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The Market Risk Premium and Empirical Tests of Asset Pricing Models with Higher Moments

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Abstract

Recent research has examined the importance of skewness in the pricing of risky assets, finding the results of such tests to be influenced by the market risk premium. The purpose of this paper is to show, empirically, why the market risk premium may influence tests of asset pricing models with higher moments. In asset pricing models with higher moments, the market risk premium enters the pricing equation in a nonlinear fashion and is implicit in the estimation of each moment's coefficient. The empirical evidence in this paper illustrates that failure to account for this interaction may lead to erroneous conclusions regarding the empirical results of such models.

The Market-Risk Premium and Empirical Tests of Asset Pricing Models with Higher Moments

I. Introduction

Following the work of Markowitz [16], Sharpe [26], Lintner [15] and Mossin [19] developed the first formulations of the mean-variance capital asset pricing model (CAPM). Subsequent modifications to the theory were made by Fama [5], Brennan [4], and Black [2] as well as others. Proponents of the CAPM note its simplicity and potential for testability; however, the model has not been empirically validated in the tests of Black, Jensen and Scholes [3], Miller and Scholes [17], Fama and MacBeth [6] and many others.

Efforts to respecify the pricing equation have gone in several directions. The direction that is of interest in this paper is the research that has expanded the investor's utility function beyond the second moment to examine the importance of higher moments. There has been recent interest in the importance of higher moments as evidenced in a paper by Scott and Horvath [22] which develops a utility theory of preference for all moments under rather general conditions. The third moment (skewness) has already received some attention in the literature (see Arditti and Levy [1], Ingersoll [8], Jean [9, 10, 11], Kane [12], Lee [14] and Schweser [23, 24]).[1] Following the work of Rubinstein [21], Kraus and Litzenberger (KL) [13] derived and tested a linear three moment CAPM, finding the additional variable (co-skewness) to explain the empirical anomalies of the two moment CAPM. The KL model was re-examined by Friend and Westerfield (FW) [7] with mixed results. The FW study found some, but not conclusive evidence of the importance of skewness in the pricing of assets.

In a recent paper, Sears and Wei (SW) [25] present a theoretical argument as to why the market risk premium $(\bar{R}_M - R_f)$ may influence empirical tests of asset pricing models with higher moments. They find that when skewness is added to a pricing model developed within the usual two-fund separation assumptions, the market risk premium enters the pricing equation in a nonlinear fashion and is implicit in the estimation of each moment's coefficient. They also argue that unless this nonlinearity is recognized, incorrect conclusions regarding the empirical tests of such models may result.

Whereas the KL and FW studies focused on the predictive content of the linear three moment CAPM, the purpose of this paper is to empirically examine the SW nonlinear formulation to determine whether tests of the importance of skewness can be influenced by the presence of the market risk premium in each moment's coefficient. In Section II, different versions of the three moment CAPM are reviewed and the SW model is extended to the N moment case. Furthermore, it is shown, analytically, why the estimated market risk premium is biased in the two moment CAPM if the true model is a three moment CAPM. Section III presents some empirical results which illustrate the influence of the market risk premium on tests regarding the importance of skewness. In section IV, tests are presented which support the contention that the market risk premium in the two moment CAPM is biased when skewness is important. A brief summary is contained in Section V.

II. Asset Pricing and the Market Risk Premium: Analysis and Extension

A. The Three Moment Model

Using the notation developed in KL, the theoretical relationship between security excess returns $(\bar{R}_i - R_f)$, the market risk premium $(\bar{R}_M - R_f)$, systematic risk (β_i) and systematic skewness (γ_i) is given in equation (1) for the KL model and in equation (2) for the SW version:

$$(1) \qquad \bar{R}_i - R_f = [(d\bar{W}/d\sigma_W)\sigma_M]\beta_i + [(d\bar{W}/dm_W)m_M]\gamma_i$$

$$(2) \qquad \bar{R}_i - R_f = [(\bar{R}_M - R_f)/(1+K_3)]\beta_i + [K_3(\bar{R}_M - R_f)/(1+K_3)]\gamma_i$$

where: $K_3 = [(d\bar{W}/dm_W)/(d\bar{W}/d\sigma_W)](m_M/\sigma_M)$, the market's marginal rate of substitution between skewness and risk times the risk-adjusted skewness of the market portfolio

σ_M, m_M = second and third central moments about the market portfolio's return

$\bar{W}$, σ_W, m_W = first, second and third central moments about end of period wealth.

The linear empirical version of the three moment model is given in equation (3):

$$(3) \qquad \bar{R}_i - R_f = b_0 + b_1\beta_i + b_2\gamma_i$$

where: b_0 = intercept, hypothesized to equal 0.

