MATHEMATICAL MEMOIRS

RESPECTING

A VARIETY OF SUBJECTS;

WITH AN

APPENDIX

CONTAINING

TABLES of THEOREMS

FOR THE

CALCULATION of FLUENTS.

VOL. I.

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PREFACE.

THE purpose of these Memoirs is to improve the various branches of science in which mathematical reasoning is necessary, by illustrating its use in treating of a variety of interesting subjects:—and, thereby, furnishing such precedents as may be the means of enabling the reader, who may be curious either in Analytics, Geometry, or Mechanics, to apply such reasoning to new subjects; and to deduce, from clear principles, with as much facility as possible, the most satisfactory conclusion the nature of the subject may admit of, in any disquisition, that he may be induced to attempt, to which any method of computation proposed to be explained in this Work may be applicable.

To promote the design of the Work, the Theorems for the calculation of Fluents in the Appendix, I presume, will be found of very considerable use: the Tables containing those theorems being, perhaps, more complete and extensive than any that are to be met with in other books on the same subject.—I do not take upon me to say, that *all* the theorems in my Tables are of my own invention;—I trust, it is enough to say, that *most of them* are:—some indeed that have been published by other writers are inserted, to make the *Set* the more complete.

How the principal theorems in the Tables are investigated may be collected from the Memoirs relative thereto, which will appear in the course of the Work.

These Papers, besides furnishing precedents for facilitating computations, will, it is presumed, be found to afford many new geometrical and philosophical Improvements of considerable importance.

The new demonstration of the property of the straight Lever, and some other articles, in the first Memoir, I conceive, may probably give pleasure and satisfaction to some readers, who may be particularly curious in such disquisitions.—But I almost assure myself, there are few geometricians who will not be pleased with the discovery of the theorem in the second Memoir, which enables us *to assign the length of any arc of any conic hyperbola by means of two elliptic arcs:*—a thing which, I am inclined to believe, had not even been thought possible by former writers on the properties of those curves; and of whose use there are many remarkable instances in the following pages.

The new theorems in the third and sixth Memoirs, respecting the motion of a Pendulum, I am induced to think, will not be unacceptable to readers who understand what had before been published concerning such motion.—And I am persuaded that the new method of computation, whereby the sums of many series are obtained, in the fifth Memoir, will engage the attention of the intelligent analyst; and incite him to exercise his skill in the farther application of that method.

With regard to such matters (in these Memoirs) as have been considered by other writers, I must particularly acknowlege, that my fourth Memoir is on a subject upon which the *celebrated Mr.* D'ALEMBERT, as I have lately found, has written at some length, in his *Opuscules Mathematiques:* yet, I flatter myself, that the critical reader will find, in that Memoir, a great number of new and interesting articles, amongst which are some very remarkable

markable inferences reſpecting the gyration of certain bodies there ſpecified.

We may underſtand by what is ſaid in the fifth Tome of the *Opuſcules* juſt now named, that, after the peruſal of what had been written on the ſubject, a doubt remained with ſome mathematician, whoſe name is not there mentioned,—*whether there be any ſolid, beſides the ſphere, in which any line whatever, paſſing through its center of gravity, will be a permanent axis of rotation?*—Upon that point (which ſome perhaps may think is not very ſatisfactorily explained by Mr. D'ALEMBERT *) I have touched a little in the *Philoſophical Tranſactions* for the year 1777; but, in the Memoir here alluded to, I have, it is preſumed, ſo fully explained the matter as to obviate or remove every doubt concerning it.

In the ſeventh, eighth, and ninth Memoirs, in which the motion of a projectile is conſidered, the reader will find ſome propoſitions before conſidered by many authors; nevertheleſs I perſuade myſelf, that what I have written reſpecting thoſe propoſitions will not be deemed trite and uninſtructive.—Moreover, there are (in thoſe Memoirs) ſome *new* reſearches concerning the motion of bodies, which the ingenious mathematician may poſſibly find not unworthy of his regard.

It may be obſerved, that the common doctrine of centripetal forces comprehends only a part of what is neceſſary to be underſtood in order to determine in general the path of a projectile and its motion therein, from a knowlege of its velocity and direction at any given time,

* He has not given one inſtance of a body having the remarkable property in queſtion!

and

and of the force or forces acting thereon whilst moving;—there being innumerable cases in which such force or forces may not continually urge the body towards a certain center as the said common doctrine supposes.—The deficiency of that doctrine in the books published on that subject, I have endeavoured in some measure to supply in the three Memoirs last mentioned:—and, as a farther application of the principal theorems in those Memoirs may be requisite to explain sufficiently the general doctrine of a projectile's motion, I purpose to make such application in some subsequent Memoirs respecting propositions too intricate to be considered amongst the examples which I thought proper to be given in the Memoirs wherein those principal theorems are investigated.

Supposing the reader to know what is meant by any term commonly used in mathematical writings, I shall seldom formally define the technical terms I may make use of in these Memoirs; but, that I may not be misunderstood, I shall endeavour particularly to explain my meaning, where, by using a new term or an ambiguous old one, I apprehend some explanation is necessary:—my design being to treat the subjects I may take upon me to consider—perspicuously, but not prolixly;—in the prosecution of which, I shall lay down the principles requisite in computations, and shew how from thence conclusions may be readily deduced, as well in such propositions as are generally reckoned abstruse as in such as are esteemed more easily investigable.

J. LANDEN.

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OF

VOLUME I.

MATHE-

MATHEMATICAL MEMOIRS.

MEMOIR I.

Of the Mechanic Powers, so far as relates to Equilibriums.

WRITERS on the mechanic powers have, generally, founded their demonstrations of the properties of those powers, on a principle which has been objected to, as obscure and unnatural. For, in treating of Equilibriums (where no moving bodies act on each other, or are any way concerned in the enquiry), they have neglected the proper principles of that doctrine, and have borrowed a foreign, less evident, one from a consideration of motion. They infer from the doctrine of motion, that "as those bodies are equipollent in the congress and reflexion, whose velocities are reciprocally as their innate "forces: so, in the use of mechanic instruments, those "agents are equipollent and mutually sustain each the "contrary pressure of the other, whose velocities, esti-

" mated according to the determination of the forces, are " reciprocally as the forces." This, properly underſtood, is indeed true; and being admitted, renders the buſineſs of the writer on the ſubject of thoſe inſtruments very eaſy: yet, as it is not a clear and natural inference, but rather a theorem, wanting a demonſtration, aſſumed as a principle; and many have expreſſed a diſſatisfaction at the manner in which this ſubject is uſually treated; it may be of uſe to conſider the matter in a different light, and to build our demonſtrations on principles more natural and evident. Such, I preſume, are thoſe upon which, without any regard to the doctrine of motion, I purpoſe to eſtabliſh the fundamental parts of this doctrine.

As we ſhall all along, in this memoir, have frequent occaſion to conſider the tenſions of ſtrings drawn over pullies, it will beſt ſuit our purpoſe to begin with the explaining the properties of thoſe inſtruments.

1. The tenſion of a ſtring, by what means ſoever it be ſtretched, is ſaid to be equal to that weight which would ſtretch it juſt as much, being faſtened to one end of it and ſuſtained by it, whilſt the ſtring itſelf is ſuſpended at reſt, in a vertical poſition, by its other end.

2. Any ſtring, which is no where faſtened to any thing but at its ends, will, in every part of it, be equally ſtretched by any force acting upon it, ſuppoſing it void of gravity.

Plate I. Fig. 1. 3. If a ſtring ABCDEFG be drawn over the immoveable pullies B, C, D, E, F; and one end of it (G) being faſtened to the fixed point G, a weight A be faſtened to the other end; the ſtring will in every part be equally ſtretched, and the parts AB, BC, CD, EF, FG, of the ſtring being all ſuppoſed perpendicular to the horizon, the point

point G will be pulled upwards by a force equal to (A) the ſtring's tenſion; and, the reaction of the point G being equal to (A), the action thereon, it evidently follows, that each of the lower pullies (C, E) will be pulled upwards, and each of the upper pullies (B, D, F) pulled downwards, by twice that force.

4. The reaction of the point G being equal to A, the action thereon, and the reaction of each of the pullies being equal to 2 A; if, inſtead of faſtening the end G to an immoveable obſtacle, a weight equal to A were ſuſpended by it; or if any pulley, inſtead of being faſtened to an immoveable obſtacle, were pulled againſt the ſtring ABCDEFG by a ſuſpended weight equal to 2 A, the reſiſtance oppoſing the action of that ſtring would be the ſame, and the ſtring would be ſtretched as before, and remain at reſt. Fig. 2. Fig. 3.

5. Any of the lower pullies, with or without the end G, being faſtened to one weight only; it follows, that if that weight be to the weight A as the number of parts (BC, CD, &c.) of the ſtring paſſing between the greater of the two weights and the upper pullies is to unity, the ſtring will have the ſame tenſion as if the pullies and end G were all faſtened to the immoveable obſtacle; and it will likewiſe remain without motion, its action being equally reſiſted in both caſes. Fig. 4. Fig. 5.

Thus it appears, that, by drawing a ſtring over ſeveral pullies, it may be made to ſuſtain various weights, whilſt its tenſion remains the ſame; and that, by ſuch means, a very great weight may be made to reſt in equilibrio with a much leſſer one.

6. If a ſtring ABCD be drawn over the pullies B and C, and, one end of it being faſtened to a fixed point D, Fig. 6.

a weight A be ſuſpended at the other end; the pulley B being an immoveable one, and C faſtened to one end *c* of another ſtring *cde*, which is drawn over an immoveable pulley *d*, and has its other end faſtened to a fixed point *e*; and the parts AB, BC, CD, *cd*, *de*, of the ſtrings being all perpendicular to the horizon; the tenſion of the ſtring ABCD will be equal to A, and the pulley C will be pulled upwards by twice that force. Conſequently the ſtring *cde* oppoſing the aſcent of that pulley (C) will have its tenſion equal to 2A, and the pulley *d* will be pulled upwards by a force equal to 4A.

Fig. 7. Therefore, according to what has been before obſerved, if inſtead of the pulley *d* being faſtened to an immoveable obſtacle, a weight equal to 4A be ſuſpended by it, the action of the ſtring *cde* will be reſiſted as before; and the weights A and 4A, it is evident, will reſt in equilibrio.

The application of this method of reaſoning to any combination of pullies (whether one, two, or more ſtrings be employed) is ſo eaſy, that I think it unneceſſary to inſiſt any longer on this head.

Fig. 8. AB being an inflexible rod (conſidered without weight) on which three forces act, at right angles thereto, by means of three parallel ſtrings A*a*, B*b*, C*c*, faſtened at any three points A, B, C, of the rod, two of them on one ſide oppoſing the third on the other ſide, keeping the rod at reſt: I now propoſe to inveſtigate (without any regard to the doctrine of motion) the ratio of the two forces acting on the ſame ſide of the rod; whoſe ſum, it is plain, muſt be equal to the other force acting on the contrary ſide.

Fig. 9. 7. If the weight W be ſuſpended from the middle of the inflexible horizontal rod BD, which is itſelf ſuſpended by

by two parallel ſtrings AB, CD; theſe ſtrings will be equally ſtretched, and each will bear half the weight W. And CD being faſtened at C, and AB drawn over the pulley A, the weight N appending to this ſtring, requiſite to keep an equilibrium with W, muſt be equal to half W.

8. The weight W being ſuſpended from any point P of the inflexible horizontal rod BD, which is itſelf ſuſpended by two parallel ſtrings AB, CD; if the tenſion of the ſtring AB be denoted by T, the tenſion of CD will be expreſſed by W − T; and the ratio of T to W − T will be the ſame, let W be what it will; for the tenſion of each ſtring will be increaſed or diminiſhed in the ſame proportion in which W ſhall be increaſed or diminiſhed. Fig. 10.

9. If BQ be equal to DP, any force acting at Q at right angles to BD will affect the tenſions of the ſtrings AB, CD reſpectively, in the ſame proportion as that in which the tenſions of the ſtrings CD, AB would be reſpectively affected by the application of any force at P in a parallel direction.

10. Therefore, if a ſtring QR, parallel to the other three, having one end faſtened to the rod at Q, be drawn over the pulley R by ſuch a weight N hung to the other end of it, that the ſame be juſt ſufficient, with the ſtring CD, to ſuſtain the rod, with the weight W appending thereto, in its former poſition; the ſtring AB then having no tenſion, the weight which it bore being now borne by the ſtring QR, together with part of the weight which before was borne by CD; T, the decreaſe of tenſion in AB, will be to N − T the decreaſe of tenſion in CD, as W − T to T. Conſequently, W − T will be to W as T to N.

DP

DP is here confidered as lefs than half BD, that confideration being fufficient for our purpofe.

Fig. 11. 11. If DP be a fubmultiple of BP, that is, if, n being fome integer, $n \times BQ$ be equal to BP; let BQ′, Q′Q″, Q″Q‴, &c. be each equal to DP. Then applying ftrings Q′R′, Q″R″, &c. fucceffively to the points Q′, Q″, &c. of the lever, as defcribed in the preceding article, and denoting the refpective tenfions of thofe ftrings as they are applied by N′, N″, &c. we fhall have.

$$W - T : W :: T : N' = \frac{TW}{W - T},$$

$$W - N' : W :: N' : N'' = \frac{N'W}{W - N'} = \frac{TW}{W - 2T},$$

$$W - N'' : W :: N'' : N''' = \frac{N''W}{W - N''} = \frac{TW}{W - 3T},$$

&c. &c. &c.

And, continuing the operation to the neceffary length, we have $N^{n-1} = \frac{TW}{W - \overline{n-1} \times T}$, which (by art. 7.) will be equal to half W: whence $n \times T = W - T$, or $T : W - T :: 1 : n$.

Thus it appears, that, in cafe BP be any multiple of PD, the tenfions of the ftrings AB, CD (which are denoted by T and W − T) will be as 1 to n, that is, as DP to BP.

12. From what has been faid it evidently follows, that, DP being any diftance whatever, if BQ, inftead of being equal thereto (as hitherto fuppofed) be equal to q, any fubmultiple of $r - q$, r being the whole length of the rod BD; $r - q$ will be to q, as T, the decreafe of tenfion in AB, to N − T, the decreafe of tenfion in CD, upon applying a ftring QR at Q as above. Whence $r - q : r :: T : N$.

13. We

13. We will now see what will be the ratio of the tensions of the strings AB and CD, when BP is not a multiple of DP. First let us suppose DP and BP commensurable, and equal to $m \times p$ and $n \times p$ respectively, p being a common measure of BP, DP. Then conceiving a string, as QR, applied successively to the points whose distances from B are p, $2p$, $3p$, &c. till it comes to a point equidistant from P with the point D, and determining the several successive values of N′, N″, &c. (the tensions of Q′R′, Q″R″, &c.) in terms of T (the tension of AB) and known quantities, the last of them will (by art. 7.) be equal to $\frac{W}{2}$: from which equation the ratio of T to W − T will appear. Thus, applying the string Q′R′ to the point Q′, whose distance from B is p, we shall, by the preceding article, have $r - p : r :: T : \frac{rT}{r-p} = N'$, the weight then borne by Q′R′. Applying the string Q″R″ to the point whose distance from B is $2p$, we shall have $r - 2p : r - p :: \frac{rT}{r-p} : \frac{rT}{r-2p} = N''$, the weight borne by Q″R″ in that situation. And it is obvious, that the weight which the string QR will bear, when applied to the point whose distance from B is $\overline{n-m} \times p$ (that is, when its distance from P is equal to DP) will be $\frac{rT}{r - \overline{n-m} \times p} = \frac{\overline{m+n} \times T}{2m}$, which (by art. 7.) must be equal to $\frac{W}{2}$. Consequently T will be to W as m to $m+n$, and T to W − T as m to n. Therefore, T being the weight borne by the string AB before the application of Q′R′, Q″R″, &c. and W − T the weight borne by the string CD at the same

Fig. 12.

ſame time, it is manifeſt, that the tenſions of thoſe ſtrings (AB, CD) when they ſuſtain any weight W, appending to the point P, will be reciprocally as their diſtances from that point, though DP be not a ſubmultiple of BP, if thoſe diſtances be commenſurable.

Fig. 13. 14. Suppoſe now that DP and BP are incommenſurable: and if the tenſion of AB be not to the tenſion of CD as DP to BP, let it be as DP to bP greater or leſs than BP. Let Dm, leſs than Bb, be a ſubmultiple of DP: and Dn being the leaſt multiple of Dm which exceeds BD, if bP be ſuppoſed greater than BP; or the greateſt multiple of Dm which is leſs than BD, if bP be ſuppoſed leſs than BP; Bn will in either caſe be leſs than Dm, and n will fall between B and b in both caſes. Now, by what has been proved, if the ſtring AB were at n, its tenſion to that of CD would be as DP to nP: therefore, if d be put for the difference of the tenſions of AB, when applied at B, and when applied at n; and t be its tenſion when at B, and T that of CD at the ſame time; we ſhall have $t \mp d : T \pm d$ :: DP : nP, and $\frac{t \mp d}{T \pm d} \times n\text{P} = \text{DP}$. But by hypotheſis, $t : T$ :: DP : bP, and $\frac{t}{T} \times b\text{P} = \text{DP}$; and conſequently $\frac{t \mp d}{T \pm d} \times n\text{P}$ ſhould be $= \frac{t}{T} \times b\text{P}$. This, it is plain, is impoſſible: for $\frac{t \mp d}{T \pm d}$ is leſs or greater than $\frac{t}{T}$, according as nP is leſs or greater than bP. The ratio of the tenſions of the ſtrings can then be no other than reciprocally as their diſtances from P.

15. Inſtead of any force whatever acting on a body by other means than that of a ſtring, we may conceive another

another force to be ſubſtituted, which pulling by a ſtring in the ſame direction, the ſame effect ſhall be produced; and forçes being meaſured by the effects they produce, this force muſt be eſteemed equal to that in whoſe ſtead it may be ſo ſubſtituted. Therefore, whatever we infer concerning the ratio of the tenſions of ſtrings, by means whereof any body is kept at reſt, the ſame may be inferred of the ratio of any other forces acting in the ſame directions, and producing the ſame or an equal effect. Conſequently, a ſtraight inflexible rod being kept at reſt by three forces acting thereon, at right angles thereto, whether thoſe forces act by means of ſtrings or otherwiſe, the two forces oppoſing the third (which muſt neceſſarily act between them) muſt be reciprocally as the diſtances of their points of action from the point at which the third force acts.

Hence the properties of ſtraight levers, the power and weight acting at right angles thereto, are evident.

16. AB being a lever of the firſt kind, F the fulcrum, and M and N two weights ſuſpended from A and B reſpectively, and ſuſtaining each other in equilibrio with AB in a horizontal poſition; the lever may be conſidered as kept at reſt by three forces M, N, and M + N acting at A, B, and F reſpectively in the manner above deſcribed. For the reaction or reſiſtance of the fulcrum is of the ſame effect with reſpect to ſuſtaining M and N as an active force equal to thoſe two would be of, if it oppoſed them by pulling or preſſing at right angles to the lever at the point F. Therefore, by the preceding article, M muſt be to N reciprocally as AF to BF. And in the ſame manner it appears, by changing the place of the fulcrum, that any forces whatever, which, acting on a ſtraight lever of the ſecond or third kind at right angles thereto, are equipol- Fig. 14. Fig. 15.

lent, are reciprocally as the diftances at which they act from the fulcrum.

Fig. 16. 17. Inftead of confidering the lever as a fingle ftraight line, let us conceive the fulcrum F to be at a point in an extended plane of any form (without weight), and that the faid plane is freely moveable about F as a center. Then, if the weights M and N, acting at the points B and D in the faid plane at right angles to the right line BFD, reft in equilibrio; they will, it is obvious, likewife reft in equilibrio, if, inftead of acting at B and D, they act at any other points *b* and *d* refpectively, in the lines B*b*, D*d*; thefe lines being the directions in which the forces act, whether B and D or *b* and *d* be the points of action.

Moreover, the weights M and N will alfo reft in equilibrio, if, FE being equal to FD, the force N, inftead of acting at D in any direction D*d*, act at E in the direction E*e*, the angles FE*e*, FD*d* being equal.

Now, by what is proved in the preceding article, BF × M will be = DF × N. Join F*d*, and fuppofe that, inftead of the weight N acting at *d* in the direction *d*N, a weight equal to $\frac{DF \times N}{dF}$ act at *d* at right angles to *d*F. Then, by what is already proved, will this laft mentioned weight reft in equilibrio with the weight M acting at B or *b* as before. Therefore, with regard to preferving an equilibrium with any oppofing force, the weight acting at *d* on the lever *d*F in the direction *d*D, or D*d*, is to the weight acting at the fame point of the fame lever at right angles thereto, as N to $\frac{DF \times N}{dF}$; that is, as *d*F to DF, or as radius to the fine of the angle which the oblique direction makes with the ray from the fulcrum to the point of action.

18. Let

18. Let two forces, G and H, act obliquely at the points B and D, on the straight lever BFD, moveable about the fulcrum F; and BA, DC, being the directions in which those forces act, let the sines of the angles ABF, CDF, to the radius 1, be denoted by p', p'', respectively. Then, by what is said in the preceding article, $p' \times G$ and $p'' \times H$ will be the respective equivalent forces considered as acting at the same points (B and D) at right angles to the lever. And, the lever being supposed to be kept in equilibrio, $p' \times G \times BF$ will be equal to $p'' \times H \times DF$. Fig. 17.

19. Considering now three forces, G, H, I, as acting at the point P in an extended plane of any form (without weight), in the directions PA, PB, PC respectively, so as to preserve an equilibrium; the ratio of those forces will be found as follows, by means of what is done above. Continue BP to any point F, and draw dFe at right angles thereto. Then, seeing that it is the same thing with regard to keeping the plane at rest, whether the forces (G, H, I) act at the points d, F, e, in the said plane respectively, or at P, as before mentioned, their directions being the same in both cases; we have, by the preceding article, $p' \times G \times dF$ equal to $p''' \times I \times eF$, p' and p''' being the sines of the angles AdF, CeF respectively, to the radius 1. Whence it is evident, that the forces G and I must be as $p''' \times eF$ and $p' \times dF$. But if q', q''' be the respective cosines of AdF, CeF, or the sines of dPF, ePF; dF will be $= \frac{q' \times FP}{p'}$, and $eF = \frac{q''' \times FP}{p'''}$. It appears therefore, by substitution, that the force G must be to the force I as q''' to q'. Fig. 18.

And, continuing AP to E, and denoting the sine of ePE by q'', it appears by the same way of reasoning, that

 the

the force H muſt be to the force I as q'' to q'; the angle BPE being manifeſtly equal to the angle dPF. Conſequently the forces G, H, I, muſt be as q''', q'', q'; that is, drawing BE parallel to PC, as the ſines of the angles EBP, BEP, BPE; or, which is the ſame thing, thoſe forces muſt be reſpectively as the ſides EP, BP, BE of the triangle BEP which are in, or parallel to, their reſpective directions.

Fig. 19. 20. If three forces (G, H, I) acting at any three points of a moveable plane ſuſtain the ſame in equilibrio, the directions in which thoſe forces act (unleſs they be parallel to each other) will interſect each other in one point only. For let the points of action be *d*, *e*, *f*; and the directions *d*A, *e*B, *f*C. Then if theſe directions be ſuppoſed not to interſect each other in one point, let them be ſuppoſed to interſect at *q*, *r*, *s*, as in our ſcheme: and let the point of action of the force G be conſidered as at *p* (in *dqr*), inſtead of *d*; which change, it is plain, will not cauſe any alteration in the effect of the forces upon the plane. Now, the point *p* being between *q* and *r*, the forces H and I acting in the directions *sre*B, *qsf*C will both urge the plane to turn the ſame way about the point *p*; and conſequently will cauſe the plane to move. Therefore the ſuppoſition is abſurd, and the plane cannot remain at reſt whilſt three forces act upon it at different points, unleſs their directions (being not parallel to each other) interſect each other in only one point.

21. By means of the laſt two articles we may readily find the quantity and direction of one ſingle force, which ſhall have the ſame effect as any two forces whoſe quantities and directions are given: and, on the contrary, one ſingle force being given with its direction, we may find two forces, which, acting together in any propoſed directions,

directions, shall be equivalent to that single force. Thus, PA, PB denoting the quantities and directions of two forces, draw AQ, BQ parallel to PB, PA respectively; and the diagonal PQ will denote the quantity and direction of a single force equivalent to the other two. And, PQ denoting the quantity and direction of a single given force; draw at pleasure QA, PB parallel to each other; and PA, QB also parallel to each other; then will PA, PB denote the quantities and directions of two forces, which acting together will be equivalent to the said single given force. Fig. 20.

22. The wheel AB being fixed to the cylinder C*n*, with its center C at the center of a circular section of the said cylinder, and the wheel itself in the plane of that section; and *mn* being a radius of another circular section of the cylinder, whose center is *n*; any weight P pulling at A, at right angles to the radius AC, will have the same effect in opposing the action of another weight W, pulling at *m* at right angles to *mn*, (and urging the cylinder and wheel to turn the contrary way,) let the distance of C from *n* be what it will. Therefore, it being manifest that, if C coincided with *n*, the two radii *mn*, CA would form a lever *mn*A; the weights P and W will (by art. 17.) rest in equilibrio, when the former is to the latter reciprocally as AC to *mn*, whether the radii, to which they act at right angles, have the same position with respect to the horizon or not, the machine being considered as only moveable about the axis (C*n*) of the cylinder. Fig. 21.

23. AB being an immoveable inclined plane, if a weight W rest thereon in equilibrio with another weight P pulling in the direction WB, parallel to the plane, by means of a string PBW passing over the immoveable pulley B, the weight Fig. 22.

weight W may be confidered as kept at reft by three forces acting thereon, one whereof is its own gravity urging it in the direction W*m* directly downwards, another the repreffure of the plane urging it in the direction W*n* perpendicular to AB, and the third the tenfion of the ftring (or the weight P) pulling it in the direction WB. Therefore, BC being a vertical line, and CD perpendicular to AB, BCD will be a triangle having its fides in, or parallel to, the directions of thofe three forces. Which forces will, therefore, (by art. 19.) be refpectively as BC, CD, and BD; that is, as AB, AC, and BC, fuppofing AC parallel to the horizon.

Fig. 23. 24. The weight W refting on the immoveable inclined plane AB, in equilibrio with another weight P pulling it in the horizontal direction W*b*, by means of a proper contrivance; if *n*W*o* be perpendicular to AB, and W*m* perpendicular to the horizontal line A*mo*C; the weight W, the reaction of the plane in the direction W*n*, and the weight P will (by art. 19.) be to each other refpectively as W*m*, W*o*, and *mo*; that is, if BC be perpendicular to AC, as AC, AB, and BC. Therefore P muft be equal to $\frac{BC}{AC} \times W$.

Fig. 24. 25. But if the weight W be kept at reft on the plane by P acting at the point E of a horizontal lever EWF, at right angles thereto, in a direction parallel to AC; F being the lever's fulcrum, the value of P in this cafe muft, by the property of the lever, be to its value in the former cafe (confidered in the laft article) reciprocally as EF to WF. Confequently in this cafe P muft be equal to $\frac{WF \times BC}{EF \times AC} \times W$.

26. When

26. When a weight is raiſed by means of a ſcrew, one part of the ſcrew riſes with the weight, and the aſcent of that part of the ſcrew is effected by ſliding its ſpiral threads up the ſpiral threads of the other part of the ſcrew. A weight therefore, when it is ſuſtained in equilibrio by any power acting on the ſcrew, may be conſidered as reſting on an inclined plane, whoſe inclination to the horizon is the ſame as that of a particle of the ſpiral whereon it reſts. Which inclination will be the ſame as that of AB to AC, if AC be the circumference of a circular ſection of the cylinder on whoſe ſuperficies the ſpiral is deſcribed; and BC, a perpendicular on AC, be the diſtance of two contiguous threads of the ſpiral meaſured according to the cylinder's length. Fig. 25.

Moreover, the ſcrew being worked by a horizontal lever whoſe fulcrum is at the axis thereof, and the whole force of a weight raiſed or ſuſtained by this inſtrument being incumbent on the ſpiral threads, and the diſtance of every point of thoſe threads from the axis being equal to the radius of a circular ſection of the cylinder whereon they are deſcribed; if that radius be denoted by r, the weight will, with regard to the lever, act at the diſtance r from the fulcrum. Therefore, the ſcrew being loaded with the weight W, and d being the diſtance from the ſcrew's axis at which a power P acts on the lever, in a horizontal direction at right angles thereto, ſuſtaining W in equilibrio; P muſt, by the laſt article, be equal to $\frac{r \times BC}{d \times AC} \times W$, d and r being written inſtead of EF and WF. But c being the circumference of a circle whoſe radius is unity, AC will here be equal to $c \times r$. Therefore P will be equal to

BC

$\frac{BC}{c \times d} \times W$, or P to W as BC to ($c \times d$) the circumference of a circle whoſe radius is d.

We have here ſuppoſed the ſpiral threads of the ſcrew to be cut but a very little depth into the cylinder they are deſcribed upon. But it appearing by our demonſtration that, if d and BC remain the ſame, P will have the ſame advantage with regard to ſuſtaining W, let r be what it will; it is plain, that, how deep ſoever the threads of the ſcrew be cut, P muſt be to W in the ratio above aſſigned.

27. If we would enquire when there will be an equilibrium between a power preſſing againſt the back of a wedge, and a force acting againſt its ſides in oppoſition to that power, we muſt firſt conſider in what direction the wedge is reſiſted by that oppoſing force; for the queſtion cannot be anſwered in general terms.

Some writers ſuppoſe the wedge reſiſted by a force acting at right angles to its ſides; others conſider the reſiſting force as acting in a direction parallel to the back of the wedge: whence it is, that their concluſions are different.

Fig. 26. A wedge ABC, preſſed downwards by the force of a weight P, being reſiſted by two forces W and W (equal to each other) preſſing perpendicularly againſt its ſides, and ſuſtaining P in equilibrio; thoſe three forces will be as AB, AC, and BC reſpectively: and conſequently P will be to (2W) the ſum of the other two forces, or the whole reſiſtance oppoſing the progreſs of the wedge, as (AB) the back of the wedge to (AC + BC) the ſum of its ſides, or as half the back of the wedge to one of its ſides. For, DE, FG being perpendicular to AC, BC reſpectively, draw

draw DF parallel to GF, and EF parallel to the vertical line CP, a perpendicular on AB. Then EF being put to denote the force P preſſing the wedge directly downwards, DE, DF will (by art. 19.) denote the equal forces preſſing againſt AC and BC. But the angles ACP, CEF will be equal, and conſequently DEF equal to CAP. And, DE, DF being equal, the angles DEF, DFE will be equal to each other, and each to each of the angles CAB, CBA. Therefore the iſoceles triangle DEF will be ſimilar to CAB, and the forces will be in the ratio we have aſſigned.

In making the experiment, repreſented by our ſcheme, the weight of the wedge, it is obvious, muſt be conſidered as part of P. And part of the weight of the ſliders Q and R, which reſt on the inclined planes S and T, acting conjunctly with the wedge in oppoſing the forces W and W, urging the ſaid ſliders againſt its ſides; an equivalent weight muſt be added to W: which equivalent weight muſt (by art. 23.) be to the weight of the reſpective ſlider as half EF to DE; or, which is the ſame thing, as AP to AC.

28. If the forces oppoſing the progreſs of the wedge, inſtead of acting as ſuppoſed in the laſt article, act in directions parallel to its back; the reſt being as in that article, the ratio of the power urging the wedge forward to the equipollent reſiſtance may be determined as follows. The two forces preſſing the ſides of the wedge acting by means of ſliders as repreſented by our ſcheme, the wedge will preſs the end E of the ſlider Q in the direction ED, and $\frac{ED}{EF} \times P$ will (by the preceding article) denote the force of that preſſure. Moreover, (by art. 21.) the efficacy of that Fig. 27.

pressure in a direction parallel to HD will be expressed by $\frac{DH}{EF} \times P$; being to the said pressure in the direction ED, as DH to DE; DH being drawn parallel to AB, to which QE is now supposed parallel. Now W, the force urging the slider Q against AC, being equal to $\frac{DH}{EF} \times P$; (as it is evident it must be, to resist the pressure of the wedge, so as to sustain it in equilibrio;) the equal force W urging the slider R against the other side of the wedge will likewise be expressed by $\frac{DH}{EF} \times P$. Therefore, the power P will be to 2W, the whole resistance opposing the progress of the wedge, as P to $2 \times \frac{DH}{EF} \times P$; that is, as EF to twice DH; or, which is the same thing, as (AP) half the back of the wedge to (CP) its height.

In cleaving of wood, the resistance opposing the force of the mallet (supposing the sides of the wedge perfectly polished, and its edge a line without breadth) is the attraction of cohesion of the particles of the wood which are about to be separated; and this being a kind of pressive force acting against the sides of the wedge, it is extremely absurd to attempt to compare it with the percussive force of a mallet, as some writers have done. For the greatest finite pressive force must give way to the least percussive one, and there cannot possibly be an equilibrium between two such different forces. Any percussive force acting on a moveable body generates a finite quantity of motion in an indefinitely small particle of time; but the time will be finite in which any given pressive force whatever, acting on the same body, can generate or destroy the same quantity of motion. Therefore, a body being urged in a certain

certain direction by any pressive force whatever, and in the contrary direction by any percussive one, the pressive force will be some finite time in destroying the quantity of motion which the percussive one generated in an instant. Consequently how great soever the pressive force may be, and how small soever the percussive one, the body will be moved (at least for some short time) by this last force. Indeed, after the stroke is given, the pressive force may quickly prevail and force back the body which the impulse of the other force had driven forward. And so it would frequently be with respect to cleaving wood, if the sides of the wedge were perfectly smooth. For, after the stroke of the mallet, the wedge, unless its weight were equivalent to the force of the attraction of the parts of the wood about to be separated, would presently be forced back from the place whereto the mallet had driven it. And it is chiefly the roughness of the sides of the wedge, and of the parts of the wood in contact therewith, which, in that operation keeps the wedge from receding. It is that roughness too, and the bluntness of the edge, which sometimes prevent the wedge from being moved by the stroke of the mallet. For, were it not obstructed by such roughness and bluntness, it would, according to what we just now observed, be always driven forward even by the least percussive force.

Having established the doctrine of equilibriums on its own proper principles, and explained the same so far as relates to an equipollence between the power and weight in the use of the instruments commonly called the Mechanic Powers; I shall conclude this Memoir with an article or two, farther explaining the doctrine particularly inculcated above; and moreover exemplifying the appli-

 cation

cation of the theorem borrowed from the doctrine of motion, to which recourse is commonly had in treating of equilibriums.

Fig. 28. 29. If the wedge (or inclined plane) ABC, placed with one of its fides (BC) on a polifhed horizontal plane BD, be urged in the direction BCD by a weight P acting at the angular point C by means of a ftring paffing over the immoveable pulley D, whilft the weight W appending to the ftring ENW (having one of its ends faftened to the fixed point E, and the other to the faid weight W) preffes againft the other fide (AC) of the wedge: P and W will reft in equilibrio when, MOW being a vertical line, and MN, NO at right angles to AC, MW refpectively, P is to W as NO to MW, let the length of the ftring EW be what it will, and the point E where it will between the vertical line WM and the fide CA, or their continuations. For then W will be kept at reft by its weight, the repreffure of the plane AC, and the tenfion of the ftring ENW; which (by art. 19.) will be to each other refpectively as MW, MN, and NW. The preffure of W, in a direction parallel to MN, will therefore be equal to $\frac{MN}{MW} \times W$: and (by art. 21.) the efficacy of that preffure, in a direction parallel to ON, will be expreffed by $\frac{NO}{MW} \times W$; which muft, in the cafe of an equilibrium, be equal to P, let the weight of the wedge be what it will. Confequently P will in that cafe be to W as $\frac{NO}{MW} \times W$ to W; that is, as NO to MW.

If the ftring ENW be parallel to AC, and AF be a perpendicular from A on BC; NW will be equal to AF

$\frac{AF}{AC} \times MW$, and NO equal to $\frac{AF \times CF}{\overline{AC}^2} \times MW$: and therefore, in this case, NO to MW (that is, P to W) as $AF \times CF$ to $\overline{AC}^2$. Mr. Ferguson (in his Lectures) erroneously asserts that, in this case, P will be to W as AF to AC; and he endeavours to support his assertion by experiments, but deduces therefrom (by not rightly considering the effect of friction in the different experiments) a conclusion most egregiously absurd, purporting that, in two cases of equilibriums, the powers will be equal, whereas in truth (their ratio being as CF to AC) the power in one case may be ten thousand times greater than in the other case!

When, supposing the power and weight (in any machine) to be moved, the ascent of one and descent of the other (estimated vertically) do not always retain an invariable ratio to each other, it will be wrong, in applying the theorem regarding the ratio of the velocities mentioned at the beginning of this memoir, to take the ratio of the contemporaneous ascent and descent at the end of any certain time whatever for the ratio of the velocities of the power and weight. For instance, in the case of our wedge and weights, whilst P descends in a vertical line, W will describe a circular arc WXY about the center E, and QX drawn from AC to the said arc being parallel to the horizon, and Wq perpendicular thereto, the descent of P will be to the ascent of W as QX to Wq, which being a variable ratio, no use can be made of it in ascertaining the ratio of the velocities of P and W, unless we consider the arc WX as indefinitely small. Now, considering that arc as an indefinitely small particle of the arc WXY, Wg a tangent to that arc at W, CG parallel to that tangent, and

AgG,

AgG parallel to BFCD, Wq will be to QX as AF to AG, that is, (in the case of an equilibrium) as P to W. But (the triangles ACG, MNW being similar) AF is to AG as NO to MW. It is evident therefore that the conclusion thus deduced agrees with that which is deduced above from the proper principles of equilibriums, whether EW be parallel to AC or not.

30. If, instead of the weight W being fastened to the string EW, it be perforated and moveable upon a string passing through the perforation and fastened at H and K, the rest being as before, when the power P descends the weight W will describe an ellipsis (whose foci are H and K); to which let Wg be a tangent at W, (that is, let Wg be perpendicular to the line bisecting the angle HWK,) and let CG be parallel to Wg: then, in the case of an equilibrium, P will be to W as AF to AG; as appears by properly applying the theorem before mentioned, regarding the velocities of P and W when moved: —and the same conclusion is deducible from the proper principles of equilibriums, the line bisecting the angle HWK being the direction of the force on the weight W arising from the tension of the string HWK, and the effect of that weight on the wedge the same as if it were fastened to the end of a string coinciding with, and having its other end fastened to a fixed point somewhere in that bisecting line.

MEMOIR II.

Of the Ellipsis and Hyperbola.

SOME of the theorems given by mathematicians for the calculation of fluents by means of elliptic and hyperbolic arcs requiring, in the application thereof, the difference to be taken between an arc of a hyperbola and its tangent; and such difference being not directly attainable when such arc and its tangent both become infinite, as they will do when the *whole* fluent is wanted, although such fluent be at the same time finite; those theorems therefore in that case fail, a computation thereby being then impracticable without some farther help.

The supplying that defect I considered as a point of some importance in geometry, and therefore I earnestly wished and endeavoured to accomplish that business; my aim being to ascertain, by means of such arcs as above mentioned, the *limit* of the difference between the hyperbolic arc and its tangent, whilst the point of contact is supposed to be carried to an infinite distance from the vertex of the curve, seeing that, by the help of that *limit*, the computation would be rendered practicable in the case wherein, without such help, the before-mentioned theorems fail. The result of my endeavours respecting that point

point appears in this Memoir: which, amongst other matters, contains the investigation of a general theorem for finding the length of any arc of any conic hyperbola by means of two elliptic arcs. A discovery (first published by me in the *Philos. Transact.* for 1775,) whereby we are enabled to bring out very elegant conclusions in many interesting enquiries, as well mechanical as purely geometrical.

Fig. 29. 1. Suppose the curve ADEF be a conic *hyperbola*, whose semi-transverse axis AC is $= m$, and semi-conjugate $= n$. Let CP, perpendicular to the tangent DP, be called p; and put $f = \frac{m^2 - n^2}{2m}$, $z = \frac{p^2}{m}$. Then will DP − AD be $=$ the fluent of $\frac{-\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, p and z being each $= m$ when AD is $= 0$. For, denoting the semidiameter CD by r, and its semi-conjugate diameter by s, we have (by the nature of the curve) $r^2 - s^2 = m^2 - n^2 = 2fm$, and $ps = mn$. Whence $s^2 = r^2 - 2fm = \frac{m^2n^2}{p^2} = \frac{mn^2}{z}$; and consequently $r^2 = \frac{2fmz + mn^2}{z}$, and DP $= \sqrt{r^2 - p^2} = \frac{\sqrt{mn^2 + 2fmz - mz^2}}{z^{\frac{1}{2}}}$. Hence the fluxion of DP is found

$$= -\frac{m^{\frac{1}{2}}n^2\dot{z} + m^{\frac{1}{2}}z^2\dot{z}}{2z^{\frac{3}{2}}\sqrt{n^2 + 2fz - z^2}}.$$

Now it is obvious that the fluxion of the curve AD is to $\dot{r}$ as r to $\sqrt{r^2 - p^2}$: therefore the fluxion of AD is $= \frac{r\dot{r}}{\sqrt{r^2 - p^2}}$, which by substitution appears to be $=$

— m

$\frac{-m^{\frac{1}{2}}n^2\dot{z}}{2z^{\frac{3}{2}}\sqrt{n^2+2fz-z^2}}$. Consequently the difference of the fluxions of DP and AD is $=\frac{-\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$.

2. Suppose the curve adefg to be a quadrant of an *ellipsis*, whose semi-transverse axis cg is $=\sqrt{m^2+n^2}$, and semi-conjugate ac $=n$. Let ct be perpendicular to the tangent dt, and let the abscissa cp be $=n\times\left(\frac{z}{m}\right)^{\frac{1}{2}}$. Then will the said tangent dt be $=m\times\left(\frac{mz-z^2}{n^2+mz}\right)^{\frac{1}{2}}$; and the fluxion thereof will be found $=\frac{1}{2}mn^2z^{-\frac{1}{2}}\dot{z}\times\frac{(m-z)^{\frac{1}{2}}}{(n^2+mz)^{\frac{3}{2}}}$ $-\frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$. Fig. 30.

3. In the expression $\frac{y^{q-1}\dot{y}}{(a+by)^r\times(c+dy)^s}$, let $\frac{c+dy}{a+by}$ be supposed $=z$. Then will $\frac{az-c}{d-bz}$ be $=y$, and the proposed expression will be $=\frac{(ad-bc)^{1-r-s}\times z^{-s}\dot{z}}{(az-c)^{1-q}\times(d-bz)^{1+q-r-s}}$.

4. Taking, in the last article, r and s each $=\frac{1}{2}$, $q=\frac{3}{2}$, $a=-d=\frac{n^2}{m}$, $b=1$, and $c=n^2$, we have

$$\frac{y^{\frac{1}{2}}\dot{y}}{\left(\frac{n^2}{m}+y\right)^{\frac{1}{2}}\times\left(n^2-\frac{n^2}{m}y\right)^{\frac{1}{2}}}\ \left(=\frac{m^{\frac{1}{2}}n^{-1}y^{\frac{1}{2}}\dot{y}}{\sqrt{n^2+2fy-y^2}}\right)=-mnz^{-\frac{1}{2}}\dot{z}$$

$\times\frac{(m-z)^{\frac{1}{2}}}{(n^2+mz)^{\frac{3}{2}}}$. It appears therefore, that, y being $=n^2\times$

$\frac{m - z}{n^2 + mz}$, $- \frac{\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $= \frac{1}{2} m n^2 z^{-\frac{1}{2}} \dot{z}$ $\times \frac{\overline{m - z}|^{\frac{1}{2}}}{\overline{n^2 + mz}|^{\frac{3}{2}}} - \frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$; which, by Art. 2. is $=$ the *fluxion of the tangent* dt.

Consequently, taking the fluents by Art. 1, and correcting them properly, we find

$$DP - AD + FR - AF = L + dt.$$

CP, in fig. 29, being $= m^{\frac{1}{2}} z^{\frac{1}{2}}$; cp, in fig. 30, $= n \times \frac{z}{m}|^{\frac{1}{2}}$; CR, perpendicular to the tangent FR, $= m^{\frac{1}{2}} y^{\frac{1}{2}}$;

$$DP - AD = \text{the } \textit{fluent} \text{ of } \frac{- \frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}};$$

$$FR - AF = \text{the } \textit{fluent} \text{ of } \frac{- \frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}};$$

and L the *limit* to which the difference DP − AD, or FR − AF, approaches upon carrying the point D, or F, from the vertex A *ad infinitum*.

5. Suppose y equal to z, and that the points D and F then coincide in E, the points d and p being at the same time in e and q respectively. Then cv being perpendicular to the tangent ev, that tangent will be a *maximum* and equal to cg − ac $= \sqrt{m^2 + n^2} - n$; the tangent EQ (in the hyperbola) will be $= \sqrt{m^2 + n^2}$; the abscissa BC $= m \times \sqrt{1 + \frac{n}{\sqrt{m^2 + n^2}}}$; the ordinate BE $= n \times \sqrt{\frac{n}{\sqrt{m^2 + n^2}}}$; and it appears, that L is $=$ 2EQ − 2AE − ev $= n + \sqrt{m^2 + n^2} - 2$AE. Thus the *limit* which I proposed to ascertain is investigated, m and n being any right lines what-

whatever. Another expreſſion for ſuch limit will be found in a ſubſequent article in this Memoir.

6. The *whole fluent* of $\frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$, generated whilſt z from o becomes $= m$, being equal to L; and the fluent of the ſame fluxion (ſuppoſing it to begin when z begins) being in general equal to L + AD − DP = FR − AF − dt; it appears that k being the value of z correſponding to the fluent L + AD − DP, $\frac{mn^2-n^2k}{n^2+mk}$ will be the value of z correſponding to the fluent L + AF — FR, and FR — AF will be the part generated whilſt z from $\frac{mn^2-n^2k}{n^2+mk}$ becomes $=m$. It follows therefore, that the *tangent* dt, together with the fluent of $\frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$ generated whilſt z from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion generated whilſt z from $\frac{mn^2-n^2k}{n^2+mk}$ becomes $=m$; cp being taken $= n \times \overline{\frac{k}{m}}|^{\frac{1}{2}}$.

Suppoſe $k = \frac{mn^2-n^2k}{n^2+mk}$; its value will then be $\frac{n}{m}\sqrt{m^2+n^2} - \frac{n^2}{m}$. Conſequently the fluent of $\frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2+2fz-z^2}}$ generated whilſt z from o becomes $= \frac{n}{m}\sqrt{m^2+n^2} - \frac{n^2}{m}$, together with the quantity $\sqrt{m^2+n^2} - n$, is equal to the fluent of the ſame fluxion generated whilſt z from $\frac{n}{m}\sqrt{m^2+n^2} - \frac{n^2}{m}$ becomes $= m$: and theſe two parts of the whole fluent

 being

being denoted by M and N respectively; M will be $= n - AE$, and $N = \sqrt{m^2 + n^2} - AE$.

7. The fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ being $= L + AD - DP$, the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} + DP - AD - L$ will be $= 0$. Therefore, the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ + the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ being $=$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \overline{\frac{n^2 + mz}{m - z}}\Big|^{\frac{1}{2}}$, it is obvious, that the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $= DP - AD - L +$ the fluent of $\frac{1}{2} z^{-\frac{1}{2}} \dot{z} \times \overline{\frac{n^2 + mz}{m - z}}\Big|^{\frac{1}{2}} = DP - AD - L$ + the *elliptic arc* dg (Fig. 30.) whose abscissa cp is $= n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$.

Consequently, putting E for $\frac{1}{4}$ of the periphery of that ellipsis, it appears that the *whole fluent* of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes $= m$, is equal to $E - L = E + 2AE - n - \sqrt{m^2 + n^2}$.

8. By taking, in Art. 3. q, r, and s, each $= \frac{1}{2}$; and $a = -d = \frac{n^2}{m}$, $b = 1$, and $c = n^2$; we find, that, if y be $= \frac{mn^2 - n^2 z}{n^2 + mz}$, $\frac{z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} + \frac{y^{-\frac{1}{2}} \dot{y}}{\sqrt{n^2 + 2fy - y^2}}$ will be $= 0$. It is obvious therefore, that the fluent of $\frac{z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, gene-

generated whilst z from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst z from $\frac{mn^2 - n^2k}{n^2 + mk}$ becomes $= m$.

Now, supposing $k = \frac{mn^2 - n^2k}{n^2 + mk}$, its value will be $\frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$. Consequently the fluent of $\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, generated whilst z from o becomes $= \frac{n}{m}\sqrt{m^2 + n^2} - \frac{n^2}{m}$, is equal to *half* the fluent of the same fluxion, generated whilst z from o becomes $= m$; which *half fluent* is known by the preceding article.

9. It appears by Art. 4. that $\frac{\frac{1}{2}m^{\frac{1}{2}}y^{\frac{1}{2}}\dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $= -$ the *fluxion of the tangent* dt; and it appears by the last article, that $\frac{\frac{1}{2}m^{-\frac{1}{2}}n^2y^{-\frac{1}{2}}\dot{y}}{\sqrt{n^2 + 2fy - y^2}} + \frac{\frac{1}{2}m^{-\frac{1}{2}}n^2z^{-\frac{1}{2}}\dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $= 0$; $mn^2 - n^2y - n^2z - myz$ being $= 0$. Therefore, by addition, we have $\frac{1}{2}y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{n^2 + my}{m - y}}\Big|^{\frac{1}{2}} + \frac{1}{2}z^{-\frac{1}{2}}\dot{z} \times \overline{\frac{n^2 + mz}{m - z}}\Big|^{\frac{1}{2}} = -$ the *fluxion of the tangent* dt. Consequently, by taking the correct fluents, we find the *tangent* dt (= the *tangent* fw) = the *arc* ad — the *arc* fg! the abscissa cp being $= n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$, the abscissa cr $= n \times \overline{\frac{y}{m}}\Big|^{\frac{1}{2}}$, and their relation expressed by the equation $n^6 - n^4u^2 - n^4v^2 - m^2u^2v^2 = 0$, u and v being put for $n \times \overline{\frac{z}{m}}\Big|^{\frac{1}{2}}$ and $n \times \overline{\frac{y}{m}}\Big|^{\frac{1}{2}}$ respectively. Moreover the tangents dt, fw, will each be $= \frac{m^2uv}{n^3}$, and $\text{ct} \times \text{cw} = \overline{\text{cv}}^2 = \text{ac} \times \text{cg}$.

If for the semi-transverse axis cg we substitute h instead of $\sqrt{m^2 + n^2}$, the relation of u to v will be expressed by the equation $n^6 - n^4u^2 - n^4v^2 - \overline{h^2 - n^2} \times u^2v^2 = 0$, and dt (= fw) will be $= \frac{h^2 - n^2}{n^3} \times uv$.

If u and v be respectively put for fr and dp, their relation will be expressed by the equation $h^6 - h^4u^2 - h^4v^2 + \overline{h^2 - n^2} \times u^2v^2 = 0$, and dt (= fw) will be $= \frac{h^2 - n^2}{h^3} \times uv$.

10. Suppose y equal to z; (that is, $v = u$;) and that the points d and f coincide in e, In which case the tangent dt will be a *maximum*, and = cg − ac. It appears then that the *arc* ae − the *arc* eg is = cg − ac. Consequently, putting E for the quadrantal arc ag, we find that

$$\text{the } arc \text{ ae is } = \frac{E + h - n}{2}!$$

$$\text{the } arc \text{ eg } = \frac{E - h + n}{2}!$$

There are, I am aware, some other parts of the arc ag whose lengths may be assigned by means of the whole length (ag) with right lines; but to investigate such other parts is not to my present purpose.

11. Taking m and n each = 1; that is, ac (= AC) = 1, and cg = $\sqrt{2}$; let the arc ag be then expressed by e: put c for *one fourth* of the periphery of the circle whose radius is 1; and let the *whole fluents* of $\frac{\frac{1}{2}z^{\frac{1}{2}}\dot{z}}{\sqrt{1 - z^2}}$ and $\frac{\frac{1}{2}z^{-\frac{1}{2}}\dot{z}}{\sqrt{1 - z^2}}$, generated whilst z from 0 becomes = 1, be denoted by F and G respectively. Then, by what is said above, F + G will be = e; and, by Part X. of my *Math. Lucubrat.* it appears

pears that F × G is $= \frac{1}{2}c$. From which equations we find $F = \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$, and $G = \frac{1}{2}e + \frac{1}{2}\sqrt{e^2 - 2c}$.

But m and n being each $= 1$, L is = F; therefore $1 + \sqrt{2} - 2AE$, the value of L from Art. 5, is, in this case, $= \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$. Consequently, in the equilateral hyperbola, the *arc* AE, whose abscissa BC is $= \sqrt{1 + \frac{1}{\sqrt{2}}}$, will be $= \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{4}e + \frac{1}{4}\sqrt{e^2 - 2c}$, by what is said in the article last mentioned. Hence the *rectification* of *that arc* may be effected by means of the *circle* and *ellipsis!*

12. By substituting $a - b$, $2 \times \overline{ab}|^{\frac{1}{2}}$, and $\overline{a-b}|^2 - t^2$, for m, n, and mz respectively, in Art. 1. it appears, that, if (in the hyperbola) the semi-transverse axis AC be $= a - b$, the semi-conjugate $= 2 \times \overline{ab}|^{\frac{1}{2}}$, and the perpendicular $CP = \overline{\overline{a-b}|^2 - t^2}|^{\frac{1}{2}}$; the difference (DP — AD) between the tangent DP and the arc AD will be equal to the fluent of $\overline{\frac{\overline{a-b}|^2 - t^2}{\overline{a+b}|^2 - t^2}}|^{\frac{1}{2}} \times \dot{t}$. Fig. 29.

13. It is well known, that, in any ellipsis whose semi-transverse axis is h, and semi-conjugate n, if x be the abscissa, measured from the center upon the transverse axis, and Q the arc between the conjugate axis and the ordinate corresponding to x, $\overline{\frac{b^2 - gx^2}{b^2 - x^2}}|^{\frac{1}{2}} \times \dot{x}$ will be $= \dot{Q}$, g being $= \frac{b^2 - n^2}{b^2}$.

Hence, by substituting $a + b$, $2 \times \overline{ab}|^{\frac{1}{2}}$, and $\frac{a+b}{a-b} \times t$,

for

Fig. 31. for h, n, and x respectively; it appears, that, in the ellipsis aed whose semi-transverse axis cd is $= a + b$, semi-conjugate ca $= 2 \times \overline{ab}|^{\frac{1}{2}}$, and abscissa cb (corresponding to the ordinate be) $= \frac{a+b}{a-b} \times t$, the arc ae (denoted by Q) will be equal to the fluent of $\frac{\overline{a+b}|^2 - t^2}{\overline{a-b}|^2 - t^2}\Big|^{\frac{1}{2}} \times \dot{t}$.

Fig. 32. 14. In the ellipsis *aed*, the semi-transverse axis *cd* being $= a$, the semi-conjugate $ca = b$, and the abscissa *cb* (corresponding to the ordinate *be*) $= x$; if *ep*, the tangent at *e*, intercepted by a perpendicular (*cp*) drawn thereto from the center *c*, be denoted by t; $gx \times \frac{a^2 - x^2}{a^2 - gx^2}\Big|^{\frac{1}{2}}$ (as is well known) will be $= t$, g being $= \frac{a^2 - b^2}{a^2}$.

Whence we have $x^2 = \frac{a^2 g + t^2}{2g} - \frac{\overline{\overline{a^2 - b^2}|^2 - 2 \times \overline{a^2 + b^2} \times t^2 + t^4}|^{\frac{1}{2}}}{2g}$.

From which equation, by taking the fluxions, we have

$$x\dot{x} = \frac{t\dot{t}}{2g} + \frac{\overline{a^2 + b^2} \times t\dot{t} - t^3\dot{t}}{2g\sqrt{\overline{a^2 - b^2}|^2 - 2 \times \overline{a^2 + b^2} \times t^2 + t^4}}$$

$$= \frac{t\dot{t}}{2g} + \frac{\overline{a^2 + b^2} \times t\dot{t} - t^3\dot{t}}{2g\sqrt{\overline{a-b}|^2 - t^2 \times \overline{a+b}|^2 - t^2}}.$$

But $\dot{R}$, the fluxion of the arc *ae*, being $= \frac{a^2 - gx^2}{a^2 - x^2}\Big|^{\frac{1}{2}} \times \dot{x}$, according to the preceding article, it follows that $\frac{gx\dot{x}}{t}$ is $= \dot{R}$. It is obvious therefore that

$\dot{R}$ is

$$R \text{ is } = \frac{\dot{t}}{2} + \frac{\overline{a^2+b^2} \times t\dot{t} - t^2\dot{t}}{2\sqrt{\overline{a-b}|^2 - t^2 \times \overline{a+b}|^2 - t^2}}.$$

$$= \frac{\dot{t}}{2} + \frac{\overline{\overline{a-b}|^2 - t^2} \times \dot{t}}{4\sqrt{\overline{a-b}|^2 - t^2 \times \overline{a+b}|^2 - t^2}} + \frac{\overline{\overline{a+b}|^2 - t^2} \times \dot{t}}{4\sqrt{\overline{a-b}|^2 - t^2 \times \overline{a+b}|^2 - t^2}}$$

$$\doteq \tfrac{1}{2}\dot{t} + \tfrac{1}{4} \times \left|\frac{\overline{a-b}|^2 - t^2}{\overline{a+b}|^2 - t^2}\right|^{\frac{1}{2}} \times \dot{t} + \tfrac{1}{4} \times \left|\frac{\overline{a+b}|^2 - t^2}{\overline{a-b}|^2 - t^2}\right|^{\frac{1}{2}} \times \dot{t}.$$

From whence, by taking the fluents ,according to Art. 12. and 13. we find $R = ae$ (Fig. 32.) $= \frac{1}{2}t + \frac{DP - AD}{4}$ (Fig. 29.) $+ \frac{Q}{4}$. Consequently the hyperbolic arc AD is $= DP + Q - 4R + 2t$.

Thus, beyond my expectation, I find, that the *hyperbola* may in general be rectified by means of two *ellipses!*

If $p'e'$, $p''e''$ be equal tangents to the ellipsis $ae'e''d$; the arc ae' (denoted by R) will (by Art. 9.) be equal to the arc de'' + the tangent $p''e''$, or $p'e'$ (denoted by t). Therefore, substituting for t its value found by this last equation, it appears that AD is $= DP + Q - 2R - 2 \times de''$ $= DP + Q + 2 \times e'e'' - 2E''$, and $DP - AD = 2E'' - 2 \times e'e'' - Q$; E'' being put for the quadrantal arc ad. It is observable that, when t is $= a - b$, e'' (by Art. 5.) coincides with e' and $e'e''$ is $= 0$; Q (by Art. 13.) = the quadrantal arc ad; and (by Art. 12) DP and AD both become infinite. Consequently writing E for that quadrantal arc (ad), and L for the *limit* of the difference $DP - AD$ whilst the point of contact (D) is supposed to be carried to an infinite distance from the vertex A of the hyperbola, we find $2E'' - E = L$. Fig. 32.

15. Exterminating a, b, and t, by means of the equations $a - b = m$, $2 \times \overline{ab}|^{\frac{1}{2}} = n$, and $\overline{a - b}|^2 - t^2 = mz$; and writing f for $\frac{m^2 - n^2}{2m}$ as in Art. 1. it appears that

(V) the fluent of $\frac{\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$ is $= 2 \times e'e''$ — de generated whilst z increases from 0. Moreover the fluent of $\frac{1}{2} m^{\frac{1}{2}} z^{-\frac{1}{2}} \dot{z} \times \overline{\frac{\frac{n^2}{m} + z}{m - z}}|^{\frac{1}{2}}$, or of its equal $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z} + \frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}}$, being = de, as observed in Art. 7. we find by subtraction

(W) the fluent of $\frac{\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z}}{\sqrt{n^2 + 2fz - z^2}} = 2 \times \overline{\text{de} - e'e''}$.

The semi-transverse axis of the ellipsis aed being now $= \sqrt{m^2 + n^2}$, the semi-conjugate $= n$; the abscissa cb $= \overline{\frac{m^2 + n^2}{m}}|^{\frac{1}{2}} \times \overline{m - z}|^{\frac{1}{2}}$, and the ordinate be $= n \times \overline{\frac{z}{m}}|^{\frac{1}{2}}$.

The semi-transverse axis (a) of the ellipsis $ae'e''d = \frac{\sqrt{m^2 + n^2}}{2} + \frac{m}{2}$; the semi-conjugate ($b$) $= \frac{\sqrt{m^2 + n^2}}{2} - \frac{m}{2}$; the tangents $e'p'$, $e''p''$, intercepted by perpendiculars (cp', cp'') drawn thereto from the center c, each $= m^{\frac{1}{2}} \times \overline{m - z}|^{\frac{1}{2}}$; and the abscissa ($cb'$, or cb'') on cd, corresponding to the point e', or e'', of the curve, is determined by the expression

$$\frac{\sqrt{\overline{m^2 + n^2}|^{\frac{1}{2}} + m - z \mp \sqrt{z^2 + \frac{n^2}{m} z}}}{2^{\frac{1}{2}} \times \overline{m^2 + n^2}|^{\frac{1}{4}}} \times a.$$

From what is done above, many other new theorems for the calculation of fluents are deducible: the most remarkable

markable of which, I intend to inſert in the Appendix to be annexed to theſe Memoirs.

16. Taking m and n each equal to 1, we find by the above theorem (marked W) the whole fluent of $\frac{\frac{1}{2}z^{-\frac{1}{2}}\dot{z}}{\sqrt{1-z^2}}$ $= 2 \times \overline{e-d}$; e denoting one fourth of the periphery of an ellipſis whoſe ſemi-tranſverſe and ſemi-conjugate axes are $\sqrt{2}$ and 1; and d one fourth of the periphery of another ellipſis whoſe ſemi-tranſverſe and ſemi-conjugate axes are $\frac{1}{\sqrt{2}} + \frac{1}{2}$ and $\frac{1}{\sqrt{2}} - \frac{1}{2}$. But, by Art. 11. the ſame fluent is found $= \frac{e}{2} + \frac{1}{2}\sqrt{e^2 - 2c}$, c denoting $\frac{1}{4}$ of the periphery of the circle whoſe radius is 1. Conſequently $2 \times \overline{e-d}$ being $= \frac{e}{2} + \frac{\sqrt{e^2 - 2c}}{2}$, we have, from that equation, $d = \frac{3}{4}e - \frac{1}{4}\sqrt{e^2 - 2c}$; or $e = \frac{3}{2}d + \frac{1}{2}\sqrt{d^2 - 2c}$. Thus it appears that the periphery of one certain ellipſis may be found by means of the periphery of another ellipſis and the periphery of a circle!

Before Mr. MACLAURIN publiſhed his excellent Treatiſe of Fluxions, ſome very eminent mathematicians imagined, that the *elaſtic curve* could not be conſtructed by the quadrature or rectification of the conic ſections. But that gentleman has ſhewn, in that treatiſe, that the ſaid curve may, in every caſe, be conſtructed by the rectification of the hyperbola and ellipſis; and he has obſerved, that, by the ſame means, we may conſtruct the curve along which, if a heavy body deſcended, it would recede equally in equal times from a given point. Which laſt mentioned

 curve

curve Mr. James Bernouilli constructed by the rectification of the elastic curve, and Mr. Leibnitz and Mr. John Bernouilli by the rectification of a geometrical curve of a higher kind than the conic sections. It is observable, that Mr. Maclaurin's method of construction, just now adverted to, though very elegant, is not without a defect. The difference between the hyperbolic arc and its tangent being necessary to be taken, the method (for the reason mentioned at the beginning of this Memoir) always fails when some principal point in the figure is to be determined; the said arc and its tangent then both becoming infinite, though their difference be at the same time finite. The contents of this Memoir, properly applied, will evince, that both the *elastic curve* and the *curve of equable recess from a given point* (with many others) may be constructed by the rectification of the ellipsis only, without failure in any point.

MEMOIR

MEMOIR III.

Of the Descent of a Body in a Circular Arc.

1. LET $lpqn$ be a semi-circle perpendicular to the horizon, whose highest point is l, lowest n, and and center m. Let ps, qt, parallel to the horizon, meet the diameter lmn in s and t; and let the radius lm (or mn) be denoted by r; the height ns by d; and the distance st by x. Then, putting h for ($16\frac{1}{12}$ feet) the space a heavy body, descending freely from rest, falls through in one second of time; and supposing a pendulum, or other heavy body, descending by its gravity from p, along the arc pqn, to have arrived at q; the fluxion of the time of descent will (as is well known) be $= \frac{\frac{1}{2} r h^{-\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\sqrt{2dr - d^2 - 2.\overline{r - d}.x - x^2}}$. Fig. 33.

The fluent whereof, or the time of descent from p to q, is, by Art. 15. of the preceding Memoir, $= \frac{2r}{h^{\frac{1}{2}} \times \overline{2r - d}} \times \overline{de - e'e''}$: m, in the theorem referred to in that Memoir, being taken $= d^{\frac{1}{2}}$, $n = \overline{2r - d}|^{\frac{1}{2}}$, $z = \frac{x}{d^{\frac{1}{2}}}$; and accordingly the axes $\sqrt{m^2 + n^2}$ and n equal to $\sqrt{2r}$ and $\sqrt{2r - d}$, and the axes $\frac{\sqrt{m^2 + n^2}}{2} + \frac{m}{2}$ and $\frac{\sqrt{m^2 + n^2}}{2} - \frac{m}{2}$ equal to $\frac{r}{2}|^{\frac{1}{2}} + \frac{d^{\frac{1}{2}}}{2}$

and

and $\left(\frac{r}{2}\right)^{\frac{1}{2}} - \frac{d^{\frac{1}{2}}}{2}$; cb (Fig. 31.) $= \left(\frac{2r}{d}\right)^{\frac{1}{2}} \times \overline{d-x}^{\frac{1}{2}}$, and $e'p'$, $e''p''$, (Fig. 32.) each $= \overline{d-x}^{\frac{1}{2}}$.

Hence it appears, that the *whole* time of descent from p to n is $= \frac{2r}{b^{\frac{1}{2}} \times \overline{2r-d}} \times \overline{E - E''}$; when, in Fig. 31. and 32. the semi-axes are taken according to the values of m and n, just now specified: E and E″ denoting the quadrantal arcs aed, *aed*, respectively.

2. If pqn be a quadrant; that is, if d be $= r$; the *whole* time of descent from p to n will be $= \frac{2}{b^{\frac{1}{2}}} \times \overline{E - E''}$, by what is found in the preceding article. Which time will also be $= \frac{1}{2b^{\frac{1}{2}}} \times \overline{E + \sqrt{E^2 - 2c}}$ $\left(= 1.31102877 \times \left(\frac{r}{b}\right)^{\frac{1}{2}}\right)$, c being $\frac{1}{4}$ of the periphery of the circle whose radius is r; as appears by writing rz for x in $\left(\frac{\frac{1}{2} r b^{-\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\sqrt{r^2 - x^2}}\right)$ the fluxion of the time of descent, and referring to Art. 11. of the preceding Memoir for the *whole* fluent of $\left(\frac{\frac{1}{2} r^{\frac{1}{2}} z^{-\frac{1}{2}} \dot{z}}{b^{\frac{1}{2}} \sqrt{1 - z^2}}\right)$ the resulting expression: E here denoting the quadrantal arc of the ellipsis, whose semi-transverse and semi-conjugate axes are $\overline{2r}^{\frac{1}{2}}$ and $r^{\frac{1}{2}}$; and E″ the quadrantal arc of another ellipsis, whose semi-transverse and semi-conjugate axes are $\left(\frac{r}{2}\right)^{\frac{1}{2}} + \frac{1}{2} r^{\frac{1}{2}}$ and $\left(\frac{r}{2}\right)^{\frac{1}{2}} - \frac{1}{2} r^{\frac{1}{2}}$.

3. The semi-axes of the ellipsis (Fig. 31.), of which E denotes the quadrantal arc, being $\sqrt{2r}$ and $\sqrt{2r - d}$; and

and the ſemi-axes of the ellipſis (Fig. 32.) of which E″ denotes the quadrantal arc, being $\left(\frac{r}{2}\right)^{\frac{1}{2}} + \frac{d^{\frac{1}{2}}}{2}$ and $\left(\frac{r}{2}\right)^{\frac{1}{2}} - \frac{d^{\frac{1}{2}}}{2}$: thoſe ellipſes become circles when d is $= 0$, and E and E″ are then quadrantal arcs of the circles whoſe radii are $\sqrt{2r}$ and $\left(\frac{r}{2}\right)^{\frac{1}{2}}$ reſpectively. Therefore, c being $\frac{1}{4}$ the periphery of the circle whoſe radius is r, the laſt mentioned quadrantal arcs are equal to $2 \times \frac{c}{\sqrt{2r}}$ and $\frac{c}{\sqrt{2r}}$ reſpectively. Conſequently, it appears by ſubſtitution, that the *limit* of $\left(\frac{2r}{b^{\frac{1}{2}} \times \sqrt{2r-d}} \times \overline{E - E''}\right)$ the *whole* time of deſcent from p to n, taking d leſs and leſs, is $= \frac{c}{\sqrt{2br}}$: which may be conſidered as the time of deſcent in a very ſmall arc; and $\frac{2^{\frac{1}{2}}c}{\sqrt{br}}$, or its equal $\left(\frac{2r}{b}\right)^{\frac{1}{2}} \times q$, may accordingly be conſidered as the time of vibration of a pendulum deſcribing ſuch ſmall arc in its deſcent and a ſimilar one in its aſcent, q being the quadrantal arc of the circle whoſe radius is 1.

4. Having joined lp, pt, make the angle lpv equal to the angle ltp, and draw rv parallel to the horizon, interſecting the circle in r, and the diameter lmn in v. Then the pendulum or other heavy body, deſcending by its gravity from p along the arc $pqrn$, will paſs over the arcs pq and rn exactly in equal times: and therefore, qt and rv coinciding when lt is equal to lp, it follows, that the time of deſcent from p to q will then be preciſely equal to *half* the time of deſcent from p to n.

For-

For m, n, and z being as specified in Art. 1. $\frac{mn^2 - n^2z}{n^2 + mz}$ is $= \frac{2dr - d^2 - 2rx + dx}{2r - d + x}$, which by our construction is $= sv$. Therefore, by Art. 8. of the preceding Memoir, the parts of the fluent of the fluxion in Art. 1. (of this Memoir) corresponding to the times of descent from p to q and from r to n are equal.

5. If pn be an arc of 120°, q and r will coincide in the point of the arc 90° above n; and, by the preceding article, the descending body will be just as long in passing over the quadrantal arc between q and n as in passing over the first 30° between p and q. Moreover, t and v then coinciding in the center m, it is obvious, that the vertical descent in the *first half* of the time (of the whole descent) will be equal to *half* the vertical descent in the *other half* of the time.

6. In general, the vertical descent in the *first half* of the time will be to the vertical descent in the *other half* of the time, as $\sqrt{2r - d}$ to $\sqrt{2r}$. It is therefore manifest, that, in the circle, the vertical descents corresponding to those two equal parts of the time (of the whole descent) cannot in any case be equal, as they always are in the cycloid.

MEMOIR IV.

Of the centrifugal Force of the Particles of a Body, arising from its rotation about a certain Axis passing through its center of Gravity.

1. LET p be a particle of matter firmly connected with the plane DOEFQG, in which the line OCQ is situated; and, pq being a perpendicular from p to the said plane, let the distance pq be denoted by u; also, the line ql being at right angles to OlCQ, let the distance pl be denoted by h. Then, the said plane with the particle p being made to revolve about OlCQ as an axis, with the angular velocity e, measured at the distance a from the said axis, the velocity of p will be $= \frac{he}{a}$, and its centrifugal force from l will (by a well known theorem) be $= \frac{he^2}{a^2} \times p$. Whence, by resolving that force into two others, one in the direction qp, and the other in a direction parallel to lq, it appears that, in consequence of the said centrifugal force of p, the point l of the plane DOEFQG will be urged in a direction at right angles to that plane by a force $= \frac{ue^2}{a^2} \times p$, let the distance lq be what it will. Fig. 34.

2. The

2. The particle p being connected with the plane DOEFQG as mentioned in the preceding article, and the diſtance Cl being denoted by v; if p be urged directly from the ſaid plane by a force $= fu \times p$, the efficacy of that force to turn the ſaid plane about the line HCI, therein drawn at right angles to OCQ, will (by the property of the lever) be equivalent to the force $\frac{fuv \times p}{g}$ acting on the ſaid line OCQ, at right angles to the ſaid plane, at the diſtance g from the point C.

Moreover it is obvious, that, *cæteris paribus*, the efficacy will be the ſame, let the diſtance of q from l be what it will.

Fig. 35. Let q coincide with l; and let Ck be a line in the plane Clp continued, (which plane will be at right angles to the plane DOEFQG); alſo, pk being at right angles to Ck, let thoſe lines pk and Ck be denoted by w and $x - k$ reſpectively, k and x reſpectively denoting the diſtances of the points C and k from ſome given point V in the line paſſing through thoſe two points. Then, the ſine and coſine of the angle kCO (to the radius 1) being reſpectively denoted by m and n, the force $\frac{fuv \times p}{g}$ will be

$$= \frac{f \times p}{g} \times \overline{mn \times \overline{w^2 - \overline{x - k}^2} - \overline{m^2 - n^2} \times w . \overline{x - k}}.$$

Conſequently, if each particle of any ſolid body, through which a line HCI and a plane DOEIFQGH may be conceived to paſs, be urged from that plane by a force expreſſed by $fu \times p$ as above; the force which, acting on the line OCQ at the diſtance g from C, would be equivalent to the efficacy of all the forces acting on the ſeveral particles of that body, to turn the ſame about the line

line HCI, will be obtained by computing the fum of all the forces $\frac{f\times p}{g}\times mn\times\overline{w^2-\overline{x-k}^2-\overline{m^2-n^2}\times w.\overline{x-k}}$ acting on the faid body.

The computation of fuch equivalent force will, in many cafes, be abridged by obferving, that, if pk be continued to p'' fo that kp'' be $= kp$, the efficacy of the force on the particle p'', to turn the body about the line HCI in oppofition to the force on the particle p, will be reprefented by the equivalent force $\frac{f\times p''}{g}\times \overline{mn\times\overline{x-k}^2-w^2-\overline{m^2-n^2}\times w.\overline{x-k}}$ acting on the line OCQ at the diftance g from C; and that therefore the efficacy of the two forces on p and p'', to turn the body about HCI, will be reprefented by the equivalent force $\frac{2f\times p}{g}\times mn\times\overline{w^2-\overline{x-k}^2}$ acting on the line OCQ, at right angles to the plane DOEIFQGH, at the diftance g from C.

3. The body being of any fuch fhape that the fection thereof hi, paffing through p and p'' at right angles to the line Ck, is a circle whofe center is k; and every other fection thereof, parallel to the faid fection hi, a circle whofe center is in the line paffing through C and k; the ordinates correfponding to the abfciffæ kp, kp'', in the faid circular fection hi, will each be parallel to that diameter (HCI) of the circular fection paffing through C about which the body will be urged to turn, C being the center of gravity of the body*: and each of thofe ordinates will be Fig. 36.

