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MATHEMATICAL

PROBLEMS AND EXAMPLES,

ARRANGED ACCORDING TO SUBJECTS,

FROM THE

SENATE-HOUSE EXAMINATION PAPERS,

1821 TO 1836 INCLUSIVE.

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PREFACE.

It is scarce necessary to observe, that during the student's preparation for mathematical honors in the University of Cambridge, he is obliged to have frequent recourse to papers of examination questions, in order to satisfy himself and his instructors as to the degree of proficiency he may have attained.

With the same object in view, it has now become the universal practice of those engaged in mathematical tuition, to draw up papers for the pupils under their charge; and in doing this, it is most important that the questions should be carefully chosen, and as much as possible of the same character as those usually proposed to students when they ultimately present themselves candidates for mathematical honors.

The papers of questions which have actually been set from time to time, at the Senate-House examinations, have long been in use among the members of the university for the above purposes; it being justly argued, that the importance of the object for which they were originally formed, must ensure more judgment in the selection, and accuracy in the wording, than could be expected in those more hastily drawn up for the usual preparatory examinations.

The Senate-House papers, however, so applied, labour under one great objection; namely, that as they have been formed for the examination of students, supposed to have a competent knowledge of the whole mathematical course pursued at the university, they contain questions in all the subjects, mingled together indiscriminately, so that a student endeavouring by means of them to examine himself in any particular branch, will find that a majority of those before him are foreign to his immediate purpose, or even depending upon investigations, of which, as yet, he may be entirely ignorant.

To remedy this inconvenience, and to render so valuable a collection of problems and examples more generally available, they are in the present volume arranged in sections, corresponding to the various branches of science whence they are derived. These sections are placed, as much as possible, in the order in which the subjects naturally follow each other; and that it may be apparent what questions in each branch have been proposed in any one year, the dates are printed in the margin.

Thus, at page 24, we see that the number of questions set in 1834, in "Arithmetic and Algebra, not including the general nature of Equations," was eighteen; and at page 60, that the number set in "Statics," in the same year, was sixteen.

Such are the advantages it is hoped the present volume will be found to possess for the student; while the man of science, who devotes himself particularly to one or two branches of mathematical research, will be enabled by it to form a pretty just estimate of the attention applied at the university to his own favourite study, and the degree of proficiency which our most promising students are expected to attain in it. In this point of view, some description of the Senate-House examination may be useful to the reader.

The students having resided the usual time in the university, and declaring themselves, through the tutors of their respective colleges, candidates for mathematical honors, are arranged in four classes, according to the degree of proficiency they are *presumed* to have attained. This arrangement, however, does not in any way influence the rank or class they may ultimately acquire; as the examination is, for the most part, the same for all the four classes; and the way in which they acquit themselves in it, is the only circumstance which determines their places in the list of mathematical honors, afterwards inserted in the public prints.

The examination is conducted entirely by means of printed papers, prepared by the examiners; the particular questions to be asked being quite unknown to the students previous to their assembling at the appointed time to write their answers to them. When they are seated in the place appropriated to the examination, the examiner, whose turn it is to preside, delivers to each a copy of the questions he is to give his answers to, and generally from two to three hours are devoted to answering each paper. When the time allowed for answering the questions is past, each candidate delivers to the examiner the produce of his labours, leaving it to him and his fellow-examiners to affix to each a certain value dependent on its merit, which value is usually expressed in numbers called *marks.*

The number of marks which each question, if correctly answered, entitles the candidate to, has been previously determined upon; and this number, in valuing the answers, is diminished, more or less, according to the degree of inaccuracy any particular one may betray. After a certain time intended for relaxation, the candidates return to the appointed place, when another paper is presented to them, and so on to the end of the examination, which on the whole employs them about twenty-eight hours.

The examiners having valued the answers, and summed up the number of marks gained by each candidate, arrange their names in the order of these numbers, and the list so formed is published in the Cambridge Calendar and in the newspapers.

Ten papers of questions in succession are usually presented to each candidate, of which, four consist of original problems and questions, answers to which could not be directly obtained from any published books; these four being specially termed *problem papers.* The remaining six contain questions, the solutions of which the candidate is supposed to have acquired during his previous reading; and these, in contradistinction, are called *book-work papers.*

The great responsibility of conducting such an examination, when so much depends upon its result, must at once be apparent to any one acquainted with the mathematical sciences; and the vast range which these sciences have at present attained, renders great care necessary in selecting the questions, so as to give no undue preference to any particular branch.

The limited time allowed scarce admits of a full examination in each of the subjects, and still, limited as it is, the fatigue and anxiety endured by those meritorious students who aim at the highest honors is found in many instances to be so great, that any extension of it would probably produce evils which it is most important to avoid; constitutional strength would have too great a part in the contest, and might gain for mediocrity a reward intended only for superior acquirements. Our safeguard must be in judiciously choosing such questions as embrace as much as possible of the very essentials of science, and in so putting them as to exhibit clearly the intention of the examiner, and yet to give some exercise to the ingenuity of the candidates.

There was a time when the scantiness of our mathematical resources, and the difficulty and uncertainty with which our investigations could be applied to the problems presented to our contemplation by the operations of nature, made it necessary to illustrate our course very fully by problems entirely of a hypothetical kind: thus, as examples of the theory of central forces, our books were filled with investigations respecting laws of force, which, to say the least, are not known to exist in nature; and although the conclusions deduced might be perfectly satisfactory on the assumed hypothesis, they could not be considered of the slightest use in a philosophical point of view. They were mere exercises, by which it was intended to ensure the student's right understanding of the propositions they were meant to exemplify.

In this way they may still be valuable, provided they follow sufficiently directly from the general theory to enable the student himself to deduce them from it; but if they are such as to require his particular and undivided attention in themselves, or are so far removed from the general course of his studies, that he must obtain his competency in these particular cases by careful perusal of the works of others, they lose their value as examples or illustrations, as far as he is concerned; and therefore, if they are unconnected with the operations of nature, the time devoted to them must be considered thrown away.

It may be urged that they have a use simply as exercises of his reasoning faculties; but, in the present state of science, so many problems and illustrations may be presented to him, which, besides this use, will give him considerable insight into the connexion of mathematical science with the operations of nature, that surely the latter class should always be preferred.

Before the problem of the three bodies was understood with any degree of accuracy, we wanted illustrations of the theory of central forces. The problems depending on this subject presented to us by nature, were of too complicated and abstruse a kind to act as such, our course of pure analysis being quite incompetent for their solution.

To supply this deficiency the investigations of the orbits commonly called Cotes' Spirals, were introduced, and for a long time were treated as a necessary part of our academic course; but since the Lunar and Planetary theories have been more fully developed, while our knowledge of pure analysis has been so much extended as to enable us to make very considerable progress in these natural illustrations of the theory of central forces, Cotes' Spirals have been properly considered in the light of mere mathematical curiosities, and as such, injudicious applications of the student's valuable time.

In like manner, now that hydrostatical science enables us to gain considerable insight into the theory of sound, the student should never be encouraged to apply his time to *learning to find* the different positions in which a triangle, if kept edgewise in a fluid, would float in it; nor in general would it be advantageous that those exertions which might be devoted fruitfully and agreeably to the theory of light, achromatism, and all their beautiful phenomena should be lost in finding caustics to reflecting and refracting surfaces, which as yet we are not able to make experiments with.

It is gratifying to see, by inspection of the following examination questions, that the views above stated have been acted upon to a considerable extent; and perhaps they would have even more influence on our course of reading, were they to be more formally and generally recognised. The main thread of our investigations would not (as is sometimes the case in books published for students in the University) be continually interrupted by problems of too difficult a nature to act as illustrations, and serving rather to puzzle than enlighten the reader.

If we can make sure that the student clearly apprehends the nature of the general processes and essential theorems of a science, by frequently presenting to him sufficiently full, yet easy examples of them, we have gained a great step, and he may safely be trusted to pursue those higher branches which are more directly connected with the philosophy of nature; while, by constantly calling upon him to get up problems of a useless yet extremely difficult kind in the lower branches, we run the risk of encumbering his memory without invigorating his mind, and stopping him short in his course long before he has caught a glimpse of the real beauties of mathematical science, or seen one instance in which it is made available to unravel the secret operations of nature, and display the beautiful contrivances by which the universe is adapted to our wants.

It must be admitted, that to arrive at the degree of mathematical proficiency requisite for the investigations of physical astronomy and other natural applications, will under the most favorable circumstances require considerable industry and mental exertion; and it is from this fact, coupled with the too frequent misapplication of the student's time, that so many persons are to be found who decry the study of mathematics as uninteresting and practically useless.

The course of *pure mathematics*, must, necessarily, be extensive and frequently abstruse; the extreme complication of the causes by which the phenomena of the universe are effected, demands of us an industrious, and as far as possible, complete investigation of all those propositions which concern magnitudes generally. Nature frequently presents to us problems, which would require for their solution, an analysis, even more powerful than any yet known; whence it is justly inferred, that no investigations which tend to enlarge our knowledge in pure analysis can be deemed, *a priori*, void of utility; but even here it may be advantageous in a course intended to instruct and open the mind, to include at first only such parts as can be fairly applied to the researches of natural philosophy.

With respect to what are termed the *mixed mathematics*, the case is far different. It may be boldly asserted of many of the difficult hypothetical problems contained in some elementary works, not only that they are at present inapplicable to natural cases, but that they must always remain so; and surely these can never be considered part of the essentials of science, and consequently should nev- form subjects for questions in our book-worl. papers.

The Editor trusts, that a perusal of the present compilation will shew, that these views are not peculiar to himself, and that the conviction he entertains of their beneficial influence upon the course of study pursued at the University, and of the advantage that would result from their being still further acted upon, will be deemed a sufficient excuse for his offering the above observations on the subject.

EXAMINATION QUESTIONS.

SECTION I.

QUESTIONS IN EUCLID.

1. In a given circle to inscribe an equilateral and equi- 1828
angular pentagon.

2. If a straight line be at right angles to a plane, every plane passing through that straight line is at right angles to the same plane.

3. In a circle the angle in a semicircle is a right angle, but 1829
the angle in a segment greater than a semicircle is less than a right angle, and the angle in a segment less than a semicircle is greater than a right angle.

4. If two triangles have the sides about equal angles reciprocally proportional, they are equal.

5. If a straight line be at right angles to two straight lines at their point of intersection, it is at right angles to the plane passing through them.

6. In any right-angled triangle, the square which is described 1830
upon the side subtending the right angle, is equal to the sum of the squares described upon the sides containing the right angle.

7. The sides about the equal angles of equiangular triangles are proportional, and those which are opposite to the equal angles are homologous sides.

8. If two straight lines meeting one another be parallel to two others which meet one another, but are not in the same plane with the first two, the plane which passes through them, is parallel to the plane passing through the others.

1831 9. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

10. Describe an equilateral and equiangular pentagon about a given circle.

11. Triangles and parallelograms of the same altitude are to one another as their bases.

12. Planes to which the same straight line is perpendicular are parallel to one another.

13. Every prism having a triangular base, may be divided into their pyramids that have triangular bases, and are equal to one another.

1832 14. The angles which one straight line makes with another upon one side of it are either two right angles, or are together equal to two right angles.

15. Draw a straight line from a given point either without or in the circumference which shall touch a given circle.

16. If the sides of a rectangle be a and b, what is meant by saying that the rectangle $= ab$? Illustrate this by considering the case where $a = 5$ and $b = 6$.

What two propositions of Euclid may then be adduced to prove the area of a triangle to be equal to $\frac{\text{base . altitude}}{2}$?

17. In right-angled triangles the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

18. If a straight line be at right angles to a plane, every plane passing through it shall be at right angles to that plane.

1833 19. In the demonstration of the fourth proposition of the First Book of Euclid, is it assumed that two straight lines cannot have a common segment? Give a definition of a straight line which shall supersede the necessity of any axiom respecting it.

20. One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

21. If an angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

22. The perimeters of similar rectilinear figures are in the simple ratio, and the areas in the duplicate ratio of their homologous sides.

23. Segments of right lines intercepted between parallel planes are proportional.

24. Through a given point draw a straight line parallel to 1834
a given straight line.

25. If any two points be taken in the circumference of a circle, the straight line which joins them falls within the circle.

26. Triangles and parallelograms of the same altitude are to one another as their bases. State Euclid's definition of proportion, and the algebraic definition, and shew that they coincide.

27. If two straight lines be at right angles to the same plane, shew that they are parallel to one another. What is Euclid's definition of parallel straight lines, and what other definition has been proposed?

28. If a circle be described touching the base of a triangle and the sides produced, and a second circle be inscribed in the triangle; prove that the points where the circles touch the base are equidistant from its extremities, and that the distance between the points where they touch either one of the sides is equal to the base.

29. The opposite side and angles of parallelograms are 1835
equal to one another, and the diameter bisects them. If both the diameters be drawn, the parallelogram will be divided into four equal parts.

30. Upon the same base and upon the same side of it there cannot be two similar segments of circles not coinciding with one another.

31. In equal circles sectors have the same ratio which the circumferences on which they stand have to one another.

32. If a solid angle be contained by three plane angles, any two of them are greater than the third. How may the magnitudes of two solid angles be compared?

1836 33. It is impossible to divide a quadrilateral figure (except it be a parallelogram) into equal triangles by lines drawn from a point within it to its four corners.

34. Similar triangles are to each other in the duplicate ratio of their homologous sides.

35. In a given rectangle inscribe another, whose sides shall bear to each other a given ratio.

36. If ab, cd be chords of a circle at right angles to each other, prove that the sum of the arcs ac, bd is equal to half the circumference.

37. A common tangent is drawn to two circles which touch each other externally; if a circle be described on that part of it which lies between the points of contact, as diameter, this circle will pass through the point of contact of the two circles, and will touch the line which joins their centres.

38. The diameter of a semicircle is divided into two parts, on each of which as diameter a semicircle is described, and on the same side as the given one. If a circle be described touching each of these three semicircles, the distance of its centre from their common diameter is equal to twice its radius.

39. Define a parallelogram. Describe one which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

40. Describe a square about a given circle.

SECTION II.

QUESTIONS IN ARITHMETIC AND ALGEBRA, NOT INCLUDING THE GENERAL NATURE OF EQUATIONS.

1. THE number 803 expressed in a different scale of notation becomes 30203; required the radix of the scale. 1821

2. Prove that

$$n^n - n.(n-1)^n + n\frac{n-1}{2}.(n-2)^n - \&c. = 1.2.3\ldots.n.$$

3. Find x from the equation

$$\sqrt{x} + \sqrt{a+x} = \frac{2a}{\sqrt{a+x}}.$$

4. The common difference of 4 numbers in arithmetical progression is 1, and their product 120; find the numbers.

5. Insert three harmonic means between a and b.

6. Investigate the rule for transposing a number from one scale of notation to another.

7. If

$$\frac{\mathrm{P}}{(x+a)^n\mathrm{Q}} = \frac{\mathrm{A}}{(x+a)^n} + \frac{\mathrm{A}_1}{(x+a)^{n-1}}\ldots. + \frac{\mathrm{A}_{n-1}}{(x+a)} + \frac{\mathrm{P}'}{\mathrm{Q}}$$

where P and Q are rational functions of x, to determine the values of $\mathrm{A}, \mathrm{A}_1, \mathrm{A}_2 \ldots \mathrm{A}_{n-1}$ and P'.

8. A tetrahedron, whose base is an equilateral triangle, the side of which is equal to one half of each of the remaining edges, is thrown upon a horizontal table: what is the probability of its resting upon its base, excluding all consideration of the mechanical action, which arises from the rotation of the solid?

9. If a and b be prime numbers, the number of numbers prime to $a\,b$ and less than $a\,b$, is equal to

$$(a-1)\,(b-1),$$

unity being considered as one of them.

10. Given the logarithm of n, to find the logarithm of $n+1$.

11. A person paid a tax of 10 per cent upon his income; what must his income have been, when, after he had paid the tax, he had £1250 remaining?