Empirical studies by KL and FW measured the importance of risk (β_i) and skewness (γ_i) by b_1 and b_2. While the KL study found γ_i to be an important explanatory variable, FW found empirical tests of the KL linear model to be "...especially sensitive to the relationship between the market rate of returns (R_M) and the risk-free rate (R_f)..." [7, p. 899]. As shown in equation (2), these studies did not remove the interaction of the market risk premium ($\overline{R}_M - R_f$) in evaluating the significance of β_i and γ_i. Because of this interaction, SW argue that the importance of risk is more properly measured by ($\overline{R}_M - R_f$) = $b_1 + b_2$, rather than b_1, and the importance of skewness, K_3, is measured on a relative basis by b_2/b_1, rather than b_2.

This interaction of ($\overline{R}_M - R_f$) with coefficients in asset pricing models with higher moments becomes compounded when moments higher than skewness are included. The theoretical N moment pricing model is:[2]

$$(4) \qquad \overline{R}_i - R_f = (\overline{R}_M - R_f) \sum_{n=2}^{N} [(K_n \gamma_i^n) / \sum_{n=2}^{N} K_n]$$

where: $K_n = [(d\overline{W}/dm_{n,W})/(d\overline{W}/dm_{2,W})](m_{n,M}/m_{2,M})$ and $K_2 = 1$.

$m_{n,M}$ = the n^{th} central moment about the market portfolio's rate of return, where $m_{2,M} = \sigma_M$ and $m_{3,M} = m_M$ as given in equation (2)

$m_{n,W}$ = the n^{th} central moment about the investor's end of period wealth, where $m_{2,W} = \sigma_W$ and $m_{2,W} = m_W$ as given in equation (2)

γ_i^n = the systematic portion of the n^{th} moment for asset i, where $\gamma_i^2 = \beta_i$ and $\gamma_i^3 = \gamma_i$ as given in equation (2)

As seen in (4), ($\overline{R}_M - R_f$) appears in each of the N moments' coefficients and tests of the importance of the n^{th} moment (K_n) should assess the preference tradeoffs in the market between the n^{th} moment and the second moment (risk).

B. Skewness Preference and Empirical Tests of the Two Moment CAPM

Empirical tests of the two moment CAPM by Black, Jensen and Scholes [3], Miller and Scholes [17], Fama and MacBeth [6] and others have typically found a positive intercept and a slope less than its theoretical value, $(\overline{R}_M - R_f)$. If the three moment model is the correct pricing mechanism, then the omission of γ_i from the two moment model should explain, empirically, the two moment model's results. Explicit consideration of $(\overline{R}_M - R_f)$ in each coefficient in the three moment model (equation (2)) gives a linkage between the two moment and three moment CAPM models and provides for an empirical test of the theoretical conditions under which the omission of γ_i is consistent with the two moment empirical results.

The two moment CAPM is given by equation (5):

$$(5) \qquad \overline{R}_i - R_f = b_0^* + b_1^* \beta_i$$

where b_0^* and b_1^* are hypothesized to equal 0 and $(\overline{R}_M - R_f)$, respectively. Under the hypothesis that the three moment model is correct:

$$b_1^* = \mathrm{cov}[(\overline{R}_i - R_f),\beta_i]/\mathrm{var}(\beta_i)$$

$$= \mathrm{cov}[(b_0 + b_1\beta_i + b_2\gamma_i),\beta_i]/\mathrm{var}(\beta_i)$$

$$(6) \qquad = (\overline{R}_M - R_f)[(1 + \alpha K_3)/(1+K_3)]$$

where $\alpha = \mathrm{cov}(\beta_i,\gamma_i)/\mathrm{var}(\beta_i)$, the slope of the regression of γ_i against β_i

Equation (6) provides analytical support for KL's "heuristic rationale" of skewness preference [13, (p. 1098] and their empirical results since if $\alpha > 1$ when $K_3 < 0$ ($m_M > 0$), there will be a specification bias in the two moment model since $b_1^* < (\bar{R}_M - R_f)$ and $b_0^* > 0$. The empirical evidence provided by KL and FW indicates considerable correlation between β_i and γ_i when $m_M > 0$ as well as when $m_M < 0$.[3] Furthermore, it seems reasonable that $var(\gamma_i) > var(\beta_i)$. Together, these imply that $\alpha > 1$ and the empirical results of the two moment CAPM are consistent with a market preference for positive skewness when $m_M > 0$. However, note that skewness preference also implies a specification bias in the two moment model when $b_1^* < (R_M - \bar{R}_f)$ and $b^* > 0$ if $K_3 > 0$ ($m_M < 0$) and $\alpha < 1$. Thus, a preference for positive skewness when $m_M < 0$ requires higher β_i's to be associated with proportionately smaller γ_i's.

