$$\mathrm{B}_a + \mathrm{C}_a + \frac{\beta}{c}\mathrm{C}_{i_1} + \frac{\gamma}{b}\,.\,\mathrm{B}_{k_1} - \frac{\alpha\beta}{c\,.\,\mathrm{AI}_1}(\mathrm{B}_{i_2} + \mathrm{C}_{i_2}) - \frac{\alpha\gamma}{b\,.\,\mathrm{AK}_1}(\mathrm{B}_{k_2} + \mathrm{C}_{k_2})$$
$$= \frac{\beta}{c}\mathrm{B} + \frac{\gamma}{b}\,.\,\mathrm{C} - \left(\frac{\alpha\beta}{c\,.\,\mathrm{AI}_1} + \frac{\alpha\gamma}{b\,.\,\mathrm{AK}_1}\right)\mathrm{A}.$$

Also,
$$\mathrm{BK}_1 \,.\, \mathrm{B}_{k_1} + \beta(\mathrm{A}_{m_{11}} + \mathrm{C}_{m_{11}}) = \beta\mathrm{B},$$
$$\mathrm{CI}_1 \,.\, \mathrm{C}_{i_1} + \gamma(\mathrm{A}_{l_{11}} + \mathrm{B}_{l_{11}}) = \gamma\mathrm{C},$$

and
$$\mathrm{AL}_{11} \,.\, \mathrm{A}_{l_{11}} + \alpha(\mathrm{B}_{l_{12}} + \mathrm{C}_{l_{12}}) = \alpha\mathrm{A},$$
$$\mathrm{AM}_{11} \,.\, \mathrm{A}_{m_{11}} + \alpha(\mathrm{B}_{m_{12}} + \mathrm{C}_{m_{12}}) = \alpha\mathrm{A},$$

from which we get, by substitution,

$$\mathrm{B}_a + \mathrm{C}_a - \frac{\alpha\beta}{c\,.\,\mathrm{AI}_1}(\mathrm{B}_{i_2} + \mathrm{C}_{i_2}) - \frac{\alpha\gamma}{b\,.\,\mathrm{AK}_1}(\mathrm{B}_{k_2} + \mathrm{C}_{k_2}) - \frac{\beta\gamma}{b\,.\,\mathrm{BK}_1}\,.\,\mathrm{C}_{m_1}$$
$$- \frac{\beta\gamma}{c\,.\,\mathrm{CI}_1}\,.\,\mathrm{B}_{l_{11}} + \frac{\alpha\beta\gamma}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{AM}_{11}}(\mathrm{B}_{m_{12}} + \mathrm{C}_{m_{12}})$$
$$+ \frac{\alpha\beta\gamma}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{AL}_{11}}(\mathrm{B}_{l_{12}} + \mathrm{C}_{l_{12}})$$
$$= \frac{\beta}{c}\mathrm{B} + \frac{\gamma}{b}\,.\,\mathrm{C} - \frac{\alpha\beta}{c\,.\,\mathrm{AI}_1}\mathrm{A}$$
$$- \frac{\alpha\gamma}{b\,.\,\mathrm{AK}_1}\mathrm{A} - \frac{\beta\gamma}{b\,.\,\mathrm{BK}_1}\,.\,\mathrm{B} - \frac{\beta\gamma}{c\,.\,\mathrm{CI}_1}\,.\,\mathrm{C}$$
$$+ \alpha\beta\gamma\left[\frac{1}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{AM}_{11}} + \frac{1}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{AL}_{11}}\right]\mathrm{A}.$$

Thus, as far as the sixth term,

$$\mathrm{B}_a + \mathrm{C}_a = -\alpha\left(\frac{\beta}{c\,.\,\mathrm{AI}_1} + \frac{\gamma}{b\,.\,\mathrm{AK}_1}\right)$$
$$+ \mathrm{B}\left(\frac{\beta}{c} - \frac{\beta\gamma}{b\,.\,\mathrm{BK}_1}\right) + \mathrm{C}\left(\frac{\gamma}{b} - \frac{\beta\gamma}{c\,.\,\mathrm{CI}_1}\right).$$

Proceeding in this way, we find that as far as the fourteenth term,

$$B_a + C_a = -\alpha\left[\frac{\beta}{c\,.\,\mathrm{AI}_1} + \frac{\gamma}{b\,.\,\mathrm{AK}_1} - \alpha\beta\gamma\left(\frac{1}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{AM}_{11}} + \frac{1}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{AL}_{11}}\right)\right]\mathrm{A}.$$

$$+\mathrm{B}\left[\frac{\beta}{c} - \frac{\beta\gamma}{b\,.\,\mathrm{BK}_1} + \frac{\alpha\beta^2}{c\,.\,\mathrm{AI}_1\,.\,\mathrm{BI}_2} + \frac{\alpha\beta\gamma}{b\,.\,\mathrm{AK}_1\,.\,\mathrm{BK}_2} + \frac{\beta^2\gamma}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{BL}_{11}}\right]$$

$$+\mathrm{C}\left[\frac{\gamma}{b} - \frac{\beta\gamma}{c\,.\,\mathrm{CI}_1} + \frac{\alpha\gamma^2}{b\,.\,\mathrm{AK}_1\,.\,\mathrm{CK}_2} + \frac{\alpha\beta\gamma}{c\,.\,\mathrm{AI}_1\,.\,\mathrm{CI}_2} + \frac{\gamma^2\beta}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{CM}_{11}}\right].$$

By continuing this process we may find any number of terms of the expression for $B_a + C_a$.

Now, if E is the charge on A, we have

$$\mathrm{A} = B_a + C_a + \frac{\mathrm{E}}{\alpha},$$

or

$$\mathrm{E} = \mathrm{A}\alpha - (B_a + C_a).$$

$$\therefore\ \mathrm{E} = \mathrm{A}\left[\alpha + \alpha^2\left(\frac{\beta}{c\,.\,\mathrm{AI}_1} + \frac{\gamma}{b\,.\,\mathrm{AK}_1}\right) - \alpha^2\beta\gamma\left(\frac{1}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{AM}_{11}} + \frac{1}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{AL}_{11}}\right) + \dots\right]$$

$$-\mathrm{B}\left[\frac{\alpha\beta}{c} - \frac{\alpha\beta\gamma}{b\,.\,\mathrm{BK}_1} + \frac{\alpha^2\beta^2}{c\,.\,\mathrm{AI}_1\,.\,\mathrm{BI}_2} + \frac{\alpha^2\beta\gamma}{b\,.\,\mathrm{AK}_1\,.\,\mathrm{BK}_2} + \frac{\alpha\beta^2\gamma}{c\,.\,\mathrm{CI}_1\,.\,\mathrm{BL}_{11}} + \dots\right]$$

$$-\mathrm{C}\left[\frac{\alpha\gamma}{c} - \frac{\alpha\beta\gamma}{c\,.\,\mathrm{CI}_1} + \frac{\alpha^2\gamma^2}{b\,.\,\mathrm{AK}_1\,.\,\mathrm{CK}_2} + \frac{\alpha^2\beta\gamma}{c\,.\,\mathrm{AI}_1\,.\,\mathrm{CI}_2} + \frac{\alpha\gamma^2\beta}{b\,.\,\mathrm{BK}_1\,.\,\mathrm{CM}_{11}} + \dots\right].$$

Thus the first terms of the expressions for q_{11}, q_{12}, q_{13} have been determined and the corresponding terms of q_{22}, q_{33}, q_{23} may be got by changing the letters. We can get the formula for two spheres from this by making b, AK_1, BK_1, AL_{11}, BL_{11} all infinite and $\mathrm{C}=0$.

Thus

$$\mathrm{E}=\mathrm{A}\left[\alpha+\frac{\alpha^2\beta}{c\,.\,\mathrm{AI}_1}+\ldots\right]-\mathrm{B}\left[\frac{\alpha\beta}{c}+\frac{\alpha^2\beta^2}{c\,.\,\mathrm{AI}_1\,.\,\mathrm{BI}_2}+\ldots\right],$$

or, since $\mathrm{AI}_1=\frac{c^2-\beta^2}{c}$, and $\mathrm{BI}_2=c-\frac{\alpha^2 c}{c^2-\beta^2}$,

$$\mathrm{E}=\mathrm{A}\left[\alpha+\frac{\alpha^2\beta}{c^2-\beta^2}+\ldots\right]-\mathrm{B}\left[\frac{\alpha\beta}{c}+\frac{\alpha^2\beta^2}{c(c^2-\beta^2-\alpha^2)}+\ldots\right].$$

As an example, suppose we have three small equal conducting spheres, whose centres are at the corners of an equilateral triangle the length of whose side is c. The terms we have found will give q_{11}, q_{12}, q_{13} correctly to $\left(\frac{\alpha}{c}\right)^3$.

$$q_{11}=q_{22}=q_{33}=\alpha\left(1+\frac{2\alpha^2}{c^2}-\frac{2\alpha^3}{c^3}\right),$$

$$q_{12}=q_{13}=q_{23}=-\frac{\alpha^2}{c}\left(1-\frac{\alpha}{c}+\frac{2\alpha^3}{c^2}\right),$$

from which we obtain the coefficients of potential

$$p_{11}=p_{22}=p_{33}=\frac{1}{3\alpha}\left(1-\frac{2\alpha^2}{c^2}+\frac{2\alpha^3}{c^3}\right),$$

$$p_{12}=p_{13}=p_{23}=\frac{1}{3\alpha}\left(1+\frac{3\alpha}{c}+\frac{\alpha^2}{c^2}-\frac{16\alpha^3}{c^3}\right).$$

The energy of the system, each sphere being supposed to have unit charge, is

$$\frac{1}{\alpha}\left(\frac{3}{2}+\frac{3\alpha}{c}-\frac{15\alpha^3}{c^3}\right),$$

and the force acting on one of them is

$$\sqrt{3}\left(\frac{1}{c^2}-\frac{15\alpha^2}{c^4}\right),$$

the force being $\frac{\sqrt{3}}{c^2}$ in the case of point charges.

By the above method the potential due to two charged conducting spheres at an external point can be written down easily. Let the centres of the spheres be A and B, their radii a and b, the distance between the centres c, and the potentials U and V. Let the image of any external point P in B be P_1, the image of P_1 in A, P_2, the image of P_2 in B, P_3, and so on. Also, let the image of P in A be Q_1, the image of Q_1 in B, Q_2, the image of Q_2 in A, Q_3, and so on.

The potential at P is

$$\mathrm{U}\Big(\frac{a}{\mathrm{AP}}-\frac{ab}{\mathrm{BP}\,.\,\mathrm{AP}_1}+\frac{a^2b}{\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{AQ}_2}$$

$$-\frac{a^2b^2}{\mathrm{BP}\,.\,\mathrm{AP}_1\,.\,\mathrm{AP}_3\,.\,\mathrm{BP}_2}+\frac{a^3b^2}{\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{BQ}_3\,.\,\mathrm{AQ}_2\,.\,\mathrm{AQ}_4}$$

$$-\frac{a^3b^3}{\mathrm{BP}\,.\,\mathrm{AP}_1\,.\,\mathrm{AP}_3\,.\,\mathrm{AP}_5\,.\,\mathrm{BP}_2\,.\,\mathrm{BP}_6}+\ldots\Big)$$

$$+\mathrm{V}\Big(\frac{b}{\mathrm{BP}}-\frac{ab}{\mathrm{AP}\,.\,\mathrm{BQ}_1}+\frac{ab^2}{\mathrm{BP}\,.\,\mathrm{AP}_1\,.\,\mathrm{BP}_2}$$

$$-\frac{a^2b^2}{\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{BQ}_3\,.\,\mathrm{AQ}_2}+\frac{a^2b^3}{\mathrm{BP}\,.\,\mathrm{AP}_1\,.\,\mathrm{AP}_3\,.\,\mathrm{BP}_2\,.\,\mathrm{BP}_4}$$

$$-\frac{a^3b^3}{\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{BQ}_3\,.\,\mathrm{BQ}_5\,.\,\mathrm{AQ}_2\,.\,\mathrm{AQ}_6}+\ldots\Big).$$

If, now, the image of A in B is I_1, the image of I_1 in A, I_2, the image of I_2 in B, I_3, and so on, and if the image of B in A is J_1, the image of J_1 in B, J_2, the image of J_2 in A, J_3, and so on, it is easy to show from similar triangles that

$$\mathrm{BP}\,.\,\mathrm{AP}_1=c\,.\,\mathrm{PI}_1,$$

$$\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{AQ}_2=c\,.\,\mathrm{PI}_2\,.\,\mathrm{AI}_1,$$

$$\mathrm{BP}\,.\,\mathrm{AP}_1\,.\,\mathrm{BP}_2\,.\,\mathrm{AP}_3=c\,.\,\mathrm{PI}_3\,.\,\mathrm{BI}_2\,.\,\mathrm{AI}_1,$$

$$\mathrm{AP}\,.\,\mathrm{BQ}_1\,.\,\mathrm{AQ}_2\,.\,\mathrm{BQ}_3\,.\,\mathrm{AQ}_4=c\,.\,\mathrm{PI}_4\,.\,\mathrm{AI}_3\,.\,\mathrm{BI}_2\,.\,\mathrm{AI}_1,$$

$$\ldots\ldots\ \&\mathrm{c}.$$

Hence the potential is

$$\mathrm{U}\Big(\frac{a}{\mathrm{AP}}-\frac{ab}{c\,.\,\mathrm{PI}_1}+\frac{a^2b}{c\,.\,\mathrm{PI}_2\,.\,\mathrm{AI}_1}$$

$$-\frac{a^2b^2}{c\,.\,\mathrm{PI}_3\,.\,\mathrm{BI}_2\,.\,\mathrm{AI}_1}+\frac{a^3b^2}{c\,.\,\mathrm{PI}_4\,.\,\mathrm{AI}_3\,.\,\mathrm{BI}_2\,.\,\mathrm{AI}_1}$$

$$-\frac{a^3b^3}{c\,.\,\mathrm{PI}_5\,.\,\mathrm{BI}_4\,.\,\mathrm{AI}_3\,.\,\mathrm{BI}_2\,.\,\mathrm{AI}_1}+\ldots\Big)$$

$$+\mathrm{V}\Big(\frac{b}{\mathrm{AP}}-\frac{ab}{c\,.\,\mathrm{PJ}_1}+\frac{ab^2}{c\,.\,\mathrm{PJ}_2\,.\,\mathrm{BJ}_1}-\frac{a^2b^2}{c\,.\,\mathrm{PJ}_3\,.\,\mathrm{AJ}_2\,.\,\mathrm{BJ}_1}+\ldots\Big),$$

which shows that the potential outside the spheres has the same value as that due to a series of point charges at

A, B, I_1, J_1, I_2, J_2, I_3, J_3, I_4, J_4, &c., equal respectively to

$$Ua,\ Vb,\ -\frac{Uab}{c},\ -\frac{Vab}{c},\ \frac{Ua^2b}{c\,.\,AI_1},\ \frac{Vab^2}{c\,.\,BJ_1},\ -\frac{Ua^2b^2}{c\,.\,BI_2\,.\,AI_1},$$

$$-\frac{Va^2b^2}{c\,.\,AJ_2\,.\,BJ_1},\ \frac{Ua^3b^2}{c\,.\,AI_3\,.\,BI_2\,.\,AI_1},\ \frac{Va^2b^3}{c\,.\,BJ_3\,.\,AJ_2\,.\,BJ_1},\ \&c.\ldots,$$

which are the image charges in the usual way of treating the subject.

Note.—The above paper was written before the one that appeared in the March number of the Philosophical Magazine on the same subject. It is, perhaps, unfortunate that the word "image" has been used for "inverse point." The method has, of course, nothing to do with electrical images.

XLV. *Some Two-Dimensional Potential Problems connected with the Circular Arc.* *By* W. G. Bickley, *B.Sc.*[*]

§ 1. IN this paper a method of dealing with potential problems in two dimensions, depending on the use of functions of a complex variable and of the method of images, is applied to the solution of problems connected with an infinitely long lamina, the section of which is a circular arc. The results obtained are interpreted in terms of electricity and hydrodynamics.

§ 2. The first step in the investigation is the determination of the transformation by which the two sides of the arc in the z-plane become the real axis in the plane of an auxiliary variable $\zeta\ (=\xi+\iota\eta)$. The arc is taken as that part of the circle $z=-\iota e^{\iota\theta}$ for which $-\alpha\leq\theta\leq\alpha$, so that the angle subtended at the centre is 2α. For any point on this circle the ratio $(1+\iota z)/(z+\iota)$ is purely real, its value being $\cot\frac{\theta}{2}$, so that when θ lies within the above limits, the values of

$$\tan\frac{\alpha}{2}\,\frac{1+\iota z}{z+\iota}+\sqrt{\tan^2\frac{\alpha}{2}\left(\frac{1+\iota z}{z+\iota}\right)^2-1}\quad .\ .\ (1)$$

are purely real, but become complex when θ lies outside them. Also, when $z\rightarrow-\iota$, the expression (1) tends to 0 or ∞ according as the root is taken negatively or positively; and when $\theta=\pm\alpha$, the expression has the values ± 1.

[*] Communicated by the Author.

Hence the transformation

$$\zeta=\frac{\sin\frac{\alpha}{2}(1+\iota z)+\sqrt{\sin^2\frac{\alpha}{2}(1+\iota z)^2-\cos^2\frac{\alpha}{2}(z+\iota)^2}}{\cos\frac{\alpha}{2}(z+\iota)} \quad (2)$$

transforms the two sides of the z arc into the real axis of the ζ-plane, the extremities of the arc into the points $\zeta=\pm1$. This relation may also be written

$$z=-\iota\frac{\zeta^2\cos\frac{\alpha}{2}+2\iota\zeta\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\zeta^2\cos\frac{\alpha}{2}-2\iota\zeta\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}} \quad . \quad . \quad . \quad . \quad (3)$$

It is easily found from (3) that $z=\infty$ corresponds to

$$\zeta=\iota\left(\sin\frac{\alpha}{2}\pm1\right)\Big/\cos\frac{\alpha}{2},$$

only one of which is in the upper half of the ζ-plane.

§ 3. Suppose that in the ζ-plane there is a charge at the point

$$\zeta=\iota\left(1+\sin\frac{\alpha}{2}\right)\Big/\cos\frac{\alpha}{2},$$

and that the real axis is a conductor. This will correspond to a charge at infinity in the z-plane, with the arc a conductor, or, what is practically the same, to the charged conducting arc in space. Let w, as usual, denote the potential ϕ together with ι times the conjugate function ψ. Then in the ζ-plane, the method of images gives

$$w=\log\left\{\left(\zeta-\iota\frac{1+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right)\Bigg/\left(\zeta+\iota\frac{1+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right)\right\}, \quad . \quad . \quad (4)$$

or, what is the same thing,

$$\zeta=-\iota\frac{1+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\cdot\frac{e^w+1}{e^w-1}. \quad . \quad . \quad . \quad . \quad (5)$$

The elimination of ζ between (3) and (5) then gives, after some reduction,

$$z=-\iota\frac{1+e^{-w}\sin\frac{\alpha}{2}}{1+e^w\sin\frac{\alpha}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

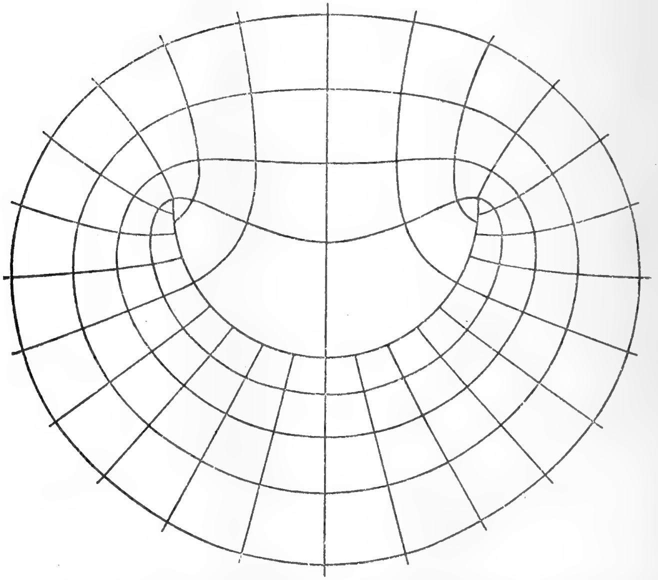
$$w = -\log\left[\frac{1}{2}\iota\,\mathrm{cosec}\,\frac{\alpha}{2}\left\{(z+\iota)+\sqrt{z^2+2\iota z\cos\alpha-1}\right\}\right]. \quad (7)$$

With regard to the branch of the two-valued function to be taken, the arc may be considered as the branch line joining the two branch points (its extremities) and that branch is to be taken which, *inter alia,* is equal to $+\iota$ when $z=0$, and tends to $+z$ as $z\rightarrow\infty$.

Equations (6) and (7) then give the potential due to the charged arc. Separating the real and imaginary parts of (6), after writing $-(\phi+\iota\psi)$ for w (since the real part is to be negative) we obtain

$$\left.\begin{aligned} x &= \frac{2\sin\frac{\alpha}{2}\cosh\phi\sin\psi+\sin^2\frac{\alpha}{2}\sin 2\psi}{1+2e^{-\phi}\sin\frac{\alpha}{2}\cos\psi+e^{-2\phi}\sin^2\frac{\alpha}{2}} \\ y &= -\frac{1+2\sin\frac{\alpha}{2}\cosh\phi\cos\psi+\sin^2\frac{\alpha}{2}\cos 2\psi}{1+2e^{-\phi}\sin\frac{\alpha}{2}\cos\psi+e^{-2\phi}\sin^2\frac{\alpha}{2}} \end{aligned}\right\} \quad . \quad . \quad (8)$$

Fig. 1.



Using (8), the equipotentials and lines of force are easily plotted. In fig. 1 they are drawn for the case of a semicircle, with intervals ·25 of ϕ, and $\frac{\pi}{12}$ of ψ. These show distinctly the physical features, which of course were anticipated in a general way. The charge resides for the most part on the outer face, the surface-density becomes infinite at the edges, and at a distance the equipotentials are approximately circular. One point of some interest appears in the figure, namely, the approximate uniformity of the field on the axis, just beyond the centre.

If we expand (6) on the supposition that $|w|$ (and consequently $|z|$) is large, we obtain

$$z \sim -\iota e^{-w}\left(\sin\frac{\alpha}{2} + e^{w}\cos^2\frac{\alpha}{2}\right),$$

i. e..

$$z + \iota\cos^2\frac{\alpha}{2} \sim -\iota e^{-w}\sin\frac{\alpha}{2}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

showing that the field at a great distance is approximately that due to an equal charge at the point $-\iota\cos^2\frac{\alpha}{2}$. It is seen that this point, the "centre of charge," is the mid point of that portion of the central radius cut off by the chord of the arc. This result may also be obtained by integration from the expressions for the surface-density now to be obtained.

§ 4. On the lamina, $w = \iota\psi$, while $z = -\iota e^{\iota\theta}$, and from (6) or (7) we obtain the relation

$$\sin\left(\psi - \frac{\theta}{2}\right)\sin\frac{\alpha}{2} = \sin\frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad (10)$$

Now the surface-density is proportional to $\frac{d\psi}{d\theta}$, and so

$$\left.\begin{aligned} \sigma &\propto \left(\frac{\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} + 1\right) \text{convex face,} \\ \sigma &\propto \left(\frac{\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} - 1\right) \text{concave face.} \end{aligned}\right\}. \quad (11)$$

The values of σ, multiplied by the factor $\sin\frac{\alpha}{2}$ (which makes the mean of the values at the pole equal to 1, and so facilitates comparison), are tabulated for the values 10°, 30°, 90°, and 170°, of α, for the central point and for four more dividing the semi-arc into five equal parts. The fractions of charge on the two faces are added, and also the values for a plane lamina. The third column in each case gives the mean of the first two, and even for $\alpha=30°$ shows no great difference from the values for the plane.

Table of Surface Densities.

Plane.	$\alpha=10°$.			$\alpha=30°$.			$\alpha=90°$.			$\alpha=170°$.		
1·000	·913	1·087	1·000	·741	1·259	1·000	·293	1·707	1·000	·0038	1·9962	1·0000
1·020	·933	1·107	1·020	·761	1·279	1·020	·305	1·720	1·013	·0042	1·9962	1·0002
1·091	1·003	1·177	1·090	·828	1·346	1·087	·358	1·772	1·065	·0055	1·9966	1·0010
1·250	1·162	1·336	1·249	·981	1·500	1·240	·454	1·868	1·161	·0097	2·0017	1·0057
1·667	1·577	1·753	1·664	1·383	1·901	1·642	·749	2·163	1·456	·0281	2·0198	1·024
	·472	·528		·417	·583		·25	·75		·0278	·9732	

These may be compared with the values obtained by Lord Kelvin for the analogous problem of the spherical bowl.

§ 5. Of course, the above results may also be interpreted as applying to the motion of an incompressible fluid circulating round the lamina, but this is passed over as of relatively small importance. The more interesting problem of the flow past such a lamina when placed in a steady stream, or the motion in an incompressible fluid due to a motion of translation of the lamina, may be solved in a similar manner, by taking a double source in the ζ-plane. In view of the final hydrodynamical interpretation it will be convenient to make w purely real along the real axis and to choose the strength and inclination of the source, so that $w \rightarrow -ze^{-\iota\beta}$ as $z \rightarrow \infty$.

This gives the value of w as a function of ζ:—

$$w=-\frac{2\sin\frac{\alpha}{2}\left(1+\sin\frac{\alpha}{2}\right)}{\cos\frac{\alpha}{2}}\left\{\frac{e^{-\iota\beta}}{\zeta-\iota\dfrac{1+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}+\frac{e^{\iota\beta}}{\zeta+\iota\dfrac{1+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}\right\}$$

$$+2\sin\frac{\alpha}{2}\sin\beta, \quad (12)$$

where the last term has been added to give the final result a neater form. The elimination of ζ between (3) and (12) is rather lengthy, but not difficult. We obtain

$$\iota z w^2 + \iota w(z+\iota)(ze^{-\iota\beta} - \iota e^{\iota\beta}) + (z+\iota)^2 - \sin^2\frac{\alpha}{2}(ze^{-\iota\beta} + \iota e^{\iota\beta})^2 = 0, \quad (13)$$

from which we have as the value of w

$$w = -\frac{(z+\iota)(ze^{-\iota\beta} - \iota e^{\iota\beta}) + (ze^{-\iota\beta} + \iota e^{\iota\beta})\sqrt{(z^2 + 2\iota z\cos\alpha - 1)}}{2z}, \quad (14)$$

where the same branch of the multiform function is to be taken as in § 3 above. The real and imaginary parts of w give the velocity potential and stream function of the motion due to the disturbance of a uniform stream by the lamina, the undisturbed velocity of the stream being unity and its direction inclined at an angle β to the axis of x. The corresponding values for the motion of the lamina in a liquid at rest at infinity are obtained by adding $ze^{-\iota\beta}$ to the above value of w. Owing to the computation of square roots of complex quantities being a laborious process in practice, the stream-lines have not been plotted, though of course, (13) could be used if it were required to do so. In the above the radius of the arc, and the velocity, have been taken as unity. The only modifications to include the case of a velocity U and radius a are that in (14) a factor U must be introduced on the left-hand side, and z, w replaced by z/a, w/a respectively.

§ 6. To obtain the "impulse" of the second of the above motions, we have only to expand in terms of $1/z$ and take $-2\pi\rho$ times the coefficient of $1/z$ (ρ=density of liquid). The required coefficient is found to be

$$-\tfrac{1}{2}\{e^{\iota\beta}(1-\cos\alpha) - \tfrac{1}{2}\sin^2\alpha e^{-\iota\beta}\},$$

so for the components of impulse we have

$$\left.\begin{aligned} \mathrm{X} &= 2\pi\rho\cos\beta\left\{\sin^2\frac{\alpha}{2} - \frac{1}{4}\sin^2\alpha\right\} = 2\pi\rho\cos\beta\sin^4\frac{\alpha}{2}; \\ \mathrm{Y} &= 2\pi\rho\sin\beta\left\{\sin^2\frac{\alpha}{2} + \frac{1}{4}\sin^2\alpha\right\} = 2\pi\rho\sin\beta\sin^2\frac{\alpha}{2}\left(1+\cos^2\frac{\alpha}{2}\right). \end{aligned}\right\} \quad (15)$$

On introducing the velocity U and radius a, these become

$$\mathrm{X} = 2\pi\rho a^2\sin^4\frac{\alpha}{2}.\,\mathrm{U}\cos\beta, \quad \mathrm{X} = 2\pi\rho a^2\sin^2\frac{\alpha}{2}\left(1+\cos^2\frac{\alpha}{2}\right).\,\mathrm{U}\sin\beta. \quad (15')$$

We may proceed to the limiting case of a plane lamina of width $2b$ by making $a\to\infty$ and $\alpha\to 0$ in such a manner that $a\sin\alpha\to b$. We obtain

$$\mathrm{X}=0, \quad \mathrm{Y} = \pi\rho b^2\mathrm{U}\sin\beta, \quad . \quad . \quad . \quad . \quad (16)$$

a known result. Also by making $\alpha=\pi$, we have the case of a complete cylinder, supposed to be filled with liquid of the same density as surrounds it :—

$$X=2\pi\rho a^2 U\cos\beta,\quad Y=2\pi\rho a^2 U\sin\beta,\ .\ .\ \ (17)$$

another known result. For the kinetic energy, we easily derive from (15′)

$$T=\pi\rho a^2 U^2\left\{\cos^2\beta\sin^4\frac{\alpha}{2}+\sin^2\beta\sin^2\frac{\alpha}{2}\left(1+\cos^2\frac{\alpha}{2}\right)\right\}.\quad (18)$$

Equations (15′) and (18) show that the lamina behaves as if it had a mass depending on the direction of motion, compounded of masses m_x, m_y, the coefficients of $U\cos\beta$, $U\sin\beta$, respectively in (15′).

§ 7. Next to determine the resultant of the pressures on the lamina. Substituting $z=-\iota e^{\iota\theta}$ in (14), w is found to be real (as it should be) when $|\theta|<\alpha$, and so we have

$$\phi=-2\left\{\sin\frac{\theta}{2}\cos\left(\frac{\theta}{2}-\beta\right)\pm\sin\left(\frac{\theta}{2}-\beta\right)\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}\right\},\quad (19)$$

the ambiguous sign referring to opposite faces of the lamina, + to the concave side. The velocity q at any point is given by $-\dfrac{\partial\phi}{\partial\theta}$, so that

$$q=\left\{\cos(\theta-\beta)\pm\cos\left(\frac{\theta}{2}-\beta\right)\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}\right.$$
$$\left.\mp\frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\left(\frac{\theta}{2}-\beta\right)}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}}\right\}.\ .\ .\ .\quad (20)$$

But $p=\text{const}-\frac{1}{2}\rho q^2$, so denoting the excess of pressure on the concave face at any point Δp, we have

$$\frac{\Delta p}{\rho}=\frac{2\cos(\theta-\beta)\left\{\cos\left(\frac{\theta}{2}-\beta\right)\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\left(\frac{\theta}{2}-\beta\right)\right\}}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}}.$$
$$.\ .\ .\ .\ .\ .\quad (21)$$

Let the total thrust, which evidently must act through the centre, have components P_x, P_y, then

$$\left.\begin{aligned} P_x &= \int_{-\alpha}^{\alpha} \Delta p \sin\theta \, d\theta = \pi\rho \sin^2\alpha \cos\alpha \sin\beta \cos\beta, \quad . \quad . \quad . \quad . \quad . \quad . \\ P_y &= -\int_{-\alpha}^{\alpha} \Delta p \cos\theta \, d\theta = -\pi\rho \sin^2\alpha \left(\cos^2\beta \sin^2\frac{\alpha}{2} + \sin^2\beta \cos^2\frac{\alpha}{2} \right) \end{aligned}\right\} . \quad (22)$$

These results are apparently in contradiction with the known fact that the resultant of the pressures should reduce to a couple. The explanation is that the velocity at the edge becomes infinite, and consequently the pressure also, and though the thickness of the lamina is assumed to be infinitely small, yet considered as the limiting case of a thin lamina, with finite velocities and pressures, the resultant pressure at the end tends to a finite limit as the thickness decreases. To see this, consider the case of the flow round a semi-infinite plane, given by $w = Az^{\frac{1}{2}}$. Suppose the fluid inside one of the parabolic stream lines is solidified. The motion of the remaining fluid is unaffected, and when we calculate the resultant thrust exerted by it on the solid, we obtain a force $\frac{\pi}{4}\rho A^2$ parallel to the axis, tending to drag the solid further into the liquid. The magnitude of this force is independent of the particular stream-line selected. Turning now to the problem in hand, the flow in the vicinity of the edge is of the type just mentioned, for on substituting $e^{\iota\alpha}(-\iota + \zeta)$ for z, where ζ is small, we obtain from (14)

$$w = \text{const} - \sqrt{2 \sin\alpha} \, . \, \sin\left(\frac{\alpha}{2} - \beta\right) \zeta^{\frac{1}{2}} + \text{higher powers of } \zeta. \quad (23)$$

Hence there is a force along the tangent at the end, of magnitude $\frac{\pi}{2}\rho \sin\alpha \sin^2\left(\frac{\alpha}{2} - \beta\right)$, and consequently at the other end, one of magnitude $\frac{\pi}{2}\rho \sin\alpha \sin^2\left(\frac{\alpha}{2} + \beta\right)$. The resultant of these has components

$$\left.\begin{aligned} P_x' &= \frac{1}{2}\pi\rho \sin\alpha \left\{ \sin^2\left(\frac{\alpha}{2} - \beta\right) - \sin^2\left(\frac{\alpha}{2} + \beta\right) \right\} \cos\alpha, \\ &= -\pi\rho \sin^2\alpha \cos\alpha \sin\beta \cos\beta ; \\ P_y' &= \frac{1}{2}\pi\rho \sin\alpha \left\{ \sin^2\left(\frac{\alpha}{2} - \beta\right) + \sin^2\left(\frac{\alpha}{2} + \beta\right) \right\} \sin\alpha, \\ &= \pi\rho \sin^2\alpha \left\{ \cos^2\beta \sin^2\frac{\alpha}{2} + \sin^2\beta \cos^2\frac{\alpha}{2} \right\} ; \end{aligned}\right\} . \quad (24)$$

and passes through the point $-\iota \sec \alpha$. Now combining (22) and (24), the resultant of all the pressures is seen to reduce to a couple of moment $-\frac{1}{2}\pi\rho \sin^2 \alpha \sin 2\beta$.

