Very

Lunar and Terrestrial
Albedoes

Occasional Scientific Papers
of the
Westwood Astrophysical Observatory
Number I.

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By
FRANK W. VERY

BOSTON: THE FOUR SEAS COMPANY

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HISTORICAL NOTE

Founded in 1906, the Westwood Astrophysical Observatory owes its inception to aid from Percival Lowell. In beginning a special series of its publications, the writer wishes to place on record his indebtedness to the warm sympathy and encouragement of a faithful friend. Himself an ardent lover of freedom, Dr. Lowell never interfered with the writer's free and independent ordering of the researches conducted at Westwood, but with a rare disinterestedness he placed at his disposal numerous spectrograms taken at the Lowell Observatory by the skilful hands of Dr. V. M. Slipher for measurement with apparatus of the writer's design. Nevertheless, the gain was mutual, for the results throw unexpected light on some of Lowell's own researches and demonstrate that complete independence in respect to control and motives of action is not incompatible with a consistent working together for a common end.

Lowell had been greatly interested in the research which forms the subject of the present communication, with its obvious bearing on the problem of planetary temperature. In his "Temperature of Mars," he had adopted 0.75 for the albedo of a half clouded earth, and I, in my "Greenhouse Theory and Planetary Temperature," had taken 0.70 for the same datum, differing but little from the value now found for the geometrical albedo of the earth, which is 0.72.

Let me also place on record as a result of my intimate association with him, my recognition of the fact that his theories were based on an elaborate accumulation of unsurpassed evidence, that he was always open-minded to new evidence, and that, while presenting some revolutionary new conceptions, he did not hesitate to modify his own ideas when convinced that they could be improved. It is this willingness to revise that constitutes the true man of science. That there was very little for him to change as his researches progressed, is a testimony to Lowell's thoroughness and to his deep insight into nature's mysteries.

With gratitude to God for the gift of a friend—generous, thoughtful for others, and noble in his ideals, keenly critical, but kindly appreciative, learned, but modest—

I dedicate these researches

To the Memory of

Percival Lowell

THE WESTWOOD ASTROPHYSICAL OBSERVATORY

THE WESTWOOD ASTROPHYSICAL OBSERVATORY is situated in Westwood, Massachusetts. Its approximate position and altitude (derived from the topographical map of the United States Geological Survey) are

Latitude = 42° 12′ 58″ North.
Longitude = 71° 11′ 58″ West.
Altitude = 190 feet above sea level.

Its publications hitherto have been in current scientific periodicals, especially, *Lowell Observatory Bulletin, American Journal of Science, Astrophysical Journal, Science, Astronomische Nachrichten,* and *Bulletin Astronomique.*

The Observatory possesses special instruments for the study of solar radiation and atmospheric transmission, for delicate heat measurements, the utilization of solar radiation and study of the "greenhouse" effect, photometry, spectral line and band comparator, etc. For several years it had the use of a fine silver-on-glass concave mirror of 12 inches aperture and 10 feet focal length, which was loaned by its maker, Dr. J. A. Brashear. The mirror was used in researches on the transmission of terrestrial radiation by the aqueous vapor of the atmosphere.

Special researches are being actively prosecuted on atmospheric transmission and the solar constant, quantitative measurements of the intensity of spectral lines, planetary atmospheres and temperatures, greenhouse theory, contributions to the theory of nebulæ and novæ, measurements of the earth's albedo and of that of the moon for all parts of the visible spectrum. The latter researches form the subject of the present communication.

LUNAR AND TERRESTRIAL ALBEDOES

Introduction.

THE word *albedo* (derived from the Latin *albus,* white) has been used by astronomers to designate the fraction of the sun's luminous rays reflected by a planet at full phase, allowance being made for the distances of the planet from sun and earth and for the dimensions of the reflecting body. If the planet were a smooth sphere with perfect specular reflection, it would be itself invisible, but would present within the diminutive limits of its disk a complete picture of the surrounding heavens, distorted by spherical aberration, but otherwise exact; and within this image the reflection of the sun would surpass in brilliancy all other objects, shining like a star at a point on the planet's disk distant from the center by the radius of the disk multiplied by the cosine of half the elongation of the planet from the sun. But whatever specular surfaces there may be on the planets of our solar system, they are of too limited extent to be recognized as such; and the planetary reflection of light is to be classed under the head of a generally diffusive one, though not necessarily an equable one in all directions; and in fact there are diversities in the distribution of the reflected light to different parts of the sphere which must be considered in getting the phase-curve of the illumination, and which are not entirely without influence even if we confine our attention to the reflection sent earthward at full phase, while they are vital to the determination of the complete reflection to the sphere.

Since all of the planets, except possibly some of the smaller asteroids, are spheroidal bodies, it is not necessary for purposes of intercomparison to refer their albedoes to the standard specific reflectivity of a flat surface; but it *is* desirable to distinguish clearly between the only thing which is certainly measurable in most cases,—which is (1) the geometrical albedo at full phase, or the amount of light sent earthwards at the planet's full phase, compared with that which would be sent by a sphere of the same size and at the same distance, which possesses perfect diffusive reflectivity;—and (2) that integration of the reflection to the entire sphere, or the *spherical* albedo, whose determination requires a knowledge of the phase-law. This law is very imperfectly known, except in the case of the moon, and hence there are rival hypotheses which give more than one kind of "spherical" albedo. There is even a diversity of usage in regard to what shall be called the "geometrical" albedo, although there need be no discrepancies in the facts of observation on which it is based. A very few words will suffice to make the fundamental distinctions plain as to their general principles; but the remoter consequences of the acceptance of the diverse points of view lead to discussions of some complexity whose complete unfolding can not be exhibited in the limits of this paper, but enough will be presented to give an intelligible conception of the subject.

If we measure the amount of light received by the eye from the full moon, that is to say, if we find the reflection of sunlight by a spheroidal surface to a point (since the pupil of the eye is virtually a point), we shall get the same value whether the moon is near the horizon or in the zenith (after correcting for the absorption by the earth's atmosphere); and it seems natural to take this constant light-quantity as the basis of *the geometrical albedo referred to a definite point in space,* comparing it with the quantity of light which would be given if the whole sky were filled with moons of perfectly diffusive reflecting quality, and viewed by turning the eye progressively to all parts of the sky and summing the successive impressions. This geometrical ratio of the *reflection to a point* compared with the perfectly diffuse reflection at that point from an ideal body of the same size and in the same situation, is the one considered in this paper and is what is meant by the geometrical albedo.

But if, instead of this, we take the *illumination of an extended surface* by the hypothetical sky full of moons, it is necessary to take into account the diminution of superficial illumination from those rays which are at low angles to the surface, and even supposing an absence of atmosphere, the *surface* illumination produced by a sky full of moons will only be half as great as the sum of the illuminations supposing each moon to be successively transported to the zenith. Thus the "surface illumination" is one half of the geometrical albedo.

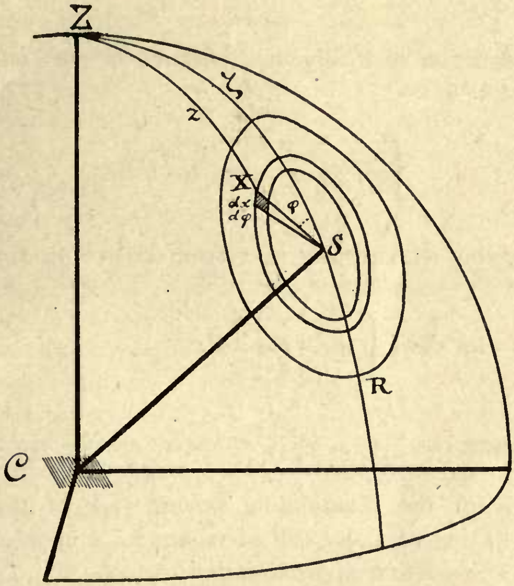

FIGURE 1

Lambert showed in his *"Photometria"* (*cap.* II.) that if we seek the illuminating power (L) of a circular luminary of radius $SR = r$ (Fig. 1), whose center is at any point (S) of zenith-distance $SZ = \zeta$, upon a surface at C, we may obtain L by summing a series of annuli concentric with S and of radius $SX = x$, where, if an element $(dx, d\varphi)$ of the annulus is at the angle $ZSX = \varphi$ from the vertical through S, the area of the element is $dx \cdot d\varphi \sin x$. Hence

$$L = \int\int \sin x \cos \zeta \, d\varphi \, dx,$$

since the illumination of the surface at C varies in proportion to $\cos \zeta$.

By spherical trigonometry, if z is the zenith-distance of any point X on the annulus,

$$\cos z = \cos \zeta \cos x + \sin \zeta \sin x \cos \varphi,$$

and

$$L = \int\int dx \cdot d\varphi \sin x \, [\cos \zeta \cos x + \sin \zeta \sin x \cos \varphi].$$

A first integration relatively to φ between $\varphi = 0°$ and $\varphi = 360°$, or 2π, gives

$$L = \int dx \sin x \cdot 2\pi \cdot \cos \zeta \cos x.$$

Integrating this with respect to x from $x = 0°$ to $x = r°$,

$$L = 2\pi \cos \zeta \int \sin x \cos x \, dx$$

$$= 2\pi \cos \zeta \left(\frac{1 - \cos 2r}{4}\right) = \pi \cdot \cos \zeta \cdot \sin^2 r.$$

This gives for the illuminating power (L) of the moon at the zenith to that of a sky full of moons (L') upon an extended surface at C, as a first approximation,

$$L : L' = \pi \cos 0° \; (\sin^2 15' \; 33'') : \pi \cos 0° \; (\sin^2 90°)$$
$$= 1 : 48{,}875.$$

But if we consider the luminous effect upon a point, such as the eye, or the heating effect upon the bulb of a thermometer which may likewise be taken as a point, instead of that communicated to an extended surface, then, neglecting atmospheric absorption, it is necessary to find the ratio of illuminations by taking the ratio of the area of the apparent lunar disk to the hemispherical sky area, a ratio which is half as great as the one just given. For a disk as small as that of a planet, the area may be taken $= \pi \sin^2 \varrho \cdot r^2$, where ϱ is the angular value of the radius of the disk and r is the distance of the planet. Comparing this with the area of the hemisphere, $2\pi r^2$, the latter exceeds the former in the ratio, $2 : \sin^2 \varrho$, which, for the moon's semi-diameter, $\varrho = 15' \; 33''$, gives for the ratio of the light reflected to a point from the two sources,

$$97{,}750 : 1$$

with a similar degree of approximation to the preceding value.

