MANUAL OF LOGARITHMS

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TREATED IN CONNECTION WITH ARITHMETIC, ALGEBRA, PLANE TRIGONOMETRY, AND MENSURATION, FOR THE USE OF STUDENTS PREPARING FOR ARMY AND OTHER EXAMINATIONS

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PREFACE.

THIS Manual is intended to supply a want that has daily become more apparent during many years' experience of preparing pupils for examination. In the elementary text-books on Algebra and Trigonometry the subject is treated too shortly for practical purposes, and there is a great scarcity of examples. These failings I have endeavoured to remedy; and, to give the student accuracy and facility in his work, a very large number of examples, over 1300 in all, have been introduced, among which will be found the more important of those that have been set during the last ten years in the examinations for entrance to Sandhurst, Woolwich, and the Staff College. A few typical examples are worked out at full length in the course of the bookwork to assist the student and spare the tutor. The subject has been treated in connection with Arithmetic, Algebra, Plane Trigonometry, and Mensuration. Notwithstanding the care with which the examples have been worked out, there must necessarily be many errors in a work of this nature. I shall therefore esteem it a great favour if notification of these be made either to the publishers or myself.

It is with many thanks that I acknowledge valuable suggestions from my friend and former college tutor Mr. J. D. H. Dickson, who so kindly consented to read through proof-sheets and to assist in making the book more useful to the student and the class-room.

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LOGARITHMS.

CHAPTER I.

Definitions of Logarithm, Characteristic, Mantissa, Arithmetical Operations.

1. If a be a *positive real quantity greater than unity* the function a^x can be shown to be a *continuous* function of x, susceptible of all positive values between 0 and $+\infty$. By this is meant that whatever real value, positive or negative, integral or fractional, be given to x, then (provided positive real roots only be admitted when x is fractional) a^x will always equal some *positive* real quantity and will change in value continuously with x; in other words, when any indefinitely small change is made in the value of x, there will always be a corresponding indefinitely small change in the value of a^x; and conversely, for every indefinitely small change in the value of a^x there will always be a corresponding indefinitely small change in the value of x.

Putting $a^x = y$, these results may be conveniently stated thus :—"When a is a constant real positive quantity greater than unity, x varies continuously with y, and conversely."

Since y is here expressed as a function of x, it follows that x must also be some function of y, and that, as the value of y is determinate when x is known, so also, when y has this value given to it, among the corresponding values of x will be found that which determines the said value of y. x is in fact the index of that power of a which is equal to y, and we require some symbol to express this new function of y.

The constant quantity being called the **base**, the new function is called the **logarithm** of y to the base a and is written $\mathbf{\log_a y}$.

Thus $x = \log_a y$, and it is evident that

$$\left.\begin{aligned} a^x &= y \\ \text{and}\quad x &= \log_a y \end{aligned}\right\}$$

express one and the same functional relation between the variables x and y.

Further, since $\log_a y$ or x is the index of the power of a that is equal to y, it follows that

$$a^{\log_a y} = y \text{ is an algebraical identity.}$$

We therefore have the following **Definition of a logarithm** :—"**The logarithm of a number to a given base is the index of that power of the base which is equal to the number.**"

For example, the square of 3 is 9 ($3^2=9$), and therefore 2 is the logarithm of 9 to the base 3; or again, the logarithm of 10 to the base 100 is $\frac{1}{2}$, since the square root of 100 or $100^{\frac{1}{2}}$ is equal to 10.

2. Now a is greater than unity and positive, so that a^x increases continuously with x and *is positive whatever x may be.*

But when x is negative and infinitely large numerically, a^x or y is positive but indefinitely small, i.e. $\log_a 0 = -\infty$.

Also, so long as x is negative a^x is always less than unity, and when $x=0$ it equals unity, for $a^0=1$. Therefore, since x and y vary continuously, as x increases from $-\infty$ to 0, a^x passes through all positive proper fractional values from 0 to 1, and equals 1 when $x=0$, i.e. $\log_a 1 = 0$. Again, as x increases from 0 to $+\infty$, a^x increases constantly from unity without limit, passing through all positive real values greater than unity till we have

$$\log_a(+\infty) = +\infty.$$

Hence, though x, or the logarithm of the number y to the base a, may have any value between $-\infty$ and $+\infty$ i.e., may be either positive or negative, a^x, or the number y, *must be positive;* in other words, though we can always find the number corresponding with a given logarithm positive or negative, we can only find the logarithms of positive numbers.

It is also clear that, whatever be the base, the *logarithms of all numbers greater than unity are positive, less than unity negative.* At the same time we see that, when the number is equal to the base, the logarithm is 1, for $a^1=a$ or $\log_a a = 1$; and when the number equals the reciprocal of the base, i.e. $\frac{1}{a}$, the logarithm is -1, for $a^{-1}=\left(\frac{1}{a}\right)$ or $\log_a\left(\frac{1}{a}\right) = -1$.

The above relations between the values of numbers and their respective logarithms may be conveniently viewed in the following table—

Numbers (+ve.),	$y = a^x$	$=0$	<1	$\frac{1}{a}$	1	>1	a	$+\infty$.
Logarithms (+ve. or −ve.),	$x=\log_a y$	$=-\infty$	−ve.	-1	0	+ve.	1	$+\infty$.

3. It is seen above that $a^x=y$ and $x=\log_a y$ express the same functional relation between x and y.

Of these two identical equations, the one gives the valve of y in terms of x and in exponential form, the other the value of x in terms of y and in logarithmic form.

As it is useful to be able to convert readily from the one form to the other, it may be observed that

(i) The logarithmic is deduced from the exponential by reading the equation $a^x=y$ in the order indicated, starting with the exponent: thus x is the logarithm of y to the base a, or $x=\log_a y$.

(ii) The exponential is deduced from the logarithmic by raising the base to the power of the logarithm and equating to the number: thus $\log_r p = q$ gives $r^q=p$.

When $x = \log_a y$, y is sometimes called the **anti-logarithm** of x with reference to the base a and written $\log_a^{-1} x$,

thus $\log_a^{-1} x = a^x$ identically.

EXAMPLES. I.

1. Express in logarithmic form (i) $2^x = y$, (ii) $p^2 = q^3$, (iii) $10^{.30103} = 2$, (iv) $\sqrt[3]{a} = b$, (v) $7 = 10^{.845098}$, (vi) $.5 = 2^{-1}$.

2. Express in exponential form (i) $\log_{10} 25 = 1.39794$, (ii) $4 = \log_y x^2$, (iii) $\log_{\sqrt{x}} 49 = 4$, (iv) $\log_{r^2} p = 8$, (v) $0 = \log_3 1$, (vi) $\log_{b^2} \sqrt{a} = 3$.

3. Prove from the definition of a logarithm that (i) $a^x = e^{x \log_e a}$, (ii) $\log_3 27 = 3$, (iii) $\log_{10} .01 = -2$, (iv) $\log_2 \sqrt{32} = 2.5$, (v) $2^{\log_4 x} = \sqrt{x}$, (vi) $\log_{b^n} a^n = \log_b a$, (vii) $n \log_{a^n} a^m = m$, (viii) $\dfrac{\log_r a^m}{\log_r a^n} = \dfrac{m}{n}$, (ix) $\log_x a^2 \times \log_x a^3 = 6(\log_x a)^2$, (x) $\log_x(\log_a a^x) = 1$.

4. Given $\log_x 6\frac{1}{4} = 2$, find x.

5. Find to 3 places of decimals the numbers whose logs. to the base 10 are $.\dot{6}$, $.25$, $.1\dot{6}$, 1.5, $.\dot{3}$, and $1.\dot{3}$.

6. If the logarithms of all numbers in the tables were doubled, to what base would they then be the logarithms of the same numbers as before?

7. Write down the numbers whose logarithms (i) to the base 25 are $.5$, 3, $-\frac{1}{2}$, (ii) to the base $2\sqrt{2}$ are 2, $\frac{1}{3}$, $-\frac{1}{6}$.

8. To what base will (i) 2 be the logarithm of 100, (ii) $-\frac{1}{2}$ be the logarithm of 2, (iii) -3 " " " 8.

9. If the logarithms of a, b, c be respectively p, q, r, prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$.

4. Logarithms are usually expressed in the decimal form and, when not entirely integral, are always so arranged as to consist partly of a proper fraction that is *positive*. When arranged in this way the fractional portion of any logarithm is called its **Mantissa**, the integral portion its **Characteristic**; and while the mantissa must be positive, if not zero, the characteristic may be either positive or negative or zero.

Positive logarithms (i.e. the logarithms of all numbers greater than unity), are of course already arranged in proper form when expressed in the ordinary way, for, being wholly positive, the fractional portions, if any, are positive and the mantissæ of their respective logarithms.

Negative logarithms, on the other hand (i.e. the logarithms of all numbers less than unity), when partly or wholly fractional, require re-arrangement, for the fractional portions are negative so long as the logarithms are expressed as negative decimals in the ordinary way.

These, then, must be transformed so as to consist in every case partly of a proper fractional portion that is positive and becomes the mantissa of the logarithm, the characteristic being that negative integer which together with the positive mantissa makes up the given negative logarithm.

[Since a negative fraction $= -1 +$ a positive fraction, it is clear that the characteristic of a negative logarithm can never be zero, but must always be a negative integer, for it consists of this -1 with or without an additional negative integer.]

We will show that the required transformation for negative logarithms can always be effected.

Suppose $\log x$ to be a negative decimal lying in value between $-n$ and $-(n+1)$, where n is zero or any positive integer, i.e. suppose

$\log x = -(n+F)$, where F is a proper fraction,

then $\log x = -n - F = -n - (1 - F')$, where $F + F' = 1$, so that F' is also a proper fraction,

$= -(n+1) + F'$.

It is thus seen that any negative logarithm $-(n+F)$, consisting of a negative fraction $-F$ and of $-n$, a negative integer or zero, can be made to consist of the positive fraction F' (where $F + F' = 1$) and the negative integer $-(n+1)$.

The characteristic of the logarithm is $-(n+1)$, the mantissa F'.

Example. Given $\log x = -.32146872$, arrange the logarithm in proper form.

$$\begin{aligned}\log x &= -3.2146872\\ &= -3 - .2146872\\ &= -3 - (1 - .7853128) \qquad \text{Characteristic} = \bar{4}.\\ &= -4 + .7853128 \qquad \text{Mantissa} = .7853128.\\ &= \bar{4}.7853128\end{aligned}$$

[Observe that the minus sign of the characteristic is written above and not before it, to avoid the confusion that would otherwise arise, since the decimal fraction that follows it is positive and not negative.]

It is evident that when a logarithm is given as a decimal *wholly negative*, the characteristic of the logarithm is negative and arithmetically greater by unity than the portion of the negative logarithm to the left of the decimal point, while the mantissa is positive and equal to that decimal fraction which *makes up unity* together with the portion of the negative logarithm to the right of the decimal point.

And the converse rule is readily deduced when a negative logarithm, properly expressed, is to be reduced to the wholly negative form.

The decimal fraction that together with any other decimal fraction makes up unity may be written down at once from left to right by *making up nines* with every figure except the last which is not zero, when the figure is written down which makes up 10.

Thus a logarithm expressed as a decimal wholly negative may be written down at once in proper logarithmic form, and *vice-versâ.*

E.g.

$$\begin{aligned}-.2468719 &= \bar{1}.7531281\\ -1.1104030 &= \bar{2}.8895970\\ \bar{3}.2415602 &= -2.7584398\\ \bar{1}.3010300 &= -.6989700\end{aligned}$$

5. Since negative logarithms are expressed as partly negative and partly positive, it is well to be able to perform the simple operations of addition, subtraction, multiplication, and division upon them while still retaining them in this form. The simple rule which guides all these operations is to "treat the mantissæ *arithmetically*, the characteristics *algebraically*."

(i) **Addition of negative logarithms.**

Rule: Place the logarithms one under the other and proceed to add in the ordinary way. On arriving at the decimal point, the tens are carried on and added in *algebraically* with the several positive and negative characteristics, giving altogether a positive, negative, or zero result as the case may be.

Examples. (i)
$$\begin{array}{r} 2.7864007 \\ \bar{3}.2419515 \\ .1152404 \\ \bar{1}.8413432 \\ \hline \bar{1}.9849358 \end{array}$$

(ii)
$$\begin{array}{r} \bar{4}.8491642 \\ 3.6523150 \\ 1.9324719 \\ \bar{1}.5005313 \\ \hline 1.9344824 \end{array}$$

(ii) **Subtraction of negative logarithms.**

Rule: Place the logarithms one under the other and proceed to subtract the mantissae *arithmetically*, borrowing as usual in the ordinary way, when necessary, on reaching the decimal point. The characteristics are then subtracted *algebraically* the one from the other, having previously paid back any borrowing that has taken place, by an algebraical addition of unity to the characteristic subtracted.

Examples. (i)
$$\left.\begin{array}{r} \bar{3}.2801562 \\ \bar{2}.7863278 \\ \hline \bar{2}.4938284 \end{array}\right\}$$
(paying back -2 becomes -1, and subtracting -1 from -3 the result is -2).

i)
$$\left.\begin{array}{r} 2.8763405 \\ 4.4452862 \\ \hline \bar{2}.4310543 \end{array}\right\}$$
(4 subtracted from 2 gives -2).

(iii) **Multiplication of negative logarithms** by positive integers.

Rule: Multiply the mantissa, and then the negative characteristic, adding in *algebraically* with this latter product the tens, if any, that are carried on from the multiplication of the mantissa.

Examples. (i) $\bar{2}.7864258 \times 24$
$$\begin{array}{r} \bar{2}.7864258 \\ 4 \\ \hline \bar{5}.1457032 \\ 6 \\ \hline \overline{30}.8742192 \end{array}$$
$(-8+3=-5)$.

(ii) $\bar{3}.5117062 \times 341$
$$\begin{array}{r} .5117062 \\ 341 \\ \hline 5117062 \\ 20468248 \\ 15351186 \\ \hline 174.49181 \end{array}$$
$(-3 \times 341)+174=-849$,
$\therefore\ \bar{3}.5117062 \times 341 = \overline{849}.49181$.

Obs. *If the multiplier be negative*, multiply by the corresponding positive number and change the sign of the result:

E.g.
$$\bar{3}.5117062 \times -341 = -(\bar{3}.5117062 \times 341) = -(\overline{849}.49181)$$
$$= 849 - .49181 = 848.50819.$$

(iv) **Division of negative logarithms** by positive integers.

Rule: If the negative characteristic be exactly divisible by the divisor, divide out at once in the ordinary way, the integral portion of the result being negative and the rest positive.

If the negative characteristic be not exactly divisible, split it up into two portions, one negative and the other positive, the negative portion being the next integer arithmetically greater than the characteristic that is exactly divisible by the divisor. The quotient obtained by dividing this negative portion is then the negative characteristic of the result, while the compensatory positive portion is taken with the positive mantissa of the dividend to give on division the positive mantissa of the result.

Examples. (i) $\bar{6}.2513248 \div 3$. (ii) $\bar{5}.0741213 \div 4$. (iii) $\overline{23}.1175056 \div 17$.

$$3)\underline{\bar{6}.2513248}$$
$$\bar{2}.0837749$$

$$4)\underline{-8+3.0741213}$$
$$\bar{2}.7685303$$

$$\frac{\overline{23}.1175056}{17} = \frac{-34+11.117056}{17}$$
$$= -2+.6539709 = \bar{2}.6539709.$$

Obs. *If the divisor be negative*, divide by the corresponding positive number, and change the sign of the result after division, or of the dividend before division.

6. Multiplication and division of approximate decimals.

Except in the few cases in which logarithms are wholly integral, they are given in the tables to a certain degree of approximation, and generally to 7 places of decimals.

Now when *approximate* decimals are submitted to arithmetical operations, the results so obtained can never be correct to a greater degree of accuracy than are *all* those decimals that enter into the calculation. By the degree of accuracy is here meant the number of correct figures given, taking all the figures into consideration and not merely those that follow the decimal point. Hence, suppose a number correct to 5 figures is to be multiplied by another given correct to 8, the resulting product can be correct to 5 figures only, and the last three figures of the number given correct to 8 are useless and may be neglected in the multiplication. Again, if these two decimals, composed of 5 figures each, be multiplied together in the ordinary way, it will be found that certain columns are deficient, that is, that figures are absent from these columns, figures that would have appeared, had more figures been given in both the approximate decimals used. Hence the results obtained by adding up these deficient columns are useless.

It naturally occurs to us that the process of multiplication might have been shortened, and that we might have adopted some method giving all the complete columns and excluding all the *superfluous* figures occurring in incomplete columns. And this is the case. So, too, in the process of the division of approximate decimals, we are able to leave out superfluous figures and still obtain a result correct to the same degree of accuracy as are both divisor and dividend, and to the greatest degree of accuracy obtainable under the circumstances.

We will illustrate these processes by applying them to the multiplication and division of approximate logarithms.

As regards addition and subtraction the end is attained by simply leaving out those figures that would appear in deficient columns, and the process needs no explanation.

Logarithms, and other decimals, expressed *exactly*, and *not* approximately, may of course be treated as accurate to any number of figures, the correct figures not written being all of them zeros.

(i) Multiplication of two positive logarithms.

Example. Multiply 1.8836614 by 2.6180481.

(1) Short Method.	(2) Long Method.	
1.8836614	1.8836614...	
2.6180481	2.6180481...	
3.7673228	1	*8836614*
1.1301968	150	*692912.*
188366	753	*46456...*
150692	150692	*912......*
753	188366	*14.......*
150	11301968	*4........*
1	37673228	....
4.9315158	4.9315158	....

Rule: Multiply by the figures of the multiplier in order, *beginning on the extreme left* instead of the right, and cut off the figures of the multiplicand from the right, one by one after each multiplication. In writing down the several products *allowance must be made for the figures cut off*, to the extent of carrying on the tens that would have been carried on had no figures been cut off, and the first figure written down in each line must be placed always in the same column.

As regards the fixing of the decimal point, its position can be calculated in any line in the usual way by adding together the number of decimal figures in the multiplier and multiplicand that produce that line (of course neglecting the figures cut off), but perhaps this can be most conveniently done when multiplying by the units figure of the multiplier, when one exists, for then we simply mark off as many decimal figures as are contained by that portion of the multiplicand multiplied.

The last figure of the result obtained by this short method of multiplication *may* differ by a few units from the true figure in consequence of the omission of the next column, but this is the extent of the error.

(ii) Division of two positive logarithms.

Example. Divide 4.9315158 by 2.6180481.

2.6180481)	4.9315158(1.8836614
	2 6180481
	2 3134677
	2 0944384
	2190293
	2094438
	95855
	78541
	17314
	15708
	1606
	1570
	36
	26
	10
	10

Rule: Start the division in the ordinary way, and after the first step in the division cut off the figures of the dividend from the right, one by one at each successive step. In writing down the products *allowance must be made* as before *for the figures cut off.*

(iii) Multiplication and division of negative logarithms.

When either or both of the logarithms are negative they must be converted into the wholly negative form previously to multiplication or division. The product or quotient will then be positive or negative according as both, or only one, is negative.

EXAMPLES. II.

1. Convert the following logarithms into the wholly negative form:
   (i) $\bar{1}.0072815$, (ii) $\bar{3}.1241793$, (iii) $\bar{4}.2840617$, (iv) $\bar{2}.5351600$.

2. Express the following negative decimals in logarithmic form:
   (i) $-.3124765$, (ii) -2.9190618, (iii) -3.5, (iv) $-2\frac{1}{8}$.

3. Find the value of

(i) $\bar{1}.3876420+2.8561247+\bar{3}.7201504+\bar{5}.3876004+3$,
(ii) $4.2861720+3.1174628+\bar{6}.5407106+2.5160208+\bar{3}$,
(iii) $.1876789+\bar{1}.4027512+\bar{2}.6171840+3.8405816$,
(iv) $4.6378315+2.8516720+\bar{3}.4116712+\bar{3}.9192117$,
(v) $5.7168497+\bar{1}.3840795+2.1197197+\bar{4}.2084006$,
(vi) $3.1196117+\bar{4}.8533162+\bar{3}.4024814+2.8461620$,

(vii) $4.6281479-6.2861504$,
(viii) $3.1750462-\bar{2}.1686128$,
(ix) $2.0041060-\bar{3}.1171628$,
(x) $\bar{3}.4281025-\bar{4}.5651526$,
(xi) $.3176212-2.8607127$,
(xii) $\bar{4}.1271616-3.2870122$,

(xiii) $2.8176404-\bar{3}.4688182-\bar{2}.6415287+\bar{1}.4114850$,
(xiv) $3.9641867-2.8451521-\bar{1}.0067167-\bar{3}.8976719$,
(xv) $\bar{4}-\bar{3}.4684254-\bar{2}.6104602+\bar{3}$,
(xvi) $3.2876406+\bar{5}.3158452+\bar{2}.1876717-4.7606156$,

(xvii) $\bar{2}.4165314\times 7$,
(xviii) $3.5090067\times 2\frac{3}{4}$,
(xix) $\bar{1}.9877617\times 11$,
(xx) $\bar{4}.2076842\times 72$,
(xxi) $\bar{2}.8681184\times 113$,
(xxii) $\bar{3}.2424860\times -99$,

(xxiii) $\overline{6}.4282007\div 3$,
(xxiv) $\bar{2}.5176861\div 7$,
(xxv) $\overline{18}.2150267\div -5$,
(xxvi) $\bar{1}.6899129\times 3\frac{1}{3}$,
(xxvii) $\bar{7}.2408717\div 2\frac{1}{6}$,
(xxviii) $\bar{4}.4465142\div 3.3$,

(xxix) $\bar{2}.5176502\times 2.4045416$,
(xxx) $3.2164112\div\bar{3}.7176407$,
(xxxi) $\bar{1}.1171115\div\bar{2}.8406712$,
(xxxii) $3.2082207\times 2.1176891$,
(xxxiii) $\bar{2}.2461172\times\bar{3}.8406002$,
(xxxiv) $\bar{4}.2895165\div 3.4671008$.

CHAPTER II.

Fundamental Properties of the Logarithm.

7. Having defined the logarithm, we proceed to establish certain properties of the function, properties that render it invaluable as a means of facilitating arithmetical processes.

We shall take b to represent the base, so that where the base is omitted from the logarithmic function it will be understood to be b.

Prop. I. "The logarithm of a product of numbers equals the sum of the logarithms of the several factors."

We have identically

$$x = b^{\log x},$$
$$y = b^{\log y},$$
$$z = b^{\log z},$$
$$\ldots\ldots\ldots\ldots, \text{ and so on.}$$

By multiplication $x \cdot y \cdot z \ldots = b^{\log x + \log y + \log z + \ldots}$

$$\therefore \quad \log (x \cdot y \cdot z \ldots) = \log x + \log y + \log z + \ldots \qquad \text{Q.E.D.}$$

Prop. II. "The logarithm of a quotient of numbers equals the logarithm of the dividend diminished by that of the divisor."

We have $\left.\begin{matrix} x = b^{\log x} \\ y = b^{\log y} \end{matrix}\right\}$. By division $\dfrac{x}{y} = b^{\log x - \log y}$;

$$\therefore \quad \log \left(\frac{x}{y}\right) = \log x - \log y. \qquad \text{Q.E.D.}$$

Prop. III. "The logarithm of any power or root of a number equals the logarithm of the number multiplied by the index of that power or root."

We have $x = b^{\log x}$, and $x^m = b^{m \log x}$.

$$\therefore \quad \log x^m = m \log x. \qquad \text{Q.E.D.}$$

Prop. IV. "The logarithm of the base itself is always unity."

We have $b^1 = b$, $\therefore \ \log_b b = 1$. Q.E.D.

Prop. V. "The logarithm of unity is zero to any base."

We have $b^0 = 1$, $\therefore \ \log_b 1 = 0$. Q.E.D.

8. In Props. I, II, III are involved the important properties of the logarithmic function, and the argument for its great utility in facilitating the operations of multiplication and division and in finding the powers and roots of numbers.

To render this part of the subject clearer we will connect these properties of logarithms more directly with the corresponding theorems in indices, with which they are in reality identical, for logarithms, being simply and purely indices, admit as such of all the algebraical simplifications to which indices are subject.

By the laws of indices, $b^m \times b^n \times b^p \times \ldots = b^{m+n+p+\ldots}$ ....(i)

$b^m \div b^n = b^{m-n}$ ....(ii)

$(b^m)^n = b^{mn}$ ....(iii)

Reading these indices as the logarithms of numbers to the base b; since b^m is the number whose logarithm to the base b is equal to m, we deduce the following results.

(i) The product of those numbers whose logarithms to the base b are m, n, p, etc., equals that number whose logarithm to the same base is $m+n+p+\ldots$ (Prop. I.)

(ii) The quotient of the numbers whose logarithms are m, n, equals that number whose logarithm is $m-n$. (Prop. II.)

(iii) The power or root (index n) of the number whose logarithm is m equals that number whose logarithm is m multiplied by the index of the power or root. (Prop. III.)

These results explain how logarithms may be utilized to effect products and quotients, and to find powers and roots of numbers.

By (i) *To find the product of certain numbers*—Add the logarithms of these numbers together, and the required product will be the number whose logarithm is this sum of logarithms.

By (ii) *To find a quotient of numbers*—Subtract the logarithm of the divisor from that of the dividend, and the required quotient will be the number whose logarithm is this difference of logarithms.

By (iii) *To find a power or root of a number*—Multiply the logarithm of the number by the index of the power or root, and the required power or root will be the number whose logarithm is this product.

In making practical use of these logarithmic properties we must of course be supplied with tables in which the logarithms of numbers are given to some degree of approximation and to a constant base. We then proceed as in the following examples (in which use is made of a table of logarithms calculated to the base 10).

Examples. (i) Find the product $32 \times 16 \times 35$.

$$\log 32 = 1.5051500$$
$$\log 16 = 1.2041200$$
$$\log 35 = \underline{1.5440680}$$

$\therefore$ by addition, $\log(32 \times 16 \times 35) = 4.2533380$

But $\log 17920 = 4.2533380$, $\therefore\ 32 \times 16 \times 35 = 17920$.

(ii) Find the value of $18069 \div 57$.

$$\log 18069 = 4.2569341$$
$$\log \quad 57 = \underline{1.7558749}$$

$\therefore$ by subtraction, $\log(18069 \div 57) = 2.5010592$

But $\log 317 = 2.5010592$, $\therefore\ 18069 \div 57 = 317$.

(iii) Find the 4th power of 17. $\log 17 = 1.2304489$
$$\underline{\qquad\qquad 4}$$
$\therefore$ multiplying by 4 (the index of 4th power), $\log 17^4 = 4.9217956$

But $\log 83521 = 4.9217956$, $\therefore\ 17^4 = 83521$.

(iv) Find the 10th root of 59049. $\log 59049 = 10\overline{)4.7712125}\ (\times \frac{1}{10})$

$\therefore$ multiplying by $\frac{1}{10}$ (the index of the 10th root), $\log \sqrt[10]{59049} = .4771213$

But $\log 3 = .4771213$, $\therefore\ \sqrt[10]{59049} = 3$.

In the extraction of roots the logarithmic function is of particular value, for the usual arithmetical processes extend only to square and cube roots and roots that may be extracted by a succession of these operations, and the Binomial Theorem is only practicable over a small range of numbers, whereas any roots whatever may be obtained approximately, and with great readiness, by the application of logarithms.

Also, we shall find that the function enables us to solve approximately a certain class of equations called Exponential Equations, and thereby to effect the solution of a variety of questions in percentage and interest.

9. We will conclude this chapter with a few easy problems that depend for their solution directly upon the fundamental properties of the logarithm above explained, all logarithms requisite for the purposes of the questions being given.

(1) *To find the logarithms of products.*

Example. Find $\log_{10}8064$, given $\left.\begin{array}{l}\log_{10}2 = .3010300\\ \log_{10}3 = .4771213\\ \log_{10}7 = .8450980\end{array}\right\}$

$$\begin{aligned}\log_{10}8064 &= \log(7\times 3^2\times 2^7)\\ &= \log_{10}7 + 2\log_{10}3 + 7\log_{10}2 \qquad \text{(Props. I, III.)}\\ &= .8450980 + .9542426 + 2.1072100\\ &= 3.9065506.\end{aligned}$$

EXAMPLES. III.

1. Given $\log_{10}2 = .3010300$, $\log_{10}3 = .4771213$, $\log_{10}7 = .8450980$, $\log_{10}11 = 1.0413927$, find

(i) $\log_{10}1728$, (ii) $\log_{10}98$, (iii) $\log_{10}675$, (iv) $\log_{10}1372$,
(v) $\log_{10}588$, (vi) $\log_{10}2857680$, (vii) $\log_{10}5625$, (viii) $\log_{10}392$,
(ix) $\log_{10}1875$, (x) $\log_{10}25200$, (xi) $\log_{10}500$, (xii) $\log_{10}98784$,
(xiii) $\log_{10}630$, (xiv) $\log_{10}1078$, (xv) $\log_{10}66825$, (xvi) $\log_{10}6048$,
(xvii) $\log_{10}19965$, (xviii) $\log_{10}44000$, (xix) $\log_{10}14553$, (xx) $\log_{10}29282$.

[*When the base is 10,* $\log 10 = 1$ by Prop. IV, and $\log 2 + \log 5 = \log 10 = 1$, so that $\log 5 = 1 - \log 2$.]

2. Given $a^2+b^2=1$, $\left.\begin{aligned}\log 2 &= .3010300\\ \log(1+a) &= .1928998\\ \log(1+b) &= .2622226\end{aligned}\right\}$, show that $\log(1+a+b)=.3780762$.

3. If $\log\dfrac{1025}{1024}=\alpha$, $\log 2=\beta$, show that $\log 4100=\alpha+12\beta$.

(2) *To find the logarithms of quotients.*

Examples. (i) Find $\log_{10}10\frac{2}{25}$, given $\left.\begin{aligned}\log_{10}2 &= .3010300\\ \log_{10}3 &= .4771213\\ \log_{10}7 &= .8450980\end{aligned}\right\}$

$$\begin{aligned}\log_{10}10\frac{2}{25} = \log_{10}\frac{252}{25} &= \log_{10}\frac{7\times 2^2\times 3^2}{5^2}\\ &= \log_{10}7+2\log_{10}2+2\log_{10}3-2\log_{10}5 \qquad \text{(Props. I, II, III.)}\\ &= .8450980+.6020600+.9542426-1.3979400\\ &= 1.0034606.\end{aligned}$$

(ii) Find $\log_{10}.015$, given $\left.\begin{aligned}\log_{10}2 &= .3010300\\ \log_{10}3 &= .4771213\end{aligned}\right\}$

$$\begin{aligned}\log_{10}.015 = \log_{10}\frac{15}{1000} &= \log_{10}\frac{3\times 5}{10^3}\\ &= \log_{10}3+\log_{10}5-3\log_{10}10\\ &= .4771213+.6989700-3\\ &= \bar{2}.1760913.\end{aligned}$$

[In such examples as (ii), *when the base is 10*, it will subsequently be found sufficient to treat the number as a whole number, neglecting the decimal point, and then merely changing the characteristic of the result so obtained. *Vide* Props. VII, VIII, IX.]

EXAMPLES. IV.

1. Given $\left.\begin{aligned}\log_{10}2 &= .3010300, & \log_{10}3 &= .4771213, & \log_{10}7 &= .8450980,\\ \log_{10}11 &= 1.0413927, & \log_{10}13 &= 1.1139434, & \log_{10}17 &= 1.2304489,\end{aligned}\right\}$ find

(i) $\log_{10}2.1$, (ii) $\log_{10}34.3$, (iii) $\log_{10}.125$, (iv) $\log_{10}.0008$,

(v) $\log_{10}.005$, (vi) $\log_{10}\dfrac{4}{343}$, (vii) $\log_{10}3\frac{1}{6}$, (viii) $\log_{10}\dfrac{2.8}{1.08}$,

(ix) $\log_{10}3.\dot{7}$, (x) $\log_{10}2.01\dot{6}$, (xi) $\log_{10}28.9\dot{5}\dot{4}$, (xii) $\log_{10}\dfrac{3.6}{65}$,

(xiii) $\log_{10}.42\dot{6}$, (xiv) $\log_{10}22\frac{3}{4}$, (xv) $\log_{10}\dfrac{1}{30.\dot{8}5\dot{1}}$, (xvi) $\log_{10}\dfrac{3.38}{23.1}$,

(xvii) $\log_{10}8.\dot{7}\dot{2}$, (xviii) $\log_{10}\dfrac{.47\dot{7}\dot{2}}{1.\dot{9}2307\dot{6}}$, (xix) $\log_{10}17\frac{56}{75}$, (xx) $\log_{10}\dfrac{.14739}{95.\dot{3}}$.

2. Given $\log_e 2 = .69314718$, $\log_e 3 = 1.09861229$, find

(i) $\log_e 4\frac{1}{2}$, (ii) $\log_e .5$, (iii) $\log_e \frac{3e}{8}$, (iv) $\log_e \frac{1}{6e^2}$.

3. Given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$, find

(i) $2\log_{10}\frac{3}{5} + 4\log_{10}\frac{4}{7} + 5\log_{10}\frac{35}{12}$,

(ii) $2\log_{10}6\frac{3}{5} + 5\log_{10}\frac{14}{9} + \log_{10}\frac{5}{121} + 6\log_{10}\frac{15}{14}$,

(iii) $7\log_{10}\frac{21}{22} - 3\log_{10}\frac{1.4}{11} + 4\log_{10}2\frac{5}{14} - 3\log_{10}7.5$.

4. If A, B, C be in H.P., then $\log(A+C)$, $\log(A-C)$, and $\log(A+C-2B)$ are in A.P.

(3) *To find the logarithms of powers and roots.*

Examples. (i) Find $\log_{10}\sqrt[15]{.084}$, given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$.

$$\log_{10}\sqrt[15]{.084} = \log_{10}(.084)^{\frac{1}{15}} = \frac{1}{15}\log_{10}.084 \qquad \text{(Prop. III.)}$$
$$= \frac{1}{15}(\log_{10}7 + \log_{10}3 + 2\log_{10}2 - 3\log_{10}10)$$
$$= \frac{1}{15}(\bar{2}.9242793) = \bar{1}.9282853.$$

(ii) Find $\log_{10}18\sqrt[3]{7\sqrt{12\sqrt{5}}}$, given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$.

$$\log_{10}18\sqrt[3]{7\sqrt{12\sqrt{5}}} = \log_{10}(18 \,.\, 7^{\frac{1}{3}} \,.\, 12^{\frac{1}{6}} \,.\, 5^{\frac{1}{12}})$$
$$= \log_{10}18 + \tfrac{1}{3}\log_{10}7 + \tfrac{1}{6}\log_{10}12 + \tfrac{1}{12}\log_{10}5$$
$$= 1.7750829.$$

EXAMPLES. V.

1. Given the same logarithms as in question 1, Ex. IV, find the following logarithms to the base 10:

(i) $\log(.0147)^2$, (ii) $\log\sqrt[5]{126}$, (iii) $\log\sqrt[15]{2.1}$,

(iv) $\log\left(\frac{64}{35}\right)^{\frac{5}{6}}$, (v) $\log\sqrt[7]{7\frac{1}{2}}$, (vi) $\log\sqrt{5}$,

(vii) $\log 2^{-\frac{1}{10}}$, (viii) $\log 5^{-\frac{1}{5}}$, (ix) $\log\sqrt{\frac{1}{2}}$,

(x) $\log\sqrt[3]{\frac{1}{3}}$, (xi) $\log(3\frac{1}{5})^{-\frac{1}{3}}$, (xii) $\log\sqrt[3]{.0343}$,

(xiii) $\log(2.1)^5$, (xiv) $\log\sqrt[5]{11.\dot{3}}$, (xv) $\log\sqrt[3]{.00020736}$,

(xvi) $\log(1.75)^{\frac{1}{2}}$, (xvii) $\log(147)^2$, (xviii) $\log\left(\frac{49}{216}\right)^{\frac{1}{3}}$,

(xix) $\log \dfrac{1}{\sqrt[3]{.0245}}$, (xx) $\log \sqrt[3]{.0000024}$, (xxi) $\log \sqrt[7]{.000735}$,

(xxii) $\log \sqrt[3]{.01}$, (xxiii) $\log \frac{1}{3} \sqrt[3]{\frac{1}{3}\sqrt{\frac{1}{2}}}$, (xxiv) $\log (363 \sqrt[3]{2})^{\frac{1}{5}}$,

(xxv) $\log 7 \sqrt[5]{6\sqrt{5}}$, (xxvi) $\log \dfrac{1}{\sqrt[7]{1.92\dot{5}}}$, (xxvii) $\log \sqrt[3]{75\sqrt{2}}$,

(xxviii) $\log \sqrt[6]{252\sqrt[3]{4}}$, (xxix) $\log \frac{1}{2}\sqrt[5]{3\sqrt{4\sqrt{5}}}$, (xxx) $\log 5\sqrt[3]{\frac{3}{4}}$,

(xxxi) $\log \dfrac{2}{3}\sqrt[3]{\dfrac{\sqrt{2}}{5}}$, (xxxii) $\log \sqrt[5]{\dfrac{2\sqrt{7}}{3\sqrt{11}}}$, (xxxiii) $\log 6\left(\dfrac{2\sqrt{13}}{3\sqrt{7}}\right)^2$,

(xxxiv) $\log \sqrt[4]{(15)^{\frac{3}{5}} \times (51)^{\frac{2}{3}}}$, (xxxv) $\log \sqrt{13\sqrt{5} \div \sqrt{7}}$, (xxxvi) $\log \left\{\left(\dfrac{14}{25}\right)^{\frac{2}{3}} \times \sqrt[5]{18} \times \sqrt[3]{2.1}\right\}$

(xxxvii) $\log \left\{\left(\dfrac{3}{4}\right)^{\frac{2}{3}} \times 3\sqrt[6]{\dfrac{2}{7}} \div \sqrt{2.5}\right\}$, (xxxviii) $\log \left[(1.87)^{\frac{5}{4}} \times \dfrac{3.3}{14\sqrt{18.59}}\right]^{\frac{1}{2}}$,

(xxxix) $\log \left\{\dfrac{(2.7)^3 \times (.81)^{\frac{4}{5}}}{(90)^{\frac{5}{4}}}\right\}$, (xl) $\log \left\{\dfrac{10\sqrt{2} \div 51\sqrt[5]{6}}{\frac{3}{4}\left(\dfrac{63}{2\sqrt[3]{2}}\right)^{\frac{2}{3}}}\right\}$.

2. Given $\log_e 2 = .69314718$, $\log_e 3 = 1.09861229$, find the logarithms to the base e of

(i) $\dfrac{8}{\sqrt{27}}$, (ii) $\dfrac{3e^2}{512}$, (iii) $\sqrt{\dfrac{2}{3}} \times \sqrt[3]{\dfrac{9}{16}} \times \sqrt[4]{\dfrac{64}{27}}$.

3. Given $\log_a 2 = .1704321$, find the logarithms to the base a of

(i) $\dfrac{\sqrt[3]{2a^{-4}}}{a^{\frac{5}{4}}}$, (ii) $\dfrac{\sqrt[3]{.5a^2}}{\frac{1}{4}\sqrt{a}}$, (iii) $\sqrt{\dfrac{1}{2a}} \times \left(\dfrac{a}{8}\right)^{\frac{5}{7}} \times \dfrac{a^2\sqrt{2}}{\sqrt[7]{a^{11}}}$.

4. Given $\log x = 1.7140628$, $\log y = \bar{1}.4255632$, find

(i) $\log \dfrac{\sqrt[4]{x^3 y^{-2}}}{\frac{1}{x}\sqrt{y^3}}$, (ii) $\log \dfrac{x^2 y \sqrt[3]{xy^2}}{y^2 \sqrt[5]{\dfrac{x}{y^3}}}$.

5. Prove that $\log_e\left[\log_e\left\{\log_e e^{e^e}\right\}\right] = 1$.

(4) *To find the logarithms of factors.*

Example. Given
$$\left.\begin{aligned} \log_{10} \tfrac{18}{49} &= \bar{1}.5650765 \\ \log_{10}\sqrt{21} &= .6611096 \\ \log .112 &= \bar{1}.0492180 \end{aligned}\right\}, \text{ find } \log_{10}2,\ \log_{10}3,\ \log_{10}7.$$

These equations give
$$\left.\begin{aligned} \log_{10}2 + 2\log_{10}3 - 2\log_{10}7 &= -.4349235 \\ \tfrac{1}{2}\log_{10}3 + \tfrac{1}{2}\log_{10}7 &= .6611096 \\ 4\log_{10}2 + \log_{10}7 &= 2.0492180 \end{aligned}\right\}$$

Put $\log_{10}2 = x$, $\log_{10}3 = y$, $\log_{10}7 = z$; then
$$\left.\begin{aligned} x + 2y - 3z &= -.4349235 \\ y + z &= 1.3222192 \\ 4x + z &= 2.0492180 \end{aligned}\right\} \quad \text{whence} \quad \left.\begin{aligned} x \text{ or } \log_{10}2 &= .3010300 \\ y \text{ or } \log_{10}3 &= .4771213 \\ z \text{ or } \log_{10}7 &= .8450980 \end{aligned}\right\}$$

EXAMPLES. VI.

1. Given $\left.\begin{array}{l}\log_{10}648 = 2.8115750\\ \log_{10}864 = 2.9365137\end{array}\right\}$, find $\log_{10}3$ and $\log_{10}5$.

2. Given $\left.\begin{array}{l}\log_{10}\ 18 = 1.2552725\\ \log_{10}125 = 2.0959100\\ \log_{10}\ 21 = 1.3222193\end{array}\right\}$, find the logarithms of the numbers from 2 to 9 inclusive.

3. Given $\left.\begin{array}{l}\log_{10}.012 = \bar{2}.0791812\\ \log_{10}.018 = \bar{2}.2552725\end{array}\right\}$, find $\log_{10}2$ and $\log_{10}3$.

4. Given $\left.\begin{array}{l}\log_{10}.013 = \bar{2}.1139434\\ \log_{10}.637 = \bar{1}.8041394\end{array}\right\}$, find $\log_{10}7$.

5. Given $\left.\begin{array}{l}\log_{10}20 = 1.3010300\\ \log_{10}30 = 1.4771213\end{array}\right\}$, find $\log_{10}360$.

6. Given $\left.\begin{array}{l}\log_{10}\ 8 = \ .9030900\\ \log_{10}28 = 1.4471580\end{array}\right\}$, find $\log_{10}50$ and $\log_{10}350$.

7. Given $\log_{10}125 = 2.0969100$, find $\log_{10}16$ and $\log_{10}(.0002)^5$.

8. Given $\left.\begin{array}{l}\log_{10}4.2 \quad = \ .6232493\\ \log_{10}\ .012 \ = 2.0791813\\ \log_{10}\ .0441 = \bar{2}.6444386\end{array}\right\}$, find the logarithms of the nine digits to the base 10.

9. Given $\left.\begin{array}{l}\log_{10}\frac{1}{2} = \bar{1}.6989700\\ \log_{10}\frac{1}{3} = \bar{1}.5228787\end{array}\right\}$,
find the logarithms to the base 10 of $\sqrt[5]{6}$, $\frac{2}{3}\sqrt[3]{14.4}$, and $\frac{72}{125}\sqrt{270}\times\frac{3}{16}\sqrt[3]{625}$.

10. Given $\left.\begin{array}{l}\log_{10}1.4 = .1461280\\ \log_{10}1.5 = .1760913\end{array}\right\}$,
find $\log_{10}.000315$ and the value of $2\log_{10}\frac{2}{81} + 3\log_{10}\frac{3}{7} - 5\log_{10}1\frac{2}{3} + 5\log_{10}1.\dot{6}$.

11. Given $\left.\begin{array}{l}\log_{10}\ 2 \quad = \ .3010300\\ \log_{10}13 \quad = 1.1139434\\ \log_{10}40.\dot{3} = 1.6056641\end{array}\right\}$, find $\log_{10}.000208$ and $\log_{10}2.01\dot{6}$.

12. Given $\left.\begin{array}{l}\log_{10}\frac{1}{2} = \bar{1}.6989700\\ \log_{10}\frac{1}{3} = \bar{1}.5228787\\ \log_{10}\frac{1}{7} = \bar{1}.1549020\end{array}\right\}$, find $\log_{10}1.5$, $\log_{10}2.5$, and $\log_{10}3.5$.

13. Given $\left.\begin{array}{l}\log_{10}\sqrt{2.4} = .1901057\\ \log_{10}\sqrt[3]{5.4} = .2441313\end{array}\right\}$, find $\log_{10}1.\dot{1}$, $\log_{10}7\frac{1}{5}$, and $\log_{10}\sqrt[4]{.0012}$.

14. Express $\log_{10}.003125$ in terms of $\log_{10}2$.

15. Given $\left.\begin{array}{l}\log_{10}18 = 1.2552726\\ \log_{10}25 = 1.3979400\end{array}\right\}$, find the logarithms to the base 10 of 5, 6, 3, 450, ·075, and 16.

16. Given $\left.\begin{array}{l}\log_{10}5.76 \quad = \ .7604226\\ \log_{10}2 \qquad = \ .3010300\\ \log_{10}\ .0105 = \bar{2}.0211893\end{array}\right\}$, find the logarithms of the digits above 2 to the base 10.

17. If $\left.\begin{array}{l}\log_e .9 \quad = -x\\ \log_e .96 \quad = -y\\ \log_e 1.0125 = z\end{array}\right\}$, find $\log_e 2$, $\log_e 3$, and $\log_e 5$.

18. Given $\left.\begin{array}{l}\log_{10}2673 = 3.42700\\ \log_{10}3267 = 3.51415\end{array}\right\}$, find $\log_{10}11$.

19. Given $\left.\begin{array}{l}\log_{10}1.76 = .2455127\\ \log_{10}8.91 = .9498777\end{array}\right\}$, find $\log_{10}150$ and $\log_{10}396$.

20. Given $\left.\begin{array}{l}\log_{10}156 = 2.1931246\\ \log_{10}\ 65 = 1.8129134\end{array}\right\}$, find $\log_{10}24$ and $\log_{10}\frac{13}{48}$.

21. Given $\left.\begin{array}{l}\log\sqrt{ab^3} = \bar{2}.0962321\\ \log\sqrt{a^3b} = \bar{1}.1255217\end{array}\right\}$, find $\log\frac{a}{b}$ and $\log ab$.

22. Given $\left.\begin{array}{l}\log_e 54\sqrt{e} = 4.4889840\\ \log_e\frac{2}{\sqrt[3]{e}} = \ .3598139\end{array}\right\}$,

find the logarithms to the base e of $\sqrt{2e}$ and $\left\{\left(\frac{3}{e}\right)^{\frac{1}{3}} \div \frac{2\sqrt[3]{e}}{9e^2}\right\}$.

23. Given $\log_{10}20 = 1.3010300$, find $\log_{10}.000125$ and the logarithm to the base 10 of

$$\frac{.2 \times .4 \times .8 \times \ldots \text{to 10 factors}}{.5 \times .2.5 \times 12.5 \times \ldots \text{to 6 factors}}.$$

(5) *To find the values of logarithms and logarithmic expressions, no logs being given.*

Prop. VI. To show that $\log_{a^n}a^m = \frac{m}{n}$.

$$\log_{a^n}a^m = \log_{a^n}(a^n)^{\frac{m}{n}} = \frac{m}{n}\log_{a^n}a^n, \text{ by Prop. III,}$$

$$= \frac{m}{n} \text{ by Prop. IV.}$$

Examples. (i) Find the value of $\log_8 2\sqrt{2}$.

$$\log_8 2\sqrt{2} = \log_{2^3}2^{\frac{3}{2}} = \frac{\frac{3}{2}}{3} = \tfrac{1}{2} = .5.$$

ii) Find the value of $6\log_{10}\frac{2}{3} - 4\log_{10}1\tfrac{1}{9} + 2\log_{10}\frac{25}{6}$.

$$6\log_{10}\frac{2}{3} - 4\log_{10}1\tfrac{1}{9} + 2\log_{10}\frac{25}{6} = \log_{10}\left\{\left(\frac{2}{3}\right)^6 \div \left(1\tfrac{1}{9}\right)^4 \times \left(\frac{25}{6}\right)^2\right\}$$

$$= \log_{10}\left(\frac{\cancel{2^6}}{\cancel{3^6}} \times \frac{\cancel{3^8}}{\cancel{2^4}.\cancel{5^4}} \times \frac{\cancel{5^4}}{\cancel{2^2}.\cancel{3^2}}\right)$$

$$= \log_{10}1 = 0.$$

[When no logs are given, the only logs we are allowed to assume are log 1 which is always zero, and the logarithm of the base itself which is unity.]