12 The difference between any number and that number inverted, is divisible by 9.

13. Prove that

$$\log\,(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\,\&c.$$

14. Shew how the logarithms of the natural numbers from 1 to 12 may be computed.

15. Prove the rule for the multiplication of duodecimals.

16. Represent $\sqrt{2n\sqrt{-1}}$ as a binomial surd.

17. Find two numbers such that their sum, product, and the difference of their squares may be all equal.

18. How many different ways may £100 be paid in crowns and guineas?

19. In the expansion of $(a+b+c+\&c.)^w$, where $w=p+q+r+\&c.$; find the coefficient of the term involving $a^p\,.\,b^q\,.\,c^r\,.\,\&c.$

20. Apply the duodenary scale of notation to find the solidity of a cube, the side of which is $13^{\text{ft}}.\ 7^{\text{in}}.\ 7^{\text{pts}}.$

1822 21. Find a series of fractions converging to $\sqrt{17}$.

22. The first term of a geometric series continued *in infinitum* is 1, and any term is equal to the sum of all the succeeding terms. Required the series.

23. Given the sum of $2n$ quantities in arithmetical progression and the sum of their squares, to find the quantities themselves.

24. Shew that $y^{mn} - 1$ is divisible by either of the quantities $y^m - 1$ and $y^n - 1$ without a remainder.

25. Given the m^{th} and n^{th} terms of an harmonical progression, to find the $(m + n)^{\text{th}}$ term.

26. Prove that

$$\log (n+1) = \log n + 2 \log \left(\frac{2n+2}{2n+1}\right) + \log \frac{(2n + 1)^2}{(2n + 1)^2 - 1}.$$

27. A and B can do a piece of work in m days; B and C in n days: in what time can A and C do the same, it being supposed that A can do p times as much as B in a given time?

28. Prove that the cube of any number and the number itself, being divided by 6, leave the same remainder.

29. Find the present worth of £P due n years hence, at r per cent. discount.

30. Investigate the rule for the extraction of the square root in whole numbers; and determine, *generally*, the limit which the remainder after any operation cannot exceed.

31. Prove the rule for single position: state to what limitations it is subject; and apply it to find such a number that when divided by 3, 4, and 5, respectively, the sum of the quotients may be 94.

32. Find a quantity, which when multiplied into $a^3 - b^{\frac{1}{3}}$ renders the product rational.

33. Four persons, A, B, C, D, in order, cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective probabilities of success?

34. A given annuity which is to continue $3n$ years, is left equally between A and B; A receives the whole for n years, and B the whole for the remainder of the time; it is required to find the present worth of the annuity, and the rate of compound interest.

35. Given the sum of three quantities in geometrical progression, and the sum of their reciprocals, to find the quantities themselves.

36. In what time will the amount of £P at r per cent. simple interest be equal to p times the interest of the same sum, and what is the rate per cent. when the required time is q years?

37. Of the two quantities $a^6+a^4b^2+a^2b^4+b^6$ and $(a^3+b^3)^2$, shew which is the greater.

38. The sum of a series of quantities in geometrical progression wanting the first term, is equal to the sum of all the terms except the last, multiplied by the common ratio. Required a proof.

39. Prove that $(\mathrm{A}a + \mathrm{B}b + \mathrm{C}c + \ldots.)^2 = (\mathrm{A} + \mathrm{B} + \mathrm{C} + \ldots\ldots.)(\mathrm{A}a^2 + \mathrm{B}b^2 + \mathrm{C}c^2 + \ldots\ldots) - \mathrm{AB}(a-b)^2 - \mathrm{AC}(a-c)^2 - \mathrm{BC}(b-c)^2 - \ldots.$

40. A and B are at play together, and the latter having lost p stakes, is determined to play till he has won them again; find the probability that this never takes place, supposing the play to continue without limitation; his number of chances (b) for winning any assigned game being less than (a) that for the contrary.

41. Find the greatest term of the expansion of $(a + b)^n$.

42. In every geometrical progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

43. Find the sum of $1 + \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{4} + \ldots.$

$\alpha, \beta, \gamma, \ldots.$ being the coefficients of the expansion of $(a+b)^n$.

44. There are four numbers, the first three of which are in arithmetical, and the last three in harmonical progression; it is required to prove that the first has to the second the same ratio which the third has to the fourth.

45. Extract the fourth root of $m^2(m^2-3n^2) + n^2(n^2-3m^2) + 4(m-n)(m+n)mn\sqrt{-1}$.

46. Find three fractions having different prime denominators, whose sum shall be $1\frac{23}{30}$.

47. The area of a floor is 403ft. 7in. and length 27ft. 10in. find the width by duodecimals. 1824

48. If a be greater than b, $a^n - b^n$ is greater than $nb^{n-1}(a-b)$, and less than $na^{n-1}(a-b)$.

49. If a be the approximate square root of any number n, and $n - a^2 = \pm b$, then will

$$\sqrt{n} = a \pm \frac{2ab}{4a^2 \pm b} \text{ very nearly.}$$

50. Let $\frac{A'}{A}, \frac{B'}{B}, \frac{C'}{C} \ldots\ldots$ be the 1st, 2d, 3d, ... approximations to the value of a fraction, when the continued fraction terminates; then will the fraction

$$= \frac{A'}{A} + \frac{1}{AB} - \frac{1}{BC} + \frac{1}{CD} - \frac{1}{DE} + \ldots\ldots$$

51. From a bag containing two balls, a white ball is drawn twice following, the ball having been replaced in the bag after the first drawing; required the probability that both balls are white, and that the ball being a second time replaced, a white ball will be drawn at the third trial.

52. Prove that $\log x = n(x^{\frac{1}{n}} - 1)$ nearly, when n is very great.

53. If $N = n^{th}$ term of the expansion of a^x, determine n when the series reckoned from that term begins to converge; and shew that the sum of all the terms which follow N is less than $\frac{Nn}{1 - x \log a}$.

54. Transform 8978 from a local value 11, and 3256 from a local value 7, to a system in which the local value is 12; and multiply the numbers together in that system.

55. Prove that $(1 + x)^m$ may in all cases be expressed by a series of the form

$$1 + ax + bx^2 + \ldots\ldots$$

56. If $\frac{A}{B}$ be a fraction in its lowest terms, B greater than A, and of the form $B' \cdot 2^m \cdot 5^n$; the quotient will be a mixed cir-

culating decimal, and the higher of the indices m, n will be the number of figures in the part which does not recur.

1825 57. Represent a million acording to the duodenary scale of notation.

58. Divide 29^2 into two other square integer numbers.

59. Required the present worth of £75. due 15 months hence at 5 per cent. per annum.

60. Shew that if the sum of the digits in the odd places be subtracted from any number expressed in decimal notation, and the sum of the digits in the even places be added to the same number, the result is divisible by 11.

61. The difference of the means of four numbers in geometrical progression is 2 and the difference of the extremes is 7. Required the numbers.

62. The present value of an annunity of £1. to continue x years is £10. and the present value of an annuity of £1. to continue $2x$ years is £16. What is the rate of interest?

63. Explain the principal advantages of Briggs' system of logarithms.

64. What is the present value of a freehold estate of £150. a year, allowing the purchaser 6 per cent. compound interest?

65. Shew that the greater two consecutive numbers are, the less is the difference between their logarithms.

66. The sum of two numbers is 6, and the sum of their cubes 72. Required the numbers.

67. Upon a given straight line as an hypothenuse, describe a right-angled triangle which shall have its three sides in continued proportion.

68. Find a series of fractions converging to $\frac{41}{72}$.

69. Sum the following series:

$$\frac{1}{2} + \frac{3}{4} + 1 + \dots \text{ to 8 terms.}$$

70. If seven balls be drawn from a bag, in which are four white balls and eight black, what is the probability that three white balls will be taken?

71. Investigate the rule for finding the least common mul- 1826
tiple of any two quantities, and apply it to find the least common multiple of 174 and 336.

72. Having given the n^{th} term of an arithmetic series and also the sum of n terms, find the series.

73. A sum of money £P is left among A, B and C, in such a manner that at the end of a, b and c years, when they respectively come of age, they are to possess equal sums: required the share of each at compound interest.

74. Prove that 1, 3, 5, 7, ... is the only arithmetical progression beginning from unity, in which the sum of the first half of any even number of terms has to the sum of the second half the same constant ratio; and find that ratio.

75. Shew that the number of permutations of m things taken n together is equal to $m(m-1)\dots(m-r+1)$ times the number of permutations of $m-r$ things taken $n-r$ together.

76. In how many different ways is it possible to pay £10 in crowns, seven shillings, and moidores?

77. Required the discount of £100. during three years hence, at 5 per cent. per annum, compound interest.

78. Find the values of x, y and z, which satisfy the equations:

$$\begin{aligned} xy &= a(x+y), \\ xz &= b(x+z), \\ yz &= c(y+z). \end{aligned}$$

79. If a fraction in its lowest terms be converted into a recurring decimal, prove that the number of figures which recur is always less than the denominator of the fraction.

80. Insert six harmonic means between 1 and 20.

81. If P_1 and P_2 be two sums, due respectively at the end of times t_1 and t_2, prove that the equated time of payment is expressed by

$$\frac{P_1 t_1 + P_2 t_2}{P_1 + P_2} - \frac{P_1 P_2}{(P_1 + P_2)^2}(t_1 - t_2)^2 r, \text{ very nearly,}$$

where r is the interest of £1. for 1 year.

82. Find the least whole numbers which satisfy the equation

$$11x - 18y = 63.$$

83. Prove that $n(n-1)(n-2) \dots (n-r)$ is divisible by $1.2.3 \dots (r+1)$, n being a whole number.

1827 84. There are two heaps of cards, the one of which contains three black and four red cards, and the other five black and two red; what is the probability that a person who takes up one card will draw a red one?

85. Sum the series $15 + \frac{44}{3} + \frac{43}{3} + \dots$ to 16 terms,

$$x^{\frac{5}{2}} - ax + \frac{a^2}{\sqrt{x}} - \dots \text{ to } n \text{ terms,}$$

and find the value of the recurring decimal $1.23434 \dots$

86. Solve the equations

$$\left.\begin{aligned} (a^2)^x - (b^2)^y &= c \\ a^x - b^y &= d \end{aligned}\right\}; \quad x - 3 = \frac{3 + 4\sqrt{x}}{x}.$$

87. Extract the square roots of $7 + 4\sqrt{3}$ and $2\sqrt{-2} - 1$.

88. In how many ways can an equivalent for thirteen dollars, at three shillings, be given in English crowns and seven shilling pieces?

89. How may the square of 50973 be found approximately by a table of squares of whole numbers from 1 to 1000?

90. Compare the probabilities of taking an odd or an even number of balls from a heap containing a given number.

91. Explain the manner of using Gunter's logarithmic scales.

92. Find two whole numbers of which the product shall be divisible by the sum.

93. Prove the rule for multiplying decimals together without any reference to vulgar fractions.

94. The length of a floor being 10 feet 6 inches, and the breadth 9 feet 3 inches, find the area by duodecimal multiplication, and define the several terms of the product.

95. The second and third terms of a geometrical progression are together equal to 24, and the two next to 216: what is the first?

96. Investigate the rule of Alligation, in which the prices of the ingredients and of the mixture are given, to find the proportions of the former.

97. If n be any prime number greater than 3, $n^2 - 1$ is divisible by 12.

98. Continue in both directions the harmonic progression of 1828
which 4 and 6 are adjacent terms.

99. If four quantities of the same kind be proportional, the greatest and least together are greater than the other two.

100. What is the amount of 37cwt. 2qrs. 14lbs., at £7. 10s. 9d. per cwt.?

101. Find $\sqrt{3\sqrt{3} + 2\sqrt{6}}$, in the form of a binomial surd.

102. Find an expression for the sum of a decreasing geometric series, and explain clearly the possibility of an infinite number of terms having a finite sum.

103. Find the value of an annuity of £100. to commence 10 years hence and to continue for ever, allowing compound interest.

104. Determine the number of permutations of n things taken all together, supposing the same quantities to recur.

105. Prove the binomial theorem when the index is fractional. Apply it to expand $(a^2 - \frac{2}{3}x^2)^{\frac{3}{2}}$ to five terms.

106. Transform the continued fraction $$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \&c.}}}$$

into a series of converging fractions, and prove that each of the latter approaches the value of the original fraction more nearly than the preceding.

107. What is the logarithm of any number? Why is the common system selected? What is its base? Explain and prove the rule for proportional parts.

108. The sums of the coefficients of the even and odd terms of any power of $a + b$ are equal.

109. Find the n quantities x_1, x_2, x_3, ... x_n, from the n equations:

$$\begin{aligned} x_1 + x_2 \dots\dots + x_n &= A, \\ a_1x_1 + a_2x_2 \dots\dots + a_nx_n &= 0, \\ a_1^2x_1 + a_2^2x_2 \dots + a_n^2x_n &= 0, \\ \dots\dots\dots\dots & \\ a_1^{n-1}x_1 + a_2^{n-1}x_2 \dots + a_n^{n-1}x_n &= 0; \end{aligned}$$

and obtain symmetrical expressions for x_1, x_2, &c.

110. Shew that

$$\frac{1}{(a+x)^m} = (p+1) \text{ terms of the expansion of } (a+x)^{-m}$$

$$+ \frac{x^{p+1}}{(a+x)^m} \times m \text{ terms of the expansion of } (\overline{a+x}-a)^{-\overline{p+1}}.$$

111. A throws 6 dice, B throws 12, C throws 18. Compare the chances of A throwing one six, B two sixes, and C three sixes.

112. Investigate the rule for determining the greatest common measure of any two quantities, and apply it to find the greatest common measure of

$$x^3 - 11x^2 + 39x - 45 \text{ and } 3x^2 - 22x + 39.$$

113. In how many years will a sum of money treble itself at $4\frac{1}{2}$ per cent. compound interest?

$$\log 3 = .4771213,$$
$$\log 1.045 = .0191163.$$

114. Find the least whole positive numbers which will satisfy the equation $7x - 9y = 29$.

115. Assuming the expansion of a^x, deduce a converging series for the logarithm of any number; and apply it to com-

pute the *Napierian* logarithm of 5 to seven places of decimals, that of 2 being .6931471; and shew how these two logarithms determine the modulus of *Briggs'* system.

116. If to the square of any number not divisible by 3 the number 2 be added, the result is divisible by 3.

117. Reduce $\sqrt[4]{-1} + b\sqrt[6]{-1}$ to the form of $a + \beta\sqrt{-1}$.

118. If n be greater than 3, shew that $\sqrt{n} > \sqrt[3]{n+1}$.

119. Prove the rule for transforming a number from one scale of notation to another. Transform 1828 to local value 3.

120. In a lottery of m tickets, n of which are prizes, if p tickets be drawn at each time, what is the probability that all the prizes will be drawn after q drawings.

121. Extract the square root of 235.6 to two places of decimals: and the cube root of .000079507. 1829

122. Expand a^x in a series ascending by the powers of x.

123. Prove that the *Napierian* logarithm of $N + z$

$$= \text{Nap. log } N + 2\left\{\frac{z}{2N+z} + \frac{1}{3}\frac{z^3}{(2N+z)^3} + \&c.\right\};$$

and from this formula shew how the logarithm of a number of six places of figures may be found from a table computed only to five places of figures.

124. Prove the rule for finding the greatest common measure of two algebraical quantities, and apply it to find the greatest common measure of

$$x^3 - 8x^2 - 12x + 144 \text{ and } 3x^2 - 16x - 12.$$

125. Prove the binomial theorem for any value of the index.

126. A ratio of greater inequality is diminished and of less inequality increased, by adding any quantity to both its terms.

127. The reciprocals of quantities in harmonical progression are in arithmetical progression.

128. Find the present value of an annuity to be paid for n years, allowing compound interest.

129. Every square number is of the form $5n$ or $5n \pm 1$.

130. Find the value of the circulating decimal 3.42753753 &c. also, perform the same operation with the fraction 45.2534534 &c. where the radix is 6.

131. If there be n bags containing each a white and b black balls, and a ball be drawn out of each successively, shew how to determine beforehand what is the most probable number of white balls that will be drawn; and apply the process to the case of 12 bags containing each 2 black and 7 white balls.

132. Investigate a true rule for the equated time of payment of two sums due at different times. Who is the gainer by the common rule?