* In a fpheroid, cylinder, cone, or any other body conformable, in regard to fections, to this under confideration, (which is called *a folid of revolution*,

be $= \sqrt{y^2 - w^2}$, y being the radius of ſuch ſection. Therefore, writing $2x' \sqrt{y^2 - w^2} \times w'$ inſtead of p, it follows that $\frac{4Af}{g} \times mnx' \times \overline{\frac{y^4}{4} - \overline{x-k}^2 . y^2}$, the *whole* fluent of $\frac{4f\sqrt{y^2 - w^2}}{g} \times mnx' \times \overline{w^2 - \overline{x-k}^2} \times \dot{w}$, generated whilſt w ($= kp = kp''$) from o becomes equal to the radius y, (both x and y being conſidered as invariable,) will expreſs the value of (E) the force which, acting on the line OCQ at the diſtance g from C, would be equivalent to the force of all the particles in the ſaid ſection, whoſe thickneſs is denoted by the indefinitely ſmall quantity x'; the diſtance Ck being denoted by $x - k$, and A being put for (.78539) the area of a quadrant of a circle whoſe radius is 1.

Fig. 37. 4. In the ſpheroid whoſe proper axis is $2b$, and equatorial diameter $2r$; taking k equal to b, we have $y^2 = \frac{r^2}{b^2} \times \overline{2bx - x^2}$, and $\frac{y^4}{4} - \overline{x-k}^2 . y^2 = \frac{r^4}{4b^4} \times \overline{2bx - x^2}^2 - \frac{r^2}{b^2} \times \overline{x-b}^2 \times \overline{2bx - x^2} = \frac{r^2 \times \overline{r^2 + 4b^2}}{4b^4} \times \overline{4b^2x^2 - 4bx^3 + x^4} - r^2 \times \overline{2bx - x^2}$. Conſequently, the fluent of $\frac{r^2 \times \overline{r^2 + 4b^2}}{4b^4} \times \overline{4b^2x^2 - 4bx^3 + x^4} \times \dot{x} - r^2 \times \overline{2bx - x^2} \times \dot{x}$, generated

volution, becauſe it may be conceived to be generated by the revolution of ſome line about the proper axis Ck,) the ſum of the forces ariſing from the rotation about OCQ, to turn it about a diameter at right angles to HCI in the circular ſection whoſe center is C, it is obvious, will be $= 0$; it being manifeſt that the force of any one particle p, urging it to turn about ſuch diameter, will be counter-balanced by the force of another particle acting in the oppoſite direction.

whilſt

whilſt x from o becomes $= 2b$, being $= \frac{4br^2}{15} \times \overline{r^2 - b^2}$, we find $\frac{16Afbr^2}{15g} \times mn \times \overline{r^2 - b^2} = \frac{fmn}{5g} \times \overline{r^2 - b^2} \times S$ the the value of (E) the force which, acting at the diſtance g from C the center of the ſpheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the ſpheroid to turn it about a diameter of its equator, S being $(= \frac{16Abr^2}{3})$ the maſs or content of the ſpheroid.

5. In the half of the ſpheroid mentioned in the preceding article, (cut off at the equator,) taking k equal to $\frac{5b}{8}$, we have $\frac{y^4}{4} - \overline{x - k}^2 . y^2 = \frac{r^4}{4b^4} \times \overline{2bx - x^2}^2 - \frac{r^2}{b^2} \times \overline{x - \frac{5b}{8}}^2 \times \overline{2bx - x^2}$. Conſequently, the fluent of $\overline{\frac{y^4}{4} - \overline{x - k}^2 . y^2} \times \dot{x}$, generated whilſt x from o becomes $= b$, being $= \frac{br^2}{480} \times \overline{64r^2 - 19b^2}$, we obtain $\frac{fmn}{320g} \times \overline{64r^2 - 19b^2} \times M$ for the value of (E) the force which, acting at the diſtance g from the center of gravity of the hemiſpheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the hemiſpheroid to turn it about a diameter of the circular ſection in which the ſaid center is ſituated, M being the maſs or content of the hemiſpheroid.

6. In the parabolic conoid, the equation of whoſe generating curve is $px = y^2$; if the height be $= b$, and V be at the vertex, k will be $= \frac{2b}{3}$, and $\frac{y^4}{4} - \overline{x - k}\rvert^2 . y^2 = \frac{p^2x^2}{4} - px$

$\times x$

$\times \overline{x - \frac{2b}{3}}\big|^2$. Confequently, the fluent of $\overline{\frac{p^2x^2}{4} - px \times \overline{x - \frac{2b}{3}}\big|^2}$ $\times \dot{x}$, generated whilft x from o becomes $= b$, being $= \frac{pb^2}{36}$ $\times \overline{3p - b}$, the force E will be $= \frac{fmn}{18g} \times \overline{3pb - b^2} \times M =$ $\frac{fmn}{18g} \times \overline{3r^2 - b^2} \times M$, r being the radius of the bafe, and M $(= 2Apb^2)$ the mafs or content of the body.

7. In the folid confifting of two parabolic conoids joined together at their bafes; it the dimenfions of each conoid be denoted as in the preceding article, and k be taken $=$ o, y^2 will be $= p.\overline{b - x}$, and $\frac{y^4}{4} - \overline{x - k}\big|^2.y^2 = \frac{p^2.\overline{b-x}\big|^2}{4} - px^2.\overline{b - x}$. Confequently, twice the fluent of $\overline{\frac{p^2.\overline{b-x}\big|^2}{4} - px^2.\overline{b - x}} \times \dot{x}$, generated whilft x from o becomes $= b$, being $= \frac{pb^3}{6}$ $\times \overline{p - b}$, the force E will be $= \frac{fmn}{6g} \times \overline{r^2 - b^2} \times N$, the mafs of the double conoid being denoted by N.

8. In the cone, the radius of whofe bafe is $= r$, and perpendicular height $= b$; if V be at the vertex, k will be $= \frac{3b}{4}$; and, y being $= \frac{rx}{b}$, $\frac{y^4}{4} - \overline{x - k}\big|^2.y^2$ will be $= \frac{r^4x^4}{4b^4}$ $- \frac{r^2x^2}{b^2} \times \overline{x - \frac{3b}{4}}\big|^2$. Confequently, by taking the fluent of $\overline{\frac{r^4x^4}{4b^4} - \frac{r^2x^2}{b^2} \times \overline{x - \frac{3b}{4}}\big|^2} \times \dot{x}$, (generated whilft x from o becomes $= b$,) and multiplying it by $\frac{4Afmn}{g}$, (as in the preceding

ceding examples,) we find the force $E = \frac{Afmnr^2b}{20g} \times \overline{4r^2 - b^2}$ $= \frac{3fmn}{80g} \times \overline{4r^2 - b^2} \times M$, $\left(\frac{4Ar^2b}{3}\right)$ the content of the body being denoted by M.

9. In the ſolid conſiſting of two cones joined together at their baſes; if the dimenſions of each cone be denoted as in the preceding article, the force E, by proceeding as above, will be found $= \frac{fmn}{20g} \times \overline{3r^2 - 2b^2} \times N$, the content of the double cone being denoted by N.

10. In the cylinder whoſe length is b and diameter $2r$; taking k equal to $\frac{b}{2}$, we have $\frac{r^2}{4} - \overline{x - k}|^2 . y^2 = r^2 \times \overline{\frac{r^2}{4} - \overline{x - \frac{b}{2}}|^2}$, y being $= r$. Conſequently, the fluent of $\overline{\frac{r^2}{4} - \overline{x - \frac{b}{2}}|^2} \times \dot{x}$, generated whilſt x from o becomes $= b$, being $= \frac{r^2b}{4} - \frac{b^3}{12}$, we find, by multiplying that fluent by $\frac{4Afmnr^2}{g}$, the value of $E = \frac{Afmnr^2b}{3g} \times \overline{3r^2 - b^2} = \frac{fmn}{12g} \times \overline{3r^2 - b^2} \times M$, the content of the cylinder being denoted by M.

Theſe equivalent forces are diſtinguiſhed by the name of *motive* forces; the correſpondent *accelerative* forces are computed in the following manner.

11. The body being a ſpheroid whoſe center is C, and whoſe proper axis PN is $= 2b$ and equatorial diameter $AB = 2r$; let F be the accelerative force of a particle of the Fig. 37.

the body at the diſtance g from the axis about which the body is urged to turn, which axis is a diameter of its equator when the motive force is ſuch as we have conſidered above. Denote C*k* and *ki* by $x - k$ and y as above; and let the abſciſſa *kp*, and its correſpondent ordinate (parallel to the laſt mentioned axis) in the circle whoſe radius is *ki* be denoted by w and t reſpectively. Then, conſidering the body as urged to turn about that diameter of its equator which is at right angles to AB, the accelerative force of every particle in the ſaid ordinate will be $= \frac{\sqrt{w^2 + \overline{x-k}|^2}}{g} \times F$, and the motive force of all the particles in the ſame ordinate will be $= \frac{\sqrt{w^2 + \overline{x-k}|^2}}{g} \times Ftw'x' = \frac{\sqrt{w^2 + \overline{x-k}|^2}}{g} \times Fw'x' \sqrt{y^2 - w^2}$: to which (by the property of the lever) a motive force $= \frac{w^2 + \overline{x-k}|^2}{g^2} \times Fw'x' \sqrt{y^2 - w^2}$, acting at the diſtance g from the center C, at right angles to a ray therefrom, would be equivalent. Therefore, conſidering x and y as invariable, and w only as variable, $\frac{4Fx'}{g^2} \times$ the *whole* fluent of $\dot{w} \sqrt{y^2 - w^2} \times \overline{w^2 + \overline{x-k}|^2}$ will denote a force which, acting at the diſtance g from C, would be equivalent to the motive force of all the particles in the circular ſection *hi*, whoſe radius is *ki* and thickneſs the indefinitely ſmall quantity x'. Which fluent is $= A \times \overline{\frac{y^4}{4} + \overline{x-k}|^2 . y^2} = \frac{Ar^2}{b^2} \times \overline{2bx - x^2} \times \frac{r^2}{4b^2} \times \overline{2bx - x^2 + \overline{x-b}|^2}$ in our ſpheroid, k being taken equal to CN $= b$. Conſequently

quently $\frac{4Ar^2F}{b^2g^2}$ × the whole fluent of $\dot{x} \times \overline{2bx - x^2} \times \overline{\frac{r^2}{4b^2} \times \overline{2bx - x^2} + \overline{x - b}|^2}$, generated whilſt x from o becomes $= 2b$, will denote a motive force which, acting at the diſtance g from C, at right angles to a ray therefrom, would be equivalent to the whole motive force urging the ſpheroid to turn as above mentioned. Such equivalent force will therefore be $= \frac{F}{5g^2} \times \overline{r^2 + b^2} \times S$: and this being put $= \frac{fmn}{5g} \times \overline{r^2 - b^2} \times S$, (the value of the ſame force found in Art. 4.) we find $F = fgmn \times \frac{r^2 - b^2}{r^2 + b^2}$; which will be $= \frac{gmne^2}{a^2} \times \frac{r^2 - b^2}{r^2 + b^2}$, if f be taken $= \frac{e^2}{a^2}$, agreeably to the computation in Art. 1. and 2.

Or F will be denoted by $\frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$; if 1 be to e as m to d, and as n to c; a and g being each $= r$.

12. It is evident from what is done in the preceding article, that the circle hi, whoſe radius is ki $(= y)$, being the ſection of any ſolid of revolution whoſe proper axis coincides with Ck, if C be the center of gravity of the body, and Ck be $= x - k$, $\frac{4AF\dot{x}}{g^2} \times \overline{\frac{y^4}{4} + \overline{x - k}|^2 . y^2}$ will denote a force which, acting at the diſtance g from C, as above deſcribed, would be equivalent to the motive force of all the particles in the ſaid circular ſection, whoſe radius is y and thickneſs the indefinitely ſmall quantity $\dot{x}$. It follows then, from Art. 4. and what is here ſaid, that $\frac{4AF}{g^2}$ ×

 the

the fluent of $\dot{x} \times \overline{\frac{y^4}{4} + \overline{x-k}|^2 . y^2}$ will be $= \frac{4Afmn}{g} \times$ the fluent of $\dot{x} \times \overline{\frac{y^4}{4} - \overline{x-k}|^2 . y^2}$: whence we have the theorem

$$F = fgmn \times \frac{\text{the fluent of } \dot{x} \times \overline{\frac{y^4}{4} - \overline{x-k}|^2 . y^2}}{\text{the fluent of } \dot{x} \times \overline{\frac{y^4}{4} + \overline{x-k}|^2 . y^2}};$$

the value of k, and the said fluents, being so taken as to comprehend the whole of the body under consideration, according to the examples given above.

13. Computing by the theorem just now investigated, it appears that,

in the hemi-spheroid, F will be $= \frac{cd}{r} \times \frac{64r^2 - 19b^2}{64r^2 + 19b^2}$;

in the parabolic conoid, F $= \frac{cd}{r} \times \frac{3r^2 - b^2}{3r^2 + b^2}$;

in the double conoid, F $= \frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$;

in the cone, F $= \frac{cd}{r} \times \frac{4r^2 - b^2}{4r^2 + b^2}$;

in the double cone, F $= \frac{cd}{r} \times \frac{3r^2 - 2b^2}{3r^2 + 2b^2}$;

in the cylinder, F $= \frac{cd}{r} \times \frac{3r^2 - b^2}{3r^2 + b^2}$;

a and g being each taken $= r$, $f = \frac{e^2}{r^2}$; 1 to e as m to d, and as n to c: and reference being had to the respective articles above for the quantities denoted by e, m, n, b, and r.

14. Seeing that the force which we have computed above will be = 0, when,

in

in the hemi-ſpheroid, the height (b) is $= \frac{8r}{\sqrt{19}}$;

in the parabolic conoid, the height (b) is $= 3^{\frac{1}{2}} r$;

in the double conoid, the half length (b) is $= r$;

in the cone, the height (b) is $= 2r$;

in the double cone, the half length (b) is $= \overline{\frac{3}{2}}\big|^{\frac{1}{2}} r$;

in the cylinder, the length (b) is $= 3^{\frac{1}{2}} r$;

it is manifeſt, that each of thoſe bodies will (with reſpect to its own particles) undiſturbedly revolve about any axis whatever paſſing through its center of gravity, as will a ſphere! And it is obvious that, by means of what is done above, other bodies (being ſolids of revolution, or fruſtums of ſuch ſolids) of various forms, may be found having the like property! No more being requiſite thereto than that the dimenſions of the body be ſo proportioned that the fluent of $\dot{x} \times \overline{\frac{y^2}{4} - \overline{x-k}|^2} \cdot y^2$ be $= 0$; k being taken equal to the diſtance from the center of gravity of the body to one end of its proper axis, and x being conſidered as flowing from 0, till it becomes equal to the whole length of that axis.

15. Any axis about which a body will ſo undiſturbedly revolve, I call *a permanent axis of rotation.*

SECOND PART.

Having hitherto only ſhewn how to compute the values of E and F when the body is a ſolid of revolution, I purpoſe now to ſhew how thoſe values may be computed when the body is not ſuch a ſolid; and likewiſe how the forces E″ and F″, at right angles to the direction of the forces E and F, may be computed: which forces E″ and F″, though always = o in a solid of revolution, are not generally = o in other bodies.

Fig. 34. 16. By resolving the force $\frac{be^2}{a^2} \times p$, mentioned in Art. 1. into two others, as specified in that article, we found that, in consequence of the centrifugal force of the particle p, (revolving as there described,) the point l of the plane DOEFQG will be urged in a direction at right angles to that plane by a force $= \frac{ue^2}{a^2} \times p$, let the distance lq be what it will. Now it is farther observable, that, at the same time, the same point (l) will, in consequence of the same centrifugal force, be urged in the direction lq by a force $= \frac{u''e^2}{a^2} \times p$, let the distance pq (denoted by u) be what it will, u'' being put for $\sqrt{h^2 - u^2} = lq$. And it follows, from the property of the lever, that, Cl being denoted by v (as before), if the same point l be urged in the said direction lq by a force $= fu'' \times p$, the efficacy of that force, to turn the said plane edgeways about an axis passing through the point C at right angles to HCI, will be equivalent to the force $\frac{fu''v}{g} \times p$ acting on the line OCQ, in a direction parallel to lq, at the distance g from the point C.

17. Ck being a line in the plane OlCQkp'' at right angles to the plane DOEFQG, let the plane $k\beta\gamma\delta\gamma''\delta''$, or a section of the body at right angles to Ck, be conceived to pass through the

the points $k, \beta, p, p'', \gamma, \delta, \gamma'', \delta''$; and let $\gamma\beta p\gamma''$, $\delta k\beta\delta''$ (drawn in the last mentioned plane) be at right angles to each other. Call the abscissa $\delta\beta$ and its correspondent ordinates $\beta\gamma$, $\beta\gamma''$; x, b', b'', respectively: and call the distances $k\delta$, $k\delta''$, and βp; a', a'', and y respectively. Also, VCV″ being the other dimension of the body, measured upon the line VCkV″, let the distances CV, CV″, and Vk be called k', k'', and z respectively. Moreover, the the sine and cosine of the angle kCl (to the radius 1) being respectively denoted by m and n (as before), let the sine and cosine of the angle $\beta k p''$ (to the same radius) be denoted by s and t respectively; pq, $p''l$ being each perpendicular to the plane DOEFQG, in which the points l and q are situated; and pp'' parallel to lq. Then will

$$v \text{ be} = mt.\overline{x - a'} + ms.\overline{y - b'} + n.\overline{z - k'},$$
$$u \quad = nt.\overline{x - a'} + ns.\overline{y - b'} - m.\overline{z - k'},$$
$$u'' \quad = \quad t.\overline{y - b'} - s.\overline{x - a'}.$$

It appears therefore, by substitution, that, E and E″ being respectively put for the values of the motive forces which, acting at the distance g from C on the line OCQ, in directions parallel to lp'', lq respectively, would be equivalent to the force of all the particles of the body, to turn it about a line passing through the point C in a plane at right angles to OCQ;

$$\text{E } (= \frac{f}{g} \times \text{the sum of all the } uv \times p) \text{ will be}$$
$$= \frac{f}{g} \times mn \times \overline{\text{A}''t^2 + \text{B}''s^2 + 2\text{A}st - \text{K}''} + \overline{n^2 - m^2} \times \overline{\text{B}s + \text{K}t},$$
$$\text{E}'' (= \frac{f}{g} \times \text{the sum of all the } u''v \times p)$$
$$= \frac{f}{g} \times m \times \overline{\text{A}.\overline{t^2 - s^2} + \text{D}'st} + n \times \overline{\text{B}t - \text{K}s};$$

A being

A being = the ſum of all the $p \times \overline{x-a'}.\overline{y-b'}$,
B = the ſum of all the $p \times \overline{y-b'}.\overline{z-k'}$,
K = the ſum of all the $p \times \overline{x-a'}.\overline{z-k'}$,
A″ = the ſum of all the $p \times \overline{x-a'}|^2$,
B″ = the ſum of all the $p \times \overline{y-b'}|^2$,
K″ = the ſum of all the $p \times \overline{z-k'}|^2$,
and D′ = B″ − A″.

Which ſums may commonly be computed in the following manner.

1ſt. To find the value of A; take the fluent of $\overline{x-a'}.\overline{y-b}.\dot{y}$, generated whilſt y from o becomes $= b' + b''$, confidering y only as variable: which fluent is $= \frac{1}{2}.\overline{x-a'}.\overline{b'^2 - b''^2}$.

2dly. Having ſubſtituted, in that fluent, the values of b' and b'' in terms of x; multiply by $\dot{x}$ and take the fluent, generated whilſt x from o becomes $= a' + a''$, conſidering x only as variable.

3dly. Having ſubſtituted, in the ſecond fluent, the values of a' and a'' in terms of z; multiply by $\dot{z}$ and take the fluent, generated whilſt z from o becomes $= k' + k''$, conſidering z only as variable.

Then will ſuch third fluent expreſs the ſum of all the $p \times \overline{x-a'}.\overline{y-b'}$: and in the ſame manner the values of B, K, A″, B″, K″ are to be computed.

18. It is obvious that, if b' be every where $= b''$, $(\frac{1}{2}.\overline{b'^2 - b''^2})$ the whole fluent of $\overline{y-b'}.\dot{y}$, generated whilſt y from o becomes $= b' + b''$, will be = o; therefore A and B will then be each = o.

19. If

19. If the line joining the point k and the center of gravity of the respective section $k\beta\gamma\delta\gamma''\delta''$ be always at right angles to the line $\delta k\beta\delta''$; the whole fluent of $\dot{x}.\overline{x - a'}.\overline{b' + b''}$, generated whilst x from o becomes $= a' + a'$, will, by the property of such center, be $= 0$: therefore K will, in such case, be $= 0$, whether b' be $= b''$ or not.

20. When all the ordinates $\gamma\beta\gamma'$ can be bisected by a right line (not coincident with $\delta\beta\delta''$), $b' - b''$ will be $= q' + q''.\overline{x - a'}$, q' and q'' being some invariable quantities: therefore, $\frac{1}{2}.\overline{b'^2 - b''^2}$ (the whole fluent of $\overline{y - b'}.\dot{y}$) being $= \frac{1}{2}q'.\overline{b' + b''} + \frac{1}{2}q''.\overline{x - a'}.\overline{b' + b''}$; B, by what is said in the preceding article, will be $= 0$, if the point k be situated as described in that article and q' be $= 0$.

If q'' be $= 0$, (i. e. if the bisecting line be parallel to $\delta k\beta\delta''$), and k be so situated; A and K will each be $= 0$.

Such remarks as these in the last three articles serve to abridge the expressions for the values of E and E''.

21. If A, B, and K be each $= 0$, and $A'' = B'' = K''$; $A''t^2 + B''s^2$ will be $= A''$; and both E and E'' will vanish, let m and s be what they will: therefore, in that case, any line whatever passing through C, the center of gravity of the body, will be a permanent axis of rotation.

22. If, A, B, and K being each $= 0$, A'' be $=$ B''; E'' will vanish, let m and s be what they will; and E will vanish when m is $= 1$: therefore, in such case, any line whatever passing through C, in a plane at right angles to Ck, will be a permanent axis of rotation. Moreover E being

being alſo made to vaniſh by taking $m = 0$, Ck will alſo be a permanent axis of rotation.

23. If, A, B, and K being each $= 0$, A″ be $=$ K″ and $s = 0$; E and E″ will both vaniſh, let m be what it will: therefore, in ſuch caſe, any line whatever paſſing through C, in the plane $\delta k \beta \delta''$C, will be a permanent axis of rotation. And, as E and E″ will alſo both vaniſh when m and s are each $= 1$; the body will alſo have a permanent axis of rotation (paſſing through C) perpendicular to the ſaid plane $\delta k \beta \delta''$C, in which the other ſuch axes will be ſituated.

24. If, A, B, and K being each $= 0$, B″ be $=$ K″ and $s = 1$; E and E″ will both vaniſh, let m be what it will: therefore, in ſuch caſe, any line whatever paſſing through C, in a plane Ck at right angles to the plane $\delta k \beta \delta''$C, will be a permanent axis of rotation. And, as E and E″ will alſo both vaniſh when m is $= 1$ and $s = 0$, the body will alſo have a permanent axis of rotation (paſſing through C) parallel to $\delta k \beta \delta''$.

25. If, B and K being each $= 0$, A″ be $=$ B″, A″ $-$ K″ $\pm$ A $= 0$, and $s^2 = t^2$; E and E″ will both vaniſh let m be what it will: therefore any line whatever paſſing through C, in a certain plane, will then be a permanent axis of rotation. Which plane will paſs through C and k, and make an angle of 45° with the plane $\delta k \beta \delta''$C, ſo that s ſhall be $= \sqrt{\frac{1}{2}}$ or $= -\sqrt{\frac{1}{2}}$, according as the upper or lower of the two ſigns prefixed to the term A takes place. And, as E and E″ will alſo both vaniſh when m is $= 1$ and

$s^2 =$

$s^2 = t^2$, the body will alſo have a permanent axis of rotation (paſſing through C) perpendicular to the plane in which the other ſuch axes will be ſituated.

26. If B and K be each = o, E and E″ will both vaniſh if m be = o: and they will alſo both vaniſh if m be = 1, and $D'st - A.\overline{s^2 - t^2} = 0$. Now $\frac{s}{t}$ being, by this laſt equation, $= \frac{D'}{2A} + \frac{\sqrt{D'^2 + 4A^2}}{2A}$, or $= \frac{D'}{2A} - \frac{\sqrt{D'^2 + 4A^2}}{2A}$; (which can neither become equal, nor imaginary values;) and the ſum of the two arcs whoſe tangents are thoſe values of (the tangent) $\frac{s}{t}$, being (as may be eaſily proved) = 90°; it is evident that the body will, at leaſt, have *three* permanent axes of rotation; whereof Ck will be one, and the other two will each paſs through C and be perpendicular to Ck and to each other.

Moreover, B and K being each = o, E and E′ will both vaniſh when $A''t^2 + B''s^2 - K'' + 2Ast$ is = o, and $D'st + A.\overline{t^2 - s^2} = 0$ at the ſame time: in which caſe m may expreſs the ſine of any angle whatever. Therefore, at firſt ſight it ſeems poſſible, that, when A, A″, B″, and K″ have a certain relation amongſt themſelves, the two values of $\frac{s}{t}$ in the one of thoſe equations may correſpond to two values of $\frac{s}{t}$ in the other of thoſe equations; and that, in ſuch caſe, the body may have an infinite number of permanent axes of rotation (paſſing through C) in *two* different planes. But, by exterminating $\frac{s}{t}$ by means of the two laſt equations, it appears that, for thoſe to be true equations,

equations, D'' muſt be $= D'''$, or $A^2 = D''D'''$; D'' being put for $K' - A'$, and D''' for $K'' - B''$. If D'' be $= D'''$, A' will be $= B'$, and $D' \pm A = 0$; and our caſe the ſame as that conſidered in the preceding article. If A^2 be $= D'D'''$; $\frac{s}{t}$, in the equation $D'st + A.\overline{t^2 - s^2} = 0$, will be $= -\frac{D''}{A}$, or $= \frac{D'''}{A}$: at the ſame time $\frac{s}{t}$ in the other equation $(A't^2 + B's^2 - K' + 2Ast = 0)$ will have only one value $= -\frac{A}{D'''}$. Conſequently, $-\frac{A}{D'''}$ being $= -\frac{D''}{A}$, but not $= \frac{D'''}{A}$; the values of E and E′ can only vaniſh together when, m being $= 0$, $\frac{s}{t}$ is of any value whatever; or, when $\frac{m}{n}$ being of any value whatever, $\frac{s}{t}$ is $= -\frac{D''}{A}$; or, m being $= 1$, $\frac{s}{t}$ is $= \frac{D'''}{A}$. Therefore it is evident, that, beſides the ſingle permanent axis of rotation determined by the equations $m = 1$ and $\frac{s}{t} = \frac{D'''}{A}$, the body (in the caſe here conſidered) can only have an infinite number of ſuch axes in *one* certain plane determined by the equation $\frac{s}{t} = -\frac{D''}{A} \left(= -\frac{A}{D'''}\right)$: to which laſt mentioned plane, the ſaid ſingle axis will be perpendicular; the ſum of the two arcs whoſe tangents are $\frac{D''}{A}$ and $\frac{D'''}{A}$ being 90°.

27. Suppoſing the general values of E and E′ each $= 0$, and exterminating m and n by means of thoſe two equations, we get

$$P\frac{s^3}{t^3} + Q\frac{s^2}{t^2} + R\frac{s}{t} + S = 0;$$

P being $= A^2B + AD'''K - BK^2$,
Q $= 2B^2K - A^2K - K^3 - D'D'''K - AB.\overline{D' + D'}$,
R $= 2BK^2 - A^2B - B^3 + BD'D' + AK.\overline{D' - D'''}$,
S $= A^2K + ABD' - B^2K$.

Now, by our equation so found, it appears that $\frac{s}{t}$ will, at least, have *one* real value; and consequently that $\frac{m}{n}$, which according to our supposition is $= \frac{Bt - Ks}{A\,\overline{s^2 - t^2} - D'st}$, will also, at least, have *one* real value. Therefore, in enquiring concerning the number of permanent axes of rotation in any body, we may suppose OCQ to coincide with C*k*; and then, that the values of E and E′ may each vanish upon taking $m = 0$, our expressions for those values will become

$$E = \frac{f}{g} \times mn \times \overline{A't^2 + B's^2 - K' + 2Ast},$$
$$E' = \frac{f}{g} \times m \times \overline{A.\overline{t^2 - s^2} + D'st};$$

B and K being necessarily each $= 0$.

28. Consequently, considering C*k* as a permanent axis of rotation, it appears, by art. 26. that any body whatever will, at least, have *three* such axes, situated as described in that article:

And, by art. 22. that, if A and D′ be each $= 0$, any line passing through C, in a plane at right angles to C*k*, will be a permanent axis of rotation:

 Also,

Alſo, by art. 23. 24. 25. and 26. that, if A and D' be each $= 0$; or A and D''' each $= 0$; or D' and $D' \pm A$ each $= 0$; or $A^2 - D''D''' = 0$; any line paſſing through C, in a certain plane Ckl, (determined as ſhewn above,) will be a permanent axis of rotation; the body, at the ſame time, having ſuch an axis (paſſing through C) perpendicular to the ſaid plane:

Moreover, by art. 21. that, if A, D', and D' be each $= 0$; any line whatever paſſing through the center of gravity (C) will be a permanent axis of rotation.

29. The accelerative forces F and F', correſponding to the motive forces E and E', will, after what has been ſaid, be readily found: it being now obvious enough, that

$$\text{E will be} = \frac{F}{g^2} \times \text{the ſum of all the } p \times \overline{v^2 + u^2}$$

$$= \frac{F}{g^2} \times \overline{A't^2 + B's^2 + K' + 2Ast}$$

$$= \frac{f}{g} \times \overline{mn.\overline{A''t^2 + B''s^2 - K'} + 2Ast + \overline{n^2 - m^2}.\overline{Bs + Kt}},$$

$$E'' \ldots\ldots = \frac{F''}{g^2} \times \text{the ſum of all the } p \times \overline{v^2 + u''^2}$$

$$= \frac{F''}{g^2} \times \overline{m^2.\overline{A'' + B''} + n^2.\overline{A''s^2 + B''t^2 + K''} - 2Ast + 2mn.\overline{Bs + Kt}}$$

$$= \frac{f}{g} \times \overline{m.D'st + A.\overline{t^2 - s^2} + n.\overline{Bt - Ks}}.$$

Whence the values of F and F'' may be immediately obtained.

30. Two or three examples will, I preſume, ſufficiently explain the method of computing by the theorems above inveſtigated.

Example

Example 1. Let the body be *a parallelopipedon*, whose dimensions are a, b, and k. Let the section $k\beta\gamma\delta\gamma''\delta''$ be parallel to one face thereof, whose length is a and breadth b: and, Ck being conceived to pass through the middle point (or center of gravity) of the said section $k\beta\gamma\delta\gamma''\delta''$, and of every other section parallel thereto; let the line $\delta k\beta\delta''$ divide the section wherein it is drawn (which will be a parallelogram) into two equal parallelograms, so that the length and breadth of each shall be a and $\frac{b}{2}$.

Then, by our remarks above, A, B, and K will each be $= 0$; and consequently

$$E = \frac{f}{g} \times mn.\overline{A''t^2 + B''s^2 - K''},\quad E'' = \frac{f}{g} \times mD'st;$$

where A″ will be $= \frac{a^3bk}{12}$*, $B'' = \frac{ab^3k}{12}$, and $K'' = \frac{abk^3}{12}$.

If a be $= b$, A″ will be $=$ B″, $D' = 0$, $E = \frac{f}{g} \times mn.\overline{A'' - K''}$, and $E'' = 0$. In which case, (that is when the body is a square prism,) any line passing through the center of gravity of the body, in a plane to which the permanent axis of rotation Ck is perpendicular, will be such an axis.

If a be $= b = k$, E and E″ will each be $= 0$, let m and s be what they will. It appears therefore, that, in *a cube*, (as in a sphere,) any line whatever passing through the center

* The value of A″ is found in the following manner.

1st. Take the fluent of $\dot{y}.\overline{x - a'}^2$: which fluent is $= 2b'.\overline{x - a'}^2$.

2dly. Take the fluent of $2b'\dot{x}.\overline{x - a'}^2$: which fluent is $= \frac{4a'^3b'}{3} = \frac{a^3b}{12}$; a' being $= \frac{1}{2}a$, $b' = \frac{1}{2}b$.

3dly. Take the fluent of $\frac{a^3b\dot{z}}{12}$: which fluent is $= \frac{a^3bk}{12} = A''$.

In like manner the values of B″ and K″ are found.

center of gravity of the body will be a permanent axis of rotation.

Example 2. Let the body be *a triangular prifm,* whofe ends are ifofceles triangles; the bafe of each of which triangles being $= b$, the perpendicular thereon from the angle made by the two equal fides (in each) $= k$, and the length of the body $= a$.

Then, conceiving the fection $k\beta\gamma\delta\gamma''\delta''$ (whereof k is the middle point) to be parallel to that fide of the prifm whofe length is a and breadth b, and fuppofing the line $\delta k\beta\delta''$ to divide fuch fection (which will be a parallelogram) lengthwife into two equal parallelograms; A, B, and K will each be $= 0$; and confequently $E = \frac{f}{g} \times mn.\overline{A''t^2 + B''s^2 - K''}$, and $E'' = \frac{f}{g} \times mD'st$; as in the preceding example: but now A'' will be $= \frac{a^3bk}{24}$ *, $B'' = \frac{ab^3k}{48}$, and $K'' = \frac{abk^3}{36}$.

If b be $= 2^{\frac{1}{2}}a$, any line paffing through the center of gravity of the body, in a plane to which the permanent axis of rotation Ck is perpendicular, will be fuch an axis.

* The value of A'' is computed in the following manner.

1ft. Take the fluent of $\dot{y}.\overline{x - a'}|^2$: which fluent is $= 2b'.\overline{x - a'}|^2$.

2dly. Take the fluent of $2b'\dot{x}.\overline{x - a'}|^2$: which fluent is $= \frac{4a'^3b'}{3} = \frac{a^3bz}{12k}$;

a' being $= \frac{1}{2}a$, $b' = \frac{bz}{2k}$.

3dly. Take the fluent of $\frac{a^3bz\dot{z}}{12k}$: which fluent is $= \frac{a^3bk}{24} = A''$.

In like manner the values of B'' and K'' are computed.

If

If k be $= \left(\frac{3}{2}\right)^{\frac{1}{2}} a$; any line passing through the center of gravity of the body, in a plane bisecting the angle made by the two equal sides of the triangle at each end of the prism, will be a permanent axis of rotation; and the body will also have such an axis (passing through its center of gravity) perpendicular to the said bisecting plane.

If k be $= \frac{3^{\frac{1}{2}} b}{2}$; that is, if the ends of the prism be equilateral triangles; any line passing through the center of gravity of the body, in a plane parallel to the planes of the said triangles, will be a permanent axis of rotation: and the body (as is very obvious) will also have such an axis passing through the center of gravity of the triangle at each end of the prism.

If b be $= 2^{\frac{1}{2}} a$ and $k = \left(\frac{3}{2}\right)^{\frac{1}{2}} a$, E and E″ will each be $= 0$, let m and s be what they will: therefore, in such a prism, (whose ends will be equilateral triangles,) any line whatever passing through the center of gravity of the body, will be a permanent axis of rotation.

Example 3. Let the body be *a pyramid*; whose base is a parallelogram, the length and breadth whereof are a and b; and the perpendicular height of the body $= k$.

Then, conceiving the section $k\beta\gamma\delta\gamma''\delta''$ (whereof k is the middle point) to be parallel to the base, and supposing the line $\delta k\beta\delta''$ to divide such section (which will be a parallelogram) lengthwise into two equal parallelograms; A, B, and K will each be $= 0$; and the values of E and E″ will be expressed as in the preceding examples:

amples: but here A″ will be $= \frac{a^3bk}{60}$*, $B'' = \frac{ab^3k}{60}$, and K″ $= \frac{abk^3}{80}$.

If k be $= \frac{2a}{\sqrt{3}}$; any line paſſing through the center of gravity of the body, in a plane paſſing through the vertex of the pyramid and biſecting its baſe lengthwiſe, will be a permanent axis of rotation: and the body will alſo have ſuch an axis (paſſing through its center of gravity) perpendicular to the ſaid biſecting plane.

If a be $= b$; that is, if the body be a ſquare pyramid of any height whatever; any line paſſing through the center of gravity of the body, in a plane to which the permanent axis of rotation Ck is perpendicular, will be ſuch an axis.

If b be $= a$, and $k = \frac{2a}{\sqrt{3}}$, Ė and E″ will each be $= 0$, let m and s be what they will: therefore it appears, that, in a ſquare pyramid whoſe height is to the ſide of its baſe as 2 to $\sqrt{3}$, any line whatever paſſing through the center of gravity of the body will be a permanent axis of rotation.

* The value of A″ is computed as follows.

1ſt. Take the fluent of $\dot{y}.\overline{x-a'}|^2$: which fluent is $= 2b'.\overline{x-a'}|^2$.

2dly. Take the fluent of $2b'\dot{x}.\overline{x-a'}|^2$: which fluent is $= \frac{4a'^3b'}{3} = \frac{a^3bz^4}{12k^4}$;

a' being $= \frac{az}{2k}$, $b' = \frac{bz}{2k}$.

3dly. Take the fluent of $\frac{a^3bz^4\dot{z}}{12k^4}$: which fluent is $= \frac{a^3bk}{60} = A''$.

The values of B″ and K″ are computed in the ſame manner.

It

It likewiſe appears by computation, that the *tetrahedron* and the *octahedron* have each the laſt mentioned property.

31. That the axis of rotation of any revolving body, under no reſtraint in regard to its rotatory motion, will always paſs through the center of gravity of the body, is aſſumed above as a well-known truth. Indeed it is a very obvious truth: for it appears by what is ſaid above concerning the centrifugal force of a particle of a revolving body, that *the ſum of all the* $p \times \overline{x - a'}$ and *the ſum of all the* $p \times \overline{y - b'}$ muſt each be $= 0$*: or elſe the joint centrifugal force of all the particles of the body would give motion to that point thereof about which the body is, by that ſame force, urged to turn; that is, the body would be moved entirely out of its place by an internal force ariſing from its own rotatory motion, without being acted on by any external force to give it a progreſſive motion; which is abſurd. But, by the property of the center of gravity, when *the ſum of all the* $p \times \overline{x - a'}$ and *the ſum of all the* $p \times \overline{y - b'}$ are each $= 0$, our point C is that center.

* By what is ſaid above, the centrifugal force of the particle p urges the point l, in the direction lp'', with a force

$$fpu = fp \times \overline{nt.\overline{x - a'} + ns.\overline{y - b'} - m.\overline{z - k'}};$$

and in the direction lq, (at right angles to lp'',) with a force

$$fpu'' = fp \times \overline{t.\overline{y - b'} - s.\overline{x - a'}}:$$

which equations, upon ſuppoſing (as we may) Ck coincident with CO, and $\delta k \beta \delta''$ parallel to lp'', become $fpu = fp.\overline{x - a'}$ and $fpu'' = fp.\overline{y - b'}$; m and s being then each $= 0$, and n and t each $= 1$. Therefore, that the joint centrifugal force of all the particles of the body may not give a progreſſive motion to it, *the ſum of all the* $p \times \overline{x - a'}$ and *the ſum of all the* $p \times \overline{y - b'}$ muſt each be $= 0$.

32. It is obſervable, that our method of computing the forces E, E″, F, and F″ holds true when the body is reſtrained from revolving freely about an axis paſſing through its center of gravity, and is made to revolve about any point C which is not that center; excepting ſuch inferences as are expreſsly derived from the property of ſuch center.

When the force compounded of the two forces E and E″, (or F and F″,) which we have been computing, is not = 0, it will diſturb the rotatory motion of the body ſo as to cauſe it to change its axis of rotation every inſtant, and endeavour to revolve about a new one: the compound motion ariſing from ſuch perturbation, and likewiſe from the perturbation cauſed by the action of an external force, I have, in ſome meaſure, explained in the *Philoſ. Tranſact.* for the year 1777; and I intend to treat of the ſame ſubject more fully in ſome ſubſequent Memoir.

MEMOIR

MEMOIR V.

A new Method of obtaining the Sums of certain Series.

1. THE fluxion of the circular arc z, whose radius is 1 and cosine x, being $=\frac{-\dot{x}}{\sqrt{1-x^2}}$; $n\dot{z}$ is $=\frac{-n\dot{x}}{\sqrt{1-x^2}}=\frac{-\dot{c}}{\sqrt{1-c^2}}$, c being the cosine of (nz) n times the arc z. Whence it appears that $\frac{\dot{c}}{n}$ is $=\frac{s\dot{x}}{\sqrt{1-x^2}}$; s denoting $\sqrt{1-c^2}$, the sine of nz.

Moreover $\dot{s}$ is $=\frac{-c\dot{c}}{\sqrt{1-c^2}}=nc\dot{z}=\frac{-nc\dot{x}}{\sqrt{1-x^2}}$. Hence $\frac{\dot{s}}{n}$ $=\frac{-c\dot{x}}{\sqrt{1-x^2}}$.

2. $\frac{n\dot{x}}{\sqrt{x^2-1}}$ being assumed $=\frac{\dot{y}}{\sqrt{y^2-1}}$; we get, by taking the fluents,

$$\text{Log. of } \overline{x+\sqrt{x^2-1}}\big|^n = \text{Log. of } \overline{y+\sqrt{y^2-1}},$$

$$\text{or } \overline{x+\sqrt{x^2-1}}\big|^n = y+\sqrt{y^2-1},$$

supposing $x=1$ when y is $=1$.

$$\text{Whence } y = \frac{\overline{x+\sqrt{x^2-1}}\,|^n + \overline{x-\sqrt{x^2-1}}\,|^n}{2},$$

$$\text{and } \sqrt{1-y^2} = \frac{\overline{x+\sqrt{x^2-1}}\,|^n - \overline{x-\sqrt{x^2-1}}\,|^n}{2\sqrt{-1}}.$$

But $\frac{n\dot{x}}{\sqrt{x^2-1}}$ being $= \frac{\dot{y}}{\sqrt{y^2-1}}$, $\frac{n\dot{x}}{\sqrt{1-x^2}}$ will be $= \frac{\dot{y}}{\sqrt{1-y^2}}$: whereof the equation of the fluents is

$n \times$ *circ. arc*, rad. 1, cosine $x =$ *circ. arc*, rad. 1, cosine y; where x is $= 1$, when y is $= 1$, agreeable to the supposition we made above when we took the fluents of the assumed equation by logarithms. Therefore, if A be put for the least arc whose cosine is y, and C for the whole circumference, to the radius 1; y being the cosine of A, A + C, A + 2C, A + 3C, &c. x will be the cosine of $\frac{A}{n}$, $\frac{A+C}{n}$, $\frac{A+2C}{n}$, &c. Consequently the values of $\sqrt{1-y^2}$ and y, found above, are respectively equal to the sine and cosine of (nz) n times the arc whose cosine is x.

3. Let s', s'', s''', &c. c', c'', c''', &c. denote the sines and cosines of z, $2z$, $3z$, &c. respectively. Then,

if x be $= 1$;
s', s'', s''', s^{iv}, &c. will be 0, 0, 0, 0, &c.
c', c'', c''', c^{iv}, &c. . . . . 1, 1, 1, 1, &c.

if x be $= 0$;
s', s'', s''', s^{iv}, &c. will be 1, 0, -1, 0, &c.
c', c'', c''', c^{iv}, &c. . . . . . . 0, -1, 0, 1, &c.

if x be $= -1$;
s', s'', s''', s^{iv}, &c. will be 0, 0, 0, 0, &c.
c', c'', c''', c^{iv}, &c. . . . . -1, 1, -1, 1, &c.

if

if x be $= \frac{1}{2}$;

$s', s'', s''', s^{iv}, s^{v}, s^{vi}$, &c. will be $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$, &c.

$c', c'', c''', c^{iv}, c^{v}, c^{vi}$, &c. ...... $\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1$, &c.

if x be $= -\frac{1}{2}$;

$s', s'', s''', s^{iv}, s^{v}, s^{vi}$, &c. will be $\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$, &c.

$c', c'', c''', c^{iv}, c^{v}, c^{vi}$, &c. ...... $-\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, 1$, &c.

&c. &c.

4. The Log. of $1 + u$ being $= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4}$ &c. we from thence, by substituting $\frac{1}{u}$ instead of u, have

Log. of $1 + \frac{1}{u} =$ Log. of $1 + u -$ Log. of $u = u^{-1} - \frac{u^{-2}}{2} + \frac{u^{-3}}{3}$ &c.

and, by subtraction, get

Log. of $u = u - u^{-1} - \frac{u^2 - u^{-2}}{2} + \frac{u^3 - u^{-3}}{3} - \frac{u^4 - u^{-4}}{4}$ &c.

Now, if u be supposed $= x + \sqrt{x^2 - 1}$, $\frac{\dot{u}}{u}$ will be $= \frac{\dot{x}}{\sqrt{x^2 - 1}}$ $= \frac{\dot{x}}{\sqrt{-1} \times \sqrt{1 - x^2}} = \frac{-\dot{x}\sqrt{-1}}{\sqrt{1 - x^2}}$; and the fluent of $\frac{\dot{u}}{u}$, or Log. of u, $= z\sqrt{-1}$, z denoting the circular arc whose radius is 1 and cosine x.

Therefore, by substitution, we have, after dividing by $2\sqrt{-1}$, the series $\frac{\overline{x + \sqrt{x^2 - 1}} - \overline{x - \sqrt{x^2 - 1}}}{2\sqrt{-1}} - \frac{\overline{x + \sqrt{x^2 - 1}}|^2 - \overline{x - \sqrt{x^2 - 1}}|^2}{2.2\sqrt{-1}}$ $+ \frac{\overline{x + \sqrt{x^2 - 1}}|^3 - \overline{x - \sqrt{x^2 - 1}}|^3}{3.2\sqrt{-1}}$ &c.

$$\text{or its equal } s' - \frac{s''}{2} + \frac{s'''}{3} - \frac{s^{iv}}{4} \&c. = \frac{z}{2};$$

except when (the cosine) x is $= -1$.

Example. Let x be $= \frac{1}{2}$, and let a denote one fourth of the periphery of the circle whose radius is 1. Then, s', s'', s''', &c. being $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$, &c. as observed in the preceding article; we have

$$\frac{\sqrt{3}}{2} \times \overline{1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} \&c.} = \frac{a}{3}.$$

Whence it appears that $\frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8}$ &c. is $= \frac{2a}{3^{\frac{3}{2}}}$.

5. Putting a to denote the same quadrantal arc as in the preceding example, $2a - z$ will denote the arc whose cosine is $= -x$ $(= -c')$; and the sines of z, $2z$, $3z$, $4z$, &c. being s', s'', s''', s^{iv}, &c. the sines of $2a - z$, $2 \times \overline{2a - z}$, $3 \times \overline{2a - z}$, $4 \times \overline{2a - z}$, &c. will be $s', -s'', s''', -s^{iv}$, &c. respectively. Therefore, by substitution, it appears that

$$s' + \frac{s''}{2} + \frac{s'''}{3} + \frac{s^{iv}}{4} \&c. \text{ is } = a - \frac{z}{2},$$

except when x is $= 1$:

and, from this and the theorem in the preceding article, it follows that

$$s'' + \frac{s^{iv}}{2} + \frac{s^{vi}}{3} \&c. \text{ is } = a - z,$$

$$\text{and } s' + \frac{s'''}{3} + \frac{s^{v}}{5} \&c. \quad = \frac{a}{2};$$

except when x is $= 1$, or $= -1$.

Example.

Example. Taking $x = \frac{1}{2}$; the correspondent values of s', s'', s''', &c. in art. 3. being properly substituted in our theorem, we find

$$\frac{\sqrt{3}}{2} \times \overline{1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17}\ \&c.} = \frac{a}{2}:$$

$$\text{whence } \frac{1}{1.5} + \frac{1}{7.11} + \frac{1}{13.17} + \frac{1}{19.23}\ \&c. = \frac{a}{4\sqrt{3}}.$$

6. Denoting the sine and cosine of qz by $\overset{(q)}{s}$ and $\overset{(q)}{c}$ respectively; the fluents of $\frac{\overset{(q)}{s}\dot{x}}{\sqrt{1-x^2}}$ and $\frac{-\overset{(q)}{c}\dot{x}}{\sqrt{1-x^2}}$ will, by art. 1. be respectively equal to $\frac{\overset{(q)}{c}}{q}$ and $\frac{\overset{(q)}{s}}{q}$.

Therefore, from our equation

$$s' - \frac{s''}{2} + \frac{s'''}{3} - \frac{s^{\text{iv}}}{4}\ \&c. = \frac{z}{2}, \text{ (found in art. 4.)}$$

by multiplying one side by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and the other by its equal $-\dot{z}$, and taking the fluents, we get

$$c' - \frac{c''}{2^2} + \frac{c'''}{3^2} - \frac{c^{\text{iv}}}{4^2}\ \&c - p'' = -\frac{z^2}{4};$$

p'' denoting the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ &c.

and the equation being so adjusted that each side shall vanish when x (or c') is $= 1$ and $z = 0$.

Whence, by taking $x = 0$, and $z = a$; (c', c'', c''', c^{iv}, c^{v}, &c. being, according to art. 3. equal to 0, -1, 0, 1, 0, &c. respectively;) we have

$$\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2}\ \&c. - p'' = -\frac{a^2}{4}.$$

But

But $\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{8^2}$ &c. is evidently $= \frac{p''}{4}$: consequently $\frac{p''}{4} - p''\ (= -\frac{3p''}{4})$ is $= -\frac{a^2}{4}$, and $p'' = \frac{a^2}{3}$.

$$\text{Let } P'' \text{ denote } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \text{ \&c.}$$

$$\text{and } Q'' \ldots\ldots\ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \text{ \&c.}$$

Then $Q'' + \frac{P''}{4}$ being manifestly $= P''$, and $Q'' - \frac{P''}{4} = p''$; it follows that P'' is $= \frac{2a^2}{3}$, and $Q'' = \frac{a^2}{2}$.

Thus, with great ease, are the sums of those series obtained: and with equal facility our method may be pursued in finding the sums of a great number of other series; the obtaining of which sums has generally been considered as a business of some difficulty.

7. To proceed with perspicuity, let us put

$$P''' = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} \text{ \&c.}$$

$$p''' = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} \text{ \&c.}$$

$$Q''' = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} \text{ \&c.}$$

$$q''' = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \text{ \&c.}$$

$$P^{iv} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \text{ \&c.}$$

$$p^{iv} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} \text{ \&c.}$$

$$Q^{iv} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} \text{ \&c.}$$

$$q^{iv} = 1 - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} \text{ \&c.}$$

&c. &c.

Then,

Then, from our equation

$$c' - \frac{c''}{2^2} + \frac{c'''}{3^2} - \frac{c^{iv}}{4^2} \text{ \&c.} - p'' = -\frac{z^2}{4},$$

by multiplying one fide by $\frac{-\dot{x}}{\sqrt{1-x^2}}$ and the other by its equal $\dot{z}$, and taking the fluents, we get

$$s' - \frac{s''}{2^3} + \frac{s'''}{3^3} - \frac{s^{iv}}{4^3} \text{ \&c.} - p''z = -\frac{z^3}{12}:$$

and, by repeating the operation, we from thence obtain

$$c' - \frac{c''}{2^4} + \frac{c'''}{3^4} - \frac{c^{iv}}{4^4} \text{ \&c.} + \frac{p''z^2}{2} - p^{iv} = \frac{z^4}{48}.$$

And fo we may proceed as far as we pleafe, obtaining a new feries and its fum at each fucceffive operation.

8. From the laft equation but one, (s', s'', s''', s^{iv}, s^{v}, &c. being refpectively equal to 1, 0, −1, 0, 1, &c. when x is = 0 and $z = a$,) we find

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \text{ \&c.} - ap'' = -\frac{a^3}{12};$$

or, ap'' being found $= \frac{a^2}{3}$,

$$q''' = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} \text{ \&c.} = \frac{a^3}{4}.$$

9. The equation $c' - \frac{c''}{2^4} + \frac{c'''}{3^4} - \frac{c^{iv}}{4^4} \text{ \&c.} + \frac{p''z^2}{2} - p^{iv} = \frac{z^4}{48}$, upon taking $x = 0$ and $z = a$, becomes

$$\frac{1}{2^4} - \frac{1}{4^4} + \frac{1}{6^4} - \frac{1}{8^4} \text{ \&c.} + \frac{a^2p''}{2} - p^{iv} = \frac{a^4}{48}.$$

But $\frac{1}{2^4} - \frac{1}{4^4} + \frac{1}{6^4} - \frac{1}{8^4}$ &c. is manifeftly $= \frac{p^{iv}}{2^4}$:

confequently $\frac{p^{iv}}{16} + \frac{a^2p''}{2} - p^{iv}$ $\left(= \frac{a^4}{6} - \frac{15p^{iv}}{16}\right)$ is $= \frac{a^4}{48}$,

and $p^{iv} = \frac{7a^4}{45}$.

L More-

Moreover, $Q^{iv} + \frac{P^{iv}}{2^4}$ being $= P^{iv}$, and $Q^{iv} - \frac{P^{iv}}{2^4} = p^{iv}$; it follows that P^{iv} is $= \frac{8a^4}{45}$, and $Q^{iv} = \frac{a^4}{6}$.

10. The arc whose cosine is x being denoted by z, $2a - z$ will denote the arc whose cosine is $-x$; and the cosines of z, $2z$, $3z$, &c. being c', c'', c''', &c. the cosines of $2a - z$, $2 \times \overline{2a - z}$, $3 \times \overline{2a - z}$, &c. will be $-c'$, c'', $-c'''$, c^{iv}, &c. Therefore, by substituting accordingly in the theorem deduced in art. 6. it appears that

$$c' + \frac{c''}{2^2} + \frac{c'''}{3^2} \text{ \&c. is } = \frac{2a^2}{3} - az + \frac{z^2}{4}:$$

and, from this and the theorem just now mentioned, it follows that

$$c'' + \frac{c^{iv}}{2^2} + \frac{c^{vi}}{3^2} \text{ \&c. is } = \frac{2a^2}{3} - 2az + z^2,$$

$$\text{and } c' + \frac{c'''}{3^2} + \frac{c^{v}}{5^2} \text{ \&c. } = \tfrac{1}{2}a.\overline{a - z}.$$

from whence other theorems may be derived by our method pursued above.

11. Let F denote the fluent of $\frac{\dot{u}}{u} \times$ Log. of $\overline{1 + u}$, which from art. 4. is found

$$= u - \frac{u^2}{2^2} + \frac{u^3}{3^2} - \frac{u^4}{4^2} \text{ \&c. } - p''$$

$$\text{or } = -u^{-1} + \frac{u^{-2}}{2^2} - \frac{u^{-3}}{3^2} + \frac{u^{-4}}{4^2} \text{ \&c. } + \frac{U^2}{2} + p'';$$

U denoting the Log. of u,

and F being supposed $= 0$ when u is $= 1$.

Let G′ denote the fluent of $\dot{u}F$,

G″ ..... the fluent of $\frac{\dot{u}}{u}G'$.

Then

Then will

$$G' \text{ be } = \frac{u^2}{1^2.2} - \frac{u^3}{2^2.3} + \frac{u^4}{3^2.4} \text{ \&c.} - p''u + p'' - R'$$

$$\text{or } = -\frac{u^{-1}}{1.2^2} + \frac{u^{-2}}{2.3^2} - \frac{u^{-3}}{3.4^2} \text{ \&c.} + u + p''u - p'' - 1 + R''$$
$$+ \frac{uU^2}{2} - uU - U;$$

$$G'' = \frac{u^2}{1^2.2^2} - \frac{u^3}{2^2.3^2} + \frac{u^4}{3^2.4^2} \text{ \&c.} - p''u + \overline{p'' - R'}.U + p'' - S$$

$$\text{or } = \frac{u^{-1}}{1^2.2^2} - \frac{u^{-2}}{2^2.3^2} + \frac{u^{-3}}{3^2.4^2} \text{ \&c.} + 3u + p''u - p'' - 3 - S$$
$$+ \frac{uU^2}{2} - \frac{U^2}{2} - 2uU - \overline{p'' + 1 - R''}.U;$$

p'' being put for $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ &c. $= \frac{a^2}{3}$,

R' . . . . for $\frac{1}{1^2.2} - \frac{1}{2^2.3} + \frac{1}{3^2.4} - \frac{1}{4^2.5}$ &c.

R'' . . . . for $\frac{1}{1.2^2} - \frac{1}{2.3^2} + \frac{1}{3.4^2} - \frac{1}{4.5^2}$ &c.

S . . . . for $\frac{1}{1^2.2^2} - \frac{1}{2^2.3^2} + \frac{1}{3^2.4^2} - \frac{1}{4^2.5^2}$ &c.

where it is obſervable that $R' + R''$ is $= 2p'' - 1 = \frac{2a^2}{3} - 1$,
and $R' - R'' \quad = S$.

Whence, by multiplying each ſide of the equation ariſing from the two values of G'' by $\frac{u^{-\frac{1}{2}}}{2\sqrt{-1}}$, bringing the two ſeries together in order to form only one ſeries, and ſubſtituting according to the method explained in art. 4. we find

$$\frac{s^{(\frac{3}{2})}}{1^2.2^2} - \frac{s^{(\frac{5}{2})}}{2^2.3^2} + \frac{s^{(\frac{7}{2})}}{3^2.4^2} - \frac{s^{(\frac{9}{2})}}{4^2.5^2} \text{ \&c.} = s^{(\frac{1}{2})} \times \overline{\frac{2a^2}{3} - \frac{z^2}{2} + 3} - 2\,c^{(\frac{1}{2})}z;$$

$s^{(\frac{1}{2})}$,

$\overset{(\frac{1}{2})}{s}$, $\overset{(\frac{3}{2})}{s}$, $\overset{(\frac{5}{2})}{s}$, &c. and $\overset{(\frac{1}{2})}{c}$, $\overset{(\frac{3}{2})}{c}$, $\overset{(\frac{5}{2})}{c}$, &c. denoting the sines and cosines of $\frac{1}{2}z$, $\frac{3}{2}z$, $\frac{5}{2}z$, &c. respectively.

12. The sines of $\frac{1}{2} \times \overline{2a-z}$, $\frac{3}{2} \times \overline{2a-z}$, $\frac{5}{2} \times \overline{2a-z}$, &c. being $\overset{(\frac{1}{2})}{c}$, $-\overset{(\frac{3}{2})}{c}$, $\overset{(\frac{5}{2})}{c}$, $-\overset{(\frac{7}{2})}{c}$, &c. respectively; and the cosine of $\frac{1}{2} \times \overline{2a-z}$ equal to $\overset{(\frac{1}{2})}{s}$; we find by substitution

$$\frac{\overset{(\frac{3}{2})}{c}}{1^2.2^2} + \frac{\overset{(\frac{5}{2})}{c}}{2^2.3^2} + \frac{\overset{(\frac{7}{2})}{c}}{3^2.4^2} \&c. = 2\overset{(\frac{1}{2})}{s}.\overline{2a-z} + \overset{(\frac{1}{2})}{c}.\overline{\frac{4a^2}{3} - 2az + \frac{z^2}{2} - 3}.$$

Example. If x ($= c'$) be $= 1$, z and $\overset{(\frac{1}{2})}{s}$ will each be $= 0$; and $\overset{(\frac{1}{2})}{c}$, $\overset{(\frac{3}{2})}{c}$, $\overset{(\frac{5}{2})}{c}$, &c. each equal to 1. Therefore we have by our theorem

$$\frac{1}{1^2.2^2} + \frac{1}{2^2.3^2} + \frac{1}{3^2.4^2} \&c. = \frac{4a^2}{3} - 3.$$

13. F denoting the fluent of $\frac{\dot{u}}{u} \times$ Log. of $1 + u$ as in art. 11. let

H′ denote the fluent of $u\dot{u}$F,

H″ . . . . . the fluent of $\frac{\dot{u}}{u}$H′,

H‴ . . . . . the fluent of $u\dot{u}$H″,

H^{iv} . . . . . the fluent of $\frac{\dot{u}}{u}$H‴.

Then will

$$H' \text{ be} = \frac{u^3}{1^2.3} - \frac{u^4}{2^2.4} + \frac{u^5}{3^2.5} \&c. - \frac{p''u^2}{2} + \frac{p''}{2} - R'''$$

or

$$\text{or } = \frac{u^{-1}}{1.3^2} - \frac{u^{-2}}{2.4^2} + \frac{u^{-3}}{3.5^2} \text{ \&c.} - u + \frac{u^2}{8} + \frac{p''u^2}{2}$$
$$+ \frac{u^2U^2}{4} - \frac{u^2U}{4} + \frac{U}{4} - \frac{p''}{2} + \frac{7}{8} - R^{iv};$$

$$H'' = \frac{u^2}{1^2.3^2} - \frac{u^4}{2^2.4^2} + \frac{u^3}{3^2.5^2} \text{ \&c.} - \frac{p''u^2}{4} + \overline{\frac{p''}{2} - R'''}.U + \frac{p''}{4} - S''$$

$$\text{or } = -\frac{u^{-1}}{1^2.3^2} + \frac{u^{-2}}{2^2.4^2} - \frac{u^{-3}}{3^2.5^2} \text{ \&c.} - u + \frac{3u^2}{16} + \frac{p''u^2}{4}$$
$$+ \frac{u^2U^2}{8} + \frac{U^2}{8} - \frac{u^2U}{4} - \overline{\frac{p''}{2} - \frac{7}{8} + R^{iv}}.U - \frac{p''}{4} + \frac{13}{16} + S'';$$

$$R''' \text{ being } = \frac{1}{1^2.3} - \frac{1}{2^2.4} + \frac{1}{3^2.5} \text{ \&c.}$$
$$R^{iv} \ldots\ldots = \frac{1}{1.3^2} - \frac{1}{2.4^2} + \frac{1}{3.5^2} \text{ \&c.}$$
$$S'' \ldots\ldots = \frac{1}{1^2.3^2} - \frac{1}{2^2.4^2} + \frac{1}{3^2.5^2} \text{ \&c.}$$

where it is obſervable, that

$$R''' + R^{iv} \text{ is } = \tfrac{3}{8}, \text{ and } R''' - R^{iv} = 2S''.$$

Whence, by multiplying each ſide of the equation ariſing from the two values of H'' by $\frac{u^{-1}}{2}$, bringing the two ſeries together, and ſubſtituting according to our method, we find

$$\frac{c''}{1^2.3^2} - \frac{c'''}{2^2.4^2} + \frac{c^{iv}}{3^2.5^2} - \frac{c^{v}}{4^2.6^2} \text{ \&c.} = \frac{s'z}{4} + c'.\overline{\frac{a^2}{6} - \frac{z^2}{8} + \frac{3}{16}} - \frac{1}{2}.$$

Moreover, the coſines of $\overline{2a-z}$, $2 \times \overline{2a-z}$, $3 \times \overline{2a-z}$, &c. being $-c'$, c'', $-c'''$, c^{iv}, &c. as obſerved in art. 10. we get by ſubſtitution

$$\frac{c''}{1^2.3^2} + \frac{c'''}{2^2.4^2} + \frac{c^{iv}}{3^2.5^2} \text{ \&c.} = \frac{s'}{4}.\overline{2a-z} + c'.\overline{\frac{a^2}{3} - \frac{az}{2} + \frac{z^2}{8} - \frac{3}{16}} - \frac{1}{2}.$$

And, from this and the preceding theorem, we have

c''

$$\frac{c''}{1^2.3^2}+\frac{c^{iv}}{3^2.5^2}+\frac{c^{vi}}{5^2.7^2}\text{ \&c.}=\frac{s'a}{4}+\frac{c'a}{4}.\overline{a-z}-\frac{1}{2},$$

$$\frac{c'''}{1^2.2^2}+\frac{c^{v}}{2^2.3^2}+\frac{c^{vii}}{3^2.4^2}\text{ \&c.}=4s'.\overline{a-z}+c'.\overline{\frac{4a^2}{3}-4az+2z^2-3}.$$

Example 1. If x be $=1$, s' and z will each be $=0$; and c', c'', c''', &c. each $=1$: therefore it appears by substitution, that

$$\frac{1}{1^2.3^2}-\frac{1}{2^2.4^2}+\frac{1}{3^2.5^2}\text{ \&c. is }=\frac{a^2}{6}-\frac{5}{16},$$

$$\frac{1}{1^2.3^2}+\frac{1}{2^2.4^2}+\frac{1}{3^2.5^2}\text{ \&c. }=\frac{a^2}{3}-\frac{11}{16},$$

$$\frac{1}{1^2.3^2}+\frac{1}{3^2.5^2}+\frac{1}{5^2.7^2}\text{ \&c. }=\frac{a^2}{4}-\frac{1}{2}.$$

Example 2. If x be $=0$, s' will be $=1$, $z=a$; and c', c'', c''', c^{iv}, &c. respectively equal to 0, -1, 0, 1, &c. therefore, by substituting accordingly, it appears that

$$\frac{1}{1^2.3^2}-\frac{1}{3^2.5^2}+\frac{1}{5^2.7^2}-\frac{1}{7^2.9^2}\text{ \&c. is }=\frac{1}{2}-\frac{a}{4}:$$

and, from this and the theorem next above, that

$$\frac{1}{1^2.3^2}+\frac{1}{5^2.7^2}+\frac{1}{9^2.11^2}\text{ \&c. is }=\frac{1}{8}.\overline{a^2-a}.$$

14. By proceeding as in the preceding article, the values of H''' and H^{iv} will be obtained: and then, by multiplying each side of the equation arising from the two values of H^{iv} by $\frac{u^{-2}}{2}$, bringing the two series together, and substituting as in that article, it will appear that

c'''

$$\frac{c'''}{1^2.3^2.5^2} - \frac{c^{iv}}{2^2.4^2.6^2} + \frac{c^{v}}{3^2.5^2.7^2} - \frac{c^{vi}}{4^2.6^2.8^2} \text{ \&c. is} = \begin{cases} \frac{3s''z}{128} - \frac{c'}{9} + \\ c''.\overline{\frac{a^2}{96} - \frac{z^2}{128} + \frac{23}{32.32}} \\ + \frac{a^2}{48} - \frac{z^2}{64} + \frac{1}{64}. \end{cases}$$

Hence, by substitution; the cosines of $\overline{2a-z}$, $2 \times \overline{2a-z}$, $3 \times \overline{2a-z}$, &c. being $-c'$, c'', $-c'''$, c^{iv}, &c. and the sine of $2 \times \overline{2a-z}$ being $= -s''$, as observed above; we find

$$\frac{c'''}{1^2.3^2.5^2} + \frac{c^{iv}}{2^2.4^2.6^2} + \frac{c^{v}}{3^2.5^2.7^2} \text{ \&c.} = \begin{cases} \frac{3s''}{128}.\overline{2a-z} - \frac{c'}{9} + \\ c''.\overline{\frac{a^2}{48} - \frac{az}{32} + \frac{z^2}{128} - \frac{23}{32.32}} \\ + \frac{a^2}{24} - \frac{az}{16} + \frac{z^2}{64} - \frac{1}{64}. \end{cases}$$

And, from these last two theorems, we have

$$\frac{c'''}{1^2.3^2.5^2} + \frac{c^{v}}{3^2.5^2.7^2} + \frac{c^{vii}}{5^2.7^2.9^2} \text{ \&c.} = \frac{3s''a}{128} - \frac{c'}{9} + \frac{c''a}{64}.\overline{a-z} + \frac{a^2-az}{32},$$

$$\frac{c^{iv}}{1^2.2^2.3^2} + \frac{c^{vi}}{2^2.3^2.4^2} + \frac{c^{viii}}{3^2.4^2.5^2} \text{ \&c.} = \begin{cases} \frac{3s''}{2}.\overline{a-z} + \frac{2a^2}{3} - 2az + z^2 - 1 \\ + c''.\overline{\frac{a^2}{3} - az + \frac{z^2}{2} - \frac{23}{16}}. \end{cases}$$

Example 1. If x be $= 1$, s'' and z will each be $= 0$; and c', c'', c''', &c. each $= 1$: it appears therefore, by substitution, that

$$\frac{1}{1^2.3^2.5^2} - \frac{1}{2^2.4^2.6^2} + \frac{1}{3^2.5^2.7^2} \text{ \&c. is} = \frac{a^2}{32} - \frac{673}{9.32.32},$$

$$\frac{1}{1^2.3^2.5^2} + \frac{1}{2^2.4^2.6^2} + \frac{1}{3^2.5^2.7^2} \text{ \&c.} \quad = \frac{a^2}{16} - \frac{1375}{9.32.32}.$$

$$\frac{1}{1^2.3^2.5^2}+\frac{1}{3^2.5^2.7^2}+\frac{1}{5^2.7^2.9^2}\ \&c. = \frac{3a^2}{64}-\frac{1}{9},$$

$$\frac{1}{1^2.2^2.3^2}+\frac{1}{2^2.3^2.4^2}+\frac{1}{3^2.4^2.5^2}\ \&c. = a^2-\frac{39}{16}.$$

Example 2. If x be $=0$, s'' will be $=0$, $z=a$; and c', c'', c''', &c. reſpectively equal to 0, -1, 0, 1, &c. therefore it follows, that

$$\frac{1}{1^2.2^2.3^2}-\frac{1}{2^2.3^2.4^2}+\frac{1}{3^2.4^2.5^2}\ \&c.\ \text{is} = \frac{7}{16}-\frac{a^2}{6}.$$

And, by addition, we have

$$\frac{1}{1^2.2^2.3^2}+\frac{1}{3^2.4^2.5^2}+\frac{1}{5^2.6^2.7^2}\ \&c. = \frac{5a^2}{12}-1:$$

and, by ſubtraction,

$$\frac{1}{2^2.3^2.4^2}+\frac{1}{4^2.5^2.6^2}+\frac{1}{6^2.7^2.8^2}\ \&c. = \frac{7a^2}{12}-\frac{23}{16}.$$

15. The fluent of $\frac{\dot{u}}{1+u^2}$, beginning when u is $=1$,

$$\text{is} = u-\frac{u^3}{3}+\frac{u^5}{5}-\frac{u^7}{7}\ \&c. -\frac{a}{2},$$

$$\text{or} = -u^{-1}+\frac{u^{-3}}{3}-\frac{u^{-5}}{5}+\frac{u^{-7}}{7}\ \&c. +\frac{a}{2}.$$

From which equation, by bringing the two ſeries together and ſubſtituting according to our method, we have

$$c'-\frac{c'''}{3}+\frac{c^{v}}{5}-\frac{c^{vii}}{7}\ \&c. = \frac{a}{2};$$

except when x $(=c')$ is negative, or $=0$.

Hence, by proceeding as in art. 7. many other theorems may be readily deduced: and more theorems may be derived from the two values of the fluent of $\frac{\dot{u}}{1+u^2}$, by proceeding as in art. 11.

16. By

16. By art. 4. we have

$$2\text{ Log. }\overline{1+u}-\text{Log. }u=\begin{cases} u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}\text{ \&c.}\\ +u^{-1}-\frac{u^{-2}}{2}+\frac{u^{-3}}{3}-\frac{u^{-4}}{4}\text{ \&c.}\end{cases}$$

Hence, by fubftituting $x+\sqrt{x^2-1}$ for u, according to our method purfued above, we get

$$\tfrac{1}{2}\text{ Log. }\overline{1+x}+\tfrac{1}{2}\text{ Log. }2=c'-\frac{c''}{2}+\frac{c'''}{3}-\frac{c^{\text{iv}}}{4}\text{ \&c.}$$

And, writing $-x$ inftead of x; and $-c'$, $-c'''$, $-c^{\text{v}}$, &c. inftead of c', c''', c^{v}, &c. refpectively, agreeably to what is faid in art. 10. we have

$$\tfrac{1}{2}\text{ Log. }\overline{1-x}+\tfrac{1}{2}\text{ Log. }2=-c'-\frac{c''}{2}-\frac{c'''}{3}\text{ \&c.}$$

Therefore it is obvious, that

$$\tfrac{1}{2}\text{ Log. }\frac{1}{\sqrt{1-x^2}}-\tfrac{1}{2}\text{ Log. }2\text{ is }=\frac{c''}{2}+\frac{c^{\text{iv}}}{4}+\frac{c^{\text{vi}}}{6}\text{ \&c.}$$

$$\text{and }\tfrac{1}{4}\text{ Log. }\frac{1+x}{1-x}=c'+\frac{c'''}{3}+\frac{c^{\text{v}}}{5}\text{ \&c.}$$

By which laft theorem, and that in the preceding article, we find

$$\tfrac{1}{8}\text{ Log. }\frac{1+x}{1-x}+\frac{a}{4}=c'+\frac{c^{\text{v}}}{5}+\frac{c^{\text{ix}}}{9}\text{ \&c.}$$

$$\text{and }\tfrac{1}{8}\text{ Log. }\frac{1+x}{1-x}-\frac{a}{4}=\frac{c'''}{3}+\frac{c^{\text{vii}}}{7}+\frac{c^{\text{xi}}}{11}\text{ \&c.}$$

when x is a pofitive quantity.

Example. If x be the cofine of 45°; c', $-c'''$, $-c^{\text{v}}$, c^{vii}, c^{ix}, &c. will each be $=\frac{1}{\sqrt{2}}$; and confequently

$$1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} \text{ \&c.} = \frac{a + \text{Log.}\ \overline{1+\sqrt{2}}}{2^{\frac{3}{2}}},$$

$$\text{and } \frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} \text{ \&c.} = \frac{a - \text{Log.}\ \overline{1+\sqrt{2}}}{2^{\frac{3}{2}}}:$$

which series, it may be observed, are respectively equal to the fluents of $\frac{\dot y}{1+y^4}$ and $\frac{y^2 \dot y}{1+y^4}$, generated whilst y from o becomes equal to 1.

17. The fluent of $\frac{u^{m-1}\dot u}{\overline{1+u^n}|^e}$

$$\text{is } = \frac{u^m}{m} - \frac{e u^{m+n}}{m+n} + \frac{e'' u^{m+2n}}{m+2n} \text{ \&c.} - \text{M}',$$

$$\text{or } = -\frac{u^{m-en}}{en-m} + \frac{e u^{m-en-n}}{en+n-m} - \frac{e'' u^{m-en-2n}}{en+2n-m} \text{ \&c.} + \text{M}'';$$

e'' being $= \frac{e.e+1}{2}$, $e''' = \frac{e.e+1.e+2}{2.3}$, &c.

and M′ and M″ respectively denoting

the series $\frac{1}{m} - \frac{e}{m+n} + \frac{e''}{m+2n}$ &c.

and $\frac{1}{en-m} - \frac{e}{en+n-m} + \frac{e''}{en+2n-m}$ &c.

which are the respective fluents of $\frac{u^{m-1}\dot u}{\overline{1+u^n}|^e}$ and $\frac{u^{en-m-1}\dot u}{\overline{1+u^n}|^e}$, generated whilst u from o becomes equal to 1.