The integration of the total light reflected by the illuminated hemisphere of the planet in all directions requires the introduction of hypotheses. The first is Lambert's hypothesis of uniform diffuse reflection, of which the following account is substantially that of Müller.

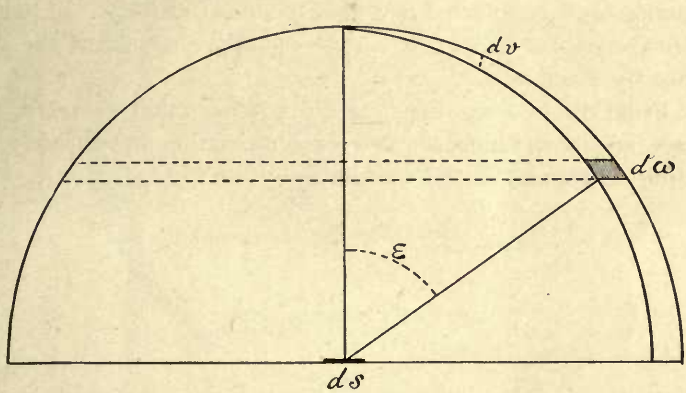

FIGURE 2

Consider a diffusely reflecting surface-element, ds (Fig. 2) illuminated under any angle of incidence, i. If L is the quantity of light which falls normally on the unit of surface, then ds receives $L\ ds \cos i$, of which a certain fracton $cL\ ds \cos i$ is reflected normally, and in any other direction, such as that of the emanation angle ε, the light-quantity $dq = cL \cos i\ ds \cos \varepsilon$ is reflected, *provided the surface reflects as well at one angle as at another.*

Construct a hemisphere with radius 1 about ds of which an element $d\omega$ receives the fractional light-quantity

$$dQ = dq\ d\omega = cL\ ds \cos i \cos \varepsilon\ d\omega.$$

Then since the element $d\omega$ has the width $dv \sin \varepsilon$ and the height $d\varepsilon$, or the angular area $d\varepsilon \sin \varepsilon\ dv$, the total light-quantity has the value

$$Q = cL\ ds \cos i \int_0^{\pi/2} \cos \varepsilon \sin \varepsilon\ d\varepsilon \int_0^{2\pi} dv,$$

and since $\int_0^{\pi/2} \cos \varepsilon \sin \varepsilon \, d\varepsilon = \frac{1}{2}$, and $\int_0^{2\pi} dv = 2\pi$,

$$Q = L \, ds \cos i \cdot A,$$

where the factor A is a fraction which tells how much of the incoming light is reflected to a hemisphere of radius 1. A, which is always smaller than 1, is simply called the albedo of the substance by Lambert.

From the two equations for Q, it follows that $c = A/\pi$, and thence we have Lambert's law of illumination by diffusely reflecting substances in the well known form:

$$dq = \frac{A}{\pi} L \, ds \cos i \cos \varepsilon,$$

or

$$dq = \Gamma_1 \, ds \cos i \cos \varepsilon$$

if $\Gamma_1 = AL/\pi$.

Calculation of the quantity of light sent to the earth at different phases of a reflecting planet requires some further slight modifications.

Let a plane be drawn through the middle point of the planet at right angles to the earth-planet line; and let its intersection with the planet's surface be represented by the circle $ABCD$ (Fig. 3). The perpendicular to this plane in the direction of the earth is shown diagrammatically by ME. MS is drawn in the directon of the sun. The arc of a great circle ES is the phase-angle α, taken from full phase. An element ds of the visible hemisphere of the planet is connected with E and S by great circles of the sphere. Arc $S - ds = i$, arc $E - ds = \varepsilon$. Latitude of $ds = \psi = F - ds$. Longitude from $E = EF = \omega$.

From the right-angled spherical triangles $FSds$ and $FEds$, we have the relations:

$$\cos i = \cos \psi \cos (\omega - \alpha),$$

$$\cos \varepsilon = \cos \psi \cos \omega.$$

If the semi-diameter of the planet is ϱ, the linear dimensions of ds are $\varrho \, d\psi$ in the direction of the meridian and $\varrho \, d\omega \cos \psi$ along a parallel. Hence

$$\text{surface of } ds = \varrho^2 \cos \psi \, d\omega \, d\psi.$$

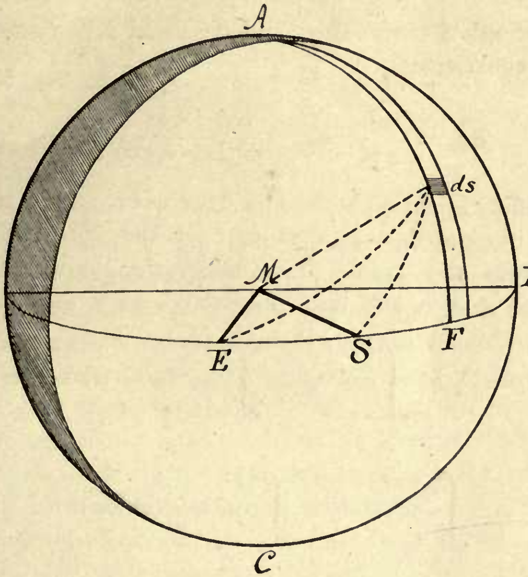

FIGURE 3

We now introduce two rival hypotheses, or laws of reflection, (1) that reflection is uniform in all directions, (2) that it varies according to a definite law, and get

(1)

Lambert's Law:

1a. $dq_1 = \Gamma_1 \varrho^2 \cos^3 \psi \, d\psi \cos (\omega - \alpha) \cos \omega \, d\omega,$

Lommel-Seeliger Law:

1b. $dq_2 = \Gamma_2 \varrho^2 \cos^2 \psi \dfrac{\cos \omega \cos (\omega - \alpha)}{\cos (\omega - \alpha) + \lambda \cos \omega} d\omega,$

where $\Gamma_2 = \dfrac{L}{4\pi} \dfrac{\mu}{k}$ and $\lambda = \dfrac{k}{k_1}$, k being the coefficient of absorption of the rays which enter into the interior of the substance of the planet's surface, k_1 the coefficient of interior absorption of the returning rays on the way to emission, and μ the diffusive power of the body. In general $k_1 < k$, because the outgoing rays have lost their more absorbable ingredients. If the material is strongly colored, k may be very much larger than k_1.

The derivation of the Lommel-Seeliger equation which takes account of interior reflection and diffusion is very complicated. The final equation is

$$dq_2 = \frac{\mu L}{4\pi k} \cdot ds \cdot \frac{\cos i \cos \varepsilon}{\cos i + \lambda \cos \varepsilon}.$$

Confining attention here to the Lambert equation, the formula must be integrated over that part of the illuminated surface visible from the earth. The integration limits for ψ are $-\pi/2$ and $+\pi/2$, and those for ω are $-\pi/2+\alpha$ and $+\pi/2$,

whence $q_1 = \Gamma_1 \varrho^2 \int_{-\pi/2}^{\pi/2} \cos^3 \psi \, d\psi \int_{\alpha-\pi/2}^{\pi/2} \cos(\omega - \alpha) \cos \omega \, d\omega.$

But

$$\int_{-\pi/2}^{\pi/2} \cos^3 \psi \, d\psi = \int_{-\pi/2}^{\pi/2} \cos^2 \psi \, d(\sin \psi)$$

$$= \int_{-\pi/2}^{\pi/2} [1 - \sin^2 \psi] \, d(\sin \psi) = \frac{4}{3}.$$

And

$$\int_{\alpha-\pi/2}^{\pi/2} \cos(\omega - \alpha) \cos \omega \, d\omega$$

$$= \tfrac{1}{2} \int_{-\pi/2}^{\pi/2} \cos \alpha \, d\omega + \tfrac{1}{2} \int_{\alpha-\pi/2}^{\pi/2} \cos(2\omega - \alpha) \, d\omega$$

$$= \tfrac{1}{2} [(\pi - \alpha) \cos \alpha + \sin \alpha].$$

Therefore

$$(2) \qquad q_1 = \Gamma_1 \varrho^2 \cdot 2/3 \, [\sin \alpha + (\pi - \alpha) \cos \alpha].$$

For the full phase, when sun, earth and planet stand in a straight line, $\alpha = 0$ and the reflected light is $q_1^{(0)} = \Gamma_1 \varrho^2 \cdot 2/3\pi$. Hence we have for the ratio

Light at phase-angle α : Light at full phase,

$$(3) \qquad q_1/q_1^{(0)} = \frac{1}{\pi} [\sin \alpha + (\pi - \alpha) \cos \alpha].$$

A similar integration for the Lommel-Seeliger law gives

$$(4) \qquad q^2 = \frac{\Gamma_2 \varrho^2 \pi}{2} [1 - \sin \alpha/2 \tan \alpha/2 \log \cot \alpha/4].$$

But for $\alpha = 0$, $q_2^{(0)} = (\Gamma_2 \varrho^2 \pi)/2$, and

$$(5) \qquad q_2/q_2^{(0)} = 1 - \sin \alpha/2 \tan \alpha/2 \log \cot \alpha/4.$$

The distribution of light over the apparent disk of the planet varies according to the adopted law. Euler's law would demand uniform light, except for a narrow strip of sudden diminution at the terminator and an excessively narrow, but exceedingly bright rim at the illuminated limb. Nothing of the sort is observed, and this law may be dismissed at once. Moreover, Euler considered nothing but superficial reflection, just as Bouguer did, whereas the penetration of the light into a thin surface layer, even in the most opaque substances, is of great importance.