III. An Empirical Test of the Nonlinear Three Moment CAPM

The purpose of this section is to present some empirical results which illustrate how the failure to separate $(\bar{R}_M - R_f)$ from K_3 can lead to incorrect conclusions regarding the importance of skewness. Since the importance of skewness, as measured by K_3, is the ratio of the two random variables b_1 and b_2 (but not necessarily uncorrelated random variables), significance tests of K_3 will be difficult when single stationary, cross-sectional regressions are used. Nevertheless, even though the exact sampling properties for the ratio of two random variables are not known, approximate values can be derived.

For equation (3), assume that $E(b_1) = \bar{b}_1$, $E(b_2) = \bar{b}_2$, $Var(b_1) = \sigma_1^2$, $Var(b_2) = \sigma_2^2$ and $Cov(b_1, b_2) = \sigma_{12}$. From Mood, Graybill and Boes [18, p. 178-181]:

(7a) $$\hat{R}_M - R_f = E(b_1 + b_1) = \bar{b}_1 + \bar{b}_2$$

(7b) $$Var(R_M - R_f) = Var(b_1 + b_2) = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

(8a) $$\hat{K}_3 = E(b_2/b_1) \approx \bar{b}_2/\bar{b}_1 - \sigma_{12}/\bar{b}_2^2 + \bar{b}_2\sigma_1^2/\bar{b}_1^3$$

(8b) $$Var(K_3) = Var(b_2/b_1) \approx (\bar{b}_2/\bar{b}_1)^2(\sigma_2^2/\bar{b}_2^2 + \sigma_1^2/\bar{b}_1^2 - 2\sigma_{12}/\bar{b}_2)$$

A single Ordinary Least Squares (OLS) regression of equation (3) provides estimates of $\bar{b}_1$, $\bar{b}_2$, σ_1^2, σ_2^2, and σ_{12}. Thus, equations (7a)-(8b) could be used to perform significance tests on the market risk premium, $\bar{R}_M - R_f$, and the importance of skewness, K_3.

However, in the traditional tests of the two moment CAPM (e.g., Fama and MacBeth [6]) and the linear three moment CAPM (KL [13], and FW [7]), each cross-sectional regression is estimated in a given time period to provide the regression coefficients for that period. The time series values are then used to compute the mean and variance for each regression coefficient. Finally, t-statistics for each regression coefficient are calculated from the ratio of the mean to its standard deviation. This nonstationary approach assumes that the time series values of the regression coefficients are independent over time. Since the nonlinear parameters of (2) are identified in terms of the linear parameters of (3), the above nonstationary approach can be used to test separate hypotheses about K_3 and $(\bar{R}_M - R_f)$.

Using the methodology suggested in KL and FW, one month Treasury bill rates and monthly deflated excess rates of returns are used to estimate beta (β_i) and gamma (γ_i) values for each stock that was continuously listed on the NYSE for the 60 months from January 1931 through December 1935. Although a value-weighted index is theoretically preferable to an equal-weighted index, the CRSP equal-weighted index is used in order to provide results which are comparable to both the KL and FW studies. Stocks are then rank ordered by beta and gamma values into 25 portfolios along the lines suggested in FW. The monthly portfolio deflated excess returns in each of the subsequent 12 months (January 1936 through December 1936) are then calculated for each of the 25 portfolios. This procedure is repeated for the 60 month period beginning each January, with the final period being January 1977-December 1981. This provides 47 years (January 1936 through December 1982) of monthly deflated excess returns for each of the 25 portfolios.

For testing purposes, the entire 47 year series of portfolio returns is partitioned in several ways. First, the data is divided into 5 year sub-periods which include those examined in FW. The initial (January 1936 through December 1941) and final (January 1977 through December 1982) sub-periods are 6 years in length to enable usage of the entire data set. Second, longer sub-periods corresponding to those used by KL (January 1936 through June 1970), FW (January 1952 through December 1976) as well as the entire period (January 1936 through December 1982) are also examined.

For each sub-period examined, portfolio betas and gammas are reestimated and non-stationary statistical tests (see KL and FW) are conducted utilizing the monthly deflated excess returns. In each month during a particular sub-period, equation (3) is estimated and its resultant parameters, $\hat{b}_0$, $\hat{b}_1$, $\hat{b}_2$, $\hat{K}_3$ and $b_1 \hat{+} b_2$ are computed. The time series average values of these parameters along with their associated t values are presented in Table 1, along with other summary statistics.[4,5]

Put Table 1 here

Several things should be noted about the results presented in Table 1. First, similar to the results reported in FW, in most of the periods, the three moment model estimate of the market risk premium, $b_1 + b_2$, is insignificant. According to the theory presented in equation (2), this can result in insignificance for both b_1 and b_2 in linear tests of the model, thus leading to the conclusion that neither risk nor skewness is important in the pricing of assets.