Introducing the velocity and radius, we have

$$\text{Couple} = -\tfrac{1}{2}\pi\rho a^2 \mathrm{U}^2 \sin^2 \alpha \sin 2\beta. \quad . \quad . \quad (25)$$

This becomes zero in the case of the complete cylinder, $\alpha = \pi$, and reduces, in the case of a plane, to

$$-\tfrac{1}{2}\pi\rho b^2 \mathrm{U}^2 \sin 2\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which are known to be correct results.

§ 8. In (23) above, it follows that if $\beta = \frac{\alpha}{2}$, there is no term in $\zeta^{\frac{1}{2}}$. This corresponds to the case in which the stream-line $\psi = 0$ divides at the edge of the lamina. If this may be applied to the case of an aeroplane wing, it is seen to lead to the conclusion that the entering edge should point downwards, at an angle equal to the angle of attack of the chord of the (cambered) plane. Observation and photographs would seem to show that in practice such a condition has been found most favourable.

P.S.—Since the majority of the above results were obtained, a paper has appeared (Dr. J. G. Leathem, Phil. Mag. (6) xxxv. Jan. 1918) in which it is shown by a general method, that (14) above is deducible from (6), and, in fact, when applied, does give the result (14) obtained independently above.

Loughborough,
Feb. 5th, 1918.

XLVI. *Molecular Frequency and Molecular Number.* Part II. *The Frequency of the Longer Residual Rays.* *By* H. Stanley Allen, *M.A., D.Sc., University of London, King's College*[*].

A METHOD of studying the frequency of vibration of the atoms in compounds is afforded by the "residual rays" obtained by repeated reflexions from the surfaces of solids, and studied by Rubens and his collaborators. It may be assumed that the frequency of such infra-red radiation corresponds with the frequency of vibration of an electrically charged ion, and, in certain cases at least, it is to be expected that the ion in question may be identified with one of the

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[*] Communicated by the Author.

atoms contained in the compound. The writer has shown* that the characteristic frequency, ν, of the atom of a solid element is connected with the atomic number N by the relation

$$N\nu = n\nu_A \quad \text{or} \quad N\nu = (n+\tfrac{1}{2})\nu_A,$$

where n is an integer, and ν_A a constant frequency having a value not far from 21×10^{12} sec.$^{-1}$. Seeing that the forces which control the vibrations of the atoms in solids are the same whether it be a question of the specific heat of the solid or the reflexion of radiation, it might be anticipated that the same relation would hold in connexion with the residual rays.

According to the theory developed by Nernst † the heat energy of a compound in the solid state is made up of the energy due to the motion of the molecules relative to one another and that due to the vibrations of the atoms in the molecule. The first contribution is calculated by the formula of Debye, the second by the formula of Einstein. Each calculation involves a knowledge of the corresponding characteristic frequency; the first, ν_1, is given, approximately at least, by Lindemann's formula; the second, ν_2, is found from the residual rays of Rubens. Nernst has pointed out that in the case of KCl and NaCl, the value of ν_1 is almost 0·75 times the value of ν_2; and in other cases, although the molecular frequency is slower than the internal frequency, the two are not very different. Indeed, in his earlier work Nernst obtained fairly good agreement by calculating the values of the atomic heat on the assumption of only one frequency, ν. The writer has shown‡ that the molecular frequency is connected with the "molecular number" by a relation of the same form as that already quoted for the atomic frequency and the atomic number. The collective results suggest that the internal frequency may conform to the same relation. In the following pages the experimental results are examined for a number of compounds, and it is found that there is a strong presumption in favour of the establishment of the relation suggested.

NaCl. In the case of rock salt§ two rays of different strengths were observed, a stronger at $53·6\,\mu$ and a weaker at $46·9\,\mu$. The corresponding frequencies are $5·60\times10^{12}$ and $6·40\times10^{12}$ sec.$^{-1}$. Assuming the former

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* H. S. Allen, Roy. Soc. Proc. vol. xciv. p. 100 (1917); Phil. Mag. vol. xxxiv. p. 478 (1917).

† Nernst, 'The Theory of the Solid State' (London, 1914).

‡ H. S. Allen, Phil. Mag. vol. xxxv., April 1918.

§ Rubens and Hollnagel, *Preuss. Akad. Berlin*, vol. iv. p. 26 (1910).

to correspond with the atom of sodium ($N=11$), and the latter with the atom of chlorine ($N=17$), we find for the sodium atom $N\nu=3\times 20{\cdot}5\times 10^{12}$ and for the chlorine atom $N\nu=5\times 21{\cdot}7\times 10^{12}$. If, however, we associate the chlorine atom with the stronger line and the sodium atom with the weaker, we obtain for the products $95{\cdot}1\times 10^{12}$ and $70{\cdot}4\times 10^{12}$, which are not integral multiples of 21×10^{12}.

KCl. For sylvin† similar results are found, but in this case the stronger line ($62{\cdot}0\mu$ or $4{\cdot}84\times 10^{12}$ sec.$^{-1}$) must be associated with chlorine, the weaker ($70{\cdot}3\mu$ or $4{\cdot}27\times 10^{12}$ sec.$^{-1}$) with potassium ($N=19$). In this case we find $N\nu=4\times 20{\cdot}6\times 10^{12}$ and $N\nu=4\times 20{\cdot}3\times 10^{12}$ respectively. The products obtained when the atoms are interchanged are $91{\cdot}9\times 10^{12}$ and $72{\cdot}6\times 10^{12}$, which are not integral multiples of 21×10^{12}.

KBr. The stronger line* is at $86{\cdot}5\mu$ ($\nu=3{\cdot}47\times 10^{12}$ sec^{-1}), the weaker at $75{\cdot}6\mu$ ($\nu=3{\cdot}97\times 10^{12}$ sec.$^{-1}$). The former must be associated with the potassium atom, giving $N\nu=3\times 21{\cdot}9\times 10^{12}$, the latter with the bromine atom, giving $N\nu=6\frac{1}{2}\times 21{\cdot}4\times 10^{12}$. The occurrence of $\frac{1}{2}$ in the latter case is noteworthy, as it is supported by independent evidence in connexion with specific heats.

It may be noted that for these three compounds the stronger line is associated with the element of smaller atomic number.

KI. The residual radiation has not been separated into two rays of different strengths, but it is stated by the experimenters † that such a constitution is possible. In such a case we may call ν_1 and ν_2 the frequencies corresponding to elements having atomic numbers N_1 and N_2, and put

$$N_1\nu_1=n_1\nu_A, \qquad N_2\nu_2=n_2\nu_A.$$

By addition,

$$N_1\nu_1+N_2\nu_2=(n_1+n_2)\nu_A.$$

But if ν_1 and ν_2 are not very different, we may replace either frequency by the arithmetic mean $\frac{1}{2}(\nu_1+\nu_2)$, and so obtain

$$(N_1+N_2)\frac{\nu_1+\nu_2}{2}\doteqdot(n_1+n_2)\nu_A,$$

The observed wave-length for KI is $96{\cdot}7\mu$, so that we have $\frac{1}{2}(\nu_1+\nu_2)=3{\cdot}10\times 10^{12}$ sec.$^{-1}$.

For N_1+N_2 we may substitute N the "molecular number." Taking $N_1=19$ and $N_2=53$, the product is found to be $11\times 20{\cdot}3\times 10^{12}$, which is in satisfactory agreement with the suggested relation.

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* Rubens and Hollnagel, *loc. cit.*

† Rubens and Hollnagel, *loc. cit.*

AgCl. The average wave-length of the residual radiation* was found to be $81{\cdot}5\,\mu$ ($\nu=3{\cdot}68\times10^{12}$). The value of $(N_1+N_2)\nu=11\times21{\cdot}5\times10^{12}$.

$PbCl_2$. The average wave-length in this case* was $91{\cdot}0\mu$ ($\nu=3{\cdot}30\times10^{12}$), giving $(N_1+2N_2)\nu=18\times21{\cdot}2\times10^{12}$. When such large integers are involved, it is always possible to find an integer which will give a concordant value for ν_A.

HgCl. The average wave-length observed* was $98{\cdot}8\mu$ ($\nu=3{\cdot}04\times10^{12}$). Taking $N_1=80$, $N_2=17$, we find

$$(N_1+N_2)\nu=14\times21{\cdot}0\times10^{12}.$$

AgBr. For this salt* $\lambda=112{\cdot}7\mu$ ($\nu=2{\cdot}66\times10^{12}$), and $(N_1+N_2)\nu=10\frac{1}{2}\times20{\cdot}8\times10^{12}$.

$CaCO_3$. The case of calc spar† is interesting as it contains three elements, whilst only two bands have been recorded, a strong band at $93{\cdot}0\mu$ ($\nu=3{\cdot}23\times10^{12}$) and a weaker at $116{\cdot}1\mu$ ($\nu=2{\cdot}58\times10^{12}$). It is found that the strong band must be assigned to calcium, giving $N\nu=3\times21{\cdot}5\times10^{12}$, the weaker to oxygen, giving $N\nu=1\times20{\cdot}7\times10^{12}$.

In a later paper Rubens and Wartenberg‡ have given the wave-lengths for two ammonium salts and three compounds of thallium.

NH_4Cl. The observed wave-lengths are very nearly the same as those for NaCl, being $54{\cdot}0\mu$ ($\nu=5{\cdot}56\times10^{12}$) and $46{\cdot}3\mu$ ($\nu=6{\cdot}48\times10^{12}$) respectively. This is what might be expected if the group NH_4 be regarded as a compound radicle replacing Na, for the "molecular number" of ammonium $(7+4)$ is the same as the atomic number of sodium (11). Taking the first line as associated with the NH_4 group, $N\nu=3\times20{\cdot}4\times10^{12}$, whilst the second line gives for the chlorine atom $N\nu=5\times22{\cdot}0\times10^{12}$.

NH_4Br. Here again two wave-lengths have been observed, $62{\cdot}3\mu$ ($\nu=4{\cdot}82\times10^{12}$) and $55{\cdot}3\mu$ ($\nu=5{\cdot}42\times10^{12}$). The former line must be associated with bromine, giving $N\nu=8\times21{\cdot}1\times10^{12}$; the latter with the ammonium group, yielding $N\nu=3\times20{\cdot}0\times10^{12}$.

TlCl. The mean wave-length recorded is $91{\cdot}6\mu$ ($\nu=3{\cdot}27\times10^{12}$). The atomic number of thallium being 81 and that of chlorine 17, we find $N\nu=15\times21{\cdot}4\times10^{12}$.

TlBr. For the bromide the mean wave-length is $117{\cdot}0\mu$ ($\nu=2{\cdot}56\times10^{12}$). In this case we find $N\nu=14\times21{\cdot}2\times10^{12}$.

TlI. For the iodide the mean wave-length is $151{\cdot}8\mu$ ($\nu=1{\cdot}98\times10^{12}$), giving for $N\nu$ the value $13\times20{\cdot}4\times10^{12}$.

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* Rubens, *Preuss. Akad. Berlin*, vol. xxviii. p. 513 (1913).

† Rubens, *D. P. G. V.* vol. xiii. p. 102 (1911).

‡ Rubens and Wartenberg, *Preuss. Akad. Berlin*, p. 169 (1914).

Thus for the halogen derivatives of thallium the interesting result is found that the "frequency numbers" (15, 14, 13) diminish by unity in passing from chloride to bromide and from bromide to iodide.

It must be stated that the suggestion which attributes the two absorption bands of NaCl, KCl, KBr to separate atoms is not a new one*. Nernst is of opinion that it was a mere coincidence—"a very curious and misleading one indeed"—that calculations of the specific heat on that supposition gave quite good results. Rubens was unable to find two bands in the case of AgCl and $PbCl_2$, and came to the conclusion that the two apparent bands were due simply to water vapour, which has a great number of absorption bands. If this conclusion be accepted the results quoted above will require modification in the sense that the "molecular number" must be employed instead of the atomic number; but the proposed relation will still hold good. Thus for NaCl we find $N\nu = 8 \times 20{\cdot}6 \times 10^{12}$, for KCl $N\nu = 8 \times 20{\cdot}1 \times 10^{12}$, and for KBr $N\nu = 9 \times 21{\cdot}6 \times 10^{12}$.

It would appear probable that the relation here discussed applies only in the case of the longer residual rays, having a wave-length greater than say 20μ. For shorter wave-lengths, corresponding to a higher frequency, the value of the product, $N\nu$, is so large that no real test of the proposed relation can be obtained. It is unfortunate for our present purpose that although the residual rays from quartz in the region of 9μ and 13μ have been measured with considerable accuracy, the longer waves have not as yet been determined accurately; we know only that quartz shows strong selective absorption for the region between 60 and 80μ.

It may be worthy of mention that water vapour has an absorption band at $14{\cdot}3\mu$, which is the wave-length corresponding to a frequency $21{\cdot}0 \times 10^{12}\,\text{sec.}^{-1}$, that is the frequency here denoted by ν_A. Further, the vapour of carbon dioxide has an absorption band at $14{\cdot}1\mu$. It would be of interest to know whether other substances show absorption in the same region.

The possibility of deducing the wave-length of the infra-red radiation from the elastic properties of the solid has been discussed by Madelung and by Sutherland. By considering a cubical space-lattice the former obtained for the wave-length the expression

$$\lambda = C_1 \sqrt{\left(\frac{M_1 M_2}{(M_1 + M_2)^{\frac{4}{3}}} K \sqrt[3]{D}\right)},$$

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* *Cf.* Nernst, 'The Theory of the Solid State,' p. 80 (1914).

where M_1, M_2 are the masses of the atoms, K is the compressibility, and D the density. This may be compared with Einstein's formula. Rubens and Wartenberg have shown that this formula gives results in moderately good agreement with their observations.

They have also obtained fair agreement by employing a modified form of the equation of Lindemann, viz.

$$\lambda = C_2\sqrt{\left(\frac{M_1 M_2}{M_1+M_2}\frac{V^{\frac{2}{3}}}{T_s}\right)},$$

where V is the molecular volume and T_s the melting-point.

In both formulæ the constant must be determined empirically.

The results of this and preceding papers support the following conclusions:—

(1) The forces binding the atoms in the molecule are similar in character to those which bind the molecules of the solid, that is the forces of chemical affinity are of the same nature as the forces of molecular cohesion*.

(2) There must be something of a discrete character in the nature of these forces, in order to account for the occurrence of integral values of n. The simplest hypothesis is to assume that the forces arise from the presence of valency electrons. As it is probable that these forces act only in definite directions, it is a plausible suggestion that the linkages between the atoms are constituted by Faraday tubes of force, which would then be regarded as physical entities. The fundamental frequencies, ν_A and ν_E, would depend on the properties of the unit tube of force. It has been pointed out by Prof. Nicholson† that such a view seems to be required in order to explain the relations between the frequencies of spectral series. Attention may also be drawn to an important article by Sir J. J. Thomson on the Forces between Atoms and Chemical Affinity‡, in which chemical valency is discussed from the same standpoint.

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* Compare Nernst, 'The Theory of the Solid State,' pp. 4–9 (1914); Langmuir, Ann. Chem. Soc. Journ. vol. xxxviii. p. 2221 (1916).

† Nicholson, Phil. Mag. vol. xxvii. p. 541, vol. xxviii. p. 90 (1914).

‡ J. J. Thomson, Phil. Mag. vol. xxvii. p. 757 (1914).

XLVII. *On Wood's Criticism of Wien's Distribution Law.*

By HAROLD JEFFREYS, *M.A., D.Sc.*[*]

IN the February number of the Philosophical Magazine, pp. 190–203, Mr. F. E. Wood offers a criticism of Wien's law of the distribution of energy in the spectrum of a radiating gas. If λ denote the wave-length and θ the absolute temperature, Wien's law is that the energy of the part of the radiation with wave-lengths between λ and $\lambda+d\lambda$ is

$$\phi(\lambda,\theta)d\lambda=\frac{C}{\lambda^5}e^{-k/\lambda\theta}d\lambda,$$

where C and k are constants. Wood shows from the same assumptions as Wien that the omission of a factor by Wien in his first equation led to an error in the final result and that the correct formula based on these hypotheses is

$$\phi(\lambda,\theta)d\lambda=\frac{C}{\lambda^{\frac{13}{2}}}e^{-k/\lambda\theta}d\lambda.$$

With reference to this law it must be noted that the total radiation is obtained by integrating $\phi d\lambda$ from zero to infinity, and as this must by Stefan's law be proportional to θ^4, it can easily be shown that $Ck^{-\frac{11}{2}}$ in Wood's equation is not a constant, but is proportional to $\theta^{-\frac{3}{2}}$. As Wood's argument is involved and in places somewhat obscure[†], I offer an alternative proof which is shorter and appears to be equally satisfactory subject to substantially the same assumptions.

If N be the total number of molecules in a mass of gas, the number whose absolute velocities lie between v and $v+dv$ is

$$dN=4N\pi^{-\frac{1}{2}}\alpha^{-3}v^2e^{-v^2/\alpha^2}dv, \quad . \quad . \quad . \quad . \quad (1)$$

where α is a velocity whose square is proportional to the temperature. Then Wien's first assumption is that the wave-length and intensity of the radiation emitted by a molecule depends on v alone. If ϵ be the rate of emission

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[*] Communicated by the Author.

[†] The argument at the foot of p. 197 and the top of the next page implies that the number of molecules with velocities between v and $v+dv$ is $kv^2\theta^{-\frac{3}{2}}e^{-lv^2/\theta}dv^2$ instead of $kv^2\theta^{-\frac{3}{2}}e^{-lv^2/\theta}dv$.

The introduction of an ideal type of gas on p. 198, with the notion of corresponding velocities, is unnecessary; so is the use of the meaningless law $\lambda\theta=$constant, which requires such careful interpretation that it is more difficult to apply than the second and third hypotheses of the present paper, expressed in equations (5) and (14), which are equivalent to it.

of energy by a molecule of velocity v, that emitted by all the molecules whose velocities lie between v and $v+dv$ is

$$dR=\phi(\lambda,\ \theta)d\lambda=4N\pi^{-\frac{1}{2}}\alpha^{-3}v^2\epsilon\, e^{-v^2/\alpha^2}dv, \quad . \quad . \quad (2)$$

where ϵ is a function of v only.

Now the total radiation for all wave-lengths together is obtained by integrating this from $v=0$ to $v=\infty$, and must by Stefan's law be proportional to θ^4, and therefore to α^8. We have thus an integral equation for ϵ. Assume that $\epsilon(v)$ can be expanded in powers of v for all values of v, so that

$$\epsilon(v)=\sum_0^\infty a_n v^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and that the series $\sum\limits_0^\infty a_n v^{n+2} e^{-v^2/\alpha^2}$ can be integrated term by term.

Then R can be expressed as a power series in α, thus:

$$\frac{R\pi^{\frac{1}{2}}}{4N}=\sum_{\kappa=1}^{\infty}\tfrac{1}{2}\kappa\,!\ a_{2\kappa-1}\alpha^{2\kappa-1}+\sum_{\kappa=1}^{\infty}\frac{1\,.\,3\ldots(2\kappa-1)}{2^{\kappa+1}}\pi^{\frac{1}{2}}a_{2\kappa-2}\alpha^{2\kappa-2}.$$

By hypothesis R is equal to $Nc^2\alpha^8$, where c is an absolute constant, and therefore all the a's are zero except a_8. Hence

$$Nc^2=\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2^4}Na_8,$$

and therefore

$$dR=\frac{64\pi^{-\frac{1}{2}}}{945}Nc^2\alpha^{-3}v^{10}e^{-v^2/\alpha^2}dv. \quad . \quad . \quad . \quad (4)$$

Thus the distribution of energy with regard to the molecular velocities is completely found from the first assumption alone, and all that is now required is to determine the relation of the velocity to the wave-length.

At this stage a further assumption is needed; and this will be that the graph of ϕ against λ for any temperature can be derived from that for any other temperature by homogeneous strain in two dimensions. In other words, if the temperature be changed from θ to $k^2\theta$, or, what is the same thing, if α be replaced by $k\alpha$, two numbers a and b will exist, so that

$$\phi(a\lambda,\ k^2\theta)=b\phi(\lambda,\ \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for all values of λ, where a and b are functions of k alone.

Now it is evident from (4), by putting $v^2=f(\lambda)$, that $\phi(\lambda,\ \theta)$ is of the form

$$\alpha^{-3}F(\lambda)e^{-f(\lambda)/\alpha^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where F and f are at present unknown, but are connected by a differential relation.

Hence we must have, by (5),

$$\frac{bk^3\mathrm{F}(\lambda)}{\mathrm{F}(a\lambda)}\exp-\frac{1}{\alpha^2}\left\{f(\lambda)-\frac{1}{k^2}f(a\lambda)\right\}\equiv 1. \quad . \quad (7)$$

Now k, a, b and λ are perfectly independent of α, and therefore this transformation cannot be possible for other values of the temperature unless

$$f(a\lambda)=k^2f(\lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

for all values of λ.

Hence $$f(aa'\lambda)=k^2(a')f(a\lambda)=k^2(a')k^2(a)f(\lambda),$$

and also $$=k^2(aa')f(\lambda).$$

Therefore $k(aa')=k(a)k(a')$, and thence it is easily shown that $k(a)$ must be equal to $a\log k(e)$, where e is the base of the Napierian logarithms, and $k(e)$ is an unknown constant. Log $k(e)$ may therefore be put equal to a further constant h.

Therefore $$f(a\lambda)=a^{2h}f(\lambda), \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and $$\frac{f(a\lambda)}{(a\lambda)^{2h}}=\frac{f(\lambda)}{\lambda^{2h}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (10)$$

whatever a may be, showing that

$f(\lambda)/\lambda^{2h}$ is an absolute constant.

Hence $$v \text{ is proportional to } \lambda^h, \quad . \quad . \quad . \quad . \quad (11)$$

making

$$\alpha^{-3}\mathrm{F}(\lambda)=\frac{64\pi^{-\frac{1}{2}}}{945}\mathrm{N}c^2\alpha^{-3}v^{10}\frac{dv}{d\lambda}$$

$$=\mathrm{N}\theta^{-\frac{3}{2}}g\lambda^{11h-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where g is another absolute constant.

Altogether

$$\phi(\lambda,\theta)=\mathrm{N}g\theta^{-\frac{3}{2}}\lambda^{11h-1}e^{-l\lambda^{2h}/\theta}, \quad . \quad . \quad . \quad . \quad (13)$$

where g, h, and l are unknown constants.

Wien introduces the further assumption that the number a is equal to k^{-2}, making

$$h=-\tfrac{1}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and v^2 proportional to $1/\lambda$. In this case the distribution law reduces to Wood's form as modified in the first paragraph of this paper, namely

$$\phi(\lambda,\theta)=\mathrm{N}g\theta^{-\frac{3}{2}}\lambda^{-\frac{13}{2}}e^{-l/\lambda\theta}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

XLVIII. *The Resolution of Mixed Colours by Differential Visual Diffusivity.* *By* HERBERT E. IVES, *Ph.D., Captain, Sig. R.C., U.S.A.*[*]

1. Introduction.
2. Experimental procedure.
3. Results with purple light.
4. Results with monochromatic and compound yellow.
5. Discussion.

1. *Introduction.*

UNDER the title "Visual Diffusivity," the writer described some time ago[†] experiments on the lagging of different colour impressions relatively to each other. These were predicted on the basis of a theory developed to explain the behaviour of the flicker photometer[‡]. An essential part of this theory was the ascription to each colour impression of a characteristic rate of transmission along the visual channel. In developing the theory it was assumed that each colour acted quite independently.

In two instances it appeared necessary to modify this last assumption. One was the case of alternated colours exposed for unequal lengths of time. Here the flicker-photometer theory apparently called for much greater effects of dissymmetry than experiment showed. The other was the failure of an attempt to resolve a purple into its constituents by moving it across the field of view. It was concluded in the light of these experiments that some mutual action of colours in the act of transmission must take place.

Later work on the flicker-photometer theory showed that when the complete equations were employed, dispensing with the earlier assumption that the effect of varying speed could be neglected, the case of unequal exposures did not require the assumption that the individual colours affect each other in transmission. There remained then to be explained the failure to resolve purple in the manner described. In view of the very satisfactory quantitative success of the flicker-photometer theory as demonstrated by the more recent work with the polarization design, it appeared highly desirable to

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[*] Communicated by the Author.

[†] "Visual Diffusivity," Ives, Phil. Mag. Jan. 1917, p. 18.

[‡] "Theory of the Flicker Photometer," Ives and Kingsbury, Phil. Mag. Nov. 1914, p. 708, and April 1916, p. 290. "A Polarization Flicker Photometer and some Data of Theoretical Bearing obtained with it," Ives, Phil. Mag. April 1917, p. 360. "Hue Difference and Flicker Photometer Speed," Ives, Phil. Mag. August 1917, p. 99.

go more fully into the question of visual diffusivity with mixed colours. The outcome of this further study has been to establish the experimental conditions pre-requisite to the detection of the resolution of mixed colours by the different visual diffusivity of the components. With the proper conditions, not only has it been possible to resolve purple into red and blue, but observations have been made on both monochromatic and compound yellows, with results of interest in colour-vision theory.

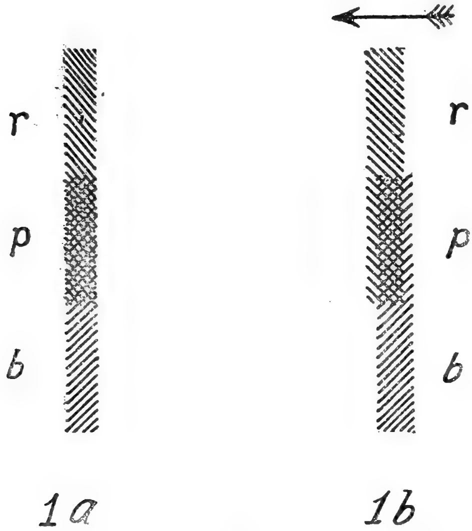
2. *Experimental procedure.*

In making the experiments recorded in the paper on "Visual Diffusivity," full use was made of the fact that the eye is very sensitive to a break in the continuity of a straight line. The two coloured strips under observation were set slightly out of line, and then passed before the eye at a speed such that they appeared to form a continuous instead of a broken line. In attempting to resolve a purple into red and blue this peculiarity of the eye was not taken advantage of. A purple strip was moved across the visual field and evidences of widening or duplication were looked for. To this the eye is much less sensitive.

On reconsidering the problem recently it was realized that the criterion of linear continuity should, if possible, be used to judge of a resolution of the kind expected. The arrangement of coloured areas finally decided upon was that of a continuous slit—red at one end, blue at the other, with purple in the middle, the purple being the mixture of the two end colours. This arrangement is shown in fig. 1 *a*. When this strip was moved across the field it was hoped that not only would the red and blue be displaced, in agreement with the previous experiments, but that it would be evident that the displacements overlapped, in the manner shown in fig. 1 *b*. That is, instead of looking for the duplicity of the portion *p*, which could not be detected previously, attention would be directed to a possible widening which would continue the straight line of both parts, *r* and *b*.

The production of such an overlapping red and blue strip, which could be placed in the disk apparatus formerly used, was first attempted by photographs on an autochrome plate, but the purity of the resultant colours was not satisfactory. The final apparatus used was a simple colour-projection scheme, the requisite motion of the colour patches being provided for by the movement of a mirror before the eye. A plan of it is shown in fig. 2, where *a* is a box containing two point-source tungsten lamps, controlled by separate resistances *b*. The light from these lamps passes through diaphragms *c* and coloured glasses *d* to the matte white screen *e*, which is viewed through the slit *f* by the eye at *h*, after being reflected from the mirror *g*, arranged to rotate about a vertical axis. At *j* are movable screens of short vertical length, which can be so placed that either top or bottom of the screen *e* is illuminated by one colour of light alone. With this arrangement it is possible to have the slit *f* appear of either colour given by the glasses at *d*, half of one colour and half of the other, or else part one colour, part the other colour, and part the mixture of the two as in fig. 1 *a*.

Fig. 1.



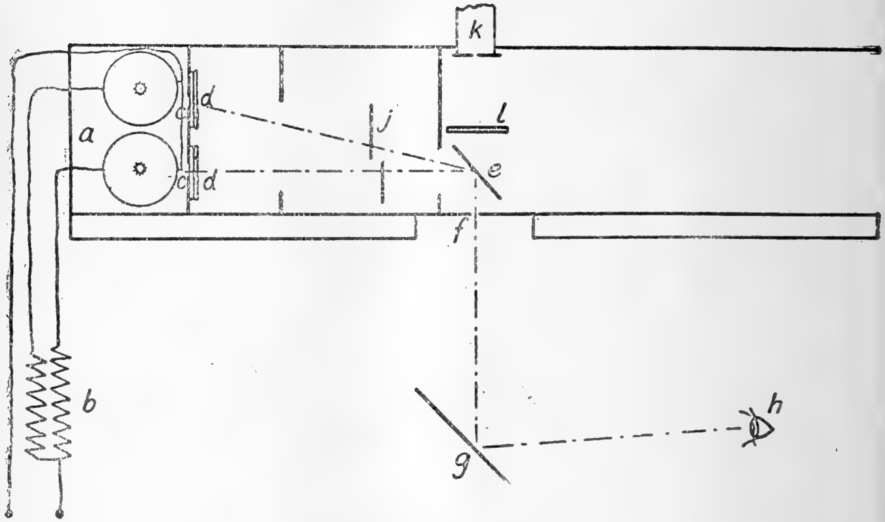
1 *a*. Stationary slit coloured red (*r*) at one end, blue (*b*) at the other, and the purple formed by their mixture (*p*) in the middle.
1 *b*. Appearance of slit when in motion in the direction of the arrow.

3. *Results with purple light.*

Four possible behaviours of the purple mixture region of the strip shown in fig. 1 *a* may be imagined. On lateral movement either (1) the purple may move bodily as the red constituent, or (2) as the blue constituent, or (3) it may exhibit an intermediate shift, or (4) it may spread out into red and blue. Either (1), (2), or (3) would be possible interpretations of the previous trial.

On making the experiment with the new arrangement, it was found that the appearance presented was unmistakably that of fig.1 *b*—that is, *the mixed colour when passed across the field of vision was resolved into its components, just as it would have been if viewed through a weak prism.* By careful adjustment of the intensities the effect was made so clear-cut as to be immediately apparent, and was verified by a number of observers. This effect is shown very strikingly if the slit image is oscillated by the proper mirror movement. In this case the two colours appear to slide over each other at their overlapping portion. The appearance here is as though a purple slit of moderate length were viewed through a prism set with its edge at right angles to the direction of the slit, and then rotated back and forth with the line of sight as an axis.

Fig. 2.



Arrangement of Apparatus.

a, box containing two point-source tungsten lamps.
b, resistance for controlling current through lamps.
c, *c*, diaphragms.
d, *d*, coloured glasses.
e, matte white surface.
f, slit.
g, mirror.
h, observer's eye.
j, movable screens of short vertical length.
k, monochromatic illuminator.
l, opal glass.

It appears from this that we have here *a new method of colour analysis*, dependent on the properties of the eye and not on those of an inanimate physical instrument. It is of interest to apply this new method to other mixed colours—in particular to yellow.

4. *Results with monochromatic and compound yellow.*

Yellow is of peculiar interest in the study of colour vision because the same sensation may be produced in two different ways. One is by the pure yellow of the spectrum, the other is by the mixture of red and green light. Unlike the blue-green of the spectrum, which may be matched similarly by a mixture of the colours to either side, yellow gives a sensation totally distinct in character from its components. So definitely is this so that the psychological elementary colours, not recognizable as mixtures, are red, *yellow*, green, and blue. MacDougall, Schenck, and others consider it probable that the red and green sensations have been developed from a more primitive yellow sensation, the latter one of the two sensations (warm and cold), into which primitive monochromatic vision first separated.

Trial by this new method (provided it proved sensitive enough to give definite results) offered the possibility of answering the following questions, suggested by the nature of yellow light, as just discussed:—

(1) Does a mixed yellow become resolved into its constituent red and green?

(2) Does a pure yellow become resolved into a red and a green?

If the answer to both these questions is affirmative, it might be interpreted as meaning that the action of both kinds of yellow light is to break down two substances, red and green, at the surface of incidence, which then travel back at different speeds to a point where the double product either combines into a yellow substance or is interpreted as yellow.

If the answer to both these questions is negative, it might be interpreted as meaning that either kind of yellow light breaks down a yellow substance, which travels back with its own individual velocity, *i. e.*, that colour fusion takes place at the surface of incidence. A third possible answer—namely, that the mixed colour is resolved and the pure is not—might be interpreted to mean that distinct red, yellow, and green substances are broken down by the light, travel back with their appropriate velocities, and that red and green fusion occurs after this transmission. The answer to these questions can be obtained only by experiment.

For the yellow light tests the apparatus as above described was provided with red and green glasses of narrow spectral transmissions for the mixed yellow, and a monochromatic illuminator *k* was added, illuminating an opal glass *l*, which could be viewed through an opening in the screen *e*. By this means either red light or green light or their mixture, or a monochromatic yellow matching the mixed yellow may be obtained, disposed in any desired way along the slit *f*.

The slit was first arranged to show half red and half green, and the intensities adjusted until on movement of the mirror the lag of green behind red was well shown. This occurs at a rather low intensity, and care must be taken to have absolutely no stray light in the room. (In these experiments the slit *f* was placed over a hole in a door between two rooms, so that an assistant could make the necessary manipulations, without any stray light reaching the observer.) When this condition was found the screens *j* were placed so that the middle of the slit showed the compound yellow. *This compound yellow was clearly resolved* by lateral motion across the field of view. In fact, the resolution was even more satisfactory than it had been with purple, because the two edges of the centre patch were seen on several occasions to be red and green. This greater success is probably due to the very distinct difference in appearance between yellow and its constituents.