Lambert's law appears to work fairly well where the reflecting medium is of the nature of cloud with internal diffusion and multiple reflection from innumerable widely dispersed and finely divided particles, such as ice crystals, or dust, or the liquid water particles of ordinary cloud. The Lommel-Seeliger Law is more appropriate for extended solid surfaces at various inclinations to the incident light. A composite inter-mingling of solid surface and cloud requires a mixture of the two laws.

In the following Table are given the computed values of the functions of the phase-angle, $\varphi(\alpha)$, for intervals of 5° according to several theories. These quantities are then multiplied by others proportional to the areas of the corresponding zones, $\sin \alpha \cdot \Delta\alpha = \Delta(1 - \cos \alpha)$, to give the values in the last three columns which, being summed, produce the proportional factors for the spherical albedo. If the intervals had been taken small enough, the sum of the differences of versine α, $\Sigma\, [\Delta(1 - \cos \alpha)]$, would have been exactly 2 which, multiplied by 2π, gives the area of the sphere of unit radius. The average reflection by a planet to the sphere has an intensity ¼ if the reflection is perfect (normal specific reflectivity $= 1$), this being the mean between the quantities (½ and 0) sent in the directions of source and antipodal point. Calling $\Sigma\, [\Delta(1 - \cos \alpha) \cdot \varphi(\alpha)]$ the *spherical factor,* its values given in the last line of the table, are

0.75 by Lambert's Law.

1.50 by the Lommel-Seeliger Law.

0.35 by the lunar phase-curve.

While the spherical albedo can not exceed unity, there may be various distributions of light to the sphere. Thus, for perfect reflection, the diverse spherical factors obtained from the summations in the table are consistent with geometrical albedoes of 0.50, 1.33, 0.67, and 2.86, the last being for the phase-law of the moon where the reflection at full phase is extraordinarily large.

Limits of phase-angle $\alpha_1 — \alpha_2$	Δ(1-cos α)	$\varphi(\alpha)$ Lambert	$\varphi(\alpha)$ Lommel-Seeliger	$\varphi(\alpha)$ Moon	$\varphi(\alpha) \times \Delta(1 - \cos\alpha)$		
					Lambert	Lommel-Seeliger	Moon
0°— 5°	.0038	.996	.999	.957	.0038	.0038	.0036
5 — 10	.0114	.986	.994	.869	.0112	.0113	.0099
10 — 15	.0189	.972	.985	.783	.0184	.0186	.0148
15 — 20	.0262	.953	.974	.703	.0250	.0255	.0184
20 — 25	.0334	.928	.961	.628	.0310	.0321	.0210
25 — 30	.0403	.898	.947	.557	.0362	.0382	.0224
30 — 35	.0469	.863	.931	.494	.0405	.0437	.0232
35 — 40	.0531	.824	.915	.440	.0438	.0486	.0234
40 — 45	.0589	.781	.898	.392	.0460	.0529	.0231
45 — 50	.0643	.734	.880	.348	.0472	.0566	.0224
50 — 55	.0692	.684	.862	.310	.0473	.0597	.0215
55 — 60	.0736	.633	.844	.275	.0466	.0621	.0202
60 — 65	.0774	.581	.826	.243	.0450	.0639	.0188
65 — 70	.0806	.531	.808	.213	.0428	.0651	.0172
70 — 75	.0832	.481	.790	.185	.0400	.0657	.0154
75 — 80	.0852	.434	.772	.159	.0370	.0658	.0135
80 — 85	.0865	.388	.755	.136	.0336	.0653	.0118
85 — 90	.0872	.342	.738	.115	.0298	.0644	.0100
90 — 95	.0872	.298	.721	.096	.0260	.0629	.0084
95 —100	.0865	.256	.705	.080	.0221	.0610	.0069
100 —105	.0852	.218	.690	.066	.0186	.0588	.0056
105 —110	.0832	.183	.675	.054	.0152	.0562	.0045
110 —115	.0806	.150	.661	.044	.0121	.0533	.0035
115 —120	.0774	.120	.648	.035	.0093	.0502	.0027
120 —125	.0736	.094	.636	.028	.0069	.0468	.0021
125 —130	.0692	.073	.625	.021	.0050	.0433	.0015
130 —135	.0643	.055	.614	.016	.0035	.0395	.0010
135 —140	.0589	.040	.605	.012	.0024	.0356	.0007
140 —145	.0531	.028	.596	.010	.0015	.0316	.0005
145 —150	.0469	.019	.589	.008	.0009	.0276	.0004
150 —155	.0403	.015	.582	.006	.0006	.0235	.0002
155 —160	.0334	.012	.577	.005	.0004	.0193	.0002
160 —165	.0262	.010	.572	.005	.0003	.0150	.0001
165 —170	.0189	.008	.569	.004	.0002	.0108	.0001
170 —175	.0114	.006	.567	.004	.0001	.0065	.0001
175 —180	.0038	.004	.566	.004	.0000	.0022	.0000
Sums	2.0202				0.7503	1.4874	0.3491

LUNAR AND TERRESTRIAL ALBEDOES

Discussion of the Observations[1]

In the *Astrophysical Journal* for April, 1916, Professor H. N. Russell has discussed some recent observations of the writer on the earth-shine, from which the earth's albedo had been obtained indirectly. The observations are of two sorts—(1) direct visual comparisons of parts of the moon (lit by the sun's rays) and of other parts of similar quality, lit by the earth-shine, with the light of a flame seen through blue glass; and (2) comparisons of the relative intensities of all the colors in the spectrum between violet and red from their relative photographic effect on spectrograms.

The earth-shine spectrograms,[2] along with similar ones of the moon and of the sky, were measured at the Westwood Observatory by means of the comparator originally designed for quantitative measures of the intensities of atmospheric bands on the Lowell Observatory spectrograms of Mars and the Moon. With this instrument I have already obtained approximate determinations of the amounts of the water vapor and oxygen in the atmosphere of Mars, and I am now engaged in measuring the intensities of the Fraunhofer lines in the solar spectrum with an accuracy which has not been approached hitherto. These facts show that on the score of precision, the comparator is capable of excellent work, though, like all instruments dependent on photography for the registration of intensities, it involves the complexities of photographic laws. These complexities, however, I have

[1] The essential features of this discussion were presented before the American Astronomical Society at its Nineteenth Meeting at Swarthmore, Pennsylvania, August 31, 1916.

[2] The spectrograms were made for me through the kindness of Dr. Percival Lowell by Dr. V. M. Slipher at Flagstaff. Several of them were taken under exceptionally favorable atmospheric conditions. See Frank W. Very—"The Photographic Spectrography of the Earth-Shine," *Astronomische Nachrichten,* Nr. 4819-20, Bd. 201, s. 353-400, November, 1915; also "Atmospheric Transmission," *Science,* N. S., Vol. XLIV., No. 1127, pages 168-171, August 4, 1916.

endeavored to minimize, and I have in large part succeeded in eliminating them by an extensive study of the photographic problem for each wave-length and a wide range of exposures on more than one kind of plate. Professor Russell's criticisms of my work with the spectral line and band comparator are largely founded on misapprehensions. He has taken unwarrantable liberties with my figures and by so doing has rejected my work on the spectrograms on insufficient grounds. When correctly reduced, the two methods give results which are in good agreement, but on the whole, those from the spectrograms are the more reliable.[1]

In his first article,[2] Russell adopts 465,000 : 1 for the ratio of sunlight to full-moon light; and on page 184 of the April *Journal* he expresses preference for the ratio 9,000 : 1 [3] between sunlight and full-earth light on the moon. According to this, the ratio of full-earth light to full-moon light is

$$465{,}000 : 9{,}000 = 51.7 : 1,$$

[1] The visual photometric values of the earth-shine which are described in my paper on "The Earth's Albedo" (*Astronomische Nachrichten*, Nr. 4696, Bd. 196, s. 269-290, November, 1913) were obtained with a special earth-shine photometer which might be improved in the light of the experience gained with it. Although free from photographic difficulties, the method has difficulties of its own, as may be recognized from the elaborate researches which were required in establishing the constants of the various absorbent pieces. Owing to the faintness of the earth-shine, the low altitude of the crescent moon when the measures have to be made, and the varying transparency of the atmosphere, there are further difficulties which Professor Russell generously allows for in his criticism, but he has not understood some of the minor details.

It must be remembered that the spectrograms were made with an analyzing spectroscope, and that the values obtained relate to the intrinsic brightness of definite regions on the moon where the reflecting quality of the surface is far from uniform, and the range of luminous values with the phase is wide, so that small displacements on the surface may give considerable alteration of light. The observed differences are due to these unavoidable vicissitudes, rather than to errors of observation; but such differences as these tend to average out from the general mean. In the earth-shine exposures, the slit was placed half on and half outside the dark limb of the moon to give the sky spectrum needed in the reductions.