EXAMPLES. VII.

1. Find the values of

(i) $\log_7 343$,	(ii) $\log_{\sqrt{2}} 4$,	(iii) $\log_{27} 9$,	(iv) $\log_{49} \sqrt[3]{7}$,
(v) $\log_{10} .01$,	(vi) $\log_{2.25} 3.375$,	(vii) $\log_{25} 125$,	(viii) $\log_4 8\sqrt{2}$,
(ix) $\log_8 \sqrt{2}$,	(x) $\log_2(\frac{1}{4})$,	(xi) $\log_{3\sqrt{5}} 25$,	(xii) $\log_{\sqrt{2}} \sqrt[3]{2\sqrt{2}}$,
(xiii) $\log_{27} 9\sqrt{3}$,	(xiv) $\log_4 \sqrt[3]{16}$,	(xv) $\log_4 \sqrt[4]{.5}$,	(xvi) $\log_4 \sqrt[3]{.015625}$,
(xvii) $\log_5 .04$,	(xviii) $\log_3 .\dot{1}$,	(xix) $\log_{2.\dot{7}} 1.\dot{6}$,	(xx) $\log_{3.375}(\frac{3}{2})$.

2. Find the values of (i) $10 \log_{10}\frac{3}{2} + 7 \log_{10}\frac{5}{18} + 4 \log_{10}\frac{48}{25}$,

(ii) $3 \log_{10}\frac{3}{5} + 2 \log_{10} 2\frac{1}{3} + 4 \log_{10}\frac{5}{14} - \log_{10}\frac{15}{784}$,

(iii) $\dfrac{\log_{10}\sqrt{27} + \log_{10} 8 - \log_{10}\sqrt{1000}}{\log_{10} 1.2}$,

(iv) $\frac{1}{6}\sqrt{\left\{\dfrac{3 \log_{10} 1728}{1 + \frac{1}{2}\log_{10} .36 + \frac{1}{3}\log_{10} 8}\right\}}$,

(v) $6 \log_{10}\frac{2}{3} - 4 \log_{10}\frac{10}{9} + 2 \log_{10}\frac{25}{6}$,

(vi) $\dfrac{\log_{10}\sqrt{54} - \log_{10}\left(\dfrac{7\frac{1}{5}}{27}\right)^2 + \log_{10}\frac{8}{3}\sqrt{.\dot{6}}}{\log_{10} 15}$.

3. Show that $\log_{10}\frac{75}{16} - 2 \log_{10}\frac{5}{9} + \log_{10}\frac{32}{243} = \log_{10} 2$

(6) *To determine the characteristic of a logarithm.*

The mantissae of logarithms are always positive; therefore, when a logarithm is not entirely integral, its characteristic is always the *algebraically smaller* of the two successive integers (whether positive or negative) between which the logarithm lies: but these integers are by Prop. III respectively the logarithms of the same powers of the base; hence, the characteristic of the logarithm of any number is the *algebraically smaller* of the indices of those successive powers of the base between which the number lies.

Examples. (i) Find the characteristic of $\log_6 2062$.

$$\begin{array}{r} 1 \\ 6 \\ \hline 6 = 6^1 \\ 6 \\ \hline 36 = 6^2 \\ 6 \\ \hline 216 = 6^3 \\ 6 \\ \hline 1296 = 6^4 \\ 6 \\ \hline 7776 = 6^5 \end{array}$$

2062 lies between 6^4 and 6^5, therefore 4 is the characteristic of $\log_6 2062$.

[This method applies to the logarithms of all numbers greater than unity.]

(ii) Find the characteristic of $\log_{12}.00023$.

$$\begin{array}{l} 12)1. \\ 12).08333 = 12^{-1} \\ 12).00694 = 12^{-2} \\ 12).00057 = 12^{-3} \\ \quad .00004 = 12^{-4} \end{array}$$

.00023 lies between 12^{-3} and 12^{-4}, therefore $\bar{4}$ is the characteristic of $\log_{12}.00023$.

[This method applies to the logarithms of numbers less than unity.]

(iii) Find the characteristic of $\log_3 \sqrt[5]{.0007}$.

Since $\log_3 \sqrt[5]{.0007} = \frac{1}{5} \log_3 .0007$, we find the characteristic of $\log_3 .0007$ and then divide it by 5.

Now, as in example (ii), the characteristic of $\log_3 .0007$ is $\bar{7}$; therefore, since $-7 = -10 + 3$, dividing -10 by 5 we find that the characteristic of $\log_3 \sqrt[5]{.0007}$ is $\bar{2}$.

EXAMPLES. VIII.

1. Find the characteristics of

(i) $\log_7 5473$,	(ii) $\log_5 .017$,	(iii) $\log_{10} 2147$,	(iv) $\log_3 .084$,
(v) $\log_2 21.84$,	(vi) $\log_3 .12$,	(vii) $\log_{2\cdot 5} .006$,	(viii) $\log_{25} .0002$,
(ix) $\log_{12} \sqrt[3]{350}$,	(x) $\log_7 \sqrt{.017}$,	(xi) $\log_3 3214$,	(xii) $\log_7 .00015$,
(xiii) $\log_2 \sqrt[5]{.014}$,	(xiv) $\log_{\sqrt{2}} 46$,	(xv) $\log_{1\cdot 5} (13.2)^{\frac{1}{3}}$,	(xvi) $\log_{\sqrt{17}} \frac{1}{\sqrt{2}}$,
(xvii) $\log_{2\cdot 5} \sqrt{.00017}$,	(xviii) $\log_3 \frac{1}{\sqrt[3]{.007}}$,	(xix) $\log_5 \left(\frac{.2}{132}\right)^{\frac{1}{3}}$,	(xx) $\log_{12} \frac{1}{9\sqrt[5]{70}}$.

2. How many positive integers are there whose logs. to the base 3 have 6 for a characteristic?

(7) *To solve exponential equations.*

Exponential equations, soluble by means of logarithms, are of two classes; (1) those in which we may proceed at once by taking logarithms, (2) those which must be reduced before taking logarithms. In the first class the signs + and − occur, if at all, only among the exponents: in the second class these signs occur between terms of the equation.

Examples. (i) Solve the equation $3^x \cdot 2^x = 4^{x+1}$, given $\left.\begin{array}{l}\log_{10}2 = .3010300\\ \log_{10}3 = .4771213\end{array}\right\}$.

Equating the logarithm of the left hand side to that of the right, we have

$$x \log_{10}3 + x \log_{10}2 = (x+1) \log_{10}4.$$

$$x (\log_{10}3 + \log_{10}2 - \log_{10}4) = \log_{10}4,$$

$$x = \frac{\log_{10}4}{\log_{10}3 + \log_{10}2 - \log_{10}4} = \frac{2 \log_{10}2}{\log_{10}3 - \log_{10}2}$$

$$= \frac{.6020600}{.1760913} = 3.41902.$$

(ii) Solve $4^x + 2^x = 12$, given $\log_{10}2$ and $\log_{10}3$.

We have $$2^{2x} + 2^x - 12 = 0,$$

$$\therefore \quad (2^x+4)(2^x-3) = 0 \text{ and } 2^x = 3 \text{ or } -4.$$

Now 2^x must be positive and cannot equal −4,

$$\therefore \quad 2^x = 3, \text{ i.e. } x \log_{10}2 = \log_{10}3$$

and $$x = \frac{\log_{10}3}{\log_{10}2} = 1.58496.$$

EXAMPLES. IX.

1. Given $\log_{10}2 = .3010300$, $\log_{10}3 = .4771213$, $\left.\begin{array}{l}\log_{10}7 = .8450980,\\ \log_{10}11 = 1.0413927,\end{array}\right\}$ solve the equations

(i) $3^{2x} \cdot 4^{3x-1} = 7$, (ii) $21^x = 20$, (iii) $\sqrt[x]{3 \cdot 2} = \frac{1}{3}$,
(iv) $2^{-5x} = 6^{2x+3}$, (v) $3^x \cdot 15^{x+1} = 14^{2x-1} \cdot 7$, (vi) $5^{x+2} = 8^{2x-1}$,
(vii) $8^x \cdot 125^{2-x} = 2^{4x+3} \cdot 5^x$, (viii) $3^{2x} \cdot 5^{3x-4} = 7^{x-1} \cdot 11^{2-x}$, (ix) $3^{x+3} = \sqrt[x+4]{4}$,
(x) $(\frac{1}{2})^{x+4} = 25^{3x+2}$, (xi) $\log_{10}2^{x+3} = 1.2221818$, (xii) $\left.\begin{array}{l}2^{3x+2y} = 5\\ 4^{2x} = 2^{2y+3}\end{array}\right\}$,
(xiii) $4^{2x} - 8(2)^{2x} + 12 = 0$, (xiv) $9 \cdot 3^{2x} - 7 \cdot 3^{x+1} + 6 = 0$,
(xv) $3^x - 6 \cdot 3^{-x} = 5$, (xvi) $\left.\begin{array}{l}18y^x - y^{2x} = 81\\ 3^x = y^2\end{array}\right\}$.

2. Given $\log_{10}\frac{1}{2} = \bar{1}.69897$, find x from the equation $20^x = 100$.

3. Given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$, and that $\log_{10}5277 = 3.722387$, solve the equations
$$\left.\begin{array}{l}6^x = 5y\\ 7^x = 3y\end{array}\right\}.$$

4. Solve the equation $\log_x 36 = 1.3678$, given $\log_{10}2$, $\log_{10}3$, and that $\log 13734.546 = 4.1378144$.

5. Solve the equations (i) $\frac{4^x}{2^{x+y}} = 8$, $x = 3y$; (ii) $4096^x = 8(64^{-x})$, (iii) $5^x = 6\frac{1}{4}(2^x)$.

6. Find the value of x from the equation $18^{8-4x} = (54\sqrt{2})^{3x-2}$, using the base $3\sqrt{2}$.

7. Solve the equations
$$\left.\begin{array}{l}a^x = 2b^y c^z\\ b^y = 3c^z a^x\\ c^z = 4a^x b^y\end{array}\right\}.$$

8. Find x from the equation $\log_{n^2+n} x = 1 - \log_{n^2+n}(x+1)$.

9. Solve $a^x(a^x - 1) = 1$.

10. Solve the equations $\left.\begin{array}{l} y^{\log z} z^{\log y} = a^2 \\ z^{\log x} x^{\log z} = b^2 \\ x^{\log y} y^{\log x} = c^2 \end{array}\right\}.$

11. Given $(a+b)^{2x}(a^4 - 2a^2b^2 + b^4)^{x-1} = (a-b)^{2x}$, find x.

12. Find x and y from the equations $\left.\begin{array}{l} x^{x+y} = y^{4a} \\ y^{x+y} = x^{a} \end{array}\right\}.$

13. What is the smallest integral value of x for which $(\frac{101}{100})^x$ is greater than a million? given $\log_{10} 10.1 = 1.0043214$.

14. How many factors, each equal to $\frac{1}{2}$, must be multiplied together that the product may be less than .000001? given $\log_{10} 2 = .3010300$.

15. How many factors $3^1 . 3^2 . 3^3 \ldots$ must be taken that the product may just exceed 100,000? given $\log_{10} 3 = .4771213$.

16. Find very nearly a 4th proportional to the 6th root of 9, the 4th root of 7, and the 5th root of 5; given $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$, $\left.\begin{array}{l} \log_{10} 7 = .84510, \\ \log_{10} 155.6 = 2.19201. \end{array}\right\}$

17. The 1st and 13th terms of a G.P. are 3 and 65 respectively; find the common ratio; given $\log_{10} 65 = 1.8129134$, $\log 1292.1592 = 3.1113160$, $\log_{10} 3 = .4771213$.

18. Given $a^1 . a^3 . a^5 \ldots = p$, find the number of factors a^1, a^3, a^5, etc.

19. Given $a^1 . a^2 . a^3 \ldots a^n = p$, find the value of n.

CHAPTER III.

The Selection of a Base.

10. The selection of the base in compiling a system of logarithms might be quite arbitrary, but certain considerations tend to give prominence to two systems, called severally the Napierian and Common Systems.

(I) The Napierian System.

11. This system, which derives its name from Napier, the inventor of logarithms, is calculated to the base e, where e is the sum of a certain infinite series whose limiting value lies between 2 and 3. It is also called the **Natural System**, because its logarithms are the first that are met with in investigating a method for the compilation of tables. We will proceed to show how Napierian logarithms are the first to present themselves in our theoretical investigations.

Now e is defined as the limiting value of $\left(1+\frac{1}{n}\right)^n$ when n is indefinitely increased, a value that can be shown to be the same as that of the infinite series $1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots$,

$$\therefore \quad e^x=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x, \text{ when } n=\infty,$$

$$=\left(1+\frac{1}{n}\right)^{nx}$$

$$=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\cdot\frac{1}{n^3}+\ldots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\ldots$$

$$=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\left(\text{Since } n=\infty, \text{ and } \therefore \text{ the fractions } \frac{1}{n}, \frac{2}{n}, \ldots \text{ are indefinitely small}\right).$$

Calling this series y, we have $e^x=y$ or $x=\log_e y$, and it is seen that we are here supplied with a rudimentary method of calculating the number y whose logarithm *to the base* e is equal to x. But this series will only be of use when x is a very small fraction (so that the terms of the series may rapidly diminish), and consequently the only numbers that can be calculated from it are those that are much less than e, and *a fortiori* less than 3. Also, the evaluation of the terms of the series is laborious when many significant figures are required in the logarithms, and moreover we require the logarithms of given numbers rather than the numbers corresponding with given logarithms. For these reasons we proceed further in our investigations.

We have $$1+x=e^{\log_e(1+x)},$$

$$\therefore\quad (1+x)^m=e^{m\log_e(1+x)}$$

$$=1+m\log_e(1+x)+\frac{\overline{m\log_e(1+x)}|^2}{\lfloor 2}+\ldots, \text{ by (i).}$$

But by the Binomial Theorem $(1+x)^m=1+mx+\dfrac{m(m-1)}{1\,.\,2}x^2+\dfrac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^3+\ldots$

Now these two values of $(1+x)^m$ must be identically equal, therefore the coefficient of m in the one equals that of m in the other; that is,

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots \quad\ldots\ldots\ldots\text{(ii)}$$

Changing the sign of x, $$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots \quad\ldots\ldots\ldots\text{(iii)}$$

Subtracting (iii) from (ii), $$\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right). \quad\ldots\ldots\ldots\text{(iv)}$$

Putting $$\frac{1+x}{1-x}=\frac{n+1}{n},\ \textit{i.e.}\ x=\frac{1}{2n+1},$$

$$\log_e\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\},$$

i.e. $$\log_e(n+1)=\log_e n+2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\}. \quad\text{(v)}$$

Here we have at length a formula from which can be computed the logarithms to the base e of all the natural numbers from unity upwards; for, provided n be greater than unity, the terms of the series in (v) are all small fractions and rapidly diminish, so that, in finding the logarithms to any required degree of accuracy, only a certain number of the terms need be retained and calculated.

We will explain the application of series (v).

$\log_e 1=0$; therefore, putting $n=1$,

$$\log_e 2=2\left\{\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\ldots\right\}.$$

This determines $\log_e 2$; then, putting $n=2$,

$$\log_e 3=\log_e 2+2\left\{\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\ldots\right\},$$

and, by giving to n in succession the values 3, 4, ..., the Napierian logarithms of all the natural numbers may be calculated, every logarithm being equal to the one previously computed plus the sum of a certain rapidly converging series. Only the logarithms of prime numbers need be calculated by direct use of series (v), those of all numbers which are not prime being obtained readily from logarithms which will have been previously computed by application of Prop. I.

It is thus seen that the logarithms first met with in investigating a method of compiling complete tables are those to the base e. Hence the importance of the Napierian system, for its logarithms *must* be first calculated before those to any other base can be obtained.

This system is also sometimes called the **Hyperbolic System.**

It will presently appear in Chap. V., when we come to discuss the relations existing between logarithms to different bases, that the logarithms to any other base x may be obtained by multiplying the corresponding Napierian logarithms by a certain constant multiplier, called the **modulus** of the new system. This modulus is $\frac{1}{\log_e x}$ or $\log_x e$, commonly written μ.

For logarithms to the base 10, $\mu = \frac{1}{\log_e 10} = .43429448$.

Putting $\frac{1}{x}$ for x in (iv) we have $\log_e \frac{x+1}{x-1} = 2\left\{\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right\}$,

or $\log_e(x+1) = \log_e(x-1) + 2\left\{\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right\}$. ....(vi)

Collecting the series of this article, they stand thus

I. $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$

[For calculating powers of e not numerically greater than unity, or the numbers whose logs. to the base e are not numerically greater than unity.]

II. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

[For calculating logs. of numbers between 1 and 2. x +ve. and < 1.]

III. $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

[For calculating logs. of numbers between 1 and 0. x +ve. and < 1.]

IV. $\log_e(1+x) = \log_e(1-x) + 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\}$.

[For calculating $\log_e(1+x)$ from $\log_e(1-x)$. $x < 1$.]

V. $\log_e(x+1) = \log_e(x-1) + 2\left\{\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right\}$.

[For calculating $\log_e(x+1)$ from $\log_e(x-1)$. $x > 1$.]

VI. $\log_e(x+1) = \log_e x + 2\left\{\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \ldots\right\}$.

[For calculating $\log_e(x+1)$ from $\log_e x$. $x \not< 1$.]

[Logarithms to any other base may be substituted for the Napierian logarithms in the above, provided the series be multiplied by μ.]

EXAMPLES. X.

1. Compute to 7 places of decimals the 15th and 25th roots of e.
2. Find to 5 places of decimals the numbers whose Napierian logarithms are .05, .125, $\bar{1}.998$.
3. Calculate to 7 places of decimals the Napierian logarithms of
   (i) 1.007, (ii) 1.03, (iii) $1.0\dot{0}\dot{1}$, (iv) $.99\dot{8}$, (v) .999, (vi) $.98\dot{3}$.
4. Given that $\mu = .43429$ for base 10; calculate $\log_{10}999$ and $\log_{10}1001$.
5. Find the Napierian logarithm of $\frac{1001}{999}$ correct to 16 places of decimals.
6. Find $\log_e(\frac{201}{200})$ to 7 places of decimals, and deduce $\log_e(\frac{199}{200})$.
7. Show that $\log_e 101 - \log_e 99 = \frac{1}{50}$ very nearly.
8. Given $\log_e 10 = 2.3025851$; calculate $\log_e 12$ to the same number of figures.
9. Find $\log_{10}7$, $\log_{10}11$, and $\log_{10}13$; given $\left.\begin{array}{r}\log_{10}2 = .30103000\\ \mu = .43429448\end{array}\right\}$.
10. Given $\log_e 10 = 2.3025851$, find $\log_{10}1.1$ to 6 places of decimals.
11. Find $\log_{10}3.001$ by a series, given $\left.\begin{array}{r}\log_{10}3 = .4771213\\ \log_{10}e = .4342945\end{array}\right\}$.
12. Solve the equation $10^x = 101$ to 5 places of decimals; given $\log_e 10 = 2.30258$.

(II) The Common System.

12. In this system, also called the **Briggsian System**, the base is 10, this number being the radix of the common scale of notation. There are great advantages in adopting for the base of a practical system of logarithms the number upon which our system of numerical notation is based. The advantages are these:—

(i) The mantissa of the logarithm is independent of the position of the decimal point in the number, and is the same for all numbers composed of the same significant figures in the same order.

(ii) The characteristic of the logarithm is determinable at once by inspection of the position of the decimal point in the number. Hence

(iii) Our table of logarithms, complete in every way for practical use, need only give the mantissae corresponding with certain collections of significant figures in the numbers, without regard to the position of the decimal point; and characteristics need not be tabulated. A table so formed will give not only the logarithms of integral numbers, but will, at one and the same time, supply the logarithms of all numbers partly or wholly fractional when expressed in the decimal form, numbers whose logarithms to any other base either would require separate tabulation or must be obtained by a subtraction of the logarithms of the numerator and denominator of the corresponding vulgar fraction. Now it will presently appear that, when the logarithms of all the natural numbers from 1 to 100,000 have been calculated to 7 places of decimals, by the application of a certain principle called the *theory of proportional parts* the logarithms of numbers composed of any number of figures may be readily calculated to the same degree of approximation. Hence, by the simple tabulation of 100,000 logarithms, we are supplied with the logarithms, correct to the 7th figure after the decimal point, of all numbers (integral or fractional) composed of any number of figures whatever.

13. We will proceed to prove the above important properties of common logarithms.

Prop. VII. "The mantissae are the same for the common logarithms of all numbers which differ only in the position of the decimal point."

Let C be the characteristic (integral), and M the mantissa (fractional), of the logarithm of any number x, so that $\log_{10}x = C + M$; then any number which differs from x only in the position of the decimal point may be represented by $x \times 10^n$, where n is some positive or negative integer. Now

$$\log_{10}(x \times 10^n) = \log_{10}x + \log_{10}10^n = C + M + n$$
$$= (C+n) + M = C' + M, \quad C' \text{ being integral and } M \text{ fractional.}$$

Hence the characteristic of $\log_{10}(x \times 10^n)$ is C' $(= C+n)$, and its mantissa M the same as that of $\log_{10}x$. Q.E.D.

Prop. VIII. "The characteristic of the common logarithm of a decimal number, *partly or wholly integral*, is zero or positive, and is *one less than the number of digits* in the integral portion."

Let x be a decimal number having n digits in its integral portion, so that it is not less than 10^{n-1} nor as great as 10^n, where n is some positive integer; then

$\log_{10}x$ is not less than $\log_{10}10^{n-1}$ nor as great as $\log_{10}10^n$,

i.e. $\log_{10}x$,, ,, $n-1$,, ,, n,

$\therefore$ $\log_{10}x = (n-1) + F$ (where F is zero or some positive proper fraction).

Hence the characteristic of $\log_{10}x$ is $n-1$, *i.e.* is zero or positive, and is one less than n, the number of digits in the integral portion of x. Q.E.D.

Prop. IX. "The characteristic of the common logarithm of a decimal number, *wholly fractional*, is negative and numerically *one more than the number of ciphers* preceding the first significant figure."

Let x be a decimal number, wholly fractional, having n ciphers preceding its first significant figure, so that it is not less than $\frac{1}{10^{n+1}}$ nor as great as $\frac{1}{10^n}$, where n is zero or some positive integer; then

$\log_{10}x$ is not less than $\log_{10}10^{-(n+1)}$ nor as great as $\log_{10}10^{-n}$,

i.e. $\log_{10}x$,, ,, $-(n+1)$,, ,, $-n$,

$\therefore$ $\log_{10}x = -(n+1) + F$ (where F is zero or some positive proper fraction).

Hence the characteristic of $\log_{10}x$ is $-(n+1)$, *i.e.* is negative and numerically one more than n, the number of ciphers preceding the first significant figure in the decimal value of x.

Q.E.D.

14. To sum up the results given by the above propositions we have

(i) The following rule for *determining by inspection the characteristic* of the common logarithm of any decimal number :—

Rule: When the decimal point does not come first in the number, the characteristic is positive and one less than the number of figures preceding the decimal point; when the decimal point does come first, it is negative and numerically one more than the number of ciphers immediately following the decimal point.

(ii) The means of writing down the common logarithm of any decimal number when that of a number is given which differs from the former only in the position of the decimal point. The mantissa is, by Prop. VII., the same as that of the given logarithm, and the proper characteristic is prefixed in accordance with the above rule for characteristics.

15. We now see why it is sufficient, in seeking the common logarithm of any decimal number, to find the logarithm of the integral number composed of its significant figures.

Example. Given $\left.\begin{array}{l}\log_{10}2 = .3010300\\ \log_{10}3 = .4771213\end{array}\right\}$, find $\log_{10}.00048$ and $\log_{10}4800$.

$$\log_{10}48 = 4\log_{10}2 + \log_{10}3 = 1.6812413,$$

$$\therefore \quad \log_{10}.00048 = \bar{4}.6812413, \qquad \log_{10}4800 = 3.6812413.$$

16. We can also determine by inspection the characteristic of the common logarithm of any root of a decimal number.

Example. Find the characteristic of $\log_{10}\sqrt[3]{.000427}$.

$$\log_{10}\sqrt[3]{.000427} = \tfrac{1}{3}\log_{10}.000427.$$

Now the characteristic of $\log_{10}.000427$ is $\bar{4}$, and putting $-4 = -6 + 2$, for purposes of division by 3, we see that the characteristic of $\log_{10}\sqrt[3]{.000427}$ is $\bar{2}$.

17. By means of Props. VIII. and IX. we are also able to solve certain questions as to the position of the decimal point in the value of any numerical expression consisting of products and quotients.

Example i. Given $\log_{10}3 = .4771213$, find the number of digits in the integral portion of $3(2.7)^{50}$.

Let $x = 3(2.7)^{50}$, then $\log_{10}x = \log_{10}3 + 50\log_{10}2.7 = 22.045316$.

Hence, since the characteristic of $\log_{10}x$ is 22, by Prop. VIII. the number of digits in the integral portion of x must be 23.

Example ii. Given $\log_{10}2 = .3010300$, find the position of the first significant figure in the decimal value of $\sqrt[3]{(.0016)^{20}}$.

Let $x = \sqrt[3]{(.0016)^{20}}$, then $\log_{10}x = \frac{20}{3}\log_{10}(.0016) = \overline{19}.360800$.

Hence, since the characteristic of $\log_{10}x$ is $\overline{19}$, by Prop. IX. the first significant figure must be the 19th after the decimal point (there being 18 ciphers).

EXAMPLES. XI.

1. Find, by inspection, the characteristics of the following common logarithms:—

(i) $\log 31.7$, (ii) $\log 2467000$, (iii) $\log 52115.32$,
(iv) $\log .0024$, (v) $\log 8.925$, (vi) $\log 85000.9$,
(vii) $\log 2008$, (viii) $\log .00007$, (ix) $\log .0067$,
(x) $\log .75$, (xi) $\log \sqrt[5]{.07}$, (xii) $\log \sqrt{.000053}$,
(xiii) $\log \left(\frac{.013}{1267.5}\right)^{\frac{1}{3}}$, (xiv) $\log \sqrt[6]{\frac{867}{(.06)^3}}$, (xv) $\log \sqrt[5]{\frac{9}{(10000)^6}}$,
(xvi) $\log \sqrt{\frac{3}{(.007)^3}}$, (xvii) $\log \sqrt[18]{.1^{100}}$, (xviii) $\log \frac{\sqrt[3]{.07}}{(300)^{\frac{3}{4}}}$.

2. Given $\log_{10}86750 = 4.947519$; write down $\log_{10}867.5$, $\log_{10}8.675$, and $\log_{10}.08675$.

3. Given $\log_{10}812.13 = 2.9096256$; write down $\log_{10}81.213$, $\log_{10}81213000$, and $\log_{10}.00081213$.

4. Given $\log_{10}2 = .3010300$, $\log_{10}3 = .4771213$, $\log_{10}7 = .8450980$; find

(i) $\log_{10}3.75$, (ii) $\log_{10}.5625$, (iii) $\log_{10}.0625$, (iv) $\log_{10}14.4$,
(v) $\log_{10}2.45$, (vi) $\log_{10}22.4$, (vii) $\log_{10}.0000021$, (viii) $\log_{10}6.75$.

5. Given $\log_{10}8.1617 = .9117806$; find the numbers whose common logarithms are $\bar{1}.9117806$, 3.9117806, $\bar{2}.0882194$, 4.0882194.

6. How many figures are there in the integral portions of the numbers whose common logarithms are 3.00271, $.28467$, 6.98015, $\frac{.367}{.05}$?

7. What is the position of the first significant figure in the numbers whose common logarithms are $\bar{1}.34816$, 4, $-\left(\frac{21.63}{4.2}\right)$?

8. How many digits are there in (i) 2^{64}, given $\log_{10}2 = .3010300$;
and in the integral portions of (ii) $\sqrt[7]{(2.25)^{60}}$, given $\log_{10}150 = 2.1760913$;
(iii) $\sqrt[3]{(2.5)^{32}}$, given $\log_{10}2 = .3010300$?

9. What is the position of the first significant figure in

(i) $(.12)^{24}$, given $\log_{10}5 = .6989700$, $\log_{10}1.5 = .1760913$;
(ii) $\sqrt[3]{(.007)^{34}}$, given $\log_{10}7 = .8450980$;
(iii) $\frac{(.0024)^{12}}{\sqrt[5]{.0000003}}$, given $\log_{10}1.2 = .0791812$, $\log_{10}1.6 = .2041200$?

10. Given $\log_{10}2$, $\log_{10}3$; find the integral values between which x must lie that the integral part of $(1.08)^x$ may contain 4 digits.

11. The integral part of $(3.981)^{100,000}$ contains 60,000 digits. Find $\log_{10}3981$ correct to five decimal places.

12. Show that $\left(\frac{21}{20}\right)^{100}$ is greater than 100, given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$.

CHAPTER IV.

Tables. Their Application.

18. We have seen in the last chapter how a table of logarithms is compiled. The logarithms of numbers are first calculated to the base e, and then those to any other base are obtained by multiplying the former by a certain constant modulus. For common logarithms this modulus is .43429448.

It was also there stated that, when the logarithms of the natural numbers up to a certain point have been calculated, by the application of a certain principle, called the Theory of Proportional Parts, those of all other numbers can be deduced to a degree of approximation that will depend upon the magnitude of the numbers to which, and the range over which, the principle is applied.

We will first prove the theory as applied to the logarithms of numbers, and then discuss its accuracy and mode of application in the case of common logarithms.

Prop. X. To show that, *when the differences are small compared with the number*, the change in the logarithm is approximately proportional to the change in the number.

We have
$$\log_x(n+d) - \log_x n = \log_x \frac{n+d}{n} = \log_x\left(1+\frac{d}{n}\right)$$
$$= \mu\left\{\frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \dots\right\}.$$

From this it is evident that, when d is so small when compared with n that all the powers of $\frac{d}{n}$ after the first may be neglected in the series,
$$\log_x(n+d) - \log_x n = \frac{\mu}{n} \cdot d,$$
i.e.,
$$\log_x(n+d) - \log_x n \propto d,$$
in other words, the increase in the logarithm is proportional to the increase in the number.

Q.E.D.

19. Applying the proposition of the preceding article to common logarithms, we will suppose that our table contains the logarithms of numbers from 1 to 100,000 (so that n contains 5 figures), and that d is not greater than unity; then, since μ is less than $\frac{1}{2}$ for common logarithms,

$\frac{\mu d^2}{2n^2}$ is less than $\frac{1}{4}\left(\frac{1}{10,000}\right)^2$, and *a fortiori* less than ·000000003;

$\frac{\mu d^3}{3n^3}$ is less than $\frac{1}{10,000}$th of this, and so on;

each term of the series being less than $\frac{1}{10,000}$th of the preceding term.

Hence, at least *as far as seven places of decimals*,

$$\log_{10}(n+d) - \log_{10}n = \frac{\mu}{n} \cdot d.$$

[For a table of logarithms from 1 to 1000 the theory of proportional parts will give results true at least to 3 places of decimals, while logarithms from 1000 to 10,000 will give results true to 5 places of decimals.]

Taking d to be any decimal fraction, so that $n+d$ is a mixed decimal number, we are able, by applying the principle of proportional parts, to obtain to seven places of decimals the logarithms of numbers in which the decimal point comes after the fifth figure, and thence, by merely altering the characteristics of the results, the logarithms of numbers with the decimal point holding any position in those numbers. Or since, as should clearly be the case, the theory of proportional parts is not vitiated when d and n are both multiplied by any one and the same power of 10, the logarithms of integers containing seven or even eight figures can be calculated from the logarithms of numbers having the same first five figures and ciphers affixed to make the total number of figures the same in the two numbers, though d in this case is greater than unity: the characteristics can then be altered to suit the positions of the decimal point when the numbers are not integral.

The theory of proportional parts is utilised not only for finding the logarithms of given numbers, but also for finding the numbers corresponding with given logarithms. In this latter case, when the logarithms are given to seven places of decimals, we can always get 7-figure results, and when the differences are large between the successively tabulated logarithms we may get 8 figures, but after this point additional figures in the number do not affect the logarithms to seven places of decimals, so that, with 7-figure logarithms, numbers having given logarithms can never be found correct to more than 8 figures.

[With 5-figure logarithms we can get 5-figure results always, and never more than 6 figures.]

20. To show how this principle is applied in practice

(i) To find the logarithm of a number of not more than 8 significant figures,

(ii) To find the number corresponding with any logarithm not given exactly in the tables,

we will take an example of each case, making use of tables that give the logarithms of numbers from 1 to 100,000 to 7 places of decimals.

Example (i). Given $\left.\begin{array}{l}\log 34567 = 4.5386617 \\ \log 34566 = 4.5386491\end{array}\right\}$ find log 345.66269.

We have 3 additional figures in the number whose logarithm is required, and therefore affix 3 ciphers to each of the given numbers, which will not affect their mantissae.

The mantissae of the logarithms of 34566000 and 34567000 are .5386491 and .5386617 respectively: therefore the mantissa of the logarithm of 34566269, which lies between 34566000 and 34567000, will have some value between .5386491 and .5386617; its first four figures will be 5386 and the remaining three will compose some number between 491 and 617; call it $491+d$. Now arrange the numbers and their corresponding logarithms in the parallel columns, in ascending or descending order, and couple the quantities *in the same way in each column*, one coupling on each side. The four differences placed outside the couplings are then four numbers in proportion. It does not really matter how the couplings are arranged, provided only they are made in the same way and in the same order in both columns, but it is *advantageous* to couple the two extremes together, and the mean with that extreme which will give a difference d outside the coupling, i.e. with the smallest of the three occurring in the d column.

$$\begin{array}{ccc} \text{No.} & \qquad & \text{Log.} \\ 1000\left(\begin{array}{l|l} 3456 & 7000 \\ 3456 & 6269 \\ 3456 & 6000 \end{array}\right)269 & & 126\left(\begin{array}{l|l} \cdot 75386 & 617 \\ \cdot 75386 & 491+d \\ \cdot 75386 & 491 \end{array}\right)d \end{array}$$

Thus, $\qquad 1000 : 269 :: 126 : d,$

or $\qquad d = \dfrac{269}{1000}\text{ths of } 126 = \left(\dfrac{2}{10}+\dfrac{6}{100}+\dfrac{9}{1000}\right)126 = 25+8+1 = 34.$

Hence $\qquad 491+d = 525,$

$\therefore \qquad \log 34566269 = 7.5386525,$

and $\qquad \log 345.66269 = 2.5386525.$

In an example of this kind it is not usual to work as fully as in the above illustration. The figures to the left of the lines indicated may be omitted, being the same in every line, and our calculation is much facilitated by the use of the table of differences generally given in a column in the logarithmic tables by the side of those logarithms to which the differences severally apply. On referring to the tables, we find 126 in the table of differences, and under this number its *tenth parts* worked out ready for use. The hundredths and thousandths are obtained by cutting off figures successively from the given tenths.

	126
1	13
2	25
3	38
4	50
5	63
6	76
7	88
8	101
9	113

[Here we have the table of tenths for difference 126. It will be observed that in tabulating the tenths, when the first figure left out is greater than 4, the figure at which we stop is increased by unity, but there is no such increase when the first figure omitted is less than 5. The reason for this is that by this process the value that is nearer to the true value is always taken, and that though the values taken are sometimes too large, sometimes too small, in the end the deficiencies and excesses tend to compensate one another.]

From the above table we see that

$$\begin{array}{lr} \frac{2}{10}\text{ths of the difference} & = 25 \\ \frac{6}{100}\text{ths} \quad ,, \qquad ,, & = 8 \\ \frac{9}{1000}\text{ths} \quad ,, \qquad ,, & = \underline{1} \\ \qquad \text{Total correction} & = 34 \end{array}$$

The working is generally written down thus:

$$\begin{array}{rrl} & \log 34566000 = & 7.5386491 \\ & 2 & \qquad\quad 25 \\ & 6 & \qquad\quad\ 8 \\ & \underline{9} & \qquad\quad\ \underline{1} \\ \therefore & \log 34566269 = & 7.5386525 \end{array}$$

Example (ii). Given $\left.\begin{array}{l}\log 86554=4.9372871\\ \log 86553=4.9372821\end{array}\right\}$ find the number whose logarithm is $\bar{2}.9372847$.

The mantissa of the logarithm whose number is required lies between the two given mantissae: therefore the number lies between 86553 and 86554, and consists of 86553 with additional figures. Now the difference between the two given mantissae is 50, and we can only get two additional figures, since thousandths of the difference 50 will not affect the 7th figure of the mantissa. Call the number composed of these additional figures d, and arrange in parallel columns as before, affixing two ciphers.

$$\begin{array}{ccc} \text{No.} & \text{Log.} & \\ 100\left(\begin{array}{l}400\\300+d\\300\end{array}\right)d & 50\left(\begin{array}{c}71\\47\\21\end{array}\right)26 & \therefore\ 50d=2600,\ \text{i.e. } d=52, \end{array}$$

and the required number is .08655352, the decimal point being so placed that the characteristic may be $\bar{2}$.

Here also if we are supplied with tables of differences our working is facilitated.

	50
1	5
2	10
3	15
4	20
5	25
6	30
7	35
8	40
9	45

[The difference between the logarithm whose number is required and the smaller of the two given logarithms is 26. Now the nearest number to this in the table of difference 50 is 25, and this is seen to be $\frac{5}{10}$ths of the difference; therefore 5 is the first additional figure required. Also $26-25=1$, and this is $\frac{2}{100}$ths of the difference ($\frac{2}{10}$ths with one figure cut off); therefore 2 is the next figure required. Hence $d=52$.]

21. We will conclude the chapter by applying the processes of the last article to one or two examples.

Example (i). Given $\log 2=.3010300$, $\log 7=.8450980$, $\log 90762=4.9579041$, $\log 90763=4.9579088$; find the value of $\sqrt[3]{\left(\frac{294\times125}{42\times32}\right)^2}$ to 6 places of decimals.

Let $x=\sqrt[3]{\left(\frac{294\times125}{42\times32}\right)^2}$; then

$$\begin{aligned}\log x &= \tfrac{2}{3}(\log 294+\log 125-\log 42-\log 32)\\ &= \tfrac{2}{3}(3\log 5+\log 7-5\log 2)\\ &= .9579053.\end{aligned}$$

$$100\left(\begin{array}{l}300\\200+d\\200\end{array}\right)d \qquad 47\left(\begin{array}{c}88\\53\\41\end{array}\right)12 \qquad \begin{aligned}47d&=1200,\\ d&=26,\\ \therefore\ x&=9.076226.\end{aligned}$$

Example (ii). Find a 3rd proportional to .0024 and 27; given
$\log 2=.3010300$, $\log 30375=4.4825163$,
$\log 3=.4771213$, $\log 30376=4.4825306$.

Let x be the required proportional;

then $$.0024:27::27:x,$$

$\therefore$ $$x=\frac{27^2}{.0024},$$

and $$\begin{aligned}\log x &= 2\log 27-\log .0024\\ &= 2.8627278-\bar{3}.3802113\\ &= 5.4825165.\end{aligned}$$

$$1000\left(\begin{array}{l}6000\\5000+d\\5000\end{array}\right)d \qquad 143\left(\begin{array}{c}306\\165\\163\end{array}\right)2 \qquad \begin{aligned}143d&=2000,\\ d&=14,\\ \therefore\ x&=303750.14.\end{aligned}$$

Example (iii). If $13^x = 2147.6827$, find x to 6 places of decimals;

$$\begin{array}{ll} \text{given } \log 13 = 1.1139434 & \log 21477 = 4.3319736 \\ & \log 21476 = \underline{4.3319534} \\ & \qquad\qquad\qquad\quad 202 \end{array}$$

$$\begin{array}{rrcr} & \log 2147.6000 & = & 3.3319534 \\ & 8 & & 162 \\ & 2 & & 4 \\ & \underline{\quad 7} & & \underline{\quad 1} \\ \therefore & \log 2147.6827 & = & 3.3319701 \end{array}$$

Taking logs. in the given equation,

$$x \log 13 = \log 2147.6827,$$

$$x = \frac{\log 2147.6827}{\log 13}$$

$$= \frac{3.3319701}{1.1139434} = 2.991149.$$

EXAMPLES. XII.

1. (i) Given $\left.\begin{array}{l}\log 56500 = 4.7520484 \\ \log 56501 = 4.7520561\end{array}\right\}$, find $\log .005650076$, and the number whose logarithm is $.7520516$.

(ii) Given $\left.\begin{array}{l}\log .82673 = \bar{1}.9173637 \\ \log .82672 = \bar{1}.9173584\end{array}\right\}$, find $\log 82672.38$, and the number whose logarithm is $\bar{3}.9173600$.

(iii) Given $\log 37.186 = 1.5703795$, diff. $= 117$; construct a table of proportional parts and use it to find $\log .37186378$, and the number whose logarithm is $\bar{2}.5703713$.

(iv) Given $\log .41556 = \bar{1}.6186337$, diff. $= 105$; find $\log 41.55578$, and the number whose logarithm is 3.6186384.

(v) Given $\left.\begin{array}{l}\log 20.867 = 1.3194600 \\ \log 20.866 = 1.3194392\end{array}\right\}$, find $\log .020866327$, and the number whose logarithm is $\bar{1}.6805452$.

(vi) Given $\left.\begin{array}{l}\log 535.35 \;\;= 2.7286378 \\ \log .053536 = \bar{2}.7286459\end{array}\right\}$, find $\log 53535.87$, and the number whose logarithm is $\bar{2}.7286403$.

(vii) Given $\left.\begin{array}{l}\log 81.22 = 1.9096630 \\ \log 81.23 = 1.9097165\end{array}\right\}$, find $\log .8122365$, and the number whose logarithm is 2.9096844.

(viii) Given $\left.\begin{array}{l}\log 13.145 = 1.1187606 \\ \log 131.46 = 2.1187936\end{array}\right\}$, find $\log 13145125$, and the number whose logarithm is 3.1187804.

(ix) Given $\left.\begin{array}{l}\log 367200 = 5.5649027 \\ \log 367100 = 5.5647844\end{array}\right\}$, find $\log .003671453$, and the number whose logarithm is $\bar{2}.5648815$.

(x) Given $\log 2 = .3010300$ and $\log 2000.1 = 3.3010517$; construct a table of proportional parts, and find $\log .2000088$.

(xi) The mantissae of the logarithms of 79531 and 79532 are respectively .9005364 and .9005419; find the logarithm of 795.314, and find the number of which the logarithm is $\bar{2}.900539$.

(xii) Find the number whose logarithm is -1.8753145; given $\left.\begin{array}{l}\log 1.3325 = .1246672 \\ \log 1.3326 = .1246998\end{array}\right\}$.

(xiii) Given $\log 31204 = 4.4942103$, $\log 31203 = 4.4941964$; construct the table of proportional parts, and thence find $\log 31203.25$.

(xiv) Given $\log 1.0007 = .0003039$, find $\log 100024$.

(xv) Given $\left.\begin{array}{l}\log \frac{131}{132} = \bar{1}.9966974 \\ \log \frac{132}{133} = \bar{1}.9967223\end{array}\right\}$, find $\log \frac{1315}{1325}$, and the number whose logarithm is $.0032857$.

(xvi) The following is an extract from a logarithm book :

N.	0	1	2	3	4	5	6	7	8	9	Dif.
1905	2798950	9178	9406	9634	9862	$\bar{0}090$	$\bar{0}317$	$\bar{0}545$	$\bar{0}773$	$\bar{1}001$	228

What is the logarithm of $190\frac{1}{2}$, and of 190595.7? Divide the product of these two quantities by 19057.5, using logarithms to obtain the result to two places of decimals.

2. Find the values of (i) $\dfrac{403.09 \times .002317 \times 17}{18.543}$

$\log 17 = 1.2304489$
$\log 4.0309 = .6054020$
$\log 2.317 = .3649260$
$\log 1.8543 = .2681800$
$\log 8.5624 = .9325955$, $D = 51$

(ii) $\left(\dfrac{4^3 \times 25^2}{2^6 \times 10^2}\right)^{\frac{1}{2}}$ $\log 25 = 1.3979400$

(iii) $\sqrt[9]{.00007}$ $\log 7 = .8450980$
$\log 9.824394 = .9923057$

(iv) $\sqrt[5]{.07}$ $\log 7 = .8450980$ $\log 58751 = 4.7690153$
$\log 58752 = 4.7690227$

(v) $\sqrt[6]{9\sqrt{3\sqrt{2}}}$ $\log 2 = .3010300$ $\log 1626 = 3.21112$
$\log 3 = .4771213$ $\log 1627 = 3.21139$

(vi) $(.001)^{.001}$ $\log 9.9328 = .9960323$
$\log 9.9329 = .9970367$

(vii) $\sqrt[7]{100}$ $\log 193.06 = 2.2856923$
$\log 19307 = 4.2857148$

(viii) $\sqrt[7]{23}$ $\log 23 = 1.3617278$ $\log 15650 = 4.1945143$
$\log 15651 = 4.1945421$

(ix) $\sqrt[3]{\dfrac{1}{19.053}}$ $\log 3.7440 = .5733358$, $D = 116$
$\cancel{\log 1}9053 = 4.2799634$

(x) $\sqrt[3]{315}$ $\log 3150 = 3.4983106$
$\log 3.1598 = .4996596$
$\log 3.1599 = .4996733$

(xi) $\sqrt[5]{\dfrac{(.3796)^3}{(.2984)^2}}$ $\log 3796 = 3.5793262$
$\log 2984 = 3.4747988$
$\log 90714 = 4.9576743$
$\log 90715 = 4.9576791$

(xii) $\sqrt{\dfrac{3\sqrt[3]{138}}{\sqrt[5]{.01}}}$ $\log 3 = .4771213$
$\log 1.38 = .1398791$
$\log 6.2403 = .7952055$, $D = 69$

(xiii) $\dfrac{1089 \times .01881 \times .405}{\sqrt[3]{(729)^4}}$

$\log 4 = .6020600$
$\log 60 = 1.7781513$
$\log 1.1 = .0413927$
$\log 19 = 1.2787536$
$\log .012644 = \bar{2}.1018845$, $D = 343$

(xiv) $\sqrt[5]{\dfrac{6300 \times .00117 \times 42.9}{\frac{1}{2}(2197)^{\frac{1}{3}}}}$

$\log 2 = .3010300$ $\log 7 = .8450980$
$\log 13 = 1.1139434$ $\log .011 = \bar{2}.0413927$
$\log 90 = 1.9542425$
$\log 217.47 = 2.3373994$, $D = 200$

3. Calculate the product of the 10th root of 5 by the 5th root of 10.

log 2 = .3010300 log 18616 = 4.2698864
log 18617 = 4.2699097

4. Find a 4th proportional to the 5th power of 11, the 4th power of 7, and the 5th power of 5; and calculate to 4 places of decimals the value of $\frac{(330 \times \frac{1}{49})^4}{\sqrt[3]{22 \times 70}}$.

log 2 = .3010300 log 17814 = 4.2507614
log 3 = .4771213 log 17815 = 4.2507858
log 7 = .8450980 log 46588 = 4.6682741
log 11 = 1.0413927 log 46589 = 4.6682834

5. If $\frac{\log x}{\log 26} = .8567$, what is the value of x?

log 2 = .3010300 log .13 = $\bar{1}$.1139434
log 16300 = 4.2121876, $D = 266$

6. If $3^x = 7175.37$, find x.

log 3 = .4771213 log 71.753 = 1.8558401, $D = 60$

7. If $7^x = 823542.4$, find x.

log 7 = .8450980 log 8.2354 = .9156847, $D = 53$

8. Calculate the value of $1 + e + e^2 + \ldots + e^9$, when $e = 2.71828$.

log 27182 = 4.4342814, $D = 160$
log 22026 = 4.342936, $D = 19$

9. If the side of a cube be 8, find the side of another cube of exactly double the volume of the former. log 2 = .3010300 log 10079 = 4.003417, $D = 43$

10. A solid cube of lead weighs 126.44 lbs. 998 ozs. of water occupy one cubic foot, and a cubic foot of lead is 11.352 times as heavy as a cubic foot of water. Find the length of a side of the cube of lead correctly to 6 places of decimals of a foot.

log .012644 = $\bar{2}$.1018845 log 56.311 = 1.7505932, $D = 78$
log 1.1 = .0413927 log 129 = 2.1105897
log 49.9 = 1.6981005

CHAPTER V.