The next point taken up was the behaviour of the pure yellow, adjusted to be a subjective match with the compound yellow, and arranged to exactly take its place between the red and green. It was at once apparent that *pure yellow does not separate into red and green.* This fact is strikingly shown by arranging the slit so as to be all compound yellow, except a small portion of pure yellow. When stationary the slit appears alike thoughout its whole length in brightness, hue, and definition. But upon moving the image sideways, or oscillating it, the compound yellow immediately broadens out and becomes ill-defined, the pure yellow remaining narrow and sharp. The appearance is identical with that produced by a weak prism, and again demonstrates that this phenomenon provides a new, if rough, method of spectrum analysis.

5. *Discussion.*

The results obtained with purple light are of chief interest as justifying the assumption made in the flicker-photometer theory, that each colour is transmitted with its own characteristic speed, irrespective of whether it occurs alone or with other colours. The results from the yellow light experiments are, however, of great interest from the standpoint of theories of colour vision.

If we consider these experiments without reference to any other work on colour vision, it is evident that the explanation called for is one in which wave-length composition rather than appearance or colour-sensation analysis determines behaviour. Recently Houstoun has suggested a theory of colour vision according to which colours do maintain their physical wave-length identity in transmission to the brain. He supposes the spectral colours cause retinal vibrators "to set up waves in the nerves and that the nerves carry these waves to the brain," the dominant wave-length of the transmitted disturbance being that of the incident light. The results of the present paper might be interpreted as supporting this theory. A place at which the new phenomena might at first sight appear to fit into Houstoun's theory may be noted as suggestive. If the writer understands the theory correctly, it calls for the production of *fluorescence* at the retina, which illuminates the brain through the *transparent* nerves (for the "waves in the nerves" must, because of their frequency, be light waves). Now it is one of the elementary facts of physical optics that a reduced rate of transmission of light waves occurs in a medium of high refractive index, and that a different rate of transmission of colours, in the order of wave-length, occurs in a medium possessing, as most transparent media do, an absorption-band in the ultra-violet. Thus the velocity of light in carbon bisulphide is less than in air, and red light travels faster than blue. If, then, to the transparent nerves, which appear to be a part of Houstoun's theory, we ascribe a high refractive index varying in the usual manner through the spectrum, we have a system which would give the differential speed effects found by the present experiments. Unfortunately the necessary refractive index to reduce the speed of light to the order of magnitude such that even a hundredth of a second lag of blue light would be possible in any distance available in the human head, is enormous and fantastic. Apart from this quantitative difficulty, the parallelism between the behaviour of light in the eye and in the highly dispersive medium is close. There is, however, no obvious explanation along this line of the variation of speed of transmission with the intensity of the stimulus.

Whether or not the visual diffusivity phenomena agree with the Houstoun theory, the latter requires, in the writer's opinion, much more support before it is profitable to attempt to harmonize new observations with it. The objections presented to it or any "wave-length" theory by the facts of colour mixture and colour blindness are very great. In particular, the production of white by various pairs of complementaries of different centres of gravity, and the neutral points in the spectra of the two types of red-green blind, present serious difficulty of interpretation.

A great deal of evidence points to the probability of vision being a process of photochemical decomposition, in which either the decomposition products or electric currents set up by the process pass along the nerve tracks at the speeds characteristic of nerve impulses. Apart from the facts of colour mixture and colour blindness there is no reason why the number of separate decomposable retinal substances should not be very large. There is nothing in the present experimental work to indicate that the number is restricted to the red, yellow, green, and blue experimented with, although the method is so crude as a means of spectrum analysis that it would be difficult to show conclusively that a much greater number of speeds of transmission than four are exhibited in the length of the visible spectrum. As the method stands, it is inferior as a means of colour analysis to the single prism and wide slit first employed to exhibit the spectrum. For an answer to the question of how many reacting substances of different velocities of transmission exist, the resolving power should be more nearly that required to show the Fraunhofer lines. What is required to effect an improvement comparable with the instrumental advance quoted is some drug which will slow down the rate of transmission of the visual impressions, or the "time machine" of Mr. H. G. Wells.

Actually the mixture phenomena point to the existence of three, and only three, such reacting substances—a red, a green, and a blue. The question of paramount interest then is, how can the different behaviour of the spectrally pure yellow and its subjectively equivalent mixed yellow be harmonized with three-colour theory? It is evident that the "three different types of nerve fibre" postulated by Young, or the "three different, independent, and mutually unopposed elementary activities" in the statement of the theory by Helmholtz, must be located (if existent) at some distance along the conducting path between receiving surface and brain. By this new method of analysis we can actually see red and green being transmitted a certain distance at different speeds* before combining to make yellow, and we

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* Or possibly with the same speed for different distances. In this connexion see Koenig, "Ueber die lichtempfindliche Schicht in der Netzhaut des menschlichen Anges," *Ges. Abh.* p. 333.

can see yellow being similarly transmitted, as yellow. Red and green light acting together do not cause a yellow reaction or the breaking down of a "yellow substance" at the surface of incidence. Neither does yellow light cause at this surface the breaking down of a red substance and a green substance. *If a trichromatic mechanism is necessary, as it appears to be from the phenomena of colour mixture and colour blindness, then the present experiments call for the additional complication of an antecedent transmitting process where colours retain their physical (spectral) individuality.*

Along the lines of the Schenck modification of the Young-Helmholtz theory the diffusivity results might be interpreted as showing that the primitive yellow substance is still present in the retina, as well as the red and green substances which have developed from it. Only upon the red and green substances reaching a certain depth is their equivalence with the yellow substance established. It may be imagined that this isolation is due to their going along separate channels. This interpretation practically amounts to assuming four reacting substances in the retina. The fourth is superfluous from the standpoint of colour mixture, and so appears in contradiction to the bases of the three-colour theory, which the Schenck modification is not. This throws us back, if we hold to the three-colour theory, upon the above suggestion that the diffusivity phenomena take place before the trichromatic mechanism is reached. Unless, therefore, a theory appears which is in accord with all the facts of colour mixture, but calls for a wave-length basis of colour vision, these new facts appear merely to add complications to an already too complex problem. Unfortunately, this has been the almost uniform history of research in vision.

Physical Laboratory,
The United Gas Improvement Company,
Philadelphia, Pa., Sept. 1917.

Note added on correction of proof.

The transmission of colour impressions in order of wave-length, which is indicated as taking place antecedent to the trichromatic mechanism, has every appearance of being a purely optical effect. It could be accounted for by the assumption of strong chromatic aberration in annular focussing elements in the retina, concentrating the incident light on photo-sensitive fibres lying in the axes of the elements. The different periods of transmission of the various colours would then be due to the different distances the resultant decomposition products had to travel along these fibres. Further discussion of this possible explanation is deferred for the present.

XLIX. *An Astronomer on the Law of Error.*
By Professor F. Y. EDGEWORTH, *F.B.A.*[*]

THE law of error has been disputed by the Astronomer Royal for Scotland [†] on two distinct grounds.

I. He finds that the law is not perfectly fulfilled by astronomical observations, not even by those made by Bradley which Bessel tested, still less by others. This verdict does not disconcert the statistician who (after Laplace) grounds his expectation of the law on the interaction of numerous independent causes. Where that condition is imperfectly fulfilled there is no reason to expect the law of error to be realised perfectly, any more than we expect a body attracted to another according to the law of gravitation to move in a perfect conic when there is a resisting medium. In the case of astronomical observations we have reason to believe that the conditions will be sometimes badly, sometimes fairly well fulfilled [‡]. The like is true of physical observations generally, with which may be classed shots aimed at an object [§]. Similarly the neighbouring class of statistics not grouped about an objective thing (*e. g.* statures of a population) fulfil the law more or less perfectly. Greater perfection may be expected in those classes of phenomena to which Laplace and Poisson confined the application of the law, namely magnitudes each of which is an *average*—or more generally a linear function—of numerous observations or statistics of the two classes above mentioned and (exemplifying such functions [‖]) occurrences at games of chance. The most perfect fulfilment is presumably presented in a molecular medley by velocities considered as the resultants of innumerable compositions.

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[*] Communicated by the Author.

[†] "On the Law of Distribution of Errors." By R. A. Sampson. Fifth International Congress of Mathematicians, 1912, vol. ii. p. 163 *et seq.*

[‡] Fulfilment of the law is not to be expected where the observations are affected by a few dominant sources of error, as pointed out by Morgan Crofton in some instructive remarks on the nature of errors in astronomical observations," Philosophical Transactions, 1870, p. 177.

[§] In the class of *shots* may be included guesses *e. g.* as to the age of an individual and even estimates of a less objective magnitude, such as the worth of an examination paper; as to which see Phil. Mag. August 1890. As to the classification of phenomena obeying the law of error, see article on "Probability," *Encyclopædia Britannica,* 11th ed., § 117 *et seq.* *Cp.* as to imperfect fulfilment of the conditions, § 157.

[‖] *E. g.* a great number of dice being tossed, the frequency with which certain faces turn up. If the dice are not perfectly symmetrical (as Weldon found, Phil. Mag. vol. 50. 1900, p. 168), the data have some affinity to statistics not representing an objective magnitude.

II. The ground for expecting that the law of error will be fulfilled more or less approximately in the preceding cases would be cut away if Dr. Sampson's objection to the proof of the law were held valid. His attack (*loc. cit.* p. 167) on the proof given by Poisson after Laplace strikes at all the applications of the law [*]; it cannot be limited to the particular class of astronomical (or more generally physical) observations—to which indeed Laplace [†] and Poisson did not propose to apply the law. The objection here combated is not based on the want of that *independence* which the law postulates; whether as between the (total) errors of successive observations, considered as *not* "accidental" (Sampson, *loc. cit.* p. 168), or between the "small errors" (p. 166) the components [‡] of the entities which may be expected to fulfil the law [§]. It is here conceded that so far as such independence is not present perfect fulfilment of the law is not to be expected; the case will present imperfection of the kind admitted under head I. But Dr. Sampson in his attack on the theory of Laplace and Poisson (*loc. cit.* p. 167) does not dispute the initial stages of the proof in which this independence is implied. It is implied that if one component error is distributed according to the frequency-function $f_1(x)$ [||] and another according to $f_2(x)$, the probability that the respective component errors x_1 and x_2 should concur is proportionate to $f_1(x_1) \times f_2(x_2)$. What the point of Dr. Sampson's objection is may be shown by a free version of his argument in a simplified case. Let us suppose that

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[*] Including the method of *sampling* which is becoming so important in social statistics; as to the theory of which see Bowley's Presidential Address to Section F of the British Association, 1906, and as to the practice 'Livelihood and Poverty,' 1915, by Bowley and another.

[†] Glaisher more than once remarks on the fact that Laplace did not employ his theorem to establish a presumption that observations themselves—as distinguished from *averages* thereof—fulfil the law of error. See 'Memoirs of the Astronomical Society,' vol. xxxix. pp. 104, 106; and 'Monthly Notices of the Astronomical Society,' vol. xxxiii. p. 397, par. 3 (1873).

[‡] Where the entities are *averages* their components consist of the original data, observations or statistics of the kinds described above (divided by the number thereof).

[§] For an example of the *first* sort of interdependence, see Article on "Probability" (*Enc. Brit.*), § 157. Where the figures grouped were each an *average* of a *set* of consecutive observations of the kind instanced, the materials would illustrate interdependence of the *second* kind.

[||] Meaning that the number of observations which occur between x and $x+\Delta x = \mathrm{N}f_1(x)\Delta x$, where N is the total number of observations: $\Sigma f_1(x)\Delta x = 1$; Δx is small, so that for the purpose in hand it may be replaced by dx.

the law of frequency is one and the same for all the components, say $f(x)$. Let this function be symmetrical about a point which is taken as the origin. Further, let the weights, called by Dr. Sampson after Todhunter* γ, be equal. Then in the place of ρ_i in Dr. Sampson's (and Todhunter's) notation we have

$$\rho = \int_{-a}^{a} f(x) \cos \alpha x d\alpha,$$

where a and $-a$ are the limits which the errors cannot exceed †. Then, since

$$\int_{-a}^{a} f(x) dx = 1 \ddagger,$$

$$\rho = 1 - \tfrac{1}{2}\alpha^2 \int_{-a}^{a} x^2 f(x) dx + \ldots$$

$$= 1 - \alpha^2 \tfrac{1}{2} k_2 + \ldots,$$

if we put k_2 for the mean square of deviation from the mean, that is *twice* Dr. Sampson's h^2 §. If s is the number of the components, we have

$$\mathrm{R} = \rho^s = (1 - \alpha^2 \tfrac{1}{2} k^2 + \ldots)^s; \quad \log \mathrm{R} = -\alpha^2 s \tfrac{1}{2} k^2 + \ldots$$

"Then approximately," as Todhunter has it in his version of Poisson's reasoning ‖,

$$\mathrm{R} = e^{-\frac{1}{4}c^2\alpha^2}, \quad \text{where} \quad c^2 = 2sk_2.$$

Accordingly the required expression for the frequency of observations between assigned values of the abscissa is given by multiplying R upon a certain function of α and the said values of the abscissa, and integrating the expression thus formed with respect to α between limits ∞ and 0. To this procedure it is objected by Dr. Sampson that "the terms

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* See Todhunter, "History of the Theory of Probability," Art. 1002.

† In Todhunter's version respectively a and b.

‡ This condition seems to obviate Dr. Sampson's objection. "If the arbitrary distributions $f_i(x)$ have any zero—and this is not excluded by the process of demonstration—I do not see how they can fail to reappear as zero in the product R" (*loc. cit.* p. 168, par. 1).

§ *Loc. cit.* p. 167, par. 3; "h" is used differently elsewhere, p. 167, par. 1, pp. 169–172.

‖ *Op. cit.* p. 565; Todhunter's Y corresponding to Dr. Sampson's and our R, and Todhunter's κ^2 to our $\frac{1}{2}c^2$ and to twice Dr. Sampson's Σh_i^2 (when the h's are identical).

in α^4 in the separate factors have been omitted"[*]; meaning no doubt that had the terms been taken into account the coefficient of α^4 in R would *not* have proved approximately equal to the corresponding coefficient in the expansion of $e^{-\frac{1}{4}c^2}$.

Now this is exactly what was maintained by the present writer in the Philosophical Magazine for 1883[†]. There, and elsewhere subsequently[‡], he has instanced forms of $f(x)$ such that when any number of components are superposed in the manner of Laplace, the compound does *not* conform to the law of error[§]. Dr. Sampson, then, is right so far as he teaches that the *independence*[||] above postulated is not sufficient by itself and without any additional conditions to secure the fulfilment of the law of error. But he should have added that commonly and practically such additional conditions are present.

What those conditions are may be shown by continuing the expansion of R. Put k_4 for the mean fourth power of deviations for one of the components from the mean (identical with Todhunter's k''' now that the mean is coincident with the origin). We have then for the logarithm of ρ,

$$-\tfrac{1}{2}k_2\alpha^2+\alpha^4\left(\frac{1}{4\,!}k_4-\tfrac{1}{8}k_2{}^2\right)$$

(k_2 as before denoting the mean square of deviation for any one of the component elements). Whence

$$\log \mathrm{R}=-\alpha^2 s\tfrac{1}{2}k_2+\alpha^4\frac{1}{4\,!}(k_4-3k_2{}^2).$$

In the coefficient of α^4 in the expansion of R the remainder after the first term tends to be negligible in comparison with that term as s increases. For the first term contains s^2 con-

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[*] So the *quæsitum* is defined by Todhunter after Poisson. But it seems much simpler on the lines of Laplace to investigate the probability of an observation occurring *at* (or in the immediate neighbourhood of) a particular point; as to which conception see "Law of Error" (by the present writer), Camb. Phil. Trans. vol. xx. p. 131 (1905). *Cp.* p. 40, *et passim.*

[†] Vol. xvi. pp. 304 & 307.

[‡] "Law of Error," Camb. Phil. Trans. vol. xiv. p. 140 (1885).

[§] The functions are of the family specified below (p. 429) as "reproductive." They may be expanded in ascending powers of x—by a series of integrations with respect to a between limits ∞ and 0.

[||] Absence of interdependence or correlation, both between the several component small errors which make up a total or composite error of observation and also between successive composite errors; these terms being used in a wide sense so as to cover the case where the compound is an *average* (or other linear function) and the components are (errors of) observations or the same divided by n the number of observations (*cp.* above, p. 423 note); and where the "error" is not a deviation from an *objective* magnitude.

stituents of the order of magnitude k_4 (or k_2^2). Whereas the remainder tends to contain only s constituents of that order; since the central term in the expansion of k_4, viz. $6k_2k_2\,\frac{s(s-1)}{2}$, is cancelled by the main term in the expansion of $3(sk_2)^2$ *. It must be postulated that the mean second and fourth powers of deviation are finite: a postulate which may be taken for granted when the range of the component frequency-functions is finite. The reasoning may be extended—if the postulate also is—to further terms in the expansion of R. Accordingly Laplace was quite right when, referring to symmetrical identical frequency-functions with limited range, he affirmed that "taking hyperbolic logarithms, we have very approximately (*à trés-peu près*), when s is a large number" † for the (Napierian) logarithm of R (in the notation above used) an expression equivalent to that which has been given above.

The conclusion may be extended to the case of frequency-functions not identical; provided that their mean powers of deviation are of the same order. The conditions have been stated elsewhere by the present writer with more precision ‡. It is needless to reproduce that statement here, since the conditions (for *even* frequency-functions) have been adequately stated in a treatise to which Dr. Sampson has referred, Poincaré's *Calcul des Probabilités* §. Poincaré supposes "the functions even, or in other words no systematic errors" ‖ (*Calcul*, p. 184); "that the errors are independent"

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* For a fuller exposition, see the present writer's article on the "Law of Error" in the Transactions of the Cambridge Philosophical Society, vol. xx. p. 42 *et seq.* (1905).

† *Théorie analytique des Probabilités,* liv. 2, Art. 18, p. 336, National edn. 1847.

‡ In the 1905 Paper referred to in the penultimate note.

§ The proof of the law of error by way of approximate identity between the mean powers of the representative function and those of the actual locus was put forward by the present writer in 1905 without acknowledgment, because without knowledge, of Poincaré's similar proof for the case of *even* functions, published in 1896. Priority may still be claimed for the essay of 1905 as having extended that proof to *odd* functions, having proposed several collateral proofs, and having carried the approximation beyond the stage at which it was left by Poisson (the "second approximation" referred to below note p. 428).

‖ It may be worth while to recall that "symmetrical" and "systematic" errors are not necessarily coincident. The centre of a symmetrical group may not coincide with the true point, and the centroid of an unsymmetrical group *may.* In the case of symmetry as well as of asymmetry something must be known or presumed as to the relation of the true point to the group as a whole. *Cp.* Phil. Mag. vol. xvi. p. 373 (1883) and Journal of the Royal Statistical Society, vol. lxxi. p. 500 (1908).

(*op. cit.* p. 182); "that the individual errors, though not following the same law, are practicably (*sensiblement*) of the same order of magnitude, and each contributes little to the total error" (*op. cit.* p. 183). Recapitulating these conditions he concludes that "in this case the resultant error *suivra sensiblement la loi de Gauss*"*.

The reasoning whereby the (proper or normal †) law of error has been found as an approximation to the actual locus (which results from the composition of several independent elements) may be extended to obtain a closer approximation by taking into account the first of the terms which have been neglected.

Put $c^2 = \mathrm{S}2k_2$, S denoting summation with respect to all the elements and k_2 the mean square of deviation (not now identical for all the elements). And for $\mathrm{S}(k_4 - 3k_2^2)$ put K_2. Then R may be written

$$\int_0^\infty e^{-\frac{1}{4}c^2\alpha^2}\left(1 + \frac{1}{4!}\alpha^4\,\mathrm{K}_2 \ldots\right)\cos\alpha x dx.$$

The first and main term of the integral is the normal error-function

$$\frac{1}{c\sqrt{\pi}}\exp - \frac{x^2}{c^2}, \text{ say } y.$$

The second term, the sought correction, is

$$\int_0^\infty e^{-\frac{1}{4}c^2\alpha^2}\frac{1}{4!}\alpha^4\,\mathrm{K}_2\cos\alpha x d\alpha;$$

which integral may be replaced by $\frac{1}{4!}\mathrm{K}_2\frac{d^4y}{dx^4}$. The process may be extended to terms of smaller orders so as to form a descending series, the "generalized law of error" ‡ for even functions.

The transition to *odd* components is effected by obtaining likewise an odd descending series; the whole of the odd series—like the part of the even series which remains after the first main term—tending to vanish as the number of the

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* *Op. cit.* § 4, Leçon xvi. *Cp.* end of § 11, Leçon xiv.

† The term "law of error" in this Paper may be understood according to the context either as the "generalized law," or only the first and main term of that approximation, the so-called "Gaussian," or "normal," law.

‡ Camb. Phil. Trans. *loc. cit.* (1905). *Cp.* Journal of the Royal Statistical Society, 1906, "Law of Great Numbers," by the present writer.

components is increased. Thus the first term of the odd series may be written

$$-\frac{1}{3!}\mathrm{K}_1\frac{d^3y}{dx^3},$$

where $\mathrm{K}_1, = \mathrm{S}k_3$*, is the mean third power of deviation for any element from its mean value; the mean value being taken as the origin for each of the elements, (and for their sum) †. When the differentiation is performed it is seen that the term is affected with a coefficient which diminishes as the number of components, say s, is increased. For example, if each element is a binomial assuming the value 0 or a with respective probabilities q and p,

$$k_3 = \tfrac{1}{2}(q-p)sa^3/(\tfrac{1}{2}spq)^{\frac{3}{2}}a^3,$$

that is of the order $1/\sqrt{spq}$. Thus not only Laplace's proof of the law of error for even frequency-functions, but also Poisson's proof thereof for odd functions, together with his determination of a second approximation ‡, are found to hold good upon certain conditions. One who considers that those conditions are very generally realized will regard as misleading the Astronomer Royal's statement: "The conclusion is therefore unwarranted and there is no proof at all that peculiarities of the functions f efface themselves in the final result" (*loc. cit.* pp. 167–8).

III. It remains to notice two other proofs § of Laplace's law ‖ which have been likewise approached and missed by Dr. Sampson.

There is first Morgan Crofton's proof—either by way of a partial differential equation¶, or more directly**—that the continued superimposition of frequency-curves of any form will result in the normal law of error. Dr. Sampson applies Morgan Crofton's method of composition to certain

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* It will be noticed that the subscripts of the k's correspond to *powers* (of the component errors). The subscripts of the K's correspond to *corrections* of the normal function.

† *Mutatis mutandis*, if the elements are *weighted.*

‡ Given by Todhunter, *op. cit.* pp. 567–8.

§ Two among several variant proofs which are referred to in the article on "Probability" (*Ency. Brit.*) §§ 104–111.

‖ Laplace rather than Gauss deserves to be the eponym of the law of error when, as throughout in this paper, it is considered as resulting from the random combination of numerous independently fluctuating constituents.

¶ Cited in the article on "Probability" (*Ency. Brit.*), § 109.

** See Phil. Trans. 1870.

frequency-curves obtained by grafting an oscillating element on the normal law, as thus

$$\phi(x)=\frac{1}{\sqrt{\pi}(1+\mathrm{A})}e^{-h^2x^2}(1+a\cos hx)\ ^{*}.$$

He shows that when curves of this type are compounded the divergence from the normal law tends to disappear. This *curiosum* may have some bearing on astronomical observations. But it may be doubted whether observations extending to infinity—other than the law of error—have much concrete significance.

The result of continued composition may often be more advantageously contemplated by means of Laplace's method above indicated. Let the frequency-function for one element be $\phi_1(x)$, for another $\phi_2(x)$, ranging between given limits, which may be infinite. If the functions are even †, put $\rho_1=\phi_1(\alpha x)\cos\alpha x$. And let $\int\rho_1 dx$ (between extreme limits) $=\theta_1(\alpha)$. Let $\theta_2(\alpha)$ be formed likewise from $\phi_2(x)$. Then for the compound of the two frequencies we have

$$\int_0^\infty \theta_1(\alpha)\times\theta_2(\alpha)\cos\alpha x d\alpha.$$

This method may be employed to obtain an answer to a question which Dr. Sampson has raised, but has not rightly answered: namely, what frequency-curves enjoy the property that when two of a family are compounded the result belongs to the same family. Dr. Sampson appears to think that the normal law of error is the only curve which possesses this property in perfection, "with any generality" (*loc. cit.* p. 170). He is not aware that the property appertains to a wide class of which the normal law is a species: namely, the class for which $\theta(\alpha)$ (as above defined) is of the form $\exp-\alpha^t$. There is here disclosed a variant proof of the law of error. If there is a final form resulting from continued composition, it must be reproductive; and reproduction is the *proprium* of curves for which $\theta(\alpha)$ is of the form $\exp-\alpha^t$. A *further* condition (borrowed from Morgan Crofton) which must be fulfilled by the sought form limits t to the value 2. Using the value of $\theta(\alpha)$ thus obtained, viz. $\exp-\alpha^2$, we obtain the normal law ‡.

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* A simple case of the more general type proposed by Dr. Sampson (*loc. cit.* p. 170). Note that a is less than unity, and A is taken so that the integral of the function between limits $+\infty$ and $-\infty$ is *unity*.

† As in the case above instanced. For the case of odd functions see Camb. Phil. Trans. *loc. cit.* p. 53 (1905).

‡ See Camb. Phil. Trans. 1905, "Law of Error," Part 1. § 4, and Appendix, § 6.

Dr. Sampson is thrown off the track by a little slip in a mathematical operation such as the best may incur when not put on their guard by prior knowledge of the subject. Dr. Sampson experiments on frequency-functions of the form $\frac{1}{\pi(1+x^2)}$ (or the more general form which is presented when we take a for the parameter of x *). Here it may be observed $\theta(x)$ is of the form e^{-a} (or $e^{-\alpha a}$), and accordingly we know *a priori* that the continued superposition of a components of the type $\frac{1}{\pi(1+x^2)}$ will have for result $\frac{a}{a^2+x^2}$. Dr. Sampson, employing Morgan Crofton's method of composition by way of integration, finds this result true for *two* components; but not true in general, for any number of components. That is, in our notation, if the frequency-function $\frac{1}{\pi(1+x^2)}$ is compounded with $\frac{a}{\pi(a^2+x^2)}$, the result is

$$\frac{1}{\pi}\frac{a+1}{(a+1)^2+x^2},\ \text{when } a=1\,;$$

but not for larger values of a.

In Dr. Sampson's own symbols, Θ the abscissa, and Φ the form of the resultant frequency-curve,

$$\text{“}\Phi(\Theta)=a\pi^{-2}\int_{-\infty}^{\infty}\frac{d\Delta'}{a^2+\Delta'^2}\cdot\frac{1}{1+(\Theta-\Delta')^2}$$

$$=\frac{2a\pi^{-1}\Theta^2}{\Theta^4+2\Theta^2(a^2+1)+(a^2-1)^2}\text{” } (\textit{loc. cit.}\ \text{p. } 170).$$

This result is evidently untenable; since it imports that if we put together two frequency-curves, each symmetrical about a central point and thence descending continuously, the central ordinate for the compound will be zero! But if we resolve the expression above equated to $\Phi(\Theta)$ into two rational fractions according to the general rule for integration, we shall find that while the *denominator* of the result is rightly given by Dr. Sampson, the *numerator* ought to have been $(a+1)[\Theta^2+(a^2-1)^2]$; the expression in square brackets being a *factor* of the denominator. Dividing out we obtain, *as we ought*, for the result

$$\frac{1}{\pi}\frac{a+1}{\Theta^2+(a+1)^2}.$$

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* Substituting in the last-written expression for x, x/a and dividing the expression by a (substituting for dx, dx/a).

This is not the only passage in the paper under consideration where an inappropriate conception has led to inexact work*. In connexion with his peculiar notion as to the nature of an error of observation, the writer appears to hold that if the extent of an error varies with the time according to the law $y=\sin t$, the frequency of error between the extreme values which it can assume is constant †. Not so, surely; the number of errors per unit of time being constant, the number of errors per unit of space ‡ is proportioned to $\frac{dt}{d\xi}$, that is to $\frac{1}{\sqrt{1-\xi^2}}$. The frequency-curve is not a straight line but a symmetrical curve of the fourth degree with an infinite ordinate at the centre and contact with the abscissa at the points distant t from the centre in either direction.

In this connexion it is remarked by Dr. Sampson, "it seems probable that Laplace and Poisson were on the wrong lines." To those who have followed the preceding comments it will seem more probable that Dr. Sampson is on the wrong lines.

L. *On Transpiration from Leaf-Stomata.*

By HAROLD JEFFREYS, *M.A., D.Sc.*§

WITH most of Sir Joseph Larmor's note in the April Philosophical Magazine I am in agreement. It is clear that the amount of water evaporated from the stomata of a leaf in a given time must be less than that from the whole surface of a wet leaf of the same size; it must also be less than the amount that could be evaporated from all the stomata if their total number were the same and all were so much separated that they could be treated as isolated. Hitherto only the latter criterion has been used to indicate the upper limit, and the important question is to decide whether the former imposes a further restriction or not. When the air is at rest the question is simple, for both limits

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* The following criticism is due to Professor Bowley.

† "For $y=\sin t$ .... the distribution is represented by $\eta=1$ for $-1<\xi<1$ and $\eta=0$ beyond those limits (fig. 3)." Fig. 3 and the context bear out the interpretation here given.

‡ The number which multiplied upon $\Delta\xi$ a small fraction of (the unit of) the abscissa, ξ gives the (proportionate) number of observations occurring between x and $x+\Delta\xi$ (or $x+\frac{1}{2}\Delta x$ and $x-\frac{1}{2}\Delta\xi$).

§ Communicated by the Author.

can be well determined; the wet-leaf limit is then found to be the smaller and therefore gives the more useful criterion. When the air is in motion the question is much more difficult, for while the evaporation from isolated stomata remains practically constant with ordinary wind velocities, that from a wet leaf increases rapidly with the velocity and may come to exceed the isolated-stomata limit. In most particular cases it is probably best to find the wet-leaf limit by direct experiment, for its theoretical value depends on the wind velocity and the amount of turbulence, which are more difficult to determine. The isolated-stomata limit can be calculated without difficulty from the formula I gave.

I stated in my paper that to decide which limit was the more important in a wind depended on the particular circumstances considered. When a wind is blowing over a fixed flat surface the air behaves differently in two regions. In contact with the surface the velocity is zero, and within a certain small distance, estimated by Major Taylor as $40/U$ centimetres, where U is the velocity at a great distance measured in centimetres per second, the motion is purely laminar. The velocity at the edge of this layer is a considerable fraction of U, say $\frac{1}{2}U$ or $\frac{1}{3}U$. Within this layer heat is communicated by heat-conduction, momentum by ordinary viscosity, and gaseous constituents by ordinary diffusion. Outside of it the motion ceases to be laminar and becomes turbulent. The turbulence causes masses of air to be transported bodily for considerable distances in all directions, and their mixture with surrounding air causes a great increase in the ease of transference of heat, momentum, and gases. The effect of this is to add to the coefficients of kinematic viscosity, thermometric conductivity, and diffusion the same quantity, called the eddy conductivity, which is much greater than any of their values in non-turbulent motion. Now the problem of evaporation consists of two parts: first, the diffusion across the layer of laminar motion, and second, through the turbulent region. Sir Joseph Larmor neglects the resistance to diffusion in this region, thus practically taking the eddy conductivity as infinite, which appears to be justifiable in the case he considers. In the notation I used previously the rate of evaporation on the isolated-stomata hypothesis is $2\pi n^2 k\rho c V_0 A$, and for a circular leaf of radius l and circular stomata of radius a this becomes $4\pi n^2 k\rho a l^2 V_0$. The wet-leaf hypothesis gives $3{\cdot}95\rho V_0 (KUl^3)^{\frac{1}{2}}$, where K is the eddy conductivity. The ratio of these is practically $\frac{\pi n^2 k a l^{\frac{1}{2}}}{(KU)^{\frac{1}{2}}}$. With $n^2 = 33000/\text{cm.}^2$, $a = 5 \times 10^{-4}$ cm., $l = 5$ cm., $k = 0{\cdot}24$ cm.^{2}/sec.,

$K=1000$ cm.^{2}/sec., $U=400$ cm./sec., this becomes 0·05 roughly. Thus the isolated-stomata law gives much the smaller limit and is therefore correct. For indoor experiments, however (and most transpiration experiments have been done indoors), K and U are both much smaller than is above assumed, and the limits will occur in the other order, so that the wet-leaf law will apply and evaporation from individual stomata will accordingly be restricted.

Consider then a leaf in air moving sufficiently slowly for the wet-leaf law to hold. What happens when the stomata contract? At first they are capable of sending into the air more water vapour than the turbulent air can carry away; and as long as this remains true it seems to me that vapour will stay in the layer of shearing, and practically saturate it; then the rate of evaporation remains nearly constant, for it depends almost entirely on the outer region, which is in the same state throughout. As soon as the contraction reduces the possible supply of vapour to below the maximum amount that the turbulence can remove, the rate of evaporation will diminish, finally reaching the limit zero when the stomata are quite closed. Thus most of the reduction to zero will take place in this last stage. I do not see in what respect this result is paradoxical, and it does appear to be well confirmed by experiment.

The amount of transpiration possible on the wet-leaf hypothesis seems to be quite adequate. Thus, consider a leaf of 5 cm. radius, in an atmosphere containing 0·04 per cent. of carbon dioxide by volume. Then even in still air the volume of CO_2 absorbed per hour could be 13 c.c., or 0·17 c.c. per sq. cm. Thoday finds experimentally that *Helianthus annuus* in the open air can at the utmost absorb 0·14 c.c. of carbon dioxide per sq. cm. per hour. On the question of evaporation I have seen no quantitative statement amenable to numerical calculation except Renner's definite assertion that the evaporation from a leaf is the same as that from a water surface of the same size.