133. If $\frac{N}{N'}$ and $\frac{P}{P'}$ be successive approximations to the value of a continued fraction,

$$NP' - PN' = \pm 1.$$

134. A banker borrows money at $3\frac{1}{2}$ per cent. per annum, and pays the interest at the end of the year: he lends it out at the rate of 5 per cent. per annum, but receives the interest quarterly, and by this means gains £200. a year. How much does he borrow?

135. A and B sit down to cards, the former having p shillings and the latter q, and they agree each to stake a shilling on every game, and to play till one has lost all his money. Find the value of the expectation of each before they begin to play, supposing the skill of A : that of B :: $m : n$.

136. If I have 9 half-guineas and 6 half-crowns in my purse, how may I pay a debt of £4. 11s. 6d.?

137. If there be a chances for an event happening and b for its failing in one trial, find the probability of its happening t times *at least* in n trials.

138. If $5x + 21y = 2000$, find x and y, and the number of positive integer solutions which the equation admits of.

1830 139. Divide 1532 feet $9\frac{9}{12}$ inches by 81 feet 9 inches.

140. Find the amount of an annuity A in n years, and the present worth of £140. per annum for ever at 5 per cent.

141. In any number, if a point be placed over every third digit, beginning with the one on the right hand, shew that the number of digits in the cube root is equal to the number of such points; and extract the cube root of 318.61199 to two places of decimals.

142. Investigate a rule for finding the greatest common measure of any two algebraical quantities, and shew that a actor of any divisor, which is not contained in the corresponding dividend, may be removed without affecting the result.

143. Solve the following equations:

$$(1)\quad \sqrt{x+16} = 2 + \sqrt{x},$$

$$(2)\quad \left.\begin{aligned} x^3 + y^3 &= 1001 \\ x + y &= 11 \end{aligned}\right\},$$

$$(3)\quad \sqrt[m]{(1+x)^2} - \sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}.$$

144. The coefficients of the terms of an expanded binomial are whole numbers, when the index is a whole number; and the coefficients of the terms equidistant from the extremes are equal.

145. Prove that in any system of logarithms

$$\log \mathrm{MN} = \log \mathrm{M} + \log \mathrm{N},\ \log \frac{\mathrm{M}}{\mathrm{N}} = \log \mathrm{M} - \log \mathrm{N},$$

$$\text{and } \log \mathrm{M}^n = n \log \mathrm{M}.$$

Explain the advantages of *Briggs'* system, and having a table constructed for one system, give the method of constructing a table for a different system.

146. Investigate a rule for transforming numbers from one scale of notation to another, and transform 42.36 from the denary scale to the scale of 5.

147. In the expansion of $(a + b + c + \ldots\ldots)^m$, find the coefficients of $a^p.\ b^q.\ c^r.$ &c.

148. If m be a prime number, and a a number not divisible by m, then $a^{m-1} - 1$ is divisible by m.

149. A person spends in the first year m times the interest of his property; in the second $2m$ times that of the remainder; in the third $3m$ times that at the end of the second, and so on; and at the end of $2p$ years he has nothing remaining; shew

that in the p^{th} year he spends as much as he has left at the end of that year.

150. If $\frac{A_1}{B_1}, \frac{A_2}{B_2}, \ldots \frac{A_n}{B_n}$ be a series of fractions approximating to $\frac{A}{B}$, shew that

$$\frac{A}{B} - \frac{A_n}{B_n} < \frac{1}{B_n^2} \text{ and } > \frac{1}{B_n(B_n + B_{n+1})}.$$

151. Shew how to find the number corresponding to a given logarithm not found exactly in the tables.

152. If a, b, and c be three whole numbers taken in succession, prove that Nap. log b

$$= \tfrac{1}{2}\text{Nap.}\log a + \tfrac{1}{2}\text{Nap.}\log c + \left(\frac{1}{2ac+1} + \tfrac{1}{3}\,\frac{(2ac+1)^3}{1} + \&c.\right);$$

and shew the peculiar use of this formula in finding the logarithms of prime numbers.

153. From the equation

$$y^3 - ay + x = 0,$$

find y in a series ascending by the powers of x, by the *Reversion of Series*.

154. a and b are respectively the first term and common difference of an arithmetic series,

S_n the sum of n terms,
S_{n+1} . . . . $(n+1)$ terms,
. . . .

prove that $S_n + S_{n+1} + S_{n+2} + \ldots$ to n terms

$$= (3n-1)\,n\,\frac{a}{2} + (7n-2)(n-1)\,n\,\frac{b}{6}.$$

155. If n be a whole number, prove that $\frac{n^3+5n}{6}$ is also a whole number.

156. The ratio between the area of an equilateral and equiangular decagon described about a circle, and that of another within the same circle is equal to $\cfrac{8}{7 + \cfrac{1}{4 + \cfrac{1}{4 + \ldots}}}$.

157. Find the present value of an annuity of £1. to be continued during the life of an individual of a given age, allowing compound interest for the money.

158. S_1, S_2, S_3 . . . S_n being the sums of n geometric series continued *in infinitum*, the first term of which is 1, and the common ratio $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \ldots \frac{1}{r^n}$ respectively. Required the sum of their reciprocals,

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \cdots + \frac{1}{S_n}.$$

159. A bag contains m white balls and n black balls; find the probability of taking out a white ball at least p times in r trials, the ball being replaced after each trial.

160. What sum must be paid for £439. 12*s.* 5*d.*, $3\frac{1}{2}$ per 1831
cent. stock, at $92\frac{7}{8}$ per cent.?

161. Shew that $\dfrac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = 16.746, \ldots$; also prove that the cube root of any number containing $3n$, $3n-1$, or $3n-2$ digits, contains n digits.

162. Reduce the fraction $\dfrac{30a^2x^4 - 5a^3x^3 + 5a^5x}{9ax^3 - a^3x + 2a^4}$ to its lowest terms.

163. Solve the following equations:

$$\left.\begin{aligned} &x + y - 8 = 0 \\ &\frac{x-y}{2} + \frac{2x-y}{3} + \frac{4}{3} = 0 \end{aligned}\right\},$$

$$\frac{x+3}{x} + 4x = 14,$$

$$x^2 + \sqrt{x^2 + 11} = 31.$$

164. Investigate the binomial theorem, and apply it to extract the square root of $\dfrac{1}{a^2 - b^2}$ to four terms.

165. Prove that $\log(m + n)$

$$= \log m + 2M\left\{\frac{n}{2m+n} + \frac{1}{3}\left(\frac{n}{2m+n}\right)^3 + \frac{1}{5}\left(\frac{n}{2m+n}\right)^3 + \cdots\right\}.$$

166. The r^{th} root of a binomial, one or both of whose factors are possible quadratic surds, may sometimes be expressed by a binomial of that description.

167. Shew how the fraction $\frac{A}{B}$ may be represented in the form of a continued fraction, and apply the process to find a fraction which shall be nearly equal to $\frac{1593}{7667}$, and in lower terms.

168. Find all the solutions in positive whole numbers of the equation $11x + 15y = 1031$.

169. Determine which is the greatest term of the expansion of $(a + b)^n$.

170. Shew that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$, and that it differs from the latter fraction by a quantity less than $\frac{1}{2 \times 305 \times 1292}$.

171. Construct a table of proportional parts, and explain its use;

$$\begin{aligned} &\text{and if } \log 67833 = 4.8314410, \\ &\text{and } \log 67832 = 4.8314346, \\ &\text{find } \log 678328. \end{aligned}$$

172. If there be (a) chances of an event happening, and (b) of its failing in one trial, find the probability of its happening (f) times at least in (n) trials.

173. If n be any whole number, one of the three n^2, $n^2 + 1$, $n^2 + 2$, is divisible by 5 without remainder.

174. A debt of £a. accumulating at compound interest, is discharged in n years by annual payments of £$\frac{a}{m}$; prove that $n = -\frac{\log(1 - mr)}{\log(1 + r)}$, r being the interest of £1. for one year.

175. Find the present value of an annuity, to commence at the expiration of p years, and to continue q years; compound interest.

176. If the sides of a rectangle be a and b, what is meant by saying that the rectangle $= ab$? Illustrate this by considering the case where $a = 5$, and $b = 6$. 1832

What two propositions of Euclid may then be adduced to prove the area of a triangle to be equal to $\frac{\text{base . altitude}}{2}$?

177. Find the amount of £420. in four years at three per cent. compound interest.

178. Extract the square root of 5.1 to two places of decimals, and find the cube root of 29.7910.

179. Find the greatest common measure of $3x^3 - 2x^2 - x$, and $6x^2 - x - 1$; and the least common multiple of 14, 18, and 84

180. The sum of an arithmetic series is 72, the first term 17, the common difference -2: find the number of terms, and explain the double answer.

181. If $a + b\sqrt{-1} = c + d\sqrt{-1}$, prove that $a = c$, and $b = d$; and extract the square root of $5 - 12\sqrt{-1}$.

182. Define a logarithm, and the base of a system of logarithms.

Prove that $\log \frac{m}{n} = \log m - \log n$,

and Nap. $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots$

183. If there be a chances of an event happening, and b of its failing in one trial, find the probability of its happening b times exactly in n trials.

184. The number of different combinations of n things taken $1, 2, 3, \ldots. n$, at a time, of which there are p of one sort, q of another, and r of another, is $= (p + 1)(q + 1)(r + 1) - 1$.

185. Thirteen persons are required to take their places at a round table by lot; shew that it is 5 to 1, that two particular persons do not occupy contiguous seats.

186. Find the number of combinations of n things taken r together, and find r when the number of such combinations is greatest.

187. A person being asked what o'clock it was, answered that it was between nine and ten, and that the hour and minute hands were together; required the time of day.

188. The amount of £1. in n years at compound interest is given within less than one farthing by the four first terms of the expansion of £$(1 + r)^n$, where r is the interest of £1. for a year, and the rate of interest not greater than 4 per cent., and n not greater than 10.

189. If the number of persons born in any year be $\frac{1}{45}$th of the whole population at the commencement of that year, and the number of those who die $\frac{1}{60}$th of it, find in how many years the population will be doubled; having given

$$\begin{aligned} \log\ 2 &= .301030, \\ \log 180 &= 2.255272, \\ \log 181 &= 2.257679. \end{aligned}$$

190. Find two fractions whose denominators shall be 7 and 9, and their sum $\frac{57}{63}$. Show that if the *difference* of the fractions be $\frac{57}{63}$, the number of solutions is unlimited.

191. Shew how to resolve $\frac{a}{b}$ into a continued fraction, and prove that the converging fractions are alternately greater and less than $\frac{a}{b}$.

192. Expand $(ax - x^2)^{\frac{5}{2}}$ to five terms.

193. The reciprocals of quantities in harmonic progression are in arithmetic progression.

194. Prove that if

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}, \quad \text{and also} \quad \frac{a+mb}{a-nb} = \frac{c+md}{c-nd}.$$

195. Extract the square root of $7 - 2\sqrt{10}$, and the cube root of 15 to two places of decimals. 1833

196. Define discount, and find the discount of £100. for one year at 5 per cent., and the interest on this discount for the same time.

197. Prove that $a^m \times a^n = a^{m+n}$, when m and n are integers. Is this theorem assumed or proved when m and n are fractional? Give the theory.

198. Find the least common multiple of $x^3 - 1$, $x^2 + x^2 - 2$, and prove the rule.

199. A person buys a certain number of oxen for £80. and finds that if he had bought four fewer for the same money, they would have cost £1. a piece more: how many oxen did he buy? Explain the negative answer.

200. Find the number of combinations of n things taken r together, and shew that it is the same as when they are taken $n - r$ together.

201. Having given $\log (1 + n) = \frac{n}{1} - \frac{n^2}{2} + \frac{n^3}{3} + \ldots$ to investigate a converging series for calculating the logarithm of any number, and apply it to calculate log 11 to 5 places of decimals, log 10 being $= 2.302585$.

202. Find the number of shot in a pile the base of which is a rectangle, m and n being the number in each of the sides of the base.

203. The n^{th} roots of unity form a geometrical progression, of which the sum $= 0$.

204. The converging fractions to any continued fraction are in their lowest terms. Express the square root of 28 in the form of a continued fraction.

205. In converting a proper fraction $a \div b$, whose denominator is prime to 10, into a decimal, shew that on arriving at a remainder $= b - a$, half the circulating period has been

obtained, and that the remaining digits may be found by subtracting, in order, each of those already obtained from 9.

206. After how many terms will the series for $(1+x)^n$ converge, x being less than unity? Exemplify in $\left(1+\frac{7}{8}\right)^{10}$.

207. Investigate the general term of $(a+bx+cx^2+\ldots)^n$ for all values of n.

208. Multiply 56.92 by 7.35 in the duodenary scale; and if the units in the factors be feet, find the number of square yards, feet, inches, &c. in the product.

209. If the ratios $a:p$, $b:q$, $c:r$, be all *ultimately* equal to $m:n$, shew that the *ultimate* ratio of $a+b+c:p+q+r$ is also $m:n$.

210. Find the discount on £275 6*s.* 8*d.* due 18 months hence, at $4\frac{1}{2}$ per cent.

1834 211. Reduce to the simplest form

$$\frac{1}{x-1}-\frac{1}{2x+2}-\frac{x+3}{2x^2+2}.$$

212. If a be prime to b, there is no fraction $\frac{c}{d}$ equal to $\frac{a}{b}$ whose terms are not equimultiples of a and b.

213. Prove the binomial theorem when the index is a positive whole number, and determine the sum of the coefficients. Write down the coefficient of x^4 in the expansion of $(1+x+x^2)^6$.

214. Form the continued fraction corresponding to $\frac{87968}{277288}$, and determine the first five converging fractions.

215. Define the terms Logarithm, and System of Logarithms. Shew how to pass from one system to another with a different base; and assuming that

$$l_e\, x = 2\left\{\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\ldots\right\},$$

prove that

$$l_e(x+1) = 2l_e\, x - l_e(x-1) - 2\left\{\frac{1}{2x^2-1}+\frac{1}{3}\frac{1}{(2x^2-1)^3}+\ldots\right\}.$$

216. Define the mathematical meaning of the term chance; and prove that the chance of an event contingent upon other events is the continued product of the chances of the separate events. What is the probability of throwing an ace once only in three trials?

217. Prove that the difference of the squares of any two odd numbers is divisible by 8; and that some term of the progression,

$$r + r^2 + r^3 + \dots\dots r^n - 1,$$

diminished by unity, is divisible by n, r being an integer not divisible by n.

218. Find the present value of an annuity to continue (n) years, allowing simple interest upon each sum from the time it becomes due; and explain why the present value of an annuity to continue for ever cannot be estimated on these principles.

219. A person about to purchase the lease of an estate, is able continually to invest money at the rate of four per cent. per annum, receiving the interest half-yearly; shew that if the tenant pays his rent half-yearly, the value of the lease to the purchaser is 1.01 times, what would be its value if the tenant paid his rent yearly.

220. Reduce $\{(a^2 + c^2)^2 - (a^2 - c^2)^2 - (a^2 + c^2 - b^2)^2\}^{\frac{1}{2}}$ to a formula adapted to logarithmic calculation; and find the logarithms of 8 and 9 from those of 6 and 15 supposed given.

221. Find the value of a corn rent of 6 quarters of wheat, Winchester measure, wheat being at 50 shillings a quarter, imperial measure; supposing 32 imperial equal to 33 Winchester gallons.

222. Having given the number of combinations, which can be formed with (n) quantities taken (r) together, find the number which can be formed with ($n + 1$) quantities also taken (r) together, with reference to any general formula. Find also the continued product of the former set of combinations.

223. Investigate a rule for finding the least common multiple of any series of numbers, and apply it to reduce $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{21}$, and $\frac{1}{60}$ to their least common denominator.

224. Two persons travel at different uniform rates along the same road, in the same direction, starting simultaneously from points at a given distance from each other; find where they will be together, and explain the result when it is negative.

225. Find, with the help of the tables, what must be the annual payment for the whole life of an individual to begin immediately, in order to secure the reversion of a given sum at his death.

226. Shew how to find the root of $x^n - 1 = 0$, n being a prime number, without Trigonometry.

227. Express the product of .2727 and 1.166 by a circulating decimal; also obtain the cube root of $74 + 23\sqrt{11}$ under the form of a binomial surd.