Variable Base.

22. We have hitherto confined ourselves to questions involving a constant base; we will now investigate the relation that exists between the logarithms of numbers to different bases.

Prop. XI. To prove $\log_b a \times \log_c b = \log_c a$.

$$\text{We have}\quad \left.\begin{aligned} b^{\log_b a} &= a, \quad \ldots\ldots\ldots \text{(i)} \\ c^{\log_c b} &= b, \quad \ldots\ldots\ldots \text{(ii)} \\ c^{\log_c a} &= a, \quad \ldots\ldots\ldots \text{(iii)} \end{aligned}\right\} \quad \text{Hence}\quad \begin{aligned} b^{\log_b a \,.\, \log_c b} &= a^{\log_c b} && \text{by (i)} \\ &= c^{\log_c a \,.\, \log_c b} && \text{by (iii)} \\ &= b^{\log_c a} && \text{by (ii)} \end{aligned}$$

$$\therefore \quad \log_b a \,.\, \log_c b = \log_c a. \qquad \text{Q.E.D.}$$

This result gives us $\log_b a = \log_c a \times \left(\frac{1}{\log_c b}\right)$, from which it is evident that when the logarithms of numbers have been calculated to any base c, those to any other base b are got by multiplying the former by the constant quantity $\frac{1}{\log_c b}$.

23. In the same way as in Prop. XI it can be shown that

$$\log_b a \,.\, \log_c b \,.\, \log_d c = \log_d a,$$

and so on, for any number of logarithmic factors in which the number for each successive logarithm in the product is the base for the immediately preceding one.

This result is easily remembered and applied, in consequence of the analogy it bears to the result obtained by compounding any number of ratios in which each consequent becomes the next antecedent. Thus

$$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \ldots \times \frac{x}{y} \times \frac{y}{z} = \frac{a}{z},$$

and reading the numerators as numbers, and the corresponding denominators as bases of logarithms, we have $\log_b a \,.\, \log_c b \,.\, \log_d c \ldots \log_y x \,.\, \log_z y = \log_z a.$

Again, by Prop. XI,
$$\log_b a = \frac{\log_c a}{\log_c b},$$

and this correspons with the identity
$$\frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}}.$$

24. **Prop. XII.** To prove $\log_b a \,.\, \log_a b = 1$.

This of course follows from Prop. XI by putting $c = a$, since $\log_a a = 1$; but it can be proved independently thus:

$$\left.\begin{array}{l} b^{\log_b a} = a, \;\ldots\ldots\ldots\; \text{(i)} \\ a^{\log_a b} = b, \;\ldots\ldots\ldots\; \text{(ii)} \end{array}\right\} \quad \text{Hence } b^{\log_b a \,.\, \log_a b} = a^{\log_a b} \text{ by (i)}$$
$$= b \quad \text{by (ii)}$$
$$\therefore \quad \log_b a \,.\, \log_a b = 1. \qquad \text{Q. E. D.}$$

25. *Example* (i). Given $\left.\begin{array}{l}\log_{10}2 = .3010300 \\ \log_{10}3 = .4771213\end{array}\right\}$, find $\log_{25}18$.

$$\log_{25}18 = \frac{\log_{10}18}{\log_{10}25} = \frac{2\log_{10}3 + \log_{10}2}{2\log_{10}5} = \frac{1.2552725}{1.3979400} = .8979445.$$

Example (ii). Given $\left.\begin{array}{l}\log_8 9 = 1.056642 \\ \log_8 5 = \;\; .7739760 \\ \log_5 7 = 1.209062\end{array}\right\}$, calculate the common logs. of the nine digits.

It will be sufficient to calculate $\log_{10}2$, $\log_{10}3$, and $\log_{10}7$, for the logarithms of the nine digits are easily expressed in terms of these.

We have
$$\left.\begin{array}{ll} \dfrac{2\log_{10}3}{3\log_{10}2} = 1.056642, \;\ldots\ldots\ldots & \text{(i)} \\ \dfrac{1-\log_{10}2}{3\log_{10}2} = \;.7739760, \ldots\ldots\ldots & \text{(ii)} \\ \dfrac{\log_{10}7}{1-\log_{10}2} = 1.209062, \;\ldots\ldots\ldots & \text{(iii)} \end{array}\right\} \quad \begin{array}{l} \text{From (ii) } \log_{10}2 = .3010300 \\ \text{substituting in (i) } \log_{10}3 = .4771213 \\ \text{substituting in (iii) } \log_{10}7 = .8450980 \end{array}$$

whence
$$\begin{array}{lll} \log_{10}4 = & 2\log_{10}2 & = .6020600, \\ \log_{10}5 = & 1 - \log_{10}2 & = .6989700, \\ \log_{10}6 = & \log_{10}2 + \log_{10}3 & = .7781513, \\ \log_{10}8 = & 3\log_{10}2 & = .9030900, \\ \log_{10}9 = & 2\log_{10}3 & = .9542426. \end{array}$$

Example (iii). By what must logarithms to the base 2 be multiplied to find them to the base 8?

Since $\log_8 x = \log_2 x \times \log_8 2$, the required multiplier is $\log_8 2$ or $\log_{2^3}2$, i.e. $\frac{1}{3}$.

EXAMPLES. XIII.

1. Prove that (i) $x = \sqrt[\log_x a]{a}$, (ii) $\log_{a^r} x = \log_a \sqrt[r]{x}$, (iii) $\dfrac{\log_a n}{\log_{ma} n} = 1 + \log_a m$,

(iv) $\left(\dfrac{\log_a m}{\log_b n}\right)^2 = \dfrac{m}{n}$, (v) $\dfrac{\log_a\{\sqrt{\log_a b}\}}{\sqrt{\log_a b}} + \dfrac{\log_b\{\sqrt{\log_b a}\}}{\sqrt{\log_b a}} = 0$.

2. Given $\log_{10}2 = .3010300$, $\log_{10}3 = .4771213$, $\log_{10}7 = .8450980$; find
(i) $\log_3 8$, (ii) $\log_7 25$, (iii) $\log_{2.5}.05$, (iv) $\log_{27}3.2$,
(v) $\log_6 7$, (vi) $\log_{12.5}20$, (vii) $\log_{12}.7$, (viii) $\log_5 18$,
(ix) $\log_{\frac{5}{2}\sqrt{3}}.6$, (x) $\log_5 .0021$, (xi) $\log_{1.4}\sqrt{\frac{1}{2}}$, (xii) $\log_{8\sqrt{2}}\sqrt{\dfrac{2\sqrt{3}}{\sqrt[3]{7}}}$,
(xiii) $\log_{\sqrt{3}}.7$, (xiv) $\log_{\sqrt{\frac{3}{2}}\frac{1}{3}}\left\{\left(\dfrac{2.7}{4.9}\right)^{\frac{1}{2}} \times \left(\dfrac{1.5}{1.6}\right)^6\right\}$.

3. Given $\left.\begin{array}{l}\log_5 3 = \;\;.6826063 \\ \log_3 \frac{1}{2} = \bar{1}.3690703\end{array}\right\}$, find $\log_{10}2$, $\log_{10}3$.

4. Given $\log_{10}5 = .6989700$, find $\log_{25}40$ and $\log_{40}25$.

5. Given $\log_{10}35 = 1.5440680$, find $\log_{1000}(\frac{343}{8})$.

6. Given $\log_{10}5 = .6989700$, find $\log_{2.5}10$.

7. Given $\left.\begin{array}{l}\log_{10} 5.6 = .7481880 \\ \log_{10} 73.5 = 1.8662873 \\ \log_{10} 10.8 = 1.0334238\end{array}\right\}$, find $\log_5 12$ and $\log_6 .07$.

8. Given $\left.\begin{array}{l}\log_8 9 = a \\ \log_3 5 = b\end{array}\right\}$, find the logarithms to the base 10 of the first 4 digits.

9. Given $\left.\begin{array}{l}\log_8 3 = m \\ \log_{25} 24 = n\end{array}\right\}$, find the common logarithm of 45.

10. Given $\log_{10} x = \bar{3}.6102407$, $\log_{10} y = 2.2481883$; find

(i) $\log_x \dfrac{100\sqrt[3]{xy}}{\sqrt[2]{x^3 y^3}}$, (ii) $\log_y\left\{\sqrt[5]{\left(\dfrac{y}{x}\right)^2} \div \sqrt{\left(\dfrac{x}{y}\right)^5}\right\}$.

11. Given $\left.\begin{array}{l}\log_{10} x = \bar{2}.6483117 \\ \log_{10} 2 = .3010300\end{array}\right\}$, find $\log_4 x$.

12. The logarithm of a number to the base 4 is .35184, find its logarithm to the base 16.

13. By what must logarithms to the base $\sqrt{2}$ be multiplied to find them to the base $\sqrt[3]{3}$? given $\log_e 2 = .6931472$, $\log_e 3 = 1.0986123$.

14. If x be the logarithm of a to the base b, what is the logarithm of a^m to the base b^n?

15. Show that the logarithm of any number to the base a^n is a mean proportional between its logarithms to the bases a and a^{n^2}.

16. Solve the equation $\frac{1}{3}\log_x 8 = 3 \log_8 x$.

17. If a, b, c be in G.P., prove that $\log_a N$, $\log_b N$, $\log_c N$ will be in H.P.

18. If a, b, c be in G.P., and $\log_c a$, $\log_b c$, $\log_a b$ in A.P., then the common difference of the latter is $1\frac{1}{2}$.

19. If a, b, c be respectively the two sides and the hypotenuse of a right-angled triangle, then $\log_{b+c} a + \log_{c-b} a = 2 \log_{b+c} a \cdot \log_{c-b} a$.

20. From the formula $\log_e\left(\dfrac{1}{1-x}\right) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots$, calculate $\log_{10} 5$ to 5 places of decimals. Given $\log_{10} e = .43429$.

21. If $x_3 = \log_{x_1} x_2$, $x_4 = \log_{x_2} x_3$, ..., $x_n = \log_{x_{n-2}} x_{n-1}$, $x_1 = \log_{x_{n-1}} x_n$, then $x_1 \cdot x_2 \ldots x_n = 1$.

CHAPTER VI.

Interest. Annuities Certain.

I. Interest.

26. When a sum of money is lent for a time, the borrower pays to the lender a certain sum for the use of it. The sum lent is spoken of as the *Capital* or *Principal:* the sum paid for the use of it is called the *Interest.* The *Amount* is the Principal plus the total Interest at the end of the time for which the money was lent.

27. (i) **Simple Interest.** The interest paid for the use of money is said to be *Simple* when it consists of a certain *fixed* sum paid at regular periods. It is generally reckoned at so much per cent. per annum. Thus 5 per cent. (5°/ₒ) per annum means £5 paid annually on every £100 of Capital. If there be p periods of payment in the course of a year, 5 per cent. per annum would signify £$\frac{5}{p}$ paid every period on each £100 of Capital.

Let P be the Principal, A the amount at the end of n years, and I the Interest accruing in the same time at 100 r per cent. per annum; so that r is the interest on £1 in 1 year; then, for *Simple Interest*,

$$A = P(1+nr), \quad \ldots\ldots\ldots\ldots (1)$$

$$I = nrP \quad \ldots\ldots\ldots\ldots (2)$$

28. (ii) **Compound Interest.** Sometimes the borrower pays at the end of the whole time for which the money was lent a *single* sum to cover both Principal and Interest. In this case since the lender loses the use of the Interest as it accrues, it is clear that the borrower should pay interest on this also; in other words, that this Interest should be added to the Principal as it becomes due, and that the borrower should proceed to pay Interest on the Principal so increased. In such a case the Principal or money lent is said to bear *Compound* Interest. It is reckoned in the same way as Simple Interest.

When the Interest becomes due it is said to be *convertible* (into Principal), and the period between two successive times at which Interest becomes convertible is called the *conversion-period.*

Let P be the Principal, A the Amount at the end of n periods, and I the Interest accruing in the same time; while r is the interest on £1 for one conversion-period, and $R = 1 + r$; then, for *Compound Interest,*

$$A = PR^n \text{ or } P = AR^{-n}, \quad \ldots\ldots (1)$$
$$I = P(R^n - 1) \quad \ldots\ldots (2)$$

[The formulae given above for Simple and Compound Interest are applicable to questions of Present Worth and Discount, P being the Present Worth of a debt A, due in time n, and I being the Discount.]

29. *Example* (i). Find the amount of £1000 invested for 21 years at 3% per annum, Compound Interest, convertible half-yearly.

r (the half-yearly interest on £1) = .015

$\therefore \quad R = 1.015, \quad$ and $n = 42$.

Hence $\quad A = PR^n = 1000(1.015)^{42}$

and taking logs. $\quad \log A = \log 1000 + 42 \log 1.015$,

$$\begin{array}{rl} \log 1.015 = & .0064660 \\ & \underline{42} \\ & 129320 \\ & \underline{258640} \\ 42 \log 1.015 = & .271572 \end{array}$$

$\log 1000 = 3$,

$\therefore \quad \log A = 3.271572$

Now $\quad \log 1868.8 = \underline{3.271563}$

$D = 23 \quad)90(4$

$\underline{92} \quad \therefore \quad A = £1868.84.$

Example (ii). How long will it take for a sum of money to double itself at 6% per annum, Compound Interest, convertible annually?

Here $\quad P = 1, \; A = 2, \; R = 1.06.$

Let n years be the required time, then $2 = (1.06)^n$,

and taking logs. $\quad \log 2 = n \log 1.06$,

$$\therefore \quad n = \frac{\log 2}{\log 1.06} = \frac{.3010300}{.0253059} = 12 \text{ years.}$$

Example (iii). If the number of births and deaths be 3.5 and 1.2 per cent. respectively of the population at the beginning of each year, after how many years will the population be trebled?

The annual increase is 2.3 per cent. or .023 of the population at the commencement of the year. Let n years be the required time,

then $\quad (1.023)^n = 3$,

and taking logs. $\quad n \log 1.023 = \log 3, \qquad n = \dfrac{\log 3}{\log 1.023} = 48.3$ years.

EXAMPLES. XIV.

[*Compound Interest is understood unless the contrary is stated.*]

1. Find the amount of £1000 in 10 years, allowing 5 per cent. per annum interest.
   $\log 2 = .3010300, \quad \log 7 = .8450980,$
   $\log 3 = .4771213, \quad \log 1.627 = .211893.$ *Sandhurst.*

2. In what time will a sum of money treble itself at 5 per cent. per annum? Given log 2, log 3, and log 7. *Sandhurst.*

3. Find, correct to a farthing, the present value of £10000 due 8 years hence at 5 per cent. per annum.
Given log 2, log 3, log 7, $\log 67683 = 4.8304796$,
$\log 67684 = 4.8304860$. *Woolwich.*

4. Find the amount of £5500 in 15 years at 5 per cent. per annum, giving the result in £'s and the decimal of a £.
Given log 2, log 3, log 7, $\log 11 = 1.0413927$,
$\log 1.1434 = .0581982$,
$\log 1.1435 = .0582362$. *Woolwich.*

5. Find by logarithms what the annual income will be if £2700 stock be sold at 90 and re-invested in the 3 per cents. at 125.
$\log 27 = 1.4313638$, $\log 25 = 1.3979400$,
$\log 90 = 1.9542425$, $\log 972 = 2.9876663$. *Woolwich.*

6. Find the total interest, payable half-yearly, on £100 for 12 years at 5 per cent. per annum.
$\log 2 = .3010300$, $\log 18087 = 4.257367$,
$\log 41 = 1.6127839$, $\log 18088 = 4.257391$.

7. Find the present value of £1000 due 10 years hence, reckoning interest at 4 per cent. per annum.
$\log 2 = .3010300$, $\log 67557 = 4.829670$,
$\log 130 = 2.1139434$, $\log 67556 = 4.829664$.

8. At what rate per cent. must money be lent that it may be doubled in 10 years?
$\log 2 = .3010300$, $\log 10717 = 4.0300732$, $D = 406$.

9. How long will it take for £1000 to amount to £2500 at 5 per cent. per annum, convertible half-yearly?
$\log 2 = .3010300$, $\log 41 = 1.6127839$.

10. A sum is laid out at 10 per cent. per annum, convertible annually, and another sum of double the amount at 5 per cent. per annum, convertible half-yearly. In what time will the two amounts be equal?
$\log 11 = 1.0413927$, $\log 1025 = 3.0107239$.

11. In how many years will the Interest on a given sum amount to double the Principal at $3\frac{1}{2}$ per cent. per annum?
$\log 3 = .4771213$, $\log 115 = 2.0606978$.

12. Find the amount of £1000 at the end of 10 years, allowing 10°/ₒ per annum interest, convertible half-yearly.
$\log 10225 = 4.0096633$, $\log 15605 = 4.193264$.

13. A country trebles its population in a century. What is the increase in one year per million?
Given log 2, log 3, $\log 67 = 1.8260748$.

14. What is the amount of one farthing for 500 years at 3 per cent. per annum?
$\log 103 = 2.0128372$, $\log 26218 = 4.4185996$. $D = 165$.

15. At what rate per cent. per annum will a given sum increase 11-fold in a century?
$\log 11 = 1.0413927$, $\log 10242 = 4.0103848$, $D = 424$.

16. A sum of money when put out to interest, payable half-yearly, amounts to £2316. 10s. in 5 years, and £2708. 5s. in 9 years. What is the rate of interest?
$\log 23165 = 4.3648323$,
$\log 27082 = 4.4326807$, $D = 161$,
$\log 10197 = 4.0084724$, $D = 426$.

17. On the birth of an infant £1000 is invested in the Funds ($2\frac{3}{4}$°/ₒ payable half-yearly). Calculate what it will be worth when the child is 21 years old to the nearest shilling.
$\log 10137 = 4.0059094$, $D = 429$,
$\log 17745 = 4.249076$, $D = 25$.

18. A person borrowed £11000 for two months at 5 per cent. per annum. At the end of the time the interest was added on, and the debt renewed for another two months. This was continually repeated till at the end of 2 years the debt and interest were paid. How much did this amount to?

$\log 2 = .3010300$, $\log .011 = \bar{2}.0413927$,
$\log 90 = 1.9542425$, $\log 121.51 = 2.0846120$, $D = 357$. *Woolwich.*

19. Show that money will increase more than 50-fold in a century at 4 per cent. per annum interest.

$\log 2 = .3010300$. $\log 13 = 1.113943$.

20. The number of births in a town is 25 in every thousand of the population annually, and the deaths 20 in every thousand. In how many years will the population double itself?

Given $\log 2$, $\log 3$, $\log 67 = 1.8260748$.

21. A man borrows £1500 for 6 months and accepts a bill for £1650 from a money lender. The bill is not met but is renewed every half-year at an increase of 20 per cent. After what time will the bill have amounted to at least £7500?

Given $\log 2$, $\log 3$, $\log 11$.

22. A person puts out £1000 at 5 per cent. per annum interest, payable half-yearly, and each time that interest is paid adds one half of the same to his capital. Find to the nearest shilling the amount at the end of 20 years.

Given $\log 2$, $\log 3$, $\log 16436 = 4.215796$. $D = 27$.

23. A cottage at the beginning of a year was worth £250, but it was found that, by dilapidations, at the end of each year it had lost 10 per cent. of its value at the beginning of the year. After what number of years would the value of the cottage be reduced below £25. $\log 3 = .4771213$. *Woolwich.*

24. A young man on coming into his fortune at the age of 21 invests it in a bank which allows 5 per cent. per annum interest. At the end of each year he withdraws for his expenses a sum equal to $\frac{20}{19}$ths of his 1st year's interest. At what age will he be penniless?

Given $\log 2$, $\log 3$, $\log 7$.

25. A quantity of water contained in a cubical cistern is found to lose by evaporation .04 of its volume in a day. The depth of the cistern is 6 ft., and a cubic foot of water weighs 1000 oz. Assuming the loss to take place by evaporation only, find to one decimal place what weight of water will be left in the cistern at the end of 10 days.

Given $\log 2$, $\log 3$, $\log 14360 = 4.1571544$,
$\log 14361 = 4.1571847$. *Sandhurst.*

26. If from a barrel full of spirit 5 per cent. be drawn and the deficiency made up with water, and the operation be repeated again and again, how soon will there be more water than spirit, and what will be the proportion of water to spirit then?

$\log 2 = .3010300$, $\log 48762 = 4.68808$.
$\log 19 = 1.27875$,

27. A person with a capital of £10000, for which he receives interest at the rate of 5 per cent. per annum, spends £900 yearly. In how many years will he be ruined?

$\log 7 = .8450980$, $\log 15 = 1.1760913$.

28. A man commences saving with the intention of putting by, every year, half as much again as he did the year before, and investing the same at 3 per cent. per annum. If he save £10 the first year, find how much he will have accumulated in 10 years, and the amount of his savings the last year.

$\log 2 = .3010300$, $\log 23305 = 4.367449$, $D = 19$,
$\log 3 = .4771213$, $\log 97669 = 4.9897567$, $D = 45$,
$\log 47 = 1.6720979$, $\log 11983 = 4.0785656$, $D = 362$,
$\log 103 = 2.0128372$, $\log 38443 = 4.5848173$, $D = 113$.

29. On January 1, 1880, I started saving a shilling every day, investing these savings at the end of every year at $2\frac{1}{2}$ per cent. per annum. What amount will have accumulated by the end of 21 years?

$\log 1025 = 3.0107239$, $\log 16795 = 4.225180$, $D = 26$,
$\log 18087 = 4.257367$, $D = 24$,
$\log 11038 = 4.042890$, $D = 40$.

II. Annuities Certain.

30. An annuity is an annual payment of a given sum of money. An *Annuity Certain* may continue for a fixed number of years, when it is said to be *terminable*; or may be vested in an individual and his heirs for ever, when it is said to be *perpetual.*

Payments may, of course, be made periodically, so that the year consists of any number of equal periods. When this is the case it must be taken into account in the same way as the conversion-period is in questions of compound interest.

The *Accumulated Value* of a forborne annuity is the amount of the several instalments plus the compound interest on each for the period during which it has been forborne.

The *Purchase Price* of an annuity is the sum of the present values of the several instalments.

The *Number of Years' Purchase* of an annuity is the ratio of the purchase price to the *annual* instalment of the annuity.

An annuity is *Deferred* or *Not Deferred* according as the first instalment is not or is due after the expiration of one period.

(I) Forborne Annuities.

31. Problem. To find the accumulated value of a forborne annuity.

Suppose the annuity to be £A for n successive periods of time, and r to be the interest on £1 for one period, and suppose the last instalment to have been due x periods ago.

Now the nth instalment + interest for x periods $= AR^x$,
the $(n-1)$th ,, ,, $(x+1)$,, $= AR^{x+1}$,
....
the 1st ,, ,, $(x+n-1)$,, $= AR^{x+n-1}$.

Hence, by addition, V (the accumulated value) $= AR^x + AR^{x+1} + \dots + AR^{x+n-1}$

$$= AR^x(1 + R + \dots + R^{n-1})$$

$$= \frac{AR^x(R^n - 1)}{R - 1}.$$

[COROLLARY. If the last instalment be only just due, putting $x=0$, we get $V=\dfrac{A(R^n-1)}{R-1}$.]

Example. What is the accumulated value of an annuity of £120 during 10 years, that lapsed 7 years ago, reckoning 4 per cent. per annum interest and half-yearly payments?

In this case $A=60, \quad x=14, \quad n=20, \quad R=1.02,$

$$\therefore \quad V=\frac{60R^{14}(R^{20}-1)}{R-1}.$$

To find $R^{20}-1$:

$$\begin{array}{rl} \log R = & .0086002 \\ & \underline{\quad\quad 20} \\ \therefore \quad \log R^{20} = & .172004 \\ \text{Now} \quad \log 1.4859 = & \underline{.171990} \\ D=29 & \quad)40(1 \\ & \quad \underline{29} \end{array}$$

$\therefore \quad R^{20}=1.48591,$
and $R^{20}-1=.48591.$

Hence we have $$V=\frac{60(1.02)^{14}(.48591)}{.02},$$

$$\therefore \quad \log V=\log 60+14\log 1.02+\log .48591-\log .02.$$

$$\begin{array}{rl} \log 1.02 = & .0086002 \\ & \underline{\quad\quad\quad 14} \\ & 344008 \\ & \underline{86002} \\ 14\log 1.02 = & .110403 \\ \log 60 = & 1.778151 \\ \log .48591 = & \underline{\bar{1}.686556} \\ & 1.575110 \\ \log .02 = & \underline{\bar{2}.301030} \\ \therefore \quad \log V = & 3.274080 \\ \text{Now} \quad \log 1879.6 = & \underline{3.274065} \\ D=23 & \quad)150(7 \\ & \quad \underline{161} \end{array}$$

$\therefore \quad V=£1879.67.$

(II) Terminable Annuities.

32. PROBLEM. To find V the purchase value, and P the number of years' purchase.

(A) **Deferred:** Suppose the annuity to consist of n periodical payments of £A, and the first payment to be made x periods hence; and, as before, let $R=1+r$, where r is the interest on £1 for one period.

Then Present value of 1st instalment due in x periods $=\dfrac{A}{R^x}$,

" " 2nd " " $(x+1)$ " $=\dfrac{A}{R^{x+1}}$,

....

" " nth " " $(x+n-1)$ " $=\dfrac{A}{R^{x+n-1}}$.

Hence, by addition,

$$V\text{ (the purchase value)}=\frac{A}{R^x}+\frac{A}{R^{x+1}}+\ldots+\frac{A}{R^{x+n-1}}=\frac{A}{R^x}\left(1+\frac{1}{R}+\ldots+\frac{1}{R^{n-1}}\right)$$

$$=\frac{A}{R^x}\left(\frac{1-\dfrac{1}{R^n}}{1-\dfrac{1}{R}}\right)=\frac{A}{R^{x+n-1}}\left(\frac{R^n-1}{R-1}\right). \quad \ldots\ldots\ldots(1)$$

If there be p payments in the course of one year,

$$P \text{ (the no. of years' purchase)} = \frac{V}{pA} = \frac{R^n - 1}{pR^{x+n-1}(R-1)}. \quad \text{(2)}$$

If the payments be annual, $p=1$, and $P = \dfrac{R^n - 1}{R^{x+n-1}(R-1)}$. (3)

[V is, of course, the amount of money which will realize the same as the annuity payments provided the investments are made at the rate of interest reckoned in the sale of the annuity; in other words, the rate reckoned in calculating V is the rate of interest the purchaser will make of his money, while replacing his capital, provided he can reinvest at the same rate.]

(B) **Not deferred:** Putting $x=1$ in (A), we have

$$V = \frac{A}{R^n}\left(\frac{R^n - 1}{R-1}\right) \text{ or } \frac{A(1-R^{-n})}{R-1}, \quad \text{(1)}$$

$$P = \frac{V}{pA} = \frac{1-R^{-n}}{p(R-1)}, \quad \text{(2)}$$

and, *for annual payments*, $P = \dfrac{1-R^{-n}}{R-1}$. (3)

(III) Perpetual Annuities.

33. Problem. To find V the purchase value, and P the number of years' purchase.

(A) **Deferred:** Putting $n=\infty$ in (II),

$$V = \frac{A}{R^x}\left(\frac{1}{1-\frac{1}{R}}\right) = \frac{A}{R^{x-1}(R-1)}, \quad \text{(1)}$$

$$P = \frac{V}{pA} = \frac{1}{pR^{x-1}(R-1)}, \quad \text{(2)}$$

and, *for annual payments*, $P = \dfrac{1}{R^{x-1}(R-1)}$. (3)

(B) **Not deferred:** Putting $n=\infty$ in (II)

$$V = \frac{A}{R-1} = \frac{A}{r}, \quad \text{(1)}$$

$$P = \frac{V}{pA} = \frac{1}{pr}, \quad \text{(2)}$$

and, *for annual payments*, $P = \dfrac{1}{r}$. (3)

[Since rate per cent. $= 100r$, it is clear from (3) that the number of years' purchase of a perpetual annuity with annual payments, to begin running at once, is $\dfrac{100}{\text{rate per cent.}}$; and, conversely, rate per cent. $= \dfrac{100}{\text{number of years' purchase}}$. Also it is evident that the present value of a deferred perpetuity is the amount of money which, laid out at compound interest at the same rate, will purchase the perpetuity when possession is to be obtained.]

III. Renewal of Leases.

34. If, when p years of a lease have to run, the tenant wishes to renew for a term $p+n$ years, the sum he must pay is called the "**fine** for renewing n years of the lease."

Supposing £A to be the net annual value of the estate, the fine clearly equals the present value of an annuity £A, to vest after p years, and to continue for n years.

Hence $$\text{the fine} = \frac{A}{R^{p+n}}\left(\frac{R^n - 1}{R - 1}\right).$$

If the object be merely to renew the original lease of $p+q$ years,

$$\text{fine} = \frac{A}{R^{p+q}}\left(\frac{R^q - 1}{R - 1}\right).$$

35. *Example* (i). Find the price that should be paid for an annuity of £250 to commence in 3 years, and to continue for 10 years, allowing interest at 6 per cent.

By (II), (A), $$V = \frac{A}{R^{x+n-1}}\left(\frac{R^n - 1}{R - 1}\right).$$

Now $A = 250, \quad x = 4, \quad n = 10, \quad R = 1.06,$

$$\therefore \quad V = \frac{250}{(1.06)^{13}}\left\{\frac{(1.06)^{10} - 1}{.06}\right\} = \frac{250}{.06}\{(1.06)^{-3} - (1.06)^{-13}\}.$$

To find $(1.06)^{-3}$:

$$\begin{aligned}
\log 1.06 &= .0253059 \\
&\quad\ \ -3 \\
\log(1.06)^{-3} &= -.0759177 \\
\text{or} &= \bar{1}.9240823 \\
\text{Now}\quad \log .83961 &= \bar{1}.9240776 \\
D = 52 &\qquad)470(9 \\
&\qquad\ \ 468
\end{aligned}$$

$\therefore \quad (1.06)^{-3} = .839619.$

To find $(1.06)^{-13}$:

$$\begin{aligned}
\log 1.06 &= .0253059 \\
&\quad\ -13 \\
&\quad 759177 \\
&\quad 253059 \\
\log(1.06)^{-13} &= -.328977 \\
\text{or} &= \bar{1}.671023 \\
\text{Now}\quad \log .46883 &= \bar{1}.671015 \\
D = 9 &\qquad)80(9 \\
&\qquad\ \ 81
\end{aligned}$$

$\therefore \quad (1.06)^{-13} = .468839.$

Hence $(1.06)^{-3} - (1.06)^{-13} = .839619 - .468839 = .37078,$

$$\therefore \quad V = \frac{250}{.06}(.37078), \quad \log V = \log 250 + \log .37078 - \log .06$$

$$\begin{aligned}
&= 3.1889050 \\
\log 1544.9 &= 3.1889004 \\
D = 281 &\qquad)460(2 \\
&\qquad\ \ 562
\end{aligned}$$

$\therefore \quad V = £1544.92.$

Example (ii). A person borrows £1000; what will be the amount of each instalment that both debt and interest may be repaid by 12 equal monthly instalments, allowing interest at the rate of 10 per cent. per annum?

By (II), (B), $V = \dfrac{A(1 - R^{-n})}{R - 1}$, where A = monthly instalment required.

Now $V = £1000, \ n = 12, \ R = 1 + r = \frac{61}{60}$ (since r = interest on £1 in 1 month),

$$\therefore \quad 1000 = \frac{A\left(1 - \overline{\tfrac{61}{60}}|^{-12}\right)}{\frac{1}{60}}, \quad \text{i.e.} \quad A = \frac{100}{60\left(1 - \overline{\tfrac{61}{60}}|^{-12}\right)}.$$

To find $1 - \overline{\frac{61}{60}}|^{-12}$.

$$\begin{aligned}
\log 61 &= 1.7853298 \\
\log 60 &= 1.7781513 \\
\log \tfrac{61}{60} &= \ \ .0071785 \\
&\qquad\ \ -12 \\
\therefore\quad \log(\tfrac{61}{60})^{-12} &= -.086142 \\
\text{or} &= \bar{1}.913858 \\
\text{Now}\quad \log .82008 &= \bar{1}.913856 \\
D = 5 &\qquad)20(4 \\
&\qquad\ \ 20
\end{aligned}$$

$\therefore \quad (\frac{61}{60})^{-12} = .820084,$

and $1 - (\frac{61}{60})^{-12} = .179916.$

Hence we have $A=\dfrac{100}{60(.179916)}$, $\log A=\log 100-\log 60-\log .179916=1.9667789.$

But $\log 92.636=1.9667798$ $\therefore$ $A=£92.636.$

Example (iii). How long may I expect to live, if the reversionary interest in the fee simple of an estate that I hold for life producing £200 a year be sold for £1500, allowing 5 per cent. interest?

By (III), (A), $V=\dfrac{A}{R^n(R-1)}$, where my expectation of life is n years.

Now $V=1500$, $A=200$, $R=1.05$,

$$\therefore \quad 1500=\frac{200}{(.05)(1.05)^n}, \text{ i.e. } (1.05)^n=\frac{2\cancel{00}}{15\cancel{00}\times .05}.$$

Taking logs. $n\log(1.05)=\log 2-\log 15-\log .05$,

$$\therefore \quad n=\frac{\log 2-\log 15-\log .05}{\log 1.05}=\frac{.4259687}{.0211893}=20.1 \text{ years.}$$

EXAMPLES. XV.

1. What is the accumulated value of a forborne annuity of £150, that lapsed 2 years ago and should have been paid in half-yearly payments during 8 years?
   $\log 1025=3.0107239$, $\log 16386=4.214473$, $D=26$,
   $\log 11038=4.042890$, $D=39$.

2. What perpetuity will £2000 purchase so that possession may be had in 10 years, allowing interest at $4\frac{1}{2}$ per cent.? $\log 9=.954243$, $\log 1045=3.0191163$,
   $\log 13976=4.145383$, $D=31$.

3. In how many years will a debt of £753. 10s. be discharged by annual payments of £100; interest at 8 per cent.?
   $\log 108=2.033424$, $\log 39712=4.598922$.

4. Find the present value of an annuity of £75 to vest in 10 years and then to continue for 15; interest at $4\frac{1}{2}$ per cent.
   $\log 1045=3.0191163$, $\log 33273=4.522092$,
   $\log 64392=4.808832$, $D=7$.

5. Find the present value at 4 per cent. per annum of a Fellowship of £300 a year for 6 years, payable half-yearly, the first payment being due in 6 months' time.
   $\log 102=2.0086002$, $\log 78849=4.896796$, $D=6$.

6. Find the present worth and the number of years' purchase of the Reversion to a Freehold Estate of £1200 a year after 30 years, reckoning interest at 6 per cent.
   $\log 2=.3010300$, $\log 34822=4.541854$,
   $\log 12=1.0791812$, $\log 29018=4.462668$.
   $\log 106=2.0253059$,

7. If I pay $13\frac{1}{2}$ years' purchase for a life-annuity, after how many years shall I be reimbursed, allowing interest at 5 per cent?
   $\log 105=2.0211893$, $\log 325=2.5118834$.

8. If $4\frac{1}{2}$ per cent. be the rate of interest reckoned, what sum must be paid now to receive a Freehold Estate of £300 a year 12 years hence?
   $\log 2=.3010300$, $\log 1045=3.0191163$,
   $\log 3=.4771213$, $\log 41080=4.613630$.

9. How much must be paid annually that a debt of £650 may be discharged in 20 years, allowing interest at 4 per cent.?
   $\log 2=.3010300$, $\log 47828=4.679682$, $D=9$,
   $\log 13=1.1139434$, $\log 45638=4.659327$, $D=9$,
   $\log 54361=4.735287$, $D=8$.

10. Find the number of years' purchase and the present value of the Fee Simple of a Freehold Estate producing £1315 per annum net, reckoning $4\frac{1}{2}$ per cent. interest?

log 2 = .3010300, log 29222 = 4.4657099, $D = 149$,
log 9 = .9542426, log 22222 = 4.3467831, $D = 196$.
log 1315 = 3.1189258,

11. After how many years may I expect to acquire the Reversion to a Freehold Estate if I pay 5 years' purchase for it now, allowing 4 per cent.?

log 2 = .3010300. log 104 = 2.0170333.

12. A man 48 years old can buy an annuity of £150 for £1812 16s. Determine what is considered the expectation of life at 48, interest allowed at 5 per cent.

log 2 = .3010300, log 7 = .8450980,
log 3 = .4771213, log 11872 = 4.0745239.

13. Supposing a perpetuity to be worth 27 years' purchase, what must be paid for an annuity of £500 to continue for 10 years?

log 27 = 1.4313638, log 69511 = 4.842054, $D = 6$.
log 28 = 1.4471580,

14. An annuity of £300 vests in 10 years' time: find the equivalent annuity vesting immediately and continuing for the same period, interest at 5 per cent.

log 3 = .4771213, log 18417 = 4.265219, $D = 24$.
log 155 = 2.0211893,

15. The reversion of an estate in fee simple producing £60 a year is made over for the discharge of a debt of £577 4s. 5d. How soon ought the creditor to take possession, if he be allowed 5 per cent. per annum interest for his debt?

log 2 = .3010300, log 105 = 2.0211893,
log 3 = .4771213, log 13853 = 4.1415438, $D = 314$

16. What is the value of the reversionary interest of an annuity of £150 for 12 years after the next 8, $5\frac{1}{2}$ per cent. interest being allowed?

log 1055 = 3.0233525, log 65039 = 4.813174, $D = 7$,
log 34115 = 4.532945, $D = 13$.

17. If two joint proprietors have an equal interest in a freehold estate worth £2500 per annum, what annuity must the one allow the other during a term of 12 years that he may buy him out and thus purchase to himself the whole freehold, allowing interest at 5 per cent. per annum?

log 105 = 2.0211893, log 55683 = 4.745723, $D = 8$,
log 125 = 2.0969100, log 44316 = 4.646561, $D = 6$,
log 28206 = 4.450342, $D = 15$.

18. What will be the amount of an annuity of £720 left unpaid for 26 years, allowing interest at 4 per cent. per annum, an instalment being just due?

log 104 = 2.0170333, log 27724 = 4.442856, $D = 16$,
log 180 = 2.2552725, log 17724 = 4.2485617, $D = 245$,
log 31904 = 4.5038451, $D = 136$.

19. How much must be paid annually that a debt of £1000 may be discharged in 20 years, interest at 5 per cent.?

log 105 = 2.0211893, log 37689 = 4.576215,
log 5 = .6989700, log 62311 = 4.794565,
log 80243 = 4.904407.

20. What difference does it make in the year whether a person receive his salary of £600 quarterly or monthly, interest at 4.8 per cent.?

log 1012 = 3.0051805, $\left.\begin{array}{l}\text{log } 10488 = 4.020693 \\ \text{log } 10490 = 4.020776\end{array}\right\} D = 41.$
log 1004 = 3.0017337,

21. A loan of £1000 is to be paid off in two years by equal quarterly payments. What is the amount of each payment, allowing interest at $10\,^{\circ}/_{\circ}$?

$\log 82074 = 4.914206$, $D = 5$,
$\log 5 = .6989700$, $\log 17925 = 4.253459$, $D = 24$,
$\log 1025 = 3.0107239$, $\log 13946 = 4.144450$, $D = 31$.

22. The lease of an estate is granted for 7 years at a pepper-corn rent, with the condition that the tenant at the expiration of the lease may renew the same on paying a fine of £100. What is the value of the landlord's interest in the estate immediately after any such renewal, allowing interest at the rate of 5 per cent. per annum?

$\log 105 = 2.0211893$, $\log 4071 = 3.60970$,
$\log 14071 = 4.148325$, $\log 24564 = 4.39030$.

23. How many years' renewal will £1009. 4s. purchase of a 40 years' lease of an estate worth £350 a year at the expiration of 10 years, allowing $5\,^{\circ}/_{\circ}$ interest?

$\log 105 = 2.0211893$. $\log 23137 = 4.364307$, $D = 19$.
$\log 87206 = 4.940546$,

24. If a perpetual annuity be worth $22\frac{1}{2}$ years' purchase, what annuity to continue for 8 years will £2000 purchase?

$\log 2 = .3010300$, $\log 70618 = 4.848915$, $D = 6$,
$\log 3 = .4771213$, $\log 29381 = 4.468067$, $D = 15$,
$\log 47 = 1.6720979$, $\log 30253 = 4.480768$, $D = 14$.

25. If I have to pay £2150 when I am 21 years of age for an annuity of £100 during my life, how long may I expect to live, 3 per cent. being the rate of interest reckoned?

$\log 2 = .30103$, $\log 71 = 1.85126$, $\log 103 = 2.01284$.

26. Find to the nearest £ how much should be paid now for an annuity of £500, the first instalment of which is paid to the annuitant five years hence, and the last instalment fifteen years hence, interest at $5\,^{\circ}/_{\circ}$.

$\log 1.05 = .0211893$, $\log 8.22702 = .915243$,
$\log 4.81027 = .682160$.

27. In the case of a 30 years' lease of an estate whose annual rental is £720, what fine must be paid in order to renew the lease after the expiration of 8 years, allowing interest at $6\,^{\circ}/_{\circ}$?

$\log 109 = 2.0253059$, $\log 277504 = 5.443270$,
$\log 17411 = 4.240823$.

28. I buy the remainder of a lease, with 15 years to run, at 8 years' purchase. If I am only able to invest at 4 per cent., what interest shall I realise on the purchase money?

$\log 32 = 1.505150$, $\log 18009 = 4.255490$, $D = 24$,
$\log 104 = 2.0170333$, $\log 10630 = 4.026533$, $D = 41$.
$\log 80094 = 4.903600$,

29. How much money must be invested at Compound Interest that in 21 years it may purchase the Fee Simple of a freehold of £200 net annual income, reckoning 4 per cent. in each case?

$\log 5 = .6989700$, $\log 21941 = 4.341256$, $D = 20$.
$\log 104 = 2.0170333$,

30. An estate whose clear annual value is £1800 is let on a 21 years' lease, renewable every seven years on payment of a fine; what is the amount of the fine, allowing interest at 5 per cent.?

$\log 105 = 2.0211893$, $\log 50506 = 4.703343$, $D = 9$,
$\log 35894 = 4.555022$, $D = 12$.

CHAPTER VII.

Application of Logarithms to Plane Trigonometry.

36. The trigonometrical ratios of angles are *abstract numbers* and *continuous* functions of the angle, that is, change *continuously* in value as the angle changes through any interval however small; hence logarithms can be applied to trigonometrical functions, so that we can treat of the logarithms of the trigonometrical ratios of angles; and, since the logarithms of numbers vary continuously with the numbers when the base is positive and greater than unity (the numbers being then also positive), the logarithms of the *positive* trigonometrical ratios of angles are *continuous* functions of the angle and change continuously as the angle changes.

These logarithms are called **logarithmic ratios,** e.g. the logarithm of the sine of A is called the logarithmic sine of A and written $\log \sin A$. The base adopted is 10, the base for common logarithms.

37. Now the trigonometrical ratios of angles are not always positive. For positive angles less than 90°, i.e. for angles in the first quadrant, they are all positive, and we can therefore speak of the logarithms of *all* the trigonometrical ratios of angles in the first quadrant.

But, corresponding with an angle of any magnitude, positive or negative, a positive angle less than 360° always exists whose trigonometrical ratios have all severally the very same values as those of the given angle, the bounding lines of the two angles being in the very same position.

Again, corresponding with this positive angle less than 360°, an angle can always be found in the first quadrant whose trigonometrical ratios have all of them severally the same arithmetical values, though some of them will be of different sign. Hence whatever be the angles involved in a trigonometrical expression, the expression can always be reduced at once to one having the same form and involving only angles in the first quadrant, so that there will be no difficulty in applying logarithms to a trigonometrical expression involving any angles whatever, provided only the expression be adapted to logarithmic computation and be on the whole positive in value when angles lying in the first quadrant have been substituted for those occupying other positions. If the expression, after the reduction here spoken of, assumes a negative sign upon the whole, its logarithm cannot be taken; but, if the problem be to find the value of the given expression, that of the corresponding positive expression can be found by means of logarithms, and, changing the sign of the result, we

D

have the required value. Should the negative expression be one side of an equation to be solved by logarithms, the other side must necessarily be also negative, and signs are changed on both sides before taking logarithms. Hence, finally, whatever angles be involved in a trigonometrical expression, it can always be reduced to another, equally adapted to logarithms, in which the angles are all of them positive and less than 90°, so that for all requisite purposes a table of logarithmic ratios need only give those of positive angles less than 90°.

To effect the necessary reduction for any angle that is not positive and less than 90°, calculate the smallest positive angle having the same position and consequently all the same ratios. Call it A, an angle got by adding or subtracting 360° again and again; then

(i) if A lies between 0° and 90°, it will be found in the tables;

(ii) if A lies between 90° and 180°, $180° - A$ will lie between 0° and 90°, and we have
$$\sin A = \sin(180 - A),$$
$$\cos A = -\cos(180 - A),$$
$$\tan A = -\tan(180 - A).$$

(iii) if A lies between 180° and 270°, $A - 180°$ will lie between 0° and 90°, and we have
$$\sin A = -\sin(A - 180),$$
$$\cos A = -\cos(A - 180),$$
$$\tan A = \tan(A - 180).$$

(iv) if A lies between 270° and 360°, $360° - A$ will lie between 0° and 90°, and we have
$$\sin A = -\sin(360 - A),$$
$$\cos A = \cos(360 - A),$$
$$\tan A = -\tan(360 - A).$$

38. The logarithmic ratios are positive or negative according as the trigonometrical ratios are greater or less than unity. Hence, among logarithmic ratios there will be as many negative as positive values, for, while numerically the sine and cosine cannot be greater than unity, the secant and cosecant cannot be less than unity, and of the tangent and cotangent, when one is greater the other is less than unity.

Now, when the logarithmic ratios are negative, in accordance with the usual method of expressing negative logarithms they will have negative characteristics with positive mantissae. But, in the case of the logarithms of the trigonometrical ratios, we are unable to determine the characteristics by inspection as we could for the logarithms of decimal numbers; hence these characteristics must be tabulated and it must also be stated whether they are positive or negative. To avoid the recurrence of negative characteristics, the logarithmic ratios are all increased by 10 before being tabulated, becoming thus **tabular logarithmic ratios**, which are positive over almost the whole range 0° to 90°, and at least between 10′ and 89° 50′.

The tabular logarithm is written L, e.g. the tabular logarithmic tangent of 20° would be written $L \tan 20°$.