Note by Sir JOSEPH LARMOR.

Dr. JEFFREYS agrees that the standard discussions on this subject are not in error, though there is a question of how closely they conform to the conditions of any particular experiment. The idea of an eddy conductivity may prove useful with due limitation; but I find it difficult to attach any definite value to K. It may be of use to sum up in other terms the broad essentials, which seem to include all the calculation that the facts are definite enough to warrant.

The sensitive surfaces of a leaf are protected from injury, by being the walls of interior chambers which are connected with the outer air by narrow necks called stomata. Their efficiency as absorbers of carbon dioxide or as transpirers of water vapour depends of course inversely on the narrowness of these necks. The reciprocal of this efficiency is the quantity that can be discussed directly; for it is proportional to the total resistance of the channel of diffusion, which is made up by addition from that of the chamber, that of the neck itself, and that of the tube of diffusion outside extending to where the atmosphere becomes normal as regards its carbon dioxide or its vapour. The length that may be assumed for this latter tube depends on circumstances involving the degree of stillness of the air: and it is this that Dr. Jeffreys proposes to elucidate by introducing the modulus K. Perhaps with moderate motion of the leaf it has practically no length at all, the air sweeping over the leaf and being renewed even close to its surface. If a large proportion of the wall of a chamber is fully efficient, the air in the chamber is, except near the neck, practically free from carbon dioxide or saturated with vapour, as the case may be: this follows from the usual electrostatic analogy. If both these conditions hold good, the one just stated on the inside and free sweep of air close to the leaf on the outside, the total resistance is that of the neck alone including the corrections usual in electric problems for its ends, there being none for its outer end if the air is renewed constantly close over that end. If the stoma were a mere aperture in a thin plate, its resistance would then reduce to this correction for the inner end alone, as Dr. Horace Brown remarks: it would be inversely as the linear dimensions, being for a circular aperture measured by the reciprocal of its diameter, multiplied by the specific resistance for air. This is the reason that a large number of very small stomata can permit almost free interchange between the interior of the leaf and the outside.

LI. *A Method of obtaining General Reaction-Velocity Curves for complete Homogeneous Gas Reactions at Constant Pressure.* *By* GEORGE W. TODD, *D.Sc.* (*Birm.*), *B.A.* (*Camb.*) *.

THE writer has already worked out general curves for homogeneous complete reactions between two substances when the volume of the reaction space remains constant †. In many binary gas reactions a continuous change of volume takes place during the reaction at constant pressure. The concentrations of the reacting gases therefore not only depend on the amounts transformed but also on the volume of the products. Since many technical gas-reactions take place at constant pressure, sets of general curves for such reactions should prove of great value. The paper gives a method of obtaining the curves.

Bi-molecular Reactions.

Take a reaction of the type A+B→1 or more resultants. The resultants need not necessarily be gaseous. We will assume that B is in excess.

Let a gm. mols. of A react with b gm. mols. of B.

Let x gm. mols. of either reactant be changed in time t.

Let v be the the total volume of reactants and resultants.

The velocity of the reaction is given by

$$\frac{d}{dt}\left(\frac{x}{v}\right)=k\,.\,\frac{a-x}{v}\,.\,\frac{b-x}{v},$$

where k=velocity constant, or

$$\frac{1}{v}\frac{dx}{dt}-\frac{x}{v^2}\frac{dv}{dt}=\frac{k}{v^2}(a-x)(b-x). \quad . \quad . \quad . \quad (1)$$

Put $\frac{x}{a}=\mathrm{X}$, *i. e.* the fraction of A changed.

If the pressure is maintained constant throughout the reaction, then the volume at any time will be proportional to the number of gas molecules present.

The initial volume when $t=0$ is

$$v_0 \propto 2a+(b-a),$$

$(b-a)$ is the excess of B present.

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* Communicated by the Author.

† Phil. Mag. vol. xxxv. p. 281.

The volume after a time t, when the fraction of A changed is X, is

$$v \propto 2a(1+\alpha X)+(b-a),$$

where α is a constant for the particular reaction.

If the volume increases α is positive, if the volume decreases α is negative.

We get

$$v=v_0\frac{2a(1+\alpha X)+(b-a)}{2a+(b-a)}^{*},$$

and

$$\frac{dv}{dt}=\frac{2av_0\alpha}{2a+(b-a)}\cdot\frac{dX}{dt}.$$

. Equation (1) now becomes

$$\frac{a}{v}\frac{dX}{dt}-\frac{aX}{v^2}\frac{2av_0\alpha}{a+b}\frac{dX}{dt}=k\frac{a^2}{v^2}(1-X)\left(\frac{b}{a}-X\right),$$

or

$$\frac{dX}{dt}=k\cdot\frac{a}{v_0}\cdot(1-X)(p-X),$$

which is independent of α. Hence a volume-change at constant pressure does not affect $\frac{dX}{dt}$, so that the general curves given in the paper previously alluded to will do for the case of bi-molecular reactions at constant pressure as well as at constant volume.

Ter-molecular Reactions.

Many binary gas-reactions are of the type 2A + B→1 or more resultants. Let us work out this case assuming first that *B is in excess.*

Let $2a$ gm. mols. of A react with b gm. mols. of B.

At time t, $2x$ gm. mols. of A and x gm. mols. of B will be changed.

Let v be the total volume at time t.

The velocity of reaction is given by

$$\frac{d}{dt}\left(\frac{x}{v}\right)=k\cdot\left(\frac{a-x}{v}\right)^2\cdot\left(\frac{b-x}{v}\right),$$

or

$$\frac{1}{v}\cdot\frac{dx}{dt}-\frac{x}{v^2}\frac{dv}{dt}=\frac{k}{v^3}(a-x)^2(b-x). \quad . \quad . \quad . \quad (2)$$

(N.B. The concentration of the gas A has been taken as *half* the number of A molecules per unit volume.)

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* If there is also initially present m molecules of an inert gas, then

$$v=v_0\frac{2a(1+\alpha X)+(b-a)+m}{2a+(b-a)+m},$$

but this does not affect the final expression for $\frac{dX}{dt}$.

Put $\frac{x}{a}=\mathrm{X}$, the fraction of A changed. The volume at any time being proportional to the number of gas-molecules present, we have

$$v_0 \propto 3a+(b-a),$$

and $$v \propto 3a(1+\alpha\mathrm{X})+(b-a),$$

whence $$v=v_0\frac{3a(1+\alpha\mathrm{X})+(b-a)}{3a+(b-a)}\text{*},$$

and $$\frac{dv}{dt}=\frac{3av_0\alpha}{3a+(b-a)}\cdot\frac{d\mathrm{X}}{dt}.$$

Substituting in equation (2) gives

$$\frac{d\mathrm{X}}{dt}=k\left(\frac{a}{v_0}\right)^2\frac{(1-\mathrm{X})^2\left(\frac{b}{a}-\mathrm{X}\right)}{1+\frac{3a\alpha\mathrm{X}}{2a+b}},$$

or, putting $\frac{b}{a}=p$,

$$\frac{d\mathrm{X}}{dt}=k\left(\frac{a}{v_0}\right)^2\frac{(1-\mathrm{X})^2(p-\mathrm{X})}{1+\frac{3\alpha}{2+p}\mathrm{X}}.$$

$$\therefore \qquad k\left(\frac{a}{v_0}\right)^2 t=\int_0^{\mathrm{X}}\frac{1+\frac{3\alpha}{2+p}\mathrm{X}}{(1-\mathrm{X})^2(p-\mathrm{X})}d\mathrm{X}. \qquad (3)$$

This is the general equation for B in excess.

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* If in addition to the $2a$ molecules of A and the b molecules of B, there are also initially present na molecules of an inactive gas, then

$$v=v_0\frac{3a(1+\mathrm{a}\mathrm{X})+(b-a)+na}{3a+(b-a)+na},$$

so that

$$\frac{d\mathrm{X}}{dt}=k\left(\frac{a}{v_0}\right)^2\frac{(1-\mathrm{X})^2\left(\frac{b}{a}-\mathrm{X}\right)}{1+\frac{3a\mathrm{a}\mathrm{X}}{2a+b+na}};$$

and equation (3) becomes

$$k\left(\frac{a}{v_0}\right)^2 t=\int_0^{\mathrm{X}}\frac{1+\frac{3\alpha}{2+p+n}\mathrm{X}}{(1-\mathrm{X})^2(p-\mathrm{X})}d\mathrm{X}.$$

When *A is in excess* we have by similar reasoning,

since $v_0 \propto 3b+2(a-b)$,

and $v \propto 3b(1+\alpha\mathrm{X})+2(a-b)$,

$$k\left(\frac{b}{v_0}\right)^2 t=\int_0^{\mathrm{X}} \frac{1+\dfrac{3\alpha}{1+2p}\mathrm{X}}{(p-\mathrm{X})^2(1-\mathrm{X})}d\mathrm{X}, \quad . \quad . \quad . \quad (4)$$

p in this case being $\frac{a}{b}$.

If nb molecules of an inactive gas are also present in the initial mixture, equation (4) becomes

$$k\left(\frac{b}{v_0}\right)^2 t=\int_0^{\mathrm{X}} \frac{1+\dfrac{3\alpha}{1+2p+n}\mathrm{X}}{(p-\mathrm{X})^2(1-\mathrm{X})}d\mathrm{X}.$$

Reactions with B in excess.

Let the volume-change when neither reactant is in excess be given by

2 vols. + 1 vol. → 2 vols.

i. e. $\alpha=-\frac{1}{3}$.

Equation (3) becomes

$$k\left(\frac{a}{v_0}\right)^2 t=\int_0^{\mathrm{X}} \frac{1-\dfrac{1}{2+p}\mathrm{X}}{(1-\mathrm{X})^2(p-\mathrm{X})}d\mathrm{X}$$

$$=\frac{1}{p+2}\left\{\frac{2}{(p-1)^2}\log_e\frac{p(1-\mathrm{X})}{p-\mathrm{X}}+\frac{p+1}{p-1}\,\frac{\mathrm{X}}{1-\mathrm{X}}\right\}.$$

For $p=1$.

X ……………	0	·2	·4	·5	·6	·7	·8	·9	1·0
$k\left(\frac{a}{v_0}\right)^2 t$ ……	0	·271	·815	1·333	2·253	4·147	9·333	36·00	∞

$p=1\frac{1}{4}$.

X ……………	0	·3	·5	·6	·7	·8	·9	1·0
$k\left(\frac{a}{v_0}\right)^2 t$ ……	0	·371	·98	1·58	2·70	5·30	17·0	∞

$p=2$.

X ……………	0	·3	·5	·7	·8	·9	1·0
$k\left(\frac{a}{v_0}\right)^2 t$ ……	0	·226	·547	1·363	2·45	5·90	∞

$p=3$.

X	0	·5	·7	·8	·9	·95	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·349	·839	1·470	3·406	7·339	∞

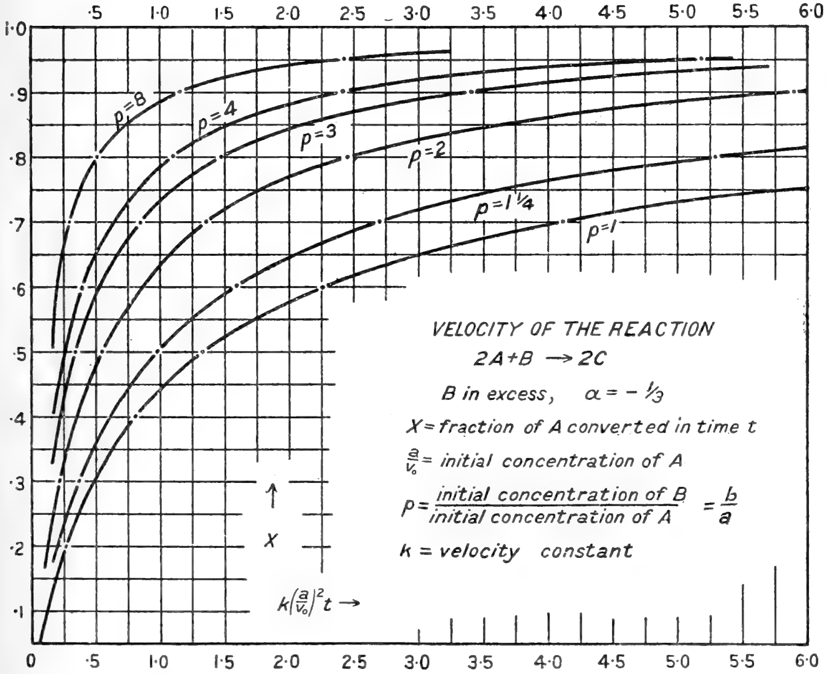
$p=4$.

X	0	·6	·8	·9	·95	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·389	1·060	2·424	5·180	∞

$p=8$.

X	0	·7	·8	·9	·95	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·295	·507	1·148	2·43	∞

Fig. 1.



These figures have been plotted in fig. 1. The curves obtained are applicable to all ter-molecular gas-reactions of the type 2A + B→ at constant pressure, the volume-change being such that $\alpha=-\frac{1}{3}$, and the gas B in excess. Examples of such reactions are

$$2H_2+O_2\rightarrow 2H_2O \text{ oxygen in excess.}$$
$$2CO+O_2\rightarrow 2CO_2 \quad ,, \quad ,,$$
$$2NO+O_2\rightarrow 2NO_2 \quad ,, \quad ,,$$

Now let the volume-change, when neither reactant is in excess, be given by

2 vols. + 1 vol. → 1 vol.

i. e. $\alpha=-\frac{2}{3}$.

Equation (3) becomes

$$k\left(\frac{a}{v_0}\right)^2 t=\int_0^X \frac{1-\frac{2}{p+2}X}{(1-X)^2(p-X)}dX$$
$$=\frac{1}{p+2}\left\{\frac{p-2}{(p-1)^2}\log_e\frac{p-X}{p(1-X)}+\frac{p}{p-1}\,\frac{X}{1-X}\right\}.$$

For $p=1$.

X	0	·2	·4	·5	·6	·7	·8	·9	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·261	·751	1·167	1·874	3·235	6·666	22·50	∞

$p=1\frac{1}{4}$.

X	0	·3	·5	·6	·7	·8	·9	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·358	·864	1·342	2·17	3·99	10·04	∞

$p=2$.

X	0	·3	·5	·6	·7	·8	·9	·95	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·214	·500	·750	1·166	2·000	4·500	9·500	∞

$p=3$.

X	0	·4	·6	·7	·8	·9	·95	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·218	·485	·747	1·265	2·797	5·830	∞

$p=5$.

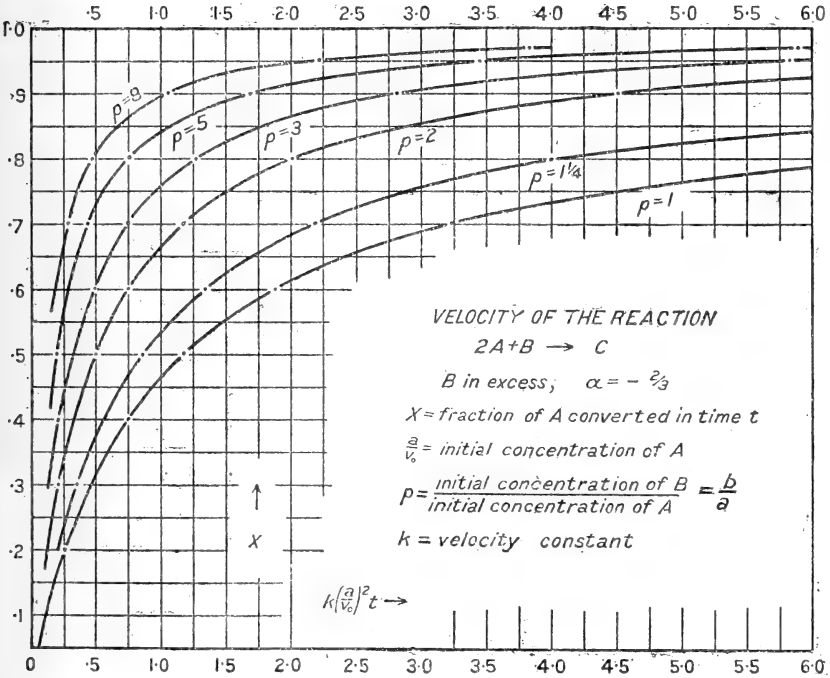
X	0	·5	·7	·8	·9	·95	·97	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·195	·444	·753	1·667	3·465	5·868	∞

$p=8$.

X	0	·7	·8	·9	·95	·97	1·0
$k\left(\frac{a}{v_0}\right)^2 t$	0	·281	·475	1·055	2·205	3·731	∞

These numbers give the curves in fig. 2. The curves are applicable to all ter-molecular gas-reactions of the type

Fig. 2.



$2A+B \rightarrow$ at constant pressure when the volume-change is such that $\alpha=-\frac{2}{3}$ and gas B is in excess. Such a reaction is $2NO+O_2 \rightarrow N_2O_4$ oxygen in excess.

Reactions with A in excess.

Let us reconsider these reactions in which $\alpha=-\frac{1}{3}$ and $\alpha=-\frac{2}{3}$ when the A gas is in excess.

Take first $\alpha=-\frac{1}{3}$.

Equation (4) becomes

$$k\left(\frac{b}{v_0}\right)^2 t=\int_0^X \frac{1-\frac{1}{1+2p}X}{(p-X)^2(1-X)}\,dX$$

$$=\frac{1}{1+2p}\left\{\frac{2p}{(p-1)^2}\log_e\frac{p-X}{p(1-X)}-\frac{p+1}{p(p-1)}\,\frac{X}{p-X}\right\}.$$

For $p=1$.

X	0	·2	·4	·5	·6	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·271	·815	1·333	2·253	4·147	9·333	36·00	∞

$p=1\frac{1}{10}$.

X	0	·3	·5	·6	·7	·75	·8	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·31	·97	1·64	2·74	3·7	5·58	∞

$p=1\frac{1}{4}$.

X	0	·3	·5	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·199	·558	1·45	2·62	5·78	∞

$p=1\frac{1}{2}$.

X	0	·4	·6	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·294	·657	·995	1·578	2·881	∞

$p=2$.

X	0	·5	·6	·7	·8	·9	·95	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·224	·319	·455	·679	1·115	1·706	∞

The curves obtained from these numbers are given in fig. 3.

Now take $\alpha=-\frac{2}{3}$.
Equation (4) becomes

$$k\left(\frac{b}{v_0}\right)^2 t=\int_0^{\mathrm{X}}\frac{1-\dfrac{2}{1+2p}\mathrm{X}}{(p-\mathrm{X})^2(1-\mathrm{X})}d\mathrm{X}$$

$$=\frac{1}{1+2p}\left\{\frac{2p-1}{(p-1)^2}\log_e\frac{p-\mathrm{X}}{p(1-\mathrm{X})}\quad\frac{1}{p(p-1)}\frac{\mathrm{X}}{p-\mathrm{X}}\right\}.$$

For $p=1$.

X	0	·2	·4	·5	·6	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·261	·751	1·167	1·874	3·235	0·666	22·50	∞

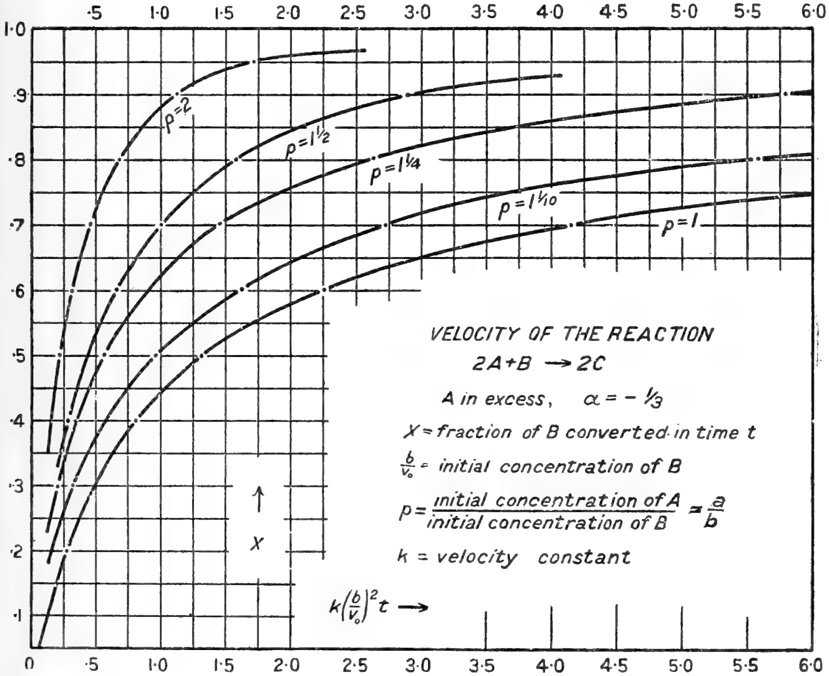
$p=1\frac{1}{10}$.

X	0	·4	·5	·6	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·56	·86	1·40	2·22	4·09	9·6	∞

$p=1\frac{1}{4}$.

X	0	·3	·5	·6	·7	·8	·9	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·271	·641	·956	1·465	2·395	4·73	∞

Fig. 3.



$p=1\frac{1}{2}$.

X	0	·6	·7	·8	·9	·95	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·588	·854	1·31	2·27	3·39	∞

$p=2$.

X	0	·4	·6	·8	·9	·95	·97	1·0
$k\left(\frac{b}{v_0}\right)^2 t$	0	·147	·292	·592	·938	1·318	1·616	∞

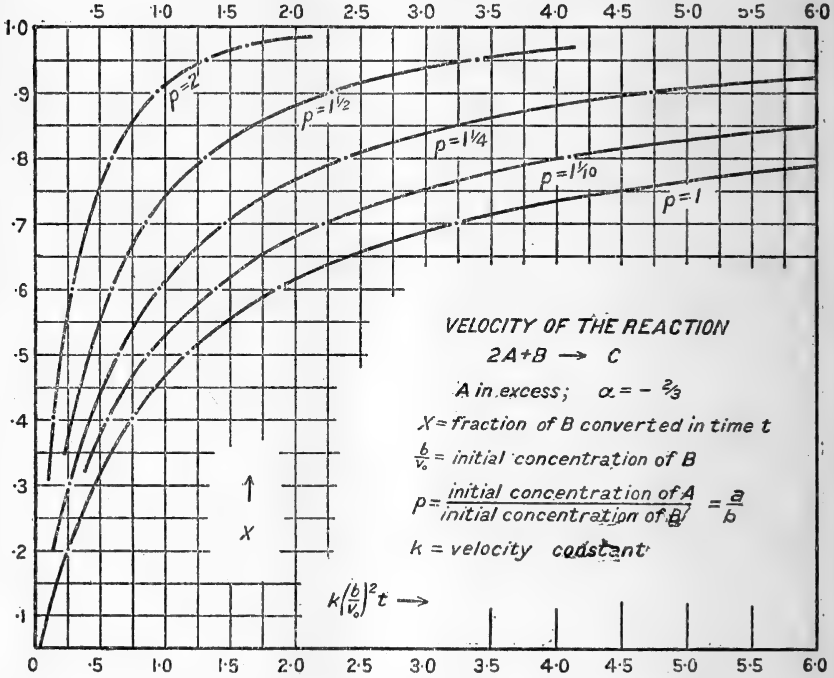
These numbers are plotted in fig. 4.

Curves for reactions in which α has other values can easily be worked out from equations (3) and (4). Values of α other than $-\frac{1}{3}$ are uncommon. When the resultants of the reaction are solid or liquid, $\alpha=-1$.

General Curves for Reactions of Higher Orders.

Gas-reactions of higher orders are rarely met with (except in the case of combustion of many hydrocarbons), therefore

Fig. 4.



examples have not been worked out. General curves for any particular type of reaction can be obtained by following a procedure similar to that used in this paper.

London, February, 1918.

LII. *Notices respecting New Books.*

Annuaire du Bureau des Longitudes pour l'année 1918. Pp. viii + 869. Paris: Gauthier-Villars. Price 2 francs.

THIS excellent almanack contains, in addition to the usual astronomical data, many other tables of interest relating to meteorology, terrestrial magnetism, and kindred subjects, together with a very useful collection of physical constants. Besides the tables there are special articles, written by good authorities, on the following subjects: Sundials, the Egyptian Calendar, Time at Sea, the Sun and Terrestrial Magnetism, and an obituary notice on Professor Gaston Darboux by M. Picard.

The book is embellished with star maps and figures, and is a very useful and wonderfully cheap publication.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1918.

LIII. *Molecular Frequency and Molecular Number.*—Part III. *Inorganic Compounds. Lindemann's Formula. By* H. STANLEY ALLEN, *M.A., D.Sc., University of London, King's College**.

IN the present communication are given the values of the characteristic molecular frequency, ν, calculated by Lindemann's formula for those inorganic compounds for which data are available. The formula in question may be written

$$\nu=k\sqrt{\left(\frac{T_s}{MV^{\frac{2}{3}}}\right)},$$

where T_s is the absolute temperature of the melting-point, M is the molecular weight, and V the molecular volume. The coefficient k has been assumed constant, and Nernst's empirical value, $3{\cdot}08\times10^{12}$, has been employed.

For each compound is tabulated the value of the molecular number, N, and of the product $N\nu\times10^{-12}$. In the majority of cases it has been found possible to express $N\nu$ either in the form $n\nu_A$ or in the form $(n+\frac{1}{2})\nu_A$, where n is an integer and ν_A is a constant frequency having a value about 21×10^{12} sec.$^{-1}$. In the few exceptional cases where neither of the above forms is applicable the product has been expressed as $(n+\frac{1}{4})\nu_A$ or $(n+\frac{3}{4})\nu_A$, but no special significance is at present attached to such results. All the results here given must be considered in the light of the general considerations affecting Lindemann's formula put forward in Part I. of this paper†.

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* Communicated by the Author.

† Phil. Mag. vol. xxxv. p. 338, April 1918.

Compounds containing water of crystallization have not been included, as in such cases it is often difficult to decide whether the recorded temperature is a true melting-point or the temperature at which the solid dissolves in the water of crystallization.

GROUP I.

Of inorganic compounds for which the density and the melting-point are known, a larger proportion belongs to the first group than to any other group in the Periodic Table. With a small number of exceptional or doubtful cases, the results show good agreement with the suggested relations. It is noteworthy that when the temperature of the melting-point is known accurately, the agreement is better than when this temperature is uncertain.

For several of the lithium compounds the melting-point has not been determined with great accuracy, yet the values of ν_A do not differ widely from the mean value which is $20{\cdot}7 \times 10^{12}$ sec.$^{-1}$ The mean value for the elements, given in a former paper*, was $20{\cdot}9 \times 10^{12}$ sec.$^{-1}$, when the same factor was employed in Lindemann's formula as that here used. It has been pointed out previously† that the chance of these results for the lithium compounds (including $Li_3PO_4 . H_2O$) agreeing accidentally was about 1 in 280.

GROUP I.

Lithium Compounds (N=3).

Name.	Formula.	N.	$\nu \times 10^{-12}$.	$N\nu \times 10^{-12}$.
LITHIUM				
amide	$LiNH_2$	12	6·073	$3\frac{1}{2} \times 20{\cdot}8$
bromide	$LiBr$	38	3·234	$6 \times 20{\cdot}5$
carbonate	Li_2CO_3	36	3·435	$6 \times 20{\cdot}6$
chloride (*a*)...........	$LiCl$	20	5·079	$5 \times 20{\cdot}3$
fluoride	LiF	12	9·207	$5\frac{1}{2} \times 20{\cdot}1$
iodide	LiI	56	2·227	$6 \times 20{\cdot}8$
nitrate..................	$LiNO_3$	34	2·821	$4\frac{1}{2} \times 21{\cdot}3$
perchlorate............	$LiClO_4$	52	1·742	$4\frac{1}{2} \times 20{\cdot}1$
silicate (*b*)	Li_2SiO_3	44	3·756	$8 \times 20{\cdot}8$
sulphate acid	$LiHSO_4$	52	1·636	$4 \times 21{\cdot}3$
sulphate	Li_2SO_4	54	2·680	$7 \times 20{\cdot}7$

(*a*) Richards and Meldrum (1917) give the melting-point of pure lithium chloride as 613° C. This makes for better agreement, giving $N\nu \times 10^{-12} = 5 \times 20{\cdot}6$.

(*b*) The melting-point of lithium silicate is given as a standard temperature (1201° C.) in the Smithsonian Physical Tables.

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* H. S. Allen, Phil. Mag. vol. xxxiv. p. 478 (1917).

† H. S. Allen, Phil. Mag. vol. xxxv. p. 338 (1918).

Sodium Compounds (N=11).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Sodium				
borate tetra-	$Na_2B_4O_7$	98	1·596	7½×20·9
bromate	$NaBrO_3$	70	1·801	6 ×21·0
bromide	$NaBr$	46	3·037	6¾×20·7
carbonate	Na_2CO_3	52	2·878	7 ×21·4
chlorate	$NaClO_3$	52	1·962	5 ×20·4
chloride (*a*)	$NaCl$	28	4·407	6 ×20·6
fluoride	NaF	20	6·794	6½×20·9
hydroxide	$NaOH$	20	6·787	6½×20·9
iodide	NaI	64	2·268	7 ×20·7
nitrate	$NaNO_3$	42	2·417	5 ×20·3
nitrite (*b*)	$NaNO_2$	34	2·724	4½×20·6
phosphate meta	$Na_4P_4O_{12}$	200	0·829	8 ×20·7
sulphate	Na_2SO_4	70	2·337	8 ×20·4
sulphate acid (*c*)	$NaHSO_4$	60	1·859	5½×20·3

(*a*) Melting-point of sodium chloride (801° C.) has been used as a standard temperature.

(*b*) Melting-point of sodium nitrite, 271° C. (Divers, 1899),

(*c*) Melting-point of $NaHSO_4$ "over 315° C." (Gmelin-Kraut).

Potassium Compounds (N=19).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Potassium				
arsenate acid	KH_2AsO_4	86	1·364	5½×21·3
bromate	$KBrO_3$	78	1·708	6½×20·5
bromide	KBr	54	2·575	6¾×20·6
carbonate	K_2CO_3	68	2·310	7½×21·0
chlorate	$KClO_3$	60	1·886	5½×20·6
chloride	KCl	36	3·493	6 ×20·9
chromate	K_2CrO_4	94	1·882	8½×20·8
dichromate	$K_2Cr_2O_7$	142	0·974	6½×21·4
fluoride	KF	28	4·789	6½×20·6
formate	$KCHO_2$	42	1·955	4 ×20·5
hydroxide	KOH	28	3·431	4½×21·3
iodate	KIO_3	96	1·609	6 ×20·4
iodide	KI	72	1·988	7 ×20·4
iodide tri-	KI_3	178	0·543	4½×21·5
nitrate	KNO_3	50	2·080	5 ×20·8
perchlorate	$KClO_4$	68	2·046	6½×21·4
periodate	KIO_4	104	1·488	7½×20·6
phosphate dihydrogen (*a*)	KH_2PO_4	68	1·308	4¼×20·9
sulphate (*b*)	K_2SO_4	86	2·121	9 ×20·3
,, acid	$KHSO_4$	68	1·536	5 ×20·9
sulphocyanate	$KCNS$	48	1·778	4 ×21·3

(*a*) Melting-point of the anhydrous salt, 96° C., Tilden (1884).

(*b*) The melting-point of potassium sulphate (1070° C.) is used as a standard temperature.

The results for sodium and potassium compounds are specially interesting, and should be compared with one another where that is possible. There are eleven cases in which compounds of sodium are represented in the second list by the corresponding potassium salts. For five of these (the bromide, chloride, fluoride, iodide, nitrate) the frequency number is the same for the sodium as for the potassium salt. For the bromate, carbonate, chlorate, the frequency number for the potassium compound is greater by $\frac{1}{2}$ than that for the sodium compound. For normal potassium sulphate $N\nu \times 10^{-12} = 9 \times 20{\cdot}3$, as compared with $8 \times 20{\cdot}4$ for the sodium salt. The results for the acid sulphates and the hydroxides are less reliable.

The bromides of sodium and potassium require special consideration. For NaBr the melting-points recorded fall between the early value of Carnelley (708° C.) and that of Ruff and Plato (765° C.). Taking the latter value, $N\nu = 139{\cdot}7 \times 10^{12}$. For KBr Carnelley gave the value 699° C., whilst Ruff and Plato in 1903 found 750° C. Using the highest value according to the principle adopted generally, it is found that $N\nu = 139{\cdot}0 \times 10^{12}$. Thus if the melting-points of Ruff and Plato are correct, the values of $N\nu$ for NaBr and KBr are in close agreement with one another; but they cannot be represented by using frequency numbers of the form n or $n + \frac{1}{2}$ unless high values for ν_A are employed, the product for NaBr being equal to $6\frac{1}{2} \times 21{\cdot}5 \times 10^{12}$ and that for KBr being equal to $6\frac{1}{2} \times 21{\cdot}4 \times 10^{12}$. A redetermination of the density and melting-point for each of these salts is to be desired. It may be noticed that both LiBr and RbBr give normal results, the value of $N\nu$ for the former salt being $6 \times 20{\cdot}5 \times 10^{12}$, for the latter $7 \times 20{\cdot}5 \times 10^{12}$; the melting-point of CsBr is not recorded.

Assuming that the value of $N\nu \times 10^{-12}$ for NaBr is written in the form $6\frac{1}{2} \times 21{\cdot}5$, calculation of the probability by the formula of Laplace shows that the chance of an accidental concordance in the values of ν_A for the sodium salts is about 1 in 36.

The agreement between the values of ν_A in the case of the compounds of rubidium and cæsium is only moderately good. Perhaps it may be safe to predict from a comparison of the results with those already given that the melting-point of cæsium iodide will prove to be higher than the recorded value, 621° C., so as to give a frequency number 8 instead of $7\frac{1}{2}$. It will be noticed that in general the frequency numbers for the compounds of the alkali metals tend to increase as the atomic number of the metal increases.

