MANUAL OF ADVANCED OPTICS

BY

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INTRODUCTORY NOTE

Anyone who has not used the methods of measurement which are based upon the interference of light waves will find it difficult to appreciate the high degree of accuracy which can thereby be attained. That these methods are not more commonly used seems to be due in large measure to the fact that they have hitherto not received adequate treatment in the texts which are in general use in physical laboratories. For this reason this "Manual of Advanced Optics," in which these methods are for the first time presented in text-book form to students of physics, is very timely, and should prove a valuable aid in making these very practical and useful optical methods familiar to all who may at any time find it necessary to make measurements of great precision. It is also hoped that the book will serve to promote interest in the general study of experimental optics. Those who desire to enter into optical investigation can not get a better foundation for future work than by studying the optical theories here presented, and performing the experiments described.

A. A. MICHELSON.

THE UNIVERSITY OF CHICAGO,
November 20, 1902.

PREFACE

That the practical study of optics has been somewhat crowded out of physical laboratories by the demands of electricity is attested by the fact that there have already been published many excellent manuals of the latter branch of science, whereas a practical treatise on optics has not yet appeared. To be sure, the theory of optics needs no better treatment than it has received at the hands of Mascart, Drude, Bassett, Preston, or of the authors of the Winkelmann's *Handbuch der Physik*. But from what book can the student find out how to determine, for example, that most important constant of the spectrometer, the resolving power? Or where can he learn to study practically the methods of using the phenomena of interference for exact measurements?

This small manual is an attempt to meet the needs of the more advanced students of optics. It has been written primarily for the use of the author's classes in the University of Chicago, and it covers the work done by them during three months of their senior year. It is hoped, however, that it will be found useful at other universities and will serve as a stimulus to a more extensive study of this most fascinating branch of science.

Every chapter begins with a brief discussion of the theory of the experiments which follow. In this discussion the attempt has been made to avoid as much as possible the use of mathematics—to present rather the physical ideas involved, and to use those ideas in building up a concrete conception of the phenomena with which we are dealing. This has often resulted in a lack of rigor of demonstration, e.g., in the case of the grating. In all such cases references have been added so that those who wish the complete and rigorous demonstration can satisfy themselves of the correctness of the conclusions. This course has been followed because the author believes that clear conceptions of fundamental ideas are absolutely indispensable to the physicist, and that the mathematical discussion, though often very elegant and convenient, adds nothing essential to these conceptions, but tends, rather, if used too freely, to cause one to forget the real essence of the subject.

It is hoped that the descriptions of the manipulations are not so detailed as to reduce the student to a mere mechanical copyist. He should not be allowed to forget that there may be other methods of adjustment and manipulation which may be better than those suggested. The numerical examples are all taken from the note-books of the students who have taken the course.

The apparatus needed for the experiments is not expensive. A spectrometer with such accessories as a grating, a pair of Fresnel mirrors and a bi-prism, a pair of Nicol prisms, and some doubly refracting crystals can be used for a large number of the exercises. The other necessary pieces are an interferometer and a polariscope or saccharimeter. Every well equipped laboratory should have all of these things, and so the course avoids the objection of requiring elaborate and expensive apparatus.

The author's own work is so arranged that a course of lectures on optical theory accompanies the laboratory work outlined in this manual. The two courses are independent, and of such a nature that either can be taken without the other. It is, however, very advantageous to take both. For the benefit of those who take the laboratory work only, the last two chapters on the development of optical theory and the trend of modern optics have been added. Though such chapters are not usual in a laboratory manual, it is believed that they will serve the purpose of adding unity to the course, by placing each experiment in its proper connection with the general body of the subject, so that the whole will appear in good perspective.

The author does not claim any great originality for his work. He has followed to a large extent the method of presentation developed by Mr. A. A. Michelson and used by him in his lectures. He is also greatly indebted to the papers of Lord Rayleigh, and to the treatises mentioned above, as well as to Kayser's monumental work, *Handbuch der Spectroscopie.* He is also under obligations to his colleagues at the University of Chicago. He received the course, somewhat in the shape in which he has now written it out, from his predecessor, Mr. S. W. Stratton, and has received during the five years in which it has been growing into its present form many suggestions from the other members of the Physics Department. He further wishes especially to thank Mr. F. B. Jewett for making the drawings for the illustrations.

Ryerson Physical Laboratory
The University of Chicago,
September 15, 1902.

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MANUAL OF ADVANCED OPTICS

I

LIMIT OF RESOLUTION

Theory

Since diffraction phenomena play a very important part in all optical instruments, it will be well to begin the study of practical optics with a discussion of some of the cases of diffraction which occur most frequently. The simplest method of observing these diffraction phenomena is that devised by Fraun-

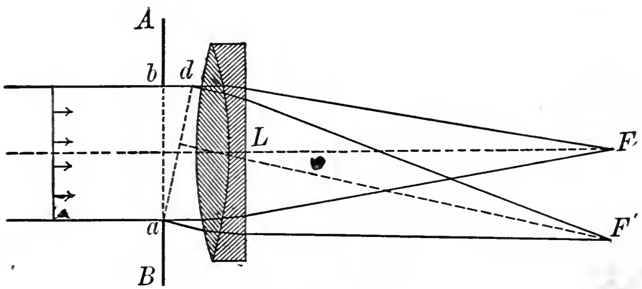

FIGURE 1

hofer.* The observation is most easily made by placing before a telescope, focused for parallel rays, a screen containing the opening whose diffraction pattern is to be studied, and directing the telescope at a distant point or line source. The advantage of

*Fraunhofer, *Neue Modification des Lichtes,* Denks. d. k. Akad. d. Wiss, zu München, Vol. VIII., p. 1, 1821. *Gesammelte Werke,* p. 53; also Harper's *Scientific Memoirs,* II, *Prismatic and Diffraction Spectra,* p. 13, 1898.

this procedure is apparent because only those rays which are parallel to each other in front of the screen come to a focus at a given point, and therefore the measurement of the diffraction pattern reduces to a measurement of angle only.

The simplest case is that of a rectangular opening.

Let AB (Fig. 1) represent a section of the screen by a plane perpendicular to two of the sides of the rectangular opening, and ab the section of the opening. Suppose that the source emits monochromatic light only, and is a bright line perpendicular to the plane of the drawing and so placed, with reference to the screen, that the rays when they reach the screen form a parallel beam which falls normally upon the opening. L represents the objective of the telescope, and FF' its principal focal plane.

The image of the line source formed in this focal plane FF' will consist of a central bright band bordered on either side by alternate dark and bright bands, these latter decreasing rapidly in intensity as they are more remote from the central bright band.

Let F represent the position of the center of the central bright band, and F' that of the center of the first dark band. We wish to find the value of the angle $FLF' = \theta$ subtended at the center of the lens L by F and F'. Through a draw ab, perpendicular to LF, and ad, perpendicular to LF'. Then ab represents the wave front of that plane wave which is brought to a focus at F. All points of this wave front are in the same phase, and hence a bright band is produced at F. Along ad, however, the various points will be in different phases, the variation depending on the angle which ad makes with ab. If now $bd = \lambda$ (λ = wave length), the phases of the successive points along ad will vary from λ to 0, that is, the phase at d will be a whole period ahead of that of the wave front ab, all parts of which are in the same phase, while at a the phases of the two coincide. Hence every point in one half of ad will have a corresponding point in the other half of ad whose phase differs from its own by $\frac{1}{2}\lambda$. Therefore, when the vibrations from all the points along *ad* are brought together at the focus F', they destroy each other in pairs, and darkness is the result.

Now the sine of the angle between the wave fronts *ab* and *ad* is

$$\sin bad = \frac{bd}{ab}.$$

But, for the first dark band, $bd = \lambda$, and hence, placing $ab = a$ and letting the angle $bad = FLF' = \theta$, we have

$$\sin \theta = \frac{\lambda}{a}. \qquad (1)$$

The illumination will again reach a minimum when $bd = 2\lambda$, $3\lambda, \ldots m\lambda$, so that the general condition for a dark band in the diffraction pattern formed in parallel light by a rectangular opening is

$$a \sin \theta = m\lambda, \qquad (2)$$

in which m stands for any whole number.

If, however, $bd = \frac{3}{2}\lambda$, the phases of the points along *ad* will vary from 0 to $\frac{3}{2}\lambda$, and we may conceive the beam of light to be divided into three sections of equal width by planes perpendicular to *ad*, each section containing points whose phases vary over half a period. The corresponding points of two of these sections will be in opposite phases, and will therefore be in condition to destroy each other, while the third section will transmit light to the focus of the lens and produce a maximum of illumination there.

Similarly, when *bd* is any odd multiple of half a wave, that is, $bd = \frac{2m+1}{2}\lambda$, the phases of the consecutive points along *ad* will vary from 0 to $\frac{2m+1}{2}\lambda$, and we may conceive *ad* divided into $2m + 1$ equal parts by planes perpendicular to *ad*. Then $2m$ of these parts are in condition to destroy each other in pairs, leaving one part to send light to the focus of the telescope. Hence the condition for a maximum of illumination is $bd = \frac{2m+1}{2}\lambda$, or,

$$a \sin \theta' = \frac{2m+1}{2}\lambda. \quad (3)$$

It is to be noted that, in determining the positions of the maxima and minima of illumination, it is necessary to consider only the width of the opening and the difference in phase of the two rays passing its edges.

If the source consists of a pair of bright lines parallel to each other, the diffraction pattern in the focal plane of the telescope will be one which results from the superposition of the two patterns which correspond to the two sources respectively. Let ϕ (Fig. 2) denote the angle subtended by the centers of the two

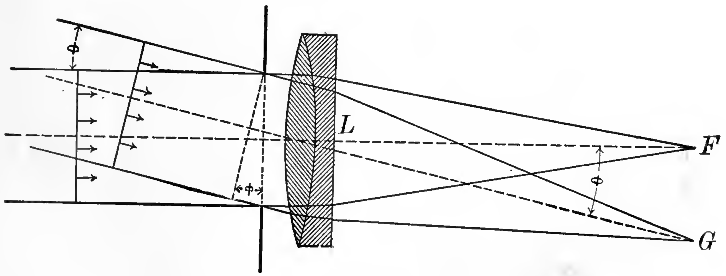

FIGURE 2

sources when viewed from the center of the lens L, and F the position of the center of the central bright fringe due to one source, and G that of the center of the central bright fringe due to the other source. Then $FLG = \phi$, and it is evident that if $\phi > \theta$ the central fringe at G, due to the second source, will be separated from that at F, due to the first source, by a dark line, and it will be possible to distinguish them from each other. If $\phi = \theta$, the two central bright fringes will just touch each other, and it may, or it may not, be possible to perceive that we are observing a double source. If $\phi < \theta$, the two central bright fringes will overlap, and it will be impossible to recognize that the source is double. Hence $\phi = \theta = \frac{\lambda}{a}$ is called the *limit of resolution.*

It is to be noted that this limit of resolution depends only on the width of the opening and the wave length of the light used. Also that, excepting the central band, the angular separation between a maximum and its adjacent minima is

$$\sin \theta' - \sin \theta = \frac{\lambda}{2a}.$$

Experiments

I. Prove the Equation $\sin \theta = \frac{\lambda}{a}$

Apparatus.—The experiment is most easily performed with a regular spectrometer.

Adjustments.—Telescope and collimator should be focused for parallel rays and their axes arranged in line so that the image of the slit is seen in the center of the field of view of the telescope. The slit should be as narrow as the intensity of the source will permit. The screen containing the rectangular opening to be studied should be placed in front of the objective of the telescope. The diffraction pattern will then appear in the field of view.

It is, of course, not necessary to use a spectrometer. A telescope and a sufficiently distant slit source will give satisfactory results.

Measurements.—The direct measurement of λ may be avoided by using a source of monochromatic light (sodium burner, mercury or cadmium vacuum tube). However, inasmuch as all sources of monochromatic light are comparatively faint, much better results will be obtained by using sunlight or the electric arc. A light which is sufficiently monochromatic for this purpose may be obtained from sunlight by allowing the solar spectrum to fall across the slit end of the collimator, so that the Fraunhofer lines are parallel to the slit. The light which passes through the slit will be sufficiently monochromatic and at the same time of sufficient intensity to form very clear diffraction patterns.

If the solar spectrum has been so placed that one of the Fraunhofer lines nearly falls upon the slit, we may assume with sufficient accuracy for this purpose that the wave-length which forms the pattern is the same as that of the Fraunhofer line which has been placed near the slit. If the slit has been placed at random in the spectrum of the sun or the electric arc, the wave length may be measured by introducing between the collimator and the telescope a small diffraction grating having a known number of lines to the millimeter.

The width of the opening may be measured on the dividing engine or with a micrometer microscope in the usual way. In order that θ may be large enough to be measured with accuracy, a should be small, say 0.2 to 0.5 mm.

If a spectrometer has been used to perform the experiment, the angle θ may be read directly from the graduated circle of the instrument. If only a telescope has been used, the angle may be measured by fastening a small mirror to the telescope and measuring with a telescope and scale the angle through which this mirror is turned. In case the vernier on the graduated circle of the spectrometer does not read to at least 5″ it is better to measure θ in any case with a telescope and scale. Since the angle between two successive dark bands or between two successive bright bands is also θ [cf. equation (2) and (3)], it is well to measure the angle which corresponds to ten or more bands rather than that which corresponds to a single one.

EXAMPLE

Angular width of 7 bands..............	57′ 29.7″
Angular width of 1 band (θ)...........	8′ 12.8″
Wave length (λ) (measured by grating)..	.0005866 mm.
Width of the opening (a)..............	0.245 mm.

$$\frac{\lambda}{a} = .00239$$

$$\sin \theta = .00239$$

II. Determination of the Limit of Resolution of an Opening of Width a.

Apparatus.—An ordinary telescope and a rather coarse grating are all that is needed.

Adjustments.—Set up before a source of monochromatic light the grating consisting of several slits about 1 mm. wide and 1 mm. apart. Such a grating is most easily obtained by the method devised by Fraunhofer. Two machine screws are set in a plate of metal so as to be parallel to each other. A wire whose diameter is about half the distance between the threads of the screw is wound tightly about the two screws and soldered to them. When the wires are cut from one side of the pair of screws, the remaining wires form a very good grating. A telescope, before the objective of which a screen containing an opening of width a has been placed, should be focused upon the grating.

Measurements.—Place the telescope with the screen before its objective rather near the grating and draw it gradually away, being careful to keep the edges of the slits of the grating in focus. Find the point at which the observer first ceases to be able to distinguish that the source at which he is looking is made up of lines, i.e., the point at which the source appears to be uniformly illuminated. At this point the limit of resolution has been reached. If now d is the distance between the centers of the line sources, and D the distance between the grating and the screen before the objective of the telescope, then, according to the theory given above,

$$\frac{d}{D} = \frac{\lambda}{a}.$$

The distance D may be measured with a tape, and the other quantities as in the experiment above.

This experiment is largely qualitative. Different observers will differ by 5% or more in determining the exact point at which the grating ceases to be resolved by the telescope.

EXAMPLE

Distance between openings (d)	0.138 cm.
Distance from source to screen (D)	358 cm.
Width of opening in screen (a)	0.157 cm.
Wave length (sodium) (λ)	.0000589 cm.

$$\frac{d}{D} = .000385$$

$$\frac{\lambda}{a} = .000375 \text{ *}$$

* In connection with the above the student will do well to read Schwerd, *Die Beugungserscheinungen aus den Fundamentalgesetzen der Undulationstheorie analytisch entwickelt,* Mannheim, 1835. Verdet, *Leçons d'optique physique,* Vol. 1, p. 309 seq. Rayleigh, *Collected Works,* Vol. 1, p. 488. Phil. Mag. (5) Vol. 10, p. 116, 1880.

II

THE DOUBLE SLIT

Theory

In the preceding chapter the diffraction pattern produced by a single rectangular opening was discussed. Let us now pass to the case of two such openings or slits of equal width and parallel to each other. It is, in the first place, quite evident that each one of these slits will produce maxima and minima of illumination in accordance with the above conditions, namely, equations (2) and (3). But in addition to these, other maxima and minima

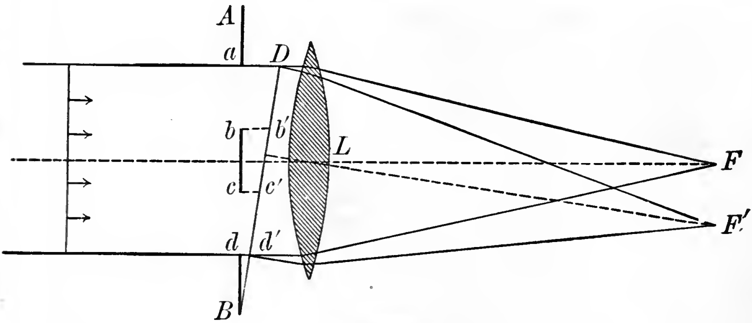

FIGURE 3

will be formed because of the action of the rays from the two slits upon each other. The conditions which determine the formation of these latter fringes may be obtained by the same process employed above.

The other conditions remaining the same as in Fig. 1, let ab and cd (Fig. 3) represent the two slits, each of width a. Call the distance bc between the neighboring edges of the two slits b, and let $a + b = d$.

Let F be the point at which the wave front ad is brought to a focus, and F' the corresponding point for some other wave front, as BD. Draw BD from B perpendicular to LF', the point B being determined by the condition $aB = 2d$. If now $aD = \lambda, 3\lambda, \ldots (2m+1)\lambda$, it will readily be seen, since $cc' = \frac{1}{2} aD$, that the points along Db' will be in opposite phase to the corresponding points along $c'd'$. The conditions by which the minima due to the interaction of the two slits are determined will, therefore, be

$$aD = \lambda,\ 3\lambda,\ \ldots\ (2m+1)\lambda.$$

If we call the angle between the wave fronts θ, this is equivalent to

$$2d \sin \theta = (2m+1)\lambda. \qquad (4)$$

Similarly, the conditions by which the maxima are determined are

$$aD = 2m\lambda,$$

or

$$2d \sin \theta' = 2m\lambda. \qquad (5)$$

It will be noted that the angle between a maximum and its adjacent minima is $\sin \theta' - \sin \theta = \frac{\lambda}{2d}$. These maxima and minima which are determined by equations (4) and (5) are called those of the second order.

It has thus far been assumed that the source was a bright line, that is, that it was infinitely narrow. As all physically attainable sources have an appreciable width, they must be looked on as a series of line sources tangent to one another along their length. Each of these line sources will produce its own set of bands in the focal plane of the telescope, so that the diffraction pattern actually observed is really made up by the superposition of a large number of such diffraction bands. If the angular width of the source, when viewed from the center of the lens L (Fig. 3), is equal to the angle between the central bright fringe and its first adjacent dark fringe as determined by equation (4), the central bright fringe formed by the light which comes from one edge of the source will fall on the first dark band formed by light from the other edge of the source, and all trace of the fringes will disappear. Hence it is seen that these interference bands produced by two slits,—for that is what these maxima and minima of the second order really are,—may be used to measure small angular magnitudes. That they enable us to measure wave length also is evident from equations (4) and (5).

As a clear understanding of these phenomena of interference is desirable for what is to follow, it will be necessary to introduce here a more detailed discussion of them.

In order that two transverse vibrations, such as the light waves are, may produce interference, it is essential that they have the same period. If they so interfere as to produce darkness, they must in addition have the same plane of vibration and the same amplitude. Since only those rays which proceed from the same vibrating particle can fulfil these requirements, it is the fundamental principle of all interference phenomena that only those rays which proceed from the same point of the source can interfere with each other. In order, therefore, to produce interference, it is necessary to divide each ray into two, to lead the two over paths of different lengths and to reunite them. Two rays which result from this operation, and which are, therefore, capable of producing interference, are called congruent rays.

Let us consider two congruent rays which are traveling in the same direction along the same straight line. If we let the line be the axis of x, and consider that the vibrations take place in the xy plane, the displacement of a particle at the point x due to one of these rays will be given by

$$y_1 = A_1 \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right),$$

and the displacement of the same particle due to the other ray by

$$y_2 = A_2 \cos 2\pi \left(\frac{t}{T} - \frac{x+\delta}{\lambda}\right),$$

in which the symbols have the usual well-known significance.

The displacement of the particle due to both rays will, therefore, be

$$y = y_1 + y_2 = A_1 \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + A_2 \cos 2\pi \left(\frac{t}{T} - \frac{x+\delta}{\lambda}\right).$$

This becomes by expanding the last term

$$y = \left(A_1 + A_2 \cos 2\pi \frac{\delta}{\lambda}\right) \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) + A_2 \sin 2\pi \frac{\delta}{\lambda} \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right).$$

If in this equation we now introduce two new constants, A and D, determined by the conditions

$$A_1 + A_2 \cos 2\pi \frac{\delta}{\lambda} = A \cos 2\pi \frac{D}{\lambda}$$

$$A_2 \sin 2\pi \frac{\delta}{\lambda} = A \sin 2\pi \frac{D}{\lambda}$$

it reduces to

$$y = A \cos 2\pi \left(\frac{t}{T} - \frac{x+D}{\lambda}\right).$$

This equation tells us that the superposition of two simple harmonic vibrations which are traveling in the same direction, results in a simple harmonic vibration which has the same period as the two, but a different amplitude and phase.

If we solve the equations which define A and D for A, we obtain

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos 2\pi \frac{\delta}{\lambda}.$$

In the special case in which A_1 and A_2 are equal, this becomes

$$A^2 = 4A_1^2 \cos^2 \pi \frac{\delta}{\lambda}. \qquad (6)$$

Hence we see that if the difference of path δ is a whole number of wave lengths, i.e., if $\frac{\delta}{\lambda} = 1, 2, 3$, etc., A^2 will have its maximum value. But if δ is an odd number of half wave lengths, i.e., if $\frac{\delta}{\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc., A will be zero.

Let Ob (Fig. 4) represent the optical axis of a telescope, and S_1 and S_2 two slits which are perpendicular to the plane of the drawing. O represents the center of a slit source parallel to the two slits S_1 and S_2, and separated from their plane by a distance $bO = d$. Consider O as the center of a coördinate system whose x

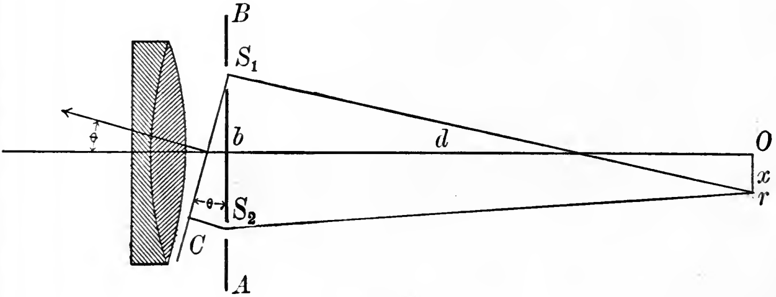

FIGURE 4

axis lies in the plane of the drawing and is perpendicular to bO, and whose y axis is parallel to the slits, i.e., perpendicular to the page at O. Let r be any point of the source on the axis of x, θ the angle CS_1S_2, and b the distance between the centers of the slits. The difference of path D of the two rays S_1r and CS_2r, which come to a focus in a direction θ, will be determined by

$$D = Cr - S_1r = CS_2 + S_2r - S_1r.$$

But, since $Or = x$,

$$\overline{S_1r}^2 = d^2 + (\tfrac{1}{2}b + x)^2, \qquad \overline{S_2r}^2 = d^2 + (\tfrac{1}{2}b - x)^2.$$

Therefore

$$\overline{S_2r}^2 - \overline{S_1r}^2 = (S_2r + S_1r)\,(S_2r - S_1r) = -2bx.$$

But $S_2r + S_1r = 2d$, nearly, and $CS_2 = b\theta$, therefore

$$D = b\left(\theta + \frac{x}{d}\right).$$

But $\frac{x}{d}$ is the angle subtended by Or at the center of the lens. If we call this angle ξ, we shall have, as the difference of phase between the two rays,

$$\delta = 2\pi \frac{b(\theta + \xi)}{\lambda} = 2p\pi(\theta + \xi),$$

in which p represents the number of waves in the distance b between the slits.

The intensity in the field of view in the direction θ may be obtained from equation (6). If we denote by $\phi(\xi)$ the intensity of the light which comes from an elementary band of the source at r, of angular width $d\xi$, we have, as the intensity due to this band alone,

$$I_r = 2\phi(\xi)\cos^2 p\pi(\theta + \xi)\, d\xi = \phi(\xi) + \phi(\xi)\cos 2p\pi(\theta + \xi)\, d\xi.$$

The total intensity in the direction θ is the integral of this expression, that is, if we let

$$P = \int \phi(\xi)\, d\xi$$

$$C = \int \phi(\xi) \cos 2p\pi\xi d\xi$$

$$S = \int \phi(\xi) \sin 2p\pi\xi d\xi,$$

our equation representing the intensity becomes

$$I = P + C\cos 2p\pi\theta - S\sin 2p\pi\theta.$$

The first term of the right-hand side of this equation, when taken between the proper limits, represents the total amount of light emitted from the source. The values of θ which correspond to the maximum and minimum values of I are determined by the condition $\frac{dI}{d\theta} = 0$, or,

$$C\sin 2p\pi\theta + S\cos 2p\pi\theta = 0.$$

From this we easily deduce as the equation for the intensity at the maxima and minima,

$$I = P \pm \sqrt{C^2 + S^2}.$$

It has been shown above that as the width of the source changes, the fringes in the field of the telescope pass through various stages of visibility, being sometimes very distinct, sometimes totally lost. We may define the visibility of the fringes as the difference in the intensities of a maximum and a minimum divided by the sum of the same intensities. Thus if I_1 represent the intensity of a maximum, and I_2 that of a minimum, and V the visibility, we may write

$$V = \frac{I_1 - I_2}{I_1 + I_2}, \tag{7}$$

or, from the above,

$$V^2 = \frac{C^2 + S^2}{P^2}. \tag{8}$$

If the source is symmetrical with respect to the axis of y, $S = O$, and we have

$$V = \frac{C}{P}. \tag{9}$$

If the source is, as we have supposed it above, a uniformly illuminated rectangular opening or slit, $\phi(\xi)$ = constant, and hence, if we denote by α the angular width of the source when viewed from the center of the lens,

$$V = \frac{\left[C\right]_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}}}{\left[P\right]_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}}} = \frac{1}{\alpha}\int_{-\frac{\alpha}{2}}^{+\frac{\alpha}{2}} \cos 2p\pi\xi d\xi = \frac{\sin p\pi\alpha}{p\pi\alpha}.$$

If the linear width of the source is a, $\alpha = \frac{a}{d}$, and we have

$$V = \frac{\sin \pi \frac{ba}{\lambda d}}{\pi \frac{ba}{\lambda d}}. \tag{10}$$

The fringes will disappear whenever $\frac{ba}{\lambda d}$ is a whole number m, that is, when

$$\frac{a}{d} = \frac{m\lambda}{b}. \tag{11}$$

Since $\frac{a}{d}$ is the angular width of the source, and $\frac{\lambda}{b}$ the limit of resolution of an opening of width b, this condition is the same as the one previously mentioned (p. 20).

If we have as a source a pair of uniformly illuminated slits placed symmetrically with respect to O (Fig. 4), each of angular width α and separated from each other by an angular distance γ, the expression above must be integrated twice, once between the limits $-\frac{\gamma}{2}-\frac{\alpha}{2}, -\frac{\gamma}{2}+\frac{\alpha}{2}$, and again between $+\frac{\gamma}{2}-\frac{\alpha}{2}, +\frac{\gamma}{2}+\frac{\alpha}{2}$. The result is

$$V=\frac{\sin p\pi\alpha}{p\pi\alpha}\cos p'\pi\gamma, \qquad (12)$$

in which p' is written to show that it differs from p.

Thus, if the linear distance between the centers of the two slits be denoted by c, $\gamma=\frac{c}{d}$, and therefore the fringes will disappear not only in accordance with equation (11) but also when $p'\gamma=\frac{1}{2}, \frac{3}{2} \cdot \cdot \cdot \cdot \frac{2m-1}{2}$, that is, when

$$\frac{c}{d}=\frac{2m-1}{2}\frac{\lambda}{b'}, \qquad (13)$$

where $b'(=p'\lambda)$ is written to distinguish it from the b in equation (11) and m is the order of the disappearance.

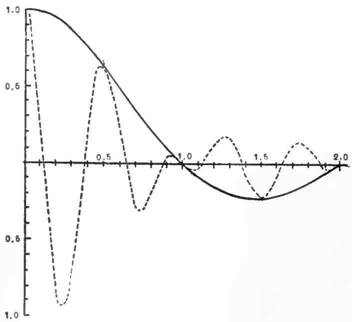

FIGURE 5

Fig. 5 represents the two curves expressed by equations (10) and (12). The dotted curve corresponds to (12), and we see that from it we infer the separation of the two sources. The full curve corresponds to (10), and from it the width of each individual source may be obtained. It is also to be noted that the full curve is an envelope of the maxima of the dotted curve.*

Experiments

Measure the Width of a Narrow Slit

Apparatus.—To get the best results in making measurements with the double slit it is advisable to use a large lens of rather long focus. A four or five inch telescope objective with a focal length of from one to two meters answers very well. The lens should be mounted fifteen or twenty meters from the source whose nature is to be investigated. The double slit should be mounted directly in front of the lens. These slits must either be movable so that their distance apart can be varied, or they may have a fixed distance apart and the nature of the source may be changed. In the former case the slits must be mounted on a plate and fitted with a right and left screw or with some other mechanism which allows them to be moved symmetrically with respect to the center of the lens, while in the latter case they may be cut from a card. Slits about 1 mm. wide and 4 or 5 cm. apart are very satisfactory.

Adjustments.—Sunlight or some other bright light is allowed to pass through the openings whose dimensions are to be determined and to fall upon the lens before which the slits have been placed. The image of the source when observed with the eyepiece is found to consist of a series of fringes. Care must be taken to find the interference fringes, which are narrower and clearer than the diffraction fringes and which lie in the center of the entire pattern.

Measurements.—The observation may be made either by separating the slits or by varying the nature of the source until the fringes disappear. In both cases it is necessary to measure

*For similar solutions for sources of other shapes see Michelson, Phil. Mag. (5) Vol. 30, p. 1, 1890. Mascart, *Traite d'Optique*, Vol. 3, p. 567 seq., Paris, 1893.

the distance b between the slits and the width a of the source. If the source is double, the distance between the two apertures which compose it must also be measured. The distance d from the double slit to the source must also be known. Since the wave length λ of the light used appears in the equation, it is well to filter the light used so that the rays which reach the lens have approximately one wave length. If white light is used the disappearance observed will be that of the fringes of greatest brightness which correspond to the wave length .00055 mm.

EXAMPLES

1. A single slit was set up at a distance of 2080 cm. from the double slit. Sunlight was used. The width of the slit was set at random and the two slits moved until the fringes disappeared. The following two sets of observations were made. Only the first disappearance of the fringes was taken.

Wave length λ	.000055 cm.
Width of slit a as measured directly	0.049 cm.
Distance between slits b	2.335 cm.
Distance d to source	2080 cm.

$$\frac{\lambda}{b} = .0000236$$

$$\frac{a}{d} = .0000236.$$

2. The width of the slit was then changed so as to be .074 cm. and the distance between the double slits was found to be 1.545 cm. Then, as above

$$\frac{\lambda}{b} = .0000356,$$

$$\frac{a}{d} = .0000356.$$

3. Using sunlight as before, the two slits were set at a fixed distance of 4.075 cm. apart. The width of the source was then

varied and measured when the fringes disappeared. In this way five disappearances were observed as follows, m denoting the order of the disappearance:

$m = 1,$	$a =$ .277 mm.	$\frac{a}{m} =$ .277 mm.
$m = 2,$	" .567	" .283
$m = 3,$	" .858	" .286
$m = 4,$	" 1.150	" .287
$m = 5,$	" 1.412	" .282
		mean .283 mm.

The distance d was in this case as above. Therefore

$$\frac{\lambda}{b} = .0000135,$$

$$\frac{a}{d} = .0000136.$$

4. A double source was used whose two components were 0.053 cm. apart, i.e., $c = 0.053$. The distance d remained 2080 cm. Disappearance of the fringes was observed for the following values of b':

$m = 1,$	$b' =$ 1.085 cm.	$\frac{2}{2m-1}b' =$ 2.170 cm.
$m = 2,$	" 3.324	" 2.216
$m = 3,$	" 5.580	" 2.232
$n = 4,$	" 8.015	" 2.290
		mean 2.227 cm.

$$\frac{c}{d} = 0.0000255, \qquad \frac{2m-1}{2}\frac{\lambda}{b'} = 0.0000247.$$

III

THE FRESNEL MIRRORS

Theory

In the preceding chapters the nature of the measurements which can be made by observing the interference phenomena which arise when the light from a source is viewed with a telescope, before the objective of which one or two slits have been placed, has been discussed. This method of producing interference is encumbered with two serious objections. First, the rays which interfere cross at a rather large angle, which results in making the fringes very narrow; and, second, the amount of light received by the observer is small because the source must be very narrow in order to see fringes at all. These two drawbacks may be obviated by suitable changes in the form of the apparatus.

It is evident that, in the formation of these bands, that portion of the objective which is covered plays no part. Hence, it is

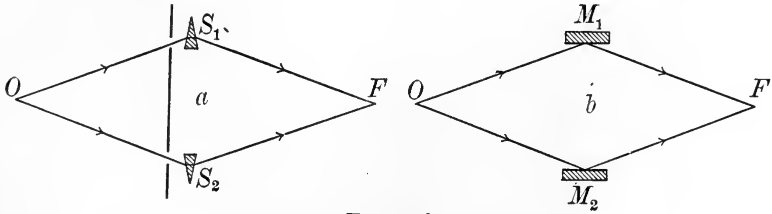

FIGURE 6

possible to dispense with the entire objective excepting those portions which are immediately behind the slits. These portions need not, for this purpose, be curved, but may be replaced either by prisms as in Fig. 6*a* or by mirrors as in Fig. 6*b*.