However, as previously noted, the importance of skewness is more properly measured as $K_3(b_2/b_1)$ rather than b_2. Since b_2 measures the interaction between the model's estimate of $(\overline{R}_M - R_f)$ and K_3, time series significance tests for b_2 can be inconclusive regarding the importance of skewness. This can be seen in Table 1 for the results in periods 1942-46, 1952-56 and 1936-6/1970 (KL period) where b_2 is insignificant yet K_3 is significant and with the theoretically correct sign (opposite in sign of m_M). Thus, it is important to separate the joint fluctuations in $(R_M - R_f)$ and K_3 in assessing the importance of skewness.

Since most of the sub-periods result in an insignificant estimate of $(\overline{R}_M - R_f)$, the importance of the interaction between $(\overline{R}_M - R_f)$ and K_3 can be illustrated more dramatically in another way. Following the procedures set forth in FW, the three longer periods (1936-82, 1936-6/70 and 1952-76)are each divided into two sets of regressions: those months where $R_M > R_f$ and those months where $R_M < R_f$. Essentially, this eliminates the time series fluctuations in the sign of $(R_M - R_f)$ and results in $(\overline{R}_M - R_f)$ being highly significant. As seen in Table 2, the impact of this division upon the significance levels and signs of the parameters is striking.

Put Table 2 here

First, as indicated by $b_1 + b_2$, risk is now very significant in all periods. Second and more importantly, the significant estimate of $(\overline{R}_M - R_f)$ by the model results in b_2 being significant in all cases except 1952-76 $(R_M > R_f)$. However, removing the strong effect of a significant $(\overline{R}_M - R_f)$ reveals that K_3 is insignificant in all cases except 1936-6/70 $(R_M > R_f)$. According to the theory, the significance of $(\overline{R}_M - R_f)$ can be so large so as to result in a significant b_2; however, separating the interaction between $(\overline{R}_M - R_f)$ and K_3 in the determination of b_2 reveals K_3 to be insignificant in most cases. Furthermore, the 1936-6/70 $(R_M < R_f)$ and 1952-76 $(R_M < R_f)$ periods illustrate how the impact of a negative market risk premium can result in b_2 and K_3 having different signs as well as significance levels. In only one of the six cases (1936-6/70, $R_M > R_f$), do b_2 and K_3 have the same sign and are both significant.

As the results in Tables 1 and 2 indicate, the interaction between $(\overline{R}_M - R_f)$ and K_3 can result in substantial differences between the signs and significance levels of b_2 and K_3. In some instances (Table 1) b_2 is insignificant while K_3 is significant; however, other tests (Table 2) indicate that the opposite may also occur. Furthermore, in many instances the signs of the two parameters may differ. For most of these results, K_3 does have the theoretically correct sign (opposite in sign from m_M). Interestingly, all of these results show $|K_3| < 1$, indicating a market marginal rate of substitution between skewness and risk of less than one.

IV. Skewness Preference and the Specification Bias of the Two Moment CAPM

As discussed in section II(B), the nonlinear (market risk premium) form of the three moment CAPM provides for a test of the conditions under which the two moment CAPM empirical results are consistent with a preference for positive skewness. Specifically, the test involves running the following regression:

$$(9) \qquad \gamma_i = a_0 + \alpha\beta_i$$

The regression coefficient α in equation (9) is defined in equation (6). Table 3 presents the empirical results of equation (9) and other information about the portfolio data used in Table 1.

Put Table 3 here

In section II(B), it is shown, analytically, that the traditional empirical results for the two moment CAPM ($b_0^* > 0$ and $b_1^* < (\bar{R}_M - R_f)$) are consistent with a preference for positive skewness if $\alpha > 1$ when $K_3 < 0 (m_M > 0)$.[6] As shown in Table 3, there are six sub-periods where these conditions occur: 1936-41, 1952-56, 1972-76, 1936-82, 1936-6/70 and 1952-76. In all of these cases except 1936-82 (where $K_3 > 0$ when $m_M > 0$ and $\alpha > 1$), whenever $K_3 < 0$ and $m_M > 0$, $\alpha > 1$. Furthermore, in the remaining six sub-periods where $m_M < 0$, $\alpha < 1$ in all periods except 1977-82 and $K_3 > 0$ in all periods except 1962-66. Thus, during periods of positive market skewness ($m_M > 0$), portfolio betas are associated with proportionately higher portfolio gammas ($\alpha > 1$) and when $m_M < 0$, portfolio betas are associated with proportionately smaller gammas ($\alpha < 1$). Even though K_3 is not always significant, these results provide evidence of the preference for positive skewness and that this preference is consistent with the empirical findings of the two moment CAPM.