Rubidium Compounds (N=37).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Rubidium				
bromide	RbBr	72	1·990	7 ×20·5
chloride (*a*)	RbCl	54	2·310	6 ×20·8
fluoride	RbF	46	3·021	6½×21·4
hydroxide	RbOH	46	2·297	5 ×21·1
iodide	RbI	90	1·616	7 ×20·8
oxide di-	Rb_2O_2	90	1·674	7 ×21·5
pentasulphide.........	Rb_2S_5	154	0·685	5 ×21·1

(*a*) Melting-point of rubidium chloride, 714° C. (Richards & Meldrum, 1917).

Cæsium Compounds (N=55).

Cæsium				
chloride (*a*)............	CsCl	72	2·065	7 ×21·2
iodide	CsI	108	1·480	7½×21·3
nitrate..................	$CsNO_3$	86	1·542	6½×20·4
oxide tri-	Cs_2O_3	134	1·076	7 ×20·6
„ tetr-	Cs_2O_4	142	1·073	7½×20·3
pentasulphide..	Cs_2S_5	190	0·612	5½×21·1

(*a*) Melting-point of cæsium chloride, 645° C. (Richards & Meldrum, 1917).

Only a few copper compounds can be included ; it is of interest to note that the frequency number for cupric chloride is 6½ as compared with 6 for the cuprous salt.

Copper Compounds (N=29).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Cuprous				
bromide	CuBr	64	2·267	7 ×20·7
chloride	CuCl	46	2·726	6 ×20·9
iodide	CuI	82	2·048	8 ×21·0
sulphide	Cu_2S	74	3·002	11 ×20·2
Cupric				
chloride	$CuCl_2$	63	2·136	6½×20·7
oxide	CuO	37	5·458	10 ×20·2

Silver Compounds (N=47).

Silver				
bromide	AgBr	82	1·935	7½×21·2
chlorate	$AgClO_3$	88	1·421	6 ×20·8
chloride	AgCl	64	2·343	7 ×21·4
fluoride	AgF	56	2·609	7 ×21·0
iodide	AgI	100	1·657	8 ×20·7
nitrate (*a*)	$AgNO_3$	78	1·543	6 ×20·1
phosphate ortho......	Ag_3PO_4	188	1·309	15 ×20·6
„ pyro	$Ag_4P_2O_7$	274	0·756	10 ×20·7
sulphate	Ag_2SO_4	142	1·378	8½×20·8
sulphide	Ag_2S	110	2·004	10½×21·0
telluride	Ag_2Te	146	1·686	12 ×20·5

(*a*) Silver nitrate changes from the rhombic to the hexagonal-rhombohedric form at 159°·8 C. The density of the rhombic form has been used in the calculation.

The salts of silver show fair agreement with the proposed relation. It is doubtful whether silver nitrate should have been included, as the density corresponds to a crystalline form different from that to which the melting-point applies.

No results are available for compounds of gold.

The melting-point and the density are known for the greater number of the halogen derivatives of Group I., so that it is possible to institute a comparison between the frequency numbers for this group of compounds. The results are collected in the Table following, which contains the atomic or molecular number in italics and the product $N\nu \times 10^{-12}$ in roman numerals. The values for the thallium compounds have been added as they resemble closely the corresponding rubidium compounds*. A close examination of the table will reveal many interesting relationships; the close resemblance between the sodium and the potassium salts is at once evident; the frequency numbers for these compounds exceed the corresponding numbers for the lithium salts by unity (the bromide excepted); the frequency numbers for cuprous chloride, bromide, iodide form the sequence 6, 7, 8; the frequency numbers for silver chloride, bromide, iodide form the sequence 7, 7½, 8. Further, it will be noticed that silver chloride, cuprous bromide, sodium iodide have the same frequency number, 7; cæsium chloride, rubidium bromide, potassium iodide form a similar chain with frequency number, 7.

Monohalides of Group I.

Element.	Fluoride F=*9*.	Chloride Cl=*17*.	Bromide Br=*35*.	Iodide I=*53*.
Li=*3*	*12* 5½×20·1	*20* 5×20·3	*38* 6×20·5	*56* 6×20·8
Na=*11*	*20* 6½×20·9	*28* 6×20·6	*46* 6¾×20·8	*64* 7×20·7
K=*19*	*28* 6½×20·6	*36* 6×20·9	*54* 6¾×20·6	*72* 7×20·4
Cu=*29*	*38* ——	*46* 6×20·9	*64* 7×20·7	*82* 8×21·0
Rb=*37*	*46* 6½×21·4	*54* 6×20·8	*72* 7×20·5	*90* 7×20·8
Ag=*47*	*56* 7×21·0	*64* 7×21·4	*82* 7½×21·2	*100* 8×20·7
Cs=*55*	*64* ——	*72* 7×21·1	*90* ——	*108* 7½×21·3
Tl=*81*	*90* ——	*98* 6×21·1	*116* 8×21·3	*134* 8×20·9

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[*] Tutton, Roy. Soc. Proc. vol. lxxix. p. 351 (1907).

GROUP II.

On the whole the results for compounds containing elements belonging to the second group are less reliable than those already considered. This arises partly from the fact that in many cases there is uncertainty as to the true value of the melting-point, no values later than those of Carnelley being recorded for several compounds[*]; further, the product $N\nu$ is in general greater for these compounds than it is for the corresponding compounds containing elements of the first group, so that larger frequency numbers are required. Consequently the frequency numbers given in the following Tables are not to be regarded in all cases as final; some of them—especially amongst the larger numbers—may require revision when more exact data have been obtained.

In spite of such uncertainties a comparison between the compounds in these Tables, and also between these and the corresponding compounds for the first group, will show many suggestive relationships.

GROUP II.

Beryllium Compounds (N=4).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
BERYLLIUM				
fluoride	BeF_2	22	5·213	$5\frac{1}{2}\times20{\cdot}9$
iodide	BeI_2	110	1·339	$7\times21{\cdot}0$

Magnesium Compounds (N=12).

MAGNESIUM				
chloride *C*	$MgCl_2$	46	2·807	$6\times21{\cdot}5$
fluoride *C*	MgF_2	30	5·446	$8\times20{\cdot}4$

Calcium Compounds (N=20).

CALCIUM				
aluminate	$CaAl_2O_4$	78	3·011	$11\frac{1}{2}\times20{\cdot}4$
bromide	$CaBr_2$	90	1·783	$8\times20{\cdot}1$
chloride	$CaCl_2$	54	2·302	$6\times20{\cdot}7$
fluoride	CaF_2	38	4·757	$9\times20{\cdot}1$
iodide	CaI_2	126	1·464	$9\times20{\cdot}5$
nitrate (*a*) *C*	$Ca(NO_3)_2$	82	1·715	$7\times20{\cdot}1$
nitride.................	Ca_3N_2	74	2·532	$9\times20{\cdot}8$
oxide	CaO	28	7·696	$10\frac{1}{2}\times20{\cdot}5$
silicate..................	$CaSiO_3$	58	3·531	$10\times20{\cdot}5$
sulphate	$CaSO_4$	68	2·979	$10\times20{\cdot}3$

(*a*) Highest density recorded, 2·472 (Landolt-Börnstein).

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[*] These cases are indicated in the tables by the letter C following the name of the compound.

Strontium Compounds (N=38).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
STRONTIUM				
bromide *C*	$SrBr_2$	108	1·512	8 ×20·4
chloride	$SrCl_2$	72	2·201	7½×21·1
fluoride *C*	SrF_2	56	3·037	8 ×21·3
hydroxide	$Sr(OH)_2$	56	2·204	6 ×20·6
iodide	SrI_2	144	1·197	8½×20·3
nitrate *C*	$Sr(NO_3)_2$	100	1·549	7½×20·7
oxide	SrO	46	6·195	14 ×20·4
silicate	$SrSiO_3$	76	2·940	11 ×20·3

Barium Compounds (N=56).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
BARIUM				
bromide	$BaBr_2$	126	1·531	9 ×21·3
carbonate (*a*)	$BaCO_3$	86	2·485	10½×20·4
chloride	$BaCl_2$	90	1·982	8½×21·0
fluoride	BaF_2	74	2·768	9 ×20·5
iodide	BaI_2	162	1·153	9 ×20·7
nitrate *C*	$Ba(NO_3)_2$	118	1·298	7½×20·4
silicate	$Ba(SiO_3)$	94	2·418	11 ×20·7

(*a*) "Schmilzt in CO_2 bei 1380° noch nicht" (Boeke).

Comparing together the compounds of calcium, strontium, and barium we find that, in general, the frequency number corresponding to a particular radicle tends to increase with an increase in the atomic number of the metal. Again, comparing the chloride, bromide, and iodide for a particular metal, it is noticed that the frequency number for the bromide is greater than that for the chloride, and the frequency number for the iodide is at least as great as that for the bromide. In the case of the fluoride, however, the frequency number is about the same as that for the iodide.

Zinc Compounds (N=30).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
ZINC				
acetate	$Zn(C_2H_3O_2)_2$	92	1·113	5 ×20·5
bromide *C*	$ZnBr_2$	100	1·408	7 ×20·1
chloride (*a*)	$ZnCl_2$	64	1·849	5½×21·5
fluoride *C*	ZnF_2	48	3·410	8 ×20·5
iodide *C*	ZnI_2	136	1·113	7½×20·5
sulphide	ZnS	46	3·936	9 ×20·1

(*a*) Melting-point of zinc chloride, 365° C. "All lower values are obtained from impure material."

Cadmium Compounds (N=48).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
CADMIUM				
bromide	$CdBr_2$	118	1·412	8 ×20·8
chloride	$CdCl_2$	82	1·846	7½×20·2
fluoride *C*	CdF_2	66	2·500	8 ×20·6
iodide	CdI_2	154	1·027	7½×21·1
sulphate	$CdSO_4$	96	2·153	10 ×20·7

Mercury Compounds (N=80).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
MERCURIC				
bromide	$HgBr_2$	150	0·927	6½×21·4
chloride	$HgCl_2$	114	1·200	6½×21·1
iodide	HgI_2	186	0·798	7 ×21·2
MERCUROUS				
iodide (*a*)	HgI	133	1·157	7¼×21·2

(*a*) Melting-point of mercurous iodide, 290° C., Yvon (1873).

Group III.

The compounds met with in connexion with elements of the third group do not require much discussion. The melting-points of boron oxide and boric acid are determinations of Carnelley in 1878, whilst those for aluminium bromide and aluminium iodide are of an even earlier date, for these latter compounds the degree of molecular association is uncertain also.

The results for the compounds of thallium form a fairly consistent series, and it is of interest to compare the frequency numbers with those previously given for the compounds of the alkali metals. The mean value of ν_A deduced from the thallium salts is $21{\cdot}05 \times 10^{12}$ sec.$^{-1}$, and the chance that the agreement should be purely accidental is about 1 in 39.

Group III.

Boron Compounds (N=5).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Boron				
oxide *C*	B_2O_3	34	3·185	5 × 21·7
sulphide tri	B_2S_3	58	1·613	4½ × 20·8
,, penta	B_2S_5	90	1·271	5½ × 20·8
Boric				
acid *C*	H_3BO_3	32	2·387	3½ × 21·8

Aluminium Compounds (N=13).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Aluminium				
bromide	Al_2Br_6	236	0·858	5 × 20·3
iodide	Al_2I_6	344	0·682	5½ × 21·3
oxide	Al_2O_3	50	4·949	12 × 20·6
sulphide	Al_2S_3	76	2·212	8 × 20·5

Yttrium Compound (N=39).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Yttrium				
chloride	$YtCl_3$	90	1·114	5 × 20·1

Lanthanum Compound (N=57).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Lanthanum				
chloride	$LaCl_3$	108	1·693	9 × 20·3

Cerium Compound (N=58).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Cerous				
chloride	$CeCl_3$	109	1·645	9 × 20·0

Thallium Compounds (N=81).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Thallium				
bromide	$TlBr_2$	116	1·466	8 × 21·3
carbonate	Tl_2CO_3	192	0·825	7½ × 21·1
chloride	$TlCl$	98	1·623	6 × 21·1
iodide	TlI	134	1·247	8 × 20·9
nitrate	$TlNO_3$	112	1·137	6 × 21·2
oxide (-ic)	Tl_2O_3	186	1·067	9½ × 20·9
perchlorate	$TlClO_4$	130	1·310	8 × 21·3
sulphate	Tl_2SO_4	210	0·981	10 × 20·6

GROUP IV.

The compounds of carbon are so numerous as to require separate consideration. It is only necessary to mention for the sake of comparison with compounds containing other elements of the same group, that for carbon tetrabromide (N=146) the value of Nν is $5\times 20{\cdot}5\times 10^{12}$, and for carbon trichloride (C_2Cl_6) the value is $4\frac{1}{2}\times 20{\cdot}6\times 10^{12}$. For silicon, titanium, and zirconium only a few results can be given.

GROUP IV.

Silicon Compound (N=14).

Name.	Formula.	N.	$\nu\times 10^{-12}$.	$N\nu\times 10^{-12}$.
SILICON				
tetrabromide	$SiBr_4$	154	0·553	4×21·3

Titanium Compounds (N=22).

Name.	Formula.	N.	$\nu\times 10^{-12}$.	$N\nu\times 10^{-12}$.
TITANIUM				
tetrafluoride	TiF_4	58	1·848	5×21·4
tetrabromide (*a*)......	$TiBr_4$	162	0·545	4×22·0
dioxide	TiO_2	38	5·536	10×21·0

(*a*) Melting-point, 39° C., Duppa (1856).

Zirconium Compound (N=40).

Name.	Formula.	N.	$\nu\times 10^{-12}$.	$N\nu\times 10^{-12}$.
ZIRCONIUM				
dioxide	ZrO_2	56	5·283	14×21·1

Disregarding the high value for ν_A in the case of titanium bromide, for which no modern determination of the melting-point is recorded, the values of ν_A for the compounds of this group are generally higher than those for the earlier groups. In the case of the lead salts, however, a lower value is found for the chloride, bromide, and iodide; but the close agreement between the values of the product Nν for these three salts deserves special notice, as also the agreement for the oxide and the sulphide.

Tin Compounds (N=50).

Name.	Formula.	N.	$\nu\times 10^{-12}$.	$N\nu\times 10^{-12}$.
STANNIC				
fluoride	SnF_4	86	2·049	$8\frac{1}{2}\times 20{\cdot}7$
iodide	SnI_4	262	0·491	6 ×21·4
oxide	SnO_2	66	3·353	$10\frac{1}{2}\times 21{\cdot}1$
STANNOUS				
bromide	$SnBr_2$	120	1·075	6 ×21·5
sulphide	SnS	66	2·780	9 ×20·4

Lead Compounds (N=82).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
LEAD				
bromide	$PbBr_2$	152	1·067	8 ×20·3
chloride	$PbCl_2$	116	1·414	8 ×20·5
iodide	PbI_2	188	0·863	8 ×20·3
oxide	PbO	90	2·443	10½×21·0
selenide	$PbSe$	116	2·029	11 ×21·4
sulphate	$PbSO_4$	130	1·795	11 ×21·2
sulphide	PbS	98	2·274	10½×21·2

Thorium Compound (N=90).

THORIUM				
chloride..................	$ThCl_4$	158	1·214	9×21·3

GROUP V.

Almost all the compounds so far considered have been metallic salts. In the fifth group a number of non-metallic compounds are met with; and although many of these conform to the proposed relation with frequency numbers n or $n+\frac{1}{2}$, several require the use of values $n+\frac{1}{4}$ or $n+\frac{3}{4}$. It is at present impossible to decide whether this means that the relation fails to hold in these two cases, or whether the apparent failure is to be attributed to our ignorance of the degree of molecular association in such compounds. It may be pointed out that the compounds in question belong to the "non-polar" type*. According to Langmuir solid polar compounds are built up of atoms bound together by secondary valencies, whilst solid non-polar compounds consist of "group molecules" in which the atoms are usually held together by primary valencies; these group molecules in turn are bound together by secondary valencies to form a large "crystal molecule." As data for determining the molecular frequency are available for only a small number of inorganic non-polar compounds, further discussion may be deferred till organic compounds are considered.

GROUP V.

Nitrogen Compounds (N=7).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
NITROGEN				
chlorophosphide ...	$N_3P_3Cl_6$	168	0·580	4½×21·6
pentoxide	N_2O_5	54	1·278	3¼×21·2
sulphide	N_4S_4	92	1·117	5 ×20·5

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* See Langmuir, Amer. Chem. Soc. Journ. vol. xxxviii. p. 2221 (1916).

Phosphorus Compounds (N=15).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
PHOSPHORIC				
acid ortho-	H_3PO_4	50	1·470	$3\frac{1}{2}\times21{\cdot}0$
PHOSPHOROUS				
acid hypo-	H_3PO_2	34	1·855	$3\ \times21{\cdot}0$
„ ortho-	H_3PO_3	42	1·794	$3\frac{1}{2}\times21{\cdot}5$
PHOSPHORUS				
oxide tri-	P_4O_6	108	0·761	$4\ \times20{\cdot}5$
oxybromide	$POBr_3$	128	0·706	$4\frac{1}{2}\times20{\cdot}1$
oxybromdichloride	$POBrCl_2$	92	0·807	$3\frac{1}{2}\times21{\cdot}2$
oxychloride	$POCl_3$	74	0·920	$3\frac{1}{4}\times20{\cdot}9$
sulphobromide	$PSBr_3$	136	0·659	$4\frac{1}{4}\times21{\cdot}1$
sesquisulphide	P_4S_3	108	0·913	$4\frac{3}{4}\times20{\cdot}7$
pentasulphide	P_2S_5	110	1·024	$5\frac{1}{2}\times20{\cdot}5$

Vanadium Compound (N=23).

VANADIUM				
pentoxide	V_2O_5	86	1·841	$7\frac{1}{2}\times21{\cdot}1$

Columbium Compound (N=41).

COLUMBIUM				
pentachloride	$CbCl_5$	126	0·876	$5\frac{1}{2}\times20{\cdot}1$

Arsenic Compounds (N=33).

ARSENIC				
disulphide	As_2S_2	98	1·299	$6\ \times21{\cdot}2$
ARSENOUS				
bromide	$AsBr_3$	138	0·688	$4\frac{1}{2}\times21{\cdot}1$
iodide	AsI_3	192	0·629	$6\ \times20{\cdot}1$
oxide	As_4O_6	180	0·712	$6\ \times21{\cdot}3$
selenide	As_2Se_3	168	0·910	$7\frac{1}{2}\times20{\cdot}4$
sulphide	As_2S_3	114	1·144	$6\ \times21{\cdot}7$

Antimony Compounds (N=51).

ANTIMONY				
bromide tri-	$SbBr_3$	156	0·702	$5\ \times21{\cdot}9$?
chloride tri-	$SbCl_3$	102	0·907	$4\frac{1}{2}\times20{\cdot}6$
fluoride tri-	SbF_3	78	1·602	$6\ \times20{\cdot}8$
iodide tri-	SbI_3	210	0·615	$6\ \times21{\cdot}5$

Bismuth Compounds (N=83).

BISMUTH				
bromide	$BiBr_3$	188	0·749	$7\ \times20{\cdot}1$?
chloride di-	$BiCl_2$	117	0·999	$5\frac{1}{2}\times21{\cdot}2$
chloride tri-	$BiCl_3$	134	0·943	$6\ \times21{\cdot}1$

Tantalum Compounds (N=73).

TANTALUM				
chloride	$TaCl_5$	158	0·777	$6\ \times20{\cdot}5$
fluoride	TaF_5	118	0·930	$5\ \times21{\cdot}9$?

GROUP VI.

In the sixth group of the Periodic Table only a dozen compounds need be considered. Several of these are oxides, and it is noticeable, both here and elsewhere, that these in general necessitate large values for ν_A.

It is interesting to find that salts of the heaviest metal, uranium, appear to conform to the same rule as salts of the lightest metal, lithium. Thus lithium fluoride, with molecular number 12, has for $N\nu$ the value $5\frac{1}{2}\times 21{\cdot}0\times 10^{12}$, uranium hexafluoride, with molecular number 146, has the value $5\times 21{\cdot}0\times 10^{12}$; lithium iodide ($N=56$) gives $N\nu=6\times 20{\cdot}8\times 10^{12}$, whilst uranium tetra-iodide, with the large molecular number of 304, gives $N\nu=9\times 20{\cdot}7\times 10^{12}$.

GROUP VI.

Sulphur Compounds ($N=16$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Sulphur trioxide β	$(SO_3)_2$	80	0·816	$3\times21{\cdot}7$
Sulphuric acid pyro	$H_2S_2O_7$	90	0·890	$4\times20{\cdot}0$

Selenium Compounds ($N=34$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Selenium dioxide	SeO_2	50	2·473	$6\times20{\cdot}6$
oxychloride	$SeOCl_2$	76	0·984	$3\frac{1}{2}\times21{\cdot}4$
Selenic acid	H_2SeO_4	68	1·269	$4\times21{\cdot}6$

Tellurium Compound ($N=52$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Tellurium tetrabromide	$TeBr_4$	192	0·792	$7\frac{1}{2}\times20{\cdot}3$

Chromium Compounds ($N=24$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Chromium trioxide	CrO_3	48	2·030	$4\frac{1}{2}\times21{\cdot}7$
Chromic oxide	Cr_2O_3	72	3·876	$13\times21{\cdot}5$
Chromous fluoride	CrF_2	42	4·300	$8\frac{1}{2}\times21{\cdot}2$

Molybdenum Compound ($N=42$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Molybdenum trioxide	MoO_3	66	2·576	$8\times21{\cdot}2$

Uranium Compounds ($N=92$).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
Uranium fluoride hexa-	UF_6	146	0·719	$5\times21{\cdot}0$
iodide tetra-	UI_4	304	0·614	$9\times20{\cdot}7$
oxide di-	UO_2	108	3·182	$16\times21{\cdot}5$

GROUP VII.

In the seventh group the only metallic compounds represented are those of manganese. The four salts in question give concordant values for ν_A. The chance of an accidental agreement is about 1 in 180.

GROUP VII.

Manganese Compounds (N=25).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
MANGANESE				
chloride	$MnCl_2$	59	2·252	$6\frac{1}{2}\times20{\cdot}4$
fluoride	MnF_2	43	3·756	$8\ \times20{\cdot}2$
silicate..................	$MnSiO_3$	63	3·057	$9\frac{1}{2}\times20{\cdot}3$
sulphate	$MnSO_4$	73	2·107	$7\frac{1}{2}\times20{\cdot}5$

In the same group are a number of compounds containing chlorine or iodine. Most of these require the provisional use of frequency numbers of the form $n+\frac{1}{4}$ or $n+\frac{3}{4}$.

Chlorine and Iodine Compounds.

Chlorosulphonic acid...	$ClSO_2 . OH$	58	1·335	$3\frac{3}{4}\times20{\cdot}6$
Iodine monochloride α.	ICl	70	1·125	$3\frac{3}{4}\times21{\cdot}0$
Iodine tetrachloride ...	ICl_3	104	0·836	$4\frac{1}{4}\times20{\cdot}5$
Iodine fluoride	IF_5	98	0·869	$4\ \times21{\cdot}3$
Iodic acid	HIO_3	78	1·342	$5\ \times20{\cdot}9$

GROUP VIII.

In the eighth group the data are scanty, only six compounds of iron being represented, two of cobalt, and one each of nickel, ruthenium, and osmium. Rhodium, palladium, platinum, and iridium are not represented at all. The results, as far as they go, tend to show that the compounds of this group fall into line with the metallic salts of the earlier groups, the only serious discrepancy being the case of cobalt carbonyl where it is possible that molecular association occurs.

GROUP VIII.

Iron Compounds (N=26).

Name.	Formula.	N.	$\nu\times10^{-12}$.	$N\nu\times10^{-12}$.
IRON				
disulphide	FeS_2	58	3·748	10½×20·7
phosphide	Fe_2P	67	3·653	12 ×20·4
FERRIC				
chloride	FeCl	83	1·498	6 ×20·7
oxide	Fe_2O_3	76	3·330	12 ×21·1
FERROUS				
sulphide	FeS	42	4·856	10 ×20·4
FERROSO-FERRIC				
oxide	Fe_3O_4	110	2·762	15 ×20·3

Cobalt Compounds (N=27).

COBALT				
carbonyl	$Co(CO)_4$	83	0·927	3¾×20·5
COBALTOUS				
sulphate	$CoSO_4$	75	2·477	9 ×20·6

Nickel Compound (N=28).

NICKEL				
sulphide mono-	NiS	44	3·914	8½×20·3

Ruthenium Compound (N=44).

RUTHENIUM				
tetroxide	RuO_4	76	1·399	5 ×21·3

Osmium Compound (N=76).

OSMIUM				
tetroxide	OsO_4	108	1·079	5½×21·2

In the foregoing pages a number of cross-correspondences between the values of the frequency numbers for related compounds have been noticed. But instances might also be cited in which the frequency number has not the value which might have been predicted for it by analogy. For example, the chlorides of potassium and rubidium have the same frequency number 6, but though the frequency number for calcium chloride is 6 that for strontium chloride is 7½; again the iodides of calcium, strontium, barium give the unexpected sequence of numbers 9, 8½, 9. In view of the hypothesis that has been put forward relating the frequency number to the number of valency electrons, it may be suggested that the frequency number determined from the melting-point may not have, in all cases, a unique value, In consequence of the presence of traces of impurity, or through imperfect crystallization, it is at least possible that fusion may sometimes occur when the number of valency electrons is smaller than the normal value.

It is noticeable that for a large number of compounds the frequency number is one of the series – – $4\frac{1}{2}$, 6, $7\frac{1}{2}$, 9, – – ; suggesting that the factor 3 plays an important part in the determination of its value. It may not be entirely accidental that a number belonging to this series frequently occurs when the compound contains an element of valency 3 (*e. g.* amongst the compounds of As, Sb, Bi, and Fe). It is not unlikely that the factor 3 should occur in such cases if it is supposed that the chemical valency is associated with a certain number of valency electrons.

The facts recorded in this paper are regarded as proving that the characteristic frequency calculated for a metallic compound by means of Lindemann's formula can, in general, be expressed in the form $n\nu_A$ or in the form $(n+\frac{1}{2})\nu_A$, where n is an integer and ν_A is approximately 21×10^{12} sec.$^{-1}$. The evidence for this may not be considered conclusive when compounds of one single metal are examined, but the evidence in the present case is cumulative. The degree of probability of this result may be small when compounds of a single element are found to conform to the rule; but it is larger when the compounds of all the elements in a group are found to give concordant values for ν_A, and becomes very great when the number of compounds is extended to include all the groups in Mendeléeff's classification. Further, these results for compounds and those recorded previously for the elements mutually support one another, and lead to the conclusion that we have to deal with a property that is fundamental and characteristic of the solid state of matter. It is to be borne in mind that the formula of Lindemann is looked upon as giving only an approximate value for the characteristic frequency, and the final justification for the proposed relation must come from observations on the specific heat of compounds at low temperatures.

LIV. *On the Operator ∇ in Combination with Homogeneous Functions.* Second Paper. *By* Frank L. Hitchcock, *Massachusetts Institute of Technology, Cambridge, Mass.**

1. *INTRODUCTION.*—In a former paper (Phil. Mag. vol. xxix. May 1915, p. 700), two theorems were developed, which yield properties of any homogeneous vector in connexion with the operators $S\nabla$ and $V\nabla$. I now propose to study in a somewhat similar manner the Laplacean operator $\triangle$, or $-\nabla^2$, defined by

$$\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in three dimensions (with analogous definition for two dimensions). The results obtained are of two sorts. First, as a more elementary matter, I shall show how a large number of reduction formulas may be written down, useful in the evaluation of multiple integrals. Later I shall prove a theorem similar to those of the first paper, giving the form of the function by which various homogeneous scalars or vectors differ from harmonic functions of the same degree.

2. *The fundamental reduction formula.*—Let us first take, as in the former paper, $F(\rho)$ a function of the point-vector ρ, homogeneous of degree m. In this case, however, I restrict m to be positive, and shall at first suppose $F(\rho)$ to be a scalar function. Write $r=T\rho$, and let $f(r)$ be a scalar function of r. Consider the problem of integrating the product $f(r).F(\rho)$ over the volume of a sphere of radius a with centre at the origin of coordinates: I shall prove the following reduction formula,—

Formula (A)

$$\iiint f(r).F_m(\rho)dV=\frac{1}{ma^{m+1}}\int_0^a r^{m+2}f(r)dr.\iiint\triangle F_m(\rho)dV, \quad (A)$$

where the triple integrations are to be taken over the volume of the sphere. We begin with Gauss's theorem,

$$\iint(lX+mY+nZ)dS=\iiint\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)dV, \quad . \quad (2)$$

where X, Y, and Z are scalars, l, m, and n are the direction cosines of the outward normal to a closed surface, and the integrations are carried over the surface and throughout its

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* Communicated by the Author.

volume, respectively. On the surface of the sphere l, m, and n become $\frac{x}{a}, \frac{y}{a}$, and $\frac{z}{a}$. Now let $\mathrm{X}=\frac{\partial \mathrm{F}}{\partial x}$, $\mathrm{Y}=\frac{\partial \mathrm{F}}{\partial y}$, and $\mathrm{Z}=\frac{\partial \mathrm{F}}{\partial z}$. Substituting in (2) we have

$$\frac{1}{a}\iint\left(x\frac{\partial \mathrm{F}}{\partial x}+y\frac{\partial \mathrm{F}}{\partial y}+z\frac{\partial \mathrm{F}}{\partial z}\right)d\mathrm{S}=\iiint\Delta\mathrm{F}\,d\mathrm{V}. \quad . \quad (3)$$

The left side simplifies by Euler's theorem giving

$$\frac{m}{a}\iint\mathrm{F}_m d\mathrm{S}=\iiint\Delta\mathrm{F}_m d\mathrm{V}, \quad . \quad . \quad . \quad . \quad (4)$$

where F_m is any homogeneous scalar function of positive degree m and the integrations are with reference to the sphere of radius a with centre at the origin.

Next start afresh with the left member of (A). By the form of the definition of a homogeneous function which was laid down in the first paper, $\mathrm{F}_m(\rho)=r^m\mathrm{F}(u)$, where $\mathrm{F}(u)$ denotes a function of $\mathrm{U}\rho$, that is, a function of a point on a unit sphere. Therefore by performing the integration with respect to r (writing $d\mathrm{S}=r^2d\mathrm{S}_0$ and $d\mathrm{V}=r^2d\mathrm{S}_0dr$), we shall obtain

$$\iiint f(r)\,.\,\mathrm{F}_m(\rho)d\mathrm{V}=\int_0^a r^{m+2}f(r)dr\,.\iint\mathrm{F}(u)d\mathrm{S}_0, \quad . \quad (5)$$

where the double integral is carried over the surface of a sphere of unit radius. But on the surface of the sphere of radius a we have r constant and equal to a, hence

$$\iint\mathrm{F}(u)d\mathrm{S}_0=\frac{1}{a^{m+2}}\iint\mathrm{F}_m(\rho)d\mathrm{S}, \quad . \quad . \quad . \quad . \quad (6)$$

the integral on the right being taken over the sphere of radius a. The formula (A) now follows at once by elimination of the double integrals between (4), (5), and (6).

By (A) we may solve a variety of simple problems. As an example, if the density at a point in the sphere is equal to the distance from the centre, the moment of inertia about OZ is found by writing $f(r)=r$, and $\mathrm{F}(\rho)=x^2+y^2$, when (A) becomes

$$\iiint r(x^2+y^2)d\mathrm{V}=\frac{1}{2a^3}\int_0^a r^5dr\,.\iiint 4d\mathrm{V}=\frac{a^3}{12}\cdot\frac{16}{3}\pi a^3=\frac{4}{9}\pi a^6.$$

3. *Extension to vector functions.*—In proving (A) by the aid of Gauss's theorem, $(\mathrm{F}\rho)$ was a scalar function of ρ. There is nothing in the result itself to require this restriction.

In fact, since $\triangle$ is a scalar operator, the formula can be applied in turn to the scalar components of a vector $F(\rho)$, hence is true for the vector. That is, $F(\rho)$ in formula (A) may be either vector, scalar, or quaternion, $f(r)$ remaining a scalar.

4. *Extension by geometrical transformation.*—Any problem which can be worked out by (A) for the sphere may be made to yield an indefinite number of other examples by performing geometrical transformations upon both members of (A). Let $\gamma(\rho)$ be a vector function of (ρ). If we replace ρ by $\gamma(\rho)$ and if $d\gamma(\rho)=\phi\, d\rho$, the Jacobian of the transformation is Hamilton's third invariant for the linear vector function ϕ which in this case I shall call J. The Laplacean operator $\triangle$, which is the same as $-\nabla^2$, is transformed into $-(\phi'^{-1}\nabla)^2$, that is to say the operator ∇ is transformed by the reciprocal conjugate of the operator ϕ. And r is transformed into $T\gamma(\rho)$, the tensor of the transformed point-vector. Formula (A) now becomes, supposing that we began with a sphere of radius unity,

$$\iiint f(T\gamma\rho)\,.\,F_m(\gamma\rho)\,.\,J(\rho)dV$$
$$=-\frac{1}{m}\int_0^1 r^{m+2}f(r)\,dr\,.\iiint(\phi'^{-1}\nabla)^2F_m(\gamma\rho)\,.\,J(\rho)dV,$$

where both triple integrals are taken throughout the transformed surface. On the right we note that ∇, as indicated by the stop, acts on $F(\gamma\rho)$ but not on $J(\rho)$. In general ∇ acts on the constituents of ϕ. The propriety of the integrations is assumed. In case $\gamma\rho$ is a homogeneous function of ρ of degree n, $F_m(\gamma\rho)$ is homogeneous of degree mn, and may therefore be taken as any homogeneous function which gives the problem a meaning.