[2] *Astrophysical Journal* for March, 1916, p. 125.

[3] On page 194 (*op. cit.*), this ratio is attributed to me, but I have not given it, and prefer the ratio 10,000 : 1.

which, since the angular area of the earth as seen from the moon is 13.4 times that of the moon seen from the earth, makes the earth's albedo $51.7/13.4 = 3.86$ times that of the moon. My own determination of this ratio is considerably larger, namely, 4.8 : 1.

The ratio of sunlight to moonlight is not easily measured with precision on account of the wide range of intensities involved and the uncertainties of atmospheric absorption. Whether we take the ratio 465,000 : 1, difference of magnitude = 14.17 (Russell), or 618,000 : 1, difference of magnitude = 14.48 (Zöllner), or even a value as large as Wollaston's (801,000 : 1) we shall still be inside the actual divergences of some very good observers. A critical examination of sources of error will improve this result greatly.

Zöllner's explanation of the peculiar efficacy of the lunar mountains in emphasizing the peak of the lunar phase-curve at the full, with Searle's emendation which notes the contribution to the same effect by crevices which retain the sunshine away from observation until the short interval when the rays strike their floors, suffices for the anomalies of this phase-curve. Whether the elevations are mountains, as is usually assumed, or innumerable crystalline facets, that is, whether the roughness is on a large or a small scale, is a matter of indifference. Zöllner's diagrammatic figure is well characterized by Russell as "artificial," but the fact of a general excessive roughness of the lunar surface is probable enough and natural enough, even though it may not be as obvious as the mountains are to telescopic vision. Nevertheless, although Zöllner's hypothetical moon behaves in some respects like the actual moon near the time of full, the analogy would fail if pushed to its limit, especially in the early crescent phases, and the title "true" which was used by Zöllner to designate the "albedo" of this hypothetical body is a misnomer when applied to the actual moon. Russell's criticism by questioning the foundation of an almost unanimous acceptance of Zöllner's value and terminology, performs a much needed use. In this he has followed Guthnick more or less. Müller, also, had previously characterized Zöllner's value of the mean (52°) slope of the lunar upheavals as "illusory."

In getting the lunar albedo, Zöllner allowed for the rotundity of the moon, and by an unfortunate misapplication of Lambert's law of reflection from a uniformly diffusing sphere[1] found that the full moon should reflect

$$\frac{1}{p}=\frac{1}{(3/2)\times 48{,}980}=\frac{1}{73{,}470}$$

of sunlight, if it were such a sphere. The corresponding value of the albedo he called the "apparent albedo" (scheinbare Albedo)—a term which is misleading, since it implies that the moon's reflection which is actually observed is this quantity given by computation, and that the reflection up to this point follows Lambert's law, which is not true.

The *total* radiation of the moon, including both reflected and emitted rays, does appear to follow pretty nearly a sequence which can be derived from Lambert's law, multiplied by the factor 2/3, a number which seems to turn up on every hand in this research. This may be seen from the phase-curve given in my "Prize Essay on the Distribution of the Moon's Heat and its Variation with the Phase," where I found

At first quarter, total radiation = 17.7% of radiation from full moon.

At last quarter, total radiation = 24.8% of radiation from full moon.

Mean (quadrature) total radiation = 21.25% of radiation from full moon.

Lambert's law, multiplied by 2/3 = 21.22% of radiation from full moon.

Here the radiation which is measured, is made up of two parts—one which is wholly reflected, and the other an emission from a heated surface whose temperature varies from a maximum at the center of the disk to a minimum at the limb, while the surfaces of equal temperature are concentric zones. The *luminous* reflection, at any rate, follows the opposite law and is greater at the limb, and probably the non-luminous rays are reflected in the same way. The smaller total radiation at first quarter is of course due to the fact that the moon is getting hotter and the energy is being expended in modifying the subsurface

[1] The law was designed to give the *spherical* albedo. What was needed here was simply the geometrical albedo.

thermal gradient, while the heat thus retained is given out again in the lunar afternoon, or at last quarter. The point which I wish to make clear is that, as far as the reflection of the moon's *light* is concerned, the introduction of Lambert's law at this or any other stage of the computation was a mistake.

The value "$\mu = 0.1195$," which Zöllner calls "die scheinbare Albedo des Mondes," is a hypothetical value which is not immediately given by observation, but is obtained by restricting the definition of albedo to diffuse reflection from a nonexistent smooth sphere on the supposition that a factor 2/3 must be introduced. I will return to this later, but will note here that the moon does not reflect much after the approved fashion of a sphere, but acts, to all appearance, more like a flat surface, or even like one a little dished at the margin; for whereas a diffusive sphere should send out light from the marginal zones at full into a rear hemisphere, whereby these zones as seen from the front should be considerably fainter than the center, it is found, on the contrary, that the limb in the actual moon is in fact the brighter of the two. Both front and rear reflections from the limb are exceptionally large, so that these portions of the lunar surface are but little heated by the sun's rays.

The arguments by which Zöllner persuaded himself that he had arrived at the "true" value of the moon's albedo from his "apparent" albedo are somewhat involved. One of the first needs is that our procedures and definitions may be clarified and simplified. Let us begin by considering the Lambert law.

On page 176 of his second article, Professor Russell says correctly in speaking of the reflective function of the phase-angle, $\varphi(\alpha)$: "The whole amount of light reflected by the planet to the celestial sphere will be proportional to

$$\int_0^{\pi} \varphi(\alpha) \sin \alpha \, d\alpha.$$

If it shone in all directions with the brightness of the full phase, the emitted light would be 2.0 on the same scale." But this being so, the factor q which transforms from the geometrical albedo at opposition to the spherical albedo should be the integral just given, and not twice that quantity, which is Russell's equation (7), because the value of the integral alone without the coefficient 2 is 2.0, if $\varphi(\alpha) = 1$ everywhere. For Lambert's law,

$$(6) \qquad q = \int_0^\pi \varphi(\alpha) \sin \alpha \, d\alpha = 0.75,$$

and this is the Lambert factor for spherical albedo, or it is the spherical albedo if the geometrical albedo at full phase is unity, instead of $q=1.5$ as given by Russell. Similarly, for the lunar phase-curve, $q=0.35$, which is to be multiplied into the observed geometrical albedo from the full moon to give the lunar spherical albedo. Since the values of q have been taken two times too large, all of the numbers in Russell's Table I., *op. cit.*, page 179, should be divided by two. On the other hand, in getting q for the Lommel-Seeliger law, Russell has inconsistently dropped the factor 2, which would make his value the same as mine, were it not that there is a further inaccuracy in the integration by which he gets "$q=1.6366$." His equation (7) should give $q = 3.0$.

Lambert's formula for spherical albedo,

$$L = (1/\pi) \; [\sin \alpha - \alpha \cos \alpha] + \cos \alpha,$$

where α is the moon's elongation, or phase-angle from conjunction, or as Russell prefers to put it, employing *phase-angles from opposition;* (in which respect I shall follow his procedure)

$$\varphi(\alpha) = (1/\pi) \; [\sin \alpha + (\pi - \alpha) \cos \alpha],$$

gives the light at quadrature, $L_{90} = 1/\pi = 0.318$, when the light at the full phase is unity. For the emission of its own radiation combined with the reflection, the radiant observation already quoted would seem to favor the value, $R_{90} = 1/(1.5\pi) = 0.212$. Here, however, we must note that, though the emission may be independent of its direction, the distribution of temperature is not uniform, and the result is a complex of two different functions of α.

Under these circumstances we can attach little significance to this special value. It is difficult to see how a particular numerical factor can survive this double vicissitude. If the factor 2/3 occurs undisguised in both functions, it will become 4/9 in their product; if in opposite senses, it will cancel out;

and if the factor is found in one case, but a different one is substituted for it in the other, it could not be so easily recognized.

By Lambert's law, at the full, $L_0 = \pi \times L_{90} = 3.14\ L_{90}$.

For the planet Venus, reflection $= L_0 = (1/.283)\ L_{90} = 3.53 L_{90}$.

Lunar emitted and reflected radiation $= R_0 = (3/2)\pi \times R_{90} = 4.71 R_{90}$.

Lunar reflected light $= L_0 = (1/.105)\ L_{90} = 9.52 L_{90}$.

The factors connecting the observed light at full phase with the light at quadrature are evidently empirical. Those connecting spherical and geometrical albedoes are probably equally empirical. The appearance of the factor $(3/2)\ \pi$ in my lunar radiation observation, to which I have called attention, is rather striking, yet it is probably no better than a coincidence. Zöllner must have been under a great misapprehension when he attempted to introduce the factor 3/2 into the discussion of his lunar observations, for it does not fit the facts.

There is, it is true, a universal usage for which the factor 3/2 is appropriate, namely: The reflection from a sphere of any ordinarily diffusive material is 2/3 of that returned with perpendicular incidence and reflection from a plane surface of the same substance, and in comparing the reflection of the entire spherical body of a planet with that from a plane surface of some terrestrial substance, it would be appropriate, provided we could be sure that the light is diffusively reflected, to multiply the reflection from the sphere by 3/2 to put it on terms of equality with the recognized reflective power, or specific reflectivity, of the given terrestrial material in the form of a flat surface when this is viewed normally. If the flat surface reflects the light at an angle of 45°, its reflection must be multiplied by $\cos 45° = 0.707$, and in this case the two reflectors are already on terms of approximate equality. It does not appear that the factor 3/2 was introduced by Zöllner with any such end as this in view, but simply because it occurs in the Lambert formula. As applied by Zöllner in his lunar theory, this use of the factor 3/2 has been a stumbling block in the path of subsequent research on the subject. The whole of this cloudy lucubration should be swept aside. Zöllner's original lunar observations are among the best that have ever been made, and they deserve to be rescued from the scandalous treatment of his theoretical discussion.