Both these forms enable us to increase the width of the fringes by changing the inclination of the rays which meet at F. In broadening the fringes by these methods it is necessary either to

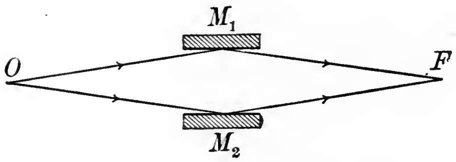

FIGURE 7

make the distance M_1F or S_1F large, which is objectionable on practical grounds, or to bring the mirrors or prisms closer together. This latter procedure changes the telescope with which we began,

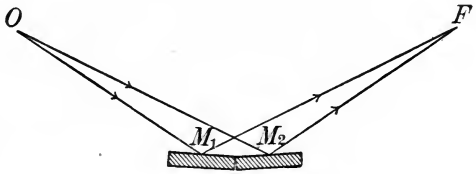

FIGURE 8

into one of the two well-known forms of interference apparatus, the Fresnel mirrors or the bi-prism. The mirrors are shown diagrammatically in Figs. 7 and 8, the bi-prism in Fig. 9. It is evident

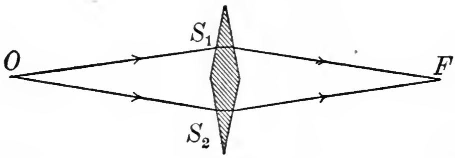

FIGURE 9

that these two forms of interference apparatus are limited in their possible applications because the various parts can not be moved sufficiently with reference to each other.

The angle between the rays which interfere may be made smaller and the fringes thereby enlarged by inserting at F a plane parallel plate of glass, as in Fig. 10. The ray M_1F is transmitted

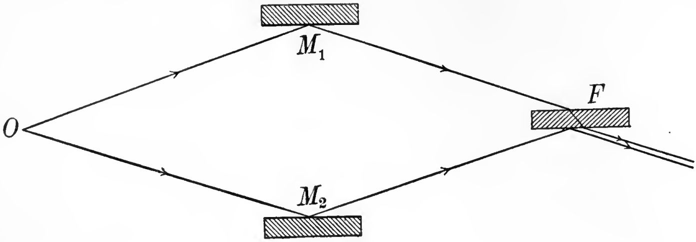

FIGURE 10

and the ray M_2F reflected by the plate F, so that by a suitable inclination of this plate the angle between the rays may be made as small as desired, and hence the fringes may be indefinitely enlarged.

A further great improvement can be made by replacing the slit by another piece of the plane parallel plate inserted at O, as in Fig. 11. With this arrangement the light is not limited to a

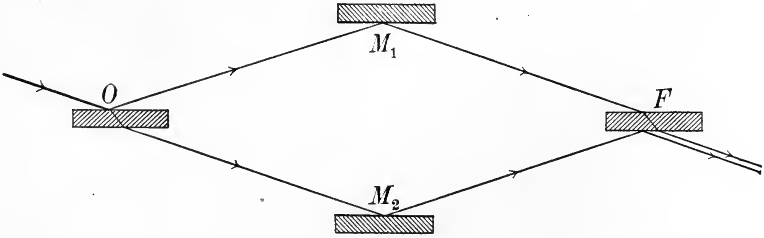

FIGURE 11

beam that has passed through a slit, but may come from any broad source. The gain in illumination obtained by this simple device is enormous. The instrument sketched in Fig. 11 is an *interferometer*. It is capable of many variations of form, and can be used for a large variety of delicate measurements. By thus converting the telescope into an interferometer all definition and resolution are lost, but a great gain in accuracy is the result.*

We will begin the discussion of the various forms of interferometer with the simpler cases of the Fresnel mirrors and bi-prism.† Let M_1M_2, Fig. 12, represent the projection of

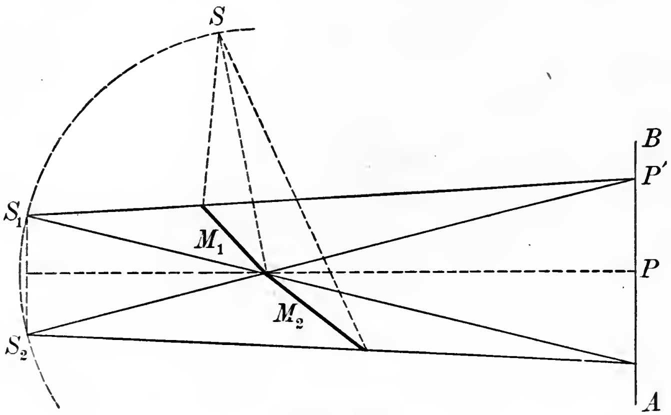

FIGURE 12

the two mirrors on a plane perpendicular to their line of intersection, and S the projection of a line source of monochromatic light, that source being parallel to the intersection of the mirrors. Then S_1 and S_2 will represent the two virtual images of S formed by M_1 and M_2 respectively. Let AB represent a screen parallel to the line joining the virtual sources S_1 and S_2.

The illumination upon the screen AB, due to light reflected by the mirrors, will be identical with that due to two separate sources, S_1 and S_2, each of the same intensity as S. Since P is on the perpendicular erected at the middle of the line S_1S_2, it is equidistant from S_1 and S_2, and therefore each pair of congruent rays from the source will arrive at P in the same phase and

* Michelson, Am. Jour. Sci. (3) 39, p. 115, 1890.

† Fresnel: *Mèmoire sur la Diffraction de la Lumière*, §§63-64. *Oeuvres*, Vol. I, p. 329 seq. *Mèm. de l'Acad.*, V., p. 414, 1826.

produce a maximum of illumination there. It is necessary to find what will be the illumination at any other point P' on the screen. This will clearly depend upon the difference in length of the paths of the rays which arrive at P' from S_1 and S_2 respectively. This difference of path is expressed in the notation of page 23 by

$$S_2P' - S_1P' = \frac{bx}{d};$$

but the illumination at P' is a maximum when this difference of path is a whole number of wave lengths, $m\lambda$. Hence the condition for a maximum is

$$m\lambda = \frac{bx}{d},$$

that is, we have a bright fringe at distances x from the center P equal to $\frac{d}{b}\lambda$, $\frac{2d}{b}\lambda$, $\frac{3d}{b}\lambda$, etc. The distance between the successive maxima is seen to be constant. If we call it e we have

$$e = \frac{d}{b}\lambda. \qquad (14)$$

We can use the mirrors then to determine wave lengths, for d and e are easily measured directly, and if the distance from the source to the intersection of the mirrors be called f, and the angle between the mirrors α, it is readily seen that

$$b = 2f \sin \alpha;$$

hence

$$\lambda = \frac{f}{d} 2e \sin \alpha. \qquad (14')$$

The measurements can be further simplified by placing the source at an infinite distance by means of a lens. Then f and d become infinite together and their ratio is unity. Under these conditions we have

$$\lambda = 2e \sin \alpha. \qquad (15)$$

In order to obtain good results with the mirrors the conditions assumed in the above discussion must be accurately fulfilled. The two mirrors must touch each other along one edge, and that edge must be made to coincide with the intersection of their reflecting surfaces.* We have already shown that when the angular width of the source, viewed from the intersection of the mirrors, is equal to the angular width of a fringe, viewed from the same point, the fringes disappear. Therefore, a narrow source is necessary for the production of distinct measurable fringes. The effects of diffraction on the phenomena have been discussed by Weber† and by Struve.‡

Let us now consider the effect of introducing a transparent plate in the path of one of the rays. Let M_1 and M_2 (Fig. 13)

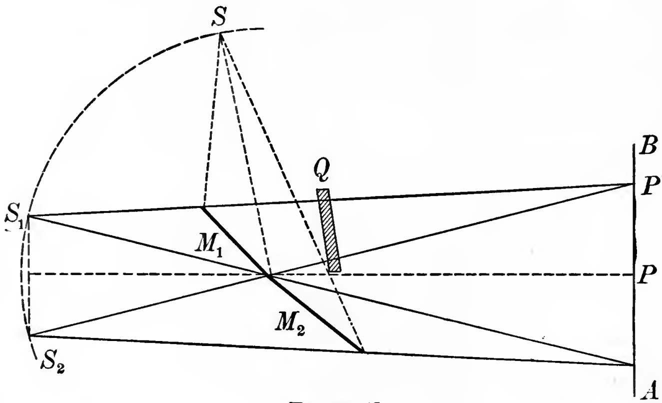

FIGURE 13

represent the mirrors, and S_1 and S_2 the virtual sources as in Fig. 12. Suppose we introduce into the path of one of the rays S_1P a plane parallel plate Q of some transparent substance, whose index of refraction for the monochromatic light used is μ,

* Feussner, Winkelmann, *Handbuch der Physik*, Vol. II, Pt. 1, p. 528.
† Weber, *Wied. Ann.*, Vol. VIII, p. 407, 1879.
‡ Struve, *Wied. Ann.*, Vol. XV, p. 49, 1882.

and whose thickness is t. The change D in the optical length of the path S_1P produced by the plate Q will be

$$D = (\mu - 1)t.$$

The difference between the optical paths of the two rays at any point of the screen P' will then no longer be expressed by $S_2P' - S_1P' = \frac{bx}{d}$, but by

$$S_2P' - S_1P' = \frac{bx}{d} - (\mu - 1)t.$$

This difference of optical path is zero when

$$x = (\mu - 1)\frac{td}{b},$$

by which the distance through which the central fringe of the system is shifted from its former position P by the introduction of the plate Q is determined. But at this distance x we had, before the insertion of the plate, a fringe determined by the equation

$$p\lambda = \frac{bx}{d},$$

in which p stands for any number, whole or fractional. If in this equation the value of x found above be substituted for x, we see that the central fringe of the shifted system takes the place of that fringe of the original system whose order is

$$p = \frac{(\mu - 1)t}{\lambda} = \frac{D}{\lambda}. \qquad (16)$$

It has thus far been assumed that the source emitted monochromatic light only. If this is not the case, but waves of various lengths are sent out, each set of waves will form its own set of fringes in accordance with equation (14). Since the distance between the fringes which correspond to each wave length is proportional to that wave length, the resultant figure on the screen will consist of the superposition of a number of systems of fringes of unequal breadth. If the mirrors are so arranged, as in Fig. 12, that the rays from the two sources travel entirely in air before reaching the screen, the position of the central fringe, which is determined by the condition, $S_2P' - S_1P' = 0$, will, since this condition is independent of the wave length, be the same for all colors. Hence all systems of fringes will agree in having a bright fringe at that point P which is determined by this equation. If white light be emitted from the source, the central fringe will be white, free from all trace of color. Hence it is called the achromatic fringe. Since the distance from P of any other bright fringe is proportional to the corresponding wave length, the various systems of fringes will correspond with each other at no other point. Hence the adjacent fringes will be colored, their color depending on how the various systems happen to overlap at the point considered.

If the symmetry of the optical paths be disturbed by the introduction of a transparent plate into the path of one of the rays, as in Fig. 13, the central fringe of each system will be shifted, as we have seen, an amount $(\mu - 1)\frac{td}{b}$, which will be different for each wave length, because μ varies. In this case there will, in general, be no point at which the congruent rays of all wave lengths will arrive in the same phase. There will, therefore, be no absolutely achromatic fringe. Nevertheless, a system of colored fringes may be obtained whose central band appears nearly achromatic. The determining condition here is not that the difference in the optical paths S_2P' and S_1P' equals zero, which is manifestly impossible because of the dispersion of the plate Q, but that the change in phase at the point P', corresponding to a change in wave length, be a minimum. Let x be the distance from P, the position of the achromatic fringe when the plate Q is out, to the point P', where the central fringe of the colored system appears when the plate Q is in. The optical difference of path of the two rays has been shown to be

$$S_2P' - S_1P' = \frac{bx}{d} - (\mu - 1)t.$$

But $\frac{bx}{d}$ is the apparent or geometrical retardation, which we will call D', and $(\mu - 1)t = D$, which we will denote now by $f(\lambda)$. The difference of phase at P' for any wave length will, therefore, be

$$2\pi \frac{D' - f(\lambda)}{\lambda}.$$

In order that this difference of phase be a minimum, its derivative with respect to λ must be zero. Performing the operation we get

$$D' = f(\lambda) - \lambda f'(\lambda) = D\left(1 - \frac{\lambda f'(\lambda)}{D}\right).$$

Now equation (16) tells us how much the central fringe of the system corresponding to any wave length λ is shifted by the introduction of the plate Q, that is, it is displaced till it coincides with a fringe of the order

$$p = \frac{D}{\lambda} = \frac{f(\lambda)}{\lambda}.$$

But the central fringe of the colored system is thereby shifted till it coincides with that fringe of the system produced by wave length λ, whose order is

$$p' = \frac{D'}{\lambda} = p - f'(\lambda).$$

Hence the center of the colored system is displaced from the shifted center of the system corresponding to λ by the number of fringes expressed by the equation

$$p' - p = -f'(\lambda).$$

This displacement can readily be calculated if we know the form of the function of λ for the substance of which the plate Q

consists. It has been found that that form of function first proposed by Cauchy satisfies the experimental facts very well for the visible spectrum. Assuming this equation, we have

$$\mu = A + \frac{B}{\lambda^2},$$

but

$$f(\lambda) = D = (\mu - 1)t;$$

hence

$$f(\lambda) = (A + \frac{B}{\lambda^2} - 1)t,$$

$$f'(\lambda) = -\frac{2Bt}{\lambda^3};$$

hence

$$p' - p = \frac{2Bt}{\lambda^3}. \qquad (17)$$

Experiments

Measurement of Wave Lengths with the Mirror

Apparatus.—The observations with the Fresnel mirrors can be made in several ways. In every case it is, of course, necessary to have a pair of mirrors properly mounted. The mirrors which are usually furnished with the optical bench are entirely satisfactory. They are mounted as follows: A brass plate, which can be fastened upon one of the uprights of the optical bench, serves as a mounting for both mirrors. One mirror is so fastened to this plate that it rests upon three screws, and can thus be adjusted so as to be parallel to the other mirror, which is rigidly fastened upon a slide and can be moved by means of a micrometer screw in a direction at right angles to its surface.

Simple and effective mirrors can be made according to Quincke* in the following way: Select a piece of best plate glass about 10 cm. long, 2.5 cm. wide, and 3 mm. thick. Cut it in the middle into two pieces each 5 cm. long. Blacken the rear

* Quincke, Pogg. Ann. 132, p. 41, 1867.

surfaces with shellac containing lamp black, in order to destroy the reflection from the rear surface. Plane a heavy block of wood smooth and flat, and arrange six balls of soft wax of equal size upon the block in such a way that each of the two pieces of plate glass, when placed upon them, will be supported upon one ball along the edge where the two touch and upon two balls along the edge which is farthest from the edge of contact. Dust the surfaces carefully and lay upon them another piece of carefully dusted plate glass about 20 cm. long, 5 cm. wide, and 3 mm. thick. Press firmly with one finger upon the larger plate of glass directly over the line of contact of the two smaller pieces. The upper glass will bend enough to set the lower pair of plates at a small angle with each other, and will at the same time keep their edges of contact together. The original apparatus used by Fresnel was of this nature.

In case it is desired to perform the experiment with divergent light, i.e., according to equation (14′), it is necessary to have in addition to the mirrors a slit, a micrometer microscope, and a telescope and scale. If the experiment is to be performed with parallel light, i.e., according to equation (15), an ordinary spectrometer may be used to advantage, the mirrors taking the place of the prism. The angle a can then be measured directly upon the graduated circle of the instrument, and the distance between the fringes can be determined by removing the objective of the telescope, and measuring the angle which the fringes subtend, and the distance from the line of contact of the mirrors to the focal plane of the eyepiece. The description of the experiment, as given below, applies to the optical bench. Its adaptation to other methods is left to the student.

Adjustments.—First, the centers of the slit, the mirrors, and the micrometer should be brought into the same horizontal plane by measuring their distance from the table upon which the apparatus rests. Second, the slit and the cross-hairs of the micrometer should be made vertical. The slit may be made vertical with the aid of a plumb line. The cross-hairs must then be adjusted so as to be parallel to the slit by forming by means of a lens of short focal length an image of the slit in the plane of the cross-hairs and then rotating the micrometer about a horizontal axis. Third, the mirrors should be adjusted so that their line of intersection coincides with the edges which are in contact. This can be approximately accomplished by observing the image of a straight edge in the mirrors and adjusting until this image is an unbroken line. The accurate adjustment is made with the help of the fringes. The mirrors are then set in place so as to reflect the light from the slit to the micrometer. The angle between the slit and the plane of the mirrors should not be less than 10° in order to avoid complications due to diffraction at the edges of the mirrors. On looking into the micrometer, the slit being illumined with monochromatic light, and altering slowly the angle between the mirrors, the fringes will appear if the preliminary adjustment has been carefully made. If the fringes do not appear, the plate which carries the mirrors should be turned about a horizontal axis to bring the intersection of the mirrors parallel to the slit, i.e., vertical. If the fringes even then do not appear it indicates that the intersection of the planes of the surfaces of the mirrors does not coincide with their common edge. This may be caused by not having adjusted the mirrors so that both are vertical, or by allowing one to protrude in front of the other. The fringes may then be found by turning the mirror which is supported on three screws, about a horizontal axis, or by moving the other mirror, which can be displaced in a direction perpendicular to its surface, or by both operations. When the fringes appear they will probably not be parallel to the cross-hairs of the micrometer. Their centers may be made parallel to the cross-hairs by tilting the adjustable mirror about a horizontal axis. They may be made parallel to the cross-hairs throughout their entire length by tilting the plate which carries both mirrors, about a horizontal axis. Having thus obtained straight monochromatic fringes parallel to the cross-hairs, the fringes in white light are found by displacing the mirror which is movable in a direction normal to its surface.

The collimation axis of the micrometer should then be brought into parallelism with the line from the intersection of the mirrors to the micrometer. Open the slit rather wide and reflect, by means of a small mirror, a beam of light through the lower half of the slit. Turn the micrometer about a vertical axis until the reflection of the slit upon the front lens of the micrometer is seen through the upper half of the slit.

MEASUREMENTS.—The distance between the fringes is measured with the micrometer. It is well to measure the distance over which the thread moves in passing ten to twenty fringes.

The distance between the intersection of the mirrors and the slit, and between that intersection and the cross-hairs of the micrometer, can be measured with a large pair of dividers with a sliding scale, or with a fine tape.

The angle between the mirrors may be measured with a telescope and scale in the usual way.

In order to attain accuracy with the Fresnel mirrors it is necessary to use a bright source of light, and to make the angle between the mirrors large enough to allow the formation of twenty or more fringes. As these fringes will be narrow a rather high-power micrometer is necessary in order to count them accurately.

EXAMPLES

1. Using as a source of light a sodium burner the following measurements were obtained:

Distance between the fringes.................. $e = .012$ cm.
Distance from the mirrors to the slit.......... $f = 29.5$ cm.
Distance from the mirrors to the micrometer... $h = 51.1$ cm.

Hence. . . . . $d = f + h =$ 80.6 cm.
Distance from mirrors to scale. . . . . $D =$ 163 cm.
Deflection, by observing the image of the scale in
first one mirror and then the other. . . . . $a =$ 2.16 cm.

Hence. . . . . $\sin \alpha = \frac{a}{2D} =$ .006625

$$\frac{2ef}{d} = .00878$$

$$\lambda = .0000582$$

2. The mirrors were mounted on a large spectrometer in place of the prism. Sodium light was used as a source. The light passed through the collimator, and was, therefore, parallel when it fell on the mirrors. The following measurements were obtained:

Angle between the mirrors, being ½ the angle between the two images of the slit, $\alpha = 11' \ 28''$.

The lens of the telescope was then removed and the angle subtended in the focal plane of the eyepiece by 28 fringes measured. This was found to be 18′ 25″. The distance from the focal plane of the eyepiece to the intersection of the mirrors, which should coincide with the axis of the instrument, was 46.21 cm. Hence the angular width of a fringe was 39″.46, and the linear width $e = .008835$ cm. Hence $\lambda = 2e \sin \alpha = .00005897$ cm.

IV

THE FRESNEL BI-PRISM

Theory

We will now pass to the consideration of the Fresnel bi-prism. Attention has already been called to this form of interferometer, its derivation from the telescope being sketched in Fig. 9. Since measurements with the bi-prism are most easily made when the source is at an infinite distance, the formulae will be developed for this case only. Let ABC, Fig. 14, represent the projection

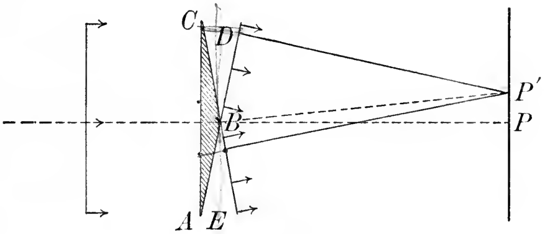

FIGURE 14

of the prisms upon a plane perpendicular to their refracting edges, and PP' a screen parallel to AC. Suppose a beam of parallel light to fall normally upon the prism faces AC. Upon leaving the prisms the beam will be divided into two whose wave fronts are represented by BD and BE. From B draw BP perpendicular to PP'. The point P being equidistant from the two wave fronts, will evidently be a position of maximum illumination. It is necessary to find what will be the illumination at some other point P' of the screen. Let $PP' = x$ and $BP' = d$. As in the case of the mirrors, since all points along BD and BE are in the same phase, the illumination at P' will be a maximum if

$$EP' - DP' = m\lambda.$$

But $EP' = d \sin EBP'$, and $DP' = d \sin DBP'$. Hence

$$EP' - DP' = 2d \cos \frac{DBE}{2} \sin P'BP.$$

But if we denote by α the angle between the wave fronts, $DBE = \pi - \alpha$; and $\sin P'BP = \frac{x}{d}$. Therefore, since α is small,

$$EP' - DP' = x \sin \alpha.$$

We, therefore, have a maximum of illumination when

$$x = \frac{\lambda}{\sin \alpha}, \frac{2\lambda}{\sin \alpha}, \text{ etc.}$$

The distance between the successive maxima, denoted by e, is, therefore,

$$e = \frac{\lambda}{\sin \alpha},$$

from which we get for the wave length

$$\lambda = e \sin \alpha. \qquad (18)$$

The same relations between the width of the source and the visibility of the fringes hold for the bi-prism as for the mirrors.

Experiments

Measurement of Wave Lengths with the Bi-Prism

Apparatus.—The apparatus needed for this experiment is the same as that used in the previous experiment, with the exception of the mirrors, these being now replaced by a bi-prism.

Adjustments.—The adjustments of the bi-prism are much simpler than those of the mirrors. The slit should be parallel to the common base of the prisms, and their common face should be perpendicular to the path of the light from the slit to them. This latter adjustment is made by reflecting a beam of light through the lower half of the slit and revolving the bi-prism about a vertical axis until the reflected light is seen through the upper half of the slit.

The measurements are most easily and accurately made with the help of a spectrometer. Telescope and collimator are adjusted for parallel light and arranged so that the image of the slit falls upon the cross-hairs of the telescope. The bi-prism is then placed upon the prism table of the spectrometer and adjusted, either in the way described above, by reflecting light through the slit, or with the help of a Gauss eyepiece, so that it is perpendicular to the common collimation axis of the collimator and telescope. Upon removing the objective of the telescope the fringes will appear in the field of view. If the slit is not parallel to the common base of the prisms the fringes will not be clear and evenly spaced throughout their entire length. The slit should then be rotated about a horizontal axis until the fringes are clear and evenly spaced. In this case, as in the case of the mirrors, the fringes are very narrow, so that a rather high-power micrometer is necessary to measure them accurately, and a bright light is indispensable. In both of these experiments the solar spectrum across the slit will be found to give the greatest satisfaction as a source of light.

Measurements.—If a spectrometer is used the only measurements needed are the distance between the fringes and the angle between the beams behind the bi-prism. To obtain the former the angle through which the telescope turns when the cross-hair passes over a counted number of fringes, and the distance between the bi-prism and the focal plane of the micrometer are measured. From these two observed quantities the linear distance between the fringes is at once calculated. The angle between the beams is measured by replacing the objective of the telescope and setting the cross-hair on first one and then the other of the images of the slit; the angle through which the telescope has turned is the desired angle. If the graduations upon the circle of the spectrometer are not sufficiently fine to allow of reading the angles accurately, a small mirror, such as is used for galvanometers, should be mounted upon the telescope and the angle read in this mirror with a telescope and scale in the usual way.

EXAMPLE

The bi-prism was placed in the spectrometer in place of the prism, and the angle between the wave fronts measured. The value was

$$a = 18' \, 21''.$$

The objective of the telescope was then removed and the angle subtended in the focal plane of the eyepiece by 16 fringes in sodium light was found to be 13′ 10″. Thus the angular width of one fringe was 49″.4. The distance from the focal plane of the eyepiece to the axis of the instrument was 46.21 cm. Hence the linear width of a fringe was $e = .01107$ cm. Hence

$$\lambda = .00005904 \text{ cm.}$$

V

THE MICHELSON INTERFEROMETER

Theory

We will now pass to the consideration of the interferometer as shown in Fig. 11. This form of instrument may be further simplified by making one plate perform the functions both of separating and reuniting the beam. To accomplish this it is only necessary to turn the two mirrors into the positions *CD* as shown in Fig. 15. This is known as the Michelson* interferometer. In

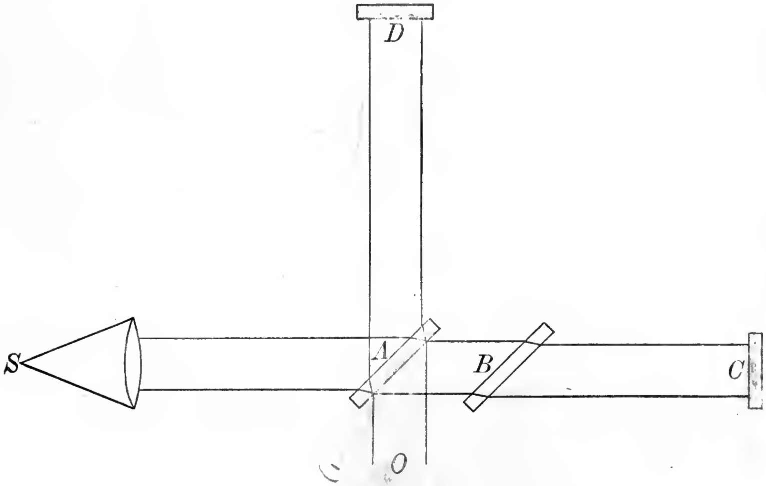

FIGURE 15

its simplest and most efficient form it consists merely of four glass plates. Let *A*, *B*, *C*, *D* (Fig. 15) represent the projections of the four plates on a plane perpendicular to their surfaces.

* Michelson, Phil. Mag. (5) *13*, p. 236.

Light from a source S falls upon the plate A at an angle of incidence of approximately 45°. Plates A and B are polished on both sides and should, to obtain the best results, be of best optical glass, free from all strains. They should, furthermore, be cut from the same piece, so as to insure their having the same optical thickness, and their surfaces should be as plane and as nearly parallel to each other as it is possible to make them. The rear surface of A is coated with a semi-transparent film of silver or platinum, which should, if the clearest interference bands are desired, be of such a thickness that it reflects half the light incident upon it to D, and transmits the other half to C. D and C are two plane mirrors, coated on the front surface with a thick coat of platinum or silver, and so adjusted as to reflect the light incident upon them back over nearly the same path. These two reflected beams meet again on the rear surface of the plate A in a condition suitable to the production of interference bands. The plate B is inserted to make the two paths optically identical.

The observer looks into the apparatus from O. He will see the plate D directly, and an image of the plate C reflected by the plate A. This image of C will appear in the direction of the plate D as far behind A as the mirror C really is in front of it. The instrument is thus seen to consist essentially of a film of air inclosed between the plate D and the virtual image of the plate C. Hence in discussing the interference phenomena produced by this instrument it is necessary to consider only this film of air.

Let om_1, om_2 (Fig. 16) represent the two plane mirrors C and D whose intersection is projected at o, and whose mutual inclination is ϕ. The illumination at any point P, not necessarily in the plane of the figure, will depend on the mean difference of phase of all the pairs of congruent rays which reach P after reflection from the mirrors.

If the source of light is sufficiently broad, the illumination

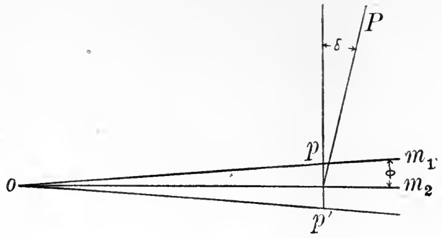

FIGURE 16

at P will be independent of its distance, form, or position. Let us suppose that it is a plane luminous surface which coincides with om_1. Then its image in om_1 will coincide with om_1; and its image in om_2 will be a plane symmetrical to om_1 with respect to om_2. For every point p of the first surface there is a corresponding point p' of the second, which is symmetrically placed, and in the same phase of vibration. If we suppress now the source of light and the mirrors, and replace them by the images, the effect at any point P is unaltered. Consider now a pair of points p, p'. Let δ be the angle formed by the line joining P and p (or p') with the normal to the surface om_2, δ and ϕ being both supposed small. The difference of optical path D will be

$$D = Pp' - Pp = pp' \cos \delta$$

to quantities of the second and higher order. If we denote by $2t$ the distance between the images at the point where they are cut by the line Pp, we will have to a close degree of approximation

$$D = 2t \cos \delta. \tag{19}$$

The difference of phase Δ at P is $2\pi \frac{D}{\lambda}$. Since $\cos \delta$ is nearly equal to unity, if $2t$ is increased by λ by gradually separating the images, D is also increased by λ, and the difference of phase Δ at P passes through a complete period.

In order to find the form of the interference fringes, let *cdef*, *c'd'e'f'* (Fig. 17) represent the two images, and let their intersection be parallel to *cf*, and their inclination be 2ϕ. Let P be the

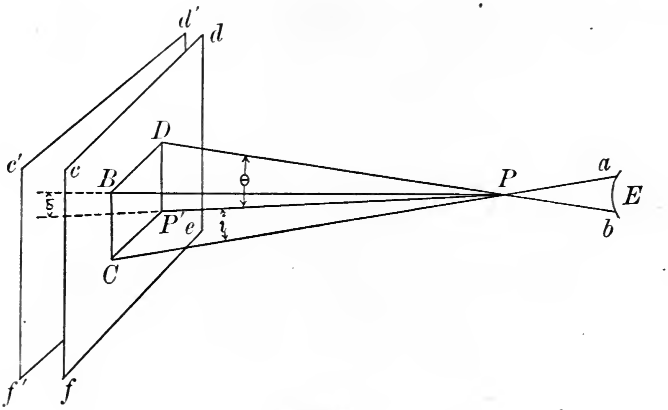

FIGURE 17

point considered; P' the projection of P on the surface *cdef*; and PB the line forming with PP' the angle δ. Draw $P'D$ parallel to *cf*, and $P'C$ at right angles, and complete the rectangle $BDP'C$. Let $P'PC = i$, and $DPP' = \theta$. Let $PP' = P$; and call the distance between the surfaces at the point P', $2t_0$. We have then,

$$t = t_0 + CP' \tan\phi = t_0 + P \tan\phi \tan i,$$

and

$$D = 2\ (t_0 + P \tan\phi \tan i) \cos\delta,$$

or

$$D = 2 \frac{t_0 + P \tan\phi \tan i}{\sqrt{1 + \tan^2 i + \tan^2\theta}}. \qquad (20)$$

We see that in general D has all possible values; and hence all phenomena of interference would be obliterated.

But to an eye placed at P the interference fringes will be visible under certain conditions. Let ab (Fig. 18) represent the pupil of the eye. Since it has appreciable size it will receive light not only from p and p', but also from other points as p_1 and p_1'. Consider the ray pa to enter the pupil at one end of a diameter, and the ray p_1b, parallel to pa, to enter it at the other end of the

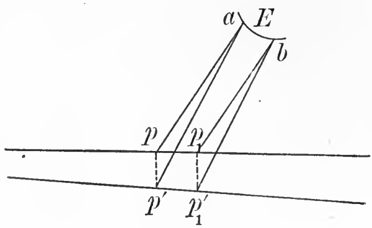

FIGURE 18

same diameter. Since these rays are parallel they will come to a focus at the same point on the retina. The difference of phase at this point of the congruent rays ap and ap' will be

$$\Delta = \frac{4\pi t \cos \delta}{\lambda}.$$

The difference of phase of the other pair of congruent rays bp_1 and bp_1' will be

$$\Delta_1 = \frac{4\pi t_1 \cos \delta}{\lambda}.$$

If now $\Delta - \Delta_1 = \frac{1}{2}\lambda$, the interference phenomena which are produced by each pair of congruent rays are in opposite phase; that is, if one pair would produce a maximum of illumination at the common focus, the other would produce a minimum. The resulting sensation would be a combination of the two. Hence, in order that interference bands may be observed by an eye at P, conditions must be so arranged that the difference in the Δs for those pairs of congruent rays which come to a focus at the same point on the retina, be less than half a wave. This may be accomplished in two different ways: First, we may observe through a small enough opening, the eye being focused on the point p. By this simple device the clearness of the interference phenomena can frequently be materially increased. Second, by making the Δs the same for every pair of congruent rays which are focused at the same point on the retina. This is accomplished by making $t = t_1$, and focusing the eye for parallel rays. The second method is generally to be preferred for reasons to be given presently.