Whence, by multiplying by $u^{en-2m-1}\dot u$ and taking the fluents, we have

u^{en}

$$\frac{u^{en-m}}{m.\overline{en-m}} - \frac{eu^{en+n-m}}{\overline{m+n}.\overline{en+n-m}} + \frac{e''u^{en+2n-m}}{\overline{m+2n}.\overline{en+2n-m}} \&c. - \frac{M'u^{en-2m}}{en-2m} + \frac{M'}{en-2m}$$

$$= \frac{u^{-m}}{m.\overline{en-m}} - \frac{eu^{-m-n}}{\overline{m+n}.\overline{en+n-m}} + \frac{e''u^{-m-2n}}{\overline{m+2n}.\overline{en+2n-m}} \&c. + \frac{M''u^{en-2m}}{en-2m} - \frac{M''}{en-2m}.$$

And hence, bringing the two ſeries together, after multiplying by $\frac{u^{m-\frac{1}{2}en}}{2\sqrt{-1}}$ and ſubſtituting $x + \sqrt{x^2 - 1}$ for u, &c. according to our method, we find

$$\frac{M' + M''}{en - 2m}.\overset{(\frac{1}{2}en-m)}{s} =$$

$$\frac{\overset{(\frac{1}{2}en)}{s}}{m.\overline{en-m}} - \frac{e\overset{(\frac{1}{2}en+n)}{s}}{\overline{m+n}.\overline{en+n-m}} + \frac{e''\overset{(\frac{1}{2}en+2n)}{s}}{\overline{m+2n}.\overline{en+2n-m}} - \frac{e'''\overset{(\frac{1}{2}en+3n)}{s}}{\overline{m+3n}.\overline{en+3n-m}} \&c.$$

$\overset{(\frac{1}{2}en-m)}{s}$, $\overset{(\frac{1}{2}en)}{s}$, $\overset{(\frac{1}{2}en+n)}{s}$, $\overset{(\frac{1}{2}en+2n)}{s}$, &c. denoting the ſines of $\overline{\frac{1}{2}en - m}.z$, $\frac{1}{2}enz$, $\overline{\frac{1}{2}en + n}.z$, $\overline{\frac{1}{2}en + 2n}.z$, &c. reſpectively; of which arcs, the coſines will be denoted by $\overset{(\frac{1}{2}en-m)}{c}$, $\overset{(\frac{1}{2}en)}{c}$, $\overset{(\frac{1}{2}en+n)}{c}$, $\overset{(\frac{1}{2}en+2n)}{c}$, &c. reſpectively; and the ſines and coſines of other arcs, which are multiples or ſubmultiples of the arc z, will be expreſſed in like manner.

Now it is obſervable that, if y be $= \frac{1}{u}$, $\frac{u^{m-1}\dot{u}}{\overline{1 + u^n}|^e}$ will be $= -\frac{y^{en-m-1}\dot{y}}{\overline{1 + y^n}|^e}$: therefore, y decreaſing from 1 to 0 whilſt u increaſes from 1 to infinity, it follows, that the fluent of $\frac{u^{en-m-1}\dot{u}}{\overline{1 + u^n}|^e}$, generated whilſt u from 0 becomes equal to 1, will be equal to the fluent of $\frac{u^{m-1}\dot{u}}{\overline{1+u^{n}}|^{e}}$, generated whilſt u from 1 becomes infinite; and conſequently, that (M′+M″) the ſum of the fluents of $\frac{u^{m-1}\dot{u}}{\overline{1+u^{n}}|^{e}}$ and $\frac{u^{en-m-1}\dot{u}}{\overline{1+u^{n}}|^{e}}$, generated whilſt u from 0 becomes equal to 1, will be equal to the *whole* fluent of $\frac{u^{m-1}\dot{u}}{\overline{1+u^{n}}|^{e}}$, or of $\frac{u^{en-m-1}\dot{u}}{\overline{1+u^{n}}|^{e}}$, generated whilſt u from 0 becomes infinite. Conſequently, denoting this laſt mentioned *whole* fluent by M, our ſeries which we found equal to

$$\frac{M'+M''}{en-2m}\cdot\overset{(\frac{1}{2}en-m)}{s} \text{ will be} = \frac{M}{en-2m}\cdot\overset{(\frac{1}{2}en-m)}{s}.$$

18. If nz be $= 2a$; $\overset{(\frac{1}{2}en)}{s}$, $-\overset{(\frac{1}{2}en+n)}{s}$, $\overset{(\frac{1}{2}en+2n)}{s}$, $-\overset{(\frac{1}{2}en+3n)}{s}$, &c. will each be equal to the ſine of ea; and $\overset{(\frac{1}{2}en-m)}{s}$ equal to the ſine of $\frac{en-2m}{n}a$: therefore it appears, that

$$\frac{M}{en-2m}\cdot\frac{\overset{(\frac{1}{2}en-m)}{s}}{\overset{(\frac{1}{2}en)}{s}} \text{ will then be} =$$

$$\frac{1}{m.en-m}+\frac{e}{m+n.en+n-m}+\frac{e''}{m+2n.en+2n-m}+\frac{e'''}{m+3n.en+3n-m}\ \&c.$$

$\overset{(\frac{1}{2}en)}{s}$ and $\overset{(\frac{1}{2}en-m)}{s}$ being as juſt now mentioned.

But this ſeries is equal to $\frac{N'-N''}{en-2m}$, N′ and N″ being the fluents of $\frac{u^{m-1}\dot{u}}{\overline{1-u^{n}}|^{e}}$ and $\frac{u^{en-m-1}\dot{u}}{\overline{1-u^{n}}|^{e}}$ reſpectively, generated

whilſt

whilſt u from o becomes equal to 1. Hence therefore we have the remarkable theorem

$$\text{Sine of } \frac{en-2m}{n}a \times M = \text{ſine of } ea \times \overline{N'-N''},$$

which is of conſiderable uſe in the calculation of fluents.

Example. If e be $=1$, and $m=rn$; $\overset{(\frac{1}{2}en)}{s}$ will be $=1$, $\overset{(\frac{1}{2}en-m)}{s} = \overset{(m)}{c}$, and (by our Appendix) $M = \frac{2a}{n\overset{(m)}{s}}$. Conſequently

$$\frac{2a}{n}\cdot\frac{\overset{(m)}{c}}{\overset{(m)}{s}}, \text{ will be } = N'-N'',$$

the difference of the fluents of $\frac{u^{m-1}\dot{u}}{1-u^{n}}$ and $\frac{u^{n-m-1}\dot{u}}{1-u^{n}}$, generated whilſt u from o becomes equal to 1; and

$$\frac{2a}{1-2r}\cdot\frac{\overset{(m)}{c}}{\overset{(m)}{s}} = \frac{1}{r.1-r} + \frac{1}{1+r.2-r} + \frac{1}{2+r.3-r} + \frac{1}{3+r.4-r} \text{ \&c.}$$

$\overset{(m)}{s}$ and $\overset{(m)}{c}$ being the ſine and coſine of $2ra$ reſpectively.

19. Multiplying each ſide of the general theorem in art. 17. by $\frac{\dot{x}}{\sqrt{1-x^{2}}}$, and taking the fluents, we have

$$\frac{M}{\overline{en-2m}^{2}} \times \overline{\overset{(\frac{1}{2}en-m)}{c} - 1} + S = \text{the ſeries}$$

$$\frac{\overset{(\frac{1}{2}en)}{c}}{m.en.en-m} - \frac{e\overset{(\frac{1}{2}en+n)}{c}}{m+n.en+2n.en+n-m} + \frac{e''\overset{(\frac{1}{2}en+2n)}{c}}{m+2n.en+4n.en+2n-m} \text{ \&c.}$$

$$S \text{ being } = \frac{1}{m.en.en-m} - \frac{e}{m+n.en+2n.en+n-m} \text{ \&c.}$$

20. If

20. If nz be $= 2a$; $\overset{(\frac{1}{2}en)}{c}$, $-\overset{(\frac{1}{2}en+n)}{c}$, $\overset{(\frac{1}{2}en+2n)}{c}$, $-\overset{(\frac{1}{2}en+3n)}{c}$, &c. will be equal to each other; therefore it follows, that

$$\frac{\text{M}}{\overline{en-2m}|^2} \times \overline{\overset{(\frac{1}{2}en-m)}{c} - 1} + \text{S} \text{ will then be} =$$

$$\overset{(\frac{1}{2}en)}{c} \times \overline{\frac{1}{m\cdot en\cdot \overline{en-m}} + \frac{e}{\overline{m+n}\cdot\overline{en+2n}\cdot\overline{en+n-m}} + \frac{e^{2}}{\overline{m+2n}\cdot\overline{en+4n}\cdot\overline{en+2n-m}} \text{ \&c.}}$$

Example. If e be $= 1$, and $m = rn$; the cosine $\overset{(\frac{1}{2}en-m)}{c}$ will be $= \overset{(m)}{s}$, and the other cosines $\overset{(\frac{1}{2}en)}{c}$, $\overset{(\frac{1}{2}en+n)}{c}$, &c. each $= 0$: consequently M being then $= \frac{2a}{n \cdot \overset{(m)}{s}}$,

$$\frac{1-\overset{(m)}{s}}{\overset{(m)}{s}} \times \frac{2a}{\overline{1-2r}|^2} \text{ will be } (= \text{S}) =$$

$$\frac{1}{1\cdot r\cdot\overline{1-r}} - \frac{1}{3\cdot\overline{1+r}\cdot\overline{2-r}} + \frac{1}{5\cdot\overline{2+r}\cdot\overline{3-r}} - \frac{1}{7\cdot\overline{3+r}\cdot\overline{4-r}} \text{ \&c.}$$

$\overset{(m)}{s}$ being the sine of $2ra$.

21. $\dfrac{\overline{x+\sqrt{x^2-1}}|^p + \overline{x-\sqrt{x^2-1}}|^p}{2}$ and $\dfrac{\overline{x+\sqrt{x^2-1}}|^p - \overline{x-\sqrt{x^2-1}}|^p}{2\sqrt{-1}}$ being respectively equal to $\overset{(p)}{c}$ and $\overset{(p)}{s}$, series whose sums are the values of $\frac{\overset{(p)}{c}}{\overline{\overset{(q)}{c}}|^t}$, $\frac{\overset{(p)}{s}}{\overline{\overset{(q)}{c}}|^t}$, $\frac{\overset{(p)}{c}}{\overline{\overset{(q)}{s}}|^t}$, and $\frac{\overset{(p)}{s}}{\overline{\overset{(q)}{s}}|^t}$ may be directly obtained by means of the binomial theorem: and by recollecting what is proved in art. 1. we may, by means of those series, deduce most of the theorems investigated in the preceding articles, and many others, without deriving them

them (by ſubſtitution) from other algebraic expreſſions as we have done above.

By the binomial theorem,

$$\left(2\overset{(\frac{n}{2})}{c}\right)^{-e} = \left(\overline{x+\sqrt{x^2-1}}^{\,\frac{n}{2}} + \overline{x-\sqrt{x^2-1}}^{\,\frac{n}{2}}\right)^{-e}$$

$$\text{is} = \overline{x+\sqrt{x^2-1}}^{\,\frac{1}{2}en} - e.\overline{x+\sqrt{x^2-1}}^{\,\frac{1}{2}en+n} + e''.\overline{x+\sqrt{x^2-1}}^{\,\frac{1}{2}en+2n}\ \&c.$$

$$\text{or} = \overline{x-\sqrt{x^2-1}}^{\,\frac{1}{2}en} - e.\overline{x-\sqrt{x^2-1}}^{\,\frac{1}{2}en+n} + e''.\overline{x-\sqrt{x^2-1}}^{\,\frac{1}{2}en+2n}\ \&c.$$

whence, by multiplying the firſt ſeries by $\overline{x+\sqrt{x^2-1}}^{\,m-\frac{1}{2}en}$ and the ſecond by $\overline{x-\sqrt{x^2-1}}^{\,m-\frac{1}{2}en}$, and ſubtracting one of the products from the other, we have, after dividing by $2\sqrt{-1}$,

$$\frac{\overset{(m-\frac{1}{2}en)}{s}}{\left(2\overset{(\frac{1}{2}n)}{c}\right)^{e}} = \overset{(m)}{s} - e\overset{(m+n)}{s} + e''\overset{(m+2n)}{s} - e'''\overset{(m+3n)}{s}\ \&c.$$

And, by adding thoſe products together, we have, after dividing by 2,

$$\frac{\overset{(m-\frac{1}{2}en)}{c}}{\left(2\overset{(\frac{1}{2}n)}{c}\right)^{e}} = \overset{(m)}{c} - e\overset{(m+n)}{c} + e''\overset{(m+2n)}{c} - e'''\overset{(m+3n)}{c}\ \&c.$$

e'', e''', &c. being as in art. 17.

From theſe theorems, others (of conſiderable uſe in calculations) may be eaſily deduced in the following manner.

22. From the equation $\frac{\overset{(m-\frac{1}{2}en)}{s}}{\left(2\overset{(\frac{1}{2}n)}{c}\right)^{e}} = \overset{(m)}{s} - e\overset{(m+n)}{s}$ &c. by multiplying by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents according to what is proved in art. 1. we have

$$dG + \text{fl.}\ \overline{\frac{\dot{x}}{\sqrt{1-x^2}} \cdot \frac{\overset{(m-\frac{1}{2}en)}{s}}{\overline{2\overset{(\frac{1}{2}n)}{c}}|^{e}}} = \frac{\overset{(m)}{c}}{m} - \frac{e\overset{(m+n)}{c}}{m+n} + \frac{e''\overset{(m+2n)}{c}}{m+2n}\ \&c.$$

which is adjuſted ſo that the fluent is ſuppoſed to begin when z is $= \frac{2a}{n}$; x (the coſine of z) being then the greateſt root of the equation $\overline{x + \sqrt{x^2 - 1}}|^{\frac{1}{2}n} + \overline{x - \sqrt{x^2 - 1}}|^{\frac{1}{2}n} = 0$, whereof the other roots are the coſines of $\frac{6a}{n}$, $\frac{10a}{n}$, &c. ſo long as theſe arcs are leſs than $2a$: d denoting the coſine of $\frac{2ma}{n}$, and G the ſum of the ſeries $\frac{1}{m} + \frac{e}{m+n} + \frac{e''}{m+2n}$ &c. which is equal to the *whole* fluent of $\frac{x^{m-1}\dot{x}}{\overline{1-x^n}|^{e}}$, generated whilſt x from 0 becomes equal to 1. For, $\overset{(m)}{c}$ being $= d$ when x is the coſine of $\frac{2a}{n}$, the coſines $\overset{(m+n)}{c}$, $\overset{(m+2n)}{c}$, $\overset{(m+3n)}{c}$, &c. will be equal to $-d$, d, $-d$, &c. reſpectively.

23. When the coſine x is $= 1$, z is $= 0$, and $\overset{(m)}{c}$, $\overset{(m+n)}{c}$, $\overset{(m+2n)}{c}$, &c. each $= 1$: conſequently, denoting the coſine of $\frac{2a}{n}$ by c, and putting H for the ſum of the ſeries $\frac{1}{m} - \frac{e}{m+n} + \frac{e''}{m+2n} - \frac{e'''}{m+3n}$ &c. which is equal to the fluent of $\frac{x^{m-1}\dot{x}}{\overline{1+x^n}|^{e}}$, generated whilſt x from 0 becomes equal

to

to 1; the fluent of $\frac{\dot{x}}{\sqrt{1-x^2}}\cdot\frac{s^{(m-\frac{1}{2}en)}}{\overline{2c^{(\frac{1}{2}n)}}|^{e}}$, generated whilst x from c becomes equal to 1, will be $= H - dG$.

24. If m be $=\frac{1}{2}en$, $s^{(m-\frac{1}{2}en)}$ will be $=0$, and fl. $\frac{\dot{x}}{\sqrt{1-x^2}}\cdot\frac{s^{(m-\frac{1}{2}en)}}{\overline{2c^{(\frac{1}{2}n)}}|^{e}}$ $=0$; therefore it appears that dG will then be $= H$.

In this easy manner we discover, that

the cosine of ea is to radius,

as the series $\frac{1}{e}-\frac{c}{e+2}+\frac{c''}{e+4}-\frac{c'''}{e+6}$ &c.

to the series $\frac{1}{e}+\frac{c}{e+2}+\frac{c''}{e+4}+\frac{c'''}{e+6}$ &c.

that is, as the fluent of $\frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1+x^n}|^{e}}$ generated whilst x from 0 becomes equal to 1, is to the fluent of $\frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1-x^n}|^{e}}$ generated in the same time. Which fluent of the first written fluxion is equal to *half* the *whole* fluent of the same fluxion, generated whilst x from 0 becomes infinite*.

* If y be $=\frac{1}{x}$, $\frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1+x^n}|^{e}}$ will be $=-\frac{y^{\frac{1}{2}en-1}\dot{y}}{\overline{1+y^n}|^{e}}$: therefore, y decreasing from 1 to 0 whilst x increases from 1 to infinity, it is evident that the fluent of $\frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1+x^n}|^{e}}$, generated whilst x from 1 becomes infinite, will be equal to the fluent of the same fluxion, generated whilst x from 0 becomes equal to 1.

25. If $m - \frac{1}{2}en$ be $= f$, e will be $= \frac{2m - 2f}{n}$, and $\frac{\overset{(m - \frac{1}{2}en)}{s}}{\sqrt{1 - x^2}}$ $= fx^{f-1} + f'''x^{f-3}.\overline{x^2 - 1} + f^{v}x^{f-5}.\overline{x^2 - 1}|^2$ &c. $= \overline{2x}|^{f-1}$ $- \overline{f - 2}.\overline{2x}|^{f-3} + \frac{f-3.f-4}{2}.\overline{2x}|^{f-5} - \frac{f-4.f-5.f-6}{2.3}.\overline{2x}|^{f-7}$ $+ \frac{f-5.f-6.f-7.f-8}{2.3.4}.\overline{2x}|^{f-9}$ &c. till the exponent of the power of $2x$ becomes 0 or 1: which series will both terminate if f be a positive integer; f''' being $= \frac{f.f-1.f-2}{2.3}$, $f^{v} = \frac{f.f-1.f-2.f-3.f-4}{2.3.4.5}$, &c. Therefore it follows, that

$$dG + \text{the fluent of } \frac{\dot{x} \times \overline{\overline{2x}|^{f-1} - \overline{f-2}.\overline{2x}|^{f-3}\ \&c.}}{\overline{\overline{x + \sqrt{x^2 - 1}}|^{\frac{1}{2}n} + \overline{x - \sqrt{x^2 - 1}}|^{\frac{1}{2}n}}|^{e}}$$

$$\text{will be} = \frac{\overset{(m)}{c}}{m} - \frac{e\ \overset{(m+n)}{c}}{m+n} + \frac{e''\ \overset{(m+2n)}{c}}{m+2n} - \frac{e'''\ \overset{(m+3n)}{c}}{m+3n}\ \&c.$$

the fluent being supposed to begin when x is equal to the cosine of $\frac{2a}{n}$.

But if the fluent be supposed to begin when x is $= 1$,

$$H + \text{the fluent of } \frac{\dot{x} \times \overline{\overline{2x}|^{f-2} - \overline{f-2}.\overline{2x}|^{f-3}\ \&c.}}{\overline{\overline{x + \sqrt{x^2 - 1}}|^{\frac{1}{2}n} + \overline{x - \sqrt{x^2 - 1}}|^{\frac{1}{2}n}}|^{e}};$$

$$\text{will be} = \frac{\overset{(m)}{c}}{m} - \frac{e\ \overset{(m+n)}{c}}{m+n} + \frac{e''\ \overset{(m+2n)}{c}}{m+2n} - \frac{e'''\ \overset{(m+3n)}{c}}{m+3n}\ \&c.$$

Example 1. If e be $= 1$, $m = 2$, and $n = 2$; f will be $= 1$, and

$\frac{1}{2}$ Log.

$$\tfrac{1}{2}\,\text{Log. } 2 + \tfrac{1}{2}\,\text{Log. } x = \frac{c''}{2} - \frac{c^{iv}}{4} + \frac{c^{vi}}{6} \text{ \&c.}$$

$$\text{or Log. } x = c'' - \frac{c^{iv}}{2} + \frac{c^{vi}}{3} \text{ \&c.} - \text{Log. } 2.$$

H, in this case, being $= \tfrac{1}{2}$Log. 2.

Example 2. m being $= e + f$ when n is $= 2$, we have

$$H + 2^{f-e-1} x^{f-e} \times \overline{\frac{1}{f-e} - \frac{\overline{f-2.2x}|^{-2}}{f-e-2} \text{ \&c.}} - 2^{f-e-1} \times \overline{\frac{1}{f-e} - \frac{f-2.2^{-2}}{f-e-2} \text{\&c.}}$$

$$= \frac{\overset{(e+f)}{c}}{e+f} - \frac{e\,\overset{(e+f+2)}{c}}{e+f+2} + \frac{e''\,\overset{(e+f+4)}{c}}{e+f+4} \text{ \&c.}$$

where H is $= \frac{1}{e+f} - \frac{e}{e+f+2} + \frac{e''}{e+f+4}$ &c.

Hence, by taking $x = 0$, it appears that, f being greater than e,

$$\text{H is} = dG + 2^{f-e-1} \times \overline{\frac{1}{f-e} - \frac{\overline{f-2}\,2^{-2}}{f-e-2} \text{ \&c.}}$$

d being here the cosine of $\overline{e+f}.a$,

$$\text{and } G = \frac{1}{e+f} + \frac{e}{e+f+2} + \frac{e''}{e+f+4} \text{ \&c.}$$

26. From the theorem $H + F = \frac{\overset{(m)}{c}}{m} - \frac{e\,\overset{(m+n)}{c}}{m+n}$, &c. in the preceding article, we get, by multiplying by $\frac{-\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents,

$$Hz - \text{fl. } \frac{\dot{x}F}{\sqrt{1-x^2}} = \frac{\overset{(m)}{s}}{m^2} - \frac{e\,\overset{(m+n)}{s}}{\overline{m+n}|^2} + \frac{e''\,\overset{(m+2n)}{s}}{\overline{m+2n}|^2} \text{ \&c.}$$

 the

the fluent beginning when x is $= 1$ and $z = 0$; and F denoting the fluent of $\dfrac{\dot{x} \times \overline{2x}|^{f-1} - \overline{f-2}.\overline{2x}|^{f-3} \text{ \&c.}}{\overline{\overline{x+\sqrt{x^2-1}}|^{\frac{1}{2}n} + \overline{x-\sqrt{x^2-1}}|^{\frac{1}{2}n}}|^{e}}$.

Likewise, by multiplying the last theorem by $\dfrac{\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents again, we get

$$-\tfrac{1}{2}Hz^2 - \overline{\text{fl.}\frac{\dot{x}}{\sqrt{1-x^2}}\,\text{fl.}\frac{\dot{x}F}{\sqrt{1-x^2}}} + Q_{'''} = \frac{\overset{(m)}{c}}{m^3} - \frac{e\overset{(m+n)}{c}}{\overline{m+n}|^3} + \frac{e''\overset{(m+2n)}{c}}{\overline{m+2n}|^3} \text{ \&c.}$$

$Q_{'''}$ denoting the series $\dfrac{1}{m^3} - \dfrac{e}{\overline{m+n}|^3} + \dfrac{e''}{\overline{m+2n}|^3}$ &c.

And, by repeating the operation, other theorems may be obtained.

If we consider the fluents as beginning when x is $= c$, we, by the like operation, obtain, from the theorem

$dG + F = \dfrac{\overset{(m)}{c}}{m} - \dfrac{e\overset{(m+n)}{c}}{m+n}$ &c. in the preceding article,

$$dG.\overline{z - \frac{2a}{n}} - \text{fl.}\frac{\dot{x}F}{\sqrt{1-x^2}} + bQ_{''} = \frac{\overset{(m)}{s}}{m^2} - \frac{e\overset{(m+n)}{s}}{\overline{m+n}|^2} + \frac{e''\overset{(m+2n)}{s}}{\overline{m+2n}|^2} \text{ \&c.}$$

$Q_{''}$ denoting the series $\dfrac{1}{m^2} + \dfrac{e}{\overline{m+n}|^2} + \dfrac{e''}{\overline{m+2n}|^2}$ &c.

and b the sine of $\dfrac{2ma}{n}$, of which d is the cosine.

And from hence, by proceeding in the same manner, other theorems may be deduced.

When x is $= 1$, z is $= 0$; and $\overset{(m)}{s}$, $\overset{(m+n)}{s}$, $\overset{(m+2n)}{s}$, &c. each $= 0$: therefore it appears that

$bQ_{''}$

$$bQ_{,,} \text{ is} = \frac{2adG}{n} + \text{the fluent of } \frac{\dot{x}F}{\sqrt{1-x^2}},$$

generated whilſt x from c becomes equal to 1.

Example 1. If m be $= \frac{1}{2}en$, f and F will each be $= 0$; and conſequently $Q_{,,} = \frac{2adG}{bn} = \frac{2aH}{bn}$;

$$\text{or } \frac{2b}{an} \times \overline{\frac{1}{e^2} + \frac{e}{\overline{e+2}|^2} + \frac{e'}{\overline{e+4}|^2} \ \&c.}$$

$$= dG \left(= d \times \text{the } \textit{whole} \text{ fluent of } \frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1-x^n}|^{e}}\right)$$

$$= H = \text{the fluent of } \frac{x^{\frac{1}{2}en-1}\dot{x}}{\overline{1+x^n}|^{e}},$$

generated whilſt x from 0 becomes $= 1$;

which is a very uſeful theorem, for computing the values of ſuch fluents in numbers.

Taking e and m each $= \frac{1}{2}$, and $n = 2$; b will be $= d$ $= \frac{1}{\sqrt{2}}$: conſequently we find

$$1 + \frac{1}{2.5^2} + \frac{1.3}{2.4.9^2} + \frac{1.3.5}{2.4.6.13^2} \ \&c.$$

$$= \frac{1}{4}Q_{,,} = \frac{aG}{4} = \frac{a}{4} \times \overline{E + \sqrt{E^2 - 2a}};$$

G being (in this caſe) the *whole* fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$; the value whereof, as is ſhewn in the Appendix, is $E + \sqrt{E^2 - 2a}$, E denoting the quadrantal arc of an ellipſis whoſe ſemi-axes are $\sqrt{2}$ and 1.

Example

Example 2. If e be $= \frac{1}{2}$, $m = \frac{1}{2}$, and $n = 2$; b will be $= \frac{1}{\sqrt{2}}$, $d = -\frac{1}{\sqrt{2}}$, $c = 0$, $f = 1$, $\dot{F} = \frac{\dot{x}}{\sqrt{2x}}$, and $F = \sqrt{2x}$, fuppofing the fluent to begin when x begins: confequently we have

$$\frac{1}{3^2} + \frac{1}{2.7^2} + \frac{1.3}{2.4.11^2} + \frac{1.3.5}{2.4.6.15^2} \text{ \&c.}$$

$$= \tfrac{1}{4}Q_{,,} = \frac{2F'' - aG}{4} = \frac{2-a}{4} \times \overline{E - \sqrt{E^2 - 2a}};$$

G being now the *whole* fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$, and F'' equal to the fame fluent.

Example 3. Taking $e = 1$, $m = 2$, and $n = 2$; we have $f = 1$, $\dot{F} = \frac{\dot{x}}{2x}$, and

$$Hz - \tfrac{1}{2}\,\text{fl.}\,\overline{\frac{\dot{x}}{\sqrt{1-x^2}}\cdot \text{fl.}\,\frac{\dot{x}}{x}} = \frac{s''}{2^2} - \frac{s^{\text{IV}}}{4^2} + \frac{s^{\text{VI}}}{6^2} \text{ \&c.}$$

the fluent being fuppofed to begin when x is $= 1$, and H being $= \frac{1}{2}$Log. 2 = the fluent of $\frac{x\dot{x}}{1+x^2}$ generated whilft x from 0 becomes equal to 1.

But fl. $\overline{\frac{\dot{x}}{\sqrt{1-x^2}}\cdot \text{fl.}\,\frac{\dot{x}}{x}}$ is $= -z\,\text{Log.}\,x - \text{fl.}\,\frac{\dot{x}}{x}\,.\,\text{fl.}\,\overline{\frac{\dot{x}}{\sqrt{1-x^2}}}$

$$= q_{,,} + \overline{a - z}\,.\text{Log.}\,x - x - \frac{x^3}{2.3^2} - \frac{1.3x^5}{2.4.5^2} - \frac{1.3.5x^7}{2.4.6.7^2} \text{ \&c.}$$

$$q_{,,} \text{ being } = 1 + \frac{1}{2.3^2} + \frac{1.3}{2.4.5^2} + \frac{1.3.5}{2.4.6.7^2} \text{ \&c.}$$

Therefore

$\frac{1}{2}z$

$$\tfrac{1}{2}z\,\text{Log.}\,2-\tfrac{1}{2}\overline{a-z}.\text{Log.}\,x-\tfrac{1}{2}q_{\prime\prime}+\tfrac{1}{2}\times\overline{x+\frac{x^3}{2.3^2}\,\&c.}\ \text{is}=\frac{s''}{2^2}-\frac{s^{iv}}{4^2}\,\&c.$$

Hence, by taking $x=0$, it appears that $q_{\prime\prime}$ is $=a$ Log. 2; z being then $=a$, $\overline{a-z}$.Log. $x=0$, and s'', s^{iv}, &c. each $=0$. Confequently, after fubftituting for $q_{\prime\prime}$ its value fo found, it follows, that

$$x+\frac{x^3}{2.3^2}+\frac{1.3x^5}{2.4.5^2}\,\&c.\ \text{is}=\overline{a-z}.\text{Log.}\,2x+\tfrac{1}{2}\times\overline{s''-\frac{s^{iv}}{2^2}+\frac{s^{vi}}{3^2}\,\&c.}$$

If x be $=\frac{1}{\sqrt{2}}$, z will be $=\frac{a}{2}$; and s'', s^{iv}, s^{vi}, s^{viii}, &c. equal to 1, 0, -1, 0, &c. refpectively: therefore it is evident, that

$$\sqrt{2}\times\overline{1+\frac{2^{-1}}{2.3^2}+\frac{1.3.2^{-2}}{2.4.5^2}\,\&c.}\ \text{is}=\tfrac{1}{2}a\,\text{Log.}\,2+1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}\,\&c.$$

Which laft feries being known to be equal to fl. $\overline{\frac{\dot{x}}{\sqrt{1-x^2}}\cdot\text{fl.}\,\frac{\dot{x}}{\sqrt{1+x^2}}}$, or $=$ fl. $\overline{\frac{\dot{x}}{x}\cdot\text{fl.}\,\frac{\dot{x}}{1+x^2}}$, each generated whilft x from 0 becomes equal to 1; and the feries $1+\frac{2^{-1}}{2.3^2}$ &c. being equal to $\frac{1}{2^{\frac{3}{2}}}$ fl. $\overline{\frac{\dot{x}}{x}\cdot\text{fl.}\,\frac{\dot{x}}{\sqrt{2x-x^2}}}$, generated in the fame time; it follows, that

$$\text{fl.}\,\overline{\frac{\dot{x}}{\sqrt{1-x^2}}\cdot\text{Log.}\,\overline{x+\sqrt{1+x^2}}}\ \text{is}=\text{fl.}\,\frac{t\dot{x}}{x}=\tfrac{1}{2}\,\text{fl.}\,\frac{v\dot{x}}{x_{\prime}}-\tfrac{1}{2}a\,\text{Log.}\,2,$$

$$\text{and fl.}\,\overline{v-2t}.\frac{\dot{x}}{x}=a\,\text{Log.}\,2=\text{fl.}\,\overline{a-z}.\frac{\dot{x}}{x};$$

thefe fluents being all generated in the time before mentioned,

tioned, and t denoting the circular arc whose radius is 1 and tangent x, and v an arc (of the same circle) whose versed sine is x.

Example 4. If e be $= \frac{1}{2}$, $m = 2$, and $n = 4$; b will be $= 1$, $d = 0$, $c = \frac{1}{\sqrt{2}}$, $f = 1$, and $\dot{F} = \frac{\dot{x}}{2^{\frac{1}{2}}\sqrt{2x^2 - 1}}$: it follows therefore, that

$$2^{\frac{1}{2}} \text{fl.} \overline{\frac{\dot{x}}{\sqrt{1-x^2}} \cdot \text{fl.} \frac{\dot{x}}{\sqrt{2x^2-1}}},$$ generated whilst x from c becomes $= 1$,

$$\text{is} = 1 + \frac{1}{2 \cdot 3^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7^2} \ \&c.$$

which, by the preceding example, is $= a$ Log. 2.

Example 5. If m be $= e$ and $n = 2$, f and F will each be $= 0$; therefore, taking z equal to a, we have

$$Q_{(e)} = \frac{1}{e^3} - \frac{e}{\overline{e+2}|^3} + \frac{e''}{\overline{e+4}|^3} \ \&c. = \tfrac{1}{2}a^2 H + d \times \overline{\frac{1}{e^3} + \frac{e}{\overline{e+2}|^3} + \frac{e''}{\overline{e+4}|^3} \ \&c.}$$

d being the cosine of ea.

and $H = \text{fl.} \frac{x^{e-1}\dot{x}}{\overline{1+x^2}|^e}$, generated whilst x from 0 becomes $= 1$;

which fluent, by art. 24, is $= d \times$ the *whole* fluent of $\frac{x^{e-1}\dot{x}}{\overline{1-x^2}|^e}$.

When e is $= 1$, d is $= 0$, and $H = \frac{1}{2}a$; and we have, in that case, $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}$ &c. $= \frac{1}{4}a^3$, as in art. 8.

27. From

27. From the equation $\frac{\overset{(m-\frac{1}{2}en)}{c}}{\overline{2\overset{(\frac{1}{2}n)}{c}}|^{e}} = \overset{(m)}{c} - e\overset{(m+n)}{c}$ &c. in art. 21. by multiplying by $\frac{-\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents according to what is shewn in art. 1. we get

$$bG - \text{fl.}\ \overline{\frac{\dot{x}}{\sqrt{1-x^2}} \cdot \frac{\overset{(m-\frac{1}{2}en)}{c}}{\overline{2\overset{(\frac{1}{2}n)}{c}}|^{e}}} = \frac{\overset{(m)}{s}}{m} - \frac{e\overset{(m+n)}{s}}{m+n} + \frac{e''\overset{(m+2n)}{s}}{m+2n} \text{ \&c.}$$

the fluent being supposed to begin when x is $(=c)$ the cosine of $\frac{2a}{n}$; the sines $\overset{(m)}{s}$, $\overset{(m+n)}{s}$, $\overset{(m+2n)}{s}$, $\overset{(m+3n)}{s}$, &c. being then equal to b, $-b$, b, $-b$, &c. respectively.

When x is $= 1$, the sines $\overset{(m)}{s}$, $\overset{(m+n)}{s}$, $\overset{(m+2n)}{s}$, &c. are each $= 0$: consequently bG is $=$ fl. $\overline{\frac{\dot{x}}{\sqrt{1-x^2}} \cdot \frac{\overset{(m-\frac{1}{2}en)}{c}}{\overline{2\overset{(\frac{1}{2}n)}{c}}|^{e}}}$, generated whilst x from c becomes equal to 1.

If the fluent be considered as beginning when x is $= 1$, the theorem will stand (without the term bG)

$$-\text{fl.}\ \overline{\frac{\dot{x}}{\sqrt{1-x^2}} \cdot \frac{\overset{(m-\frac{1}{2}en)}{c}}{\overline{2\overset{(\frac{1}{2}n)}{c}}|^{e}}} = \frac{\overset{(m)}{s}}{m} - \frac{e\overset{(m+n)}{s}}{m+n} + \frac{e''\overset{(m+2n)}{s}}{m+2n} \text{ \&c.}$$

Example 1. Taking $e = 1$, and m and n each equal 2, we have $z = s'' - \frac{s^{iv}}{2} + \frac{s^{vi}}{3}$ &c. when x is positive: from

G whence,

whence, by multiplying by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents, other theorems may be easily deduced, as in art. 6. and 7.

Example 2. If e be $= \frac{1}{3}$, $m = 1$, and $n = 6$; b will be $= \frac{1}{2}$, $c = \frac{\sqrt{3}}{2}$, and

$$\tfrac{1}{2}\mathrm{G} - \text{the fluent of } \frac{2^{-\frac{1}{3}} x^{-\frac{1}{3}} \dot{x}}{\overline{4x^2-3}|^{\frac{1}{3}} \times \sqrt{1-x^2}}$$

$$= s' - \frac{s^{\text{vii}}}{3.7} + \frac{4 s^{\text{xiii}}}{3.6.13} - \frac{4.7 s^{\text{xix}}}{3.6.9.19} \ \&c.$$

G denoting the whole fluent of $\frac{\dot{x}}{\overline{1-x^6}|^{\frac{1}{3}}}$, or of $\frac{\frac{1}{3} y^{-\frac{2}{3}} \dot{y}}{\overline{1-y^2}|^{\frac{1}{3}}}$, generated whilst x, or y, from o becomes equal to 1: the value whereof is assigned, in Table IV. of the Appendix, by means of a *circular* and an *elliptic* arc.

The series $s' - \frac{s^{\text{vii}}}{3.7}$ &c. vanishing when x is $= 1$, the fluent of $\frac{2^{-\frac{1}{3}} x^{-\frac{1}{3}} \dot{x}}{\overline{4x^2-3}|^{\frac{1}{3}} \times \sqrt{1-x^2}}$, generated whilst x from $\frac{\sqrt{3}}{2}$ becomes equal to 1, is $= \frac{1}{2}$G.

Example 3. If m be $= e$ and $n = 2$,

$$b\mathrm{G} - \text{fl.} \frac{\dot{x}}{2^e x^e \sqrt{1-x^2}} \text{ will be} = \frac{s^{(e)}}{e} - \frac{e\, s^{(e+2)}}{e+2} + \frac{e'' s^{(e+4)}}{e+4} \ \&c.$$

and the *whole* fluent of $\frac{\dot{x}}{2^e x^e \sqrt{1-x^2}} = b\mathrm{G} = b \times$ the *whole* flu. of $\frac{x^{e-1}\dot{x}}{\overline{1-x^2}|^e}$

$= \frac{b}{d}$

$= \frac{b}{d} \times$ the contemporary fluent of $\frac{x^{e-1}\dot{x}}{\overline{1+x^2}|^e}$, by art. 24.

b and d being the fine and cofine of ea.

Example 4. Taking $m = e + 1$ and $n = 2$, we have

$$bG - \text{fl.}\,\frac{x^{1-e}\dot{x}}{2^e\sqrt{1-x^2}} = \frac{s^{(e+1)}}{e+1} - \frac{e\,s^{(e+3)}}{e+3} + \frac{e''\,s^{(e+5)}}{e+5}\ \&c.$$

and $bG =$ the *whole* fluent of $\frac{x^{1-e}\dot{x}}{2^e\sqrt{1-x^2}}$.

Moreover, G, by Ex. 2. art. 25. being $= \frac{1}{d} \times \overline{H - \frac{1}{2^e.\overline{1-e}}}$, it appears that

$\frac{b}{d} \times \overline{H - \frac{1}{2^e.\overline{1-e}}}$ is alfo = the fame *whole* fluent;

b and d being here the fine and cofine of $\overline{e+1}.a$,

$\therefore$ G = the *whole* fluent of $\frac{x^e\dot{x}}{\overline{1-x^2}|^e}$

$=$ the feries $\frac{1}{e+1} + \frac{e}{e+3} + \frac{e''}{e+5} + \frac{e'''}{e+7}$ &c.

and H = the contemporary fluent of $\frac{x^e\dot{x}}{\overline{1+x^2}|^e}$

$=$ the feries $\frac{1}{e+1} - \frac{e}{e+3} + \frac{e''}{e+5} - \frac{e'''}{e+7}$ &c.

Other theorems may be deduced from the general theorem in this article, by multiplying by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents, as in the preceding article.

 28. By

28. By divifion we have

$$\frac{1}{\overline{x-\sqrt{x^2-1}}|^{\frac{1}{2}n}+y.\overline{x+\sqrt{x^2-1}}|^{\frac{1}{2}n}}=\overline{x+\sqrt{x^2-1}}|^{\frac{n}{2}}-y.\overline{x+\sqrt{x^2-1}}|^{\frac{3n}{2}}$$
$$+y^2.\overline{x+\sqrt{x^2-1}}|^{\frac{5n}{2}} \text{ \&c.}$$

and $$\frac{1}{\overline{x+\sqrt{x^2-1}}|^{\frac{1}{2}n}+y.\overline{x-\sqrt{x^2-1}}|^{\frac{1}{2}n}}=\overline{x-\sqrt{x^2-1}}|^{\frac{n}{2}}-y.\overline{x-\sqrt{x^2-1}}|^{\frac{3n}{2}}$$
$$+y^2.\overline{x-\sqrt{x^2-1}}|^{\frac{5n}{2}} \text{ \&c.}$$

whence, by fubtracting one equation from the other, and dividing by $2\sqrt{-1}$, we get

$$\overline{1-y}.\frac{s^{(\frac{1}{2}n)}}{1+y^2+2yc^{(n)}}=s^{(\frac{n}{2})}-y.s^{(\frac{3n}{2})}+y^2 s^{(\frac{5n}{2})} \text{ \&c.}$$

and, by taking n equal to 2,

$$\overline{1-y}.\frac{\overline{1-x^2}|^{\frac{1}{2}}}{\overline{1-y}|^2+4yx^2}=s'-ys'''+y^2s^{v} \text{ \&c.}$$

Hence, by multiplying by $\frac{\dot{x}}{\sqrt{1-x^2}}$, taking the fluents, and writing y^2 inftead of y, we find

$$\tfrac{1}{2}\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{2xy}{1-y^2}=c'y-\frac{c'''y^3}{3}+\frac{c^{v}y^5}{5} \text{ \&c.}$$

where both x and y may be confidered as variable, independent of each other.

Example 1. Taking $x=\frac{1}{\sqrt{2}}$, we have c', c''', c^{v}, c^{vii}, &c. equal to $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, &c. refpectively; and

$\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \text{ Circ. Arc, rad. } 1, \text{ tang. } \frac{2^{\frac{1}{2}}y}{1-y^2} = y + \frac{y^3}{3} - \frac{y^5}{5} - \frac{y^7}{7} \;\&c.$$

$$= \text{fl. } \frac{\dot{y}}{1+y^4} + \text{fl. } \frac{y^2\dot{y}}{1+y^4}.$$

Moreover, the ſum of theſe two fluents, generated whilſt y from o becomes equal to 1, being, by art. 17. equal to the *whole* fluent of $\frac{\dot{y}}{1+y^4}$, or of $\frac{y^2\dot{y}}{1+y^4}$, generated whilſt y from o becomes infinite; it appears that each of theſe *whole* fluents is $= \frac{a}{\sqrt{2}}$.

Example 2. If x be $= \frac{1}{2}$; c', c''', c^{v}, c^{vii}, c^{ix}, &c. will be equal to $\frac{1}{2}$, -1, $\frac{1}{2}$, $\frac{1}{2}$, -1, &c. reſpectively; and

$$\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{y}{1-y^2} \quad y + \frac{2y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} - \frac{2y^9}{9} - \frac{y^{11}}{11} \;\&c.$$

$$= \text{fl. } \frac{\dot{y}}{1+y^6} + 2 \text{ fl. } \frac{y^2\dot{y}}{1+y^6} + \text{fl. } \frac{y^4\dot{y}}{1+y^6}:$$

or, fl. $\frac{y^2\dot{y}}{1+y^6}$ being $= \frac{1}{3}$ Circ. Arc, rad. 1, tang. y^3,

$$\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{y}{1-y^2} - \frac{2}{3} \text{ Circ. Arc, rad. } 1, \text{ tang. } y^3$$

$$= y - \frac{y^7}{7} + \frac{y^{13}}{13} - \frac{y^{19}}{19} \;\&c. + \frac{y^5}{5} - \frac{y^{11}}{11} + \frac{y^{17}}{17} - \frac{y^{23}}{23} \;\&c.$$

$$= \text{fl. } \frac{\dot{y}}{1+y^6} + \text{fl. } \frac{y^4\dot{y}}{1+y^6}.$$

Moreover, the ſum of theſe two laſt written fluents, generated whilſt y from o becomes equal to 1, being, by art. 17. equal to the *whole* fluent of $\frac{\dot{y}}{1+y^6}$, or of $\frac{y^4\dot{y}}{1+y^6}$, generated whilſt y from o becomes infinite; it follows, that each of theſe *whole* fluents is $= \frac{2a}{3}$.

29. Add-

29. Adding together the firſt two equations in the preceding article, and dividing by 2, we have

$$\frac{\overline{1+y}\cdot c^{\left(\frac{n}{2}\right)}}{1+y^2+2y\,c^{(n)}} = c^{\left(\frac{n}{2}\right)} - c^{\left(\frac{3n}{2}\right)} y + c^{\left(\frac{5n}{2}\right)} y^2 \text{ \&c.}$$

and, by taking n equal to 2,

$$\frac{\overline{1+y}\cdot x}{\overline{1-y}|^2 + 4yx^2} = c' - c'''y + c^{v}y^2 \text{ \&c.}$$

Hence, by multiplying by $\frac{-\dot{x}}{\sqrt{1-x^2}}$, taking the fluents, and writing y^2 inſtead of y, we have

$$\frac{1}{4}\text{Log.}\,\frac{1+y^2+2y\sqrt{1-x^2}}{1+y^2-2y\sqrt{1-x^2}} = s'y - \frac{s'''y^3}{3} + \frac{s^{v}y^5}{5} \text{ \&c.}$$

Example 1. Taking $x = \frac{1}{\sqrt{2}}$, we have s', s''', s^{v}, s^{vii}, &c. equal to $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, &c. reſpectively, and

$$\frac{1}{2^{\frac{3}{2}}}\text{Log.}\,\frac{1+y^2+2^{\frac{1}{2}}y}{1+y^2-2^{\frac{1}{2}}y} = y - \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} \text{ \&c.}$$

$$= \text{fl.}\,\frac{\dot{y}}{1+y^4} - \text{fl.}\,\frac{y^2\dot{y}}{1+y^4}.$$

But we have found, in Ex. 1. of the preceding article, that the ſum of theſe two laſt written fluents is $\frac{A}{\sqrt{2}}$. Conſequently we find

$$\text{fl.}\,\frac{\dot{y}}{1+y^4} = \frac{A+L}{2^{\frac{3}{2}}}; \text{ and fl.}\,\frac{y^2\dot{y}}{1+y^4} = \frac{A-L}{2^{\frac{3}{2}}};$$

A de-

A denoting the Circ. Arc, rad. 1, tang. $\frac{2^{\frac{1}{2}}y}{1-y^2}$,

and L the Log. of $\overline{\frac{1+y^2+2^{\frac{1}{2}}y}{1+y^2-2^{\frac{1}{2}}y}}\Big|^{\frac{1}{2}} = \text{Log.}\ \frac{1+y^2+2^{\frac{1}{2}}y}{\sqrt{1+y^2}}$.

Example 2. If x be $=\frac{1}{2}$; s', s''', s^{v}, s^{vii}, will be equal to $\frac{\sqrt{3}}{2}$, 0, $-\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, &c. respectively; and

$$\frac{1}{2\sqrt{3}}\ \text{Log.}\ \frac{1+y^2+3^{\frac{1}{2}}y}{1+y^2-3^{\frac{1}{2}}y} = y - \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^{11}}{11}\ \&c.$$

$$= \text{fl.}\ \frac{\dot{y}}{1+y^6} - \text{fl.}\ \frac{y^4\dot{y}}{1+y^6}.$$

Hence, and from Ex. 2. of the preceding article, we have fl. $\frac{\dot{y}}{1+y^6} = \frac{A}{2} - \frac{A''}{3} + \frac{L}{4\sqrt{3}}$, and fl. $\frac{y^4\dot{y}}{1+y^6} = \frac{A}{2} - \frac{A''}{3} - \frac{L}{4\sqrt{3}}$; A being = Circ. Arc, rad. 1, tang. $\frac{y}{1-y^2}$; A″ = Circ. Arc, rad. 1, tang. y^3; and L = Log. $\frac{1+y^2+3^{\frac{1}{2}}y}{1+y^2-3^{\frac{1}{2}}y}$.

30. Multiplying the first equation in art. 28. by $\overline{x+\sqrt{x^2-1}}\big|^{\frac{n}{2}}$, and the second by $\overline{x-\sqrt{x^2-1}}\big|^{\frac{n}{2}}$, we have

$$\frac{\overline{x+\sqrt{x^2-1}}\big|^{\frac{n}{2}}}{\overline{x-\sqrt{x^2-1}}\big|^{\frac{n}{2}} + y.\overline{x+\sqrt{x^2-1}}\big|^{\frac{n}{2}}} = \overline{x+\sqrt{x^2-1}}\big|^{n} - y.\overline{x+\sqrt{x^2-1}}\big|^{2n}\ \&c.$$

and
$$\frac{\overline{x-\sqrt{x^2-1}}\big|^{\frac{n}{2}}}{\overline{x+\sqrt{x^2-1}}\big|^{\frac{n}{2}} + y.\overline{x-\sqrt{x^2-1}}\big|^{\frac{n}{2}}} = \overline{x-\sqrt{x^2-1}}\big|^{n} - y.\overline{x-\sqrt{x^2-1}}\big|^{2n}\ \&c.$$

Whence,

Whence, by ſubtracting one equation from the other, and dividing by $2\sqrt{-1}$, we get

$$\frac{\overset{(n)}{s}}{1+y^2+2y\overset{(n)}{c}} = \overset{(n)}{s} - \overset{(2n)}{s}\,y + \overset{(3n)}{s}\,y^2 - \overset{(4n)}{s}\,y^3 \text{ \&c.}$$

Hence, by multiplying by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents, we, upon adjuſting the equation, find

$$\tfrac{1}{2}\text{Log.}\,1+y^2+2y\overset{(n)}{c} = \overset{(n)}{c}\,y - \frac{\overset{(2n)}{c}\,y^2}{2} + \frac{\overset{(3n)}{c}\,y^3}{3} - \frac{\overset{(4n)}{c}\,y^4}{4} \text{ \&c.}$$

31. Writing $-y$ inſtead of y, we have (from the laſt theorem)

$$\tfrac{1}{2}\text{Log.}\,1+y^2-2y\overset{(n)}{c} = -\overset{(n)}{c}\,y - \frac{\overset{(2n)}{c}\,y^2}{2} - \frac{\overset{(3n)}{c}\,y^3}{3} \text{ \&c.}$$

and conſequently (from the laſt two theorems)

$$\tfrac{1}{4}\text{Log.}\,\frac{1+y^2+2y\overset{(n)}{c}}{1+y^2-2y\overset{(n)}{c}} = \overset{(n)}{c}\,y + \frac{\overset{(3n)}{c}\,y^3}{3} + \frac{\overset{(5n)}{c}\,y^5}{5} \text{ \&c.}$$

$$\text{and } \tfrac{1}{4}\text{Log.}\,\frac{1}{\overline{1+y^2}\,^2 - 4y^2\cdot\overline{\overset{(n)}{c}}\,^2} = \frac{\overset{(2n)}{c}\,y^2}{2} + \frac{\overset{(4n)}{c}\,y^4}{4} + \frac{\overset{(6n)}{c}\,y^6}{6} \text{ \&c.}$$

Example. If x be $= \frac{1}{2}$; c', c''', c^{v}, &c. will be as in Ex. 2. art. 28. and therefore, from the laſt theorem but one, n being therein taken equal to 1, we have

$$\tfrac{1}{2}\text{Log.}\,\frac{1+y^2+y}{1+y^2-y} = y - \frac{2y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} - \frac{2y^9}{9} + \frac{y^{11}}{11} \text{ \&c.}$$

= fl.

$$= \text{fl.}\ \frac{\dot{y}}{1-y^6} - 2\ \text{fl.}\ \frac{y^2\dot{y}}{1-y^6} + \text{fl.}\ \frac{y^4\dot{y}}{1-y^6}:$$

or, $\text{fl.}\ \frac{y^2\dot{y}}{1-y^6}$ being $= \frac{1}{6}\,\text{Log.}\ \frac{1+y^3}{1-y^3}$,

$$\tfrac{1}{2}\,\text{Log.}\ \frac{1+y^2+y}{1+y^2-y} + \tfrac{1}{3}\,\text{Log.}\ \frac{1+y^3}{1-y^3}$$

$$= y + \frac{y^7}{7}\ \&c. + \frac{y^5}{5} + \frac{y^{11}}{11}\ \&c. = \text{fl.}\ \frac{\dot{y}}{1-y^6} + \text{fl.}\ \frac{y^4\dot{y}}{1-y^6}.$$

32. Taking n equal to 1, we have from the preceding article and art. 28.

$$\tfrac{1}{8}\,\text{Log.}\ \frac{1+y^2+2xy}{1+y^2-2xy} + \tfrac{1}{4}\,\text{Circ. Arc, rad. 1, tang.}\ \frac{2xy}{1-y^2}$$

$$= c'y + \frac{c^{v}y^5}{5} + \frac{c^{ix}y^9}{9}\ \&c.$$

and $\frac{1}{8}\,\text{Log.}\ \frac{1+y^2+2xy}{1+y^2-2xy} - \frac{1}{4}\,\text{Circ. Arc, rad. 1, tang.}\ \frac{2xy}{1-y^2}$

$$= \frac{c'''y^3}{3} + \frac{c^{vii}y^7}{7} + \frac{c^{xi}y^{11}}{11}\ \&c.$$

Example 1. Taking x equal to 1, we have

$$\tfrac{1}{4}\,\text{Log.}\ \frac{1+y}{1-y} + \tfrac{1}{2}\,\text{Circ. Arc, rad. 1, tang.}\ y$$

$$= y + \frac{y^5}{5} + \frac{y^9}{9}\ \&c. = \text{fl.}\ \frac{\dot{y}}{1-y^4};$$

and $\frac{1}{4}\,\text{Log.}\ \frac{1+y}{1-y} - \frac{1}{2}\,\text{Circ. Arc, rad. 1, tang.}\ y$

$$= \frac{y^3}{3} + \frac{y^7}{7} + \frac{y^{11}}{11}\ \&c. = \text{fl.}\ \frac{y^2\dot{y}}{1-y^4};$$

Circ. Arc, rad. 1, tang. $\frac{2y}{1-y^2}$ being = 2 Circ. Arc, rad. 1, tang. y.

Example 2. If x be $= \frac{1}{2}$; c', c''', c^{v}, &c. will be as in Ex. 2. art. 28. and

P $\frac{1}{4}$ Log.

$\frac{1}{4}$ Log. $\frac{1+y^2+y}{1+y^2-y} + \frac{1}{2}$ Circ. Arc, rad. 1, tang. $\frac{y}{1-y^2}$

$= y + \frac{y^5}{5} - \frac{2y^9}{9} + \frac{y^{13}}{13} + \frac{y^{17}}{17} - \frac{2y^{21}}{21}$ &c.

$= \text{fl.}\,\frac{\dot{y}}{1-y^{12}} + \text{fl.}\,\frac{y^4\dot{y}}{1-y^{12}} - 2\,\text{fl.}\,\frac{y^8\dot{y}}{1-y^{12}}$;

and $\frac{1}{4}$ Log. $\frac{1+y^2+y}{1+y^2-y} - \frac{1}{2}$ Circ. Arc, rad. 1, tang. $\frac{y}{1-y^2}$

$= -\frac{2y^3}{3} + \frac{y^7}{7} + \frac{y^{11}}{11} - \frac{2y^{15}}{15} + \frac{y^{19}}{19} + \frac{y^{23}}{23}$ &c.

$= -2\,\text{fl.}\,\frac{y^2\dot{y}}{1-y^{12}} + \text{fl.}\,\frac{y^6\dot{y}}{1-y^{12}} + \text{fl.}\,\frac{y^{10}\dot{y}}{1-y^{12}}$.

But by Ex. 1. fl. $\frac{y^2\dot{y}}{1-y^{12}}$ and fl. $\frac{y^8\dot{y}}{1-y^{12}}$ are equal to

$\frac{1}{12}$ Log. $\frac{1+y^3}{1-y^3} + \frac{1}{6}$ Circ. Arc, rad. 1, tang. y^3,

and $\frac{1}{12}$ Log. $\frac{1+y^3}{1-y^3} - \frac{1}{6}$ Circ. Arc, rad. 1, tang. y^3, respectively. Consequently we have

$\frac{1}{4}$ Log. $\frac{1+y^2+y}{1+y^2-y} + \frac{1}{6}$ Log. $\frac{1+y^3}{1-y^3} + \frac{1}{2}$ Circ. Arc, rad. 1, tang. $\frac{y}{1-y^2}$

$- \frac{1}{3}$ Circ. Arc, rad. 1, tang. $y^3 = \text{fl.}\,\frac{\dot{y}}{1-y^{12}} + \text{fl.}\,\frac{y^4\dot{y}}{1-y^{12}}$;

and $\frac{1}{4}$ Log. $\frac{1+y^2+y}{1+y^2-y} + \frac{1}{6}$ Log. $\frac{1+y^3}{1-y^3} - \frac{1}{2}$ Circ. Arc, rad. 1, tang. $\frac{y}{1-y^2}$

$+ \frac{1}{3}$ Circ. Arc, rad. 1, tang. $y^3 = \text{fl.}\,\frac{y^6\dot{y}}{1-y^{12}} + \text{fl.}\,\frac{y^{10}\dot{y}}{1-y^{12}}$.

33. Adding together the first two equations in Art. 30. and dividing by 2, we have

$$\frac{y + \overset{(n)}{c}}{1+y^2+2y\overset{(n)}{c}} = \overset{(n)}{c} - y\overset{(2n)}{c} + y^2\overset{(3n)}{c} - y^3\overset{(4n)}{c} \text{ \&c.}$$

Hence,

Hence, by taking n equal to 1, multiplying by $\frac{-y\dot{x}}{\sqrt{1-x^2}}$ and taking the fluents, we find

$$\frac{\pi}{2} - \text{Circ. Arc. rad. } 1, \text{ tang. } \frac{1-y}{1+y} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$$

$$= \text{Circ. Arc, rad. } 1, \text{ tang. } \frac{y\sqrt{1-x^2}}{xy+1} = s'y - \frac{s''y^2}{2} + \frac{s'''y^3}{3} \text{ \&c.}$$

34. Writing $-y$ instead of y, we have

$$\frac{\pi}{2} - \text{Circ. Arc, rad. } 1, \text{ tang. } \frac{1+y}{1-y} \times \overline{\frac{1-x}{1+x}}\Big|^{\frac{1}{2}}$$

$$= \text{Circ. Arc, rad. } 1, \text{ tang. } \frac{y\sqrt{1-x^2}}{xy-1} = -s'y - \frac{s''y^2}{2} - \frac{s'''y^3}{3} \text{ \&c.}$$

and consequently (from the last two theorems)

$$\tfrac{1}{2}\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{2y}{1-y^2}\sqrt{1-x^2} = s'y + \frac{s'''y^3}{3} + \frac{s^{v}y^5}{5} \text{ \&c.}$$

and

$$\tfrac{1}{2}\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{2xy^2}{1+y^2-2x^2y^2}\sqrt{1-x^2} = \frac{s''y^2}{2} + \frac{s^{iv}y^4}{4} + \frac{s^{vi}y^6}{6} \text{\&c.}$$

Example. If x be $= \frac{1}{2}$; s', s''', s^{v}, &c. will be equal to $\frac{\sqrt{3}}{2}$, 0, $\frac{-\sqrt{3}}{2}$, &c. respectively; and

$$\frac{1}{3^{\frac{1}{2}}}\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{3^{\frac{1}{2}}y}{1-y^2}$$

$$= y - \frac{y^5}{5} + \frac{y^7}{7} - \frac{y^{11}}{11} \text{ \&c.} = \text{fl. } \frac{\dot{y}}{1-y^6} - \text{fl. } \frac{y^4\dot{y}}{1-y^6}.$$

Hence, and from art. 31. we have fl. $\frac{\dot{y}}{1-y^6} =$

$$\tfrac{1}{4}\text{Log. } \frac{1+y^2+y}{1+y^2-y} + \tfrac{1}{6}\text{Log. } \frac{1+y^3}{1-y^3} + \frac{1}{2\sqrt{3}}\text{Circ. Arc, rad. } 1, \text{ tang. } \frac{3^{\frac{1}{2}}y}{1-y^2};$$

P 2 and

and fl. $\frac{y^4\dot{y}}{1-y^6}=$

$\frac{1}{4}\text{Log.}\frac{1+y^2+y}{1+y^2-y}+\frac{1}{6}\text{Log.}\frac{1+y^3}{1-y^3}-\frac{1}{2\sqrt{3}}\text{Circ. Arc, rad. 1, tang.}\frac{3^{\frac{1}{2}}y}{1-y^2}.$

In the ſame manner, from the 3d theorem in art. 31. and the 3d theorem in this article, we find fl. $\frac{y\dot{y}}{1-y^6}=$

$\frac{1}{4}\text{Log.}\,1+y^2+y^4-\frac{1}{6}\text{Log.}\,1-y^6+\frac{1}{2\sqrt{3}}\text{Circ. Arc, rad. 1, tang.}\frac{3^{\frac{1}{2}}y^2}{2+y^2};$

and fl. $\frac{y^3\dot{y}}{1-y^6}=$

$\frac{1}{4}\text{Log.}\,1+y^2+y^4-\frac{1}{6}\text{Log.}\,1-y^6-\frac{1}{2\sqrt{3}}\text{Circ. Arc, rad. 1, tang.}\frac{3^{\frac{1}{2}}y^2}{2+y^2}:$

or fl. $\frac{\dot{y}}{1-y^3}=$

$\frac{1}{2}\text{Log.}\,1+y+y^2-\frac{1}{3}\text{Log.}\,1-y^3+\frac{1}{\sqrt{3}}\text{Circ. Arc, rad. 1, tang.}\frac{3^{\frac{1}{2}}y}{2+y};$

and fl. $\frac{y\dot{y}}{1-y^3}=$

$\frac{1}{2}\text{Log.}\,1+y+y^2-\frac{1}{3}\text{Log.}\,1-y^3-\frac{1}{\sqrt{3}}\text{Circ. Arc, rad. 1, tang.}\frac{3^{\frac{1}{2}}y}{2+y}.$

35. From the preceding article and art. 29. we have

$$\frac{1}{4}\text{Circ. Arc, rad. 1, tang.}\frac{2y}{1-y^2}\sqrt{1-x^2}+\frac{1}{8}\text{Log.}\frac{1+y^2+2y\sqrt{1-x^2}}{1+y^2-2y\sqrt{1-x^2}}$$
$$=s'y+\frac{s^{v}y^5}{5}+\frac{s^{ix}y^9}{9}\ \&c.$$

and

$$\frac{1}{4}\text{Circ. Arc, rad. 1, tang.}\frac{2y}{1-y^2}\sqrt{1-x^2}-\frac{1}{8}\text{Log.}\frac{1+y^2+2y\sqrt{1-x^2}}{1+y^2-2y\sqrt{1-x^2}}.$$
$$=\frac{s'''y^3}{3}+\frac{s^{vii}y^7}{7}+\frac{s^{xi}y^{11}}{11}\ \&c.$$

Example.

Example. Taking x equal to $\frac{1}{2}$, we have s', s''', s^{v}, &c. as in the preceding example; and

$$\frac{1}{2\sqrt{3}}\text{ Circ. Arc, rad. 1, tang. }\frac{3^{\frac{1}{2}}y}{1-y^2}+\frac{1}{4\sqrt{3}}\text{ Log. }\frac{1+y^2+3^{\frac{1}{2}}y}{1+y^2-3^{\frac{1}{2}}y}$$

$$=y-\frac{y^5}{5}+\frac{y^{13}}{13}-\frac{y^{17}}{17}\ \&c.=\text{fl. }\frac{\dot{y}}{1-y^{12}}-\text{fl. }\frac{y^4\dot{y}}{1-y^{12}};$$

and

$$\frac{1}{2\sqrt{3}}\text{ Circ. Arc, rad. 1, tang. }\frac{3^{\frac{1}{2}}y}{1-y^2}-\frac{1}{4\sqrt{3}}\text{ Log. }\frac{1+y^2+3^{\frac{1}{2}}y}{1+y^2-3^{\frac{1}{2}}y}$$

$$=\frac{y^7}{7}-\frac{y^{11}}{11}+\frac{y^{19}}{19}-\frac{y^{23}}{23}\ \&c.=\text{fl. }\frac{y^6\dot{y}}{1-y^{12}}-\text{fl. }\frac{y^{10}\dot{y}}{1-y^{12}}.$$

Hence, and from Ex. 2. art. 32. the fluents of $\frac{\dot{y}}{1-y^{12}}$, $\frac{y^4\dot{y}}{1-y^{12}}$, $\frac{y^6\dot{y}}{1-y^{12}}$, and $\frac{y^{10}\dot{y}}{1-y^{12}}$ may be readily obtained.

36. Multiplying the first equation in art. 28. by $\overline{x+\sqrt{x^2-1}}\big|^{m-\frac{n}{2}}$ and the 2d. equat. by $\overline{x-\sqrt{x^2-1}}\big|^{m-\frac{n}{2}}$, bringing the products together according to our method, writing y^q instead of y, and afterwards multiplying by $y^{p-1}\dot{y}$ and taking the fluents, we have

$$\text{fl. }y^{p-1}\dot{y}\times\frac{\overset{(m)}{s}-\overset{(n-m)}{s}\ y^q}{1+2\overset{(n)}{c}\ y^q+y^{2q}}=\frac{\overset{(m)}{s}\ y^p}{p}-\frac{\overset{(m+n)}{s}\ y^{p+q}}{p+q}+\frac{\overset{(m+2n)}{s}\ y^{p+2q}}{p+2q}\ \&c.$$

$$\text{and fl. }y^{p-1}\dot{y}\times\frac{\overset{(m)}{c}+\overset{(n-m)}{c}\ y^q}{1+2\overset{(n)}{c}\ y^q+y^{2q}}=\frac{\overset{(m)}{c}\ y^p}{p}-\frac{\overset{(m+n)}{c}\ y^{p+q}}{p+q}+\frac{\overset{(m+2n)}{c}\ y^{p+2q}}{p+2q}\ \&c.$$

Whence, by considering y only as variable, we may obtain all the theorems which we, by considering x only as variable, have deduced in art. 28. 29. 30. and 33. and we may

may likewise obtain, from these two, some other theorems not unworthy notice; the values of fl. $y^{p-1}\dot{y} \times \frac{\overset{(m)}{s} - \overset{(n-m)}{s} y^q}{1 + 2\overset{(n)}{c} y^q + y^{2q}}$ and fl. $y^{p-1}\dot{y} \times \frac{\overset{(m)}{c} + \overset{(n-m)}{c} y^q}{1 + 2\overset{(n)}{c} y^q + y^{2q}}$ being always assignable by means of circular arcs and logarithms, as will be shewn in the Appendix.

Taking p and q equal to m and n respectively, we have

$$\text{fl. } y^{m-1}\dot{y} \times \frac{\overset{(m)}{s} - \overset{(n-m)}{s} y^n}{1 + 2\overset{(n)}{c} y^n + y^{2n}} = \text{fl. } \frac{-y^m \dot{x}}{\sqrt{1-x^2}} \times \frac{\overset{(m)}{c} + \overset{(n-m)}{c} y^n}{1 + 2\overset{(n)}{c} y^n + y^{2n}}$$

$$= \frac{\overset{(m)}{s} y^m}{m} - \frac{\overset{(m+n)}{s} y^{m+n}}{m+n} + \frac{\overset{(m+2n)}{s} y^{m+2n}}{m+2n} \ \&c.$$

$$\text{and fl. } y^{m-1}\dot{y} \times \frac{\overset{(m)}{c} + \overset{(n-m)}{c} y^n}{1 + 2\overset{(n)}{c} y^n + y^{2n}} = \text{fl. } \frac{y^m \dot{x}}{\sqrt{1-x^2}} \times \frac{\overset{(m)}{s} - \overset{(n-m)}{s} y^n}{1 + 2\overset{(n)}{c} y^n + y^{2n}}$$

$$= \frac{\overset{(m)}{c} y^m}{m} - \frac{\overset{(m+n)}{c} y^{m+n}}{m+n} + \frac{\overset{(m+2n)}{c} y^{m+2n}}{m+2n} \ \&c.$$

Hence the theorems in art. 33. and 30. may be obtained by taking m and n each equal to 1; and the theorems in art. 29. and 28. by taking $m = 1$ and $n = 2$.

By expanding $\left(\left(x - \sqrt{x^2-1}\right)^{\frac{n}{2}} + y.\left(x + \sqrt{x^2-1}\right)^{\frac{n}{2}}\right)^{-e}$ and $\left(\left(x + \sqrt{x^2-1}\right)^{\frac{n}{2}} + y.\left(x - \sqrt{x^2-1}\right)^{\frac{n}{2}}\right)^{-e}$ by the binomial theorem, we may extend our analysis still farther.

37. It

37. It may be obſerved, that, as the value of the expreſſion $\frac{x^{p+1}-c^{p+1}}{p+1}$ becomes = Log. $\frac{x}{c}$ when p is $= -1$, ſo, when p has that value (-1), the value of the expreſſion $\frac{s^{(p+1)}}{p+1}$ becomes $= z$, and the value of the expreſſion $\frac{c^{(p+1)}-1}{\overline{p+1}^2}$ becomes $= \frac{z^2}{2}$: as appears by applying the rule derived from the doctrine of fluxions for finding the value of an algebraic fraction when its numerator and denominator both vaniſh together; the fluxion of $s^{(p+1)}$ being $= z\,c^{(p+1)}\dot{p}$, and the fluxion of $c^{(p+1)} = z\,s^{(p+1)}\dot{p}$, p only being conſidered as variable. Which being recollected, our theorems will be found uſeful in caſes wherein they may without ſuch recollection ſeem to fail.

I preſume, I have now ſufficiently explained this *new analytical Improvement:* by which the intelligent analyſt may obtain the ſums of many other ſeries, and the fluents of many other fluxions, perhaps with greater facility than by any other method.

POSTSCRIPT.

The ſeries conſidered in the following articles having relation to ſome of thoſe in the preceding articles, it is thought not improper to take notice of them here, though their ſums are obtained by a method different from our new method explained above.

38. It

38. It is well known that

$$\text{Log. } \frac{1}{1-x} \text{ is } = x + \frac{x^2}{2} + \frac{x^3}{3} \text{ \&c.}$$

and, by ſubſtituting $\frac{x}{x-1}$ for x, the ſame

$$\text{Log. } \frac{1}{1-x} \text{ is found} = -\overline{1-x^{-1}}^{\,-1} - \frac{\overline{1-x^{-1}}^{\,-2}}{2} - \frac{\overline{1-x^{-1}}^{\,-3}}{3} \text{ \&c.}$$

From which equations, by ſubſtituting $\frac{x-1}{x}$ (or $1-x$) for x, we have

$$\text{Log. } x = 1 - x^{-1} + \frac{\overline{1-x^{-1}}^{\,2}}{2} + \frac{\overline{1-x^{-1}}^{\,3}}{3} \text{ \&c.}$$

$$\text{or } = -\overline{1-x} - \frac{\overline{1-x}^{\,2}}{2} - \frac{\overline{1-x}^{\,3}}{3} \text{ \&c.}$$

It follows therefore that

$$\text{fl. } \overline{\frac{\dot{x}}{x} \text{ Log. } \frac{1}{1-x}} \text{ is } = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} \text{ \&c.,}$$

$$\text{and fl. } \overline{\frac{\dot{x}}{1-x} \text{ Log. } x} = 1 - x + \frac{\overline{1-x}^{\,2}}{2^2} + \frac{\overline{1-x}^{\,3}}{3^2} \text{ \&c.} - \frac{2a^2}{3},$$

the ſeries $1 + \frac{1}{2^2} + \frac{1}{3^2}$ &c. being found $= \frac{2a^2}{3}$.

But the ſum of the two fluents on one ſide is manifeſtly equal to Log. x × Log. $\frac{1}{1-x}$; which therefore is

$$= \begin{cases} x + \frac{x^2}{2^2} + \frac{x^3}{3^2} \text{ \&c.} \\ + 1 - x + \frac{\overline{1-x}^{\,2}}{2^2} + \frac{\overline{1-x}^{\,3}}{3^2} \text{ \&c.} - \frac{2a^2}{3}. \end{cases}$$

Hence, by taking $x = \frac{1}{2}$, it appears that, when x has that value

x +

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} \text{ \&c. is} = \frac{a^2}{3} - \tfrac{1}{2} \text{ fq. Log. } 2.$$

39. Seeing that $\frac{\dot{x}}{x^2}$ is the fluxion of $\frac{x-1}{x}$, we, from our two values of Log. x in the preceding article, get (by multiplying by $\frac{\dot{x}}{x} + \frac{\dot{x}}{1-x}$, or its equal $\frac{-\dot{x}}{x^2} \times \frac{x}{x-1}$, and taking the fluents)

$$\left.\begin{array}{l} 1 - x^{-1} + \frac{\overline{1-x^{-1}}|^2}{2^2} + \frac{\overline{1-x^{-1}}|^3}{3^2} \text{ \&c.} \\ + 1 - x + \frac{\overline{1-x}|^2}{2^2} + \frac{\overline{1-x}|^3}{3^2} \text{ \&c.} \end{array}\right\} = -\tfrac{1}{2} \text{ fq. Log. } x:$$

and from hence the value of $x + \frac{x^2}{2^2} + \frac{x^3}{3^2}$ &c. when x is $= \frac{1}{2}$, may be found as above.

Moreover, taking $1 - x = \overline{\frac{x-1}{x}}|^2$, we have

$$\left.\begin{array}{r} \tfrac{5}{4} \times \overline{x^2 + \frac{x^4}{2^2} + \frac{x^6}{3^2} \text{ \&c.}} \\ - x - \frac{x^3}{3^2} - \frac{x^5}{5^2} \text{ \&c.} \end{array}\right\} = -\tfrac{1}{2} \text{ fq. Log. } x;$$

x^2 being $= 1 - x = \frac{3-\sqrt{5}}{2}$, and $x = \frac{\sqrt{5}-1}{2}$.
Therefore it appears that then

$$x^2 + \frac{x^4}{2^2} + \frac{x^6}{3^2} \text{ \&c. is} = \tfrac{4}{5} \times \overline{x + \frac{x^3}{3^2} + \frac{x^5}{5^2} \text{ \&c.}} - \tfrac{2}{5} \text{ fq. Log. } x.$$

It alſo appears, by our value of Log. $x \times$ Log. $\frac{1}{1-x}$ in the preceding article, that, when x^2 is $= 1 - x$,

$$\left.\begin{array}{r}\frac{5}{4}\times\overline{x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\&c.}\\ +\overline{x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\&c.}\end{array}\right\}\text{ is }=\frac{2a^2}{3}-2\text{ sq. Log. }x.$$

Consequently we have, in this case,

$$x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\text{ \&c.}$$

$$=\tfrac{8}{15}a^2-\tfrac{8}{5}\text{ sq. Log. }x-\tfrac{4}{5}\times\overline{x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\text{ \&c.}}$$

$$=\tfrac{4}{5}\times\overline{x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\text{ \&c.}}-\tfrac{2}{5}\text{ sq. Log. }x,$$

our former value of the same series.