At the start, Zöllner has evidently adopted the incorrect idea that the fraction of light from the sun received upon the moon's surface and which has to be considered, is the fraction of the *total* luminous output of the sun to the entire sphere, and he thus gets for the denominator the number 48,980. He sees that this number is too small for his observations and makes the hypothesis that it must be multiplied by 3/2, giving 73,470, to which he assigned the symbol p. But this is in turn too small, and he introduces the further hypothesis of the lunar mountains with slopes of 52°, getting a new factor $\varkappa$, and $\varkappa p = 107{,}300$, with which his value of the so-called "true" albedo, 0.174, was obtained by comparing it with the observed ratio of sunlight to moonlight. If he had started out with the correct conception that the moon receives a certain fraction of the light emitted by the *half* sphere of the sun which is turned toward the planet, he would have obtained at once the number 97,960 (slightly different from the one which I have used, because we have adopted slightly different values of the moon's semidiameter) and he would have obtained at once for the geometrical albedo of the moon, $97{,}960/618{,}000 = 0.1585$.

In short, Zöllner started with a wrong number, multiplied this by 3/2 and then by nearly another 3/2, or in all by nearly 2¼, instead of by 2 exactly, as he should have done, and thus by a threefold error he reached a result which was nearly right, but solely by accident.

Russell starts with the same erroneous conception that the sun's complete spherical emission should be considered, but immediately abandons it (though without noting the fact) for another, not necessarily incorrect, but different from mine, since he does not introduce 4 into the numerator of his expression for p, nor yet the number 2 which would give what I call the geometrical albedo, but multiplies by 1, whence his p is one half of the geometrical albedo, given by eye observation, and represents *surface illumination* as I have shown in the Introduction. He then makes the reverse change by introducing 2 into his value of q, outside the integral, where it does not belong if by q is meant the spherical factor, as in my equation (6), so that his q is two times mine and through the cancellation of these opposite transformations we finally reach similar results for the spherical albedo.

The geometrical albedo, A_2, which, like Russell's p, "depends only on the geometrical and photometric relations of the planet as observed at the full phase" [1] is correctly given by equation (7),

$$A_2 = \frac{2M_0 \sin^2 S}{\sin^2 s_0 \sin^2 \sigma_0}, \tag{7}$$

where S is the apparent semi-diameter of the sun as seen from the earth, s_0 and σ_0 are the semi-diameters of the sun as seen from the planet and of the planet as seen from the earth at the time of opposition, and M_0 is the ratio of the light received from the planet at mean opposition, to the light of the sun as observed from the earth. This equation is the same as the middle one of Müller's (14)[2] and is a special case derived from the Lommel-Seeliger theory, which, although it is a theory of spherical reflection, gives the geometrical albedo at this particular point,[3] which Russell's equation (5) does not do.

Professor Russell says: "Let r be the mean radius of the planet's disk, and R its distance from the sun, and M_0 be the ratio of the apparent brightness of the planet at the full phase, and at distance Δ from the earth, to that of the sun at unit distance. The fraction of the sun's whole radiation which the planet intercepts is $r^2/4R^2$." [4] This is all true, but it is not what we want to know. The fraction of the sun's light emitted by the solar hemisphere which is visible from the planet, and which the planet intercepts, is $\pi r^2/2\pi R^2 = r^2/2R^2$; and M_0 being the ratio of the observed planetary light at full phase to sunlight and Δ the distance from the earth at the time of observation, so that if the other things remain the same, $M_0\Delta^2 =$ const., an alternative expression for the geometrical albedo is

$$A_2 = M_0\Delta^2 / \frac{r^2}{2R^2} = \frac{2R^2 M_0 \Delta^2}{r^2}.$$

It is not necessary in this problem of the reflection of the sun's

[1] H. N. Russell, in *Astrophysical Journal* for April, 1916, p. 177.

[2] *Photometrie der Gestirne*, p. 65.

[3] The geometrical albedo is in fact the same as the spherical albedo on the assumption that the spherical factor is unity ($q = 1$).

[4] *Astrophysical Journal*, April, 1916, p. 176.

rays to consider the sun's invisible hemisphere. The reflection would be the same if this were dark. For the particular problem under discussion, the other side of the sun is as if it were non-existent, and it has nothing to do with the question. That there may be no misunderstanding, I repeat that Russell's final expression for what he denotes under the symbol p, and which I call surface illumination, must be multiplied by 2 in order to obtain the quantity A_2 or the geometrical albedo; and therefore his factor "q" which reduces to spherical albedo is to be divided by 2, so that, apart from all other considerations, that is, granting the reliability of the original data, there should be no difference between us in respect to the spherical albedoes ("A" of Russell's Table V), provided we could agree in regard to the best phase-law to be adopted.[1]

The quantity "p" is defined in two ways in Russell's paper:

"The factor p may also be defined as the ratio of the actual brightness of the planet at the full phase to that of a self-luminous body of the same size and position, which radiates as much light from each unit of its surface as the planet receives from the sun under normal illumination" (*Op. cit.*, pp. 177-178).

By this definition, as interpreted by Russell, the quantity p is proportional to Müller's M_0, since all of the planetary values in Russell's equation have been reduced to unit distance, and M_0 is a ratio to sunshine at unit distance. But from (7)

$$M_0 = \tfrac{1}{2} A_2 \times const.,$$

or Russell's p, like Müller's M_0, is proportional to the half of the geometrical albedo.

The writer would interpret the definition itself differently, because a planet of radius r and distance from the sun R, will receive from the sun, if L is the sun's total spherical emission of light, the light-quantity $L/4\pi R^2$ on each unit of normally exposed surface. But the planet "receives" light from the sun on only one half of the planetary surface, and hence, if it radiates from "each *unit* of its surface" self-luminously (or what amounts to the same thing, if the light falling normally on a plane surface equal to the planet's section, is wholly emitted by a hemispherical surface of that planet) the emitted light is given out through

[1] This will be considered in a separate paper.

a surface twice as great as the receiving surface, and should be on the average intrinsically half as intense as the received light. If the intrinsic brightness of the emitted light is

$$J = \tfrac{1}{2}(L/4\pi R^2),$$

the total light emitted by the sphere is

$$L' = 4\pi J r^2 = \frac{L}{2}\frac{r^2}{R^2}.$$

But this does not represent the real conditions, because the definition itself is faulty, since the planet does not receive light "under normal illumination" on every part of its exposed hemisphere. If, therefore, we make my parenthetical substitution, and omit the stipulation that each unit of surface shall radiate as much light as it receives "under normal illumination," the total light emitted by the *hemisphere* is

$$L' = 2\pi J r^2 = \frac{\mathrm{L}}{4}\frac{r^2}{\mathrm{R}^2},$$

and we have returned to the previous proposition relating to the fraction of the sun's whole spherical emission which the planet intercepts. This, as I have said, does not concern us in the problem of reflection, where the planet reflects light solely from the visible hemisphere of the sun, and the invisible solar hemisphere with its emission need not be considered, since none of its light reaches the planet.

We have, therefore, for the reflection-ratio, if all of the sunlight is reflected,

$$L'/\frac{L}{2} = 2\frac{L'}{L} = \frac{r^2}{2R^2},$$

and the geometrical albedo is the ratio of the observed reflection to this value; whence Russell's first definition, if modified so as to bring it into accord with the actual conditions, agrees with my definition of the geometrical albedo.

On the other hand, Russell's second definition can not be thus reconciled. It reads as follows:

"The factor p may be defined verbally as the ratio of the observed brightness of the planet at full phase to that of a flat disk of the same size and in the same position, illuminated and viewed normally, and reflecting all the incident light in accordance with Lambert's law" (*op. cit.*, p. 188).

If the sphere shines "in all directions with the brightness of the full phase" (*op. cit.*, p. 176), the quantity which I call q (eq. 6) is

$$q = \int_0^\pi \varphi(\alpha) \sin \alpha \, d\alpha = 2.0.$$

But if the sphere shines according to Lambert's law, the value of the integral is 3/4. If, however, the further proviso be made that p is "the ratio of the observed brightness of the planet at full phase to that of a flat disk," since the ratio of reflection from a sphere illuminated and viewed from the front (or that reflection corresponding to its geometrical albedo), is to the reflection from a flat disk, illuminated and viewed normally, as 2/3 : 1, the 3/4 must be multiplied by 2/3 giving 1/2 as the maximum "flat-disk" value of the spherical albedo. The same fraction expresses the relation between the intregral computed for $\varphi(\alpha)$ as given by Lambert's law (q_1) and by the Lommel-Seeliger law (q_2), namely,

$$q_1 : q_2 = 0.75 : 1.50 = 1 : 2.$$

Since, however, the total reflection, or spherical albedo, must be the same and equal to the incident light on either hypothesis *with complete and diffusive reflection,* it follows that whatever differences result from these hypotheses must fall upon q and p equably and oppositely in order that their product, $A = qp$, may remain the same for the given planet. That is to say, whatever variations there may be in the distribution of light to different zones by reflection according to the rival theories, the sum total, or spherical albedo, must be the same for either if the reflection is complete. Hence if q is obtained by Lambert's law, p must be twice as large as it would be if q were given by the Lommel-Seeliger law. There appears, therefore, to be an inconsistency in Russell's second definition, and his "p" matches the condition demanded by the Lommel-Seeliger law, while my doubled value, though given as a particular case of the Lommel-Seeliger formula, has to be combined with the Lambert value of q, if $A = qp$ is to be kept constant. We thus reach the curious dilemma that neither of these two rival theories can dispense with the other. As if to enforce this point, the actual phase-curve of Venus follows a mixture of the two laws. It becomes exceedingly difficult to "mind your *p*'s and *q*'s," under these circumstances.