The exact investigation of the form of the fringes is a matter of considerable complexity. An approximate conception of their appearance to an eye placed at a point P may be obtained from equation (20). Thus let $cdef$ (Fig. 17) represent the xy plane, and P' the center of a system of rectangular coördinates whose axis of y is parallel to the intersection of the mirrors. Then $P'C = x$, $P'D = y$, $\tan i = \frac{x}{P}$ and $\tan \theta = \frac{y}{P}\cdot$ Let $\tan \phi = k$. Our equation then becomes

$$D = \frac{2t_0 + 2kx}{\sqrt{1 + \frac{x^2}{P^2} + \frac{y^2}{P^2}}}.$$

Since $\frac{x}{P}$ and $\frac{y}{P}$ are small, this equation reduces to

$$D\left(1 + \frac{x^2}{2P^2} + \frac{y^2}{2P^2}\right) = 2t_0 + 2kx,$$

or

$$\left(x - \frac{2P^2k}{D}\right)^2 + y^2 = \frac{4P^2t_0}{D} + \frac{4P^4k^2}{D^2} - 2P^2.$$

This is the equation of a circle whose center is at the point $\left(\frac{2P^2k}{D}, O\right)\cdot$ Hence the fringes are always approximately circles whose centers lie on the axis of x at a distance from the origin determined by the inclination of the mirrors, the difference of optical path, and the distance P to the point of observation.

Two cases are of special practical interest. First, if $t = t_0$, that is, if the mirrors are parallel, $k = 0$, and our equation reduces to

$$x^2 + y^2 = \frac{4P^2t}{D} - 2P^2.$$

In this case the center of the circles lies at the origin itself and their radii are given by

$$r^2 = \frac{4P^2t}{D} - 2P^2.$$

If the difference of the optical paths between the successive rings counted from the center be $n\lambda$, we shall have

$$n\lambda = 2t - D,$$

or

$$D = 2t - n\lambda.$$

Hence the radii of the successive rings are given approximately by

$$r = P\sqrt{\frac{2n\lambda}{2t - n\lambda}}. \qquad (21)$$

Second, if the intersection of the mirrors passes through the origin, $D = 0$ at that point. In this case the distance to the center of the circles, $\frac{2P^2k}{D}$, becomes infinite, and the fringe through the origin is a straight line parallel to the axis of y. This particular fringe, since it is the only absolutely straight one, serves as a convenient mark from which to begin measurements. It is called the central fringe of the system.

If the mirror om_2 (Fig. 16) is moved perpendicular to itself till it passes to the other side of the mirror om_1, D, since its value passes through 0, changes sign, and the center of the circles passes from $+\frac{2P^2k}{D}$ to $-\frac{2P^2k}{D}$. Hence on opposite sides of the central fringe the curvature of the fringes has opposite signs; that is, on both sides they appear convex toward the central fringe. This fact is of great assistance in locating the central fringe.

Experiments

I. Measure the Wave Length of Sodium Light

Apparatus.—From the discussion above it is evident that the essential parts of the interferometer are four plates of glass arranged as shown in Figs. 15 and 19, and an arrangement for

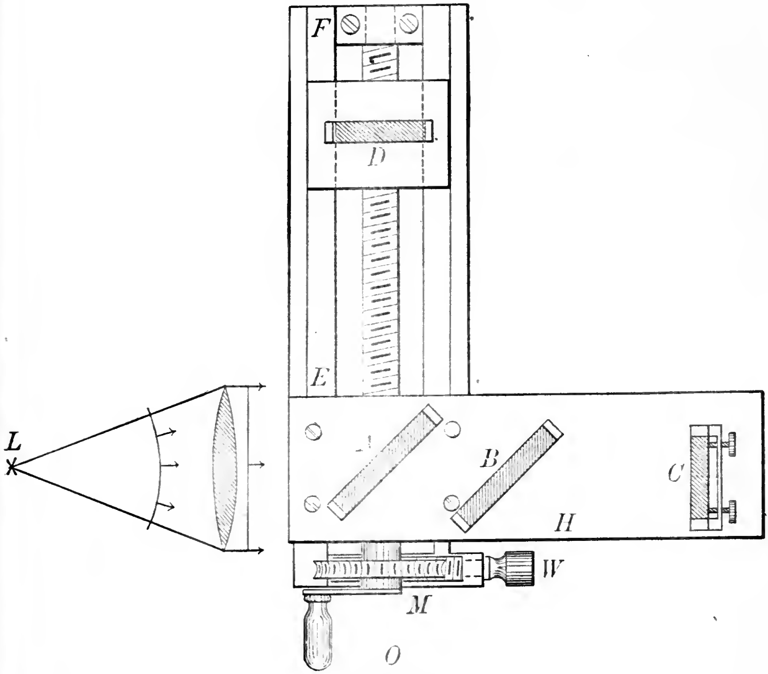

Figure 19

moving one of the mirrors in a direction normal to its surface. This motion is effected by mounting the mirror *D* upon a slide, which can be moved by a screw along the ways *EF*. These ways must be accurately straight, so that in its motion the mirror remains strictly parallel to its original position. The screw carries at its front end a worm wheel *M* which in turn is driven by a worm *W*. The worm can be disengaged from the wheel when a rapid motion of the screw is desired. The worm wheel is graduated upon its front face.

The dividing plate *A* is mounted firmly in a metal frame upon a plate of brass *H* which is screwed to one end of the ways. That side of *A* which carries the silver half-film should be turned toward the plate *B*. This plate *B* should be so mounted upon the brass plate *H* as to allow of a small motion about a vertical axis. This motion is necessary in order to be able to adjust the two plates *A* and *B* so that they are strictly parallel. It is also useful in measuring differences in phase or small fractions of a wave accurately. The mirror *D* is rigidly mounted upon the slide, while the mirror *C* rests upon three adjusting screws, which are set in a plate, which is perpendicular to the plate *H* and firmly fastened to it. Small springs hold this mirror in place against the three screws.

ADJUSTMENTS.—Measure roughly the distance from the silver half-film upon the rear of the plate *A* to the front of the mirror *C*. Set the mirror *D*, by turning the worm wheel *M*, so that its distance from the rear of *A* is the same as that of *C* from *A*. This need not be done accurately. It is suggested because it is easier to find the fringes when the distance between the mirror *D* and the virtual image of the mirror *C* is small. This distance will hereafter be called the distance between the mirrors.

Now place a sodium burner, or some other source of monochromatic light, at *L*, in the principal focus of a lens of short focus. It is not necessary that the incident beam be strictly parallel. Hold some small object, such as a pin or the point of a pencil, between *L* and *A*. On looking into the instrument from *O*, three images of the small object will be seen. One image is formed by reflection at the front surfaces of *A* and *D*; the second is formed by the reflection at the rear surface of *A* and the front surface of D; the third is formed by reflection from the front surface of C and the rear surface of A. Interference fringes in monochromatic light are found by bringing this third image into coincidence with either of the other two by means of the adjusting screws upon which the mirror C rests. If, however, it is desired to find the fringes in white light, the second and third of these images should be brought into coincidence, because then the two paths of the light in the instrument are symmetrical, i.e., each is made up of a given distance in air and a given thickness of glass. When the paths are symmetrical, the fringes are always approximately arcs of circles as described above. If, however, the first and third images are made to coincide, then the two optical paths are unsymmetrical, i.e., the path from A to C has more glass in it than that from A to D, and in this case the fringes may be ellipses or equilateral hyperbolae, because of the astigmatism which is introduced by the two plates A and B. It is quite probable that the fringes will not appear when the two images of the small object seem to have been brought into coincidence. This is simply due to the fact that the eye can not judge with sufficient accuracy for this purpose when the two are really superposed. To find the fringes then it is only necessary to move the adjusting screws slightly back and forth. As the instrument has been here described, the second image lies to the right of the first.

Having found the fringes the student should practice adjustment until he can produce at will the various forms of fringes described on page 54. Thus the circles appear when the distance between the mirrors is not zero, and when the mirror D is strictly parallel to the virtual image of C. The accuracy of this adjustment may be tested by moving the eye sideways and up and down while looking at the circles. If the adjustment is correct, any given circle will not change its diameter, as the eye is thus moved. To be sure, the circles appear to move across the plates because their center is at the foot of the perpendicular dropped from the eye to the mirror D, but their apparent diameters are independent of the lateral motion of the eye. For this reason it is advisable to use the circular fringes whenever possible.

To find the fringes in white light, adjust so that the monochromatic fringes are arcs of circles. Move the carriage rapidly by intervals of a quarter turn or so of the wheel M. When the region of the white-light fringes has been passed, the curvature of the fringes will have changed sign, i.e., if the fringes were convex toward the right, they will now be convex toward the left. Having thus located within rather narrow limits the position of the mirror D, which corresponds to zero difference of path, it is only necessary to replace the sodium light by a source of white light, and move the mirror D by means of the worm slowly through this region until the fringes appear.

These white-light fringes are strongly colored with the colors of Newton's rings. The central fringe,—the one which indicates exactly the position of zero difference of path,—is, as in the case of Newton's rings, black. This black fringe will be entirely free from color, i.e., perfectly achromatic, if the plates A and B are of the same piece of glass, are equally thick, and are strictly parallel. If they are matched plates, i.e., if they are made of the same piece of glass and have the same thickness, their parallelism should be adjusted until the central fringe of the system is perfectly achromatic. When this is correctly done, the colors of the bands on either side of the central one will be symmetrically arranged with respect to the central black fringe.

Measurements.—An accurate scale graduated to tenths of a millimeter is set upon the slide behind the mirror D. Over this a micrometer microscope is placed and focused on the scale. The microscope should be rigidly attached to the base of the instrument so that it does not move relatively to the interferometer during the observation. The cross-hair of the micrometer is set upon one of the tenth millimeter divisions. The mirror D is then slowly moved with the worm W, and the number of fringes which pass when the cross-hair of the microscope moves over one-tenth of a millimeter are counted. The circular fringes should be used because, as stated above, their phase is independent of the position of the eye, so that if the eye moves during the observation, no error will be introduced. It will be necessary to look from time to time through the microscope so as to note when the cross-hair reaches the next tenth millimeter mark. Since a motion of the mirror of 0.1 mm. introduces a difference of path of about 340 waves, it is safe to count 300 without looking at the microscope. Having obtained the number of fringes which pass when the mirror D moves through 0.1 mm., then, since the difference in path introduced by this motion is 0.2 mm., the wave length sought is $\lambda = \frac{0.2}{N}$ mm., in which N denotes the number of fringes counted.

EXAMPLE

Sodium light was used as a source and the number of waves which passed while the carriage moved 0.3 mm. were counted. This number was found to be 1018. Hence

$$\lambda = \frac{0.6}{1018} \text{ mm.} = 5894 \cdot 10^{-7} \text{ mm.}$$

II. Determine the Ratio of the Wave Lengths of the Sodium Lines D_1 and D_2

Apparatus and adjustment as in Experiment I.

Measurements.—Since the yellow radiation of sodium contains two vibrations of different periods, two different sets of fringes will be formed by it. As the mirror D is moved these sets of fringes, since they are formed by waves of different lengths, will move at different rates. Thus in certain positions of the mirror D the bright fringes of one set will fall upon the dark fringes of the other set, and the field of view will be almost evenly illuminated. The fringes do not disappear entirely because the light produced by the shorter of the waves is more intense than that produced by the longer. At certain other positions of the mirror D, the bright fringes of one set will coincide with the bright fringes of the other set, and there will be strong contrast between the bright and the dark fringes in the field of view. This contrast in the intensity of the fringes is called their *visibility*. This subject of visibility is treated at length in the next chapter. Here it is sufficient to note that in the interval between two positions of greatest contrast of the fringes, i.e., between two positions of maximum visibility, there must be one more of the shorter waves than of the longer. In order to determine the ratio of the wave lengths, then, it is necessary to measure the distance the mirror D moves in passing between two positions of maximum or of minimum visibility. If this distance be divided by the shorter wave length we obtain the number of shorter waves in the interval. This number minus one will be the number of longer waves in the interval, and the ratio of the wave lengths will be the inverse of the ratio of the number of waves.

In making the observations it will be found impossible to determine accurately the position of any one maximum or minimum. An accurate result may, however, be obtained in the following way: Draw the mirror D as far forward as is possible without causing the fringes to disappear entirely. Then move the mirror backward by jumps of about one-twentieth of a turn of the worm wheel. Take the reading on the worm wheel at the points which appear to be either maxima or minima. In this way about twenty readings of the positions of the maxima and twenty of those of the minima can be obtained. The average should then be taken by subtracting the first reading in each set from the eleventh, the second from the twelfth, etc., and then taking the mean of these averages. In making the calculation it is to be noted that the difference of path is twice the distance through which the mirror has moved.

EXAMPLE

The distance between the positions of the maximum clearness of the fringes was determined in the way described above. In all, thirty maxima and thirty minima were read. The mean value of the interval was found to be 0.5802 mm. Hence

$$n_2 = \frac{0.5802}{.0005890} = 985,$$

$$n_1 = n_2 - 1 = 984,$$

hence
$$\frac{\lambda_1}{\lambda_2} = \frac{985}{984},$$

in which λ_1 is the wave length of D_1, and λ_2 that of D_2.

If we wish the difference between the wave lengths in millimeters we may proceed thus:

$$\frac{.5802}{\lambda_2} - \frac{.5802}{\lambda_1} = 1, \quad \text{or} \quad \frac{1}{\lambda_2} - \frac{1}{\lambda_1} = \frac{1}{.5802}.$$

Therefore, $$\lambda_1 - \lambda_2 = \frac{\lambda^2}{.5802} = 5.98 \cdot 10^{-7} \text{ mm}.$$

III. Determine the Index of Refraction and the Dispersion of a Piece of Glass

Apparatus and adjustment as in Experiment I.

Measurements.—In order to perform this experiment in a theoretically rigorous way there should be added to the interferometer two extra metal frames similar to those used to hold the mirrors, one in front of each of the mirrors *D* and *C*. These frames should be set upon pivots so that they can be rotated about a vertical axis. The one in front of the mirror *C* should be arranged with a worm wheel or a tangent screw so that it can be rotated slowly and steadily. Two pieces of the glass whose index of refraction and dispersion are to be determined should be used. These pieces should, of course, have the same thickness. One of them is waxed to each of the movable frames so as to cover half —the same half—of the field of view. The interferometer must then be adjusted for the white-light fringes. The piece of glass which is in front of the mirror *D* is then turned through any angle, say 15° to 20°. In this way extra glass is introduced into the path of the light between *A* and *D*. This extra glass should then be compensated for by turning the other piece, that between *A* and *C*, through the same angle. When the angles through which the two pieces have been turned are the same, the white-light fringes will appear in the half of the field of view which is covered by the two plates. This turning of the second plate of glass should be done slowly with the worm wheel, and the fringes which pass during the operation should be counted. The angle through which this plate turns must be measured by fastening to the frame which carries it a small mirror, and reading the angle through which this mirror turns with an ordinary telescope and scale. Before measuring this angle, care must be taken to have the plate perpendicular to the beam passing through it. This can be done by rotating the plate through the position in which it is at right angles to the beam, and noting the point at which the fringes reverse their direction of motion; for it is evident that when the plate is normal to the beam its optical thickness is a minimum, and therefore a turning of the plate in either direction will increase the optical thickness, and cause the fringes to move in one particular direction. It will, however, be found that the plate can be turned through a considerable angle before the fringes move appreciably. Therefore, to obtain the scale reading which corresponds to the normal position of the plate, turn the plate in one direction until two or three fringes have passed, and take the reading on the scale. Then turn it in the other direction until the same number of fringes has passed, and take the reading. The mean of these two readings will then be the reading which corresponds accurately to the normal position of the plate. Having then counted the fringes which pass while the plate is turning through the angle i, and having measured that angle, the index of refraction is obtained as follows: Let t represent the thickness of the plates of glass, and $2N$ the number of fringes counted while turning through the angle i. Let, further, AB (Fig. 20) represent the direction of the light, $MNOP$

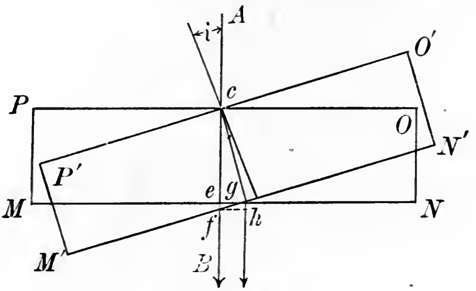

FIGURE 20

the plate in its position perpendicular to the beam, and $M'N'O'P'$ its position after it has been turned through the angle i. Let the two surfaces OP and $O'P'$ intersect at c, and draw through f, the intersection of the surface $M'N'$ with AB, the line fh parallel to MN. The light incident along AB upon $O'P'$ will, when the plate has been turned, travel along the path cgh. It is evident that before the turning the optical distance between the planes OP and fh consisted of a distance $ce = t$ in glass, and a distance $ef = \frac{t}{\cos i} - t$ in air. After the turning, the optical distance between these two planes consists of a distance $cg = \frac{t}{\cos r}$ in glass, and of a distance $gh = \frac{t \sin (i - r)}{\cos r} \tan i$ in air, in which r denotes the angle of refraction. The numbers of waves in these various distances are obtained by dividing the distances in air by λ, the wave length, and those in glass by $\frac{\lambda}{\mu}$, in which μ stands as usual for the index of refraction. The difference between the number of waves in the optical path between the planes OP and fh before the turning and the number in the path after the turning is half of the number which has been counted, because in the observation the light has passed twice through the plate. Therefore the following equation is obtained:

$$\frac{t\mu}{\cos r} + \frac{t \sin (i-r)}{\cos r} \tan i - t\mu - \frac{t}{\cos i} + t = N\lambda.$$

If this equation be reduced with the help of the equation $\frac{\sin i}{\sin r} = \mu$, and solved for μ, there results, neglecting the term $\frac{N^2\lambda^2}{2t}$ in the numerator,

$$\mu = \frac{(t - N\lambda)(1 - \cos i)}{t(1 - \cos i) - N\lambda}. \qquad (22)$$

The thickness t of the glass may be measured with the calipers or in any other accurate way.

It was shown in the chapter on the Fresnel mirrors that if the optical symmetry of the two paths over which the light travels is disturbed by the introduction of a plate of some transparent substance, the fringes in white light no longer possess a truly achromatic central fringe, but one which may seem fairly achromatic, and which is displaced from the true position of the central fringe of the set of fringes which correspond to the wave length λ by a number of fringes [cf. equation (17)]

$$p' - p = \frac{2Bt'}{\lambda^3},$$

in which

$$t' = \frac{t}{\cos r} - t. \qquad (23)$$

Now, when the glass which was added to the path AD by turning the plate in front of the mirror D, is compensated for, by rotating the plate in front of the mirror C, we add extra glass to the path AC also. The two paths are thus made finally symmetrical, so that the achromatic light fringe to which we count indicates the true position of the central band of the monochromatic system. By such a count, then, we obtain the true number of fringes through which the monochromatic system has been shifted, namely p. It is possible, however, to compensate for the extra glass in the path AD in another way, namely, by drawing the mirror D toward A. If we do this and count the fringes which pass, we count to the shifted position of the white-light fringes, i.e., we count the number p'. If then we make the count both ways, once by turning the plate in front of the mirror C, and once by drawing up the mirror D, the plate in front of C being perpendicular to the beam, we determine both p and p' of the above equation. Since t and λ are also known, it is in this way possible to determine the B of the Cauchy dispersion equation (p. 39). With this value of B and the value of μ determined from equation (22), it is then possible to determine the A of the dispersion equation. Thus with a single source of monochromatic light it is possible to determine both the index of refraction and the dispersion of a plate of glass.

EXAMPLE

Two pieces of optical glass were mounted in the instrument as described above. The thickness of the glass was 6.81 mm., i.e., $t = 6.81$ mm. One of the plates was then turned through an angle i, which was measured with a telescope and scale, and found to be $i = 16° \ 41' \ 30''$. The other piece was then turned, and the sodium fringes counted until the fringes in white light appeared over the whole field. This number was 342. Since the light traverses the plates twice, the number N in the formula is half of this, i.e., $N = 171$. Hence the index of refraction for sodium is $\mu_{Na} = 1.5180$. The compensating plate was then turned back till it was normal to the path of the light, and, the first piece of glass remaining inclined at the angle i, the movable mirror was drawn up and the fringes again counted until the white fringes appeared in their first position. This number of fringes was $2p' = 355$.

Hence $2(p'-p) = 355 - 342 = 13$ or $p'-p = 6.5$.

From equation (23), $t' = .1254$ mm. Hence from equation (17)

$$B = 53 \cdot 10^{-10},$$

and therefore (cf. p. 39), $A = 1.5027$.

With these values of A and B the index of refraction for the green line of mercury was calculated ($\lambda = 5461 \cdot 10^{-7}$). The result was $\mu_{Hg} = 1.5205$. The fringes were then counted in mercury light, and the result was $2N = 370$. With this N we get from equation (22) $\mu_{Hg} = 1.5204$.

IV. Determine the Change of Phase Produced by Perpendicular Reflection at a Silver Surface

Apparatus and adjustments as in Experiment I.

Measurements.—One of the mirrors C or D of the interferometer must be removed and freshly silvered over three-quarters of its surface. This is best accomplished by keeping one-half of the mirror covered with a piece of glass while it is in the silvering solution. When the silver is sufficiently deposited so that the film is perfectly opaque, the solution is poured off, the mirror and the tray which holds it rinsed with distilled water, the piece of glass on the surface of the mirror turned through 90°, and a fresh solution poured on. In this way the surface of the mirror is coated with silver over three-quarters of its surface, as shown in Fig. 21. The quarter marked a has upon it two layers of silver, b and c each has one layer, and d has none. If now the mirror be replaced in the interferometer, and the fringes found, it will be noted that where the fringes cross the boundaries of these four sections of the surface, they are displaced. Thus a fringe which passes from a to b will not be a straight line or an arc of a circle, but that portion of it which is over the surface a will be displaced with respect to that over the surface *b* by an amount which depends upon the thickness of the film *ac*. If this displacement is measured, we thereby determine the thickness of the film *ac*.

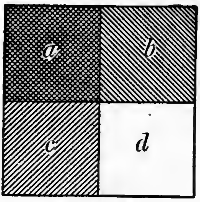

FIGURE 21

In order to measure it, the white-light fringes must first be found so as to determine, by means of the central black fringe, in which direction the shifting has taken place. Having thus determined the direction of the shifting, its amount is measured by the compensator. There should be fastened to one end of the frame which holds the compensator, a small spiral spring. The other end of the spring should be fastened to a string which may be wound about a pin. The pin must carry a graduated drum or circle, so that its position may be read and thereby the tension of the spiral spring determined. The tension of this spring is opposed by the elasticity of the stud by which the compensator frame is fastened to the plate *H*. If the pin is turned so as to tighten or loosen the spiral spring, the compensator will turn through a small angle, and this angle will be proportional to the amount of the turning of the pin. To measure the difference of phase, the pin is turned until one fringe has passed, and the angle through which the pin has turned is read upon the graduated head. The pin should then be turned until the shifted part of the fringe comes to the position of the unshifted part, and the angle through which it has turned is read upon the graduated head. The ratio of the angles through which the pin has turned in these two operations is then the fraction of a wave by which the fringe is shifted, and half of this fraction is the thickness of the film *ac* in wave lengths. Of course this measurement should be made in monochromatic light, the white-light fringes being used merely to recognize the direction of the shift. Having thus measured the thickness of the film *ac*, it is only necessary to measure the shift in a fringe at the junction between *c* and *d*, in order to be able to calculate the change of phase due to the reflection at the silver surface. In calculating this change of phase, account must be taken of the fact that half a wave is lost at the reflection upon glass, and also of the direction of the shift after allowance has been made for the thickness of the film *ac*. If the shift is in the same direction as that in the fringe across *ab*, then, since *a* is nearer the observer than *b*, the wave from the silver surface is ahead of that from the glass.

The difference of phase between the light reflected at a surface glass-air, and that reflected at a surface glass-silver, is very easily obtained with the aid of a plane parallel plate of optical glass which has been silvered over half of one surface. This plate must be introduced into the interferometer in place of the mirror *D*, with the silver side away from the observer. A second piece of the same plate of glass must then be introduced in front of the mirror *C* in order to compensate for the extra glass added by the introduction of the first plate into the path *AD*. The white-light fringes having been found upon the rear of the plate in front of *D*, and been set perpendicular to the dividing line between the silvered and the unsilvered portions of the plate, the displacement of the central fringe is measured as described above. Since the light is reflected from the glass-air surface without change of phase, the shifting of the fringe indicates a retardation, i.e., a loss of part of a wave.

EXAMPLES

1. One of the mirrors of the interferometer was coated with a double film of silver as illustrated in Fig. 21. The displacement

of the fringe across *ab* was measured in sodium light and found to be 0.26. Hence the thickness of the film *ac* was 0.13λ. The fringe across *cd* was displaced 0.17 of a fringe. Since the reflection upon the glass surface *d* produces a change of phase of 0.5 of a wave, the retardation produced by the silver is $0.5 + 0.17 - 0.26 = 0.41\lambda$.

2. The displacement of the fringe on the glass-silver surface was found to be 0.28λ.

For further study of the applications which can be made of the interferometer the student is referred to the following:

Michelson and Morley, "On the Relative Motion of the Earth and Ether," Am. Jour. Sci. (3) *22*, p. 120, 1881; *34*, p. 333, 1887; Phil. Mag. (5) *24*, p. 449, 1887. "On the Effect of the Motion of the Medium upon the Velocity of Light," Am. Jour. Sci. (3) *31*, p. 377, 1886. "On a Method of Using the Wave Length of Sodium Light as a Practical Standard of Length," Am. Jour. Sci. (3) *34*, 427, 1887; Phil. Mag. (5) *24*, p. 463; Am. Jour. Sci. (3) *37*, p. 181, 1889.

Michelson, "Light Waves and Their Applications to Meteorology," Nature, *49*, p. 56, 1893. "Valeur du Mètre en longeurs d'ondes lumineuse," Trav. et Mém. Bur. Internat. Poids et Mes. XI, p. 1, 1894; "On the Relative Motion of the Earth and Ether," Am. Jour. Sci. (4) *3*, p. 475, 1897.

Morley and Rogers, "On the Measurement of the Expansion of Metals by the Interferential Method," Phys. Rev. *4*, pp. 1 and 106, 1896.

Wadsworth, "On the Application of the Interferometer to the Measurement of Small Deflections of a Suspended System," Phys. Rev. *4*, p. 480, 1897.

Hull, "On the Use of the Interferometer in the Study of Electric Waves," Phys. Rev. *5*, p. 231, 1897.

Johonnott, "On the Thickness of the Black Spot on Liquid Films," Phil. Mag. (5) *47*, p. 501, 1899.

Earhart, "On Sparking Distances between Plates," Phil. Mag. (6) *1*, p. 147, 1901.

Gale, "On the Relation between Density and Index of Refraction of Air," Phys. Rev. *14*, p. 1, 1902.

VI

THE VISIBILITY CURVES

Theory

In Chapter II it has been shown that it is possible to determine the width of a rectangular source and the distance between two such sources by observations made with the double slit. In these experiments the fringes disappeared when the distance between the slits was such that the angle subtended by the sources was one wave divided by that distance, and also when that distance was such that the angular width of each single source was equal to a wave length divided by it. In Chapter III, the two slits have been converted into an interferometer, and in Chapter V we have used the interferometer to measure the ratio of the wave lengths of the two sodium lines, by determining the number of waves in the change which takes place in the distance between the mirrors in passing from one position of maximum visibility to the next. The close similarity between the two experiments must be evident at once,—the difference lying in the fact that in the case of the two slits we have angles to resolve, while with the interferometer we have differences in wave lengths, or rather in numbers of vibrations, to determine. The resemblance becomes even closer if we conceive the spectral source to be resolved as far as possible by an ordinary spectroscope. The sodium lines, for example, would then appear as two line sources, i.e., they would very much resemble the double source consisting of a pair of parallel slits as treated above.

We might expect then that the equations which connect the visibility curves with the distribution of light in the source would be very similar in the two cases, and would differ only in the fact that angles in the case of the two slits would be replaced in the case of the interferometer by wave lengths or numbers of vibrations.

The solution of a visibility curve is very difficult. It will help us much in obtaining such solutions if we begin by the inverse process of assuming a known distribution of light and plotting the corresponding visibility curve. Fig. 22 gives a series of such curves. The nature of the distribution in the source is shown at the left, and the actual vibrations are plotted, the visibility curve being the envelope of the curve. The abscissae of the curves represent distance traveled, and the ordinates intensity. Thus in Fig. 22 the curve 1 represents the resultant of two trains of homogeneous waves of the same amplitude but with slightly different periods which start in the same phase at *a*. When they have traveled a distance *ab*, they are seen to be in opposite phase, and the visibility curve comes to zero. It is quite clear that the distance they have to travel before they come into opposite phases depends upon the difference of their periods. So we can already guess that a determination of the distance *ab* would lead to some knowledge of that difference in the periods.

In the case of the interferometer we have formed by the two trains of waves two separate sets of fringes, and when the movable mirror is displaced, these sets travel across the field at different rates, as was shown on page 59. When a certain difference of path has been introduced, represented by *ab* in curve 1, these two sets of fringes overlap so as to present an evenly illuminated field of view and the visibility curve comes to zero. As the difference of path is further increased, the fringes soon come into such positions that one set has overtaken the other by one whole fringe, and then we have a maximum of visibility as indicated at *c*. Thus if we interpret the vibrations which unite to form the curve 1 as fringes, i.e., as periodic variations of intensity, and consider that the distance traveled is replaced in the interferometer by difference of path, then the envelope represents the variations in the visibility of the resultant set of fringes as the two separate sets pass by in the field of view. Hence the envelopes of the curves in Fig. 22, are, in the case of the interferometer, the visibility

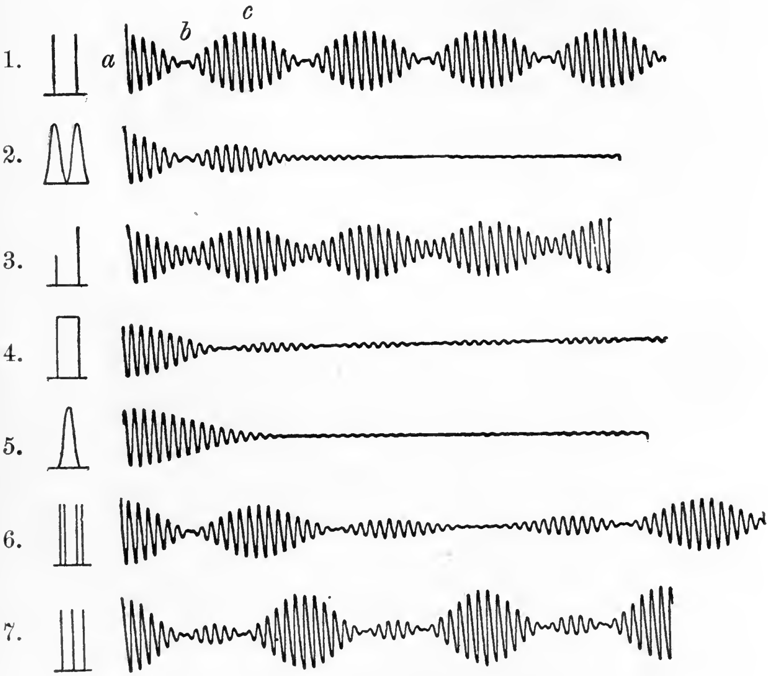

FIGURE 22

curves, and from them we can draw conclusions as to the nature of the source.

Thus curve 2 represents the visibility curve which corresponds to a double source, each of whose components is broad, i.e., does not send out waves of one definite period only, but waves whose lengths vary between the limits λ and $\lambda + d\lambda$. It will be noted that the distance between the centers of these two sources is the same as that of curve 1, so that the positions of zero visibility are not changed. The effect of broadening the source is seen to be a decrease in the visibility at each successive maximum, so that the fringes soon disappear altogether.

Curve 3 corresponds to two homogeneous radiations of unequal amplitudes, and curve 4 represents a single, broad, uniformly illuminated source. The other curves are easily understood.

Let us now take up the analytical discussion of the subject. According to equation (6), page 22, the intensity of illumination produced at any point by two congruent rays of equal brightness is expressed by

$$A^2 = 4A_1^2 \cos^2 \pi \frac{\delta}{\lambda} = 2A_1^2 \left(1 + \cos 2\pi \frac{\delta}{\lambda}\right).$$

Now $2A_1^2$ represents twice the intensity of each of the two rays. This intensity may be regarded as a function of the wave length, so that we may replace $2A_1^2$ by $\psi(\lambda)$ and our equation becomes

$$I_\lambda = \psi(\lambda) + \psi(\lambda) \cos 2\pi \frac{\delta}{\lambda}.$$

Since each of the separate vibrations is independent, the resultant intensity I will be the integral of this expression taken between the limits λ_1 and λ_2. Let now λ_0 represent a wave length which is intermediate between λ_1 and λ_2. Then

$$\frac{1}{\lambda} = \frac{1}{\lambda_0}(1 + x); \qquad \delta = p\lambda_0,$$

in which p represents the number of waves in the distance δ. The intensity may then be regarded as a function of x between the limits $-x_1$ and $+x_2$. Hence we may replace $\psi(\lambda)\, d\lambda$ by $\phi(x)\, dx$, and there results for the total intensity

$$I = \int \phi(x)\, dx + \int \phi(x) \cos 2p\pi (1 + x)\, dx.$$

If now we expand this cosine and introduce the notation,

$$P = \int \phi(x)\, dx$$

$$C = \int \phi(x) \cos 2p\pi x dx$$

$$S = \int \phi(x) \sin 2p\pi x dx,$$

there results

$$I = P + C \cos 2p\pi - S \sin 2p\pi.$$

Now as long as the interval from x_1 to x_2 is small, the variation of the values of C and S corresponding to a change in p is small. Hence as on page 24, the maxima and minima of I are determined by $\frac{dI}{dp} = 0$, i.e., by

$$C \sin 2p\pi + S \cos 2p\pi = 0.$$

Hence

$$I = P \pm \sqrt{C^2 + S^2}.$$

The visibility may then be defined by equations (7), (8), and (9), page 25. Thus in the case of a single uniformly illuminated source ϕ (x) = const., which sends out waves for which x varies within the limits $\pm \frac{a}{2}$, we have

$$V = \frac{\Big[C\Big]_{-\frac{a}{2}}^{+\frac{a}{2}}}{\Big[P\Big]_{-\frac{a}{2}}^{+\frac{a}{2}}} = \frac{1}{a}\int_{-\frac{a}{2}}^{+\frac{a}{2}} \cos 2p\pi x dx = \frac{\sin p\pi a}{p\pi a},$$

as on page 25, but in this case p is equal to the number of waves in the difference of path, and a is a small fraction which determines the width of the source. It will be noted that the visibility is equal to zero when $pa = 1, 2, 3$, etc., i.e., when

$$p = \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \text{ etc.}$$

The more important case is that in which the distribution of light in the source is represented by

$$\phi(x) = e^{-k^2x^2},$$

which distribution is in accord with Maxwell's Law deduced from the theory of probability.