V. Conclusions

Recent research regarding the importance of skewness has found the results to be sensitive to the market risk premium, $(\bar{R}_M - R_f)$. Sears and Wei [25] present a theoretical argument as to why this may be true. This paper has empirically examined the Sears and Wei (1985) nonlinear model. The empirical results underscore the importance of isolating the market risk premium in evaluating the importance of risk and skewness. Furthermore, explicit consideration of $(\bar{R}_M - R_f)$ in the three moment CAPM provides for an analytical examination and an empirical test of the conditions under which the specification bias of the two moment CAPM is consistent with a preference for positive skewness.

Footnotes

[1] In this paper, the word "skewness" refers to the third moment of the return distribution. Many authors use the term "skewness" as the third moment divided by the standard deviation cubed.

[2] The derivation of equation (4) is available upon request from the authors.

[3] For example, see FW [7, p. 902, Fn. 15] and KL [13, p. 1098, Table III].

[4] Exact significance levels cannot be stated without knowledge of the distributions of the underlying variables. Under the assumption that all of the variables are normally distributed, significance at the .05 level in a one-tailed test requires a $|t|$ value greater than 1.67 for the shorter sub-periods and a $|t|$ value greater than 1.65 for the three longer periods.

[5] It is instructive to note that K_3 which equals b_2/b_1 is the ratio of two random variables. Therefore, the average value of the ratio over time will not necessarily equal the ratio of the two time series average values (its value can be approximated via a Taylor series expansion; see Mood, Graybill and Boes [18, p. 181]. In some circumstances, because of large fluctuations in the ratio, its sign may even differ from the sign of the ratio of the two averages (e.g., 1947-51). However, in all instances in which K_3 is opposite in sign from b_2/b_1, K_3 is never significant.

[6] Theoretically, when $m_M > 0$, $K_3 < 0$. However, empirically, $m_M > 0$ does not imply that $K_3 < 0$.

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Table 1

Risk Return Regressions for Common Stocks Grouped to Minimize Measurement Errors
January 1936-December 1982: Nonstationary Estimates Using Equal Weighted Stock Index[a,b,c]

Period	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{K}_3$	$\hat{b}_1+\hat{b}_2$	$\bar{R}^2$	$\overline{R_M-R_f}(.01)$	$m_M^3(.0001)$
1936-41	-.071	.757	-.106	-.112	.651	.427	.568	5.283
	(-.013)	(.481)	(-.209)	(-.984)	(.473)			
1942-46	.712	1.810	.011	.167	1.821	.340	2.684	-.492
	(1.638)	(2.017)	(.050)	(3.287)	(2.031)			
1947-51	.936	.647	-.370	.126	.276	.316	1.080	-.331
	(3.274)	(.904)	(-1.259)	(.751)	(.425)			
1952-56	.616	.728	-.002	-.135	.726	.292	1.197	.090
	(2.232)	(1.403)	(-.014)	(-2.299)	(1.528)			
1957-61	1.908	-.945	.113	.108	-.833	.332	1.032	-.116
	(5.242)	(-1.594)	(.645)	(.267)	(-1.410)			
1962-66	-.402	1.095	-.076	-.785	1.019	.384	.425	-.238
	(-.770)	(1.508)	(-.311)	(-1.596)	(1.350)			
1967-71	-.012	.694	.033	.058	.727	.452	.715	-.176
	(-.028)	(.816)	(.275)	(.873)	(.856)			
1972-76	.249	-.320	.544	-.024	.223	.405	.235	4.405
	(.395)	(-.201)	(.603)	(-.083)	(.200)			
1977-82	-.254	1.068	-.025	.094	1.043	.428	.924	-1.067
	(-.662)	(1.040)	(-.039)	(.079)	(1.408)			
1936-82	.298	.758	-.034	.052	.724	.343	.947	.717
	(1.481)	(2.092)	(-.338)	(.330)	(2.248)			
1936-6/70	.380	.738	-.043	-.144	.696	.310	1.036	.597
	(1.667)	(1.831)	(-.614)	(-2.185)	(1.866)			
1952-76	.521	.216	.132	-.046	.348	.368	.721	.652
	(2.390)	(.528)	(.906)	(-.268)	(.967)			

[a]The coefficient values represent the time series averages of the monthly cross-sectional regressions of the form: $\bar{R}_i = b_0 + b_1\beta_i + b_2\gamma_i = b_0 + [(\bar{R}_M-R_f)/(1+K_3)]\beta_i + [K_3(\bar{R}_M-R_f)/(1+K_3)]\gamma_i$ where

$\hat{b}_0 = 1/T\sum_t \hat{b}_{0,t}$, $(\bar{R}_M - R_f) = 1/T\sum_t(\hat{b}_{1t} + \hat{b}_{2t})$, $\hat{K}_3 = 1/T\sum_t \hat{b}_{2t}/\hat{b}_{1t}$ and T is the number of months.