Hence we find

$$x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\text{ \&c.}=\frac{a^2}{3}-\tfrac{3}{4}\text{ sq. Log. }x,$$

$$x\text{ being }=\frac{\sqrt{5}-1}{2}.$$

Now the value of this series being found, we, by means thereof, find

$$x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\text{ \&c.}=\tfrac{4}{15}a^2-\text{sq. Log. }x,$$

$$\text{or }\tfrac{1}{4}\times\overline{x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\text{ \&c.}}=\frac{a^2}{15}-\tfrac{1}{4}\text{ sq. Log. }x.$$

$$\left.\begin{array}{r}\text{But }x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\text{ \&c.}\\ +\tfrac{1}{4}\times\overline{x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\text{ \&c.}}\end{array}\right\}\text{ is }=x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}\text{ \&c.}$$

$$\left.\begin{array}{r}\text{and }x+\frac{x^3}{3^2}+\frac{x^5}{5^2}\text{ \&c.}\\ -\tfrac{1}{4}\times\overline{x^2+\frac{x^4}{2^2}+\frac{x^6}{3^2}\text{ \&c.}}\end{array}\right\}=x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}\text{ \&c.}$$

There-

Therefore it is evident, that

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} \text{ \&c. is } = \tfrac{2}{5}a^2 - \text{ſq. Log. } x,$$

$$\text{and } x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} \text{ \&c. } = \tfrac{4}{15}a^2 - \tfrac{1}{2}\text{ſq. Log. } x;$$

$$x \text{ being } = \frac{\sqrt{5}-1}{2}.$$

Mr. JOHN BERNOULLI, Mr. EULER, and ſome other authors have found the ſums of the ſeries $y \pm \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c. and $y + \frac{y^3}{3^2} + \frac{y^5}{5^2}$ &c. when y is $= 1$; and the laſt mentioned gentleman, in his *Inſtit. Calc. Integ.* has alſo given the value of the ſeries $y + \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c. when y is $= \frac{1}{2}$: which value I had before given in the *Philoſ. Tranſact.* for the Year 1760; together with the values of $y \pm \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c. and $y + \frac{y^3}{3^2} + \frac{y^5}{5^2}$ &c. in the firſt mentioned caſe.

In this Memoir, the value of the ſeries $y + \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c. is aſſigned, not only in both thoſe caſes, but alſo when y is $= \frac{\sqrt{5}-1}{2}$, or $= \frac{3-\sqrt{5}}{2}$: its value being here found equal to $\frac{2a^2}{3}$, $\frac{a^2}{3} - \frac{1}{2}$ſq. Log. 2, $\frac{4a^2}{15} -$ ſq. Log. $\frac{\sqrt{5}-1}{2}$, or $\frac{2}{15}a^2 -$ ſq. Log. $\frac{\sqrt{5}-1}{2}$, according as y is taken equal to 1, $\frac{1}{2}$, $\frac{3-\sqrt{5}}{2}$, or $\frac{\sqrt{5}-1}{2}$.

The value of $y + \frac{y^3}{3^2} + \frac{y^5}{5^2}$ &c. which before, I believe, had been affigned only when y is $= 1$, is found above, not only in that cafe, but likewife when y is $= \frac{\sqrt{5} - 1}{2}$: and as the value of this laft mentioned feries (as alfo the value of $y \pm \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c.) is ufeful, as well in affigning the fums of fome other feries of different forms, as in the calculation of certain fluents; I think it worth while to proceed to fhew how the values of the faid feries $y + \frac{y^3}{3^2} + \frac{y^5}{5^2}$ &c. are obtained in two other cafes.

40. It is obvious that

$$\text{fl.}\ \overline{\frac{\dot{x}}{x} \cdot \text{fl.}\ \frac{\dot{x}}{1 - x^2}} = \tfrac{1}{2}\,\text{Log.}\ \frac{x}{g} \times \text{Log.}\ \frac{1 + x}{1 - x} - \text{fl.}\ \overline{\frac{\dot{x}}{1 - x^2} \cdot \text{fl.}\ \frac{\dot{x}}{x}}$$

$$\text{is} = x + \frac{x^3}{3^2} + \frac{x^5}{5^2}\ \text{\&c.}$$

Let y be $= \frac{1 - x}{1 + x}$; then will x be $= \frac{1 - y}{1 + y}$, $\dot{x} = \frac{-2\dot{y}}{\overline{1 + y)}^2}$, $\frac{\dot{x}}{x} = \frac{-2\dot{y}}{1 - y^2}$, $1 - x^2 = \frac{4y}{\overline{1 + y)}^2}$, and $\frac{\dot{x}}{1 - x^2} = \frac{-\dot{y}}{2y}$: and therefore it follows, that

$$\text{fl.}\ \overline{\frac{\dot{x}}{1 - x^2} \cdot \text{fl.}\ \frac{\dot{x}}{x}}\ \text{will be} = \text{fl.}\ \overline{\frac{\dot{y}}{y} \cdot \text{fl.}\ \frac{\dot{y}}{1 - y^2}} = y + \frac{y^3}{3^2} + \frac{y^5}{5^2}\ \text{\&c.} - \frac{a^2}{2}$$

$$- g\ \text{Log.}\ \frac{1 + x}{1 - x}.$$

Confequently

$$x + \frac{x^3}{3^2} + \frac{x^5}{5^2}\ \text{\&c. will be} = \tfrac{1}{2}\,\text{Log.}\ x \times \text{Log.}\ \frac{1 + x}{1 - x} + \frac{a^2}{2}$$

$$- y - \frac{y^3}{3^2} - \frac{y^5}{5^2}\ \text{\&c.}$$

and

$$\left.\begin{array}{l}\text{and } x + \frac{x^3}{3^2} + \frac{x^5}{5^2} \text{ \&c.} \\ + \frac{1-x}{1+x} + \frac{\overline{1-x}|^3}{3^2.\overline{1+x}|^3} + \frac{\overline{1-x}|^5}{5^2.\overline{1+x}|^5} \text{ \&c.}\end{array}\right\} = \frac{a^2}{2} + \frac{1}{2}\text{Log.}\, x \times \text{Log.}\, \frac{1+x}{1-x}.$$

Hence, by taking $x = \sqrt{2} - 1$, we find that, when y has that value,

$$y + \frac{y^3}{3^2} + \frac{y^5}{5^2} \text{ \&c. is} = \frac{a^2}{4} - \frac{1}{2}\text{ sq. Log. } y:$$

and, by taking $x = \frac{\sqrt{5}-1}{2}$, (and referring to the preceding article for the value of $x + \frac{x^3}{3^2} + \frac{x^5}{5^2}$ &c.) we find that, when y is $= \sqrt{5} - 2$,

$$y + \frac{y^3}{3^2} + \frac{y^5}{5^2} \text{ \&c. is} = \frac{a^2}{6} + \frac{3}{4}\text{ sq. Log. } x - \frac{1}{2}\text{ Log. } x \times \text{Log. } y.$$

By means of our conclusions respecting the values of the series $y + \frac{y^3}{3^2} + \frac{y^5}{5^2}$ &c. and $y \pm \frac{y^2}{2^2} + \frac{y^3}{3^2}$ &c. some useful theorems for the calculation of fluents may be obtained: for instance, it being known that the *whole* fluent of $\overline{\frac{\dot{y}}{\sqrt{r^2-y^2}} \cdot \text{fl.} \frac{\dot{y}}{\sqrt{1-by^2}}}$, generated whilst y from o becomes equal to r, is equal to $r + \frac{br^3}{3^2} + \frac{b^2r^5}{5^2}$ &c. we, by referring to what is proved in this and the preceding article, easily infer that, the *whole* fluent of $\frac{x\dot{y}}{\sqrt{r^2-y^2}}$, generated in the time before mentioned, is equal to $\frac{a^2}{3} - \frac{3}{4}$ sq. Log. r, $\frac{a^2}{4} - \frac{1}{2}$ sq. Log. r, or $\frac{a^2}{6} + \frac{3}{4}$ sq. Log. $\frac{1+r}{2} - \frac{1}{2}$ Log. $r \times$ Log. $\frac{1+r}{2}$,

$\frac{1+r}{2}$, according as r is equal to $\frac{\sqrt{5}-1}{2}$, $\sqrt{2}-1$, or $\sqrt{5}-2$, refpectively; z denoting the circular arc whofe radius is 1. and fine y, and a the quadrantal arc of the fame circle. Other theorems relating to fl. $\frac{\dot{x}}{x}$ Log. $\overline{1 \pm x}$ and fl. $\frac{\dot{x}}{x}$ Log. $\frac{1+x}{1-x}$ are more obvious.

41. By proceeding a ftep farther according to the method purfued in art. 38. and 39.

$$1 + \frac{1}{3^3} + \frac{1}{5^3} \text{ \&c. } (= \tfrac{7}{8} \times \overline{1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} \text{ \&c.}})$$

is found $= \frac{1}{6}$ Cube Log. $x - \frac{a^2}{3}$ Log. $x + x + \frac{x^2}{2^3} + \frac{x^3}{3^3}$ &c.

and $1 + \frac{1}{2^3} + \frac{1}{3^3}$&c. $= \frac{5}{6}$ Cube Log. $y - \frac{2a^2}{3}$ Log. $y + \frac{5}{4} \times \overline{y^2 + \frac{y^4}{2^3} + \frac{y^6}{3^3} \text{\&c.}}$

x being $= \frac{1}{2}$, $y = \frac{\sqrt{5}-1}{2}$, $y^2 = \frac{3-\sqrt{5}}{2}$, and a as in the preceding article.

NOTE. All the *Logarithms* mentioned in this Memoir are of the *hyperbolic* kind.

MEMOIR

MEMOIR VI.

A remarkable new Property of the Cycloid discovered, which suggests a new Method of regulating the Motion of a Clock.

1. LET A′B′P′E′Q′, A″B″P″E″Q″ be two similar, curved, small tubes, situated exactly alike in a vertical plane; let a small ball be supposed to be put into each tube; and, both the balls P′, P″ being equal, let them be conceived to be connected by a perfectly flexible line, without weight, passing from P′ up the tube wherein it is put to the top A′, and from thence to the top A″ of the other tube, then down that other tube to P″: let that flexible line (P′A′A″P″) be equal to E′B′A′A″B″E″; and A′A″, B′B″, E′E″, Q′Q″ being horizontal lines, let B′E′, E′Q′, B″E″, E″Q″ be all equal, that, the balls being moved, P″ may be at Q″, E″, or B″, when P′ shall be at B′, E′, or Q′ respectively. Then the ball P′ being raised to B′, and left to descend from thence in the tube A′B′E′Q′; and the ball P″, during the descent of P′, being drawn up the other tube from Q″, by means of the said connecting line; it is proposed to find the nature of the curve into which the tubes must be bent, that the time of descent of the ball P′ (so connected with the ball P″) from B′ to Q′ may always be the same, let the height B′E′ be what it will. Fig. 38.

Put

Put a for the length of the part B′E′ of the tube into which the ball P′ is ſuppoſed to be put; b for the vertical height of B′ above E′; z for the ſpace paſſed over by P′ in the tube in its deſcent; x for the vertical deſcent of P′; y for the vertical aſcent of P″; v for the velocity of each of the balls; t for the time elapſed during the deſcent of P′; and g for $32\frac{1}{6}$ feet, the accelerative force of gravity: then will $\frac{g\dot{x}P'}{\dot{z}}$ be the motive force by which the velocity v will be accelerated, $\frac{g\dot{y}P''}{\dot{z}}$ the motive force by which v will be retarded, and $\frac{1}{2}g \times \frac{\dot{x}-\dot{y}}{\dot{z}}$ the actual accelerating force of each ball. Now, that P′ may always arrive at E′ in the ſame time, let the diſtance B′E′ be what it will, its accelerating force muſt be always as the ſpace to be paſſed over during ſuch deſcent*; that is, $\frac{1}{2}g \times \frac{\dot{x}-\dot{y}}{\dot{z}}$ muſt be $= c \times \overline{a-z}$, c being ſome invariable quantity not yet known.

* Let s be any ſpace to be paſſed over, z a part of that ſpace; and ſuppoſe that, in the time t, the moving body has paſſed over that part z, and has acquired a velocity v by the continued action of an accelerating force $= c \times \overline{s-z}$. Then will $c \times \overline{s-z} \times \frac{\dot{z}}{v}$ be $= \dot{v}$, and conſequently $csz - \frac{1}{2}cz^2 = \frac{1}{2}v^2$, v being $= 0$ when z is $= 0$. Moreover $\dot{t}$ $(= \frac{\dot{z}}{v})$ will be $= \frac{\dot{z}}{c^{\frac{1}{2}}\sqrt{2sz - z^2}}$: and hence t is found $= \frac{1}{c^{\frac{1}{2}}} \times$ Circ. Arc, radius 1, verſed ſine $\frac{z}{s}$. Conſequently, taking z equal to s, we find the *whole* time of paſſing over the ſpace $s = \frac{1}{c^{\frac{1}{2}}} \times$ the quadrantal arc of the circle whoſe radius is 1; which, c being given, is always the ſame, let s be what it will.

Whence

Whence we have $\frac{g}{c}\times\overline{\dot{x}-\dot{y}}=2a\dot{z}-2z\dot{z}$; and, by taking the fluents, we find $\frac{g}{c}\times\overline{x-y}=2az-z^2$.

2. Let ABPERQN be a ſemi-cycloid inverted, the diameter $MHpIrKN$ of whoſe generating circle is d: let AB be $=e$, BE $=$ EQ $=a$, HI $=b$, H$p=x$, K$r=y$, and BP $=$ QR $=z$; BH, Pp, EI, Rr, and QK being each parallel to the horizontal line AM. We ſhall then, Fig. 39.

by the nature of the curve, have AN $=2d$, HN $=\frac{\overline{2d-e}|^2}{4d}$,

N$p=\frac{\overline{2d-e-z}|^2}{4d}$, NK $=\frac{\overline{2d-e-2a}|^2}{4d}$, N$r=\frac{\overline{2d-e-2a+z}|^2}{4d}$,

$\frac{\overline{2d-e}|^2-\overline{2d-e-z}|^2}{4d}=$ HN $-$ N$p=x$, and $\frac{\overline{2d-e-2a+z}|^2-\overline{2d-e-2a}|^2}{4d}$

$=$ Nr $-$ NK $=y$. Hence, it appearing by ſubtraction that $x-y$ is $=\frac{4az-2z^2}{4d}$, we have $\frac{g}{c}\times\overline{x-y}=\frac{g}{c}\times\frac{4az-2z^2}{4d}$;

which, if c be $=\frac{g}{2d}$, will be $=2az-z^2$, and the equation the ſame as that which we have deduced in the preceding article. It appears therefore, that our cycloid is the curve required: and, the accelerating force of the ball

P′ being $=\frac{g}{2d}\times\overline{a-z}$, the time of its deſcent from B′ to Q′

($=$ twice the time of deſcent from B′ to E′) $=\overline{\frac{2d}{g}}|^{\frac{1}{2}}\times$ ſemicircle, rad. 1, which, d being given, will be the ſame, let B′E′ be what it will; and will be equal to twice the time of free deſcent, from B to N, in the ſame cycloid; or the limit of the time of vibration (in a circular arc) of a pendulum whoſe length is $2d$.

It is obvious that the consequence will be the same, if P′, P″ be similar, slender chains perfectly flexible.

When P′ shall have descended from B′ to Q′; P″, having been drawn up from Q″ to B″, will begin to descend from the last mentioned point and draw P′ upwards, so that a vibratory motion will ensue, which will be such, that, abstracting from friction, the time of vibration will be the same, from what point soever P′ may begin to move, and whatever may be the length of the line connecting the balls or chains. By means of which line, a rod applied to a clock may be made to vibrate in any plane whatever: and, only small parts of the cycloidal tubes being requisite, the mechanism may, in a little room, be so adapted, by taking the diameter *d* of a proper length, (agreeable to what is proved above,) that any given number of vibrations shall be performed in a given time.

The evolute of the cycloid being a similar cycloid, the balls (P′, P″) may be easily made to describe any cycloidal arcs by evolution: and, by substituting evolutes instead of tubes, the friction of the movement may be diminished; but it will then take up more room.

MEMOIR

MEMOIR VII.

Of the Motion of a Body, keeping always in the ſame given Plane, whilſt acted on by any Force, or Forces, urging it continually to change its Direction in that Plane.

1. LET a body (B) be ſuppoſed to deſcribe the trajectory ABb about the center C: let (CB) its diſtance from C be denoted by y: let $B \times f$ denote a motive force urging it continually towards the ſaid center: let $B \times g$ denote another ſuch force always acting on it at right angles to (CB) the radius vector, or ray drawn from C to the body: let v denote the velocity of the body from the center C, in the variable direction CB; (which is ſometimes called its *paracentric velocity*); u its angular velocity about C, meaſured at the diſtance r therefrom; and w its velocity at B in the direction Bd, at right angles to CB; (which is ſometimes called its *circulatory velocity.*) Then, the right line BP being a tangent to the trajectory at B; if the right line CP, drawn in any manner from C to that tangent, be denoted by p'; and the ſine of the angle CPB (to the radius 1) by s; the ſine of the angle CBP will be $=\frac{p's}{y}$, and the abſolute velocity of the body, in the direction of the tangent PBb, (or particle Bb of the curve,) will be $=\frac{wy}{p's}$: which would continue in- Fig. 40.

R 2 variable

variable if no force acted on the body. In which case it would continue to move in the right line PBb, and p' and s would remain invariable. Therefore $\frac{\dot{w}y + w\dot{y}}{p's}$, the fluxion of its velocity, would, in that case, be $= 0$, and $\dot{w} = -\frac{w\dot{y}}{y} = -\frac{u\dot{y}}{r}$, w being $= \frac{uy}{r}$. But the motive forces $B \times f$ and $B \times g$ (answering to the accelerating or retarding forces f and g)* acting on the body, $\dot{w}$ will be $= \frac{\dot{u}y + u\dot{y}}{r}$. Consequently, the directions BC, Bd being at right angles to each other, $\frac{\dot{u}y + 2u\dot{y}}{r}$ (the excess of $\frac{\dot{u}y + u\dot{y}}{r}$, the fluxion of w when the forces act, above $-\frac{u\dot{y}}{r}$, which would be the fluxion of w if the forces ceased to act) will be the fluxion of the velocity generated or destroyed by the action of the force $B \times g$ only: and, the motive force into $\left(\frac{\dot{y}}{v}\right)$ the fluxion of the time being equal to the fluxion

* The quantity here called an accelerating or retarding force is the measure of the force causing acceleration or retardation; and is denoted or expressed by the *velocity* that would be generated or destroyed in the same or an equal body, in a given time, by the action of the same or an equal force uniformly continued in the direction in which the body, during such unform action, might move.

The measure of such a force may also, in comparing the effects of forces, be denoted or expressed by the *space* which a body would be made to pass over in a given time by the action of the same or an equal force uniformly continued; such space being always the *half* of the space which a body would pass over with the velocity acquired in that time by the action of such force, and the halves of quantities being to each other as the whole quantities are to each other respectively: but due care must be taken, that these two ways of expressing the measure of such force be not confounded and both used in the same process.

of

of the quantity of motion generated or deftroyed by that force, $B \times g \times \frac{\dot{y}}{v}$ will be $= B \times \frac{\dot{u}y + 2u\dot{y}}{r}$, and $g = \frac{v}{r} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}}$.

Moreover $\sqrt{v^2 + w^2}$, the velocity of the body in the direction PB*b*, would be invariable if the forces ceafed acting: therefore $\frac{v\dot{v} + w\dot{w}}{\sqrt{v^2 + w^2}}$ (the fluxion of that velocity) would then be = o, and $\dot{v} = -\frac{w\dot{w}}{v} = \frac{u^2 y\dot{y}}{r^2 v}$, w (in that cafe) being $= \frac{uy}{r}$ and $\dot{w} = -\frac{u\dot{y}}{r}$, by what is faid above. But, the motive forces $B \times f$ and $B \times g$ continuing to act on the body, the fluxion of v will in general be expreffed by $\dot{v}$. Confequently (the directions BC, B*d* being at right angles to each other) $\frac{u^2 y\dot{y}}{r^2 v} - \dot{v}$ will be the fluxion of the velocity deftroyed or generated by the motive force $B \times f$; and therefore *(f)* the retarding or accelerating force produced by the action of the faid motive force (being equal to the laft mentioned fluxion divided by the fluxion of the time) will be equal to $\frac{u^2 y\dot{y} - r^2 v\dot{v}}{r^2 \dot{y}}$.

The general theorems

$$f = \frac{u^2 y\dot{y} - r^2 v\dot{v}}{r^2 \dot{y}}, \text{ and } g = \frac{v}{r} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}},$$

which we have inveftigated (with others that will be fuggefted by the particular nature or circumftances of the propofition to be confidered) will be fufficient for the purpofe of determining every thing that may be required refpecting reſpecting the trajectory and the motion of the body therein; as every force which can act on the body, in any direction (in our given plane) different from BC, Bd, (the two directions in which we have ſuppoſed the forces B $\times f$, B $\times g$ to act,) may be reſolved into two forces acting in thoſe two particular directions.

2. It is obvious that our two forces f and g may be reſolved into two others h and k; the former in the direction of the tangent to the trajectory, retarding or accelerating the abſolute velocity of the body; and the latter in a direction at right angles to the ſaid tangent, neither retarding nor accelerating the abſolute velocity of the body, but only changing its direction. Which forces found by ſo reſolving the forces above inveſtigated are

$$h = \frac{-v}{\sqrt{r^2v^2 + u^2y^2}} \times \frac{r^2v\dot{v} + u\dot{u}y^2 + u^2y\dot{y}}{r\dot{y}},$$

$$k = \frac{1}{\sqrt{r^2v^2 + u^2y^2}} \times \frac{r^2v^2\dot{u}y - r^2v\dot{v}uy + 2r^2v^2u\dot{y} + u^2y^2\dot{y}}{r^2\dot{y}}.$$

3. It is likewiſe obvious that one ſingle force (F) compounded of the forces f and g, or h and k, is ſufficient to cauſe a projectile to deſcribe any trajectory whatever, with a velocity varying in any poſſible manner; the direction in which F muſt act being properly varied at every inſtant: which ſingle force is $= \sqrt{f^2 + g^2}$; the direction in which it acts at every inſtant making angles with the radius vector and tangent to the trajectory, ſuch that their ſines ſhall be to radius as g to F, and $\frac{fuy + grv}{\sqrt{r^2v^2 + u^2y^2}}$

to

to F, and their cofines to radius as f to F, and $\frac{frv - guy}{\sqrt{r^2v^2 + u^2y^2}}$ to F refpectively.

4. If $\dot{z}$ be the fluxion of the circular arc defcribed by a point in the radius vector, at the diftance r from the center C, v will be to $\frac{uy}{r}$ as $\dot{y}$ to $\frac{y\dot{z}}{r}$, and $u\dot{y} = v\dot{z}$.

5. If the trajectory, inftead of being referred to the center C, be referred to a bafe; and, y being the ordinate correfponding (at right angles) to the abfciffa x, v be the velocity of the body from the bafe, (in the direction of the faid ordinate,) and u its velocity in a direction parallel to the bafe; the force (f') urging it towards the bafe will be $= -\frac{v\dot{v}}{\dot{y}}$, and the force (g') urging it in a direction parallel to the bafe will be $= \frac{v\dot{u}}{\dot{y}}$. Moreover $u\dot{y}$ will be $= v\dot{x}$.

6. Which two laft mentioned forces may be refolved into two others, h' and k': whereof the former, in the direction of the tangent to the trajectory, (retarding or accelerating the velocity of the body,) will be $= -\frac{v^2\dot{v} + vu\dot{u}}{\sqrt{v^2 + u^2} \times \dot{y}}$; and the latter, in a direction at right angles to the faid tangent, (changing the direction of the body,) $= \frac{v^2\dot{u} - v\dot{v}u}{\sqrt{v^2 + u^2} \times \dot{y}}$.

The ufe of the theorems, obtained above with fuch facility, is far more extenfive than the common doctrine of of centripetal forces; as will in ſome meaſure appear by the following articles.

7. If only the centripetal force $B \times f$ act on the body; g being $= 0$, we have $\dot{u}y + 2u\dot{y} = 0$; and, by taking the fluents, after multiplying by y, we get $uy^2 = a^2b$, b being the value of u when y is $= a$. Therefore, u being $= \frac{a^2b}{y^2}$, $w \left(= \frac{uy}{r}\right)$ will be $= \frac{a^2b}{ry}$; and (U) the abſolute velocity of the body in its trajectory $= \frac{\sqrt{a^4b^2 + r^2v^2y^2}}{ry}$, the value of $\sqrt{v^2 + w^2}$: which (being to w as y to p) is alſo $= \frac{a^2b}{rp}$, p denoting the perpendicular from C, (the center of force) to the tangent to the trajectory at the point where the body ſhall then be. Moreover the force

$$f\left(= \frac{u^2y}{r^2} - \frac{v\dot{v}}{\dot{y}}\right) \text{ will be} = \frac{a^4b^2}{r^2y^3} - \frac{v\dot{v}}{\dot{y}}$$

$$= \frac{a^4b^2}{r^2y^3} - \frac{a^4b^2}{2\dot{y}} \times \text{the fluxion of } \frac{\dot{y}^2}{y^4\dot{z}^2} = \frac{a^4b^2\dot{p}}{r^2p^3\dot{y}},$$

$$v \text{ being} = \frac{u\dot{y}}{\dot{z}} = \frac{a^2b\dot{y}}{y^2\dot{z}} = \frac{a^2b\sqrt{y^2 - p^2}}{rpy}.$$

Hence, by ſubſtitution, the requiſite force may be found, when the nature or equation of the curve is given: or, if the force be given in terms of p, v, u, y, or z; the nature of the curve deſcribed by the projectile, and its motion therein, may be determined.

Example 1. If, r being taken $= a$, the centripetal force f be $= \frac{a^2b^2}{y^3} \left(= \frac{u^2y}{r^2}\right) =$ the centrifugal force ariſing from the circulatory

circulatory velocity of the projectile; $\dot{v}$ will be $= 0$, and $\dot{z}\left(=\frac{u\dot{y}}{v}=\frac{u\dot{y}}{c}\right)=\frac{a^2b\dot{y}}{cy^2}$: whence $z=\frac{ab}{c}\times\frac{y-a}{y}$.

It is remarkable, that, when y is infinite, p and z will each be $=\frac{ab}{c}$.

Example 2. Let f be supposed $=\frac{Av\dot{v}}{B\dot{y}}$; as it will be in the case of a body B revolving on a horizontal plane, about a given point (C), and drawing another body A, on the same plane, directly towards that point, by means of a string connecting the bodies*.

In this instance, f, which we above found equal to $\frac{a^4b^2}{r^2y^3}-\frac{v\dot{v}}{\dot{y}}$, being $=\frac{Av\dot{v}}{B\dot{y}}$, we have, from that equation, $\frac{a^4b^2\dot{y}}{r^2y^3}=\frac{A+B}{B}v\dot{v}$: whence, by taking the fluents, we get $\frac{a^2b^2}{r^2}\times\frac{y^2-a^2}{y^2}=\frac{A+B}{B}v^2$, v being supposed $=0$ when y is $=a$ and $u=b$. Therefore v will be $=\frac{bP^{\frac{1}{2}}}{y}\times\sqrt{y^2-a^2}=\frac{a^2b\dot{y}}{y^2\dot{x}}$, the value of v found above; and $\dot{z}$, in consequence, $=\frac{a^2}{P^{\frac{1}{2}}}\times\frac{y^{-1}\dot{y}}{\sqrt{y^2-a^2}}$; r being taken equal to a, and P being put for $\frac{B}{A+B}$. By taking the fluents again, z is found

* In this case f is $=\frac{T}{B}$, T being the tension of the string: and it is obvious that $(T\dot{t}=)\ \frac{T\dot{y}}{v}$ is $=A\dot{v}$: therefore f is $=\frac{Av\dot{v}}{B\dot{y}}$.

$= \frac{1}{P^{\frac{1}{2}}} \times$ *circ. arc*, rad. a, sec. y: and t the time elapsed (computed from the time of y being $= a$) is, from the equation $\dot{t} = \frac{\dot{z}}{u} = \frac{\dot{y}}{v} = \frac{y\dot{y}}{bP^{\frac{1}{2}}\sqrt{y^2-a^2}}$, found $= \frac{\overline{y^2-a^2}|^{\frac{1}{2}}}{bP^{\frac{1}{2}}}$.

It is observable that, z being $\left(= \frac{da}{P^{\frac{1}{2}}}\right) = \frac{1}{P^{\frac{1}{2}}} \times$ the quadrantal arc of the circle whose radius is a, when y is infinite; the body B, after having made a number of revolutions $\left(= \frac{1}{4P^{\frac{1}{2}}} = \frac{1}{4} \times \overline{\frac{A+B}{B}}\Big|^{\frac{1}{2}}\right)$ about the center C, will fly off to an infinite distance from that point; approaching continually to a rectilineal asymptote, whose distance from C is $= \overline{\frac{A+B}{B}}\Big|^{\frac{1}{2}} \times a$, parallel to the ultimate direction of the radius vector: which direction will be known, the ultimate value of z being (manifestly $= \overline{\frac{A+B}{B}}\Big|^{\frac{1}{2}} \times da$) as above mentioned.

Example 3. Let $f = \frac{a^2b^2}{y^3} - \frac{a^4b^2}{2y} \times$ the fluxion of $\frac{\dot{y}^2}{y^4\dot{z}^2}$ be supposed $= \frac{M}{y^m} + \frac{N}{y^n}$, r being taken $= a$. Then, by multiplying by $2\dot{y}$ and taking the fluents, we have $b^2 + c^2 - \frac{a^2b^2}{y^2} - \frac{a^4b^2\dot{y}^2}{y^4\dot{z}^2} = \frac{2Ma^{1-m}}{m-1} + \frac{2Na^{1-n}}{n-1} - \frac{2My^{1-m}}{m-1} - \frac{2Ny^{1-n}}{n-1}$, c being the value of v when y is $= a$.

Whence

Whence $\dot{z}$ is found $= \dfrac{a^2 b y^{-1} \dot{y}}{\sqrt{K y^2 + \frac{2 M y^{3-m}}{m-1} + \frac{2 N y^{3-n}}{n-1} - a^2 b^2}}$,

$$K \text{ being } = b^2 + c^2 - \frac{2 M a^{1-m}}{m-1} - \frac{2 N a^{1-n}}{n-1}.$$

And it follows from what is done above, that

$$p \text{ will be } = \frac{a b}{\sqrt{K + \frac{2 M y^{1-m}}{m-1} + \frac{2 N y^{1-n}}{n-1}}};$$

$$u = \frac{a^2 b}{y^2},\ v = \sqrt{K + \frac{2 M y^{1-m}}{m-1} + \frac{2 N y^{1-n}}{n-1} - a^2 b^2 y^{-2}};$$

$\sqrt{v^2 + w^2}$ (the absolute velocity of the body in its trajectory)

$$= \sqrt{v^2 + \frac{u^2 y^2}{a^2}} = \sqrt{v^2 + \frac{a^2 b^2}{y^2}} = \sqrt{K + \frac{2 M y^{1-m}}{m-1} + \frac{2 N y^{1-n}}{n-1}};$$

and $\dot{t}$ (the fluxion of the time) $= \dfrac{y \dot{y}}{\sqrt{K y^2 + \frac{2 M y^{3-m}}{m-1} + \frac{2 N y^{3-n}}{n-1} - a^2 b^2}}$.

Now, though we cannot in general determine the trajectory, and the motion of the projectile therein, from these equations, without sometimes having recourse to infinite series; yet there are certain cases wherein the curve, and the motion of the body describing it, may be determined by means of logarithms, or circular or elliptic arcs: concerning some of which cases, I purpose to subjoin a few remarks.

Remark 1. If m be $= 2$ and $n = 3$, $\dot{z}$ will be equal to $\dfrac{a^2 b y^{-1} \dot{y}}{\sqrt{K y^2 + 2 M y + N - a^2 b^2}}$, K being $= b^2 + c^2 - 2 M a^{-1} - N a^{-2}$:

 and

and it appears by our Appendix, that, in all poſſible caſes, the fluent z will be aſſigned by logarithms or circular arcs.

If K be $= \frac{M^2}{N - a^2b^2}$, $\dot{z}$ will be $= \frac{a^2b\sqrt{N - a^2b^2}}{M} \times \frac{y^{-1}\dot{y}}{y + \frac{N - a^2b^2}{M}}$,

and $z = \frac{a^2b}{\sqrt{N - a^2b^2}} \times \text{Log.}\ \frac{aM}{aM + N - a^2b^2} \times \frac{y - \frac{N - a^2b^2}{-M}}{y}$;

in which caſe c^2 is $= \frac{M^2a^{-2}}{N - a^2b^2} \times \overline{a + \frac{N - a^2b^2}{M}}\Big|^2$.

Conſequently, if, whilſt the body is continually urged *towards* the center C by a force $\frac{N}{y^3}$, it be continually urged *from* that center by a force $\frac{M}{y^2}$, N being greater than a^2b^2; the projectile, ſuppoſing y to increaſe after being $= a$, (that is, ſuppoſing c to be poſitive,) will revolve in a ſpiral about C, and for ever recede therefrom; yet never will arrive at the periphery of the circle whoſe center is C, and radius $= \frac{N - a^2b^2}{M}$, a given quantity greater than a: or, ſuppoſing y to decreaſe after being $= a$, (that is, ſuppoſing c negative,) the body will revolve about C, and continually approach nearer and nearer to that point; yet never will get within the circle whoſe center is C and radius $= \frac{N - a^2b^2}{M}$, a given quantity leſs than a: M being accordingly leſs than $\frac{N - a^2b^2}{a}$ when the body recedes from the center, or greater than $\frac{N - a^2b^2}{a}$ when it approaches towards the center.

Remark

Remark 2. If m be $= 2$ and $n = 4$, $\dot{z}$ will be equal to $\frac{a^2 b y^{-\frac{1}{2}} \dot{y}}{\sqrt{Ky^3 + 2My^2 - a^2b^2y + \frac{2}{3}N}}$, K being $= b^2 + c^2 - 2Ma^{-1} - \frac{2}{3}Na^{-3}$: and it appears by our Tables, that the fluent z will always be aſſigned by elliptic or circular arcs, or by logarithms, or by algebraic quantities.

If M be $= 0$, $\dot{z}$ will be $= \frac{a^2 b y^{-\frac{1}{2}} \dot{y}}{\sqrt{Ky^3 - a^2b^2y + \frac{2}{3}N}}$; which will be $= \frac{3^{\frac{1}{2}} a A y^{-\frac{1}{2}} \dot{y}}{\sqrt{y^3 - 3A^2y + 2A^3}} \left(= \frac{3^{\frac{1}{2}} a A y^{-\frac{1}{2}} \dot{y}}{\overline{y - A}.\sqrt{y + 2A}} \right)$ when K is $= \frac{N}{3A^3}$, A being $= \frac{N}{a^2b^2}$. Hence, by taking the fluents, we find

$$z = a \times \text{Log.} \frac{y - A}{a - A} \times \frac{a + 2A + \sqrt{6aA + 3a^2}}{y + 2A + \sqrt{6Ay + 3y^2}}.$$

Conſequently, if $\sqrt{\frac{b^2}{3aA^2} \times \overline{a - A}^2 \times \overline{a + 2A}}$ (the value of c) be poſitive, the projectile will deſcribe a ſpiral about C, and for ever recede therefrom; yet never will arrive at the periphery of the circle whoſe center is C and radius $\frac{N}{a^2b^2}$: or, if the value of c be negative, the body will revolve about C, and continually approach nearer and nearer to that point; yet never will get within the circle whoſe center is C and radius $\frac{N}{a^2b^2}$: N being greater or leſs than a^3b^2, according as the body recedes from, or approaches towards the center.

If K be $= -\frac{4M^2}{3a^2b^2}$ and N $= \frac{1}{4}\frac{a^4b^4}{M}$,

$\dot{z}$ will

$\dot{z}$ will be $= \frac{3^{\frac{1}{2}} A a y^{-\frac{1}{2}} \dot{y}}{\overline{A-y}|^{\frac{1}{2}}}$; and, by Theorem III. Table I.

$$z = 2.3^{\frac{1}{2}} a \times \overline{\frac{y^{\frac{1}{2}}}{\overline{A-y}|^{\frac{1}{2}}} - \frac{a^{\frac{1}{2}}}{\overline{A-a}|^{\frac{1}{2}}}}, \text{ A being } = \frac{1}{2}\frac{a^2 b^2}{M}.$$

Consequently, if, M being a positive quantity less than $\frac{1}{2} a b^2$, $\left(\frac{b \times \overline{A-a}|^{\frac{1}{2}}}{3^{\frac{1}{2}} a^{\frac{1}{2}} A}\right)$ the value of c be taken positive, the projectile will describe a spiral having a circular asymptote whose center is C and radius $\frac{1}{2}\frac{a^2 b^2}{M}$.

Remark 3. If, n being $= 4$, m be $= 3$; or n being $= 5$, m be equal to 2, 3, or 4; the value of z will, by our Tables, be always assigned by elliptic or circular arcs, or by logarithms.

If, n being $= 5$, M be $= 0$,

$\dot{z}$ will be $= \frac{a^2 b \dot{y}}{\sqrt{K y^4 - a^2 b^2 y^2 + \frac{1}{2} N}}$;

which, when K is $= \frac{1}{2}\frac{N}{A^4}$, will be $= \frac{2^{\frac{1}{2}} A a \dot{y}}{y^2 - A^2}$, A being $= \frac{N^{\frac{1}{2}}}{ab}$. Hence z is found $= \frac{a}{2^{\frac{1}{2}}} \times \text{Log.} \frac{A+a}{A-a} \times \frac{A-y}{A+y}$.

Consequently, if $\sqrt{\frac{b^2}{2 a^2 A^2} \times \overline{a^2 - A^2}|^2}$ (the value of c) be taken either positive or negative, the projectile will describe a spiral having a circular asymptote whose center is C and radius $\frac{N^{\frac{1}{2}}}{ab}$; N being greater or less than $a^4 b^2$, according

cording as the body recedes from, or approaches towards, the center.

8. The remarkable circumſtance of the trajectory continually approaching to a circular aſymptote, I believe, was firſt taken notice of by Mr. MACLAURIN, in the caſe wherein (g being $=0$) f is $=\frac{N}{y^5}$: afterwards Mr. SIMPSON obſerved, that the ſame thing will happen in an infinity of other caſes, g being $=0$, n greater than 3, and $f=\frac{N}{y^n}$. It is farther obſervable, that ſuch circumſtance always takes place when, g being $=0$, the value of f is expreſſed in any manner whatever: provided the values of b, c, and y, determined from the equations $\overline{b^2+c^2}.y^2-a^2b^2-2y^2\times$ the fluent of $f\dot{y}=0$ and $\dot{y}=\dot{p}$ $\left(=\frac{abf\dot{y}}{\overline{b^2+c^2-2\text{ fluent of }f\dot{y}}|^{\frac{3}{2}}}\right)$, be real; the fluent mentioned in theſe expreſſions being generated whilſt y from being $=a$ becomes $=A=$ the value of y in the equation ($fy^3=a^2b^2$) reſulting from thoſe two equations. Which equations are obtained by conſidering C as the center of curvature of the trajectory when the value of v is $=0$. From whence it follows, that (A) the value of y in the equation $fy^3=a^2b^2$ will be the radius of the aſymptotic circle; the value of f when y is $=a$ being ſo adapted, that A ſhall be greater or leſs than a, according as the body is conſidered as aſcending or deſcending towards ſuch aſymptote, that is, according as c is taken poſitive or negative.

9. If only the force g act on the projectile; f being $=0$, we have $\frac{u^2}{r^2}=\frac{v\dot{v}}{y\dot{y}}$: by means of which equation and that

that of the proposed curve, the requisite value of $\frac{v}{r} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}}$ (=g) may be readily computed; or, g being given, the trajectory, and the motion of the body therein, may be determined.

Example 1. To find the force requisite (at every instant) to cause a projectile to describe a logarithmic, or equiangular spiral, by acting thereon always in a direction at right angles to the radius vector?

Let p be $= my$, $m^2 + n^2$ being $= 1$; then will $\dot{z}$ be $= \frac{mr\dot{y}}{ny} = \frac{u\dot{y}}{v}$: whence $u = \frac{mrv}{ny}$. Therefore $\frac{v\dot{v}}{y\dot{y}}$ $\left(= \frac{u^2}{r^2}\right)$ will be $= \frac{m^2v^2}{n^2y^2}$, and $\frac{\dot{v}}{v} = \frac{m^2}{n^2} \cdot \frac{\dot{y}}{y}$: hence v is found $= c.\left(\frac{y}{a}\right)^{\frac{m^2}{n^2}}$. Consequently u will be $= b.\left(\frac{y}{a}\right)^{\frac{m^2-n^2}{n^2}}$, and g $= \frac{b^2}{mna} \cdot \left(\frac{y}{a}\right)^{\frac{2m^2-n^2}{n^2}}$; r being taken $= a$, and nb being then $= mc$.

If m be $= \frac{1}{\sqrt{3}}$ and $n = \sqrt{\frac{2}{3}}$, g will be equal to the invariable quantity $\frac{3b^2}{2^{\frac{1}{2}}a}$.

Example 2. Let the trajectory be a conic section, whereof C is a focus, and whose semi-axes are $\frac{d}{1 \backsim e^2}$ and $\frac{d}{\sqrt{1 \backsim e^2}}$; the curve being a circle when e is $= 0$; and an ellipsis, a parabola, or a hyperbola, according as e is less, equal to, or greater than 1.

Then

Then will p be $= \frac{dy^{\frac{1}{2}}}{\sqrt{2d + \overline{e^2 - 1}.y}} = \frac{uy^2}{\sqrt{r^2v^2 + u^2y^2}}$: and, exterminating $v\dot{v}$, we find

$$\frac{\dot{u}}{u} = \frac{d\dot{y}}{\overline{e^2 - 1}.y^2 + 2dy - d^2} - \frac{2\dot{y}}{y};$$

whence, by taking the fluents, u is found $= \frac{a^2b}{y^2} \times \overline{\frac{\overline{e+1}.y - d \times \overline{e-1}.a + d}{\overline{e-1}.y + d \times \overline{e+1}.a - d}}\Big|^{\frac{1}{2e}}$. Consequently v will be equal to

$$\frac{a^2b}{dry} \times \overline{\frac{\overline{e-1}.a + d}{\overline{e+1}.a - d}}\Big|^{\frac{1}{2e}} \times \frac{\overline{\overline{e+1}.y - d}\big|^{\frac{1+e}{2e}}}{\overline{\overline{e-1}.y + d}\big|^{\frac{1-e}{2e}}}.$$

Which being known, we are thereby enabled to obtain, by substitution, the requisite value of the force

$$g = \frac{a^4b^2}{r^2y^2} \times \overline{\frac{\overline{e-1}.a + d}{\overline{e+1}.a - d}}\Big|^{\frac{1}{e}} \times \frac{\overline{\overline{e+1}.y - d}\big|^{\frac{2-e}{2e}}}{\overline{\overline{e-1}.y + d}\big|^{\frac{2+e}{2e}}}.$$

By which it appears, that the body cannot begin to move from the vertex of the curve unless e be $= 2$. In which case the curve is an equilateral hyperbola, whose semi-axes are each $= d$; and g must then be $= \frac{a^4b^2}{r^2y^2} \times \overline{\frac{a + d}{3a - d}}\Big|^{\frac{1}{2}} \times \frac{1}{y + d}$. Therefore, taking $b^4 = \overline{3a - d}\big|^{\frac{1}{2}}$ and $r = a = \frac{d}{3}$, it follows, that the body resting at the vertex of the curve may be moved from thence by the force g, (at first $= \frac{3^{\frac{1}{2}}}{2d^{\frac{1}{2}}}$,) and be afterwards made to describe such hyperbola by the continued action of the force g $= \frac{2d^{\frac{3}{2}}}{3^{\frac{3}{2}}y^2\cdot\overline{y+d}}$.

Example 3. Let u be always $= \frac{Mb}{N+y^2}$; as it will be in the case of a ball moving within a tube (or a ring upon a rod) revolving in a horizontal plane about one of its ends; the ball and tube, (or ring and rod) after the first percussive impulse, being disturbed by no force but their mutual pressure against each other*.

Then will $\frac{v\ddot{v}}{y\dot{y}}\left(=\frac{u^2}{r^2}\right)$ be $= \frac{M^2b^2}{r^2\cdot\overline{N+y^2}|^2}$, and $v\dot{v} = \frac{M^2b^2y\dot{y}}{r^2\cdot\overline{N+y^2}|^2}$: hence v is found $= \sqrt{c^2 + \frac{M^2b^2}{r^2\cdot\overline{N+a^2}} - \frac{M^2b^2}{r^2\cdot\overline{N+y^2}}}$. Consequently g will be $= \frac{2MNb}{r^2\cdot\overline{N+y^2}|^2} \times \sqrt{r^2c^2 + \frac{M^2b^2}{N+a^2} - \frac{M^2b^2}{N+y^2}}$, and $\dot{z}\left(=\frac{u\dot{y}}{v}\right) = \frac{Mrb\overline{N+y^2}|^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{r^2c^2+Mb^2}\times\overline{N+y^2}-M^2b^2}}$, $\frac{M}{N+a^2}$ being $=1$. Which value of $\dot{z}$ becomes $= \frac{\frac{1}{2}Mrbx^{-\frac{1}{2}}.\overline{x-N}|^{-\frac{1}{2}}\times\dot{x}}{\sqrt{\overline{r^2c^2+Mb^2}\times x - M^2b^2}}$, upon substituting x for $N+y^2$: and therefore it appears by our Appendix, that the fluent z will always be assigned by elliptic arcs, or by logarithms.

* It will appear by a subsequent Memoir, that, in this case, when the tube (or rod) is very slender, N $(= M - a^2)$ is $= \frac{l^2T}{3B}$; l being the length of the tube (or rod), T its weight, and B the weight of the ball (or ring).

If,

If, taking $r = a$, N be $= \frac{a^2 b^2}{c^2 - b^2}$; M will be $= \frac{a^2 c^2}{c^2 - b^2}$, $\dot{z} = \frac{a N^{\frac{1}{2}} y^{-1} \dot{y}}{\sqrt{N + y^2}}$, $z = \frac{1}{2} a \times \text{Log.} \frac{b+c}{b-c} \times \frac{N^{\frac{1}{2}} - \sqrt{N + y^2}}{N^{\frac{1}{2}} + \sqrt{N + y^2}}$, $\dot{t} \left(= \frac{\dot{z}}{u}\right) = \frac{a N^{\frac{1}{2}}}{b M} y^{-1} \dot{y} \times \sqrt{N + y^2}$, and $t = \frac{bz}{c^2} + \frac{\sqrt{c^2 - b^2}}{c^2} \times \sqrt{N + y^2} - \frac{a}{c}$. Whence it is evident, that, their first velocities being adapted accordingly, the tube and ball within it will so revolve about the center C, to which one end of the tube is fixed, that the ball will continually approach nearer and nearer to that center; yet never will arrive at it.

In other cases, if the ball first moves towards that center, it will, in a finite time, make its nearest approach thereto; and afterwards it will continually recede therefrom till it comes to the revolving end of the tube: or, c being greater than $\overline{\frac{M}{N}}\Big|^{\frac{1}{2}} \times b$, it will, in a finite time, arrive at the center, with a velocity $= \sqrt{c^2 - \frac{Mb^2}{N}}$, r being taken $= a$.

10. In general, we may, by means of our theorems, either find the requisite force, or forces, from the nature or equation of the curve and some circumstance respecting the motion of the projectile or the action of the force, or forces, thereon; or, from some such circumstance and a knowlege of the force, or forces, acting on the body, its trajectory may be found, and every thing else that may be required relative to its motion.

Example 1. To find the force and its direction requisite (at every instant) to cause a body to revolve in a trajectory, so that it shall, in equal times, recede equal spaces from, and describe equal angles about, a given point; that is, that it shall describe the *spiral of Archimedes* with an invariable angular velocity?

In this case v and u being invariable, we have, by art. 1. $f = \frac{b^2 y}{r^2}$ and $g = \frac{2bc}{r}$: therefore $\sqrt{f^2 + g^2}$, the force sought, must be $= \frac{b}{r^2}\sqrt{4r^2c^2 + b^2y^2}$; and, f being to g as $\frac{by}{2r}$ to c, if a perpendicular to the radius vector, drawn from the center C to the tangent to the trajectory at the point (B) where the body shall at any time be, be bisected by a right line drawn from that same point of contact, the line from B at right angles to that bisecting line will be the direction in which that force must act.

Example 2. To find the force, or forces, requisite (at every instant) to cause a projectile to describe an ellipsis with an invariable angular velocity about one of its foci?

In any elliptic orbit, v is $= \frac{uy}{dr} \times \sqrt{\overline{e^2 - 1}.y^2 + 2dy - d^2}$; the semi-axes being $\frac{d}{1 - e^2}$ and $\frac{d}{\sqrt{1 - e^2}}$, and $p = \frac{dy^{\frac{1}{2}}}{\sqrt{2d + \overline{e^2 - 1}.y}}$: therefore, first supposing u, the angular velocity about the focus C, to be invariable; we have $v^2 = \frac{b^2y^2}{d^2r^2} \times \overline{\overline{e^2 - 1}.y^2 + 2dy - d^2}$; and, by substitution,

$f =$

$$f=\frac{b^2y}{r^2}-\frac{v\dot{v}}{\dot{y}}=\frac{b^2y^2}{dr^2}-\frac{2b^2y}{d^2r^2}\times\overline{\overline{e^2-1}.y^2+2dy-d^2},$$

$$g=\frac{2bv}{r}=\frac{2b^2y}{dr^2}\times\sqrt{\overline{e^2-1}.y^2+2dy-d^2};$$

$$F=\sqrt{f^2+g^2}=\frac{b^2y^2}{dr^2}\times\sqrt{1+2.\frac{1-e^2}{d^2}.\frac{y''-y}{y}\times\overline{\overline{1-e^2}.yy''-d^2}};$$

y being the distance of the body from the focus C, and $y''\left(=\frac{2d+\overline{e^2-1}.y}{1-e^2}\right)$ its distance from the other focus.

Secondly: supposing C to be one focus of the orbit, and the angular velocity about the other focus to be invariable; u will be $=\frac{b''y''}{y}$ and $v=\frac{b''y''}{dr}\times\sqrt{\overline{1-e^2}.yy''-d^2}$, b'' denoting the said angular velocity measured at the distance r from the respective focus: consequently, by substitution, we find in this case

$$f=\frac{u^2y}{r^2}-\frac{v\dot{v}}{\dot{y}}=\frac{b''^2y''^2}{dr^2}-\frac{b''^2y''}{d^2r^2}.\frac{y''-y}{y}\times\overline{\overline{1-e^2}.yy''-d^2},$$

$$g=\frac{v}{r}.\frac{\dot{u}y+2u\dot{y}}{\dot{y}}=\frac{b''^2y''}{dr^2}.\frac{y''-y}{y}\times\sqrt{\overline{1-e^2}.yy''-d^2},$$

$$F=\sqrt{f^2+g^2}=\frac{b''^2y''^2}{dr^2}\times\sqrt{1+2.\frac{e^2-1}{d^2}.\frac{y''-y}{y''}\times\overline{\overline{1-e^2}.yy''-d^2}}.$$

Remark. It is observable that $\overline{1-e^2}.yy''-d^2$ is $=0$ when the body is at either end of the transverse axis, and $y''-y=0$ when it is at either end of the conjugate axis, let the excentricity of the orbit be what it will: and when the excentricity is small, $\frac{y''.\overline{y''-y}}{y}\times\overline{\overline{1-e^2}.yy''-d^2}$ is always a very small quantity; therefore it may then be rejected.

rejected as inconsiderable; and, g being at the same time inconsiderable, the only force to be considered, as affecting the motion of the body, will be $f = \frac{b''^2 y''^2}{dr^2}$, in our second case.

Now, if a body be made to revolve in the same ellipsis by the action of a centripetal force at C, that force will be $= \frac{b^2 d^3}{\overline{1+e}|^4 . r^2 y^2}$, b denoting the angular velocity of the body at the apsis nearest to the focus C; and the angular velocity of the body about the other focus will, at the same time, be to b as $1 - e$ to $1 + e$. Moreover, if b'' be assumed in that proportion to b, our force $f = \frac{b''^2 y''^2}{dr^2}$ will become $= \frac{\overline{1-e}|^2 . b^2 y''^2}{\overline{1+e}|^2 . dr^2}$; which is to the centripetal force $\frac{b^2 d^3}{\overline{1+e}|^4 . r^2 y^2}$, just now mentioned, as $y^2 y''^2$ to $\frac{d^4}{\overline{1-e^2}|^2}$. But $y^2 y''^2$ is $= \frac{d^4}{\overline{1-e^2}|^2}$ when the body is at either apsis; and, if the excentricity be small, e^2 being then very small, and the focal distances y and y'' differing very little from the semi-parameter d, $y^2 y''^2$ will always be nearly $= \frac{d^4}{\overline{1-e^2}|^2}$. Therefore, in such an orbit, the motion of the body made to describe it by the action of the centripetal force $\frac{b^2 d^3}{\overline{1+e}|^4 . r^2 y^2}$, which is reciprocally as the square of the distance of the body from the focus C, will be nearly the same as the motion of a body revolving in the same orbit with

with the invariable angular velocity $\frac{1-e}{1+e} \times b$ about the other focus.

Thus, without any regard to the area of the trajectory described, we see the physical ground of the *celebrated Bishop* WARD's method of determining the place of a Planet in its orbit: which method, it appears by what is here shewn, is not merely an hypothesis, as it is commonly called; nor is it indeed, in any elliptic orbit, strictly true; but it may properly be considered as an useful approximation in orbits nearly circular.

Example 3. To find how a ball will move within a straight slender tube, revolving in a vertical plane, with an invariable angular velocity, about an axis at the point C of the tube?

The tube being continued both ways from C, let the circular arc z, whose radius is r and sine x, measure the angle of elevation BCH, (above the horizontal line CH,) of the ascending branch CB of the tube, in which the ball is supposed at first to move. Then, putting $m = 16\frac{1}{12}$ feet, the force of gravity urging the ball towards the axis C will be $= \frac{2mx}{r}$; which will be $= f = \frac{b^2 y}{r^2} - \frac{v\dot{v}}{\dot{y}}$, b denoting the angular velocity of the tube about the said axis, measured at the distance r therefrom. Moreover $\dot{t}$ will be $= \frac{\dot{z}}{b} = \frac{\dot{y}}{v}$: therefore v will be $= \frac{b\dot{y}}{\dot{z}}$, and $\dot{v} = \frac{b\ddot{y}}{\dot{z}}$, $\dot{z}$ being considered as invariable; and, by substituting accordingly, we have $\frac{2mx}{r} = \frac{b^2 y}{r^2} - \frac{b^2 \ddot{y}}{\dot{z}^2}$, and consequently $\frac{2mrx\dot{z}^2}{b^2} = y\dot{z}^2 - r^2\ddot{y}$. Fig. 41.

Hence,

Hence, by multiplying by $N^{\frac{z}{r}}$ and taking the fluents according to what is shewn in the Appendix, we get.

$$\frac{mr}{b^2}\times\overline{xN^{\frac{z}{r}}\dot{z}-rN^{\frac{z}{r}}\dot{x}}=yN^{\frac{z}{r}}\dot{z}-rN^{\frac{z}{r}}\dot{y}-\overline{a-d}.\dot{z};$$

d being $=\frac{cr}{b}-\frac{mr^2}{b^2}$, N = the number whose hyperbolic logarithm is 1; and a and c being the values of y and v respectively, when x and z are each $=0$, $\dot{x}=\dot{z}$, and $\dot{y}=\frac{c\dot{z}}{b}$; c being considered as positive when the ball at first moves in the ascending branch of the tube so that y (then $=a$) increases.

From which equation of the fluents, we have

$$r\dot{y}N^{-\frac{z}{r}}-yN^{-\frac{z}{r}}\dot{z}=\frac{mr}{b^2}\times\overline{r\dot{x}N^{-\frac{z}{r}}-xN^{-\frac{z}{r}}\dot{z}}-\overline{a-d}.N^{-\frac{2z}{r}}\dot{z};$$

from whence, by again taking the fluents, we get

$$yN^{-\frac{z}{r}}=\frac{mrxN^{-\frac{z}{r}}}{b^2}+\tfrac{1}{2}.\overline{a-d}.N^{-\frac{2z}{r}}+\tfrac{1}{2}.\overline{a+d}.$$

Consequently the general equation of the curve described by the ball is

$$y=\frac{mrx}{b^2}+\tfrac{1}{2}.\overline{a-d}.N^{-\frac{z}{r}}+\tfrac{1}{2}.\overline{a+d}.N^{\frac{z}{r}};$$

and v will be $=\frac{m\sqrt{r^2-x^2}}{b}-\frac{b}{2r}.\overline{a-d}.N^{-\frac{z}{r}}+\frac{b}{2r}.\overline{a+d}.N^{\frac{z}{r}}$.

Remark 1. If a be $=0$, and $c=\frac{mr}{b}$; that is, if the ball begins to move from C, with the horizontal velocity

city $\frac{mr}{b}$, in the ascending branch of the tube; the equation of the curve described by the ball becomes $y=\frac{cx}{b}$. Which, answering to a circle, whose diameter is $\frac{cr}{b}$, touching the horizontal line passing through C, suggests this *remarkable inference:* the ball being put in motion at C, with the velocity $\frac{mr}{b}$ as just now mentioned, it will revolve uniformly in a circle (whose diameter is $\frac{cr}{b}$) standing upon the horizontal line with which the tube at first is supposed to coincide! and it will continue so to revolve (moving up and down alternately in the different branches of the tube) so long as the uniform motion of the tube is continued, making two complete revolutions whilst the tube makes one revolution! and the uniform velocity $\left(\frac{mr}{b}\right)$ wherewith the ball so revolves in such circle will be to its velocity along the tube every where as r to $\sqrt{r^2-x^2}$.

That the ball, in the case adverted to in this Remark, will describe a circle, will, without finding the general equation of the trajectory, readily appear upon enquiring what angular velocity the tube must have, that the ball, when a is $=0$, shall describe a given circle, or part of a given circle, touching the horizontal line at C. Thus: y being $=2x$, and v to u as $\dot{y}$ to $\frac{r\dot{y}}{\sqrt{4r^2-y^2}}$, by the nature of the circle, when r is the radius of the given one as well as of that upon which z is measured; we, by substituting properly in the equation $\frac{2mx}{r}=\frac{u^2y}{r^2}+\frac{v\dot{v}}{\dot{y}}$ $(=f)$, have

U $\frac{my}{r}$

$$\frac{my}{r} = \frac{2u^2 y}{r^2} - \frac{4u\dot{u}}{\dot{y}} + \frac{u\dot{u}y^2}{r^2\dot{y}}:$$

whence it plainly appears, that u may be equal to the invariable quantity $\overline{\frac{mr}{2}}|^{\frac{1}{2}}$ $(= b)$; which, c being $= \overline{2mr}|^{\frac{1}{2}}$ $(= 2b)$ is agreeable to what is ſaid above.

Or, by taking the fluents, after multiplying by $\dot{y}$ and bringing the fluxionary equation into a convenient form, we find

$$u \text{ muſt be} = \frac{\sqrt{mry^4 - 8mr^3y^2 + 32b^2r^4}}{2^{\frac{1}{2}} \times \overline{4r^2 - y^2}};$$

b being the initial value of u, and $c = 2b$.

If the velocity u were regulated according to this equation, and b were leſs than $\overline{\frac{mr}{2}}|^{\frac{1}{2}}$, the ball could only deſcribe a part of the given circle; of which part, the chord would be $= 2r \times \sqrt{1 - \overline{\frac{mr - 2b^2}{mr}}|^{2}}$.

In a ſimilar manner, we may find how the angular velocity of the tube muſt be regulated, that the ball ſhall deſcribe a given ellipſis, or part of a given ellipſis, or other figure, touching the horizontal line at C.

Remark 2. If a be $= \frac{cr}{b} - \frac{mr^2}{b^2}$, our equation of the trajectory becomes

$$y = aN^{\frac{x}{r}} - \overline{\frac{a}{r} - \frac{c}{b}} \cdot x;$$

and v will then be $= \overline{aN^{\frac{x}{r}} - \overline{\frac{a}{r} - \frac{c}{b}} \cdot \sqrt{r^2 - x^2}} \times \frac{b}{r}$.

Remark

Remark 3. If a be $= \frac{mr^2}{b^2} - \frac{cr}{b}$, the equation of the ſpiral deſcribed by the ball will be

$$y = aN^{-\frac{x}{r}} + \overline{\frac{a}{r} + \frac{c}{b}} \cdot x;$$

and v will be $= \overline{\overline{\frac{a}{r} + \frac{c}{b}} \cdot \sqrt{r^2 - x^2} - aN^{-\frac{x}{r}}} \times \frac{b}{r}$.

In this caſe, ſuppoſing both a and c poſitive, the ball will revolve in a ſpiral B′B″B‴ &c. having a circular aſymptote CD; and will make two revolutions in ſuch ſpiral whilſt the tube makes one revolution: which revolutions in the ſpiral will be alternately without and within the circle whoſe diameter is $a + \frac{cr}{b}$, ſtanding upon, and touching at the point C, the horizontal line paſſing through that point; to the periphery of which circle, the ball, going alternately into and out of it at the ſaid point C, will continually approach nearer and nearer, but never can abſolutely revolve in it! Fig. 42.

In each of the ſpirals which, in a preceding article, we found to have circular aſymptotes, the deſcribing projectile either keeps always without or always within the aſymptotic circle. The ſpiral here deſcribed is perhaps ſtill of a more extraordinary nature, being in its revolutions alternately without and within its aſymptotic circle; and, at its ingreſs and egreſs, always interſecting the periphery of that circle in the ſame point C And this ſpiral is a curve, that (abſtracting from reſiſtance within the tube) a ball would actually be made to deſcribe by the force of gravity, whilſt carried about C by the uniform motion of the tube revolving in a vertical plane as above explained.

The ſpiral deſcribed by the ball will be of the ſame kind, when a and c are not both poſitive, provided $a + \frac{cr}{b}$ be equal to the poſitive quantity $\frac{mr^2}{b^2}$; but it may ſometimes make more than one revolution before it enters the aſymptotic circle.

Remark 4. The angular velocity about C being invariable, the force B × g wherewith the tube, by the nature of the motion, would neceſſarily urge the ball in a direction at right angles thereto, if gravity did not act,

$$\text{is} = \frac{2Bbv}{r} = 2Bm.\frac{\sqrt{r^2 - x^2}}{r} + \frac{Bb^2}{r^2}.\overline{a + d}.N^{\frac{x}{r}} - \frac{Bb^2}{r^2}.\overline{a - d}.N^{-\frac{x}{r}}:$$

to which adding $2Bm.\frac{\sqrt{r^2 - x^2}}{r}$, the preſſure againſt the tube ariſing from the gravity of the ball B, we have

$$4Bm.\frac{\sqrt{r^2 - x^2}}{r} + \frac{Bb^2}{r^2}.\overline{a + d}.N^{\frac{x}{r}} - \frac{Bb^2}{r^2}.\overline{a - d}.N^{-\frac{x}{r}},$$

the quantity expreſſing the whole preſſure ariſing from the gravity of the ball and the mutual action or re-action of the ball and tube againſt each other.

Which preſſure, it may be obſerved, is $= 2Bm + \frac{2Bbc}{r}$, the inſtant the tube is moved from a horizontal poſition; and it is the ſame let a be what it will: whereas the preſſure, if the tube reſted in that poſition, would be only $2Bm$; which likewiſe will be the preſſure when it is firſt moved, if c be $= 0$.

Remark

Remark 5. If the ball be ſuppoſed to deſcend from the point C along a revolving plane, (inſtead of being included in a tube,) whilſt the plane itſelf moves uniformly about that point, from a horizontal poſition; the preſſure againſt the plane will, by taking a and c each $= 0$, be found by our theorem $= Bm \times \overline{\frac{4\sqrt{r^2-x^2}}{r} - N^{\frac{z}{r}} - N^{-\frac{z}{r}}}$. Which being $= 0$ when the angle meaſured by the arc z is $= 47° 11' 54''$; it follows that the ball ſo deſcending will quit the plane when, by revolving as juſt now mentioned, it comes to make that angle ($47° 11' 54''$) with the horizon, let the invariable angular velocity of the plane be what it will.

Remark 6. Retaining the plane (continued above the axis) inſtead of the tube, if a be $= \frac{mr^2}{b^2}$, and $c = 0$; that is, if the ball be laid at a diſtance $= \frac{mr^2}{b^2}$ from the axis of motion, upon the aſcending part of the plane, afterwards made to revolve (from a horizontal poſition) as above; the preſſure will be $= 2Bm \times \overline{\frac{2\sqrt{r^2-x^2}}{r} - N^{-\frac{z}{r}}}$. Which will be $= 0$ when $2\sqrt{r^2-x^2}$ is $= rN^{-\frac{z}{r}}$; that is, when the angle meaſured by the arc z becomes $= 83° 17' 21''$; and conſequently the ball, at that inſtant, will quit the plane, y being at the ſame time $= 1.2268 \times a$: the ball at firſt aſcending up the plane, and continuing to do ſo till y becomes $= 1.2361 \times a$; $\sqrt{r^2-x^2}$ being then $= rN^{-\frac{z}{r}}$, and the inclination of the plane to the horizon $74° 3' 58''$.

Remark

Fig. 43. *Remark 7.* If the ſtraight ſlender tube DBPE be faſtened to the lever CP, and the tube with a ball in it be made to revolve about the axis C, in a vertical plane, ſo that the angular velocity of CP ſhall be invariable; we may, in ſuch caſe, find how the ball will move, by obſerving that its relative velocity from the point P within the tube will be the ſame, let the length of the lever CP (which we will ſuppoſe at right angles to the tube) be what it will: provided the angular velocity of CP, the firſt poſition of the tube, the firſt diſtance of the ball from P, and its firſt velocity towards D, be reſpectively the ſame.

Accordingly, having reſpect to a tube ſo moved, and ſuppoſing b now to denote the velocity of a point in CP at the diſtance r from C; z the circular arc deſcribed by ſuch point; x the ſine of that arc; Y the variable diſtance of the ball from P; A the value of Y when x and z are each $= 0$; V the variable relative velocity of the ball within the tube towards the end D thereof; and C its relative velocity, towards the ſame end, at the commencement of the motion, when the tube is ſuppoſed to be parallel to the horizontal line CH:

$$Y = \sqrt{y^2 - P^2} \text{ will be} = \frac{mrx}{b^2} + \tfrac{1}{2}.\overline{A + D}.N^{\frac{z}{r}} + \tfrac{1}{2}.\overline{A - D}.N^{-\frac{z}{r}};$$

$$V = \frac{b\dot{Y}}{\dot{z}} = \frac{m\sqrt{r^2 - x^2}}{b} + \frac{b}{2r}.\overline{A + D}.N^{\frac{z}{r}} - \frac{b}{2r}.\overline{A - D}.N^{-\frac{z}{r}};$$

v, the velocity of the ball from the center C, $= \frac{\sqrt{y^2 - P^2}}{y} \times V$;

U, the abſolute velocity of the ball in the curve it will deſcribe,

$$= \frac{\sqrt{b^2y^2 + r^2V^2 - 2brPV}}{r};$$

and

and the pressure $= \overline{\frac{2m\sqrt{r^2-x^2}}{r} + \frac{2bV}{r} - \frac{b^2P}{r^2}} \times B$;

D being $= \frac{rC}{b} - \frac{mr^2}{b^2}$, and P = the perpendicular CP.

Remark 8. If A be = o and $C = \frac{mr}{b}$, y^2 will be = $P^2 + \frac{C^2x^2}{b^2}$: and the ball will describe a geometrical oval, which will be a line of the fourth order, whose equation is

$$W^2 = \tfrac{1}{2}P^2 + RX - X^2 \pm \tfrac{1}{2}P\sqrt{P^2 + 4RX - 4X^2};$$

X being the abscissa, measured upwards from C on a line at right angles to CH, and W the correspondent ordinate parallel to CH; and R being put for $\frac{rC}{b} = \frac{mr^2}{b^2}$.

The oval described by the ball will be as in Figure 44, 45, or 46; according as R is greater, equal to, or less than P; R, when less than P, being greater than $\frac{1}{2}$P. Fig. 44. Fig. 45. Fig. 46.

If R be less than $\frac{1}{2}$P, the oval will be every where concave towards C.

Remark 9. If A + D be = o, the ball will describe a spiral whose equation is $y^2 = P^2 + \frac{C^2x^2}{b^2} + \frac{2ACxN^{-\frac{x}{r}}}{b} + A^2N^{-\frac{2x}{r}}$; of which spiral, one or other of the ovals mentioned in the preceding Remark will be an asymptote.

Remark 10. It is obvious that, by making the tube revolve uniformly in some certain plane between the horizontal and vertical, the sine of whose inclination to the horizon (to the radius r) shall be s', we may make the ball describe a curve whose equation is

$$Y = \frac{ms'x}{b^2} + \tfrac{1}{2}.\overline{A+D}.N^{\frac{x}{r}} + \tfrac{1}{2}.\overline{A-D}.N^{-\frac{x}{r}};$$

where

where $\frac{ms'}{b^2}$, the coefficient of x, may be of any value between o and $\frac{mr}{b^2}$; and where D is $= \frac{rC}{b} - \frac{mrs'}{b^2}$.

Remark 11. If s' be taken $=$ o, the equation of the curve which the ball will defcribe becomes

$$Y = \sqrt{y^2 - P^2} = \tfrac{1}{2}.\overline{A + \tfrac{rC}{b}}.N^{\frac{x}{r}} + \tfrac{1}{2}.\overline{A - \tfrac{rC}{b}}.N^{-\frac{x}{r}};$$

$$\text{and V will be} = \frac{bA + rC}{2r}.N^{\frac{x}{r}} - \frac{bA - rC}{2r}.N^{-\frac{x}{r}}.$$

Which equations relate to the curve defcribed by a ball within a ftraight tube at the lever CP, revolving in a horizontal plane about the center C.

And if $Ab - rC$ be $=$ o, the equation of the curve becomes

$$y^2 = P^2 + A^2N^{\frac{2x}{r}}.$$

Remark 12. If, s' being $=$ o, C be $= -\frac{Ab}{r}$; that is, if the ball at firft moves towards the point P of the tube, with the relative velocity $\frac{Ab}{r}$; our equation of the curve defcribed by the ball becomes

$$y^2 = P^2 + A^2N^{-\frac{2x}{r}}; \text{ and V will be} = -\frac{AN^{-\frac{x}{r}}}{r}.$$

By which it appears that the ball, in that cafe, will revolve about the center C in a fpiral, continually approaching nearer and nearer to the circumference of the circle whofe center is C and radius $=$ P, yet never will arrive at it.

In

In other cases, (the tube revolving in a horizontal plane,) if the ball at first moves towards the point P of the tube with a relative velocity less than b, r being taken $=A$, it will, in a finite time, make its nearest approach to that point; (when Y will be $=\frac{A\sqrt{b^2-C^2}}{b}$;) and afterwards it will continually recede therefrom: or, C being less than $-b$, the ball will, in a finite time, arrive at P with the relative velocity $\sqrt{C^2-b^2}$; and, after touching the circumference of the circle just now mentioned, continually recede from it.

Remark 13. The relation between (y) the radius vector and (V) the relative velocity of the ball moving in any straight tube, revolving uniformly in a horizontal plane about C, being (by what is said above) expressed by the equation $b^2y\dot{y}=r^2V\dot{V}$, let the perpendicular P and the angle made by the tube and radius vector be what they will; it follows, that the same equation will express the relation between the radius vector and the relative velocity of a ball moving in any curved tube, carried about the center C in a horizontal plane so that the angular velocity of the lever CQ (to which we suppose the curved tube to be fastened) shall be expressed by the invariable quantity b. From which equation, by taking the fluents, we get Fig. 47.

$$b^2.\overline{y^2-a^2}=r^2.\overline{V^2-C^2};$$

and consequently $V=\sqrt{C^2+\frac{b^2}{r^2}.\overline{y^2-a^2}}$;

C being the value of V when y is $=a$.

Therefore $\dot{z} = \frac{b\dot{Z}}{V}$ will be $= \frac{b\dot{Z}}{\sqrt{C^2 + \frac{b^2}{r^2}.y^2 - a^2}}$;

$\dot{Z}$ denoting the fluxion of the curve QB, or (which is the same thing) of the relative space passed over by the ball within such curved tube.

Whence the trajectory of the ball may be found, and its motion therein; $\dot{Z}$ being given in terms of y and $\dot{y}$.

Fig. 48. *Remark* 14. If the form of the tube be the circumference of a circle whose radius is r, and C be a point therein, $\dot{Z}$ will be $= \frac{2r\dot{y}}{\sqrt{4r^2 - y^2}}$; and consequently

$$\dot{z} = \frac{2br\dot{y}}{\sqrt{4r^2 - y^2} \times \sqrt{C^2 + \frac{b^2}{r^2}.y^2 - a^2}}.$$

Therefore, by what is shewn in Table XII. of the Appendix, the motion of a ball in a circular tube, revolving uniformly in a horizontal plane about a point in such tube, will be determined by means of elliptic arcs.

When, supposing $a = 0$, C (the velocity of the ball at the point C) is $= 2b$; the time in which the tube will make one revolution, about the axis C, will be to the time in which the ball will make one revolution in the tube, as $4q$ to $e + \sqrt{e^2 - 2q}$; that is, as 3.14159265 to 1.31102877771; q denoting the quadrantal arc of a circle whose radius is 1, and e the quadrantal arc of an ellipsis whose semi-axes are $2^{\frac{1}{2}}$ and 1.

Example 4. Let the body B' describe any trajectory whatever about the center C, by means of known forces (f and g) acting thereon; and let another body B″ de-

scribe

ſcribe ſuch a trajectory about the ſame center, that the bodies ſhall always be equidiſtant therefrom, and the angular velocity of the former to that of the latter (about that center) as u to $e+mu$: to find the force, or forces, requiſite to retain the body B″ in ſuch a trajectory, e and m being invariable quantities?

The forces acting on B′ being

$$f = \frac{u^2 y}{r^2} - \frac{v\dot{v}}{\dot{y}}, \text{ and } g = \frac{v}{r} \times \overline{2u + \frac{\dot{u}y}{\dot{y}}};$$

thoſe acting on B″ will be

$$f'' = \frac{\overline{e+mu}\,^2 . y}{r^2} - \frac{v\dot{v}}{\dot{y}}, \text{ and } g'' = \frac{v}{r} \times \overline{2.\overline{e+mu} + \frac{m\dot{u}y}{\dot{y}}}:$$

the additional forces therefore muſt be

$$f'' - f = \frac{e^2 + 2em + \overline{m^2 - 1}.u^2}{r^2} \times y,$$

$$\text{and } g'' - g = \frac{v}{r} \times \overline{2.e + \overline{m-1}.u + \frac{\overline{m-1}.\dot{u}y}{\dot{y}}}.$$

Remark 1. If e and g be each = o, the only additional force requiſite will be $f'' - f = \frac{\overline{m^2-1}.u^2 y}{r^2} = \frac{\overline{m^2-1}.a^4 b^2}{r^2 y^3}$;

u being then $= \frac{a^2 b}{y^2}$, by art. 7.

which agrees with the concluſion deduced by Sir ISAAC NEWTON and others, relative to the requiſite additional force in this caſe.