39. It has been seen that the logarithmic ratios for the range 0° to 90° are

sufficient for all purposes; it will now appear that really the tabulation of the range 0° to 45° gives all that is required, for any ratio of an angle A between 45° and 90° is equal to some ratio of the complementary angle $90° - A$, which will lie between 0° and 45°.

We have

$$\begin{aligned} L\sin A &= L\cos(90 - A),\\ L\cos A &= L\sin(90 - A),\\ L\tan A &= L\cot(90 - A),\\ L\cot A &= L\tan(90 - A),\\ L\sec A &= L\operatorname{cosec}(90 - A),\\ L\operatorname{cosec} A &= L\sec(90 - A). \end{aligned}$$

The subjoined extract from the tables will show how the above formulae are utilized in abbreviating our tabulation; for instance, $L\tan 63°\ 3'$ is given as 10.2937716, and is the same as $L\cot 26°\ 57'$; $L\cos 63°\ 1'$ and $L\sin 26°\ 59'$ are both equal to 9.6567987.

26 DEG.

′	Sine.	Diff.	Cosec.	Tang.	Diff.	Cotang.	Secant.	Diff.	Cosine.	′
0	9.6418420	2589	10.3581580	9.6881818	3205	10.3118182	10.0463398	617	9.9536602	60
1	9.6421009	2587	10.3578991	9.6885023	3204	10.3114977	10.0464015	616	9.9535985	59
2	9.6423596	2586	10.3576404	9.6888227	3203	10.3111773	10.0464631	618	9.9535369	58
3	9.6426182	2583	10.3573818	9.6891430	3201	10.3108570	10.0465249	617	9.9534751	57
57	9.6563021	2484	10.3436979	9.7062284	3126	10.2937716	10.0499262	643	9.9500738	3
58	9.6565505	2482	10.3434495	9.7065410	3125	10.2934590	10.0499905	643	9.9500095	2
59	9.6567987	2481	10.3432013	9.7068535	3124	10.2931465	10.0500548	643	9.9499452	1
60	9.6570468	2478	10.3429532	9.7071659	3122	10.2928341	10.0501191	644	9.9498809	0
′	Cosine.	Diff.	Secant.	Cotang.	Diff.	Tang.	Cosec.	Diff.	Sine.	′

63 DEG.

It will be observed that in the tables the degrees of an angle lying within the range 0° to 45° appear in the left hand corner at the top of the page, while those of the complementary angle within the range 45° to 90° appear at the bottom in the right hand corner. Again, for angles from 0° to 45°, the names of the ratios are at the top and the minutes are read downwards, while from 45° to 90° the ratios are at the bottom and the minutes are read upwards.

40. As the angle increases so also do its sine, secant, and tangent; but the cosine, cosecant, and cotangent (the co-ratios) decrease as the angle increases.

Hence also $\left.\begin{array}{r} \log\sin A,\ \log\sec A,\ \log\tan A \\ L\sin A,\ \ L\sec A,\ \ L\tan A \end{array}\right\}$ increase as the angle increases,

but $\left.\begin{array}{r} \log\cos A,\ \log\operatorname{cosec} A,\ \log\cot A \\ L\cos A,\ \ L\operatorname{cosec} A,\ \ L\cot A \end{array}\right\}$ decrease as the angle increases.

41. Certain trigonometrical ratios are reciprocal, i.e. have a product unity. In these cases the tabular logarithms will be together equal to 20.

For $\sin A \times \operatorname{cosec} A = 1$, $\therefore$ taking logs. $\log \sin A + \log \operatorname{cosec} A = 0$,
$\cos A \times \sec A = 1$, $\log \cos A + \log \sec A = 0$,
$\tan A \times \cot A = 1$, $\log \tan A + \log \cot A = 0$,

or, expressing in tabular logs., $L \sin A + L \operatorname{cosec} A = 20$,
$L \cos A + L \sec A = 20$,
$L \tan A + L \cot A = 20$.

Since the tabular logarithms of two reciprocal ratios have always a constant sum, viz. 20, their differences over any range must be the same, only in one case the difference will be an increase, while in the other it will be an equal decrease. It is for this reason that in the tables there is only one column of differences for each pair of reciprocal ratios.

42. When logarithms are applied to trigonometrical expressions, *tabular* logarithms are written down at once in every case of a logarithmic ratio, but compensation must be made with 10's if the sum of the coefficients of the tabular logarithms affected with the plus sign be not equal to that of the tabular logarithms having the minus sign. In the case of an equation the sum of the coefficients must be the same on both sides of the equation, otherwise compensation will be necessary on that side on which there is a deficiency.

Thus, $$\log\left(\frac{2 \sin^2 x}{3 \cos x}\right) = \log 2 + 2L \sin x - \log 3 - L \cos x - 10,$$

and, taking logs. throughout the equation,

$$\tfrac{1}{2} \sin^2 x = \sqrt{\tfrac{2}{13}} \tan 16^\circ,$$

we have $$2\,L \sin x - \log 2 = \tfrac{1}{2}(\log 2 - \log 13) + L \tan 16^\circ + 10.$$

43. From the foregoing articles it is clear that, when a tabular logarithmic ratio of an angle is known, we also know or can find at once

(i) the *same* tabular logarithmic ratio of all angles for which the ratio has the same value as that of the given angle;

(ii) the tabular logarithms of the *reciprocal* ratios of all the same angles;

(iii) the tabular logarithms of the *complementary* ratios of angles *complementary* to those in (i) and (ii).

For example, given $L \sin 22^\circ 18' = 9.5791616$, find $L \sin 157^\circ 42'$ ....(i)

$\left.\begin{matrix} L \operatorname{cosec} 22^\circ 18' \\ L \operatorname{cosec} 157^\circ 42' \end{matrix}\right\}$ ....(ii)

$\left.\begin{matrix} L \cos 67^\circ 42' \\ L \sec 67^\circ 42' \end{matrix}\right\}$ ....(iii)

We have $$\left.\begin{matrix} L \sin 157^\circ 42' \\ L \cos 67^\circ 42' \end{matrix}\right\} = L \sin 22^\circ 18' = 9.5791616,$$

$$\left.\begin{matrix} L \operatorname{cosec} 22^\circ 18' \\ L \operatorname{cosec} 157^\circ 42' \end{matrix}\right\} = 20 - L \sin 22^\circ 18' = 10.4208384,$$

$$\begin{aligned} L \sec 67^\circ 42' &= 20 - L \cos 67^\circ 42' \\ &= 20 - L \sin 22^\circ 18' = 10.4208384. \end{aligned}$$

It is seen that when there is a change to the reciprocal ratio, the angle remaining unaltered, the required tabular logarithm is obtained by subtracting that given from 20; but when there is a change both to the complementary ratio and complementary angle, the tabular logarithm does not alter.

EXAMPLES. XVI.

1. Given $\log 2 = .3010300$, $\log 3 = .4771213$; find the tabulated logarithms of
   (i) $\sin 30°$, (ii) $\sin 45°$, (iii) $\sec 30°$, (iv) $\sec 60°$,
   (v) $\tan 30°$, (vi) $\cot 45°$, (vii) $\sec 45°$, (viii) $\sin 90°$.

2. Given $\left.\begin{aligned} L\sin 60° &= 9.9375306 \\ L\cos 60° &= 9.6989700 \end{aligned}\right\}$, find $\log 2$, $\log 3$.

3. Given that $\left.\begin{aligned} 4\sin 18° \,.\, \sin 54° &= 1 \\ L\sin 18° &= 9.4899824 \\ \log 2 &= .3010300 \end{aligned}\right\}$, find $L\sin 54°$.

4. If $L\sin 15° = 9.4129962$ and $\log 2 = .3010300$, find $L\cos 15°$.

5. Write down the values of $\log_3 \cot 60°$, $L_2 \sin 30'$.

6. (i) Given $L\tan 35° \, 22' = 9.8511285$, find $\left.\begin{aligned} &L\tan 54° \, 38' \\ &L\cot 54° \, 38' \\ &L\cot 35° \, 22' \end{aligned}\right\}$.

   (ii) Given $L\operatorname{cosec} 117° \, 46' = 10.0531293$, find $\left.\begin{aligned} &L\operatorname{cosec} 62° \, 14' \\ &L\sin 62° \, 14' \\ &L\sec 27° \, 46' \\ &L\cos 27° \, 46' \end{aligned}\right\}$.

   (iii) Given $L\sin 44° \, 53' = 9.8485989$, find $\left.\begin{aligned} &L\sin 135° \;\; 7' \\ &L\operatorname{cosec} 44° \, 53' \\ &L\sec 45° \;\; 7' \\ &L\cos 45° \;\; 7' \end{aligned}\right\}$.

7. Transform the following equations into others involving tabular logarithms:

   (i) $\sqrt{2}\sec A = \frac{1}{5}\sqrt[3]{\frac{\cos^2 B}{\sin C}}$, (ii) $\sin^2 A \sqrt{2\tan A} = 3$,

   (iii) $\frac{1}{2}\tan^3 A = \frac{\sin^2 B \sin^2 C}{\sin A}$, (iv) $\frac{\sin A}{\sqrt[3]{\cos^2 A}} = \frac{2}{\sqrt{3} \,.\, \cot^3 A}$,

   (v) $\tan A \,.\, \sqrt{\tan B} = 3\sec^2\frac{A}{2}$, (vi) $\frac{\sin^2 A \,.\, \cos B}{\sin^3(A+B)} = 2\sin B \sqrt[3]{\cos^2 A \,.\, \sin B}$.

CHAPTER VIII.

Tables of Logarithmic Ratios.

44. From the brief description of the tables in the last chapter it is seen how the angle corresponding with any given logarithmic ratio, or the logarithmic ratio of any given angle, is found at once whenever the given quantity is contained exactly in the tables. When the given logarithmic ratio or angle does not appear exactly in the tables, but lies between two successive tabulations, then, as in the case of the logarithms of numbers, the corresponding angle or logarithmic ratio respectively can be found by the application of the principle of proportional parts, provided that over a small range the changes in the tabular logarithmic ratios are approximately proportional to the change in the angle. This, we will proceed to show, is the case, if only the angle is not *very nearly* 0° or 90°.

45. It can be shown by Trigonometry that, if d be the circular measure of a very small angle (e.g. an angle not greater than 1′) so that $\sin d = \tan d = d$ and $\cos d = 1$ very approximately, and if θ be an angle not very near 0° or 90°, we have

$$\sin(\theta + d) - \sin\theta = \sin d \cos\theta - \sin\theta(1 - \cos d) = d\cos\theta, \text{ approximately};$$
$$\cos(\theta + d) - \cos\theta = -\sin d \sin\theta - \cos\theta(1 - \cos d) = -d\sin\theta, \text{ approximately};$$
$$\tan(\theta + d) - \tan\theta = \frac{\tan d \sec^2\theta}{1 - \tan d \tan\theta} = d\sec^2\theta, \text{ approximately}.$$

From these we get

$$\left.\begin{aligned} \frac{\sin(\theta + d)}{\sin\theta} &= 1 + d\cot\theta \\ \frac{\cos(\theta + d)}{\cos\theta} &= 1 - d\tan\theta \\ \frac{\tan(\theta + d)}{\tan\theta} &= 1 + 2d\,\mathrm{cosec}\,2\theta \end{aligned}\right\}$$

and taking logs.

$$\left.\begin{aligned} L\sin(\theta + d) - L\sin\theta &= \log_{10}(1 + d\cot\theta) \\ L\cos(\theta + d) - L\cos\theta &= \log_{10}(1 - d\tan\theta) \\ L\tan(\theta + d) - L\tan\theta &= \log_{10}(1 + 2d\,\mathrm{cosec}\,2\theta) \end{aligned}\right\}$$

$\therefore$ approximately, neglecting higher powers of d than the first, since d is very small,

$$\left.\begin{aligned}
L\sin(\theta+d)-L\sin\theta &= \mu d\cot\theta \\
L\cos(\theta+d)-L\cos\theta &= -\mu d\tan\theta \\
L\tan(\theta+d)-L\tan\theta &= 2\mu d\,\mathrm{cosec}\,2\theta
\end{aligned}\right\}
\begin{aligned}
&\textit{terms neglected—}\\
&\left[-\frac{d^2\cot^2\theta}{2}+\frac{d^3\cot^3\theta}{3}-\ldots\right]\\
&\left[-\frac{d^2\tan^2\theta}{2}-\frac{d^3\tan^3\theta}{3}-\ldots\right]\\
&\left[-\frac{4d^2\mathrm{cosec}^2 2\theta}{2}+\frac{8d^3\mathrm{cosec}^3 2\theta}{3}+\ldots\right]
\end{aligned}$$

whence also, approximately, by Art. 41,

$$\left.\begin{aligned}
L\,\mathrm{cosec}(\theta+d)-L\,\mathrm{cosec}\,\theta &= -\mu d\cot\theta \\
L\sec(\theta+d)-L\sec\theta &= \mu d\tan\theta \\
L\cot(\theta+d)-L\cot\theta &= -2\mu d\,\mathrm{cosec}\,2\theta
\end{aligned}\right\}$$

From these results it is clear that, if d be the circular measure of a very small angle, and θ be not very nearly 0° or 90°, the tabular logarithmic ratios of $(\theta+d)$ differ from those of θ by quantities that are approximately proportional to d, or, since the circular measure of an angle is proportional to its measurement in any other unit, *for small increments in the angle the changes in the logarithmic ratios are approximately proportional to the changes in the angle.*

46. It has been said that the principle of proportional parts cannot be applied when the angle is very small or very nearly a right angle. The reason is that, in these portions of the tables, the differences in the logarithmic ratios for small increments in the angle are either irregular, or both insensible and irregular. *Irregularity* would be owing to the fact that, in obtaining the results of the last article, terms have been neglected that are of the same order as those retained, while there is *insensibility* when these latter are themselves very small.

For $\left.\begin{matrix} L\sin\theta \\ L\,\mathrm{cosec}\,\theta\end{matrix}\right\}$ the differences are $\left.\begin{matrix}\textit{insensible}\\ \text{and } \textit{irregular}\end{matrix}\right\}$ near 90°, *irregular* near 0°;

for $\left.\begin{matrix} L\cos\theta \\ L\sec\theta\end{matrix}\right\}$ the differences are $\left.\begin{matrix}\textit{insensible}\\ \text{and } \textit{irregular}\end{matrix}\right\}$ near 0°, *irregular* near 90°;

for $L\tan\theta$ the differences are *irregular* near 90°;

for $L\cot\theta$ „ „ „ „ 0°.

47. We will conclude this chapter with a few examples showing how the principle of proportional parts may be applied in practice.

Example (i). Given $\left.\begin{aligned} L\tan 29°12' &= 9.7473194 \\ L\tan 29°13' &= 6.7476160\end{aligned}\right\}$, find $L\tan 29°12'18''$, and the angle whose $L\tan$ is 9.7475285.

(a) $L\tan 29°12'18''$ has an 8-figure value whose first four figures are the same as those of the given logarithmic tangents, while the last four compose some number lying between 3194 and 6160. Call this $3194+d$.

Write down the angles and logarithmic ratios in two columns in ascending or descending order of magnitude (or at least such portions of them as are not the same throughout the column), and couple the quantities in the same way in both columns, placing the differences outside their respective couplings. These differences are then four proportionals.

Thus
$$60''\left(\begin{array}{l}13'\\12'18''\\12'\end{array}\right)18'' \quad 2966\left(\begin{array}{l}6160\\3194+d\\3194\end{array}\right)d \quad \text{or} \quad 18''\left(\begin{array}{l}12'\\12'18''\\13'\end{array}\right)60'' \quad d\left(\begin{array}{l}3194\\3194+d\\6160\end{array}\right)2966$$

$$\therefore \quad 60d = 2966 \times 18,$$
$$d = 890,$$
and $$3194 + d = 4080.$$
Hence $$L\tan 29°12'18'' = 9.7474080.$$

[It is the simplest plan so to arrange the couplings as to have a difference d outside one coupling, and all the other differences purely numerical (i.e. not containing d). This is effected if we couple together the two extremes, and the mean with the *smaller* of the two extremes *in the d column*. Of course this is not essential, but it renders the finding of d more convenient.]

(β) The angle whose $L\tan$ is 9.7475285 lies between 29°12′ and 29°13′. Call it $29°12'd''$ and proceed with two columns as before.

$$60''\left(\begin{array}{l}13'\\12'd''\\12'\end{array}\right)d'' \quad 2966\left(\begin{array}{l}6160\\5285\\3194\end{array}\right)2091 \qquad \therefore \quad 2966d = 2091 \times 60,$$
$$d = 42'',$$
and the required angle $= 29°12'42''$.

Example (ii). Given $\left.\begin{array}{l}L\cos 6°13'10'' = 9.9974363\\ L\cos 6°13'20'' = 9.9974340\end{array}\right\}$, find $L\cos 6°13'12''$, and the angle whose $L\cos$ is 9.9974351.

(α)
$$10''\left(\begin{array}{l}20''\\12''\\10''\end{array}\right)8'' \quad 23\left(\begin{array}{l}40\\40+d\\63\end{array}\right)d \qquad \therefore \quad 10d = 23 \times 8,$$
$$d = 18,$$
and $$40 + d = 58.$$
Hence $$L\cos 6°13'12'' = 9.9974358.$$

(β) The required angle lies between 6°13′10″ and 6°13′20″. Call it $6°13'10'' + d''$.

$$10''\left(\begin{array}{l}20''\\10''+d''\\10''\end{array}\right)d'' \quad 23\left(\begin{array}{l}40\\51\\63\end{array}\right)12 \qquad \therefore \quad 23d = 120,$$
$$d = 5,$$
and the required angle $= 6°13'15''$.

Example (iii). Given $\left.\begin{array}{l}L\sin 59°18' = 9.9344238\\ \text{diff. } 1' = 750\end{array}\right\}$, find $L\sin 59°18'20''$, and the angle whose $L\sin$ is 9.9343724.

(α) Since $L\sin$ increases with the angle, $L\sin 59°18'20''$ must be greater than $L\sin 59°18'$. Let the number composing its last four figures be $4238 + d$, then, since those of $L\sin 59°19'$ are the number $4238 + 750$ or 4988, we have

$$60''\left(\begin{array}{l}19'\\18'20''\\18'\end{array}\right)20'' \quad 750\left(\begin{array}{l}4988\\4238+d\\4238\end{array}\right)d$$
whence $d = 250$,
and $L\sin 59°18'20'' = 9.9344488$.

(β) Since $L\sin$ increases with the angle, the angle whose $L\sin$ is 9.9343724 must be less than 59°18′ and lies between 59°18′ and 59°17′. Call it $59°17'd''$; then

$$60''\left(\begin{array}{l}18'\\17'd''\\17'\end{array}\right)d'' \quad 750\left(\begin{array}{l}4238\\3724\\3488\end{array}\right)236$$
whence $d = 19$,
and the required angle $= 59°17'19''$.

Example (iv). Given $\left.\begin{array}{l}L\cot 82°44'30'' = 9.1050462\\ \text{diff. } 10'' = 1680\end{array}\right\}$, find $L\cot 82°44'33''$, and the angle whose $L\cot$ is 9.1049630.

(α) Since $L\cot$ decreases as the angle increases, and *vice versa*, $L\cot 82°44'40'' = 9.1048782$ $(50462 - 1680 = 48782)$. Call the number composing the last five figures of $L\cot 82°44'33''$ $48782 + d$; then we have

$$10''\begin{pmatrix}40''\\33''\\30''\end{pmatrix}7'' \qquad 1680\begin{pmatrix}48782\\48782+d\\50462\end{pmatrix}d \qquad \begin{matrix}\text{whence} & d = 1176,\\ \text{and} & L\cot 82°44'33'' = 9.1049958.\end{matrix}$$

(β) The angle whose $L\cot$ is 9.1049630 must be greater than $82°44'30''$, and lies between $82°44'30''$ and $82°44'40''$. Call it $82°44'30'' + d''$; then

$$10''\begin{pmatrix}40''\\30''+d''\\30''\end{pmatrix}d'' \qquad 1680\begin{pmatrix}48782\\49630\\50462\end{pmatrix}832 \qquad \begin{matrix}\text{whence} & d = 5,\\ \text{and the required angle} & = 82°44'35''.\end{matrix}$$

48. By applying the principle of proportional parts we are able to find all the remaining logarithmic ratios corresponding with any given one, without first finding the angle belonging to the same, often a very convenient process.

Example. If $L\sin\theta = 9.8146828$, find all the other trigonometrical ratios of θ. Given

$$\left.\begin{aligned}L\sin 40°44' &= 9.8146067\\ L\sin 40°45' &= 9.8147534\\ L\cos 40°44' &= 9.8795287\\ L\cos 40°45' &= 9.8794199\end{aligned}\right\}$$

Since, in this example, we have logarithmic cosines given besides the two logarithmic sines between which $L\sin\theta$ lies, we will begin by finding $L\cos\theta$; the remaining logarithmic ratios can then all be found from these two.

Since θ lies between $40°44'$ and $40°45'$, $L\cos\theta$ must have some value between 9.8795287 and 9.8794199. Let the number composed of its last four figures be $4199 + d$; then

$$\begin{matrix}L\sin. & & L\cos. \\ 1467\begin{pmatrix}6067\\6828\\7534\end{pmatrix}706 & & 1088\begin{pmatrix}5287\\4199+d\\4199\end{pmatrix}d\end{matrix} \qquad \begin{aligned}\text{whence}\quad 1467d &= 1088 \times 706,\\ d &= 524,\\ \therefore\quad 4199 + d &= 4723,\\ \text{and}\quad L\cos\theta &= 9.8794723.\end{aligned}$$

Now the remaining logarithmic ratio can be found at once. Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we have

$$\begin{aligned}L\tan\theta &= L\sin\theta - L\cos\theta + 10 = 9.9352105,\\ L\sec\theta &= 20 - L\cos\theta = 10.1205277,\\ L\operatorname{cosec}\theta &= 20 - L\sin\theta = 10.1853172,\\ L\cot\theta &= 20 - L\tan\theta = 10.0647895.\end{aligned}$$

EXAMPLES. XVII.

1. Given $\left.\begin{aligned}L\cos 22°17' &= 9.9662920\\ L\cos 22°18' &= 9.9662402\end{aligned}\right\}$, find $L\cos 22°17'32''$, and the angle whose $L\cos$ is 9.9662585.

2. Given $\left.\begin{aligned}L\sec 68°48' &= 10.4417421\\ L\sec 68°47' &= 10.4414165\end{aligned}\right\}$, find $L\sec 68°47'12''$, and the angle whose $L\cos$ is 9.5584911.

3. Given $\left.\begin{aligned}L\tan 77°12' &= 10.6436023\\ \text{diff. } 1' &= 5851\end{aligned}\right\}$, find $L\tan 77°12'24''$, and the angle whose $L\tan$ is 10.6440212.

4. Given $\left.\begin{aligned}L\cos 63° &= 9.6570468\\ \text{diff. } 1' &= 2478\end{aligned}\right\}$, find $L\cos 63°0'45.5''$, and the angle whose $L\cos$ is 9.6571240.

5. Given $\left.\begin{aligned}L\cot 5°31'20'' &= 11.0146611\\ \text{diff. } 10'' &= 2198\end{aligned}\right\}$, find $L\cot 5°31'27''$, and the angle whose $L\cot$ is 11.0147209.

6. Given $L\cot 44^\circ 59' = 10.0002527$, find $L\cot 45.0152^\circ$, and the angle whose $L\cot$ is 10.0001214.

7. Given $\left.\begin{array}{r} L\sin 30^\circ 1' = 9.6991887 \\ \log 2 = .3010300 \end{array}\right\}$, find $L\sin 30^\circ 0' 22''$, and the angle whose $L\cos$ is 9.6992008.

8. Given $\left.\begin{array}{r} L\sin 84^\circ 29' = 9.9979838 \\ \text{diff. } 10'' = 20 \end{array}\right\}$, find $L\sin 84^\circ 28' 58''$, and the angle whose $L\sin$ is 9.9979850.

9. Given $\left.\begin{array}{r} L\tan 16^\circ 21' = 9.4674127 \\ L\tan 16^\circ 22' = 9.4678802 \end{array}\right\}$, find $L\cot 73^\circ 38' 17.2''$, and the angle whose $L\cot$ is 10.5322862.

10. Given $\left.\begin{array}{r} L\sin 15^\circ 30' = 9.426899 \\ \text{diff. } 1' = .000455 \end{array}\right\}$, find $L\sin 15^\circ 30' 36''$, and the angle whose $L\cos$ is 9.427263.

11. Given $\left.\begin{array}{r} L\sin 36^\circ 18' = 9.7723314 \\ L\sin 36^\circ 19' = 9.7725033 \end{array}\right\}$, find $\left.\begin{array}{l} L\ \sin\ 36^\circ 18' 25'' \\ L\ \cos\ 53^\circ 41' 16'' \\ L\,\text{cosec}\, 36^\circ 18' 38'' \end{array}\right\}$.

12. Given $\left.\begin{array}{r} L\sin 30^\circ 1' = 9.6991887 \\ L\cos 30^\circ 1' = 9.9374577 \\ \log 2 = .3010300 \\ \log 3 = .4771213 \end{array}\right\}$, find all the tabular logarithmic ratios of $30^\circ 0' 40''$.

13. Given $\left.\begin{array}{ll} L\sin 11^\circ 42' = 9.3070407, & L\sin 11^\circ 43' = 9.3076503, \\ L\cos 11^\circ 42' = 9.9908815, & L\cos 11^\circ 43' = 9.9908553, \end{array}\right\}$ find $L\cos\theta$ and $L\tan\theta$ when $L\sin\theta = 9.3071520$.

14. If $L\cos\theta = 9.8310328$, find all the other tabular logarithmic ratios of θ.

$$L\cos 47^\circ 20' = 9.8310580, \quad \text{diff. } 1' = 1371,$$
$$L\tan 47^\circ 20' = 10.0354119, \quad \text{diff. } 1' = 2535.$$

15. Find $L\sin 2\theta$, when $L\tan\theta = 10.5872917$.

$$L\sin 75^\circ 29' = 9.9859089, \quad D = 327,$$
$$L\cos 75^\circ 29' = 9.3990878, \quad D = 4882,$$
$$\log 2 = .3010300.$$

16. Find the values of

(i) $2.1078\cos^3 3A$, when $A = 27^\circ 10'$;
$$\log 21078 = 4.3238294$$
$$\log 68066 = 4.8329302, \quad D = 64$$
$$L\cos 81^\circ 30' = 9.1697021$$

(ii) $.02845\cos^3\dfrac{A}{2}$, when $A = 35^\circ 15'$;
$$\log\ 2845 = 3.4540823$$
$$\log 24628 = 4.3914291, \quad D = 177$$
$$L\cos 17^\circ 37' = 9.9791397, \quad D = 401$$

(iii) $\dfrac{.0076829\sin^2\dfrac{A}{2}}{\cos A}$, when $A = 35^\circ 17'$;
$$\log 76829 = 4.8855252$$
$$\log 86444 = 4.9367382 \quad D = 50$$
$$L\cos 35^\circ 17' = 9.9118528$$
$$L\sin 17^\circ 38' = 9.4813342 \quad D = 3973$$

(iv) $\dfrac{1}{3}\tan^{-1}\dfrac{3\sqrt{2}}{10}$; given $\log 2$, $\log 3$, $L\tan 22^\circ 59' = 9.6275006$, $D = 3513$

(v) $\dfrac{2}{5}\text{cosec}^{-1}10\dfrac{2}{3}$; given $\log 2$, $\log 3$, $L\sin 5^\circ 22' 40'' = 8.9718424$, diff. $10'' = 2236$

(vi) $2\cos^{-1}\left(\dfrac{3}{4}\right)^{\frac{1}{4}}$; given $\log 75 = 1.8750613$, $L\cos 21^\circ 28' = 9.9687773$, $D = 497$

(vii) $\cot^{-1}\left\{\left(\dfrac{1}{2}\right)^{\frac{1}{2}}\left(\dfrac{1}{3}\right)^{\frac{1}{3}}\right\}$; given $\log 2$, $\log 3$, $L\cot 63^\circ 53' = 9.6904226$, $D = 3196$

17. Find the smallest positive values of the angles satisfying the equations

(i) $\sin^3\theta = \sqrt{\frac{2}{7}}$; given $\log 2$, $\log 7$, $L\sin 55°14' = 9.9092371$, $D = 910$.

(ii) $8\tan x = 3\cos x$; given $\log 3$, $L\sin 19°28' = 9.5227811$, $D = 3572$.

(iii) $\tan^3\theta = \frac{5}{12}$; given $\log 2$, $\log 3$, $L\tan 36°45' = 9.8731668$, $D = 2634$.

(iv) $3\tan\theta = 8\cot\theta$; given $\log 2$, $\log 3$, $L\cot 58°32' = 9.7867520$, $D = 2837$.

(v) $3\sin^2 2\theta = 2\sqrt{2}$; given $\log 2$, $\log 3$, $L\sin 76°9' = 9.9871860$, $D = 311$.

(vi) $2\cos^4\theta = \frac{1}{3}\sec\theta$; given $\log 2$, $\log 3$, $L\cos 45°41' = 9.8442432$, $D = 1293$.

(vii) $3\sin^2\theta + 2\sin\theta = 1$, given $\log 3$, $L\sin 19°28' = 9.5227811$, $D = 3572$.

(viii) $\left.\begin{array}{l}\sin\theta\cos\phi = \frac{1}{3}\\ \sin\phi\cos\theta = \frac{1}{5}\end{array}\right\}$; given $\log 2$, $\log 3$, $L\sin 32°13' = 9.7268269$, $D = 2004$,
$L\sin 7°39' = 9.1242477$, $D = 9395$.

(ix) $\left.\begin{array}{l}\sin x = 2\sin y\\ \cos x = \frac{1}{2}\cos y\end{array}\right\}$; given $\log 2$, $L\sin 26°33' = 9.6502868$, $D = 2527$,
$L\sin 63°26' = 9.9515389$, $D = 631$.

18. Given $L\tan 54°15'20'' = 10.1428185$, and that the tabular difference for $10'' = .0000444$, find x from the equation $10\tan x = (\tan 54°15'29'')^8$.

19. Show that the smallest positive value of θ which satisfies the equation $7\tan^2\theta + 8\sqrt{3}\tan\theta = 1$ is $3°59'16.2''$, having given

$$\begin{aligned} \log 2 &= .3010300 \\ L\sin 33°59' &= 9.7473743 \\ L\sin 34° &= 9.7475617 \end{aligned}$$

CHAPTER IX.

Reductional Formulae.

49. In order that expressions may be adapted to logarithmic computation they must be expressed as consisting of products and quotients. Hence, when logarithms are to be applied to trigonometrical expressions, these latter will frequently have to undergo reduction into a suitable form before any computation can take place.

It will therefore be well to give a few of the simpler reductional formulae, all of which can be easily verified by the student, and which will assist him in working the more complicated examples.

(A)
$$1-\sin^2 A=\cos^2 A \quad \text{or} \quad 1-\cos^2 A=\sin^2 A,$$
$$1+\tan^2 A=\sec^2 A \quad \text{or} \quad \sec^2 A-1=\tan^2 A,$$
$$1+\cot^2 A=\operatorname{cosec}^2 A \quad \text{or} \quad \operatorname{cosec}^2 A-1=\cot^2 A.$$

(B)
$$\sin A\cos B\pm\cos A\sin B=\sin(A\pm B),$$
$$\cos A\cos B\mp\sin A\sin B=\cos(A\pm B),$$
$$\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}=\tan(A\pm B),$$
$$\frac{\cot A\cot B\mp 1}{\cot B\pm\cot A}=\cot(A\pm B).$$

From these we get

(i)
$$\tan A\pm\tan B=\frac{\sin(A\pm B)}{\cos A\cos B}, \qquad \cot B\pm\cot A=\frac{\sin(A\pm B)}{\sin A\sin B},$$
$$\cot A\mp\tan B=\frac{\cos(A\pm B)}{\sin A\cos B}, \qquad 1\pm\cot A\tan B=\frac{\sin(A\pm B)}{\sin A\cos B},$$
$$1\mp\tan A\tan B=\frac{\cos(A\pm B)}{\cos A\cos B}, \qquad \cot A\cot B\mp 1=\frac{\cos(A\pm B)}{\sin A\sin B}.$$

(ii) Putting A or B equal to $45°$,
$$\cos A+\sin A=\sqrt{2}\sin(A+45°),\ \ \sqrt{2}\cos(45°-A),\ \text{ or } \sqrt{2}\cos(A-45°);$$
$$\cos A-\sin A=\sqrt{2}\cos(A+45°),\ \text{ or } \sqrt{2}\sin(45°-A);$$
$$\sin A-\cos A=\sqrt{2}\sin(A-45°).$$
$$\frac{\tan A\pm 1}{1\mp\tan A}=\tan(A\pm 45°), \qquad \frac{1\pm\tan A}{1\mp\tan A}=\tan(45°\pm A),$$

$$\frac{\cot A \mp 1}{1 \pm \cot A} = \cot(A \pm 45^\circ), \qquad \frac{\cot A \mp 1}{\cot A \pm 1} = \cot(45^\circ \pm A),$$

$$\frac{\sin A \pm \cos A}{\cos A \mp \sin A} = \tan(A \pm 45^\circ).$$

(iii) Putting A or B equal to 30° or 60°,

$$\cos A + \sqrt{3}\sin A = 2\sin(A + 30^\circ) \text{ or } 2\cos(60^\circ - A),$$
$$\cos A - \sqrt{3}\sin A = 2\sin(30^\circ - A) \text{ or } 2\cos(A + 60^\circ),$$
$$\sqrt{3}\sin A - \cos A = 2\sin(A - 30^\circ),$$
$$\sin A + \sqrt{3}\cos A = 2\cos(A - 30^\circ),\ 2\cos(30^\circ - A), \text{ or } 2\sin(A + 60^\circ),$$
$$\sin A - \sqrt{3}\cos A = 2\sin(A - 60^\circ),$$
$$\sqrt{3}\cos A - \sin A = 2\cos(A + 30^\circ) \text{ or } 2\sin(60^\circ - A).$$

$$\frac{1 \pm \sqrt{3}\tan A}{\sqrt{3} \mp \tan A} = \tan(30^\circ \pm A), \qquad \frac{\sqrt{3}\tan A \pm 1}{\sqrt{3} \mp \tan A} = \tan(A \pm 30^\circ),$$

$$\frac{\sqrt{3} \pm \tan A}{1 \mp \sqrt{3}\tan A} = \tan(60^\circ \pm A), \qquad \frac{\tan A \pm \sqrt{3}}{1 \mp \sqrt{3}\tan A} = \tan(A \pm 60^\circ).$$

$$\frac{\sqrt{3}\cot A \mp 1}{\cot A \pm \sqrt{3}} = \cot(30^\circ \pm A), \qquad \frac{\sqrt{3}\cot A \mp 1}{\sqrt{3} \pm \cot A} = \cot(A \pm 30^\circ),$$

$$\frac{\cot A \mp \sqrt{3}}{\sqrt{3}\cot A \pm 1} = \cot(60^\circ \pm A), \qquad \frac{\cot A \mp \sqrt{3}}{1 \pm \sqrt{3}\cot A} = \cot(A \pm 60^\circ).$$

(iv) Putting $B = A$, $\cos^2 A - \sin^2 A = \cos 2A$,

$$\text{whence } \begin{cases} 1 - 2\sin^2 A = \cos 2A \text{ or } 1 - \cos 2A = 2\sin^2 A, \\ 2\cos^2 A - 1 = \cos 2A \text{ or } 1 + \cos 2A = 2\cos^2 A. \end{cases}$$

$$\frac{2\tan A}{1 - \tan^2 A} = \tan 2A \text{ or } 1 - \tan^2 A = \frac{2\tan A}{\tan 2A},$$

$$\frac{\cot^2 A - 1}{2\cot A} = \cot 2A \text{ or } \cot^2 A - 1 = 2\cot A \,.\, \cot 2A.$$

(v) Putting $B = 2A$,

$$3\sin A - 4\sin^3 A = \sin 3A \text{ or } 3 - 4\sin^2 A = \frac{\sin 3A}{\sin A},$$

$$4\cos^3 A - 3\cos A = \cos 3A \text{ or } 4\cos^2 A - 3 = \frac{\cos 3A}{\cos A}.$$

$$\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan 3A \text{ or } \frac{3 - \tan^2 A}{1 - 3\tan^2 A} = \frac{\tan 3A}{\tan A},$$

$$\frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1} = \cot 3A \text{ or } \frac{\cot^2 A - 3}{3\cot^2 A - 1} = \frac{\cot 3A}{\cot A}.$$

(C)

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\cos B - \cos A = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

From these we get

$$\left.\begin{array}{l}\sin^2 A - \sin^2 B \\ \text{or} \quad \cos^2 B - \cos^2 A\end{array}\right\} = \sin(A+B)\sin(A-B), \qquad \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B),$$

$$\tan^2 A - \tan^2 B = \frac{\sin(A+B)\sin(A-B)}{\cos^2 A \cos^2 B}, \qquad \cot^2 B - \cot^2 A = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A \sin^2 B}.$$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}, \qquad \frac{\cos B + \cos A}{\cos B - \cos A} = \cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B),$$

$$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan\tfrac{1}{2}(A \pm B), \qquad \frac{\sin A \pm \sin B}{\cos B - \cos A} = \cot\tfrac{1}{2}(A \mp B).$$

(D)

$$\cot A + \tan A = 2\operatorname{cosec} 2A, \qquad \cot A - \tan A = 2\cot 2A,$$

$$\frac{\cot A + \tan A}{\cot A - \tan A} = \sec 2A.$$

$$\frac{2\tan A}{1+\tan^2 A} = \sin 2A, \qquad \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A, \qquad \frac{2\cot A}{1+\cot^2 A} = \sin 2A.$$

$$\cos^4 A - \sin^4 A = \cos 2A.$$

(E)

$$\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} = \tan\frac{A}{2},$$

$$\frac{\sin A}{1-\cos A} = \frac{1+\cos A}{\sin A} = \cot\frac{A}{2},$$

$$\frac{\cos A}{1+\sin A} = \frac{1-\sin A}{\cos A} = \tan\left(45^\circ - \frac{A}{2}\right) \text{ or } \cot\left(45^\circ + \frac{A}{2}\right),$$

$$\frac{\cos A}{1-\sin A} = \frac{1+\sin A}{\cos A} = \cot\left(45^\circ - \frac{A}{2}\right) \text{ or } \tan\left(45^\circ + \frac{A}{2}\right).$$

From these we get

$$1+\cos A = \sin A \cot\frac{A}{2}, \qquad 1-\cos A = \sin A \tan\frac{A}{2},$$

$$1+\sin A = \cos A \tan\left(45^\circ + \frac{A}{2}\right), \qquad 1-\sin A = \cos A \cot\left(45^\circ + \frac{A}{2}\right).$$

$$\frac{1-\cos A}{1+\cos A} = \tan^2\frac{A}{2}, \qquad \frac{1-\sin A}{1+\sin A} = \cot^2\left(45^\circ + \frac{A}{2}\right).$$

$$\operatorname{cosec} A - \cot A = \tan\frac{A}{2}, \qquad \operatorname{cosec} A + \cot A = \cot\frac{A}{2},$$

$$\sec A \pm \tan A = \tan\left(45^\circ \pm \frac{A}{2}\right).$$

Subsidiary Angles.

50. Expressions may sometimes be adapted to logarithmic computation by the introduction of subsidiary angles.

(1) To adapt $\sqrt{a^2+b^2}$ to logarithmic computation.

Put $\qquad b = a\tan\theta, \quad$ i.e. $\quad \tan\theta = \dfrac{b}{a}. \qquad$ .... (i)

This can always be done, since the tangent can have any value between 0 and ∞. Then

$$\sqrt{a^2+b^2}=\sqrt{a^2(1+\tan^2\theta)}=a\sec\theta. \quad \text{.......(ii)}$$

$L\tan\theta$ can be calculated from (i); the corresponding value of $L\sec\theta$ is then substituted in (ii), and the required value computed.

Example. Compute the value of $\sqrt{a^2+b^2}$ when $\left.\begin{array}{l}a=713.541\\ b=562.337\end{array}\right\}.$

Given
$$\begin{array}{ll}\log 56233=4.7499913 & D=77\\ \log 71354=4.8534183 & D=61\\ \log 90849=4.9583202 & D=47\end{array}$$

$$\begin{array}{ll}L\tan 38°14'=9.8964517 & L\sec 38°14'=10.1048555\\ L\tan 38°15'=9.8967116 & L\sec 38°15'=10.1049550\end{array}$$

$$\begin{array}{llll} & \log 562.33 \;=2.7499913 & & \log 713.54\;=2.8534183\\ & \qquad\quad 7 \qquad\qquad\quad 54 & & \qquad\quad 1 \qquad\qquad\quad 6\\ \therefore & \log b \text{ or } \log 562.337=2.7499967 & \therefore & \log a \text{ or } \log 713.541=2.8534189\end{array}$$

Hence $\qquad L\tan\theta=\log b-\log a+10=9.8965778.$

To find $L\sec\theta$:

$$L\tan \qquad\qquad\qquad L\sec$$

$$2599\begin{pmatrix}7116\\5778\\4517\end{pmatrix}1261 \qquad 995\begin{pmatrix}9550\\8555+d\\8555\end{pmatrix}d$$

$$\begin{aligned}\therefore\quad 2599d&=1261\times 995\\ d&=483\\ 8555+d&=9038\end{aligned}$$

Hence $L\sec\theta=10.1049038.$

Therefore $\qquad \log\sqrt{a^2+b^2}=\log a+L\sec\theta-10=2.8534189+10.1049038-10$

$$=2.9583227$$

Now
$$\begin{array}{l}\log 908.49=2.9583202\\ \qquad D=47\quad)250(5\\ \qquad\qquad\qquad\; \underline{235}\end{array} \qquad \therefore\quad \sqrt{a^2+b^2}=908.495.$$

(2) To adapt $\sqrt{b^2+c^2-2bc\cos A}$ to logarithmic computation.

Let $\sqrt{b^2+c^2-2bc\cos A}=x$;

then (i)
$$x^2=b^2+c^2-2bc\cos A=b^2+c^2-2bc\left(2\cos^2\frac{A}{2}-1\right)$$
$$=(b+c)^2-4bc\cos^2\frac{A}{2}=(b+c)^2\left\{1-\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right\}.$$

Now, $\frac{4bc}{(b+c)^2}$ being necessarily a proper fraction, we can put

$$\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}=\sin^2\theta \text{ or } \cos^2\theta, \quad \text{.......(i)}$$

in which case we have $\quad x=(b+c)\cos\theta$ or $(b+c)\sin\theta$, respectively. ....(ii)

$L\sin\theta$ or $L\cos\theta$ can be found from (i), and the corresponding value of $L\cos\theta$ or $L\sin\theta$ respectively is then substituted in (ii) to determine x.

Or thus (ii),
$$x^2=b^2+c^2-2bc\left(1-2\sin^2\frac{A}{2}\right)=(b-c)^2+4bc\sin^2\frac{A}{2}$$
$$=(b-c)^2\left\{1+\frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}\right\}.$$

And putting $\qquad \frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}=\tan^2\theta,$ ....(i)

we have $\qquad x=(b-c)\sec\theta,$ ....(ii)

and the value of x can be determined from (i) and (ii) as before.

Example. Compute the value of $\sqrt{b^2+c^2-2bc\cos A}$ when $b=8214$, $c=3732$, and $A=61°53'$.

$$\begin{array}{lll} \text{Given} & \log 4 = .6020600 & \log 8214 = 3.9145547 \\ & \log 3732 = 3.5719416 & \log 7246 = 3.8600983 \\ & & \log 11946 = 4.0772225 \\ & L\sin 52°39' = 9.9003367 & L\cos 52°39' = 9.7829614 \\ & L\sin 52°40' = 9.9004331 & L\cos 52°40' = 9.7827958 \\ & & L\cos 30°56' = 9.9333688 \qquad D=757 \end{array}$$

Since logarithmic tangents are not given we adopt the first mode.

Put $\dfrac{4bc}{(b+c)^2}\cos^2\dfrac{A}{2}=\sin^2\theta$, so that $x=(b+c)\cos\theta$.

$$\begin{array}{rrl} & \log 4 = & .6020600 \\ \log b = & \log 8214 = & 3.9145547 \\ \log c = & \log 3732 = & 3.5719416 \\ 2L\cos\dfrac{A}{2} = & 2L\cos 30°56'30'' = & \underline{19.8666618} \\ & & 26.9552181 \\ 2\log(b+c) = & 2\log 11946 = & \underline{8.1544450} \\ \therefore & 2L\sin\theta = & 18.8007731 \\ & L\sin\theta = & 9.4003866 \end{array}$$

$$\begin{array}{rl} L\cos 30°56' & = 9.9333688 \\ \text{diff. for } 30'' & = \underline{-.0000379} \\ \therefore\ L\cos 30°56'30'' & = 9.9333309 \\ & \underline{\qquad\qquad 2} \\ & 19.8666618 \end{array}$$

To find $L\cos\theta$:

$$L\sin \quad 964\begin{pmatrix}4331\\3866\\3367\end{pmatrix}465 \qquad L\cos \quad 1656\begin{pmatrix}7958\\7958+d\\9614\end{pmatrix}d$$

$$\therefore\ 964d = 1656\times 465, \quad d = 799, \quad 7958+d = 8757$$

Hence

$$\begin{array}{rl} L\cos\theta = & 9.7828757 \\ \log(b+c) = & 4.0772225 \\ & \underline{-10.\qquad\quad} \\ \therefore\quad \log x = & 3.8600982, \text{ and } x = 7246. \end{array}$$

(3) To solve the equation $a\sin x+b\cos x=c$.

Put $\left.\begin{array}{l}a=r\cos\phi\\ b=r\sin\phi\end{array}\right\}$ so that $\tan\phi=\dfrac{b}{a}$, ...(i)

and $r=\sqrt{a^2+b^2}$.

the equation then becomes $r\sin(x+\phi)=c$. ....(ii)

The values of ϕ and $x+\phi$ can be calculated from (i) and (ii) respectively, and x is then the difference between these computed values.

EXAMPLES. XVIII.

1. Express in forms adapted to logarithmic computation,

(i) $\sec^2 A+\operatorname{cosec}^2 A$, (ii) $\tan^2\theta-\sin^2\theta$,

(iii) $\dfrac{\cos 5A-\cos 7A}{\sin 8A-\sin 2A}$, (iv) $\dfrac{\sin 7A-\sin A}{\sin 8A-\sin 2A}$,

(v) $\dfrac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}-\dfrac{\cos A-\cos 3A}{\sin 3A-\sin A}$, (vi) $\dfrac{\sin 3\theta+2\sin 5\theta+\sin 7\theta}{\sin 5\theta+2\sin 7\theta+\sin 9\theta}$,

(vii) $\dfrac{\sin(\alpha+3\beta)+\sin(3\alpha+\beta)}{\sin 2\alpha+\sin 2\beta}$, (viii) $\sin 3A+\sin 2A-\sin A$,

(ix) $\cos A+\cos 2A+\cos 3A$, (x) $\cos x\cos(y+z)-\cos y\cos(x+z)$,

(xi) $\cos 3x\cos 2x+\sin 4x\sin x$, (xii) $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$,

(xiii) $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$, (xiv) $\sqrt{\dfrac{2\sin A-\sin 2A}{2\sin A+\sin 2A}}$,

(xv) $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}$, (xvi) $\tan A-\tan\dfrac{A}{2}$,

(xvii) $\tan 3A-\tan 2A-\tan A$, (xviii) $\tan A+\tan(60^\circ+A)+\tan(120^\circ+A)$,

(xix) $\cot\theta\sec^2\theta-\cos\theta\operatorname{cosec}\theta$, (xx) $\sin^3 A+\sin^3(120^\circ+A)+\sin^3(240^\circ+A)$,

(xxi) $\sin 3A\cos^3 A+\cos 3A\sin^3 A$, (xxii) $1+\sin A+\cos A$,

(xxiii) $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$, (xxiv) $1+\tan x\tan\dfrac{x}{2}$,

(xxv) $\dfrac{\cot^2\theta-6+\tan^2\theta}{\cot^2\theta+2+\tan^2\theta}$.