$\beta_i = \sum_t(R_{it}-\bar{R}_i)(R_{M,t}-\bar{R}_M)/\sum(R_{M,t}-\bar{R}_M)^2$, $\gamma_i = \sum_t(R_{it}-\bar{R}_i)(R_{Mt}-\bar{R}_M)^2/\sum_t(R_{Mt}-\bar{R}_M)^3$, $R_{it} = ((R_{it}-R_{ft})/R_{ft}) \times 100$,

$R_{Mt} = ((R_{Mt}-R_{ft})/R_{ft}) \times 100$ and $\bar{R}_i$, $\bar{R}_M$ and $\bar{R}_f$ are the average monthly deflated excess return relatives on stock i, the market index and one month Treasury Bills over the respective sub-period. m_M^3 represents market skewness and equals $1/T\sum_t(\bar{R}_{M,t}-R_M)^3$.

[b]The t-test statistics are indicated by ( ).

[c]$\bar{R}^2$ values are averages of the monthly $\bar{R}^2$ values.

Table 2

Risk Return Regressions for Common Stocks Grouped to Minimize Measurement Errors
January 1936-December 1982: Nonstationary Estimates Using Equal Weighted Stock Index[a,b]

Period	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{K}_3$	$\hat{b}_1+\hat{b}_2$	$\bar{R}^2$	$\overline{R_M-R_f}(.01)$	$m_M^3(.0001)$	Number of Observations
1936-82									
$R_M > R_f$	-.148	5.233	-.412	.033	4.820	.324	4.633	2.721	331
	(-.538)	(13.418)	(-3.322)	(.144)	(13.931)				
$R_M < R_f$	.932	-5.597	.503	.078	-5.094	.368	-4.223	-1.725	233
	(3.236)	(-13.856)	(4.574)	(.395)	(-14.782)				
1936-6/70									
$R_M > R_f$	.010	4.754	-.298	-.175	4.457	.283	4.422	2.939	253
	(.034)	(11.346)	(-3.116)	(-2.313)	(11.502)				
$R_M < R_f$	.960	-5.572	.358	-.095	-5.214	.351	-4.301	-2.104	161
	(2.735)	(-11.441)	(4.018)	(-.785)	(-11.804)				
1952-76									
$R_M > R_f$	.450	3.715	-.086	-.023	3.629	.343	3.684	1.342	174
	(1.500)	(8.129)	(-.394)	(-.081)	(9.001)				
$R_M < R_f$	.619	-4.615	.432	-.078	-4.183	.403	-3.620	-.391	126
	(1.978)	(-9.456)	(2.581)	(-.704)	(-10.991)				

[a] The t-test statistics are indicated by ( ).

[b] $\bar{R}^2$ values are averages of the monthly $\bar{R}^2$ values.

Table 3

Testing for the Specification Bias in the Two Moment CAPM[a,b]

Period	$\hat{a}_0$	$\hat{\alpha}$	$\sqrt{Var(\beta_i)}$	$\sqrt{Var(\gamma_i)}$	$Corr(\beta_i,\gamma_i)$	Sign of m_M^3	Sign of $\hat{K}_3$
1936-41	-1.360 (-8.26)	2.226 (13.06)	0.345	0.819	0.939	+	-
1942-46	2.129 (5.16)	-0.849 (-1.84)	0.280	0.664	-0.358	-	+
1947-51	0.100 (0.73)	0.906 (6.48)	0.276	0.312	0.804	-	+
1952-56	-0.459 (-1.66)	1.330 (4.97)	0.276	0.510	0.720	+	-
1957-61	0.261 (1.36)	0.970 (4.79)	0.281	0.386	0.706	-	+
1962-66	0.721 (5.30)	0.312 (2.15)	0.247	0.188	0.409	-	-
1967-71	1.228 (3.04)	-0.289 (-0.66)	0.272	0.572	-0.137	-	+
1972-76	-0.230 (-2.20)	1.175 (10.24)	0.236	0.306	0.906	+	-
1977-82	-0.196 (-1.88)	1.101 (11.01)	0.266	0.337	0.917	-	+
1936-82	-2.074 (-5.86)	2.888 (7.64)	0.252	0.859	0.847	+	+
1936-6/70	-3.335 (-7.27)	4.133 (8.46)	0.266	1.265	0.870	+	-
1952-76	-0.394 (-2.24)	1.272 (6.73)	0.246	0.385	0.814	+	-

[a]These are cross-section regressions of the form: $\gamma_i = a_0 + \alpha\beta_i$.