Remark 2. If m be = 1 and g = o,

$f'' - f$ will be $= \frac{e^2 + 2eu}{r^2} \times y = \frac{e^2 y}{r^2} + \frac{2a^2 be}{r^2 y}$, and $g'' - g = \frac{2ev}{r}$;

u being as in the preceding Remark.

Example 5. Let the projectile be ſuppoſed to move near the earth's ſurface, in a medium, the reſiſtance whereof is MDU^n; D denoting the density of the medium, M some invariable quantity, and U the absolute velocity of the body in the path it may describe.

Fig. 49. Then the resistance in a direction parallel to the horizontal base ACD will be $= \frac{MDU^n \dot{x}}{\dot{z}}$; and in a direction perpendicular to the same base $= \frac{MDU^n \dot{y}}{\dot{z}}$; x, y, and z, respectively denoting the abscissa AC (parallel to the horizon), the ordinate CB (at right angles thereto,) and the length AB of the trajectory. Therefore, by art. 5.

$$f' = -\frac{v\dot{v}}{\dot{y}} \text{ will be} = 2m + \frac{MDU^n \dot{y}}{\dot{z}} = 2m + \frac{MDv^n \dot{z}^{n-1}}{\dot{y}^{n-1}},$$

$$g' = \frac{v\dot{u}}{\dot{y}} \;\ldots\ldots = -\frac{MDU^n \dot{x}}{\dot{z}} = -\frac{MDvu^{n-1}\dot{z}^{n-1}}{\dot{x}^{n-2}\dot{y}},$$

$$\frac{\dot{z}}{U} = \frac{\dot{x}}{u} = \frac{\dot{y}}{v} = \dot{t};$$

v being the velocity of the body from the base ACD, u its velocity in a direction parallel to the said base, and $2m$ the accelerative force of gravity in the medium.

Whence, by exterminating v and $\dot{v}$, we have $2m\dot{x}^2 = -u^2\ddot{y}$; $\dot{x}$ being considered as invariable.

From which equations every thing relative to the trajectory, and the motion of the projectile therein, may be determined.

Remark 1. If n be $= 1$, and MD equal to the invariable quantity d; we have, from the values of g', $\dot{u} = -d\dot{x}$: whence, by taking the fluents, we find $u = b - dx$, b being the value of u when x is $= 0$. Consequently we have, by sub-

substitution, $2m\dot{x}^2 = -\overline{b-dx}|^2 . \ddot{y}$, or $\ddot{y} = -\frac{2m\dot{x}^2}{\overline{b-dx}|^2}$.
Hence, by taking the fluents, we have $\dot{y} = e\dot{x} - \frac{2m\dot{x}}{d.\overline{b-dx}}$; and, by taking the fluents again, y is found $= ex + \frac{2m}{d^2} \times$ Log. $\frac{b-dx}{b}$; e being $= \frac{c}{b} + \frac{2m}{bd}$, where c denotes the initial vertical velocity of the projectile.

Remark 2. If n be $= 2$, and MD as in the preceding remark; we have, from the values of g′, $\frac{\dot{u}}{u} = -d\dot{z}$: whence, by taking the fluents, we get Log. $\frac{u}{b} = -dz$; and consequently $u = bN^{-dz}$, N being the number whose hyperbolic logarithm is 1. Therefore, by substitution, we have $2m\dot{x}^2 = -b^2\ddot{y}N^{-2dz}$. Let $s\dot{x}$ be $= \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: then will $\dot{y}$ be $= \dot{x}\sqrt{s^2-1}$, $\ddot{y} = \frac{s\dot{s}\dot{x}}{\sqrt{s^2-1}}$, $\frac{2m\dot{z}}{s} = -\frac{b^2 s\dot{s}}{\sqrt{s^2-1}} \times N^{-2dz}$, and $\frac{b^2 s^2 \dot{s}}{\sqrt{s^2-1}} = -2m\dot{z}N^{-2dz}$.

Hence $\frac{b^2}{2} \times \overline{s\sqrt{s^2-1} + \text{Log.}\frac{s+\sqrt{s^2-1}}{q+\sqrt{q^2-1}}} = \frac{m}{d}N^{-2dz} - \frac{m}{d}$; q being $= \frac{C}{b}$, where C denotes the initial absolute velocity of the projectile.

Now, from the equation so found, expressing the relation of s and z, the value of this last mentioned quantity may be found in terms of s; and consequently the values of $\dot{x}$ $(= \frac{\dot{z}}{s})$ and $\dot{y}$ $(= \frac{\dot{z}\sqrt{s^2-1}}{s})$ will be had in terms of s and $\dot{s}$: from whence the values of x and y may be found in terms of s.

ME-

MEMOIR VIII.

Of the Motion of a Body in (or upon) a Spherical Surface; in (or upon) which it is retained by some Force urging it towards the Center of the Sphere, whilst it is continually impelled by some other Force, or Forces, to change its Direction in (or upon) that Surface.

Fig. 50. 1. LET the body B be supposed to describe the curve Bb in (or upon) the spherical surface CPBbd, wherein C is a given point: let the fine of the arc CB (part of a great circle passing through C and B) be denoted by y: let $B \times f$ denote a motive force continually urging the body towards C, in the direction of the tangent to the said arc CB at the point B, where the body is supposed to be: let $B \times g$ denote another such force always acting on it, in a direction Bd at right angles to such tangent, and to the plane of the great circle CB: let v denote the velocity of the body from C, in the variable direction of the said tangent; u the angular velocity of a plane considered as revolving with the point B about the diameter from C, measuring such velocity at the distance r (= the radius of the sphere) from the sphere's center; and w the velocity of the body at B, in the direction Bd. Then, BP being a great circle touching the track (Bb) of the body in B; if the fine of the arc CP (of another great circle) be denoted by p, and the fine of the angle CPB (to the radius

radius 1) by s; the sine of the angle CBP will be $= \frac{p's}{y}$, and the absolute velocity of the body in the direction (Bb) of the tangent to its track at B will be $= \frac{wy}{p's}$: which would continue invariable, and the body would describe the great circle PBb, if no force acted thereon but that which, by urging it towards the center of the sphere, would be requisite to retain it in (or upon) the surface thereof. In which case, the great circles BP and CP keeping their positions, p' and s would remain invariable; and $\frac{\dot{w}y + w\dot{y}}{p's}$, the fluxion of the velocity of the body, would be $= 0$: therefore $\dot{w}$ would then be $= -\frac{w\dot{y}}{y} = -\frac{u\dot{y}}{r}$, w being $= \frac{uy}{r}$. But the forces $B \times f$ and $B \times g$ acting on the body, $\dot{w}$ will be $= \frac{\dot{u}y + u\dot{y}}{r}$. Consequently, the directions in which those forces act being at right angles to each other, $\frac{\dot{u}y + 2u\dot{y}}{r}$ (the excess of $\frac{\dot{u}y + u\dot{y}}{r}$, the fluxion of w when those forces act, above $\frac{-u\dot{y}}{r}$, which would be the fluxion of w if those forces ceased to act) will be the fluxion of the velocity generated or destroyed by the action of the force $B \times g$ only: and, the motive force into $\left(\frac{r\dot{y}}{v\sqrt{r^2 - y^2}}\right)$ the fluxion of the time being equal to the fluxion of the quantity of motion generated or destroyed by that force, $\frac{Bgr\dot{y}}{v\sqrt{r^2 - y^2}}$ will be $= B \times \frac{\dot{u}y + 2u\dot{y}}{r}$, and $g = \frac{v\sqrt{r^2 - y^2}}{r^2} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}}$.

Moreover $\sqrt{v^2 + w^2}$, the velocity of the body in its track Bb, would be invariable if the forces $B \times f$ and $B \times g$ ceased ceafed acting; and it would then (as obferved above) defcribe a great circle: therefore $\frac{v\dot{v}+w\dot{w}}{\sqrt{v^2+w^2}}$ (the fluxion of that velocity) would, in that cafe, be $= 0$; and $\dot{v} = -\frac{w\dot{w}}{v} = \frac{u^2 y\dot{y}}{r^2 v}$; w being then $= \frac{uy}{r}$ and $\dot{w} = -\frac{u\dot{y}}{r}$, by what is faid above. But the faid forces continuing to act on the body, the fluxion of v will in general be expreffed by $\dot{v}$. Confequently (the forces acting in directions at right angles to each other) $\frac{u^2 y\dot{y}}{r^2 v} - \dot{v}$ will exprefs the fluxion of the velocity deftroyed or generated by the motive force $B \times f$; and therefore (f) the retarding or accelerating force produced by the action of the faid motive force (being equal to the laft mentioned fluxion divided by the fluxion of the time) will be $= \frac{\sqrt{r^2-y^2}}{r} \times \frac{u^2 y\dot{y} - r^2 v\dot{v}}{r^2\dot{y}}$, $\frac{r\dot{y}}{v\sqrt{r^2-y^2}}$ expreffing the fluxion of the time, as before obferved.

The forces

$$f\left(= \frac{\sqrt{r^2-y^2}}{r} \times \overline{\frac{u^2 y}{r^2} - \frac{v\dot{v}}{\dot{y}}}\right) \text{ and } g\left(= \frac{v\sqrt{r^2-y^2}}{r^2} \times \overline{2u + \frac{\dot{u}y}{\dot{y}}}\right),$$

according to the method purfued in art. 2. of the preceding Memoir, may be refolved into two others h and k; the former in the direction of the tangent to the track of the body, retarding or accelerating its abfolute velocity; and the latter in a direction at right angles to fuch tangent, changing the direction of the body. Which forces h and k being to the forces denoted by the fame fymbols in the article juft now mentioned manifeftly as $\sqrt{r^2-y^2}$ to r refpectively, I omit fpecifying their values particularly,

particularly, and proceed to explain the use of the theorems already investigated.

2. If only the force $B \times f$ act on the body: g being $= 0$, we have $\dot{u}y + 2u\dot{y} = 0$; whence we get $uy^2 = a^2 b$, b being the value of u when y is $= a$. Therefore, u being $= \frac{a^2 b}{y^2}$, $w\ (= \frac{uy}{r})$ will be $= \frac{a^2 b}{ry}$; and (U) the absolute velocity of the body $= \frac{\sqrt{a^4 b^2 + r^2 v^2 y^2}}{ry}$. Moreover the force

$$f\left(= \frac{\sqrt{r^2 - y^2}}{r} \times \overline{\frac{u^2 y}{r^2} - \frac{v\dot{v}}{\dot{y}}}\right) \text{ will be } = \frac{\sqrt{r^2 - y^2}}{r} \times \overline{\frac{a^4 b^2}{r^2 y^3} - \frac{v\dot{v}}{\dot{y}}}$$

$$= \frac{\sqrt{r^2 - y^2}}{r} \times \overline{\frac{a^4 b^2}{r^2 y^3} - \frac{a^4 b^2 r^2}{2\dot{y}} \times \text{the fluxion of } \frac{y^{-4}\dot{y}^2}{\dot{x}^2 . \overline{r^2 - y^2}}},$$

$$v \text{ being} = \frac{ru\dot{y}}{\dot{x}\sqrt{r^2 - y^2}} = \frac{a^2 b r y^{-2}\dot{y}}{\dot{x}\sqrt{r^2 - y^2}};$$

where $\dot{x}\ (= u\dot{t})$ denotes the fluxion of the spherical angle BCP, the arc CP being supposed to keep its position.

Example 1. If the body describes a circle whose radius is a, in a plane at right angles to the diameter of the sphere drawn from C, v will be $= 0$, and y always $= a$: consequently the force f must, in that case, be $= \frac{ab^2\sqrt{r^2 - a^2}}{r^3} = \frac{d^2\sqrt{r^2 - a^2}}{ar}$, d denoting the invariable velocity of the body in the circle it describes.

Which theorem is of considerable use in enquiries relative to the motion of a point, or body, in (or upon) the

 surface

ſurface of a ſphere, as will appear in a ſubſequent Memoir.

Example 2. Let the body be ſuppoſed to deſcribe a *loxodromic*, making an angle, whoſe ſine (to the radius 1) is s, with the great circles paſſing through the pole C.

In which caſe v will be $= \frac{\sqrt{1-s^2}.uy}{rs} = \frac{a^2 b\sqrt{1-s^2}}{rsy}$, and $\dot{v} = -\frac{a^2 b\sqrt{1-s^2}\times \dot{y}}{rsy^2}$: conſequently the force f muſt then be $= \frac{\sqrt{r^2-y^2}}{r} \times \overline{\frac{a^4 b^2}{r^2 y^3} + \frac{a^4 b^2.\overline{1-s^2}}{r^2 s^2 y^3}} = \frac{a^4 b^2 \sqrt{r^2-y^2}}{r^3 s^2 y^3}$.

This caſe coincides with that in the preceding example, if s be taken equal to 1; y being then always equal to the invariable quantity a.

Example 3. One end of a ſtring (whoſe length is r) being faſtened to an immoveable point, let a ſmall ball be faſtened to the other end thereof; and, after ſtretching the ſtring ſtraight out in any direction, and putting the ball in motion in a direction making a given angle with the horizon, let it be left to move in ſuch a track as the action of its gravity and the tenſion of the ſtring ſhall cauſe it to deſcribe.

Then, ſuppoſing C to be the loweſt point of the ſpherical ſurface in which the ball will move, and putting $2m$ (= $32\frac{1}{6}$ feet) for the accelerative force of gravity,

$$f \text{ will be} = \frac{2my}{r} = \frac{\sqrt{r^2-y^2}}{r} \times \overline{\frac{a^4 b^2}{r^2 y^3} - \frac{v\dot{v}}{\dot{y}}}:$$

from which equation we have $v\dot{v} = \frac{a^4 b^2 \dot{y}}{r^2 y^3} - \frac{2my\dot{y}}{\sqrt{r^2-y^2}}$; and hence, by taking the fluents,

$v =$

$$v = \sqrt{4m \times \overline{\sqrt{r^2 - y^2} - \sqrt{r^2 - a^2}} - \frac{a^4 b^2}{r^2 y^2} + c^2 + \frac{a^2 b^2}{r^2}};$$

a, b, and c being the refpective values of y, u, and v, at the commencement of the motion.

Therefore

$$\frac{-r\dot{y}}{v\sqrt{r^2 - y^2}} = \dot{t} \text{ will be} = \frac{-r.\overline{r^2 - y^2}|^{-\frac{1}{2}} \times \dot{y}}{\sqrt{4m \times \overline{\sqrt{r^2 - y^2} - \sqrt{r^2 - a^2}} - \frac{a^4 b^2}{r^2 y^2} + n^2}},$$

n^2 being put for $c^2 + \frac{a^2 b^2}{r^2}$;

or, fubftituting x for $\sqrt{r^2 - y^2} - \sqrt{r^2 - a^2}$,

$$\dot{t} \text{ will be} = \frac{r^2 \dot{x}}{\sqrt{\overline{4mr^2 x + n^2 r^2} \times \overline{a^2 - 2\sqrt{r^2 - a^2}.x - x^2} - a^4 b^2}}:$$

and, by our Appendix, t will always be affigned by means of the arcs of the conic fections.

Remark 1. If c be $= 0$, $\dot{t}$ will be

$$= \frac{\frac{1}{2} r m^{-\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\sqrt{a^2 - \frac{a^2 b^2 \sqrt{r^2 - a^2}}{2mr^2} - \frac{a^2 b^2 + 8mr^2 \sqrt{r^2 - a^2}}{4mr^2}.x - x^2}};$$

and the value of t may always be affigned by means of elliptic arcs; or, by what is done in Mem. III. the time t may be compared with the time of defcent of a common pendulum in a circular arc.

Let R denote the length of fuch a pendulum, D the vertical height from which its bob defcends, x its vertical defcent, and T the time of its defcent:

then will t be $= \frac{r\text{T}}{\text{R}}$;

$$R \text{ being } = \sqrt{P^2+Q^2}, \text{ and } D = \sqrt{P^2+Q^2} - Q,$$

when $a^2 - \frac{a^2b^2.\overline{r-b}}{2mr^2}$ the value of P^2 is poſitive;

$$\text{or } R = \frac{Q+\sqrt{P^2+Q^2}}{2}, \text{ and } D = Q - \sqrt{P^2+Q^2},$$

when, $\left(\frac{ab}{r}\right)$ the initial velocity of the ball being greater than $a.\overline{\frac{2m}{r-b}}\big|^{\frac{1}{2}}$, P^2 is negative;

$$Q \text{ being put for } \frac{a^2b^2}{8mr^2} + r - h = \frac{4r^2 - 3a^2 - P^2}{4.\overline{r-b}},$$

and h for $r - \sqrt{r^2 - a^2}$, the height of the ball at the commencement of the motion above the horizontal plane touching the ſpherical ſurface at its loweſt point C.

The aſcent and deſcent of the ball will, it is obvious, be limited by two horizontal circles; in which the places of the apſides will continually vary: but I have not found that z, the fluent of $\frac{a^2bry^{-2}\dot{y}}{v\sqrt{r^2-y^2}} = u\dot{t}$, (which meaſures, on a horizontal plane, the angle deſcribed by the ball about the vertical diameter of the ſurface in which it moves,) can be aſſigned without having recourſe to an infinite ſeries, or to ſome curve of a higher kind than the conic ſections.

Remark 2. If, c being $=$ o, the initial velocity $\left(\frac{ab}{r}\right)$ of the ball, in a horizontal direction at right angles to the ſtring, be $= \sqrt{8m.\overline{h-r}}$,

$$\dot{t} \text{ will be } = \frac{\frac{1}{2}m^{-\frac{1}{2}}rx^{-\frac{1}{2}}\dot{x}}{\sqrt{4r^2-3a^2-x^2}}.$$

And

And it follows, that the time of revolving from apſis to apſis (that is, from the higheſt point of its track to the loweſt, or from the loweſt to the higheſt) will be to the time of deſcent (of another body) in the quadrantal arc, whoſe loweſt point is C and radius r, as $r^{\frac{1}{2}}$ to $\overline{4r^2 - 3a^2}|^{\frac{1}{4}}$: which time of deſcent will be $= \overline{\frac{r}{m}}|^{\frac{1}{2}} \times \frac{e + \sqrt{e^2 - 2q}}{2} =$ $1.31102877 \times \overline{\frac{r}{m}}|^{\frac{1}{2}}$, by Mem. III. q denoting the quadrantal arc of a circle whoſe radius 1, and e the quadrantal arc of an ellipſis whoſe ſemi-axes are $2^{\frac{1}{2}}$ and 1.

Remark 3. If b be $= 0$, the ball will move in a vertical plane, and

$$\dot{t} \text{ will be} = \frac{\frac{1}{2} m^{-\frac{1}{2}} r \dot{x}}{\sqrt{\frac{c^2}{4m} + x \times \overline{2r - b + x} \times \overline{b - x}}}.$$

And it appears by our Tables, that t will then be aſſigned by means of elliptic arcs: except c be $= \sqrt{4m.\overline{2r - h}}$; when,

$$\dot{t} \text{ being} = \frac{\frac{1}{2} m^{-\frac{1}{2}} r \dot{x}}{\overline{2r - b + x}.\sqrt{b - x}},$$

$$t \text{ will be} = \frac{r^{\frac{1}{2}}}{2^{\frac{1}{2}} m^{\frac{1}{2}}} \times \text{Log.} \frac{\sqrt{2r} - \sqrt{b}}{\sqrt{2r} + \sqrt{b}} \times \frac{\sqrt{2r} + \sqrt{b - x}}{\sqrt{2r} - \sqrt{b - x}}.$$

Remark 4. If, b being $= 0$, c be $= \sqrt{8mr}$ and $h = 2r$;

$$\dot{t} \text{ will be} = \frac{\frac{1}{2} m^{-\frac{1}{2}} r x^{-\frac{1}{2}} \dot{x}}{\sqrt{4r^2 - x^2}};$$

and the ball will revolve in a vertical circle, ſo that the time of revolution will be to the time of deſcent (of another

other body) in the quadrantal arc, whose lowest point is C and radius r, as $2^{\frac{1}{2}}$ to 1; which time of descent is specified in the preceding Remark.

Remark 5. Taking v and $\dot{v}$ each $=0$, in the value of f above written, we get

$$\frac{2ma}{r} = \frac{ab^2\sqrt{r^2-a^2}}{r^3}, \text{ or } \frac{2mr^2}{\sqrt{r^2-a^2}} = b^2 = \frac{d^2r^2}{a^2},$$

d denoting the initial velocity of the ball.

Whence we have $d = a \times \overline{\frac{2m}{\sqrt{r^2-a^2}}}\Big|^{\frac{1}{2}}$: and therefore it follows, that the ball, having a velocity given it equal to that particular value of d, in a horizontal direction at right angles to the string, will revolve uniformly in a circle in a horizontal plane; as has been demonstrated in a different manner by Mr. Huygens and others.

Remark 6. When the initial velocity of the ball is nearly $= a \times \overline{\frac{2m}{r-b}}\Big|^{\frac{1}{2}}$ its track (projected on a horizontal plane) will be nearly a circle, and the time of its revolution therein nearly $= \frac{4q\sqrt{r-b}}{\overline{2m}|^{\frac{1}{2}}}$, q being as in Remark 2. Moreover the *limit* of the time of moving from apsis to apsis, upon taking the initial velocity nearer and nearer to $a \times \overline{\frac{2m}{r-b}}\Big|^{\frac{1}{2}}$, appears, by such comparison as is mentioned in Remark 1. to be $= \frac{2qr\sqrt{r-b}}{\overline{2m}|^{\frac{1}{2}} \times \sqrt{4r^2-3a^2}}$. Consequently, when the track differs but little from a circle, the angle between apsis and apsis will be nearly $= \frac{r.180^\circ}{\sqrt{4r^2-3a^2}}$; as found

found by Mr. EULER, (in his *Mechanics*,) by a very different method.

3. If only the force g act on the body; f being $= 0$, $\frac{u^2}{r^2}$ will be $= \frac{v\dot{v}}{y\dot{y}}$: and, by means of that equation, the value of the force g $\left(= \frac{v\sqrt{r^2 - y^2}}{r^2} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}}\right)$ may be readily computed, when the equation of the curve described by the body is given.

Example. If the proposed curve be the loxodromic specified in Example 2. of the preceding article; u being $= \frac{rsv}{\sqrt{1 - s^2 . y}}$, $\frac{v\dot{v}}{y\dot{y}}$ $\left(= \frac{u^2}{r^2}\right)$ will be $= \frac{s^2v^2}{1 - s^2.y^2}$, $\frac{\dot{v}}{v} = \frac{s^2}{1 - s^2}.\frac{\dot{y}}{y}$, and $v = c.\frac{y}{a}\Big|^{\frac{s^2}{1 - s^2}}$. Consequently u will be $= b.\frac{y}{a}\Big|^{\frac{2s^2 - 1}{1 - s^2}}$, and $g = \frac{ab^2\sqrt{r^2 - y^2}}{r^3s\sqrt{1 - s^2}}.\frac{y}{a}\Big|^{\frac{3s^2 - 1}{1 - s^2}}$, crs being $= ab\sqrt{1 - s^2}$.

$$\text{If } s \text{ be} = \frac{1}{\sqrt{3}}, \text{ g must be} = \frac{3ab^2\sqrt{r^2 - y^2}}{2^{\frac{1}{2}}r^3}.$$

4. In general, from proper data, we may, by means of our theorems investigated in art. 1. find not only the requisite force or forces, but also every thing else that may be required relative to the track of the body and its motion therein.

Example 1. Suppose the body to describe with an invariable velocity, the loxodromic specified in Ex. 2. art. 2. and referred to in the last example.

Then,

Then, d being the given velocity of the body, v will be $= d\sqrt{1-s^2}$, and $\frac{uy}{r} = ds$. Consequently, by substitution, we have $f = \frac{d^2 s^2 \sqrt{r^2-y^2}}{ry}$, and $g = \frac{d^2 s \sqrt{1-s^2}.\sqrt{r^2-y^2}}{ry}$. The single force requisite to cause the body so to move in the proposed curve is $\sqrt{f^2+g^2} = \frac{d^2 s \sqrt{r^2-y^2}}{ry}$; the variable direction in which it must act being always at right angles to the track of the body.

Example 2. If the body describe the loxodromic referred to in the preceding example, so that the velocity u shall always be equal to the invariable quantity b; s will be to $\frac{by}{r}$ as $\sqrt{1-s^2}$ to $v = \frac{b\sqrt{1-s^2}.y}{rs}$. Consequently, by substitution, we have in this case,

$$f = \frac{b^2.\overline{2s^2-1}.y\sqrt{r^2-y^2}}{r^3 s^2}, \text{ and } g = \frac{2b^2\sqrt{1-s^2}.y\sqrt{r^2-y^2}}{r^3 s}.$$

The requisite single force is $\sqrt{f^2+g^2} = \frac{b^2 y\sqrt{r^2-y^2}}{r^3 s^2}$; and its variable direction such, that the angle it must make with the great circle passing through the place of the body and the pole C shall always be bisected by the track of the body.

Example 3. Let u and the absolute velocity of the body be supposed invariable.

Then, those velocities being denoted by b and d respectively, $v^2 + \frac{b^2y^2}{r^2}$ will be $= d^2$, $v = \sqrt{d^2 - \frac{b^2y^2}{r^2}}$, and $\dot{t} = \frac{\dot{x}}{b}$

=

$= \frac{r\dot{y}}{v\sqrt{r^2-y^2}} = \frac{r^2\dot{y}}{\sqrt{r^2-y^2} \times \sqrt{d^2r^2-b^2y^2}}$. And it appears, by our Appendix, that t and z will always be affigned by means of elliptic arcs: except when d is $= b$; and then, $\dot{t}\ (= \frac{\dot{z}}{b} = \frac{r\dot{y}}{v\sqrt{r^2-y^2}})$ being $= \frac{r^2\dot{y}}{b.r^2-y^2}$, $t\ (= \frac{z}{b})$ will be $= \frac{r}{2b} \times \text{Log.}\ \frac{r-a}{r+a} \cdot \frac{r+y}{r-y}$; and it follows, that, in this particular cafe, the body (defcribing a fpiral) will continually approach nearer and nearer to the great circle of which C is a pole, yet never can arrive thereat.

The fingle force requifite to caufe the body to move according to our fuppofition is $\sqrt{f^2+g^2} = \frac{2bd\sqrt{r^2-y^2}}{r^2}$; and the variable direction in which it muft act will always be at right angles to the track of the body.

Example 4. Suppofing the velocities u and v to be invariable, and equal to b and c refpectively: it appears by our theorems, that

$$f \text{ muft be} = \frac{b^2y\sqrt{r^2-y^2}}{r^3}, \text{ and } g = \frac{2bc\sqrt{r^2-y^2}}{r^2}:$$

and the requifite fingle force $\sqrt{f^2+g^2}$, compounded of thofe two forces, muft be $= \frac{b\sqrt{r^2-y^2} \times \sqrt{4c^2r^2+b^2y^2}}{r^3}$; the direction in which it muft act being inclined to the plane of the great circle BC (paffing through the place of the body and the pole C) in an angle whofe fine fhall be to radius as $2c$ to $\frac{\sqrt{4c^2r^2+b^2y^2}}{r}$, and in a plane at right angles to the plane of the faid great circle.

Remark. Mr. SIMPSON, in Lem. 2. pag. 3. of his *Mifcell. Tracts*, computes (by a different method) the value of the force g (making it as above) upon the fuppofition that, if u be invariable, v will likewife be fo; without taking any notice that v will not be invariable when u is fo, unlefs the body be acted on by a force $f = \frac{b^2 y \sqrt{r^2 - y^2}}{r^3}$ as well as by the force $g = \frac{2bc\sqrt{r^2 - y^2}}{r^2}$.*

Indeed he has confidered the velocity b as very fmall: and then the force f will be very fmall, but not abfolutely $= 0$, nor yet indefinitely fmall; for b (though fmall) being finite, the value of f will be alfo finite.

Mr. DE LA LANDE, in his *Aftronomy* (art. 3547.), propofing to explain Mr. SIMPSON's computation, has obferved, that, only the force g acting on the body, v will not be invariable when u is fo. But, without computing the requifite force f, or the exact value of the force g when f is $= 0$, he neglects a part of the force g, and entirely neglects the force f, as being what he calls *infiniments petits du troifieme ordre*; whereas they are not generally *infiniments petits* of any order whatever, being affignable finite quantities which may in fome cafes be confiderable, and therefore fhould not be neglected without firft computing their values, and fhewing that, in the cafe in queftion, they are really inconfiderable.

Example 5. If, u being invariable, the force f be $= 0$; $x^2 v\dot{v}$ will be $= b^2 y\dot{y}$, and $v = \frac{\sqrt{c^2 r^2 + b^2 y^2}}{r}$, c being the

* If only this force g act on the body, neither v nor u will be invariable; u being, in that cafe, $= \frac{2bc\dot{y}}{\dot{u}y + 2u\dot{y}}$, and $u^2 = \frac{r^2 v\dot{v}}{y\dot{y}}$, as appears by the theorems in art. 1.

value

value of v when y is $= 0$. Therefore it follows, that g will, in that case, be $= \frac{2b\sqrt{r^2 - y^2} \times \sqrt{c^2r^2 + b^2y^2}}{r^4}$,

$$\text{and } \dot{t} = \frac{\dot{x}}{b} = \frac{r\dot{y}}{v\sqrt{r^2 - y^2}} = \frac{r^2\dot{y}}{\sqrt{r^2 - y^2} \times \sqrt{c^2r^2 + b^2y^2}}:$$

and, by our Appendix, it appears that t and z will always be assigned by means of elliptic arcs.

Remark 1. If c be $= b$, the time in which the revolving great circle (CB) will make one revolution about the diameter drawn from the pole C, will be to the time in which the body will make one revolution in that circle, as $2q$ to $e + \sqrt{e^2 - 2q}$; that is, as 1.57079632 to 1.3110287771; e and q being as specified in Rem. 2. Ex. 3. Art. 2.

Remark 2. The consequence is obvious when a number of bodies, kept from flying from the spherical surface by any force whatever tending to its center, follow one another in the revolving circle CB from the pole C, each having the same velocity (c) when at that pole. If only the force $g = \frac{2b\sqrt{r^2 - y^2} \times \sqrt{c^2r^2 + b^2y^2}}{r^3}$ act on each, they will always be found (forming a kind of ring) in one and the same great circle CB, supposed to revolve uniformly about the diameter drawn from C; but they will not revolve uniformly in such circle, unless each be continually acted on by the two forces f and g, whose values are computed in the 4th Example.

Example 6. If a small ball move within a slender circular tube, or a small ring upon a slender circular rod,

 whilst

whilſt ſuch tube or rod revolves uniformly about a diameter thereof perpendicular to the horizon; u being invariable, and $f = \frac{2my}{r}$, $r^2 v\dot{v}$ will be $= b^2 y\dot{y} - \frac{2mr^2 y\dot{y}}{\sqrt{r^2 - y^2}}$,

and $v = \frac{\sqrt{c^2 r^2 - 4mr^3 + b^2 y^2 + 4mr^2 \sqrt{r^2 - y^2}}}{r}$;

c being the value of v when y is $= 0$.

Therefore

$$\dot{t} = \frac{\dot{x}}{b} = \frac{r\dot{y}}{v\sqrt{r^2 - y^2}} \text{ will be} = \frac{r^2 \overline{r^2 - y^2}|^{-\frac{1}{2}} \dot{y}}{\sqrt{c^2 r^2 - 4mr^3 + b^2 y^2 + 4mr^2 \sqrt{r^2 - y^2}}}:$$

which, by ſubſtituting x for $\sqrt{r^2 - y^2}$, will be adapted to our Tables, and we have another inſtance of their uſe in aſſigning the fluent by means of elliptic or circular arcs, or logarithms, as the caſe may require.

This caſe coincides with the preceding, if, abſtracting from gravity, m be conſidered as $= 0$.

MEMOIR

MEMOIR IX.

Of the Motion of a Body in any variable Plane.

1. LET the plane BCD, in which the body (B) is always to be found, be supposed to revolve about an immoveable axis CD, with the angular velocity u measured on the circular arc z, described about the center C by a point in that plane at the distance r from the said axis: let v denote the velocity, and y the distance of the body from that axis; v its velocity, and $x - k$ the distance it has moved in the direction Bβ parallel to the same axis: and let B $\times f$ denote a motive force continually urging the body towards the said axis CD; B $\times$ g another such force urging it in a direction at right angles to the said plane BCD; and B $\times h$ a third force urging it in the said direction Bβ. Fig. 51.

Then, by the method pursued in Memoir VII.

$$f \text{ is found} = \frac{u^2 y\dot{y} - r^2 v\dot{v}}{r^2 \dot{y}}, \quad g = \frac{v}{r} \times \frac{\dot{u}y + 2u\dot{y}}{\dot{y}},$$

$$h = \frac{v\dot{\mathrm{v}}}{\dot{y}} = \frac{\mathrm{v}\dot{\mathrm{v}}}{\dot{x}};$$

and $\dot{t}$, the fluxion of the time, will be $= \frac{\dot{x}}{\mathrm{v}} = \frac{\dot{y}}{v} = \frac{\dot{z}}{u}$.

From these theorems others may be readily derived, by the resolution or composition of forces or motion, to suit

ſuit particular propoſitions: and the theorems ſo derived, with others that the particular circumſtances of the propoſition to be conſidered may ſuggeſt, will enable the intelligent analyſt to proceed in determining the path of the body and its motion therein in any poſſible caſe whatever; as every force that can act on the body in any direction different from the particular directions in which we have ſuppoſed the forces $B \times f$, $B \times g$, $B \times h$ to act may be reſolved into two or three others acting in two or three of thoſe particular directions.

2. If the force g be $= 0$; $\dot{u}y + 2u\dot{y}$ being $= 0$, we have $uy^2 = a^2 b$, b being the value of u when y is $= a$: and f will be $= \frac{a^4 b^2}{r^2 y^3} - \frac{v\dot{v}}{\dot{y}}$.

Let B, the place of the body, be projected in B″ by a perpendicular on a plane CB″ at right angles to the axis CD; and let $\dot{Z}$ be the fluxion of the line in which the point B″ ſhall be found, U the velocity of that point in that line, and p the perpendicular from C on the tangent to the ſame line at the point B″ correſponding to the place of the body: then will U be $= \frac{\sqrt{a^4 b^2 + r^2 v^2 y^2}}{ry}$,

$$f = \frac{a^4 b^2}{r^2 y^3} - \frac{v\dot{v}}{\dot{y}} = \frac{a^4 b^2}{r^2 y^3} - \frac{a^4 b^2}{2\dot{y}} \times \text{the fluxion of } \frac{\dot{y}^2}{y^4 \dot{z}^2} = \frac{a^4 b^2 \dot{p}}{r^2 p^3 \dot{y}},$$

$$v \text{ being} = \frac{U\dot{y}}{\dot{Z}} = \frac{u\dot{y}}{\dot{z}} = \frac{a^2 b \dot{y}}{y^2 \dot{z}} = \frac{a^2 b \sqrt{y^2 - p^2}}{rpy}.$$

Fig. 52. 3. Suppoſing the body to move near the earth's ſurface in the concave ſuperficies of a ſolid of revolution, generated by the line DB revolving about the axis DC; let the abſciſſa Db, and the correſpondent ordinate bB at right

right angles thereto, be denoted by x and y respectively; and let $\dot{\xi}$ denote the fluxion of the said line DB.

Then, DbC being perpendicular to the horizon, g will be $=0$; and $\frac{b\dot{x}}{\dot{\xi}}$ will denote the force upwards, in the direction of the tangent to the line DB, arising from the action of the force $B \times h$; $\frac{f\dot{y}}{\dot{\xi}}$ the force downwards, in the direction of the said tangent, arising from the action of the force $B \times f$; and $\frac{f\dot{y} - b\dot{x}}{\dot{\xi}}$ $\left(= \frac{a^4 b^2 \dot{y}}{r^2 y^3 \dot{\xi}} - \frac{v\dot{v}}{\dot{\xi}} - \frac{v\dot{v}}{\dot{\xi}} = \frac{a^4 b^2 \dot{y}}{r^2 y^3 \dot{\xi}} - \frac{V\dot{V}}{\dot{\xi}} = -\frac{W\dot{W}}{\dot{\xi}}\right)$ the actual force downwards in the said direction; V denoting the velocity of the body upwards in that same direction, and W its absolute velocity in the path it describes.

Which last mentioned force will be $= \frac{2m\dot{x}}{\dot{\xi}}$, the force arising from the gravity of the body in the direction of the said tangent, $2m$ denoting ($32\frac{1}{6}$ feet) the accelerating force of gravity directly downwards. Therefore

$$\frac{a^4 b^2 \dot{y}}{r^2 y^3} - V\dot{V} \text{ will be } = 2m\dot{x};$$

and consequently $V = \sqrt{d^2 + \frac{a^2 b^2}{r^2} + 4mk - 4mx - \frac{a^4 b^2}{r^2 y^2}}$,

d being the value of V when x is $= k$ and $y = a$.

Whence we have

$$\dot{t} = \frac{\dot{\xi}}{V} = \frac{\dot{\xi}}{\sqrt{d^2 + \frac{a^2 b^2}{r^2} + 4mk - 4mx - \frac{a^4 b^2}{r^2 y^2}}},$$

and

and $\dot{z} = \frac{a^2 b \dot{t}}{y^2} = \frac{a^2 b y^{-2} \dot{\xi}}{\sqrt{d^2 + \frac{a^2 b^2}{r^2} + 4mk - 4mx - \frac{a^4 b^2}{r^2 y^2}}}$:

and hence, the relation of x and y being given, the values of t and z may be found; by which means the place of the body at any time will be determined.

$\dot{V}$ being $= 0$ when, y being $= a$, $\frac{a^4 b^2 \dot{y}}{r^2 y^3}$ is $= 2m\dot{x}$; it follows, that, if the initial velocity of the body be $= \overline{\frac{2amx}{\dot{y}}}\Big|^{\frac{1}{2}}$ and in a horizontal direction, the body will revolve uniformly in a circle parallel to the horizon.

Fig. 53. *Example* 1. Let the superficies in which the body moves be that of an inverted cone, the radius of whose base is to its perpendicular height as r to q.

Then, x being supposed to begin at the vertex D, $\frac{rx}{q}$ will be $= y$, $\frac{aq}{r} = k$,

$$V = \sqrt{d^2 + \frac{a^2 b^2}{r^2} + \frac{4amq}{r} - 4mx - \frac{a^4 b^2 q^2}{r^4 x^2}},$$

$$\dot{t} = \frac{r^2 s x \dot{x}}{q\sqrt{\overline{d^2 r^4 + a^2 b^2 r^2 + 4amqr^3} \times x^2 - 4mr^4 x^3 - a^4 b^2 q^2}},$$

$$\text{and } \dot{z} = \frac{a^2 b q s x^{-1} \dot{x}}{\sqrt{\overline{d^2 r^4 + a^2 b^2 r^2 + 4amqr^3} \times x^2 - 4mr^4 x^3 - a^4 b^2 q^2}};$$

s being put for $\sqrt{q^2 + r^2}$:

and, by the help of our Tables, the value of t may be assigned by means of elliptic arcs: the value of z also may *sometimes* be so assigned; for instance, when, d being $= 0$ and $r = a$, the initial velocity of the body is $= 2\sqrt{2^{\frac{1}{2}} - 1} \times \sqrt{mq}$.

Remark

Remark 1. If d be $=$ o,

$$V \text{ will be } = x^{-1}\sqrt{4m \times \overline{x - \frac{aq}{r}} \times \overline{R' - x} \times \overline{R'' + x}};$$

$$R' \text{ being } = \frac{\sqrt{a^4 b^4 + 16a^3 b^2 mqr} + a^2 b^2}{8mr^2},$$

$$R'' \ldots\ldots = \frac{\sqrt{a^4 b^4 + 16a^3 b^2 mqr} - a^2 b^2}{8mr^2}.$$

Hence it appears, that the aſcent and deſcent of the body will be limited by two horizontal circles; whoſe heights above D, in this caſe, will be $\frac{aq}{r}$ and R′: and the places of the apſides (in thoſe circles) will vary every revolution.

Remark 2. If the initial velocity of the body be $= \overline{\frac{2amq}{r}}\Big|^{\frac{1}{2}}$, and in a direction parallel to the horizon, the two limiting circles (mentioned in the preceding Remark) will coincide, and the body will revolve in the circle of coincidence, always keeping the ſame diſtance from the vertex of the cone; the value of $\dot{V}$ being then $=$ o, as well as V $=$ o.

When the initial velocity is nearly $= \overline{\frac{2amq}{r}}\Big|^{\frac{1}{2}}$, and in a direction parallel to the horizon, the path of the body will be nearly a circle: and the limit of the angle deſcribed between apſis and apſis may be found by the method commonly purſued in ſuch caſes; or by means of the limit of the time of moving from apſis to apſis, as in the like caſe in the preceding Memoir.

Remark 3. If, abſtracting from gravity, m be conſidered as $=$ o,

$$\dot{t} \text{ will be } = \frac{r^2 s x \dot{x}}{q\sqrt{d^2r^4 + a^2b^2r^2 \times x^2 - a^4b^2q^2}},$$

$$\dot{z} \ldots\ldots = \frac{a^2bqsx^{-1}\dot{x}}{\sqrt{d^2r^4 + a^2b^2r^2 \times x^2 - a^4b^2q^2}},$$

$$t = \frac{s}{d^2r^2 + a^2b^2} \times \sqrt{\frac{d^2r^4 + a^2b^2r^2}{q^2} \times x^2 - a^4b^2} - adr,$$

$$\text{and } z = \begin{cases} \text{Circ. arc, rad. } s, \text{ sec. } \frac{rs\sqrt{d^2r^2 + a^2b^2}}{a^2bq} \times x \\ -\text{Circ. arc, rad. } s, \text{ sec. } \frac{s\sqrt{d^2r^2 + a^2b^2}}{ab}; \end{cases}$$

or, supposing $d = 0$,

$$t \text{ will be } = \frac{s}{ab} \times \sqrt{\frac{r^2x^2}{q^2} - a^2},$$

$$\text{and } z = \text{circ. arc, rad. } s, \text{ sec. } \frac{rsx}{aq}.$$

Fig. 54. *Example* 2. If the superficies wherein the body moves be that of a parabolic conoid, generated by the revolution of the parabola whose equation is $qx = y^2$ about its own axis; x being supposed to begin at D, k will be $= \frac{a^2}{q}$,

$$V = \sqrt{d^2 + \frac{a^2b^2}{r^2} + \frac{4a^2m}{q} - 4mx - \frac{a^4b^2}{qr^2x}},$$

$$\dot{t} = \frac{\frac{1}{2}q^{\frac{1}{2}}r\dot{x}\sqrt{q+x}}{\sqrt{d^2qr^2 + a^2b^2q + 4a^2mr^2 \times x - 4mqr^2x^2 - a^4b^2}},$$

$$\text{and } \dot{z} = \frac{\frac{1}{2}a^2bq^{-\frac{1}{2}}rx^{-1}\dot{x}\sqrt{q+x}}{\sqrt{d^2qr^2 + a^2b^2q + 4a^2mr^2 \times x - 4mqr^2x^2 - a^4b^2}};$$

and in this Example, as well as in the preceding one, the value of t may, by having recourse to our Tables, be assigned.

assigned by means of elliptic arcs: the value of z also may *sometimes* be so assigned; for instance, when, d being $=0$, the initial velocity of the body is $= 2a \times \overline{\frac{mq}{q^2 \pm a^2}}\Big|^{\frac{1}{2}}$.

Moreover it is obvious, that remarks respecting the motion of the body, similar to those in the preceding example, may be here made: except that, if m were here considered as $=0$, the values of t and z would not be assigned by the like means as in that example; but the value of t would be assigned by means of a logarithm, and the value of z by means of a circular arc and a logarithm.

4. If the body be acted on by the force F, urging it towards the center C in the axis CD; and by the force G, urging it towards the plane CAB″ passing through C at right angles to the said axis, and always in a direction perpendicular to that plane; Fig. 55.

$$\text{g being} = 0,\ f = \frac{a^4 b^2}{r^2 y^3} - \frac{v\dot{v}}{\dot{y}} \text{ will be} = \frac{y}{\sqrt{x^2+y^2}} \times \text{F},$$

$$\text{and } h = \frac{v\dot{v}}{\dot{x}} = -\text{G} - \frac{x}{\sqrt{x^2+y^2}} \times \text{F}.$$

Hence, when the values of the forces F and G are given in terms of x and y, the path of the body and its motion therein may be determined.

Example. Let the forces F and G be supposed to be as the distances of the body from the center C and from the plane CAB″ respectively; that is, let F be supposed $= m\sqrt{x^2+y^2}$ and $\text{G} = nx$, m and n being invariable quantities.

Then

Then will $\frac{a^4 b^2 \dot{y}}{r^2 y^3} - v\dot{v}$ be $= my\dot{y}$, and $\mathrm{v}\dot{\mathrm{v}} = -\overline{m+n}.x\dot{x}$:

whence we have $v = \sqrt{ma^2 + \frac{a^2 b^2}{r^2} - my^2 - \frac{a^4 b^2}{r^2 y^2}}$,

$$\mathrm{v} = \sqrt{e^2 + \overline{m+n}.k^2 - \overline{m+n}.x^2},$$

$$\dot{t} = \frac{\dot{x}}{\sqrt{e^2 + \overline{m+n} \times \overline{k^2 - x^2}}} = \frac{ry\dot{y}}{\sqrt{\overline{ma^2 r^2 + a^2 b^2} \times y^2 - mr^2 y^4 - a^4 b^2}},$$

$$\text{and } \dot{z} = \frac{a^2 bry^{-1}\dot{y}}{\sqrt{\overline{ma^2 r^2 + a^2 b^2} \times y^2 - mr^2 y^4 - a^4 b^2}};$$

v being ſuppoſed $= 0$ and $\mathrm{v} = e$, when x is $= k$ and $y = a$.

Therefore p will be $= \frac{a^2 b}{\sqrt{ma^2 r^2 + a^2 b^2 - mr^2 y^2}}$: and it follows, that the curve of projection will be an ellipſis whoſe center will be C, and ſemi-axes $\frac{ab}{m^{\frac{1}{2}} r}$ and a; which curve becomes a circle whoſe radius is a, when, v being $= 0$, the initial velocity of the body is $= \sqrt{ma^2 + e^2}$.

From thoſe fluxional equations we find, by taking the fluents,

$$t = \frac{1}{\overline{m+n}|^{\frac{1}{2}} \times a} \times \text{circ. arc, rad. } a, \text{ coſ. } a \times \frac{\overline{m+n}.kx + e\sqrt{e^2 + \overline{m+n}.\overline{k^2 - x^2}}}{e^2 + \overline{m+n}.k^2}$$

$$= \frac{1}{m^{\frac{1}{2}} a} \times \text{circ. arc, rad. } a, \text{ cosine } \overline{\frac{a^2 b^2 - mr^2 y^2}{b^2 - mr^2}}\Big|^{\frac{1}{2}},$$

$$\text{and } z = r \times \text{circ. arc, rad. } 1, \text{ cosine } \frac{1}{y} \times \overline{\frac{a^2 b^2 - mr^2 y^2}{b^2 - mr^2}}\Big|^{\frac{1}{2}}.$$

It is evident that, the ordinate bB″ being at right angles to the ſhorter ſemi-axis CbA (denoted by a), the abſciſſa

Cb

Cb will be $= \overline{\frac{a^2b^2 - mr^2y^2}{b^2 - mr^2}}\Big|^{\frac{1}{2}}$; and, if that abſciſſa be repreſented by w,

$$t \text{ will be} = \frac{1}{\overline{m+n}\big|^{\frac{1}{2}} \times a} \times \text{circ. arc, rad. } a, \text{ coſine } w:$$

therefore, from comparing this value of t with that above found in terms of x, it follows, that

$$\frac{\overline{m+n}.kx + e\sqrt{e^2 + \overline{m+n}.k^2 - x^2}}{e^2 + \overline{m+n}.k^2} \text{ will be}$$

$$= \frac{1}{a} \times \text{coſine of } \overline{\frac{m+n}{m}}\Big|^{\frac{1}{2}} \text{ times the arc, rad. } a, \text{ coſine } w;$$

which, by art. 2. Mem. V. is

$$= \frac{\overline{w + \sqrt{w^2 - a^2}}\big|^{q} + \overline{w - \sqrt{w^2 - a^2}}\big|^{q}}{2a^q}, \; q \text{ being put for } \overline{\frac{m+n}{m}}\Big|^{\frac{1}{2}}.$$

Conſequently the value of x will from hence be obtained in terms of the abſciſſa w.

The points in which the path of the body interſects the plane CAB″ being diſtinguiſhed by the name of the *nodes*; let the body, when y is $= a$, be ſuppoſed to be *below* the ſaid plane and approaching towards an aſcending node. Then, x being $= 0$ at the time the body comes to the node, w will at the ſame time be the coſine of $\frac{A}{q}$, A denoting the arc whoſe radius is a and coſine $\frac{ae}{\sqrt{e^2 + \overline{m+n}.k^2}}$; and, when the body comes to the next deſcending node, w will be the coſine of $\frac{A''}{q}$, A″ denoting the arc (of the

ſame

fame circle) whofe cofine is $\frac{-ae}{\sqrt{e^2 + \overline{m+n}.k^2}}$, the quantity expreffed by $\sqrt{e^2 + \overline{m+n}.k^2 - x^2}$ becoming negative after being equal to o, as it will be when the body fhall be at the greateft diftance from the faid plane CAB″: moreover, when the body comes again to an afcending node, w will be the cofine of $\frac{A'''}{q}$, A‴ denoting the arc whofe radius is a and cofine $\frac{ae}{\sqrt{e^2 + \overline{m+n}.k^2}}$, which is the fame as the cofine of the arc A. Therefore, denoting the quadrantal arc of the circle whofe radius is a by Q, it evidently follows, that the fucceffive places of the nodes (varying in antecedentia) will be determined by taking (in the ellipfis of projection) the abfciffa w equal to the cofine of $\frac{A}{q}$, $\frac{A+2Q}{q}$, $\frac{A+4Q}{q}$, $\frac{A+6Q}{q}$, &c. fucceffively.

The pofition of the ray CB″ when the body fhall beat the greateft diftance from the plane CAB″ is found by taking the value of v equal to o, and confidering $\frac{\sqrt{e^2 + \overline{m+n}.k^2}}{\sqrt{m+n}}$, the value of x at fuch time, as alternately pofitive and negative. By which means it appears, that the fucceffive pofitions of the faid ray CB″, when the body fhall be at the greateft diftance from the faid plane, will be determined by taking (in the faid ellipfis of projection) the abfciffa w equal to the cofine of $\frac{A+Q}{q}$, $\frac{A+3Q}{q}$, $\frac{A+5Q}{q}$, &c. fucceffively; $\frac{\overline{m+n}|^{\frac{1}{2}} \times ak}{\sqrt{e^2 + \overline{m+n}.k^2}}$ (= ∓ the fine of the arc A) being

being the cosine of A + Q, considering k as negative, agreeable to the supposition above.

It is manifest that, if q $\left(=\overline{\frac{m+n}{m}}\right|^{\frac{1}{2}}\Big)$ be a rational number $=\frac{N}{D}$ in its least terms, the nodes, after $\frac{1}{2}$D or D revolutions of the body, (according as D is even or odd,) will successively fall in the same points as before.

The theorems in the preceding Memoir, it is obvious, might be deduced from those in art. 1. of this Memoir: and many other instances might be given of the use of the principal theorems in these last three Memoirs; some other instances of their use may probably appear hereafter in the course of this Work, when we come to consider some particular subjects in distinct Memoirs.

END OF THE FIRST VOLUME.

APPENDIX:

CONTAINING

TABLES of THEOREMS

FOR THE

CALCULATION of FLUENTS.

TABLES of THEOREMS

FOR THE

CALCULATION of FLUENTS.

TABLE I.

THEOREM I.

$$\dot{F} = x^n \dot{x}.$$

$$F = \frac{x^{n+1} - c^{n+1}}{n+1}.$$

c the quantity to which x is equal when $F = 0$.

NOTE. When n is $= -1$, the expreſſion for the value of F becomes $= \text{Log.}\ \frac{x}{c}$.

THEOREM II.

$$\dot{F} = \overline{a^n + bx^n}|^{p} \times x^{n-1}\dot{x}.$$

$$F = \frac{\overline{a^n + bx^n}|^{p+1} - \overline{a^n + bc^n}|^{p+1}}{bn.\overline{p+1}}.$$

c the value of x when $F = 0$.

NOTE. When p is $= -1$, the expreſſion for the value of F becomes $= \frac{1}{bn} \times \text{Log.}\,\frac{a^n + bx^n}{a^n + bc^n}$.

THEOREM III.

$$\dot{F} = \frac{x^{np-1}\dot{x}}{\overline{a^n + bx^n}|^{p+1}}.$$

$$F = \frac{1}{npa^n} \times \overline{\frac{x^{np}}{\overline{a^n + bx^n}|^{p}} - \frac{c^{np}}{\overline{a^n + bc^n}|^{p}}}.$$

c the value of x when $F = 0$,

NOTE. When p is $= 0$, the expreſſion for the value of F becomes $= \frac{1}{na^n} \times \text{Log.}\,\frac{x^n}{c^n} \times \frac{a^n + bc^n}{a^n + bx^n}$.

THEO-

THEOREM IV.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{\sqrt{2ax^n + x^{2n}}}.$$

$$F = K + \frac{1}{n} \times \text{Log.}\ \overline{a + x^n + \sqrt{2ax^n + x^{2n}}}.$$

Note. K here, and in the following theorems, denotes ſome invariable quantity; which will be determined in any equation whereto it may appertain, by properly ſubſtituting therein any known contemporary values of the variable quantities. And when K is ſo determined the equation may be ſaid to be fitly adjuſted, or corrected.

THEOREM V.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{\sqrt{ba^{2n} + x^{2n}}}.$$

$$F = K + \frac{1}{n} \times \text{Log.}\ \overline{x^n + \sqrt{ba^{2n} + x^{2n}}}.$$

THEOREM VI.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{a^{2n} - x^{2n}}.$$

$$F = K + \frac{1}{2na^n} \times \text{Log.}\ \frac{a^n + x^n}{a^n - x^n}.$$

THEO-

THEOREM VII.

$$\dot{F} = \frac{x^{-1}\dot{x}}{\sqrt{a^{2n} + bx^{2n}}}.$$

$$F = K + \frac{1}{2na^n} \times \text{Log.} \frac{a^n - \sqrt{a^{2n} + bx^{2n}}}{a^n + \sqrt{a^{2n} + bx^{2n}}}.$$

THEOREM VIII.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{\sqrt{2a^n x^n - x^{2n}}}.$$

$$F = K + \frac{1}{na^n} \times \text{Circ. Arc, rad. } a^n, \text{ verſed ſine } x^n.$$

THEOREM IX.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{\sqrt{a^{2n} - x^{2n}}}.$$

$$F = K + \frac{1}{na^n} \times \text{Circ. Arc, rad. } a^n, \text{ ſine } x^n.$$

THEOREM X.

$$\dot{F} = \frac{x^{n-1}\dot{x}}{a^{2n} + x^{2n}}.$$

$$F = K + \frac{1}{na^{2n}} \times \text{Circ. Arc, rad. } a^n, \text{ tang. } x^n.$$

THEO-

THEOREM XI.

$$\dot{F} = \frac{x^{-1}\dot{x}}{\sqrt{x^{2n} - a^{2n}}}.$$

$$F = K + \frac{1}{na^{2n}} \times \text{Circ. Arc, rad. } a^n\text{, secant } x^n.$$

TABLE II.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^m \dot{x}}{x+a}.$$

m any positive integer.

$$F = K + a^m \times \overline{\frac{x^m}{ma^m} - \frac{x^{m-1}}{m-1.a^{m-1}} + \frac{x^{m-2}}{m-2.a^{m-2}} \, (m)^* \pm \text{Log.}\, x+a}.$$

* + or − according as m is even or odd.

THEOREM II.

$$\dot{F} = \frac{x^{-m} \dot{x}}{x+a}.$$

m any positive integer.

$$F = K + \frac{1}{a^m} \times \overline{\frac{a^{m-1}}{m-1.x^{m-1}} - \frac{a^{m-2}}{m-2.x^{m-2}} + \frac{a^{m-3}}{m-3.x^{m-3}} (m-1)^* \pm \text{Log.} \frac{x+a}{x}}.$$

* + or − according as $m-1$ is even or odd.

THEO-

THEOREM III.

$$\dot{F} = \frac{x^m \dot{x}}{x^2 + a^2}.$$

m any even positive number.

$$F = K + a^{m-2} \times \overline{\frac{x^{m-1}}{m-1.a^{m-2}} - \frac{x^{m-3}}{m-3.a^{m-4}} + \frac{x^{m-5}}{m-5.a^{m-6}} \left(\frac{m}{2}\right)^{*} \pm A}.$$

A = Circ. Arc, rad. *a*, tang. *x*.

* + or — according as $\frac{m}{2}$ is even or odd.

THEOREM IV.

$$\dot{F} = \frac{x^m \dot{x}}{x^2 - a^2}.$$

m any even positive number.

$$F = K + a^{m-1} \times \overline{\frac{x^{m-1}}{m-1.a^{m-1}} + \frac{x^{m-3}}{m-3.a^{m-3}} + \frac{x^{m-5}}{m-5.a^{m-5}} \left(\frac{m}{2}\right) + \frac{1}{2} \text{Log.} \frac{a-x}{a+x}}.$$

THEOREM V.

$$\dot{F} = \frac{x^m \dot{x}}{x^2 + a^2}.$$

m any odd positive number.

$$F = K + a^{m-1} \times \overline{\frac{x^{m-1}}{m-1.a^{m-1}} - \frac{x^{m-3}}{m-3.a^{m-3}} + \frac{x^{m-5}}{m-5.a^{m-5}} \left(\frac{m-1}{2}\right)^{*} \pm \frac{1}{2} \text{Log.} \overline{x^2 + a^2}}.$$

a^2 either positive or negative.

* + or — according as $\frac{m-1}{2}$ is even or odd.

THEOREM VI.

$$\dot{F} = \frac{x^{-m}\dot{x}}{x^2 + a^2}.$$

m any even positive number.

$$F = K - \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1.x^{m-1}} - \frac{a^{m-3}}{m-3.x^{m-3}} + \frac{a^{m-5}}{m-5.x^{m-5}}\left(\frac{m}{2}\right)^{*} \pm A}.$$

$A =$ Circ. Arc, rad. 1, tang. $\frac{a}{x}$.

* + or − according as $\frac{m}{2}$ is even or odd.

THEOREM VII.

$$\dot{F} = \frac{x^{-m}\dot{x}}{x^2 - a^2}.$$

m any even positive number.

$$F = K + \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1.x^{m-1}} + \frac{a^{m-3}}{m-3.x^{m-3}} + \frac{a^{m-5}}{m-5.x^{m-5}}\left(\frac{m}{2}\right) + \tfrac{1}{2}\,\text{Log.}\frac{x-a}{x+a}}.$$

THEOREM VIII.

$$\dot{F} = \frac{x^{-m}\dot{x}}{x^2 + a^2}.$$

m any odd positive number.

$$F = K - \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1.x^{m-1}} - \frac{a^{m-3}}{m-3\,x^{m-3}} + \frac{a^{m-5}}{m-5.x^{m-5}}\left(\frac{m-1}{2}\right)^{*} \pm \text{Log.}\frac{x^2+a^2}{x^2}}.$$

a^2 either positive or negative.

* + or − according as $\frac{m-1}{2}$ is even or odd.

TABLE

TABLE III.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}.$$

$$F = K + \frac{4}{a^{\frac{1}{2}}} \times \overline{de - e'e''} = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{de + DP - AD - L}.$$

THEOREM II.

The fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$, generated whilſt x from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt x from $a \times \frac{a-k}{a+k}$ becomes equal to a.

NOTE. All the theorems in this Table refer to the Scheme at the end of it, for the values of the quantities required.

THEO-

THEOREM III.

The fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$, generated whilſt x from o becomes equal to $\overline{2^{\frac{1}{2}} - 1} \times a$, is $= \frac{M}{a^{\frac{1}{2}}}$.

THEOREM IV.

The *whole* fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$ is $= \frac{2M}{a^{\frac{1}{2}}}$.

THEOREM V.

$$\dot{F} = \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}.$$

$$F = \dot{K} + \frac{2}{a^{\frac{1}{2}}} \times \overline{2e'e'' - \text{de}} = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{L + AD - DP}.$$

THEOREM VI.

The tangent eo $\left(= \overline{ax}|^{\frac{1}{2}} \times \overline{\frac{a - x}{a + x}}|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{\frac{1}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$, generated whilſt x from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt x from $a \times \frac{a - k}{a + k}$ becomes equal to a.

THEO-

THEOREM VII.

The fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$, generated whilſt x from o becomes equal to $\overline{2^{\frac{1}{2}} - 1} \times a$, is $= \frac{L}{a^{\frac{1}{2}}} - \overline{2^{\frac{1}{2}} - 1} \times a^{\frac{1}{2}}$.

THEOREM VIII.

The *whole* fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2 - x^2}}$ is $= \frac{2L}{a^{\frac{1}{2}}}$.

THEOREM IX.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times \overline{ae - E'' + e'e''} = K + \frac{2}{a^{\frac{3}{2}}} \times \overline{ae + AD - DP}.$$

$$x = \frac{a^2}{y}.$$

THEOREM X.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}$, generated whilſt y from a becomes equal to $\overline{2^{\frac{1}{2}} + 1} \times a$, is $= \frac{M}{a^{\frac{3}{2}}}$.

THEOREM XI.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}$ is $= \frac{2M}{a^{\frac{3}{2}}}$.

THEO-

THEOREM XII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{DP + ae + 2e'e'' - 2E''} = K + \frac{2}{a^{\frac{1}{2}}} \times AD.$$

$$x = \frac{a^2}{y}.$$

THEOREM XIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}$, generated whilst y from a becomes equal to $\overline{2^{\frac{1}{2}} + 1} \times a$, is $= \overline{2^{\frac{1}{2}} + 1} \times a^{\frac{1}{2}} - \frac{L}{a^{\frac{1}{2}}}$.

Note. The *whole* fluent is infinite.

THEOREM XIV.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{a^2 + y^2}}.$$

$$F = K + \frac{2^{\frac{5}{2}}}{a^{\frac{3}{2}}} \times \overline{ae + e'e'' - E''} = K + \frac{2^{\frac{1}{2}}}{a^{\frac{3}{2}}} \times \overline{ae + AD - DP}.$$

$$x = \overline{a^2 + y^2}|^{\frac{1}{2}} - y.$$

THEOREM XV.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{a^2 + y^2}}$, generated whilst y from o becomes equal to a, is $= \frac{2^{\frac{1}{2}}}{a^{\frac{3}{2}}} \times M$.

THEO-

THEOREM XVI.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{a^2+y^2}}$ is $= \frac{2^{\frac{1}{2}}}{a^{\frac{3}{2}}} \times \mathrm{M}$.

THEOREM XVII.

$$\dot{\mathrm{F}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a^2+y^2}}.$$

$$\mathrm{F} = \mathrm{K} + \frac{2^{\frac{3}{2}}}{a^{\frac{1}{2}}} \times \overline{\mathrm{ae} + 2e'e'' - 2\mathrm{E}'' + \tfrac{1}{2}\mathrm{DP}} = \mathrm{K} + \frac{2^{\frac{3}{2}}}{a^{\frac{1}{2}}} \times \overline{\mathrm{AD} - \tfrac{1}{2}\mathrm{DP}}.$$

$$x = \overline{a^2+y^2}|^{\frac{1}{2}} - y.$$

THEOREM XVIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a^2+y^2}}$, generated whilst y from o becomes equal to a, is $= \overline{\sqrt{2a} - \frac{2}{a}}|^{\frac{1}{2}} \times \mathrm{L}$.

NOTE. The *whole* fluent is infinite.

THEOREM XIX.

$$\dot{\mathrm{F}} = \frac{\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{4}}}.$$

$$\mathrm{F} = \mathrm{K} + \frac{2}{a^{\frac{1}{2}}} \times \overline{2\mathrm{E}'' - 2e'e'' - \mathrm{ae}} = \mathrm{K} + \frac{2}{a^{\frac{1}{2}}} \times \overline{\mathrm{DP} - \mathrm{AD}}.$$

$$x = \sqrt{a^2-y^2}.$$

THEO-

THEOREM XX.

The fluent of $\frac{\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2^{\frac{3}{2}} - 2} \times a$, is $= \frac{L}{a^{\frac{1}{2}}} + \overline{2^{\frac{1}{2}} - 1} \times a^{\frac{1}{2}}$.

THEOREM XXI.

The *whole* fluent of $\frac{\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}$ is $= \frac{2L}{a^{\frac{1}{2}}}$.

THEOREM XXII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}.$$

$$F = K + 2^{\frac{1}{2}} \times \overline{2E'' - 2e'e'' - ae} - (ay - y^2)^{\frac{1}{2}} \times \left(\frac{a-y}{a+y}\right)^{\frac{1}{2}}$$

$$= K + 2^{\frac{1}{2}} \times \overline{DP - AD} - (ay - y^2)^{\frac{1}{2}} \times \left(\frac{a-y}{a+y}\right)^{\frac{1}{2}}.$$

$$x = a \times \left(\frac{a-y}{a+y}\right)^{\frac{1}{2}}.$$

THEOREM XXIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\frac{a}{\sqrt{2}}$, is $= \frac{L}{\sqrt{2}}$.

THEOREM XXIV.

The *whole* fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}$ is $= 2^{\frac{1}{2}}L$.

THEO-

THEOREM XXV.

$$\dot{F} = \frac{\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}.$$

$$F = K + \frac{2^{\frac{3}{2}}}{a^{\frac{1}{2}}} \times \overline{ae + 2e'e'' - 2E'' + \tfrac{1}{2}DP} = K + \frac{2^{\frac{3}{2}}}{a^{\frac{1}{2}}} \times \overline{AD - \tfrac{1}{2}DP}.$$

$$x = y - \sqrt{y^2 - a^2}.$$

THEOREM XXVI.

The fluent of $\frac{\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$, generated whilst y from a becomes equal to $2^{\frac{1}{2}}a$, is $= 2^{\frac{1}{2}} \times \overline{a^{\frac{1}{2}} - \frac{L}{a^{\frac{1}{2}}}}$.

NOTE. The *whole* fluent is infinite.

THEOREM XXVII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}.$$

$$F = K + de - 2e'e'' + \frac{\overline{y^2 - a^2}|^{\frac{1}{2}*}}{y^{\frac{1}{2}}}$$

$$= K + DP - AD - L + \frac{\overline{y^2 - a^2}|^{\frac{1}{2}*}}{y^{\frac{1}{2}}}.$$

* A 3d proportional to DP, CP.

$$x = \frac{a\sqrt{y^2 - a^2}}{y}.$$

THEOREM XXVIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{4}}}$, generated whilst y from a becomes equal to $\overline{\frac{2^{\frac{1}{2}} + 1}{2}}|^{\frac{1}{2}} \times a$, is $= \frac{1}{2} \times \overline{a - \mathrm{L}}$.

Note. The *whole* fluent is infinite.

THEOREM XXIX.

$$\dot{\mathrm{F}} = \frac{\dot{y}}{\overline{a^2 + y^2}|^{\frac{1}{4}}}.$$

$$\mathrm{F} = \mathrm{K} + \frac{2}{a^{\frac{1}{2}}} \times \overline{ae \mp 2e'e'' - 2\mathrm{E}'' + \mathrm{DP}} = \mathrm{K} + \frac{2}{a^{\frac{1}{2}}} \times \mathrm{AD}.$$

$$x = \frac{a^2}{\sqrt{a^2 + y^2}}.$$

THEOREM XXX.

The fluent of $\frac{\dot{y}}{\overline{a^2 + y^2}|^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2 + \sqrt{2}} \times a$, is $= \overline{2^{\frac{1}{2}} + 1} \times a^{\frac{1}{2}} - \frac{\mathrm{L}}{a^{\frac{1}{2}}}$.

Note. The *whole* fluent is infinite.

THEO-

THEOREM XXXI.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\overline{a^2 + y^2}|^{\frac{3}{4}}}.$$

$$F = K + de - 2e'e'' + \frac{y^{\frac{3}{2}}}{\overline{a^2+y^2}|^{\frac{1}{4}}}{}^{*}$$

$$= K + DP - AD - L + \frac{y^{\frac{3}{2}}}{\overline{a^2+y^2}|^{\frac{1}{4}}}{}^{*}.$$

* A 3d proportional to DP, CP.

$$x = \frac{ay}{\sqrt{a^2 + y^2}}.$$

THEOREM XXXII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\overline{a^2+y^2}|^{\frac{3}{4}}}$, generated whilst y from o becomes equal to $\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}} \times a$, is $= \frac{1}{2} \times \overline{a - L}$.

Note. The *whole* fluent is infinite.

THEOREM XXXIII.

$$\dot{F} = \frac{\dot{y}}{\overline{a^2 - y^2}|^{\frac{3}{4}}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times \overline{ae + e'e'' - E''} = K + \frac{2}{a^{\frac{3}{2}}} \times \overline{ae + AD - DP}.$$

$$x = \sqrt{a^2 - y^2}.$$

 THEO-

THEOREM XXXIV.

The fluent of $\frac{\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2 - \sqrt{2}} \times a$, is $= \frac{M}{a^{\frac{1}{2}}}$.

THEOREM XXXV.

The *whole* fluent of $\frac{\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}$ is $= \frac{2M}{a^{\frac{1}{2}}}$.

THEOREM XXXVI.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}.$$

$$F = K + \frac{2^{\frac{1}{2}}}{a^2} \times \overline{ae + \acute{e}e'' - E''} = K + \frac{2^{\frac{3}{2}}}{a^2} \times \overline{ae + AD - DP}.$$

$$x = a \times \overline{\frac{a - y}{a + y}}|^{\frac{1}{2}}.$$

THEOREM XXXVII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\frac{a}{\sqrt{2}}$, is $= \frac{2^{\frac{1}{2}}}{a^2} \times M$.

THEOREM XXXVIII.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}$ is $= \frac{2^{\frac{3}{2}}}{a^2} \times M$.

THEO-

THEOREM XXXIX.

$$\dot{F} = \frac{\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}.$$

$$F = K + \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}} \times \overline{ac + e'e'' - E''} = K + \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}} \times \overline{ac + AD - DP}.$$

$$x = y - \sqrt{y^2 - a^2}.$$

THEOREM XL.

The fluent of $\frac{\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$, generated whilst y from a becomes equal to $2^{\frac{1}{2}} \times a$, is $= \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}} \times$ M.

THEOREM XLI.

The *whole* fluent of $\frac{\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$ is $= \overline{\frac{2}{a}}|^{\frac{1}{2}} \times$ M.

THEOREM XLII.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}.$$

$$F = K + \frac{4}{a^2} \times \overline{de - e'e''} = K + \frac{2}{a^2} \times \overline{de + DP - AD - L}.$$

$$y = \frac{a\sqrt{y^2 - a^2}}{y}.$$

THEO-

THEOREM XLIII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{y^2-a^2}|^{\frac{1}{2}}}$, generated whilſt y from a becomes equal to $\overline{\frac{1}{2^{\frac{1}{2}}}+\frac{1}{2}}|^{\frac{1}{2}} \times a$, is $= \frac{M}{a^2}$.

THEOREM XLIV.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{y^2-a^2}|^{\frac{1}{2}}}$ is $= \frac{2M}{a^2}$.

THEOREM XLV.

$$\dot{F} = \frac{\dot{y}}{\overline{a^2+y^2}|^{\frac{1}{2}}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times \overline{ae + e'e'' - E''} = K + \frac{2}{a^{\frac{3}{2}}} \times \overline{ae + AD - DP}.$$

$$x = \frac{a^2}{\sqrt{a^2+y^2}}.$$

THEOREM XLVI.

The fluent of $\frac{\dot{y}}{\overline{a^2+y^2}|^{\frac{1}{2}}}$, generated whilſt y from o becomes equal to $\sqrt{2+\sqrt{2}} \times a$, is $= \frac{M}{a^{\frac{1}{2}}}$.

THEO-

THEOREM XLVII.

The *whole* fluent of $\frac{\dot{y}}{\overline{a^2+y^2}|^{\frac{5}{4}}}$ is $= \frac{2M}{a^{\frac{3}{2}}}$.

THEOREM XLVIII.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2+y^2}|^{\frac{5}{4}}}.$$

$$F = K + \frac{4}{a^2} \times \overline{de - e'e''} = K + \frac{2}{a^2} \times \overline{de + DP - AD - L}.$$

$$x = \frac{ay}{\sqrt{a^2+y^2}}.$$

THEOREM XLIX.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2+y^2}|^{\frac{5}{4}}}$, generated whilst y from o becomes equal to $\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}} \times a$, is $= \frac{M}{a^2}$.

THEOREM L.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{a^2+y^2}|^{\frac{5}{4}}}$ is $= \frac{2M}{a^2}$.

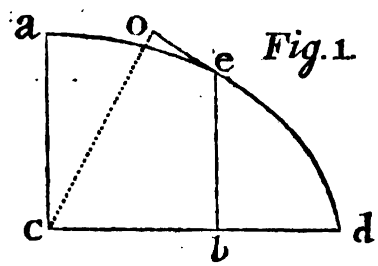
SCHEME.

S C H E M E

FOR

T A B L E III.

$d \begin{cases} = \frac{1}{4} \text{ of the periphery of a circle whose radius is } 1. \\ = 1.57079632. \end{cases}$



aed (Fig. 1.) is a quadrantal arc of an ellipsis = E′.

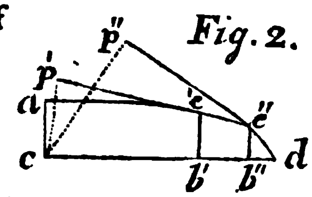
Semi-transverse axis cd = $2^{\frac{1}{2}}a$.

Semi-conjugate axis ac = a.

Abscissa cb = $2^{\frac{1}{2}}\sqrt{a^2 - ax}$.

Ordinate be = $\sqrt{ax}$.

$e \begin{cases} = \text{the value of E}' \text{ when } a \text{ is } = 1. \\ = 1.91009889. \end{cases}$



$ae'e''d$ (Fig. 2.) is a quadrantal arc of another ellipsis = E″.

Semi-transverse axis $cd = \overline{\frac{1}{\sqrt{2}} + \frac{1}{2}} \times a$.

Semi-conjugate axis $ac = \overline{\frac{1}{\sqrt{2}} - \frac{1}{2}} \times a$.

$e'p'$ and its equal $e''p''$ (each $= \sqrt{a^2 - ax}$) are tangents, to which cp', cp'' are perpendiculars.