If we knew the mean temperature of a planet for all latitudes from the equator to the poles, we should no doubt find some relation between the thermal quantity corresponding to this temperature and the spherical albedo of the planet. In general, we may anticipate that the greater the spherical albedo is, the less will be the heat, but not necessarily in any exact proportionality, because the blanketing action of the planet's atmosphere is the principal factor in the retention of any heat which the surface may receive. Many geologists believe that a continually cloudy atmosphere, which would certainly reflect most of the sun's rays, but would retain terrestrial heat, was largely responsible for the growth of a tropically luxuriant vegetation within the Arctic circle in past ages. Morever, some highly important climatic properties, such as the melting of snow in high latitudes and the possibility or nonpossibility of a permanent ice-cap, depend on the maximum summer temperature, rather than on the mean temperature. The maximum temperature at the sub-solar point is somewhat intimately related to the reflection of *total* radiation spherically, and to some extent the latter follows the luminous albedo. The winter snows of Mars reach nearly as low a latitude as on earth, but no lower. The feebler sunshine of Mars is better conserved because it suffers a smaller reflective loss in passing through a clearer and rarer atmosphere. In fact, the albedo of Mars is so much smaller than the earth's that Mars would have the hotter climate of the two, in spite of greater distance from the sun, were it not that a rare atmosphere permits an easier escape of Martian surface radiation. On a planet with hardly any air, but having a long period of insolation and approximation to a steady state of thermal equilibrium, the sub-solar effect of the sun's rays must be nearly equal to the solar constant multiplied by one minus the spherical reflection of solar rays of every wavelength ($1 - A_1^{(t)}$). Thus, for the moon, the reflection of total radiation in connection with the temperature (both of which are measurable) has a bearing on the problem of the solar constant, although it may not be possible to utilize the information fully for lack of other data.

If a completely and uniformly diffusive reflecting sphere of indefinitely great radius be drawn about the sun as a center and including the earth, and if the central radius of the segment of this sphere in view from the night side of the earth be in the prolongation of the line from the sun to the earth, or what amounts to the same thing, if we imagine an ideal night sky to be "packed" with perfectly reflecting full moons, such a segment of the sphere (embraced in a hemisphere as viewed from the earth), or such a sky, should reproduce sunlight at the earth's distance. The moon sends us 1/98,317 part of the light reflected from such a hemisphere to a point,[1] and the full moon, which may be likened to a circular disk of 15′32.″7 radius cut out from this surface, should send us that fraction of sunlight, if it were not that it does not reflect in that way, but absorbs all but a small part of the combined luminous and nonluminous radiation received from the sun. Since, however, when equilibrium is attained, the combined emission and diffuse reflection of rays of every wave-length must equal the total of solar radiation received (unless there is some exceptional specular reflection in a particular direction, of which there is no evidence, and except for a slight retention of heat to be radiated away during the night) a measurement of the heat received from the total radiations of sun and moon, respectively, should approximate to this ratio. Such a measurement was made by Director Langley and myself at the Allegheny Observatory from which the fraction 1/96,509 was obtained, which seems to be in sufficiently close agreement with the theory.

Combining the above-named theoretical value with the observed ratio of $\frac{\text{sunlight}}{\text{moonlight}}$ at full moon, we have the following

Values of the geometrical lunar albedo:

By Zöllner's observation, $$A_2 = \frac{98,317}{618,000} = 0.159,$$

[1] Namely, light from surface of hemisphere : light from lunar disk $= \frac{2}{\sin^2 \sigma} \times \frac{\sin^2 S}{\sin^2 s} = \frac{2}{\sin^2 15' \, 32''.7} \times 1.00515 = 98,317 : 1.$

With Russell's adopted ratio, $A_2 = \frac{98,317}{465,000} = 0.211$,[1]

With Müller's adopted ratio, $A_2 = \frac{98,317}{569,500} = 0.173$.

Mean $A_2 = 0.181$.

Müller points out that numerical values will differ according to the definitions of albedo of which there are several. Of three theories given in his book with much detail, neither one is even remotely applicable to the moon, except in so far as the particular values by the Lommel-Seeliger and Euler theories *for full phase* do coincide with the geometrical albedo, on account of the aforesaid identity of the equations for this special case. The albedoes "by Seeliger's definition" which are set down by Müller are obtained on the limiting assumption that the coefficients of absorption of incoming and outgoing rays have the ratio $\lambda = 1$, which makes the coefficient in the Seeliger formula for $A_2 = 2$, or the same as in the formula for geometrical albedo. Otherwise, if λ differs from unity, we have

$\lambda = 1$, numerical coefficient = 2.0000
$\lambda = 2$, numerical coefficient = 1.8924
$\lambda = 3$, numerical coefficient = 1.8679
$\lambda = 4$, numerical coefficient = 1.8628
$\lambda = 5$, numerical coefficient = 1.8639
$\lambda = 6$, numerical coefficient = 1.8673
$\lambda = 10$, numerical coefficient = 1.8846

The albedoes "by Lambert's definition" are spherical albedoes derived from the geometrical albedoes by applying the factor $q = 0.75$, which is obtained by integration of the Lambert phase-curve. Müller leaves the reader to choose for himself between these values of the moon's albedo:

"$A_1 = 0.129$ (by Lambert's definition),
$A_2 = 0.172$ (by Seeliger's definition)."[2]

The use of the Lambert theory and of the constant factor 0.75 in passing from A_2 to A_1, prevents the values of A_1 in Müller's book from being regarded as spherical albedoes, except in those cases where Lambert's law may possibly be followed approximately.

[1] Russell himself, as already noted, divides this by 2 to get his "p," obtaining $p = 0.105$, and for Zollner's value, $p = 0.08$.

[2] *Photometrie der Gestirne*, p. 343.

With my value of the earth : moon ratio and Müller's geometrical albedo of the moon, the earth's geometrical albedo is

$$A_{e2} = 4.8 \times 0.172 = 0.826.$$

The ratio 4.8 : 1 applies only to the geometrical albedoes. The spherical albedoes adopted by Russell, namely,

$$\text{“}A\text{”} = A_{m1} = 0.073 \text{ for the moon,}$$

$$\text{“}A\text{”} = A_{e1} = 0.45 \text{ for the earth,}$$

have the larger ratio $A_{e1} : A_{m1} = 6.16 : 1$, and I shall show presently that this ratio ought to be still further increased. On the other hand Russell's values of p which are proportional to geometrical albedoes have a ratio smaller than mine, namely:

$$p^{(e)} : p^{(m)} = A_{e2} : A_{m2} = 3.86 : 1,$$

and one which does not agree with his adopted ratio of sunlight to moonlight. A revision on this account is certainly required.

Russell's lunar value, "$p = 0.105$," if it represents the "reflecting power" of the lunar surface,[1] would require that the moon should be composed of something almost as dark as dark grey slate, or nearly like trachyte lava, 0.10, according to the figures which he quotes from Wilsing and Scheiner. But excluding the very brightest and darkest spots which are of relatively small area, there are extensive dark regions on the moon whose average total-radiation reflection (bolometrically determined by measuring the transmission of lunar radiation through a glass plate which cuts off practically all of the emitted rays and distinguishes between these and the reflected ones) is from 10 to 12 per cent., while that of correspondingly situated bright regions (similarly determined) is from 20 to 25 per cent. Since the moon's surface is about equally divided between such "dark" and "bright" areas, a mean total-radiation reflection of 0.15 to 0.185 (average = 0.168) is indicated by my bolometric measures which form a useful check on my photometric results.[2]

[1] Russell says (*op. cit.*, p. 192): "Wilsing and Scheiner have determined the reflecting power of many ordinary rocks, using an approximately flat, rough, natural surface normal to the incident and reflected rays. Their formula of reduction gives exactly the quantity which has been designated by p." To the writer, it looks as if p, the planetary illuminating power, should be multiplied by 3/2 before making this comparison.

[2] Some samples of these are to be found in my "Photometry of a Lunar Eclipse," *Astrophysical Journal*, November, 1895, p. 299-300.

If the sum total of reflected rays of every wave-length agrees approximately with unaltered solar radiation, the preceding fraction must be increased a little to represent the result as it would be found outside the earth's atmosphere; because the solar reflected rays are of shorter wave-length than the rays emitted by the moon, and they are differently modified in passing through the atmosphere, which alters the relative values of the terms of the comparison. Except for certain bands of selective absorption, the longer waves are more readily transmitted by the air. It becomes increasingly evident that the solar constant is about 3.5 (C. G. Min.), but this is reduced to 1.5 at sea-level, so that the real transmission of solar rays by the atmosphere is 3/7. In a seasonally comparable observation of the moon, 48 per cent. of its emitted radiation entered through the air. Reducing to conditions outside the atmosphere by these values,

$$\frac{\text{radiation reflected by the moon}}{\text{radiation absorbed by the moon}} = \frac{16.8 \times (7/3)}{83.2 \times (10/4.8)} = \frac{1}{4.42},$$

and the true percentage of total solar radiation reflected from the moon is $100/5.42 = 18.5\%$, which differs little from a mean of the three results quoted for the reflected light of the moon, $A_2 = 18.1\%$. These, however, as I shall show, need to be diminished somewhat.