228. When n is even, find for what value of r the number of combinations of n things taken r together is the greatest possible.

1835 229. Required the amount of £819. 4*s.* in 6 years, allowing £12. 10*s.* per cent. both simple and compound interest.

230. Investigate the rules for the multiplication and division of algebraical fractions; and reduce

$$\frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} - \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}$$

to its simplest form.

231. If a be prime to b, there is no other fraction equal to $\frac{a}{b}$ whose terms are not equimultiples of a and b.

232. When four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth to their difference.

233. Shew how the logarithm of a number consisting of 6 digits may be found from a table calculated for numbers not exceeding 5 digits, and having given log 33819 = 4.5291608, log 33818 = 4.5291479, find log 338185.

234. Express $\dfrac{a + b\sqrt{-1}}{c + d\sqrt{-1}}$ and $\log_e (a + b\sqrt{-1})$ in the form $A + B\sqrt{-1}$.

235. Shew how to transfer a number from one scale of notation to another, and express 123.45 in a scale whose radix is 5.

236. If the diameter of the earth, supposing it a sphere, be 7916 miles, find the length of a French metre which is one ten-millionth part of a fourth of its circumference.

237. A number is divisible by 9, if the sum of its digits is divisible by 9; and by 11, if the sum of the 1st, 3rd, 5th .... digits is equal to the sum of the 2nd, 4th, 6th, ... digits.

238. Explain the method of solving the indeterminate equation $ax + by = c$ by continued fractions; and shew how the number of solutions may be determined.

239. Supposing $\dfrac{ad - bc}{a - b - c + d} = \dfrac{ac - bd}{a - b - d + c}$, each of them $= \dfrac{a + b + c + d}{4}$.

240. Reduce $\dfrac{\sqrt[3]{5.12} + \sqrt[3]{.03375}}{\sqrt[3]{80} - \sqrt[3]{.01}}$ to its equivalent simple decimal.

241. The coefficients of the $(r + 1)^{th}$ term of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of the r^{th} and $(r + 1)^{th}$ terms of $(1 + x)^n$; prove this, and express $(1 + x)^n$ in a continued fraction.

242. Every number consisting of n digits will have $2n$ or $2n - 1$ digits in its square. Hence explain the rule for pointing in the extraction of the square root of a number.

243. If a body a inches long weighs m pounds, find the length of a similar body that weighs n pounds.

244. In the expansion of $(a + b)^n$ where n is an integer, the coefficients of terms equidistant from the two extremes are equal. Write down the p^{th} term of expansion of $(1 - x)^{\frac{1}{2}}$.

245 If there be a chances of an event's happening in any one trial, b of another, and c of a third; find the probability of the first event's happening p times, the second q times, and the third r times, in $p + q + r$ trials.

1836 246. Find the volume of a cube whose edge is 13 feet 8 inches.

247. Determine the present value of p pounds, due n years hence, at a given rate of interest.

Ex. £1000. 10*s*. due 5 years 4 months hence, at $4\frac{1}{2}$ per cent. per annum.

248. Define the least common multiple of two quantities; and prove that it measures every other common multiple of them. Find the least common multiple of

$$6x^3 - 11x^2 + 5x - 3 \text{ and } 9x^3 - 9x^2 + 5x - 2.$$

249. Reduce to the simplest form

$$\frac{1}{2x+2} - \frac{4}{x+2} + \frac{9}{2(x+3)}.$$

250. Investigate the law of formation of the product of a series of binomial factors, $(x + a)$, $(x + b)$, ... and deduce the coefficient of x in the product

$$(x + 2) . (x + 6) . (x + 10) . (x + 14).$$

251. Shew how to pass from one system of logarithms to another. What is the advantage of the common system? Assuming the expansion of a^x, calculate the value of the base of the Napierian system to six places of decimals.

252. Shew, that if n and N be nearly equal,

$$\left(\frac{N}{n}\right)^{\frac{1}{2}} = \frac{N}{N+n} + \frac{1}{4} \cdot \frac{n+N}{n} \text{ very nearly.}$$

In using this formula, we may sufficiently know the degree of correctness attained, by observing that if

$$\frac{N}{N+n} \text{ and } \frac{1}{4} . \frac{n+N}{n}$$

have their p first decimal figures equal, the approximation may be relied upon to $2p$ decimals at least. Prove this, and apply the formula to find an approximate value of $\sqrt{30}$ true to eight decimal places.

253. The corners of a common die are filed away till the faces which before were squares become regular octagons. Compare the respective probabilities, when the die is thrown, of turning up a triangular and an octagonal face. (Considerations of a dynamical nature to be neglected.)

254. Expand $\log_e(1+x)$ in a series ascending by powers of x. Shew that no hypothesis, which renders nugatory a preceding step, has been introduced in the demonstration.

255. If $\frac{p}{q}, \frac{p'}{q'}$ be consecutive converging fractions, then $pq' - p'q = \pm 1$. Prove this, and apply it to find all the solutions, in positive integers, of the equation

$$33x + 17y = 743.$$

256. Find the least number, the product of which by 882 shall be a perfect cube.

257. By selling a given quantity of a certain article for 10*s.* the seller loses 5 per cent.; what will be the loss or gain when it is sold for 12*s.* 6*d.*?

258. Explain the popular meaning of the term *ratio*: shew that it may be represented algebraically by a fraction. On this assumption prove that the geometrical definition of proportion is a consequence of the algebraical.

259. The product of any r consecutive integers is divisible by $1.2.3 \ldots . r$.

260. Prove that the n^{th} roots of unity form a geometrical progression. Interpret the values of $a.(1)^{\frac{1}{n}}$, when a represents a line.

261. Find the mean proportional between .016 and 2.704. Express 1000 in the scale whose radix is 11.

262. Prove that the number of combinations of n things taken r together is equal to the number when taken $n-r$ together. What is the n^{th} term of $(a^2 - x^2)^{-\frac{1}{3}}$?

SECTION III.

QUESTIONS ON THE NATURE OF ALGEBRAIC EQUATIONS.

1821 1. Solve $x^3 - \frac{5}{2}x^2 + x - \frac{1}{6} = 0$, the roots being in harmonical progression.

2. If we dènote the sum, sum of the squares, cubes, . . n^{th} powers of the m quantities a, b, c, d, &c., by $S_1, S_2, S_3, \ldots S_n$, then the sum of the n^{th} powers of the differences of a, b, c, d, &c. or

$$(a-b)^n + (b-a)^n + (a-c)^n + \&c.$$

$$= mS_n - nS_1 S_{n-1} + \frac{n(n-1)}{1 \cdot 2} S_2 S_{n-2} - \&c.$$

3. Every equation has at least as many changes of sign from $+$ to $-$ and from $-$ to $+$, as it has positive and possible roots; and as many continuations of sign from $+$ to $+$, and from $-$ to $-$, as it has negative and possible roots.

4. Give *Clairaut's* approximation to the solution of a cubic equation in the irreducible case.

1822 5. Transform $x^3 - 2x^2 + 2x - 4 = 0$ into an equation, the roots of which are the squares of the roots of the original equation.

6. Prove that *Waring's* solution of a biquadratic equation fails when all the roots are impossible, and of the form

$$a \pm b\sqrt{-1}.$$

7. The roots of the equation $6x^4 - 43x^3 + 107x^2 - 108x + 36 = 0$, are of the form a, b, $\frac{a}{b}$ and $\frac{b}{a}$, find them; and shew what relation exists between the coefficients of a cubic whose roots are of the form $+a$, $-a$, and $\pm b$.

8. Shew that if an equation have two equal roots, and the terms are multiplied by the terms of an arithmetic progression, the result will $= 0$.

9. In an equation of n dimensions, the second and third terms may be taken away by the same transformation when the square of the sum of the roots : the sum of their squares :: $n : 1$. Required a proof.

10. The roots of the equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ are of the form $a+1$, $a-1$, $b+1$, $b-1$; find them.

11. Shew that *Cardan's* rule for the solution of a cubic equation is applicable when all the roots are possible and two of them equal; and by means of it find the roots of the equation $x^3 + 6x^2 - 32 = 0$.

12. Transform the equation $x^3 - px^2 + qx - r = 0$ whose 1823
roots are a, b, c, into one whose roots are

$$\left(\frac{a}{b} + \frac{b}{a}\right), \quad \left(\frac{a}{c} + \frac{c}{a}\right), \quad \left(\frac{b}{c} + \frac{c}{b}\right).$$

13. If a be a root of Des Cartes's reducing cubic, then will the four roots of the equation $x^4 + qx^2 + rx + s = 0$ be

$$-\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} + \frac{r}{2\sqrt{a}}\right)},$$

$$\text{and } +\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2} - \frac{a}{4} - \frac{r}{2\sqrt{a}}\right)}.$$

14. Take away the third term of the equation

$$x^3 - 6x^2 + 9x - 20 = 0.$$

15. In the equation $x^3 - px^2 + qx - r = 0$, prove that the sum of the products of the roots and their reciprocals taken three and three together : product of the roots :: $1 + p^2 + q^2 + r^2 : r^2$.

16. If three roots of an equation be nearly equal to one another, and much less than all the others, shew that an approximation may be made to them by the solution of a cubic.

1824 17. Find the sum of the m^{th} powers of the reciprocals of the roots of an equation in terms of the inferior powers.

18. If a be an approximate value of x in any equation, and b, c be the results, when a is substituted for x in the original and in the limiting equation; then will $x = a - \frac{b}{c}$ nearly.

19. If $x^n + Ax^{n-1} \ldots - Px^p \ldots - Sx^s \ldots + Tx + V = 0$, where P is the greatest and S the last negative coefficient, then $\dfrac{V^{\frac{1}{s}}}{V^{\frac{1}{s}} + P^{\frac{1}{s}}}$ is an inferior limit of the positive roots.

1825 20. Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$, the roots being in arithmetical progression.

21. Find one of the roots of the equation $3x^3 - 26x^2 + 34x - 12 = 0$ by the method of divisors.

22. The roots of the equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \ldots = 0$ are in geometrical progression beginning from unity; given $p = 15$, $q = 70$. Required n, r, &c.

23. If the terms of an equation, all whose roots are possible, be multiplied by the terms of the arithmetical progression 0, 1, 2, 3, &c. the resulting equation will be a limiting equation to the former, with this exception, that no root of the limit will lie between the positive and negative roots of the proposed equation.

24. The equation $x^3 - 7x^2 + 16x - 12 = 0$ has two equal roots. Find all the roots.

1826 25. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are α, β, γ, into one whose roots are $\alpha^2 + \beta^2$, $\alpha^2 + \gamma^2$, and $\beta^2 + \gamma^2$.

26. Prove that *Des Cartes'* solution of a biquadratic equation succeeds when all the roots are possible and two of them equal, and apply it to solve the equation

$$x^4 - 6x^3 + 8x^2 + 6x - 9 = 0.$$

27. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. + \mathrm{Q}x^2 - \mathrm{P}x + \mathrm{L} = 0$$

be in harmonical progression, then will the greatest and least be respectively

$$\frac{n\sqrt{n+1}\,\mathrm{L}}{\sqrt{n+1}\,\mathrm{P} - \sqrt{3\,(n-1)^2\,\mathrm{P}^2 - 6n\,(n-1)\,\mathrm{QL}}},$$

and

$$\frac{n\sqrt{n+1}\,\mathrm{L}}{\sqrt{n+1}\,\mathrm{P} + \sqrt{3\,(n-1)^2\,\mathrm{P}^2 - 6n\,(n-1)\,\mathrm{QL}}}:$$

required a proof.

28. In any recurring equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. + qx^2 - px + 1 = 0,$$

whose roots are a, b, c, &c.; prove that

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} + \&c.$$

$$= (p^2 - 2q + \sqrt{n})\,(p^2 - 2q - \sqrt{n}).$$

29. If an equation has n equal roots, the equation formed 1827
by multiplying the terms by the terms of an arithmetical progression has $n - 1$ of them.

30. In the equation $x^4 + 8x^3 + x^2 - x - 10 = 0$ take away 1828
the second term, and then find the reducing cubic.

31. Find the sum of the sixth powers of the roots of the equation

$$x^3 - x - 1 = 0.$$

32. Shew that *Cardan's* solution applies only to those cases in which the equation has two impossible roots, unless two of the roots be equal.

33. The coefficient of the second term of an equation with its proper sign, is the sum of the roots with their signs changed; the coefficient of the third term is the sum of the products of every two roots with their signs changed; the coefficient of the fourth term is the sum of the products of every three roots with their signs changed, &c. &c.

34. Shew that the limiting equation has at least as many possible roots as the original equation, wanting one; and determine the nature of the roots of the equation

$$x^7 - a^5 x^2 + c^7 = 0.$$

35. Approximate to the greatest root of the equation

$$x^3 - 7x = 1.$$

36. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$, whose roots are α, β, γ, into the equation whose roots are

$$\frac{1}{\alpha^2 + \beta^2}, \quad \frac{1}{\alpha^2 + \gamma^2}, \quad \frac{1}{\beta^2 + \gamma^2}.$$

1829 37. Every equation whose roots are possible has as many changes of sign from $+$ to $-$ and from $-$ to $+$, as it has positive roots; and as many continuations of the same sign from $+$ to $+$ and from $-$ to $-$, as it has negative roots.

38. If the equation $x^3 - px^2 + qx - r = 0$ has two equal roots, one of them is $\frac{9r - pq}{6q - 2p^2}$; but the converse is not necessarily the case.

39. Explain the method of finding those roots of an equation which are whole numbers, by the *Method of Divisors*, and apply it to solve the equation

$$x^3 - 9x^2 + 22x - 24 = 0.$$

40. Shew that any recurring equation of $2m$ or $2m + 1$ dimensions may be solved by an equation of m dimensions.

41. In the solution of a biquadratic by *Des Cartes*' method, whatever root of the reducing cubic is employed, the same values of the roots of the biquadratic will be obtained.

42. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$\frac{c}{a + b - c}, \quad \frac{b}{a + c - b}, \quad \frac{a}{b + c - a}.$$

43. Explain *Newton's* rule for discovering impossible roots in any equation.

44. In the equation $x^2 - px + q = 0$, find the sum of the n^{th} powers of the roots in terms of the coefficients.

45. If a, b, c, &c. be the roots of an equation, find the value of

$$a^2b + a^2c + b^2a + \&c.$$

46. Find the sum of the m^{th} powers of the roots of an equation in terms of the coefficients and the sums of the inferior powers.

47. Shew the method of extracting the cube root of the binomial surd $a + \sqrt{b}$, and apply it to the solution of the equation $x^3 - 3x - 18 = 0$, by *Cardan's* rule.

48. Solve by *Cardan's* rule $x^3 + 3x^2 + 9x - 13 = 0$.

49. If two magnitudes, when substituted for the unknown quantity in an equation, give results affected with different signs, an odd number of roots lies between them; but if they give results with the same sign, either no root or an even number of roots lies between them. 1830

50. An equation of m dimensions has n equal roots, shew how to find them; and solve the equation

$$x^4 + 13x^3 + 33x^2 + 31x + 10 = 0,$$

which has three equal roots.

51. Explain *Newton's* method of approximating to the roots of an equation, and shew that its accuracy does not depend upon the ratio of the quantity assumed to the root, but upon its being nearer to one root than to any other.

52. The roots of the equation $x^3 - px^2 + qx - r = 0$ are a, b, and c; transform it into one the roots of which are

$$\frac{a}{b} + \frac{b}{a}, \quad \frac{a}{c} + \frac{c}{a}, \quad \frac{b}{c} + \frac{c}{b}.$$

53. Investigate *Waring's* rule for the solution of a biquadratic equation.

54. Impossible roots enter equations by pairs. 1831

55. Investigate Waring's rule for the solution of a biquadratic equation, and shew that the reducing cubic equation is soluble by Cardan's rule, only in the case where two roots of the given equation are possible and two impossible.

56. If two roots of the cubic $x^3 - qx + r = 0$

be $a + b\sqrt{-3}$ and $a - b\sqrt{-3}$,

$$\text{then will } -\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = (b - a)^3.$$

57. Required the conditions to be satisfied, that in the division of $x^4 + qx^2 + rx + s$ by $x^2 + ax + b$, the remainder may be 0 independently of the value of x, after three terms of the quotient are obtained; and when these conditions are satisfied, obtain an equation for finding b from q, r, s, given, and shew how to solve it.