2. Compute the values of

(i) $\dfrac{1+\cot^2 A}{\sqrt{2}\cos^2 A}$, when $A=24^\circ 20'$; given $\log 2=.3010300$
$\log 50165=4.7004008$ $D=87$
$L\sin 48^\circ 40'=9.8755706$

(ii) $.0085627\dfrac{1-\tan^2 A}{1+\tan^2 A}$, when $2A=37^\circ 45'$; given $\log 85627=4.9326107$
$\log 67.704=4.8306143$ $D=64$
$L\sin 52^\circ 15'=9.8980060$

(iii) $.00023715\dfrac{\sin A-\cos A}{\sin A+\cos A}$, when $A=73^\circ 10' 20''$; $\log 23715=4.3750231$
$\log 12701=4.1038379$ $D=342$
$L\tan 28^\circ 10'=9.7287161$ $D=3035$

(iv) $.00017\dfrac{1+\tan^2 A}{1-\sin^2 A}$, when $A=135^\circ 15' 45''$; given $\log 17=1.2304489$
$\log 66770=4.8245814$ $D=65$
$L\cos 44^\circ 44'=9.8514969$ $D=1252$

(v) $250000.9\dfrac{1+\cot^2 A}{1+\tan^2 A}$, when $2A=77^\circ$; given $\log 25000=4.3979400$ $D=174$
$\log 39512=4.5967290$ $D=110$
$L\cot 38^\circ 30'=10.0993948$

(vi) $1.00076(1+\tan^2 A)$, when $A=125^\circ$; given $\log 10007=4.0003039$ $D=434$
$\log 30419=4.4831449$ $D=143$
$L\cos 55^\circ=9.7585913$

(vii) $\sqrt[3]{\dfrac{\sin A(1-\cos A)}{.0070639}}$, when $A=215^\circ$; given $\log 35319=4.5480084$ $D=123$
$\log 52861=4.7231354$ $D=82$
$L\sin 35^\circ=9.7585913$
$L\sin 72^\circ 30'=9.9794195$

(viii) $(\tan A+\cot A)\dfrac{\sin^2\frac{A}{2}}{3\cdot 5}$, when $A=105^\circ 27'$; given $\log 175=2.2430380$
$\log 70457=4.8479241$ $D=62$
$L\sin 30^\circ 54'=9.7105753$
$L\sin 52^\circ 43'=9.9007219$ $D=962$

(ix) $32.574(\tan A+\tan B)$, when $\left.\begin{array}{l}A=71^\circ 32'\\ B=25^\circ 18'\end{array}\right\}$ given $\log 32574=4.5128711$
$\log 11293=4.0528093$ $D=385$
$L\sin 83^\circ 10'=9.9969040$
$L\cos 71^\circ 32'=9.5007206$
$L\cos 25^\circ 18'=9.9562081$

(x) $\sqrt{\cos^4 A-\sin^4 A}$, when $A=34^\circ 16'$; given $\log 60494=4.7817123$ $D=72$
$L\cos 68^\circ 32'=9.5634335$

(xi) $\dfrac{11.315\sin^2 A}{1+\cos A}$, when $A=115^\circ 45'$; given $\log 56575=4.7526246$
$\log 16230=4.2103185$ $D=268$
$L\sin 64^\circ 15'=9.9545793$
$L\cos 57^\circ 52'=9.7258229$ $D=2012$

(xii) $\dfrac{\tan^2 A - \tan^2 B}{\sin^2 A - \sin^2 B}$, when $\left.\begin{array}{l}A = 74°\ 8' \\ B = 53°28'\end{array}\right\}$ given $\log 37752 = 4.5769400 \quad D = 115$
$L \cos 74°\ 8' = 9.4367980$
$L \cos 53°28' = 9.7747288$

(xiii) $\sqrt{\dfrac{3 \sin A - \sin 3A}{3 \cos A + \cos 3A}}$, when $A = 85°$; given $\log 38643 = 4.5870708 \quad D = 113$
$L \tan 85° = 11.0580482$

(xiv) $\dfrac{\sin 70° + \sin 85°}{\cos 50° + \cos 105°}$; given $\log 50417 = 4.7025770 \quad D = 86$
$L \tan 77°30' = 10.6542448$
$L \cos 7°30' = 9.9962686$
$L \cos 27°30' = 9.9479289$

(xv) $\left(\dfrac{\operatorname{cosec} A - \cot A}{\operatorname{cosec} A + \cot A}\right)\left(\dfrac{\sec A - \tan A}{\sec A + \tan A}\right)$, when $A = 63°15'$;
given $\log 21439 = 4.3312045 \quad D = 203$
$L \tan 31°37' = 9.7893023 \quad D = 2829$
$L \tan 13°22' = 9.3758810 \quad D = 5613$

(xvi) $\left(\dfrac{\sin A + \sin B}{\sin A - \sin B}\right)\left(\dfrac{\cos A + \cos B}{\cos A - \cos B}\right)$, when $\left.\begin{array}{l}A = 128°12' \\ B = 48°28'\end{array}\right\}$
given $\log 14337 = 4.1564583 \quad D = 303$
$L \tan 39°52' = 9.9217602$

(xvii) $\dfrac{\sin 35° + \sin 55° + \sin 75°}{\sin 45° + \sin 65° + \sin 85°}$; given $\log 90383 = 4.9560868 \quad D = 48$
$L \sin 55° = 9.9133645$
$L \sin 65° = 9.9572757$

(xviii) $\frac{1}{4}(1 - \tan A)^4$, when $A = 16°8'32''$; given $\log 63732 = 4.8043575 \quad D = 69$
$L \cos 16°8' = 9.9825506 \quad D = 366$
$L \cos 61°8' = 9.6837430 \quad D = 2293$

3. Find the values of the positive angles less than 180° that satisfy the following equations:

(i) $\sin x + \cos x = 1.2$, given $\log 2$, $\log 3$, $L \sin 58°\ 3' = 9.9286571$, $D = 787$.

(ii) $\sin x - \cos x = .3$, given $\log 2$, $\log 3$, $L \sin 12°14' = 9.3261174$, $D = 5823$.

(iii) $\cos x - \sin x = .2118$, given $\log 2$, $\log 2118 = 3.3259260$,
$L \cos 81°23'20'' = 9.1753004$, diff. $10'' = 1390$.

(iv) $\sin x - \sqrt{3} \cos x = \frac{1}{3}$, given $\log 2$, $\log 3$, $L \sin 9°35' = 9.2213671$, $D = 7476$.

(v) $\sqrt{3} \sin x = 1\frac{1}{4} - \cos x$, given $\log 625 = 2.7958800$,
$L \cos 51°20' = 9.7957330$, $D = 1579$.

(vi) $3 \sin x + 4 \cos x = 4.3$, given $\log 2$, $\log 3$, $\log 86 = 1.9344985$,
$L \sin 59°19' = 9.9344988$,
$L \tan 53°\ 7' = 10.1247266$, $D = 2632$.

(vii) $4 \sin x - 5 \cos x = 1$, given $\log 2$, $\log 41 = 1.6127839$,
$L \sin 8°59' = 9.1935341$, diff. $10'' = 1331$,
$L \tan 51°20' = 10.0968034$, $D = 2590$.

(viii) $\tan \theta + \cot \theta = 3\frac{1}{2}$, given $\log 2$, $\log 7$, $L \sin 34°51' = 9.7569630$, $D = 1815$.

(ix) $\tan x - \cot x = 2\frac{1}{4}$, given $\log 2$, $\log 3$, $L \cot 41°38' = 10.0511557$, $D = 2544$.

(x) $1 - \tan^2 x = 7 \tan x$, given $\log 2$, $\log 7$, $L \tan 15°56' = 9.4555857$, $D = 4784$.

(xi) $1 + \tan^2 x = 5 \tan x$, given $\log 2$, $L \sin 23°34' = 9.6018600$, $D = 2895$.

(xii) $1 - 2 \sin A + \cos A = 0$, given $\log 2$, $L \tan 26°33' = 9.6986847$, $D = 3159$.

(xiii) $1 - \sin A = \frac{3}{8}\cos A$, given $\log 2$, $\log 3$, $L\tan 33^\circ 41' = 9.8237981$, $D = 2738$.

(xiv) $\sec(x+\theta) + \sec(x-\theta) = 2\sec\theta$, when $\theta = 140^\circ$;
given $\log 2$, $L\cos 40^\circ = 9.8842540$,
$L\cos 57^\circ 12' = 9.7337654$, $D = 1961$.

(xv) $\left.\begin{array}{l}\sin x + \sin y = 1.24\\ \cos x + \cos y = \;\;.65\end{array}\right\}$ given

$$\begin{array}{rll} \log 62 = & 1.7923917 & \\ \log 65 = & 1.8129134 & \\ \log 124 = & 2.0934217 & \\ L\cos 45^\circ 35' = & 9.8450181 & D = 1289 \\ L\tan 62^\circ 20' = & 10.2804451 & D = 3073 \\ L\sin 62^\circ 20' = & 9.9472689 & D = 663 \end{array}$$

4. Find the acute angle whose tangent $= \sqrt[3]{\dfrac{\sin^2 45^\circ - \sin^2 35^\circ}{\sin^2 35^\circ - \sin^2 25^\circ}}$,

given

$$\begin{array}{rll} L\sin 60^\circ = & 9.9375306 & \\ L\sin 80^\circ = & 9.9933515 & \\ L\tan 46^\circ 13' = & 10.0184499 & D = 2529 \end{array}$$

5. If $2a = 3b$, find the acute angles satisfying the equation $a\cos\theta + b\sin\theta = \dfrac{a+b}{\sqrt{2}}$; given $\log 2$, $\log 3$, $L\tan 33^\circ 41' = 9.8237981$, $D = 2738$.

6. If $\left.\begin{array}{l}\sin\theta = m\sin\phi\\ \tan\theta = n\tan\phi\end{array}\right\}$, find the principal trigonometrical ratios of θ and ϕ in forms adapted to logarithmic computation.

7. Given $\sin(\theta + \alpha) = m\sin\theta$, find θ in terms adapted to logarithmic computation.

8. Find, by means of subsidiary angles, the values of

(i) $\sqrt{a^2 + b^2}$, when $a = 30.4025$, $b = 21.7856$. Given

$$\begin{array}{ll} \log 21785 = 4.3381576 & D = 199 \\ \log 30402 = 4.4800645 & D = 143 \\ \log 37241 = 4.5710213 & D = 117 \\ L\tan 35^\circ 48' = 9.8580694 & L\sec 35^\circ 48' = 10.0909450 \\ L\tan 35^\circ 49' = 9.8583357 & L\sec 35^\circ 49' = 10.0910361 \end{array}$$

(ii) $\sqrt{a^2 + b^2}$, when $a = 87.079$, $b = 129.384$. Given

$$\begin{array}{ll} \log 87079 = 4.9399134 & \\ \log 12938 = 4.1118671 & D = 336 \\ \log 15595 = 4.1929854 & D = 278 \\ L\cot 56^\circ 3' = 9.8281696 & L\sec 56^\circ 3' = 10.2530008 \\ L\cot 56^\circ 4' = 9.8278969 & L\sec 56^\circ 4' = 10.2531885 \end{array}$$

(iii) $\sqrt{a^2 + b^2}$, when $a = .35991$, $b = .24376$. Given

$$\begin{array}{lll} \log 24376 = 4.3869624 & L\tan 34^\circ 6' = 9.8306213 & D = 2721 \\ \log 35991 = 4.5561939 & L\cos 34^\circ 6' = 9.9180620 & D = 856 \\ \log 43468 = 4.6381697 & D = 99 & \end{array}$$

(iv) $\sqrt{b^2 + c^2 - 2bc\cos A}$, when $b = 17.14$, $c = 32.36$, $A = 48^\circ 22'$. Given

$$\begin{array}{lll} \log 2 = .3010300 & \log 1714 = 3.2340108 & \\ \log 495 = 2.6946052 & \log 3236 = 3.5100085 & \\ L\cos 24^\circ 11' = 9.9601088 & \log 24575 = 4.3904935 & D = 177 \\ L\sin 60^\circ 13' = 9.9384747 & L\cos 60^\circ 13' = 9.6961130 & \\ L\sin 60^\circ 14' = 9.9385470 & L\cos 60^\circ 14' = 9.6958922 & \end{array}$$

(v) $\sqrt{b^2+c^2-2bc\cos A}$, when $b=2139$, $c=5817$, $A=115°28'$. Given

$\log\ 4 = .6020600$	$\log\ 2139 = 3.3302108$	
$\log 7956 = 3.9006948$	$\log\ 5817 = 3.7646991$	
$L\cos 57°44' = 9.7274278$	$\log 70080 = 4.8455941$	$D=62$
$L\cos 61°44' = 9.6753896$	$L\sin 61°44' = 9.9448541$	
$L\cos 61°45' = 9.6751546$	$L\sin 61°45' = 9.9449220$	

(vi) $\sqrt{b^2+c^2-2bc\cos A}$, when $b=104.28$, $c=217.54$, $A=80°30'$. Given

$\log\ 4 = .6020600$	$\log 10428 = 4.0172010$	
$\log 11326 = 4.0540766$	$\log 21754 = 4.3375391$	
$L\sin 40°15' = 9.8103159$	$\log 22499 = 4.3521632$	$D=193$
$L\tan 59°46' = 10.2344857$	$D=2904$	
$L\sec 59°46' = 10.2979810$	$D=2168$	

9. By introducing subsidiary angles adapt to logarithms the expressions

(i) $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$, (ii) $\frac{a-b)^{\frac{1}{2}}}{a+b}+\frac{(a+b)^{\frac{1}{2}}}{a-b}$.

CHAPTER X.

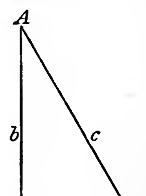
Solution of Triangles.

51. Logarithms are applied to the Solution of Triangles, that is, are used for finding the remaining sides or angles when certain of them are given. To solve a triangle completely, of the six parts (3 sides and 3 angles), three must be known, but these must have values that are independent of one another. Now it is known that the three angles of a triangle are not independent in value, for they are always together equal to two right angles; therefore it will not be sufficient to have only the three angles given, but a complete solution will be possible when (i) the three sides, (ii) two angles and a side, and (iii) two sides and an angle are given.

The angles of a triangle are generally called A, B, C, and the sides respectively opposite to them a, b, c.

52. We will first discuss the case of **right-angled triangles.**

Let C be the right angle, so that $C = 90°$; then



(1) $a^2 = c^2 - b^2$ or $a = \sqrt{(c+b)(c-b)}$
$b^2 = c^2 - a^2$ or $b = \sqrt{(c+a)(c-a)}$

(2) $\left.\begin{matrix}\sin A\\ \cos B\end{matrix}\right] = \dfrac{a}{c}$ or $a = \left[\begin{matrix} c\sin A\\ c\cos B\end{matrix}\right.$

(3) $\left.\begin{matrix}\sin B\\ \cos A\end{matrix}\right] = \dfrac{b}{c}$ or $b = \left[\begin{matrix} c\sin B\\ c\cos A\end{matrix}\right.$

i.e. each side equals the hypotenuse multiplied by the *sine* of the angle *opposite* to, or the *cosine* of the angle *adjacent* to the former.

(4) $\left.\begin{matrix}\tan A\\ \cot B\end{matrix}\right] = \dfrac{a}{b}$ or $a = \left[\begin{matrix} b\tan A\\ b\cot B\end{matrix}\right.$

(5) $\left.\begin{matrix}\tan B\\ \cot A\end{matrix}\right] = \dfrac{b}{a}$ or $b = \left[\begin{matrix} a\tan B\\ a\cot A\end{matrix}\right.$

i.e. each side equals the other side multiplied by the *tangent* of the angle *opposite* to, or the *cotangent* of the angle *adjacent* to the former.

In the right-angled triangle, one angle C $(=90°)$ is known; hence, in addition, of the remaining sides and angles two only need be given for the complete solution of the triangle, but these must not be the two angles.

Case (i). *Given the two sides* a, b.

Either A or B is found from (4) or (5) and the other angle is then known since $A + B = 90°$. The hypotenuse c is then found from (2) or (3).

Case (ii). *Given one side and the hypotenuse, e.g.* a, c.

Either A or B is found from (2), then $A+B=90°$ gives the other angle. The other side b can then be found from (3), (4), or (5); or independently of the angles from (1).

Case (iii). *Given one angle and side, e.g.* a, A.

$B=90°-A$. The other side b is found from (4) or (5), and the hypotenuse c from (2) or (3).

Case (iv). *Given one angle and the hypotenuse, e.g.* c, A.

$B=90°-A$; then a is found from (2), and b from (1), (3), (4), or (5).

53. When the **triangle** is **not right-angled** we have the following cases:—

(i) *Given the three sides.*
(ii) *Given two sides and the included angle.*
(iii) *Given two sides and an angle not included.*
(iv) *Given two angles and a side.*

In the solution of these triangles the following formulae are employed:—

$$(1)\quad \begin{cases} \sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}} \\ \cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}} \\ \tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \end{cases} \qquad s=\tfrac{1}{2}(a+b+c).$$

$$(2)\quad \tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}.$$

[Putting $\tan\theta=\dfrac{b}{c}$ this formula becomes $\tan\tfrac{1}{2}(B-C)=\tan(\theta-45°)\cot\dfrac{A}{2}$.

Putting $\cos\phi=\dfrac{c}{b}$ it becomes $\tan\tfrac{1}{2}(B-C)=\tan^2\dfrac{\phi}{2}\cot\dfrac{A}{2}$.]

$$(3)\quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

(4) $a^2=b^2+c^2-2bc\cos A$, this formula being adapted to logarithms by introducing a subsidiary angle as in Art. 50.

[In the above formulae the letters a, b, c are of course interchangeable, provided A, B, C be also interchanged in like manner.]

Case I.

54. A. *Given* the three sides a, b, c.

B. *Solution.* One of the formulae (1) is used to determine each of two of the angles A, B, C; and the third angle is then known since $A+B+C=180°$.

[Obs. When all the angles are required the tangent-formula is the most convenient to use since fewer logarithms are then required (4 instead of 6 on the right hand side); but, if only one angle be wanted, there is no such advantage. Of course, when logs. are given for the purposes of any question, our selection of the formula must be guided by the data.]

C. *Example.* If $a=217$, $b=192$, $c=89$; find all the angles.

Given $\log 32 = 1.5051500$ $\quad L\tan 46°55' = 10.0290779 \quad D=2532$
$\log 57 = 1.7558749$
$\log 160 = 2.2041200$ $\quad L\tan 30°59' = 9.7784875 \quad D=2862$
$\log 249 = 2.3961993$

To find A:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{57 \cdot 160}{249 \cdot 32}}$$

$$\begin{array}{r} 217 = a \\ 192 = b \\ 89 = c \\ \hline 2)498 \\ \hline 249 = s \end{array}$$

$$\therefore \quad L\tan\frac{A}{2} = \tfrac{1}{2}(\log 57 + \log 160 - \log 249 - \log 32) + 10$$

$$= 10.0293228$$

But $\quad L\tan 46°55' = 10.0290779$

$$\begin{array}{r} 2449 \\ 60 \\ \hline D = 2532)146940(58.0 \\ 12660 \\ \hline 20340 \\ 20256 \\ \hline 840 \end{array}$$

Hence $\dfrac{A}{2} = 46°55'58.0''$

and $\quad A = 93°51'56''$.

To find B:

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{32 \cdot 160}{249 \cdot 57}},$$

$$\therefore \quad L\tan\frac{B}{2} = \tfrac{1}{2}(\log 32 + \log 160 - \log 249 - \log 57) + 10$$

$$= 9.7785979$$

But $\quad L\tan 30°59' = 9.7784875$

$$\begin{array}{r} 1104 \\ 60 \\ \hline D = 2862)66240(23.1 \\ 5724 \\ \hline 9000 \\ 8586 \\ \hline 4140 \end{array}$$

Hence $\dfrac{B}{2} = 30°59'23.1''$

and $\quad B = 61°58'46''$.

To find C:

$$\begin{array}{rr} A = & 93°51'56'' \\ B = & 61°58'46'' \\ \hline \therefore \quad A+B = & 155°50'42'' \\ & 180°\ 0'\ 0'' \\ \hline \text{and} \qquad C = & 24°\ 9'18'' \end{array}$$

[Obs. In finding the values of A, B, C to the nearest second, $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ must be calculated to the nearest tenth of a second, since the multiplication of these tenths by 2 may affect the seconds' units in the values of A, B, C. It is also evident that we need only know the ratios of the sides a, b, c to one another, and that the numbers expressing the ratios $a:b:c$ may be taken as the values of the sides themselves.]

Case II.

55. A. *Given* two sides and the included angle, e.g. b, c, A.

B. *Solution.* Formula (2) of Art. 53 gives $\frac{1}{2}(B-C)$; then $\frac{1}{2}(B+C)=90°-\frac{A}{2}$. Adding and subtracting these, B and C are found. The third side a can then be found by means of formula (3) using the value just obtained of B or C, or independently by means of formula (4).

C. *Example.* If $b=23.46$, $c=7.85$, $A=73°14'$; find the remaining angles and side.

Given $\log 1561=3.1934029$	$L\sin 73°14' = 9.9811331$	
$\log 2346=3.3703280$	$L\cot 36°37' = 10.1289428$	
$\log 3131=3.4956831$	$L\tan 33°51' = 9.8265323$	$D=2730$
$\log 22488=4.3519508$	$L\sin 87°14'20'' = 9.9994955$	$D\,10''=10$
$D=193$		

To find B and C: $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{15.61}{31.31}\cot 36°37'$,

$\therefore\ L\tan\frac{1}{2}(B-C)=\log 15.61-\log 31.31+L\cot 36°37'$

$=9.8266626$

But $L\tan 33°51'=9.8265323$

$$\begin{array}{r}1303\\60\\ D=2730)78180(28.6\\5460\\ \hline 23580\\21840\\ \hline 17400\\16380\end{array}$$

Hence $\frac{1}{2}(B-C)=33°51'28.6''$

and $\frac{1}{2}(B+C)=53°23'\ \left(90°-\frac{A}{2}\right)$

$\therefore$ adding $B=87°14'28.6''$

and subtracting $C=19°31'31.4''$

To find a: $\frac{\sin A}{a}=\frac{\sin B}{b}$, (Taking the b, a portion of the formula, since $\log b$ is given and not $\log c$.)

$a=\frac{b\sin A}{\sin B}$,

$\therefore\ \log a=\log b+L\sin A-L\sin B$

$=\log 23.46+L\sin 73°14'-L\sin 87°14'28.6''$

$=1.3519647$

But $\log 22.488=1.3519508$

$$\begin{array}{r}D=\ 193)1390(7\\1351\end{array}$$

Hence $a=22.4887$.

[OBS. In finding the angles B and C, only the ratio $b:c$ need be given, in which case the numbers expressing the ratio can be used for the sides themselves; but in finding the third side a, the actual values of the sides must be given. This same method of solution is applicable when two sides b, c are given and the difference $B-C$ between their opposite angles. Formula (2) determines A, and thence $B+C$ which $=180°-A$. B and C are then obtained by addition and subtraction of $B+C$ and $B-C$.]

Case III.

56. A. *Given* two sides and an angle not included, e.g. a, b, A.

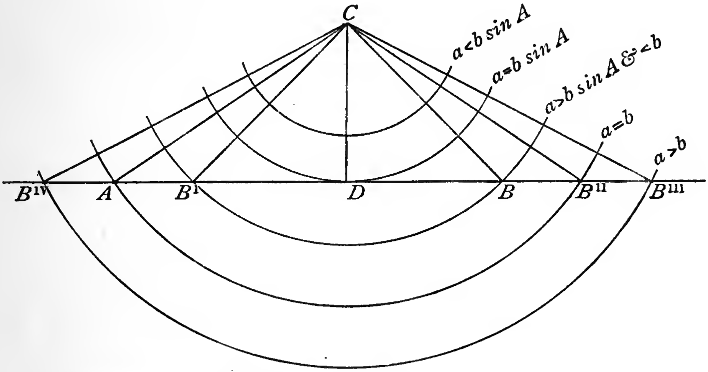
B. *Solution.* Since the sides given are a and b, we take the a, b portion of formula (3) and thence determine the angle B. Then $C=180°-(A+B)$; and the third side c is obtained by using the c portion of formula (3).

This is the case in which the solution may be *ambiguous*, that is, in which there may be two triangles with the given parts.

[The Ambiguous Case.]

First consider the angle A acute.

Draw $CA=b$, $C\hat{A}X=\hat{A}$, and with centre C and radius $=a$ describe a circle; there is ambiguity only when this circle cuts AX in two distinct points lying on the same side of A, neither of which coincides with A. The two triangles in the ambiguous case are ABC, $AB'C$



Now the perpendicular $CD=b\sin A$. Therefore we have the following results:

(1) **a<b.**	(i) $a<b\sin A$.		No solution possible.
	(ii) $a=b\sin A$.	Right-angled triangle ACD.	Not ambiguous.
	(iii) $a>b\sin A$.	Two solutions ABC, $AB'C$.	Ambiguous.
(2) **a = b.**		One solution $AB''C$.	Not ambiguous.
(3) **a>b.**		One solution $AB'''C$.	Not ambiguous.

(In this case $AB^{iv}C$ has no angle A; $C\hat{A}B^{iv}=180°-A$.)

It will be seen from the above that

(i) when $a=b$, there is no ambiguity;

(ii) when a does not equal b, *the solution is ambiguous when the given angle* (A) *is opposite to the smaller side unless the triangle is right-angled.* [When the triangle is right-angled the angle first found in the process of solution (i.e. B) comes out 90°, so that $L\sin B=10$.]

When the given angle is right or obtuse, taking the angle to be $A\hat{D}C$ or $A\hat{B}'C$ respectively, it is clear from the figure that there can be no ambiguity, and in these cases the angle given is opposite to the *greater* side.

In the ambiguous case there will be double values for each of the required parts. The acute value found for B ($A\hat{B}C$) is taken from 180° to obtain its second value ($A\hat{B}'C$). In each case $A+B$ is taken from 180° to determine the third angle C ($A\hat{C}B$ or $A\hat{C}B'$). To find the third side c (AB or AB'), either formula (3) of Art. 53 is used, or AD and DB can be calculated from the equations $AD=b\cos A$, $DB=a\cos B$, the two values of c being then the sum and difference of AD and DB.

C. *Example.* If $a=47$, $b=53$, $A=36°42'$; find the remaining angles and side.

Given $\log 47 = 1.6720979$	$L\sin 36°42' = 9.7764289$	
$\log 53 = 1.7242759$	$L\sin 42°22' = 9.8285778$	$D = 1385$
$\log 77218 = 4.887719$	$L\sin 79°\ 4' = 9.9920445$	$D = 244$
$\log 77704 = 4.89044$	$L\sin\ 5°40'10'' = 8.9947089$	$D\ 10'' = 2121$

This is presumably an ambiguous case since the angle given is opposite to the *smaller* side.

To find B: $$\frac{\sin A}{a} = \frac{\sin B}{b},$$

$$\sin B = \frac{b \sin A}{a},$$ (It may be necessary here to reduce $\frac{b}{a}$ before taking logarithms to suit data.)

$$\therefore\ L\sin B = \log b - \log a + L\sin A$$
$$= \log 53 - \log 47 + L\sin 36°42'$$
$$= 9.8286069$$

But $L\sin 42°22' = 9.8285778$

$$\begin{array}{r} 291 \\ 60 \\ D = 1385)\overline{17460}(12.6 \\ \underline{1385} \\ 3610 \\ \underline{2770} \\ 8400 \end{array}$$

Hence $B = 42°22'12.6''$ or $137°37'47.4''$.

[The solution is ambiguous since B has not come out $90°$, and therefore we get a second value of B by taking the first value from $180°$.]

To find C: $$A+B = 79°\ 4'12.6'' \text{ or } 174°19'47.4'',$$
$$\therefore\ C = 100°55'47.4'' \text{ or } 5°40'12.6''.$$

To find c: $$\frac{\sin A}{a} = \frac{\sin C}{c}, \qquad c = \frac{a \sin C}{\sin A},$$
$$\therefore\ \log c = \log a + L\sin C - L\sin A.$$

(i) For the larger value of c we must take the larger value of C,

$$\therefore\ \log c = \log 47 + L\sin 100°55'47.4'' - L\sin 36°42'$$
$$= \log 47 + L\sin\ 79°\ 4'12.6'' - L\sin 36°42'$$
$$= 1.8877186.$$

Hence $c = 77.218$.

(ii) The smaller value of c can be shown in like manner to be equal to 7.7704.

[Obs. In finding the angles B, C only the ratio $a:b$ need be given; but for the third side c the actual values must be known.]

Case IV.

57. A. *Given* two angles and a side, e.g., A, B, a.

B. *Solution.* $C = 180° - (A+B)$. The remaining sides b, c are determined by using formula (3) in the same way as in the example in the last article.

58. The following table gives a list of the formulae used in the solution of triangles, and in finding their areas and the radii of their circumscribed, inscribed, and escribed circles.

Δ = Area of triangle,
R = Radius of circumscribed circle,
r = Radius of inscribed circle,
r_a = Radius of escribed circle, opposite to angle A.

I. Given 3 Sides.	II. Given 2 Sides and Included Angle.	III. Given (I) 2 Angles and a Side; (II) 2 Sides and Angle not Included.
$\begin{cases} \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}} \\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}} \\ \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \end{cases}$	$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}$	$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$
$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$	$\frac{1}{2}bc\sin A$	$\frac{a^2\sin B\sin C}{2\sin A}$
$R=\frac{abc}{4\Delta}$ or $\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$		$\frac{a}{2\sin A}$
$r=\frac{\Delta}{s}$ or $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$		$\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$
$r_a=\frac{\Delta}{s-a}$ or $\sqrt{\frac{s(s-b)(s-c)}{s-a}}$		$\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

59. The following reductional formulae will be found useful in applying logarithms to the properties of triangles.

(A)
$$\begin{matrix}\sin\\ \cos\\ \tan\end{matrix}\begin{bmatrix}B+C\\ C+A\\ A+B\end{bmatrix}=\begin{matrix}\sin\\ -\cos\\ -\tan\end{matrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix}\qquad \begin{matrix}\sin\\ \cos\\ \tan\end{matrix}\begin{bmatrix}\frac{B+C}{2}\\ \frac{C+A}{2}\\ \frac{A+B}{2}\end{bmatrix}=\begin{matrix}\cos\\ \sin\\ \cot\end{matrix}\begin{bmatrix}\frac{A}{2}\\ \frac{B}{2}\\ \frac{C}{2}\end{bmatrix}.$$

(B)
$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$
$$\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},$$
$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$
$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{180^\circ-A}{4}\cos\frac{180^\circ-B}{4}\cos\frac{180^\circ-C}{4},$$
$$\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{180^\circ+A}{4}\cos\frac{180^\circ+B}{4}\cos\frac{180^\circ-C}{4},$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$
$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$
$$\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C,$$
$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

(C) $$\frac{c}{a+b} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}, \qquad \frac{c}{a-b} = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)},$$
$$\cos A + \cos B = \frac{2(a+b)}{c}\sin^2\frac{C}{2}, \qquad b\cos C + c\cos B = a,$$
$$b\cos C - c\cos B = \frac{(b+c)(b-c)}{a}, \qquad b\cos B + c\cos C = a\cos(B-C),$$
$$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C, \qquad b^2\cos 2A - a^2\cos 2B = (b+a)(b-a).$$

EXAMPLES. XIX.

Sandhurst and Militia (1-24).

1. $a=3$, $b=2.75$, $c=1.75$; find B.
 $\log 2 = .30103$ $\quad L\tan 32°19' = 9.8008365$ $\quad D = 2796$

2. The sides of a triangle are 4, 10, 11; find the greatest angle.
 $\log 2 = .3010300$ $\quad L\cos 46°47' = 9.8355378$
 $\log 3 = .4771213$ $\quad$ diff. $1' = 1345$

3. Find the greatest angle of a triangle whose sides are 5, 8, 11 respectively.
 $\log 7 = .8450980$ $\quad L\sin 56°47' = 9.9225205$
 $L\sin 56°48' = 9.9226032$

4. The sides BC, CA, AB of a triangle are as 4 : 5 : 6. Find B.
 $\log 2 = .3010300$ $\quad L\cos 27°53' = 9.9464040$
 $L\cos 27°54' = 9.9463371$

5. $b=9$, $c=6$, $A=60°$; find the other angles.
 $\log 2 = .30103$ $\quad L\tan 19°6' = 9.5394287$
 $\log 3 = .47712$ $\quad L\tan 19°7' = 9.5398371$

6. Two sides of a triangle are 540 yds. and 420 yds., and the included angle is 52°6'. Find the remaining angles. $\log 2 = .3010300$ $\quad L\tan 14°20' = 9.4074189$
 $L\tan 26°3' = 9.6891430$ $\quad L\tan 14°21' = 9.4079453$

7. Two sides of a triangle are 9 and 7 feet respectively, and the angle between them is 60°; find the other angles. $\log 2 = .3010300$ $\quad L\tan 12°12' = 9.3348711$
 $L\cot 30° = 10.2385606$ $\quad L\tan 12°13' = 9.3354823$

8. Two sides of a triangle are as 5 : 9, and the included angle is a right angle. Find the other angles. $\log 2 = .3010300$ $\quad L\tan 19°26' = 9.7446051$
 $\log 3 = .4771213$ $\quad L\tan 19°27' = 9.7448497$

9. Two sides of a triangle are 1.5 and 13.5 respectively, and the included angle is 65°; find the remaining angles. $\log 2 = .3010300$ $\quad L\tan 51°28' = 10.0988763$
 $L\cot 32°30' = 10.1958127$ $\quad L\tan 51°29' = 10.0991355$

10. Two sides of a triangle are 9 and 7, and the included angle is 38°56'32.8"; find the base and the remaining angles. $\log 2 = .3010300$ $\quad L\tan 19°29' = 9.5487471$
 $L\tan 19°28' = 9.5483452$

11. In a triangle ABC, $b=14$, $c=11$, $A=60°$; find the other angles.

$\log 2=.3010300$ $\quad L\tan 11°44'29''=9.3177400$
$\log 3=.4771213$

12. $b=2$ ft. 6 in., $c=2$ ft., $A=22°20'$; find the other angles; and then show that the side a is very approximately a foot.

$\log 2=\ .30103$ $\quad L\tan 29°22'20''=9.75038$
$\log 3=\ .47712$ $\quad L\tan 29°22'30''=9.75043$
$L\cot 11°10'=10.70465$ $\quad L\sin 22°20'\ 0''=9.57977$
$L\sin 49°27'34''=9.88079$

13. The sides of a triangle are 9 and 3, and the difference of the angles opposite to them is 90°. Find the base and all the angles.

$\log 2=.3010300$ $\quad \log 75894=4.8802074$ $\quad L\tan 26°33'=9.6986847$
$\log 3=.4771213$ $\quad \log 75895=4.8802132$ $\quad L\tan 26°34'=9.6990006$

14. $b=8.4$ inches, $c=12$ inches, $B=37°36'$. Find A.

$\log 7=.8450980$ $\quad L\sin 37°36'=9.7854332$
$L\sin 60°39'=9.9403381$ $\quad$ diff. $1'=711$

15. $A=40°$, $a=140.5$, $b=170.6$. Find B and C.

$\log 1405=3.1476763$ $\quad L\sin 40°\ 0'=9.8080675$
$\log 1706=3.2319790$ $\quad L\sin 51°18'=9.8923342$
$L\sin 51°19'=9.8924354$

16. $a=9$, $b=12$, $A=30°$. Find the values of c.

$\log\ 12=1.07918$ $\quad L\sin\ 30°\ 0'\ 0''=9.69897$
$\log\ 9=\ .95424$ $\quad L\sin\ 11°48'39''=9.31168$
$\log 171=2.23301$ $\quad L\sin\ 41°48'39''=9.82391$
$\log 368=2.56635$ $\quad L\sin 108°11'21''=9.97774$

17. $a=145$, $b=178$, $B=41°10'$. Find A.

$\log 178=2.2511513$ $\quad L\sin 41°10'\ 0''=9.8183919$
$\log 145=2.1613680$ $\quad L\sin 32°21'54''=9.7286086$

18. Two sides of a triangle are 9 and 7 inches, and the angle opposite the latter is $51°3'27.15''$. Find the remaining angles and the logarithm of the base.

$\log 2=.3010300$ $\quad L\sin 51°4'=9.8909113$
$\log 3=.4771213$ $\quad L\sin 51°3'=9.8908092$
$\log 7=.8450980$

19. Two sides of a triangle in a survey are found to be 1404 and 960 yards respectively, while an angle opposite to one of them is 32°15'; find the angle the two given sides include.

$\log\ 2=\ .3010300$ $\quad L\operatorname{cosec} 32°15'=10.2727724$
$\log\ 3=\ .4771213$ $\quad L\ \sin\ 21°23'=\ 9.5621316$
$\log 13=1.1139434$ $\quad L\ \sin\ 51°18'=\ 9.8923236$

20. In the triangle ABC, $BC=1652$, $A\hat{B}C=26°30'$, $A\hat{C}B=47°15'$. Find AB and AC.

$\log\ 1652=3.2180100$ $\quad L\sin 73°45'=9.9822938$
$\log\ 7678=3.8852481$ $\quad D=57$ $\quad L\sin 47°15'=9.8658868$
$\log 12636=4.1016096$ $\quad D=344$ $\quad L\sin 26°30'=9.6495274$

21. One of the sides of a right-angled triangle is $\frac{4}{7}$ths of the hypotenuse; find the other angles.

$\log 2=.301030$ $\quad L\sin 14°11'=9.455921$
$\log 7=.845098$ $\quad L\sin 14°12'=9.456031$

22. If $\tan\theta=\dfrac{2\sqrt{ab}\sin\frac{C}{2}}{a-b}$, find θ from the following data:

$a=5$, $b=2$, $C=120°$ $\quad L\tan 61°17'=10.261329$
$\log 3=.477121$ $\quad L\tan 61°18'=10.261629$

23. If $3c = 7b$ and $A = 6°37'24''$, use the formula $\tan\left(\frac{A}{2} + B\right) = \frac{c+b}{c-b} \tan \frac{A}{2}$ to find the other angles. $\log 2 = .3010300$
$L \tan 3°18'42'' = 8.7624069$ $\quad L \tan 8°13'50'' = 9.1603083$ $\quad$ diff. $10'' = 1486$

24. Find the vertical angle of the isosceles triangle, of which the base is 10 feet, and which contains 60 sq. feet. $\log 24 = 1.380211$ $\quad L \tan 22°37' = 9.619720$
diff. $15'' = .000089$

Woolwich (25-41).

25. $a = 40$, $b = 51$, $c = 43$. Find A. $\log 128 = 2.1072100$ $\quad L \tan 24°44'16'' = 9.6634465$
$\log 603 = 2.7803173$

26. $a = 32$, $b = 40$, $c = 66$. Find the angle C.
$\log 207 = 2.3159703$ $\quad L \cot 66°18' = 9.6424342$ $\quad D = 3431$
$\log 1073 = 3.0305997$

27. Find the least angle of the triangle whose sides are 24, 22, 14.
$L \tan 17°33' = 9.500042$ $\quad$ diff. $1' = .000439$

28. Find the greatest angle of the triangle whose sides are 50, 60, 70.
$\log 6 = .7781513$ $\quad L \cos 39°14' = 9.8890644$ $\quad$ diff. $1' = 1032$

29. $a = 13$, $b = 7$, $C = 60°$. Find A and B.
$\log 3 = .4771213$ $\quad L \tan 27°27' = 9.7155508$ $\quad$ diff. $1' = 3087$

30. Two sides of a triangle which are respectively 250 and 200 yards long contain an angle of $54°36'24''$; find the other two angles.
$\log 3 = .4771213$ $\quad L \cot 27°18'\ 0'' = 10.2872338$ $\quad$ diff. $1' = 3100$
$L \tan 12°\ 8'50'' = 9.3329292$

31. One side of a triangle is double of another and the included angle is a right angle. Find the other angles. $\log 2 = .3010300$ $\quad L \cot 26°34' = 10.3009994$
diff. $1' = .0003159$

32. Given $a = 2b$, $C = 120°$; find A, B.
$\log 3 = .4771213$ $\quad L \tan 10°53' = 9.2839070$ $\quad$ diff. $1' = 6808$

33. The included angle is $70°30'$ and the ratio of the containing sides is $5:3$; find the other angles. $\log 2 = .3010300$ $\quad L \cot 35°15'\ 0'' = 10.1507464$
$L \tan 19°28'50'' = 9.5486864$

34. $a = 1.56234$, $b = .43766$, $C = 58°42'6''$. Find A and B.
$\log 56234 = 4\frac{3}{4}$ $\quad \log \cot 29°21' = .250015$
$\log \cot 29°22' = .249715$

35. If $a = 3$, $b = 1$, $C = 53°7'48''$; find c without determining A and B.
$\log 2 = .3010300$ $\quad L \cos 26°33'54'' = 9.9515452$
$\log 25298 = 4.4030862$ $\quad L \tan 26°33'54'' = 9.6989700$
$\log 25299 = 4.4031034$

36. $a = 5$, $b = 4$, $A = 45°$. Find the other angles.
$\log 2 = .3010300$ $\quad L \sin 33°29' = 9.7520507$
$L \sin 33°30' = 9.7530993$

37. $a = 250$, $b = 240$, $A = 72°4'48''$. Find B, C.
$\log 2.5 = .3979400$ $\quad L \sin 72°\ 4' = 9.9783702$
$\log 2.4 = .3802112$ $\quad L \sin 72°\ 5' = 9.9784111$
$L \sin 65°59' = 9.9606739$

38. $AB = 250$ ft., $BC = 200$ ft., and $A = 30°$; find the smaller value of AC.

$\log 2 = .3010300$ $\quad L\sin 38°41' = 9.7958800$
$\log 6.0389 = .7809578$ $\quad L\sin 8°41' = 9.1789001$
$\log 6.0390 = .7809650$

39. If the sides of a triangle be 7.152 in., 8.263 in., 9.375 in.; find its area.

$\log 1.2395 = .0932465$ $\quad \log 3.02 = .4800069$
$\log 5.243 = .7195799$ $\quad \log 2.8477 = .4544942$ $\quad D = 152$
$\log 4.132 = .6161603$

40. The angles A, B, C of a triangle ABC are 40°, 60°, 80° respectively, and CD is drawn from C to the base bisecting the angle ACB; find CD.

$AB = 100$ inches $\quad L\sin 40° = 9.8080675$
$\log 2 = .3010300$ $\quad L\sin 50° = 9.8842540$
$\log 5.73979 = .7588951$ $\quad L\sin 60° = 9.9375306$

41. If b be to c as 11 to 10 and $A = 35°25'$, use the formula $\tan\frac{1}{2}(B - C) = \tan^2\frac{\phi}{2}\cot\frac{A}{2}$ to find B and C.

$\log 1.1 = .041393$ $\quad L\tan 12°18'36'' = 9.338891$
$L\cos 24°37'12'' = 9.958607$ $\quad L\cot 17°42'30'' = 10.495800$
$L\tan 8°28'56.5'' = 9.173582$

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42. $a = 3$, $b = 7$, $c = 8$. Find C. $\quad \log 75 = 1.8750613$ $\quad L\cot 49°6'22'' = 9.9375306$

43. The sides of a triangle are 7, 11, 14; find the smallest angle.

$\log 2 = .3010300$ $\quad L\tan 14°46' = 9.4209275$
$\log 3 = .4771213$ $\quad L\tan 14°45' = 9.4204196$

44. $a = 12$, $b = 17$, $c = 23$. Find A. $\quad \log 364 = 2.5611014$ $\quad L\cos 15°14' = 9.9844660$
$\log 391 = 2.5921768$ $\quad$ diff. $1' = 344$

45. $a = 7$, $b = 10$, $c = 5$. Find A. $\quad \log 2 = .3010300$ $\quad \log 11 = 1.0413927$
$\log 3 = .4771213$ $\quad L\cot 20°16' = 10.4326795$ $\quad D = 3886$

46. The sides of a triangle are 32, 40, 66 ft. respectively; find the greatest angle.

$\log 207 = 2.3159703$ $\quad L\cot 66°18' = 9.6424342$
$\log 1073 = 3.0305997$ $\quad$ diff. $1' = 3433$

47. The sides of a triangle are 25, 26, 27; find the largest angle.

$\log 2 = .3010300$ $\quad L\tan 31°57'\ 0'' = 9.7949455$
$\log 3 = .4771213$ $\quad L\tan 31°56'50'' = 9.7948986$
$\log 7 = .8450980$

48. $b = 5$, $c = 3$, $A = 120°$; find the other angles.

$\log 4.8 = .6812412$ $\quad L\tan 8°12' = 9.1586706$ $\quad$ diff. $60'' = .0008940$

49. Two sides of a triangle are respectively 200 ft. and 115.462 ft. and the included angle is 30°; find the other angles.

$\log 4.2269 = .6260220$
$\log 1.57731 = .1979695$ $\quad L\tan 15° = 9.4280525$

50. $a = 55$, $b = 40$, $C = 120°$; find the other angles.

$\log 3 = .4771213$ $\quad L\cot 84°47'20'' = 8.9600075$
$\log 19 = 1.2787536$ $\quad$ diff. $10'' = 2328$

51. $a = 7$, $b = 5$, $C = 44°24'36''$; find A and B.

$\log 2 = .3010300$ $\quad L\tan 22°12' = 9.6107586$
$\log 3 = .4771213$ $\quad L\tan 22°13' = 9.6111196$

52. $a=17$, $b=13$, $C=40°7'20''$; find A and B.

$\log 2 = .3010300$

$\log 3 = .4771213$ $\quad L\tan 20°3' = 9.5622439$ $\quad D = 3921$

53. $b=25$, $c=7$, $A=73°44'$; find B and C.

$\log 75 = 1.8750613$ $\quad L\tan 36°52' = 9.8750102$

$L\tan 36°53' = 9.8752734$

54. $b=19$, $a=35$, $C=57°7'30''$; find A and B.

$\log 15 = 1.1760913$ $\quad L\cot 28°33' = 10.2643323$ $\quad$ dift. $1' = 3008$

55. $a=14$, $b=11$, $C=13°41'8''$; find A and B.

$\log 120 = 2.0791812$ $\quad L\cot 6°50'40'' = 10.9207117$ $\quad$ diff. $10'' = 1780$

56. In a triangle ABC the angle A is $86°44'$, and the sides containing it are 11 ft. and 21 ft. Find the side opposite to A.

$\log 2 = .3010300$ $\quad L\sin 40°42' = 9.8143131$

$\log 231 = 2.3636120$ $\quad L\cos 40°42' = 9.8797462$

$\log 24255 = 4.3848013$ $\quad L\sin 40°43' = 9.8144600$

$\log 24256 = 4.3848192$ $\quad L\cos 40°43' = 9.8796375$

$L\cos 43°22' = 9.8367447$

57. If $A=30°$, $AB=5$, $BC=3$, find the remaining angles.

$\log 12 = 1.0791812$ $\quad L\sin 56°26' = 9.9207717$ $\quad D=838$

58. Find the length of the side a of the triangle ABC, having given $A=65°30'$, $B=70°40'$, $c=123$.

$\log 123 = 2.0902581$ $\quad L\sin 65°30' = 9.9590229$

$\log 1.6174 = .2088174$ $\quad L\sin 43°50' = 9.8404593$

$\log 1.6175 = .2088443$

59. Use the formulae

$$\left\{\begin{aligned}&\cos\frac{A-B}{2}=\frac{(a+b)\sin\theta}{2\sqrt{ab}}\\&\cos\frac{A+B}{2}=\frac{c\sin\theta}{2\sqrt{ab}},\text{ where }\cos\theta=\frac{a-b}{c}\end{aligned}\right\}$$

to find the angles of the triangle whose sides a, b, c are respectively 10, 8, 4.