As perhaps the simplest illustration, let $\gamma\rho=\phi\rho$, a self-conjugate linear vector function. The unit sphere is carried into an ellipsoid. The scalar $f(r)$ becomes a function whose level surfaces are similar to and concentric with the bounding ellipsoid. Since $n=1$, $F_m(\gamma\rho)$ may be any homogeneous function of ρ under the same restrictions of continuity as in Gauss's theorem. The constant Jacobian cancels out. Hence

$$\iiint f(T\phi\rho)\,.\,F_m(\rho)dV=\frac{1}{m}\int_0^1 r^{m+2}f(r)dr\,.\iiint(\phi^{-1}\nabla)^2F_m(\rho)dV,$$

where ϕ is any self-conjugate linear vector function with constant constituents, and the triple integrals imply integration over the ellipsoid $T^2\phi\rho=1$.

To take a specific example, let us extend to the volume of the ellipsoid the problem worked at the close of Art. 2, the equation of the surface being $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density equal to the square root of the left side of this equation, and the moment of inertia about OZ required. The transformation is given by $\phi\rho=i\frac{x}{a}+j\frac{y}{b}+k\frac{z}{c}$. The Laplacean operator $\triangle$ transforms into $-(\phi^{-1}\nabla)^2$, that is into

$$a^2\frac{\partial}{\partial x^2}+b^2\frac{\partial}{\partial y^2}+c^2\frac{\partial}{\partial z^2},$$

and the transformed equation reads

$$\iiint(x^2+y^2)\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\,d\mathrm{V}$$

$$=\tfrac{1}{2}\int_0^1 r^5dr\,.\iiint\left(a^2\frac{\partial}{\partial x^2}+b^2\frac{\partial}{\partial y^2}+c^2\frac{\partial}{\partial z^2}\right)(x^2+y^2)d\mathrm{V},$$

where the volume integrals are taken over the volume of the ellipsoid. The right side by inspection is $\frac{1}{6}\iiint(a^2+b^2)d\mathrm{V}$, or $\frac{2}{9}\pi(a^2+b^2)abc$.

5. *Inverse method of transformation.*—When a surface is given, and when we can transform it into a sphere, we can apply (A) directly if the integrand is homogeneous after the transformation. The transformation is then the inverse of $\gamma\rho$ in the last article. This method, while resting on the same principles as the foregoing, differs in using an untransformed $\triangle$. To illustrate, let $\rho=\lambda(\rho')=\gamma^{-1}(\rho')$ be the transformation which carries a given surface into a sphere. Let the Jacobian of the transformation λ be J', and let λ be homogeneous of degree n. We then obtain by (A)

$$\iiint f(\mathrm{T}\gamma\rho)\,.\,\mathrm{F}_m(\rho)d\mathrm{V}$$

$$=\iiint f(r')\,.\,\mathrm{F}_m(\lambda_n\rho)\,.\,\mathrm{J}'(\rho')d\mathrm{V}', \text{ by the trf. } \rho=\lambda\rho',$$

$$=\frac{1}{pa^{p+1}}\int_0^a r^{p+2}f(r)dr\,.\iiint\triangle[\mathrm{F}_m(\lambda_n\rho')\,.\,\mathrm{J}'(\rho')]d\mathrm{V}',$$

where $p=mn+3(n-1)$, the degree of the integrand after the transformation. The integration on the left is carried over the volume of the original surface, the others are over a sphere of radius a and centre at the origin of coordinates into which the given surface is carried by the transformation $\rho=\lambda_n(\rho')$, or simply $\lambda\rho'$. We note that $\triangle$, as indicated by the brackets, operates on J' as well as F. To take an example, the problem worked at the close of Art. 4 would read by the present method

$$\iiint(x^2+y^2)\mathrm{T}\gamma\rho\, d\mathrm{V}=\tfrac{1}{2}\int_0^1 r^5 dr\,.\iiint\triangle(a^2x'^2+b^2y'^2)\,abc\, d\mathrm{V}',$$

where the right-hand integral is over a unit sphere, leading to the same answer as before. Here $\lambda\rho'=iax'+jby'+kcz'$, so that the transformation is equivalent to putting ax' for x, &c.

6. *Analogue in two dimensions.*—By parallel reasoning we may work problems of the same character in a plane. Thus if F be homogeneous in a plane point-vector ρ, that is in x and y,

and if $\triangle=\dfrac{\partial}{\partial x^2}+\dfrac{\partial}{\partial y^2}$,

Formula (B)

$$\iint f(r)\,.\,\mathrm{F}_m(\rho)d\mathrm{A}=\frac{1}{ma^m}\int_0^a r^{m+1}f(r)dr\,.\iint\triangle\mathrm{F}_m(\rho)d\mathrm{A},\ .\quad \text{(B)}$$

where the double integrals are to be taken over the area of a circle of radius a with centre at the origin of ρ.

As an example easy to verify by other methods, let it be required to carry the integral $\displaystyle\iint\frac{x^2+y^2}{2ax}d\mathrm{A}$ over the area included within the circle $x^2+y^2=2ax$. The transformation $x=x'^2$, $y=x'y'$, carries the area within the circle into the area included between the axis of X′ and the new circle $x'^2+y'^2=2a$, on either side of OX′. The Jacobian is $2x'^2$. The transformed integral is $\displaystyle\iint\frac{r'^2}{a}x'^2 d\mathrm{A}'$, which if taken over the whole of the new circle gives twice the original quantity. We therefore have, dropping accents and applying (B), (remembering that the new radius is $\sqrt{2a}$),

$$\frac{1}{2a}\cdot\frac{1}{2.2a}\int_0^{\sqrt{2a}} r^5 dr\,.\iint\triangle x^2 d\mathrm{A}=\tfrac{2}{3}\pi a^2,$$

the double integral being over the whole circle.

7. *Application to a hemisphere.*—By the combined use of (A) and (B) we may obtain a reduction process for integrating a homogeneous function over a hemispherical volume. The relation (5) is unchanged in form, but we now carry the

triple and the double integrals over the volume and *curved* surface of the hemisphere, respectively. In employing Gauss's theorem, however, we must add a term to the left member of (2) corresponding to integration over the base of the hemisphere. If the base is taken in the xy plane l and m vanish and $n=1$. If $Z=\frac{\partial F}{\partial z}$ as before, the term to be added to the left of the equations (2)–(4) is $\iint \frac{\partial F}{\partial z}\,dA$, where the integration is over a circle of radius a in the xy plane with centre at the origin. Eliminating the integration over the curved surface as before we now have

Formula (C)

$$\iiint f(r)\,.\,F_m(\rho)dV=\frac{1}{ma^{m+1}}\int_0^a r^{m+2}f(r)dr\,.\left[\iint\frac{\partial F}{\partial z}dA+\iiint\Delta F\,dV\right] \quad . \quad . \quad . \quad (C)$$

where the triple integrals are carried over the volume of the hemisphere, the double integral over its base, the latter to be further reduced by (B) if necessary.

As an elementary example, let us find the gravitational moment of the hemisphere with respect to its base, the density being any function of the radius. We have

$$\iiint f(r)\,.\,z\,dV=\frac{1}{a^2}\int_0^a r^3 f(r)\,dr\,.\,\left[\iint dA+0\right]=\pi\int_0^a r^3 f(r)\,dr.$$

Such formulas and examples can be multiplied in great number. Thus by quite similar reasoning we have the two-dimensional analogue of (C),

Formula (D)

$$\iint f(r)\,.\,F_m(\rho)\,dA=\frac{1}{ma}\int_0^a r^{m+1}f(r)\,dr\left[\int_{-a}^{+a}\frac{\partial F}{\partial y}dx+\iint\Delta F\,dA\right] \quad . \quad . \quad . \quad (D)$$

where the notation is as in (B), and the double integrals are over a semicircle of radius a and centre at the origin, having its diameter on the x axis. The single integral is taken along this diameter, that is, y is set equal to zero before integrating.

These formulas may be extended to cover cones, cylinders, segments of parabolas, and even triangles. As a final example of the sort let us use (D) to find the area between the parabola $x^2-16x+4y=0$ and the axis of x. By writing $\frac{1}{4}y^2$ for y and leaving x unchanged we carry the segment of the parabola into a semicircle. We need not move the origin, since the new integrand does not contain x. The area is therefore given by integration over a semicircle of radius 8 standing on the x axis, by (D),

$$\iint \tfrac{1}{2}y\,dA = \tfrac{1}{2}\cdot\tfrac{1}{8}\int_0^8 r^2 dr\Big[\int_0^{16} dx - 0\Big] = \frac{1}{16}\cdot\frac{8^3}{3}.16 = \frac{512}{3}.$$

It may fairly be said, however, that we are here intruding on ground properly belonging to ordinary rectangular coordinates. Yet the method may with equal ease be applied to problems of much greater complexity; and has some theoretic interest in that we use a process of differentiation to arrive at the definite integral.

8. *A more general problem.*—We may now travel a little further afield and bring the foregoing integrals in touch with the theory of the potential function. As a first step in this direction let it be required to extend formula (A) so as to evaluate

$$\iiint \frac{1}{r'} f(r) \,.\, F(\rho)\, dV, \quad . \quad , \qquad . \quad . \quad (7)$$

where r' is the distance from the variable point ρ to some fixed point other than the origin. This is the same as finding the potential, at the fixed point, due to the attraction of the sphere if its density at ρ is $f(r)\,.\,F(\rho)$. The homogeneous function $F(\rho)$ is, however, assumed at present to be a polynomial.

9. *Theorem on the expansion of a polynomial in terms of harmonics.*—The solution of the above problem depends on the following theorem:—

Any homogeneous polynomial $F_m(\rho)$ *differs from an harmonic function by a sum of terms, each of which consists of an even power of the radius vector multiplied by an harmonic function of lower degree.*

In symbols this theorem may be stated as

Formula (E)

$$F_m = H_m + r^2 H_{m-2} + r^4 H_{m-4} + \ldots + r^m H_0 \quad . \quad . \quad (E)$$

if m is even; while if m is odd the last term is $r^{m-1}H_1$. Here the H's are harmonic polynomials of degrees indicated by their subscripts.

To prove this result we have first to find the effect of ∇ on any term of the form $r^k t$, where t is any homogeneous scalar function of degree n in ρ. By direct expansion, (remembering $\nabla r = u$),

$$\nabla(r^k t) = kr^{k-1}ut + r^k \nabla t = kr^{k-2}\rho t + r^k \nabla t, \quad . \quad . \quad (8)$$

and by operating a second time with ∇

$$\nabla^2(r^k t) = k(k-2)r^{k-4}\rho^2 t - 3kr^{k-2}t + 2kr^{k-2}\mathrm{S}\rho\nabla t + r^k\nabla^2 t$$

$$= r^k\nabla^2 t - k(2n+k+1)r^{k-2}t, \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

because $\rho^2 = -r^2$, and $\mathrm{S}\rho\nabla t = -nt$ by Euler's theorem. If we prefer to put $\nabla^2 = -\triangle$ we may state this identity as

Formula (F)

$$\triangle(r^k t) = r^k\triangle t + k(2n+k+1)r^{k-2}t, \quad . \; . \quad (\mathrm{F})$$

where t is homogeneous of degree n.

If t is harmonic we may conveniently write

Formula (G)

$$\triangle(r^k \mathrm{H}_n) = cr^{k-2}\mathrm{H}_n, \quad . \; . \; . \; . \; , \; , \quad (\mathrm{G})$$

where c denotes the constant $k(2n+k+1)$.

We are now able to prove the existence of the expansion (E) inductively, by showing that if it exists for all polynomials F_p of degree p it exists for all polynomials of degree $p+2$. Let $m=p+2$. Then $\triangle\mathrm{F}_m$ is of degree p. If the theorem is true for degree p we may write

$$\triangle\mathrm{F}_m = \mathrm{H}_p + r^2\mathrm{H}_{p-2} + r^4\mathrm{H}_{p-4} + \ldots ; \quad . \; . \; . \; . \quad (10)$$

but by operating with $\triangle$ on both sides of (E)

$$\triangle\mathrm{F}_m = c\mathrm{H}_{m-2} + c'r^2\mathrm{H}_{m-4} + c''r^4\mathrm{H}_{m-6} + \ldots, \quad . \quad (11)$$

where the c's are positive constants by formula (G). Since $p=m-2$ we may take as a possible set of values

$$\mathrm{H}_{m-2} = \frac{1}{c}\mathrm{H}_p\,; \quad \mathrm{H}_{m-4} = \frac{1}{c'}\mathrm{H}_{p-2}\,; \quad \mathrm{H}_{m-6} = \frac{1}{c''}\mathrm{H}_{p-4}, \;\&\mathrm{c}.$$

By substituting these values in (E), since F_m is known, H_m is known. It is therefore evident that the expansion (11) is known by comparison with (10). Hence (E) is known. Now a constant and a linear expression are always harmonic. That is, (E) exists when $m=0$ and when $m=1$. Hence it exists when $m=2$ and when $m=3$, and so universally.

The same inductive argument shows that the expansion (E) is unique; for if expansion in the form (10) is uniquely possible (11) is uniquely possible. But the expansion is unique for polynomials of degree 0 and 1, hence for all polynomials.

As a simple example let it be required to expand x^4 in the form

$$x^4 = \mathrm{H}_4 + r^2\mathrm{H}_2 + r^4\mathrm{H}_0,$$

where the subscripts denote the degrees of the harmonics, Operating with $\triangle$ and determining the numerical coefficients on the right by (F),

$$\triangle x^4 = 12x^2 = 14\mathrm{H}_2 + 20r^2\mathrm{H}_0\,; \qquad \triangle^2 x^4 = 24 = 120\mathrm{H}_0,$$

whence

$$\mathrm{H}_0 = \tfrac{1}{5}, \quad \mathrm{H}_2 = \tfrac{1}{7}(6x^2 - 2r^2), \quad \mathrm{H}_4 = x^4 - \tfrac{1}{7}r^2(6x^2 - 2r^2) - \tfrac{1}{5}r^4.$$

It appears that, in general, each of the H's will be expressed in terms of all the H's of lower subscript.

10. *Term by term evaluation of the integral.*—Let us now use the elementary theory of the potential function to evaluate the integral

$$\iiint \frac{1}{r'} f(r) \,.\, \mathrm{H}_n d\mathrm{V}$$

over a sphere of radius a with centre at the origin. This is the same as finding the potential at a point O′ due to a volume-density of $f(r) \,.\, \mathrm{H}_n$ throughout the sphere. Suppose first that O′ is outside the sphere. Following Maxwell's notation*, we may write $\mathrm{H}_n = r^n\mathrm{Y}_n$, where Y_n is a surface harmonic. The potential at an external point due to a surface-distribution Y_n over a sphere of radius a with centre at the origin is known to be $\dfrac{4\pi a^{n+2}\mathrm{Y}_n}{(2n+1)r^{n+1}}$, that is $\dfrac{4\pi a^{n+2}\mathrm{H}_n}{(2n+1)r^{2n+1}}$, and, since $\mathrm{H}_n = \mathrm{Y}_n a^n$ *at the surface*, a distribution $f(a)\mathrm{H}_n$ over the surface will give an external potential $\dfrac{4\pi a^{2n+2} f(a)\mathrm{H}_n}{(2n+1)r^{2n+1}}$. For the external potential due to the solid sphere we therefore have

Formula (H)

$$\iiint \frac{1}{r'} f(r) \,.\, \mathrm{H}_n d\mathrm{V} = \frac{4\pi\mathrm{H}_n}{(2n+1)r^{2n+1}} \int_0^a r^{2n+2} f(r)\, dr, \quad . \quad \text{(H)}$$

the accents being dropped after the integration, i. e. r is put for T(OO′). We can now evaluate the integral propounded

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* Elect. and Mag. 3rd Ed. vol. i. Art. 131 *a*.

in Art. 8 when the point O′ is outside the sphere by applying (E) and (H) in succession, viz.

$$\iiint \frac{1}{r'} f(r) \,.\, \mathrm{F}_m d\mathrm{V} = \frac{4\pi \mathrm{H}_m}{(2m+1)r^{2m+1}} \int_0^a r^{2m+2} f(r)\,dr + \frac{4\pi \mathrm{H}_{m-2}}{(2m-3)r^{2m-3}} \int_0^a r^{2m} f(r)\,dr + \&\text{c.}$$

To illustrate, let us find the external potential when the density of the solid sphere varies as the fourth power of the distance from a diametral plane. By the expansion of x^4 already obtained

$$\iiint \frac{x^4}{r'}\,d\mathrm{V} = \frac{4\pi \mathrm{H}_4}{9r^2} \cdot \frac{a^{11}}{11} + \frac{4\pi \mathrm{H}_2}{5r^5} \cdot \frac{a^9}{9} + \frac{4\pi \mathrm{H}_0}{r} \cdot \frac{a^7}{7},$$

where the values of the H's are those calculated in Art. 9 for the expansion of x^4, and r is written for T(OO′) on the right.

11. *The potential inside the sphere.*—It is known that a surface-distribution Y_n over a sphere of radius a and centre at the origin causes a potential inside the sphere $\frac{4\pi r^n \mathrm{Y}_n}{(2n+1)a^{n-1}}$ or $\frac{4\pi \mathrm{H}_n}{(2n+1)a^{n-1}}$. The potential at a point inside the solid sphere may be regarded as due to two additive causes, first a solid sphere of radius r on whose surface the point lies, second a shell of thickness $a-r$ fitting outside the first. The first part of the resulting potential is, by (H),

$$\frac{4\pi \mathrm{H}_n}{(2n+1)r^{2n+1}} \int_0^r r^{2n+2} f(r)\,dr;$$

and the second part, due to a surface-density $f(r)\mathrm{H}_n dr$ on each infinitesimal shell of radius greater than r, is by the formula quoted at the beginning of this article,

$$\frac{4\pi \mathrm{H}_n}{(2n+1)} \int_r^a r f(r)\,dr.$$

Hence the potential inside the solid sphere due to volume-distribution $f(r\mathrm{H})_m$ is given by

Formula (A′)

$$\iiint \frac{1}{r'} f(r)\,\mathrm{H}_n d\mathrm{V} = \frac{4\pi \mathrm{H}_n}{(2n+1)} \left\{ \frac{1}{r^{2n+1}} \int_0^r r^{2n+2} f(r)\,dr + \int_r^a r f(r)\,dr \right\}, \quad . \;\; . \;\; . \quad (\mathrm{H}')$$

whence by means of the expansion (E) we can find the value of (7) when the point is inside the sphere.

Completing the application to a distribution x_4 we have the internal potential

$$\iiint \frac{x^4}{r'}\,dV = \frac{4\pi H_4}{9}\left\{\frac{r^2}{11}+\frac{a^2-r^2}{2}\right\}+\frac{4\pi H_2}{5}\left\{\frac{r^4}{9}+\frac{a^4-r^4}{4}\right\}$$
$$+\frac{4\pi H_0}{1}\left\{\frac{r^6}{7}+\frac{a^6-r^6}{6}\right\}.$$

As a verification, the internal and external expansions coincide if we write $r=a$. The external potential satisfies Laplace's equation while the other yields $4\pi x^4$ when operated on with $\triangle$, agreeing with Poisson's equation*.

LV. *On Graphical Methods of correcting Telescopic Objectives.* *By* A. O. Allen, *Lecturer in Optics, The University of Leeds* †.

HAVING had occasion recently to use the N.P.L. tables relating to small objectives, it occurred to me that the information there furnished, as well as much more of equal or even greater importance, could be given in a very small compass by means of a few formulæ, in combination with graphical methods. It is true that this means substituting calculation for a direct extraction of values from tables, but a number of considerations may be set off against this. First, the calculations I propose are quite simple; most of them are also fairly short. Such as they are, they are not likely to act as a deterrent; for it must be remembered that both the tables and the equivalent calculations lead to figures such as no manufacturer with a reputation to keep up would employ. It may safely be assumed that in future all lens-makers will use the services of an expert computer, and the labour of computing is so great in any case that a little more at the outset will not be objected to, especially if that little extra work saves a great deal of labour further on. Again, the tables are only for a few selected glasses; no tables of reasonable bulk could include all available glasses, and even with these few it is necessary to apply sundry corrections for variations of refractive index and dispersive

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* In general the polynomial

$$\frac{r^2 H_m}{2(2m+3)}+\frac{r^4 H_{m-2}}{4(2m+1)}+\frac{r^6 H_{m-4}}{6(2m-1)}+\dots$$

yields the right member of (E) when operated on by Δ, (proof by (F)).

† Communicated by the Author.

power. All the lenses are achromatic doublets for C and F, whereas in practice the objective will often be required to have some chromatic error. All the lenses are cemented, so that they are corrected either for sphericity or for coma, but not for both; no tables of moderate bulk could include the possibility of air-gaps.

Again, all the lenses are computed for an object at infinity; whereas reading-telescopes should be computed for a comparatively near object. Finally, the tables refer only to doublets, whereas the methods given below can be applied also to triplets; or, for that matter, to systems with any number of thin components, but combinations of four or five lenses as telescopic objectives are to be regarded as mere scientific bizarreries. Now all these variables (refractive index, dispersive power, position of object, air-gaps, number of components) are taken account of below, and without any serious addition to the labour involved. But it must always be remembered that the results arrived at in this way (or by the tables) ought never to be seen by the lens-grinder; they are simply intended to give the computer a favourable start.

The assumptions made are: (1) that the thickness of each lens or gap is negligible; (2) that all the angles in the calculation are so small that the excess of any angle above its sine is exactly equal to a sixth of the cube of the angle. In other words, the rays could all travel within a capillary tube lying along the axis of the lens.

The symbols employed below are chosen to suit the present problem, and would not necessarily fit into a more general scheme. Focal lengths, radii, and intersection-distances are avoided; it is the reciprocals of these quantities which are more important. The four curvatures are c_1, c_2, c_3, c_4, from left to right; the powers of the two lenses are p_1, p_2, and it is assumed that the power of the combination is chosen as a unit, so that $p_1+p_2=1$. So far as this paper is concerned there is no condition whatever connecting p_1 and p_2; they may be quite independent, or may be chosen to give achromatism between any two colours, or to give a desired chromatic error, or to satisfy some other condition not stated; they may be of like or unlike signs. If a ray incident on the system is converging toward a point beyond the system, the reciprocal of its intersection-distance will give the initial convergence, u_1.

After the ray has passed through the first surface the quantity u_1 becomes u_1'; and as the thickness of the lens is neglected, $u_1'=u_2$, and so on. All these c's, u's and p's are to be thought of as "angles per unit height of incidence." The excess of

c_3 above c_2, called g, is the air-gap between the two lenses; when it is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ the gap is $\left\{\begin{matrix}\text{marginal}\\ \text{axial}\end{matrix}\right\}$. The refractive index of the first lens is N; of the second n; these may be taken to refer, for instance, to the brightest part of the spectrum. Clearly

$$c_2=c_1-\frac{p_1}{\mathrm{N}-1},\quad c_3=c_1-\frac{p_1}{\mathrm{N}-1}+g,\quad c_4=c_1-\frac{p_1}{\mathrm{N}-1}-\frac{p_2}{n-1}+g,$$

so that any one of the c's determines the rest; c_1 will be used as the independent one.

The problem is to express (1) the spherical aberration, (2) the offence against the sine-condition, as functions of N, n, c_1, u_1, g. The only rational way to express the first is by means of the angle between the emergent ray (in air) and the line joining the point of emergence to the ideal image-point. The other aberration can also be expressed as an angle. If the sine-condition were fulfilled the locus of the intersections of corresponding incident and emergent rays would be a circle of curvature u_1+u_4' (see Steinheil & Voit, p. 66; Southall, p. 409; Cheshire, J. R. Micr. Soc.). The excess of the actual curvature above u_1+u_4' is the angle required. The expressions I obtain are these:—

(1) The spherical aberration is

$$\mathrm{A}c_1^2+\mathrm{B}u_1^2+\mathrm{C}g^2+\mathrm{D}c_1u_1+\mathrm{E}c_1g+\mathrm{F}u_1g+\mathrm{G}c_1+\mathrm{H}u_1+\mathrm{K}g+\mathrm{L}; \quad .\ .\ .\ (1)$$

(2) The sine-error is

$$\mathrm{P}c_1+\mathrm{Q}u_1+\mathrm{R}g+\mathrm{S}, \quad .\ .\ .\ .\ .\ (2)$$

where

$$\mathrm{A}=\tfrac{1}{2}p_1\left(1+\frac{2}{\mathrm{N}}\right)+\tfrac{1}{2}p_2\left(1+\frac{2}{n}\right);\quad \mathrm{B}=\tfrac{1}{2}p_1\left(3+\frac{2}{\mathrm{N}}\right)+\tfrac{1}{2}p_2\left(3+\frac{2}{n}\right);$$

$$\mathrm{C}=\tfrac{1}{2}p_2\left(1+\frac{2}{n}\right);\quad \mathrm{D}=-p_1\left(2+\frac{2}{\mathrm{N}}\right)-p_2\left(2+\frac{2}{n}\right);$$

$$\mathrm{E}=p_2\left(1+\frac{2}{n}\right);\quad \mathrm{F}=-p_2\left(2+\frac{2}{n}\right);$$

$$\mathrm{G}=-\tfrac{1}{2}p_1^2\frac{\left(2+\frac{1}{\mathrm{N}}\right)}{\left(1-\frac{1}{\mathrm{N}}\right)}-\tfrac{1}{2}p_2^2\frac{\left(2+\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)}-\frac{p_1p_2}{n}\left(2n+2+\frac{n+2}{\mathrm{N}-1}\right);$$

$$H=\tfrac{1}{2}p_1^2\frac{\left(3+\frac{1}{N}\right)}{\left(1-\frac{1}{N}\right)}+\tfrac{1}{2}p_2^2\frac{\left(3+\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)}+\frac{p_1p_2}{n}\left(3n+2+\frac{2n+2}{N-1}\right);$$

$$K=-\tfrac{1}{2}p_2^2.\frac{\left(2+\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)}-\frac{p_1p_2}{n}\left(2n+2+\frac{n+2}{N-1}\right);$$

$$L=\tfrac{1}{2}p_1^3.\frac{1}{\left(1-\frac{1}{N}\right)^2}+\tfrac{1}{2}p_2^3.\frac{1}{\left(1-\frac{1}{n}\right)^2}+\tfrac{1}{2}p_1^2p_2.\frac{\left(3+\frac{2}{n}-\frac{2}{N}\right)}{\left(1-\frac{1}{N}\right)^2}$$

$$+\tfrac{1}{2}p_1p_2^2\frac{\left(3+\frac{1}{n}-\frac{1}{N}\right)}{\left(1-\frac{1}{N}\right)\left(1-\frac{1}{n}\right)};$$

$$P=p_1\left(1+\frac{1}{N}\right)+p_2\left(1+\frac{1}{n}\right);\quad Q=-p_1\left(2+\frac{1}{N}\right)-p_2\left(2+\frac{1}{n}\right);$$

$$R=p_2\left(1+\frac{1}{n}\right);\quad S=-p_1^2.\frac{1}{1-\frac{1}{N}}-p_2^2.\frac{1}{1-\frac{1}{n}}-\frac{p_1p_2}{n}\left(2n+1+\frac{n+1}{N-1}\right).$$

These expressions have perhaps a forbidding aspect; but it is to be noticed that owing to the constant occurrence of $\frac{1}{N}$ and $\frac{1}{n}$ along with *integers*, the computation is not nearly as toilsome as it looks; even L and S yield quite readily. The expression (1) is merely an expansion of Seidel's first term, $\frac{1}{2}S_1$. His second term, $\frac{1}{2}S_2$, expands into a form which, if S_1 is zero, contains expression (2) as a factor; so that the expression is only to be regarded as measuring the departure from the sine-condition, and the amount of coma, *provided there is no spherical aberration*; and the meaning of Cheshire's "intersection-surface" is subject to the same limitation. More generally, the connexion with the Seidel terms may be stated thus:—If the object-point is at a finite distance, and at a height y_1 above the axis, then the spherical aberration of a ray from it, striking the objective at height h, is $(y_1-h)^3(p_1+p_2)^3.\frac{S_1}{2}$, and the comatic error is $-3y_1(y_1-h)^2(p_1+p_2)^3 . \frac{S_2}{2}$, and expression (2) is $(S_2-S_1)u_1$; but if the object is at infinity, and the ray from it strikes at height h and slope θ, the spherical aberration is then $\{\theta-h(p_1+p_2)\}^3 . \frac{S_1}{2}$, the comatic error is $3\theta\{\theta-h(p_1+p_2)\}^2 . \frac{S_2}{2}$, and expression (2) is $(S_1-S_2)(p_1+p_2)$; and in either case expression (1) is $\frac{1}{2}S_1$. Of course only rays in a meridian plane through the object are here considered.

For the purpose of merely calculating the aberrations, without discussion, I find it very convenient to follow Professor Conrady's plan of introducing a fictitious air-gap between all cemented surfaces, and then computing (by slide-rule) the aberrations for each lens, rather than for each surface. The ray is first taken through the system by the schedule $i_1=c_1-u_1$, $Ni_1'=i_1$, $u_1'=c_1-i_1'$, $u_2=u_1'$, and so on. Then the form taken by S_1 is $\left(1-\frac{1}{N}\right)\Sigma\{i_1^2(i_1-u_1')+(i_2')^2(u_2-i_2')\}$, while the expression $\left(1-\frac{1}{N}\right)\Sigma\{i_1(i_1-u_1')+i_2'(u_2-i_2')\}$ is S_2-S_1 multiplied by u_1 or by $-P$ as the case may be; P denotes the power of the whole combination, p_1+p_2.

There is no difficulty about modifying the coefficients A, B, &c. to allow for a third component, especially as most examples of this class would be cemented, so that all terms in g_1 or g_2 would disappear.

I now proceed to deduce some consequences of the above formulæ. As (2) and (1) are respectively of the same shape as the equations of a plane and a conicoid, it is natural that a number of familiar expressions should occur in the deductions from them.

We notice first that (2) is linear. Therefore when two of the variables (say c_1 and u_1) have been fixed to remove spherical aberration, there is only one value of the third which will remove coma, and there always is one; *i. e.*, with the form of the leading lens fixed, as well as the position of the object, there is only one air-gap, and therefore only one form appropriate for the second lens. And a cemented lens cannot be free from coma (except by good luck) if c_1 and u_1 have been fixed by other considerations. The graph of the comatic error is, for spherically corrected combinations, a straight line, whether it is plotted against g, or u_1, or any of the c's. For all other combinations, as soon as two variables are selected, the graph of the comatic error against the third is a parabola.

Expression (1) is quadratic in c_1, u_1, or g; when any two of the variables have been fixed, there are *two* values of the third which will remove spherical aberration (though these two may coincide, or be imaginary). With g and u_1 fixed, the graph of the aberration against any one of the c's is a parabola with its axis vertical; its vertex will be downward if p_1 and p_2 are both positive, and also if p_1 is positive and p_2 negative, provided the combination is a converging one. The matter upon which above all I would lay stress is that the latus rectum of this parabola, being the reciprocal of A, is entirely independent of c_1, u_1, and g; in other words, so long as we keep to the same two glasses, and to the same two powers, we may vary the curves and the air-gap and the position of the object as much as we please, but we shall always get the same parabola; all that will change will be the position of its vertex. Therefore we need only calculate the latus rectum (1/A), plot the parabola once for all (either by the focus-directrix property with compasses, or from the equation $y^2=cx$ with logarithms or a slide-rule), and then cut out a templet of this parabolic form. Thereafter, in any problem on lenses of these two glasses, and of powers proportionate to p_1 and p_2 all we need do is to calculate the position of the vertex, lay the templet in position and draw the parabola (I), and immediately we see all the possibilities within reach by variations of the four curves. We can see what curves (if any) will remove spherical error (or introduce a desired amount), and whether these curves will also correct for coma, and if not, how bad the coma will be. The vertex is to be found thus:—Knowing u_1 and g, calculate c_1 from the equation

$$2\mathrm{A}c_1+\mathrm{D}u_1+\mathrm{E}g+\mathrm{G}=0. \quad . \quad . \quad . \quad . \quad (3)$$

This gives the abscissa; then calculate the ordinate (the aberration) by substituting for c_1, u_1, and g in (1).

But it may be asked, how can the method be made to include *all* values of u_1 and g, when the above graph is for a particular u_1 and g? Suppose in the first place that we do not object to having a fixed value for u_1, but wish to exhibit all the possibilities which follow from varying g as well as c_1. We begin by finding *the locus of the vertices* of all such parabolas as the above, as g is varied with u_1 fixed. It is easy to see what this locus will be; it means plotting the aberration (1) with the restriction (3) imposed upon it; and as (3) is linear, the graph will still be a parabola, although of a new shape (II). It is easy to show that the reciprocal of the new latus rectum is $A\left(4\frac{AC}{E^2}-1\right)$. At the vertex of this parabola we not only satisfy equation (3), but also

$$Ec_1+Fu_1+2Cg+K=0. \quad . \quad . \quad . \quad . \quad (4)$$

We therefore find the vertex by solving (3) and (4) together (u_1 is given), which gives the abscissa c_1, while the aberration is calculated from (1). Having the vertex and the latus rectum the locus is quickly drawn. It would be used as follows. The position of the object is given by u_1, and we wish to know what can be done with a series of values for g. Begin with one of them, insert it and u_1 in (3), and find c_1. Pick out the point on parabola II which has this c_1. Lay the templet for parabola I with its vertex at this point, and immediately we see all the possible aberrations with that air-gap. Now change g, make the necessary change in c_1 (by simple proportion, as we see from (3)), shift the templet to this new vertex, and so on.

Had it been desired to vary u_1 while g was fixed, the only difference would be that we should get parabola III; its latus rectum is the reciprocal of $A\left(4\frac{AB}{D^2}-1\right)$, and its vertex is given by using

$$DC_1+2Bu_1+Fg+H=0, \quad . \quad . \quad . \quad . \quad (5)$$

instead of (4).

Finally, suppose we wish to generalize parabola II so as to include all values of u_1; we must first find the locus of the vertices of all such parabolas as II while u_1 varies. It will again be a parabola (IV); its latus rectum is the reciprocal of $\frac{(4AC-E^2)}{(2CD-EF)^2}(4ABC+DEF-AF^2-BE^2-CD^2)$, and its vertex is found by solving (3), (4), and (5) for c_1, and then finding the ordinate from (1).