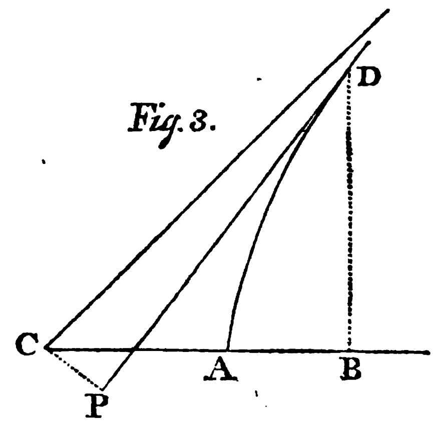
The abscissa cb', or cb'', corresponding to the ordinate $b'e'$, or $b''e''$, is $= \frac{2^{\frac{1}{2}} + 1}{2^{\frac{7}{4}}} \times a^{\frac{1}{2}} \sqrt{2^{\frac{1}{2}}a + a - x \mp \sqrt{ax + x^2}}$.

$f \begin{cases} = \text{the value of E}'' \text{ when } a \text{ is } = 1. \\ = 1.2545845059. \end{cases}$

SCHEME for TABLE III. continued.

AD (Fig. 3.) is an equilateral hyperbola, whoſe vertex is A and center C.

DP is a tangent, to which CP is perpendicular.



$AC = a.$

$CP = \sqrt{ax}.$

$DP = \left(\frac{a}{x}\right)^{\frac{1}{2}} \times \left(a^2 - x^2\right)^{\frac{1}{2}}.$

Abſciſſa CAB (correſponding to the ordinate BD)

$= a \times \left(\frac{a + x}{2x}\right)^{\frac{1}{2}}.$

L, the *limit* of DP–AD, is $= 2E'' - E' = a \times \overline{2f - e} = \frac{a}{2} \times \overline{e - \sqrt{e^2 - 2d}}$

$= .5990701173 \times a.$

$M = 2 \times \overline{E' - E''} = 2a \times \overline{e - f} = \frac{a}{2} \times \overline{e + \sqrt{e^2 - 2d}}$

$= 1.3110287771 \times a.$

NOTE. *All the Theorems in* TABLE III. *refer to this Scheme.*

TABLE IV.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

$$F = K - \tfrac{3}{2}a^{-\frac{2}{3}}B.$$

$$z = \frac{a^{\frac{2}{3}} - x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{b - y}{y}.$$

THEOREM II.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{2}{3}}P}{3^{\frac{1}{2}} + 1}$.

NOTE. The neceſſary explanation, reſpecting the values of the quantities concerned in the theorems in this Table, is given at the end of it.

THEO-

THEOREM III.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

$$F = K - \tfrac{3}{2}a^{-\frac{1}{2}}D.$$

$$z = \frac{a^{\frac{2}{3}} - x^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{b - y}{b}.$$

THEOREM IV.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{1}{2}}P}{3^{\frac{1}{2}} + 1}$.

THEOREM V.

$$\dot{F} = \frac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{\frac{1}{3}} \times \overline{C - D}.$$

$$\dot{z} = \frac{a^{\frac{2}{3}} - x^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{b - y}{b}.$$

THEOREM VI.

The *whole* fluent of $\frac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$ is $= 3^{\frac{1}{4}}a^{\frac{1}{2}}Q$.

THEO-

THEOREM VII.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{4}a^{\frac{2}{3}} \times \overline{A + B} - \tfrac{3}{2}x^{-\frac{1}{3}}\sqrt{a^2 - x^2}.$$

$$z = \frac{a^{\frac{2}{3}} - x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{b - y}{y}.$$

THEOREM VIII.

The *whole* fluent of $\frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$ is $= \frac{3^{\frac{1}{4}}a^{\frac{2}{3}}Q}{2}$.

THEOREM IX.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{-\frac{2}{3}}D.$$

$$z = \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{y - b}{y}.$$

THEOREM X.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{2}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEO-

THEOREM XI.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{-\frac{1}{3}}B.$$

$$z = \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{y - b}{b}.$$

THEOREM XII.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{1}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEOREM XIII.

$$\dot{F} = \frac{x^{\frac{1}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{\frac{1}{3}} \times \overline{A + B}.$$

$$z = \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{y - b}{b}.$$

NOTE. The *whole* fluent is infinite.

THEO-

THEOREM XIV.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{4}a^{\frac{2}{3}} \times \overline{C - D} + \tfrac{3}{2}x^{-\frac{1}{3}}\sqrt{x^2 - a^2}.$$

$$z = \frac{x^{\frac{2}{3}} - a^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{y - b}{y}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XV.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

$$F = K - \tfrac{3}{2}a^{-\frac{2}{3}}D.$$

$$z = \frac{a^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{b + y}{y}.$$

THEOREM XVI.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$ is $= \frac{2.3^{\frac{1}{4}}a^{-\frac{2}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEO-

THEOREM XVII.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{-\frac{1}{3}}D.$$

$$z = \frac{a^{\frac{2}{3}} + x^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{b + y}{b}.$$

THEOREM XVIII.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$ is $= \frac{2 \cdot 3^{\frac{1}{4}}a^{-\frac{1}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEOREM XIX.

$$\dot{F} = \frac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

$$F = K + \tfrac{3}{2}a^{\frac{1}{3}} \times \overline{C - D}.$$

$$z = \frac{a^{\frac{2}{3}} + x^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{b + y}{b}.$$

NOTE. The *whole* fluent is infinite.

THEO-

THEOREM XX.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

$$F = K - \tfrac{3}{4}a^{\frac{2}{3}} \times \overline{C - D} + \tfrac{3}{2}x^{-\frac{1}{3}}\sqrt{a^2 + x^2}.$$

$$z = \frac{a^{\frac{2}{3}} + x^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{b + y}{y}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXI.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{3}}}.$$

$$F = -\text{fl.}\ \frac{w^{-\frac{1}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{2}}},$$ to be found by theorem XVII.

$$w = \frac{a^2 - x^2}{x} = \frac{b^3 - y^3}{y^{\frac{3}{2}}}.$$

THEOREM XXII.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}}$ is $= \frac{2^{\frac{2}{3}}3^{\frac{1}{4}}a^{-\frac{1}{3}}P}{3^{\frac{1}{2}} + 1}.$

THEO-

THEOREM XXIII.

$$\dot{F} = \frac{\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}} = \frac{\frac{1}{2}y^{\frac{1}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{2}{3}}}.$$

$$F = -\text{fl.}\ \frac{w^{\frac{1}{2}}\dot{w}}{\overline{a^2 - w^2}|^{\frac{1}{2}}},$$ to be found by theor. v.

$$w = \overline{a^2 - x^2}|^{\frac{1}{3}} = \overline{b^3 - y^3}|^{\frac{1}{2}}.$$

THEOREM XXIV.

The *whole* fluent of $\frac{\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}}$ is $= 3^{\frac{1}{4}} a^{\frac{1}{3}} Q.$

THEOREM XXV.

$$\dot{F} = \frac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2}y\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{3}}}.$$

$$F = -\tfrac{3}{4}w^{\frac{2}{3}} + \tfrac{1}{2}\text{fl.}\ \frac{w^{\frac{2}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{2}}},$$ to be found by theor. xx.

$$w = \frac{a^2 - x^2}{x} = \frac{b^3 - y^3}{y^{\frac{3}{2}}}.$$

THEOREM XXVI.

The *whole* fluent of $\frac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}}$ is $= \frac{3^{\frac{1}{4}} a^{\frac{2}{3}} Q}{2^{\frac{1}{2}}}.$

THEOREM XXVII.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

$F = \dfrac{a^{\frac{1}{3}}w^{\frac{5}{3}}}{\overline{a^2 + w^2}|^{\frac{1}{2}}} - \frac{2}{3}a^{\frac{1}{3}}\,\text{fl.}\,\dfrac{w^{\frac{2}{3}}\dot{w}}{\overline{a^2 + w^2}|^{\frac{1}{2}}}$, to be found by theor. XX.

$$w = \frac{ax}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}y^{\frac{3}{2}}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

THEOREM XXVIII.

The *whole* fluent of $\dfrac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}}$ is $= \dfrac{2^{\frac{1}{3}}aQ}{3^{\frac{1}{4}}}$.

THEOREM XXIX.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{3}}}.$$

$F = \text{fl.}\,\dfrac{w^{-\frac{2}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{3}}}$, to be found by theor. XVII.

$$w = \frac{x^2 - a^2}{x} = \frac{y^3 - b^3}{y^{\frac{3}{2}}}.$$

THEOREM XXX.

The *whole* fluent of $\dfrac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{3}}}$ is $= \dfrac{2^{\frac{2}{3}}3^{\frac{5}{4}}a^{-\frac{1}{3}}P}{3^{\frac{1}{2}} + 1}$

THEO-

THEOREM XXXI.

$$\dot{F} = \frac{\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2} y^{\frac{1}{2}} \dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{3}}}.$$

$F = \text{fl.}\ \dfrac{w^{\frac{1}{3}} \dot{w}}{\overline{a^2 + w^2}|^{\frac{1}{2}}}$, to be found by theor. XIX.

$$w = \overline{x^2 - a^2}|^{\frac{1}{2}} = \overline{y^3 - b^3}|^{\frac{1}{2}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXXII.

$$\dot{F} = \frac{x^{\frac{1}{3}} \dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2} y \dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{3}}}.$$

$F = \frac{3}{4} w^{\frac{4}{3}} + \frac{1}{2}\ \text{fl.}\ \dfrac{w^{\frac{2}{3}} \dot{w}}{\overline{a^2 + w^2}|^{\frac{1}{2}}}$, to be found by theor. XX.

$$w = \frac{x^2 - a^2}{x} = \frac{y^3 - b^3}{y^{\frac{3}{2}}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXXIII.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

$$F = \frac{a^{\frac{1}{3}}w^{\frac{5}{3}}}{\overline{w^2 - a^2}|^{\frac{1}{2}}} - \tfrac{2}{3}a^{\frac{1}{3}}\,\text{fl.}\,\frac{w^{\frac{2}{3}}\dot{w}}{\overline{w^2 - a^2}|^{\frac{1}{2}}},$$ to be found by theor. XIV.

$$w = \frac{ax}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}y^{\frac{3}{2}}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXXIV.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

$$F = \pm\,\text{fl.}\,\frac{w^{-\frac{1}{3}}\dot{w}}{\overline{w^2 - 4a^2}|^{\frac{1}{2}}},$$ to be found by theor. XI.

$$w = \frac{a^2 + x^2}{x} = \frac{b^3 + y^3}{y^{\frac{3}{2}}}.$$

THEOREM XXXV.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$ is $= \frac{2^{\frac{2}{3}}3^{\frac{1}{4}}a^{-\frac{1}{3}}p}{3^{\frac{1}{2}} + 1}.$

THEO-

THEOREM XXXVI.

$$\dot{F} = \frac{\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2} y^{\frac{1}{2}} \dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{3}}}.$$

$$F \doteq \text{fl.}\ \frac{w^{\frac{1}{3}} \dot{w}}{\overline{w^2 - a^2}|^{\frac{1}{2}}},$$ to be found by theor. XIII.

$$w = \overline{a^2 + x^2}|^{\frac{1}{2}} = \overline{b^3 + y^3}|^{\frac{1}{2}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXXVII.

$$\dot{F} = \frac{x^{\frac{1}{3}} \dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2} y \dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{3}}}.$$

$$F = \frac{3}{4} w^{\frac{2}{3}} \pm \frac{1}{2}\ \text{fl.}\ \frac{w^{\frac{2}{3}} \dot{w}}{\overline{w^2 - 4a^2}|^{\frac{1}{2}}},$$ to be found by theor. XIV.

$$w = \frac{a^2 + x^2}{x} = \frac{b^3 + y^3}{y^{\frac{3}{2}}}.$$

NOTE. The *whole* fluent is infinite.

THEO-

THEOREM XXXVIII.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{1}{3}}}.$$

$$F = \frac{a^{\frac{1}{3}}w^{\frac{5}{3}}}{\overline{a^2 - w^2}|^{\frac{1}{2}}} - \frac{2}{3}a^{\frac{1}{3}}\,\text{fl.}\,\frac{w^{\frac{2}{3}}\dot{w}}{\overline{a^2 - w^2}|^{\frac{1}{2}}},$$ to be found by theor. VII.

$$w = \frac{ax}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}y^{\frac{3}{2}}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM XXXIX.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}} = \frac{\frac{1}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{2}{3}}}.$$

$$F = a^{-\frac{1}{3}}\,\text{fl.}\,\frac{w^{-\frac{2}{3}}\dot{w}}{\overline{a^2 + w^2}|^{\frac{1}{6}}},$$ to be found by theor. XV.

$$w = \frac{ax}{\overline{a^2 - x^2}|^{\frac{1}{2}}} = \frac{b^{\frac{1}{2}}y^{\frac{1}{2}}}{\overline{b^3 - y^3}|^{\frac{1}{2}}}.$$

THEOREM XL.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}}$ is $= \frac{2.3^{\frac{1}{4}}a^{-1}P}{3^{\frac{1}{2}} + 1}$.

THEO-

THEOREM XLI.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}} = \frac{\frac{3}{2}\dot{y}}{\overline{b^3 - y^3}|^{\frac{1}{3}}}.$$

$$F = -\text{fl.}\ \frac{w^{-\frac{2}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{2}}},$$ to be found by theor. XV.

$$w = \frac{a^2 - x^2}{x} = \frac{b^3 - y^3}{y^{\frac{3}{2}}}.$$

THEOREM XLII.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{3}}}$ is $= \frac{2^{\frac{1}{3}}3^{\frac{1}{4}}a^{-\frac{2}{3}}p}{3^{\frac{1}{2}} + 1}$.

THEOREM XLIII.

$$\dot{F} = \frac{\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{1}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{2}{3}}}.$$

$$F = -\text{fl.}\ \frac{w^{-\frac{1}{3}}\dot{w}}{\overline{a^2 - w^2}|^{\frac{1}{2}}},$$ to be found by theor III.

$$w = \overline{a^2 - x^2}|^{\frac{1}{2}} = \overline{b^3 - y^3}|^{\frac{1}{2}}.$$

THEOREM XLIV.

The *whole* fluent of $\frac{\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{1}{3}}p}{3^{\frac{1}{2}} + 1}$.

THEO-

THEOREM XLV.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{b^3 - y^3}|^{\frac{2}{3}}}.$$

$$F = -\tfrac{3}{2}w^{\frac{2}{3}} + \tfrac{1}{2}\,\text{fl.}\,\frac{w^{\frac{1}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{2}{3}}},$$ to be found by theor. XIX.

$$w = \frac{a^2 - x^2}{x} = \frac{b^3 - y^3}{y^{\frac{3}{2}}}.$$

THEOREM XLVI.

The *whole* fluent of $\frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 - x^2}|^{\frac{2}{3}}}$ is $= 2^{\frac{1}{3}}3^{\frac{1}{4}}a^{\frac{1}{3}}Q.$

THEOREM XLVII.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{2}{3}}}.$$

$$F = -a^{-\frac{1}{3}}\,\text{fl.}\,\frac{w^{-\frac{2}{3}}\dot{w}}{\overline{w^2 - a^2}|^{\frac{1}{2}}},$$ to be found by theor. IX.

$$w = \frac{ax}{\overline{x^2 - a^2}|^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}y^{\frac{3}{2}}}{\overline{y^3 - b^3}|^{\frac{1}{2}}}.$$

THEOREM XLVIII.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}}$ is $= \frac{3^{\frac{1}{2}}a^{-1}P}{3^{\frac{1}{2}} + 1}.$

THEO-

THEOREM XLIX.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}\dot{y}}{\overline{y^3 - b^3}|^{\frac{2}{3}}}.$$

$F = \text{fl.}\ \frac{w^{-\frac{2}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{3}}}$, to be found by theorem XV.

$$w = \frac{x^2 - a^2}{x} = \frac{y^3 - b^3}{y^{\frac{3}{2}}}.$$

THEOREM L.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}}$ is $= \frac{2^{\frac{1}{3}}3^{\frac{1}{4}}a^{-\frac{1}{3}}p}{3^{\frac{1}{4}} + 1}$.

THEOREM LI.

$$\dot{F} = \frac{\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{1}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{2}{3}}}.$$

$F = \text{fl.}\ \frac{w^{-\frac{1}{3}}\dot{w}}{\overline{a^2 + w^2}|^{\frac{1}{2}}}$, to be found by theorem XVII.

$$w = \overline{x^2 - a^2}|^{\frac{1}{3}} = \overline{y^3 - b^3}|^{\frac{1}{3}}.$$

THEOREM LII.

The *whole* fluent of $\frac{\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}}$ is $= \frac{2 \cdot 3^{\frac{1}{4}}a^{-\frac{1}{3}}p}{3^{\frac{1}{2}} + 1}$.

THEOREM LIII.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{x^2 - a^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{3}{2}}\dot{y}}{\overline{y^3 - b^3}|^{\frac{2}{3}}}.$$

$$F = \tfrac{3}{2}w^{\frac{1}{3}} + \tfrac{1}{2}\text{fl.}\frac{w^{\frac{1}{3}}\dot{w}}{\overline{4a^2 + w^2}|^{\frac{1}{2}}},$$ to be found by theor. XIX.

$$w = \frac{x^2 - a^2}{x} = \frac{y^3 - b^3}{y^{\frac{3}{2}}}.$$

NOTE. The *whole* fluent is infinite.

THEOREM LIV.

$$\dot{F} = \frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{-\frac{1}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{2}{3}}}.$$

$$F = a^{-\frac{1}{3}}\text{fl.}\frac{w^{-\frac{2}{3}}\dot{w}}{\overline{a^2 - w^2}|^{\frac{1}{2}}},$$ to be found by theor. I.

$$w = \frac{ax}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{b^{\frac{3}{2}}y^{\frac{3}{2}}}{\overline{b^3 + y^3}|^{\frac{1}{2}}}.$$

THEOREM LV.

The *whole* fluent of $\frac{x^{-\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}}$ is $= \frac{3^{\frac{1}{4}}a^{-1}P}{3^{\frac{1}{2}} + 1}$.

THEO-

THEOREM LVI.

$$\dot{F} = \frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}\dot{y}}{\overline{b^3 + y^3}|^{\frac{2}{3}}}.$$

$$F = \pm \text{fl.}\ \frac{w^{-\frac{2}{3}}\dot{w}}{\overline{w^2 - 4a^2}|^{\frac{1}{2}}},$$ to be found by theorem IX.

$$w = \frac{a^2 + x^2}{x} = \frac{b^3 + y^3}{y^{\frac{3}{2}}}.$$

THEOREM LVII.

The *whole* fluent of $\frac{x^{-\frac{1}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}}$ is $= \frac{2^{\frac{1}{3}}3^{\frac{1}{4}}a^{-\frac{2}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEOREM LVIII.

$$\dot{F} = \frac{\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{1}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{2}{3}}}.$$

$$F = \text{fl.}\ \frac{w^{-\frac{1}{3}}\dot{w}}{\overline{w^2 - a^2}|^{\frac{1}{2}}},$$ to be found by theorem XI.

$$w = \overline{a^2 + x^2}|^{\frac{1}{2}} = \overline{b^3 + y^3}|^{\frac{1}{2}}.$$

THEOREM LIX.

The *whole* fluent of $\frac{\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}}$ is $= \frac{3^{\frac{1}{4}}a^{-\frac{1}{3}}P}{3^{\frac{1}{2}} + 1}$.

THEOREM LX.

$$\dot{F} = \frac{x^{\frac{2}{3}}\dot{x}}{\overline{a^2 + x^2}|^{\frac{2}{3}}} = \frac{\frac{3}{2}y^{\frac{1}{2}}\dot{y}}{\overline{b^3 + y^3}|^{\frac{2}{3}}}.$$

$$F = \tfrac{3}{2}w^{\frac{1}{3}} \pm \tfrac{1}{2}\,\text{fl.}\,\frac{w^{\frac{1}{3}}\dot{w}}{\overline{w^2 - 4a^2}|^{\frac{1}{2}}},$$ to be found by theorem XIII.

$$w = \frac{a^2 + x^2}{x} = \frac{b^3 + y^2}{y^{\frac{1}{2}}}.$$

NOTE. The *whole* fluent is infinite.

EXPLA-

EXPLANATION of TABLE IV.

A denotes the fluent of $\frac{z^{\frac{1}{2}}\dot{z}}{\sqrt{3+3z+z^2}}$, B the fluent of $\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{3+3z+z^2}}$,

C ..... the fluent of $\frac{z^{\frac{1}{2}}\dot{z}}{\sqrt{3-3z+z^2}}$, D the fluent of $\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{3-3z+z^2}}$;

whereof the general values are aſſigned in TABLE XII. by means of the arcs of the conic ſections.

$$P \text{ is } = p + \sqrt{p^2 - \overline{3^{\frac{1}{2}} + 1}.d} = 4.366205,$$

$$Q \ = p - \sqrt{p^2 - \overline{3^{\frac{1}{2}} + 1}.d} = .982889;$$

d being $(= 1.570796)$ the quadrantal arc of a circle whoſe radius is 1,

p ..... $(= 2.674547)$ the quadrantal arc of an ellipſis whoſe ſemi-axes are $3^{\frac{1}{2}}$ and $\frac{3^{\frac{1}{2}}.3^{\frac{1}{2}}+1}{2^{\frac{1}{2}}}$.

The *whole* fluent (mentioned in any theorem) is generated whilſt x from 0 becomes equal to a, whilſt x from a becomes infinite, or whilſt x from 0 becomes infinite; according as the denominator of the reſpective fluxional expreſſion is ſome power or root of $a^2 - x^2$, $x^2 - a^2$, or $a^2 + x^2$.

TABLE

TABLE V.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{a^n + x^n}.$$

$$F = \frac{2a^{m-n}}{n} \times \begin{cases} \text{fl.}\,\frac{M'b\dot{z}}{b^2+z^2} + \text{fl.}\,\frac{M''c\dot{z}}{c^2+z^2} + \text{fl.}\,\frac{M'''d\dot{z}}{d^2+z^2}\ \&c. \\ -\,\text{fl.}\,\frac{N'z\dot{z}}{b^2+z^2} - \text{fl.}\,\frac{N''z\dot{z}}{c^2+z^2} - \text{fl.}\,\frac{N'''z\dot{z}}{d^2+z^2}\ \&c. \end{cases}$$

m any poſitive integer leſs than the integer n.

$$z = \frac{x-a}{x+a}.$$

$$\left.\begin{matrix} M' \text{ and } N' \\ M'' \text{ and } N'' \\ M''' \text{ and } N''' \end{matrix}\right\}\ \text{ſine and coſine of}\ \begin{cases} \frac{m \times 180°}{n}. \\ \frac{3m \times 180°}{n}. \\ \frac{5m \times 180°}{n}. \end{cases}$$

&c. &c.

$b =$ tang. of $\frac{90°}{n}$, $c =$ tang. of $\frac{3 \times 90°}{n}$, $d =$ tang. of $\frac{5 \times 90°}{n}$, &c.

ſo long as theſe arcs are leſs than 90°.

Radius $= 1$.

THEO-

THEOREM II.

The fluent of $\frac{x^{m-1}\dot{x}}{a^n+x^n}$, generated whilst x from being equal to any quantity k becomes equal to $\frac{a^2}{k}$, is

$$=\frac{4a^{m-n}}{n}\times\left\{\begin{array}{l}M'\times \text{Circ. Arc, rad. 1, tang. } \frac{g}{b}\\ +M''\times \text{Circ. Arc, rad. 1, tang. } \frac{g}{c}\\ +M'''\times \text{Circ. Arc, rad. 1, tang. } \frac{g}{d}\\ \&c.\end{array}\right.$$

m, n, M′, M″, &c. b, c, &c. as in the preceding theor. $g=\frac{a-k}{a+k}$.

THEOREM III.

The fluent of $\frac{x^{m-1}\dot{x}}{a^n+x^n}$, generated whilst x from being equal to any quantity k becomes equal to a, — the fluent of the same fluxion, generated whilst x from a becomes equal to $\frac{a^2}{k}$, is

$$=\frac{a^{m-n}}{n}\times\overline{N'\text{ Log. }\frac{b^2+g^2}{b^2}+N''\text{ Log. }\frac{c^2+g^2}{c^2}\ \&c.}$$

m, n, N′, N″, &c. b; c, &c. as in theor. 1. $g=\frac{a-k}{a+k}$.

THEO-

THEOREM IV.

The fluent of $\frac{x^{m-1}\dot{x}}{a^n + x^n}$, generated whilst x from o becomes equal to a, is

$$= \frac{a^{m-n}}{n} \times \overline{\frac{Q}{s} + N' \text{ Log. } \frac{1 + b^2}{b^2} + N'' \text{ Log. } \frac{1 + c^2}{c^2} \ \&c.}$$

m, n, N′, N″, &c. b, c, &c. as in theorem 1.

Q = quadrantal arc of the circ. rad. 1. s = sine of $\frac{2mQ}{n}$.

THEOREM V.

The *whole* fluent of $\frac{x^{m-1}\dot{x}}{a^n + x^n}$, generated whilst x from o becomes infinite, is $= \frac{2a^{m-n}}{ns} \times Q$.

m any positive integer or fraction less than the integer or fraction n.

Q and s as in the preceding theorem.

THEOREM VI.

$$\dot{F} = \overline{1 + a^{2m-n}x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n + x^n}.$$

$$F = \frac{4a^{m-n}}{n} \times \overline{\text{fl. } \frac{M'b\dot{z}}{b^2 + z^2} + \text{fl. } \frac{M''c\dot{z}}{c^2 + z^2} \ \&c.}$$

m, n, z, M′, M″, &c. b, c, &c. as in theorem 1.

THEO-

THEOREM VII.

The fluent of $\overline{1 + a^{2m-n}x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n + x^n}$, generated whilst x from o becomes equal to a, or whilst x from a becomes infinite, is $= \frac{2a^{m-n}}{ns} \times Q$.

m and n as in theorem v. Q and s as in theorem IV.

THEOREM VIII.

$$\dot{F} = \overline{1 - a^{2m-n}x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n + x^n}.$$

$$F = -\frac{4a^{m-n}}{n} \times \overline{\text{fl.}\ \frac{N'z\dot{z}}{b^2 + z^2} + \text{fl.}\ \frac{N''z\dot{z}}{c^2 + z^2}\ \&c.}$$

m, n, z, N′, N″, &c. b, c, &c. as in theorem I.

THEOREM IX.

The fluent of $\overline{1 - a^{2m-n}x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n + x^n}$, generated whilst x from o becomes equal to a, is

$$= \frac{2a^{m-n}}{n} \times \overline{N' \text{Log.}\ \frac{1 + b^2}{b^2} + N'' \text{Log.}\ \frac{1 + c^2}{c^2}\ \&c.}$$

m, n, N′, N″, &c. b, c, &c. as in theorem I.

THEOREM X.

$$\dot{F} = \overline{a+x}|^{n-r} \times \frac{x^{m-1}\dot{x}}{a^n + x^n}.$$

$$F = \frac{2^{n-r+1}}{na^{r-m}} \times \begin{cases} \text{fl.}\ \frac{N'b\dot{z}}{b^2+z^2} - \text{fl.}\ \frac{N''c\dot{z}}{c^2+z^2} + \&c. \\ -\ \text{fl.}\ \frac{M'z\dot{z}}{b^2+z^2} + \text{fl.}\ \frac{M''z\dot{z}}{c^2+z^2} - \&c. \end{cases}$$

r any positive integer not greater than the integer n.

m any positive integer less than r.

$$z = \frac{x-a}{x+a}.$$

$M' = P' \times$ sine of $\frac{r-2m}{n}.90°$. $N' = P' \times$ cosine of $\frac{r-2m}{n}.90°$.

$M'' = P'' \times$ sine of $3.\frac{r-2m}{n}.90°$. $N'' = P'' \times$ cosine of $3.\frac{r-2m}{n}.90°$.

$M''' = P''' \times$ sine of $5.\frac{r-2m}{n}.90°$. $N''' = P''' \times$ cosine of $5.\frac{r-2m}{n}.90°$.

&c. &c. &c. &c.

$b =$ tang. of $\frac{90°}{n}$, $c =$ tang. of $\frac{3 \times 90°}{n}$, $d =$ tang. of $\frac{5 \times 90°}{n}$, &c.

so long as these arcs are less than 90°.

Radius $= 1$.

$P' = \overline{1+b^2}|^{\frac{r-n}{2}}$, $P'' = \overline{1+c^2}|^{\frac{r-n}{2}}$; $P''' = \overline{1+d^2}|^{\frac{r-n}{2}}$, &c.

THEO-

THEOREM XI.

$$\dot{F} = \overline{a+x}|^{n-r} \times \frac{x^{\frac{1}{2}r-1}\dot{x}}{a^n + x^n}.$$

$$F = \frac{2^{n-r+1}}{na^{\frac{r}{2}}} \times \overline{\text{fl.}\ \frac{P'b\dot{z}}{b^2+z^2} - \text{fl.}\ \frac{P''c\dot{z}}{c^2+z^2} + \&c.}$$

n, z, b, c, &c. P', P'', &c. as in the preceding theorem.
r any even poſitive number not greater than n.

THEOREM XII.

$$\dot{F} = \overline{a+x}|^{n-r} \times \frac{x^{r-1}\dot{x}}{a^n + x^n}.$$

$$F = \text{fl.}\ \frac{\dot{z}}{1-z} + \frac{2^{n-r+1}}{na} \times \left\{ \begin{array}{l} \text{fl.}\ \frac{N'b\dot{z}}{b^2+z^2} - \text{fl.}\ \frac{N''c\dot{z}}{c^2+z^2} + \&c. \\ + \text{fl.}\ \frac{M'z\dot{z}}{b^2+z^2} - \text{fl.}\ \frac{M''z\dot{z}}{c^2+z^2} + \&c. \end{array} \right.$$

$r = 0$, or any poſitive integer not greater than the integer n.
z, b, c, &c. P', P'', &c. as in theorem X.

$M' = P' \times$ ſine of $\frac{r}{n}.90^\circ$. $N' = P' \times$ coſine of $\frac{r}{n}.90^\circ$.

$M'' = P'' \times$ ſine of $\frac{3r}{n}.90^\circ$. $N'' = P'' \times$ coſine of $\frac{3r}{n}.90^\circ$.

$M''' = P''' \times$ ſine of $\frac{5r}{n}.90^\circ$. $N''' = P''' \times$ coſine of $\frac{5r}{n}.90^\circ$.

&c. &c. &c. &c.

THEOREM XIII.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{a^n - x^n}.$$

$$F = \frac{2a^{m-n}}{n} \times \left\{ \begin{array}{r} \text{fl. } \frac{M'c\dot{z}}{c^2+z^2} + \text{fl. } \frac{M''d\dot{z}}{d^2+z^2} \text{ \&c.} \\ - \text{fl. } \frac{\frac{1}{2}\dot{z}}{z} - \text{fl. } \frac{N'z\dot{z}}{c^2+z^2} - \text{fl. } \frac{N''z\dot{z}}{d^2+z^2} \text{ \&c.} \end{array} \right.$$

m any poſitive integer leſs than the integer n.

$$z = \frac{x-a}{x+a}.$$

$$\left. \begin{array}{l} M' \text{ and } N' \\ M'' \text{ and } N'' \\ M''' \text{ and } N''' \end{array} \right\} \text{ſine and coſine of } \left\{ \begin{array}{l} \frac{2m \times 180^\circ}{n}. \\ \frac{4m \times 180^\circ}{n}. \\ \frac{6m \times 180^\circ}{n}. \end{array} \right.$$

&c. &c.

$c =$ tang. of $\frac{180^\circ}{n}$, $d =$ tang. of $\frac{2 \times 180^\circ}{n}$, $e =$ tang. of $\frac{3 \times 180^\circ}{n}$, &c.

ſo long as theſe arcs are leſs than 90°.

Radius = 1.

THEOREM XIV.

$$\dot{F} = \overline{1 - a^{2m-n}x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n - x^n}.$$

$$F = \frac{4a^{m-n}}{n} \times \overline{\text{fl. } \frac{M'c\dot{z}}{c^2+z^2} + \text{fl. } \frac{M''d\dot{z}}{d^2+z^2} \text{ \&c.}}$$

m, n, z, M', M'', &c. c, d, &c. as in the preceding theorem.

THEO-

THEOREM XV.

The fluent of $\overline{1 - a^{2m-n} x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n - x^n}$, generated whilst x from o becomes equal to a, or whilst x from a becomes infinite, is $= \frac{2a^{m-n}}{nt} \times Q$.

m and n as in theor. v. Q as in theor. iv. $t =$ tang. of $\frac{2mQ}{n}$.

THEOREM XVI.

$$\dot{F} = \overline{1 + a^{2m-n} x^{n-2m}} \times \frac{x^{m-1}\dot{x}}{a^n - x^n}.$$

$$F = -\frac{4a^{m-n}}{n} \times \overline{\text{fl. } \frac{\frac{1}{2}\dot{z}}{z} + \text{fl. } \frac{N'z\dot{z}}{c^2 + z^2} + \text{fl. } \frac{N''z\dot{z}}{d^2 + z^2} \text{ \&c.}}$$

m, n, z, N′, N″, &c. c, d, &c. as in theorem XIII.

THEOREM XVII.

$$\dot{F} = \overline{a + x}|^{n-r} \times \frac{x^{m-1}\dot{x}}{a^n - x^n}.$$

$$F = \frac{2^{n-r+1}}{na^{r-m}} \times \begin{cases} \quad + \text{fl.}\ \frac{M'c\dot{z}}{c^2+z^2} - \text{fl.}\ \frac{M''d\dot{z}}{d^2+z^2} + \&c. \\ - \text{fl.}\ \frac{\frac{1}{2}\dot{z}}{z} + \text{fl.}\ \frac{N'z\dot{z}}{c^2+z^2} - \text{fl.}\ \frac{N''z\dot{z}}{d^2+z^2} + \&c. \end{cases}$$

r any positive integer not greater than the integer n.

m any positive integer less than r.

$$z = \frac{x-a}{x+a}.$$

$M' = P' \times$ sine of $\frac{r-2m}{n} \cdot 180°$. $N' = P' \times$ cosine of $\frac{r-2m}{n} \cdot 180°$.

$M'' = P'' \times$ sine of $2 \cdot \frac{r-2m}{n} \cdot 180°$. $N'' = P'' \times$ cosine of $2 \cdot \frac{r-2m}{n} \cdot 180°$.

$M''' = P''' \times$ sine of $3 \cdot \frac{r-2m}{n} \cdot 180°$. $N''' = P''' \times$ cosine of $3 \cdot \frac{r-2m}{n} \cdot 180°$.

&c. &c. &c. &c.

$c =$ tang. of $\frac{180°}{n}$, $d =$ tang. of $\frac{2 \times 180°}{n}$, $e =$ tang. of $\frac{3 \times 180°}{n}$, &c.

so long as these arcs are less than 90°.

Radius = 1.

$P' = \overline{1 + c^2}|^{\frac{r-n}{2}}$, $P'' = \overline{1 + d^2}|^{\frac{r-n}{2}}$, $P''' = \overline{1 + e^2}|^{\frac{r-n}{2}}$, &c.

THEO-

THEOREM XVIII.

$$\dot{F} = \overline{a+x}|^{n-r} \times \frac{x^{\frac{1}{2}r-1}\dot{x}}{a^n - x^n}.$$

$$F = -\frac{2^{n-r+1}}{na^{\frac{1}{2}r}} \times \overline{\text{fl.}\,\frac{\frac{1}{2}\dot{z}}{z} - \text{fl.}\,\frac{P'z\dot{z}}{c^2+z^2} + \text{fl.}\,\frac{P''z\dot{z}}{d^2+z^2} - \&c.}$$

n, z, c, d, &c. P′, P″, &c. as in the preceding theorem.

r any even positive number not greater than n.

THEOREM XIX.

$$\dot{F} = \overline{a+x}|^{n-r} \times \frac{x^{r-1}\dot{x}}{a^n - x^n}.$$

$$F = \text{fl.}\,\frac{\dot{z}}{z-1} - \frac{2^{n-r+1}}{n} \times \left\{ \begin{array}{l} + \text{fl.}\,\frac{M'c\dot{z}}{c^2+z^2} - \text{fl.}\,\frac{M''d\dot{z}}{d^2+z^2} + \&c. \\ \text{fl.}\,\frac{\frac{1}{2}\dot{z}}{z} - \text{fl.}\,\frac{N'z\dot{z}}{c^2+z^2} + \text{fl.}\,\frac{N''z\dot{z}}{d^2+z^2} - \&c. \end{array} \right.$$

r any positive integer not greater than the integer n.

z, c, d, &c. P′, P″, &c. as in theorem XVII.

$M' = P' \times$ fine of $\frac{r}{n}.180°$. $N' = P' \times$ cofine of $\frac{r}{n}.180°$.

$M'' = P'' \times$ fine of $\frac{2r}{n}.180°$. $N'' = P'' \times$ cofine of $\frac{2r}{n}.180°$.

$M''' = P''' \times$ fine of $\frac{3r}{n}.180°$. $N''' = P''' \times$ cofine of $\frac{3r}{n}.180°$.

&c. &c. &c. &c.

THEO-

THEOREM XX.

$$\dot F = \frac{y^{-1}\dot y}{\overline{gb^r + ky^r}^{\frac{m}{n}}}$$

$F = -\frac{na^{n-m}}{rb^{\frac{mr}{n}}} \times \text{fl.} \frac{x^{m-1}\dot x}{a^n - gx^n}$, to be found by Theor. I. or XIII.

$$x = \frac{ab^{\frac{r}{n}}}{\overline{gb^r + ky^r}^{\frac{1}{n}}}. \quad y = h \times \frac{\overline{a^n - gx^n}^{\frac{1}{r}}}{k^{\frac{1}{r}} x^{\frac{n}{r}}}.$$

THEOREM XXI.

The fluent of $\frac{y^{-1}\dot y}{\overline{y^r - b^r}^{\frac{m}{n}}}$, generated whilst y from being equal to h becomes equal to $2^{\frac{1}{r}}h$, is

$$= \frac{1}{rb^{\frac{mr}{n}}} \times \overline{\frac{Q}{s} - N' \text{Log.} \frac{1+b^2}{b^2} - N'' \text{Log.} \frac{1+c^2}{c^2} \text{ \&c.}}$$

m, n, N', N'', &c. b, c, &c. as in Theor. I. Q and s as in Theor. IV.

THEOREM XXII.

The *whole* fluent of $\frac{y^{-1}\dot y}{\overline{y^r - b^r}^{\frac{m}{n}}}$, generated whilst y from being equal to h becomes infinite, is $= \frac{2Q}{rsb^{\frac{mr}{n}}}$.

m, n, Q, and s as in Theorem IV.

THEO-

THEOREM XXIII.

$$\dot{F} = \frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{gb^r + ky^r}|^{\frac{m}{n}}}.$$

$F = \frac{na^{n-m}}{r} \times \text{fl.}\, \frac{x^{m-1}\dot{x}}{a^n - kx^n}$, to be found by Theorem I. or XIII.

$$x = \frac{ay^{\frac{r}{n}}}{\overline{gb^r + ky^r}|^{\frac{1}{n}}}. \quad y = h \times \frac{g^{\frac{1}{r}} x^{\frac{n}{r}}}{\overline{a^n - kx^n}|^{\frac{1}{r}}}.$$

THEOREM XXIV.

The fluent of $\frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{b^r - y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to $\frac{b}{2^{\frac{1}{r}}}$, is

$$= \frac{1}{r} \times \overline{\frac{Q}{s} + N' \text{Log.} \frac{1+b^2}{b^2} + N'' \text{Log.} \frac{1+c^2}{c^2} \&c.}$$

m, n, N', N''. &c. b, c, &c. as in Theor. I. Q and s as in Theor. IV.

THEOREM XXV.

The *whole* fluent of $\frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{b^r - y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to h, is $= \frac{2Q}{rs}$.

m, n, Q, and s as in Theorem IV.

TABLE

TABLE VI.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F}=\overline{a+bx^n}|^p \times x^{rn-1}\dot{x}.$$

$p=0$, or any positive integer.

$$F=K+\frac{a^p x^{rn}}{n}\times\overline{\frac{1}{r}+\frac{pbx^n}{r+1.a}+\frac{p.p-1.b^2x^{2n}}{2.r+2.a^2}+\frac{p.p-1.p-2.b^3x^{3n}}{2.3.r+3.a^3}(p+1)}$$

$$=K+\frac{b^p x^{sn}}{n}\times\overline{\frac{1}{s}+\frac{pa}{s-1.bx^n}+\frac{p.p-1.a^2}{2.s-2.b^2x^{2n}}+\frac{p.p-1.p-2.a^3}{2.3.s-3.b^3x^{3n}}(p+1)}$$

$$=K+\frac{x^{rn}z^p}{ns}\times\overline{1+\frac{pa}{s-1.z}+\frac{p.p-1.a^2}{s-1.s-2.z^2}+\frac{p.p-1.p-2.a^3}{s-1.s-2.s-3.z^3}(p+1)}$$

$$=K+\frac{x^{rn}z^p}{nr}\times\overline{1-\frac{pbx^n}{r+1.z}+\frac{p.p-1.b^2x^{2n}}{r+1.r+2.z^2}-\frac{p.p-1.p-2.b^3x^{3n}}{r+1.r+2.r+3.z^3}(p+1)}$$

$s=p+r.\quad z=a+bx^n.$

THEO-

THEOREM II.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}.$$

r any positive integer.

$$F = K + \frac{z^s}{nb^r} \times \overline{\frac{1}{s} - \frac{r-1.a}{s-1.z} + \frac{r-1.r-2.a^2}{2.s-2.z^2} - (r)}$$

$$= K + \frac{\overline{-a}|^{r-1} z^{p+1}}{nb^r} \times \overline{\frac{1}{p+1} - \frac{r-1.z}{p+2.a} + \frac{r-1.r-2.z^2}{2.p+3.a^2} - (r)}$$

$$= K + \frac{x^{rn-n} z^{p+1}}{bns} \times \overline{1 - \frac{r-1.a}{s-1.bx^n} + \frac{r-1.r-2.a^2}{s-1.s-2.b^2x^{2n}} - (r)}$$

$$= K + \frac{x^{rn-n} z^{p+1}}{p+1.bn} \times \overline{1 - \frac{r-1.z}{p+2.bx^n} + \frac{r-1.r-2.z^2}{p+2.p+3.b^2x^{2n}} - (r)}.$$

$$s = p + r. \quad z = a + bx^n.$$

THEOREM III.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}.$$

$p + r$ any negative integer.

$$F = K + \frac{x^{rn}}{na^t z^r} \times \overline{\frac{1}{r} - \frac{t-1.bx^n}{r+1.z} + \frac{t-1.t-2.b^2x^{2n}}{2.r+2.z^2} - (t)}$$

$$= K - \frac{\overline{-b}|^{t-1} z^{p+1}}{na^t x^{pn+n}} \times \overline{\frac{1}{p+1} - \frac{t-1.z}{p+2.bx^n} + \frac{t-1.t-2.z^2}{2.p+3.b^2x^{2n}} - (t)}$$

$$= K + \frac{x^{rn} z^{p+1}}{rna} \times \overline{1 + \frac{t-1.bx^n}{r+1.a} + \frac{t-1.t-2.b^2x^{2n}}{r+1.r+2.a^2} + (t)}$$

$$= K - \frac{x^{rn} z^{p+1}}{p+1.na} \times \overline{1 - \frac{t-1.z}{p+2.a} + \frac{t-1.t-2.z^2}{p+2.p+3.a^2} - (t)}.$$

$$t = -p - r. \quad z = a + bx^n.$$

NOTE I. When the exponent of the power of x or z in any term (taken with its prefixed factor) in these theorems is $= 0$, and the denominator of the same term is also $= 0$, the equation will be fitly adjusted by taking the value of such term according to the Note to Theorem I. II. or III. in TABLE I. K being, at the same time, taken of a proper value.

NOTE 2. For the fluent of $\overline{a+bx^n}|^p \times x^{rn-1}\dot{x}$, when p and r are such numbers that neither of the series in the values of F, in the three theorems in this Table, terminates, take any one of the values in which the series may be found to converge; and such value of F, considering the series therein as continued *ad infinitum*, will express the said fluent.

TABLE VII.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \overline{a + bx^{n}}|^{p} \times x^{rn-1}\dot{x}.$$

$$F = \frac{x^{rn}z^{p}}{ns} \times \overline{1 + \frac{pa}{s-1.z} + \frac{p.p-1.a^{2}}{s-1.s-2.z^{2}} + (v)}$$

$$+ \frac{p.p-1.p-2\,(v) \times a^{v}}{s.s-1.s-2\,(v)} \times \text{fl. } z^{p-v}x^{rn-1}\dot{x}.$$

$$= \frac{x^{rn}z^{p}}{nr} \times \overline{1 - \frac{pbx^{n}}{r+1.z} + \frac{p.p-1.b^{2}x^{2n}}{r+1.r+2.z^{2}} - (v)}$$

$$\pm \frac{p.p-1.p-2\,(v) \times b^{v}}{r.r+1.r+2\,(v)} \times \text{fl. } z^{p-v}x^{rn+vn-1}\dot{x}.$$

$$= -\frac{x^{rn-n}z^{p+1}}{bnt} \times \overline{1 + \frac{r-1.a}{t+1.bx^{n}} + \frac{r-1.r-2\,a^{2}}{t+1.t+2.b^{2}x^{2n}} + (v)}$$

$$+ \frac{r-1.r-2.r-3\,(v) \times a^{v}}{t.t+1.t+2\,(v) \times b^{v}} \times \text{fl. } z^{p}x^{rn-vn-1}\dot{x}.$$

=

$$= \frac{x^{rn-n} z^{p+1}}{p+1.bn} \times \overline{1 - \frac{r-1.z}{p+2.bx^n} + \frac{r-1.r-2.z^2}{p+2.p+3.b^2 x^{2n}} - (v)}$$

$$* \pm \frac{r-1.r-2.r-3\,(v)}{p+1.p+2.p+3\,(v) \times b^v} \times \text{fl. } z^{p+v} x^{rn-vn-1} \dot{x}$$

$$= \frac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \frac{t-1.bx^n}{r+1.a} + \frac{t-1.t-2.b^2 x^{2n}}{r+1.r+2.a^2} + (v)}$$

$$+ \frac{t-1.t-2.t-3\,(v) \times b^v}{r.r+1.r+2\,(v) \times a^v} \times \text{fl. } z^p x^{rn+vn-1} \dot{x}$$

$$= -\frac{x^{rn} z^{p+1}}{p+1.an} \times \overline{1 + \frac{s+1.z}{p+2.a} + \frac{s+1.s+2.z^2}{p+2.p+3.a^2} + (v)}$$

$$+ \frac{s+1.s+2.s+3\,(v)}{p+1.p+2.p+3\,(v) \times a^v} \times \text{fl. } z^{p+v} x^{rn-1} \dot{x}.$$

$$s = -t = p + r. \quad z = a + bx^n.$$

* + or − according as v is even or odd.

NOTE. This theorem being derived from the theorems in the preceding Table, it may sometimes be necessary to adjust the equation in the manner directed in Note 1. at the end of that Table: and the following theorems being deduced from this, the like correction in some of them may sometimes be requisite.

THEO-

THEOREM II.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a + bx^n}|^p \times x^{rn+vn-1}\dot{x}.$$

$$G = \frac{r.r+1.r+2\,(v) \times a^v}{t-1.t-2.t-3\,(v) \times b^v} \times$$

$$\overline{\overline{F - \frac{x^{rn} z^{p+1}}{anr} \times 1 + \frac{t-1.bx^n}{r+1.a} + \frac{t-1.t-2.b^2 x^{2n}}{r+1.r+2.a^2}\,(v).}}$$

$$t = -p - r. \quad z = a + bx^n.$$

THEOREM III.

The *whole* fluent of $\overline{a + bx^n}|^p \times x^{rn+vn-1}\dot{x}$,

generated whilst x from $\begin{cases} 0 \text{ becomes } = \overline{\frac{-a}{b}}|^{\frac{1}{n}}, \ ^{(1)} \\ \overline{\frac{-a}{b}}|^{\frac{1}{n}} \text{ becomes infinite}, \ ^{(2)} \\ 0 \text{ becomes infinite}, \ ^{(3)} \end{cases}$

$$\text{is} = \frac{r.r+1.r+2\,(v) \times a^v}{t-1.t-2.t-3\,(v) \times b^v} \times F':$$

(1) a, n, $p+1$, and r being positive; b negative.

(2) b, n, and $p+1$ being positive; a, and $p+r+v$ negative.

(3) a, b, n, and r being positive; $p+r+v$ negative.

F' = the contemporary fluent of $\overline{a + bx^n}|^p \times x^{rn-1}\dot{x}$.

THEO-

THEOREM IV.

$$\dot{F} = \overline{a+bx^n}\,\rceil^{p} \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a+bx^n}\,\rceil^{p} \times x^{rn-vn-1}\dot{x}.$$

$$G = \frac{t.t+1.t+2\,(v) \times b^v}{r-1.r-2.r-3\,(v) \times a^v} \times$$

$$\overline{F + \frac{x^{rn-n} z^{p+1}}{bnt} \times \overline{1 + \frac{r-1.a}{t+1.bx^n} + \frac{r-1.r-2.a^2}{t+1.t+2.b^2x^{2n}}\,(v)}}.$$

$$t = -p-r. \quad z = a+bx^n.$$

THEOREM V.

The *whole* fluent of $\overline{a+bx^n}\,\rceil^{p} \times x^{rn-vn-1}\dot{x}$,

generated whilst x from
$$\begin{cases} 0 \text{ becomes } = \overline{\frac{-a}{b}}\,\rceil^{\frac{1}{n}}, & (1) \\ \overline{\frac{-a}{b}}\,\rceil^{\frac{1}{n}} \text{ becomes infinite}, & (2) \\ 0 \text{ becomes infinite}, & (3) \end{cases}$$

$$\text{is} = \frac{t.t+1.t+2\,(v) \times b^v}{r-1.r-2.r-3\,(v) \times a^v} \times F':$$

(1) a, n, $p+1$, and $r-v$ being positive; b negative.
(2) b, n, and $p+1$ being positive; a, and $p+r$ negative.
(3) a, b, n, and $r-v$ being positive; $p+r$ negative.

F' = the contemporary fluent of $\overline{a+bx^n}\,\rceil^{p} \times x^{rn-1}\dot{x}$.

THEO-

THEOREM VI.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a + bx^n}|^{p+v} \times x^{rn-1}\dot{x}.$$

$$G = \frac{p + 1.p + 2.p + 3\,(v) \times a^v}{s + 1.s + 2.s + 3\,(v)} \times$$

$$\overline{F + \frac{x^{rn} z^{p+1}}{p + 1.an} \times \overline{1 + \frac{s + 1\,z}{p + 2.a} + \frac{s + 1.s + 2.z^2}{p + 2.p + 3.a^2}\,(v)}}.$$

$$s = p + r. \quad z = a + bx^n.$$

THEOREM VII.

The *whole* fluent of $\overline{a + bx^n}|^{p+v} \times x^{rn-1}\dot{x}$,

generated whilst x from $\begin{cases} 0 \text{ becomes} = \overline{\frac{-a}{b}}|^{\frac{1}{n}},\ ^{(1)} \\ \overline{\frac{-a}{b}}|^{\frac{1}{n}} \text{ becomes infinite},\ ^{(2)} \\ 0 \text{ becomes infinite},\ ^{(3)} \end{cases}$

$$\text{is} = \frac{p + 1.p + 2.p + 3\,(v) \times a^v}{s + 1.s + 2.s + 3\,(v)} \times F':$$

(1) a, n, $p + 1$, and r being positive; b negative.

(2) b, n, and $p + 1$ being positive; a, and $p + r + v$ negative.

(3) a, b, n, and r being positive; $p + r + v$ negative.

$F' =$ the contemporary fluent of $\overline{a + bx^n}|^p \times x^{rn-1}\dot{x}$.

THEOREM VIII.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a + bx^n}|^{p-v} \times x^{rn-1}\dot{x}.$$

$$G = \frac{s.s-1.s-2\,(v)}{p.p-1.p-2\,(v) \times a^v} \times$$

$$\overline{F - \frac{x^{rn}z^p}{ns} \times 1 + \frac{pa}{s-1.z} + \frac{p.p-1.a^2}{s-1.s-2.z^2}\,(v)}.$$

$$s = p + r. \quad z = a + bx^n.$$

THEOREM IX.

The *whole* fluent of $\overline{a + bx^n}|^{p-v} \times x^{rn-1}\dot{x}$,

generated whilst x from
- o becomes $= \overline{\frac{-a}{b}}|^{\frac{1}{n}}$, (1)
- $\overline{\frac{-a}{b}}|^{\frac{1}{n}}$ becomes infinite, (2)
- o becomes infinite, (3)

$$\text{is} = \frac{s.s-1.s-2\,(v)}{p.p-1.p-2\,(v) \times a^v} \times F':$$

(1) a, n, $p - v + 1$, and r being positive; b negative.

(2) b, n, and $p - v + 1$ being positive; a, and $p + r$ negative.

(3) a, b, n, and r being positive; $p + r$ negative.

F' = the contemporary fluent of $\overline{a + bx^n}|^p \times x^{rn-1}\dot{x}$.

THEO-

THEOREM X.

$$\dot{F} = \overline{a+bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a+bx^n}|^{p-v} \times x^{rn+vn-1}\dot{x}.$$

$$G = {}^{*}\pm \frac{r.r+1.r+2\,(v)}{p.p-1.p-2\,(v) \times b^v} \times$$

$$\overline{F - \frac{x^{rn}z^p}{nr} \times 1 - \frac{pbx^n}{r+1.z} + \frac{p.p-1.b^2x^{2n}}{r+1.r+2.z^2} - (v)}$$

$$z = a + bx^n.$$

THEOREM XI.

The *whole* fluent of $\overline{a+bx^n}|^{p-v} \times x^{rn+vn-1}\dot{x}$,

generated whilst x from
- o becomes $= \overline{\frac{-a}{b}}|^{\frac{1}{n}}$, (1)
- $\overline{\frac{-a}{b}}|^{\frac{1}{n}}$ becomes infinite, (2)
- o becomes infinite, (3)

$$\text{is} = \pm \frac{r.r+1.r+2\,(v)}{p.p-1.p-2\,(v) \times b^v} \times F':$$

(1) a, [illegible] $-v+1$, and r being positive; b negative.

(2) [illegible] and $p-v+1$ being positive; a, and $p+r$ negative.

(3) [illegible], n, and r being positive; $p+r$ negative.

F' = the contemporary fluent of $\overline{a+bx^n}|^p \times x^{rn-1}\dot{x}$.

* + or − according as v is even or odd.

THEOREM XII.

$$\dot{F} = \overline{a+bx^n}|^{p} \times x^{rn-1}\dot{x}. \quad \dot{G} = \overline{a+bx^n}|^{p+v} \times x^{rn-vn-1}\dot{x}.$$

$$G = {}^{*}\pm \frac{p+1.p+2.p+3\,(v) \times b^{v}}{r-1.r-2.r-3\,(v)} \times$$

$$\overline{F - \frac{x^{rn-n}z^{p+1}}{p+1.bn} \times \overline{1 - \frac{r-1.z}{p+2.bx^{n}} + \frac{r-1.r-2.z^{2}}{p+2.p+3.b^{2}x^{2n}} - (v)}}.$$

$$z = a + bx^n.$$

THEOREM XIII.

The *whole* fluent of $\overline{a+bx^n}|^{p+v} \times x^{rn-vn-1}\dot{x}$,

generated whilst x from
- o becomes $= \overline{\frac{-a}{b}}|^{\frac{1}{n}}$, (1)
- $\overline{\frac{-a}{b}}|^{\frac{1}{n}}$ becomes infinite, (2)
- o becomes infinite, (3)

$$\text{is} = {}^{*}\pm \frac{p+1.p+2.p+3\,(v) \times b^{v}}{r-1.r-2.r-3\,(v)} \times F':$$

(1) a, n, $p+1$, and $r-v$ being positive; b negative.

(2) b, n, and $p+1$ being positive; a, and $p+r$ negative.

(3) a, b, n, and $r-v$ being positive; $p+r$ negative.

F' = the contemporary fluent of $\overline{a+bx^n}|^{p} \times x^{rn-1}\dot{x}$.

* + or − according as v is even or odd.

THEO-

THEOREM XIV.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{H} = \overline{a + bx^n}|^{p+w} \times x^{rn+vn-1}\dot{x}.$$

$$H = \frac{p+1.p+2(w) \times a^w}{s+1.s+2\,(w)} \times \frac{x^{rn+vn} z^{p+1}}{p+1.an} \times \overline{1 + \frac{s+1.z}{p+2.a} + \frac{s+1.s+2.z^2}{p+2.p+3.a^2}(w)}$$

$$* \pm \frac{r.r+1\,(v) \times p+1.p+2\,(w) \times a^{v+w}}{p+r+1.p+r+2\,(v+w) \times b^v} \times$$

$$\overline{F - \frac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \frac{t-1.bx^n}{r+1.a} + \frac{t-1.t-2.b^2x^{2n}}{r+1.r+2.a^2}\,(v)}}.$$

$$s = p + r + v. \quad t = -p - r. \quad z = a + bx^n.$$

THEOREM XV.

The *whole* fluent of $\overline{a + bx^n}|^{p+w} \times x^{rn+vn-1}\dot{x}$,

generated whilst x from $\begin{cases} \text{o becomes} = \overline{\frac{-a}{b}}|^{\frac{1}{n}}, \ ^{(1)} \\ \overline{\frac{-a}{b}}|^{\frac{1}{n}} \text{ becomes infinite,} \ ^{(2)} \\ \text{o becomes infinite,} \ ^{(3)} \end{cases}$

$$\text{is} = * \pm \frac{r.r+1\,(v) \times p+1.p+2\,(w) \times a^{v+w}}{p+r+1.p+r+2\,(v+w) \times b^v} \times F':$$

(1) a, n, $p + 1$, and r being positive; b negative.

(2) b, n, and $p + 1$ being positive; a, and $p + r + v + w$ negative.

(3) a, b, n, and r being positive; $p + r + v + w$ negative.

F' = the contemporary fluent of $\overline{a + bx^n}|^p \times x^{rn-1}\dot{x}$.

* + or − according as v is even or odd.

THEO-

THEOREM XVI.

$$\dot{F} = \overline{a + bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{H} = \overline{a + bx^n}|^{p-w} \times x^{rn+vn-1}\dot{x}.$$

$$H = \frac{s.s-1\,(w)}{p.p-1\,(w) \times a^w} \times - \frac{x^{rn+vn}z^p}{ns} \times \overline{1 + \frac{pa}{s-1.z} + \frac{p.p-1.a^2}{s-1.s-2.z^2}\,(w)}$$

$$+ \frac{r.r+1\,(v) \times s.s-1\,(w) \times a^{v-w}}{t-1.t-2\,(v) \times p.p-1\,(w) \times b^v} \times$$

$$\overline{F - \frac{x^{rn}z^{p+1}}{anr} \times \overline{1 + \frac{t-1.bx^n}{r+1.a} + \frac{t-1.t-2.b^2x^{2n}}{r+1.r+2.a^2}\,(v)}}.$$

$$s = p + r + v. \quad t = -p - r. \quad z = a + bx^n.$$

THEOREM XVII.

The *whole* fluent of $\overline{a + bx^n}|^{p-w} \times x^{rn+vn-1}\dot{x}$,

generated whilst x from $\begin{cases} 0 \text{ becomes } = \overline{\frac{-a}{b}}|^{\frac{1}{n}}, & (1) \\ \overline{\frac{-a}{b}}|^{\frac{1}{n}} \text{ becomes infinite}, & (2) \\ 0 \text{ becomes infinite}, & (3) \end{cases}$

is $= \frac{r.r+1\,(v) \times s.s-1\,(w) \times a^{v-w}}{t-1.t-2\,(v) \times p.p-1\,(w) \times b^v} \times F'$:

(1) a, n, $p - w + 1$, and r being positive; b negative.

(2) b, n, and $p - w + 1$ being posit.; a, $p + r$, and $p + r + v - w$ negat.

(3) a, b, n, and r being positive; $p + r$, and $p + r + v - w$ negative.

F' = the contemporary fluent of $\overline{a + bx^n}|^p \times x^{rn-1}\dot{x}$.

THEO-

THEOREM XVIII.

$$\dot{F} = \overline{a+bx^n}|^{p} \times x^{rn-1}\dot{x}. \quad \dot{H} = \overline{a+bx^n}|^{p+w} \times x^{rn-wn-1}\dot{x}.$$

$$H = \frac{p+1.p+2\,(w) \times a^{w}}{s+1.s+1\,(w)} \times \frac{x^{rn-wn} z^{p+1}}{p+1.an} \times \overline{1 + \frac{s+1.z}{p+2.a} + \frac{s+1.s+2.z^2}{p+2.p+3.a^2}\,(w)}$$

$$+ \frac{t.t+1\,(v) \times p+1.p+2\,(w) \times b^{v}}{r-1.r-2\,(v) \times s+1.s+2\,(w) \times a^{v-w}} \times$$

$$\overline{\overline{F + \frac{x^{rn-n} z^{p+1}}{bnt} \times \overline{1 + \frac{r-1.a}{t+1.bx^n} + \frac{r-1.r-2.a^2}{t+1.t+2.b^2x^{2n}}\,(v)}}}.$$

$$s = p+r-v. \quad t = -p-r. \quad z = a+bx^n.$$

THEOREM XIX.

The *whole* fluent of $\overline{a+bx^n}|^{p+w} \times x^{rn-wn-1}\dot{x}$,

generated whilst x from
- o becomes $= \overline{\frac{-a}{b}}|^{\frac{1}{n}}$, (1)
- $\overline{\frac{-a}{b}}|^{\frac{1}{n}}$ becomes infinite, (2)
- o becomes infinite, (3)

$$\text{is} = \frac{t.t+1\,(v) \times p+1.p+2\,(w) \times b^{v}}{r-1.r-2\,(v) \times s+1.s+2\,(w) \times a^{v-w}} \times F';$$

(1) a, n, $p+1$, and $r-v$ being positive; b negative.

(2) b, n, and $p+1$ being positive; a, $p+r$, and $p+r-v+w$ negative.

(3) a, b, n, and $r-v$ being positive; $p+r$, and $p+r-v+w$ negat.

F' = the contemporary fluent of $\overline{a+bx^n}|^{p} \times x^{rn-1}\dot{x}$.

THEO-

THEOREM XX.

$$\dot{F} = \overline{a+bx^n}|^p \times x^{rn-1}\dot{x}. \quad \dot{H} = \overline{a+bx^n}|^{p-w} \times x^{rn-wn-1}\dot{x}.$$

$$H = \frac{s.s-1\,(w)}{p.p-1\,(w) \times a^w} \times -\frac{x^{rn-wn}z^p}{ns} \times \overline{1 + \frac{pa}{s-1.z} + \frac{p.p-1.a^2}{s-1.s-2.z^2}\,(w)}$$

$$+ \frac{t.t+1\,(v) \times s.s-1\,(w) \times b^v}{r-1.r-2\,(v) \times p.p-1\,(w) \times a^{v+w}} \times$$

$$\overline{F + \frac{x^{rn-n}z^{p+1}}{bnt} \times \overline{1 + \frac{r-1.a}{t+1.bx^n} + \frac{r-1.r-2.a^2}{t+1.t+2.b^2x^{2n}}\,(v)}}$$

$$s = p + r - v. \quad t = -p - r. \quad z = a + bx^n.$$

THEOREM XXI.

The *whole* fluent of $\overline{a+bx^n}|^{p-w} \times x^{rn-wn-1}\dot{x}$,

$$\text{generated whilst } x \text{ from} \left\{ \begin{array}{l} 0 \text{ becomes } = \overline{\frac{-a}{b}}|^{\frac{1}{n}}, \;^{(1)} \\ \overline{\frac{-a}{b}}|^{\frac{1}{n}} \text{ becomes infinite, } ^{(2)} \\ 0 \text{ becomes infinite, } ^{(3)} \end{array} \right.$$

$$\text{is} = \frac{t.t+1\,(v) \times s.s-1\,(w) \times b^v}{r-1.r-2\,(v) \times p.p-1\,(w) \times a^{v+w}} \times F':$$

(1) a, n, $p - w + 1$, and $r - v$ being pofitive; b negative.

(2) b, n, and $p - w + 1$ being pofitive; a, and $p + r$ negative.

(3) a, b, n, and $r - v$ being pofitive; $p + r$ negative.

F' = the contemporary fluent of $\overline{a+bx^n}|^p \times x^{rn-1}\dot{x}$.

THEO-

THEOREM XXII.

The *whole* fluent of $\overline{a - x^n}|^p \times x^{rn-1}\dot{x} \times \overline{P + Qx^n + Rx^{2n} \&c.}$

generated whilst x from o becomes equal to $a^{\frac{1}{n}}$,

$$\text{is} = F \times \overline{P + \frac{Qar}{s+1} + \frac{Ra^2 r.r+1}{s+1.s+2} + \frac{Sa^3 r.r+1.r+2}{s+1.s+2.s+3} \&c.}$$

a, n, $p+1$, and r being positive; $s = p + r$;

F = the contemporary fluent of $\overline{a - x^n}|^p \times x^{rn-1}\dot{x}$.

THEOREM XXIII.

The *whole* fluent of $\overline{a - x^n}|^p \times x^{rn-1}\dot{x} \times \overline{P + Q.\overline{a - x^n} + R.\overline{a - x^n}|^2 \&c.}$

generated whilst x from o becomes equal to $a^{\frac{1}{n}}$,

$$\text{is} = F \times \overline{P + \frac{Qa.p+1}{s+1} + \frac{Ra^2.p+1.p+2}{s+1.s+2} + \frac{Sa^3.p+1.p+2.p+3}{s+1.s+2.s+3} \&c.}$$

a, n, $p+1$, r, s, and F being as in the preceding theorem.

THEOREM XXIV.

The *whole* fluent of $\overline{x^n - a}|^p \times x^{rn-1}\dot{x} \times \overline{P + Qx^{-n} + Rx^{-2n} \&c.}$

generated whilst x from $a^{\frac{1}{n}}$ becomes infinite,

$$\text{is} = F \times \overline{P + \frac{Qs}{a.r-1} + \frac{Rs.s-1}{a^2.r-1.r-2} + \frac{Ss.s-1.s-2}{a^3.r-1.r-2.r-3} \&c.}$$

a, n, and $p+1$ being positive; s $(= p + r)$ negative;

F = the contemporary fluent of $\overline{x^n - a}|^p \times x^{rn-1}\dot{x}$.

THEOREM XXV.

The *whole* fluent of $\overline{x^n - a}^p \times x^{rn-1}\dot{x} \times \overline{P + Q.\frac{x^n-a}{x^n} + R.\overline{\frac{x^n-a}{x^n}}^2 \&c.}$

generated whilst x from $a^{\frac{1}{n}}$ becomes infinite,

is $= F \times \overline{P - \frac{Q.p+1}{r-1} + \frac{R.p+1.p+2}{r-1.r-2} - \frac{S.p+1.p+2.p+3}{r-1.r-2.r-3} \&c.}$

a, n, $p+1$, $p+r$, and F being as in the preceding theorem.

THEOREM XXVI.

The *whole* fluent of $\overline{a+x^n}^p \times x^{rn-1}\dot{x} \times \overline{P + \frac{Q}{a+x^n} + \frac{R}{\overline{a+x^n}^2} \&c.}$

generated whilst x from o becomes infinite,

is $= F \times \overline{P + \frac{Qs}{ap} + \frac{Rs.s-1}{a^2p.p-1} + \frac{Ss.s-1.s-2}{a^3p.p-1.p-2} \&c.}$

a, n, and r being positive; s $(= p + r)$ negative;

$F =$ the contemporary fluent of $\overline{a+x^n}^p \times x^{rn-1}\dot{x}$.

THEOREM XXVII.

The *whole* fluent of $\overline{a+x^n}^p \times x^{rn-1}\dot{x} \times \overline{P + \frac{Qx^n}{a+x^n} + \frac{Rx^{2n}}{\overline{a+x^n}^2} \&c.}$

generated whilst x from o becomes infinite,

is $= F \times \overline{P - \frac{Qr}{p} + \frac{Rr.r+1}{p.p-1} - \frac{Sr.r+1.r+2}{p.p-1.p-2} \&c.}$

a, n, r, $p+r$, and F being as in the preceding theorem.

TABLE

TABLE VIII.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\frac{1.2.3(r-1) \times \overline{p+q} + \overline{1.p+q} + \overline{2.p+q+3(r-1)}}{p + \overline{1.p} + \overline{2.p+3(r-1)} \times q + \overline{1.q} + \overline{2.q+3(r-1)}} \text{ is } =$$

$$F \times \overline{1 + \frac{pq}{r} + \frac{p.\overline{p-1} \times q.\overline{q-1}}{1.2 \times r.\overline{r+1}} + \frac{p.\overline{p-1}.\overline{p-2} \times q.\overline{q-1}.\overline{q-2}}{1.2.3 \times r.\overline{r+1}.\overline{r+2}} \ \&c.}$$

F being $= nq \times$ the *whole* fluent of $\overline{1-x^n}|^p \times x^{qn-1}\dot{x}$.

THEOREM II.

$$\frac{p + \overline{r}.\overline{p+r+1}.\overline{p+r+2}\,(q)}{r.\overline{r+1}.\overline{r+2}\,(q)} \text{ is } =$$

$$1 + \frac{pq}{r} + \frac{p.\overline{p-1} \times q.\overline{q-1}}{1.2 \times r.\overline{r+1}} + \frac{p.\overline{p-1}.\overline{p-2} \times q.\overline{q-1}.\overline{q-2}}{1.2.3 \times r.\overline{r+1}.\overline{r+2}} \ \&c.$$

 THEO-

THEOREM III.

$$\dot{F} = \overline{a - x^n}|^{p} \times x^{rn-1}\dot{x}.$$

$$F = \begin{cases} K + \dfrac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \dfrac{s+1.x^n}{r+1.a} + \dfrac{s+1.s+2.x^{2n}}{r+1.r+2.a^2}\,(v)} \\ + \dfrac{s+1.s+2\,(v)}{r.r+1\,(v) \times a^v} \times \dfrac{x^{r''n} z^p}{nr''} \times \overline{1 + \dfrac{px^n}{r''+1.z} + \dfrac{p.p-1.x^{2n}}{r''+1.r''+2.z^2}\,(w)} \\ + \dfrac{s+1.s+2.s+3\,(v)}{r.r+1.r+2\,(v) \times a^v} \times \dfrac{p.p-1.p-2\,(w)}{r''.r''+1.r''+2\,(w)} \times \dfrac{x^{r''+w} z^{p-w+1}}{an.r''+w} \end{cases}$$

$$\times \overline{1 + \frac{s''+1.x^n}{r'''+1.a} + \frac{s''+1.s''+2.x^{2n}}{r'''+1.r'''+2.a^2}\ \&c.}$$

$$= \begin{cases} K + \dfrac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \dfrac{s+1.x^n}{r+1.a} + \dfrac{s+1.s+2.x^{2n}}{r+1.r+2.a^2}\,(v)} \\ + \dfrac{s+1.s+2\,(v)}{r.r+1\,(v) \times a^v} \times \dfrac{x^{r''n} z^p}{nr''} \times \overline{1 + \dfrac{px^n}{r''+1.z} + \dfrac{p.p-1.x^{2n}}{r''+1.r''+2.z^2}\,(w)} \\ + \dfrac{s+1.s+2.s+3\,(v)}{r.r+1.r+2\,(v) \times a^v} \times \dfrac{p.p-1.p-2\,(w)}{r''.r''+1.r''+2\,(w)} \times \dfrac{2x^{2s''n}}{n.\overline{2s''+1}.z^{s''}} \end{cases}$$

$$\times \overline{y + \frac{y^3}{2s''+3} + \frac{3y^5}{2s''+3.2s''+5} + \frac{3.5y^7}{2s''+3.2s''+5.2s''+7}\ \&c.}$$

$r'' = r+v$. $r''' = r''+w$. $s = p+r$. $s'' = p+r''$. $y = \dfrac{x^n}{2a-x^n}$. $z = a-x^n$.

v and w any positive integers;

so that $w - v$, in the second value of F, be $= 2p + r + 1$.

THEO-

THEOREM IV.

The *whole* fluent of $\overline{a - x^n}|^{p} \times x^{rn-1}\dot{x}$,

generated whilst x from o becomes equal to $a^{\frac{1}{n}}$,

$$\text{is} = \frac{1.2.3\,(r)}{p+1.p+2.p+3\,(r)} \times \frac{a^{p+r}}{nr} = \frac{1.2.3\,(p)}{r+1.r+2.r+3\,(p)} \times \frac{a^{p+r}}{nr}$$

$$= \left\{ \begin{array}{l} \dfrac{1.2.3\,(r+v+y) \times t.t+1.t+2\,(w)}{p+1.p+2.p+3\,(r+w+y) \times r+1.r+2.r+3\,(v)} \times \dfrac{a^{p+r}}{nr} \\ \overline{\times\, 1 + \dfrac{p-v+w \times y}{r+v+1} + \dfrac{p-v+w.p-v+w-1 \times y.y-1}{1.2 \times r+v+1.r+v+2}\ \&c.} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \dfrac{1.2.3\,(p+v+y) \times t.t+1.t+2\,(w)}{p+1.p+2.p+3\,(v) \times r+1.r+2.r+3\,(p+w+y)} \times \dfrac{a^{p+r}}{nr} \\ \overline{\times\, 1 + \dfrac{r-v+w \times y}{p+v+1} + \dfrac{r-v+w.r-v+w-1 \times y.y-1}{1.2 \times p+v+1.p+v+2}\ \&c.} \end{array} \right.$$

$$= \text{the } \mathit{Limit} \text{ of } \frac{1.2.3\,(z) \times t.t+1.t+2\,(z)}{p+1.p+2.p+3\,(z) \times r+1.r+2.r+3\,(z)} \times \frac{a^{p+r}}{nr}$$

z increasing ad infinitum.

a, n, $p+1$, and r being positive. $t = p+r+1$.

$p+y$, or $r+y$, any positive integer.

v and w any positive integers.