$\log 2 = .3010300$ $\quad L\cos 29°22' = 9.9402670$ $\quad D=711$

$\log 15 = 1.1760913$ $\quad L\cos 78°50' = 9.2870480$ $\quad D=6404$

60. If the vertical angle of a triangle be 120°, the length of the line joining the vertex to the middle point of the base $10\sqrt{7}$ feet, and that of the line bisecting the vertical angle 24 feet; find the sides and remaining angles.

$\log 3 = .477121$ $\quad L\sin 23°24' = 9.598952$

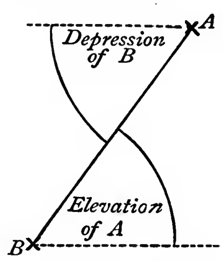
$\log 19 = 1.278754$ $\quad L\sin 23°25' = 9.599244$

CHAPTER XI.

Heights and Distances.

60. Problems in heights and distances are simply practical illustrations of the solution of triangles, and therefore the formulae used in solving them are those set forth in the last chapter.

If we consider the positions of two points A and B, one (say A) at a higher level than the other (B), the angles between the straight line AB and the horizontal lines through A and B in the same vertical plane are called respectively the *angle of depression* of B, and the *angle of elevation* of A.



These two angles are of course equal.

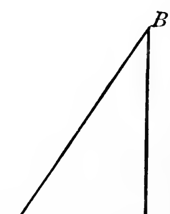
The **angle of elevation** of A (as viewed from B) is the angle through which the arm must be *elevated* from a horizontal position in order to point to A.

The **angle of depression** of B (as viewed from A) is the angle through which the arm must be *depressed* from a horizontal position in order to point to B.

The **angle subtended** at a point by any straight line is the angle contained by the two straight lines drawn from the point to the extremities of the straight line subtending it. Thus ACB is the angle subtended at C by the straight line AB.

Two points are **accessible** to one another when no obstacle prevents the measurement of the direct distance between them.

61. **Problem A. To find the height above the horizontal plane of an object standing upon the plane and accessible at its base.**



Data. Let AB be the object, its base being at A; P the point of observation in the horizontal plane through A.

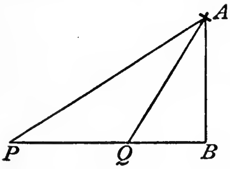
Observations. Measure PA, BPA.

Solution. BA (the required height) $= PA \tan BPA$.

and $$\log BA = \log PA + L \tan BPA - 10,$$

whence BA can be computed.

62. Problem B. To find the height above the horizontal plane of an inaccessible object.



Case (i). *By measurements in the same vertical plane with the object.*

Data. Let A be the object, P and Q two points of observation in the same vertical plane with A and mutually accessible.

Construction of figure. Draw AB perpendicular to PQ produced, taking P and Q on the same side of B.

Observations. Measure PQ, APB, AQB.

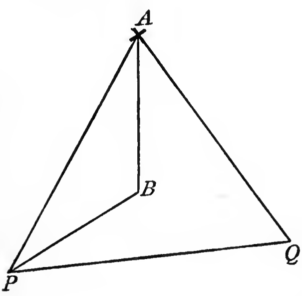
Solution.
$$PQ = PB - QB = AB\cot APB - AB\cot AQB$$
$$= AB(\cot APB - \cot AQB) = AB\frac{\sin(AQB - APB)}{\sin(APB\sin AQB)}.$$

Thus AB (required height) $= \dfrac{PQ\sin APB\sin AQB}{\sin(AQB - APB)}$,

and
$$\log AB = \log PQ + L\sin APB + L\sin AQB - L\sin(AQB - APB) - 10,$$
whence AB can be computed.

(If P and Q had been on opposite sides of B, the only difference would have been a plus instead of a minus sign in the value of AB.)

Case (ii). *By measurements not in the same vertical plane with the object.*



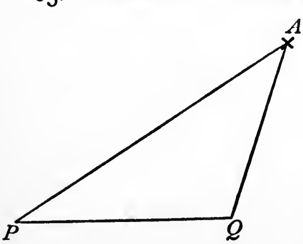
Data. Let A be the object, P and Q two points of observation not in the same vertical plane with A but mutually accessible.

Construction of figure. Draw AB perpendicular to the horizontal plane through P.

Observations. Measure PQ, APB, APQ, AQP.

Solution. $PA = PQ\dfrac{\sin AQP}{\sin PAQ}$,
$$PAQ = 180° - (APQ + AQP),$$
and
$$AB = PA\sin APB$$
therefore
$$AB = \frac{PQ\sin AQP\sin APB}{\sin PAQ},$$
and
$$\log AB = \log PQ + L\sin AQP + L\sin APB - L\sin PAQ - 10,$$
whence AB can be computed.

63. Problem C. To find the distance of an inaccessible object.



Data. Let A be the object, P and Q two points of observation mutually accessible.

Observations. Measure PQ, APQ, AQP.

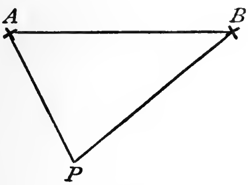
Solution. $PA = PQ\dfrac{\sin PQA}{\sin PAQ}$,
$$PAQ = 180° - (APQ + AQP),$$
$$\therefore \quad \log PA = \log PQ + L\sin PQA - L\sin PAQ,$$
whence the distance PA can be computed.

64. **Problem D. To find the distance between two accessible objects.**

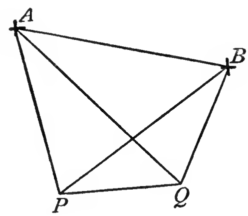
Data. Let A and B be the two objects, both accessible from the point of observation P.

Observations. Measure PA, PB, $A\hat{P}B$.

Solution. The distance AB can be computed as in Art. 55, two sides and the included angle being known in the triangle APB.



Art. 64.



Art. 65.

65. **Problem E. To find the distance between two inaccessible objects.**

Case (i). *By measurements in the same plane with the objects.*

Data. Let A and B be the two objects, P and Q two points of observation mutually accessible and in the same plane with A and B.

Observations. Measure PQ, $A\hat{P}Q$, $B\hat{P}Q$, $A\hat{Q}P$, $B\hat{Q}P$.

Solution. AP and BP can be computed from the triangles APQ and BPQ respectively by Art 57; and then, since $A\hat{P}B = A\hat{P}Q - B\hat{P}Q$, the distance AB is obtained, as in Art. 55, from two sides and the included angle in the triangle APB.

Case (ii). *By measurements not in the same plane with the objects.*

In this case, in addition to the measurements of Case (i), we shall require the angle APB: the solution is then the same as in Case (i).

EXAMPLES. XX.

1. A river PQ is 300 yards broad, and runs at the foot of a vertical cliff QR which subtends at the edge of the opposite bank an angle QPR of $25°10'$; find the height of the cliff.

$\log 3 = .4771213$
$L \tan 64°50' = 10.3280372$ $\log 1.4095 = .1490651$ $D = 308$. *Militia.*

2. A lighthouse appears to a man in a boat 300 yards from its foot to subtend an angle of $6°20'24.7''$. Find in feet the height of the lighthouse.

$\log 3 = .4771213$
$L \tan 6°20' = 9.0452836$ diff. $1' = 11507$ *Sandhurst.*

3. The shadow of a tower is observed to be half the known height of the tower, and some time after to be equal to the full height; how much will the sun have gone down in the interval? $\log 2 = .3010300$
$L \tan 63°26' = 10.3009994$ diff. $1' = 3159$ *Sandhurst.*

4. A person wanting to calculate the height of a cliff, takes its angular altitude $12°30'$, and then measures 950 yards in a direct line towards the base, when he is stopped by a river; he then takes a second altitude and finds it $69°30'$. Find the height of the cliff.

$\log 5 = .6989700$ $\quad L \sin 12°30' = 9.3353368$
$\log 19 = 1.2787536$ $\quad L \cos 33° \; 0' = 9.9235914$
$\log 2296 = 3.3610566$ $\quad L \cos 20°30' = 9.9715876$ *Sandhurst.*

5. From each of two ships, a mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52°25'15''$ and $75°9'30''$ respectively. Find the distance of the beacon from each of the ships.

$\log 1.2197 = .0862530$ $\quad L \sin 75° \; 9'30'' = 9.9852635$
$\log 1.2198 = .0862886$ $\quad L \sin 52°25'15'' = 9.8990055$ *Woolwich.*

6. AB is a horizontal line whose length is 400 yds.; from a point in the line between A and B a balloon ascends vertically, and after a certain time its altitude is taken simultaneously from A and B; at A it is observed to be $64°15'$, at B $48°20'$; find the height of the balloon when the observations are taken.

$\log 2 = .3010300$ $\quad L \sin 64°15' = 9.9545793$
$\log 2.29149 = .4646213$ $\quad L \sin 48°20' = 9.8733352$
$L \sin 67°25' = 9.9653532$ *Woolwich.*

7. A man who is walking on a level plain towards a tower observes at a certain point that the elevation of the top of the tower is $10°$, and, after going 50 yds. nearer to the tower, that the elevation is $15°$. Find the height of the tower in yards to four places of decimals.

$L \sin 15° = 9.4129962$ $\quad \log 25.783 = 1.4113334$
$L \cos 5° = 9.9983442$ $\quad \log 25.784 = 1.4113503$ *Woolwich.*

8. A ship, sailing due north, observes two lighthouses bearing respectively N.E. and N.N.E. After sailing 20 miles the lighthouses are seen to be in a line due east; find the distance in miles between the lighthouses correct to four places of decimals.

$\log 2 = .3010300$ $\quad \log 11.715 = 1.0687423$
$L \tan 22°30' = 9.6172243$ $\quad \log 11.716 = 1.0687794$ *Woolwich.*

9. The elevation of an object on the top of a tower 150 ft. high is found to be $57°38'$ at a point 120 ft. from the base of the tower. Find the height of the object.

$\log 12 = 1.0791812$ $\quad \log 18933 = 4.2772194$ $\quad D = 230$
$L \tan 57°38' = 10.1980454$

10. The centre of the base of a tower which leans to the west is O, and P is an object at the top. From two points A (due east of O) and B (due west of O) P is observed to have the same altitude, viz. $58°26'$. The observer then walks from O due south to a point K through a distance of 150 ft., and there finds that OA and OB subtend respectively at K the angles $32°53'$ and $39°21'$. Find the height of P above the ground, and its distance from the vertical line through O.

$\log 2 = .3010300$ $\quad \log 96977 = 4.9866687$ $\quad D = 44$
$\log 3 = .4771213$ $\quad \log 12299 = 4.0898698$ $\quad D = 353$
$L \tan 32°53' = 9.8105796$ $\quad \log 17901 = 4.2528773$ $\quad D = 243$
$L \tan 39°21' = 9.9137868$ $\quad \log 10998 = 4.0413137$ $\quad D = 395$
$L \tan 58°26' = 10.2115471$

11. In order to calculate the height of a cliff, an observation is taken from a fixed position, and the angular altitude is found to be $12°30'$; a second observation is taken from a point 950 yds. nearer to the cliff, and the angular altitude is found to be $69°30'$; find the height of the cliff, and the distance of the first station from its base.

$\log 2 = .3010300$
$\log 19 = 1.2787536$ $\quad L \sin 12°30' = 9.3353368$
$\log 8586 = 3.93379$ $\quad L \sin 20°30' = 9.5443253$
$\log 2296 = 3.36097$ $\quad L \cos 20°30' = 9.9715876$
$\log 2297 = 3.36116$ $\quad L \sin 57° \; 0' = 9.9235914$

12. Two straight roads intersect at an angle of 30°: from the point of junction two pedestrians A and B start at the same time, A walking along one of the roads at the rate of 5 miles an hour, B walking uniformly along the other road. At the end of 3 hours A and B are 9 miles apart. Show that there are two rates at which B may walk to fulfil the conditions, and determine the slower rate of the two.

$\log 2 = .3010300$ $\quad L\sin 56^\circ 27' = 9.9208555$
$\log 3 = .4771213$ $\quad L\sin 56^\circ 26' = 9.9207717$
$\log 8.0154 = .9039248$ $\quad L\sin 26^\circ 26' 33'' = 9.6486522$

13. A person in a balloon, which ascended vertically from the land at the sea level, finds the angle of depression of a ship at anchor to be 30°; after descending again vertically for 600 ft. he finds the angle of depression to be 15°; find the horizontal distance of the ship from the point of ascent.

$\log 3 = .4771213$
$\log 1.9392 = .2876294$ $\quad L\cot 15^\circ = 10.5719475$

14. In ascending a tower 150 ft. high a person observes from a window the depression of a point in the horizontal plane upon which the tower stands to be $48^\circ 18'$. When he reaches the top of the tower the depression of the same point is observed to be $56^\circ 20'$. Find the height of the window above the ground.

$\log 2 = .3010300$
$\log 3 = .4771213$ $\quad L\tan 33^\circ 40' = 9.8235244$
$\log 11214 = 4.04976$ $\quad L\tan 48^\circ 18' = 10.0501381$

15. After climbing 1600 yards up a mountain side towards the summit in a direction making an angle of $38^\circ 12'$ with the horizontal plane, the summit is seen at an elevation of $66^\circ 38'$. Calculate the height of the mountain, its elevation at the foot being observed to be $53^\circ 20'$. $\quad \log 2 = .3010300$ $\quad \log 26562 = 4.4242608$ $\quad D = 163$

$L\sin 13^\circ 18' = 9.3618217$ $\quad L\sin 28^\circ 26' = 9.6777309$
$L\sin 53^\circ 20' = 9.9042411$

16. The elevation of a tower at each of two points distant 100 yards from one another is $26^\circ 22'$, and at a point midway between them $30^\circ 40'$. Find the height of the tower.

$\log 2 = .3010300$ $\quad \log 45156 = 4.6547155$ $\quad D = 96$
$L\sin 26^\circ 22' = 9.6474945$ $\quad L\sin 4^\circ 18' = 8.8749381$
$L\sin 30^\circ 40' = 9.7076064$ $\quad L\sin 57^\circ 2' = 9.9237554$

17. Wishing to find the breadth of a river and being unable to walk any distance along the bank either way, I notice an object directly opposite to me on the other bank and walk a distance of 400 ft. in a direction making an angle of $28^\circ 17'$ with the bank. The object is then seen in a direction making an angle of $78^\circ 12'$ with the bank. Determine the breadth of the river.

$L\sin 49^\circ 55' = 9.8837232$ $\quad \log 2 = .3010300$
$L\sin 11^\circ 48' = 9.3106849$ $\quad \log 14965 = 4.1750767$ $\quad D = 290$

18. The angle of elevation of a tower is $28^\circ 18'$ at a point A. After walking 270 ft. in a horizontal direction from A and at right angles to the line joining A to the base of the tower the elevation is seen to be $16^\circ 34'$. Find the height of the tower.

$\log 27 = 1.4313638$ $\quad \log 96361 = 4.9839013$
$L\sin 11^\circ 44' = 9.3082590$ $\quad L\sin 28^\circ 18' = 9.6758592$
$L\sin 16^\circ 34' = 9.4550441$ $\quad L\sin 44^\circ 52' = 9.8484720$

19. The car of a balloon, C, is observed at A to have an elevation of $66^\circ 48'$. At a point B, 600 yards from A, the angle CBA is observed to be $53^\circ 27'$. CAB being $82^\circ 14'$, find the height of the balloon.

$\log 600 = 2.7781513$ $\quad L\sin 53^\circ 27' = 9.9048980$
$\log 63414 = 4.8021851$ $\quad L\sin 44^\circ 19' = 9.8442432$
$L\sin 66^\circ 48' = 9.9633795$

20. From two points A and B on the bank of a river I observe two objects C and D at some distance from the bank upon the other side. The distance between A and B is 1000 yds. At A the angles CAB, DAB are observed to be respectively $72°36'$ and $28°10'$; at B the angles CBA, DBA are found to be $43°25'$ and $124°42'$ respectively. Find the distance between C and D.

	$L\cos 22°13' = 9.9664987$	
$\log 7648 = 3.8835479$	$L\sin 27°\ 8' = 9.6590246$	
$\log 18027 = 4.2559235$	$L\sin 43°25' = 9.8371456$	
$\log 25675 = 4.4095105$	$L\sin 55°18' = 9.9149479$	
$\log 13659 = 4.1354189$	$L\sin 63°59' = 9.9535985$	
$\log 2 = .3010300$	$L\sin 57°51' = 9.9277079$	$D = 794$
	$L\cos 57°51' = 9.7260240$	$D = 2011$

21. Walking in a horizontal direction from a point A at which the elevation of an object is observed to be less than $30°$, I find on reaching B that the elevation is just doubled, and that at C it is trebled. A, B, C being in the same vertical plane with the object observed, AB 156 yards, and BC 109 yards, calculate the vertical height of the object.

$\log 78 = 1.8920946$	
$\log 109 = 2.0374265$	$\log 265 = 2.4232459$
$\log 171 = 2.2329961$	$\log 15233 = 4.1827854$

22. Standing directly in front of the centre one of three pillars of a building which are in the same vertical plane, and known to be 36 ft. apart, I observe the elevations of the pillars to be $38°26'$ and $44°14'$. What is my distance from the nearest pillar?

$\log 36 = 1.5563025$	$\log 50645 = 4.704537$
$L\sin 5°48' = 9.0045634$	$L\sin 38°26' = 9.7935135$
$L\sin 82°40' = 9.9964330$	$L\cos 44°14' = 9.8552192$

23. A tower standing on a horizontal plane leans over towards the south. At equal distances due north and south of it, the elevations of its summit are $30°$ and $32°$ respectively. Calculate the inclination of the tower to the vertical.

$L\sin 2° = 8.5428192$	$L\tan 3°46' = 8.8184608$
$L\sin 32° = 9.7242097$	diff. $1' = 19230$

24. Three objects A, B, C are visible from a station D in the same plane, at which the sides of the triangle ABC subtend equal angles. Find AD; given $AB = 12$ chains, $AC =$ 6 chains, $CAB = 46°\ 34'$.

$\log 2 = .30103$	$L\cot 53°17' = 9.87264$
$\log 3 = .47712$	$L\tan 13°57'30'' = 9.39552$
$\log 536 = 2.72916$	$L\sin 50°40'30'' = 9.88849$

CHAPTER XII.

Application to Mensuration.

66. Most of the formulae used in solving questions on the mensuration of plane and solid figures consist entirely of products and quotients, and are thus adapted to logarithmic computation. The following is a list of the more important formulae used in this branch of mathematics.

I. Triangles.

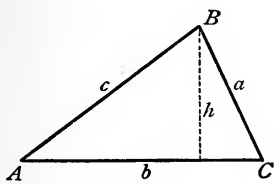
(1) *Any triangle.* Area = (i) $\frac{1}{2}bh$ (i.e. $\frac{1}{2}$ base × height),

(ii) $\frac{1}{2}bc \sin A$,

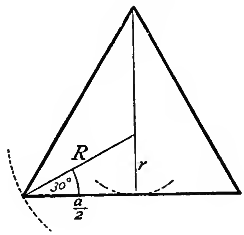
(iii) $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

(2) *Right-angled triangle.* ($C = 90°$.)

Area $= \frac{1}{2}ab$ (i.e. $\frac{1}{2}$ product of the sides containing the right angle).



(1)



(3)

(3) *Equilateral triangle.* (Side, a.) Height $= \frac{a\sqrt{3}}{2}$, area $= \frac{a^2\sqrt{3}}{4}$,

r (the radius of the inscribed circle) $= \frac{a}{2} \tan 30° = \frac{a}{2\sqrt{3}}$,

R (the radius of the circumscribed circle) $= \frac{a}{2} \sec 30° = \frac{a}{\sqrt{3}}$,

$R = 2r$.

(4) *Isosceles triangle.* (Each of equal sides $= a$.)

(i) Vertical angle θ, area $= \frac{1}{2}a^2 \sin \theta$,

(ii) Vertical angle 30° or 150°, area $= \frac{a^2}{4}$,

(iii) Vertical angle 60° or 120°, area $= \frac{a^2\sqrt{3}}{4}$.

II. Quadrilaterals.

(1) *Square.* (Side, a). Area $= a^2$, diagonal $= a\sqrt{2}$.

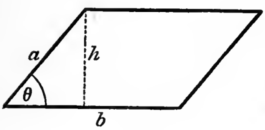
(2) *Rectangle.* (Sides, a, b.) Area $= ab$,

diagonal $= a \sec \theta$, where $\tan \theta = \frac{b}{a}$. *Vide* Art. 50.

(3) *Parallelogram.* Area $=$ (i) bh (i.e. base $\times$ height)

(ii) $ab \sin \theta$,

diagonals $= \sqrt{a^2 + b^2 \pm 2ab \cos \theta}$. *Vide* Art. 50.



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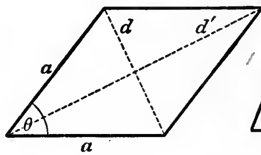
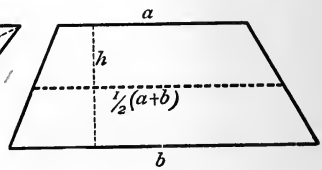


(4)



(5)

(4) *Rhombus.* Area $= \frac{1}{2}dd'$ (i.e. $\frac{1}{2}$ product of diagonals),
or as for parallelogram, putting $b = a$;

$$\text{diagonals} = 2a \sin \frac{\theta}{2}, \quad 2a \cos \frac{\theta}{2}.$$

[In the rhombus the diagonals *bisect* one another *at right angles.*]

(5) *Trapezium or Trapezoid.* Area $= \frac{1}{2}(a+b)h$ (i.e. mean length $\times$ height).

(6) The area of *any quadrilateral whose diagonals intersect at right angles* equals half the product of the diagonals.

III. Regular Polygons.

(1) *Hexagon.* (Side, a.) Area $= 6\left(\frac{a^2\sqrt{3}}{4}\right)$ (i.e. 6 equilateral triangles).

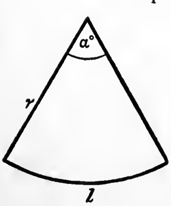
(2) *Polygon of* n *sides.* (Side, a.) Area $= \frac{na^2}{4} \cot \frac{180°}{n}$.

IV. Circles.

Circumference of circle (radius, r) $= 2\pi r$,

area of circle $= \pi r^2$.

Area of plane circular ring (radii, R, r) $= \pi(R+r)(R-r)$.



Arc of circular sector $= \frac{a°}{360}(2\pi r)$,

area of circular sector $=$ (i) $\frac{a°}{360}(\pi r^2)$,

(ii) $\frac{1}{2}lr$,

(iii) $\frac{1}{2}r^2\theta$ ($\theta =$ circular measure of $a°$),

(iv) $\frac{90 l^2}{\pi a}$.

V. Polygons and Circles.

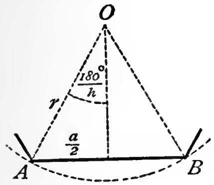
(1) *Regular polygon inscribed in circle.*

$$\text{Area of polygon} = n \text{ times } OAB = \frac{na^2}{4}\cot\frac{180^\circ}{n} \text{ or } \frac{nr^2}{2}\sin\frac{360^\circ}{n},$$

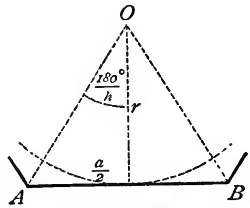
$$\text{perimeter of polygon} = n \text{ times } AB \;= na \text{ or } 2nr\sin\frac{180^\circ}{n},$$

$$a = 2r\sin\frac{180^\circ}{n}.$$

[n = number of sides of polygon,
a = side of polygon,
r = radius of circle.



(1)



(2)

(2) *Circle inscribed in regular polygon.*

$$\text{Area of polygon} = n \text{ times } OAB = \frac{na^2}{4}\cot\frac{180^\circ}{n} \text{ or } nr^2\tan\frac{180^\circ}{n},$$

$$\text{perimeter of polygon} = n \text{ times } AB \;= na \text{ or } 2nr\tan\frac{180^\circ}{n},$$

$$a = 2r\tan\frac{180^\circ}{n}.$$

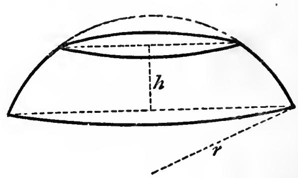
VI. Rectangular Parallelopipeds.

Volume of rectangular parallelopiped (edges, a, b, c) $= abc$,

volume of cube (edge, a) $= a^3$,

diagonal of cube $= a\sqrt{3}$.

VII. Spheres.

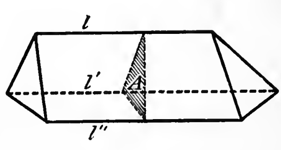


Surface of sphere (radius, r) $= 4\pi r^2$,

volume of sphere $= \frac{4}{3}\pi r^3$.

Curved Surface of spherical zone $= 2\pi rh$.

VIII. Prisms.



Volume of prism $= Bh$ (i.e. base × height).

Volume of prismatic frustum or wedge $\Big\} = \frac{1}{3}(l + l' + l'') \, . \, A$

(i.e. mean length × area of right section).

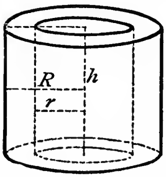
IX. Cylinders.

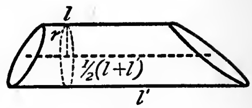
Volume of cylinder $= \pi r^2 h$,

curved surface of cylinder $= 2\pi r h$,

total surface of cylinder $= 2\pi r(r+h)$.

Volume of cylindrical shell $= \pi h(R+r)(R-r)$.





Volume of cylindrical frustum $= \frac{1}{2}(l+l')\pi r^2$ (i.e. mean length × area of right section),

Curved surface of cylindrical frustum $= (l+l')\pi r$.

X. Pyramids.

Volume of pyramid $= \frac{1}{3}Bh$ (i.e. $\frac{1}{3}$ base × height).

Volume of tetrahedron (edge, a) $= \dfrac{a^3}{6\sqrt{2}}$,

surface of tetrahedron $= a^2\sqrt{3}$.

Volume of octahedron (edge, a) $= \dfrac{a^3\sqrt{2}}{3}$,

surface of octahedron $= 2a^2\sqrt{3}$.

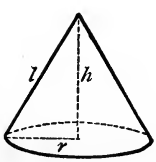
Volume of pyramidal frustum $= \dfrac{Ah}{3} \cdot \dfrac{1-\left(\frac{a}{A}\right)^{\frac{3}{2}}}{1-\left(\frac{a}{A}\right)^{\frac{1}{2}}}$, where a, A are the areas of the top and bottom of the frustum, and h is the height of the frustum.

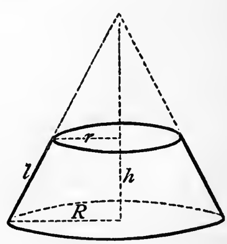
XI. Cones.

Volume of cone $= \frac{1}{3}\pi r^2 \cdot h$,

curved surface of cone $= \pi r l$,

total surface of cone $= \pi r(r+l)$.





Volume of conical frustum $= \dfrac{\pi R^2 h}{3} \cdot \dfrac{1-\left(\frac{r}{R}\right)^3}{1-\frac{r}{R}}$.

Curved surface of conical frustum $= \pi l(R+r)$

$\left.\begin{array}{r}\text{100 links}\\ \text{22 yards}\end{array}\right\} = 1$ chain.

$\left.\begin{array}{r}\text{100,000 sq. links}\\ \text{4,840 sq. yards}\\ \text{10 sq. chains}\end{array}\right\} = 1$ acre.

640 acres = 1 sq. mile.

$\log \pi = .4971499$.

EXAMPLES. XXI.

[*Tables to be used.*]

A. Mensuration of Plane Figures.

1. Two sides of a triangular field containing an obtuse angle are 127 yds. and 232 yds. respectively. Find to the nearest yard the length of the third side that the field may contain exactly an acre.

2. In a quadrangular field $ABCD$, $AB = 38.54$ chains, $BC = 24.16$ chains, $CD = 52$ chains, $DA = 35.08$ chains, and the angle ACB is a right angle. Find its area in acres.

3. An equilateral triangle is inscribed in a square with one of its angular points coinciding with an angular point of the square. Find the ratio of the area of the triangle to the area of the square to three places of decimals. *Staff College.*

4. Find to three places of decimals the side of the equilateral triangle whose area equals that of the scalene triangle whose sides are 105, 116, and 143.

5. What is the height in inches of the isosceles triangle whose area is a square foot and vertical angle the unit of circular measure?

6. On opposite sides of a base 120 yards long, two isosceles triangles are described whose vertical angles are respectively $38°15'$ and $83°42'$. Find the total area.

7. Find to the nearest sq. foot the area of a square whose side is 317.2857 feet.

8. What are the lengths of the diagonals of the rhombus whose acute angles are $64°28'$, and whose area is 27 sq. inches.

9. A rhombus whose acute angles are $38°30'$, and whose side is 12 inches long, has inscribed in it an isosceles triangle whose vertex coincides with one of these acute angles and whose base bisects the opposite sides. Find the area of this triangle.

10. Four equal rods, each 6 inches long, are hinged together so as to form a square. The rods are now turned about the hinges till opposite corners are 10 inches apart. Find the angles and area of the figure formed by the rods in this position.

11. The two parallel sides of a trapezium are 117 yds. 2 ft. and 172 yds. 1 ft. respectively, and the other sides are both 34 yds. long; find the area to the nearest square foot.

12. Find the area of the trapezium whose parallel sides are respectively 112 ft. and 154 ft., and whose other sides make angles of $52°12'$ and $37°48'$ with the greater of the given sides.

13. The two parallel sides of a trapezium are 89 feet apart, and the other sides make angles of $52°12'$ and $37°36'$ with the greater of the two parallel sides, whose length is 254 ft. Find the area to the nearest square foot.

14. What would be the perimeter and area of a regular figure of 100 sides inscribed in a circle of 100 yards radius? By how much does the area differ from that of the circle?

15. Find to five places of decimals the ratio of the areas of the regular hexagon and octagon inscribed in any circle.

16. What is the number of sides in the regular polygon, the ratio of above inscribed and circumscribed circles is most nearly equal to $\frac{99}{100}$?

17. Find the area and perimeter of the regular dodecagon inscribed in a circle of 6 ft. radius.

18. Find the area of the regular quindecagon inscribed in a circle of radius 5 ft. What ratio does it bear to that of the circumscribing quindecagon?

19. What would be the difference between the areas enclosed by 500 yards of rope when held taut by 100 and 120 posts respectively, placed at equal distances along the circumference of a circle?

20. Find in yards the radius of the circle whose area is half an acre.

21. Compute to the nearest square inch the area of a circle in which a chord 4 ft. in length subtends at the centre the angle $18°36'$. *Staff College.*

22. What is the length of the chord, in a circle of 10 ft. radius, which subtends an angle of $112°15'$ at the centre?

23. Express to the nearest second the angle which is subtended at the centre of a circle of 3 square inches area by an arc of 1 inch.

24. Compute to the nearest yard the length of that part of a circular railway curve which subtends an angle of $25°\ 36'$ to a radius of a mile.

25. Find the distance in miles between two places on the equator which differ in longitude by $6°18'$, assuming the earth's equatorial diameter to be 7925.6 miles.

26. Find to the nearest square foot the area of the complete circle, whose sector of angle $5°$ contains an arc of 10 yards.

27. A circle is inscribed in a right-angled isosceles triangle. Find the ratio of the areas of the circle and triangle. *Staff College.*

28. In a circle of 10 ft. diameter a straight line 4 ft. long is placed. Compute to the nearest inch the lengths of the segments into which the circumference of the circle is thus divided. *Staff College.*

29. In what latitude will a correction of one second in time have to be reckoned for every furlong travelled east or west, taking the earth to be a sphere whose radius is 3957 miles?

30. Find the side of the equilateral triangle that can be inscribed in a circle whose area is 14 square inches.

31. Compute to the nearest square inch the area of the smaller segment into which a circle of 100 feet radius is divided by a chord of 37.25 feet.

32. Find to the nearest inch the length of the arc subtending an angle of $35°$ at the centre of a circle whose area is 1000 square yards.

33. After walking 200 yards round a circular pond, I notice that the point from which I started and an object in the centre of the pond lie in directions inclined at $32°15'$ to each other. Compute the diameter and area of the pond.

34. Calculate the radius and area of the circle inscribed in the triangle whose sides are 131.16 ft., 373.75 ft., and 407.23 ft. respectively.

35. A circular plot of grass is surrounded by a walk 40 links wide, whose inner circumference is 2408 links; find the number of acres contained in the walk.

36. Find to the nearest square inch the area of the equilateral triangle inscribed in the circle whose radius is 13.26 feet.

37. Find the radius of the circle whose area is equal to what is left after cutting a sector of angle $44°26'$ from a circle of 31.68 feet radius.

38. What is the area contained between the arc of a circular sector and the tangents at its extremities, the arc being $18\frac{1}{2}$ inches long and the perimeter of the sector 35 inches?

39. Find the length in inches of the circumference of the circle whose area is the one-millionth of an acre.

40. What is the area of the segment of a circle of $8\frac{1}{8}$ inches radius which subtends an angle of $18°24'$ at the centre?

41. A railway curve is an arc of a circle of $\frac{1}{2}$ mile radius. What is the shortest distance between two stations whose distance apart along the line is 1000 yards?

42. Taking the latitude of St. Paul's to be $51°30'$, what is its velocity in feet per sec. due to the earth's rotation? (Diameter of earth = 7925.6 miles.)

43. Find in square inches the area of the segment of a circle, the arc being the tenth part of the whole circumference and the radius being 6 feet.

44. Compute in links the radius of the circle whose area is an acre.

45. Find the area of the segment which contains an angle of $38°12'$ on a base 8 feet long.

46. If, in a circle of 4 ft. radius, an arc of 10 ft. subtends a chord of 7.592 feet, find the value of π to three places of decimals.

47. Two chords are drawn in a circle of 12 inches radius, cutting one another at right angles and subtending angles of $156°$ and $125°$ at the centre respectively; find the area of the quadrilateral formed by joining their extremities.

48. Calculate the area and perimeter of the circle inscribed in a square the side of which is 359.5678 feet.

49. Determine the diameter of the earth in geographical miles [60 to a degree of latitude], each degree subtending $1°$ at the centre of the earth.
$\pi = 3.14159265...$ *Woolwich.*

50. It is proposed to add to a square lawn, measuring 58 feet on a side, two circular ends, the centre of each circle being the point of intersection of the diagonals of the square. How much turf will be required for the purpose? *Woolwich.*

51. What are the areas and perimeters of the two segments into which a circle of 13 ft. radius is cut by a chord of 20 ft.?

52. An isosceles triangle whose vertical angle is $48°12'$ is inscribed in a circle of 18 ft. radius; find the area between the triangle and circumference of the circle.

53. The arc of a semicircle is divided into two parts so that the chord of one is 5 times that of the other; find the ratio of these parts.

54. A triangle whose sides are 17, 23, and 30 inches respectively has a circle inscribed in it, and in this circle a similar triangle is inscribed. Find the angles and area of this latter triangle.

55. Find the expense of paving a circular court 80 feet in diameter at 3s. 4d. a square foot, leaving in the centre a space for a fountain in the shape of a hexagon, each side of which is a yard.

56. The chord of an arc of a circle is $18\frac{1}{2}$ inches, and the height of the arc is $6\frac{1}{4}$ inches; find the length of the arc.

57. The perimeters of a circle, a square, and an equilateral triangle are each of them 1 foot. Find the area of each of these figures to the nearest hundredth of a square inch.

58. The side of an equilateral triangle is 200 ft. Find the radius of the circle circumscribing the triangle, and the area of the triangle to the nearest square inch.

59. The length of the arc of a sector is 13 feet 7 inches, and the angle of the sector is $56°10'$; find the area of the sector to the nearest square inch.

60. Two circles, whose diameters are 18.34 feet and 26.12 feet respectively, cut one another at an angle of $40°$; find the length of the common chord.

B. Mensuration of Solids.

61. The three conterminous edges of a rectangular parallelopiped are 3, 2.52, and 1.523 feet in length. Find the number of cubic inches of volume. Find also the cubical space inside a box of the same external dimensions, constructed of material $\frac{1}{10}$th of a foot thick.

62. Compute the edge and diagonal of the cube whose volume is 100 cubic yards.

63. Find the length of the edge of a cubical block of stone containing 146 cub. yds. 716 cub. ins., and the number of sq. inches of surface.

64. What is the length of the side, to the nearest tenth of an inch, of a cubical cistern holding, when full, 2000 cub. ft. of water?

65. The corners of a cube whose weight is found to be 7.38 lbs. are ground down evenly and equally till the weight is reduced to 6.64 lbs. Find the surface of the solid so formed, if 1000 cub. ins. of the material weigh 12.5 lbs.

66. Find the radius of the sphere whose volume is 750 cub. ft.

67. Compute the radius of the sphere whose volume equals that of a cube of 12 inches radius.

68. How many square miles of the earth's surface lie in the tropics, i.e. between $22\frac{1}{2}°$ north and south of the equator, taking the diameter of the earth to be 7926 miles?

69. Find the edge of the cubical block of lead which, when melted down, will make a million shot .125 inches in diameter.

70. Find the amount of material required to make a spherical balloon containing 10,000 cub. ft. of gas.

71. Find the radius of the sphere (1) whose volume = 1 cub. ft., (2) whose surface = 1 sq. ft.

72. How many cub. ft. of gas will be contained in a spherical balloon formed out of 180 sq. yds. of silk?

73. What would be the diameter of a spherical balloon made out of 112 yds. of canvas, $4\frac{1}{2}$ ft. wide?

74. A right triangular prism, whose edges are all equal, and a sphere are of equal volume. Compare their external surfaces.

75. How many cub. yds. of earth have been removed in boring a tunnel 1 mile 170 yards long, whose section is a semi-circle of 14 ft. radius?

76. A right prism on a triangular base, each of whose sides is 21 inches, is such that a sphere, described within it, touches its five faces. Find the volume of the sphere, and of the space between it and the surface of the prism.

77. Find the volume of a right triangular prism, the edges of whose base are 38.7, 49.2, and 40.3 ft. respectively, and whose height is 20 ft.

78. The vertical ends of a horizontal trough are parallel equilateral triangles, with 12 inches in each side, a side of each triangle being horizontal. If the distance between the ends be 6 ft., find (1) the number of cubic feet of water the trough will contain, (2) the number of gallons it will contain, it being given that a gallon of water weighs 10 lbs. and a cubic foot of water 62.5 lbs.

79. Determine the diameter of a cylindrical gas holder to contain 10 million cubic feet of gas, supposing the height to be made equal to the diameter; and determine in tons the weight of iron plate, weighing $2\frac{1}{2}$ lbs. per sq. ft., required in the construction of the gas holder, supposing it open at the bottom, and closed by a flat top. *Woolwich.*

80. A hollow pontoon has a cylindrical body 20 ft. long, and hemispherical ends, and is made of metal $\frac{1}{8}$th of an inch thick. The outside diameter is 3 ft. 4 in. Find its weight, having given that a cubic inch of the metal weighs 4.5 oz. *Woolwich.*

81. A right cylinder open at the top, with a diameter of 24 inches, weighs 167.5 lbs. When filled with water it weighs 2131 lbs. Find the height of the cylinder, it being given that a cubic foot of water weighs 62.5 lbs.

82. What is the weight of a cylinder formed of sheet iron $\frac{1}{2}$ inch thick, with an outer circumference of 10 ft. $7\frac{2}{7}$ ins. and a length of 3 ft. 6 ins.? 240 cub. ins. of iron weigh 1000 oz.

83. A well 5 feet in diameter and 30 feet deep is to have a lining of bricks, fitting close together without mortar, 9 inches thick. Required approximately in lbs. the weight of the bricks, supposing a brick $9 \times 4\frac{1}{2} \times 3$ ins. to weigh 5 lbs.

84. A cylindrical pipe 14 feet long contains 396 cubic feet. Find its diameter, and the cost of gilding its surface at $9\frac{3}{4}$d. per sq. ft.

85. A right circular cylinder is cut by two planes inclined to one another at an angle of $32° \ 18'$, so that the areas of the two ends are each of them equal to 12 sq. feet, and the distance between their centres is 7 ft.: find the volume intercepted by the planes.

86. In a rectangular building with a wedge-shaped roof, whose ridge is parallel with the length of the building, there are cylindrical columns in a plane, at equal distances from one another and from the side walls of the building, and reaching from the ground to the roof. There are 6 of these columns, $12\frac{1}{2}\frac{5}{6}$ ft. in circumference: the height of the building is 80 ft. and of the walls 58 ft., while the width of the building is 122 ft. Find the total volume, and exposed surface, of the six columns.

87. Determine the number of cubic yards in a bank of earth on a horizontal rectangular base 60 ft. long and 20 ft. broad, the four sides of the bank sloping up to a ridge at an angle of 40° to the horizon. *Woolwich.*

88. How many cubic feet of earth must be dug out to form a trench 120 yards long, whose right section is a trapezium 7 feet deep, the inclination of the sides to the vertical being $12°20'$ and the breadth of the trench at the top 18 feet?

89. The Great Pyramid of Egypt was 481 feet high when complete, and its base was a square whose side was 764 feet long: find the volume to the nearest number of cubic yards.

90. Compute the solid content of the pyramid whose height is 6.99, each side of the triangular base being 4.18.

91. A pyramid on a square base has all its edges equal. If the exterior surface be 117.38 sq. inches, find its volume.

92. Find the total surface and volume of a pyramid on a hexagonal base, each side of which is 10 inches, the perpendicular height of the pyramid being 18 inches. Also find to the nearest second the angle of inclination to the base of the triangular faces.

93. A pyramid on a triangular base, whose sides are 12.7, 8.5, and 15.8 inches respectively, is cut by a plane parallel to the base and at a distance of 6 inches from it. If the height of the pyramid was 14 inches, find the volumes of the two portions into which the pyramid is divided.

94. A right pyramid, upon a square base whose side equals 1 foot, has its triangular faces inclined at $78°16'$ to the base. Find the inclination of the edges to the base, and the surface and volume of the pyramid.

95. A pyramid upon a regular hexagonal base, and with its triangular surfaces inclined at angles of $52°30'$ with the base, stands upon an area of 15 sq. feet. What is its volume?

96. A conical hole is bored in a sphere, whose vertex is at the centre of the sphere and whose edge is circular. If the angle made by a straight line drawn from any point on the edge to the vertex with the plane of the circular edge be $64°48'$, and the circumference of the sphere be 8 ft., find the volume removed to the nearest cubic inch.

97. What are the diameter and surface of the sphere of equal volume with the pyramid whose vertex is in one of the faces of a cube, and whose base is the opposite face: each edge of the cube being 13.7 ins.?

98. Compute the volume of the largest tetrahedron that can be formed out of a wooden sphere by planing down its surface, the circumference of the sphere being 217.64 inches.

99. Find the edge of the tetrahedron (1) whose volume, (2) whose surface equals that of a cube whose edge is 21.178 inches.

100. Compare the edges of the tetrahedron and octahedron that they may contain (1) equal volumes, (2) equal surfaces.

101. Find the edges and surfaces of the tetrahedron and octahedron that could be obtained by melting down a leaden spherical ball whose weight is 28.16 lbs., supposing a cub. in. to weigh 6.6 oz.

102. What is the volume of the octahedron whose surface is 100 sq. inches?

103. Find the volume of the cone whose vertical angle is $78°25'$, and diameter of base 8 inches.

104. The inclination of the slant height of a cone to its base is $14°25'$, and its height is 4 inches. What is the area of its curved surface?

105. What is the vertical angle of a right cone that its curved surface may be double that of the cylinder of the same base and height?

106. If the vertical angle of a cone be $43°27'$, and the diameter of its base $8\frac{1}{2}$ inches, find its volume and total external surface.

107. Compute to the nearest second the vertical angle of the cone in which the area of the curved surface is 3 times that of the plane surface.

108. What is the total surface of the right-angled cone whose volume is 394 cubic inches?

109. Find how many sq. yds. of canvas will be required to make a conical tent standing on an area of 100 sq. yds., and having its semi-vertical angle $38°30'$.

110. Find the volume and the inclination to the vertical of the slant height of the conical tent that can be made out of 100 sq. yds. of canvas standing upon 50 sq. yds. of area.

111. The vertical angle of a right cone is 124°36′, and its height is $17\frac{1}{2}$ inches. Find its curved surface and volume.

112. The curved surface of a cone is 24 sq. ft., and its base is 18 sq. ft. Find the volume of the cone to the nearest cubic inch.

113. Find the height of the cone whose volume shall be 1000 cubic inches, if it stand upon a circular base whose radius is 10 inches.

114. Find the radius of the hemispherical bowl, which contains as much as a conical vessel whose vertical angle is 42°48′ and diameter of rim 8 inches.

115. The greatest cone that can be inserted in any given sphere has its vertical angle 60°. Find the volume of the greatest cone for the sphere whose surface is 2148 sq. inches.

116. Find the volume of the largest cone that can be cut out of a sphere of 12 inches radius, the vertical angle of the cone being 72°18′.

117. Find the volume and surface of the solid generated by the revolution of an equilateral triangle about one of its sides, each side being 7.9 inches.

118. An isosceles triangle whose vertical angle is 156°40′, and whose equal sides are 15 inches long, revolves about its base. What are the volume and surface of the solid generated?

119. A regular hexagon, whose side is a foot, revolves about the straight line joining two opposite angular points. Find the volume of the solid generated in cubic inches.

120. A solid is made up of a right circular cylinder surmounted by a cone, on an equal base and of the same altitude. If the area of the common base be 10 square feet, and the vertical angle of the cone 68°30′, find the volume of the solid to the nearest cubic inch.

121. If S be the surface of a regular tetrahedron and l be the length of an edge, prove the formula
$$\log S = 2 \log l + .23856.$$
Militia.

122. If V is the volume of a sphere and A the area of its surface, prove that
$$3 \log A = 2 \log 6 + \log \pi + 2 \log V.$$
Calculate the value of A, if $V = 796.325$ cub. in. *Militia.*

MISCELLANEOUS EXAMPLES.

[*Tables to be used.*]

A.

Woolwich (1-5).

1. Find the value of (i) 52.4574×3.78472, (ii) $\dfrac{87.327 \times 784.55 \times .020868}{.61659 \times 58.844}$, (iii) $(5.7432)^{1.246}$.

2. Find (i) a 4th proportional to 1.3046, .01042, and 2.375,
(ii) a mean proportional between 33.549 and 44.642.

3. How many terms of the series .04, .08, .16, .32, ..., will amount to 41943?

4. What is the amount of £1000 in 100 years at 5 per cent. per annum compound interest?

5. If the number of persons born in any year equals $\frac{1}{45}$th of the whole population at the beginning of the year, and the number who die equals $\frac{1}{60}$th of it, find in how many years the population will be doubled.

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6. Find to three places of decimals the mean proportional between .0374 and 32310.

7. Find the cube root of .043758.

8. Compute to 5 places of decimals the value of $\sqrt{x^3+3x}$, where $x = .84729$.

9. Employ logarithms to divide 39.8765 by $\sqrt{.0000843}$, and to compute $\dfrac{1}{a^2}$ when $a = .03857$, the result in each case being given to the first place of decimals.

10. Calculate to the nearest penny the amount of £126. 8s. 6d. placed at 6 per cent. per annum, compound interest, for 20 years, convertible half-yearly.