In this case when k is large the value of the integral diminishes rapidly with increasing x, the terms near the origin being the only important ones. Hence the limits of the integration may be taken as $\pm\infty$. Then

$$V = \frac{\Big[C\Big]_{-\infty}^{+\infty}}{\Big[P\Big]_{-\infty}^{+\infty}} = \frac{\int_{-\infty}^{+\infty} e^{-k^2x^2} \cos 2p\pi x dx}{\int_{-\infty}^{+\infty} e^{-k^2x^2}} = e^{\frac{-p^2\pi^2}{k^2}}.$$

It will be noted that the curve is not periodic, but diminishes gradually as p increases. If we assume that the source is practically limited when $\phi(x)$ has reached a value equal to one-half of its maximum value, then, calling $\frac{a}{2}$ the corresponding value of x, we have

$$\frac{1}{2} = e^{-\frac{k^2a^2}{4}}, \qquad k^2a^2 = 4 \log 2.$$

Hence

$$V = e^{-\frac{p^2\pi^2a^2}{4 \log 2}}.$$

If now q represent the value of p for which $V = \frac{1}{2}$, then

$$\frac{1}{2} = e^{-\frac{q^2\pi^2a^2}{4 \log 2}}, \qquad \frac{q^2\pi^2a^2}{4 \log 2} = \log 2,$$

hence

$$\frac{a}{2} = \frac{\log 2}{\pi q} = \frac{.22}{q}.$$

If this value of a be substituted above, there results

$$V = e^{-\frac{p^2 \log 2}{q^2}},$$

or

$$V = 2^{-\frac{p^2}{q^2}}.$$

It will be noted that $\frac{a}{2}$ is a small fraction denoting parts of a wave length. If we wish to get the width of the source in millimeters we must multiply by λ. Also both p and q are numbers of waves. If we wish to have them expressed in millimeters we multiply both by λ expressed in millimeters. Thus letting $p\lambda = X$, $q\lambda = \Delta$, we get

$$V = 2^{-\frac{X^2}{\Delta^2}}, \qquad \frac{W}{2} = \frac{.22}{\Delta}\lambda^2, \qquad (24)$$

in which W represents the width of the source in millimeters.

It has been shown on page 26, that the equation of the visibility curve for a double source differs from that for a single source by the addition of a cosine factor. In general let us suppose we have a series of similar sources which lie about the origin of coördinates and whose distribution of intensity is expressed by $\phi(x)$. The expression for the number of vibrations of the waves of any source may be put in the form

$$\frac{1}{\lambda} = \frac{1}{\lambda_0}(d + x).$$

For a symmetrical distribution $(S = 0)$, the integrals P and C take the form

$$\int \phi(x) \sin 2p\pi(d + x) = C \sin 2p\pi d,$$

$$\int \phi(x) \cos 2p\pi(d + x) = C \cos 2p\pi d.$$

Hence if U represent the visibility which results from all the sources

$$U^2 = \frac{(\Sigma C \sin 2p\pi d)^2 + (\Sigma C \cos 2p\pi d)^2}{(\Sigma P)^2}.$$

Now the visibility due to each source by itself is represented by

$$V = \frac{C}{P},$$

therefore

$$U^2 = \frac{(\Sigma VP \sin 2p\pi d)^2 + (\Sigma VP \cos 2p\pi d)^2}{(\Sigma P)^2},$$

or

$$U^2 = \frac{\Sigma V^2 P^2 + 2\Sigma PP'VV' \cos 2p\pi\,(d' - d)}{(\Sigma P)^2}.$$

If the law of the distribution of the light in the separate sources is the same, while their intensities are proportional to factors r, r', r'', etc., the visibility V produced by each will be the same, but P will be proportional to r; hence, for this case

$$U^2 = \frac{\Sigma r^2 + 2\Sigma rr' \cos 2p\pi\,(d' - d)}{(\Sigma r)^2} V^2.$$

In the case of a double source, the ratio of the intensities of whose components is $r:1$, this reduces, if $d_0 = d' - d$, to

$$U^2 = \frac{1 + r^2 + 2r \cos 2p\pi d_0}{(1 + r)^2} V^2; \qquad (25)$$

and if the two sources have equal intensities, to

$$U = V \cos p\pi d_0. \qquad (26)$$

The value of d_0 may be found from the positions of zero visibility, i.e., the points at which $U = 0$. In case the two lines have not equal intensities, we may still determine the value of d_0 from the period of the curve. Thus in the case of the two sodium lines the visibility curve reaches its first minimum when $p = 492$ waves; hence, since $U = 0$ when $pd_0 = \frac{1}{2}, \frac{3}{2}$, etc., $d_0 = \frac{1}{983}$. So it follows that d_0 represents a fraction of a wave length, so that when it is multiplied by λ_0 we obtain the difference between the wave lengths of the two sources in millimeters, i.e., $\lambda_1 - \lambda_2 = \lambda_0 d_0$.

Since the period of the curve is the distance between two successive minima which correspond to differences of path p_1 and p_2, we have, denoting the number of waves in that period by p_0,

$$p_0 = p_2 - p_1 = \frac{3}{2d_0} - \frac{1}{2d_0} = \frac{1}{d_0};$$

but if D represents the length of the period in millimeters, $p_0 = \frac{D}{\lambda_0}$, and therefore

$$d_0 = \frac{\lambda_0}{D}, \qquad \lambda_0 d_0 = \frac{\lambda_0^2}{D} = \lambda_1 - \lambda_2. \tag{27}$$

In general $p = \frac{X}{\lambda}$, hence equation (25) becomes

$$U^2 = \frac{1 + r^2 + 2r \cos 2\pi \frac{X}{D}}{(1+r)^2} V^2. \tag{28}$$

Since this coefficient of V appears frequently it will be denoted symbolically by $\cos \frac{r}{D}$.

The inverse problem of determining the form of the distribution when the visibility curve is given is more difficult. The general solution is shown by Rayleigh* to depend upon both C and S. Now the visibility curve alone determines only $C^2 + S^2$. Hence the solution is not single valued unless we can obtain a second relationship between C and S. This can be done by determining the displacement of the phase of the fringes as the difference in path is increased. We may, however, obtain a fairly accurate idea of the distribution in quite a number of cases from the visibility curve alone if we assume that we are dealing with a source in which the distribution is symmetrical, for in this case $S = 0$, and the solution is definite.

The process of determining the visibility curve of a given source by observation is as follows: The light which is to be analyzed is passed into the interferometer, the two mirrors being

* Rayleigh, Phil. Mag. (5) *34*, p. 407.

near together and adjusted to be parallel to each other. The visibility of the fringes, that is, the contrast between the bright and dark fringes, is called 100. The screw of the instrument is then turned through a whole turn and the visibility again estimated. It will generally be less than 100. This estimation of visibility requires some practice. This practice may be obtained by mounting between Nicols a convex and a concave quartz lens of the same curvature. If these lenses are cut parallel to the crystallographic axis and set so that their axes are at right angles to each other, circular fringes similar to those in the interferometer will be seen. As the lenses are rotated about the line of sight as an axis, the visibility varies in a way which can be calculated from the angle of inclination of the axes of the lenses with the plane of polarization of the analyzer. For if a represent that angle, then the two extreme values I_1 and I_2 of the resultant intensity will be respectively 1 and $\cos^2 2a$, and therefore

$$V = \frac{I_1 - I_2}{I_1 + I_2} = \frac{1 - \cos^2 2a}{1 + \cos^2 2a}.$$

Having trained the eye with such an arrangement the visibility is estimated at each revolution of the screw, and these estimates are plotted as ordinates, the corresponding differences of path being the abscissae. Even if the entire curve is not worked out, considerable information can be obtained from a determination of the differences of path which correspond to the minima.

Experiment

Determine the Distribution in the Cadmium Lines

Apparatus and adjustments as in the previous chapter.

Measurements.—Using as a source of light a cadmium tube, the light is first passed through an ordinary spectroscope so that only one radiation at a time passes into the interferometer. Starting near the position of the white-light fringes, the visibilities which correspond to the gradually increasing differences of path are observed as has been described. It is well to observe the fringes through a small telescope focused for parallel rays. If a telescope can not be used, a card with a small hole in it should be mounted in front of the instrument to insure keeping the eye at the same point during the observations. The observations are then plotted as ordinates in a curve, the difference of path being the abscissae. From the curve thus obtained we find Δ the difference in path which corresponds to $V = 50$. If the curve is periodic, corresponding to a double source, we must take the envelope of the maxima for making this calculation, i.e., the part of the curve represented in our equation by $2^{-\frac{X^2}{\Delta^2}}$. The half width of the source is then $\frac{.22\lambda}{q} = \frac{.22}{\Delta}\lambda^2$. If the line is double, the distance between the sources is determined from D, the period of the curve, i.e., the distance between the maxima or the minima; for, as was shown above, $\lambda_1 - \lambda_2 = \frac{\lambda_0^2}{D}$. The ratio of the intensities of the two lines may be obtained approximately from the heights of the first maximum and minimum. Thus the visibility at the first maximum is always 100. If at the first minimum it is, say 20, and if a and b represent the two intensities, $a + b = 100$, $a - b = 20$, and, therefore, $\frac{b}{a} = \frac{2}{3} = 0.7$ approximately.

EXAMPLES

1. The red radiation from cadmium ($\lambda = 6438 \cdot 10^{-7}$) was observed, and the curve shown in Fig. 23 b obtained. Since this

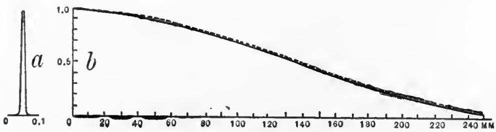

FIGURE 23

curve is not periodic we may conclude that the line is single. The value of Δ is seen to be $\Delta = 138$, hence the half width of the source is, from equation (24),

$$\frac{0.22}{\Delta}\lambda^2 = .0066 \cdot 10^{-7}\text{mm.}$$

The equation of the visibility curve would then be $V = 2^{-\frac{X^2}{(138)^2}}$. The curve marked a shows the distribution which is seen to correspond to a very nearly homogeneous source.

2. The green radiation of cadmium ($\lambda = 5086 \cdot 10^{-7}$) was observed and the curve shown in Fig. 24 b obtained. Since the

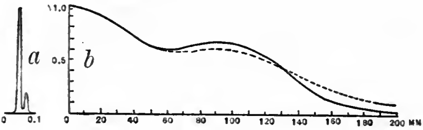

FIGURE 24

curve is periodic with a single period it corresponds to a double line. The period of the curve is seen to be $D = 115$. Hence the distance between the lines is $\lambda_1 - \lambda_2 = \frac{\lambda_0^2}{D} = .022 \cdot 10^{-7}$mm.

Further, $V = 50$ for $X = 120$, i.e., $\Delta = 120$, therefore, the half width of the line is $\frac{.22}{\Delta}\lambda^2 = .0048 \cdot 10^{-7}$mm.

The value of V at the first maximum is 100, at the first minimum 66, hence $\frac{b}{a} = .2$ nearly. Hence the equation of the curve would be represented by

$$V = 2^{-\frac{X^2}{(120)^2}} \cos\frac{.2}{115}.$$

The corresponding distribution is represented at the left of the figure.

3. The next curve, Fig. 25, represents the envelope of the visibility curve for sodium ($\lambda = 5890 \cdot 10^{-7}$). The period which deter-

mines the separation of the two lines D_1 and D_2 has already been found to be .58 mm. (cf. page 61). Hence this period is omitted

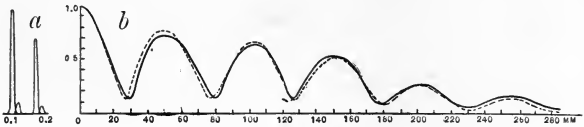

FIGURE 25

from the curve. As it stands, the curve represents the distribution of each of the sodium lines upon the supposition that it is the same for both. The curve is seen to have two periods, one of 50 and one of 150. Corresponding to the period 50, we have $\frac{\lambda^2}{D} = .069 \cdot 10^{-7}$mm., while for the period 150 we have $\frac{\lambda^2}{D} = .023 \cdot 10^{-7}$mm.

The value of Δ is seen to be 156, and so the half width of each of these lines is $.0063 \cdot 10^{-7}$mm.

The ratio of the intensities corresponding to the first period is found to be .7, and that corresponding to the second .2, hence the equation of the curve is represented by

$$V = 2^{-\frac{x^2}{(156)^2}} \cos\frac{.7}{50} \cos\frac{.2}{150}.$$

In connection with this chapter the student should read Michelson, Phil. Mag. (5) *31*, p. 338, 1891; Phil. Mag. (5) *34*, p. 280, 1892; Journal de Physique (3) *3*, p. 5, 1894; B. A. Reports, 1892, p. 170; Trav. et Mém. du Bureau Internat. des Poids et Mesures, *XI*, p. 1, 1894. Also some further developments of the same method are given by Perot and Fabry, C. R. *126*, pp. 34, 331, 407, 1561, 1624, 1706, 1779; Ann. de Chim. et de Phys. (7) *16*, pp. 115, 289; C. R. *130*, p. 653.

Some important applications of the method will be found as follows: Michelson, "On the Broadening of Spectral Lines," Astrophysical Journal, *2*, p. 251, 1894; "Radiation in the Magnetic Field," Phil. Mag. (5), *44*, p. 109, 1897; *45*, p. 348, 1898; Astrophysical Journal, *7*, p. 130, 1898.

VII

THE PRISM SPECTROMETER

Theory

To understand the conditions which must be fulfilled in order that the spectrometer may be used with the greatest efficiency, it is necessary to discuss first some of the optical properties of prisms.

An optical prism is a transparent solid, two of whose faces at least are plane surfaces, which intersect in a line. This line of intersection is called the edge of the prism. The angle inclosed by the two plane surfaces is called the refracting angle of the prism, and will be denoted in what follows by A. A plane passing through the prism parallel to the edge and perpendicular

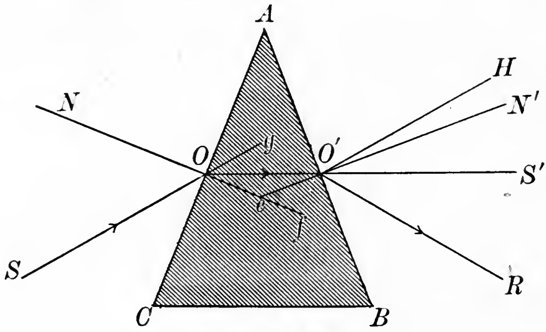

FIGURE 26

to the plane bisecting the refracting angle is called the optical base, and a plane perpendicular to the edge is called a principal plane.

Let CAB (Fig. 26) represent the section of a prism by a principal plane. Let S represent a source of light, and $SOO'R$ the path of a ray from that source through the prism. Draw NOe

and $N'O'e$ perpendicular to the faces CA and BA at the points O and O' respectively. Continue the line SO to g, and OO' to S', and draw through O', $O'H$ parallel to SO.

The angle NOS between the normal NO and the direction of the incident ray SO is called the angle of incidence, and is denoted by i. As it is measured from the normal NO, it may be either positive or negative. It is defined as positive when the incident ray SO and the refracting angle A lie on opposite sides of the normal. It is, therefore, negative when the incident ray lies between the normal and the angle of the prism.

Similarly the angle $N'O'R$ is called the angle of emergence and is denoted by i'. It is defined as positive when the emergent ray $O'R$ and the refracting angle A lie on opposite sides of the normal $N'O'$. It is, therefore, negative when the emergent ray lies between the normal and the refracting angle.

The angles eOO' and $eO'O$ are called angles of refraction, and are denoted by r and r' respectively. The angle r is positive when the i, to which it corresponds, is positive, and negative when i is negative. Similarly, r' is positive or negative according as the i', to which it corresponds, is positive or negative.

The angle $HO'R$ being the angle through which the ray is bent by its passage through the prism, is called the angle of deviation and is denoted by δ.

From Fig. 25 we see that the following relations exist:

$$HO'R = HO'S' + S'O'R = \delta$$

$$HO'S' = gOO' = i - r$$

$$S'O'R = i' - r'.$$

Hence $$\delta = i + i' - (r + r').$$

But $$feO' = r + r' = A, \qquad (29)$$

therefore, $$\delta = i + i' - A. \qquad (30)$$

The index of refraction of one medium with respect to another is defined as the ratio of the velocity of light in the one medium to that in the other. Thus, if V represent the velocity in one medium, V' that in the other, and μ the index of refraction,

$$\mu = \frac{V}{V'}.$$

As is well known, this ratio is equal to that of the sines of the angles of incidence and refraction, that is,

$$\mu = \frac{V}{V'} = \frac{\sin i}{\sin r} = \frac{\sin i'}{\sin r'}.$$

We have, then, as the fundamental equations of the prism,

$$\mu = \frac{\sin i}{\sin r},$$

$$\mu = \frac{\sin i'}{\sin r'}; \qquad (31)$$

$$r + r' = A.$$

To obtain the general equation which connects the index of refraction with the angles A, i and i' we proceed as follows: From equations (31) we have

$$\sin i' = \mu \sin r' = \mu \sin (A - r);$$

if we expand $\sin (A - r)$ and substitute for $\sin r$ its value $\frac{\sin i}{\mu}$ and for $\cos r$ its value $\frac{1}{\mu}\sqrt{\mu^2 - \sin^2 i}$, this equation reduces to

$$\sin i' = \sin A \sqrt{\mu^2 - \sin^2 i} - \cos A \sin i. \qquad (32)$$

This equation holds in general without any conditions imposed upon the quantities involved. Since, however, the quantity whose value is to be determined from measured values of the others is usually the index of refraction μ, and since it is a matter of some difficulty to measure the angle of incidence i with accuracy, it is generally advisable to use the prism in one of two particular positions.

One of these particular positions is determined by the condition

$$i' = 0,$$

that is, the ray emerges from the prism normal to its second face. In this case $\sin i' = 0$, $\delta = i - A$, or $i = \delta + A$, $r' = 0$, and $r = A$. Upon substituting these values in the first of equations (31), it readily reduces to

$$\mu = \frac{\sin (A + \delta)}{\sin A}. \qquad (33)$$

Since in this case the determining condition is $i' = 0$, and since we also have $r' = 0$ and $r = A$, therefore the fundamental equation $\sin i = \mu \sin r$ becomes, under these circumstances, $\sin i = \mu \sin A$. But $\sin i$ must be less than unity. Therefore, $\sin A < \frac{1}{\mu}$. Hence, a prism can not be used in this particular position unless its refracting angle falls within the limit prescribed by this inequality, that is, unless $\sin A < \frac{1}{\mu}$.

The other of these particular positions is determined by the condition

$$i' = i.$$

In this case $r' = r = \frac{1}{2}A$ and $\delta = 2i - A$, or $i = \frac{1}{2}(A + \delta)$, and on substituting these values in the first of equations (31) it becomes

$$\mu = \frac{\sin \frac{1}{2}(A + \delta)}{\sin \frac{1}{2}A}. \qquad (34)$$

When the prism is used under this condition, namely $i' = i$, the deviation δ produced by it is the smallest which can be obtained with a prism of given angle and index of refraction. Hence this position of the prism is known as that of minimum deviation. That it is so may be proved as follows:

Since in equation (30) A is a constant for any given prism, the value of δ will depend on that of $i + i'$. Therefore δ will be a minimum when $i + i'$ is. But an inversion of equations (31) gives

$$i = \sin^{-1} \mu \sin r, \text{ and } i' = \sin^{-1} \mu \sin (A - r), \text{ hence}$$
$$i + i' = \sin^{-1} \mu \sin r + \sin^{-1} \mu \sin (A - r).$$

To find when this value of $i + i'$ will be a minimum, differentiate this equation with respect to r, regarding μ as constant. This gives

$$\frac{d\,(i + i')}{dr} = \frac{\mu \cos r}{\sqrt{1 - \mu^2 \sin^2 r}} - \frac{\mu \cos (A - r)}{\sqrt{1 - \mu^2 \sin^2 (A - r)}}.$$

This becomes equal to zero when $r = \frac{1}{2}A$. A second differentiation with respect to r and a substitution of $\frac{1}{2}A$ for r gives

$$\frac{d^2\,(i + i')}{dr^2} = \frac{\mu^2 - 1}{\left(1 - \mu^2 \sin^2 \frac{1}{2} A\right)^{\frac{3}{2}}}\, 2\mu \sin^2 \frac{1}{2} A.$$

Since $\mu \frac{1}{2} \sin A$ is the sine of that angle of incidence which will have a corresponding angle of refraction $\frac{1}{2} A$, the value of the right-hand side of this equation will be real and positive when $\mu > 1$. When $\mu < 1$ its value will be negative, which means that the corresponding value of $i + i'$ is a maximum. But since, when $\mu < 1$, the prism is optically less dense than the surrounding medium, the deviation will not be given by equation (30), but by $\delta = A - (i + i')$. Hence in any case, the condition $r = \frac{1}{2} A$ will make δ a minimum. But $r + r' = A$. Hence the condition $r = \frac{1}{2} A$ is equivalent to $r = r'$ or to $i = i'$. Therefore, under this condition the deviation produced by the prism is a minimum.

There is a limit to the use of a prism in this case also; for, under the condition $i = i'$ we have seen that $r' = \frac{1}{2}A$. Since $\sin i' < 1$ it follows that $\sin \frac{1}{2}A < \frac{1}{\mu}$. If $\sin r' > \frac{1}{\mu}$, a thing which often occurs in practice, there is no $\sin i'$ to correspond to it. Hence the ray can not leave the prism at the surface where this occurs, but is totally reflected. The application of total reflection to the determination of indices of refraction will be discussed in the next chapter.

Consider now that instead of a single ray we have given a narrow beam of light. As before let CAB (Fig. 27) represent the

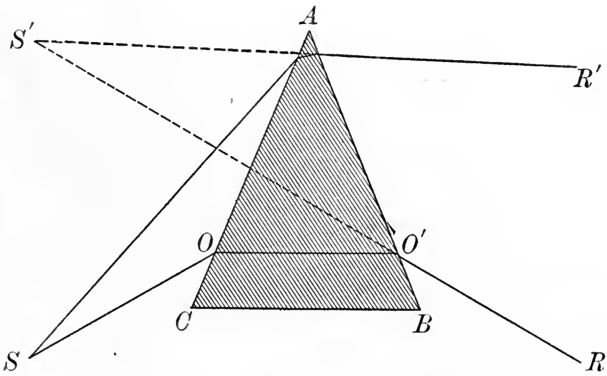

FIGURE 27

section of a prism by a principal plane, and S the projection of a narrow source upon that plane. Suppose that a narrow beam of monochromatic light SOA falls upon the prism in such a way that one boundary of the beam passes through the edge A of the prism. Let i represent the angle of incidence of the ray SO, and i' the corresponding angle of emergence, and r and r' the respective angles of refraction. Since the beam is supposed narrow, we may denote the corresponding angles for the ray SA by $i + di$, $i' + di'$, etc. We then have from equations (31)

$$\sin i' = \mu \sin (A - r)$$
$$\sin i = \mu \sin r,$$

whence, by differentiation, regarding μ as constant,

$$\cos i' di' = - \mu \cos (A - r)\, dr,$$
$$\cos i di = \mu \cos r dr.$$

Eliminating dr and substituting for $A - r$ its value r' we have

$$di' = - \frac{\cos r' \cos i}{\cos i' \cos r} di. \qquad (35)$$

From this equation it follows, since the cosine terms are always positive, that when di is positive, that is, $i + di > i$, di' is negative, and therefore $i' + di' < i'$. Hence, the two rays $O'R$ and AR', when prolonged backwards, will intersect at some point S', that is, a virtual image of the point S will be formed at S' by the prism.

Equation (35) is capable of another interpretation. If S is not a point source, but has finite width, di may be regarded as the angular width of the source when viewed from O; di' then represents the width of the virtual image when viewed from O'. From this it follows that, for a given i and i', the width of the image is proportional to the width of the source. Also the width of the image may be altered by varying i and i'. Thus if $i = 90°$, $\cos i = 0$, and $di' = 0$, or the emergent beam is parallel. If $i' = 90°$, $\cos i' = 0$, and $di' = \infty$.

When the prism is in the position of minimum deviation, $i = i'$, $r = r'$, and, therefore, $di = di'$, that is, the width of the image is equal to the width of the source. Since in practice prisms are most frequently used in this position, and since, as will be shown later, considerations of the purity of the spectrum make it desirable to have the width of the image as small as possible, the slit in a spectroscope is made infinitely narrow by placing it at an infinite distance by means of a lens. Hence the origin of the collimator.

If in equation (35) $di = 0$, then $di' = 0$, that is, a beam of monochromatic light which is parallel before falling on the prism remains so after its passage through the prism.

We have thus far been considering only those rays from the source which lie in the plane CAB (Fig. 27), i.e., in a principal plane. When we take into account also the rays which, coming from the source S, pass through the prism in some other than a principal plane, the development becomes much more complicated; especially if the source of light is a straight bright line parallel to the refracting edge of the prism. The virtual image of such a line source will, in general, be curved, owing to the passing of some of the rays through the prism obliquely to a principal plane.*

Up to this point the index of refraction μ has been regarded as constant. While this is true for a prism of any given substance so long as light of a definite wave length only is considered, it ceases to be so when the light from the source contains more than one wave length. We have, therefore, to consider now what further effects are produced by a prism when the variations of the index of refraction due to changes of wave length are taken into account, and to determine how the conditions under which the prism is used alter those effects.

The question as to how the index of refraction depends on the wave length has been the subject of a large amount of investigation. Practically it has been found that the formula first given by Cauchy† expresses the relation between the two quantities to

* M. A. Bravais, Jour. de l'éc. Polyt., *18*, p. 79, 1845. E. Reusch, Pogg. Ann., *117*, p. 241, 1862. A. Cornu, Ann. éc. norm. (2) *1*, p. 255, 1872. G. G. Stokes, Proc. Roy. Soc., *22*, p. 309, 1874. J. L. Hoorweg, Pogg. Ann., *154*, p. 309, 1875.

† Cauchy, *Mémoire sur la dispersion de la lumière*, Prague, 1836.

a fair degree of accuracy in all cases of normal dispersion within the visible spectrum.* This formula is

$$\mu = A + \frac{B}{\lambda^2}, \qquad (36)$$

in which A and B are for any given substance constants to be determined by experiment. It will be noted that when $\lambda = \infty$, $\mu = A$, i.e., A represents the index of refraction for infinitely long waves.

In order to find the change in the angle of emergence di' which results from a variation $d\mu$ in the index of refraction, it is necessary to differentiate equations (31), regarding i and A only as constant. This gives

$$0 = \mu \cos r dr + \sin r d\mu$$

$$\cos i' di' = \mu \cos r' dr' + \sin r' d\mu$$

$$dr + dr' = 0.$$

By elimination of dr and dr' these reduce to

$$di' = \frac{\sin A}{\cos i' \cos r} d\mu. \qquad (37)$$

This equation gives the relation between di' and $d\mu$ in terms of the angles A, i', and r. It is more practical to have their relation in terms of the dimensions of the prism and its adjuncts. This may be accomplished for the most useful case,† that in which the prism is in the position of minimum deviation, as follows: Since, under these circumstances we have $i = i'$, $r = r'$, $A = 2r$, equation (37) reduces to

$$di' = \frac{\sin 2r}{\cos i \cos r} d\mu = \frac{2 \sin r}{\cos i} d\mu. \qquad (38)$$

Suppose that a parallel beam of light falls upon the prism

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* Schmidt, *Die Brechung des Lichts in Gläsern*, Leipzig, 1874.

† For the general case cf. Rayleigh, Phil. Mag. (5), *7*, pp. 261, 403, 477, 1879; *9*, p. 40, 1880; *Collected Works, 1*, p. 415. Also Drude, *Theory of Optics*, p. 233, Longmans, 1902.

CAB (Fig. 28) through a rectangular opening ab, two of whose parallel sides are parallel to the edge A of the prism, and whose plane is perpendicular to the direction of propagation of the beam. Let the prism be so placed that one boundary bA of the

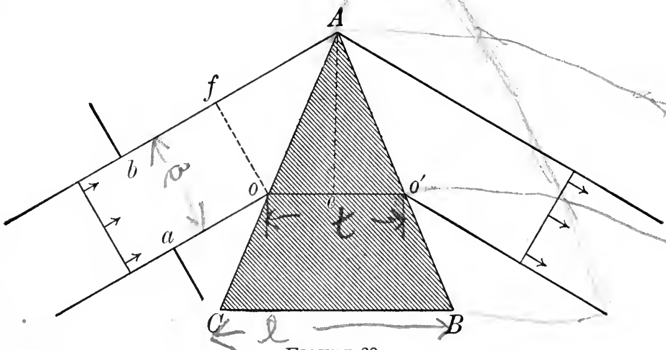

FIGURE 28

beam passes through its edge A. Let ab, the width of the beam, be represented by a, and the distance oo', which the other boundary of the beam travels in the prism, by t. Draw Ae bisecting the angle A, and of through o perpendicular to bA. Then, since the prism is in the position of minimum deviation, oo' is perpendicular to Ae, and $oe = \frac{1}{2}t$. From the figure we get

$$\sin oAe = \sin \frac{1}{2}A = \frac{oe}{oA} = \frac{t}{2oA} = \sin r,$$

$$\cos Aof = \cos i = \frac{of}{oA} = \frac{a}{oA}.$$

Substituting in equation (38) it reduces to

$$di' = \frac{t}{a}\,d\mu. \qquad (39)$$

Hence we note that the amount of separation di' of two beams of light whose wave lengths are such that the difference of their indices of refraction is $d\mu$, depends only on the width a of the beam, and on the excess of transparent substance of the prism traversed by one side of the beam over that traversed by the other.

One of the most important practical applications of the dispersive power of the prism is that of analyzing a composite beam into its components. It is, therefore, of great importance to know how far this analysis may be carried with any given prism. Thus if we have given for analysis a beam of light containing the two different wave lengths λ and $\lambda + d\lambda$, we wish to know what sort of prism is needed to accomplish the task, that is, to resolve the beam into its component parts. Equations (39) and (1) answer this question completely. Equation (39) gives the angular separation due to the prism of the two beams, and if we suppose a lens placed behind the prism to form real images of the source, (1) tells whether the two images formed by that lens will appear separated or not. Combining the two we see that the limit of resolution is expressed by the following equation:

$$di' = \frac{t}{a} d\mu = \frac{\lambda}{a},$$

or

$$t = \frac{\lambda}{d\mu}. \tag{40}$$

From this equation it appears that, if it is desired to analyze or resolve with a prism two beams of light which proceed from the same source and whose indices of refraction are μ and $\mu + d\mu$ respectively, it will be necessary to use a prism whose thickness t, as defined above, is at least equal to $\frac{\lambda}{d\mu}$.

It is often more convenient to have this expression in terms of λ and $d\lambda$ instead of λ and $d\mu$. The value of $d\mu$ in terms of $d\lambda$ may be obtained from equation (36), and when substituted in (40) there results

$$t = \frac{\lambda^4}{2Bd\lambda}. \tag{41}$$

The dispersion of a prism, denoted by D, may be defined as the ratio of a change in deviation $d\delta$ to the corresponding change in wave length $d\lambda$, that is,

$$D = \frac{d\delta}{d\lambda} = \frac{d\delta}{d\mu} \cdot \frac{d\mu}{d\lambda}.$$

But

$\delta = i + i' - A$, therefore, for constant A and i, and in consideration of equation (37),

$$\frac{d\delta}{d\mu} = \frac{di'}{d\mu} = \frac{\sin A}{\cos i' \cos r}.$$

Hence

$$D = \frac{\sin A}{\cos i' \cos r} \frac{d\mu}{d\lambda}. \qquad (42)$$

If the incident beam contains all possible wave lengths, that part of it which corresponds to each particular wave length will, on account of the dispersion of the prism, form its own particular image of the source at the focus of a lens suitably placed behind the prism. These images will be a series of parallel bright lines which overlap and form a bright band of light. Such a band is called a spectrum. The spectrum is said to be the purer, the less the successive images which unite to form it overlap. Since each of these elementary bands is produced by waves whose lengths vary over a small interval $d\lambda$, we may use this change in wave length within the band as a convenient measure of the purity, that is, we may define purity, denoted by P, as the reciprocal of $d\lambda$. Or, better, if we choose λ as the unit of measure, the purity may be defined as

$$P = \frac{\lambda}{d\lambda}. \qquad (43)$$

Let θ be the angular width of the elementary image due to wave length λ alone. The center of this image will receive light from the neighboring images whose centers lie within a range of $\frac{1}{2}\theta$ on either side of it, that is, from those images included within a region $d\delta = \theta$. Hence, since the change in wave length in this interval is $d\lambda$, the illumination in the center of each elementary image will be produced by wave lengths varying from $\lambda - \frac{1}{2}\,d\lambda$ to $\lambda + \frac{1}{2}\,d\lambda$. To get a value for $d\lambda$ divide the equation $d\delta = \theta$ by $d\lambda$. We have, since the interval is small,

$$\frac{d\delta}{d\lambda} = \frac{\theta}{d\lambda}.$$

But

$$\frac{d\delta}{d\lambda} = D.$$

Therefore,

$$\frac{1}{d\lambda} = \frac{D}{\theta},$$

and

$$\frac{\lambda}{d\lambda} = \frac{\lambda D}{\theta}. \tag{44}$$

But from equation (35)

$$\theta = di' = \frac{\cos i \cos r'}{\cos i' \cos r}\,di.$$

Therefore, in consideration of equations (42), (43), and (44),

$$P = \frac{\lambda}{di}\,\frac{\cos i' \cos r}{\cos i \cos r'}\,D = \frac{\sin A}{\cos i \cos r'}\,\frac{d\mu}{d\lambda}. \tag{45}$$

When the source is narrow, θ is the limit of resolution of the opening, that is, $\theta = \frac{\lambda}{a}$; therefore, in this case, equation (44) becomes

$$\frac{\lambda}{d\lambda} = aD.$$

Under this condition $\frac{\lambda}{d\lambda}$ is called the resolving power of the prism. Denoting it by R we have,

$$R = aD. \tag{46}$$

From this it appears that the resolving power of a prism of given thickness is directly proportional to both the width of the incident beam and the dispersion of the prism.