If III is generalized in this way, we get parabola V, with the same vertex as IV, but its latus rectum is the reciprocal of $\frac{(4AB-D^2)}{(2BE-DF)^2}(4ABC+DEF-AF^2-BE^2-CD^2)$.

To use V, we should first assign a value to g; solve (3) and (5) for c_1, and pick out the corresponding point on V; fit to that point as vertex the templet of III, and draw III. Assign a value to u_1, solve (3) for c_1, pick out the point on III, and use it as a vertex for the templet of I. All variations in aberration as the curvatures are altered are now shown for the selected g and u_1.

I propose next to illustrate the use of these methods by solving a few numerical problems, but instead of employing graphs (which would involve expensive plates) the solutions will be entirely algebraical. In a discussion of principles it it desirable to use 5-figure accuracy, although in practice 4 figures would be ample. I take the case of two glasses for which $1/N=0\cdot658805$, $1/n=\cdot61721$; also $p_1=2\cdot508$, $p_2=-1\cdot508$. These two powers were, as a matter of fact, selected for C$-$F achromatism, but we have no concern with that. The combination is one which was studied in some detail in Professor Conrady's autumn class last year. I take first a few simple questions on the front lens alone; p_2 must then be put equal to zero in the coefficients; $A=2\cdot9063$, $B=5\cdot4143$, $D=-8\cdot3206$, $G=-24\cdot5088$, $H=33\cdot7267$, $L=67\cdot7599$.

Problem (1). If the object is at infinity, what curves give least spherical aberration? Solve equation (3) with u_1 and $g=0$; $c_1=-G/2A=4\cdot2165$; c_2 is less than c_1 by $4\cdot8426$ in each example, and is here $-0\cdot6261$; the so-called "crossed lens." The aberration, according to (1), is $16\cdot089$.

Problem (2). If the lens is given, for what point will its aberration be least? For example, suppose $c_1=3\cdot0$, $c_2=-1\cdot8426$. Solve (5) with $c_1=3$ and $g=0$; $u_1=-(3D+H)/2B=-0\cdot80942$, *i. e.* the object is about 12 focal lengths in front of the lens, and the aberration is $16\cdot843$.

Problem (3). Is it possible for a single thin lens to be free from aberration? For correction, parabola I must cross the horizontal axis, so that its vertex must not be above that line. The critical positions are where parabola III crosses the line. Its vertex is given by solving (3) and (5), with

$$g=0; \text{ these give } u_1=-\frac{p_1}{2}=-1\cdot254, c_1=\frac{p_1}{2(N-1)}=2\cdot4213,$$

as common-sense shows. For these values the aberration is $16\cdot9400$. Parabola III is therefore given by: "excess aberration above $16\cdot9400$=square of excess curvature above $2\cdot4213$." Where it crosses the line, aberration $=0$, therefore $c_1=10\cdot4309$ or $-5\cdot5883$, and, by (3), $u_1=4\cdot3396$ or $-6\cdot8494$. So that aberration cannot be corrected except for object-points within about $\frac{1}{7}$ of the focal length on one side (real) and $\frac{1}{4}$ on the other (virtual), and even then the curves have to be very strong. There is no useful case. As a mere arithmetical exercise, we may take $u_1=-8$; then c_1 has to be $-8\cdot5342$, and c_2 is $-13\cdot377$.

Problem (4). To correct simultaneously for spherical aberration and for sine-error; in other words, to find the two "aplanatic points." The position of these is well-known, so that the exercise affords a useful check on the foregoing formulæ. Equate the expressions (1) and (2) to zero, and solve simultaneously, with $P=4{\cdot}1603$, $Q=-6{\cdot}6683$, $S=-18{\cdot}4354$, and $g=0$; the solutions are $u_1=4{\cdot}8426$, $c_1=12{\cdot}1932$, $c_2=7{\cdot}3506$, or $u_1=-7{\cdot}3506$, $c_1=-7{\cdot}3506$, $c_2=-12{\cdot}1932$. These tally with the familiar solutions $u_2'=c_2=p_1/(N-1)$; $u_1=c_1=-p_1/(N-1)$. In one case the rays enter the first face normally, in the other they leave the second face normally. The cases are both useless, so far as telescopic work is concerned.

I take next a few examples on the doublet. The coefficients are now :—$A=1{\cdot}22153$, $B=2{\cdot}22153$, $C=-1{\cdot}68475$, $D=-3{\cdot}44306$, $E=-3{\cdot}36951$, $F=4{\cdot}87751$, $G=-3{\cdot}73215$, $H=4{\cdot}83548$, $K=20\ 77590$, $L=1{\cdot}81770$, $P=1{\cdot}72153$, $Q=-2{\cdot}72153$, $R=-2{\cdot}43875$, $S=-2{\cdot}66771$.

The latus rectum of all parabolas of class I is $1/(1{\cdot}22153)$; II, $-1/(2{\cdot}10720)$; III, $-1/(0{\cdot}103042)$; IV, $-1/(0{\cdot}099481)$; V, $-1/(1{\cdot}19840)$.

Problem (5). With parallel light and a cemented lens, what curves give (algebraically) least aberration? Solve (3) with $u_1=0$, $g=0$: $c_1=1{\cdot}5277$, $c_2=c_3=-3{\cdot}3149$, $c_4=c_3+2{\cdot}4315=0{\cdot}8834$; the aberration is $-1{\cdot}0330$.

Problem (6). Given $u_1=0$ and $g=0$, find the aberration for $c_2=-4{\cdot}5$. Parabola (I) gives: excess aberration above $-1{\cdot}0330=1{\cdot}22153\times$ (square of excess curvature above $-3{\cdot}3149$), so that the aberration $=-1{\cdot}0330+1{\cdot}7155=0{\cdot}6824$.

Problem (7). Given $u_1=0$ and $g=0$, what curves will remove spherical aberration? We must have $1{\cdot}0330=1{\cdot}22153(c_1-1{\cdot}5277)^2$, or $c_1=0{\cdot}60806$ or $2{\cdot}4473$. Then, as usual, deduct $4{\cdot}8426$ for c_2, and add $2{\cdot}4315$ to c_3 for c_4.

Problem (8). Repeat (5), with $g=0{\cdot}1$. The vertex of the parabola is now given by $c_1=1{\cdot}6656$ instead of $1{\cdot}5277$; indeed, every $0{\cdot}1$ in g means $0{\cdot}1379$ extra in c_1; so, an *axial* gap ($g=-0{\cdot}1$) would mean $c_1=1{\cdot}3898$. With $u_1=0$, $g=0{\cdot}1$, and $c_1=1{\cdot}6656$, the aberration is $0{\cdot}4897$. As this is positive, no choice of curves with air-gap $0{\cdot}1$ (and parallel rays) can give freedom from aberration. Yet for a suitable value of u_1 this could be done.

Problem (9). Over what range of values of u_1 could it be done? To answer this, we require parabola III. Its latus rectum has been given above; its vertex is found by solving (3) and (5) with $g=0{\cdot}1$; namely, $c_1=0{\cdot}24933$, aberration $=0{\cdot}6965$.

It crosses the axis where $-0{\cdot}6965=-{\cdot}103042(c_1-0{\cdot}24933)^2$, *i. e.* where $c_1=2{\cdot}8492$ and $-2{\cdot}3505$, and (3) gives for the corresponding values $u_1=0{\cdot}8398$ to $-2{\cdot}8497$. Provided that u_1 does not lie within this range, an air-gap 0·1 is not incompatible with spherical correction.

Problem (10). What is the largest air-gap which could be substituted for 0·1 in problem (8) so as to make spherical correction possible? This time parabola II is needed. For the vertex, solve (3) and (4) with $u_1=0$; namely, $c_1=4{\cdot}2164$. aberration $=14{\cdot}201$. It crosses the axis where $-14{\cdot}201=-2{\cdot}1072(c_1-4{\cdot}2164)^2$, *i. e.* where $c_1=1{\cdot}6204$ or $6{\cdot}8124$, and (3) then gives for g the values 0·06726 and 3·8317. So that for spherical correction for a distant object the air-gap must not exceed 0·06726 (unless it has an absurdly high value).

Problem (11). If u_1 lies outside a certain range, *all* values of the air-gap are compatible with spherical correction; what is this range? Parabola IV supplies the answer. Its vertex is found by solving (3), (4), and (5) simultaneously; namely, $c_1=3{\cdot}6073$, $u_1=-0{\cdot}42546$, and $g=1{\cdot}9426$; aberration$=14{\cdot}2374$. Its equation is: (aberration $-14{\cdot}2374)=-0{\cdot}099481(c_1-3{\cdot}6073)^2$. It crosses the axis at $c_1=15{\cdot}570$ or $-8{\cdot}356$, which means, by equations (3) and (4), that $u_1=7{\cdot}931$ or $-8{\cdot}783$. For stronger convergence or divergence than this, correction is possible with any air-gap, provided the curvatures are chosen properly.

Problem (12). Returning to problem (7), let us find the comatic error for the two spherically corrected lenses there given. Expression (2) gives (for $c_1=0{\cdot}60806$) the value $-1{\cdot}62092$; and for $c_1=2{\cdot}4473$, the error is 1·5454. Equating (2) to zero, with $u_1=0$ and $g=0$, we have $c_1=1{\cdot}5496$ to comply with the sine-condition. What the manufacturer would probably do, if he were tied down to these two glasses and a cemented doublet, would be to adopt a compromise; and this again shows the difficulty of dealing with lenses by tables, for no tables can be so voluminous as to provide for compromises.

Problem (13). Giving up the idea of a cemented doublet, let us find what air-gap will remove both spherical aberration and coma for a distant object. Equate (1) and (2) to zero, and solve with $u_1=0$. The result is: $g={\cdot}06721$ (or 3·8339, which does not matter), $c_1=1{\cdot}6448$, $c_2=-3{\cdot}1978$, $c_3=-3{\cdot}1306$, $c_4=-0{\cdot}6991$. These curves, then, form a favourable starting point for calculating an achromatic aplanat. They would be improved by making small allowances for the thickness of the lenses, but this matter must be postponed. It also remains to show what actual residue of error remains in all three respects; this involves trigonometric computation. Most important of all is the question how best to utilise the above methods in order to make a much better attempt for the next trigonometric computation, instead of striking out blindly.

Two other similarly computed objectives are:—

(1) For $u_1=-0{\cdot}5$ (object equal to image): $c_1=0{\cdot}8351$, $c_2=-4{\cdot}0075$, $c_3=-3{\cdot}9539$, $c_4=-1{\cdot}5224$.

(2) For $u_1=-1{\cdot}0$ (object at focus): $c_1=0{\cdot}0414$, $c_2=-4{\cdot}8012$, $c_3=-4{\cdot}7498$, $c_4=-2{\cdot}3183$: this may be regarded as corrected for infinity, but with flint leading.

LVI. *Electrical Theories of Matter and their Astronomical Consequences with special reference to the Principle of Relativity. By* A. S. Eddington, *M.A., F.R.S., Plumian Professor of Astronomy in the University of Cambridge**.

MR. WALKER'S paper in the Philosophical Magazine for April has given a new turn to the discussion arising out of the motions of the perihelia of the planets, and perhaps some remarks on the points raised may be useful. I think that he greatly overestimates the differences of opinion between us; our views seem to coincide on what he calls "the *main* point at issue," and it is difficult to believe that relativists in general hold any other view.

Walker describes my method of dealing with the problem as depending on an unsatisfactory assumption, and concludes therefore that my treatment is invalid; but I am afraid he gives a wrong impression by not mentioning that my argument tended to *disprove* the assumption in question. I did not advocate the assumption leading to the equations of motion used in my paper; on the contrary I showed that the results disagreed with observation. The suggestion had been made that the famous discordance of Mercury could be accounted for by the variation of mass with velocity, according to the well-known hypothesis $m=m_0(1-w^2/c^2)^{-\frac{1}{2}}$, if account be taken of its interaction with the sun's motion through the æther. This hypothesis was examined, and the conclusion was unfavourable, the detailed results being irreconcilable with astronomical observation. That is to say, I attempted to disprove the hypothesis $m=m_0(1-w^2/c^2)^{-\frac{1}{2}}$ which Walker rejects more summarily. Sir Oliver Lodge also concludes that "If therefore the theory fails to give all the known perturbations correctly, something must be wrong;

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* Communicated by Sir Oliver Lodge.

and by finding out what is wrong, we may perhaps discover something instructive"*. We may therefore start from this point of general agreement.

The foregoing law of inertia corresponds to the Lorentz electron in steady motion, but the method applies equally to any other law of variation of inertia with velocity; the only difference is a numerical factor which is of no consequence to the argument. A fundamental revision of the theory is therefore necessary. Walker has stepped in with the suggestion that a trial should be made with the more general (and, I consider, more plausible) assumption that the inertia involves the acceleration as well as the velocity. I cannot predict whether his mode of developing this view will lead to an accordance with observation; but I certainly do not undertake to prove a general negative.

Walker's thesis that the Lagrangian function depends on the acceleration as well as on the velocity cannot be a point of cleavage between relativity dynamics and non-relativity dynamics—if I have rightly grasped the meaning of the statement. In Newtonian particle dynamics, and also in the quasi-stationary treatment of these problems, the Lagrangian function is supposed to consist of two parts, (1) the kinetic energy, involving the velocity only, and (2) the force-function involving position in the field of force. The contention is that this separation is inadmissible, and that there is a cross-term involving both the velocity and the force (or acceleration). Walker's standpoint tends to associate this cross-term with the kinetic energy, so that the kinetic energy differs from the value calculated without regard to the acceleration. Sir Oliver Lodge† and the writer find it more natural to group the term with the force-function, and say that the force of gravitation involves a term depending on the velocity. The distinction appears to be purely verbal.

It is essential to an out-and-out relativity theory that this cross-term should exist, and it is surprising to find relativists represented as opposed to it.

Nor can the quasi-stationary assumption be regarded as a fundamental point of difference. It would, I think, be absurd either to affirm or deny the quasi-stationary principle irrespective of the particular application proposed. The question is whether it is a legitimate approximation in a definite problem. I sympathize with Walker in demanding a justification of this approximation in the cases where it has been used—whether by relativists or others. The problem of

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* Phil. Mag. February 1918, p. 143.

† *Loc. cit.* pp. 155–156.

the motion of the planets seems to afford a good illustration of its fallibility. But it seems rather unfair to blame relativists for a method which was introduced by Abraham in a non-relativity theory. I am not sufficiently versed in the history of the subject to know how extensively relativists have followed his example; but I should have regarded it as one of the sins of our youth—due to the influence of evil associates—and long since repented.

I understand that the true relativity theory of Kaufmann's experiment (which seems to be the point in dispute) runs something like this:—Consider an electron momentarily at rest, but continually accelerated, in an electric field of force E and a magnetic field H; then the acceleration α is given by the equation

$$\mathrm{E}e = m\alpha,$$

where e/m is a certain universal constant for the negative electron; we need not inquire into its nature. This equation expresses the ordinary definition of E. Now choose new axes of coordinates with respect to which the electron has an instantaneous velocity w. Referred to these axes the electric and magnetic forces take known values E', H', and the new value of the acceleration α' is obtained by making Lorentz's transformation of the coordinates and the time. Accordingly the relativity-theory predicts that an electron moving with velocity w in an electromagnetic field E', H' will experience this acceleration α'. Keeping E' and H' constant, we find how α' depends on w. Kaufmann's experiment—or rather the recent repetitions of it—confirm the predicted relation with considerable accuracy. Not only does this give the prestige of successful prediction to the relativity theory, but it confirms that part of the hypothesis most in doubt. It is generally admitted that the Lorentz transformation holds for the differential equations of the field; the question is, Does it hold for the boundary conditions (whatever they may be) at the surface of an electron? The Kaufmann experiment, dealing with a single isolated electron, answers this in the affirmative. If then the differential equations and the boundary conditions satisfy the transformation, nothing more is needed to establish its validity*. It should be noted that the experiment does not

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* Experiments are, however, still needed to test whether the Lorentz transformation covers the phenomena of quanta, which appear to involve something outside the ordinary electromagnetic theory. The exception is of special importance because it includes the vibration of an atom, which is the simplest form of a natural clock that could be used for measuring the time in the two systems.

tell us what are the boundary conditions at the electron, but only how they are transformed by uniform motion.

There is no reference to the quasi-stationary principle in this theory. It is true also that there is no reference to the mass, energy or momentum of the electron; the motion is treated geometrically. It is, I think, inappropriate to speak of the energy or momentum *of an electron* in accelerated motion: these quantities are being radiated, and it is impossible to define the precise moment at which an element of energy or momentum ceases to be attached to the electron and passes into the general field. For uniform motion, however, the values can be clearly defined. We do not determine them directly from Kaufmann's experiment; but we arrive at them indirectly because the relativity transformation is verified. These expressions for the momentum and energy of a uniformly moving electron are of limited utility; as Walker rightly points out, it is not permissible to differentiate them.

I gather from Walker's remarks on p. 329 that he has doubts whether the Fitzgerald-Lorentz contraction should theoretically take place under circumstances such as those of the Michelson-Morley experiment; that is to say, the correlation found by Lorentz and Larmor is a possible one, but it need not necessarily be the correlation occurring in Nature. But a proof based on statistical mechanics has been put forward, which seems to be sound *. The arrangement of the particles constituting a solid is one of an infinite number of possible states, and the form taken up by the solid is that which is statistically most probable; since the possible states of the stationary and moving solid are correlated one to one, the most probable states (and therefore the actual states) satisfy the same correlation. In other words entropy is invariant for the Lorentz transformation.

The relativity principle has the great advantage that it leads directly to the law $m=m_0(1-w^2/c^2)^{-\frac{1}{2}}$ for uniform motion of *matter in bulk*, and it is unnecessary to consider the behaviour of an electron, or indeed to adopt an electrical theory of matter. I see no way of deducing from the various electrons treated by Walker the corresponding laws of mass for matter in bulk, so that the discussion of these does not seem to advance the astronomical problem except by suggesting possible analogies.

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* E. Cunningham, 'The Principle of Relativity,' p. 206. Elsewhere this book warns the reader against assuming that the correlation holds for a non-uniform translation; and indeed those who accept Einstein's latest theory assert definitely that it does not hold.

Turning now to Walker's astronomical calculations, it may be pointed out that his detailed calculations deal with a solar system at rest in the æther. The discussion therefore does not relate to Sir Oliver Lodge's suggestion as to the effects of a solar motion, nor does it throw light on the point subsequently brought out—that the motion of the solar system (if any) has no observable effect on the motions of Venus and the Earth. At the end of his investigation Walker hints that a satisfactory theory similar to Lodge's might be constructed by using the more general type of Lagrangian function. He has, however, already three unknown constants, k_1, k_2, k_3; the components of the unknown solar motion will give him three more; with six constants at disposal, he can scarcely fail to secure a forced agreement of the perihelia and eccentricities of the four inner planets, and it is difficult to find any observational test for such a theory.

On p. 337 it is stated that the observed motion of perihelion of Mercury is satisfied by supposing that the attraction depends on the velocity (*i. e.* relative velocity) according to the law

$$\mu = \mu_0(1 + \tfrac{5}{2} w^2/c^2),$$

and that Einstein implicitly introduces this comparatively large dependence on speed. Einstein's law may be transformed in a great many ways; but I do not think any possible interpretation of it reduces to this. If it is desired to put the new wine into old bottles, I think we must say that the theory involves different effects of the radial and transverse components of velocity in modifying gravitation, or to quote Walker's earlier remark "the modified Lagrangian function depends on the acceleration as well as on the speed of the system and *involves also the relative direction of these*"*. The point is perhaps not of great importance; because in any case a theory which deduces the exact motion of Mercury from a general principle stands on a different footing from theories which merely use the motion of Mercury to obtain an empirical determination of their arbitrary constants.

As a closely connected subject, the question of the alleged discordance of the node of Venus deserves some remarks. Dr. Jeffreys (Nature, April 11, p. 103) has commented on

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* Walker's method of taking this into account is to give k_1 and k_2 appropriate values, but in calculating the number $\frac{5}{2}$ he has used the quasi-stationary values, presumably as a concession to relativists. I am afraid I must reject the concession, and insist on agreeing with his true opinion on this point.

the scant attention paid to this as compared with the perihelion of Mercury. It may be well to explain why the former discordance has been considered unimportant. The residual of the node of Venus is $4\frac{1}{2}$ times its probable error *, and the theoretical chances against such an error are about 400 to 1. But it must not be forgotten that this element has been deliberately selected out of 16 elements as showing the greatest discordance. To apply the test of probable error we must select fairly and not pick out the worst cases. Let α be the probability of an error less than x, then the probability that all sixteen residuals are less than x is α^{16}. For a limit of $4\frac{1}{2}$ times the probable error this gives a probability $(\cdot 9976)^{16} = \cdot 962$, so that the chance of the largest residual being as much as $4\frac{1}{2}$ times the probable error is ·038, or about 1 in 26—an adverse probability, but not very emphatic. To put the matter another way, we find (by solving $\alpha^{16} = \frac{1}{2}$) that the largest discordance of the 16 elements should just exceed 3 times the probable error. We may therefore ask, What is the probability that Newcomb underestimated his errors in the ratio $\frac{2}{3}$ owing to unsuspected sources of error? The evidence for a genuine discordance seems very flimsy. To the astronomer, no doubt, it is an indication well worth looking into; but it would be extremely rash to build a theory on so slight a foundation.

The present state of the problem of the elements of the four inner planets appears to be as follows:—The theory given in Sir Oliver Lodge's and my own papers leads to secular perturbations of the Earth and Venus, which ought to be perceptible to observation if the sun's motion is greater than about 10 km. per sec. Since these are not observed, we conclude either that the sun's motion happens to be very small, or that there must be compensating terms in the more complete theory. Following out the second alternative, there are again two possibilities. Either the compensation is an accident due to the particular elements of the orbits and their relation to the direction of the sun's motion, or it is a general compensation. A theory can no doubt be constructed which gives an accidental cancelling for Venus and the Earth, provided it contains a sufficient number of disposable constants not otherwise determined; but such a theory cannot carry much conviction. If we suppose that the compensation is general, then we are adopting effectively a relativity theory of gravitation—that uniform motion of a gravitating system produces no observable effects. This involves a dependence

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* The discordance of the perihelion of Mercury from the Newtonian theory is 30 times its probable error.

of gravitation on the velocity of the planets; or, if preferred, the same thing may be expressed in Walker's phraseology.

In this last case the motion of the sun can have no influence on the perihelion of Mercury, and the observed excess must be ascribed either to outside causes, such as the mass or resistance* of the Zodiacal Light, or to the laws of *relative* motion. In my paper only $\frac{1}{6}$ of the observed excess† is given by the relative motion; and the difficulty is to bring it up to the required amount without extravagant or *ad hoc* assumptions. Walker's investigation appears rather to emphasize this difficulty. Nevertheless Einstein's generalized relativity theory gives the precise value required without any arbitrary constants. It may perhaps be said that the factor 6 (or 3) must be implicitly contained in the assumptions of his theory. I suppose that the results of any theory are implicitly contained in its postulates; so I cannot deny that the factor 6 is concealed in Einstein's Principle of Equivalence—"that it is impossible by any experiment to discriminate between a gravitational field and a field of force (such as the centrifugal force) arising from a transformation of the coordinates of reference." But at least it is cleverly camouflaged! Many pages of analysis are required to obtain the result for Mercury, and I do not think any simple interpretation of the occurrence of the factor can be given at present.

LVII. *On the Relation of the Audibility Factor of a Shunted Telephone to the Antenna Current as used in the reception of Wireless Signals.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN your January number you published a paper with the above title in which I criticised a previous communication from Mr. van der Pol; my paper was followed by an explanatory note in which Mr. van der Pol attempted to justify his methods and conclusions. He suggests that I had rather lost sight of the motives of his paper and of certain experimental difficulties; I wish to show, however, that, through depending too much on Prof. Love's account of Austin's work and apparently

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* Sir Oliver Lodge, 'Nature,' March 21, p. 44.

† Or $\frac{1}{3}$, if the added inertia is subject to gravitation.

failing to consult Austin's original paper, he has lost sight of the facts and has attempted to prove something which arose solely owing to an unfortunate misunderstanding on the part of Prof. Love.

Mr. van der Pol says, "In discussing some experiments by Dr. Austin, Prof. Love makes use of the audibility factor defined as $(R+S)/S$, where S is the resistance of the shunt and R the telephone *resistance*." This is true, but unfortunately Prof. Love overlooked the fact that, although Austin gives the resistance of the telephone receiver as 600 ohms, he adds a footnote to the effect that "the *inductive resistance* of each telephone used in calculating the shunt ratio was 2000 ohms." No one can doubt that by inductive resistance Austin means impedance, so that both Austin and Hogan used the impedance in calculating the audibility factors which constitute the ordinates of the points in figs. 3 & 4 which Prof. Love reproduces from Hogan's paper. Prof. Love's Table IV. based on the simple resistance of the telephone receiver gives results which are quite incomparable with the plotted results of Hogan's observations. It should also be pointed out that Hogan definitely states that in his experiments R is the *impedance* of the telephone, as correctly quoted by Prof. Love (p. 126). For this reason his adverse criticism of the conclusions reached by Austin and Hogan needs revision.

If Table IV. is recalculated using the value of R employed by Austin, viz. 2000 ohms, the following results are obtained:

Shunt S. Ohms.	Current I. 10^{-6} amp.	$I^2 . \frac{S}{R+S}$.
0·5	672	113
1	474	112
10	150	112
50	68	112·5
100	49	114
400	26	112·5
3000	13	111·5

The last column shows that there is not the slightest foundation for Prof. Love's statement that "the results of these experiments are recorded by him [Austin] in a table which does not support the conclusion that the current is proportional to the square root of the audibility factor." In view of the experimental difficulties the proportionality, as indicated by the constancy of the values given in the last column, is wonderfully exact and speaks well for the experimental skill of those who carried out the measurements.

It is seen therefore that the peculiar relationship which

Prof. Love found between the received current and the audibility factor was due entirely to this oversight; his results involve the relationship between the impedance and the resistance of the telephone and would vary with different receivers. *There can obviously be no simple relation between the antenna current and the audibility factor, unless the latter be calculated in such a way as to give correctly the ratio between the total sound-producing current and that fraction of it which passes through the telephone receiver.* It is obvious, moreover, that for large values of $(R+S)/S$, the wrongly calculated audibility factor will also vary as the square of the antenna current, but that for small values of $(R+S)/S$, that is, for weak signals, the wrongly calculated audibility factor will vary less rapidly, as is clearly shown in the table given in my paper (Phil. Mag. Jan. p. 133).

This then is the simple explanation of Prof. Love's suggestions which the experiments of Mr. van der Pol were intended to test.

Mr. van der Pol's experiments have simply confirmed the fact that if, when using such detectors, one miscalculates the audibility factor by taking the resistance instead of the impedance of the telephone receiver, the result will have the peculiarity found by Prof. Love. This was obvious, however, without any experimental confirmation.

Mr. van der Pol's remark that "it is by no means clear whether Austin or Hogan employed the true impedance of their telephones, as in their papers no references at all are given how they determined these impedances" is obviously unjust in view of the work of these two experimenters. The table given above indicates, if it does not prove, that the impedance of the receiver has been fairly accurately determined.

In reply to Mr. van der Pol's statement that "no assumptions as that made by Prof. Howe that the true impedance of the telephone under actual working conditions is equal to four times [I gave 3 and 4 as alternatives] the steady resistance has been justified by any experiments," I wish to say that the figures given were not assumptions but measured values at the frequencies quoted. It will be noticed that Austin's receiver had an inductive resistance $3\frac{1}{3}$ times the steady current resistance.

With respect to the closing paragraph of the note I think that "the uncertainty attending the constants employed by Austin and Hogan and the difficulty of determining exact values" are by no means so great as Mr. van der Pol imagines; but even were it otherwise, I cannot agree that it is "better to base the reduction of the observations on known measurements rather than on assumptions as to the ratio of impedance to resistance," unless the known measurements are of some magnitude on which the observed phenomena depend. In the case in point, no uncertainty as to the exact wave-form can be regarded as a legitimate excuse for neglecting the fact that one is dealing with an alternating or pulsating current.

Yours truly,
G. W. O. Howe.

31st January, 1918.
South Kensington.

LVIII. *A Doctrine on Material Stresses.*

By R. F. Gwyther[*].

It is intended to provide a reasoned basis for a theory of stresses, applicable in the first case to a body which satisfies the geometrical conditions for acting as a Rigid Body (not being in a state of constraint), but which shall be capable of extension to Elastic Bodies or to other approximations to Natural Bodies, without introducing the fiction of an "undisturbed" condition in which the body is assumed to be free from stress. The doctrine, for the development of which the several stages are indicated below, is the outcome of a series of papers on the "Specification of Stress," published by the Manchester Literary and Philosophical Society[†], but it cannot be described as the motive of the papers, since it has formulated itself during the progress of the series.

The body contemplated is not supposed to be crystalline, fibrous, or annealed, and not to be subject to any special conditions either locally or at the surface. The body may, in the first instance, be regarded as either at rest or in motion under the geometrical conditions which define rigidity, and any modification of those geometrical conditions, such as elastic modifications, are to be deduced from the stresses in the initial instance, and from such definitions as may prove necessary in the sequel.

The stages by which the theory is developed are stated below. No analytical expressions are used, but reference is made to such analytical expressions at all the stages. This is inevitable, since the question is, at this stage, an analytical question.

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[*] Communicated by the Author.

[†] Manchester Memoirs, No. 10, vol. lvi. (1912), No. 5, vol. lvii. (1913), No. 5, vol. lviii. (1914), No. 14, vol. lx. (1916), No. 1, vol lxii. (1917).

(1) The nine elements of mechanical stress obey well-known laws of resolution. By introducing the notion of infinitesimal rotations of the coordinate axes about their own positions, we can, for our present purposes, replace these relations by the three differential-operators*, acting upon the elements of stress, the determination of which follows from the laws of their resolution; that is, these operators are based on mechanical considerations.

(2) If we now consider any arbitrary vector (which we shall speak of conveniently as a virtual or potential displacement), we may look upon the nine first differential coefficients of its components as being replaced by the nine elements of virtual or potential strain (including among them the three rotations).

These nine virtual strains may now be affected by the same infinitesimal rotation of the axes as is employed in (1), and the three consequent differential-operators* acting upon the elements of strain may be deduced; in this case on geometrical grounds. It will be noticed that the forms of the two sets of operators are similar, and may easily be made identical.

(3) I shall now introduce the fundamental assumption on which the theory is based; that the elements of a material stress are functions of the first differential coefficients of the components of some vector quantity; in other words, functions of some set of nine virtual strains.

(4) If we now turn to the two sets of differential-operators concerning elements of stress and of virtual strain, we are able, in consequence of assumption (3), to express each element of stress in terms of the nine elements of virtual strain; or, conversely, each element of strain in terms of the nine elements of stress, by means of sets of simple partial differential equations.

In obtaining these solutions constants will be regarded as uniform and isotropic, and, consequently, we shall exclude, among other things, the possibility of a crystalline structure. We shall also suppose the body not to be in a state of constraint, or otherwise that that state has been eased. I shall, further, limit the solutions to relations of a linear form. From this it will follow that the relations between the elements of stress and of virtual strain agree, in form, with the corresponding relations familiar to us in the Theory of Elasticity.

(5) Supposing the elements of virtual strain to be expressed

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* Manchester Memoirs, No. 3, vol. ix. (1895), No. 1, vol. lxii. (1917).

in terms of the elements of stress, we may eliminate the former by differentiation, and obtain a set of relations involving only the elements of stress (stress-relations).

(6) I shall now suppose the set of three mechanical stress-equations to be introduced which correspond to the state of rest or motion of the body as a rigid body. With these equations I shall suppose the stress-relations of the preceding paragraph to be combined. From this combination we may obtain Stress-Equations, by which we may regard the elements of stress to be defined: as far as they are capable of being defined so long as the surface-traction conditions have not been considered.

(7) Up to this point I have been contemplating such cases as a cube of metal on a rough inclined plane, or a connecting-rod moving in a prescribed manner, the geometrical conditions for rigidity being preserved. The stresses are not to be regarded as undefined, but as being determinate.

(8) The Theory of Elasticity is now introduced by the definition of an Elastic Body as a body such that the strain, hitherto considered as a virtual or potential strain, is the actual strain experienced by the parts of the body. The stresses being definite, the strains become definite, and the displacement may be deduced.

The acceptance of the principles involved in this doctrine would have the effect of removing the subject of stresses from its accepted place in the Theory of Elasticity, and of making it an integral part of the Statics and Dynamics of a Rigid Body. Apart from the introduction of the cases of motion of the body, this would appear, at first sight, to be merely a matter of exposition of the Elastic Theory. But a further consequence would be that the determination of the elastic strains from the Rigid Body Stresses would be only the first stage in the Theory of Elasticity. A closer approximation to the values of the stresses would follow from the estimated alteration of the surface-traction conditions consequent on the displaced condition of the surface of the body, and the subject would become one of continued approximations on the lines of certain other subjects of Mathematical Physics, and an opportunity would be provided for a theory of permanent set and of rupture.

Lymm, Cheshire.

LIX. *On the Wolf-note in Bowed Stringed Instruments. By* C. V. RAMAN, *M.A., Sir Taraknath Palit Professor of Physics in the Calcutta University*[*].

1. IN the Phil. Mag. for June 1917 (page 536), Mr. J. W. Giltay has questioned the correctness of the remark made by me in the Phil. Mag. for Oct. 1916 (page 394), that the explanation of the effect of a "mute" on the tone of bowed stringed instruments is chiefly to be sought for in the lowering of the frequencies of resonance of the instrument produced by the loading of the bridge.

2. Before replying to the specific issues raised by Mr. Giltay, I may be permitted to point out that the view of the action of the mute suggested by me rests upon the secure foundation of mathematical analysis. The effect of adding inertia to any part of a dynamical system has been considered by Lord Rayleigh, Routh and others, and it has been shown that the natural frequencies as altered by the addition of the load are given by the roots of the equation (see Routh's 'Advanced Rigid Dynamics,' Section 76)

$$(N_1^2-n^2)(N_2^2-n^2)\ \&c.,\ -\alpha n^2(n_2^2-n^2)(n_3^2-n^2)\ \&c.=0,$$

In the above, N_1, N_2, &c. are the frequencies before the addition of load, n_1, n_2, n_3, &c. are the limiting values of the frequencies attained when the load becomes infinitely large, and α is a positive quantity proportionate to the added inertia.