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11. Find the common logarithms of the following numbers:

(i) 217.6328,	(ii) 16500.876,	(iii) 3.459125,	(iv) .000784032,
(v) .5161205,	(vi) 8761.3577,	(vii) 24.60908,	(viii) 59769.44,
(ix) 8400.827,	(x) 113.1113,	(xi) .3510689,	(xii) 2.852038,
(xiii) .002195976,	(xiv) 18030.15,	(xv) 2.5768643,	(xvi) 410428.4,
(xvii) 1779.023,	(xviii) .11737017,	(xix) 620.3151,	(xx) .000007813248.

12. Find the numbers whose common logarithms are

(i) 3.2147067, (ii) $\bar{1}.8501042$, (iii) $.9143314$, (iv) 4.2580703,
(v) 2.7116210, (vi) $\bar{2}.8517532$, (vii) $\bar{3}.2400276$, (viii) 2.0003145,
(ix) $\bar{1}.1071238$, (x) 4.4236500, (xi) 1.3021811, (xii) $.5117097$,
(xiii) 6.2361566, (xiv) $\bar{3}.9987280$, (xv) 2.1685205, (xvi) $\bar{2}.0073841$,
(xvii) -2.4625383, (xviii) -4.1047934, (xix) $-.5682002$, (xx) -1.8394216.

13. Compute the following roots to 6 significant figures:

(i) the 7th and 15th roots of $.1$, (ii) the 5th root of 1000,
(iii) the cube root of $.0000083825$, (iv) the 7th root of $(.0018423)^{10}$.
(v) the 5th root of $\sqrt[3]{2}$, (vi) the 11th root of $\dfrac{15(.318)^{\frac{1}{7}}}{16}$.

14. Find approximately the following proportionals:

the mean proportional to (i) 35.76 and $.004235$,
(ii) $.003$ and 3000000;
the 3rd proportional to (iii) 31.13 and $.02437$,
(iv) $.082$ and 7.4131;
the 4th proportional to (v) $.0081724$, 3.17245, and $.0001$,
(vi) $.0076842$, 32000, and $.5$.

15. Find the values of

(i) $\log_{10}(207.8967)^{20}$, (ii) $\log_{10}\left(\dfrac{30798 \times 56200}{71 \times .000007}\right)$,

(iii) $\log_{10}\left(\dfrac{93.2}{1117.4}\right)^{\frac{3}{4}}$, (iv) $\log_{10}\dfrac{(3.7184)^{\frac{1}{3}}}{\sqrt{.0076}}$,

(v) $\log_{10}\left(\dfrac{.001325 \times .37856}{2.81409}\right)^2$, (vi) $\log_{10}\sqrt[7]{(.1111111)^{13}}$,

(vii) $\log_{10}\left\{\dfrac{(11.3)^{50} \times (12.3)^{40}}{(13.3)^{90}}\right\}$, (viii) $\log_{10}\left\{\dfrac{(20.15)^{\frac{1}{2}} \times (.003)^{\frac{1}{3}}}{(200.3426)^5}\right\}$,

(ix) $\log_{10}\dfrac{\sqrt[3]{3.7\sqrt{.0037}}}{37^{11}}$, (x) $\log_{10}\left\{\dfrac{\sqrt{54.3117}}{\sqrt[3]{710.2584}}\right\}^{\frac{1}{2}}$,

(xi) $\log_{11.5}.00719$. (xii) $\log_{1000}\left\{\dfrac{.0175 \times 1.3125}{\sqrt[3]{32.25}}\right\}$.

16. Compute the values of

(i) $(20.009)^5$, (ii) $(151.102)^3$, (iii) $(7.3001)^{-4}$,
(iv) $(11.9116)^{-3}$, (v) $(.0186)^3$, (vi) $(.1124)^6$,
(vii) $(.2692)^{-4}$, (viii) $(.0717)^{-2}$, (ix) $(-11.04)^3$,
(x) $(-5.91)^4$, (xi) $(-2.089)^{-3}$, (xii) $(-20.21)^{-4}$,
(xiii) $(6327)^{\frac{221}{119}}$, (xiv) $(4.898)^{\frac{17}{16}}$, (xv) $\dfrac{(3.18)^{2.51}}{(17.1)^{3.6}}$,
(xvi) $(1.418)^{\frac{4}{.91}}$, (xvii) $(.00821)^{\frac{255}{312}}$, (xviii) $(.072)^{\frac{.0031}{39}}$,
(xix) $(31.17)^{-\frac{2.41}{1.39}}$, (xx) $(.00202)^{-\frac{5.4}{4.3}}$, (xxi) $\sqrt[3]{\dfrac{7}{9}}$,
(xxii) $\sqrt[15]{\dfrac{317}{290000}}$, (xxiii) $\sqrt[6]{9\sqrt{3\sqrt{2}}}$, (xxiv) $\sqrt[5]{\dfrac{8}{(3174)^{16}}}$,

(xxv) $\sqrt[3]{\dfrac{9.864 \times .01234}{.005678 \times .0000876529}}$, (xxvi) $\sqrt[3]{\left\{\dfrac{.00078 \times 3256800}{.06851178}\right\}}$,

(xxvii) $\dfrac{\sqrt[3]{24} \times \sqrt{2.4}}{\sqrt[5]{.000024}}$, (xxviii) $\dfrac{30.2846\sqrt[3]{.0007}}{.0000842065}$, (xxix) $\sqrt[3]{\dfrac{3.82179}{14.36}}$,

(xxx) $\dfrac{32.889\sqrt[3]{.03}}{.000246397}$, (xxxi) $\dfrac{.7698\sqrt[4]{.893}}{.052\sqrt[3]{.028}}$, (xxxii) $\sqrt[3]{\dfrac{3}{4}\left(\dfrac{78.9}{\sqrt{3.17}}\right)}$,

(xxxiii) $\dfrac{49 \times (21)^8 \times \sqrt[11]{750}}{(252)^5}$, (xxxiv) $\dfrac{.003768(2.00\dot{7})^{\frac{2}{5}}}{\sqrt[3]{\frac{2}{3}} \times .11\dot{3}7\dot{6}}$, (xxxv) $\dfrac{\frac{2}{3} \text{ of } \sqrt[3]{.0003172}}{30.00765\sqrt{96.74}}$,

(xxxvi) $\sqrt[5]{\left\{\dfrac{6300 \times .00117 \times 42.9}{\frac{1}{2}(2197)^{\frac{1}{3}}}\right\}}$, (xxxvii) $\dfrac{32047}{\sqrt[3]{3.789}} \div \sqrt{.026\dot{7}1\dot{5}}$,

(xxxviii) $\left\{\dfrac{\sqrt[3]{.00078165} \times \sqrt[5]{23.4}}{\sqrt{13119.\dot{7}}}\right\}^{\frac{3}{2}}$, (xxxix) $\dfrac{\frac{1}{5}\sqrt{\frac{3}{7}\sqrt[3]{.068}}}{\frac{2}{7}(.017)^{\frac{1}{7}}3\sqrt[3]{3}}$,

(xl) $\{27384 + \sqrt[3]{1762.843}\}^{-\frac{1}{5}}$.

17. Find the value of

(i) $x^2 - x - 56$, when $x = 310.427$; (ii) $2x^2 - 5x + 2$, when $x = 10.075$;
(iii) $3x^2 + 14x - 5$, when $x = 72.82\dot{3}$; (iv) $x^4 - 13x^2 + 36$, when $x = 7.39134$;
(v) $x^3 - x + 2x^2 - 2$, when $x = 21.513$; (vi) $(x+1)^2(x-1)$, when $x = 1.00008$;
(vii) $(x-2)^3(x+3)^2$, when $x = 5.3212$; (viii) $\dfrac{(x+3)^3(x-4)^5}{(x+1)^2}$, when $x = .418574$;
(ix) $(2x^2 - 5x - 12)^3$, when $x = 7.2538$; (x) $x^3(x^2 - 3x + 2)$, when $x = 23.1525$;
(xi) $x^5 - 10x^6$, when $x = 1.7744$; (xii) $4(x^4 + 1) - 17x^2$, when $x = 2.00765$;
(xiii) $\sqrt[3]{x^4 - 5x^2 + 4}$, when $x = .5768268$; (xiv) $\sqrt[3]{x^2(x+3)^5}$, when $x = 35.4848$;
(xv) $\sqrt{3189.718x^8}$, when $x = 4.10072$; (xvi) $x^4(1 + \sqrt[3]{.08216x^4})$, $x = .3625$;
(xvii) $\sqrt{.76x^4 + 63.309x^5}$, when $x = .021846$;
(xviii) $\sqrt[3]{\{22.87 - \sqrt{815.0328x^3}\}}$, when $x = 11.6038$;
(xix) $\dfrac{\sqrt{(14.753)^5x^3 - (13.142)^3x^5}}{(2068.974)^{\frac{3}{2}}}$, when $x = 2.17484$;
(xx) $3x^6 - 7x^5 + 4x^4 - 5x^3 - 2x^2 - 6x - 12$, when $x = 4.107634$.

18. Solve the following equations:

(i) $2.03^x = 10.2$, (ii) $181.2^x = .02$, (iii) $317.68^x = 74100$,
(iv) $.171^x = .051$, (v) $.001^x = 221$, (vi) $3^x \cdot 2^{x+1} = \sqrt[7]{2}$,
(vii) $2^{2x} \cdot 7^{5x} = 1882384$, (viii) $(\sqrt[3]{.000712})^{2x-1} = 13.0156$,
(ix) $\sqrt[x]{.0000286788} = .123456$, (x) $(.00761)^{\frac{1}{x}} = .1$,
(xi) $(31.8)^{3x-1} = \dfrac{37}{915}(.0076)^{2x}$, (xii) $(1.5)^{4x} + 7 = (1.5)^{-4x}$,

(xiii) $\left.\begin{aligned} xyz &= 317.24 \\ x^2y^3z^4 &= 8276.5 \\ x^3y^4z^2 &= 12347 \end{aligned}\right\}$ (xiv) $\left.\begin{aligned} x + xy + y &= .246879 \\ y + yz + z &= .453284 \\ x + xz + z &= .867091 \end{aligned}\right\}$ (xv) $\left.\begin{aligned} (z+x)(x+y) &= (2.4)^{15} \\ (x+y)(y+z) &= (3.6)^{14} \\ (y+z)(z+x) &= (4.2)^{16} \end{aligned}\right\}$

19. Find the number which, multiplied by 604327, will give 2465816904306.

20. Extract the cube root of 949862087000.

21. How many digits are there in 2^{100} and 3^{64}?

22. Find the number of digits in the integral portion of $(4506.23)^{50}$, and the position of the first significant figure in the decimal value of $\left(\frac{2}{3125}\right)^{25}$.

23. What power of 2 is equal to 131072?

24. Compute the mean proportional between the side and diagonal of a square whose area is an acre.

25. Find $\log_e 4\frac{1}{2}$ when $e = 2.71828$.

26. n things can be distributed among x persons in x^n ways. If there be half a dozen boys, how many things must be distributed that they may be given in at least a million different ways? What is the actual number of ways in this case?

27. Find the Amount at Compound Interest of £100 for

	Years.		Per cent. per ann.				Years.		Per cent. per ann.	
(i)	20	at	$4\frac{1}{2}$	(convertible annually),		(ii)	15	at	5	(convertible annually),
(iii)	50	,,	$3\frac{1}{2}$	,, ,,		(iv)	36	,,	6	,, ,,
(v)	27	,,	4	,, ,,		(vi)	18	,,	3	,, ,,
(vii)	70	,,	5	,, ,,		(viii)	17	,,	8	,, ,,
(ix)	100	,,	$4\frac{1}{2}$	,, ,,		(x)	10	,,	4	,, ,,
(xi)	23	,,	10	,, ,,		(xii)	13	,,	$3\frac{1}{2}$	,, ,,
(xiii)	28	,,	3	,, ,,		(xiv)	39	,,	5	,, ,,
(xv)	69	,,	7	,, ,,		(xvi)	81	,,	$4\frac{1}{2}$	,, ,,
(xvii)	56	,,	$4\frac{1}{2}$	(convertible half-yearly),		(xviii)	$47\frac{1}{2}$	,,	6	(convertible half-yearly),
(xix)	42	,,	5	,, ,,		(xx)	30	,,	4	,, ,,

28. What sum will amount to £1000 at Compound Interest in

	Years.		Per cent. per ann.				Years.		Per cent. per ann.	
(i)	17	at	4	(convertible annually),		(ii)	12	at	$3\frac{1}{2}$	(convertible annually),
(iii)	50	,,	5	,, ,,		(iv)	10	,,	$4\frac{1}{2}$	,, ,,
(v)	100	,,	5	,, ,,		(vi)	6	,,	3	,, ,,
(vii)	20	,,	6	,, ,,		(viii)	15	,,	4	,, ,,
(ix)	8	,,	$4\frac{1}{2}$	,, ,,		(x)	21	,,	4	,, ,,
(xi)	87	,,	7	,, ,,		(xii)	72	,,	3	,, ,,
(xiii)	35	,,	$3\frac{1}{2}$	,, ,,		(xiv)	26	,,	9	,, ,,
(xv)	61	,,	5	,, ,,		(xvi)	13	,,	8	,, ,,
(xvii)	10	,,	$4\frac{1}{2}$	(convertible half-yearly),		(xviii)	12	,,	6	(convertible half-yearly),
(xix)	$7\frac{1}{2}$	,,	5	,, ,,		(xx)	21	,,	4	,, ,,

29. At what rate per cent. per ann. will the following sums amount to £1000, viz.,

			Years.						Years.	
(i)	£530	in	7	(convertible annually),		(ii)	£100	in	10	(convertible annually),
(iii)	£425	,,	21	,, ,,		(iv)	£715	,,	14	,, ,,
(v)	£200	,,	32	,, ,,		(vi)	£350	,,	13	,, ,,
(vii)	£632	,,	$5\frac{1}{2}$	(convertible half-yearly),		(viii)	£418	,,	12	(convertible half-yearly),
(ix)	£820	,,	$8\frac{1}{2}$	,, ,,		(x)	£500	,,	$14\frac{1}{2}$	,, ,,

30. In what time will the following sums amount to £1000, viz.,

	Per cent. per ann.			Per cent. per ann.	
(i) £325 at	$5\frac{1}{2}$	(convertible annually),	(ii) £450 at	4	(convertible annually),
(iii) £512 ,,	8	,, ,,	(iv) £100 ,,	$4\frac{1}{2}$	,, ,,
(v) £748 ,,	$3\frac{3}{4}$	,, ,,	(vi) £270 ,,	10	,, ,,
(vii) £815 ,,	$3\frac{1}{2}$	(convertible half-yearly),	(viii) £630 ,,	$2\frac{1}{2}$	(convertible half-yearly),
(ix) £200 ,,	7	,, ,,	(x) £500 ,,	10	,, ,,

31. Find the time in which a sum of money will double itself at $2\frac{1}{2}$, 4, 5, $8\frac{1}{2}$, and 10 per cent. per ann., compound interest, respectively.

32. What sum of money will amount at compound interest to £1000 in 6 years, and £1250 in 8 years, and what rate of interest will be reckoned?

33. At what rate per cent., compound interest, will a sum of money quadruple itself once in a century?

34. Find the accumulated values of forborne annuities of £100 in the following cases, payable annually:

(i)	for 12	years at	$4\frac{1}{2}$	per cent.	per ann.,	(ii)	for 21	years at	5	per cent.	per ann.,
(iii)	,, 50	,,	4	,,	,,	(iv)	,, 100	,,	5	,,	,,
(v)	,, 17	,,	$4\frac{1}{2}$	,,	,,	(vi)	,, 35	,,	$3\frac{1}{2}$	,,	,,
(vii)	,, 18	,,	6	,,	,,	(viii)	,, 49	,,	3	,,	,,
(ix)	,, 73	,,	4	,,	,,	(x)	,, 29	,,	3	,,	,,
(xi)	,, 27	,,	8	,,	,,	(xii)	,, 54	,,	$3\frac{1}{2}$	,,	,,

35. Find the present value of an annuity of £100

(i)	for 10	years at	3	per cent.	per ann.,	(ii)	for 17	years at	$3\frac{1}{2}$	per cent.	per ann.,
(iii)	,, 35	,,	4	,,	,,	(iv)	,, 100	,,	5	,,	,,
(v)	,, 72	,,	$4\frac{1}{2}$	,,	,,	(vi)	,, 26	,,	$3\frac{1}{2}$	,,	,,
(vii)	,, 89	,,	3	,,	,,	(viii)	,, 44	,,	$4\frac{1}{2}$	,,	,,
(ix)	,, 51	,,	4	,,	,,	(x)	,, 60	,,	5	,,	,,
(xi)	,, 85	,,	3	,,	,,	(xii)	,, 96	,,	$3\frac{1}{2}$	,,	,,

36. Find the annuity purchaseable with £1000

(i)	for 100	years at	$3\frac{1}{2}$	per cent.	per ann.,	(ii)	for 21	years at	5	per cent.	per ann.,
(iii)	,, 15	,,	$4\frac{1}{2}$	,,	,,	(iv)	,, 25	,,	4	,,	,,
(v)	,, 80	,,	3	,,	,,	(vi)	,, 37	,,	$4\frac{1}{2}$	,,	,,
(vii)	,, 64	,,	4	,,	,,	(viii)	,, 99	,,	$3\frac{1}{2}$	,,	,,
(ix)	,, 50	,,	3	,,	,,	(x)	,, 18	,,	5	,,	,,
(xi)	,, 81	,,	5	,,	,,	(xii)	,, 76	,,	$3\frac{1}{2}$	,,	,,

37. What is the difference in value between a freehold and a 99 years' lease of a property worth £100 per annum, taking the interest of money at 5 per cent. per annum?

38. If a debt of £1000 is to be paid off in 10 years by equal annual instalments, 5 per cent. being charged each year on the outstanding debt, find the amount of each instalment to the nearest penny.

39. An annuity of £100 has remained unpaid for the last 21 years. What perpetuity is equivalent to the accumulated value, allowing 5 per cent. interest in each case?

40. For how many years has a certain annuity been unpaid if the accumulations at 5 per cent. be 21.58 times the value of the annuity.

B.

1. Find (α) the Tabular Logarithmic Sines of

(i) $114°36'54.6''$,	(ii) $35°18'28.3''$,	(iii) $27°49'32.0''$,
(iv) $18°42'24.0''$,	(v) $163°30'49.7''$,	(vi) $82°57'14.2''$;

(β) the Tabular Logarithmic Cosines of

(i) $51°19'20.8''$,	(ii) $47°38'36.4''$,	(iii) $272°27'48.5''$,
(iv) $34°43'27.0''$,	(v) $356°16'56.7''$,	(vi) $17°29'10.2''$;

(γ) the Tabular Logarithmic Tangents of

(i) $33°26'24.0''$,	(ii) $216°35'52.6''$,	(iii) $17°47'38.7''$,
(iv) $234°20'49.1''$,	(v) $78°18'30.0''$,	(vi) $87°\ 0'43.3''$;

(δ) the Tabular Logarithmic Cotangents of

(i) $53°10'40.6''$,	(ii) $10°58'25.5''$,	(iii) $47°23'46.0''$,
(iv) $254°33'51.3''$,	(v) $36°17'24.0''$,	(vi) $220°44'39.8''$.

2. Find the angles (α) whose Tabular Logarithmic Sines are

(i) 9.6872304,	(ii) 8.8645120,	(iii) 9.7928147,
(iv) 9.8847125,	(v) 9.9381029,	(vi) 9.8545278;

(β) whose Tabular Logarithmic Cosines are

(i) 9.9692136,	(ii) 9.3152164,	(iii) 9.8933790,
(iv) 9.5242812,	(v) 9.7098000,	(vi) 9.9405135;

(γ) whose Tabular Logarithmic Tangents are

(i) 9.4361278,	(ii) 10.2271613,	(iii) 10.1151415,
(iv) 9.9972367,	(v) 10.0178034,	(vi) 8.8794162;

(δ) whose Tabular Logarithmic Cotangents are

(i) 10.5863078,	(ii) 9.8119826,	(iii) 10.2207100,
(iv) 9.5823515,	(v) 8.9798217,	(vi) 10.9408238.

3. Find the values of

(i) $\dfrac{\sin^3 A}{.0342}$, when $A = 16°18'40''$;

(ii) $3.18 \sin^2\dfrac{A}{2}$, ,, $A = 37°15'22''$;

(iii) $.0054329\dfrac{1+\tan^2 A}{\cos A}$, ,, $A = 127°15'$;

(iv) $\sqrt{3.826 + .3942\cos^2 A}$, ,, $A = 51°16'$;

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(v) $\sqrt{12.118}\tan^2\dfrac{A}{3}$, ,, $A = 50°$;

(vi) $\sqrt{.036}\tan^4\dfrac{A}{2}$, ,, $A = 32°12'24''$;

(vii) $.00284\sqrt[3]{\cos^2\dfrac{A}{4}}$, ,, $\sin 2A = \frac{1}{3}$;

(viii) $\sqrt{.9156 + .48971\tan^3\dfrac{A}{3}}$, $A = 152°21'20''$;

(ix) $\cos^4 A - \sin^4 A$, ,, $A = 32°25'20''$;

(x) $\tan^2 A - \tan^2 B$, ,, $A = 127°15'$ and $B = 45°20'$;

(xi) $30.62\sqrt{\dfrac{1+\cos A}{1-\cos A}}$, ,, $A = 78°12'$;

(xii) $.09156\dfrac{1-\sin^2 A}{1-\tan^2 A}$, when $A = 227°$;

(xiii) $\dfrac{\tan^2 A}{\sqrt{3}(1-\cot^2 A)}$, ,, $A = 33°25'$;

(xiv) $\dfrac{\sin A \,.\, \cos 10A}{\cos 7A - \cos 3A}$, ,, $A = 65°24'$;

(xv) $\dfrac{.007}{\sqrt{.00006}}\left(\dfrac{1-\tan A}{1+\tan A}\right)^3$, ,, $A = 250°$;

(xvi) $\sin 3A \sin^3 A + \cos 3A \cos^3 A$, when $A = 133°17'$.

4. Find the value of $\sin A + \sin B + \sin C$, when A, B, C are the angles of the triangle described in Euc. IV, 10.

5. If $L \tan A = 10.5240134$, find $L \sin A$ and $L \cos A$.

6. Calculate, by introducing subsidiary angles, the value of $\sqrt{a^2+b^2}$ when

(i) $a = 131.573$, $b = 34.21917$;
(ii) $a = 16.0408$, $b = 18.1535$;
(iii) $a = .717242$, $b = 2.49801$.

7. Solve the equations (i) $\sqrt{\sin^3\dfrac{A}{2}} = .26814$, (ii) $\sqrt[5]{\tan\theta} = 1\frac{1}{4}$,

(iii) $\tan\theta = \sqrt{7\cot^3\theta}$, (iv) $\left.\begin{array}{l}\tan\theta = \tan^3\dfrac{\phi}{2}\\ 3\cos^2\phi = 1\end{array}\right\}$,

(v) $2\sin^2\theta + 3\cos^2\theta = 2\frac{7}{8}$, (vi) $\sqrt{3}\sin\theta + \cos\theta = \frac{5}{4}$,

(vii) $12\sin x + 5\cos x = 13$, (viii) $\left.\begin{array}{l}\tan(2A+B) = 3\\ \cos(3A-2B) = \frac{1}{3}\end{array}\right\}$.

8. Calculate to the nearest second the smallest positive angle

(i) whose tangent equals 3, (ii) whose cosine equals $-\frac{1}{3}$.

9. Given $\cos 3A = \frac{11}{13}$, find $\tan 5A$.

10. Find to the nearest second the angles of the isosceles triangle whose equal sides are double the base.

11. What acute angle has its sine to its cosecant in the ratio of 12 to 17?

12. If $\tan 2A = \dfrac{2n\sqrt{1-n^2}}{1-2n^2}$, find the value of A when $n^3 = \frac{1}{2}$.

13. Solve the equations $\left.\begin{array}{l}\dfrac{x+y}{1-xy} = \dfrac{43}{17}\\[2ex] \dfrac{x-y}{1+xy} = \dfrac{37}{29}\end{array}\right\}$ by means of the tables of logarithmic ratios.

14. A computer, in referring to the tables, reads the logarithmic tangent by mistake for the cotangent and uses a value too great by .5316768. What is the angle?

15. Find all the positive and negative values of θ less than 180° which satisfy the equation

$$4\sin^3\theta - \sin 3\theta = \frac{1}{\sqrt{2}}.$$

16. If $y = \tan^{-1}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, find the value of y to the nearest second when $x = .5127$.

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17. The sides of a triangle being 87, 93, and 100 ft. in length, compute to the nearest inch the length ot the perpendicular drawn to the longest side from the opposite angle.

18. On the same base, 20 yds. in length, and on opposite sides of it, are an equilateral triangle and an isosceles triangle with the vertical angle 30°. Compute to the nearest foot the length of the straight line joining the vertices of these triangles.

19. The radii of two intersecting circles being 1 and 2 feet, and their centres being 2 feet apart, find to the nearest inch the length of the straight line joining the points of intersection.

20. Each side of a parallelogram is 8 feet long, and its area is 46 sq. feet. Compute to the nearest minute the angles of the parallelogram.

21. Compute to the nearest second the angles of the two triangles which have two sides 17 and 12 feet long, and an angle $43°12'12''$ opposite to the shorter of these sides.

22. Compute the remaining angles of a triangle wherein one angle is $105°44'49''$, the side opposite to it 427 feet, and a side adjacent 250 feet.

23. P and Q are two points. An observer at A, where AP is perpendicular to PQ, measures the angle $PAQ=39°$. He moves 100 yds. parallel to PQ to B and measures the angle $ABP=53°$. Compute the distance between P and Q to the nearest foot.

24. The summit of a wall 20 ft. high has, to an observer in the horizontal plane through its base, the angular elevation $18°36'$. What is the distance of the point of observation from the tower?
If the observer is liable to an error of $30'$ of excess or defect in the measured elevation, within what limits can he be sure that his computed distance is correct?

25. The angular elevation of an object above the horizon is taken at different points in a straight horizontal road. Its greatest elevation is $29°17'$ and its elevation at a point in the road 200 yds. away from the former point of observation is $18°52'$. Find the height of the object above the horizontal plane to the nearest foot.

26. The sides of a triangle being 580 and 483 feet long, and the angle opposite to the latter being $48°17'23''$, find to the nearest second the two values of the angle opposite to the former side.

27. Compute to the nearest square foot the area of a triangle wherein sides 134 and 137 feet long include an angle $118°17'$.

28. The lengths of the sides of a triangle being 34 ft., 46 ft., and 65 ft., compute to the nearest second the largest angle of the triangle.

29. A and B are points in the same horizontal line 1000 yds. apart, P a visible point. At A the angle is observed $PAB=27°15'$, and at B the angle is observed $PBA=24°39'$. Find to the nearest foot the perpendicular distance of P from the line AB.

30. In a regular pentagon, whose sides are each 10 ft. long, compute to the nearest inch the length of a straight line drawn from an angular point to one of the more distant angular points.

31. A diagonal of a rectangle is 100 ft. long, and the angle which it makes with one of the sides of the rectangle is $34°18'22''$. Find to the nearest sq. ft. the area of the rectangle.

32. Two angles of a triangle being $22°18'17''$ and $47°16'18''$, and the shortest side being 222 ft. long, what is the length of the longest side?

33. Sides of a triangle, 46 and 112 ft. long, include the angle $143°29'$. Compute to the nearest second the smallest angle of the triangle.

34. ABC being a triangle wherein the angle A is a right angle, a straight line AD is drawn bisecting the right angle and meeting the opposite side BC in D. Find the length of AD to the nearest foot when AB is 34 ft. and AC is 56 ft. in length.

35. The shortest side of a right-angled triangle is 284 ft., and its smallest angle is $18°37'29''$. Find to the nearest foot the length of the hypotenuse.

36. In a plane triangle sides 320 and 562 feet in length include an angle $128°4'$. Find the other angles, each to the nearest minute.

37. Compute to the nearest foot the radius of a circle inscribed in a triangle whose sides are 32, 56, and 80 feet in length.

38. A and B are two points in a horizontal plane. At A the elevation of a point C above the plane is $19°17'$ and at B it is $16°5'$, A and B being in the same vertical plane with C and on the same side of C. The height of C above the horizontal plane is 100 feet. Find to the nearest foot the distance AB.

39. AB is a vertical object, 50 ft. high, standing on ground of uniform slope. Measure BC, 200 ft., from the foot of the object up the slope, and let the elevation of A above the horizontal plane be observed at C to be $12°15'$. Find the inclination of the ground to the horizon to the nearest minute.

40. Compute to the nearest second the acute angle A when $\tan A = 3 \sin 38°$.

41. A side of a right-angled triangle being 214 yds. long, and the angle opposite to it $34°1'21''$, find the length of the other side of the triangle to the nearest foot.

42. Compute to the nearest yard the length of the base of an isosceles triangle wherein the equal sides are each 190 yards in length, and each angle at the base is $31°15'$.

43. In the triangle ABC the side AC is 341 yards, BC is 237 yards, and the angle CAB is $18°17'15''$. Find the length of the side AB to the nearest foot.

44. The lengths of the sides of a triangle being 37, 45, and 52 chains, find its area in acres, roods and perches to the nearest perch.

Woolwich (45-59).

45. Prove that, to turn circular measure into seconds, we must multiply by 206265; and, to turn seconds into circular measure, we must multiply by .000004848, approximately. [$\pi = 3.14159265\ldots$]

46. The value of the divisions on the outer rim of a graduated circle is $5'$, and the distance between two successive divisions is .1 of an inch; find the radius of the circle.
A church spire whose height is known to be 45 feet subtends an angle of $9'$ at the eye; find its distance approximately.

47. $a = 3795$ yds., $B = 73°15'15''$, and $C = 42°18'30''$, find the other sides of the triangle.

48. $b = 130$, $c = 63$, and $A = 42°15'30''$, find the other angles and the third side of the triangle.

49. Given, in feet, $a = 10$, $b = 24$, $c = 26$, determine the angles and the area of the triangle in square feet.

50. Given $a = 5$ inches, $b = 7$ inches, $A = 31°15'$, find the area of the larger triangle with these data.

51. The base of a triangle being 7 feet, and the base angles $129°23'$ and $38°36'$, find the length of the shortest side.

52. Two sides of a triangle are 2.7402 ft. and .7401 ft. respectively, and contain an angle $59°27'5''$. Find the base and altitude of the triangle.

53. Given the difference between the angles at the base of a triangle $17°48'$ and the sides subtending these angles 105.25 ft. and 76.75 ft.; find the angle included by the given sides.

54. In a circle which has a radius of 10 feet two chords AB, CD are drawn at right angles to each other, and intersecting in O. AO and CO are three and four feet respectively; find the sides and angles of the quadrilateral $ACBD$ formed by joining the extremities of the chords.

55. From a boat the angles of elevation of the highest and lowest points of a flagstaff, 30 ft. high, on the edge of a cliff are observed to be $46°12'$ and $44°13'$; determine the height of the cliff and its distance.

56. The angular altitude of a lighthouse seen from a point on the shore is $12°31'46''$, and from a point 500 ft. nearer to it is $26°33'55''$. Required its height above the shore.

57. An observer in a balloon, when it is one mile high, observes the angle of depression of a conspicuous object on the horizontal ground to be $35°20'$, then after ascending vertically and uniformly for 20 mins. he observes the angle of depression of the same object to be $55°40'$; find the rate of ascent of the balloon in miles per hour.

58. A tower which stands on a horizontal plane is 200 ft. high, and there is a small loophole in the tower at a certain height above the ground; an observer is at a horizontal distance from the tower of 300 ft., but stands on a mound so that his eye is 12 ft. above the ground on which the tower stands, and in that position the angles subtended at his eye by the portions of the tower above and below the loophole are equal; find the height of the loophole from the ground.

59. An observer finds that from the doorstep of his house the angular elevation of the top of a church spire is 5α, and that from the roof above the doorstep it is 4α. The height of the roof above the doorstep being h, prove that the height of the top of the spire above the doorstep is equal to $h \operatorname{cosec} \alpha \,.\, \cos 4\alpha \,.\, \sin 5\alpha$, and that the horizontal distance of the top of the spire from the house is equal to $h \operatorname{cosec} \alpha \,.\, \cos 4\alpha \,.\, \cos 5\alpha$.
If h is 39 ft. and if α is equal to $7°17'39''$, calculate the height and the distance.

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60. Solve completely the following right-angled triangles, C being the right angle:
(i) $a = 127.38$, $b = 250$; (ii) $a = 10.7$, $c = 27.63$;
(iii) $b = 8.116$, $A = 34°18'24''$; (iv) $a = 1000$, $A = 72°35'$;
(v) $c = 33.57$, $B = 28°12'40''$.

61. Given in a plane triangle $A = 74°14'30''$, $B = 51°42'20''$, $c = 786.02$, calculate the side a.

62. Given in a triangle $a = 472.6$, $b = 309.4$, $C = 65°14'$, find the area and the radius of the inscribed circle.

63. In a triangle ABC, $AC = 166.5$ ft., $BC = 162.5$ ft., the angle $A = 52°19'$. Solve the triangle.

64. If $\tan\theta = \dfrac{2\sqrt{ab}\sin\dfrac{C}{2}}{a-b}$, find θ; given $a = 7$, $b = 3$, $C = 115°35'$.

65. Given that in any triangle $\sin A + \sin B + \sin C = \dfrac{4Ss}{abc}$, calculate the sum of the sines of the angles of the triangle whose sides are 31.7, 23.5, and 19.4 ft. respectively.

66. It is known that in any triangle $\left.\begin{aligned}\frac{c}{a+b}&=\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}\\ \frac{c}{a-b}&=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\end{aligned}\right\}$ Use these formulae to solve completely the triangle that can be inscribed in a circle of 10 ins. radius on a base 12 ins. long so that (i) the perimeter may be 30 ins.,
(ii) the difference of the two sides may be 4 ins.

67. Find the least angle of the triangle whose three sides are 200, 250, and 300 ft. respectively.

68. The base of an isosceles triangle is 100 ft. and the vertical angle is 125°; solve the triangle.

69. The sides of a triangle are 525 ft., 650 ft., and 777 ft. respectively. Determine its three angles.

70. Given two sides of a triangle, 102 ft. and 70 ft. long respectively, which contain an angle 99°22′, calculate the length of the base.

71. A tower 150 ft. high throws a shadow 35 ft. long on a horizontal plane. Find the sun's altitude to the nearest minute.

72. If the sides of a triangle are 51, 52, 53, find the area and the sine of the smallest angle.

73. Find to the nearest square foot the area of the rectilineal field $ABCD$, whose side AB is measured and found to be 250 yds., the following angles being observed: $DAB=90°$, $CAB=40°20'$, $DBA=35°15'$, and $CBA=72°$.

74. If the longer diagonal of a parallelogram be 72 feet, one of its sides 58 feet, and the angle between the other side and the longer diagonal 42°12′, find its greatest area.

75. The sides of a triangle are 15, 20, and 25 inches respectively. Find its area and smallest angle.

76. ABC is a triangle having the perpendiculars from A and B on opposite sides equal to 18 and 20 inches respectively. If $AB=30$ inches, find the other sides and angles of the triangle.

77. The diagonals of a parallelogram are 186 and 78 yards, and include an angle of 34°10′25″. Find the area and perimeter of the parallelogram.

78. Given that one side of a triangle is double a second and that the two include an angle of 127°24′, find the other angles.

79. If the sides of a triangle be 7.152, 8.263, and 9.375 ft. respectively; find in inches the area of the triangle and the radius of its inscribed circle.

80. One of the sides of a rectangle is 1500 ft. long and subtends an angle of 35°17′48″ at either of the opposite angles; find the length of the diagonal and the area of the rectangle.

81. A triangular plot has one side 106 ft. long, the adjacent angles being 105°16′ and 37°24′. Find the other sides.

82. Two sides of a triangle are 9 and 7 and the included angle is 38°56′32″; find the base and remaining angles.

83. The vertical angle of a triangle is 120° and the difference of the sides is $\frac{4}{9}$ths of the base; find the other angles.

84. The sides of a quadrilateral are 135, 180, 150, and 125 yds., and the angle contained by the first two is a right angle. Determine the area of the figure to the nearest sq. ft.

85. The diagonals of a rhombus are 120 ft. and 195 ft. respectively. Find its angles.

86. If the altitude of an isosceles triangle be 3 times the base, find its angles.

87. One side of a triangular lawn is 172 feet long, its inclinations to the other sides being $70°30'$ and $78°18'$. Determine the other sides and area.

88. Two parallel chords of a circle whose radius is 50 yds., lying on the same side of the centre, subtend respectively 72° and 125° at the centre. Find to the nearest inch the distance between them.

89. Find the area to the nearest square foot of the largest triangle which has two sides equal to 175 and 160 feet respectively, and the angle opposite to the latter equal to 60°.

90. Find the angles of the rhombus equal in area to one-third of a square described on an equal base.

91. Find the other angles of the triangle, two of whose sides, containing an angle of $123°12'24''$, are in the ratio of 13 to 17.

92. A triangular field has its sides 50, 60, and 70 yds. long respectively; find its area to the nearest square foot.

93. Find the greatest angle of the triangle in which the perpendiculars drawn to the sides from opposite angular points are 3, 4, and 5 feet respectively.

94. The sides of a triangle are 31, 24, and 11 feet long respectively; find the greatest angle and smallest altitude.

95. Find the angles of the right-angled triangle in which the straight line bisecting the right angle passes through a point of trisection of the hypotenuse.

96. The sides of a triangle are 112, 86, and 72 feet in length respectively. Find the greatest altitude of the triangle.

97. Two straight lines 200 and 300 ft. long include an angle of $50°26'$; find the length of the straight line joining their extremities.

98. At a point in the side of a rectangular field, 20 yds. from the corner, the opposite side and the nearer of the two adjacent sides subtend angles of $35°30'$ and $74°20'$ respectively. Calculate the area of the field to the nearest square foot.

99. Find the angles of the isosceles triangle whose equal sides are 15 ft. long and whose area is 50 square feet.

100. A rectangle is 3 times as long as it is broad; compute the angles between its diagonals to the nearest second.

101. A triangle has its base 175 feet long, and the adjacent angles $52°18'$ and $40°21'$. Find its area and shortest side.

102. The adjacent sides of a parallelogram are 75 and 115 ft., and the perpendicular from the point at which they meet to the diagonal is 45 ft. Calculate to the nearest second the angles of the parallelogram, and find its area.

103. A triangle has sides 98 and 172 feet long, and the angle opposite to the former is $20°12'$; find the third side.

104. Find the lengths of the trisecting lines of the angle of an equilateral triangle whose side is 155 feet long, and the areas of the three triangles into which the whole triangle is divided.

105. Calculate to the nearest second the smallest angle of the triangle whose sides are 20.3, 13.5, and 25.7 feet.

106. The sides of a triangle are 586 ft., 1212 ft., and 1600 ft.; find its area.

107. Compare the areas of a regular pentagon and hexagon described on equal bases.

108. A side of a triangle 118 feet long has an adjacent angle $32°9'$, and the opposite angle $54°6'$; find the longest side of the triangle.

109. Two sides of a triangle, 3071 and 2846 feet respectively, contain an angle of $52°17'$. Find its other angles and area.

110. Find the angles of the right-angled triangle, the sum and difference of whose sides are in the ratio 2 : 1.

111. The radius of a railway curve is 4 furlongs 2 chains, while the angle between the tangents at the two ends is $138°20'$. Calculate the lengths of the tangents, and the distance of the middle point of the curve from the intersection of the tangents in feet.

112. The angular altitude of a lighthouse seen from a point on the shore is $12°31'46''$, and from a point 500 ft. nearer to it is $26°33'55''$. Required its height above the shore.

113. AB is a vertical object on the horizontal plane CBD, and at the points C, D on opposite sides of the object the elevations of A are observed, $ACB = 12°18'$ and $ADB = 15°14'$. The distance CD is 400 yards: find the height of the object.

114. AB is a measured base 500 yards long, C is a visible object whose angular elevation above the horizontal plane at A is $12°16'$. The angles are observed, $CAB = 35°18'$ and $CBA = 118°12'$. Find the height of C above the horizontal plane at A to the nearest foot.

115. The angles subtended by a chimney shaft 150 ft. high, standing at one corner of a triangular yard, at the opposite corners are $25°20'$ and $38°15'$ respectively, while the distance between these corners is 100 yards; find the area of the yard.

116. From the top of a tower, whose height is 100 ft., the angles of depression of two small objects on the plain below, and in the same vertical plane with the tower, are observed and found to be $43°25'$ and $12°12'$ respectively. Find the distance between them.

117. If a tower stands at the foot of a hill whose inclination to the horizon is $10°50'$, and if from a point 100 ft. up the hill the tower subtends an angle of $55°$; find its height.

118. From the top of a hill I observe two cottages lying before me in the same direction, their angles of depression being $23°20'$ and $18°10'$ respectively. They are known to be $\frac{1}{4}$ mile apart, find the height of the hill.

119. Knowing that telegraph poles are placed at intervals of 20 yds. along the bank of a river, from a point on the opposite bank I observe that two of them, next but one to one another, lie in directions making angles $75°15'$ and $72°25'$ with the bank, one being to the right and the other to the left of the point of observation. Find the breadth of the river.

120. From a ship sailing north two lighthouses are observed to lie due east. After an hour's time they are S.E. and S.S.E. respectively. The distance between the lighthouses being known to be 8 miles, find the speed of the ship.

121. A river is 300 yards broad and runs at the foot of a vertical cliff which subtends at the edge of the opposite bank an angle of $25°10'$. Find the height of the cliff.

122. At a point in a straight road I notice that two distant church spires are in a line making an angle of $75°45'$ with the road. A mile further on, the line joining them subtends an angle of $12°30'$ while the more distant spire lies in a direction at right angles to the road; find the distance apart.

123. At noon a column in the direction E.S.E. from an observer cast a shadow, the extremity of which lay in the direction N.E. from him. The elevation of the column was found to be $45°$ and the length of the shadow 80 feet; determine the height of the column, and the altitude of the sun.

124. Wanting to know the breadth of a river I measure along the bank a base AB 250 feet long. At A the bearings of B and of a tree situated on the opposite bank are $124°4'$ and $60°33'$ E. of N. respectively; and at B the tree bears $28'$ W. of N. Compute the breadth of the river to the nearest foot.

125. What is the distance from one another of the summits of two mountains, 3 miles and 2 miles high respectively, just visible the one from the other, taking the earth to be a sphere whose radius is 3957 miles?

126. The height of the Peak of Teneriffe being 12170 ft., calculate the dip of the horizon to the nearest minute (neglecting refraction), and the distance of the visible horizon in miles.

127. Taking the earth to be a sphere of 7912 miles diameter, what will be the dip of the sea horizon to the nearest minute as seen from a mountain 3 miles high, making no allowance for terrestrial refraction.

128. From two points in the same straight line with the base of a tower, and in the same horizontal plane, the angles of elevation are observed to be $58°12'$ and $31°46'$; find the height of the tower, the distance between the points of observation being 185 feet.

129. Two straight railroads are inclined to one another at an angle of $20°16'$. At the same instant from their point of junction two engines start, one along each line. If one travel at the rate of 20 miles an hour, at what rate must the other travel so that after 3 hours the engines may be at a distance from each other of 30 miles?

130. A, B, C are three points in a straight line on a level piece of ground. A vertical pole is erected at C; the angle of elevation of its top as observed from A is $5°30'$, and as observed from B $10°45'$. The distance from A to B being 100 yds., find the distance BC and the height of the pole.

131. In order to ascertain the distance of an inaccessible object C, a person measures a length $AB = 200$ yds. in a convenient direction; at A he observes the angle $PAB = 60°$, and at B the angle $PBA = 109°20'$; find approximately the distance BP. What is the extent of the error to which the result is liable, supposing there may be an error of $1'$ in each angular measurement?

132. ABC is a triangle on a horizontal plane on which stands a column CD, whose elevation at A is $50°3'2''$. AB is 100.62 ft., and BC, AC make with AB respectively angles of $40°35'17''$ and $9°59'50''$. Find the height of CD.

133. The angular elevation of a steeple at a place due south of it is $45°$, and at another place 650 ft. west of the former station it is $14°17'$. Find the height of the steeple.

134. Two cross roads meet a canal at angles of $37°30'$ and $55°20'$ respectively, and at points distant 3000 yards from one another. What would be the length of a road cut direct to the canal from their junction, and lying between the cross roads?

135. B starts to walk in a north-east direction from a station 400 yds. to the north of A at the rate of 90 yds. a minute; how far and in what direction must A walk, starting simultaneously with B, in order to overtake him, walking at the rate of 120 yards a minute?

136. A man places a ladder against a house so that it just reaches to the top. He observes that the ladder makes an angle of $76°25'30''$ with the ground in this position, and that on removing the foot of the ladder a distance of 10 ft., while the ladder itself rests against the wall in the same vertical plane as before, the angle is diminished by $11°10'20''$. Find the height of the house.

137. From each of three points in the same horizontal plane, distant 65, 83, and 106 ft. apart, the elevation of a tower is observed to be $45°$. Find its height.

138. A hill, the sine of whose inclination is $\frac{1}{5}$, faces south; find the inclination of a road which travels up the hill in a north-easterly direction.

ANSWERS.

EXAMPLES. I.

1. (i) $x=\log_2 y$, (ii) $2=3\log_p q$ or $3=2\log_q p$, (iii) $\log_{10}2=.30103$,
(iv) $\log_a b=\frac{1}{3}$, (v) $\log_{10}7=.845098$, (vi) $\log_2 .5=-1$.

2. (i) $10^{1.39794}=25$, (ii) $x^2=y^4$, (iii) $x^2=49$,
(iv) $r^{16}=p$, (v) $3^0=1$, (vi) $b^6=\sqrt{a}$.

4. $2\frac{1}{2}$, .5, 4.642, 1.778, 1.468, 31.623, 2.154, 21.544.

6. The square root of the original base.

7. (i) 5, 15625, .2, (ii) 8, $\sqrt{2}$, $\sqrt[4]{.5}$.

8. (i) 10, (ii) .25, (iii) .5.

EXAMPLES. II.

1. (i) $-.9927185$, (ii) -2.8758207, (iii) -3.7159383, (iv) -1.4648400.

2. (i) $\bar{1}.6875235$, (ii) $\bar{3}.0809382$, (iii) $\bar{4}.5$, (iv) $\bar{3}.875$.

3. (i) $\bar{2}.3515175$, (ii) 1.4603662, (iii) 2.0481957, (iv) 2.8203864,
(v) 3.4290495, (vi) .2215713, (vii) $\bar{2}.3419975$, (viii) 5.0064334,
(ix) 4.8869432, (x) .8629499, (xi) $\bar{3}.4569085$, (xii) $\bar{8}.8401494$,
(xiii) 6.1187785, (xiv) 4.2146460, (xv) $\bar{4}.9211144$, (xvi) $\bar{8}.0305419$,
(xvii) $\overline{12}.9157198$, (xviii) 9.6497684, (xix) $\bar{1}.8653787$, (xx) $\overline{274}.95326$,
(xxi) $\overline{128}.09738$, (xxii) 272.993886, (xxiii) $\bar{2}.1427336$, (xxiv) $\bar{1}.7882409$,
(xxv) 3.5569947, (xxvi) $\bar{2}.966376$, (xxvii) $\bar{4}.9276689$, (xxviii) $\bar{2}.923186$,
(xxix) $\bar{4}.435629$, (xxx) 2.590752, (xxxi) .761551, (xxxii) 6.794014,
(xxxiii) 3.787334, (xxxiv) $\bar{2}.929802$.

EXAMPLES. III.

1. (i) 3.2375439, (ii) 1.9912260, (iii) 2.8293039, (iv) 3.1373540,
(v) 2.7693773, (vi) 6.4560138, (vii) 3.7501226, (viii) 2.5932860,
(ix) 3.2730013, (x) 4.4014006, (xi) 2.6989700, (xii) 4.9946866,
(xiii) 2.7993406, (xiv) 3.0326187, (xv) 4.8249392, (xvi) 3.7816119,
(xvii) 4.3002694, (xviii) 4.6434527, (xix) 4.1629526, (xx) 4.4666008.