The question whether the invisible and longer solar waves of radiation are better or worse reflected by the moon than the visible ones has never been definitely settled, and indeed there is diversity of opinion as to the relative reflection by the moon of different colors in the visible spectrum. We need not consider the great bands of "metallic" reflection by quartz near 9μ and the large reflection by many common terrestrial substances between 8 and 10μ, for there is very little solar radiation of these wave-lengths to suffer reflection.[1] Metals have greater specular reflection for infra-red radiation just beyond the visible spectrum than for luminous rays; but metals are not in question here. The lunar reflection is almost entirely diffusive, and we wish to know how substances which reflect diffusely behave to infra-red rays between 0.7 and 3.0μ. Eighteen years ago, I published the value of 13.1

[1] See my paper on "The Temperature Assigned by Langley to the Moon," *Science*, N. S., Vol. XXXVII, No. 964, pp. 949-957, June 20, 1913.

per cent. for the lunar reflection of total solar radiation,[1] but I now think that this should be increased to the value given above, (·18.5%), because in my former work I under-rated the absorption of solar radiation by the air.

On the other hand, on the strength of Zöllner's oft quoted, but little studied value of what he calls the "true" lunar albedo (17.4%) I had formerly supposed that luminous rays are better reflected than the visible ones from 0.7 to 3.0μ; but it now appears probable from measures which are to follow, that this relation must be reversed, and that the larger luminous reflection which would result from the lunar-solar ratios adopted by Müller and Russell, can not be accepted. In fact, in place of Zöllner's hitherto accepted albedo must be substituted the smaller value $A_2 = 0.159$, which follows from his own observations (entirely apart from any considerations whatsoever as to the shape of the moon, or as to its surface quality, or the peculiarities of its phase law).

I will now give a series of ratios of sunlight to moonlight for homogeneous radiations in the visible spectrum derived from my spectro-photometric observations, published in *Astronomische Nachrichten*, Nr. 4820 (s. 385—386) which, as there given, are corrected for atmospheric absorption only. The original values are all that is necessary for a comparison of the relative reflection of different colors by the moon, but for our present purpose they

λ	$\frac{\text{Sunlight}}{\text{Moonlight}}$	λ	$\frac{\text{Sunlight}}{\text{Moonlight}}$ [2]
μ		μ	
0.40	531,000	0.56	682,000
0.42	559,000	0.58	666,000
0.44	587,000	0.60	646,000
0.46	607,000	0.62	619,000
0.48	640,000	0.64	600,000
0.50	669,000	0.66	513,000
0.52	681,000	0.68	456,000
0.54	685,000	Mean =	609,000

[1] *Astrophysical Journal,* Vol. VIII, p. 275, December, 1898.

[2] The reduction factor to moon's mean distance from the earth, and to full moon, rests on the following data:

	Series 1	Series 2	Series 3
Mean phase-angle from full,	$a = -30°$	$a = -18°$	$a = -6°$
Light (from lunar phase-curve),	0.53	0.69	0.90
Moon's parallax,	3374″	3415″	3455″
Reduction factors,	0.515	0.687	0.916

require further reduction for the distance of the moon and for the interval to exact full moon. The original figures have been multiplied by the factor 0.70 and are fully corrected. The spectro-photometric method yields results which have one special advantage. They are entirely free from the troublesome Purkinje effect which has vitiated much of the previous measurement.

The mean ratio of sunlight to moonlight for light of every color within the visible spectrum is

$$\text{sunlight} : \text{moonlight} = 609{,}000 : 1,$$

but considering that the central region in the green affects the eye most powerfully, a mean visual ratio of 681,000 : 1 is to be preferred.

It is evident from inspection of the numbers in this table that, while the reflection of blue and violet light by the moon is larger than that in the green and yellow, there is also a large reflection in the red which increases in the direction of the infra-red. Unfortunately, there is a dearth of observations of the reflective power of ordinary terrestrial materials in the region between 0.7 and 3.0μ, where there is a great block of solar radiant energy; but I think we may conclude that this region is probably more reflected by the moon than the visible part of the spectrum. In this case, Russell's ratio (1 : 465,000) may answer well enough for the reflection of solar infra-red radiation by the moon, but it is much too large for the reflection of visible rays.

The earth can not have the same ratio of reflection for visible and infra-red rays that the moon does, because the earth's reflection is mainly atmospheric, with visible rays somewhat better reflected than the infra-red. If the moon's geometrical albedo for visible rays is $A_{m2} = 0.15$, that of the earth is $A_{e2} = 4.8 \times 0.15 = 0.72$; but the reflection of total radiation by the earth, unlike that by the moon, is smaller than for visible rays (because the infra-red rays are but little reflected by the atmosphere). It can hardly exceed $A_{e2}{}^{(t)} = 0.70$, and may be as low as 0.50. For the present I shall adopt $A_{e2}{}^{(t)} = 0.60$. *Whatever values are finally adopted ought to be consistent among themselves and with the general principles now under discussion.*

In my visual photometric work on the earth-shine, the intrinsic brightness of the moon at quadrature is taken = 0.16 times the light at full moon, giving full-moon light: full-earth light = 1,600 : 0.16 = 10,000 : 1, since I found that full earth-shine at the time of new moon must be about 1/1,600 of the light of a corresponding sunlit area of the moon near quadrature.[1] It was not necessary for this purpose that the whole illuminated surface of the moon should be measured, nor is it possible to make such a measure of the earth-shine directly at new moon; but it is essential that the lunar surfaces to be compared shall be similar, and Professor Russell's arbitrary change of my mean ratio is not admissible, even after granting the large probable error of the result.[2]

I propose to give equal weights to the results of my own measures and to those of Zöllner as now correctly reduced. Comparing the lunar-solar ratio with the moonlight: earth-shine ratio given above, we have (with Very's value)

sunlight : full-earth light = 681,000 : 10,000 = 68.1 : 1,

(with Zöllner's value)

sunlight : full-earth light = 618,000 : 10,000 = 61.8 : 1.

Allowing for the greater area of the earth as seen from the moon, these ratios become:

68.1/13.4 = 5.08, and 61.8/13.4 = 4.61, mean = 4.8.

Moon's geometrical albedo (for the visual effect)

$$\left.\begin{array}{l}\text{According to Very, } A_{m_2} = 98{,}317/681{,}000 = 0.144 \\ \text{According to Zöllner, } A_{m_2} = 98{,}317/618{,}000 = 0.159\end{array}\right\}\text{mean} = \mathbf{0.15}$$

Earth's geometrical albedo = 4.8 × 0.15 = **0.72**

Geometrical reflection of total radiation:

Moon = **0.185**, Earth = **0.60** (?)

I take the reduction factor for spherical albedo, $q = 0.35$ for the moon from the integration of its phase-curve, and twice this, or $q = 0.70$ for the earth, which is a little less than the Lambert value.

Spherical albedo of moon, $A_{m_1} = 0.35 \times 0.15 =$ **0.053**,
Spherical albedo of earth, $A_{e_1} = 0.70 \times 0.72 =$ **0.504**.

[1] *Astronomische Nachrichten*, Nr. 4696, s. 286.

[2] With the increased assurance given by the good agreement of the photographic result, I do not believe that this error can amount to as much as 10%. Some weighting of the observations is perhaps desirable.

Moon's Stellar Magnitude:

Difference of magnitude from sun = log 681,000/0.4	= + 14.58
Stellar magnitude of sun (Russell)	= — 26.72
Stellar magnitude of moon (Very)	= — 12.14
Photographic magnitude of moon (King) (with Russell's phase-curve, etc.)	= — 11.37
Moon's Color-Index (King-Very)	= + 0.77,

or a little less than that of the sun,[1] which agrees with Abney's photographic observations, confirmed by my spectro-photometric comparison of sun and moon, in showing that the moonlight is bluer than sunlight. I have shown that the moonlight is redder than sunlight in the extreme red, but these rays do not count for much either photographically or visually, and therefore do not effect the color-index as usually defined.

Earth's Stellar Magnitude (Very):

As seen from moon,	— 12.14 — 4.58 = — 16.72
As seen from sun,	— 16.72 + 12.95 = — 3.77

I make no further claim for my previously published value of the earth's albedo[2] than that its ratio to the moon's albedo has been fairly well determined. The previous figures were based upon Zöllner's published lunar albedo and must be diminished a little according to what precedes; but this will not effect the argument for a high value of the solar constant, because this rests on wholly different grounds. If I could accept in principle Professor Russell's argument that my measurement of the earth-shine, "far from being inconsistent with Abbot's value of the solar constant (1.93 calories) is actually in agreement with it," [3] since it has now been shown that Russell's p must be doubled to give the geometrical albedo, I might claim that Abbot's constant should be doubled! But unhappily this simple method of disposing of the solar-constant problem will not work. A high value of terrestrial reflection of total solar radiation is indeed inconsistent with a low value of the solar-constant, but a low value of this reflection is not necessarily inconsistent with a high value of solar radiation, because the atmospheric depletion of the sun's rays is composed of several

[1] Color-index of sun = between + 0.8 and + 0.9, that of Capella being + 1.0.

[2] *Astronomische Nachrichten*, Nr. 4820, s. 400.

[3] *Astrophysical Journal*, April, 1916, p. 195.

parts. If reflection is found to be less potent than has been supposed, this simply puts a heavier burden on the other processes of depletion.