In the position of minimum deviation equation (42) reduces to

$$D = \frac{t}{a}\frac{d\mu}{d\lambda}.$$

But $R = aD$, therefore, in consideration of equation (36),

$$R = t\frac{d\mu}{d\lambda} = t\frac{2B}{\lambda^3}. \tag{47}$$

From this it appears that, with a given width of beam, the resolving power of a prism is proportional directly to the thickness of the prism, and inversely to the cube of the wave length.

For further discussion of the properties of prisms and prism spectroscopes the student is referred to Kayser, *Handbuch der Spectroscopie, 1,* p. 253 seq. and 490 seq., Leipzig, 1900, where a complete bibliography will be found. Also important discussions by Helmholtz, *Physiologische Optik,* 2d Ed., p. 290 seq., Leipzig, 1896; Rayleigh, Phil. Mag. (5), *8,* pp. 261, 403, 477, 1879; *9,* p. 40, 1880; also in his *Collected Works, 1,* p. 415 seq.; Czapski, Winkelmann's *Handbuch II, 1,* p. 152 seq.

Experiments

I. Determine the Index of Refraction of a Prism

Apparatus.—The spectrometer consists of a collimator *SA* (Fig. 29) and a telescope *BE*, both mounted at right angles to a vertical axis *CD* and movable about that axis, and a graduated circle *C*, upon which the angle between the telescope and collimator can be read with the help of the vernier *v*. Both the telescope and the collimator should be so mounted as to be movable about horizontal axes *aa*, so that they may be adjusted to have their collimation axes parallel to each other and perpendicular to the axis *CD*. It is usually possible to clamp the collimator and the graduated circle to the base of the instrument. The clamp which fastens the telescope to the graduated circle contains a tangent screw which permits a slow motion of the telescope. The center of the graduated circle should coincide

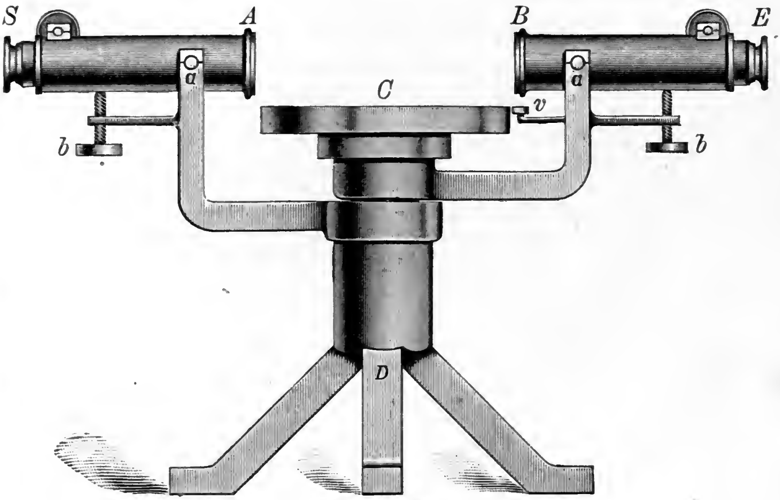

FIGURE 29

with the axis *CD*, and it is useful to have a small hole bored through that center parallel to that axis.

ADJUSTMENTS.—*The axes of the telescope and collimator must intersect the vertical axis of the instrument.* If the telescope and collimator are rigidly mounted so that they are not movable about a vertical axis, we assume that this adjustment has been properly made by the maker. In any case the adjustment can be tested with sufficient accuracy by sticking a straight metal rod into the hole in the center of the graduated circle, and observing this rod through the slit of the collimator. To adjust the telescope the eyepiece must be removed and the rod viewed as in the case of the collimator. When the adjustment is correct. the rod seems to cover the center of the objective.

Telescope and collimator must be focused for parallel rays. The telescope may be pointed out of the window and focused on a distant object, but it is better to place a prism on the prism table, set the telescope perpendicular to one of its faces, and observe the reflection upon the prism face of the cross-hairs in the telescope. In order to do this easily the telescope should be fitted with a Gauss eyepiece, or have some other provision for illuminating the cross-hairs. When the cross-hairs are illuminated and the telescope is perpendicular to the prism face, two sets of cross-hairs will be visible in the field of view. The two should be brought nearly into coincidence and the telescope so focused that when the eye is moved about behind the eyepiece the two sets show no parallax with respect to each other. Having thus focused the telescope the collimator is easily adjusted for parallel rays by observing, in the telescope, the image of the slit, and focusing the collimator until that image shows no parallax with respect to the cross-hairs. The adjustment can also be effected by placing on the prism table of the instrument a plane-parallel plate of glass in such a way that the light from the slit is reflected to the telescope at oblique incidence. The reflected image of the slit in the telescope will appear double unless both telescope and collimator are focused for parallel rays.

Telescope and collimator must be perpendicular to the axis of the instrument and the surfaces of the prism must be parallel to that axis. To attain this the prism should be mounted on a leveling tripod, so that one of its faces AB is perpendicular to the line joining two of the leveling screws. The prism so mounted is then set upon the prism table a little to one side of the center of the graduated circle, so that when the telescope and collimator are brought into line and the prism face AB is made parallel to that line, part of the field of view of the telescope is unobstructed, and it is possible to have an image of the slit formed in the focal plane of the eyepiece. The slit should then be turned so that it is horizontal, and the telescope and collimator adjusted by means of the screws *bb* until the image of the slit falls upon the intersection of the cross-hairs in the telescope. The telescope should then be turned about the axis of the instrument through any angle, and the prism turned through such an angle that the light from the slit is so reflected from the prism face *AB* that the image of the slit again appears in the field of view of the telescope. In general this image of the slit will not fall upon the intersection of the cross-hairs. It should be made to do so by adjusting the leveling screws of the prism. The telescope is then turned so as to be perpendicular to the face *AB* of the prism. In order to set it accurately perpendicular to that face, it is necessary to illuminate the cross-hairs and observe their image reflected from the face *AB*. When this image coincides with the cross-hairs themselves, the telescope is perpendicular to the face of the prism. The cross-hairs and their image should be brought into coincidence by adjusting the telescope. The telescope and prism should then be brought into their original positions, and the image of the slit observed directly. It will in general no longer fall upon the intersection of the cross-hairs. It should be made to do so by adjusting the collimator by means of the screw *b*. This operation must be repeated until the image of the slit remains on the intersection of the cross-hairs in all three of the positions. When this is accomplished the telescope and collimator are perpendicular to the axis of the instrument, and the prism face *AB* is parallel to that axis.

It is then necessary to make the other face of the prism, namely *AC*, parallel to the axis *CD*. To accomplish this it is merely necessary to set that face perpendicular to the telescope. It will be remembered that the face *AB* of the prism was set perpendicular to the line which joins two of the leveling screws of the tripod upon which the prism is mounted. The adjustment of the face AC should be made entirely with the other screw of the tripod, because a movement of this third screw of the tripod will not tip the face AB, but only rotate it about an axis perpendicular to its plane.

MEASUREMENTS.—It is first necessary to measure the angle A of the prism. The telescope is set perpendicular to the face AB of the prism, and its position read upon the graduated circle. It is then set perpendicular to the face AC of the prism, and its position read in the same way. The angle through which the telescope turns in making these two settings is the supplement of the angle of the prism. To measure the angle δ of deviation, turn the slit so that it is vertical, and illuminate it with monochromatic light,—or better, with sunlight. Place the telescope and collimator in line, bring the image of the slit upon the cross-hair, and read the position of the telescope upon the graduated circle. The prism should then be introduced and placed in such a position that the angle of incidence of the light from the collimator upon it is nearly equal to the angle of emergence. The adjustment for minimum deviation is made by rotating the prism about a vertical axis while observing the spectral image of the slit in the telescope. As the prism is turned that image moves in the field of view, and it will be found that at one particular position of the prism that image is nearer the direct image of the slit than in any other position. The cross-hair is set upon the spectral image when the prism is in this position, and the angle through which the telescope has been turned read upon the graduated circle. If sunlight is used, the cross-hair is set upon the particular Fraunhofer line for which the corresponding index is required. Having thus measured the angle of the prism and the angle of deviation, the index of refraction is easily found with the help of equation (34).

EXAMPLE

A hollow prism filled with bisulphide of carbon was used. The angle of the prism was 59° 50′ = A. The following deviations were observed for the lines of the solar spectrum as indicated:

LINE	WAVE LENGTH	DEVIATION	INDEX (CALCULATED)
B	$6870 \cdot 10^{-7}$	47° 34′ 24″	1.6060
C	$6563 \cdot 10^{-7}$	47° 53′ 0″	1.6192
D	$5890 \cdot 10^{-7}$	48° 46′ 46″	1.6285
E	$5270 \cdot 10^{-7}$	50° 0′ 12″	1.6409
F	$4861 \cdot 10^{-7}$	51° 14′ 46″	1.6532
G	$4308 \cdot 10^{-7}$	53° 14′ 28″	1.6727

The temperature was 18°C.

II. DETERMINE THE DISPERSION CURVE

Apparatus and adjustments as in Experiment I.

MEASUREMENTS.—It is merely necessary to measure the index of refraction in the way described above, for several different wave lengths, and then to plot the indices as ordinates with the corresponding wave lengths as abscissae. If it is desired to determine the constants of the Cauchy dispersion equation [equation (36)], any two values of the index of refraction with their corresponding wave lengths may be used. It is desirable as a test of the accuracy of the work to measure the indices corresponding to several of the Fraunhofer lines, and to calculate the A and B of the Cauchy equation from each pair of such indices.

EXAMPLE

From the values of μ, obtained in the example of Experiment I, the following values of A and B of equation (36) were obtained:

INTERVAL	A
$B-E$	1.5805
$C-F$	1.5786
$D-G$	1.5777
mean	1.5789 = A
$\therefore B. =$	$1.735 \cdot 10^{-8}$

III. Determine the Resolving Power of the Spectrometer

Apparatus and adjustments as in Experiment I.

Measurements.—Having determined the B of equation (36), it is in addition necessary to measure t, the thickness of the prism. If the incident beam has a rectangular cross-section, it is evident from Figure 28 that the value of t is given by

$$t = \frac{2a \sin \frac{1}{2}A}{\cos i}. \qquad (48)$$

If, however, the entire aperture of the collimator is used, and if a' represent the diameter of that aperture, then, because the cross-section of the beam is circular,

$$a = 0.82a'.*$$

Hence the equation for the determination of t is, in the case of an incident beam of circular cross-section,

$$t = 0.82 \frac{2a' \sin \frac{1}{2}A}{\cos i}. \qquad (49)$$

It is to be noted that the resolving power of a prism is different in different parts of the spectrum.

An interesting check upon equation (47) is obtained by altering a, and consequently t, until a given resolution is obtained. This is accomplished by limiting the width of the beam by two cards held upon the front face of the prism. The distance between the cards should be varied until two known lines are no longer resolved. The two sodium lines are a convenient pair for this purpose. If d is the distance between the cards upon the front face of the prism, when the sodium lines cease to appear double, then it is evident that

$$t = 2d \sin \frac{1}{2}A.$$

* Verdet, *Leçons D'optique Physique,* Vol. I, p. 301 seq.

The value of $\frac{\lambda}{d\lambda}$ in this case is about 1000. It will in general be found that the experimentally determined value of the resolving power is larger than the theoretical one. This is natural, as the theoretical value is the limit of resolution beyond which it is impossible to go, and the fact that the observed limit is somewhat higher merely goes to show the imperfection of our eyes and our instruments.

EXAMPLE

Two cards were held over the objective of the collimator and moved until the two sodium lines just ceased to be resolved. In this position their distance apart was $a = 3.5$ mm. From equation (48), $t = 5.98$, since $i = \frac{1}{2}(\delta + A) = 54° 18'$. Hence, from equation (47), using the value of B given on p. 101,

$$R = \frac{2Bt}{\lambda^3} = 1016.$$

The theoretical value of R for the two D lines is $\frac{\lambda}{d\lambda} = 983.$

To get some idea of the resolving power of the instrument without stops in different parts of the spectrum, use was made of equation (46) in connection with the observations in Experiment I. The value of R varies over the intervals chosen, hence the numbers obtained are only averages over the interval.

INTERVAL	$d\delta$		$d\lambda$	$\frac{d\delta}{d\lambda} = D$
$C - D$	$53' 46'' = 3226'' = .01564$	radians	.0000673	232
$F - G$	$1° 59' 42'' = 7182'' = .03482$	"	.0000553	630

Since in this case, if a' represent the diameter of the objective, $a = 0.82a'$, and since a' was 40 mm., $a = 32.8$, therefore,

INTERVAL	$R (= aD)$
$C - D$	7610
$F - G$	20700

The extreme values of R for the lines B and G are added. The calculation is made from equations (47) and (49).

Line	R
B	6020
G	24400

VIII

TOTAL REFLECTION

Theory

In Chapter VII we have deduced the general equation (32) for the prism. Mention was also made of the fact that when the internal angle of incidence r' becomes so large that $\sin r' = \frac{1}{\mu}$ the light can not escape from the prism, but is totally reflected. This limiting angle of total reflection may then be used to determine μ. At that angle $\sin i' = 1$, and hence (32) becomes

$$1 = \sin A \sqrt{\mu^2 - \sin^2 i} - \cos A \sin i.$$

If this equation be solved for μ^2 we get

$$\mu^2 = 1 + \left(\frac{\cos A + \sin i}{\sin A}\right)^2. \qquad (50)$$

Hence, to determine μ by this method it is necessary to measure the refracting angle A of the prism, and the angle i of incidence.

If the totally reflecting surface be covered with a liquid of index μ', then at the limiting angle of total reflection $\sin i'' = 1 = \frac{\sin i'}{\mu'}$, and equation (32) becomes for this case

$$\mu' = \sin A \sqrt{\mu^2 - \sin^2 i} - \cos A \sin i. \qquad (51)$$

Experiments

I. Determine the Index of Refraction by Total Reflection

The apparatus and adjustments are those of the last chapter. The prism used should be for convenience a total reflecting prism of 90°.

MEASUREMENTS.—Place the prism on the prism table of the spectrometer and allow diffused light to fall upon the face AB as shown in Fig. 30. Observe the reflected light with the telescope and find that position of the prism and telescope in which half of the field of view is bright and the other half less bright. Set the cross-hair of the telescope on the line of separation between these two portions of the field and read the position of the telescope on the circle. Then set the telescope perpendicular to the face BC, and again read the circle. The difference between these two readings will, because of the symmetry of the figure, be

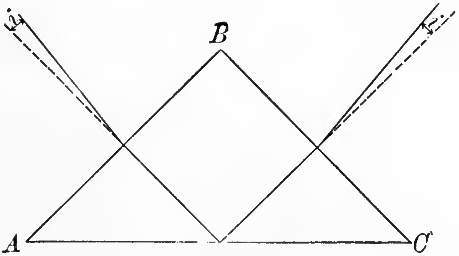

FIGURE 30

the angle i, which corresponds to the angle $i' = 90°$. Note that in the case drawn the refracting angle of the prism is at A, and therefore since it and the incident ray are on opposite sides of the normal, i is positive.

If it is desired to determine the index μ' of a liquid ($\mu' < \mu$) wet the surface AC with the liquid. It will probably be necessary to press a film of the liquid between the prism face and a piece of glass. Observe again the limiting angle. In this case the angle i will probably be negative. The index can then be calculated from equation (51).

The same prism can be used to measure indices of solid substances less dense than the prism itself by pressing between the surface AC and the substance whose index is to be determined a liquid like oil of cassia, whose index is greater than those of the substance and the prism. Equation (51) holds for this case also, the symbols having the same meaning as in the preceding case.

EXAMPLE

Using a total reflection prism and sodium light it was found that $A = 45°$, $i = 5° \, 30'$. Hence, from equation (50) . . . . $\mu = 1.5133$.

As a check the deviation δ at minimum deviation was also measured, the result being $\delta = 25° \, 47'$. Hence, from (34) . . . . $\mu = 1.5134$.

The surface AC was then coated with a water film, and the value of i determined as $i = -25° \, 50'$. Hence, from (51) . . . . $\mu' = 1.3329$.

Using a denser total reflection prism and sodium light it was found that $A = 45°$, $i = 11° \, 2'$. Hence, as above . . . . $\mu = 1.6170$.

The hypothenuse of the prism was then coated with a thin layer of cassia oil, and a plate of the glass whose index of refraction was determined with the interferometer (page 65), was placed upon the oil. The value of i was then measured as $i = -42° \, 57' \, 20''$. Hence, from (51) . . . . $\mu' = 1.5189$.

For other methods of determining indices from the angle of total reflection, the student is referred to Kohlrausch, Wied. Ann. *4*, p. 1. Abbe, "Apparate zur Bestimmung des Brechungsvermögen," Jena, 1874. Pulfrich, Zs. für Instrk. *7*, p. 55, 1887; *8*, p. 47, 1888; *15*, p. 389, 1895; *19*, p. 9, 1899. Czapski, Zs. für Instrk. *10*, p. 246, 1890.

IX

THE DIFFRACTION GRATING

Theory

The conditions under which the maxima and minima of illumination are formed by the interference of the light waves that have passed through two parallel, rectangular openings or slits, have been discussed in Chapter II and are expressed by equations (4) and (5), page 20. If we now consider, in the same way, a large number n of such equal, equidistant, parallel slits, all of which lie in one plane, we shall find, in general, that the position of a maximum will be determined by an equation similar to equation (5) in which the 2 is replaced by n; that is, the condition for a maximum will be

$$nd \sin \theta = nm\lambda. \tag{52}$$

Similarly the condition for a minimum will be

$$nd \sin \theta = (nm + 1)\lambda \tag{53}$$

Such a series of parallel slits is called a plane grating.*

In general the positions of the maxima and minima will be determined by these equations. There are certain cases, however, when one or more of the maxima may be wanting. According to equation (2) each individual opening has a minimum in a direction determined by $a \sin \theta = m\lambda$. But from equation (52) the maxima are determined by $d \sin \theta' = m'\lambda$. When the θ's are the same,

$$\frac{a}{d} = \frac{m}{m'}. \tag{54}$$

* Cf. Drude, *Theory of Optics*, p. 222 seq., Longmans, 1902. Kayser, *Handbuch der Spectroscopie, I*, p. 416 seq., Leipzig, Hirzel, 1900. Rayleigh, Phil. Mag. (4) *47*, p. 193, 1874. Quincke, Pogg. Ann. *146*, p. 1, 1872. Rowland, Phil. Mag. (5) *35*. Mascart, Ann. éc. norm. *1*, p. 219, 1864; *Traité d'optique, 1*, p. 364 seq., Paris, 1889.

Hence, when this condition happens to be fulfilled the corresponding maximum is wanting.

We have thus far considered that the incident beam fell normally upon the grating. The introduction of this condition has made the explanation of the equations which determine the positions of the maxima and minima very simple. It is, however, immaterial whether the difference of phase nm between the outside rays of the beam is introduced entirely behind the grating, or part of it before the grating and the rest behind it. Part of this difference of phase will be introduced before the grating if the incident beam fall upon it at some angle other than a right angle. Then let AB (Fig. 31) represent the grating,

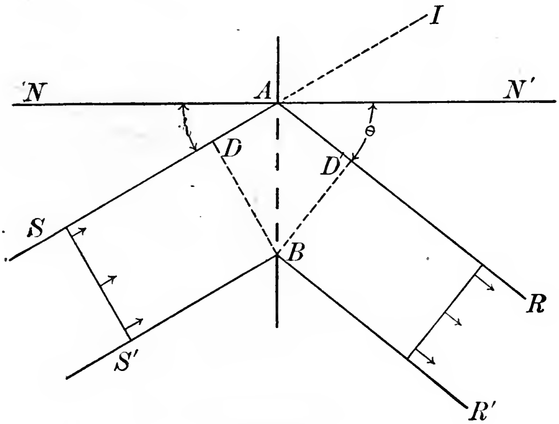

FIGURE 31

and $SABS'$ the incident beam, making an angle $NAS = i$ with the normal AN to the grating. Let AR be the direction corresponding to a maximum; call the angle which AR makes with the normal AN' to the grating, the angle of diffraction, and denote it as above by θ. The difference of path between the two outside rays AR, BR' is now represented by $DA + AD'$. This must, therefore, be substituted for AD in the deductions that lead to equation (52). But $DA = nd \sin i$, and $AD' = nd \sin \theta$, therefore,

$$DA + AD' = nd (\sin i + \sin \theta),$$

and equation (52) becomes under these circumstances

$$nm\lambda = nd (\sin i + \sin \theta).$$

Should the direction AR fall between the normal AN', and the direct ray SAI, the difference of path between the outside rays AR' and BR' would be given by $AD - BD'$, and the equation reduces to

$$nm\lambda = nd (\sin i - \sin \theta).$$

Hence, the complete equation is, after dropping the n,

$$m\lambda = d (\sin i \pm \sin \theta), \tag{55}$$

whereby it is to be remembered that the negative sign is to be used when the diffracted ray AR lies on the same side of the normal as the direct ray AI.

The dispersion of a grating, denoted by D, may be defined as in the case of the prism, as the ratio of the change in the angle of diffraction $d\theta$ to the corresponding change in the wave length $d\lambda$, that is, $D = \frac{d\theta}{d\lambda}$; or, from equation (52),

$$D = \frac{m}{d} \frac{1}{\cos \theta} \tag{56}$$

From this equation it is seen that the dispersion varies directly with m, that is, with the order of the spectrum. Hence the spectrum of the second order will be twice as long as that of the first order. Further, the dispersion is inversely proportional to the grating space d. For this reason makers of gratings have made it their aim to make d as small as possible. The dispersion also varies inversely as the cos θ. If $\cos \theta = 1$, i.e., $\theta = 0$, the diffracted ray is normal to the grating and the dispersion is a minimum. Since the cosine varies very little from the value unity when the arc varies through a small angle on either side of zero, the dispersion curve is, under this condition, most nearly a straight line. Hence the spectrum formed under this condition is called a normal spectrum.

The purity of grating spectra may be measured in the same way as that of the prismatic spectra, that is, as before, $P = \frac{\lambda}{d\lambda}$, or, when the source is infinitely narrow, the resolving power, R equals $\frac{\lambda}{d\lambda}$. The value of $\frac{\lambda}{d\lambda}$ for gratings may be determined as follows: By equation (52) the direction of a maximum for wave length $\lambda + d\lambda$ is, for normal incidence, determined by

$$nd \sin \theta = nm\ (\lambda + d\lambda),$$

and by equation (53) that of the minimum of the same order for wave length λ by

$$nd \sin \theta' = (nm + 1)\lambda.$$

The limit of resolution is reached when θ and θ' in these two equations are identical. Hence,

$$nm\ (\lambda + d\lambda) = (nm + 1)\lambda,$$

or

$$\frac{\lambda}{d\lambda} = R = mn. \tag{57}$$

Two conclusions can be drawn from this equation. First, if m is constant, that is, if we observe spectra of the same order, the resolving power of different gratings will be proportional to the number of lines which they contain.

The second conclusion may be reached as follows: From equation (52) we have, for normal incidence,

$$nm = \frac{nd}{\lambda} \sin \theta,$$

but $nd = l$ is the total width of the grating, therefore,

$$\frac{\lambda}{d\lambda} = \frac{l}{\lambda} \sin \theta.$$

If in this equation $\sin\theta$ is constant, that is, if the spectra are observed at a constant angle of diffraction, the resolving power of all gratings of the same width is the same, and is consequently independent of the number of lines.

Further, $l\cos\theta$ is the width of the diffracted beam, denoted by a', that is, $l\cos\theta = a'$; or,

$$\cos\theta = \frac{a'}{l}.$$

But by equation (56)

$$D = \frac{m}{d}\,\frac{1}{\cos\theta} = \frac{nm}{nd}\,\frac{l}{a'};$$

or, since $nd = l$, and $nm = R$,

$$D = \frac{R}{a'}, \quad \text{or,} \quad R = a'D, \tag{58}$$

exactly as in the case of the prism. It is to be noted, however, that a' is the width of the diffracted beam, not of the incident beam.

Experiments

I. Determine the Constant of the Grating

Apparatus.—The spectrometer is used in these experiments as in those described in the last chapter. The grating should be so mounted that it can be placed upon the prism table of the instrument and be adjustable about two horizontal axes, one of which is parallel, and the other perpendicular to its surface.

Adjustments.—The grating should be so set upon the prism table that its surface lies directly over the center of the graduated circle of the instrument and its lower edge is parallel to the prism table. The only further adjustment is to make its surface parallel to the axis of the instrument. This is done, if the instrument has been adjusted as described in the last chapter, by merely setting the grating perpendicular to the telescope. If the instrument has not been so adjusted, the process there described must be used, the surface of the grating taking the place of the face AB of the prism.

MEASUREMENTS. — Allow monochromatic light, — or better, sunlight,—to fall upon the slit. If a transmission grating is to be used, it is better to set it normal to the incident light. To attain this, set the cross-hairs of the telescope on the undiffracted image of the slit. Then set the grating perpendicular to the telescope by means of the reflection of the cross-hairs in the usual way. Read the position of the telescope at this point. Then turn the telescope until the first spectral image falls on the cross-hairs, and again read its position. The angle through which it has been turned in this operation will be the angle θ of diffraction for the spectrum of the first order ($m = 1$). For the sake of a check, this angle should be read on both sides of the central undiffracted image of the slit. It is also well to determine the angles of diffraction for spectra of several different orders. The constant of the grating is then calculated with the help of equation (52), the wave length of the light used being taken from the table of wave lengths. If the constant of the grating is known, the same measurements can be used to determine wave lengths. If the solar spectrum is used, of course a given Fraunhofer line is used instead of the spectral image of the slit.

If the grating used is a reflection grating, it will probably be impossible to observe with perpendicular incidence, because the spectra of lower order fall so near the collimator that they can not be observed in the telescope. In this case, the grating is set up so that the light is incident at any angle i. The telescope is then set perpendicular to the grating, and its position read on the graduate circle. The angle of incidence is determined by setting the cross-hair upon the directly reflected image of the slit, and the angles of diffraction by setting it on the spectral images. The angles of diffraction are, of course, all measured from the normal to the grating, and their sign is minus if they lie on the same side of the normal as the direct reflected image. The grating space d is then easily obtained from equation (55), the wave length being taken from the table.

EXAMPLE

Using sunlight as a source, a plane grating was adjusted in the spectrometer. The angle of incidence was $i = 48° \; 17' \; 30''$. The deviations for the first three spectra for three of the solar lines were observed as follows:

LINE	λ	θ_1	θ_2	θ_3
C	$6563 \cdot 10^{-7}$	21° 58′ 20″	0° 3′ 30″	21° 48′ 10″
D_2	$5890 \cdot 10^{-7}$	24° 19′ 20″	4° 21′ 10″	14° 58′ 10″
F	$4861 \cdot 10^{-7}$	28° 2′ 40″	11° 10′ 0″	4° 48′ 10″

Hence from equation (55) the following values of $n_0 \left(= \frac{1}{d}\right)$ were calculated:

LINE	n_0
C	567.6
D_2	568.8
F	568.9
Mean	568.4

II. FIND THE RESOLVING POWER OF THE GRATING

Apparatus and adjustments as in Experiment I.

Having determined the constant d, the resolving power is merely $m \frac{l}{d}$, in which l is the length of the ruling and m the order of the spectrum. As in the case of the prism the resolution can be tested by finding the width of beam a which is just sufficient for the resolution of a pair of known lines like the two sodium lines.

EXAMPLE

A slit was placed over the objective of the collimator and closed until the two sodium lines in the first spectrum were no longer resolved. The width of the slit was then found to be $a = 1.28$ mm. Hence, the length of grating used was $\frac{a}{\cos i} = 1.924$ mm., and the number of lines in this width is $n \cdot 1.924 = 1094 = R$, the resolving power according to equation (57).

The dispersion was then calculated from the equation $D = \frac{d\theta}{d\lambda}$.

INTERVAL	$d\theta$		$d\lambda$	D
$C - D$	$8460'' = .0410$ radians		0.0000673	610
$D - F$	$13400'' = .0664$	"	0.0001029	645

From equation (56) we get for D_2 in the first spectrum

$$D = \frac{568.4}{\cos\theta} = 613.$$

The length of grating, which was just sufficient to resolve the two sodium lines, was $l = 1.924$ mm. For the first spectrum we have $\theta = 21°\ 58'\ 20''$. Hence, since $l \cos\theta = a'$, the width of the diffracted beam, we have as the resolving power, according to equation (58),

$$R = a'D = 1.784 \times 613 = 1093.$$

The total width of the grating was 40 mm. Hence the resolving power in the first spectrum is $40 \times 568.4 = 22736$. It will be noted that this number is the same for all parts of the spectrum.

For information as to the production and errors of gratings cf. Rowland, Phil. Mag. (5) *13*, p. 469; Nature *26*, p. 211. Rayleigh, Phil. Mag. (4) *47*, p. 81, B. A. Reports, 1872; Nature *54*, p. 332.

X

THE CONCAVE GRATING

Theory

In using a plane grating, as in using a prism, lenses are necessary to make the light from the source parallel and to form an image for observation. Inasmuch as lenses absorb much of the total energy of vibration that falls upon them, it is highly desirable to avoid their use if possible. That this is possible has been beautifully demonstrated by Rowland,* and his concave gratings, by which he has been able to accomplish this end, have justly become the standard instruments for spectroscopic work.

As the name implies, a grating of this kind is made by ruling fine lines upon a concave mirror. The use of lenses in connection with the grating is thus avoided, because the concave mirror will form real images by itself.

Since the series of openings thus formed are on the surface of a sphere, they can not be treated as a series of parallel slits. The theory of the concave grating has been fully treated by Rowland and others. The following simple deduction of its leading characteristics is abridged from a complete discussion of it by Runge:†

Suppose we have a source of monochromatic light at A (Fig. 32), and a curved reflecting surface MM' at O. Let P be any point upon the surface MOM'. An image of A will be formed at A' whenever the distance $AP + PA'$ is the same for every

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* Am. Jour. Sci. (3) *26*, p. 87; Phil. Mag. (5) *16*, p. 197.

† Winkelmann's *Handbuch,* II, *1,* p. 407. Kayser, *Handbuch der Spectroscopie,* I, p. 452.

point P on the curved surface MM', or whenever the rays reflected from every point P arrive at A' in the same phase. The first of these conditions will clearly be fulfilled if the reflecting surface is an ellipsoid of revolution having A and A' as foci.

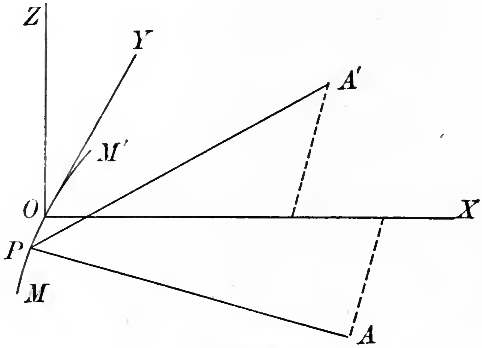

FIGURE 32

Let us now conceive a series of such ellipsoids of revolution constructed about A and A' as foci under the condition that $AP + PA'$ shall increase by $\frac{\lambda}{2}$ for each such successive surface. If the mirror be now supposed spherical, its surface will be divided by its intersections with this series of ellipsoids into a large number of zones. From any two adjacent zones light will reach A' in opposite phase, because of the condition by which the ellipsoids were drawn. If the radius of curvature of the mirror MM' and the distances AP and PA' are great with respect to λ, these zones will have very nearly equal width, and will, therefore, extinguish each other's effect at A' and produce there a minimum of illumination. If, however, this balance in the effect of the zones is partly destroyed by so scratching a line on the mirror within the region covered by every pair of adjacent zones that some light shall arrive from each pair of zones in the same phase, we shall obtain illumination there. It is to be noted that the condition for illumination at A' is that the light from every pair of zones shall arrive there in the same phase. Therefore, we may have between the trains of waves from any adjacent pairs of zones a difference of phase of any number of whole waves, that is, of $m\lambda$.

It is now necessary to find how these lines must be drawn to produce the desired effect.