EXAMPLES. IV.

1. (i) .3222193, (ii) 1.5352940, (iii) $\bar{1}.0969100$, (iv) $\bar{4}.9030900$,
(v) $\bar{3}.6989700$, (vi) $\bar{2}.0667660$, (vii) .5051500, (viii) .4137341,
(ix) .5772363, (x) .3046341, (xi) 1.4617167, (xii) $\bar{2}.7433892$,
(xiii) $\bar{1}.6300887$, (xiv) 1.3569814, (xv) $\bar{2}.5107190$, (xvi) $\bar{1}.1653048$,
(xvii) .9408786, (xviii) $\bar{1}.3947700$, (xix) 1.2491168, (xx) $\bar{3}.1892232$.

2. (i) 1.50407740, (ii) $\bar{1}.30685282$, (iii) .01917075, (iv) $\bar{4}.20824053$.

3. (i) .9085841, (ii) 1.3944794, (iii) 1.4087304.

EXAMPLES. V.

1. (i) $\bar{4}.3346346$, (ii) .4200741, (iii) .0214812, (iv) .218427,
(v) .1250087, (vi) .3494850, (vii) $\bar{1}.969897$, (viii) $\bar{1}.860206$,
(ix) $\bar{1}.849485$, (x) $\bar{1}.8409596$, (xi) 1.8316161, (xii) $\bar{1}.5117647$,
(xiii) 1.6110965, (xiv) .2108715, (xv) $\bar{2}.7722417$, (xvi) .1215190,
(xvii) 4.3346346, (xviii) $\bar{1}.7852474$, (xix) .5369447, (xx) $\bar{2}.1267371$,
(xxi) $\bar{1}.5523267$, (xxii) $\bar{1}.3333333$, (xxiii) $\bar{1}.3136666$, (xxiv) .5320500,
(xxv) 1.0706253, (xxvi) $\bar{1}.9593372$, (xxvii) .6751920, (xxviii) .4336811,
(xxix) $\bar{1}.8578305$, (xxx) .6573238, (xxxi) $\bar{1}.6410904$, (xxxii) $\bar{1}.9451523$,
(xxxiii) .6948141, (xxxiv) .4610087, (xxxv) .5204397, (xxxvi) .1905862,
(xxxvii) .1041808, (xxxviii) $\bar{1}.5387740$, (xxxix) $^{2}.7780766$, (xl) $\bar{2}.4802752$.

2. (i) .43152311, (ii) $\bar{4}.86028767$, (iii) $\bar{1}.82124095$.

3. (i) $\bar{3}.4734774$, (ii) .4507201, (iii) .2776455.

4. (i) 4.1484835, (ii) 3.5034842.

EXAMPLES. VI.

1. .4771213, .6989700.

2. $\log 2 = .3010300$, $\log 3 = .4771213$, $\log 4 = .6020600$, $\log 5 = .6989700$,
$\log 6 = .7781513$, $\log 7 = .8450980$, $\log 8 = .9030900$, $\log 9 = .9542426$.

3. .3010300, .4771213.

4. .8450980.

5. 2.5563025.

6. 1.6989700, 2.5440680.

7. 1.2041200, $\overline{19}.5051500$.

8. Same as for question 2.

9. .1556302, .2100295, 1.1810658.

10. $\bar{4}.4983106$, $\bar{5}.6811595$.

11. $\bar{4}.3180634$, .3046341.

12. .1760913, .3979400, .5440680.

13. .0457574, .8573326, $\bar{1}.2697953$.

14. $-(1+5\log 2)$.

15. .6989700, .7781513, .4771213, 2.6532126, $\bar{2}.8750613$, 1.2041200.

16. Same as for question 2.

17. $\log_e 2 = 7x - 2y + 3z$, $\log_e 3 = 11x - 3y + 5z$, $\log_e 5 = 16x - 4y + 7z$.

18. 1.0413927.

19. 2.1760913, 2.5976952.

20. 1.3802112, 1.4327022.

21. 1.0292896, $\bar{2}.6108769$.

22. .8465736, 3.2036149.

23. $\bar{4}.09691$, $\bar{3}.87827$.

EXAMPLES. VII.

1. (i) 3, (ii) 4, (iii) $.\dot{6}$, (iv) $.1\dot{6}$, (v) $\bar{2}$, (vi) 1.5,
(vii) 1.5, (viii) 1.75, (ix) $.1\dot{6}$, (x) $\bar{2}$, (xi) 6, (xii) 1,
(xiii) $.8\dot{3}$, (xiv) $.\dot{6}$, (xv) $\bar{1}.875$, (xvi) $\bar{1}$, (xvii) $\bar{2}$, (xviii) $\bar{2}$,
(xix) 2, (xx) $.\dot{3}$.

2. (i) $\bar{1}$, (ii) 0, (iii) 1.5, (iv) .5, (v) 0, (vi) 2.

EXAMPLES. VIII.

1. (i) 4, (ii) $\bar{3}$, (iii) 3, (iv) $\bar{3}$, (v) 4, (vi) $\bar{2}$, (vii) $\bar{6}$,
(viii) $\bar{3}$, (ix) 0, (x) $\bar{2}$, (xi) 7, (xii) $\bar{5}$, (xiii) $\bar{2}$, (xiv) 11,
(xv) 2, (xvi) $\bar{5}$, (xvii) $\bar{5}$, (xviii) 1, (xix) $\bar{2}$, (xx) $\bar{2}$.

2. 1458.

EXAMPLES IX.

1. (i) $x=.524252$, (ii) $x=.983974$, (iii) $x=-1.058746$, (iv) $x=-.762531$,
(v) $x=2.311457$, (vi) $x=2.078224$, (vii) $x=1.062585$, (viii) $x=1.242073$,
(ix) $\left.\begin{array}{l} x=-4.729577 \\ \text{or } -2.270423 \end{array}\right\}$, (x) $x=-.889907$, (xi) $x=1.06$, (xii) $\left.\begin{array}{l} x=.760275 \\ y=.020550 \end{array}\right\}$,
(xiii) $\left.\begin{array}{l} x=\ .5 \\ \text{or } 1.292481 \end{array}\right\}$, (xiv) $\left.\begin{array}{l} x=-1 \\ \text{or } .630930 \end{array}\right\}$, (xv) $x=1.630930$, (xvi) $\left.\begin{array}{l} x=2 \text{ or } -2 \\ y=3 \text{ or } \frac{1}{3} \end{array}\right\}$.

2. $x=1.537244$.

3. $x=-3.31381$, $y=.0005277$.

4. $x=13.734546$.

5. (i) $x=\frac{9}{2}$, $y=\frac{3}{2}$; (ii) $x=\frac{1}{8}$; (iii) $x=2$.

6. $x=\frac{22}{17}$.

7. $x=-\dfrac{\log 2\sqrt{3}}{\log a}$, $y=-\dfrac{\log 2\sqrt{2}}{\log b}$, $z=-\dfrac{\log\sqrt{6}}{\log c}$.

8. $x=n$.

9. $x=\dfrac{\log\frac{1}{2}(1+\sqrt{5})}{\log a}$.

10. $x=\log^{-1}\sqrt{\dfrac{\log b\log c}{\log a}}$, with similar values for y and z.

11. $x=\dfrac{\log(a^2-b^2)}{\log(a+b)^2}$.

12. $\left.\begin{array}{r} x+y=\pm 2a \\ x=y^2 \end{array}\right\}$.

13. 1389.

14. 20.

15. 5.

16. 1.556.

17. 1.2921592.

18. $\sqrt{\dfrac{\log p}{\log a}}$.

19. $-\frac{1}{2}\left\{1-\sqrt{1+\dfrac{8\log p}{\log a}}\right\}$.

EXAMPLES. X.

1. 1.0689391, 1.0408108.

2. 1.05127, 1.13315, .99800.

3. (i) .0069756, (ii) .0295588, (iii) .0010096, (iv) $\bar{1}.9988883$, (v) $\bar{1}.9989995$, (vi) $\bar{1}.9831929$.

4. 2.99957, 3.00043.

5. .002000000666670.

6. (i) .0049875, (ii) $\bar{1}.9949874$.

8. 2.4849067.

9. .8450980, 1.0413927, 1.1139434.

10. .041393.

11. .4772660.

12. $x=2.00432$.

EXAMPLES. XI.

1. (i) 1, (ii) 7, (iii) 4, (iv) $\bar{3}$, (v) 0, (vi) 4,
(vii) 3, (viii) $\bar{5}$, (ix) $\bar{3}$, (x) $\bar{1}$, (xi) $\bar{1}$, (xii) $\bar{3}$,
(xiii) $\bar{2}$, (xiv) 1, (xv) $\bar{5}$, (xvi) 3, (xvii) $\bar{6}$, (xviii) $\bar{3}$.

2. 2.947519, .947519, $\bar{2}$.947519.

3. 1.9096256, 7.9096256, $\bar{4}$.9096256.

4. (i) .5740313, (ii) $\bar{1}$.7501225, (iii) $\bar{2}$.7958800, (iv) 1.1583625,
(v) .3891661, (vi) 1.3502480, (vii) $\bar{6}$.3222193, (viii) .8293038.

5. .81617, 8161.7, .012252, 12252.

6. 4, 1, 7, 8.

7. 1st, 4th, 6th.

8. (i) 20, (ii) 4, (iii) 5.

9. (i) 23rd, (ii) 25th, (iii) 31st.

10. Between 89 and 120.

11. 3.59999.

EXAMPLES. XII.

1. (i) $\bar{3}$.7520543, 5.650041; (ii) 4.9173604, .00826723;

(iii) $117 \begin{array}{|c|c|c|c|c|c|c|c|c|c|} 12 & 23 & 35 & 47 & 59 & 70 & 82 & 94 & 105 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$, $\bar{1}$.5703839, .0371853;

(iv) 1.6186314, 4155.645; (v) $\bar{2}$.3194460, .4792313;

(vi) 4.7286449, .05353531; (vii) $\bar{1}$.9096825, 812.24;

(viii) 7.1187647, 1314.56; (ix) $\bar{3}$.5648380, .03671821;

(x) $217 \begin{array}{|c|c|c|c|c|c|c|c|c|c|} 22 & 43 & 65 & 87 & 109 & 130 & 152 & 174 & 195 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$, $\bar{1}$.3010491;

(xi) 2.9005386, .0795315; (xii) .0133256;

(xiii) $139 \begin{array}{|c|c|c|c|c|c|c|c|c|c|} 14 & 28 & 42 & 56 & 70 & 83 & 97 & 111 & 125 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$, 4.4941999;

(xiv) 5.0001042; (xv) $\bar{1}$.9967099, 1.0075944;

(xvi) 2.2798950, 5.2801131, 1905.21.

2. (i) .8562428, (ii) .25, (iii) .9824394, (iv) .5875158, (v) 1.62681,
(vi) .9932896, (vii) 1.930698, (viii) 1.565065, (ix) .374408, (x) 3.159818,
(xi) .907144, (xii) 6.240325, (xiii) .001264450, (xiv) 2.174774.

3. 1.861646.

4. 46.58847, 178.1415.

5. 16.30076.

6. 8.081476.

7. 7.

8. 12818.3.

9. 10.0794.

10. .563119.

EXAMPLES. XIII.

2. (i) 1.892789, (ii) 1.654175, (iii) $\bar{4}$.730587, (iv) .3529152, (v) 1.086033,
(vi) 1.186087, (vii) $\bar{1}$.8564634, (viii) 1.795889, (ix) $\bar{1}$.6514556, (x) $\bar{4}$.168962,
(xi) $\bar{1}$.3506808, (xii) .1223851, (xiii) $\bar{1}$.350681, (xiv) $\bar{9}$.20105.

3. .3010300, .4771213.

4. 1.146015, .872589.

5. .5440680.

6. .9116518.

7. 1.54396, $\bar{2}$.515839.

8. $\log 1 = 0$, $\log 2 = \frac{2}{2+3ab}$, $\log 3 = \frac{3a}{2+3ab}$, $\log 4 = \frac{4}{2+3ab}$. 9. $\frac{3+3m+12mn}{3+3m+2n}$.

10. (i) $\bar{1}.0939815$, (ii) 5.9826165. 11. $\bar{3}.754894$. 12. .17592.

13. .9463947. 14. $\frac{mx}{n}$. 16. $x = 2$. 20. .69897.

EXAMPLES. XIV.

1. £1627. 2. $22\frac{1}{2}$ years. 3. £6768. 7s. $10\frac{1}{2}$d.
4. £11434.1. 5. £97. 4s. 6. £80. 17s. $5\frac{1}{2}$d.
7. £675. 11s. $3\frac{1}{2}$d. 8. 7.177 per cent. 9. 18.6 years.
10. 15.1 years. 11. 31.9 years. 12. £1560. 10s.
13. 11046.6. 14. £2731. 0s. $10\frac{1}{2}$d. 15. 2.427 per cent.
16. 3.944 per cent. 17. £1774. 12s. 18. £12151.87.
20. 139.02 years. 21. $4\frac{1}{2}$ years. 22. £1643. 13s.
23. 22 years. 24. 83. 25. 143604.2 ozs.
26. After 14th drawing; 21 : 20. 27. In the 17th year.
28. £1198.32; £384.434. 29. £496. 9s. 8d.

EXAMPLES. XV.

1. £1604. 8s. 2. £139. 15s. 4d. 3. 12 years.
4. £518. 13s. 3d. 5. £1586. 6s. 1d. 6. £3482.4s.; 2.9018 years.
7. $23\frac{1}{2}$ years. 8. £4108. 9. £47. 16s. $6\frac{3}{4}$d.
10. £29222. 4s. 5d.; 22.2222 yrs. 11. 41 years. 12. 19 years.
13. £4115.92. 14. £184. 3s. 6d. 15. In 15 years' time.
16. £843. 7s. 11d. 17. £2820. 12s. 18. £31904. 5s. 7d.
19. £80. 4s. 10d. 20. £2. 10s. 21. £139. 9s. 4d.
22. £245. 12s. 10d. 23. 20 years. 24. £302. 10s. 7d.
25. To the age of 56. 26. £3416. 15s. 27. £1240. 14s. 7d.
28. 6.307 per cent. 29. £2194. 3s. 6d. 30. £5260. 10s.

EXAMPLES. XVI.

1. (i) 9.6989700, (ii) 9.8494850, (iii) 10.0624694, (iv) 10.3010300, (v) 9.7614394, (vi) 10, (vii) 10.1505150, (viii) 10.

2. .4771213, .3010300. 3. 9.9079576, 4. 9.9849438. 5. $\bar{1}.5$, 9.

6. (i) 10.1488715, 9.8511285, 10.1488715;
(ii) 10.0531293, 9.9468707, 10.0531293, 9.9468707;
(iii) 9.8485989, 10.1514011, 10.1514011, 9.8485989.

7. (i) $3L\sec A - 4L\cos B + 2L\sin C = 10 - 3\log 2 - 6\log 5$,
(ii) $4L\sin A + L\tan A = 50 + 2\log 3 - \log 2$,
(iii) $3L\tan A + L\sin A - 2L\sin B - 2L\sin C = \log 2$,
(iv) $6L\sin A - 4L\cos A + 18L\cot A = 200 + 6\log 2 - 3\log 3$,
(v) $4L\sec\frac{A}{2} - 2L\tan A - L\tan B = 10 - 2\log 3$,
(vi) $6L\sin A + 3L\cos B - 9L\sin(A+B) - 2L\cos A - 4L\sin B = 3\log 2 - 60$.

EXAMPLES. XVII.

1. 9.9662644, $22°17'38.8''$.
2. 10.4414816, $68°47'17''$.
3. 10.6438363, $77°12'43''$.
4. 9.6568589, $62°59'41.3''$.
5. 11.0145072, $5°31'17.3''$.
6. 9.9997695, $44°59'31.2''$.
7. 9.6990502, $59°58'56.7''$.
8. 9.9979834, $84°29'6''$.
9. 9.4677462, $16°21'38.6''$.
10. 9.427172, $74°29'12''$.
11. 9.7724030, 9.7724575, 10.2275597.
12. $L\sin = 9.6991158$, $L\text{cosec} = 10.3008842$, $L\cos = 9.9374820$, $L\sec = 10.0625180$, $L\tan = 9.7616338$, $L\cot = 10.2383662$
13. $L\cos\theta = 9.9908767$, $L\tan\theta = 9.3162753$.
14. $L\sin\theta = 9.8664913$, $L\text{cosec}\,\theta = 10.1335087$, $L\tan\theta = 10.0354585$, $L\cot\theta = 9.9645415$, $L\sec\theta = 10.1689672$
15. 9.6856149.
16. (i) .006806686, (ii) .0246287, (iii) .000864447, (iv) $7°39'47.7''$, (v) $2°9'6.3''$, (vi) $42°56'29''$, (vii) $63°52'55.9''$.
17. (i) $\theta = 55°14'56''$, (ii) $x = 19°28'16.4''$, (iii) $\theta = 36°45'21.9''$, (iv) $\theta = 58°31'4.3''$, (v) $\theta = 38°4'55''$, (vi) $\theta = 45°40'1.3''$, (vii) $\theta = 19°28'16.4''$, (viii) $\theta = 19°56'47.7''$, $\phi = 12°17'3.7''$; (ix) $x = 63°26'5.8''$, $y = 26°33'54.1''$.
18. $x = 54°15'31.1''$.

EXAMPLES. XVIII.

1. (i) $\sec^2 A\,\text{cosec}^2 A$, (ii) $\tan^2\theta\sin^2\theta$, (iii) $\tan 6A$,
(iv) $\frac{\cos 4A}{\cos 5A}$, (v) $\frac{\sin A}{\cos 2A\cos 3A}$, (vi) $\frac{\sin 5\theta}{\sin 7\theta}$,
(vii) $2\cos(\alpha+\beta)$, (viii) $4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$, (ix) $\frac{\cos 2A\sin\frac{3A}{2}}{\sin\frac{A}{2}}$,
(x) $\sin(x-y)\sin z$, (xi) $\cos x\cos 2x$, (xii) $4\sin\frac{\alpha-\gamma}{2}\sin\frac{\gamma-\beta}{2}\sin\frac{\beta-\alpha}{2}$,
(xiii) $4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$, (xiv) $\tan\frac{A}{2}$, (xv) $4\cos 2\theta\cos 4\theta$,
(xvi) $\tan\frac{A}{2}\sec A$, (xvii) $\tan 3A\tan 2A\tan A$, (xviii) $3\tan 3A$,
(xix) $\tan\theta$, (xx) $\frac{3}{4}\sin(3A+180°)$, (xxi) $\frac{3}{4}\sin 4A$,
(xxii) $\sqrt{\frac{\sin 2A\tan\left(45+\frac{A}{2}\right)}{\tan\frac{A}{2}}}$, (xxiii) $\tan\frac{\theta}{2}$, (xxiv) $\sec x$,
(xxv) $\cos 4\theta$.

2. (i) 5.01654, (ii) .00677044, (iii) .000127011, (iv) .000667708, (v) 395122, (vi) 3.041917, (vii) -5.28614, (viii) $-.7045742$, (ix) 112.9396, (x) .604946, (xi) 16.23075, (xii) 37.75256, (xiii) 38.64313, (xiv) 5.041787, (xv) .02143932, (xvi) -1.43377, (xvii) .903834, (xviii) .0637321.

3. (i) $x = 13°3'7''$, (ii) $x = 57°14'50.4''$, (iii) $x = 36°23'12''$, (iv) $x = 69°35'38.6''$, (v) $x = 8°40'55.8''$, (vi) $x = 6°11'12''$, (vii) $x = 60°19'30.3''$, (viii) $x = 17°25'30''$, (ix) $x = 69°10'59.6''$, (x) $x = 7°58'21.7''$ or zero, (xi) $x = 11°47'20.7''$, (xii) $A = 53°7'48.4''$, (xiii) $A = 22°37'11.6''$, (xiv) $x = 114°24'16''$, (xv) $x = 107°54'29.9''$, $y = 17°45'54.7''$.

4. $46°13'37''$.

5. $22°22'48.4''$ and $45°$.

6. $\sin\theta = \sqrt{\dfrac{(n+m)(n-m)}{(n+1)(n-1)}}$ $\quad \sin\phi = \dfrac{1}{m}\sqrt{\dfrac{(n+m)(n-m)}{(n+1)(n-1)}}$

$\cos\theta = \sqrt{\dfrac{(m+1)(m-1)}{(n+1)(n-1)}}$ $\quad \cos\phi = \dfrac{n}{m}\sqrt{\dfrac{(m+1)(m-1)}{(n+1)(n-1)}}$

$\tan\theta = \sqrt{\dfrac{(n+m)(n-m)}{(m+1)(m-1)}}$ $\quad \tan\phi = \dfrac{1}{n}\sqrt{\dfrac{(n+m)(n-m)}{(m+1)(m-1)}}$.

7. $\tan\left(\theta + \dfrac{\alpha}{2}\right) = \dfrac{m+1}{m-1}\tan\dfrac{\alpha}{2}$.

8. (i) 37.24144, (ii) 155.9583, (iii) .4346886, (iv) 24.57586, (v) 7008.071, (vi) 224.9947.

9. (i) If $b = a\tan\theta$, the expression $= \dfrac{2\cos\theta}{\sqrt{\cos 2\theta}}$.

(ii) If $b = a\cos\theta$ (since $a > b$) and $\tan^3\dfrac{\theta}{2} = \tan^2 x$, the expression $= \dfrac{\sec^2 x}{\sqrt{2}\sin\dfrac{\theta}{2}\tan\dfrac{\theta}{2}}$.

EXAMPLES. XIX.

1. $64°39'23''$.

2. $93°35'$.

3. $113°34'41.4''$.

4. $55°46'16''$.

5. $B = 79°\ 6'23.4''$, $C = 40°53'36.6''$.

6. $49°36'20.4''$, $78°17'39.6''$.

7. $72°12'59''$, $47°47'\ 1''$.

8. $70°33'30''$, $19°26'30''$.

9. $108°58'\ 6''$, $6°\ 1'54''$.

10. $90°$, $51°3'27.2''$, $4\sqrt{2}$.

11. $B = 71°44'29''$, $C = 48°15'31''$.

12. $B = 108°12'26''$, $C = 49°27'34''$, $\log a = .00001$.

13. $\begin{cases} 18°26'\ 5.8'', \\ 108°26'\ 5.8'', \\ 53°\ 7'48.4'',\ 7.58947. \end{cases}$

14. $81°45'2.5''$.

15. $B = 51°18'21''$ or $128°41'39''$, $C = 88°41'39''$ or $11°18'21''$.

16. 17.1, 3.68.

17. $32°21'54''$.

18. $38°56'32.9''$, $90°$, .7525750.

19. $19°3'$, $96°27'$, or $126°22'$.

20. $AB = 1263.58$, $AC = 767.72$.

21. $75°48'54''$, $14°11'\ 6''$.

22. $61°17'22''$.

23. $B = 4°55'10.6''$, $C = 168°27'25.4''$.

24. $45°14'23''$.

25. $49°28'32''$.

26. $132°34'32''$.

27. $35°5'48.6''$.

28. $78°27'52.8''$.
29. $A = 87°27'25.5''$, $B = 32°32'34.4''$.
30. $74°50'38''$, $50°32'58''$.
31. $26°33'55''$, $63°26'\ 5''$.
32. $A = 40°53'36.2''$, $B = 19°\ 6'23.8''$.
33. $74°13'50''$, $35°16'10''$.
34. $A = 105°38'57''$, $B = 15°38'57''$.
35. 2.529823.
36. $B = 33°29'30''$, $C = 101°30'30''$.
37. $B = 65°59'$, $C = 41°56'12''$.
38. 60.3893 ft.
39. 28.47717 sq. ins.
40. 57.3979.
41. $B = 80°46'26.5''$. $C = 63°48'33.5''$.
42. $98°12'44''$.
43. $29°31'34''$.
44. $30°28'12.8''$.
45. $40°32'9''$.
46. $132°34'32.2''$.
47. $63°53'46''$.
48. $B = 38°12'47.6''$, $C = 21°47'12.4''$.
49. $30°$, $120°$.
50. $A = 35°12'31.4''$, $B = 24°47'28.6''$.
51. $A = 90°\ 0'19''$, $B = 45°35'\ 5''$.
52. $A = 89°59'49''$, $B = 49°52'51''$.
53. $B = 90°\ 0'23.3''$, $C = 16°15'36.7''$.
54. $A = 89°59'48''$, $B = 32°52'42''$.
55. $A = 128°9'26''$, $B = 38°9'26''$.
56. 24.2554 ft.
57. $B = 93°33'26.3''$ or $26°26'33.7''$, $C = 56°26'33.7''$ or $123°33'26.3''$.
58. 161.7416.
59. $A = 108°12'40''$, $B = 49°27'26''$, $C = 22°19'54''$.
60. 40, 60, $20\sqrt{19}$, $23°24'48''$, $36°35'12''$.

EXAMPLES. XX.

1. 140.956 yds.
2. 100 ft.
3. $18°26'6''$.
4. 229.6 yds.
5. 1 mile, 1.219714 miles.
6. 229.149 yds.
7. 25.7834 yds.
8. 11·7157 miles.
9. 39.333 ft.
10. 179.011 ft., 13.008 ft.
11. 229.65 yds., 85.86 yds.
12. 2.6718 miles per hour.
13. 1939.2 ft.
14. 112.14 ft.
15. 2656.26 yds.
16. 45.1569 yds.
17. 1496.57 ft.
18. 96.361 ft.
19. 634.14 yds.
20. 1365.9 yds.
21. 152.33 yds.
22. 50.645 ft.
23. $3°46'4.6''$.
24. 5.36 chains.

EXAMPLES. XXI.

1. 354.415 yds.
2. 86.964 acres.
3. .464.
4. 117.772.
5. 16.235 ins.
6. 14400.84 sq. yds.
7. 100670 sq. ft.
8. 9.2542 ins., 5.8352 ins.
9. 33.6158 sq. ins.
10. $\begin{cases} 67°\ 6'53'', \\ 112°53'\ 7'', \end{cases}$ 33.1663 sq. in.
11. 26388 sq. ft.
12. 2705.253 sq. ft.
13. 14391 sq. ft.
14. $\begin{cases} 628.215 \text{ yds.}, \\ 31395.25 \text{ yds.}, 20.68 \text{ s. yd.} \end{cases}$
15. .91856.

16.	31.	17.	108 sq. ft., 37.27 ft.	18.	76.2631 sq. ft., .956773.
19.	2 sq. yds.	20.	27.7545 yds.	21.	69290 sq. ins.
22.	16.6051 ft.	23.	58°37′56″.	24.	786 yds.
25.	435.7335 miles.	26.	371277 sq. ft.	27.	.53901.
28.	49 ins., 328 ins.	29.	64°15′13″.	30.	3.65637 ins.
31.	6268 sq. ins.	32.	392 inches.	33.	198.427 yds., 30923.67 sq. yds.
34.	53.5188 ft., 8998.35 s. ft.	35.	1.013465 acres.	36.	32891 sq. ins.
37.	29.6606 ft.	38.	64.737 sq. ins.	39.	8.8783 ins.
40.	.1831 sq. ins.	41.	947.056 yds.	42.	947 ft. per sec.
43.	105.06 sq. ins.	44.	178.412 links.	45.	123.876 sq. ft.
46.	3.142.	47.	249.877 sq. ins.	48.	101543.4 s. ft., 1129.616 ft.
49.	6875.5.	50.	960.08 sq. ft.	51.	65.6 sq. ft., 465.33 sq. ft., 42.8185 ft., 78.8629 ft.
52.	615.351 sq. ft.	53.	6.9576 : 1.	54.	96° 1′ 9″, 34°18′4″, 49°40′47″, 26.37 sq. ins.
55.	£833. 17s. $2\frac{1}{2}$d.	56.	23.697 inches.	57.	11.46, 9, 6.93.
58.	1385.64 in., 2494153 s. in.	59.	13552 sq. ins.	60.	7.3554 ft.
61.	19896 cub. ins. 14851 cub. ins.	62.	4.6416 yds. 8.0395 yds.	63.	189.57 inches. 215620 sq. inches.
64.	151.2 inches.	65.	358.6682 sq. ins.	66.	5.6363 ft.
67.	7.4442 ins.	68.	75526230 sq. miles.	69.	10.075 ins.
70.	2244.66 sq. ft.	71.	7.4442 ins., 3.3851 ins.	72.	6131.21 cub. ft.
73.	21.9382 ft.	74.	1.39673 : 1.	75.	66022.3 cub. yds.
76.	933.2 c. ins., 224.4 c. ins.	77.	15197 cub. ft.	78.	2.5981 c. ft., 16.238 gals.
79.	233.509 ft.	80.	19722 oz.	81.	10 ft.
82.	6 cwt. 0 qrs. $15\frac{1}{2}$ lbs.	83.	21363 lbs.	84.	6.0012 ft., £10. 14s. $5\frac{1}{2}$d.
85.	80.6851 cub. ft.	86.	5298 cub. ft., 5298 sq. ft.	87.	169.609 cub. yds.
88.	41503 cub. ft.	89.	3466145 cub. yds.	90.	17.62825.
91.	66.3776 cub. ins.	92.	1558.846 cub. ins., 859.058 sq. ins., 64°18′23.8″.		
93.	46.8678 cub. ins. 204.3142 cub. ins.	94.	73°37′51.7″, 852.115 sq. ins., 1386.637 cub. ins.		
95.	13.5594 cub. ft.	96.	711 cub. ins.	97.	11.78548 in., 436.36 s. ins.
98.	21328.64 cub. ins.	99.	43.19568 in., 39.41669 in.	100.	1.5874 : 1, 1.4142 : 1.
101.	8.33601 ins., 5.25136 ins., 120.3586 sq. ins., 95.5288 sq. ins.				
102.	73.1152 cub. ins.	103.	82.151 cub. ins.	104.	785.37 sq. ins.

105. 151°2′42″. 106. 201.752 c.in., 210.047 s.in. 107. 38°56′32.8″.

108. 395.28 sq. ins. 109. 160.6388 sq. yds. 110. 115.1647 cub. yds., 30°.

111. 3942.32 s.in., 20361.3 c.in. 112. 21887 cub. ins. 113. 9.5493 inches.

114. 4.33837 inches. 115. 2632.8 cub. ins. 116. 2141.6 cub. ins.

117. { 387.232 cub. ins.; 339.598 sq. ins. 118. { 283.077 cub. ins.; 285.878 sq. ins. 119. 5428.67 cub. ins.

120. 60373 cub. ins. 122. 415.474 cub. ins.

MISCELLANEOUS EXAMPLES.

A.

1. (i) 198.5366, (ii) 39.405, (iii) 8.82888. 2. (i) .0189694, (ii) 38.7.

3. 20. 4. £131500. 5. 125.1 yrs. 6. 34.762.

7. .3523864. 8. 1.51751. 9. 4343.1, 672.2. 10. £412. 8s. 1d.

11. (i) 2.3377244, (ii) 4.2175071, (iii) .5389662, (iv) $\bar{4}.8943338$,
(v) $\bar{1}.7127511$, (vi) 3.9425714, (vii) 1.3910954, (viii) 4.7764792,
(ix) 3.9243220, (x) 2.0535060, (xi) $\bar{1}.5453923$, (xii) .4551553,
(xiii) $\bar{3}.3416276$, (xiv) 4.2559993, (xv) .4110916, (xvi) 5.6132374,
(xvii) 3.2501815, (xviii) $\bar{1}.0695578$, (xix) 2.7926124, (xx) 6.8928316,

12. (i) 1639.482, (ii) .7081156, (iii) 8.209777, (iv) 18116.33,
(v) 514.7792, (vi) .07108095, (vii) .001737911, (viii) 100.0724,
(ix) .1279746, (x) 26524.67, (xi) 20.05308, (xii) 3.248701,
(xiii) 1722489, (xiv) .009970755, (xv) 147.4078, (xvi) .01017148,
(xvii) .003447162, (xviii) .00007856093, (xix) .2702712, (xx) .01447366.

13. (i) .719686, .857696, (ii) 3.98107, (iii) .0203138, (iv) .000123986,
(v) 1.04729, (vi) .979467.

14. (i) .3891576, (ii) 94.8683, (iii) .000019078, (iv) 670.171,
(v) .0038819, (vi) 2082195.

15. (i) 46.356952, (ii) 12.5419025, (iii) $\bar{1}.1909054$, (iv) 1.2497119,
(v) $\bar{8}.5020252$, (vi) $\bar{2}.2278352$, (vii) $\bar{5}.10348$, (viii) $\overline{12}.3023115$,
(ix) $\overline{18}.533882$, (x) $\bar{1}.9584873$, (xi) $\bar{3}.979376$, (xii) $\bar{1}.2860980$.

16. (i) 3207207, (ii) 3449933, (iii) .000352115, (iv) .000591684,
(v) .00000643485, (vi) .0000020165, (vii) 190.414, (viii) 194.519,
(ix) − 1345.57, (x) 1219.97, (xi) − .109694, (xii) .00000599425,
(xiii) 11464820, (xiv) 5.40935, (xv) .000664211, (xvi) 4.64204,
(xvii) .0197413, (xviii) .999791, (xix) .00257122, (xx) 2420.9,
(xxi) .919641, (xxii) .634713, (xxiii) 1.626835, (xxiv) .0000000000945096,
(xxv) 62.5368, (xxvi) 33.3457, (xxvii) 37.50885, (xxviii) 31933.18,
(xxix) .643237, (xxx) 41475.24, (xxxi) 47.3916, (xxxii) 3.215162,
(xxxiii) 3.32903, (xxxiv) .0117641, (xxxv) .000154046, (xxxvi) 2.174773,
(xxxvii) 125765.8, (xxxviii) .0000587254, (xxxix) .121104, (xl) .129557.

17. (i) 95998.5, (ii) 154.63625, (iii) 16924.24, (iv) 2310.43, (v) 10858.52, (vi) .000320026, (vii) 2536.63, (viii) − 11697.97, (ix) 184864, (x) 5815380, (xi) − 294.5213, (xii) .463638, (xiii) 1.34756, (xiv) 4737.075, (xv) 4516032, (xvi) .0173972, (xvii) .00069865, (xviii) − 35.4586, (xix) .02827142, (xx) 6946.36.

18. (i) $x = 3.28004$, (ii) $x = -.752371$, (iii) $x = 1.94638$, (iv) $x = 1.68504$, (v) $x = -.7814641$, (vi) $x = -.331588$, (vii) $x = 1.29977$, (viii) $x = -.0311154$, (ix) $x = 5$, (x) $x = 2.1186153$, (xi) $x = .012487$, (xii) $x = -1.212$,
(xiii) $\begin{cases} x = 23025.4, \\ y = .00230828, \\ z = 5.96886, \end{cases}$ (xiv) $\begin{cases} x = .265667, \\ y = -.014845, \\ z = .475183, \end{cases}$ (xv) $\begin{cases} x = -529528, \\ y = 529586, \\ z = 538307. \end{cases}$

19. 3768478. 20. 9830. 21. 31, 31.

22. 183, the 80th figure after the decimal. 23. The 17th power.

24. 82.7333 yds. 25. 1.504079. 26. 8, 1679616.

27. (i) £241.171, (ii) £207.893, (iii) £558.493, (iv) £814.725, (v) £288.337, (vi) £170.243, (vii) £3042.64, (viii) £370, (ix) £8158.85, (x) £148.024, (xi) £895.43, (xii) £156.396, (xiii) £228.793, (xiv) £670.475, (xv) £10653.2, (xvi) £3535.25, (xvii) £1208.6, (xviii) £1657.81, (xix) £795.808, (xx) £328.105.

28. (i) £513.373, (ii) £661.783, (iii) £87.204, (iv) £643.928, (v) £7.6045, (vi) £837.484, (vii) £311.805, (viii) £555.265, (ix) £703.185, (x) £438.834, (xi) £2.7772, (xii) £119.047, (xiii) £299.977, (xiv) £106.393, (xv) £50.986, (xvi) £367.698, (xvii) £491.934, (xviii) £690.466, (xix) £690.466, (xx) £435.304.

29. (i) 9.4937, (ii) 2.3293, (iii) 4.1587, (iv) 2.4252, (v) 5.158, (vi) 8.4106, (vii) 8.5195, (viii) 7.4026, (ix) 2.3484, (x) 4.8379.

30. (i) 20.99 yrs., (ii) 23.59 yrs., (iii) 8.73 yrs., (iv) 52.31 yrs., (v) 7.89 yrs., (vi) 13.74 yrs., (vii) 5.9 yrs., (viii) 18.6 yrs., (ix) 23.4 yrs., (x) 7.1 yrs.

31. 28.071 yrs., 17.673 yrs., 14.207 yrs., 8.497 yrs., 7.273 yrs.

32. £512, 11.8 per cent.

33. 1.4 per cent.

34. (i) £1546.4, (ii) £3571.93, (iii) £15266.7, (iv) £261003, (v) £2474.17, (vi) £6667.4, (vii) £3090.57, (viii) £10854.1, (ix) £41289.9, (x) £4521, (xi) £8735.1, (xii) £15453.8.

35. (i) £853.02, (ii) £1265.13, (iii) £1866.46, (iv) £1984.79, (v) £2128.81, (vi) £1689.04, (vii) £3093.25, (viii) £1901.84, (ix) £2161.75, (x) £1892.93, (xi) £3063.12, (xii) £2752.03.

36. (i) £36.159, (ii) £77.996, (iii) £93.114, (iv) £64.012, (v) £33.112, (vi) £55.984, (vii) £43.538, (viii) £36.201, (ix) £38.866, (x) £85.546, (xi) £50.98, (xii) £37.765.

37. £15. 19s. 5d. 38. £129. 10s. 1d. 39. £165.33. 40. 15 years.

B.

1. (α) (i) 9.9586240, (ii) 9.7619049, (iii) 9.6691133,
(iv) 9.5061303, (v) 9.4529884, (vi) 9.9967077.
(β) (i) 9.7958362, (ii) 9.8284938, (iii) 8.6332897,
(iv) 9.9148210, (v) 9.9990851, (vi) 9.9794525.
(γ) (i) 9.6370569, (ii) 9.8707607, (iii) 9.5064387,
(iv) 10.1442807, (v) 10.6841591, (vi) 11.2823535.
(δ) (i) 9.8743056, (ii) 10.7124113, (iii) 9.9636331,
(iv) 9.4410936, (v) 10.1341239, (vi) 10.0647525.

2. (α) (i) 29°7′17.9″, (ii) 4°11′52.1″, (iii) 38°21′37.0″,
(iv) 50°4′19.9″, (v) 60° 7′52.0″, (vi) 45°40′23.3″.
(β) (i) 21°19′11.0″, (ii) 78°4′27.9″, (iii) 38°31′38.7″,
(iv) 70°27′47.7″, (v) 59°9′40.1″, (vi) 29°18′31.7″.
(γ) (i) 15°16′6.6″, (ii) 59°20′40.4″, (iii) 52°30′28″,
(iv) 44°49′3.8″, (v) 46°10′26.6″, (vi) 4°19′56″.
(δ) (i) 14°31′59.2″, (ii) 57° 1′56.0″, (iii) 31° 1′48.3″,
(iv) 69° 4′49.7″, (v) 84°32′49.4″, (vi) 6°32′14.8″.

3. (i) .647754, (ii) .324459, (iii) −.0244981, (iv) 1.995076,
(v) .3120054, (vi) .00131804, (vii) .00283829, (viii) 1.34791,
(ix) .425077, (x) .705856, (xi) 37.6779, (xii) −.283958,
(xiii) −.193769, (xiv) .448496, (xv) −.0916304, (xvi) −.000214783.

4. 2.489898.

5. 9.9813803, 9.4573669.

6. (i) 135.95, (ii) 24.22513, (iii) 2.59894.

7. (i) $A = 49°8'32''$, (ii) $\theta = 71°51'25.4''$, (iii) $\theta = 55°52'40.8''$,
(iv) $\left.\begin{array}{l}\theta = 7°53'47.7'' \\ \phi = 54°44'\ 8.2''\end{array}\right\}$, (v) $\theta = 20°42'17.3''$, (vi) $\theta = 8°40'56''$,
(vii) $x = 67°22'48.5''$, (viii) $\left.\begin{array}{l}A = 30°31'21.7'' \\ B = 10°31'10.8''\end{array}\right\}$.

8. (i) 71°33′54″, (ii) 109°28′16″.

9. 1.360022.

10. 75°31′21″, 75°31′21″, 28°57′18″.

11. 57°9′28.4″.

12. 52°31′57.7″.

13. $\left\{\begin{array}{l}x = 1.74398, \\ y = \ .145149.\end{array}\right.$

14. 61°32′.

15. 45°, 135°, −20°42′17″, −159°17′43″.

16. 7°37′12″.

17. 896 inches.

18. 164 feet.

19. 23 inches.

20. $\left\{\begin{array}{l}45°57', \\ 134°\ 3'.\end{array}\right.$

21. $\left\{\begin{array}{l}75°53'29'' \text{ or } 104°\ 6'31'', \\ 60°54'19'' \text{ or } 32°41'17''.\end{array}\right.$

22. $\left\{\begin{array}{l}34°17'55'', \\ 39°57'16''.\end{array}\right.$

23. 322 feet.

24. $\left\{\begin{array}{l}\text{59.4288 feet,} \\ \text{within 1.76 feet.}\end{array}\right.$

25. 525 feet.

26. $\left\{\begin{array}{l}63°41'39'', \\ 116°18'21''.\end{array}\right.$

27. 8083 feet.

28. 107°44′17″.

29. 728 feet. 30. 194 inches. 31. 4656 sq. feet.

32. 548.16 feet. 33. $10°24'42''$. 34. 30 feet.

35. 889 feet. 36. $33°35'$, $18°21'$. 37. 8 feet.

38. 61 feet. 39. $1°53'$. 40. $61°34'4''$.

41. 951 feet. 42. 325 yards. 43. 337 ft. or 1606 ft.

44. 81 acres 1 rd. 31 po. 46. 68.755 inches, 17189 feet.

47. 4028.5 yds., 2831.7 yds. 48. $\begin{cases} B=110°48'15'', \ C=26°56'1.5'', \\ a=93.519. \end{cases}$ 49. $\begin{cases} 22°37'11.5'', \\ 67°22'48.5'', \\ 90°, \quad 120 \text{ sq. ft.} \end{cases}$

50. 17.1064 sq. inches. 51. 20.97616 ft. 52. $\begin{cases} 2.44845 \text{ feet}, \\ .713322 \text{ feet}. \end{cases}$

53. 90°. 54. $\begin{cases} 5 \text{ ft.}, 16 \text{ ft.}, 5\sqrt{15} \text{ ft.}, 12 \text{ ft.}, \\ 128°39'13'', \ 51°20'47'', \\ 112°23'37'', \ 67°36'23''. \end{cases}$ 55. 418.4 ft., 430 ft.

56. 200 feet. 57. 3.2 miles per hour. 58. 84 feet.

59. 159.422 ft., 215.676 ft.

60. (i) $A=26°59'59''$, $B=63°0'1''$, $c=280.581$;
(ii) $A=22°47'3''$, $B=67°12'57''$, $b=25.474$;
(iii) $B=55°41'36''$, $a=5.53774$, $c=9.82528$;
(iv) $B=17°25'$, $b=313.7$, $c=1048.05$;
(v) $A=61°47'20''$, $b=15.8693$, $a=29.5823$.

61. 934.433. 62. 66386.6, 108.355. 63. $\begin{cases} B=54°10'56'' \text{ or } 125°49'\ 4'', \\ C=73°30'\ 4'' \text{ or } \ 1°51'56'', \\ c=196.8775 \text{ ft. or } 6.68 \text{ ft.} \end{cases}$

64. $62°42'53.3''$. 65. 2.34506.

66. (i) $\begin{cases} A=133°14'54'', \ B=9°52'54'', \ C=36°52'12'', \\ a=14.56775 \text{ inches}, \ b=3.43225 \text{ inches}. \end{cases}$
(ii) $\begin{cases} A=90°, \ B=53°7'48'', \ C=36°52'12'', \\ a=20 \text{ inches}, \ b=16 \text{ inches}. \end{cases}$

67. $41°24'34''$. 68. $\begin{cases} \text{base angles } 27°30', \\ \text{sides } 56.369 \text{ feet}. \end{cases}$ 69. $\begin{cases} 42°\ 0'14'', \\ 55°56'46'', \\ 82°\ 3'\ 0''. \end{cases}$

70. 132.771 feet. 71. $76°52'$, 72. 1170, .849056.

73. 342954 sq. feet. 74. 4127.98 sq. feet. 75. 150 sq. inches, $36°52'12''$.

76. $\begin{cases} 41°48'37, \ 36°52'12'', \\ 101°19'11'', \\ 20.3968 \text{ ins.}, \ 18.3571 \text{ ins.} \end{cases}$ 77. $\begin{cases} 4074.6 \text{ sq. yds.} \\ 383.46 \text{ yds.} \end{cases}$ 78. $\begin{cases} 35°39'18'', \\ 16°56'42''. \end{cases}$

79. $\begin{cases} 4100.7 \text{ sq. inches}, \\ 27.57 \text{ inches}. \end{cases}$ 80. $\begin{cases} 2596 \text{ feet}, \\ 3178182 \text{ sq. feet}. \end{cases}$ 81. $\begin{cases} 168.62 \text{ feet}, \\ 106.16 \text{ feet}. \end{cases}$

82. 5.6568, 90°, $51°3'28''$. 83. $42°50'22.5''$, $17°9'37.5''$. 84. 188900 sq. feet.

85. $\begin{cases} 63°12'54'', \\ 116°47'\ 6''. \end{cases}$ 86. $\begin{cases} 80°32'15.6'', \ 80°32'15.6'', \\ 18°55'28.8''. \end{cases}$ 87. $\begin{cases} 325.130 \text{ feet}, \ 312.984 \text{ feet}, \\ 26357.4 \text{ sq. feet}. \end{cases}$

88. 625 inches. 89. 10518 sq. feet. 90. $19°28'16''$, $160°31'44''$.

91. $24°16'25.5''$, $32°31'10.5''$. 92. 13227 sq. feet. 93. $94°56'24''$.

94. 120°, 7.3752 feet.	95. 63°26′6″, 26°33′54″.	96. 86 feet.
97. 231.437 feet.	98. 29344 sq. feet.	99. {26°23′16″, 76°48′22″, 76°48′22″.
100. {36°52′12″, 143° 7′48″.	101. {7852.7 sq. feet, 113.426 feet.	102. {59°54′20″, 120°5′40″, 7462.35 sq. feet.
103. {83.468 feet or 239.374 feet.	104. {118.717 ft., 2410.17 sq. ft., 2740.74 sq. ft., 2740.74 sq. ft.	105. 31°25′47″.
106. 301944 sq. feet.	107. .66221.	108. 145.3596 feet.
109. {59°25′41″, 68°17′19″, 3456895 sq. feet.	110. {18°26′ 6″, 71°33′54″.	111. 1054.83 ft., 193.91 ft.
112. 200 feet.	113. 48.4345 yds.	114. 629 feet.
115. 27802 sq. feet.	116. 356.833 feet.	117. 114.197 feet.
118. 603.37 yds.	119. 68.9445 yds.	120. 13.66 miles per hour.
121. 140.956 yds.	122. 3332.45 yds.	123. 61.23 feet, 37°25′45″.
124. 211 feet.	125. 280 miles.	126. 1°58′, 135 miles.
127. 2°14′.	128. 186 feet.	129. 25.97 or 11.55 miles per hr.
130. {102.91 yds., 19.5379 yds.	131. {935.763 yds., 3.06 yds.	132. 101.166 feet.
133. 171.12 feet.	134. 1503.91 yds.	135. {1260 yds., 32°1′40″ E. of N.
136. 52.859 feet.	137. 53 feet.	138. 8°13′.

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