58. Prove that in any equation, the greatest negative coefficient increased by unity is greater than the greatest root. Find also a limit less than the least positive root.

59. Find the sum of the m^{th} powers of the roots of an equation of n dimensions in terms of the sums of inferior powers.

60. A recurring equation of an even number of dimensions may be solved by means of an equation of half the number of dimensions; prove this, and solve the equation

$$x^4 + 4x^3 - 5x^2 + 4x + 1 = 0.$$

1832 61. Shew that

$$nx^{n-1} - (n-1)px^{n-2} + (n-2)qx^{n-3} \dots - Q$$

is the sum of the products of every $(n - 1)$ simple factors of

$$x^n - px^{n-1} + qx^{n-2} - \dots - Qx + R.$$

62. Solve the equation $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$, and shew that the roots are of the form $a, \frac{1}{a}, b, \frac{1}{b}$.

63. Shew that the rational roots of any numerical equation can always be found. Mention any ways of shortening the operation; and apply the method to the equation

$$3x^3 - 2x^2 - 6x + 4 = 0.$$

64. If e be the sum of two roots of the equation

$$x^4 + qx^2 + rx + s = 0,$$

shew (without consideration of the reducing cubic) that e^2 has only three *different* values.

65. Every equation of an even number of dimensions, of which the last term is negative, must have at least two possible roots, one positive, the other negative.

66. The coefficient of the second term of an equation, with its proper sign, is the sum of the roots with their signs changed; the coefficient of the third term is the sum of the products of every two roots with their signs changed, &c., and the last term is the product of all the roots with their signs changed. 1833

67. Find a number next greater than the greatest positive root, and next less than the least negative root of the equation

$$x^3 - 4x^2 - 4x + 20 = 0.$$

68. Determine the relation which exists between the coefficients of the equation $x^3 - px^2 + qx - r = 0$, when the roots are in arithmetic progression.

69. If p be the coefficient of the second term of an equation of n dimensions, shew that the sum of the roots of it and all its successive limiting equations is equal to $-p.\dfrac{n+1}{2}$.

70. No equation can have more positive roots than it has changes of sign, nor more negative roots than it has continuations of the same sign. If one term be wanting, what inference may be drawn respecting the number of possible roots?

71. Solve the equation $x^3 + 6x = 2$, by Cardan's method, and shew that the possible root is .32748, having given

$$\log 2 = .30103,$$
$$\log 1.58740 = .200686,$$
$$\log 1.25992 = .100343.$$

72. Impossible roots enter equations by pairs; make this appear also from geometrical considerations. 1834

73. Shew how to transform an equation into one which shall want its second or third term: under what circumstances may both be made to disappear by one operation? Solve the equation

$$x^3 - 3x^2 - x + 3 = 0.$$

74. Every equation whose roots are possible has as many changes of sign as it has positive roots, and as many continuations of the same sign as it has negative roots. State the proposition when the roots are not all possible. Hence shew that all the real roots of

$$x^5 + x^4 + x^2 - 25x - 36 = 0$$

lie between -10 and $+10$.

75. Solve the recurring equation $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0$; and having given that $2\frac{1}{2}$ is an approximation to a root of $x^3 - 5x - 3 = 0$, find its exact value to four places of decimals.

76. Each of a series of numbers is the sum of two roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \ldots = 0;$$

prove that the symmetrical function, formed by combining these numbers in products, three taken together, is equal to

$$p^3.\frac{(n-1)(n-2)(n-3)}{1.2.3} + p.q.(n-2)^2 + r.(n-4).$$

77. Express the sum of any powers of the roots of an equation in terms of the coefficients, and sums of the inferior powers. Shew that the roots a, b, c, &c. of the equation

$$x^n + p_1x^{n-1} + p_3x^{n-3} + p_1p_3x^{n-4} + \ldots = 0.$$

Satisfy the equations

$$(a + b + c + \ldots .)^2 = a^2 + b^2 + c^2 + \ldots .$$
$$(a + b + c + \ldots .)^4 = a^4 + b^4 + c^4 + \ldots .$$

1835 78. Supposing $a_1, a_2 \ldots a_n$ to be the n roots of an equation $X = 0$, shew that $X = (x - a_1)(x - a_2) \ldots (x - a_n)$.

79. Shew how the solution of a biquadratic equation may be made to depend upon that of a cubic; and exhibit the roots of the biquadratic in terms of the three roots of the cubic.

80. Shew that all the rational roots of an equation can always be found. Apply the general method to solve the equation

$$2x^3 - 3x^2 + 2x - 3 = 0.$$

81. Every biquadratic equation of the form $x^4 + px^3 + qx^2 + rx + s = 0$, can be solved by means of a quadratic equation, if $p^3 - 4pq + 8r = 0$.

82. Form an equation, whose roots differ by a given quantity from those of $x^n + px^{n-1} + \ldots + Q = 0$. Explain the use of this process to determine a superior limit of the roots of an equation. 1836

$$\text{Ex. } x^4 - 2x^3 - 3x^2 - 15x - 3 = 0.$$

83. Every equation of an even number of dimensions, and whose last term is negative, has at least two real roots.

84. Take away the third term from the equation

$$x^4 - 18x^3 - 60x^2 + x - 2 = 0.$$

Find the equation whose roots are the squares of the differences of the roots of the equation $x^3 + qx + r = 0$.

85. The equation $x^3 - qx^2 + r = 0$ has two impossible roots if $\frac{r}{4}$ be greater than $\frac{q^3}{27}$.

Also, in general, an equation cannot have all its roots possible if the coefficient of any term be less than a mean proportional between those of the adjacent terms.

86. Shew what conclusions may be drawn as to the number of positive and negative roots of an equation, from observing the signs of its terms. Apply them to determine the nature of the roots of the equation

$$x^5 - 2x^4 - 4x + 8 = 0.$$

SECTION IV.

QUESTIONS IN PLANE TRIGONOMETRY.

1821 1. Prove *Demoivre's* formula

$$(\cos A \pm \sqrt{-1} \,.\, \sin A)^m = \cos mA \pm \sqrt{-1} \,.\, \sin mA.$$

2. Why is *Cardan's* formula for the solution of a cubic equation inapplicable when all the roots are possible? solve the equation in this case by trigonometrical formulæ, and reduce the results for logarithmic computation.

3. Given two sides and the included angle of a plane triangle: find the remaining parts, and reduce the results to logarithmic computation.

4. The sides of a plane triangle are 3, 5, 6: compare the radii of the inscribed and circumscribed circles.

1822 5. If $1, \rho, \rho^2, \ldots\ldots \rho^{n-1}$ are the roots of the equation $x^n - 1 = 0$; find the value of

$$1^m \,.\, \rho^r + \rho^m \,.\, 1^r + \ldots\ldots + \rho^m \,.\, \rho^{2r} + \rho^{2m} \,.\, \rho^r + \ldots\ldots$$

6. Prove that

$$2(\sin^2 A \sin^2 B + \cos^2 A \cos^2 B) = 1 + \cos 2A \cos 2B.$$

7. In an isosceles plane triangle, prove *geometrically* that the versed sine of the vertical angle : radius :: the square of the base : twice the square of either side.

8. Prove that $\cos(A + B) \sin(A - B) + \cos(B + C) \sin(B - C) + \cos(C + D) \sin(C - D) + \cos(D + A) \sin(D - A) = 0$.

9. Investigate the following formulæ:

$$\cos\theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2}, \quad \sin\theta = \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2\sqrt{-1}};$$

and thence prove that

$$\frac{\sin A}{1-\cos A} = \cot\frac{A}{2}.$$

10. Given the ratios of the sines of the angles of a plane triangle, and the radius of the inscribed circle, to construct the triangle.

11. Find the values of θ which satisfy the equation 1823

$$2\sin^2 3\theta + \sin^2 6\theta = 2, \text{ radius being } = 1.$$

12. The angles of a plane triangle are A, B, C; it is required to prove that the perimeter of the triangle : the diameter of its inscribed circle :: $\text{rad}^3 : \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

13. Prove *geometrically* that

$$\text{versin } A : \sin A :: \tan\frac{A}{2} : \text{rad}.$$

14. In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.

15. Find the number of different triangles into which a polygon of n sides may be divided by lines joining the angular points.

16. If through any point O within a triangle, three straight lines be drawn from the angles A, B, C meeting the opposite sides in a, b, c, then will $\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1$.

17. The product of all the lines that can be drawn from one of the angles of a regular polygon of n sides, inscribed in a circle whose radius is a, to all the other angular points $= na^{n-1}$.

18. Extract the square root of $\cos 4A \pm \sqrt{-1}\sin 4A$.

19. If 2S be equal to the sum of the sides of a plane triangle, A, B the angles opposite to the sides a, b, respectively, then is

$$\sin^2\frac{A}{2} : \sin^2\frac{B}{2} :: \frac{a}{b} : \frac{S-a}{S-b}.$$ Required a proof.

20. In any right-angled plane triangle, prove that twice the side of the inscribed square is an harmonical mean between the sides containing the right angle.

21. Shew that $\log \sec \theta = \frac{\tan^2 \theta}{2} - \frac{\tan^4 \theta}{4} + \frac{\tan^6 \theta}{6} - \&c.$ and thence deduce the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \textit{ in infinitum.}$$

22. Prove the following formulæ:

$$\tan^2 \frac{A}{2} = \frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} \text{ and}$$

$$\tan nA = \frac{\sin A + \sin 3A + \ldots \text{ to } n \text{ terms}}{\cos A + \cos 3A + \ldots \text{ to } n \text{ terms}}; \text{ and}$$

from the former deduce the tangent of 15°.

1824 23. Find the value of A in the equation

$$\tan A + 2 \cot 2A = \sin A \left(1 + \tan A \tan \frac{A}{2}\right).$$

24. In a triangle whose sides are a, b, c, and A, B, C the angles opposite, having given B, a and the area of the triangle, find the remaining sides and angles.

25. Prove that the area of a regular polygon *inscribed* in a circle, is a *geometric* mean between the areas of an inscribed and of a circumscribed regular polygon of half the number of sides; and that the area of a regular polygon *circumscribed* about a circle, is an *harmonic* mean between the areas of an inscribed regular one of the same number of sides, and of a circumscribed regular one of half that number.

26. Investigate the expression for the area of a plane triangle in terms of the sides; and apply the result to the case where the sides are 24, 30, 18.

27. D is a point within a triangle ABC; having given AB, AC, and the angles ABD, ACD, BDC, find BC.

28. Having given the area, the base, and the sum of the angles at the base, find the difference of the angles at the base.

29. Prove that $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ to rad. 1. 1825

30. Find $\tan 5A$ in terms of $\tan A$.

31. Express a circular arc in a series in terms of its tangent.

32. If a, b, c, d be the sides of a quadrilateral inscribed in a circle, and $S = \dfrac{a+b+c+d}{2}$; prove that the area of the quadrilateral

$$= \sqrt{(S-a)(S-b)(S-c)(S-d)}.$$

33. Express the chord of 36° to radius unity in a continued fraction.

34. Express $\cos nA$ in terms of $\cos A$, n being a whole number.

35. Given two sides of a triangle and the difference of the angles opposite to them. Required the other angle.

36. Solve the equation $x^6 + 1 = 0$.

37. In the cubic equation $x^3 - qx - r = 0$, where $\dfrac{r^2}{4}$ is less 1826
than $\dfrac{q^3}{27}$, shew that the solution may be effected by means of a table of natural sines; and explain why the accuracy of this method cannot be depended upon in practice.

38. Having given the sine of 12°, find the sine of 48°.

39. Shew that $\text{vers}\,(\pi - A) = 2\,\text{vers}\,\frac{1}{2}(\pi + A)\,\text{vers}\,\frac{1}{2}(\pi - A)$, when the radius $= 1$.

40. Express the secant and cosecant of the sum and difference of two arcs in terms of the secants and cosecants of the simple arcs.

41. If a quadrantal arc be divided into two parts in the ratio of $2n : 1$, of which θ is the less; prove that

$$\sin\theta \sin 3\theta \sin 5\theta \ldots \sin(2n+1)\theta = \frac{1}{2^n}.$$

42. Divide the angle A into two parts, so that their versed sines may be in the given ratio of $m : n$.

43. Find $\sin\theta$ from the equation

$$\sin 3\theta - 2\sin 2\theta + \sin^2\theta + 4\sin^3\theta = 0.$$

44. Having given the radius of a circle, find the area and perimeter of a regular octagon inscribed in it, and compare them with the area and perimeter of the circumscribing octagon.

1827 45. Given the base, the vertical angle, and the difference of the sides of a plane triangle; find the remaining angles.

46. Expand $\sin x$ in powers of x, state whether in your result x is expressed in seconds or in parts of the radius, and convert it into the other.

47. How is the multiplication of two high numbers facilitated by a table of squares, or by one of cosines?

48. Shew that with any four lines, each of which is less than the sum of the others, it is possible to construct a trapezium which may be inscribed in a circle.

49. If AB be the diameter of a circle, C the centre, FAG a tangent at A, ACF one-third of a right angle, and FG triple of the radius, BG being joined will be very nearly equal to the half circumference.

50. In a plane triangle ABC, when the side b is much less than a, the angle B may be found by the formula

$$B = \frac{b}{a}\frac{\sin C}{\sin 1''} + \frac{b^2}{a^2}\frac{\sin 2C}{\sin 2''} + \frac{b^3}{a^3}\frac{\sin 3C}{\sin 3''} - \ldots.$$

51. Prove geometrically that if A, B be any two arcs, rad $\times$ $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

52. Given a, b, c the sides of a plane triangle, find the radius of the inscribed circle.

53. If $C_0 = \cos a\cos b\cos c \ldots$, $C_n =$ the sums of the products of all the cosines but n, multiplied by the sines of those n,

$$\cos(a + b + c + \ldots) = C_0 - C_2 + C_4 - \ldots$$
$$\sin(a + b + c + \ldots) = C_1 - C_3 + C_5 - \ldots$$

54. If $\tan\theta$ be assumed $= \dfrac{b}{a}$,

$$\sqrt[n]{a \pm b\sqrt{-1}} = (a^2 + b^2)^{\frac{1}{2n}}\left\{\cos\frac{\theta}{n} \pm \sin\frac{\theta}{n}\sqrt{-1}\right\}.$$

55. If A, B, C be the angles, a, b, c the sides of a plane triangle,

$$\sin(A - B) : \sin C :: a^2 - b^2 : c^2.$$

56. Given the sines and cosines of two arcs, find the sine of their sum and difference. 1828

57. If the side of a pentagon inscribed in a circle be 1, the radius is

$$\frac{\sqrt{5 + \sqrt{5}}}{\sqrt{10}}.$$

Prove this, and hence find the sine of 36° to five places of decimals.

58. If a, b, c be the sides of a plane triangle, and A the angle opposite to a, and $S = \frac{a + b + c}{2}$, prove the four following formulæ:

$$\sin A = \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)};$$

$$\sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}};$$

$$\cos\frac{A}{2} = \sqrt{\frac{S(S-a)}{bc}};$$

$$\tan\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}.$$

State under what circumstances each of these formulæ may be used with the greatest advantage.

59. Sum the following series:

$$\left.\begin{array}{l}\cos a + \cos 3a + \cos 5a + \ldots \\ \text{and } \tan a + 2\tan 2a + 2^2\tan 2^2 a + \ldots\end{array}\right\} \text{to } n \text{ terms.}$$

60. Solve the equation $x^3 - 6x = 4$, by Trigonometry, and obtain a numerical result.

61. Given the radius of the circumscribed circle and the three angles of a triangle; find expressions for the three sides.

62. Having given two sides and the included angle of a plane triangle, find the angles at the base: find also the third side by an independent method, and adapt the trigonometrical expressions to logarithmic computation. 1829

63. Prove the formula

$$\sin(a+b) = 2\sin a - \sin(a-b) - 4\sin a \sin^2\frac{b}{2};$$

and explain fully its use in the construction of the Trigonometrical Canon.

64. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Find the ratio of the sides, and the value of the largest angle.

65. Expand the n^{th} power of sin A in terms of sines and cosines of multiples of A; and write down the last term with its proper sign when n is of the form $4m + 2$.

66. Given $\tan 3A = n \tan A$. Find A in terms of n: find also the value of n that A may be $15°$.

67. If R, r be the radii of the circumscribed and inscribed circles of a regular polygon of m sides, and R′, r' the corresponding radii for a regular polygon of $2m$ sides and of the same perimeter as the former, then $Rr' = R'^2$ and $R + r = 2r'$.

1830 68. Find the sines of the sum and difference of two arcs in terms of the sines and cosines of the arcs themselves.

69. Given the sides of a plane triangle, find the cosine of an angle; investigate formulæ adapted to logarithmic computation for the solution of the triangle, and explain which of the methods is best in particular cases.

70. Given tan A, tan B . . . find tan (A + B + ...), and thence deduce tan 7A.

71. Find the value of $\frac{1}{(\sin\theta)^2} - \frac{1}{\theta^2}$ when $\theta = 0$.

72. Given
$$\left.\begin{aligned} \tan\theta + \tan\phi + \tan\psi &= 1 + \frac{4}{\sqrt{3}}, \\ \tan\theta\tan\phi + \tan\theta\tan\psi + \tan\phi\tan\psi &= 1 + \frac{1}{\sqrt{3}}, \\ \tan\theta\tan\phi\tan\psi &= 1, \end{aligned}\right\}$$
find θ, ϕ, and ψ; and sum the series

$$(\sec\theta)^2 + \left(\frac{1}{2}\sec\frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2}\sec\frac{\theta}{2^2}\right)^2 + \left(\frac{1}{2^3}\sec\frac{\theta}{2^3}\right)^2 + \ldots \textit{ad inf.}$$

73. In any polygon with n sides A_1A_2, A_2A_3, ... respectively represented by a_1, a_2 . . . . prove that

$$a_1 \sin A_1 - a_2 \sin (A_1 + A_2) + a_3 \sin (A_1 + A_2 + A_3) - \dots \pm a_{n-1} \sin (A_1 + A_2 + A_3 + \dots + A_{n-1}) = 0.$$

74. Resolve $x^m - 1$ into its simple and quadratic factors.

75. If a and b be the sides of a plane triangle, A and B their opposite angles, then will

$$\text{hyp. log}\, b - \text{hyp. log}\, a = \cos 2A - \cos 2B + \tfrac{1}{2}(\cos 4A - \cos 4B) + \tfrac{1}{3}(\cos 6A - \cos 6B) + \dots$$

76. Investigate an expression for the sine of an angle of a plane triangle in terms of the sides. 1831

77. Prove that

$$\cos n\theta \pm \sqrt{-1} \sin n\theta = (\cos \theta \pm \sqrt{-1} \sin \theta)^n,$$

whether n be integral or fractional.

78. Express $(\sin A)^{4n+1}$ in terms of the sines of the multiple arcs.

79. From a station B at the base of a mountain, its summit A is seen at an elevation of 60°; after walking one mile towards the summit up a plane making 30° with the horizon, to another station C, the angle BCA is observed to be 135°. Find the height of the mountain in yards.

80. An indefinite area is to be divided into similar and equal *regular* figures. Shew by what figures this can be done. Also, if three equal areas be divided into the same number of equal *regular* figures, which are respectively triangular, square, and hexagonal, shew that the sum of the lengths of the dividing lines in the cases of triangular, square, and hexagonal divisions, are to one another as $\sqrt[4]{27} : \sqrt[4]{16} : \sqrt[4]{12}$; the whole area to be divided being very great in comparison of one of the divisions.

81. If a be less than 45°, shew that

$$2 \sin\left(\frac{\pi}{4} \pm a\right) = \sqrt{1 + \cos 2a} \pm \sqrt{1 - \cos 2a},$$

$$\text{and } 2 \cos\left(\frac{\pi}{4} \pm a\right) = \sqrt{1 + \cos 2a} \mp \sqrt{1 - \cos 2a}.$$

1832 82. Prove that $\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$; and find $\sin 3A$ in terms of $\sin A$.

83. Assuming $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, deduce an expression for the area of a triangle in terms of the sides.

84. A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 55° with a horizontal line drawn from his eye; receding backwards 30 feet, he then finds it to subtend an angle of 48°. Determine the breadth of the river.

$$\log \sin 7^\circ = 9.08589, \quad \log \sin 35^\circ = 9.75859,$$
$$\log \sin 48^\circ = 9.87107, \quad \log 3 = 47712,$$
$$\log 1.0493 = .02089.$$

85. Determine the distance between two inaccessible objects, by observations made at two stations, the distance between which is known.

86. Solve the cubic $x^3 + qx + r = 0$ by trigonometrical formulæ, in the case in which the three roots are possible.

87. If a straight line bisect at right angles any side AB of a regular polygon of an odd number of sides, shew that it will pass through the point of intersection of the two sides of the polygon which are most remote from AB.

If $2n + 1$ be the number of sides, prove that the length of the part of the bisecting line within the polygon is equal to $\dfrac{AB}{2} \times \cot \dfrac{\pi}{2(2n+1)}$.

88. If the tangents of all arcs less than 45° be found, the tangents of all greater arcs can be found by addition. Prove also that $4 \sin(\theta - a) \sin(m\theta - a) \cos(\theta - m\theta)$

$$= 1 + \cos(2\theta - 2m\theta) - \cos(2\theta - 2a) - \cos(2m\theta - 2a).$$

89. Resolve $x^m + 1$ into its factors, m being odd.

90. Resolve $\sin\theta$ into its factors. Shew also that in the determination of an angle, which is nearly 90°, from its logarithmic sine, a small error in the logarithmic sine will produce a large error in the angle.

91. Find the sine of 36°. Prove the formula

$\sin A = \sin(36^\circ + A) + \sin(72^\circ - A) - \sin(36^\circ - A) - \sin 72^\circ$, and mention its use.

92. Define an angle and the sine and cosine of an angle. 1833

Prove that $\cos B - \cos A = 2 \sin \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$,

and in a plane triangle that $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, where C is an obtuse angle.

93. Determine the distance between two visible but inaccessible objects in the same plane with an observer.

94. If A, B, C be the angles of a plane triangle, a, b, c any points in the sides respectively opposite to them, prove that the lines joining A, a; B, b; C, c, respectively will intersect in a point if

$$\frac{Ac}{Ab} \cdot \frac{Ba}{Bc} \cdot \frac{Cb}{Ca} = 1.$$

95. When x is possible, shew that all the possible values of

$$(a + b\sqrt{-1})^x + (a - b\sqrt{-1})^x$$

are contained in the formula

$$2R \cos(2n\pi + \theta)^x,$$

where R is the arithmetical value of $(a^2 + b^2)^{\frac{x}{2}}$, n is any integer, and $\theta = \cos^{-1} \frac{a}{\sqrt{a^2 + b^2}}$.

96. Expand the cosine of a multiple arc in terms of the powers of the cosine of the simple arc.

97. Explain the construction of a table of logarithmic sines and cosines.

98. Prove that $\frac{501 + 80\sqrt{10}}{240}$ is a close approximation to the known numerical value of the semi-circumference of a circle whose radius is 1.

99. Find the area of an equilateral and equiangular polygon of n sides, circumscribed about a circle whose radius is r. What is its ultimate value, when n is increased indefinitely?

100. If R, r, be the radii of circles circumscribed about and inscribed in the same plane triangle, prove that the distance of the centres of these circles $= \sqrt{R^2 - 2Rr}$.

101. Shew that

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239},$$

and prove that π is an incommensurable quantity.

1834 102. Define the sine and cosine of an angle; prove that

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B;$$

and write down a general formula for all angles whose cosine $= - \cos A$.

103. Express the cosine of half an angle of a triangle in terms of the sides; and explain in what cases the formula may be used with advantage in determining the angle when the sides are given.

104. Explain the use of subsidiary angles in adapting algebraical formulæ to numerical calculation.

Reduce the expression

$$\frac{\sqrt{a-b}}{a+b} + \frac{\sqrt{a+b}}{a-b}$$

to a form adapted to logarithmic calculation.

105. Find x from the equation

$$nx = (\sqrt{1+x} - 1)(\sqrt{1-x} + 1),$$

shewing that if $n = \tan\frac{a}{2}$, $x = \sin 2a$.

Also, prove that the equation

$$x^4 + rx + s = 0$$

has equal roots when $\left(\frac{r}{4}\right)^4 = \left(\frac{s}{3}\right)^3$, and no real root when $\left(\frac{r}{4}\right)^4 < \left(\frac{s}{3}\right)^3$.

106. All the values of θ which satisfy the equation

$$\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{6}$$

are comprised in the formula $\left(n + \frac{1}{5} \pm \frac{1}{10}\right)\pi$, (n) any integer; and all or none of the values of $\log(-a)^n$ are comprised among those of $\log(+a)^n$, according as (n) is even or odd.

107. When a quadrilateral is capable of having a circle inscribed in it, the sums of the opposite sides are equal to one another; and if, besides, it is capable of having one circumscribed about it, its area equals the square root of the continued product of the sides.

108. Shew that the increment of the logarithmic sine of an angle varies nearly as the increment of the angle. State the exceptions. Find tab. log sin 17° 0′ 12″, having given tab. log sin 17° 1′ = 9.4663483, tab. log sin 17° = 9.4659353.

109. Express $(\sin\theta)^{4m+1}$ and $(\cos\theta)^{2m}$ in terms of the cosines and sines of the multiples of θ. Write down the value of $(\sin\theta)^5$.

110. The area of any triangle is to the area of the triangle, whose sides are respectively equal to the lines joining its angular points with the middle points of the opposite sides, as 4 to 3.

111. In the series

$$\cot a + \operatorname{cosec} a + \frac{1}{2}\cot\frac{a}{2} + \frac{1}{2}\operatorname{cosec}\frac{a}{2} + \frac{1}{2^2}\cot\frac{a}{2^2} + \frac{1}{2^2}\operatorname{cosec}\frac{a}{2^2}\ldots$$

shew that the terms beginning with the 3rd are alternately an arithmetic and geometric mean between the two preceding; and if $a_1, a_2, \ldots$ be the arcs of a circle, radius measured from the same point, each of which is equal to its tangent, prove that

$$a_n + \frac{1}{a_n} = (2n+1)\frac{\pi}{2} \text{ nearly.}$$

112. Determine the radius of a circle which touches each of three straight lines which cut one another in terms of the sides of the triangle which they form. How many such circles can there be?

113. Prove that

$$\cos m\theta = 2^{m-1}\sin\left(\frac{\pi}{2m}+\theta\right)\cdot\sin\left(\frac{3\pi}{2m}+\theta\right)\ldots\sin\left\{\frac{(2m-1)\pi}{2m}+\theta\right\},$$

and thence resolve $\cos\theta$ into its quadratic factors.

114. Shew how to find $\sin 1'$ by the continued bisection of an angle, and compute its numerical value, having given $\sin(60^\circ \div 2^{12}) = .00025566$. When θ is small, is it more advantageous to determine $\sin\theta$ from the formula involving $\sin 2\theta$, or $\cos 2\theta$?

1835 115. Having given two sides and the included angle of a plane triangle, find the remaining side and angles.

116. Prove that

$$(\cos a + \sqrt{-1}\sin a)(\cos b + \sqrt{\quad 1}\sin b)(\cos c + \sqrt{-1}\sin c)$$
$$= \cos(a+b+c) + \sqrt{-1}\sin(a+b+c):$$

also,

$$(\cos a + \sqrt{-1}\sin a)^{\frac{2}{3}} = \cos\left(\frac{4n\pi+2a}{3}\right) + \sqrt{-1}\sin\left(\frac{4n\pi+2a}{3}\right).$$

117. Shew that if a, b, c be in geometric progression, $\log_a N$, $\log_b N$, $\log_c N$ are in harmonic progression; and that if u represent any root of the equation $x^n - 1 = 0$, two roots of the equation $x^{2n} - 2x^n \operatorname{cosec} 2a + 1 = 0$ are represented by $(\tan a) \pm \frac{1}{n}\cdot u$.

118. Three circles whose radii are a, b, c, touch each other externally; prove that the tangents at the points of contact meet in a point, whose distance from any one of them

$$= \left\{\frac{abc}{a+b+c}\right\}^{\frac{1}{2}}.$$

119. Resolve $x^{2n} - 1$ into its quadratic factors, and write down the result when $n = 5$.

120. If θ be a small arc, $\sin\theta = \theta - \frac{\theta^3}{6}$ nearly. What alteration must be made in this formula when θ is given in seconds?

121. Eliminate θ and θ' from the equations

$$a(\sin\theta)^2 + a'(\cos\theta)^2 = a;$$

$$a'(\sin\theta')^2 + a(\cos\theta')^2 = a'; \quad a\tan\theta = a'\tan\theta';$$

and shew that

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{a} + \frac{1}{a'}.$$

122. If n be any whole number not divisible by 4; then will

$$\left\{\tan\frac{\theta}{n} + \tan\frac{2\pi+\theta}{n} + \tan\frac{4\pi+\theta}{n} + \ldots + \tan\frac{2(n-1)\pi+\theta}{n}\right\}$$

$$\times\left\{\cot\frac{\theta}{n} + \cot\frac{2\pi+\theta}{n} + \cot\frac{4\pi+\theta}{n} + \ldots + \cot\frac{2(n-1)\pi+\theta}{n}\right\} = n^2.$$

Prove it when $n = 3$.

123. Let a, b, c be the middle points of the sides of the triangle ABC; and S, s the sums of the squares of the three triangles whose bases are the sides of the triangles ABC, abc respectively, and common vertex *any* other point within or without the *plane* of ABC;

$$\text{shew that } \mathrm{S} - 4s = \frac{(\mathrm{ABC})^2}{4}.$$

124. Prove that in a plane triangle $\tan B = \dfrac{b \sin C}{a - b\cos C}$, and express B in a series proceeding according to sines of the multiples of C.

125. Compare the areas of regular octagons described in and about a circle.

126. Explain the construction of a table of logarithmic 1836
sines. If it be calculated only for angles which contain degrees and minutes, shew how to find the logarithmic sine of an angle which also contains seconds.

127. Prove that $\cos(A + B) = \cos A . \cos B - \sin A . \sin B$; and thence find $\cos 105°$.

Write down a formula for all angles, whose tangent $= -\tan A$.

128. Having given one side and the hypothenuse of a right-angled triangle, find an expression for the logarithm of the

remaining side. If the given side be nearly equal to the hypothenuse, what is the most convenient formula for determining either of the angles?

129. Find the number of degrees and minutes in an angle subtended by that circular arc which is equal to the radius.

130. The angles of a plane triangle form a geometrical progression whose common ratio is $\frac{1}{2}$, shew that the greatest side

$$= 2\,(\text{perimeter})\,.\,\sin 12^\circ\, 51'\, 25''\,\tfrac{5}{7}.$$

Find the sum of n terms of the series

$$1 + \cos x + \cos 2x + \cos 3x + \ldots.$$

Find the probable sum of the same series when the number of terms is not exactly ascertainable, but is known to be not less than p nor greater than q. Explain the two results when n, p, q, and $q - p$ are all infinite.

131. Shew that

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}} + \frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}} = \sqrt{2}.$$

Find a value of x which will render $x - \frac{2}{3}\sin 2x + \frac{1}{12}\sin 4x + \pi\cos^4 x$ a maximum.

What form does the equation

$$\sqrt{y^2 - b^2} = b\tan\left(x + \frac{\sqrt{y^2-b^2}}{a}\right)$$

assume when $b = 0$?

132. Prove the formula

$$(\cos x \pm \sqrt{-1}\,.\sin x)^{\frac{m}{n}} = \cos\left(\frac{2i\pi}{n} + \frac{mx}{n}\right) \pm \sqrt{-1}\,.\sin\left(\frac{2i\pi}{n} + \frac{mx}{n}\right);$$

m, n, i being integers. Hence write down the cube roots of $3 - 4\sqrt{-1}$, in the form $\alpha + \beta\sqrt{-1}$.