THEO-

THEOREM V.

$$\dot{F} = \overline{x^n - a}|^{p} \times x^{rn-1}\dot{x}.$$

$$F = \begin{cases} K + \dfrac{x^{rn} z^{p+a}}{an.p+1} \times \overline{1 - \dfrac{s+1.z}{p+2.a} + \dfrac{s+1.s+2.z^2}{p+2.p+3.a^2} - (v)} \\ \pm{}^{*} \dfrac{s+1.s+2\,(v)}{p+1.p+2(v)\times a^{v}} \times \dfrac{x^{rn-n} z^{p''+1}}{n.p''+1} \times \overline{1 - \dfrac{r-1.z}{p''+2.x^{n}} + \dfrac{r-1.r-2.z^2}{p''+2.p''+3.x^{2n}} - \&c.} \end{cases}$$

$$= \begin{cases} K + \dfrac{x^{rn} z^{p+1}}{an.p+1} \times \overline{1 - \dfrac{s+1.z}{p+2.a} + \dfrac{s+1.s+2.z^2}{p+2.p+3\,a^2} - (v)} \\ \pm{}^{*} \dfrac{s+1.s+2\,(v)}{p+1.p+2(v)\times a^{v}} \times \dfrac{x^{rn-n} z^{p''+1}}{n.p'+1} \times \overline{1 - \dfrac{r-1.z}{p''+2.x^{n}} + \dfrac{r-1.r-2.z^2}{p''+2.p''+3.x^{2n}} - (w)} \\ \pm\dagger \dfrac{s+1.s+2.s+3\,(v)}{p+1.p+2\,p+3\,(v)\times a^{v}} \times \dfrac{r-1.r-2.r-3\,(w)}{p''+1.p''+2.p''+3\,(w)} \times \dfrac{2z^{2s''}}{n.2s''+1.x^{s''n}} \end{cases}$$

$$\times \overline{y + \frac{y^3}{2s''+3} + \frac{3y^5}{2s''+3.2s''+5} + \frac{3.5y^7}{2s''+3.2s''+5.2s''+7}\ \&c.}$$

$p'' = p + v.$ $s = p + r.$ $s'' = p'' + r.$ $y = \dfrac{x^n - a}{x^n + a}.$ $z = x^n - a.$

v and w any poſitive integers; ſo that $w - v$ be $= p + 2r.$

* $+$ or $-$ according as v is even or odd.

† $+$ or $-$ according as $v + w$ is even or odd.

THEO-

THEOREM VI.

The *whole* fluent of $\overline{x^n - a}|^p \times x^{-rn-1}\dot{x}$,

generated whilst x from $a^{\frac{1}{n}}$ becomes infinite,

$$\text{is} = \frac{1.2.3\,(t)}{p+1.p+2.p+3\,(t)} \times \frac{a^{p-r}}{nr} = \frac{1.2.3\,(p)}{t+1.t+2.t+3\,(p)} \times \frac{a^{p-r}}{nr}$$

$$= \begin{cases} \dfrac{1.2.3\,(t+v+y) \times r.r+1.r+2\,(w)}{p+1.p+2.p+3\,(t+w+y) \times t+1.t+2.t+3\,(v)} \times \dfrac{a^{p-r}}{nr} \\ \times\, \overline{1 + \dfrac{p-v+w \times y}{t+v+1} + \dfrac{p-v+w.p-v+w-1 \times y.y-1}{1.2 \times t+v+1.t+v+2}\ \&c.} \end{cases}$$

$$= \begin{cases} \dfrac{1.2.3\,(p+v+y) \times r.r+1.r+2\,(w)}{p+1.p+2.p+3\,(v) \times t+1.t+2.t+3\,(p+w+y)} \times \dfrac{a^{p-r}}{nr} \\ \times\, \overline{1 + \dfrac{t-v+w \times y}{p+v+1} + \dfrac{t-v+w.t-v+w-1 \times y.y-1}{1.2 \times p+v+1.p+v+2}\ \&c.} \end{cases}$$

$$= \text{the } \textit{Limit} \text{ of } \frac{1.2.3\,(z) \times r.r+1.r+2\,(z)}{p+1.p+2.p+3\,(z) \times t+1.t+2.t+3\,(z)} \times \frac{a^{p-r}}{nr}$$

z increasing ad infinitum.

a, n, $p+1$, and $r-p$ being positive. $t = r-p-1$.

$p+y$, or $t+y$, any positive integer.

v and w any positive integers.

THEO-

THEOREM VII.

$$\dot{F} = \overline{a + x^n}|^{p} \times x^{rn-1}\dot{x}.$$

$$F = \left\{ \begin{array}{l} K + \dfrac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \dfrac{t-1 . x^n}{r+1 . a} + \dfrac{t-1 . t-2 . x^{2n}}{r+1 . r+2 . a^2}\ (v)}. \\ + \dfrac{t-1 . t-2\ (v)}{r . r+1\ (v) \times a^v} \times \dfrac{x^{rn+vn} z^p}{n . r+v} \times \overline{1 - \dfrac{px^n}{r''+1 . z} + \dfrac{p . p-1 . x^{2n}}{r''+1 . r''+2 . z^2} - \&c.} \end{array} \right.$$

$$= \left\{ \begin{array}{l} K + \dfrac{x^{rn} z^{p+1}}{anr} \times \overline{1 + \dfrac{t-1 . x^n}{r+1 . a} + \dfrac{t-1 . t-2 . x^{2n}}{r+1 . r+2 . a^2}\ (v)} \\ + \dfrac{t-1 . t-2\ (v)}{r . r+1\ (v) \times a^v} \times \dfrac{x^{r''n} z^p}{nr''} \times \overline{1 - \dfrac{px^n}{r''+1 . z} + \dfrac{p . p-1 . x^{2n}}{r''+1 . r''+2 . z^2} - (w)} \\ \pm^{*} \dfrac{t-1 . t-2 . t-3\ (v)}{r . r+1 . r+2\ (v) \times a^v} \times \dfrac{p . p-1 . p-2\ (w)}{r'' . r''+1 . r''+2\ (w)} \times \dfrac{2x^{2sn}}{n . 2s+1 . z^s} \end{array} \right.$$

$$\times \overline{y + \frac{y^3}{2s+3} + \frac{3y^5}{2s+3 . 2s+5} + \frac{3 . 5y^7}{2s+3 . 2s+5 . 2s+7}\ \&c.}$$

$r'' = r + v$. $s = p + r''$. $t = -p - r$. $y = \dfrac{x^n}{2a + x^n}$. $z = a + x^n$.

v and w any positive integers; so that $w - v$ be $= 2p + r + 1$.

* + or — according as w is even or odd.

THEO-

THEOREM VIII.

The *whole* fluent of $\overline{a + x^n}\big|^{-p} \times x^{rn-1}\dot{x}$,

generated whilst x from o becomes infinite,

$$\text{is} = \frac{1.2.3\,(t)}{r + 1.r + 2.r + 3\,(t)} \times \frac{a^{r-p}}{nr} = \frac{1.2.3\,(r)}{t + 1.t + 2.t + 3\,(r)} \times \frac{a^{r-p}}{nr}$$

$$= \left\{\begin{array}{l} \dfrac{1.2.3\,(t + v + y) \times p.p + 1.p + 2\,(w)}{r + 1.r + 2.r + 3\,(t + w + y) \times t + 1.t + 2.t + 3\,(v)} \times \dfrac{a^{r-p}}{nr} \\ \times\ \overline{1 + \dfrac{r - v + w \times y}{t + v + 1} + \dfrac{r - v + w.r - v + w - 1 \times y.y - 1}{1.2 \times t + v + 1.t + v + 2}\ \&c.} \end{array}\right.$$

$$= \left\{\begin{array}{l} \dfrac{1.2.3\,(r + v + y) \times p.p + 1.p + 2\,(w)}{r + 1.r + 2.r + 3\,(v) \times t + 1.t + 2.t + 3\,(r + w + y)} \times \dfrac{a^{r-p}}{nr} \\ \times\ \overline{1 + \dfrac{t - v + w \times y}{r + v + 1} + \dfrac{t - v + w.t - v + w - 1 \times y.y - 1}{1.2 \times r + v + 1.r + v + 2}\ \&c.} \end{array}\right.$$

$$= \text{the } \textit{Limit} \text{ of } \frac{1.2.3\,(z) \times p.p + 1.p + 2\,(z)}{r + 1.r + 2.r + 3\,(z) \times t + 1.t + 2.t + 3\,(z)} \times \frac{a^{r-p}}{nr}$$

z increasing ad infinitum.

a, n, r, and $p - r$ being positive. $t = p - r - 1$.

$r + y$, or $t + y$, any positive integer.

v and w any positive integers.

THEOREM IX.

The *Limit* of $1 - m^2 . 2^2 - m^2 . 3^2 - m^2 (z) \times \frac{N^{2z}}{z^{2z+1}}$ is $= \frac{2}{m}$ sine of $2md$.

z increasing ad infinitum.

THEOREM X.

The *Limit* of $1.2.3\ (z) \times \frac{N^z}{z^{z+\frac{1}{2}}}$ is $= 2d^{\frac{1}{2}}$.

THEOREM XI.

The *Limit* of $bz+a.bz+a+b.bz+a+2b\,(z) \times \frac{N^z}{2^{2z}b^z z^z}$ is $= 2^{\frac{a}{b}-\frac{1}{2}}$.

THEOREM XII.

The *Limit* of $1.3.5\ (z) \times \frac{N^z}{2^z z^z} = z+1.z+2.z+3\ (z) \times \frac{N^z}{2^{2z} z^z}$ is $= 2^{\frac{1}{2}}$.

THEOREM XIII.

The *Limit* of $1.5.9\ (z) \times \frac{N^z}{2^{2z} z^{z-\frac{1}{4}}}$ is $= \frac{\sqrt{e - \sqrt{e^2 - 2d}}}{2^{\frac{1}{2}} d^{\frac{1}{4}}}$.

THEOREM XIV.

The *Limit* of $3.7.11\ (z) \times \frac{N^z}{2^{2z} z^{z+\frac{1}{4}}}$ is $= \frac{2^{\frac{1}{2}} \sqrt{e + \sqrt{e^2 - 2d}}}{d^{\frac{1}{4}}}$.

THEO-

THEOREM XV.

The *Limit* of 1.4.7 (z) $\times \frac{N^z}{3^z z^{z-\frac{1}{6}}}$ is $= \frac{3^{\frac{1}{12}}Q^{\frac{1}{2}}}{2^{\frac{1}{9}}d^{\frac{1}{6}}}$.

THEOREM XVI.

The *Limit* of 2.5.8 (z) $\times \frac{N^z}{3^z z^{z+\frac{1}{6}}}$ is $= \frac{2^{\frac{1}{9}}3^{\frac{5}{12}}d^{\frac{1}{6}}}{Q^{\frac{1}{2}}}$.

THEOREM XVII.

The *Limit* of 1.7.13 (z) $\times \frac{N^z}{6^z z^{z-\frac{1}{3}}}$ is $= \frac{Q^{\frac{2}{3}}}{2^{\frac{1}{18}}3^{\frac{1}{2}}d^{\frac{1}{3}}}$.

THEOREM XVIII.

The *Limit* of 5.11.17 (z) $\times \frac{N^z}{6^z z^{z+\frac{1}{3}}}$ is $= \frac{2^{\frac{7}{18}}3^{\frac{1}{4}}d^{\frac{1}{3}}}{Q^{\frac{2}{3}}}$.

N = 2.718281 = the number whoſe hyp. log. is 1.

$Q = .982889 = f - \sqrt{f^2 - \overline{3^{\frac{1}{2}}+1}.d}$.

d = 1.570796 = the quadrantal arc of a circ. whoſe rad. is 1.

e = 1.910098 = the quadrantal arc of an ellipſis whoſe ſemi-axes are $2^{\frac{1}{2}}$ and 1.

f = 2.674547 = the quadrantal arc of another ellipſis whoſe ſemi-axes are $3^{\frac{1}{2}}$ and $\frac{3^{\frac{1}{4}}.3^{\frac{1}{2}}+1}{2^{\frac{1}{2}}}$.

TABLE IX.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

The *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\overline{a+x^n}|^p}$,

generated whilst x from o becomes infinite,

$$\text{is} = \frac{p-2r}{na^{p-r}} \times \frac{\text{sine of } p \times 90^\circ}{\text{sine of } \overline{p-2r} \times 90^\circ} \times \overline{\frac{1}{r.\overline{p-r}} + \frac{p}{\overline{r+1}.\overline{p-r+1}} + \frac{p''}{\overline{r+2}.\overline{p-r+2}} \text{ \&c.}}$$

a, n, r, and $p-r$ being positive.

$$p'' = \frac{p.\overline{p+1}}{1.2},\ p''' = \frac{p.\overline{p+1}.\overline{p+2}}{1.2.3},\ p^{\text{iv}} = \frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}}{1.2.3.4}, \text{ \&c.}$$

THEOREM II.

The *whole* fluent of $\frac{x^{\frac{1}{2}pn-1}\dot{x}}{\sqrt{1-x^n}}$,

generated whilst x from o becomes equal to 1,

$$\text{is} = \frac{2^{p+1} \times \text{sine of } pd}{nd} \times \overline{\frac{1}{p^2} + \frac{p}{\overline{p+2}|^2} + \frac{p.\overline{p+1}}{\overline{2.p+4}|^2} + \frac{p.\overline{p+1}.\overline{p+2}}{\overline{2.3.p+6}|^2} \text{ \&c.}}$$

p, and $2-p$ being positive.

d = the quadrantal arc of a circle whose radius is 1.

THEO-

THEOREM III.

$$F = {}^{*}\pm 2^{p} \times \frac{q-p.q-p-2.q-p-4\,(q)}{p.p+1.p+2\,(q)} \times \text{cosine of } \overline{p+q}.90^{\circ} \times G,$$

$p+q$, and $1-p$ being positive. Radius $=1$.

$$\left.\begin{matrix} F = \\ \\ G = \end{matrix}\right\}\text{the } \textit{whole} \text{ fluent of } \left\{\begin{matrix} \dfrac{x^{\frac{1}{2}pn+\frac{1}{2}qn-1}\dot{x}}{\sqrt{1-x^{n}}}, \\ \dfrac{x^{\frac{1}{2}pn+\frac{1}{2}qn-1}\dot{x}}{\overline{1-x^{n}}|^{p}}, \end{matrix}\right.$$

generated whilst x from 0 becomes equal to 1.

* + or − according as q is even or odd.

THEOREM IV.

$$F = {}^{*}\pm \frac{q-p.q-p-2.q-p-4\,(q)}{p.p+1.p+2\,(q)} \times$$

$$\overline{2^{p}H - \frac{2}{n.q-p} \times \overline{1 - \frac{p}{q-p-2} + \frac{p.p+1}{q-p-2.q-p-4} - (q)}};$$

q being $=0$, or any positive integer;

$p+q$ any positive integer, or fraction.

$$F = \text{the } \textit{whole} \text{ fluent of} \ldots\ldots \frac{x^{\frac{1}{2}pn+\frac{1}{2}qn-1}\dot{x}}{\sqrt{1-x^{n}}}.$$

$$H = \text{the contemporary fluent of } \frac{x^{\frac{1}{2}pn+\frac{1}{2}qn-1}\dot{x}}{\overline{1+x^{n}}|^{p}}.$$

* + or − according as q is even or odd.

THEO-

THEOREM V.

The *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\sqrt{1-x^{n}}}$,

generated whilst x from o becomes equal to 1,

$$\text{is} = \frac{2}{n} \times \overline{\frac{1}{r} + \frac{1}{2.r+1} + \frac{1.3}{2.4.r+2} + \frac{1.3.5}{2.4.6.r+3}}\ (r);$$

the sum of r terms of the series being exactly equal to *half* the sum of the whole series continued ad infinitum, r being any positive number whatever!

THEOREM VI.

The *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\overline{1-x^{n}}|^{p}}$,

generated whilst x from o becomes equal to 1;

the *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\overline{x^{n}-1}|^{p}}$,

generated whilst x from 1 becomes infinite;

and the *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\overline{1+x^{n}}|^{p}}$,

generated whilst x from o becomes infinite;

are to each other

as the sines of $\overline{p-r} \times 180^{\circ}$, $r \times 180^{\circ}$, and $p \times 180^{\circ}$

respectively;

$1-p$, $p-r$, and r being positive.

TABLE

TABLE X.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

The *whole* fluent of $\overline{a-x^n}|^p \times x^{rn-1}\dot{x}$, generated whilst x from o becomes equal to $a^{\frac{1}{n}}$, is to the fluent of the same fluxion, generated whilst x from o becomes equal to any quantity k, as

$$a^q \text{ to } k^{qn} + qk^{qn-n}.\overline{a-k^n} + \frac{q.\overline{q-1}}{1.2}k^{qn-2n}.\overline{a-k^n}|^2 \ (p+1):$$

a, n, $p+1$, and r being positive:

$$q = p + r.$$

When r is $= p+1$, the *whole* fluent is to the part generated whilst x from o becomes equal to $\overline{\frac{a}{2}}|^{\frac{1}{n}}$ as 2 to 1, whether $p+1$ be an integer or not.

THEO-

THEOREM II.

The *whole* fluent of $\overline{x^n - a}|^p \times x^{-rn-1}\dot{x}$, generated whilst x from $a^{\frac{1}{n}}$ becomes infinite, is to the fluent of the same fluxion, generated whilst x from $a^{\frac{1}{n}}$ becomes equal to any quantity k, as

$$k^{rn} \text{ to } \overline{k^n - a}|^r + ra.\overline{k^n - a}|^{r-1} + \frac{r.r-1}{1.2}a^2.\overline{k^n - a}|^{r-2} \ (r-p);$$

a, n, $p+1$, and $r-p$ being positive.

When r is $= 2p+1$, the *whole* fluent is to the part generated whilst x from $a^{\frac{1}{n}}$ becomes equal to $\overline{2a}|^{\frac{1}{n}}$ as 2 to 1, whether $r-p$ be an integer or not.

THEOREM III.

The *whole* fluent of $\frac{x^{rn-1}\dot{x}}{\overline{a+x^n}|^{p+1}}$, generated whilst x from 0 becomes infinite, is to the fluent of the same fluxion, generated whilst x from 0 becomes equal to any quantity k, as

$$\overline{a+k^n}|^p \text{ to } k^{pn} + pak^{pn-n} + \frac{p.p-1}{1.2}a^2k^{pn-2n} \ (p-r+1);$$

a, n, $p-r+1$, and r being positive.

When $2r$ is $= p+1$, the *whole* fluent is to the part generated whilst x from 0 becomes equal to $a^{\frac{1}{n}}$ as 2 to 1, whether $p-r+1$ be an integer or not.

TABLE

TABLE XI.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$\dot{F} = \dfrac{\dot{x}}{\overline{a+bx} \times \overline{d+ex}}$. $\dfrac{d}{e}$ greater or leſs than $\dfrac{a}{b}$.

$F = \dfrac{1}{bd-ae} \times \text{Log.}\ \dfrac{d+ec}{a+bc} \times \dfrac{a+bx}{d+ex}$.

c the value of x when $F = 0$.

THEOREM II.

$\dot{F} = \dfrac{x\dot{x}}{\overline{a+bx} \times \overline{d+ex}}$. $\dfrac{d}{e}$ greater or leſs than $\dfrac{a}{b}$.

$F = \dfrac{1}{bd-ae} \times \overline{\dfrac{d}{e}\ \text{Log.}\ \dfrac{d+ex}{d+ec} - \dfrac{a}{b}\ \text{Log.}\ \dfrac{a+bx}{a+bc}}$.

c the value of x when $F = 0$.

THEOREM III.

$\dot{F} = \dfrac{x^m\dot{x}}{\overline{a+bx} \times \overline{d+ex}}$. $\dfrac{d}{e}$ greater or leſs than $\dfrac{a}{b}$.

$F = \dfrac{1}{bd-ae} \times \overline{\text{fl.}\ \dfrac{x^m\dot{x}}{x+\frac{a}{b}} - \text{fl.}\ \dfrac{x^m\dot{x}}{x+\frac{d}{e}}}$.

THEOREM IV.

$\dot{F} = \frac{\dot{x}}{a^2 - 2kax - x^2}$. k any value positive or negative.

$F = \frac{1}{2b} \times \text{Log.} \frac{b-c}{b+c} \times \frac{b+y}{b-y}$.

$y = x + ka$, $b = a\sqrt{k^2+1}$. c the value of y when $F = 0$.

THEOREM V.

$\dot{F} = \frac{x\dot{x}}{a^2 - 2kax - x^2}$. k any value positive or negative.

$F = \frac{1}{2b} \times \overline{\overline{b+ka}.\text{Log.} \frac{b+c}{b+y} + \overline{b-ka}.\text{Log.} \frac{b-c}{b-y}}$.

y, b, and c as in the preceding theorem.

THEOREM VI.

$\dot{F} = \frac{x^m\dot{x}}{a^2 - 2kax - x^2}$. k any value positive or negative.

$F = \frac{1}{2b} \times \overline{\text{fl.} \frac{x^m\dot{x}}{x+p} - \text{fl.} \frac{x^m\dot{x}}{x+q}}$.

$b = a\sqrt{k^2+1}$. $p = ka + b$. $q = ka - b$.

THEO-

THEOREM VII.

$\dot{F} = \frac{\dot{x}}{a^2 - 2kax + x^2}$. k greater than 1, or less than -1.

$F = \frac{1}{2b} \times \text{Log.} \frac{b+c}{b-c} \times \frac{b-y}{b+y}$.

$y = x - ka$. $b = a\sqrt{k^2 - 1}$. c the value of y when $F = 0$.

THEOREM VIII.

$\dot{F} = \frac{x\dot{x}}{a^2 - 2kax + x^2}$. k greater than 1, or less than -1.

$F = \frac{1}{2b} \times \overline{\overline{b+ka}.\text{Log.}\frac{b-y}{b-c} + \overline{b-ka}.\text{Log.}\frac{b+y}{b+c}}$.

y, b, and c as in the preceding theorem.

THEOREM IX.

$\dot{F} = \frac{x^m \dot{x}}{a^2 - 2kax + x^2}$. k greater than 1, or less than -1.

$F = \frac{1}{2b} \times \overline{\text{fl.} \frac{x^m \dot{x}}{x-p} - \text{fl.} \frac{x^m \dot{x}}{x-q}}$.

$b = a\sqrt{k^2 - 1}$. $p = ka + b$. $q = ka - b$.

THEOREM X.

$\dot{F} = \frac{\dot{x}}{a^2 - 2kax + x^2}$. k lefs than 1, but greater than -1.

$F = K + \frac{1}{2ab} \times$ circ. arc whofe cofine is $\frac{2ax - k.\overline{a^2 + x^2}}{a^2 + x^2 - 2kax}$

$= K + \frac{1}{ab} \times$ circ. arc whofe tangent is $\frac{y}{b}$.

$b =$ the tangent of *half* the arc whofe fine is h and cofine k.

Radius $= 1$. $h = \sqrt{1 - k^2}$. $y = \frac{x - a}{x + a}$.

Note. The value of the fluent F, generated whilft x from o becomes equal to g, is equal to

$\frac{1}{2ab} \times$ the arc, fine k; $\frac{1}{2ab} \times \overline{Q + \text{the arc, fine } k}$;

or $\frac{1}{2ab} \times \overline{2Q + \text{the arc, fine } k}$;

according as g is equal to $\frac{1 - b}{k} \times a$, a, or $\frac{1 + b}{k} \times a$:

and the *whole* fluent F, generated whilft x from o becomes infinite, is $= \frac{1}{ab} \times \overline{Q + \text{the arc, fine } k}$:

Q being the quadrantal arc.

THEOREM XI.

$\dot{F} = \frac{x\dot{x}}{a^2 - 2kax + x^2}$. k lefs than 1, but greater than -1.

$F = \frac{1}{2}\,\text{Log.}\, \frac{a^2 - 2kax + x^2}{a^2 - 2kad + d^2} + ka \times \text{fl.}\, \frac{\dot{x}}{a^2 - 2kax + x^2}$.

d the value of x when $F = 0$.

THEO-

THEOREM XII.

$\dot{F} = \frac{x^{m+1}\dot{x}}{a^2 - 2kax + x^2}$. k less than 1, but greater than -1.

m any positive integer.

$$F = \frac{1}{h'} \text{ into } \begin{cases} a^m \overset{(m+1)}{h} \times \text{fl.} \frac{x\dot{x}}{a^2 - 2kax + x^2} - a^{m+1} \overset{(m)}{h} \times \text{fl.} \frac{\dot{x}}{a^2 - 2kax + x^2} \\ + \frac{h'.\overline{x^m - d^m}}{m} + \frac{ah''.\overline{x^{m-1} - d^{m-1}}}{m-1} + \frac{a^2h'''.\overline{x^{m-2} - d^{m-2}}}{m-2} (m). \end{cases}$$

B the arc whose radius is 1 and cosine k.

h' = the sine of B. $\overset{(m)}{h}$ = the sine of mB.

h'' = the sine of 2B. $\overset{(m+1)}{h}$ = the sine of $\overline{m+1}$.B.

&c.

d the value of x when $F = 0$.

THEOREM XIII.

$\dot{F} = \frac{x^{-m}\dot{x}}{a^2 - 2kax + x^2}$. k less than 1, but greater than -1.

m any positive integer.

$$F = \frac{1}{h'} \text{ into } \begin{cases} \frac{\overset{(m)}{h}}{a^{m+1}} \text{Log.} \frac{x}{d} - \frac{\overset{(m)}{h}}{a^{m+1}} \times \text{fl.} \frac{x\dot{x}}{a^2 - 2kax + x^2} + \frac{\overset{(m+1)}{h}}{a^m} \times \text{fl.} \frac{\dot{x}}{a^2 - 2kax + x^2} \\ + \frac{h'.\overline{d^{1-m} - x^{1-m}}}{\overline{m-1}.a^2} + \frac{h''.\overline{d^{2-m} - x^{2-m}}}{\overline{m-2}.a^3} (m-1). \end{cases}$$

d, and h', h'', h''', &c. $\overset{(m)}{h}$, $\overset{(m+1)}{h}$ as in the preceding theorem.

THEO-

THEOREM XIV.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{P + 2kx^n + x^{2n}} = \frac{x^{m-1}\dot{x}}{\overline{R' + x^n} \times \overline{R'' + x^n}} \quad k^2 \text{ greater than P.}$$

$$F = \frac{1}{2\sqrt{k^2 - P}} \times \overline{\text{fl.}\, \frac{x^{m-1}\dot{x}}{R' + x^n} - \text{fl.}\, \frac{x^{m-1}\dot{x}}{R'' + x^n}}.$$

$$R' = k - \sqrt{k^2 - P}. \quad R'' = k + \sqrt{k^2 - P}.$$

THEOREM XV.

The *whole* fluent of $\frac{x^{m-1}\dot{x}}{P + 2kx^n + x^{2n}}$, generated whilst x from o becomes infinite, is $= \frac{R'^{\frac{m-n}{n}} - R''^{\frac{m-n}{n}}}{np\sqrt{k^2 - P}} \times Q.$

k and P being both positive; and k^2 greater than P.

m any positive integer or fraction less than the integer or fraction n.

Q = the quadrantal arc of the circle whose radius is 1.

p = the sine of the arc $\frac{2m}{n}Q$.

NOTE. When k^2 is = P, R′ and R″ being each = k, the expression for the value of the *whole* fluent becomes equal to $\frac{2.\overline{n-m}.k^{\frac{m}{n}-2}}{n^2 p} \times Q.$

THEO-

THEOREM XVI.

$$\dot{F} = \frac{x^{n+m-1}\dot{x}}{P + 2kx^n + x^{2n}} = \frac{x^{n+m-1}\dot{x}}{\overline{R' + x^n} \times \overline{R'' + x^n}}. \quad k^2 \text{ greater than } P.$$

$$F = \frac{1}{2\sqrt{k^2 - P}} \times \overline{\text{fl.}\, \frac{R' x^{m-1}\dot{x}}{R' + x^n} - \text{fl.}\, \frac{R'' x^{m-1}\dot{x}}{R'' + x^n}}.$$

R′ and R″ as in theorem XIV.

THEOREM XVII.

The *whole* fluent of $\frac{x^{n+m-1}\dot{x}}{P + 2kx^n + x^{2n}}$, generated whilst x from o becomes infinite, is $= \frac{R''^{\frac{m}{n}} - R'^{\frac{m}{n}}}{np\sqrt{k^2 - P}} \times Q.$

k, P, m, n, p, and Q as in theorem XV.

NOTE. When k^2 is $= P$, R′ and R″ being each $= k$, the expreffion for the value of the *whole* fluent becomes equal to $\frac{2mk^{\frac{m}{n}-1}}{n^2 p} \times Q.$

THEO-

THEOREM XVIII.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}}.$$ *k* less than 1, but greater than -1.

$$F = \frac{a^{m-2n}}{bn} \times \begin{cases} \text{fl. } \dfrac{N'b\dot{z}}{b^2+z^2} + \text{fl. } \dfrac{N''c\dot{z}}{c^2+z^2} + \text{fl. } \dfrac{N'''d\dot{z}}{d^2+z^2} \text{ \&c.} \\ - \text{fl. } \dfrac{M'z\dot{z}}{b^2+z^2} - \text{fl. } \dfrac{M''z\dot{z}}{c^2+z^2} - \text{fl. } \dfrac{M'''z\dot{z}}{d^2+z^2} \text{ \&c.} \end{cases}$$

m any positive integer less than the *even* number $2n$.

$$z = \frac{x-a}{x+a}.$$

$$\left.\begin{matrix} M' \text{ and } N' \\ M'' \text{ and } N'' \\ M''' \text{ and } N''' \end{matrix}\right\} \text{ sine and cosine of } \begin{cases} \dfrac{n-m}{n} \times B. \\ \dfrac{n-m}{n} \times \overline{B + 360^\circ}. \\ \dfrac{n-m}{n} \times \overline{B + 2.360^\circ}. \end{cases}$$

&c. &c.

$$\left.\begin{matrix} b = \\ c = \\ d = \end{matrix}\right\} \text{ tangent of } \begin{cases} \dfrac{A}{n}. \\ \dfrac{A + 180^\circ}{n} \\ \dfrac{A + 2.180^\circ}{n}. \end{cases}$$

&c. to $\dfrac{A + \overline{n-1}.180^\circ}{n}$.

(n).

A = *half* the arc B whose sine is *h* and cosine *k*.

Radius = 1.

THEO-

THEOREM XIX.

The *whole* fluent of $\frac{x^{m-1}\dot{x}}{a^{2n}+2ka^{n}x^{n}+x^{2n}}$, generated whilst x from o becomes infinite, is $= \frac{2qa^{m-2n}}{nhp} \times Q$.

k less than 1, but greater than -1.

m, n, p, and Q as in theorem XV.

$q =$ the sine of $\frac{n-m}{n}$B; B being the arc whose sine is h and cosine k.

Radius $= 1$.

THEOREM XX.

The *whole* fluent of $\frac{x^{n+m-1}\dot{x}}{a^{2n}+2ka^{n}x^{n}+x^{2n}}$, generated whilst x from o becomes infinite, is $= \frac{2qa^{m-n}}{nhp} \times Q$.

h, k, m, n, p, B, Q, and radius as in the preceding theorem.

$q =$ the sine of $\frac{m}{n}$B.

NOTE. When m is $= 0$, the expression for the value of the *whole* fluent becomes equal to $\frac{B}{nha^{n}}$.

THEOREM XXI.

$\dot{F} = \frac{\overline{a+x}|^{2n-r} \times x^{m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}}$. k less than 1, but greater than -1.

$$F = \frac{2^{2n-r}}{nba^{r-n}} \times \begin{cases} \text{fl.}\, \frac{N'b\dot{z}}{b^2+z^2} + \text{fl.}\, \frac{N''c\dot{z}}{c^2+z^2} \text{ \&c.} \\ + \text{fl.}\, \frac{M'z\dot{z}}{b^2+z^2} + \text{fl.}\, \frac{M''z\dot{z}}{c^2+z^2} \text{ \&c.} \end{cases}$$

r any positive integer not greater than the even number $2n$.
m any positive integer less than r.

$$z = \frac{x-a}{x+a}.$$

$M' = P' \times \text{sine of } \frac{r-2m}{n}.A$ $\qquad$ $N' = P' \times \text{cos. of } \frac{r-2m}{n}.A.$

$M'' = P'' \times \text{sine of } \frac{r-2m}{n}.\overline{A+180^\circ}.$ $N'' = P'' \times \text{cos. of } \frac{r-2m}{n}.\overline{A+180^\circ}.$

$M''' = P''' \times \text{sine of } \frac{r-2m}{n}.\overline{A+2.180^\circ}.$ $N''' = P''' \times \text{cos. of } \frac{r-2m}{n}.\overline{A+2.180^\circ}.$

&c. &c. &c. &c.

$$\left.\begin{array}{l} b = \\ c = \\ d = \end{array}\right\} \text{tangent of} \left\{\begin{array}{l} \frac{A}{n}. \\ \frac{A+180^\circ}{n}. \\ \frac{A+2.180^\circ}{n}. \end{array}\right.$$

&c. to $\frac{A+\overline{n-1}.180^\circ}{n}$.

(n)

A = *half* the arc whose sine is h and cosine k.

Radius = 1.

$P' = \overline{1+b^2}|^{\frac{1}{2}r-n}$, $P'' = \overline{1+c^2}|^{\frac{1}{2}r-n}$, $P''' = \overline{1+d^2}|^{\frac{1}{2}r-n}$, &c.

THEO-

THEOREM XXII.

$$\dot{F} = \frac{\overline{a+x}|^{2n-r} \times x^{\frac{1}{2}r-1}\dot{x}}{a^{2n} - 2ka^{n}x^{n} + x^{2n}}.$$

$$F = \frac{2^{2n-r}}{nba^{\frac{1}{2}r}} \times \overline{\text{fl.}\ \frac{P'b\dot{z}}{b^2+z^2} + \text{fl.}\ \frac{P''c\dot{z}}{c^2+z^2}\ \&c.}$$

h, *k*, *n*, *z*, *b*, *c*, &c. P′, P″, &c. as in the preceding theorem.

r any even positive number not greater than 2*n*.

THEOREM XXIII.

$$\dot{F} = \frac{\overline{a+x}|^{2n-r} \times x^{r-1}\dot{x}}{a^{2n} - 2ka^{n}x^{n} + x^{2n}}.$$

$$F = \text{fl.}\ \frac{\dot{z}}{1-z} + \frac{2^{2n-r}}{nb} \times \begin{cases} \text{fl.}\ \dfrac{N'b\dot{z}}{b^2+z^2} + \text{fl.}\ \dfrac{N''c\dot{z}}{c^2+z^2}\ \&c. \\ + \text{fl.}\ \dfrac{M'z\dot{z}}{b^2+z^2} + \text{fl.}\ \dfrac{M''z\dot{z}}{c^2+z^2}\ \&c. \end{cases}$$

h, *k*, *n*, *r*, *z*, *b*, *c*, &c. A, P′, P″, &c. as in theorem XXI.

$M' = P' \times$ sine of $\frac{r}{n}.A$. $N' = P' \times$ cos. of $\frac{r}{n}.A$.

$M'' = P'' \times$ sine of $\frac{r}{n}.\overline{A + 180°}$. $N'' = P'' \times$ cos. of $\frac{r}{n}.\overline{A + 180°}$.

$M''' = P''' \times$ sine of $\frac{r}{n}.\overline{A + 2.180°}$. $N''' = P''' \times$ cos. of $\frac{r}{n}.\overline{A + 2.180°}$.

&c. &c. &c. &c

THEOREM XXIV.

$\dot{F} = \frac{x^{vn+m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}}$. k leſs than 1, but greater than -1.

$$F = \frac{1}{h'} \times \begin{cases} \text{fl. } \frac{a^{vn-n} \overset{(v)}{b} x^{n+m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}} - \text{fl. } \frac{a^{vn} \overset{(v-1)}{b} x^{m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}} \\ + \text{fl. } x^{m-1}\dot{x} \times \overline{h' x^{vn-2n} + a^n h'' x^{vn-3n} \ (v-1)}. \end{cases}$$

v any poſitive integer.

B the circ. arc whoſe rad. is 1 and coſine k.

h' = the ſine of B. $\overset{(v-1)}{h}$ = the ſine of $\overline{v-1}$.B.

h'' = the ſine of 2B. $\overset{(v)}{h}$ = the ſine of vB.

h''' = the ſine of 3B. $\overset{(v+1)}{h}$ = the ſine of $\overline{v+1}$.B.

&c. &c.

THEOREM XXV.

$\dot{F} = \frac{x^{-vn+m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}}$. k leſs than 1, but greater than -1.

$$F = \frac{1}{h'} \times \begin{cases} \text{fl. } \frac{a^{-vn} \overset{(v+1)}{b} x^{m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}} - \text{fl. } \frac{a^{-vn-n} \overset{(v)}{b} x^{n+m-1}\dot{x}}{a^{2n} - 2ka^n x^n + x^{2n}} \\ + \text{fl. } x^{m-1}\dot{x} \times \overline{\frac{h' x^{-vn}}{a^{2n}} + \frac{h'' x^{n-vn}}{a^{3n}} + \frac{h''' x^{2n-vn}}{a^{4n}} \ (v)}. \end{cases}$$

v, B, h', h'' &c. $\overset{(v)}{h}$, $\overset{(v+1)}{h}$, as in the preceding theorem.

THEO-

THEOREM XXVI.

$$\dot{F} = \frac{\dot{x}}{\sqrt{a^2 + 2bdx + d^2x^2}}.$$

$$F = K + \frac{1}{d} \times \text{Log.}\, \overline{b + dx + \sqrt{a^2 + 2bdx + d^2x^2}}.$$

THEOREM XXVII.

$$\dot{F} = \frac{\dot{x}}{\sqrt{a^2 + 2bdx - d^2x^2}}.$$

$$F = K + \frac{1}{d} \times \text{Circ. arc, rad. 1, sine } \frac{dx - b}{\sqrt{a^2 + b^2}}.$$

THEOREM XXVIII.

$$\dot{F} = \frac{x\dot{x}}{\sqrt{a^2 + 2bdx \pm d^2x^2}}.$$

$$F = \frac{\sqrt{a^2 + 2bdx \pm d^2x^2}}{\pm d^2} \mp \frac{b}{d} \times \text{fl.}\, \frac{\dot{x}}{\sqrt{a^2 + 2bdx \pm d^2x^2}}.$$

THEOREM XXIX.

$$\dot{F} = \frac{x^{mn-1}\dot{x}}{\sqrt{a^2 + 2bdx^n \pm d^2x^{2n}}}.$$

$$F = \frac{1}{n} \times \text{fl.}\, \frac{\overline{y \mp b}^{m-1}\dot{y}}{\sqrt{a^2 \mp b^2 \pm d^2y^2}} = -\frac{2}{n} \times \text{fl.}\, \frac{\overline{b + 2kv + cv^2}^{m-1}\dot{v}}{\overline{v^2 \mp d^2}^{m}}.$$

$$h = 2bd \pm cd^2. \quad k = \sqrt{a^2 + 2bcd^2 \pm c^2d^2}.$$

$c = 0$, or any quantity, so that the value of k be not imaginary.

$$v = \frac{k + \sqrt{a^2 + 2bdx^n \pm d^2x^{2n}}}{x^n - c}.$$

$$x^n = \frac{b + 2kv + cv^2}{v^2 \mp d^2} = y \mp \frac{b}{d}.$$

THEOREM XXX.

$$\dot{F} = \frac{y^{r-1}\dot{y}}{\overline{h+ky^r} \times \overline{p+qy^r}|^{\frac{m}{n}}}.$$

$$F = -\frac{na^{n-m}}{r.\overline{hq-kp}|^{\frac{m}{n}}} \times \text{fl.}\ \frac{x^{m-1}\dot{x}}{ka^n + x^n}.$$

$$x = \frac{a.\overline{hq-kp}|^{\frac{1}{n}}}{\overline{p+qy^r}|^{\frac{1}{n}}}. \quad y = \frac{\overline{\overline{hq-kp}.a^n - px^n}|^{\frac{1}{r}}}{q^{\frac{1}{r}}\, x^{\frac{n}{r}}}.$$

THEOREM XXXI.

The fluent of $\frac{y^{r-1}\dot{y}}{\overline{h+y^r} \times \overline{y^r-p}|^{\frac{m}{n}}}$, generated whilst y from being equal to $p^{\frac{1}{r}}$ becomes equal to $\overline{h+2p}|^{\frac{1}{r}}$, is

$$= \frac{1}{r\overline{h+p}|^{\frac{m}{n}}} \times \overline{\frac{Q}{s} - N'\,\text{Log.}\,\frac{1+b^2}{b^2} - N''\,\text{Log.}\,\frac{1+c^2}{c^2}\ \&c.}$$

m, n, N′, N″, &c. b, c, &c. Q, and s as in Theor. IV. Tab. V.
p and $h+p$ any positive quantities.

THEOREM XXXII.

The *whole* fluent of $\frac{y^{r-1}\dot{y}}{\overline{h+y^r} \times \overline{y^r-p}|^{\frac{m}{n}}}$, generated whilst y from being equal to $p^{\frac{1}{r}}$ becomes infinite, is $= \frac{2Q}{rs.\overline{h+p}|^{\frac{m}{n}}}$

m, n, Q and s as in Theorem IV. Tab. V
p and $h+p$ any positive quantities.

THEO-

THEOREM XXXIII.

$$\dot{F} = \frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{b+ky^r} \times \overline{p+qy^r}|^{\frac{m}{n}}},$$

$$F = \frac{na^{n-m}}{r.\overline{kp-bq}|^{\frac{m}{n}}} \times \text{fl.}\ \frac{x^{m-1}\dot{x}}{ha^n + x^n}.$$

$$x = \frac{\overline{a.kp-bq}|^{\frac{1}{n}}.y^{\frac{r}{n}}}{\overline{p+qy^r}|^{\frac{1}{n}}}.\quad y = \frac{p^{\frac{1}{r}}x^{\frac{n}{r}}}{\overline{\overline{kp-bq}.a^n - qx^n}|^{\frac{1}{r}}}.$$

THEOREM XXXIV.

The fluent of $\frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{1+ky^r} \times \overline{p-y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to $\frac{p^{\frac{1}{r}}}{\overline{kp+2}|^{\frac{1}{r}}}$, is

$$= \frac{1}{r.\overline{kp+1}|^{\frac{m}{n}}} \times \overline{\frac{Q}{s} + N' \,\text{Log.}\, \frac{1+b^2}{b^2} + N'' \,\text{Log.}\, \frac{1+c^2}{c^2}\ \text{\&c.}}$$

m, n, N′, N″, &c. b, c, &c. Q, and s as in Theor. IV. Tab. V.
p and $kp+1$ any positive quantities.

THEOREM XXXV.

The *whole* fluent of $\frac{y^{\frac{rm}{n}-1}\dot{y}}{\overline{1+ky^r} \times \overline{p-y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to $p^{\frac{1}{r}}$, is $= \frac{2Q}{rs.\overline{kp+1}|^{\frac{m}{n}}}$.

m, n, Q, and s as in Theorem IV. Tab. V.
p and $kp+1$ any positive quantities.

THEO-

THEOREM XXXVI.

$$\dot{F} = \frac{y^{r-1}\dot{y}}{\overline{h+2ky^r+ly^{2r}} \times \overline{p+qy^r}|^{\frac{m}{n}}}$$

$$F = -\frac{nqa^{n-m}}{rt^{\frac{m+n}{n}}} \times \text{fl.}\ \frac{x^{n+m-1}\dot{x}}{la^{2n} + 2.\frac{kq-lp}{t}.a^n x^n + x^{2n}}$$

$$t = \sqrt{hq^2 - 2kpq + lp^2}.$$

$$x = \frac{at^{\frac{1}{n}}}{\overline{p+qy^r}|^{\frac{1}{n}}} \qquad y = \frac{\overline{ta^n - px^n}|^{\frac{1}{r}}}{q^{\frac{1}{r}}x^{\frac{n}{r}}}.$$

THEOREM XXXVII.

The *whole* fluent of $\frac{y^{r-1}\dot{y}}{\overline{h+2ky^r+y^{2r}} \times \overline{y^r-p}|^{\frac{m}{n}}}$, generated whilst y from being equal to $p^{\frac{1}{r}}$ becomes infinite, is

$$= \frac{\overline{k+p+\sqrt{k^2-h}}|^{\frac{m}{n}} - \overline{k+p-\sqrt{k^2-h}}|^{\frac{m}{n}}}{rt^{\frac{2m}{n}}\sqrt{k^2-h} \times \text{the sine of } \frac{2m}{n}Q} \times Q.$$

k^2 greater than h. p and $k+p-\sqrt{k^2-h}$ any positive quantities.

m, n, and Q as in Theorem xv.

$$t = \sqrt{h+2kp+p^2}.$$

NOTE. When h is $= k^2$ the expression for the value of the *whole* fluent becomes $= \frac{2m}{nr.\overline{k+p}|^{\frac{m+n}{n}} \times \text{the sine of } \frac{2m}{n}Q} \times Q.$

THEO-

THEOREM XXXVIII.

The *whole* fluent of $\frac{y^{r-1}\dot{y}}{\overline{h+2ky^r+y^{2r}}\times\overline{y^r-p}|^{\frac{m}{n}}}$, generated whilst y from being equal to $p^{\frac{1}{r}}$ becomes infinite, is

$$= \frac{\text{sine of } \frac{m}{n}B}{rt^{\frac{m}{n}}\sqrt{h-k^2}\times\text{the sine of }\frac{m}{n}A} \times A.$$

k^2 less than h. p any positive quantity.
m any positive integer less than the integer n.

$$t = \sqrt{h+2kp+p^2}.$$

A = the semi-circumference of the circle whose radius is 1;
B = an arc of the same circle whose sine is $\frac{\sqrt{h-k^2}}{t}$.

NOTE. If m be $=0$, the fluent, generated whilst y from being equal to any quantity $p^{\frac{1}{r}}$ becomes infinite, will be $= \frac{B}{r\sqrt{h-k^2}}$.

THEOREM XXXIX.

The *whole* fluent of $\frac{y^{r-1}\dot{y}}{\overline{h-2py^r+y^{2r}}\times\overline{y^r-p}|^{\frac{m}{n}}}$ is

$$= \frac{Q}{r\,\overline{h-p^2}|^{\frac{m+n}{2n}}\times\text{the cosine of }\frac{m}{n}Q}$$

h greater than p^2. m, n, and Q as in Theorem XV.

NOTE. If m be $=0$, the fluent, generated whilst y from $p^{\frac{1}{r}}$ becomes infinite, will be $= \frac{Q}{r\sqrt{h-p^2}}$.

THEOREM XL.

$$\dot{F} = \frac{y^{r+\frac{rm}{n}-1}\dot{y}}{\overline{h+2ky^r+ly^{2r}}\times\overline{p+qy^r}|^{\frac{m}{n}}}.$$

$$F = \frac{npa^{n-m}}{rt^{\frac{m+n}{n}}} \times \text{fl.}\ \frac{x^{n+m-1}\dot{x}}{ha^{2n} + 2.\frac{kp-hq}{t}.a^n x^n + x^{2n}}.$$

$$t = \sqrt{hq^2 - 2kpq + lp^2}.$$

$$x = \frac{at^{\frac{1}{n}}y^{\frac{r}{n}}}{\overline{p+qy^r}|^{\frac{1}{n}}}. \quad y = \frac{p^{\frac{1}{r}}x^{\frac{n}{r}}}{\overline{ta^n - qx^n}|^{\frac{1}{r}}}.$$

THEOREM XLI.

The *whole* fluent of $\frac{y^{r+\frac{rm}{n}-1}\dot{y}}{\overline{h+2ky^r+y^{2r}}\times\overline{p-y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to $p^{\frac{1}{r}}$, is

$$= \frac{\overline{h+kp+p\sqrt{k^2-h}}|^{\frac{m}{n}} - \overline{h+kp-p\sqrt{k^2-h}}|^{\frac{m}{n}}}{rt^{\frac{2m}{n}}\sqrt{k^2-h}\times \text{the sine of } \frac{2m}{n}Q} \times Q.$$

h, p, and $h+kp$ any positive quantities, so that k^2 be greater than h. m, n, and Q as in Theorem XV.

$$t = \sqrt{h+2kp+p^2}.$$

NOTE. When h is $= k^2$ the expression for the value of the *whole* fluent becomes $= \frac{2mpk^{\frac{m-n}{n}}}{nr.\overline{k+p}|^{\frac{m+n}{n}} \times \text{the sine of } \frac{2m}{n}Q} \times Q.$

THEO-

THEOREM XLII.

The *whole* fluent of $\dfrac{y^{r+\frac{rm}{n}-1}\dot{y}}{\overline{h+2ky^r+y^{2r}}\times\overline{p-y^r}|^{\frac{m}{n}}}$, generated whilst y from o becomes equal to $p^{\frac{1}{r}}$, is

$$=\frac{h^{\frac{m}{2n}}\times\text{the fine of }\frac{m}{n}B}{rt^{\frac{m}{n}}\sqrt{h-k^2}\times\text{the fine of }\frac{m}{n}A}\times A.$$

h, p, and $h+kp$ any poſitive quantities, ſo that k^2 be leſs than h. m any poſitive integer leſs than the integer n.

$$t=\sqrt{h+2kp+p^2}.$$

A = the ſemi-circumference of the circle whoſe radius is 1;

B = an arc of the ſame circle whoſe ſine is $\dfrac{p\sqrt{h-k^2}}{h^{\frac{1}{2}}t}$.

Note. If m be = o, the fluent, generated whilſt y from o becomes equal to any quantity $p^{\frac{1}{r}}$, will be $=\dfrac{B}{r\sqrt{h-k^2}}$.

THEOREM XLIII.

The *whole* fluent of $\dfrac{y^{r+\frac{rm}{n}-1}\dot{y}}{\overline{h-\frac{2h}{p}y^r+y^{2r}}\times\overline{p-y^r}|^{\frac{m}{n}}}$ is

$$=\frac{ph^{\frac{m-n}{2n}}Q}{r\overline{p^2-h}|^{\frac{m+n}{2n}}\times\text{the coſine of }\frac{m}{n}Q}.$$

p^2 greater than h. m, n, and Q as in theorem xv.

Note. If m be = o, the fluent, generated whilſt y from o becomes equal to $p^{\frac{1}{r}}$, will be $=\dfrac{pQ}{rh^{\frac{1}{2}}\sqrt{p^2-h}}$.

THEOREM XLIV.

$$\dot{F} = y^{-1}\dot{y} \times \overline{\frac{b + ky^r}{p + qy^r}}\Big|^{\frac{m}{n}}.$$

$$F = \frac{\overline{n.bq - kp}}{pqr} \times \text{fl.}\ \frac{x^{n+m-1}\dot{x}}{\overline{x^n - \frac{b}{p}} \times \overline{x^n - \frac{k}{q}}}$$

$$x = \overline{\frac{b + ky^r}{p + qy^r}}\Big|^{\frac{1}{n}}. \quad y = \overline{\frac{px^n - b}{k - qx^n}}\Big|^{\frac{1}{r}}.$$

THEOREM XLV.

The *whole* fluent of $y^{-1}\dot{y} \times \overline{\frac{y^r - h}{p - y^r}}\Big|^{\frac{m}{n}}$, generated whilst y from being equal to $h^{\frac{1}{r}}$ becomes equal to $p^{\frac{1}{r}}$, is

$$= \frac{p^{\frac{m}{n}} - h^{\frac{m}{n}}}{rp^{\frac{m}{n}} \times \text{the sine of } \frac{m}{n}A} \times A.$$

h and p any positive quantities, so that h be less than p.

m any positive integer less than the integer n.

A as in theorem XLII.

THEO-

THEOREM XLVI.

$$\dot{F} = \frac{y^{-1}\dot{y}}{l+y^r} \times \overline{\frac{h+ky^r}{p+qy^r}}\Big|^{\frac{m}{n}}.$$

$$F = \frac{n.\overline{hq-kp}}{pr.\overline{lq-p}} \times \text{fl.} \frac{x^{s+n-1}\dot{x}}{\overline{x^n - \frac{h}{p}} \times \overline{x^n - \frac{h-kl}{p-lq}}}.$$

x and y as in Theorem XLIV.

THEOREM XLVII.

The *whole* fluent of $\frac{y^{-1}\dot{y}}{l+y^r} \times \overline{\frac{y^r-h}{p-y^r}}\Big|^{\frac{m}{n}}$, generated whilst y from being equal to $h^{\frac{1}{r}}$ becomes equal to $p^{\frac{1}{r}}$, is

$$= \frac{p.\overline{h+l}|^{\frac{m}{n}} - h.\overline{p+l}|^{\frac{m}{n}}}{lr.p.\overline{p+l}|^{\frac{m}{n}} \times \text{the fine of } \frac{m}{n}A} \times A.$$

A, m, n, h, and p as in Theorem XLV.

l positive or negative, so that it be greater than $-h$.

NOTE. If l be $= -h$, the *whole* fluent (of $\frac{y^{-1}\dot{y}}{\overline{y^r-h}|^{\frac{n-m}{n}} \times \overline{p-y^r}|^{\frac{m}{n}}}$) will be $= \frac{A}{rh^{\frac{n-m}{n}} p^{\frac{m}{n}} \times \text{the fine of } \frac{m}{n}A}$.

THEO-

THEOREM XLVIII.

$$\dot{F} = \frac{\overline{r \pm z}|^{m}\dot{z}}{\overline{1+z}|^{n} + \overline{1-z}|^{n}} = \frac{a}{2^{n-1}} \times \frac{\overline{x+a}|^{n-m-2} \times \overline{\overline{r \pm 1}.x + \overline{r \mp 1}.a}|^{m}\dot{x}}{a^{n}+x^{n}}.$$

$$F = \frac{1}{n} \times \left\{ \begin{array}{l} \text{fl.}\, \frac{M'b\dot{z}}{b^2+z^2} - \text{fl.}\, \frac{M''c\dot{z}}{c^2+z^2} + \text{fl.}\, \frac{M'''d\dot{z}}{d^2+z^2} - \text{\&c.} \\ \pm \text{fl.}\, \frac{N'z\dot{z}}{b^2+z^2} \mp \text{fl.}\, \frac{N''z\dot{z}}{c^2+z^2} \pm \text{fl.}\, \frac{N'''z\dot{z}}{d^2+z^2} \mp \text{\&c.} \end{array} \right.$$

$m = 0$, or any even positive number less than the integer $n-1$, or any odd positive number less than the integer n.

$$x = a \times \frac{1+z}{1-z}. \quad z = \frac{x-a}{x+a}.$$

$M' = P' \times$ cosine of mA'. $N' = P' \times$ sine of mA'.
$M'' = P'' \times$ cosine of mA''. $N'' = P'' \times$ sine of mA''.
$M''' = P''' \times$ cosine of mA'''. $N''' = P''' \times$ sine of mA'''.
&c. &c. &c. &c.

A', A'', A''', &c. are circular arcs, whose radius is 1 and tangents $\frac{b}{r}$, $\frac{c}{r}$, $\frac{d}{r}$, &c. respectively.

b, c, d, &c. as in Theorem 1, Table V.

$$P' = \frac{\overline{r^2+b^2}|^{\frac{1}{2}m}}{\overline{1+b^2}|^{\frac{1}{2}n-1}},\ P'' = \frac{\overline{r^2+c^2}|^{\frac{1}{2}m}}{\overline{1+c^2}|^{\frac{1}{2}n-1}},\ P''' = \frac{\overline{r^2+d^2}|^{\frac{1}{2}m}}{\overline{1+d^2}|^{\frac{1}{2}n-1}},\ \text{\&c.}$$

THEO-

THEOREM XLIX.

$$\dot{F} = \frac{\overline{r \pm z}|^{m} \dot{z}}{\overline{1+z}|^{n} - \overline{1-z}|^{n}} = -\frac{a}{2^{n-1}} \times \frac{\overline{x+a}|^{n-m-2} \times \overline{\overline{r \pm 1}.x + \overline{r \mp 1}.a}| \dot{x}}{a^{n} - x^{n}}.$$

$$F = \frac{1}{n} \times \begin{cases} \qquad \pm \text{fl.}\ \frac{M' c \dot{z}}{c^2 + z^2} \mp \text{fl.}\ \frac{M'' d \dot{z}}{d^2 + z^2} \pm \&c. \\ + \text{fl.}\ \frac{\frac{1}{2} r^{m+1} \dot{z}}{z} - \text{fl.}\ \frac{N' z \dot{z}}{c^2 + z^2} + \text{fl.}\ \frac{N'' z \dot{z}}{d^2 + z^2} - \&c. \end{cases}$$

$m = 0$, or any even positive number less than the integer n,

or any odd positive number less than the integer $n - 1$.

x and z as in the preceding theorem.

$M' = P' \times$ sine of $m A'$. $N' = P' \times$ cosine of $m A'$.

$M'' = P'' \times$ sine of $m A''$. $N'' = P'' \times$ cosine of $m A''$.

$M''' = P''' \times$ sine of $m A'''$. $N''' = P''' \times$ cosine of $m A'''$.

&c. &c. &c. &c.

A', A'', A''', &c. are circular arcs whose radius is 1 and tangents $\frac{c}{r}$, $\frac{d}{r}$, $\frac{e}{r}$, &c. respectively.

c, d, e, &c. as in Theorem XIII. Table V.

$$P' = \frac{\overline{r^2 + c^2}|^{\frac{1}{2}m}}{\overline{1 + c^2}|^{\frac{1}{2}n-1}},\quad P'' = \frac{\overline{r^2 + d^2}|^{\frac{1}{2}m}}{\overline{1 + d^2}|^{\frac{1}{2}n-1}},\quad P''' = \frac{\overline{r^2 + e^2}|^{\frac{1}{2}m}}{\overline{1 + e^2}|^{\frac{1}{2}n-1}},\ \&c.$$

THEO-

THEOREM L.

$$\dot{F} = \frac{\overline{r \pm z}|^{m}\dot{z}}{\overline{1+z}|^{2n} - 2k.\overline{1+z}|^{n}.\overline{1-z}|^{n} + \overline{1-z}|^{2n}}$$

$$= \frac{a}{2^{2n-m-1}} \times \frac{\overline{x+a}|^{2n-m-2} \times \overline{\overline{r \pm 1}.x + \overline{r \mp 1}.a}|^{m}\dot{x}}{a^{2n} - 2ka^{n}x^{n} + x^{2n}}$$

$$F = \frac{1}{2nb} \times \begin{cases} \text{fl.}\ \dfrac{M'b\dot{z}}{b^2+z^2} + \text{fl.}\ \dfrac{M''c\dot{z}}{c^2+z^2} + \text{fl.}\ \dfrac{M'''d\dot{z}}{d^2+z^2}\ \text{\&c.} \\ \pm\text{fl.}\ \dfrac{N'z\dot{z}}{b^2+z^2} \pm \text{fl.}\ \dfrac{N''z\dot{z}}{c^2+z^2} \pm \text{fl.}\ \dfrac{N'''z\dot{z}}{d^2+z^2}\ \text{\&c.} \end{cases}$$

k leſs than 1, but greater than -1.

$m = 0$, or any poſitive integer leſs than the even number $2n$.

x and z as in the preceding theorem.

$M' = P' \times$ coſine of mA'.	$N' = P' \times$ ſine of mA'.
$M'' = P'' \times$ coſine of mA''.	$N'' = P'' \times$ ſine of mA''.
$M''' = P''' \times$ coſine of mA'''.	$N''' = P''' \times$ ſine of mA'''.
&c. &c.	&c. &c.

A', A'', A''', &c. are circular arcs whoſe radius is 1 and tangents $\frac{b}{r}$, $\frac{c}{r}$, $\frac{d}{r}$, &c. reſpectively.

h, b, c, d, &c. as in theorem XVIII.

$$P' = \frac{\overline{r^2+b^2}|^{\frac{1}{2}m}}{\overline{1+b^2}|^{n-1}},\ P'' = \frac{\overline{r^2+c^2}|^{\frac{1}{2}m}}{\overline{1+c^2}|^{n-1}},\ P''' = \frac{\overline{r^2+d^2}|^{\frac{1}{2}m}}{\overline{1+d^2}|^{n-1}},\ \text{\&c.}$$

TABLE

TABLE XII.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{b^2 + 2fx - x^2}} = \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{\overline{a - x} \times \overline{\frac{b^2}{a} + x}}}.$$

$$\therefore F = K + \frac{4a^{\frac{1}{2}}}{b^2} \times \overline{\mathrm{de} - e'e''} = K + \frac{2a^{\frac{1}{2}}}{b^2} \times \overline{\mathrm{de} + \mathrm{DP} - \mathrm{AD} - \mathrm{L}}.$$

$$F = \frac{a^2 - b^2}{2a}.$$

THEOREM II.

The fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{b^2 + 2fx - x^2}}$, generated whilst x from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst x from $\frac{b^2}{a} \times \frac{a - k}{\frac{b^2}{a} + k}$ becomes equal to a.

NOTE. *All the theorems in this Table refer to the Scheme at the end of it, for the values of the quantities required.*

THEOREM III.

$$\dot{F} = \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{b^2 + 2fx - x^2}} = \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{\overline{a - x} \times \overline{\frac{b^2}{a} + x}}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{2e'e'' - de} = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{L + AD - DP}.$$

f as in the preceding theorems.

THEOREM IV.

The tangent eo $\left(= \overline{ak}\right|^{\frac{1}{2}} \times \left.\overline{\frac{a-k}{\frac{b^2}{a}+k}}\right|^{\frac{1}{2}}\Bigg)$ together with the fluent of $\frac{\frac{1}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{\sqrt{b^2 + 2fx - x^2}}$, generated whilst x from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst x from $\frac{b^2}{a} \times \frac{a-k}{\frac{b^2}{a}+k}$ becomes equal to a.

THEOREM V.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2fy - b^2}} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{y + a} \times \overline{y - \frac{b^2}{a}}}}.$$

$$F = K + \frac{4a^{\frac{1}{2}}}{b^2} \times \overline{ae + e'e'' - E''} = K + \frac{2a^{\frac{1}{2}}}{b^2} \times \overline{ae + AD - DP}.$$

$x = \frac{b^2}{y}$. f as in the preceding theorems.

THEOREM VI.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2fy - b^2}}$, generated whilst y from $\frac{b^2}{a}$ becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{b^2}{a} \times \frac{k+a}{k-\frac{b^2}{a}}$ becomes infinite.

THEO-

THEOREM VII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2fy - b^2}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{\overline{y+a} \times \overline{y - \frac{b^2}{a}}}}.$$

$F = K + \frac{2}{a^{\frac{1}{2}}} \times AD$. f and x as in the two preceding theorems.

THEOREM VIII.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2gy + b^2}} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{y+m} \times \overline{y + \frac{b^2}{m}}}}.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^2} \times \overline{ae + e'e'' - E''} = K + \frac{2m^{\frac{1}{2}}}{b^2} \times \overline{ae + AD - DP}.$$

$$a = \sqrt{m^2 - b^2}. \quad g = \frac{m^2 + b^2}{2m} \quad x = \frac{ab^2}{my + b^2}.$$

THEOREM IX.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2gy + b^2}}$, generated whilst y from o becomes equal to any quantity k, is equal to the fluent of the same fluxion generated whilst y from $\frac{b^2}{k}$ becomes infinite.

THEOREM X.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 + 2gy + b^2}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{\overline{y+m} \times \overline{y + \frac{b^2}{m}}}}.$$

$F = K + \frac{2}{m^{\frac{1}{2}}} \times \overline{DP - ae}$. a, g, and x as in theorem VIII.

THEOREM XI.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{2gy - y^2 - b^2}} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{m-y} \times \overline{y - \frac{b^2}{m}}}}.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^2} \times \overline{de - e'e''} = K + \frac{2m^{\frac{1}{2}}}{b^2} \times \overline{de + DP - AD - L}.$$

$$a = \sqrt{m^2 - b^2}. \quad g = \frac{m^2 + b^2}{2m}. \quad x = \frac{my - b^2}{a}.$$

THEOREM XII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{2gy - y^2 - b^2}}$, generated whilst y from $\frac{b^2}{m}$ becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{b^2}{k}$ becomes equal to m.

THEOREM XIII.

$$\dot{E} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{2gy - y^2 - b^2}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{\overline{m-y} \times \overline{y - \frac{b^2}{m}}}}.$$

$E = K + \frac{2}{m^{\frac{1}{2}}} \times de$. a, g, and x as in theorem XI.

THEOREM XIV.

The tangent eo $\left(= \overline{\frac{m}{k}}\Big|^{\frac{1}{2}} \times \sqrt{2gk - k^2 - b^2}\right)$ together with the fluent of $\frac{\frac{1}{2}m^{\frac{1}{2}}y^{\frac{1}{2}}\dot{y}}{\sqrt{2gy - y^2 - b^2}}$, generated whilst y from $\frac{b^2}{m}$ becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{b^2}{k}$ becomes equal to m.

THEO-

THEOREM XV.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{m-y} \times \overline{\frac{a^2}{m} - y}}} \quad y \text{ less than } \frac{a^2}{m}.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^2} \times \overline{ae + e'e'' - E''} = K + \frac{2m^{\frac{1}{2}}}{b^2} \times \overline{ae + AD - DP}.$$

$$b = \sqrt{m^2 - a^2}. \quad g = \frac{m^2 + a^2}{2m}. \quad x = \frac{a^2 - my}{a}.$$

THEOREM XVI.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}}$, generated whilst y from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{a^2 - mk}{m - k}$ becomes equal to $\frac{a^2}{m}$.

THEOREM XVII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{\overline{m-y} \times \overline{\frac{a^2}{m} - y}}} \quad y \text{ less than } \frac{a^2}{m}.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^2} \times \overline{g\,ae + m.\overline{e'e'' - E''}} = K + \frac{2m^{\frac{1}{2}}}{b^2} \times \overline{\frac{a^2}{m}ae + m.\overline{AD - DP}}.$$

b, g, and x as in the two preceding theorems.

THEOREM XVIII.

The fluent of $\frac{\frac{1}{2}m^{\frac{1}{2}}y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}}$, generated whilst y from o becomes equal to any quantity k, is equal to the tangent eo $\left(= \overline{mk}\right|^{\frac{1}{2}} \times \left.\overline{\frac{\frac{a^2}{m} - k}{m - k}}\right|^{\frac{1}{2}}\Big)$ together with the fluent of the same fluxion, generated whilst y from $\frac{a^2 - mk}{m - k}$ becomes equal to $\frac{a^2}{m}$.

THEO-

THEOREM XIX.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{\overline{y-m} \times \overline{y - \frac{a^2}{m}}}}. \quad y \text{ greater than } m.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^2} \times \overline{ae + e'e'' - E''} = K + \frac{2m^{\frac{1}{2}}}{b^2} \times \overline{ae + AD - DP}.$$

$$b = \sqrt{m^2 - a^2}. \quad g = \frac{m^2 + a^2}{2m}. \quad x = \frac{ab^2}{my - a^2}.$$

THEOREM XX.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2gy + a^2}}$, generated whilst y from m becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{mk - a^2}{k - m}$ becomes infinite.

THEOREM XXI.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{y^2 - 2gy + a^2} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{\overline{y-m} \times \overline{y - \frac{a^2}{m}}}}. \quad y \text{ greater than } m.$$

$$F = K + \frac{4m^{\frac{1}{2}}}{b^4} \times \overline{\frac{b^2}{2m}DP + g.ae + m.\overline{e'e'' - E''}}.$$

b, g, and x as in the two preceding theorems.

THEOREM XXII.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2fy + g^2}}.$$

$$F = K + \frac{2^{\frac{5}{2}}a^{\frac{1}{2}}}{b^2} \times \overline{ae + e'e'' - E''} = K + \frac{2^{\frac{3}{2}}a^{\frac{3}{2}}}{b^2} \times \overline{ae + AD - DP}.$$

$$f = \frac{a^2 - b^2}{2a}. \quad g = \frac{a^2 + b^2}{2a}. \quad x = f - y + \sqrt{y^2 - 2fy + g^2}. \quad DP = \sqrt{2ay}.$$

THEO-

THEOREM XXIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2fy + g^2}}$, generated whilst y from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $\frac{g^2}{k}$ becomes infinite.

THEOREM XXIV.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2fy + g^2}}.$$

$$F = K + \frac{2^{\frac{5}{2}}a^{\frac{1}{2}}}{b^2} \times \overline{\frac{a}{2}.ae + g.\overline{e'e'' - E''} + \frac{b^2}{4a}.DP}.$$

f, g, x, and DP as in the two preceding theorems.

THEOREM XXV.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{\overline{f \pm y} \times \overline{g^2 - y^2}}} = \frac{-\dot{x}}{\sqrt{f \pm \sqrt{g^2 - x^2}}}.$$

$$F = K + \frac{4a^{\frac{1}{2}}}{b^2} \times \overline{g.e'e'' - \frac{a}{2}.de}.$$

$$f = \frac{a^2 - b^2}{2a}. \quad g = \frac{a^2 + b^2}{2a}. \quad x = f \pm y = f \pm \sqrt{g^2 - z^2}.$$

THEOREM XXVI.

The tangent eo $\left(= a^{\frac{1}{2}} \times \overline{f + k}|^{\frac{1}{2}} \times \overline{\frac{g - k}{g + k}}|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{\frac{1}{2}b^{\frac{1}{2}}y\dot{y}}{\sqrt{\overline{f + y} \times \overline{g^2 - y^2}}}$, generated whilst y from $-f$ becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst y from $g \times \frac{g - 2f - k}{g + k}$ becomes equal to g.

THEO-

THEOREM XXVII.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{b \pm y \times \frac{1}{4}a^2 - y^2}} = \frac{-\dot{z}}{\sqrt{b \pm \sqrt{\frac{1}{4}a^2 - z^2}}}.$$

$$F = K + \frac{4a^{\frac{1}{2}}}{b^2} \times \overline{h.e'e'' - g.de}.$$

$$g = \tfrac{1}{2}a + \frac{b^2}{2a}. \quad h = \tfrac{1}{2}a + \frac{b^2}{a}. \quad x = \tfrac{1}{2}a \pm y = \tfrac{1}{2}a \pm \sqrt{\tfrac{1}{4}a^2 - z^2}.$$

THEOREM XXVIII.

The tangent eo $\left(= a^{\frac{1}{2}} \times \overline{\frac{\frac{1}{4}a^2 - k^2}{b+k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{\frac{1}{2}a^{\frac{1}{2}}y\dot{y}}{\sqrt{b+y \times \frac{1}{4}a^2 - y^2}}$, generated whilſt y from $-\frac{1}{2}a$ becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt y from $-\frac{\frac{1}{2}a^2 + bk}{b+k}$ becomes equal to $\frac{1}{2}a$.

THEOREM XXIX.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{b - y \times y^2 - \frac{b^4}{4a^2}}} = \frac{\dot{z}}{\sqrt{b - \sqrt{\frac{b^4}{4a^2} + z^2}}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times e'e''. \quad h = a + \frac{b^2}{2a}. \quad x = y - \frac{b^2}{2a} = \overline{\frac{b^4}{4a^2} + z^2}\Big|^{\frac{1}{2}} - \frac{b^2}{2a}.$$

THEOREM XXX.

The tangent eo $\left(= a^{\frac{1}{2}} \times \overline{k - \frac{b^2}{2a}}\Big|^{\frac{1}{2}} \times \overline{\frac{b-k}{\frac{b^2}{2a} + k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{\frac{1}{2}a^{\frac{1}{2}}y\dot{y}}{\sqrt{b - y \times y^2 - \frac{b^4}{4a^2}}}$, generated whilſt y from $\frac{b^2}{2a}$ becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt y from $\frac{b^2}{2a} \times \frac{b^2 + 4ab - 2ak}{b^2 + 2ak}$ becomes equal to h.

THEO-

THEOREM XXXI.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{\overline{b+y} \times \overline{y^2 - \frac{b^4}{4a^2}}}} = \frac{\dot{z}}{\sqrt{b + \sqrt{\frac{b^4}{4a^2} + z^2}}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{DP + e'e'' - E''}.$$

$$h = a + \frac{b^2}{2a}. \quad x = \frac{2ab^2}{b^2 + 2ay} = \frac{2ab^2}{b^2 + \sqrt{b^4 + 4a^2z^2}}.$$

THEOREM XXXII.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{\overline{y-f} \times \overline{y^2 - g^2}}} = \frac{\dot{z}}{\overline{\overline{g^2 + z^2}|^{\frac{1}{2}} - f}|^{\frac{1}{2}}}.$$

$$F = K + \frac{2a^{\frac{1}{2}}}{b^2} \times \overline{a.ae + 2g.\overline{e'e'' - E''} + \frac{b^2}{a}.DP}.$$

$$f = \frac{a^2 - b^2}{2a}. \quad g = \frac{a^2 + b^2}{2a}. \quad x = \frac{b^2}{y - f} = \frac{b^2}{\overline{g^2 + z^2}|^{\frac{1}{2}} - f}.$$

THEOREM XXXIII.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{\overline{y-b} \times \overline{y^2 - \frac{1}{4}a^2}}} = \frac{\dot{z}}{\overline{\overline{\frac{1}{4}a^2 + z^2}|^{\frac{1}{2}} - b}|^{\frac{1}{2}}}.$$

$$F = K + \frac{4a^{\frac{1}{2}}}{b^2} \times \overline{g.ae + h.\overline{e'e'' - E''} + \frac{b^2}{2a}.DP}.$$

$$g = \tfrac{1}{2}a + \frac{b^2}{2a}. \quad h = \tfrac{1}{2}a + \frac{b^2}{a}. \quad x = \frac{2b^2}{2y - a} = \frac{2b^2}{\overline{a^2 + 4z^2}|^{\frac{1}{2}} - a}.$$

THEOREM XXXIV.

$$\dot{F} = \frac{y\dot{y}}{\sqrt{\overline{f \pm y} \times \overline{b^2 + y^2}}} = \frac{\dot{z}}{\sqrt{f \pm \sqrt{z^2 - b^2}}}.$$

$$F = K + \frac{2}{a}\Big|^{\frac{1}{2}} \times \overline{2AD - DP}.$$

$$f = \frac{a^2 - b^2}{2a}. \quad x = \overline{b^2 + y^2}|^{\frac{1}{2}} \mp y = z \mp \sqrt{z^2 - b^2}. \quad DP = \sqrt{2a.\overline{f \pm y}}.$$

THEOREM XXXV.

$$\dot{F} = x^{-\frac{1}{2}}\dot{x} \times \overline{\frac{\frac{b^2}{a}+x}{a-x}}\Big|^{\frac{1}{2}}. \quad F = K + \frac{2}{a^{\frac{1}{2}}} \times \text{de}.$$

THEOREM XXXVI.

The tangent eo $\left(=\overline{ak}|^{\frac{1}{2}} \times \overline{\frac{a-k}{\frac{b^2}{a}+k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x} \times \overline{\frac{\frac{b^2}{a}+x}{a-x}}\Big|^{\frac{1}{2}}$, generated whilſt x from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt x from $\frac{b^2}{a} \times \frac{a-k}{\frac{b^2}{a}+k}$ becomes equal to a.

THEOREM XXXVII.

$$\dot{F} = x^{-\frac{1}{2}}\dot{x} \times \overline{\frac{a-x}{\frac{b^2}{a}+x}}\Big|^{\frac{1}{2}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}b^2} \times \overline{\overline{2a^2+b^2}.\text{de} - \overline{2a^2+2b^2}.e'e''}.$$

THEOREM XXXVIII.

The fluent of $\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x} \times \overline{\frac{a-x}{\frac{b^2}{a}+x}}\Big|^{\frac{1}{2}}$, generated whilſt x from o becomes equal to any quantity k, is equal to the tangent eo $\left(=\overline{ak}|^{\frac{1}{2}} \times \overline{\frac{a-k}{\frac{b^2}{a}+k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of the ſame fluxion, generated whilſt x from $\frac{b^2}{a} \times \frac{a-k}{\frac{b^2}{a}+k}$ becomes equal to a.