Let the spherical mirror MM' now be referred to a set of rectangular axes, so that its vertex is tangent to the xy plane at the origin O. Call the radius of curvature ρ. The equation of the mirror so placed is

$$x^2 + y^2 + z^2 - 2\rho x = 0.$$

Let A and A' lie in the xy plane, and let their coördinates be represented by a, b, and a', b', respectively. Then

$$\overline{AP}^2 = (x-a)^2 + (y-b)^2 + z^2;$$

or, by performing the indicated operations and letting $a^2 + b^2 = r^2$,

$$\overline{AP}^2 = r^2 - 2ax - 2by + x^2 + y^2 + z^2.$$

Upon substituting in this equation for $2ax$ its value $\frac{a}{\rho}(x^2 + y^2 + z^2)$ taken from above, it becomes

$$\overline{AP}^2 = r^2 - 2by + \left(1 - \frac{a}{\rho}\right)y^2 + \left(1 - \frac{a}{\rho}\right)z^2 + \left(1 - \frac{a}{\rho}\right)x^2.$$

Remembering that x is of the second order with respect to y and z, and neglecting terms of the third order, this reduces to

$$AP = r - \frac{b}{r}y + \frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right)y^2 + \frac{1}{2r}\left(1 - \frac{a}{\rho}\right)z^2.$$

The value of PA' may be obtained in precisely the same manner. It will be found to differ from that of AP only in that

a', b', r', appear in it in place of a, b, r, respectively. We may, therefore, write

$$AP + PA' = r + r' - \left(\frac{b}{r} + \frac{b'}{r'}\right)y$$
$$+ \left[\frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right) + \frac{a'}{2r'}\left(\frac{a'}{r'^2} - \frac{1}{\rho}\right)\right]y^2$$
$$+ \left[\frac{1}{2r}\left(1 - \frac{1}{\rho}\right) + \frac{1}{2r'}\left(1 - \frac{1}{\rho}\right)\right]z^2.$$

This equation reduces to a much simpler form for certain particular positions of A and A'. In order to find the conditions which determine these particular positions of A and A', let us suppose that the mirror is so limited that the terms in z^2 may be neglected. The terms in y^2 will vanish also if

$$\frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right) + \frac{a'}{2r'}\left(\frac{a'}{r'^2} - \frac{1}{\rho}\right) = 0.$$

This will be true if $r^2 = a\rho$ and $r'^2 = a'\rho$, that is, if A and A' lie on a circle whose center lies on the axis of x at a distance $\frac{\rho}{2}$ from the origin. When this simple condition is fulfilled, the equation reduces to

$$AP + PA' = r + r' - \left(\frac{b}{r} + \frac{b'}{r'}\right)y.$$

We are now in a position to answer the question as to how the lines must be drawn on the mirror to produce the required effect. Let e represent the difference in millimeters between the values of the y's which correspond to the nth and the $n+1$st lines. The distance between the centers of their adjacent zones will also be e. Since when A and A' are fixed, as we have supposed them, $r + r'$ is independent of the position of P on the mirror, the question as to the illumination at A' depends for its answer upon the value of the term containing y. The condition for illumination at A' is that the light arriving there from any zone, the nth for instance, shall differ in phase from that arriving from the next zone, the $n+1$st, by a whole number of wave lengths $m\lambda$. This difference of phase will accordingly be determined by

$$\left(\frac{b}{r}+\frac{b'}{r'}\right)(y+e)-\left(\frac{b}{r}+\frac{b'}{r'}\right)y;$$

or, since this must be equal to $m\lambda$, by

$$e\left(\frac{b}{r}+\frac{b'}{r'}\right)=m\lambda. \qquad (59)$$

Hence e, the difference in the values of the y's for two consecutive lines, is constant. That is, the lines must be equally spaced along a chord of the mirror perpendicular to the x axis.

If conditions are so arranged that the image formed at A' is observed only when A' lies upon the axis of x, b' becomes zero, and equation (59) reduces to

$$e\frac{b}{r}=m\lambda. \qquad (60)$$

But $\frac{b}{r}=\sin i$, where i is, as usual, the angle of incidence. Hence, equation (60) may be written in the form

$$e\sin i=m\lambda. \qquad (61)$$

The distance AA' is, when $b'=0$, equal to $\rho\sin i$, or taking equation (61) also into account,

$$AA'=\frac{\rho m\lambda}{e}. \qquad (62)$$

Hence AA' is proportional to the wave length λ.

The use of the concave grating under the condition $b'=0$ is especially to be recommended, because, when the point of observation lies upon the normal to the grating, the dispersion is, as in the case of the plane grating, very nearly constant throughout a considerable range on either side of the normal.

Equations (56) and (57), expressing the dispersion and resolving power of a plane grating, apply to the concave grating also.

Experiments

I. Determine the Constant of the Grating

Apparatus.—As shown above it is best to mount the concave grating in such a way that its center of curvature coincides with the point of observation. This may be accomplished in every position of the grating if the slit, the center of the grating, and the point of observation lie upon a semicircle whose diameter passes through the center of curvature and the center of the grating, and is equal in length to the radius of curvature of the grating. The best method of fulfilling this condition is that

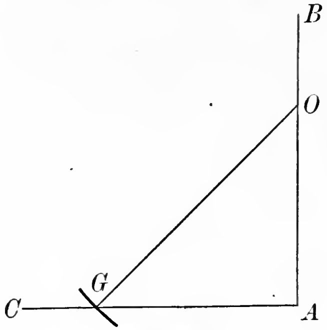

Figure 33

adopted by Rowland. The grating and the observing eyepiece are firmly mounted on the ends of a rigid arm *GO* (Fig. 33), whose length is equal to the radius of curvature of the grating. This arm is supported at each end upon a small carriage, and these carriages may be moved along the tracks *AB* and *AC*, respectively. If these two tracks are accurately perpendicular to each other, and if the slit is placed at *A*, then in any position of the arm *GO* the points *G*, *O* and *A* will lie on a semicircle, which fulfills the required conditions.

Adjustments.*—The tracks *AB* and *AC* must be straight,

*Cf. Ames, Phil. Mag. (5) *27*, p. 369. *Astronomy and Astrophys. 11*, p. 28, 1892.

level, and perpendicular to each other. These tracks are usually mounted upon heavy steel or wooden beams with adjusting screws so that they can be raised or lowered and moved laterally. The adjustment for straightness can be made by tightly stretching beside the track a fine piano wire or silk thread, and then bringing the track parallel to this either with the adjusting screws or by filing.

The tracks may be made horizontal with the help of a good spirit level, or with a cathetometer set up at some point, as *D*, and focused on the upper edge of the track.

The tracks may be made perpendicular to each other by the 3-4-5 rule, i.e., by measuring from their point of intersection a distance of 3 units on one track and of 4 units on the other; the distance along the hypothenuse between the two points thus determined, should be made to equal 5 units. One of the beams upon which the track is fastened is usually so mounted as to allow of a lateral motion, in order to permit this adjustment to be made.

The arm *GO*, mounted upon its carriages, should then be put in place. The axes about which the arm turns with respect to the carriages should be directly over the center of the track, and be marked at the top by a small hole or point. It is assumed that the maker of the carriages has attended to this. The grating should then be set in place. The center of its surface should be tangent to the axis of the carriage over which it stands. The grating holder should be adjustable in every direction, i.e., about each of the three rectangular axes of Figure 31.

The center of curvature of the grating should fall upon the axis about which the arm *GO* turns at the end *O*. To accomplish this, a Gauss eyepiece is mounted at *O*, so that its cross-hairs are directly over the hole or point which marks that axis. The centers of the eyepiece and the grating should lie in the same horizontal plane, i.e., the distance of both from the top of the track should be the same. The adjustment then divides itself into two parts: First, the normal erected at the center of the grating must intersect the axis at the observing end of the arm, and must be parallel to the plane of the tracks; and, second, the cross-hairs of the eyepiece and their image formed by the grating must coincide, i.e., the cross-hairs must lie at the center of curvature of the grating. To fulfil the first of these conditions, the grating should first be set by eye, so that the lines upon it are approximately vertical. It should then be turned about the horizontal and vertical axes, which are parallel to the plane tangent to it at its center, i.e., about the y- and the z-axes of Figure 31, until the image of the cross-hairs of the eyepiece nearly coincide laterally and vertically with the cross-hairs themselves. The arm GO must then be adjusted in length until the cross-hairs and their reflected image show no parallax. The arm GO should have been constructed to have very nearly the correct length. It should, however, have a splice in it, which allows an adjustment of a centimeter or so in length.

The slit should then be mounted over the intersection of the tracks, with its center at the same distance from the tracks as the center of the grating. If the other adjustments have been correctly made, the slit should be in focus in the eyepiece at O. In case it is not in focus it should be moved slightly forward or backward in the direction AB. If it is found necessary to move it more than a millimeter or two in order to bring it into focus, it indicates that some of the other adjustments are inaccurate, and it is advisable to verify them.

To be sure that the lines of the grating are perpendicular to the plane of the tracks move the arm GO back and forth. The spectra will pass the point O. The lines are vertical when the spectra remain at the same height above the track at O as the arm is moved.

The slit should be parallel to the lines of the grating. This adjustment is best made with the help of the solar spectrum. The slit should be illuminated with sunlight, and the grating set in such a position that a group of fine lines appears in the eyepiece. While observing these lines the slit should be rotated about a horizontal axis. When the lines are most sharply defined, the slit is parallel to the lines of the grating. This adjustment may be made easier by introducing a narrow opaque object, like a medium-sized wire, horizontally across the slit. The images of the slit in the eyepiece will then be divided into two parts, and will appear to run into points on the two sides of the division. When these points are vertically opposite each other the slit is parallel to the lines of the grating.

MEASUREMENTS.—Illuminate the slit with monochromatic light of known wave length, or better with sunlight,—and move the arm GO until the spectral image of the slit, or a known Fraunhofer line, falls upon the cross-hairs of the eyepiece. Measure the distance AO and the radius of curvature OG of the grating. Now the angle i of incidence is AOG, and the angle of diffraction is equal to zero, hence from equation (61)

$$m\lambda = e \sin i = e \frac{OA}{OG},$$

from which the value of e is readily determined.

The concave grating has been used to make absolute determinations of wave lengths. In these measurements the grating space was determined by dividing the width of the ruling by the total number of lines upon it. The results obtained by different observers by this method differ among themselves by one part in 15,000.* The grating furnishes, however, the most accurate

* The following are the most important absolute determinations of the wave length D_1. 5895.81, Ängström, corrected by Thalén, Nov. Act. Upsal. (3) *12*, p. 1. 5896.25, Müller and Kempf, Publ. Potsdam Obs. *5*. 5895.90, Kurlbaum, Wied. Ann. *33*, p. 159. 5896.20, Bell, Am. Jour. Sci. (3) *33*, p. 167; *35*. p. 265; Phil. Mag. (5) *23*, p. 265; *25*, p. 255.

means of determining relative wave lengths after it has been calibrated by known waves. Thus Rowland's relative determinations, made by this method, have justly become the standard work on the subject.* For fixing the absolute lengths he uses a mean of the absolute determinations of the lengths of D_1, as given in the footnote below. Probably the most accurate absolute determination of wave length is that of the three cadmium lines, which was made by Michelson† by the interferometer method. As a discussion of the details of the methods employed to obtain accuracy in the determinations with the grating would lead us beyond the scope of this book, the student is referred for further information to the articles mentioned in the footnotes or to Kayser's *Handbuch der Spectroscopie*, Vol. 1, p. 715 seq., where the methods are presented at length.

EXAMPLE

To determine the constant of the grating the following measurements were made upon some of the Fraunhofer lines of the solar spectrum:

LINE	$AO\ (m = 1)$	$AO\ (m = 2)$	$AO\ (m = 3)$
B	259.8 cm.	519.5 cm.	779.2 cm.
D_2	222.7 "	445.4 "	668.1 "
E	199.2 "	398.5 "	597.9 "
F	183.8 "	367.6 "	551.3 "

The radius of curvature of the grating was 640.5 cm. Hence, from equation (60) the following values of $n_0\left(=\frac{1}{e}\right)$ are obtained:

LINE	n_0
B	590.32
D_2	590.31
E	590.30
F	590.30
mean	590.31

* Am. Jour. Sci. (3) *33*, p. 182; Phil. Mag. (5) *23*, p. 257; *27*, p. 479; *36*, p. 49.

† Mém. du Bur. internat. des Poids et Mes. *11*, p. 1; C. R. *116*, p. 790.

The width of the ruling was $l = 145$ mm. Hence, the total number of lines upon the grating is $n = ln_0 = 85{,}595$; and therefore, from equation (57), the resolving power in the first spectrum is $R = 85{,}595$. Since $\theta = 0$ at O, the dispersion D is determined from equation (56) as $D = \frac{m}{e} = mn_0 = 590.31$ for the first spectrum. The width a of the beam is in this case the width of the ruling, i.e., $a = 145$ mm. Hence, by equation (58), $R = aD = 85{,}595$.

XI

POLARIZED LIGHT

Quantitative experiments in polarized light presuppose some general knowledge of the phenomena of polarization. Since experience has shown that few possess this knowledge with sufficient definiteness, the following simple exercises are suggested, and the student is advised to take any of the standard texts as a guide and to work them out before attempting the experiments in the next four chapters.

1. Make a dot on a piece of white paper and observe it perpendicularly through a crystal of Iceland spar. The line connecting the two dots seen is always parallel to the line connecting which angles of a perfect rhomb?

2. Rotate the crystal and note the effect produced. Call the ray which behaves extraordinarily the extraordinary ray *E*, and the other the ordinary ray *O*.

3. Toward which angle of the rhomb is *E* bent?

4. Prove that the two emergent rays are parallel.

5. Observe the dot through two crystals similarly placed; explain the effect.

6. Observe the dot through two crystals oppositely placed; explain the effect.

7. Place the crystals so that their axes inclose an angle of 45° and explain the effect. Call the four rays *Oo*, *Oe*, *Eo* and *Ee*, and state how you distinguish each.

8. Which of the four disappear when the axes of the crystals include an angle of 90°? Which when the axes include an angle of 180°? Since the position of the upper crystal when *Ee* disap-

pears differs by 90° from its position when *Eo* disappears, these two rays are said to be polarized in planes at right angles to each other.

9. Observe a dot obliquely through a rhomb of Iceland spar, and determine, by means of the bending of *O* and *E*, which travels the faster in the crystal.

10. Explain the construction of the Nicol prism.

11. Assume that *O* vibrates in the plane perpendicular to the principal plane of the wave, i.e., to the plane defined by the wave normal and the optic axis. By means of this assumption determine the plane of transmission of the Nicol, i.e., the plane in which the transmitted vibrations take place.

12. Using the Nicol to determine the plane of vibration, find in which plane the vibrations of the light reflected from a glass surface at the polarizing angle take place.

13. Determine the plane of maximum vibration of the light transmitted by a plate of glass when the light is incident at the polarizing angle.

14. Increase the number of plates and explain the effect produced upon both the reflected and the transmitted light.

15. Prove, experimentally, that a Wollaston prism makes the beams *O* and *E* divergent, and explain the reason.

16. Observe a sodium burner through two Nicols. Shut off the light by crossing the Nicols. Insert a thin crystal and explain the effect.

17. Rotate the crystal and locate its axes.

18. Rotate the analyzer and explain the effects.

19. Replace the sodium burner with a source of white light. Rotate the crystal and explain the effects. Rotate the analyzer and explain the effects.

20. Try the same experiment without the polarizer.

21. Replace the analyzer with a Wollaston prism and prove that the colors of the two images are complementary.

22. Replace the crystal by one thinner and then by one thicker and explain the effects.

23. Examine and explain the effects produced by passing convergent or divergent plane polarized light through a crystal cut perpendicular to the axis.

XII

ROTATION OF THE PLANE OF POLARIZATION

Theory

It has long been known that if plane polarized light be passed through certain substances, the plane of polarization is rotated through an angle which is peculiar to each such substance. On this account such substances are called *optically active.* They may conveniently be divided into three classes as follows:

First, those substances which rotate the plane of polarization only when they are in crystal form and lose their optical activity when they are melted or brought into solution. The most important of these substances is quartz, which crystallizes in the hexagonal crystal system. Since the optical activity of these substances is lost when their crystal form is destroyed, this peculiar property of theirs must depend only on the geometrical arrangement of the molecules in the crystal, and hence its investigation falls properly within the domain of physics.

Second, those substances which show this optical activity not only in the crystal form but also when melted or in solution. In this class belong several of the camphors and tartrates.

Third, those substances which are optically active only when in the liquid form or in solution. This class contains carbon compounds only. Since the members of the second and third classes retain their optical activity not only in solution but also in the vapor form, it follows that their power of rotating the plane of polarization depends on the arrangement of the atoms in the molecule, and, therefore, its study belongs properly in the domain of chemistry.

For substances of the first class, the amount of rotation varies with the thickness of the substance traversed by the light, with the wave length of the light, and with the temperature.

It has been experimentally proved that the angle, denoted by α, through which the plane of polarization is rotated by an active substance is proportional to the thickness traversed.* If l denotes this thickness, then

$$\alpha = kl,$$

in which k denotes the rotation produced by unit thickness. For quartz, the values of k for a plate 1 mm. thick cut perpendicular to the optic axis are, for light of the wave lengths corresponding to the Fraunhofer lines of the solar spectrum,†

FRAUNHOFER LINE	B	C	D	E	F	G
k	15.75°	17.31°	21.72°	27.54°	32.76°	42.59°

The dependence of the angle of rotation upon the wave length may be expressed by ‡

$$\alpha = \frac{A}{\lambda^2} + \frac{B}{\lambda^4},$$

in which A and B are constants to be determined for each substance by experiment. From the values of λ given above for quartz, A and B are found to have the following values, when λ is expressed in mm.,

$$A = 7.1083 \cdot 10^{-6} \qquad B = 0.1477 \cdot 10^{-12}.$$

The variation of α due to changes of temperature is, for substances of the first class, small. For quartz,§

$$\alpha_t = \alpha_{20}[1 + 0.000147\ (t)].$$

* Biot, Mém. de l'Acad. *2*, pp. 41, 91, 1817; Ann. chim. phys. (2) *10*, p. 63.

† Soret and Sarasin, C. R. *95*, p. 635, 1882.

‡ Boltzmann, Pogg. Ann. Jubelbd., p. 128, 1874. For another dispersion equation cf. Lommel, Wied. Ann. *14*, p. 523, 1881.

§ Gumlich, Wiss. Abh. d. Phys.-techn. Reichsanstalt, *2*, p. 230, 1895.

In considering the optical activity of substances of the second and third classes in solution, the idea of *specific rotation*, introduced by Biot, is found convenient. The specific rotation of an active substance in solution, denoted by $[\alpha]$, is defined as the angle through which the plane of polarization is rotated by a column of the solution 1 dm. long, which has in 1 cc. of its volume 1 gm. of the active substance. Thus if the solution be obtained by placing c gm. of the active substance in a flask and filling the flask till it contain 100 gm. of solution, then by definition,

$$[\alpha] = \frac{100\alpha}{lc}. \tag{63}$$

Since, as has been observed above, the optical activity of substances of the second and third classes is due to their molecular structure, it has been found useful to introduce the molecular weight into the definition. Hence, the molecular rotation is defined as the specific rotation multiplied by the molecular weight and divided by 100. The 100 is introduced merely to avoid large numbers. Thus, if $[M]$ represent the molecular rotation, and M the molecular weight,

$$[M] = \frac{M}{100}[\alpha],$$

that is, the molecular rotation is the angle through which the plane of polarization would be rotated by a column of the solution of the length of 1 mm., which contained in every cc. of its volume 1 gram molecule of the active substance.

It was at first supposed that specific rotation was a constant characteristic of a substance. Later investigation has, however, brought out the fact that it varies with the concentration, with the nature of the solvent, and with the temperature. These variations are generally small. Their cause has been shown to lie, for some substances at least, in the incomplete dissociation of the molecules in the solution into their ions. As a discussion of this subject lies beyond the scope of this work the reader who wishes to pursue the matter further is referred to Landolt, *Optische Drehungsvermögen organischer Substanzen*, 2d edition, Braunschweig, 1898.

The case of sugar, however, on account of its peculiar importance from a practical point of view, merits special attention. The value of [α] for sugar, as determined from solutions whose concentration varies from 5 to 30%, is essentially constant. From a large number of different measurements, its value at 20°C. for sodium light has been determined as

$$[\alpha]_D^{20} = 66.5^\circ.$$

Its variation with the temperature is, according to the most recent work, for values of t between 12° and 25°C,*

$$[\alpha]_D^t = [\alpha]_D^{20} - 0.0217\ (t - 20). \qquad (64)$$

Sugar possesses very nearly the same dispersive power as quartz. The following figures give the rotation corresponding to some of the Fraunhofer lines as measured in solutions whose concentrations varied from 10 to 20%, the column of the liquid used being of such length that the sodium light is rotated the same amount as by a plate of quartz 1 mm. thick.†

Fraunhofer Line	*B*	*C*	*D*	*E*	*F*	*G*
Rotation	15.20°	17.23°	21.71°	27.64°	33.08°	43.14°

A comparison of these values with those given above for quartz shows how nearly the dispersive powers of the two substances correspond. It is because of this fact that, as is done in some forms of saccharimeter, it is correct to make the measurements by compensating the rotation produced by the sugar solution by a rotation in the opposite sense produced by quartz.

* Schönrock, Zs. f. Instrk. *20*, p. 97, 1900.

† Stefan, Wien. Ber. *52*, II, p. 486.

The specific rotation of invert sugar varies much more than that of cane sugar with changes of concentration and temperature. Its value and its variations with the concentration and the temperature are,* for values of c' up to 35 and of t between 0° and 30° C.,

$$[\alpha']_D^{20} = 19°.657 + 0.0361\, c'$$

$$[\alpha']_D^{t} = [\alpha']_D^{20} - 0.304\,(t - 20), \qquad (65)$$

in which $c' = \dfrac{360}{340}c$, the fraction being the ratio of the molecular weights of the two sugars. The rotation is in this case left-handed.

Cane sugar may be converted into invert sugar by adding to 100 cc. of the sugar solution, 10 cc. of strong hydrochloric acid, and keeping the mixture at a temperature of 70° C. for ten minutes. It has, however, been shown† that the value of $[\alpha']_D^{20}$ determined from solutions thus converted is not constant, but depends somewhat on the relative amounts of sugar and acid in the solution. The following is recommended: To every 100 parts of sugar add 1 part of oxalic acid and let the mixture stand for some hours at a temperature of 50° to 60° C.

Experiment

Determine the Purity of a Sample of Sugar

Apparatus.—The essential parts of the polarimeter are two Nicols, A and B (Fig. 34), mounted on the ends of a firm horizontal bar about 30 cm. long. It should be possible to rotate the analyzing Nicol B about a horizontal axis, and to read the angle through which it has been turned upon a vertical graduated circle C, which is fastened to the base of the instrument. The accuracy

* Landolt, *Optische Drehungsvermögen*, Braunschweig, 1898, p. 526.
† Gubbe, Ber. I. deutschen Chem. Ges. *18*, p. 2210.

of the readings which can be made with this simplest form of instrument is not very great. The methods employed in increasing the attainable accuracy of setting are different in different forms of instrument. The device most frequently used is that of introducing directly behind the polarizer a second doubly refract-

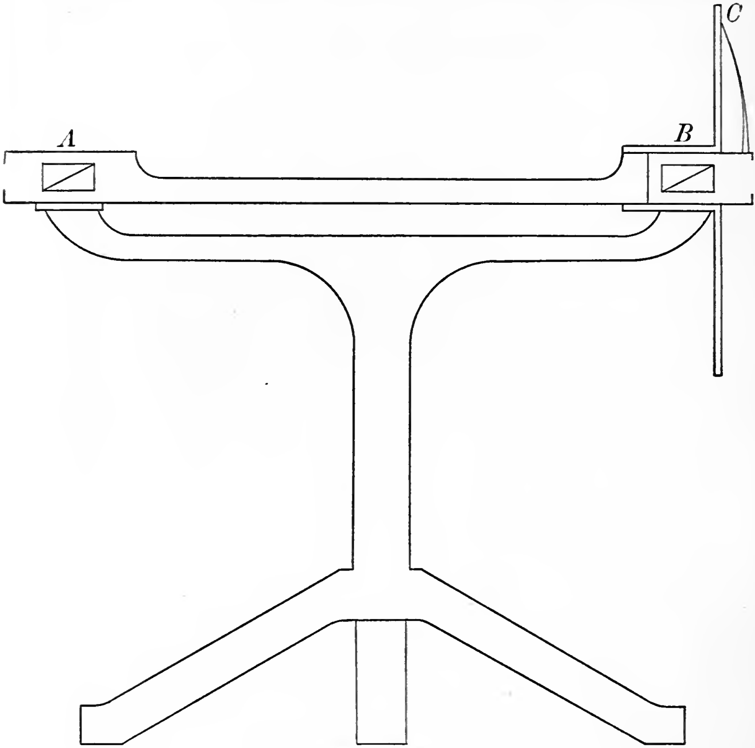

FIGURE 34

ing object, which has the effect of dividing the field of view into two or more parts whose planes of polarization make a small angle with each other. Thus in the Laurent polarimeter there is introduced behind the polarizer a thin plane-parallel plate of quartz so cut as to effect a slight rotation of the plane of polarization of the polarizer, and so placed as to cover half of the field of view. In the most recent form of instrument, this alteration of the azimuth of the plane of polarization is effected by a small Nicol, which covers half of the field of view, and whose principal plane may be set to make a small angle with that of the polarizer. The accuracy of setting is increased by introducing this small angle between the planes of polarization of the two halves of the field of view, because it is then impossible so to set the analyzer as to extinguish the light from both halves of the field of view at the same time. When the principal plane of the analyzer is perpendicular to the plane which bisects the angle between the planes of polarization of the two halves of the field of view, this field appears uniformly illuminated, i.e., the line between the two halves disappears. It has been found that the eye can judge more accurately the position of equal illumination of the two halves of the field, than it can the position of total extinction of the whole field.

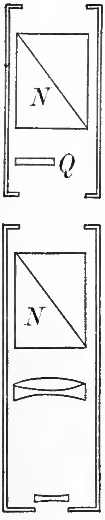

FIGURE 35

In order to make the line of division between the two halves of the field sharp, a small telescope is usually added to the observing end of the instrument. This telescope is focused upon the edge of the quartz plate of the Nicol which has been introduced. The arrangement of the optical parts of one of the more sensitive forms of the instrument is shown in Fig. 35.

In most instruments the polarizer can be turned through a small angle, thus allowing an alteration in the angle between the planes of polarization of the two halves of the field of view.

ADJUSTMENTS.—If the instrument has been properly constructed, the only adjustment needed is that of the angle between the planes of polarization of the two halves of the field of view. Up to a certain limit, the smaller this angle the more accurate the setting. It should be made as small as the brightness of the source and the absorption of the solution to be examined will permit. The student should experiment and set the angle at the point at which he can set with greatest accuracy in a given case.

MEASUREMENTS.—It is first necessary to determine the zero of the instrument. Using a sodium burner as a source of light, the empty tube which is to contain the solution to be tested is placed between the polarizer and the analyzer, and the analyzer is rotated until the field of view is uniformly dark. The position of the analyzer is then read on the graduated circle. The analyzer should then be rotated through 180°, and the position again read. These two readings are to be used as the zero readings of the instrument. The tube is then filled with the solution to be tested and again placed between the polarizer and the analyzer. The analyzer should then be rotated until the field of view appears uniformly dark, and its position read upon the graduated circle. The reading must be taken also after the analyzer has been turned through 180°. The mean of the differences between these two last readings and the corresponding zero readings is the angle through which the plane of polarization has been rotated by the solution.

If the rotation is large, it may be doubtful whether the rotation is to the right or to the left. This question may be settled either by observing with two different sources of light and remembering that the rotation for longer waves is ordinarily less than for short waves, or by making a second set of observations with a more dilute solution, which would, of course, give a smaller rotation.

The polarimeter is most frequently used in analyzing solutions of cane sugar. In this case the method of procedure is as follows: Add 10 to 15 gm. of the sugar to be tested to about 85 cc. of distilled water. After the sugar is dissolved add water till the solution weighs 100 gm. Take half of this solution, i.e., 50 cc., and add to it 5 cc. of strong hydrochloric acid. Warm this mixture to a temperature of 70°C., and let it stand at that temperature for ten minutes. The sugar will then be converted into invert sugar. Before using it should be allowed to cool to the temperature of the room.

Two tubes are usually supplied with the instrument, one 20 cm., and the other 22 cm. long. The shorter tube should be filled with the sugar solution, and the longer with the solution of the invert sugar. The rotations which the two produce are then determined, the sugar rotating to the right, the invert sugar to the left. The temperature should be noted, especially during the observations with the invert sugar.

To reduce the observations, let α represent the measured rotation of the sugar, α' that of the invert sugar, and t' the temperatures at which the respective observations were made. Then from equations (63) and (64),

$$\alpha = \left\{ [\alpha]_D^{20} - 0.0217\ (t - 20) \right\} \frac{lc}{100} + \beta,$$

in which β represents any rotation which may be produced by substances other than sugar in the solution. Similarly from equations (63) and (65),

$$\alpha' = \left\{ [\alpha']_D^{20} - 0.304\ (t' - 20) \right\} \frac{lc'}{100} - \beta,$$

in which β is negative, because active substances other than sugar are not altered by the hydrochloric acid. The sum of these two equations is, in consideration of equations (64) and (65),

$$\alpha + \alpha' = \left\{ [\alpha]_D^{t} + \frac{360}{342} [\alpha']_D^{t'} \right\} \frac{lc}{100},$$

from which the value of c, which is the number of grams of sugar in 100 gm. of solution, may be readily calculated.

EXAMPLE

13.29 gm. of cane sugar, which had been crystallized from a sugar solution, was carefully dried over sulphuric acid in vacuo and dissolved in water, so that 100 gm. of solution were obtained. Half of this was converted into invert sugar as above directed. The following observations were made, l being for the sugar solution 20 cm., and for the invert sugar solution 22 cm.:

$$\alpha = 17^\circ.35,\ \alpha' = 5^\circ.30,\ t = t' = 27^\circ\text{C}.$$

The reduction is then as follows:

$$[\alpha']_D^{20} = 19.657 + .0361c' = 20.162$$

$$.304\,(t' - 20) = 2.128$$

$$[\alpha']_D^{t'} = 18.034$$

$$\frac{360}{342}[\alpha']_D^{t'} = 18.983$$

$$[\alpha]_D^{t} = 66.352$$

$$[\alpha]_D^{t} + \frac{360}{342}[\alpha']_D^{t'} = 85.335$$

$$\therefore\ \alpha + \alpha' = 0.85335lc.$$

But, from the observations, $\alpha + \alpha' = 22^\circ.65$ and $l = 2$ dm.

$$\therefore\ 22.65 = 1.7067c$$

$$c = 13.27 \text{ gm.}$$

XIII

ELLIPTICALLY POLARIZED LIGHT

Theory

Suppose we have given a beam of plane polarized light whose vibrations take place in the plane OC (Fig. 36). Let OC (= A) represent the amplitude of the vibration. Suppose, further, that a plane-parallel plate of a doubly refracting crystal, cut parallel to the optic axis, be introduced into the path of the plane polarized beam in such a way that its faces are perpendicular to the beam,

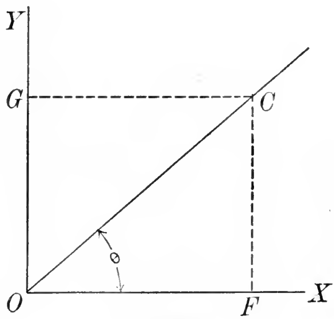

FIGURE 36

while its principal planes have the directions OX and OY. The incident vibration OC will be divided by the plate into two components OF (= a) and OG (= b), which will be equal respectively to $OC \cos \theta$ and $OC \sin \theta$ if θ represents the angle COF. These two component vibrations travel through the plate with different velocities; hence, when they emerge, though they will still be vibrating parallel respectively to OF and OG, one will be in advance of the other by an amount which depends upon the thickness of the crystalline plate and the difference of velocities of the two components in it. If then $A \cos 2\pi \frac{t}{T}$ represent the original vibration, and δ the difference in the optical paths of the two components when they emerge from the crystal, the vibrations of the two components will be represented by

$$\left.\begin{aligned} x &= a \cos 2\pi \frac{t}{T} \\ y &= b \cos 2\pi \left(\frac{t}{T} + \frac{\delta}{\lambda}\right), \end{aligned}\right\} \quad (66)$$

respectively. After passage through the plate these vibrations unite into a single one. The path which any vibrating particle pursues may be determined by eliminating t from equations (66). The result is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy \cos 2\pi \frac{\delta}{\lambda}}{ab} = \sin^2 2\pi \frac{\delta}{\lambda}. \quad (67)$$

This is the equation of an ellipse inscribed in the rectangle whose sides are $2a$ and $2b$ respectively. The position of the ellipse in the rectangle depends manifestly upon the value of δ. Two cases are of especial interest: First, when $\delta = 0$, or $\frac{\lambda}{2}$. In this case $\sin 2\pi \frac{\delta}{\lambda} = 0$, $\cos 2\pi \frac{\delta}{\lambda} = +1$, and the path of the particle is given by

$$\frac{x}{a} - \frac{y}{b} = 0;$$

i.e., the emergent light is again plane polarized, the direction of vibration being parallel to OC.

Second, when $\delta = \frac{\lambda}{4}, \frac{3\lambda}{4}$. In this case, $\sin 2\pi \frac{\delta}{\lambda} = +1$, $\cos 2\pi \frac{\delta}{\lambda} = 0$, and the equation for the path reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

i.e., the path of the vibration of the emergent light is an ellipse whose principal axes coincide in direction with the principal planes of the crystalline plate. When $\delta = \frac{\lambda}{4}$, the motion about the ellipse is in a direction contrary to that of the hands of a watch (levogyre), while when $\delta = \frac{3}{4}\lambda$, the rotation is in the same direction as that of the hands of a watch (dextrogyre). If in addition it should happen that $a = b$, the equation becomes

$$x^2 + y^2 = a^2;$$

i.e., the path of the vibration is a circle.