[b]The t-statistics are indicated by ( ).

Notes for the Reviewer

Derivation of the N Moment Capital Asset Pricing Model (Equation 4 in Text)

Extension of the KL framework to an N moment pricing model implies that the investor seeks to:

maximize: (1) $E[U(\tilde{W})] = U[\bar{W}, m_{2,W}, m_{3,W}, \cdots, m_{N,W}]$

subject to: (2) $\sum_i q_i + q_f = W_0$

where:

$E[U(\tilde{W})]$ = expected value of the utility of terminal wealth W

(3) $$\bar{W} = E(\tilde{W}) = \sum_i q_i \bar{R}_i + q_f R_f$$

(4) $$m_{2,W} = [E(\tilde{W}-\bar{W})^2]^{1/2} = [\sum_{ij}\sum q_i q_j m_{ij}]^{1/2}$$

$$m_{3,W} = [E(\tilde{W}-\bar{W})^3]^{1/3} = [\sum_{ijk}\sum\sum q_i q_j q_k m_{ijk}]^{1/3}$$

$$\vdots$$

$$m_{N,W} = [E(\tilde{W}-\bar{W})^N]^{1/N} = [\sum_{i}\sum_{j}\cdots\sum_{N} q_i q_j \cdots q_N m_{ij\cdots N}]^{1/N}$$

q_i, q_f = amount (in dollars) of initial wealth (W_0) invested in asset i and the riskless asset f

$\bar{R}_i, R_f$ = expected holding period return on i and the holding period return on f

(5) $$m_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]$$

$$m_{ijk} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)(R_k - \bar{R}_k)]$$

$$\vdots$$

$$m_{ij\cdots N} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)\cdots(R_N - \bar{R}_N)]$$

At the end of the period, the investor receives (6) $\tilde{W} = \sum_i q_i \tilde{R}_i + q_f R_f$

For the investor's portfolio, define the following terms:

(7) $$\bar{R}_p = E(R_p) = \sum_i (q_i/W_0)\bar{R}_i + (q_f/W_0)R_f$$

(8) $$\gamma^2_{ip} = m_{ip}/m^2_{2,p} = E[(R_i - \bar{R}_i)(R_p - \bar{R}_p)]/E(R_p - \bar{R}_p)^2$$

$$= \sum_j (q_j/W_0)m_{ij}/\sum_{ij}\sum (q_i q_j/W_0^2)m_{ij}$$ = the systematic risk of asset i with the investor's portfolio p

$$\gamma^3_{ip} = m_{ipp}/m^3_{3,p} = E[(R_i - \bar{R}_i)(R_p - \bar{R}_p)^2]/E(R_p - \bar{R}_p)^3$$

$$= \sum_{jk}\sum (q_j q_k/W_0^2)m_{ijk}/\sum_{ijk}\sum\sum (q_i q_j q_k/W_0^3)m_{ijk}$$ = the systematic skewness of asset i with the investor's portfolio p

$$\vdots$$

$$\gamma^N_{ip} = m_{ip\cdots p}/m^N_{N,p} = E[(R_i - \bar{R}_i)(R_p - \bar{R}_p)^{N-1}]/E(R_p - \bar{R}_p)^N$$

$$= \sum_j\cdots\sum_N (q_j\cdots q_N/W_0^{N-1})m_{ij\cdots N}/\sum_{ij}\sum\cdots\sum_N (q_i q_j\cdots q_N/W_0^N)m_{ij\cdots N}$$

= the systematic portion of the n^{th} moment for asset i with portfolio p

The Lagrangian and first order conditions are:

(9) $$L = U(\overline{W}, m_{2,W}, m_{3,W}, \ldots, m_{N,W}) - \lambda[\sum_i q_i + q_f - W_0]$$

(10) $$dL/dq_i = (dU/d\overline{W})(d\overline{W}/dq_i) + (dU/dm_{2,W})(dm_{2,W}/dq_i) + (dU/dm_{3,W})(dm_{3,W}/dq_i)$$

$$+ \ldots + (dU/dm_{N,W})(dm_{N,W}/dq_i) - \lambda = 0 \text{ for all } i$$

(11) $$dL/dq_f = (dU/d\overline{W})(d\overline{W}/dq_f) - \lambda = 0$$

(12) $$dL/d\lambda = \sum_i q_i + q_f - W_0 = 0$$

In solving for the investor's portfolio equilibrium conditions, note that:

(13) $$m_{2,W} = \sum_i q_i \gamma_{ip}^2 m_{2,p}$$

$$m_{3,W} = \sum_i q_i \gamma_{ip}^3 m_{3,p}$$

$$m_{N,W} = \sum_i q_i \gamma_{ip}^N m_{N,p}$$

Conditions (3) and (13) imply:

(14) $$d\overline{W}/dq_i = \overline{R}_i$$

(15) $$d\overline{W}/dq_f = R_f$$

(16) $$dm_{2,W}/dq_i = \gamma^2_{ip} m_{2,p}$$

$$dm_{3,W}/dq_i = \gamma^3_{ip} m_{3,p}$$

$$dm_{N,W}/dq_i = \gamma^N_{ip} m_{N,p}$$

$$dm_{n,W}/dq_f = 0 \quad \text{for } n = 2, \ldots, N$$

Conditions (11) and (15) imply:

(17) $$\lambda = (dU/d\overline{W})R_f$$

Substituting (14), (16) and (17) into (10):

(18) $$(dU/d\overline{W})(\overline{R}_i - R_f) = - (dU/dm_{2,W})\gamma^2_{ip} m_{2,p} - (dU/dm_{3,W})\gamma^3_{ip} m_{3,p} - \ldots - (dU/dm_{N,W})\gamma^N_{ip} m_{N,p} \quad \text{for all } i$$

Moving from the investor's equilibrium condition (18) to a market equilibrium requires that (18) holds for all individuals and that markets clear. For markets to clear, all assets have to be held which requires the value weighted average of all individual's portfolios equal the market portfolio M. Summing (18) across all individuals gives:

(19) $$(dU/d\overline{W})(\overline{R}_i - R_f) = - (dU/dm_{2,W})\gamma^2_i m_{2,M} - (dU/dm_{3,W})\gamma^3_i m_{3,M} - \ldots - (dU/dm_{N,W})\gamma^N_i m_{N,M} \quad \text{for all } i$$

Since (19) holds for any security or portfolio, it also holds for the market portfolio:

(20) $$(dU/d\overline{W})(\overline{R}_M - R_f) = - (dU/dm_{2,W}) m_{2,M} - (dU/dm_{3,W}) m_{3,M} - \ldots - (dU/dm_{N,W}) m_{N,M}$$

Dividing (19) by (20) gives the capital asset pricing model in terms of the N moments and the market risk premium $(\overline{R}_M - R_f)$

$$\overline{R}_i - R_f = (\overline{R}_M - R_f)[(K_2\gamma_i^2/\sum_{n=2}^{N} K_n) + (K_3\gamma_i^3/\sum_{n=2}^{N} K_n)$$

$$+ \dots + (K_N\gamma_i^N/\sum_{n=2}^{N} K_n)]$$

(21) $$\overline{R}_i - R_f = (\overline{R}_M - R_f) \sum_{n=2}^{N} [(K_n\gamma_i^n)/\sum_{n=2}^{N} K_n]$$

where:

$$K_n = [(d\overline{W}/dm_{n,W})/(d\overline{W}/dm_{2,W})](m_{n,M}/m_{2,M}) \text{ and } K_2 = 1$$

In words, equation (21) says that in equilibrium the excess return on security i, $(\overline{R}_i - R_f)$, is a function of the excess return on the market $(\overline{R}_M - R_f)$, the market-related systematic risks of variance and the higher moments (γ_i^n), and the preference tradeoffs in the market between risk and all higher moments. This is equation (4) in the text.

Special Cases: Two Moment and Three Moment Pricing Models

An investor who makes investment decisions solely upon the mean and variance of wealth seeks to maximize $E[U(\tilde{W})] = U[\overline{W}, m_{2,W}]$. Similarly, an investor who considers only the first three moments will maximize $E[U(\tilde{W})] = U[\overline{W}, m_{2,W}, m_{3,W}]$. These two versions are special cases of (21) where N = 2 and N = 3. When N = 2, we have the two moment CAPM model:

(22) $\bar{R}_i - R_f = (\bar{R}_M - R_f)\gamma_i^2$, where $\gamma_i^2 = \beta_i$

as given in equation (2) in the text and when N = 3, we obtain the three moment version:

(23) $\bar{R}_i - R_f = [(\bar{R}_M - R_f)/(1+K_3)]\gamma_i^2 + [(\bar{R}_M - R_f)K_3/(1+K_3)]\gamma_i^3$

where $\gamma_i^3 = \gamma_i$ as given in equation (2) in the text.