133. An object, six feet high, placed on the top of a tower, subtends an angle, whose tangent is .015, at a place whose horizontal distance from the foot of the tower is 100 feet; determine the tower's height.

134. If $a, b, c \ldots g, h$ be any unequal numbers, prove that

$$\frac{1}{(a-b)(a-c)\ldots(a-h)}+\frac{1}{(b-a)(b-c)\ldots(b-h)}+\ldots\ldots$$
$$+\frac{1}{(h-a)(h-b)\ldots(h-g)}=0.$$

Resolve $\cos ax - \sin bx$ into algebraic quadratic factors.

135. Express $(\cos\theta)^n$ in terms of the cosines of multiples of θ. Ex. $(\cos\theta)^6$.

136. Upon what principle are distances, measured on a line which revolves about a fixed point in it, affected with their proper algebraic signs? Upon this principle trace the sign of the secant of an arc through 360°, and of the chord through 720°.

137. Prove the formulæ

$$2\sin x = -\sqrt{1+\sin 2x}+\sqrt{1-\sin 2x}$$
$$2\cos x = -\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}.$$

Between what limits is x comprised?

138. Compare numerically the areas of regular pentagons described in and about a circle; and express the ratio in the form of a continued fraction.

SECTION V.

QUESTIONS IN SPHERICAL TRIGONOMETRY.

1821 1. Given the three angles of a spherical triangle, to find its surface.

2. In the solution of right-angled spherical triangles by *Napier's* rules, what cases are ambiguous?

3. The hypothenuse of a right-angled triangle, whether plane or spherical, being supposed invariable, to compute the corresponding variations of the two sides.

4. Prove that the sides of the polar or supplemental triangle are supplements of the angles of the given triangle.

1822 5. In a spherical triangle, having given two angles and the included side, it is required to find the other angle.

6. Find the surface of an equilateral and equi-angular spherical polygon of n sides, and determine the value of each of the angles when the surface equals half the surface of the sphere.

7. Shew how every case of oblique spherical triangles may be solved by *Napier's* rules *only*.

1824 8. Prove the properties of the *complemental* triangle; and from these properties, and the expressions for the cosine of an angle in terms of the sides, and for the cosine of a side in terms of the angles, deduce *Napier's* rules for the solution of right-angled spherical triangles.

9. If c be the hypothenuse of a right-angled spherical triangle, prove that

$$\sin^2 \frac{c}{2} = \sin^2 \frac{a}{2} \cos^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sin^2 \frac{b}{2}.$$

10. The sum of the three angles of a spherical triangle is 1825
greater than two right angles and less than six right angles. Required proof.

11. In a right-angled spherical triangle whereof c is the 1826
hypothenuse and a and b the sides, prove that

$$\tan \frac{c+a}{2} \tan \frac{c-a}{2} = \left(\tan \frac{b}{2}\right)^2.$$

12. If A, B, and C be the angles, and a, b, and c the sides of a spherical triangle, and if $b + c = \pi$, prove that

$$\sin 2B + \sin 2C = 0.$$

13. Having given six straight lines, of which each is less than the sum of any two, determine how many tetrahedrons can be formed, of which these straight lines are the edges.

14. If A, B, and C be the angles of a spherical triangle, a, b, and c the opposite sides, and δ the distance of a point on the surface of the sphere, equally distant from the angular points; prove that

$$\tan^2 \delta = \frac{\tan^2 \frac{a}{2} \tan^2 \frac{c}{2} - 2 \tan \frac{a}{2} \tan \frac{c}{2} \cos B}{\sin^2 B}.$$

15. Find the locus of the vertices of all right-angled spherical triangles having the same hypothenuse; and, from the equation obtained, prove that the locus is a circle when the radius of the sphere is infinite.

16. Draw through a given point in the side of a spherical 1827
triangle, an arc of a great circle, cutting off a given part of the triangle.

17. If S be half the sum of the sides of a spherical triangle, D its area,

$$\tan \frac{D}{4} = \sqrt{\left\{\tan \frac{S}{2} \tan \frac{S-a}{2} \tan \frac{S-b}{2} \tan \frac{S-c}{2}\right\}}.$$

18. By what modification may formulæ for spherical triangles be adapted to plane ones?

1828 19. In a spherical triangle the sines of the angles are as the sines of the opposite sides.

20. If A, B, C be the angles of any spherical triangle, and a the side opposite to A, prove that

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

21. State *Napier's* rules for the solution of right-angled spherical triangles, and prove the two cases in which the complement of the hypothenuse is the middle part.

1829 22. Find an expression in terms of the sides of a spherical triangle for the arc drawn from one angle C bisecting the opposite side c, and adapt the expression to logarithmic computation.

23. State the construction of the polar triangle, and shew that its sides and angles are respectively the supplements of the angles and sides of the original triangle.

24. Shew that in a small spherical triangle, if $\frac{1}{3}$ of the *Spherical Excess* be subtracted from each of the angles, the resulting angles will be those of a plane triangle having the same sides as those of the spherical triangle.

1830 25. Prove *Napier's* rules for the solution of right-angled triangles when one of the sides is the middle part; and having given one side and an angle opposite to it, solve the triangle and explain whether there is any ambiguity.

26. In a spherical triangle,

$$\cot a \sin b = \cos b \cos C + \sin C \cot A.$$

27. Investigate *Napier's* analogies: shew for what cases in the solution of spherical triangles they are applicable; shew also how these cases may be solved by the aid of *Napier's* rules alone.

28. In a spherical triangle, the sides of which are small compared with the radius of the sphere, having given two sides and the included angle, find the angle between the chords of those two sides.

29. Given the two sides and the included angle of a spherical triangle, required its area; and from the expression obtained, find the area of a plane triangle in corresponding terms.

30. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles. 1831

31. Having given two sides and the included angle of a spherical triangle, obtain the third side in a formula convenient for logarithmic computation.

32. Having given the sides of a spherical triangle, find the sine of one of its angles.

33. Having given the hypothenuse and one angle of a right-angled spherical triangle, determine the remaining angle and sides. Is there any ambiguity in the determination of the side opposite to the given angle? 1832

34. In a spherical triangle

$$\tan \frac{A+B}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cdot \cot \frac{C}{2}.$$

35. Having given the three sides of a spherical triangle, find the cosine of one of its angles; and having given the three angles, find the cosine of one of the sides.

36. ABC is a spherical triangle, and CD the arc of a great circle drawn from the angle C to the point of bisection of AB; prove that

$$\cos . AC + \cos . BC = 2 \cos \tfrac{1}{2} AB \times \cos . CD.$$

Shew from this expression that if ABC be a plane triangle,

$$AC^2 + BC^2 = 2AD^2 + 2CD^2.$$

37. Having given two sides and the included angle of a spherical triangle, determine its area.

38. In a spherical triangle any one side is less than the sum of the two others, and the sum of the three angles is greater than two right angles, and less than six. 1833

39. By what alteration of the circular parts are *Napier's* rules applicable to the solution of quadrantal triangles? In a quadrantal triangle, having given an angle and a side opposite to it, deduce all the other parts.

40. If a_1, a_2, a_3, a_4; D_1, D_2 represent respectively the arcs of great circles which form the sides and diagonals of a quadrilateral figure described upon the surface of a sphere, prove that the cosine of the arc joining the middle points of the diagonals is equal to

$$\frac{\cos a_1 + \cos a_2 + \cos a_3 + \cos a_4}{4 \,.\, \cos \frac{D_1}{2} \,.\, \cos \frac{D_2}{2}}.$$

41. Express the sine of an angle of a spherical triangle in terms of the sides, and adapt the result to logarithmic computation.

42. Having given two sides a, b of a triangle, plane or spherical, and the included angle C, to find the variation produced in A corresponding to a given small variation of C.

43. Find the relation between two sides and two angles of a spherical triangle, one of the angles being included by the sides; and thence deduce the corresponding relation in a plane triangle.

1834 44. Explain the construction and prove the properties of the polar triangle.

45. Enunciate *Napier's* rules for the solution of right-angled spherical triangles; and prove them when the middle part is one of the sides containing the right angle.

46. If a, b, be the radii of the inscribed and circumscribed spheres of a regular tetrahedron, r, r' the radii of spheres to which the edges, and one face and the planes of the three others produced, are respectively tangents, prove that

$$r = \sqrt{ab},\ r' = \sqrt{2ab}.$$

47. In a spherical triangle, given one of the sides adjacent to the right angle, and the angle opposite to it, find by *Napier's* rules the other parts; and shew that there are two triangles which satisfy the conditions, the unknown quantities in them being supplementary to one another.

48. If ABC be an equilateral spherical triangle, p the pole of small circle circumscribing it, Q any other point on the surface of the sphere; prove that

$$\cos QA + QB + QC = 3 \cos pA \,.\, \cos pQ.$$

49. If the two sides CA, CB of a spherical triangle CAB be quadrants, shew that C is the pole of the arc AB. 1835

50. In applying *Napier's* rules to the solution of quadrantal spherical triangles, what are to be considered the circular parts? Having given the two sides, find the angle opposite to the quadrantal side.

51. Express the cotangent of the angle of a spherical triangle in terms of another angle and the sides including it, and reduce the formula to one adapted to logarithmic computation.

52. If in a right-angled spherical triangle a, β be the arcs drawn from the right angle respectively perpendicular to, and bisecting the hypothenuse c, shew that

$$\sin\frac{c}{2} = \frac{\sin\beta}{\sqrt{1+(\sin a)^2}}:$$

determine also the cosine of a side in terms of a and β.

53. In a spherical triangle, determine the limits of the sum of the angles. Prove also that the difference between any angle and the sum of the other two is less than 180°.

54. If one angle and the hypothenuse of a right-angled spherical triangle be given, shew that there is no ambiguity in determining the other parts. In what case is the solution of a right-angled triangle ambiguous?

55. The three angles are not sufficient data for the determination of a plane triangle; explain why they are sufficient in a spherical triangle.

In a spherical triangle having given two sides and an angle opposite to one of them, determine the other parts.

56. The arc of a great circle, which joins the middle point of a side of a spherical triangle with its pole, bisects the angle formed by the arcs which join the pole with the poles of the other sides.

57. If one angle of a triangle, either plane or spherical, be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

SECTION VI.

QUESTIONS IN ANALYTICAL GEOMETRY.

1821 1. Find the equation of a straight line, which shall pass through two points whose coordinates are given.

1822 2. Given the base of a plane triangle and the difference of the angles at the base, to find the curve traced by the vertex.

3. If the area of a curve between any two values of one abscissa can be expressed in finite terms, shew that the area between two values of any other abscissa of the same curve can be found.

4. A normal drawn to a cissoid at the point where it cuts the generating circle, meets the axis produced in a certain point; prove that the line intercepted between this point and the vertex of the cissoid is divided into three equal parts by the centre and the further extremity of the diameter, of the generating circle.

5. A ship (P) begins to sail towards another (B) from a fixed point (C). At the same instant B begins to move in a direction perpendicular to P's first motion: P is always found in the line joining C and B; but can only accelerate her rate of sailing so as to retain the same distance from B as at first. What is the curve traced by P?

1823 6. AN and NP are the abscissa and ordinate of a cissoid, the diameter of whose generating circle is AP: AP is joined and NQ always taken equal to it. Prove that the whole area of the curve traced out by Q : AB^2 :: 4 : 3.

7. Inscribe a semicircle in a quadrant. 1824

8. If α, β, γ .... be the roots of the equation $X = 0$, and A, B, C .... the results when these roots are substituted for x in the limiting one; then will

$$y = \frac{X}{A(x-\alpha)}a + \frac{X}{B(x-\beta)}b + \frac{X}{C(x-\gamma)}c + \ldots.$$

be the equation of the parabolic curve which passes through the points of which α, β, γ... are the abscissas, and a, b, c... the corresponding ordinates.

9. If two lines SP, HP revolve about the points S, H, so that $SP \times HP = CS^2$, (C being the middle point of SH) then the locus of the point P is the Lemniscate of *Bernoulli*.

10. Describe a circle about a given segment of a parabola made by an ordinate perpendicular to the axis.

11. Given the hypothenuse of a right-angled triangle and the side of an inscribed square. Required the two sides of the triangle. 1825

12. Determine the locus of a point so situated within a plane triangle, that the sum of the squares of the straight lines drawn from it to the angular points is constant; if the curve has a centre, determine its position. 1826

13. What are the lines traced by the vertex and the focus of a parabola rolling on another equal to it, the vertices coinciding in one position? 1827

14. Explain the method of Geometrical Analysis, and by it solve the problem. In a given square to inscribe another square having its side equal to a given straight line. To what limitation is this line subject?

15. Draw a straight line touching a circle at a given point, without any other instruments besides a parallel ruler and a pencil.

16. If any two circles, the centres of which are given, intersect each other, the greatest line which can be drawn through either point of intersection and terminated by the circles is independent of the diameters of the circles. 1828

17. Give a construction depending upon the cycloid, for determining an arc equal to its cosine.

18. Find the cosine of the angle contained between two straight lines whose equations are

$$y = ax + b, \text{ and } y = a'x + b'.$$

1829 19. Find the equation to the curve in which the distance of any point from a given fixed point is equal to the perpendicular drawn from the same point in the curve upon a given line.

20. Having given the equation to a straight line, find the equation to another straight line drawn perpendicular to it from a given point; find also the length of the perpendicular.

21. Find the rectangular equation to the conchoid of *Nicomedes*, and draw a tangent to the curve.

22. If from two fixed points in the circumference of a circle, straight lines be drawn intercepting a given arc and meeting without the circle, the locus of their intersection is a circle.

23. Given the equations of two straight lines, find the equation to a third which shall pass through their point of intersection and make equal angles with them; and shew from the result that there are two straight lines at right angles to each other which satisfy the question.

24. In the general equation of the second degree

$$ay^2 + bxy + cx^2 + ex + fy + g = 0,$$

shew in what cases the curve will be an ellipse, hyperbola and parabola; and find the coordinates of the centre in the former case.

25. If a curve have as many asymptotes as it has dimensions, and a right line be drawn which cuts them all, the parts of the line measured from the asymptotes to the curve will together be equal to the parts measured in the same direction from the curve to the asymptotes.

1830 26. Find the equation to a straight line passing through a given point, and cutting a given straight line at a given angle. Required also the coordinates of the point of intersection.

27. If $y = ax + b$, and $y = a'x + b'$, be the equations to two straight lines in the same plane; prove that the cosine of the angle contained between them is equal to

$$\frac{1 + aa'}{\sqrt{(1 + a^2)(1 + a'^2)}}.$$

28. A straight line revolving in its own plane about a given point intersects a curve line in two points; find the curve when the rectangle of the lines intercepted between the given point and the points of intersection is constant.

29. Two straight lines, which are always tangents to a given parabola, are so inclined to the axis of x that the sum of the cotangents of the angles which they make with that axis is constant; prove that the locus of their intersections is a straight line parallel to the axis.

30. Find the magnitudes and positions of the principal axes of the curve of the second order, the equation of which is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

31. x, y and X, Y are the coordinates of a point referred to two systems of rectangular coordinates having a common origin and inclined to each other at a given angle: find the relation subsisting between x, y, X and Y.

32. Having given the equations to two straight lines in the 1831
same plane, find the tangent of the angle contained between them.

33. If a, b, c be the lengths of the chords of three arcs of a circle, which together make up a semi-circumference, and r the radius of the circle, then

$$4r^3 - (a^2 + b^2 + c^2)\,r - abc = 0.$$

34. Transform the equation to a plane curve, from one system of coordinates to another inclined at a given angle to the former, and having a different origin, both systems being rectangular; and take the example $y^2 = mx + nx^2$.

35. If a straight line be drawn from a given point perpendicular to a given straight line, find the coordinates of the point of intersection, and the length of the perpendicular.

1832 36. When a series of algebraical quantities is to be represented in one line, and each of them measured from the same point, the positive quantities being represented by lines taken in one direction, the negative quantities must be represented by lines taken in the opposite direction.

37. Shew how the cycloid is traced out, and find its equation.

38. Having given the coordinates to a point and the equation to a straight line, find the distance of a point from the straight line.

1833 39. Find the equation to a straight line which passes through a given point and cuts the axis of x at an angle of $135°$.