Conversely it is true that if any elliptical vibration is separated into components along the principal axes of the ellipse the difference in phase of those components is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

These two special cases of elliptically polarized light are of importance, because they furnish a method of finding the position of the axes of the elliptical vibration, and of measuring the relative intensity of the components along those axes. The method consists in reducing the difference in phase between the two components by $\frac{\pi}{2}$, and measuring the angle which the resulting plane polarized vibration makes with the axes of the ellipse. The method is explained in detail in what follows.

Experiment

Analyze an Elliptical Vibration

Apparatus. — The most convenient instrument for determining the nature of elliptically polarized light is an ordinary spectrometer to which has been added a pair of Nicols and three graduated circles. The circles *CCC* (Fig. 37) should be mounted on short tubes *bbb* with their planes perpendicular to the length of the tubes. Two of these tubes should be made to fit over the telescope and collimator at their objective ends, while the third fits over the eye end of the telescope. Into the end of each of these tubes fits a collar *ddd* which may be rotated freely

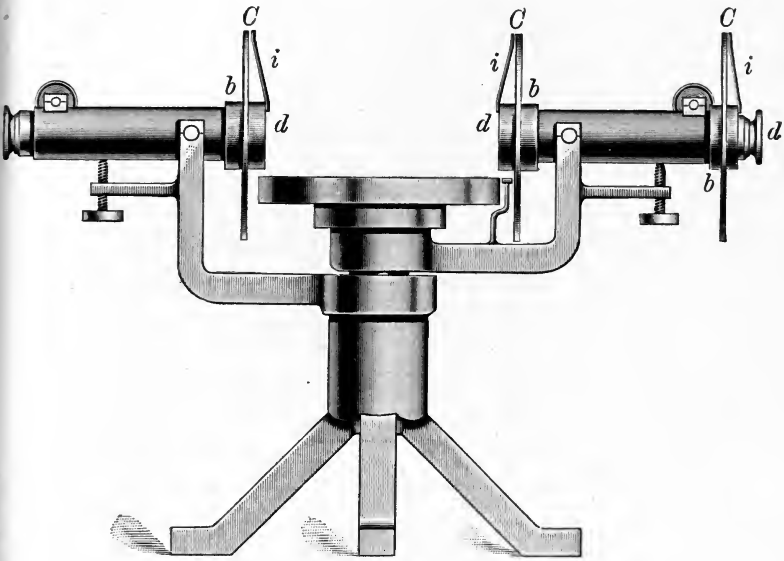

FIGURE 37

within the tube, and to which may be fitted either the Nicol prisms or the Babinet compensator or the quarter-wave plate as desired. These collars also carry indices *iii* by which their positions may be read on the graduated circles upon the tubes.

The Babinet compensator consists of a pair of quartz wedges having the same angle between their faces. One wedge is so cut that the optic axis of the quartz is parallel to the edge of the wedge, while in the other the optic axis is perpendicular to the edge and parallel to one of the faces of the wedge. These wedges are mounted in a brass case, the wedge *A* (Fig. 38) being fastened to one side of the case, while the wedge *B* can be moved back and forth by the micrometer screw C in a direction perpendicular to its edge. The head of this screw is graduated, and the mounting of this wedge B carries a scale by which its position can be determined.

Suppose light polarized in a plane which is normal to the plane defined by the two optic axes of the wedges, and which bisects the angle between those axes, to fall perpendicularly upon the wedges as indicated by the line NN (Fig. 38). On enter-

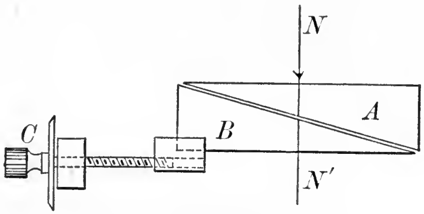

FIGURE 38

ing the wedge A, the vibration will be separated into two equal components, one parallel to the edge of the wedge, and the other perpendicular to it. These components will travel with different velocities within the wedge, and will, therefore, emerge with a difference of phase, the magnitude of which will depend upon the color of the light used and the thickness of the wedge at the place where the light has passed through. On entering the wedge B, the light again separates into two component vibrations which travel with different velocities, but because the optic axis of this wedge is perpendicular to that of the other wedge, the component vibration which traveled slower in wedge A will travel faster in wedge B, and that which traveled faster in A will travel slower in B. It is evident, then, that along that line parallel to the edges of the wedges where the light has traveled through equal thicknesses of the two wedges, it will emerge in exactly the same condition in which it entered, while along other lines its condition will depend on the difference of the thicknesses through which it has passed.

If such a pair of wedges be brought between crossed Nicols and so oriented that the optic axes make angles of 45° with the planes of transmission of the Nicols, the line along which the thicknesses of the two wedges is the same will appear as a dark line, because along it the condition of polarization of the incident light is not altered by the compensator. If the combination is illuminated with white light, colored bands will be seen on either side of this dark band, because wherever the difference in the thicknesses is such that a difference of phase of a whole wave for any particular color is introduced by the compensator, that color will be wanting in the light transmitted by the analyzer. If the combination be illuminated with monochromatic light, a series of equidistant dark bands will appear, the distance between the successive bands showing how far it is necessary to move along the wedges in order to introduce a difference of phase of a whole wave of the particular light used.

If the wedge B be moved by the screw C, the bands will move, and the constant of the instrument is merely the number of turns which must be given the screw C in order to make the dark bands move up one, i.e., the number of turns which correspond to a difference of phase of a whole wave. By turning the screw C through a fraction of the number of turns which correspond to a difference of phase of a whole wave, a difference of phase of that same fraction of a wave is introduced along the line formerly marked by the central dark band. Thus the Babinet compensator allows us to introduce any difference of phase which may be desired. It may also be used to measure a difference which has been introduced by other means, for such a difference of phase will shift the central band, and may be measured by counting the number of turns of the screw C, which are needed to cause the bands to return to their original position. This number, divided by the constant of the instrument, will be the required difference of phase. It is because of its use in this way to compensate a phase difference which has already been introduced, that the instrument has received its name.

The quarter-wave plate is a plate of crystal so cut and of such thickness that incident plane polarized light is separated by the plate into two components at right angles to each other, and one of these components is retarded over the other by a quarter-wave of the particular light used.

ADJUSTMENTS.—The tubes carrying the graduated circles having been set in place upon the collimator and telescope of the spectrometer, preferably with the zero of the circle upward, the Nicols must be mounted in these tubes so that their planes of transmission are either vertical or horizontal when the circles read zero. To attain this, place in the tube at the end of the collimator a double-image prism. On looking into the eyepiece, two images of the slit will be seen. Rotate the double-image prism until these two images are superimposed in the center of the field, one projecting above the area in which they are superimposed, the other below it. If the slit of the collimator is vertical, then, when the images overlap, as described, the vibrations of the light in one of the images will be vertical, while those in the other will be horizontal. A Nicol, suitably mounted to fit the tube, should then be placed in front of the objective of the telescope, and turned until it entirely extinguishes one of the images of the slit. The double-image prism should then be replaced by another Nicol, the index of the circle on the collimator should be set to zero, and the Nicol rotated in the collar which holds it till the image of the slit is extinguished. The plane of transmission of the Nicol on the collimator is then either vertical or horizontal when the index by which its position is read stands at zero. If the plane along which the two halves of the Nicol are cemented together is vertical, the transmitted vibrations are horizontal, etc.

This adjustment can also be made by setting the analyzer so as to extinguish the light reflected from a vertical plate of glass at the angle of complete polarization. When this takes place the plane of transmission of the analyzer is horizontal.

The Nicol in front of the objective of the telescope should then be removed to the eye end of the telescope, and a quarter-wave plate mounted in front of the objective. It is very convenient to place the Nicol at the eye end between the lenses of the eyepiece, though of course it can be mounted either in front of or behind the eyepiece. This Nicol is then crossed with the Nicol on the collimator (the polarizer), and the quarter-wave plate turned until the field is dark. The planes of transmission of the quarter-wave plate are then respectively vertical and horizontal. The readings of the circles which carry the quarter-wave plate and the Nicol on the eye end of the telescope (the analyzer) should then be taken. They are the zero positions of these two parts of the instrument.

If the Babinet compensator is to be used, it should be mounted at the eye end of the telescope directly in front of the eyepiece. A cross-hair should be fastened to the compensator to mark the position of the central black fringe, and the eyepiece focused on this cross-hair. The constant of the Babinet must then be determined for the wave length to be used. To accomplish this, the plane of vibration of the polarizer should be turned to make an angle of 45° with the horizontal, and the compensator set so that its optic axes are respectively vertical and horizontal. The analyzer should then be crossed with the polarizer. Open the slit wide and illuminate it with monochromatic light of the desired color and count the number of turns of the micrometer screw on the compensator which are necessary to make the dark bands in the field of view move up one.

In all of these experiments with polarized light, a bright source should, if possible, be used. A section of a solar spectrum, or filtered sunlight, will, in most of these cases, be sufficiently uniform.

MEASUREMENTS. — Having adjusted the instrument as has been described, introduce a thin piece of mica behind the polarizer. Let OH (Fig. 39) represent the direction and amplitude

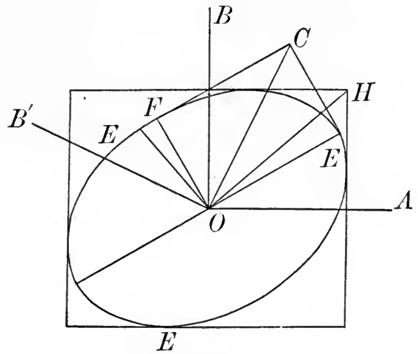

FIGURE 39

of the plane polarized light, which is incident upon the mica, and OA and OB the planes of transmission of the mica. Let EEE represent the resulting elliptical vibration which it is desired to analyze. First consider the case in which the quarter-wave plate is to be used. It is evident that the elliptical vibration will be reduced to a plane one if the directions of the planes of vibration of the quarter-wave plate coincide with those of the major axes of the ellipse, i.e., with the directions OE and OF in the figure. The plane of vibration of the resultant plane polarized light will evidently be represented by OC, and hence in order to extinguish the light, the analyzer must be turned till its plane of transmission is perpendicular to OC, i.e., till it has the direction OB'. When, therefore, the quarter-wave plate and the analyzer have been set so that the light is totally extinguished, the planes of vibration of the plate will indicate the positions of the principal axes of the elliptical vibration, and the tangent of the angle between one of those planes and the plane of transmission of the analyzer, i.e., angle $B'OF'$ (= angle COE), will determine the ratio of those axes, for

$$\tan COE = \frac{EC}{EO}. \tag{68}$$

If the compensator is used instead of the quarter-wave plate, it is merely necessary to set it so that it will introduce a difference in phase of a quarter-wave and rotate it until the central black band returns to the zero position. The analyzer must then be rotated until the central black fringe is blackest. The directions of the optic axes of the compensator show the directions of the principal axes of the ellipse, and the tangent of the angle between one of those axes, and the plane of transmission of the analyzer is the ratio of those axes.

EXAMPLE

The analyzer, quarter-wave plate and polarizer were adjusted as described above. Filtered sunlight was used. After the introduction of the mica it was necessary to turn the quarter-wave plate through an angle of 13° 20′ and the analyzer through an angle of 43° 30′ to extinguish the light. Hence, the principal axes of the elliptical vibration form made an angle of 13° 20′ with the horizontal and vertical planes respectively, and the ratio of their amplitudes was tan (43° 30′ – 13° 20′) = tan 30° 10′ = 0.581.

XIV

THE REFLECTION OF POLARIZED LIGHT FROM HOMOGENEOUS TRANSPARENT SUBSTANCES

Theory

If a beam of light of amplitude A, plane polarized in the plane of incidence, fall upon the surface of a homogeneous transparent substance, the amplitude A' of the reflected light will, according to Fresnel,* be given by

$$A' = -A\frac{\sin(i-r)}{\sin(i+r)}, \tag{69}$$

in which i and r denote, as usual, the angles of incidence and refraction respectively. When the index of refraction of the substance is greater than unity, i.e., when $\mu > 1$, $i > r$. Hence, in this case the reflected amplitude has a sign opposite to that of the incident amplitude; i.e., there is a loss of half a wave at reflection, and in absolute value $A' < A$, becoming equal to A only when $i = 90°$. At normal incidence, $i = 0$, and the right-hand side of the equation becomes indeterminate. The limiting value of the expression may then be found with the help of the consideration that, for small values of i and r, the sine may be replaced by the angle. Thus $i = nr$ and

$$A' = -A\left(\frac{n-1}{n+1}\right). \tag{70}$$

If the incident light has the amplitude B, and is polarized in

* Fresnel, *Oeuvres, I,* p. 640, Paris, 1866. Drude, *Theory of Optics,* p. 282, Longmans, 1902.

a plane perpendicular to the plane of incidence, Fresnel's expression for the amplitude of the reflected light is

$$B' = -B\frac{\tan(i-r)}{\tan(i+r)}. \tag{71}$$

When $\mu > 1$, i.e., $i > r$, and when also $i + r < 90°$, B' is negative, i.e., there is a loss of half a wave at reflection. When $i + r = 90°$, $B' = 0$. When $i + r > 90°$, the light is reflected without the loss of half a wave. For the particular case $i + r = 90°$ we have $\sin r = \cos i$, and hence, since $\frac{\sin i}{\sin r} = \mu$,

$$\tan i = \mu. \tag{72}$$

The angle determined by this equation is called the angle of total polarization. This equation, first experimentally established by Brewster, is known as Brewster's law.*

If the incident light is polarized neither in the plane of incidence nor in a plane at right angles to it, but in a plane which makes an angle θ with the plane of incidence, then, since we have adopted Fresnel's assumption that the direction of vibration is perpendicular to the plane of polarization, the component y of the amplitude in the plane of incidence will be

$$y = A \sin \theta,$$

while the component x perpendicular to that plane will be

$$x = A \cos \theta,$$

the x-axis lying in the surface, the y-axis in the plane of incidence, both perpendicular to the intersection of these two planes. The values of these components after reflection will, therefore, be

$$x' = -x\frac{\sin(i-r)}{\sin(i+r)}$$

$$y' = -y\frac{\tan(i-r)}{\tan(i+r)}.$$

* Brewster, Phil. Trans. 1815, p, 125. Jamin, Ann. de chim. et phys. (3) *29*, pp. 31 and 263; (3) *30*, p. 257. Conroy, Proc. Roy. Soc. *31*, p. 487, 1881. Rayleigh, Phil. Mag. (5) *33*, p. 1, 1892; *Collected Works*, *3*, p. 498.

These two components will unite to form a plane polarized beam whose inclination θ' to the plane of incidence is given by

$$\tan \theta' = \frac{y'}{x'};$$

or, substituting the values of x' and y' taken from above,

$$\tan \theta' = \frac{y}{x} \frac{\cos (i+r)}{\cos (i-r)} = \frac{\cos (i+r)}{\cos (i-r)} \tan \theta. \qquad (73)$$

When the light is incident normally, $i = 0$ and $\tan \theta = \tan \theta'$. As i increases, θ' becomes less than θ until i reaches the angle of complete polarization, when $\theta' = 0$. If i be further increased, θ' becomes negative, reaching the value $-\theta$ when $i = 90°$.

These reflection equations of Fresnel have been frequently verified experimentally.* This is most easily done with the help of equation (73). It is merely necessary to allow light which is plane polarized in a plane whose azimuth with the plane of incidence is known, to be reflected from the surface of a transparent medium whose index of refraction is known, and to measure the angle of incidence i, and the azimuth θ', of the plane of polarization of the reflected light.

Experiments

I. Verify Brewster's Law

The apparatus and adjustments are those described in the preceding chapter.

Measurements.—A plate of crown glass is set upon the prism table of the spectrometer, and adjusted so that its faces are parallel to the axis of the instrument, as described in Chapter VII, page 98. Filtered sunlight is allowed to pass through the colli-

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* Rood, Am. Jour. Sci. (2) *49*; *50*, 1870. Rayleigh, Nature *35*, p. 64, 1886; Proc. Roy. Soc. *41*, p. 275, 1886; *Collected Works II*, p. 522. Conroy, Proc. Roy. Soc. *35*, p. 26, 1883; *37*, p. 38, 1884; *45*, p. 101, 1888; Phil. Trans. *180*, p. 245, 1889.

mator and fall upon one of the faces of the glass plate, and the reflected light is received in the telescope, having the analyzer in front of the objective. At the angle of complete polarization all of the vibrations of the light reflected from the glass take place in a vertical plane. The analyzer should, therefore, be set so that its plane of transmission is horizontal, and the telescope and glass plate revolved about the axis of the instrument until the position is found in which the analyzer extinguishes all of the reflected light. In this position, the telescope makes with the normal to the glass plate an angle equal to that of complete polarization. This angle is then measured in the usual way. Its tangent should be equal to the index of refraction of the glass for the color used. As a check this index may be determined in the usual way. Care must be taken to have the surface of the glass to be tested perfectly clean.

EXAMPLE

The solar spectrum was allowed to fall across the slit end of the collimator and the sodium line placed on the slit. The slit was then opened wide and the light allowed to fall on a piece of the same glass whose index was determined with the interferometer (cf. page 65). The polarizer was removed from the collimator, and the analyzer set so that its plane of transmission was horizontal. It was found that the light was completely cut off when the angle of incidence was $i = 56° \ 32'$; hence, $\tan i = 1.513 = \mu$.

II. Verify Fresnel's Laws of Reflection

The apparatus and adjustments are those of Experiment I.

Measurements.—As stated above this is easily done by measuring the rotation of the plane of polarization which is produced by the reflection. Using the glass plate as a reflecting surface, as above, allow filtered sunlight to fall upon it through the collimator and polarizer. The polarizer should then be set at a definite azimuth by the graduated circle on which its position is read, and the light allowed to be reflected at a measured angle i. The azimuth of the reflected light is then read on the circle which gives the position of the analyzer, the latter being set so as to extinguish the light reflected from the prism. Having determined the index of refraction in the previous experiment, and since i and θ are measured, equation (73) furnishes a check upon Fresnel's reflection equations. In case it is found difficult to locate exactly the angle of complete polarization these observations of the rotation of the plane of polarization may be used to find it, for it is that angle at which the azimuth of the reflected vibration is 90° from the plane of incidence. Hence, if these observations be plotted with the i's as abscissae and the θ's as ordinates, the resulting curve will cross the axis of abscissae at a point corresponding to the angle of complete polarization.

EXAMPLE

The polarizer was set so that its plane of transmission was at an azimuth of 45° with the vertical plane. The light used in Experiment I was then allowed to fall upon the reflecting glass plate at measured angles of incidence i, and the following were observed as the azimuths of the reflected light:

	θ (Obs.)	θ (Calc.)
30°	33° 24′	33° 36′
35°	28° 50′	29° 3′
40°	23° 50′	23° 36′
45°	17° 20′	17° 14′
50°	10° 20′	10° 9′
55°	2° 20′	2° 31′
60°	−5° 10′	−5° 16′
65°	−13° 20′	-12° 56′
70°	−20° 20′	−20° 13′
75°	−27° 40′	−27° 0′

The observed values were plotted in a curve with the i's as abscissae. The curve crosses the axis at the point 56° 30′, which corresponds to the angle of complete polarization (cf. Experiment I). The calculated values were obtained with the help of equation (73).

XV

METALLIC REFLECTION

Theory

Experiments on the light reflected from metallic surfaces furnish the following facts:

1. When plane polarized light is reflected from a metallic surface, the reflected light is elliptically polarized, unless the plane of polarization of the incident light is either parallel or perpendicular to the plane of incidence.

2. Metallic surfaces do not possess the faculty of completely polarizing light by a single reflection at any angle.

3. If the incident light is circularly polarized, there is one particular angle of incidence for which the reflected light is plane polarized.

In discussing the phenomena of metallic reflection, it is convenient to conceive that when light falls upon such a surface, the incident vibration is resolved into two, one perpendicular and one parallel to the plane of incidence, and that each of these two components undergoes a change of phase at reflection. Experiment shows that that component which is parallel to the plane of incidence undergoes a greater change of phase than the other, and that the difference in phase between the two components after reflection is zero at normal incidence, and increases with the angle of incidence.

Since circularly polarized light becomes plane polarized by reflection at a particular angle of incidence, this angle corresponds somewhat to the angle of complete polarization of transparent substances and is called the principal angle of incidence. It is defined as the angle at which the difference in phase between the component in the plane of incidence and the one perpendicular to it amounts to a quarter of a wave length. This particular angle will be denoted by I. The azimuth of the reflected plane polarized light at this angle of incidence is called the principal azimuth, and will be denoted by θ.

The theory of metallic reflection deduces a relation between these two principal angles and the index of refraction and coefficient of absorption of the metal. The index of refraction μ of the metal is defined like that of a transparent body as the ratio of the velocity of light in vacuo to the velocity in the metal. The coefficient of absorption κ is defined as follows: If A represent the amplitude of the vibration at a given instant, and A' its amplitude after the wave has traveled one wave length in the metal, then the coefficient is defined by the equation $A : A' = 1 : e^{-2\pi\kappa}$. The relation between these optical constants and the angles I and θ is expressed by the equations*

$$\left.\begin{aligned} \kappa &= \tan 2\theta \\ \mu &= \frac{\sin I \tan I}{\sqrt{1+\kappa^2}}. \end{aligned}\right\} \quad (74)$$

Hence μ and κ can be determined from observations of I and θ.

Experiment

Determine the Optical Constants of Silver, Gold, and Platinum

The apparatus and adjustments are the same as those described in Chapter XIII.

Measurements.—The measurements may be made in two ways: First, we may allow circularly polarized light to fall on the metallic surface and determine the angle of incidence at which

* Drude, *Theory of Optics*, p. 361 seq., Longmans, 1902. Drude, Wied. Ann. *36*, p. 885, 1889; *39*, p. 481, 1890.

the reflected light is plane polarized, and the azimuth of the plane of polarization.

Second, we may allow light plane polarized at an azimuth of 45° to fall upon the metallic surface, and observe the angle of incidence at which the central fringe of a Babinet compensator set for a quarter-wave returns to the zero position. The amount which the Nicol behind the Babinet has to be rotated to make the central fringe black determines the principal azimuth.

The metallic surface is set upon the prism table of the spectrometer and adjusted to be perpendicular to the telescope.

EXAMPLE

Glass plates coated with opaque films of silver, gold, and platinum were used, also a plate of polished steel. The following observations and results were obtained, the values of μ and κ being computed with the help of equations (74):

METAL	I	θ	κ	μ
Silver	74° 38′	43° 20′	17.2	0.20
Gold	71° 40′	41° 50′	9.02	0.32
Platinum	77° 30′	32° 40′	2.17	2.01
Steel	76° 5′	28° 10′	1.50	2.56

Sunlight filtered through the red solution described in Appendix A was used.

XVI

THE SPECTROPHOTOMETER

Theory

Ordinary photometers, like the Bunsen or the Lummer-Brodhun, may be used to compare the intensities of the total radiations of two sources. It is, however, often necessary to be able to compare the intensities of radiation of two sources for each of the separate colors. This is accomplished by the spectrophotometer by separating each of the sources into a spectrum, and arranging the optical parts so that the two spectra are adjacent and can be compared at any point of their length.

One of the most convenient forms of spectrophotometer is that devised by Glan.* This instrument consists of a spectrometer whose slit is divided into two parts. The light from one source is allowed to pass through one portion of the slit, while that from the other source passes through the other portion. We thus get in the field of view two adjacent spectra, one from each source. A screen with a vertical slit in it in the eyepiece cuts off all of these spectra but one vertical band of color, half of which comes from one source, and the other half from the other.

In order to be able to measure the relative intensities of the two halves of this band of light, the apparatus is so arranged that the light in one half is polarized in one plane and that in the other in a perpendicular plane. This is accomplished by placing a double-image prism in the collimator. Such a prism will give two images of each half of the slit, and these two will be polarized

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* Glan, Wied. Ann. *1*, p. 351, 1877.

in planes at right angles to each other. Thus we get four spectra in the field of view, and of these the first and third are polarized in one plane, while the second and fourth are polarized in a perpendicular plane. The screen in the eyepiece cuts off the first and fourth spectra and leaves the second and third, one from the upper half of the slit polarized, say in a vertical plane, and one from the lower half of the slit polarized in a horizontal plane. By introducing a Nicol behind the double-image prism we are able to cut out either one or the other of these spectra. Thus when the plane of transmission of the Nicol is vertical, the spectrum from the upper half of the slit will be cut out; while when that plane is horizontal, the spectrum from the lower half of the slit disappears.

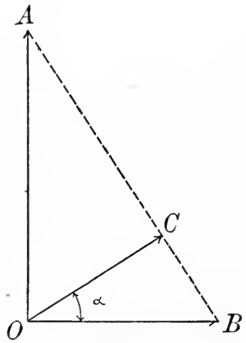

FIGURE 40

At intermediate positions of the Nicol, both spectra are visible, the intensity of each depending on the intensities of the sources and the position of the Nicol.

Let I_1 and I_2 represent the intensities of the two sources. Let, further, the amplitude of vibration of the light from source 1 be represented by OA (Fig. 40), and that of source 2 by OB. Suppose also that OA is cut out when the index on the Nicol stands at zero. The plane of transmission of the Nicol has then the direction OB. Conceive the Nicol to be rotated until the two spectra are equally bright. This will be the case when the plane of transmission OC has such a direction that the projections of OA and OB upon it are equal, i.e., when it is perpendicular to the line joining A and B. Call the angle through which this plane has been turned α, then $OB \cos \alpha = OC = OA \sin \alpha$, i.e., the ratio of the amplitudes is

$$\frac{OB}{OA} = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha.$$

Hence, the ratio of the intensities of the two sources is

$$\frac{I_1}{I_2} = \tan^2 \alpha. \tag{75}$$

Experiments

I. Compare the Radiations of Two Different Sources of Light

Apparatus.—The spectrometer, as used in the last chapters, can readily be converted into a spectrophotometer.

Adjustments.—To convert the spectrometer into a spectrophotometer it is necessary, as stated above, to place in the collimator tube a double-image prism directly in front of the objective. This prism should be so set that the two images of the vertical slit overlap to form one line. The vibrations in one image will then be vertical, while those in the other will be horizontal. A narrow card should then be fastened over the middle of the slit and so adjusted in width that the overlapping portions of the two central adjacent images, one from the lower and one from the upper half of the slit, are cut out. The center of the image of the slit in the eyepiece then appears continuous, but the vibrations of the upper half are, say vertical, while those of the lower half are horizontal. The upper and lower ends of the image of the slit should then be cut out with a screen in the eyepiece, so that the light in the entire upper half of the image vibrates in one plane, while that in the lower vibrates in a perpendicular plane.

The Nicol is then put in place in the tube *b* (Fig. 37) at the end of the collimator. The index is set at zero, and the Nicol rotated in the collar *d* until one of the spectra is extinguished.

A prism is then introduced so as to produce in the field of view two spectra, adjacent, and polarized in planes at right angles to each other.

MEASUREMENTS.—It is first necessary to determine a scale of wave lengths. To do this allow sunlight to pass through the slit and take the readings on the circle of the spectrometer for four or five of the Fraunhofer lines. Then introduce into the eyepiece a screen with a narrow vertical slit whose center coincides with the intersection of the cross-hairs. Set the telescope so that the slit in the eyepiece falls upon the red end of the spectra and take the reading on the circle of the spectrometer.

Allow the two lights which are to be compared to enter the two halves of the slit, and then turn the Nicol until both halves of the image of the slit, appear equally bright. Take the reading of this position of the Nicol. Then move the telescope along a definite amount, say 20′ or 30′. Again set the Nicol for equal illumination, and take the reading. Proceed in this way through the entire spectrum. The readings are then plotted with the angles which denote the successive positions of the telescope as abscissae, and the squares of the tangents of the angles of the Nicol as ordinates. The resulting curve represents the intensities of the various parts of the spectrum of one source, in terms of those of the other as unity. For if $I_2 = 1$, $I_1 = \tan^2 \alpha$.

Having determined the angles of the telescope which correspond to certain Fraunhofer lines, the scale of abscissae can be converted into a scale of wave lengths.

EXAMPLE

Sunlight was used to calibrate the instrument and the following readings were obtained:

Fraunhofer Line	Reading
B	14° 10′
C	14° 0′
D	13° 26′
E	12° 32′
F	11° 56′
G	10° 36′

An incandescent lamp of sixteen candle power was then compared with a Welsbach light. The following readings were obtained, the last column giving $\tan^2 \alpha$, which is the ratio of the intensity of the Welsbach to that of the incandescent lamp:

Reading	Corresponding $\lambda \cdot 10^{-6}$	α	$\tan^2 \alpha$
11° 0′	444	52°	1.64
20′	458	54°	1.90
40′	473	56°	2.19
12° 0′	490	57°	2.37
20′	508	57°	2.37
40′	528	56°	2.19
13° 0′	552	55°	2.04
20′	580	54°	1.90
40′	614	50°	1.42
14° 0′	656	46°	1.08
10′	687	38°	0.61

If these values of $\tan^2 \alpha$ be plotted as ordinates with either the readings or the corresponding wave lengths as abscissae, the curve will show that the Welsbach is relatively richer in blue and green rays.

II. Determine the Absorption of a Solution of Cyanin

Apparatus and adjustments as in Experiment I.

Measurements.—Sunlight is allowed to pass through both halves of the slit, and then the absorbing substance is placed over one of them. The method of making and plotting the observations is the same as that described above. The student should determine several such absorption curves for substances like cyanin, permanganate of potash, ruby glass, or a thin film of silver on glass.

EXAMPLE

Light from an incandescent lamp passed into the upper half of the slit directly, and into the lower half through a glass cell containing a dilute solution of cyanin. The following observations were made:

Reading	α	$\tan^2 \alpha$
11° 0′	42°	0.81
12° 0′	42°	0.81
13° 0′	42°	0.81
10′	31°	0.36
20′	14°	0.06
30′	0°	0.00
40′	16°	0.08
50′	38°	0.61
14° 0′	42°	0.81

If these values of $\tan^2 \alpha$ are plotted as ordinates with the corresponding readings as abscissae, the curve will represent the intensity of the light transmitted by the cyanin solution in terms of the intensity of the light from the incandescent lamp. It will be noted that cyanin absorbs completely the radiation corresponding to the reading 13° 30′. From the preceding example it is seen that this reading corresponds to wave length $597 \cdot 10^{-6}$ mm.

XVII

THE DEVELOPMENT OF OPTICAL THEORY

By optical theory is meant that system of ideas or conceptions in which the various phenomena of light are unified, and by which they are explained. To gain a clear idea of the meaning of this statement, we must distinguish two factors which enter into the formation of every science. In the first place, we perceive certain external events through their effect upon our senses, and, in the second place, we form conceptions by which these events are systematized, harmonized, and interpreted.

It is hardly correct to apply the word development to a mere increase in the number of phenomena observed; for, though the number of things which we perceive with regard to any object becomes greater every year, this increase may be better described by the word accretion than by the word development. This latter word seems to imply an organized increase,—an evolution. Hence it can appropriately be used only when it includes a reference to our conceptions, for it is in the organized expansion of our knowledge through these conceptions, that the life of science really lies.

This distinction becomes perfectly clear if we consider an example. Such an example might be taken at random from almost any domain of science; but, since this book treats of optics, it will, perhaps, be better to draw our illustration from this branch of physics. For the sake of brevity we will begin the discussion with the end of the seventeenth century, for it is then that optics first became prominent.

At that time only the more conspicuous phenomena of light had been noted. Thus, it was known that light seems to travel in straight lines, that it is reflected from a plane mirror in such a way that the angle of incidence is equal to the angle of reflection, that it is bent from its straight path when it passes obliquely from one medium into another of different density, and that the different parts of a beam seem to be independent of one another.

In order to describe these phenomena concisely, two conceptions were formed, one by Descartes and Newton, the other by Huygens and Hooke. The former considered that light consists of fine particles or corpuscles which are shot out by luminous bodies and travel with enormous velocity; while the latter believed that light is a form of wave motion. As is well known, the former of these conceptions prevailed and was adopted by the scientists of the eighteenth century as being more nearly correct. Newton was unable to adopt the latter mainly because waves are known to bend or be diffracted around the edges of obstacles placed in their path, and the light waves did not appear to do this. Hence he lent his energies to developing the conception of corpuscles moving in straight lines, and carried with him the world of physicists for a century or more.

It is interesting to note that of these two conceptions that of particles moving in straight lines is mechanically much simpler than that of wave motion, for it is characteristic of scientific thought to base the first conception of a phenomenon upon something crudely mechanical or which is familiar and easily conceived. This tendency is often disastrous to the healthy progress of a subject, and proved to be so in the case of optics, for that science made very little advance during the period in which the conception of light corpuscles was held.

It finally became evident to some men of science that the corpuscular idea must be abandoned, first, because it was mechanically too simple to account for the exceedingly complex and varied phenomena of light; and, second, because it seemed to them to be internally absurd, since it was found that even so commonplace an event as the passage of a beam of light from air into water, for example, required for its elucidation, additional assumptions which were somewhat ridiculous, as shown by Newton's "fits of easy reflection," etc. When the phenomena of interference and polarization became matters of observation, largely through the efforts of Young and Fresnel, then, notwithstanding the fact that the corpuscular theory had to adorn itself with many gratuitous supplementary hypotheses, the strife between the rival conceptions became fierce, and all parties to the contest sought an "*experimentum crucis*" which would finally decide between them.

Such an experiment was found in connection with refraction; for, according to the corpuscular theory, light must travel faster in the denser medium; while, according to the wave theory, the reverse is true. The controversy was then finally settled by Foucault when he proved experimentally that light travels more slowly in a denser medium, and so the conception that light is a form of wave motion came to be generally accepted.