THEO-

THEOREM XXXIX.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y - \frac{b^2}{a}}{y + a}\right)^{\frac{1}{2}}. \quad F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{DP - ae}.$$

$$x = \frac{b^2}{y}. \quad DP = \left(\frac{a}{y}\right)^{\frac{1}{2}} \times \sqrt{\overline{y + a} \times \overline{y - \frac{b^2}{a}}}.$$

THEOREM XL.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y + a}{y - \frac{b^2}{a}}\right)^{\frac{1}{2}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}b^2} \times \overline{b^2.DP + \overline{2a^2 + b^2}.ae + \overline{2a^2 + 2b^2}.e'e'' - E''}.$$

x, and DP as in the preceding theorem.

THEOREM XLI.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y + \frac{b^2}{m}}{y + m}\right)^{\frac{1}{2}}. \quad F = K + \frac{2}{m^{\frac{1}{2}}} \times AD.$$

$$a = \sqrt{m^2 - b^2}. \quad x = \frac{ab^2}{my + b^2}.$$

THEOREM XLII.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y + m}{y + \frac{b^2}{m}}\right)^{\frac{1}{2}}.$$

$$F = K + \frac{2}{m^{\frac{1}{2}}b^2} \times \overline{b^2.DP + \overline{2m^2 - b^2}.ae + 2m^2.\overline{e'e''} - E''}.$$

a, and x as in the preceding theorem. $DP = \overline{my}^{\frac{1}{2}} \times \left(\frac{y + m}{y + \frac{b^2}{m}}\right)^{\frac{1}{2}}$.

THEOREM XLIII.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y - \frac{b^2}{m}}{m - y}\right)^{\frac{1}{2}}. \quad F = K + \frac{2}{m^{\frac{1}{2}}} \times \overline{2e'e'' - de}.$$

$$a = \sqrt{m^2 - b^2}. \quad x = \frac{my - b^2}{a}.$$

THEOREM XLIV.

The tangent eo $\left(= \left(\frac{m}{k}\right)^{\frac{1}{2}} \times \left(\overline{m - k} \times \overline{k - \frac{b^2}{m}}\right)^{\frac{1}{2}}\right)$ together with the fluent of $\frac{1}{2}m^{\frac{1}{2}}y^{-\frac{1}{2}}\dot{y} \times \left(\frac{y - \frac{b^2}{m}}{m - y}\right)^{\frac{1}{2}}$, generated whilſt y from $\frac{b^2}{m}$ becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt y from $\frac{b^2}{k}$ becomes equal to m.

THEOREM XLV.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \left(\frac{m - y}{y - \frac{b^2}{m}}\right)^{\frac{1}{2}}.$$

$$F = K + \frac{2}{m^{\frac{1}{2}}b^2} \times \overline{2m^2 - b^2 . de - 2m^2 . e'e''}.$$

a, and x as in the two preceding theorems.

THEOREM XLVI.

The fluent of $\frac{1}{2}m^{\frac{1}{2}}y^{-\frac{1}{2}}\dot{y} \times \left(\frac{m - y}{y - \frac{b^2}{m}}\right)^{\frac{1}{2}}$, generated whilſt from $\frac{b^2}{m}$ becomes equal to any quantity k, is equal to the tangent eo $\left(= \left(\frac{m}{k}\right)^{\frac{1}{2}} \times \left(\overline{m - k} \times \overline{k - \frac{b^2}{m}}\right)^{\frac{1}{2}}\right)$ together with the fluent of the ſame fluxion, generated whilſt y from $\frac{b^2}{k}$ becomes equal to m.

THEO-

THEOREM XLVII.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{\frac{a^2}{m} - y}{m - y}}\Big|^{\frac{1}{2}}. \quad y \text{ lefs than } \frac{a^2}{m}.$$

$$F = K + \frac{2}{m^{\frac{1}{2}}} \times \overline{DP - AD} = K + \frac{2}{m^{\frac{1}{2}}} \times \overline{2E'' - 2.e'e'' - ae}.$$

$$b = \sqrt{m^2 - a^2}. \quad x = \frac{a^2 - my}{a}.$$

THEOREM XLVIII.

The tangent eo $\left(= \overline{mk}\big|^{\frac{1}{2}} \times \overline{\frac{\frac{a^2}{m} - k}{m - k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{1}{2}m^{\frac{1}{2}}y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{\frac{a^2}{m} - y}{m - y}}\Big|^{\frac{1}{2}}$, generated whilft y from o becomes equal to any quantity k, is equal to the fluent of the fame fluxion, generated whilft y from $\frac{a^2 - mk}{m - k}$ becomes equal to $\frac{a^2}{m}$.

THEOREM XLIX.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{m - y}{\frac{a^2}{m} - y}}\Big|^{\frac{1}{2}}. \quad y \text{ lefs than } \frac{a^2}{m}. \quad F = K + \frac{2}{m^{\frac{1}{2}}} \times ae.$$

b, and x as in the two preceding theorems.

THEOREM L.

The tangent eo $\left(= \overline{mk}\big|^{\frac{1}{2}} \times \overline{\frac{\frac{a^2}{m} - k}{m - k}}\Big|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{1}{2}m^{\frac{1}{2}}y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{m - y}{\frac{a^2}{m} - y}}\Big|^{\frac{1}{2}}$, generated whilft y from o becomes equal to any quantity k, is equal to the fluent of the fame fluxion, generated whilft y from $\frac{a^2 - mk}{m - k}$ becomes equal to $\frac{a^2}{m}$.

THEO-

THEOREM LI.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{y - \frac{a^2}{m}}{y - m}}\Big|^{\frac{1}{2}}. \quad y \text{ greater than } m. \quad F = K + \frac{2}{m^{\frac{1}{2}}} \times AD.$$

$$b = \sqrt{m^2 - a^2}. \quad x = \frac{ab^2}{my - a^2}.$$

THEOREM LII.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y} \times \overline{\frac{y - m}{y - \frac{a^2}{m}}}\Big|^{\frac{1}{2}}. \quad y \text{ greater than } m. \quad F = K + \frac{2}{m^{\frac{1}{2}}} \times \overline{DP - ae}.$$

b, and x as in the preceding theorem. $DP = \overline{my}|^{\frac{1}{2}} \times \overline{\frac{y - m}{y - \frac{a^2}{m}}}\Big|^{\frac{1}{2}}$.

THEOREM LIII.

$$\dot{F} = y^{-\frac{1}{2}}\dot{y}\,\frac{\sqrt{p^2 + 2.\overline{1 - q}.py + \overline{q^2 - q}.y^2}}{\sqrt{2p - qy}}.$$

$F = K + VW$. (Fig. 4.)

When q is negative, VW is an *hyperbola* whose semi-axes are $\frac{p}{-q}$ and $\frac{p}{\sqrt{-q}}$.

When q is $= 0$, . . . VW is a *parabola* whose semi-parameter is p.

When q is positive, VW is an *ellipsis* whose semi-axes are $\frac{p}{q}$ and $\frac{p}{\sqrt{q}}$;

which becomes a *circle* when q is $= 1$.

THEO-

THEOREM LIV.

$$\dot{F} = \frac{v^{-\frac{1}{2}}\dot{v}}{v + f \times \sqrt{cv^2 + dv + e}} = \frac{-w^{\frac{1}{2}}\dot{w}}{1 + fw \times \sqrt{c + dw + ew^2}}.$$

$$F = \begin{cases} \frac{1}{2f} \times \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{x^2 - 2dx + d^2 - 4ce}} - \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{x - d + cf + \frac{e}{f}} \\ + \frac{cf^2 - e}{2f^2} \times \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{x - d + cf + \frac{e}{f} \times \sqrt{x^2 - 2dx + d^2 - 4ce}}. \end{cases}$$

$$x = \frac{cv^2 + dv + e}{v}. \quad v = \frac{1}{w} = \frac{x - d + \sqrt{x^2 - 2dx + d^2 - 4ce}}{2c}.$$

$$w = \frac{1}{v} = \frac{x - d - \sqrt{x^2 - 2dx + d^2 - 4ce}}{2e}.$$

Case 1. $e = cf^2$.

$$F = \frac{1}{2f} \times \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{x^2 - 2dx + d^2 - 4ce}} - \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{x - d + 2cf}.$$

Case 2. $e'' = c''f''^2$.

$$F = \begin{cases} \frac{1}{2f} \times \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{x^2 - 2dx + d^2 - 4ce}} - \text{fl.} \frac{x^{-\frac{1}{2}}\dot{x}}{x - d + cf + \frac{e}{f}} \\ + \frac{cf^2 - e}{4f^2 f''} \times \text{fl.} \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{y^2 - 2d''y + d''^2 - 4c''e''}} - \text{fl.} \frac{y^{-\frac{1}{2}}\dot{y}}{y - d'' + 2c''f''}. \end{cases}$$

$$c'' = 1. \ d'' = -2d. \ e'' = d^2 - 4ce. \ f'' = cf + \frac{e}{f} - d. \ y = \frac{c''x^2 + d''x + e''}{x}.$$

In which cases F will be assigned by the *arcs of the conic sections*.

NOTE. By substituting repeatedly in the resulting term similar to the original value of $\dot{F}$, the value of F may be assigned, by means of the *arcs of the conic sections*, in other particular cases, though not in general.

THEOREM LV.

$$\dot{F} = \frac{v^{-1}\dot{v}}{\sqrt{2gv+h} \times \sqrt{cv^2+2dv+e}} = \frac{-w^{\frac{1}{2}}\dot{w}}{\sqrt{2g+hw} \times \sqrt{c+2dw+ew^2}}.$$

$$F = \begin{cases} -\frac{1}{2h^{\frac{1}{2}}} \times \text{fl.}\ \frac{g\dot{x}}{\sqrt{x+e} \times \sqrt{g^2x^2+\overline{ch^2-2dgh+2eg^2}.x+\overline{dh-eg}|^2}} - \frac{x^{-1}\dot{x}}{\sqrt{x+e}}. \\ +\frac{dh-eg}{2h^{\frac{1}{2}}} \times \text{fl.}\ \frac{x^{-1}\dot{x}}{\sqrt{x+e} \times \sqrt{g^2x^2+\overline{ch^2-2dgh+2eg^2}.x+\overline{dh-eg}|^2}}. \end{cases}$$

$$x = h \times \frac{cv^2+2dv+e}{2gv+h} - e.\quad v = \frac{1}{w} = \frac{\sqrt{g^2x^2+px+q^2}+gx-q}{ch}.$$

$$w = \frac{1}{v} = \frac{\sqrt{g^2x^2+px+q^2}-gx+q}{hx}.$$

$$p = ch^2 - 2dgh + 2eg^2.\quad q = dh - eg.$$

Case 1. $dh = eg.\quad x = \frac{chv^2}{2gv+h}.$

$$F = -\frac{1}{2h^{\frac{1}{2}}} \times \text{fl.}\ \frac{gx^{-\frac{1}{2}}\dot{x}}{\sqrt{x+e} \times \sqrt{g^2x+ch^2}} - \text{fl.}\ \frac{x^{-1}\dot{x}}{\sqrt{x+e}}.$$

Case 2. $d''h'' = e''g''.\quad x = \frac{chv^2 + \overline{2.dh-eg}.v}{2gv+h}.$

$$F = \begin{cases} -\frac{1}{2h^{\frac{1}{2}}} \times \text{fl.}\ \frac{g\dot{x}}{\sqrt{x+h''} \times \sqrt{c''x^2+2d''x+e''}} - \text{fl.}\ \frac{x^{-1}\dot{x}}{\sqrt{x+e}} \\ -\frac{dh-eg}{4h^{\frac{1}{2}}h''^{\frac{1}{2}}} \times \text{fl.}\ \frac{g''y^{-\frac{1}{2}}\dot{y}}{\sqrt{y+e''} \times \sqrt{g''^2y+c''h''^2}} - \text{fl.}\ \frac{y^{-1}\dot{y}}{\sqrt{y+e''}}. \end{cases}$$

$$c'' = g^2.\ d'' = \tfrac{1}{2}ch^2 - dgh + eg^2.\ e'' = \overline{dh-eg}|^2.\ g'' = \tfrac{1}{2}.\ h'' = e.$$

$$y = e \times \frac{c''x^2+2d''x+e''}{x+e} - e'' = \frac{eg^2x^2}{x+e}.$$

In which cases F will be assigned by the *arcs of the conic sections.*

NOTE. The value of F will be so assigned in other particular cases (though not in general) by proceeding as intimated in the Note to the preceding theorem.

THEO-

THEOREM LVI.

$$\dot{F} = \frac{v^{-2}\dot{v}}{\sqrt{2gv+h} \times \sqrt{cv^2+2dv+e}} = \frac{-w^{\frac{1}{2}}\dot{w}}{\sqrt{2g+hw} \times \sqrt{c+2dw+ew^2}}.$$

$$F = -\frac{1}{eh} \times \begin{cases} \frac{\sqrt{2gv+h} \times \sqrt{cv^2+2dv+e}}{v} - \text{fl.}\ \frac{cgv\dot{v}}{\sqrt{2gv+h} \times \sqrt{cv^2+2dv+e}} \\ + \overline{dh+eg} \times \text{fl.}\ \frac{v^{-1}\dot{v}}{\sqrt{2gv+h} \times \sqrt{cv^2+2dv+e}}\ * \end{cases}.$$

$$w = \frac{1}{v}.$$

NOTE. The value of F will be aſſigned by the *arcs of the conic ſections*, not only when $dh+eg$ is $=0$, but likewiſe when the fluent of the laſt written fluxion * can be ſo aſſigned.

THEOREM LVII.

$$\dot{F} = \frac{\dot{v}}{\sqrt{c+2dv+ev^2} \times \sqrt{f+2gv+hv^2}} = \frac{-\dot{w}}{\sqrt{cw^2+2dw+e} \times \sqrt{fw^2+2gw+h}}.$$

$$F = -\tfrac{1}{2}\,\text{fl.}\ \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{p^2+qx+rx^2}}.$$

$$p^2 = d^2 - ce. \quad q = ch + ef - 2dg. \quad r = g^2 - fh.$$

$$x = \frac{c+2dv+ev^2}{f+2gv+hv^2}. \quad v = \frac{1}{w} = \frac{d - gx + \sqrt{p^2+qx+rx^2}}{hx - e}.$$

$$\text{Or } F = -\frac{h^{\frac{1}{2}}}{2}\ \text{fl.}\ \frac{\dot{y}}{\sqrt{e+y} \times \sqrt{P^2+Qy+ry^2}}.$$

$$P^2 = \overline{dh-eg}|^2. \quad Q = ch^2 - efh - 2dgh + 2eg^2 = hq + 2er.$$

$$y = h \times \frac{c+2dv+ev^2}{f+2gv+hv^2} - e. \quad v = \frac{1}{w} = \frac{1}{h} \times \frac{P - gy + \sqrt{P^2+Qy+ry^2}}{y}.$$

NOTE. The value of F will always be aſſigned by the *arcs of the conic ſections.*

 THEO-

THEOREM LVIII.

$$\dot{F}=\frac{v\dot{v}}{\sqrt{c+2dv+ev^2}\times\sqrt{f+2gv+bv^2}}=\frac{-w^{-1}\dot{w}}{\sqrt{cw^2+2dw+e}\times\sqrt{fw^2+2gw+b}}.$$

$$F=\frac{1}{2b^{\frac{1}{2}}}\times\begin{cases}\text{fl. }\dfrac{g\dot{y}}{\sqrt{e+y}\times\sqrt{P^2+Qy+ry^2}}-\text{fl. }\dfrac{y^{-1}\dot{y}}{\sqrt{e+y}}\\ -\overline{dh-eg}\times\text{fl. }\dfrac{y^{-1}\dot{y}}{\sqrt{e+y}\times\sqrt{P^2+Qy+ry^2}}*.\end{cases}$$

P, Q, r, v, w, and y as in the preceding theorem.

NOTE. The value of F will be assigned by the *arcs of the conic sections* when dh is $= eg$, and likewise in the particular cases wherein the fluent of the last written fluxion * can be so assigned.

Case 1. $dh=eg$. $y=\dfrac{cb-ef}{f+2gv+bv^2}$.

$$F=\frac{1}{2b^{\frac{1}{2}}}\times\text{fl. }\frac{gy^{-\frac{1}{2}}\dot{y}}{\sqrt{e+y}\times\sqrt{cb^2-efb+\overline{g^2-fb}.y}}-\text{fl. }\frac{y^{-1}\dot{y}}{\sqrt{e+y}}.$$

Case 2. $\overline{ce-d^2}\times h^2=\overline{fh-g^2}\times e^2$. $y=\dfrac{ch-ef+2.\overline{dh-eg}.v}{f+2gv+bv^2}$.

$$F=\frac{1}{2b^{\frac{1}{2}}}\times\begin{cases}\text{fl. }\dfrac{g\dot{y}}{\sqrt{e+y}\times\sqrt{P^2+\frac{P^2}{e}y+ry^2}}-\text{fl. }\dfrac{y^{-1}\dot{y}}{\sqrt{e+y}}\\ +\dfrac{db-eg}{2e^{\frac{1}{2}}}\times\text{fl. }\dfrac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{P^2+z}\times\sqrt{4e^2.\overline{g^2-fb}+z}}-\text{fl. }\dfrac{z^{-1}\dot{z}}{\sqrt{P^2+z}}.\end{cases}$$

$$z=\frac{e.\overline{g^2-fb}.y^2}{y+e}.$$

THEO-

THEOREM LIX.

$$\dot{F}=\frac{v^{2}\dot{v}}{\sqrt{c+2dv+ev^{2}}\times\sqrt{f+2gv+hv^{2}}}=\frac{-w^{-2}\dot{w}}{\sqrt{cw^{2}+2dw+e}\times\sqrt{fw^{2}+2gw+h}}.$$

$$F=\frac{1}{2h^{\frac{3}{2}}}\times\begin{cases}\frac{2}{ey}\times\sqrt{e+y}\times\sqrt{P^{2}+Qy+ry^{2}}\\+2g\times\text{fl.}\frac{y^{-1}\dot{y}}{\sqrt{e+y}}-2.\overline{dh-eg}\times\text{fl.}\frac{y^{-2}\dot{y}}{\sqrt{e+y}}\\+\overline{fh-2g^{2}}\times\text{fl.}\frac{\dot{y}}{\sqrt{e+y}\times\sqrt{P^{2}+Qy+ry^{2}}}\\+\frac{fh-g^{2}}{e}\times\text{fl.}\frac{y\dot{y}}{\sqrt{e+y}\times\sqrt{P^{2}+Qy+ry^{2}}}\\+\frac{\overline{dh-eg}\times\overline{dh+eg}}{e}\times\text{fl.}\frac{y^{-1}\dot{y}}{\sqrt{e+y}\times\sqrt{P^{2}+Qy+ry^{2}}}*.\end{cases}$$

P, Q, r, v, w, and y as in the two preceding theorems.

NOTE. The value of F will be aſſigned by the *arcs of the conic ſections* when dh is $=eg$, or $dh=-eg$; and likewiſe in the particular caſes wherein the fluent of the laſt written fluxion * can be ſo aſſigned.

SCHEME

FOR

TABLE XII.

aed (Fig. 1.) is a quadrantal arc of an ellipsis $= E'$.

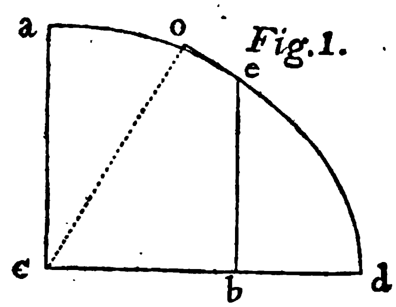


Fig. 1.

Semi-transverse axis cd $= \sqrt{a^2+b^2}$.

Semi-conjugate axis ac $= b$.

Abscissa cb $= \overline{\frac{a^2+b^2}{a}}\Big|^{\frac{1}{2}} \times \overline{a-x}\Big|^{\frac{1}{2}}$.

Ordinate be $= b \times \overline{\frac{x}{a}}\Big|^{\frac{1}{2}}$.

Tangent eo $= \overline{a-x} \times \frac{a^{\frac{1}{2}}x^{\frac{1}{2}}}{\sqrt{b^2+2fx-x^2}} = a \times \overline{\frac{ax-x^2}{b^2+ax}}\Big|^{\frac{1}{2}}$.

co being perpendicular to eo.

$ae'e''d$ (Fig. 2.) is a quadrantal arc of another ellipsis $= E''$.

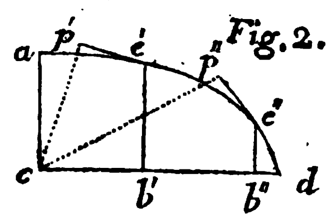


Fig. 2.

Semi-transverse axis $cd = \frac{1}{2}\sqrt{a^2+b^2} + \frac{1}{2}a$.

Semi-conjugate axis $ae = \frac{1}{2}\sqrt{a^2+b^2} - \frac{1}{2}a$.

$e'p'$ and its equal $e''p''$ (each $= \sqrt{a^2 - ax}$) are tangents, to which cp', cp'' are perpendiculars.

The abscissa cb', or cb'', corresponding to the ordinate $b'e'$,

or $b''e''$, is $= \overline{\frac{\sqrt{a^2+b^2} + a - x \mp \sqrt{\frac{b^2}{a}x + x^2}}{2\sqrt{a^2+b^2}}}\Bigg|^{\frac{1}{2}} \times cd$.

$$ae' - de'' = e'p' = e''p''.$$

SCHEME

AD (Fig. 3.) is a hyperbola, whoſe vertex is A and center C.

Semi-tranſverſe axis $= a$.

Semi-conjugate axis $= b$.

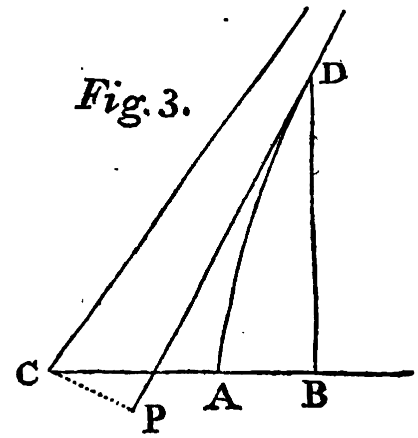
DP is a tangent, to which CP is perpendicular.

CP $= \sqrt{ax}$.

DP $= \left(\frac{a}{x}\right)^{\frac{1}{2}} \times \left(b^2 + 2fx - x^2\right)^{\frac{1}{2}}$.

$f = \frac{a^2 - b^2}{2a}$.

Fig. 3.



The abſciſſa CAB, correſponding to the ordinate BD,

is $= a \times \left(\frac{a}{x}\right)^{\frac{1}{2}} \times \left(\frac{ax + b^2}{a^2 + b^2}\right)^{\frac{1}{2}}$.

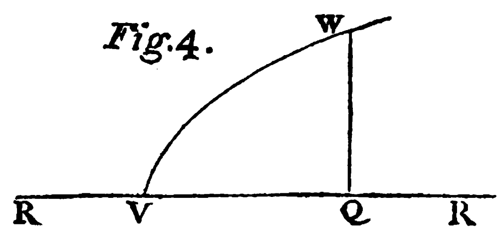
L, the *limit* of DP — AD, is $= 2E'' - E'$;

AD being always $= DP + ae + 2\acute{e}e'' - 2E''$.

VW (Fig. 4.) is an arc of a conic ſection, whoſe vertex is V.

Abſciſſa VQ, correſponding to the ordinate QW, $= y$;

Fig. 4.



RVQ, or VQR being an axis of the ſection:

which axis is $= \frac{2p}{\mp q}$.

NOTE. *All the Theorems in* TABLE XII. *(where any reference is neceſſary) refer to this Scheme.*

TABLE

TABLE XIII.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$\dot{A} = V^p z^m \dot{z}$, $\dot{B} = V^p z^{m+n} \dot{z}$, $\dot{C} = V^p z^{m+2n} \dot{z}$, &c.

$\dot{P}_{,} = V^{p+1} z^m \dot{z}$, $\dot{P}_{,,} = V^{p+1} z^{m+n} \dot{z}$, $\dot{P}_{,,,} = V^{p+1} z^{m+2n} \dot{z}$, &c.

$\dot{Q}_{,} = V^p \dot{V} z^{m+1}$, $\dot{Q}_{,,} = V^p \dot{V} z^{m+n+1}$, $\dot{Q}_{,,,} = V^p \dot{V} z^{m+2n+1}$, &c.

$$V = a + bz^n + cz^{2n} + dz^{3n} \ (t).$$

$P_{,} = aA + bB + cC \ (t)$. $Q_{,} = n \times \overline{bB + 2cC + 3dD \ (t-1)}$.

$P_{,,} = aB + bC + cD \ (t)$. $Q_{,,} = n \times \overline{bC + 2cD + 3dE \ (t-1)}$.

$P_{,,,} = aC + bD + cE \ (t)$. $Q_{,,,} = n \times \overline{bD + 2cE + 3dF \ (t-1)}$.

&c. &c. &c. &c.

$$V^{p+1} z^{m+1} = \overline{m+1} \times P_{,} + \overline{p+1} \times Q_{,}.$$

$$V^{p+1} z^{m+n+1} = \overline{m+n+1} \times P_{,,} + \overline{p+1} \times Q_{,,}.$$

$$V^{p+1} z^{m+2n+1} = \overline{m+2n+1} \times P_{,,,} + \overline{p+1} \times Q_{,,,}.$$

&c. &c.

Hence, if $t-1$ of the fluents A, B, C, &c. $P_{,}$, $P_{,,}$, $P_{,,,}$, &c. $Q_{,}$, $Q_{,,}$, $Q_{,,,}$, &c. be given, the rest will be determined.

THEO-

THEOREM II.

$$\dot{A} = V^{p} W^{q} z^{m} \dot{z}, \qquad \dot{B} = V^{p} W^{q} z^{m+n} \dot{z}, \text{ \&c.}$$

$$\dot{P}_{,} = V^{p+1} W^{q} z^{m} \dot{z}, \qquad \dot{P}_{,,} = V^{p+1} W^{q} z^{m+n} \dot{z}, \text{ \&c.}$$

$$\dot{Q}_{,} = V^{p} W^{q+1} z^{m} \dot{z}, \qquad \dot{Q}_{,,} = V^{p} W^{q+1} z^{m+n} \dot{z}, \text{ \&c.}$$

$$\dot{R}_{,} = V^{p+1} W^{q+1} z^{m} \dot{z}, \qquad \dot{R}_{,,} = V^{p+1} W^{q+1} z^{m+n} \dot{z}, \text{ \&c.}$$

$$\dot{S}_{,} = V^{p} \dot{V} W^{q+1} z^{m+1}, \qquad \dot{S}_{,,} = V^{p} \dot{V} W^{q+1} z^{m+n+1}, \text{ \&c.}$$

$$\dot{T}_{,} = V^{p+1} W^{q} \dot{W} z^{m+1}, \qquad \dot{T}_{,,} = V^{p+1} W^{q} \dot{W} z^{m+n+1}, \text{ \&c.}$$

$$V = a + bz^{n} + cz^{2n} + dz^{3n} \ (t).$$

$$W = a'' + b''z^{n} + c''z^{2n} + d''z^{3n} \ (t'').$$

$$P_{,} = aA + bB + cC \ (t). \qquad Q_{,} = a''A + b''B + c''C \ (t'').$$

$$P_{,,} = aB + bC + cD \ (t). \qquad Q_{,,} = a''B + b''C + c''D \ (t'').$$

&c. &c. &c. &c.

$$R_{,} = aQ_{,} + bQ_{,,} + cQ_{,,,} \ (t) = a''P_{,} + b''P_{,,} + c''P_{,,,} \ (t'').$$

$$R_{,,} = aQ_{,,} + bQ_{,,,} + cQ_{iv} \ (t) = a''P_{,,} + b''P_{,,,} + c''P_{iv} \ (t'').$$

&c. &c. &c. &c.

$$S_{,} = n \times \overline{bQ_{,,} + 2cQ_{,,,} + 3dQ_{iv} (t-1)}. \quad T_{,} = n \times \overline{b''P_{,,} + 2c''P_{,,,} + 3d''P_{iv} (t''-1)}.$$

$$S_{,,} = n \times \overline{bQ_{,,,} + 2cQ_{iv} + 3dQ_{v} (t-1)}. \quad T_{,,} = n \times \overline{b''P_{,,,} + 2c''P_{iv} + 3d''P_{v} (t''-1)}.$$

&c. &c. &c. &c.

$$V^{p+1} W^{q+1} z^{m+1} = \overline{m+1} \times R_{,} + \overline{p+1} \times S_{,} + \overline{q+1} \times T_{,}.$$

$$V^{p+1} W^{q+1} z^{m+n+1} = \overline{m+n+1} \times R_{,,} + \overline{p+1} \times S_{,,} + \overline{q+1} \times T_{,,}.$$

&c. &c.

Hence, if $t + t'' - 2$ of the fluents A, B, C, &c. $P_{,}$, $P_{,,}$, $P_{,,,}$, &c. $Q_{,}$, $Q_{,,}$, $Q_{,,,}$, &c. $R_{,}$, $R_{,,}$, $R_{,,,}$, &c. $S_{,}$, $S_{,,}$, $S_{,,,}$, &c. $T_{,}$, $T_{,,}$, $T_{,,,}$, &c. be given, the rest will be determined.

TABLE

TABLE XIV.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

The fluent of $\frac{\dot{x}}{x} \times$ fl. $\frac{\dot{x}}{1+x}$, generated whilst x from o becomes equal to b, is equal to $\frac{a^2}{3}$, or $\frac{4a^2}{15} - \frac{1}{2}$ sq. Log. $\frac{5^{\frac{1}{2}}-1}{2}$, according as b is equal to 1, or $\frac{5^{\frac{1}{2}}-1}{2}$.

$a =$ the quadrantal arc of the circle whose radius is 1.

THEOREM II.

The fluent of $\frac{\dot{x}}{x} \times$ fl. $\frac{\dot{x}}{1-x}$, generated whilst x from o becomes equal to b, is equal to $\frac{2a^2}{3}$, $\frac{a^2}{3} - \frac{1}{2}$ sq. Log. 2, $\frac{2a^2}{5}$ — sq. Log. $\frac{5^{\frac{1}{2}}-1}{2}$, or $\frac{4a^2}{15}$ — sq. Log. $\frac{5^{\frac{1}{2}}-1}{2}$; according as b is equal to 1, $\frac{1}{2}$, $\frac{5^{\frac{1}{2}}-1}{2}$, or $\frac{3-\sqrt{5}}{2}$.

a being as in the preceding theorem.

THEOREM III.

The *whole* fluent of $\frac{\dot{x}}{x} \times$ fl. $\frac{x^{\frac{1}{2}pn-1}\dot{x}}{\overline{1-x^n}|^p}$, generated whilst x from o becomes equal to 1, is $= \frac{aF}{2^{p-1}ns}$.

F = the contemporary fluent of $\frac{x^{\frac{1}{2}pn-1}\dot{x}}{\sqrt{1-x^n}}$.

a as in the preceding theorems.

$s =$ the sine of pa. p and $2-p$ positive.

THEO-

THEOREM IV.

The fluent of $\frac{\dot{x}}{\sqrt{b^2 - x^2}} \times$ fl. $\frac{\dot{x}}{\sqrt{1 - x^2}}$, generated whilst x from o becomes equal to b, (= the contemporary fluent of $\frac{\dot{x}}{x} \times$ fl. $\frac{\dot{x}}{1 - x^2}$,) is equal to $\frac{a^2}{2}$, $\frac{a^2}{3} - \frac{1}{4}$ sq. Log. $\frac{5^{\frac{1}{2}} - 1}{2}$, $\frac{a^2}{4} - \frac{1}{2}$ sq. Log. $2^{\frac{1}{2}} - 1$, or $\frac{a^2}{6} + \frac{1}{4}$ sq. Log. $\frac{5^{\frac{1}{2}} - 1}{2} - \frac{1}{2}$ Log. $5^{\frac{1}{2}} - 2 \times$ Log. $\frac{5^{\frac{1}{2}} - 1}{2}$; according as b is equal to 1, $\frac{5^{\frac{1}{2}} - 1}{2}$, $2^{\frac{1}{2}} - 1$, or $5^{\frac{1}{2}} - 2$.

a being as in the preceding theorems.

THEOREM V.

The *whole* fluent of $\frac{\dot{x}}{\sqrt{1 - x^2}} \times$ fl. $\frac{\dot{x}}{\sqrt{x + x^2}}$, generated whilst x from o becomes equal to 1, (= 2 × the contemporary fluent of $\frac{\dot{x}}{x} \times$ fl. $\frac{\dot{x}}{\sqrt{1 - x^2}}$,) is = $2a$ Log. 2.

a being as in the preceding theorems.

THEOREM VI.

The *whole* fluent of $\frac{x^{rn-1}\dot{x}}{(b^n - x^n)^p} \times$ fl. $\frac{x^{sn-1}\dot{x}}{(c + dx^n)^q}$, generated whilst x from o becomes equal to b, is $= \frac{F'F''}{b^{sn}c^{1-p}}$.

$$\left.\begin{array}{l} F' = \\ \\ F'' = \end{array}\right\}\text{the contemporary fluent of}\left\{\begin{array}{l} \frac{x^{rn+sn-1}\dot{x}}{(b^n - x^n)^p}. \\ \\ \frac{x^{sn-1}\dot{x}}{(c + dx^n)^{p+q-1}}. \end{array}\right.$$

$\frac{d}{c}$ greater than $-\frac{1}{b^n}$. $p + q = r + s + 1$. $1 - p$ and $r + s$ positive.

NOTE. This theorem may be of use in computing the fluent of $\frac{x^{rn-1}\dot{x}}{(b^n - x^n)^p} \times$ fl. $\frac{x^{sn+vn-1}\dot{x}}{(c + dx^n)^{q+w}}$, as the value of fl. $\frac{x^{sn+vn-1}\dot{x}}{(c + dx^n)^{q+w}}$ may be assigned in terms of fl. $\frac{x^{sn-1}\dot{x}}{(c + dx^n)^q}$ and algebraic quantities by Theorem XIV, XVI, XVIII, or XX, TAB. VII.

v and *w* being positive or negative integers.

THEOREM VII.

$\dot{F} = x^{m-1} z^{n} \dot{x}$. $z =$ the hyp. log. of $\frac{x}{c} =$ fl. $\frac{\dot{x}}{x}$.

$$F = K + \frac{x^{m}}{m} \times \overline{z^{n} - \frac{n}{m} z^{n-1} + \frac{n.n-1}{m^{2}} z^{n-3} - \frac{n.n-1.n-2}{m^{3}} z^{n-4} + \&c.}$$

Note. The fluent of $\overline{mz+n} \times x^{m-1} z^{n-1} \dot{x}$ is $= K + x^{m} z^{n}$.

THEOREM VIII.

$\dot{F} = z^{r} \dot{x}$. $z =$ fl. $\frac{a\dot{x}}{\sqrt{b + 2cx + dx^{2}}}$.

$$F = K + vz^{r} - \frac{r}{d} ayz^{r-1} + \frac{r.r-1}{d} a^{2} vz^{r-2}$$

$$- \frac{r.r-1.r-2}{d^{2}} a^{3} yz^{r-3} + \frac{r.r-1.r-2.r-3}{d^{2}} a^{4} vz^{r-4}$$

&c.

$v = x + \frac{c}{d}$. $y = \sqrt{b + 2cx + dx^{2}}$.

THEOREM IX.

$\dot{F} = x^{p} y^{q} z^{r} \dot{x}$. y and z as in the preceding theorem.

$$F = K + Pz^{r} - naQz^{r-1} + n.n-1.a^{2}Rz^{r-2} - n.n-1.n-2.a^{3}Sz^{r-3} + \&c.$$

$P =$ fl. $x^{p} y^{q} \dot{x}$, $Q =$ fl. $\frac{P\dot{x}}{y}$, $R =$ fl. $\frac{Q\dot{x}}{y}$, $S =$ fl. $\frac{R\dot{x}}{y}$, &c.

THEO-

THEOREM X.

$$Pp + Qqx + Rrx^2 + Ssx^3 + Ttx^4 \text{ \&c.}$$

$$\text{is} = PG + D'F'x + D''F''\frac{x^2}{2} + D'''F'''\frac{x^3}{2.3} \text{ \&c.}$$

$$G \text{ being} = p + qx + rx^2 + sx^3 \text{ \&c.}$$

P, Q, R, &c. p, q, r, &c. any invariable quantities.

$F' = \frac{\dot{G}}{\dot{x}}$, $F'' = \frac{\ddot{G}}{\dot{x}^2}$, $F''' = \frac{\dddot{G}}{\dot{x}^3}$, &c. $\dot{x}$ being considered as invariable.

$$D' = Q - P,$$
$$D'' = R - 2Q + P,$$
$$D''' = S - 3R + 3Q - P,$$
$$D^{\text{iv}} = T - 4S + 6R - 4Q + P,$$
&c. &c.

NOTE. If P, Q, R, &c. be equal to 1, $\frac{m+n}{n}$, $\frac{m+n.m+n+1}{n.n+1}$, $\frac{m+n.m+n+1.m+n+2}{n.n+1.n+2}$, &c. respectively; D′, D″, D‴, &c. will be respectively equal to $\frac{m}{n}$, $\frac{m.m-1}{n.n+1}$, $\frac{m.m-1.m-2}{n.n+1.n+2}$, &c.

And if p, q, r, &c. be equal to 1, $\frac{e}{f}b$, $\frac{e.e+1}{f.f+1}b^2$, $\frac{e.e+1.e+2}{f.f+1.f+2}b^3$, &c. respectively; G will be equal to $\frac{f-1}{x^{f-1} \times \overline{1-bx}|^{e-f+1}} \times \text{fl. } \overline{1-bx}|^{e-f} \times x^{f-2}\dot{x}$.

TABLE XV.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\overline{ax+by+c}\times\dot{x}+\overline{fx+gy+h}\times\dot{y}=0.$$

$$\overline{2a.x+r+\overline{b+f-q}.\overline{y+s}}\Big|^{b-f+q}\times\overline{2a.x+r+\overline{b+f+q}.\overline{y+s}}\Big|^{f-b+q}=K.$$

$$q=\sqrt{\overline{b+f}\,\big|^{2}-4ag}.\quad r=\frac{bh-cg}{bf-ag}.\quad s=\frac{cf-ah}{bf-ag}.$$

NOTE. If bf be $=ag$, x will be $=$ fl. $\frac{\dot{z}}{a}\times\frac{fz+ah}{\overline{f-b}.z+ah-bc}$.

z being $=ax+by$.

THEOREM II.

$$\overline{ax^{p}y^{q}+bx^{p-r}y^{q+r}\,\&c.}\times\dot{x}+\overline{fx^{p-s}y^{q+s}+gx^{p-t}y^{q+t}\,\&c.}\times\dot{y}=0.$$

$$\text{Fl. }\frac{\dot{x}}{x}=-\text{ fl. }\frac{\overline{fv^{q+s}+gv^{q+t}\,\&c.}\times\dot{v}}{av^{q}+bv^{q+r}\,\&c.+fv^{q+s+1}+gv^{q+t+1}\,\&c.}$$

$$v=\frac{y}{x}.$$

THEO-

THEOREM III.

If $aP^p + \overline{b'Q + b''\dot{y}} \times P^{p-1} + \overline{c'Q^2 + c''Q\dot{y} + c'''\dot{y}^2} \times P^{p-2}$ &c. be $= 0$.

$$paP^{p-1} + \overline{p-1}.b'P^{p-2}Q + \overline{p-2}.c'P^{p-3}Q^2 \text{ \&c.}$$
$$+ \overline{p-1}.b''P^{p-2}\dot{y} + \overline{p-2}.c''P^{p-3}Q\dot{y} \text{ \&c.}$$
$$+ \overline{p-2}.c'''P^{p-3}\dot{y}^2 \text{ \&c.}$$
&c.

$$+ b'P^{p-1}y^{n-m} + 2c'P^{p-2}Qy^{n-m} + 3d'P^{p-3}Q^2y^{n-m} \text{ \&c.}$$
$$+ c''P^{p-2}y^{n-m}\dot{y} + 2d''P^{p-3}Qy^{n-m}\dot{y} \text{ \&c.}$$
$$+ d'''P^{p-3}y^{n-m}\dot{y}^2 \text{ \&c.}$$
&c.

will be $= 0$.

P being $= y^{m+1}\dot{x} + nxy^m\dot{y}$, $Q = y^{n+1}\dot{x} + mxy^n\dot{y}$.

By means of which equations $\frac{\dot{x}}{\dot{y}}$ may be exterminated, and the relation of x and y determined by a *particular* equation of the fluents respecting the first equation: and the same will be the *general* equation of the fluents respecting the second equation.

Moreover $\overline{n-m}.xy^{m+n} + hy^m = ky^n$ (derived from the equation $mnx\dot{y}^2 + \overline{m+n+1}.y\dot{y}\dot{x} + y^2\ddot{x} = 0$) will be an equation of the fluents respecting the first equation; the relation of the invariable quantities h and k (by which the equation of the fluents may be *adjusted)* being expressed by the equation

$$ak^p + \overline{b'h + b''} \times k^{p-1} + \overline{c'h^2 + c''h + c'''} \times k^{p-2} \text{ \&c.} = 0.$$

NOTE. The second equation is deduced from the first by taking the fluxions of the several terms, considering $\dot{y}$ as invariable, and dividing by $y^{m-1} \times \overline{mnx\dot{y}^2 + \overline{m+n+1}.y\dot{y}\dot{x} + y^2\ddot{x}}$ $(= \dot{P} = y^{m-n}\dot{Q})$.

THEO-

THEOREM IV.

$$\left.\begin{array}{l} \text{If } \overline{ay^{2m} + b'y^{m+n} + c'y^{2n}} \times y\dot{x} \\ + \overline{nay^{2m} + \frac{m+n}{2}b'y^{m+n} + mc'y^{2n}} \times x\dot{y} \\ + \frac{b''y^{m} + c''y^{n}}{2} \times \dot{y} \end{array}\right\} \text{be} = 0,$$

$$\text{or } \overline{nay^{2m} + \frac{m+n}{2}b'y^{m+n} + mc'y^{2n}} \times xy\dot{x} + \frac{b''y^{m} + c''y^{n}}{2} \times y\dot{x}$$
$$+ \overline{n^{2}ay^{2m} + mnb'y^{m+n} + m^{2}c'y^{2n}} \times x^{2}\dot{y} + \overline{nb''y^{m} + mc''y^{n}} \times x\dot{y}$$
$$+ c'''\dot{y} \text{ be} = 0,$$

$$\left.\begin{array}{l} \overline{m-n}|^{2} \times \overline{b'^{2} - 4ac'} \times x^{2}y^{2m+2n} \\ + 2.\overline{m-n} \times \overline{\overline{b'b'' - 2ac''}.y^{m} - \overline{b'c'' - 2b''c'}.y^{n}} \times xy^{m+n} \\ + \overline{b''y^{m} + c''y^{n}}|^{2} - 4c''' \times \overline{ay^{2m} + b'y^{m+n} + c'y^{2n}} \end{array}\right\} \text{will be} = 0.$$

NOTE. This theorem is derived from the preceding one, p being therein taken equal to 2.

And it is obſervable that, though, with reſpect to the firſt equation here, the third is the *general* equation of the fluents; yet, with reſpect to the ſecond equation here, and the equation

$$a \times \overline{y^{m+1}\dot{x} + nxy^{m}\dot{y}}|^{2} + b'.\overline{\overline{y^{n+1}\dot{x} + mxy^{n}\dot{y}} + b''\dot{y}} \times \overline{y^{m+1}\dot{x} + nxy^{m}\dot{y}}$$
$$+ c'.\overline{y^{n+1}\dot{x} + mxy^{n}\dot{y}}|^{2} + c''\dot{y} \times \overline{y^{n+1}\dot{x} + mxy^{n}\dot{y}} + c'''\dot{y}^{2} = 0,$$

(from which this theorem is derived,) the ſaid third equation is only a *particular* equation of the fluents, unleſs m or n be $= 0$: and then it is *general*, as well with reſpect to the ſecond equation as to the firſt.

THEO-

THEOREM V.

If $\overline{d+by+ay^2}\times\dot{x}+\frac{1}{2}\times\overline{e-bx-cy-2axy}\times\dot{y}$ be $=0$,
or $\overline{f+cx+ax^2}\times\dot{y}+\frac{1}{2}\times\overline{e-bx-cy-2axy}\times\dot{x}=0$,

or $\frac{\dot{x}}{\sqrt{f+cx+ax^2}}=\frac{\dot{y}}{\sqrt{d+by+ay^2}}$,

$e^2-4df-\overline{2be+4cd}.x-\overline{2ce+4bf}.y-\overline{2bc+4ae}.xy$
$+\overline{b^2-4ad}.x^2+\overline{c^2-4af}.y^2$ will be $=0$.

Note. This theorem is derived from the third, p being therein taken equal to 2, $m=0$, $n=-1$, and b', b'', c', c'', c''' equal to b, $-c$, d, e, f respectively.

And it is observable that, though, with respect to the three fluxional equations here, the fourth equation is the *general* equation of the fluents; yet, with respect to the equation $a\times\overline{y\dot{x}-x\dot{y}}|^2+\overline{b\dot{x}-c\dot{y}}\times\overline{y\dot{x}-x\dot{y}}+d\dot{x}^2+e\dot{x}\dot{y}+f\dot{y}^2=0$, (from which the theorem is derived), the said fourth equation is only a *particular* equation of the fluents.

THEOREM VI.

If $a\times\overline{y^{n+1}\dot{x}+nxy^n\dot{y}}+b\times\overline{y^{m+1}\dot{x}+mxy^m\dot{y}}+c\dot{y}$ be $=0$,
$\overline{n-m}.xy^{m+n}+hy^m$ will be $=ky^n$;

the relation of the invariable quantities h and k being expressed by the equation $ak+bh+c=0$.

Note. This theorem is a particular case of the third; p being therein taken equal to 1, and b', b'' equal to b, c respectively.

THEO-

THEOREM VII.

If $pa\mathrm{P}^{p-1} + \overline{p-1}.b'\mathrm{P}^{p-2}\mathrm{Q} + \overline{p-2}.c'\mathrm{P}^{p-3}\mathrm{Q}^2$ &c.

$+ \overline{p-1}.b''\mathrm{P}^{p-2}\dot{y} + \overline{p-2}.c''\mathrm{P}^{p-3}\mathrm{Q}\dot{y}$ &c.

$+ \overline{p-2}.c'''\mathrm{P}^{p-3}\dot{y}^2$ &c.

&c.

$+ b'\mathrm{P}^{p-1}y^{n-m} + 2c'\mathrm{P}^{p-2}\mathrm{Q}y^{n-m} + 3d'\mathrm{P}^{p-3}\mathrm{Q}^2y^{n-m}$ &c.

$+ c''\mathrm{P}^{p-2}y^{n-m}\dot{y} + 2d''\mathrm{P}^{p-3}\mathrm{Q}y^{n-m}\dot{y}$ &c.

$+ d'''\mathrm{P}^{p-3}y^{n-m}\dot{y}^2$ &c.

&c.

be $=$ F,

$a\mathrm{P}^p + \overline{b'\mathrm{Q}+b''\dot{y}} \times \mathrm{P}^{p-1} + \overline{c'\mathrm{Q}^2+c''\mathrm{Q}\dot{y}+c'''\dot{y}^2} \times \mathrm{P}^{p-2}$ &c.

will be $=$ fl. F$\dot{\mathrm{P}}$.

P, Q, and $\dot{\mathrm{P}}$ ($= y^{n-m}\dot{\mathrm{Q}}$) being as in theorem III. and $\dot{y}$, in computing the value of fl. F$\dot{\mathrm{P}}$, being considered as invariable: which value will be assignable when F is a proper function of P and $\dot{y}$; and then, by means of these equations, $\frac{\dot{x}}{\dot{y}}$ may be exterminated, and the *general* relation of x and y determined; as well as when F is $= 0$.

Example. If $\frac{\dot{x}^2}{\dot{y}}$ be $= hy \times \overline{y\dot{x} - x\dot{y}}$; to apply the theorem, m may be taken $= -1$, $n = 0$, $p = 2$, $c' = \frac{1}{2}h$, and a, b', b'', c'', c''', &c. each $= 0$: then P being $= \dot{x}$, $\dot{\mathrm{P}} = \ddot{x}$, and $\mathrm{Q} = y\dot{x} - x\dot{y}$;

$$\mathrm{F}\dot{\mathrm{P}} = \frac{\dot{x}^2\ddot{x}}{\dot{y}} \text{ will be } = hy.\overline{y\dot{x} - x\dot{y}}.\ddot{x},$$

$$\text{fl. } \mathrm{F}\dot{\mathrm{P}} = \mathrm{K}\dot{y}^2 + \frac{1}{3}\frac{\dot{x}^3}{\dot{y}} = \frac{1}{2}h.\overline{y\dot{x} - x\dot{y}}\big|^2;$$

and, consequently, $\overline{3x^2 - k}\big|^2 = h \times \overline{2x^3y^3 - 6kxy^3 + hky^6}$, k being the invariable quantity whereby the equation (of the fluents) may be *adjusted.*

THEO-

THEOREM VIII.

If $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{b+mx+anx^2}}$ be $= \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{bn+my+ay^2}}$,

the *general* equation of the fluents will be

$$axy - cnx - cy \pm 2\sqrt{cm + c^2n + ab} \times x^{\frac{1}{2}}y^{\frac{1}{2}} + b = 0.$$

THEOREM IX.

If $a \times y^{m+1}\ddot{x} + \overline{n+q}.y^m\dot{x}\dot{y} + \overline{n-1}.qxy^{m-1}\dot{y}^2$
$+ b \times y^{n+1}\ddot{x} + \overline{m+q}.y^n\dot{x}\dot{y} + \overline{m-1}.qxy^{n-1}\dot{y}^2$
$+ c\dot{y}^2$ be $= 0$.

the *general* equation of the fluents will be

$$\overline{n-m}.xy^{m+n+q} + gy^{m+n} + hy^{m+q+1} = ky^{n+q+1},$$

$\dot{y}$ being invariable, and the relation of the invariable quantities h and k (by which that equation may be *adjusted*) being expressed by the equation

$$a.\overline{q-m+1}.k + b.\overline{q-n+1}.h + c = 0.$$

THEOREM X.

If $\dot{v} + v^2\dot{y} + \frac{a.\overline{n+q}.y^m + b.\overline{m+q}.y^n}{ay^{m+1} + by^{n+1}}v\dot{y} + \frac{a.\overline{n-1}.qy^{m-1} + b.\overline{m-1}.qy^{n-1}}{ay^{m+1} + by^{n+1}}\dot{y}$

be $= 0$.

the *general* equation of the fluents will be

$$v = \frac{qy^{m+n-1} - \overline{n-1}.by^{m+q} - \overline{m-1}.ky^{n+q}}{by^{m+q+1} + ky^{n+q+1} - y^{m+n}},$$

$$a.\overline{q-m+1}.k \text{ being } = b.\overline{q-n+1}.h.$$

NOTE. This theorem is derived from the preceding, by substituting v for $\frac{\dot{x}}{x\dot{y}}$.

THEOREM XI.

$$Q\dot{x}^n = Ay\dot{x}^n + Bx\dot{y}\dot{x}^{n-1} + Cx^2\ddot{y}\dot{x}^{n-2} + Dx^3\dddot{y}\dot{x}^{n-3}\ (n+1).$$

the coefficient of the last term being $=1$, Q any function of x and y, and $\dot{x}$ invariable.

$$y = K'x^{r'} + K''x^{r''} + K'''x^{r'''}\ (n) + \overset{(n)}{F}x^{\overset{(n)}{r}}.$$

r', r'', r''', &c. are the roots (r) of the equation

$$A+B.r+C.r.\overline{r-1}+D.r.\overline{r-1}.\overline{r-2}\ldots\ldots+r.\overline{r-1}.\overline{r-2}(n)=0.$$

$F'=$ fl. $x^{-r'-1}Q\dot{x}$, $F''=$ fl. $x^{r'-r''-1}F'\dot{x}$, $F'''=$ fl. $x^{r''-r'''-1}F''\dot{x}$, &c.

Note I. In deducing the theorem by repeatedly taking the fluents, after multiplying by $x^{-r'-1}$, $x^{r'-r''-1}$, $x^{r''-r'''-1}$, &c. successively, the terms $G'\dot{x}^{n-1}$, $\frac{G'x^{r'-r''}\dot{x}^{n-2}}{r'-r''} + G''\dot{x}^{n-2}$, $\frac{G'x^{r'-r'''}\dot{x}^{n-3}}{\overline{r'-r''}.\overline{r'-r'''}} + \frac{G''x^{r''-r'''}\dot{x}^{n-3}}{r''-r'''} + G'''\dot{x}^{n-3}$, &c. successively arise; the roots r', r'', r''', &c. being supposed unequal. But when r'' is $=r'$, $\overline{L'z+K''}\times x^{r'-\overset{(n)}{r}}$ arises (by such operation) instead of $K'x^{r'-\overset{(n)}{r}}+K''x^{r''-\overset{(n)}{r}}\left(=\frac{G'x^{r'-\overset{(n)}{r}}}{\overline{r'-r''}.\overline{r'-r'''}(n-1)}+\frac{G''x^{r''-\overset{(n)}{r}}}{\overline{r''-r'''}.\overline{r''-r^{iv}}(n-2)}\right)$; $G'\times$ fl. $x^{r'-r''-1}\dot{x}$ being then $=G'z$ instead of $\frac{G'x^{r'-r''}}{r'-r''}$, z being $=$ Log. x: therefore, when r' is $=r''=r'''\ldots\ldots=\overset{(m)}{r}$, $\overline{L'z^{m-1}+L''z^{m-2}+L'''z^{m-3}\,(m-1)+\overset{(m)}{K}}\times x^{r'}$ must be taken (in the value of y) instead of $K'x^{r'}+K''x^{r''}+K'''x^{r'''}\ (m)$.

Note II. If r be $=a+b\sqrt{-1}$, x^r will be $=x^a N^{bz\sqrt{-1}}$, the value whereof is shewn in the Scholium at the end of the Tables.

THEO-

THEOREM XII.

If $A + Bv + Cv^2 + Dv^3 + \overline{C+D}.\frac{x\dot{v}}{\dot{x}} + 3D\frac{xv\dot{v}}{\dot{x}} + D\frac{x^2\ddot{v}}{\dot{x}^2}$ be $= 0$,

v will be $= \frac{x\dot{y}}{y\dot{x}}$;

$\dot{x}$ being invariable, and $y = K'x^{r'} + K''x^{r''} + K'''x^{r'''}$;
where r', r'', r''' are the roots (r) of the equation
$A + Br + Cr^2 + Dr^3 = 0$.

THEOREM XIII.

If $\overline{A+Bv+Cv^2+Dv^3} \times \dot{v} + Cw\dot{v} + 3Dwv\dot{v} + Dw\dot{w}$ be $= 0$,

v will be $= \frac{x\dot{y}}{y\dot{x}}$, and $w = \frac{x\dot{v}}{\dot{x}}$;

y being as in the preceding theorem.

THEOREM XIV.

If $A\dot{v} + Bv\dot{v} + Cz\dot{v} + Dzv\dot{v} - Dv^2\dot{z} + Dz\dot{z}$ be $= 0$,

v will be $= \frac{x\dot{y}}{y\dot{x}}$, and $z = \frac{x\dot{y}}{y\dot{x}} + \frac{x^2\ddot{y}}{y\dot{x}^2}$;

$\dot{x}$ and y being as in theorem XII.

THEOREM XV.

If $A\dot{v} + \overline{B+C+D}.v\dot{v} + \overline{C+D}.z\dot{v} + Dv\dot{z} - Dv^2\dot{z} + Dz\dot{z}$ be $= 0$,

v will be $= \frac{x\dot{y}}{y\dot{x}}$, and $z = \frac{x^2\ddot{y}}{y\dot{x}^2}$;

$\dot{x}$ and y being as in theorem XII.

THEOREM XVI.

If, $\dot{z}$ being $=\sqrt{\dot{x}^2+\dot{y}^2}$ and $\dot{x}$ invariable, $\dot{z}^2+y\ddot{y}$ be $=a\dot{x}\dot{z}$;

$$v \text{ will be } = \frac{k}{1-aw},\ v-ay=k,$$

$$y\dot{z}-ay\dot{x}=k\dot{x},\ \text{and } x = \text{fl.}\ \frac{y\dot{y}}{\sqrt{k+2aky+\overline{a^2-1}.y^2}}:$$

where x is considered as the abscissa of a curve (z); y as the correspondent ordinate, at right angles to the base upon which x is measured; v as the normal to the curve, terminated by that base; and w as the sine (to the radius 1) of the angle made by the said normal and base; k being an invariable quantity serving to *adjust* the equation of the fluents, with another such quantity that must be added upon taking the value of fl. $\frac{y\dot{y}}{\sqrt{k+2aky+\overline{a^2-1}.y^2}}$.

NOTE. y is $=vw=\frac{v\dot{x}}{\dot{z}}$, $\dot{y}=\dot{v}w+v\dot{w}$, $\ddot{y}=-\frac{w.\overline{\dot{v}w+v\dot{w}}}{w.\overline{1-w^2}}$,

$$\dot{x}=w\dot{z}=\frac{\overline{w.\dot{v}w+v\dot{w}}}{\sqrt{1-w^2}},\ \text{and } \dot{z}=\frac{\dot{v}w+v\dot{w}}{\sqrt{1-w^2}}:$$

and, by properly substituting as many of these values as may be requisite, the equations of the fluents may be readily deduced from some other fluxional equations.

TABLE

TABLE XVI.

CONTAINING

THEOREMS

FOR THE

CALCULATION of FLUENTS.

THEOREM I.

$$\dot{F} = N^{rx} x^{p} \dot{x}.$$

$$F = K + \frac{N^{rx}}{r} \times \overline{x^{p} - \frac{p}{r} x^{p-1} + \frac{p \cdot p-1}{r^2} x^{p-2} - \frac{p \cdot p-1 \cdot p-2}{r^3} x^{p-3} + \&c.}$$

N being (everywhere in this Table) the number whoſe hyp. log. is 1.

NOTE. $p \times \text{fl.}\ N^{rx} x^{p-1} \dot{x} + r \times \text{fl.}\ N^{rx} x^{p} \dot{x}$ is $= K + N^{rx} x^{p}$.

THEOREM II.

$$\ddot{F} = ryN^{rx} \dot{x}^2 - \frac{\ddot{y} N^{rx}}{r} \quad \dot{x} \text{ invariable.}$$

$$\dot{F} = yN^{rx} \dot{x} - \frac{\dot{y} N^{rx}}{r} + K\dot{x}.$$

The fluent of $\dot{F} N^{-2rx} = -\frac{y N^{-rx}}{r} - \frac{K N^{-2rx}}{2r}$

THEO-

THEOREM III.

$$sy^{s-1}\dot{y} + qy^{s}\dot{z} + py^{s}x^{-1}\dot{x} = \dot{P}.$$

P and z any functions of x or y.

$$y^{s} = x^{-p}N^{-qz} \times \overline{K + \text{fl. } x^{p}N^{qz}\dot{P}}.$$

Example 1. If z be $= x^{b}$, and $P = bx^{m} + cx^{n}$ &c. $= bz^{\frac{m}{b}} + cz^{\frac{n}{b}}$ &c.
y^{s} will be $= x^{-p}N^{-qz} \times \overline{K + \frac{bm}{b} \times \text{fl. } N^{qz}z^{e}\dot{z} + \frac{bn}{b} \times \text{fl. } N^{qz}z^{f}\dot{z} \text{ \&c.}}$

$$e \text{ being} = \frac{m + p - b}{b},\ f = \frac{n + p - b}{b}.$$

Example 2. If $ax + by$ be $= Py + Q$, P and Q being functions of $\frac{\dot{x}}{\dot{y}}$ without x or y being concerned therein;

$$a\dot{x} + b\dot{y} - P\dot{y} = \overline{ar + b - P} \times \dot{y} \text{ will be} = \dot{P}y + \dot{Q},$$

$$\dot{y} - y\dot{w} = \frac{\dot{Q}}{v}, \text{ and } y = N^{w} \times \overline{K + \text{fl. } \frac{\dot{Q}}{vN^{w}}}.$$

$$r \text{ being} = \frac{\dot{x}}{\dot{y}},\ v = ar + b - P, \text{ and } w = \text{fl. } \frac{\dot{P}}{v}.$$

By which means r may be exterminated, and the equation of the fluents obtained.

Example 3. If $\ddot{w} + p\dot{w}v^{-1}\dot{v}$ be $= b\dot{H}\dot{x}$, $\dot{x}$ being invariable;
$\dot{w}\,(=y)$ will be $= v^{-p} \times \overline{K\dot{x} + \text{fl. } bv^{p}\dot{H}\dot{x}}$, and $w = \text{fl. } \frac{K\dot{x} + \text{fl. } bv^{p}\dot{H}\dot{x}}{v^{p}}$.

$$\text{And if } pv^{-1}\dot{v} \text{ be} = -\frac{2a^{2}x^{-1}\dot{x}}{a^{2} + x^{2}}, \text{ and } b\dot{H} = -\frac{a\dot{x}}{a^{2} + x^{2}};$$

$$v^{p} \text{ will be} = \frac{a^{2} + x^{2}}{x^{2}},$$

and $w = k'' + k'x + \frac{1}{2}a\,\text{Log.}\frac{a^{2} + x^{2}}{a^{2}} - k' \times$ *Circ. Arc*, rad. a, tang. x.

REMARK.

REMARK. Sometimes, when the value of fl. $x^p N^{qz} \dot{P}$ cannot be immediately obtained, it may be of uſe to ſuppoſe a new variable quantity $u = K + \text{fl.}\ x^p N^{qz} \dot{P} = y x^p N^{qz}$; and, by ſuch means, to exterminate x (or y): the relation of u and y (or x) being afterwards aſſignable, as in the following examples.

Example 4. $\dot{y} + qy\dot{z}$ being $= rN^{mz} y^n \dot{y}$,

y (by the theorem) will be $= N^{-qz} \times \overline{K + \text{fl.}\ rN^{mz+qz} y^n \dot{y}}$.

Now, ſuppoſing $u = N^{qz} y = K + \text{fl.}\ rN^{mz+qz} y^n \dot{y}$, N^{mz+qz}

$$\text{will be} = \overline{\tfrac{u}{y}}\Big|^{\frac{m+q}{q}},\ rN^{mz+qz} y^n \dot{y} = r u^{\frac{m+q}{q}} y^{\frac{nq-q-m}{q}} \dot{y} = \dot{u},$$

$$\text{and } r \times \text{fl.}\ y^{\frac{nq-q-m}{q}} \dot{y} = \text{fl.}\ u^{-\frac{m+q}{q}} \dot{u}.$$

Example 5. If $\dot{y} - y\frac{\dot{x}}{x}$ be $= bx\dot{x} + \frac{cy^2\dot{y}}{x}$,

y (by the theorem) will be $= x \times \overline{K + \text{fl.}\ b\dot{x} + \text{fl.}\ c\frac{y^2\dot{y}}{x^2}}$.

$$\text{Suppoſe } u = \frac{y}{x} = K + \text{fl.}\ b\dot{x} + \text{fl.}\ c\frac{y^2\dot{y}}{x^2}:$$

$$\text{then, } x \text{ being} = \frac{y}{u},\ \dot{x} = \frac{\dot{y}}{u} - \frac{y\dot{u}}{u^2},$$

$$\text{and } b\dot{x} + c\frac{y^2\dot{y}}{x^2} = \frac{b\dot{y}}{u} - \frac{by\dot{u}}{u^2} + cu^2\dot{y} = \dot{u},$$

$$\dot{y} - y.\frac{u^{-1}\dot{u}}{1 + \frac{c}{b}u^3} \text{ will be} = \frac{\frac{1}{b}u\dot{u}}{1 + \frac{c}{b}u^3}:$$

whence, by applying the theorem a ſecond time,

$$y \text{ is found} = \frac{u}{\overline{1 + \frac{c}{b}u^3}\big|^{\frac{1}{3}}} \times \overline{K + \tfrac{1}{b}\ \text{fl.}\ \frac{\dot{u}}{\overline{1 + \frac{c}{b}u^3}\big|^{\frac{2}{3}}}};$$

$\frac{u^{-1}\dot{u}}{1 + \frac{c}{b}u^3}$ being $= \frac{\dot{w}}{w}$, when w is $= \frac{u}{\overline{1 + \frac{c}{b}u^3}\big|^{\frac{1}{3}}}$, by Theor. I. and III. Tab. I.

THEO-

THEOREM IV.

$$Q\dot{x}^n = Ay\dot{x}^n + B\dot{y}\dot{x}^{n-1} + C\ddot{y}\dot{x}^{n-2} + D\dddot{y}\dot{x}^{n-3}\ (n+1);$$

the coefficient of the last term being $= 1$, Q any function of x and y, and $\dot{x}$ invariable.

$$y = K'N^{r'x} + K''N^{r''x} + K'''N^{r'''x}\ (n) + \overset{(n)}{F}N^{\overset{(n)}{r}x}.$$

r', r'', r''', &c. the roots (r) of the equation $A + Br + Cr^2 + Dr^3 \ldots r^n = 0$.

$$F' = \text{fl. } QN^{-r'x}\dot{x},\quad F'' = \text{fl. } N^{r'x-r''x}F'\dot{x},\quad F''' = \text{fl. } N^{r''x-r'''x}F''\dot{x},$$

$$F^{iv} = \text{fl. } N^{r'''x-r^{iv}x}F'''\dot{x},\ \text{\&c.}$$

NOTE. In deducing the theorem by repeatedly taking the fluents, after multiplying by $N^{-r'x}$, $N^{r'x-r''x}$, $N^{r''x-r'''x}$, &c. successively, the terms $G'\dot{x}^{n-1}$, $\dfrac{G'N^{r'x-r''x}\dot{x}^{n-2}}{r'-r''} + G''\dot{x}^{n-2}$, $\dfrac{G'N^{r'x-r'''x}\dot{x}^{n-3}}{r'-r''.r'-r'''} + \dfrac{G''N^{r''x-r'''x}\dot{x}^{n-3}}{r''-r'''} + G'''\dot{x}^{n-3}$, &c. successively arise; the roots r', r'', r''', &c. being supposed unequal. But when r'' is $= r'$, $\overline{L'x + K''} \times N^{r'x - \overset{(n)}{r}x}$ arises (by such operation) instead of $K'N^{r'x-\overset{(n)}{r}x} + K''N^{r''x-\overset{(n)}{r}x}$ $\left(= \dfrac{G'N^{r'x-\overset{(n)}{r}x}}{\overline{r'-r''}.\overline{r'-r'''}\,(n-1)} + \dfrac{G''N^{r''x-\overset{(n)}{r}x}}{\overline{r''-r'''}.\overline{r''-r^{iv}}\,(n-2)}\right)$; $G' \times \text{fl. } N^{r'x-r''x}\dot{x}$ being then $= G'x$ instead of $\dfrac{G'N^{r'x-r''x}}{r'-r''}$: therefore, when r' is $= r'' = r''' \ldots\ldots = \overset{(m)}{r}$, $\overline{L'x^{m-1} + L''x^{m-2} + L'''x^{m-3}\ (m-1) + \overset{(m)}{K}} \times N^{r'x}$ must be taken (in the value of y) instead of $K'N^{r'x} + K''N^{r''x} + K'''N^{r'''x}\ (m)$.

THEO-

THEOREM V.

If $A + Bv + Cv^2 + Dv^3 + C\frac{\dot{v}}{\dot{x}} + 3D\frac{v\dot{v}}{\dot{x}} + D\frac{\ddot{v}}{\dot{x}^2}$ be $= 0$.

v will be $= \frac{\dot{y}}{y\dot{x}}$;

$\dot{x}$ being invariable, and $y = K'N^{r'x} + K''N^{r''x} + K'''N^{r'''x}$; where r', r'', r''' are the roots (r) of the equation

$$A + Br + Cr^2 + Dr^3 = 0.$$

THEOREM VI.

If $Ax^3 + \overline{Bx^2 - Cx + 2D}.w + \overline{Cx - 3D}.w^2 + Dw^3 + \overline{Cx - 2D}.\frac{x\dot{w}}{\dot{x}} + 3D\frac{xw\dot{w}}{\dot{x}} + D\frac{x^2\ddot{w}}{\dot{x}^2}$ be $= 0$,

w will be $= \frac{x\dot{y}}{y\dot{x}}$; y being as in the preceding theorem.

THEOREM VII.

$$\ddot{y} - c^2x^{4m}y\dot{x}^2 = \dot{v} + v^2\dot{x} - c^2x^{4m}\dot{x} = 0. \quad v = \frac{\dot{y}}{y\dot{x}}$$

$$y = \begin{cases} \overline{K'N^{rz} + K''N^{-rz}} \times \overline{x^{-m} + \frac{Cx^{-5m-2}}{\overline{2c.2m+1}|^2} + \frac{Ex^{-9m-4}}{\overline{2c.2m+1}|^4}\ \&c.} \\ \overline{+K'N^{rz} - K''N^{-rz}} \times \overline{\frac{Bx^{-3m-1}}{2c.2m+1} + \frac{Dx^{-7m-3}}{\overline{2c.2m+1}|^3} + \frac{Fx^{-11m-5}}{\overline{2c.2m+1}|^5}\ \&c.} \end{cases}$$

$$r = \frac{c}{2m+1}. \quad z = x^{2m+1}.$$

$B = -m.\overline{m+1}$, $C = -\frac{1}{2}.\overline{3m+1}.\overline{3m+2} \times B$, $D = -\frac{1}{3}.\overline{5m+2}.\overline{5m+3} \times C$, $E = -\frac{1}{4}.\overline{7m+3}.\overline{7m+4} \times D$, $F = -\frac{1}{5}.\overline{9m+4}.\overline{9m+5} \times E$, &c.

NOTE. Both the ſeries will terminate, if, n being any integer whatever, m be $= -\frac{n}{2n \pm 1}$.

And then, from the above value of y, that of v $(= \frac{\dot{y}}{y\dot{x}})$ may be eaſily obtained.

u THEO-

THEOREM VIII.

$F' = \text{fl.}\, N^{rm} x^{p} \dot{z}$, $F'' = \text{fl.}\, N^{rm} x^{p-1} y \dot{z}$, $F''' = \text{fl.}\, N^{rm} x^{p-2} y^{2} \dot{z}$, &c.

$G' = \text{fl.}\, N^{rm} x^{p-1} \dot{z}$, $G'' = \text{fl.}\, N^{rm} x^{p-2} y \dot{z}$, $G''' = \text{fl.}\, N^{rm} x^{p-3} y^{2} \dot{z}$, &c.

$H' = \text{fl.}\, N^{rm} x^{p-2} \dot{z}$, $H'' = \text{fl.}\, N^{rm} x^{p-3} y \dot{z}$, $H''' = \text{fl.}\, N^{rm} x^{p-4} y^{2} \dot{z}$, &c.

&c.

$$y = \sqrt{b + 2cx + dx^{2}}. \quad z = \text{fl.}\, \frac{a\dot{x}}{\sqrt{b + 2cx + dx^{2}}}.$$

$$F' = \frac{x^{p} N^{rm}}{r} - \frac{pF''}{ar}.$$

$$F'' = \frac{x^{p-1} y N^{rm}}{r} - \frac{p-1.F'''}{ar} - \frac{dF'}{ar} - \frac{cG'}{ar}.$$

$$F''' = \frac{x^{p-2} y^{2} N^{rm}}{r} - \frac{p-2.F^{iv}}{ar} - \frac{2dF''}{ar} - \frac{2cG''}{ar}.$$

$$F^{iv} = \frac{x^{p-3} y^{3} N^{rm}}{r} - \frac{p-3.F^{v}}{ar} - \frac{3dF'''}{ar} - \frac{3cG'''}{ar}.$$

$(p+1)$.

$$G' = \frac{x^{p-1} N^{rm}}{r} - \frac{p-1.G''}{ar}.$$

$$G'' = \frac{x^{p-2} y N^{rm}}{r} - \frac{p-2.G'''}{ar} - \frac{dG'}{ar} - \frac{cH'}{ar}.$$

(p).

$$H' = \frac{x^{p-2} N^{rm}}{r} - \frac{p-2.H''}{ar}.$$

$$H'' = \frac{x^{p-3} y N^{rm}}{r} - \frac{p-3.H'''}{ar} - \frac{dH'}{ar} - \frac{cI'}{ar}.$$

$(p-1)$

&c.

Hence, when p is a positive integer, all the fluents F', F'', $(p+1)$; G', G'', (p); H', H'', $(p-1)$; &c. may be found.

THEO-

THEOREM IX.

The fluent of $\frac{x^{n}\dot{x}}{\sqrt{1-x^{2}}}$ is $= \frac{2^{-n}}{\overline{-1}|^{\frac{1}{2}n}} \times \text{fl.}\ \overline{N^{z\sqrt{-1}} - N^{-z\sqrt{-1}}}|^{n} \times \dot{z}$,

which is $= K + \frac{2^{1-n}}{\overline{-1}|^{\frac{1}{2}n}} \times \overline{\frac{\overset{(n)}{s}}{n} - n.\frac{\overset{(n-2)}{s}}{n-2} + \frac{n.\overline{n-1}}{2}.\frac{\overset{(n-4)}{s}}{n-4}\left(\tfrac{1}{2}n\right)} \pm {}^{*}\tfrac{1}{2}Mz$,

or $= K - \frac{2^{1-n}}{\overline{-1}|^{\frac{n-1}{2}}} \times \overline{\frac{\overset{(n)}{c}}{n} - n.\frac{\overset{(n-2)}{c}}{n-2} + \frac{n.\overline{n-1}}{2}.\frac{\overset{(n-4)}{c}}{n-4}\left(\frac{n+1}{2}\right)}$,

according as n is an even or an odd poſitive number.

$M = \frac{n.\overline{n-1}.\overline{n-2}\left(\frac{1}{2}n\right)}{1.2.3\left(\frac{1}{2}n\right)}$. $z = \text{fl.}\frac{\dot{x}}{\sqrt{1-x^{2}}} =$ Circ. Arc, rad. 1, ſine x.

$\overset{(n)}{s}$, $\overset{(n-2)}{s}$, $\overset{(n-4)}{s}$, &c. $\overset{(n)}{c}$, $\overset{(n-2)}{c}$, $\overset{(n-4)}{c}$, &c. the ſines and coſines of nz, $\overline{n-2}.z$, $\overline{n-4}.z$, &c. reſpectively.

* + or − according as $\frac{1}{2}n$ is even or odd.

SCHOLIUM.

In computations wherein exponentials are concerned, it may ſometimes be neceſſary to obſerve, that,

$$a \times \frac{N^{\frac{rz\sqrt{-1}}{a}} - N^{-\frac{rz\sqrt{-1}}{a}}}{2\sqrt{-1}} \text{ and } a \times \frac{N^{\frac{rz\sqrt{-1}}{a}} + N^{-\frac{rz\sqrt{-1}}{a}}}{2}$$

denoting the ſine and coſine of rz reſpectively, z being an arc of the circle whoſe radius is $= a$; if that ſine and coſine be alſo reſpectively denoted by s and c, $N^{\frac{rz\sqrt{-1}}{a}}$ will be $= \frac{c+s\sqrt{-1}}{a}$, and $N^{-\frac{rz\sqrt{-1}}{a}} = \frac{c-s\sqrt{-1}}{a}$.

☞ All the *Logarithms* mentioned in theſe Tables, and likewiſe in the Memoirs, are of the *hyperbolic* kind.

END OF THE APPENDIX.