I have elsewhere concluded[1] that only about 18 per cent. of the sun's rays, received upon the entire sunward hemisphere of the earth, are effective at the earth's surface in production of temperature. Out of the 82 per cent. of solar radiation lost by the sunward hemisphere of the earth in one way or another in passing through the earth's atmosphere, the measurement of the earth's spherical albedo which has just been given indicates that approximately 50 are reflected back to space by air, or by clouds, including a reflection of a few per cent. by the solid or liquid surface of the earth. The rest of the depletion is divided among agencies which go under the general name of "absorption," but this also is really a complex of several processes.

As was pointed out in my "Note on Atmospheric Radiation,"[2] a portion of the incoming solar radiation of short wave-length is used up in the upper air in ionization of atmospheric ingredients, or in the production of ozone and other highly efficient absorbents; and since there is at present no way of finding out how potent this part of the atmospheric process may be (except possibly through an interpretation of certain little understood facts made known to us in the study of atmospheric thermodynamics) it is possible, as Dr. Louis Bell has suggested to me, that more solar energy than we imagine is lost in the ionization processes; and in this case quite a little of the remaining 32 per cent. of "absorption," so-called, may be ionization by solar radiation of very short wave-length at great altitudes in the atmosphere, these rays being wholly obliterated in the process.

I have alluded to the changes which Professor Russell has introduced into my earth-shine measures as founded on misapprehensions, and must now substantiate this claim. On page 185 of his second article we are told that "Table IVA contains data derived from Very's paper"; but the mode of derivation is not consistent. The first three numbers in the last column have been obtained by *multiplying* the ratio of exposure-durations for earth-shine and for sunlit moon by the ratio of photographic intensities;

[1] *Astrophysical Journal*, Vol. XXXIV, p. 382, Dec., 1911.

[2] *American Journal of Science*, Vol. XXXIV, p. 533, Dec., 1912.

but the last two numbers have been found by *dividing* the first quantities by the second. By this means, the two sets of numbers (for January and August respectively) which, if they had been correctly derived, would have been entirely different, because as yet uncorrected for photographic peculiarities, are brought into seeming approximate agreement, and the conclusion is reached that the photographic correction which I have derived for the relatively over-exposed lunar spectrograms was unnecessary, and that my corrected values are wrong! By this wholly erroneous argument, Russell supports his reduction of my ratio of the earth's albedo to the moon's from the spectrograms, to a quantity about half as great as mine. It is needless to say that the seeming agreement of the numbers in the last column of Table IVA is wholly accidental.

On page 184 (*op. cit.*) Professor Russell says that my "conclusions regarding the relative intensity of the light of these two sources [the earth-lit and the sun-lit portions of the moon] depend on assumptions regarding the photographic action of exposures to light of different brightness." On the contrary, my results do not rest on "assumptions," but on carefully executed quantitative measurements of the photographic effects in question throughout the entire visible spectrum. When properly reduced, there is no difference between the results of the photographic and of the visual observations. The statement (*op. cit.*, p. 186) that "the photographic observations therefore make the earth-shine only half as bright as do the visual observations," is consequently entirely wrong; and the conclusion that "this is just what might be expected if the plates had followed the ordinary law for faint illumination and long exposure, and been 'less sensitive than $i \times t$'," is equally erroneous. The error has come from the incorrect reduction of my observations by Russell in the aforesaid Table IVA.

The discrepancy which Professor Russell thinks he finds between my theory and my observations in connection with the phase-curve of the moon (*op. cit.*, p. 186 to 187) does not really exist. The observations had first to be reduced to a constant unit of comparison surface, and then to a selected lunar phase-angle. As it happened, the two corrections in a particular case were of equal numerical value, but opposite sign. Limited areas of the moon were necessarily observed, and the comparison in every case in the visual measures was between "twin circular apertures in black card, each subtending 7′ of arc on the celestial sphere, one above, and the other below the horizontal line of junction of the last pair of reflecting prisms." [1] One of these apertures was illuminated by the standard light, and the other by a sample of the lunar surface. In preparing the material of *A.N.* 4696 for the press, I have omitted a remark which is needed to complete the sense. After the first equation at the top of page 286 in that paper, there should be inserted these words: "The ratio of intrinsic brightness for the particular (limited) region of the moon under observation is the inverse of that just given." A quotation from my note book will clear up the matter fully: "At $\varphi = 44°$, M = 0.090. At $\varphi = 87°$, M = 0.058. Ratio = 1.55 : 1.00. For equal moonlight E/M (for $\varphi = 44°$) must be multiplied by 1.55." The phase-reduction required *division* by the same number. This peculiarity is due to an exceptionally large reflection from the lunar substance when the angle of incidence is large and nearly equal to the angle of reflection, as was the case for the particular lunar region observed with the moon's elongation, $\varphi=44°$; but the reflection diminished as the angle of incidence of the solar rays on this spot decreased. The Lommel-Seeliger law was formulated to deal with just such peculiarities. My published result:

$$\text{``}E_{44} : E_{87} = (0.0003477 \times 1.55) : 0.0002210$$
$$= 0.0005389 : 0.0002210$$
$$= 2.438 : 1,\text{''}$$

(*op. cit.*, p. 286) still stands and is fairly comparable with the ratio for Venus, computed by me from Müller's result, namely,

$$V_{44} : V_{87} = 2.100 : 1,$$

or as Russell gives it for elongations 30° and 90°,

$$V_{30} : V_{90} = 2.74 : 1.00,$$

where Lambert's law would give 2.94 : 1.00. The agreement is close enough to show that the phase-law for the earth resembles that for Venus, approaching, however, a little more nearly to the Lambert law, and is quite different from that for the moon.

On page 189, Professor Russell says: "Very's observations of the earth-shine indicate that the mean full earth, as seen from

[1] *Astronomische Nachrichten*, Nr. 4696, s. 269.

the moon is forty times brighter than the full moon as seen from the earth," where the forty should be *sixty-eight*.

Russell's adopted value of the moon's stellar magnitude (— 12.55) is 0.41 magnitude brighter than mine (— 12.14). Zöllner's result from a comparison with the sun gave the intermediate value,— 12.24, while that from his Capella comparison, —12.18, approaches still more nearly to mine.[1]

The last line of Russell's Table V. (*op. cit.*, p. 190) which purports to give "the earth (from Very's reductions of Slipher's spectrograms)" is misleading, since he has substituted his own reduction for mine.

A final word may be permitted on the vicissitudes of the solar-lunar light-ratio. Bouguer, who obtained a ratio of 300,000 : 1 for sunlight to moonlight (*Traité d'Optique*, p. 87, 1760) was careful to observe when the full moon was near its mean distance and when both bodies were at the same altitude; but unfortunately, he thought it necessary to use identical optical means in either case, and therefore his candle had to be at a distance of 50 feet for the moon and 1⅓ feet for the sun, so that the illuminations actually measured were in the ratio of 1407 : 1, both moonlight and sunlight having been much reduced. Under these circumstances, the bluer light of the heavenly bodies being compared with reddish candle light, the moonlight, on account of its greater faintness and of the relatively greater sensitiveness of rod-vision for faint blue light as the general illumination diminished, had an undue advantage, in the candle comparison, over sunlight, as will be evident from a short table in my paper on "The Earth's Albedo." [2] Bouguer's 300,000 must be at least doubled to correct for this error. The spectrophotometric method entirely removes this difficulty.

[1] From Zöllner (*op. cit.*, p. 125 and p. 105).

Log ratio Sun : Capella	=	10.7463
Log ratio Sun : Moon	=	5.7910
Log ratio Moon : Capella	=	4.9553
Log ratio divided by 0.4	=	— 12.39
Stellar magnitude of Capella	=	+ 0.21
Stellar magnitude of Moon	=	— 12.18

[2] *Astronomische Nachrichten*, Nr. 4696, s. 276.

Dr. W. H. Wollaston, who found[1] that the sun =

$$5{,}563 \times (12)^2 = 801{,}072 \text{ moons,}$$

used better observing conditions, but he made only two readings on the moon. In one observation at full, his candle was placed at 12 feet. In the other, made at a time when, if the atmosphere had been equally transparent the light should have been 0.84 of that at full moon, the same candle-reading "12 feet," was recorded. One can not help surmising that neither reading was better than a rough approximation.

Bond's solar-lunar ratio, 471,000 : 1, is an underestimate for the same reason that Bouguer's value is too small. Zöllner's criticism of Bond's fireworks as not accurate enough for standards is also fully justified.

Zöllner's own measurements of the ratio of sunlight to moonlight appear to have been made with great care; but in reducing them he becomes lost in the mazes of an unnecessarily complex argument. With the removal of this blemish, no fault can be found with the new value deduced from the original measures. The same can not be said of Zöllner's isolated measurements of the earth-shine[2] which require unknown corrections for skylight. The earth-shine observations of Arago and Laugier have been utilized by me in conjunction with my own with which they are in good agreement.

Various other more or less aberrant values of the moon's albedo usually err from inadequate correction for changes in atmospheric transparency. As an instance of a great name attached to an extraordinarily small value which is simply impossible, may be cited that of William Thomson (Lord Kelvin): "70,000 : 1" for the ratio of sunlight to full-moon light.[2]

Whatever faults may still remain in the values which are given here, they at least have this merit, that they are consistent among themselves, which is very far from being the case with the results which have been published hitherto.

WESTWOOD ASTROPHYSICAL OBSERVATORY,

August, 1916.

[1] *Philosophical Transactions* of the *Royal Society of London*, Vol. CXIX, p. 19-27, 1829.

[2] *Poggendorff's Jubelband*, p. 624, 1874.

[3] *Nature*, Vol. XXVII, p. 279, January 18, 1883.