Thus, although the corpuscular theory is mechanically the simpler, the phenomena of light are so complex that the more intricate conception of wave motion has after all proved to be more satisfactory in that it not only furnished simple and exact descriptions of interference, diffraction, and polarization, but also enabled the scientists of that day to predict effects which led to the discovery of new laws.

It was, however, soon found that the idea that light is a wave motion had its limitations when the phenomena of dispersion were considered. In dealing with diffraction and interference it is sufficient to conceive that we have to do with a wave motion, but in the case of dispersion we must also form some notion of the nature of the medium in which the wave motion takes place. Hence the next problem which presented itself was that of forming an adequate conception of that medium,—the ether, as it has been called.

The most natural solution of this problem is to assume that the waves are elastic waves like those of sound, and that the medium which transmits them is one which possesses mechanical properties similar to those which are necessary for the propagation of elastic waves. If, however, this solution is adopted, several serious difficulties are at once encountered.

The first of these difficulties arises because of the enormous velocity of light,—300,000 kilometers a second. Since the velocity of an elastic impulse in any medium is determined by the square root of the elasticity of the medium divided by its density, it is necessary to assume that the medium which transmits light has an extraordinarily high elasticity, or a very small density, or both. It would not be so difficult to conceive such a medium if it could be thought of as a very rare gas. But the light waves are transverse waves, and the medium which transmits such waves must have rigidity, i.e., must have the properties of a solid.

Even this conception of a highly elastic and very rare solid which fills all space might not be impossible if it were not for the fact that the planets and the comets move at great velocities through it without apparent resistance, and the conception of a solid which yet offers no resistance to motion through it is rather difficult.

But there are other conceptions involved in the assumption of a medium which reacts to mechanical forces which are even more impossible. These conceptions depend upon the elastic constants of the medium. The theory of elasticity demands that six conditions be fulfilled when an elastic impulse passes the boundary between two media. These conditions are the equality on both sides of the boundary of the components of the elastic displacements, and the equality of the components of the elastic forces. In order to satisfy these six conditions it is necessary to assume both transverse and longitudinal disturbances in the second medium, for the transverse alone can at best satisfy four of these conditions. Hence it has been the burden of the elastic theories of ether to make these four constants do the work of six, since longitudinal vibrations in the ether have never been detected. A detailed account of the devices employed by Cauchy, Fresnel, Green, and others to surmount this difficulty will be found in Winkelmann's *Handbuch der Physik*, Vol. II, pt. 1, p. 641 seq.*

The fact which is of special interest to us here, is, that although the adoption of the idea that light is wave motion was a step toward conceptions of greater mechanical complexity, and hence served as a great stimulus to progress in the growth of optical theory, yet the concomitant notion of a mechanically elastic solid medium contained internal absurdities which placed serious limitations on the usefulness of the theory as a whole. Hence it appears that further development demanded an expansion or change in the conception of the nature of the ether.

The first to suggest a new conception was Faraday,† who, in 1851, while discussing the question whether the magnetic force is transferred through bodies by action in a medium external to the magnet or by action at a distance, wrote: "For my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that, in the transmission of the force, there is such an action external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations."

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*Cf. also Lloyd, *Report on Optical Theories*, B. A. Reports, 1834, p. 295 seq. Glazebrook, *Report on Optical Theories*, B. A. Reports, 1885, p. 157 seq. L'Abbé Moigno, *Repertoire d'Optique Moderne*, 4 Vols., Paris, 1847-50.

† Faraday, *Experimental Researches*, No. 3075.

This hint of Faraday's was taken up later by Maxwell,* who in 1873 published a theory of light based on the assumption that the medium which transmits light is the same as that which serves as a vehicle for the electric and magnetic forces. He further developed a conception of the process of ether wave propagation which renders this theory free from the internal absurdities and contradictions which hindered the progress of its predecessor.

Thus his fundamental assumption concerning the nature of the forces in the ether, i.e., his conception of displacement currents which produce magnetic effects, is such that it follows at once from it that electromagnetic waves in the ether are transverse.† Hence after adopting this theory we are no longer compelled to consider the ether a solid.

Furthermore, there are only four independent conditions which must be fulfilled when a train of waves passes through the boundary between two media, and hence it is not necessary to make any special assumptions to explain that passage.

Finally, the theory enables us to calculate optical constants from electrical measurements, thus bringing two distinct fields of investigation into relations which can be subjected to quantitative measurements.

Now, although the attempt to describe the properties of this two-sided medium, the ether, as if they were similar to the crudely mechanical properties of grosser matter, has since then frequently been made, science is fast coming to believe, if it has not already reached the conclusion, that this ether belongs in a category by itself,—that its properties are discretely different from those of perceptible matter. Thus Maxwell's theory has proved to be a step from the mechanical conception of an elastic solid to a wholly new and radically different idea. It thus opened at once

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* Maxwell, *Electricity and Magnetism*, Vol. II., p. 431.

† Drude, *Theory of Optics*, p. 278.

a vast domain for new investigation—it bridged the chasm between the territories of light and electricity, and has been the means of adding immensely to our store of information concerning both sciences.

It is to be noted, however, that this conclusion that the ether is discretely different from ordinary matter does not by any means indicate that the problem is completely solved. It merely places that medium in a position in which an open discussion of its properties is possible, i.e., it removes all crudely mechanical bias from our minds and shows us that we have before us an almost wholly unexplored domain which invites investigation. It thus lends enthusiasm to optical and electrical research by offering a field for the exercise of untrammeled imagination,—a field unobstructed by conceptions of a grossly mechanical nature. Some of the results of this impetus which the electromagnetic theory of light has given to optical research will be briefly discussed in the next chapter.

In brief, then, optical theory has developed from a simple conception of material particles traveling in straight lines, through the more complex conception of waves in a mechanically elastic medium, to the recognition of a wave motion in a medium whose properties are still to be discovered and classified. And while the theory has passed through these stages, the observed details in the phenomena which it is invented to describe have increased in number in a way commensurate with the expansion of the theory. And so we return to the statement with which we began, namely, that science is a vitally living thing, whose life can be traced in the development of the conceptions which we form of phenomena in the world about us.

XVIII

THE TREND OF MODERN OPTICS

It remains for us to consider briefly some of the more marked results to which the adoption of the electromagnetic theory has led. The first important step in the development of the theory was taken by Maxwell when he showed that the velocity of an electromagnetic wave in a dielectric is equal to the ratio of the electromagnetic to the electrostatic unit divided by the square root of the dielectric constant of the medium. Thus if V represent the velocity of the wave, c the ratio of the units, and k the dielectric constant,*

$$V = \frac{c}{\sqrt{k}}. \qquad (76)$$

Two important results follow from this equation. The first is derived from the fact that the dielectric constant of the ether is defined as unity. Hence the velocity of an electromagnetic wave in the ether is equal to the ratio of the electromagnetic to the electrostatic units. Now this ratio of the units has been determined by experiment and found to have the value $3 \cdot 10^{10} \frac{\text{cm.}}{\text{sec.}}$. But this is the velocity of light in the free ether, in fact the two numbers agree so closely that we can hardly regard it as a mere coincidence, but are led to believe that light is an electromagnetic vibration.

The second result relates to the index of refraction. In Chapter VII this index has been defined as the ratio of the velocity of light in ether to its velocity in the medium considered. Thus if

* Drude, *Theory of Optics*, p. 276, Longmans, 1902.

V' represent the velocity in the medium, k' the dielectric constant, and μ the index for infinitely long waves, we have from equation (76), since $k = 1$,

$$\mu = \frac{V}{V'} = \sqrt{k'},$$

i.e., the square of the index is equal to the dielectric constant. This conclusion has also been tested by experiment. The first results were, however, rather discouraging, for it was found that, while the relation proved true for some substances, it was far from correct for others. Thus for benzole we find $\mu = 1.482$, and $\sqrt{k'} = 1.49$; while for water, $\mu = 1.33$ and $\sqrt{k'} = 9.0$.

It was some years before this discrepancy was explained in a satisfactory way. It was suggested that the difficulty might be due to the fact that vibrations of short period were used in determining the index for the visible spectrum from which the index for infinitely long waves was calculated from a dispersion equation like that of Cauchy (cf. p. 91), while very long oscillations were used in measuring k'. The objection to this explanation was, that when there is a discrepancy, k' is always greater than μ^2, whereas, according to the experience of that time, the longer the wave, the smaller the index; so that for infinitely long waves the index should be even smaller than it is for the visible spectrum, thus making the discrepancy worse.

It was at length discovered that the dispersion does not always follow an equation like that of Cauchy; in short, it was found that when the medium absorbs some of the radiations which fall upon it, the dispersion law is quite different. Thus if, for example, a spectrum be formed by a prism of cyanin, which absorbs the yellow, it will be found that the red and orange rays are deflected more than the green, blue, and violet, the two halves being separated by a dark band where the yellow should be. Such dispersion is called anomalous. From this discovery it became at once clear that we can not, in the case of absorbing media, calculate, with the help of the dispersion equations which belong to transparent media, the numerical value of the index of refraction on one side of an absorption band from observations taken on the other side.

After this it was easy to surmise that substances like water, for which k' is greater than μ^2, have absorption bands which correspond to periods of vibration between those of visible light and those by which k' is measured, i.e., beyond the ultra red. To prove this it is merely necessary to measure the index of refraction for very long oscillations. This was done for water by Drude,* and the result was a complete confirmation of the theory, for he found that for such long waves the index of refraction of water has the value 9.0. Hence the original statement of Maxwell, namely $\mu = \sqrt{k'}$, has been modified to read μ can not be greater than $\sqrt{k'}$. If for any substance it is less, we believe that we can with certainty predict that that substance absorbs completely some of the radiations which correspond to the ultra red.

But this discovery of the relation between absorption and anomalous dispersion had an even greater result than that of clearing away the discrepancy in Maxwell's original theory, for the question, why should a substance absorb one particular set of waves, led for its answer to the conception that the smallest particles of such absorbing media must be free to vibrate, and have natural periods of vibration which are the same as those of the waves which they absorb. This conception was borrowed from the known phenomena of absorption by vapors as manifested in the Fraunhofer lines in the solar spectrum. But the assumption that the smallest particles are set in vibration by an electromagnetic wave whose period coincides with their own natural period involves the idea that such particles must carry elec-

* Drude, Wied. Ann. *59*, p. 17, 1896.

tric charges. Hence we reach a conception which is one of the most important in modern science, namely, this: That the particles of all substances which allow light to pass at all are charged particles which have definite natural periods of vibration. Such charged particles have been given the name *ions*.

This conception is not peculiar to optics, but is found to be necessary in describing the phenomena of electrolysis, and has proved itself equally indispensable in discussing the cathode rays and the action of magnetism on light.

But how does this idea help our understanding of the facts of normal dispersion? Because it can be shown* that the dielectric constant of a medium composed of charged particles which have natural periods of vibration, depends upon the ratio of those natural periods to that of the impressed vibration. When the value of this ratio is not unity, i.e., when the impressed period and the natural periods are not the same, the dielectric constant varies continuously with the period, so that the curve which expresses the relation between the two is continuous. Hence the curve which expresses the relation between the index of refraction and the wave length is also continuous, i.e., it agrees with the observed dispersion curve. When, however, the impressed and the natural periods coincide, then absorption takes place, and the dispersion seems to become discontinuous at that point.

But this idea of charged particles which have natural periods can be extended so as to give us some conception of the nature of the particle. For it is not difficult to conceive that if by any means we could set those ions into violent vibrations, they would become a source of electromagnetic waves of the same period as the natural periods of the particles. Now it is a well-known fact that all solids, when heated to incandescence, send out waves of all possible periods; but that when the solid has been changed to

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*Drude, *Theory of Optics*, p. 382.

a vapor, the vibrations sent out have certain particular periods which are peculiar to the substance and characteristic of it. These facts are the basis of spectrum analysis, and lead to two conceptions which bear closely upon the general theory of optics.

For when these facts are considered in relation to the difference in the volumes occupied by the same quantity of a substance in the solid and in the vapor state, we must recognize that the ions are much closer together when they send out white light than they are when they produce only certain definite vibrations. Hence the conception has been formed that the ions, which by their vibration produce light, are grouped into larger particles. When the substance is in a solid state and heated to incandescence, these larger particles are set into comparatively slow vibration. But since they are close together, they will collide with one another very frequently, i.e., each particle will not travel very far before it meets another. At every such collision between two particles each receives a blow which causes it to shiver so that the ions which compose it are set into violent vibration. Now these vibrations due to impact are not at first those corresponding to the natural period of the ion, but are forced vibrations. Hence they may have any period whatever. Such forced vibrations, however, die down very rapidly, and the ions would then continue to vibrate in their natural periods were it not for the fact that they are so close together that the forced vibrations do not have time to disappear between impacts. Hence white light may be conceived to be due to the vibrations forced upon the ions by the frequent impact of the particles in which they are grouped.

But when the substance is in the form of vapor, the particles are farther apart, so that each particle travels very much farther between impacts. In this case the time required for the forced vibrations to die away is small in comparison with the time between impacts, so that the ions send out waves of their natural periods mainly. Hence the spectrum is no longer continuous but is composed of a few lines only, namely, of those which correspond to the natural periods of the ions.

Now we know that a simple sounding body, like a vibrating string, sends out a series of vibrations, partial vibrations as they are called, whose periods stand related to one another inversely as the numbers 1, 2, 3, etc. When the geometrical form of the sounding body is not so simple, the relation between the partial vibrations becomes more complex; in fact, the theoretical solution of the case of even so simple a form as the ring has not yet been worked out. Now in simple cases we can analyze the compound sound vibrations which come to us into the partial vibrations, and from that analysis draw certain conclusions as to the nature of the sounding body. Can we do the same with the light vibrations? Can we find a numerical relation between the periods of vibration indicated by the bright lines in the spectrum of a substance, and from that relation draw conclusions as to the geometrical form of the vibrating ions? This is an extremely interesting and pertinent question, for the determination of the shape of what has been called an atom has been for ages one of the favorite problems of the philosophers.

Now experiment has shown that there is a numerical relation between the periods of vibration of the lines of each of a large number of spectra. That this is not mere chance is shown by the fact that this numerical relationship has led to the discovery of new lines whose position in the series had been calculated but which had not before been observed. Investigation has also established relationships between the groups of the Mendeleeff series. Thus the elements of the first and third groups (Li., Na., K., Rb., Cs., Mg., Ca., Sr.) are characterized by doublets, while the elements of the second group (Cu., Ag.) show series of triplets. But, although an equation has been found* which expresses

* Kayser and Runge, Abh. d. k. Preuss., Akad. d. Wiss., 1888–1893. Winkelmann's *Handbuch*, II, *1*, p. 441 seq.

numerically the relation between the periods of vibration, the geometrical form of the object that would be capable of sending out such a series of partial vibrations has not yet been determined.

The problem is, however, by no means hopeless, and every new detail discovered in the spectrum of any substance adds its share, be that little or great, to the general store of information. The importance of the detailed study of spectra is thus made clear, for the more detailed our knowledge of the vibrations of the ions, the nearer we come to being able to form a conception of their size and structure. Hence the value of increasing in every way the power of spectroscopic apparatus becomes manifest, and we can appreciate the bearing of the chapter on visibility curves, for this is at present the most refined method of light wave analysis, i.e., the method which furnishes the most detailed information as to the exact nature of any complex set of light vibrations.

Another factor which helps to emphasize the usefulness of increased spectroscopic power is the recent discovery of the action of magnetism on the vibrations of the ions. Faraday had suspected that the magnetic field should act upon the vibrations of light, and discovered the magnetic rotation of the plane of polarization, but he was unable to find that the field produced any change in the period of vibration of the light. The effect was discovered a few years ago (1897) by Zeeman, who found that each line of a spectrum seems, when the source of light is placed between the poles of a powerful magnet, to be separated into three lines, the two outer being polarized in one plane, and the middle one in a plane at right angles.

The theoretical explanation of this phenomenon is as simple as it is beautiful. For we have come to believe, as has been stated, that the light waves are caused by vibrating ions, and, in the general case, the paths which such vibrating ions follow are ellipses. But an elliptical motion can be resolved into a circular motion in one plane and a rectilinear motion in a plane at right angles. Hence we may assume that the vibration forms of the ions are thus resolved into circles in planes at right angles to the lines of force of the field, and straight lines parallel to those lines. Now a moving electric charge is equivalent to an electric current, and, as is well known, a magnet attracts or repels an electric current which flows at right angles to the lines of force of the field according to the direction of the current. Since we have resolved the vibrations into circles whose planes are perpendicular to the lines of the field and straight lines parallel to those lines, only the circular vibrations will be affected by the field. But since some of the ions are rotating in one direction and others in the opposite direction, the effect of the field upon them will be different in the two cases. For if the field act upon those rotating in one direction to repel the ion, it will act with an attraction upon those rotating in the opposite direction. If, therefore, in the first case, the circular path of the ion is made larger, and hence the period of vibration increased, in the second case the path will be contracted, reducing the period of the oscillation. Since these vibrations are in the plane perpendicular to the lines of force of the field, they will appear to an observer looking at right angles to those lines like plane vibrations. Hence, if the lines of force be horizontal, these vibrations will appear to be vertical, and the observer will perceive two plane polarized vibrations whose periods are slightly different from that of the original vibration, one being somewhat smaller, the other somewhat greater.

But the period of the vibrations parallel to the lines of force of the field is not altered by the field. Hence the observer should also perceive a plane polarized vibration whose oscillations are horizontal. Hence we can conceive how a single line can be separated into three, in two of which the vibrations are vertical and the period of vibration altered, while in the third the vibrations are horizontal and of the same period as that of the original light.

If, however, an observer looks along the lines of force of the field he should perceive two circular vibrations rotating in opposite directions. Observation confirms this conclusion also.

But while observation confirms in general these results, it has nevertheless been found that when the analysis of the vibrations is carried farther, the phenomenon is not so simple. For it has been shown* that a single line is not merely separated into three, as has been described, but that each of the three may itself be compound. Furthermore, the complexity of the lines is different for different lines in the spectrum of the same element. Thus in the case of cadmium the three lines into which the red radiation is separated are all single; while the green radiation breaks into a triplet whose vibrations are horizontal and two quadruplets whose vibrations are vertical; and in the case of the blue radiation, all the lines are doublets. These details have not yet been satisfactorily accounted for in the theory. The entire subject is one of great interest, and promises well to become an important factor in helping us to form conceptions of the geometrical construction of the ions. Since the effects here described are so small that they can be detected only by the highest spectroscopic power at present available, the value of increasing the resolving power of the spectroscope becomes again manifest. A very decided advance has recently been made in this direction by Michelson† in the invention of his echelon spectroscope.

Another very interesting conclusion has been reached from the consideration of this effect of the magnetic field upon the vibration of ions. For it is clear that the magnitude of these effects depends upon two factors, namely, upon the strength of the electric charge carried by the ion, and upon the mass of the ion. Thus the greater the charge, the greater will be the

* Michelson, Phil. Mag. (5) *44*, p. 109, 1897; *45*, p. 348, 1898.

† Michelson, Am. Jour. Sci. (4) *5*, p. 215; Astrophys. Jour. *8*, p. 37, 1898.

effect, and the greater the mass, the smaller will be the effect. Hence if e represent the charge, and m the mass of the ion, the effect will be proportional to $\frac{e}{m}$. Now there are other phenomena in the description of which this factor $\frac{e}{m}$ appears, namely, electrolysis and the cathode rays. For in the case of the former the action will evidently be more rapid the greater the charge and the less the mass of the ion; and the deflection of the cathode rays by the magnetic field will, upon the assumption that those rays are caused by ions shot out from the cathode, be greater the greater the charge and the less the mass of the ion.

Unfortunately these phenomena do not permit of an independent determination of either e or m, but lead only to a value of their ratio. Now the value of this ratio for hydrogen, as determined from electrolysis, is about 10^4, while its value, as determined from the effects of the magnetic field upon the cathode rays and the light vibrations, is about $1.7 \cdot 10^7$. The difference in the two cases may be accounted for in three ways: namely, either e is the same in the three cases, while m is greater in electrolysis; or the m's are the same, but the e's are different in the three cases; or both e and m vary in the different phenomena. Hence it is an interesting problem to find a means of obtaining independent determinations of the two quantities.

This has not yet been done without making assumptions as to the number of particles in a given volume at a given pressure. But it is found that, if we take for that number the one to which the kinetic theory of gases leads us, we find as the charge of the univalent ion in electrolysis $1.29 \cdot 10^{-10}$.* J. J. Thomson† has calculated from observations upon ions in vacuo a number of the

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*Drude, *Theory of Optics*, p. 532.

†J. J. Thomson, Phil. Mag. (5) *44*, p. 293, 1897; *46*, p. 528, 1898; *48*, p. 547, 1899.

same order, namely, $6.7 \cdot 10^{-10}$, as the value of the charge upon an ion under those conditions. Hence the conclusion seems probable that the charges are, in the two cases, nearly the same, but that the masses of the ions in electrolysis are greater than those of the ions which take part in the phenomena of the cathode rays and of the action of magnetism on light.

Hence physicists have come to believe that the particles whose vibrations cause light are parts of an atom,—that the time-honored atom, which to the chemist is indivisible, is in reality compounded of a very large number of smaller ions to which the special name *electron* has been given. The calculations are not exact, but there is good reason for the belief that the hydrogen atom is made up of something like one thousand of these electrons, and that the numbers in the other chemical atoms are roughly proportional to the atomic weights. According to this conception an atom of mercury would be composed of about 200,000 electrons.

Such ideas as these tax the powers of conception of the human mind to the limit. For, if it is difficult to form a mental image of an atom, which is so minute that we can not even with the best microscope see a group of several hundred of them, is it not bold of us to attempt to realize that such atoms are themselves highly complex?

Hence we can see that the science of optics is not a worn-out, barren field of investigation, but that it now throbs with vigorous life, and that the conceptions of Nature which it has developed under the stimulus of the electromagnetic theory have led, and probably will always lead, the human mind to the very outermost boundaries of the knowable and the conceivable.

APPENDIX

A. Sources of Light

One of the most useful sources of monochromatic light is the sodium flame. Such a flame can readily be produced by heating a piece of hard glass tubing in the flame of a Bunsen burner. The tubing can be supported in the flame by a piece of iron wire, the other end of the wire being wound around the burner. Or a piece of asbestos which has been soaked in a strong solution of one of the sodium salts can be tied about the upper end of the burner. This sort of flame is very convenient in the interferometer work, especially if there be mounted on the same base an ordinary white-light burner with a cock, so that the sodium light can be replaced by white light by merely turning the cock.

The objection to this form of sodium burner is that the light which it furnishes is too faint for experiments like those of the Fresnel mirrors, in which all the light used has to pass through a narrow slit. A brighter sodium light can be obtained by heating the hard glass in the flame of an oxyhydrogen blow-pipe.

The most satisfactory light for these experiments is, however, a portion of the solar spectrum. To obtain this it is merely necessary to pass the sunlight through an ordinary spectrometer and allow the solar spectrum to fall upon the slit which acts as the source of light. A simple device for producing such a spectrum is shown in Fig. 41. Sunlight passes through the slit S and then through the upper half of the lens L which renders it parallel. It then traverses the prism P, is reflected by the mirror M, and returns through the lower half of the prism and the lens, and forms a spectrum below the slit. A small total reflection prism p

turns the light to one side so that the spectrum is formed in a position convenient for observation. The advantage of this form of instrument is that it gives a rather large dispersion with only

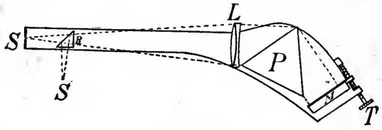

FIGURE 41

one prism, and that it allows the different parts of the spectrum to be brought upon the slit of the instrument being used, by merely changing the position of the mirror *M*. Such a spectroscope is easily constructed, as it does not require the careful adjustment usually necessary in this kind of instrument.

The vacuum tubes are useful for furnishing the cadmium and mercury light,* their success depending upon their having been sealed when the vapor pressure in them is just correct. Experience has shown that the pressure is correct when the tube, excited by the electric spark, shows stratifications about 1 mm. apart. The amount of cadmium or mercury needed is small, about the size of a pin head.

The cadmium tubes have to be heated to about 270° C. before the cadmium vaporizes. To make the heating uniform the tube should be inclosed in a heavy brass box in which is a small window covered with mica. The mercury tubes have to be warmed somewhat, not over 100° C. A metal box is not necessary with them, although it is desirable to have one to keep the temperature fairly uniform throughout the entire tube. Empty tubes can be purchased from any good glass-blower.

Filtered sunlight is sufficiently monochromatic for many

* These can be purchased from William Gaertner, 5347 Lake Avenue, Chicago, or can be readily made.

experiments, notably those in polarized light. The following solutions, recommended by Landolt,* will be found to be satisfactory:

Color	Thickness of Layer in mm.	Aqueous Solution of	Grams in 100 cc.	Average $\lambda \cdot 10^6$
Red........	20	Crystal violet, 5 BO..	0.005	656
	20	Potassium chromate.	10	
Green......	20	Copper chloride	60	533
	20	Potassium chromate.	10	
Blue.......	20	Crystal violet	0.005	448
	20	Copper sulphate	15	

B. Silvering of Optical Surfaces

The optical surfaces used in interferometer work have to be coated on their front faces with silver. One of the simplest methods of doing this is the following:

Prepare two solutions as follows:

A

Silver nitrate.................................. 5 gm.
Distilled water...........................40 cc.

To this add ammonia slowly until the precipitate which is at first formed is nearly redissolved. The success of the solution depends upon leaving an excess of the precipitate. If a drop too much ammonia has been added, a small crystal of silver nitrate must be put in to bring back traces of the precipitate. When the solution is right it will look like slightly muddy water. Then dilute to 500 cc. and filter.

B

Silver nitrate.................................. 1 gm.
Rochelle salt (sodium-potassium tartrate)....0.83 gm.
Distilled water.............................500 cc.

* Landolt, *Optische Drehungsvermögen*, Braunschweig, 1898, p. 390.

Bring the solution to a boil and filter hot. It must be cooled to the temperature of the room before it is used.

For silvering use equal parts of A and B.

The essential of success in silvering is, besides the ammonia in solution A, cleanliness. The following process of cleaning the surfaces to be silvered is recommended:

1. Remove wax, if there is any, with spirits of turpentine.
2. Wash off the turpentine with soap and water.
3. Place the surfaces to be silvered in a glass or porcelain tray, and remove any remaining silver with strong nitric acid.
4. Rinse well in running water.
5. Wash in a strong solution of caustic potash. During this washing the plates and the dish which holds them should be rubbed hard with a tuft of cotton or a piece of pure gum tubing on the end of a glass rod. The success of the process depends largely upon the thoroughness of this washing.
6. Pour off the solution of caustic potash and rinse well in running water, being careful not to touch either the surfaces or the inside of the dish with the fingers. A very minute trace of grease will make the film streaked.
7. Wash in strong nitric acid.
8. Wash in running water for five minutes or more, raising the surfaces with a glass rod to allow the water to run beneath them.
9. Rinse in several changes of distilled water.

Now mix the two solutions and pour over the surfaces. If only a thin coat is desired, the deposit must be watched, and the surface removed when the film has the necessary thickness. The opaque films should remain in the solution till it turns black. The surfaces are then removed and set up on edge on filter paper to dry. When dry they may be polished by rubbing them gently on a piece of chamois skin laid on the table and covered with jewelers' rouge. The transparent films can not be polished, for a slight touch will rub the coating off entirely. If the solution is successful the opaque films will be so hard that they can not be rubbed off with the finger.

Table 1

WAVE LENGTHS OF SOME OF THE IMPORTANT LINES

SOLAR LINE	SUBSTANCE	$\lambda \cdot 10^7$	SOLAR LINE	SUBSTANCE	$\lambda \cdot 10^7$
A		7600		Tl.	5350
B	O.	6870	E	Ca.	5270
C	H.	6563	b	Mg.	5173
	Cd.	6438		Cd.	5086
D_1	Na.	5896	F	H.	4861
D_2	Na.	5890		Cd.	4800
	Hg.	5790		Hg.	4358
	Hg.	5770	G	Fe. Ca.	4308
	Hg.	5461	H	H. Ca.	3968

Table 2

INDICES OF REFRACTION

SOLAR LINE	A	B	C	D	E	F	G	H
Water	1.3293	.3309	.3317	.3335	.3358	.3377	.3412	.3441
Alcohol	1.3586	.3599	.3606	.3624	.3647	.3667	.3705	.3736
Carbon bisulphide	1.6103	.6166	.6198	.6293	.6421	.6541	.6786	.7016
Cassia oil	1.5861	.5920	.5962	.6053	.6191	.6340	.6652	.7010
Crown glass	1.5099	.5118	.5127	.5153	.5186	.5214	.5267	.5312
Flint glass	1.7351	.7406	.7434	.7515	.7623	.7723	.7922	.8110

Table 3

USEFUL NUMBERS

Base of the natural system of logarithms $e = 2.7183$; $\log e = .43429$.

Modulus of the natural logarithms $M = \frac{1}{\log e} = 2.3026$; $\log M = .36222$.

Angle whose arc is equal to the radius $= 57^\circ.2958 = 206265''$.

logs 1.75812 5.31442.

Index of refraction of air $= 1.00029$.

Natural Sines

Angle	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Complement Difference		
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89°	
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	87	
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	86	
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0872	85	
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	84	
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	83	
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	82	
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	81	
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	1736	80	
10	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79	
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78	
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77	17
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76	
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2588	75	
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74	
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73	
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72	
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71	
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3420	70	
20	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69	
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68	
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67	
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66	16
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	4226	65	
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64	
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63	
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62	
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61	
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	5000	60	
30	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59	15
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58	
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57	
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56	
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	5736	55	
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54	
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53	14
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52	
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51	
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	6428	50	
40	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49	
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48	13
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47	
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46	
44°	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	7071	45°	
Complement		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	Angle	

Natural Cosines

Natural Sines

Angle	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Complement		Difference
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	**44°**	
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	**43**	12
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	**42**	
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	**41**	
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	7660	**40**	
50	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	7771	**39**	
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	**38**	11
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	**37**	
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	**36**	
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	8192	**35**	
55	0.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	**34**	10
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	**33**	
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	**32**	
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	**31**	
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	8660	**30**	9
60	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	**29**	
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	**28**	
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	**27**	8
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	**26**	
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	9063	**25**	
65	0.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	**24**	
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	**23**	7
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	**22**	
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	**21**	
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	9397	**20**	6
70	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	**19**	
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	**18**	
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	**17**	
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	**16**	5
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	9659	**15**	
75	0.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	**14**	
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	**13**	4
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	**12**	
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	**11**	
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	9848	**10**	
80	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	**9**	3
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	**8**	
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	**7**	
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	**6**	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	9962	**5**	
85	0.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	**4**	
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	**3**	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	**2**	
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	9998	**1**	
89°	9998	9999	9999	9999	9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	**0°**	0
Complement		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	Angle	

Natural Cosines

Natural Tangents

Angle	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Complement		Difference
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	**89°**	
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	**88**	
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	**87**	
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	**86**	
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0875	**85**	
5	0.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	**84**	
6	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	**83**	
7	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	**82**	
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	**81**	
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	1763	**80**	
10	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	**79**	18
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	**78**	
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	**77**	
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	**76**	
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	2679	**75**	
15	0.2679	2698	2717	2736	2754	2774	2792	2811	2830	2849	2867	**74**	
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	**73**	19
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	**72**	
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	**71**	
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3640	**70**	
20	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	**69**	
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	**68**	20
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	**67**	
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	**66**	
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4663	**65**	21
25	0.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	**64**	
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	**63**	
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	**62**	22
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	**61**	
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	5774	**60**	23
30	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6099	**59**	
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	**58**	24
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	**57**	
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	**56**	25
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	7002	**55**	
35	0.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	**54**	26
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	**53**	27
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	**52**	28
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	**51**	28
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	8391	**50**	29
40	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	**49**	30
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	**48**	31
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	**47**	32
43	9325	9358	9391	9424	9557	9490	9523	9556	9590	9623	9657	**46**	33
44°	9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	**45°**	34
Complement		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	Angle	

Natural Cotangents

Natural Tangents

Angle	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	Dif.
45°	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319	36
46	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686	37
47	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067	38
48	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463	40
49	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875	41
50	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305	43
51	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753	45
52	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222	47
53	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713	49
54	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229	52
55	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770	54
56	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340	57
57	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941	60
58	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577	64
59	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251	68
60	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966	72
61	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728	77
62	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542	82
63	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413	88
64	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348	94
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	10
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	11
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	12
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	13
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	14
70	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	16
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	17
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.250	19
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	22
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.700	25
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	28
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	32
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	37
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	44
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	52
80	5.67	5.73	5.79	5.85	5.91	5.98	6.04	6.11	6.17	6.24	7
81	6.31	6.39	6.46	6.54	6.61	6.69	6.77	6.85	6.94	7.03	8
82	7.12	7.21	7.30	7.40	7.49	7.60	7.70	7.81	7.92	8.03	10
83	8.14	8.26	8.39	8.51	8.64	8.78	8.92	9.06	9.21	9.36	14
84	9.51	9.68	9.84	10.0	10.2	10.4	10.6	10.8	11.0	11.2	
85	11.4	11.7	11.9	12.2	12.4	12.7	13.0	13.3	13.6	14.0	3
86	14.3	14.7	15.1	15.5	15.9	16.3	16.8	17.3	17.9	18.5	6
87	19.1	19.7	20.4	21.2	22.0	22.9	23.9	24.9	26.0	27.3	
88	28.6	30.1	31.8	33.7	35.8	38.2	40.9	44.1	47.7	52.1	
89°	57.	64.	72.	82.	95.	115.	143.	191.	286.	573.	
Angle	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	

Natural Tangents

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Logarithms

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

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