[$n_1=0$, $n_2>N_1$, $n_3>N_2$, &c., according to the theorem due to Routh].

The forced vibration due to a periodic force of frequency n (assumed to act on the system at the point at which the load is fixed) also depends on the magnitude of the expression on the left-hand side of the preceding equation, being in fact inversely proportional to it except in the immediate neighbourhood of the frequencies of resonance. The expression may, for convenience, be written in the form $(p-\alpha q)$. Assuming that the frequency n of the impressed force lies between two of the natural frequencies, say N_1 and N_2, of the system without any load, the effect of the load on the forced vibration evidently depends on whether p and q are of the same or of opposite sign. If n be less than n_2, they are of opposite signs, while if n be greater than n_2, they are of the same sign. In the former case, the load *decreases* the amplitude of the forced vibration throughout. In the

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[*] Communicated by the Author.

latter case, the vibration is *increased* by the addition of load till the stage is reached at which $p=\alpha q$, the amplitude then becoming very large. Subsequent additions of load decrease the forced vibration till it finally vanishes in the limit.

3. If, however, the point at which the load is fixed is not the same as that at which the impressed force acts on the system, the treatment is not equally simple. The expression for the forced vibration then obtained from the Lagrangian equations has the determinant for the free periods as its denominator; but the numerator contains some additional terms, the magnitude of which is proportional to the applied load. If these terms are ignored, the sequence of changes with increasing load would be exactly the same as that stated in the preceding paragraph.

4. There is no difficulty in verifying the foregoing indications of theory experimentally. In the case of the violin or 'cello, at least the first three of the natural modes of vibration of the instrument have to be taken into account to explain the phenomena produced by the mute within the ordinary range of tone of the instrument. The two first resonance-frequencies are those mentioned by Helmholtz in his work. The pitch of the first is only slightly lowered by the mute. The second is the well-known "wolf-note," and the pitch of this is depressed by about 450 cents by the mute. The pitch of the third resonance is about an octave higher than that of the second, and this also gives a marked "wolf-note." The mute lowers the pitch of this by about 700 cents. The mass of an ordinary brass mute is sufficient to make the second, third, and higher resonance-frequencies approximate to their limiting values. The effect of the mute should accordingly be to *increase* the intensity of the graver tones and harmonics of the instrument, and to *decrease* those of high pitch. This is exactly what has been found experimentally by Edwards (Physical Review, Jan. 1911).

5. Mr. Giltay's criticisms may now be easily disposed of. Experiment shows that he is incorrect in saying "I suppose that the change of pitch of the note of maximum resonance of bridge, belly, &c. will practically be the same whether the bridge be loaded at its highest point or as low as possible and near to its left foot." As a matter of fact, trial shows that the lowering of the pitch of either of the two "wolf-notes" is three to five times as much in the former case as in the latter. As the observed mute-effect is less when the load is placed at the foot of the bridge, the experiment actually furnishes a strong confirmation of the correctness of my views, and shows also that the interpretation given by

Giltay and De Haas to their observations (Proc. Roy. Soc. Amsterdam, January 1910) requires revision. As a matter of fact, it appears from my detailed observations that Giltay and De Haas were in error in assuming that the motion of the bridge in its own plane is practically that of a rigid body. Owing to the form of the bridge, the cuts in it, &c., this is very far indeed from being the case, the elastic distortions being very large. For instance, it makes all the difference in the pitch of the wolf if a load be fixed immediately above instead of immediately below the cut on the G-string side of the bridge. This fact is inconsistent with the supposition made by Giltay and De Haas that the motion of the bridge in its own plane is one of simple rotation about an axis, and proves that the theory of the action of the mute put forward by these writers is untenable.

6. In view of what has been said in para. 4, the observed muting of the high notes of the instrument which Mr. Giltay suggests as a difficulty, is easily seen to be exactly what is to be expected according to the view of the action of the mute put forward by me. In the absence of a mute, the resonance of the violin is by far the strongest at the pitch of the two wolf-notes. Theory thus indicates that the quality of violin-tone and the effect of a mute upon it may be characterized as follows: the gravest tones have a weak fundamental with strong second and third harmonics, muting increasing the fundamental at the expense of the harmonics: in the middle of the scale the tones should have strong fundamental and second harmonic with relatively weak higher harmonics, all except the fundamental being decreased by muting; the highest tones should have strong fundamental and weak upper partials, all the components being decreased by muting. The observations of Hewlett (Physical Review, Nov. 1912) and those of Edwards already quoted are in substantial agreement with the above.

7. Another interesting question which arises regarding the action of the mute is its effect on the minimum bowing pressure necessary in order to elicit a steady vibration of the usual type. I have investigated this question theoretically by considering the effect of the mute on the motion of the bridge and consequently on the minimum frictional force which should be exerted by the bow on the string in order that a steady vibration should be possible. The question has also been studied experimentally using a mechanical player in which an ordinary violin bow excites the strings of a violin under strictly controlled pressure and velocity of movement. The quantitative data obtained clearly show the

great increase in the bowing pressure which becomes necessary at the wolf-note pitch, and prove that the effect of the mute is to *increase* the bowing pressure necessary at low frequencies and to *decrease* it at high frequencies. With this mechanical player, the "cyclical" or "beating" tones obtained in certain cases (Phil. Mag. Oct. 1916 and Feb. 1917) may be steadily maintained and controlled by suitable adjustment of the bowing pressure.

Indian Association for the
Cultivation of Science,
Calcutta.

LX. *On a New Type of Rough Surface the Motion of a Heavy Particle on which is determinable by Quadratures.* By Nalinimohan Basu, *M.Sc., University Lecturer in Applied Mathematics, Calcutta*[*].

1. It is well known that the motion of a heavy particle on a rough surface is determinable by quadratures when the surface is an inclined plane, a circular cylinder, a circular cone, or a vertical cylinder standing on a logarithmic spiral as the base[†].

The object of the present paper is to make known another surface which has, in a certain sense, the same property as the four surfaces mentioned above and which, it is believed, has not been considered by any previous writer.

2. Let us consider the surface whose equation is

$$\chi \equiv z - \tan\alpha \, . \, \tan^{-1}\frac{y}{x} - (x^2+y^2-1)\mathrm{P}(z) = 0, \quad . \quad (1)$$

where $\mathrm{P}(z)$ is given by the relation

$$\frac{1}{2\mu\cot\alpha\cos\alpha}\left[\log\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} - \frac{1}{\sqrt{1-\mu^2\cot^2\alpha}} \times \log\frac{\sqrt{1-\mu\cot\alpha}+\sqrt{1+\mu\cot\alpha}\,.\,\tan\frac{\theta}{2}}{\sqrt{1-\mu\cot\alpha}-\sqrt{1+\mu\cot\alpha}\,.\,\tan\frac{\theta}{2}}\right] = z-h, \quad . \quad (2)$$

$\tan\theta$ standing for $2\mathrm{P}\cos\alpha$ and the *axis of* z being drawn vertically upwards.

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[*] Communicated by Prof. G. Prasad.

[†] For the first three cases, see any well known text-book on the "Dynamics of a Particle," *e.g.* Routh's book; for the last case, see A. Razzaboni's paper, "Sul movimento d'un punto materiale sopra una superficie non levigata." (*Giornale di matematiche*, vol. xxxiv.)

Then we proceed to investigate the path of a heavy particle moving on this surface, the coefficient of friction being μ.

3. The equations of motion are

$$\ddot{x} = l\mathrm{R} - \mu\mathrm{R}\frac{dx}{ds},$$

$$\ddot{y} = m\mathrm{R} - \mu\mathrm{R}\frac{dy}{ds},$$

$$\ddot{z} = n\mathrm{R} - \mu\mathrm{R}\frac{dz}{ds} - g,$$

where R is the normal reaction per unit mass and l, m, n are the direction-cosines of the normal to that side of the surface on which the particle lies.

Eliminating R we have

$$\frac{\ddot{x}}{l-\mu\dfrac{dx}{ds}} = \frac{\ddot{y}}{m-\mu\dfrac{dy}{ds}} = \frac{\ddot{z}+g}{n-\mu\dfrac{dz}{ds}} = \frac{\dfrac{d}{dt}\left(\dfrac{1}{2}v^2+gz\right)}{-\mu v}.$$

Thus we have

$$\mu v\ddot{x} = \left(\mu\frac{dx}{ds} - l\right)\frac{d}{dt}\left(\frac{1}{2}v^2+gz\right),$$

$$\mu v\ddot{y} = \left(\mu\frac{dy}{ds} - m\right)\frac{d}{dt}\left(\frac{1}{2}v^2+gz\right).$$

Writing

$$\ddot{x} = \frac{d}{dt}\left(v\frac{dx}{ds}\right), \quad \ddot{y} = \frac{d}{dt}\left(v\frac{dy}{ds}\right), \quad \frac{d}{dt} = v\frac{d}{ds},$$

and simplifying, we get

$$\mu v^2\frac{d^2x}{ds^2} + lv\frac{dv}{ds} + \left(l-\mu\frac{dx}{ds}\right)g\frac{dz}{ds} = 0,$$

$$\mu v^2\frac{d^2y}{ds^2} + mv\frac{dv}{ds} + \left(m-\mu\frac{dy}{ds}\right)g\frac{dz}{ds} = 0.$$

Hence we obtain

$$\frac{\mu v^2}{\mu\left(m\dfrac{dx}{ds} - l\dfrac{dy}{ds}\right)} = \frac{v\dfrac{dv}{ds}}{\left(l-\mu\dfrac{dx}{ds}\right)\dfrac{d^2y}{ds^2} - \left(m-\mu\dfrac{dy}{ds}\right)\dfrac{d^2x}{ds^2}} = \frac{g\dfrac{dz}{ds}}{m\dfrac{d^2x}{ds^2} - l\dfrac{d^2y}{ds^2}}$$

$$\therefore \quad v^2 = g\frac{dz}{ds}\cdot\frac{m\dfrac{dx}{ds} - l\dfrac{dy}{ds}}{m\dfrac{d^2x}{ds^2} - l\dfrac{d^2y}{ds^2}}, \quad \ldots\ldots \quad (3)$$

and

$$v\frac{dv}{ds}=g\frac{dz}{ds}\left\{\frac{\mu\left(\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}\right)}{m\frac{d^2x}{ds^2}-l\frac{d^2y}{ds^2}}-1\right\}.$$

The path of the particle must therefore satisfy the equation

$$\frac{d}{ds}\left\{\frac{dz}{ds}\cdot\frac{m\frac{dx}{ds}-l\frac{dy}{ds}}{m\frac{d^2x}{ds^2}-l\frac{d^2y}{ds^2}}\right\}$$

$$=2\frac{dz}{ds}\cdot\left\{\frac{\mu\left(\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}-\frac{dx}{ds}\cdot\frac{d^2y}{ds^2}\right)}{m\frac{d^2x}{ds^2}-l\frac{d^2y}{ds^2}}-1\right\}\quad . \quad . \quad (4)$$

4. The problem before us is therefore to find a curve whose equations will satisfy

(i.) The initial conditions,

(ii.) The equation (1),

and (iii.) The equation (4).

Now from (1) we have

$$\frac{\partial\chi}{\partial x}=\frac{y\tan\alpha}{x^2+y^2}-2x\mathrm{P}(z),$$

$$\frac{\partial\chi}{\partial y}=-\frac{x\tan\alpha}{x^2+y^2}-2y\mathrm{P}(z),$$

$$\frac{\partial\chi}{\partial z}=1-(x^2+y^2-1)\frac{d\mathrm{P}}{dz}.$$

$$\therefore\quad l=\frac{1}{\Omega}\left[\frac{y\tan\alpha}{x^2+y^2}-2x\mathrm{P}(z)\right],$$

$$m=-\frac{1}{\Omega}\left[\frac{x\tan\alpha}{x^2+y^2}+2y\mathrm{P}(z)\right],$$

$$n=\frac{1}{\Omega}\left[1-(x^2+y^2-1)\frac{d\mathrm{P}}{dz}\right],$$

where

$$\Omega^2=1+\frac{\tan^2\alpha}{x^2+y^2}+4P^2(x^2+y^2)-2(x^2+y^2-1)\frac{dP}{dz}$$
$$+(x^2+y^2-1)^2\left(\frac{dP}{dz}\right)^2.$$

Assuming the equations of the path to be given by

$$x=\cos(z\cot\alpha),\qquad y=\sin(z\cot\alpha),\quad .\ .\ .\ \ (5)$$

we have, for points on the curve,

$$\Omega^2 = 4P^2+\sec^2\alpha,$$
$$\Omega\,.\,l = y\tan\alpha-2xP,$$
$$\Omega\,.\,m=-(x\tan\alpha+2yP),$$

and $$\Omega\,.\,n = 1.$$

Also we find, for downward motion,

$$\frac{dz}{ds}=-\sin\alpha,$$

$$\frac{dx}{ds}= \cos\alpha\sin(z\cot\alpha),$$

$$\frac{dy}{ds}=-\cos\alpha\cos(z\cot\alpha),$$

$$\frac{d^2x}{ds^2}=-\cos^2\alpha\cos(z\cot\alpha),$$

$$\frac{d^2y}{ds^2}=-\cos^2\alpha\sin(z\cot\alpha).$$

Substituting these in the equation (4) and putting for x and y their values given by (5), we see that the equation

$$\frac{dP}{dz}=-\left\{1-\mu\cot\alpha\,.\,\cos\alpha\sqrt{4P^2+\sec^2\alpha}\right\}$$

must be true in order that the curve described by the particle may be represented by the equations (5). But differentiating (2) we find that the above equation is true.

The equations (5) also satisfy the equation (1).

Therefore it is proved that, if a particle be placed on the surface (1) *and projected with a suitable velocity along the helix* (5), *it will continue to describe that curve.*

5. The velocity at any point of the curve is found from (3) to be given by

$$v^2=2g\mathrm{P}(z).$$

We see from (2) that $z=h$ makes $\mathrm{P}(z)=0$. Thus the level of no velocity is given by $z=h$.

6. To study the nature of the surface we observe at the outset that, when $\mu=0$, *i. e.* when the surface is smooth, the limiting form of (2) becomes

$$\mathrm{P}(z)=h-z,$$

and taking $h=0$, we obtain*

$$z(x^2+y^2)=\tan\alpha\,.\,\tan^{-1}\frac{y}{x}.$$

Now consider the general equation (1). Then we find that (i.) the section by the cylinder $x^2+y^2=1$ is the helix $x=\cos(z\cot\alpha)$, $y=\sin(z\cot\alpha)$; (ii.) the section by the horizontal plane $z=h$ is the straight line $y=x\tan(h\cot\alpha)$, $z=h$; (iii.) the other horizontal sections are spirals of the form $r^2=a\phi+b$; (iv.) generally, sections by vertical circular cylinders are different from helices; (v.) the surface is not a minimal one like the helicoid. Hence we conclude that, although for small values of μ the surface has nearly the same shape as the surface discussed by Catalan, for fairly large values of μ it differs essentially from Catalan's surface as well as from the helicoid.

I wish to express my thanks to Prof. Prasad for his kind interest in the paper.

LXI. *Two-Dimensional Motion of an Infinite Liquid.*

By W. G. Bickley, *B.Sc.*†

§ 1. IN a recent paper (Phil. Mag. (6) xxxv. no. 205, p. 119, Jan. 1918) Dr. J. G. Leathem has shown how to determine the motion in two dimensions of an infinite liquid occupying the space outside a solid body bounded by a closed curve or polygon, due to prescribed motion of the boundary. The method used depends on the use of periodic conformal transformations whereby the doubly connected space outside the boundary is transformed into a semi-infinite rectangle. The solution for the case of translatory motion is neat and immediate, but this can hardly be said of the solution in the case of rotation, although it is perfectly

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[*] This is the surface discussed by Catalan (*Journal de mathématiques*, ser. 1, tome xi.).

[†] Communicated by the Author.

complete and general. When the paper appeared, the author of the present paper was engaged in an attempt, since completed, to solve a particular case of the above problem, and was led to a general method of attack which seems to give results more immediately, and with a less complicated procedure, and it is thought that an outline of the method may be of interest.

§ 2. Needless to mention, conformal representation plays a large part. Instead of the periodic transformation advocated by Dr. Leathem, it was found more convenient to use one whereby the doubly connected region of the z-plane becomes the upper half of the ζ-plane, the boundary in the z-plane becoming the real axis in the ζ-plane. If the periodic transformation is known, this may be effected by taking as the auxiliary variable $\tan\frac{\pi\zeta}{\lambda}$ in the notation of Dr. Leathem. This will for the present be denoted by $\zeta\ (=\xi+\iota\eta)$. The transformation may be written

$$z=f(\zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In particular for the ellipse, of semiaxes $c\cosh\alpha$, $c\sinh\alpha$,

$$z=c\{2\zeta\cosh\alpha+\iota(1-\zeta^2)\sinh\alpha\}/(1+\zeta^2).$$

§ 3. The method now depends on the fact that, except as to a constant, the value of the stream function is known on the boundary, and therefore on the axis of ξ. In particular, if the motion is one of uniform translation with velocity U in the direction inclined at an angle β to the x-axis, we have on the boundary

$$\psi_1=\iota\mathrm{U}\times\mathrm{I}\{ze^{-\iota\beta}\}+\text{const.} \quad . \quad . \quad . \quad . \quad (2)$$

For the case of rotation about the point z_0 with angular velocity ω, on the boundary

$$\psi_2=\tfrac{1}{2}\omega\,|z-z_0|^2+\text{const.} \quad . \quad . \quad . \quad . \quad (3)$$

On the boundary $z=f(\xi)$ since $\eta=0$, therefore we have as the values of ψ on the real axis in the ξ-plane,

$$\psi_1=\iota\mathrm{U}\times\mathrm{I}\{f(\xi)\,.\,e^{-\iota\beta}\}+\text{const.} \quad . \quad . \quad . \quad (2')$$

$$\psi_2=\tfrac{1}{2}\omega\,|f(\xi)-z_0|^2+\text{const.} \quad . \quad . \quad . \quad . \quad (3')$$

Hence, as the corresponding values of $w\ (=\phi+\iota\psi)$,

$$w_1=\frac{\iota\mathrm{U}}{\pi}\int_{-\infty}^{\infty}\mathrm{I}\{f(\xi)\,.\,e^{-\iota\beta}\}\frac{d\xi}{\zeta-\xi},$$

$$w_2=\frac{\omega}{2\pi}\int_{-\infty}^{\infty}|f(\xi)-z_0|^2\frac{d\xi}{\zeta-\xi},$$

provided these integrals converge. This may always be secured, since the boundary in the z-plane is by hypothesis finite, and so $f(\xi)$ tends to a finite limit as $\xi \rightarrow \infty$. Thus we obtain finally

$$w_1 = \frac{\iota U}{\pi}\int_{-\infty}^{\infty} I\{f(\xi) - f(\infty)e^{-\iota\beta}\}\frac{d\xi}{\zeta - \xi} \quad . \quad . \quad . \quad (4)$$

$$w_2 = \frac{\omega}{2\pi}\int_{-\infty}^{\infty} |f(\xi) - f(\infty)|^2 \frac{d\xi}{\zeta - \xi} \quad . \quad . \quad . \quad (5)$$

These integrals, which express w as a function of ζ, may be evaluated by the method of residues, and the elimination of ζ between (1) and the results give w as a function of z, as is required.

Loughborough,
March 5, 1918.

LXII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 292.]

January 9th, 1918.—Dr. Alfred Harker, F.R.S., **President**, in the Chair.

THE following communication was read:—

'The Highest Silurian Rocks of the Clun Forest District (Shropshire).' By Laurence Dudley Stamp, B.Sc., A.K.C.L.

Clun Forest is a large district—extending on both sides of the Welsh Border—in which Upper Silurian rocks crop out over a wide area, interrupted by outliers of Old Red Sandstone. The district is separated from the typical Silurian area of Ludlow, which lies some 15 miles away to the east, by the great line of disturbance that passes through Church Stretton and Old Radnor.

The classification adopted for the highest Silurian strata is as follows:—

			Thickness in feet.	
OLD RED SANDSTONE				Purplish-red sandstones.
TEMESIDE GROUP.	Temeside Shales		350	Olive-green shales with bands of micaceous green grit; a fragment-bed, with Eurypterid- and plant-remains, forms the upper limit.
	Downton Castle Sandstone Series.		110	Yellow sandstones and tilestones, with shales and *Platyschisma* Limestones.
UPPER LUDLOW GROUP.	*Chonetes* Beds	Upper	50	Green laminated flags and blue flagstones.
		Lower	300	Irregularly-bedded calcareous flagstones.
	Rhynchonella Beds		300	Grey calcareous flags with massive blue flagstones.
AYMESTRY GROUP.	*Dayia* Shales		?300	Striped laminated shales and mudstones.
	Lower Ludlow Shales			Dark-grey shales and indurated mudstones.
	Total		1410	

The distribution and characters of the beds are described. The succession compares very closely with that in the Ludlow district itself. The main differences are: (1) that the Aymestry Limestone is represented by mudstones west of the great fault-line, and (2) that all other divisions show greatly increased thicknesses.

There is no evidence of any stratigraphical break. On the contrary, the sequence is complete from the Lower Ludlow rocks up into the Old Red Sandstone, and the changes in lithology are usually quite gradual. The oncoming of the Old Red Sandstone conditions is discussed, with regard to their effect on the lithological and palæontological characters of the strata.

The extent of Old Red Sandstone, as indicated on present maps, must be greatly restricted, since most of the supposed Old Red Sandstone has been found to belong to the Temeside Group, which in this district attains a great development. The Silurian age of the beds in question is shown by the occurrence in them of *Lingula minima*, and of characteristic lamellibranchs, etc., also by comparison with similar strata in the Ludlow area.

A comparison with other districts in which Upper Silurian rocks are developed shows that deposition attained its maximum along the Welsh Border, the thickness of the formations decreasing rapidly southwards and eastwards.

On the east of the district—in the neighbourhood of the great fault-line—the strata are considerably folded along axes ranging north-north-eastwards parallel to the main fault, with minor faults following the same direction. Away from the major faults the folding is gentler in character, and a series of folds ranging nearly due east and west make their appearance. Farther west the north-north-eastward folding and fracturing reappear.

January 23rd.—Dr. Alfred Harker, F.R.S., President, in the Chair.

The following communication was read:—

'On a Flaked Flint from the Red Crag.' By Professor William Johnson Sollas, M.A., Sc.D., LL.D., F.R.S., V.P.G.S.

The remarkable specimen forming the subject of the paper was obtained by Mr. Reid Moir from the base of the Red Crag exposed in the brick-pit worked by Messrs. Bolton & Company near Ipswich.

It is a fragment of a nodule of chalk-flint, irregularly rhombic in outline, with a nearly flat base and a rounded upper surface which retains the whitish weathered crust of the original nodule.

The base was formed by a natural fracture which exposes the fresh flint bordered by its weathered crust.

Both upper and under surfaces of the specimen are scored with scratches which are mainly straight, but in some cases curvilinear.

Two adjacent sides have been flaked by a force acting from below upwards, in a manner that recalls Aurignacian or Neolithic workmanship. The two edges in which the flaked faces meet the base are marked by irregular minute and secondary chipping, such as might be produced by use. On the hypothesis that the flint has been flaked by design, these edges will correspond to the 'surface d'utilisation' of M. Rutôt, and we should expect to find on the opposite edges of the flint the 'surface d'accommodation,' as in fact we do.

A singular feature, which seems difficult to reconcile with its use as an implement, is the restriction of the flaking on one edge to the weathered crust.

The origin of the flaking is discussed, and the author, while admitting that the fashioning of the flint is not inconsistent with intelligent design, concludes that the evidence is not sufficient to establish this beyond dispute. It is eminently a case of 'not proven.'

February 6th.—Dr. Alfred Harker, F.R.S., President, in the Chair.

The following communication was read:—

'Some Considerations arising from the Frequency of Earthquakes.' By Richard Dixon Oldham, F.R.S., F.G.S.

The publication* of an abstract of twenty years' record of earthquakes in Italy gives an opportunity for studying the effect of the gravitational attraction of the sun; the period is so nearly coincident with the lunar cycles of 19 and 18·6 years that the effect of the moon may be regarded as eliminated, the record is of exceptional continuity and completeness, and the number of observations is large enough to allow of the extraction of groups sufficiently numerous to give good averages.

The distribution of the stresses is dealt with in text-books; there is a maximum upward stress, in diminution of the earth's attraction at its surface, at the two points where the sun is in the zenith or nadir, and a maximum downward stress along the great circle where it is on the horizon; but as, for the purpose of this investigation, a decrease of downward pressure is equivalent to an increase of upward, I shall take the line along which the downward stress is greatest as the zero-line, and express the amount of stress at any other time or place as a fraction of the difference between the net force of gravity along this line and at the point where the sun is in the zenith. The fraction, at any given time and place, depends solely on the zenith distance of the sun, which is continually varying with the revolution of the earth. At the equinox, when the sun is on the equator, the curve of variation between 6 A.M. and 6 P.M. is the same as in the other half of the day; at any other part of the year it is not symmetrical in the two halves of the day, but is the same during the day in the summer half of the year as during the night in the corresponding part of the winter half, when the declination of the sun is equal in amount, though opposite in direction.

This gave the first suggestion for grouping the records. The year was divided into two halves by the equinoxes, and the day into two halves at six hours before or after noon, called day and night for convenience, irrespective of the time of sunrise or sunset. The result is given in the tabular statement below, the frequency

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* Boll. Soc. Sismol. Italiana, vol. xx. (1916) p. 30.

being expressed as a ratio to the mean, of each group, taken as 100:—

DISTRIBUTION OF SHOCKS BY DAY AND NIGHT.

Italy, 1891–1910.	*Day.*		*Night.*
June–July	90	:	110
Summer half	88	:	112
Whole year	84	:	116
Winter half	81	:	119
December–January	77	:	123
Japan, 1885–1892.	*Day.*		*Night.*
Summer half	102	:	98
Whole year	97	:	103
Winter half	93	:	107
Assam Aftershocks.			
Summer half	113	:	87
Whole year	107	:	93
Winter half	101	:	99

From this statement it will be seen that the mean ratio of day to night shocks over the whole period is represented by the figures 84 : 116; for the summer half of the year they become 88 : 112, and for the winter half 81 : 119, showing that during the day the shocks are somewhat less frequent than the average in summer and somewhat more frequent in the winter, with an opposite variation during the night. Taken by itself this difference might be merely fortuitous, and further confirmation is required: this can be got in two ways. In the first place by comparison with other records, two of which, Milne's catalogue of Japanese earthquakes from 1885 to 1892*, and the aftershocks of the Indian earthquake of 1897† stood ready for use. They show a variation identical in character with that of the Italian record. A second test depends on the argument that, if the variation is in any way seasonal, the divergence should be increased at the height of each season; the figures for the months of January–February and of June–July were taken out, as representing midwinter and midsummer respectively, and found to show a divergence in each case greater than, and in the same direction as, the respective half-years.

Taken by itself the variation, as between any pair of ratios, is as likely to be in one direction as in the other, but the odds against a complete concordance throughout the whole series is 31 to 1; there is, therefore, a strong presumption that the variations are not fortuitous, but due to some common cause which tends to increase the frequency during the day and decrease it during the night in summer, with the opposite in winter.

The variation in the frequency of earthquakes may, or may not, be connected with the variation in the gravitational stresses due to the sun; but there is another line of investigation by which a connexion may be better traced, dependent on the fact that the prevailing effect of the vertical stress is in the direction of lightening the load, and the prevailing direction of the horizontal stress between east and south, during the six hours before the meridian passages at noon and midnight, and of an increase in the

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* Seismol. Journ. Japan, vol. iv. (1895).

† Mem. Geol. Surv. India, vol. xxxv. pt. 2 (1903).

downward pressure and a horizontal stress between south and west during the next six hours. The record was accordingly grouped by the successive two-hour periods from XII to XII o'clock, and the mean amount of variation in the stresses was calculated for the same periods. The result is set forth in the appended tabular statement:—

DISTRIBUTION OF STRESSES AND SHOCKS IN TWO-HOUR PERIODS, BEFORE AND AFTER MIDDAY AND MIDNIGHT.

Hours	XII	II	IIII	VI	VIII	X	XII
STRESSES. Mean range of stress in each two-hours, in Italy.							
Total stress	−·10	−·27	−·23	+·23	+·27	+·10	
Horizontal component	+·07	−·11	−·20	+·20	+·11	−·07	
Vertical component	−·14	−·27	−·13	+·13	+·27	+·14	
SHOCKS. Ratio of actual to mean frequency of each two-hour period.							
ITALY, 1891–1910	1·06	1·17	1·01	·90	·88	·99	
JAPAN, Aftershocks of Mino-Owari, Oct. 28th, 1891 ...	1·01	·95	·96	·97	1·08	1·03	
JAPAN, 1885–1890	1·00	1·11	·89	·98	1·03	·99	

From these figures it is seen that, while there is no apparent relation between the frequency and the total, or the horizontal, stress, there is a close one with the variation of the vertical stress; the greatest number of earthquakes being in the period in which there is the greatest increase of downward pressure; as the rate of increase diminishes the number of shocks is less, suffering a further diminution as the pressure begins to decrease, and reaching its minimum in the period where the decrease in pressure is greatest, increasing again in the same way to the maximum.

An attempt to apply the same method to the Japan record gave a result which was, at first sight, contradictory and also inconsistent in itself, for it gave an absolute maximum at the time when the Italian gave a minimum, with another maximum, almost as great, in coincidence with the Italian; but, in any comparison, it is necessary to allow for the contrast in the character of the two records. The Italian does not contain more than two, or at most three, great earthquakes of the type that gives rise to long-distance records (bathyseisms), and the aftershocks account for no more than a quarter of the whole record; the Japanese record, on the other hand, is dominated by bathyseisms and aftershocks. Not only does the region give origin to an unusually large number of teleseisms, or bathyseisms, but aftershocks form fully three-quarters of the record, and nearly a half consists of aftershocks of the Mino-Owari earthquake of October 28th, 1891. Taking these separately, we get a curve of frequency similar to the Italian, except that the maximum and minimum are reversed, the greatest number of shocks corresponding to the period when the load is being lightened most rapidly, indicating that these shocks are due to a general movement of elevation rather than depression, a conclusion in accord with field observations of other great earthquakes. In addition, the shocks which occurred during the period 1885–90 were taken out, as representing a more normal activity, though still one in which aftershocks form fully half of the record, and the curve was found, as might have been expected from the character of the record, to combine the features of the Mino-Owari aftershocks with those of the Italian curve of frequency, of earthquakes prevailingly of the so-called ‘ tectonic ’ type.

These results are of twofold geological interest. In the first place they confirm the conclusion drawn from a study of the Californian earthquake of 1906*, that the great earthquakes differ from the ordinary, not merely in degree but in kind. They indicate that in the latter the main stress is compressive, probably due to settlement, and in the former to elevation or tension, a conclusion which is in accord with the fact that, in those cases in which it has been possible to compare accurate measurements made before and after the earthquake, the comparison has indicated an expansion, elevation, or both, of the area affected by the disturbance.

The second point of interest is that the figures give a means of estimating the rate of growth of the strain which produces earthquakes. If we accept the hypothesis that earthquakes, in the limited sense of their orchesis, are due to the relief by fracture of a growing strain when this has reached the breaking point, it can be easily shown that a variable strain, acting in alternate periods in increase or decrease of the general growth of strain, while leaving the average rate unaltered, will give rise to a corresponding variation in the frequency of shocks in each period; and, besides that, there is a simple relation between the magnitudes of the two stresses, to which the strains are due, and the variations from the mean frequency of earthquakes. A calculation on these lines shows that the growth of strain, for Italy, is such that, accepting the published estimates that an area of the earth's crust of the magnitude of Italy would crush under its own weight if left unsupported to the extent of 1/400 of the force of gravity, the breaking strain would be reached in about 3½ years, starting from a condition of no strain. The aftershocks of the Mino-Owari earthquake give a little less than half this figure, which is again reduced to from five to six months if account is taken of the difference between the resistance of rock to tension and to compression. These figures are given for what they are worth; at the least, they are of interest as being the first authentic estimate which it has been possible to make of the time required to prepare for, and, thence, of the rate of growth of the particular tectonic process involved in the production of earthquakes.

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* Q. J. G. S. vol. lxv. (1909) p. 14.

LXIII. *Intelligence and Miscellaneous Articles.*

On Relativity and Electrodynamics.

To the Editors of the Philosophical Magazine.

Dear Sirs, 27th April, 1918.

As the result of some correspondence, Dr. G. A. Schott has detected an error of sign in my calculation of the transverse inertia of a "contracted conducting electron." In consequence of this the following numerical correction should be inserted in my recent paper on "Relativity and Electrodynamics."

For a contracted conducting electron

$$m_2 = m_0\left(1 + \frac{31}{60}\frac{v^2}{c^2}\right)$$

in place of

$$m_2 = m_0\left(1 + \frac{41}{60}\frac{v^2}{c^2}\right),$$

leading to

$$\tfrac{1}{2}k_1 - k_2 = +\frac{2}{15}$$

in place of

$$\tfrac{1}{2}k_1 - k_2 = -\frac{1}{30}.$$

There is a corresponding correction for a Bucherer electron, viz.

$$m_2 = m_0\left(1 + \frac{21}{60}\frac{v^2}{c^2}\right)$$

in place of

$$m_2 = m_0\left(1 + \frac{31}{60}\frac{v^2}{c^2}\right),$$

giving

$$\tfrac{1}{2}k_1 - k_2 = \frac{13}{60}$$

in place of

$$\tfrac{1}{2}k_1 - k_2 = \frac{1}{20}.$$

I am,

Yours faithfully,

George W. Walker.

INDEX TO VOL. XXXV.

END OF THE THIRTY-FIFTH VOLUME.

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