40. Every curve of an odd degree either in x or y has at least one infinite branch.

41. Express the sine of the angle which a straight line makes with a plane in terms of the coefficients of their equations, and hence determine the relation between the coefficients when the straight line is perpendicular to the plane.

42. Define the centre of a curve, and shew how it may be determined. In what cases has the curve defined by the general equation of the second order a centre?

1834 43. Find the equation to a straight line: determine the distances from the origin at which it cuts the axes of x and y, and the cosine of the angle which it makes with a given straight line passing through the origin.

44. If an odd number of equidistant ordinates be drawn in a curve, the area between the extreme ordinates may be found approximately by adding the first and last ordinates to four times the sum of the even ordinates, and twice the sum of the odd ones, and multiplying by $\frac{1}{3}$ their common distance.

1836 45. Find the equation to a straight line drawn through a given point in the axis of y, and making an angle of $60°$ with a given straight line.

46. In rolling a circular board on level ground at the foo of a wall, in such a manner that the plane of the board was

parallel to the wall, a nail projecting out of the wall traced a curve upon the board: required its equation. Deduce a simple method of describing the spiral of Archimedes by a continuous motion.

47. Prove the following method of drawing a tangent to any curve of the second order from a given point P without it. From P draw any two lines, each cutting the curve in two points. Join the points of intersection two and two, and let the points in which the joining lines (produced if necessary) cross each other be joined by a line which will, in general, cut the curve in two points A, B. PA, PB are tangents at A and B.

48. Investigate formulæ for transforming the equation of a curve from one system of oblique coordinate axes to another having the same origin. Transform the equation $xy = c^2$, where the axes are inclined to each other at a given angle, to one where the axes bisect the angles made by the former.

49. Define a cycloid and investigate its equation. Prove that the length of an arc measured from the vertex is equal to twice the corresponding chord of the generating circle.

SECTION VII.

QUESTIONS IN CONIC SECTIONS.

1821 1. Of all sections made by planes passing through both sides of an oblique cone, two are circles, and all the rest ellipses.

2. S being the focus of an hyperbola, and (pm) the perpendicular upon its directrix from a point (p) in the *opposite hyperbola*, $Sp : pm :: SC : AC$, a given ratio.

3. Two tangents to a parabola drawn from the same point of the directrix, are at right angles to each other.

1822 4. PSp is any parameter of a parabola whose focus is S and latus rectum L, prove that

$$4SP \,.\, Sp = L\,(SP + Sp).$$

5. The ordinate to the axis of an ellipse is produced till it equals the corresponding subtangent: find the equation to the curve thus traced out, and its area.

1823 6. Putting A and B for the sectors CAP, CAp of a rectangular hyperbola, whose semi-axis $CA = 1$, and calling the abscissas CN, Cn the *cosines*, and the ordinates PN, pn the *sines* of A and B, then will

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$
$$\text{and } \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

7. If in an ellipse there be taken three abscissas in arithmetic progression, the radius vectors drawn from the focus to the extremities of the ordinates at those points will also be in arithmetic progression.

8. If tangents drawn to any two points of an ellipse meet each other; shew that their lengths are inversely as the sines of the angles which they make with the lines drawn to either focus.

9. A triangle is described about an ellipse: prove that the products of the alternate segments of the sides, made by the points of contact, are equal.

10. If two equal parabolas have a common axis, a straight line touching the interior and bounded by the exterior will be bisected in the point of contact. 1824

11. If two lines revolving in the same plane round the points S and H, intersect one another in the point P in such a manner that (1), $SP^2 + HP^2 =$ constant quantity: (2), that SP be to HP in the given ratio of n to 1; prove that in each case the locus of the point P is a circle.

12. If a right cone whose vertical angle is 90°, be cut by a plane which is parallel to one touching the slant side, prove that the latus rectum of the section is equal to twice its distance from the vertex.

13. Describe the conic section whose equation is

$$y^2 - 2xy + x^2 - 8x + 16 = 0.$$

14. If PSp be any line drawn through the focus S of a conic section, meeting the curve in the points P and p, and SL be the semi-latus rectum, then will SP, SL, Sp be in harmonic progression.

15. From the vertex of a parabola a straight line is drawn inclined at 45° to the tangent at any point; find the equation to the curve which is the locus of their intersections.

16. In a parabola, in the focal distance SP, Sp is taken equal to the ordinate PN. Find the equation to the curve traced out by the point p. 1825

17. The centre of an ellipse coincides with the vertex of a common parabola, and the axis major of the ellipse is perpendicular to the axis of the parabola. Required the proportion of the axes of the ellipse, so that it may cut the parabola at right angles.

1826 18. Having given two conjugate diameters of an ellipse and the angle contained between them; find the magnitudes and positions of the axes.

19. The abscissa and double ordinate of a common parabola are a and b, and the diameters of its circumscribed and inscribed circles D and d; prove that $D + d = a + b$.

20. Three straight lines revolve about three given points not in the same straight line, and intersect one another in three points; prove that if the loci of two of these intersections be straight lines, the locus of the third will be a conic section.

21. If r be the radius vector of an ellipse from the centre, and θ the angle which it makes with the major axis; it is required to express r in a series of the form

$$A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \dots\dots$$

and to shew particularly how A_0, A_1, and A_2 may be determined.

22. Through any point in the straight line joining the centre and intersection of the tangents to any two points, of an ellipse, two straight lines are drawn respectively parallel to its diameters passing through the points of contact; prove that the triangles formed by these lines and the tangents are equal.

23. Shew that the parameter belonging to any diameter of a parabola varies inversely as the square of the sign of the angle at which the corresponding ordinates are inclined to it.

24. If $ay^2 + bxy + cx^2 + dy + ex + f = 0$, be the equation to a curve of the second order; prove that the angles which its principal diameters make with the axis of x, may be determined from the equation

$$\tan 2\theta = -\frac{b}{a-c}.$$

1827 25. If C be the centre of an ellipse, and in the normal to any point P, PQ be taken equal to the semi-conjugate at P, Q will trace out a circle round C.

26. Given the radius vector at any point of a parabola, and the angle it makes with the curve; find the latus rectum and the place of the vertex.

27. A pair of conjugate hyperbolas being given, find their centre.

28. Find at what angle a plane must be inclined to the side of a cone in order that the section may be a rectangular hyperbola: and determine the least vertical angle of the cone for which the problem is possible.

29. If two chords of a parabola move parallel to themselves intersecting each other, the rectangles of their segments are in a constant ratio. 1828

30. In the ellipse all the circumscribing parallelograms are equal.

31. If a right cone of which the semi-angle is γ be cut by a plane making an angle δ with its axis, the ellipse thus obtained will have its

minor-axis : major-axis :: $\sqrt{\sin(\delta+\gamma)\sin(\delta-\gamma)} : \cos\gamma$.

32. If a line be drawn through the focus of an ellipse making an angle θ with the major-axis, and tangents be drawn at the extremities of this line; these tangents will make an angle ϕ, such that $\tan\phi = \dfrac{2e\sin\theta}{1-e^2}$.

33. The sum of the squares of any two conjugate diameters in an ellipse is constant.

34. Find the equation to the curve from any point of which if two tangents be drawn to a given ellipse, the angle contained between them shall be constant.

35. If a and b be the semi-axes of an ellipse, and θ and ϕ the angles which any two conjugates make with the major-axis, prove that $\tan\theta\tan\phi = \dfrac{b^2}{a^2}$.

36. Having given the equation to an ellipse referred to its principal axes, transform it into one in which the axes are inclined at an angle θ, and in which the axis of y' makes with that of y, a given angle ϕ. Find also the relation between ϕ and θ when the transformed equation is of the same form with the original equation, and shew that in this case each of the new axes is parallel to the tangent drawn at the extremity of the other.

1829 37. In an ellipse prove that

$$CP^2 + CD^2 = AC^2 + BC^2.$$

38. Prove that the chord of curvature of an hyperbola through the focus

$$= \frac{2CD^2}{AC}.$$

39. If the distance CP in an ellipse be a mean proportional between the semi-axes, prove that the semi-conjugate CD divides the quadrant of the ellipse into two arcs, whose difference is equal to the difference of the semi-axes.

40. If a straight line DCP be made to revolve about C, and cut the curve PP_1, P_2 in as many points as it has dimensions; and if $\frac{1}{CD^2}$ be made always equal to $\frac{1}{CP^2} + \frac{1}{CP_1^2} + \ldots$ the locus of the point D will be a conic section whose centre is C.

41. If a line, intersecting an hyperbola in the point P and its asymptotes in R, r, move parallel to itself, the rectangle RP . Pr is constant.

42. The section of a right cone made by a plane will be an ellipse, hyperbola, or parabola. Prove this, and determine the position of the plane for each case.

43. Investigate the formulæ for the transformation of rectangular coordinates to oblique coordinates in the same plane; and hence, having given the equation to a parabola referred to its principal axis, find its equation referred to any other diameter; the abscissa being measured from a point in the curve, and the ordinate being parallel to the tangent at that point.

44. AP is the arc of a conic section, of which the vertex is A; PG the normal, and PK a perpendicular to the chord AP, meet the axis in G and K. Shew that GK is equal to half the latus rectum.

45. The equation to a conic section is

$$5y^2 + 2xy + 5x^2 - 12x - 12y = 0.$$

Find its centre, and the magnitudes and positions of its principal axes.

46. In the ellipse, if the distances CP, CQ be drawn at right angles to each other, prove that

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{AC^2} + \frac{1}{BC^2}.$$

47. Investigate the polar equation to the hyperbola, the focus being the pole (given that $SP - HP = 2AC$), and draw the asymptote by means of this equation.

48. If the distance of any point P from a fixed point S be 1830
in a given ratio to its distance PM from a fixed line, find the rectangular equation to the locus of P, according as SP is equal to, less than, or greater than PM.

49. In the ellipse, $\dfrac{PV.VG}{QV^2} = \dfrac{CP^2}{CD^2}$.

50. If CP and CD be semi-conjugate diameters of an hyperbola, prove that

$$CP^2 - CD^2 = AC^2 - BC^2.$$

51. In the hyperbola prove that $CD \,.\, pF = AC \,.\, BC$. 1831

52. The radius of curvature in the ellipse $= \dfrac{CD^2}{pF}$.

53. Obtain the equation to a straight line which passes through two given points of a parabola, and find what the equation becomes when the points are supposed to coincide.

54. BC, CD are two consecutive arcs of a parabola, the sagittæ of which, bisecting the chords, and parallel to the axis, are equal; prove that the chord of BCD is parallel to the tangent at C.

55. Find the equation to the ellipse,

(1) Referred to rectangular coordinates measured from the centre.

(2) . . . . polar . . . . focus.

56. Find the equation to the section of a right cone made by a plane, and determine the position of the plane when the section is a parabola.

57. Determine in what cases the equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

belongs to each conic section.

58. If S be the focus, and A the vertex of any conic section, and if LT the tangent at the extremity of the latus rectum L meet the axis in T, shew that $\frac{AS}{AT}$ = the eccentricity.

59. The distance of a point p from the circumference of a circle : its distance from a fixed diameter AB $= n : 1$. Prove that the locus of p is a conic section.

60. In a given equilateral parallelogram inscribe an ellipse of given eccentricity.

61. Prove that the locus of the points of bisection of any number of chords to an ellipse, which pass through the same point, is an ellipse; and find the magnitude and position of the axes when the coordinates to the point are given.

62. Shew that three conditions must be satisfied by the constant coefficients in the general equation to lines of the second order, that it may be the equation to two straight lines.

1832 63. In the parabola, the subtangent to any diameter is double of the abscissa.

64. Prove that the latus rectum of an ellipse or hyperbola

$$= \frac{2BC^2}{AC}.$$

65. CP and CD are semi-conjugate diameters of an ellipse, and PF is a perpendicular let fall upon CD; determine the locus of the point F.

66. In the ellipse the lines SP and HP drawn from the foci S and H make equal angles with the tangent to the ellipse at P.

67. Describe a parabola which shall touch a circle at a given point, and have its axis coincident with a given diameter of the circle.

68. The section of a right cone by a plane is an ellipse of which α and β are the axes, a and b the distances from the vertex to the points where the plane cuts the sides of the generating triangle; shew that $\alpha^2 - \beta^2 = (a - b)^2$

69. The equation to the hyperbola being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

prove that the equation to its tangent is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1.$$

70. Shew that $4y^2 + 4x^2 + 16y - 8x + 19 = 0$ is the equation to a circle, and determine the radius and position of the centre.

71. Find the positions of the asymptotes to the hyperbola; and the equation to the hyperbola referred to the asymptotes as axes.

72. In the parabola if QQ' be a chord drawn parallel to the tangent at any point P, and PV be drawn parallel to the axis, cutting QQ' in V, QQ' will be bisected in V. 1833

73. Find expressions for the chords of curvature through the focus and centre at any point of an ellipse.

74. If a right line be drawn from the extremity of any diameter of an ellipse to the focus, the part intercepted by the conjugate diameter is equal to the semi-axis major.

75. If at the extremities of any two conjugate diameters of an hyperbola, tangents be drawn so as to form a parallelogram, the areas of all such parallelograms are equal.

76. When a conic section is described by the extremity P of a straight line, whose other extremity A and a given point B in it move in straight lines intersecting at right angles in C; prove that if the rectangle ACBD be completed, and PD be joined, PD is a normal at P.

77. Shew that the slant section of a cylinder by a plane is an ellipse; and determine the axes, having given the radius of the base of the cylinder, and the inclination of the cutting plane to its axis.

78. In any conic section if two chords move parallel to themselves and intersect each other, the ratio of the rectangles of their segments is invariable.

79. Investigate the polar equation to a parabola, and apply it to determine the length of the latus rectum. 1834

80. Define conjugate diameters, and prove that in an ellipse there can be only one pair which are at right angles to each other.

81. Determine the chord of curvature through the focus at any point of an hyperbola.

82. In any conic section, SG $\propto$ SP, G being the foot of the normal, and if GL be perpendicular to SP, PL $= \frac{1}{2}$ latus rectum.

83. Assuming the equations to the tangent of an ellipse, and to the perpendicular upon it from the focus, find the locus of their intersection; and account for the presence, in the result, of the factor which is rejected.

84. Shew that the curve $y = bx + \frac{c}{x}$ has the origin of the coordinates for its centre; trace it, and find the magnitude of its axes.

85. If a right cone be cut by a plane, and a, b denote the distances from the vertex of those points of the curve of intersection which lie in a plane through the axis perpendicular to the cutting plane, prove that the distance between the foci $= a \mp b$ according as the section is an ellipse or hyperbola.

86. Two conjugate diameters are produced to intersect the same directrix of an ellipse, and from the point of intersection of each one a perpendicular is drawn on the other; prove that these perpendiculars will cut one another in the nearer focus.

87. In an ellipse shew that

$$\tan\frac{PSH}{2} \cdot \tan\frac{PHS}{2} = \frac{1-e}{1+e}.$$

1835 88. Investigate the polar equation to an ellipse, the focus being the pole; and find the position of the distance which is half the sum of the greatest and least distances.

89. If E be the point of intersection of SP and CD in an ellipse, shew that PE = AC.

90. Investigate those sections of an oblique cone which are circular.

91. In the ellipse PV . VG : QV^2 :: CP^2 : CD^2.

92. Shew that $Ay^2 + By + Cx + D = 0$ is the equation to a parabola, and determine the position of the vertex and magnitude of the latus rectum.

93. The tangent at any point of a parabola will meet the directrix and latus rectum in two points equally distant from the focus.

94. From any point Q in the line BQ, which is perpendicular to the axis CAB of a parabola, whose vertex is A, QP is drawn parallel to the axis to meet the curve in P; shew that if CA be taken equal to AB, the locus of the intersections of AQ and CP is a parabola.

95. The length of the perpendicular upon the tangent from the centre of an ellipse is equal to $a\sqrt{1 - e^2 (\cos\phi)^2}$, where ϕ is the inclination of the tangent to the axis major.

96. Find the magnitude and position of the axes of the curve whose equation is

$$3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$$

97. a, b being the axes of an ellipse, and a', b' being conju- 1836
gate diameters respectively inclined to them (viz. a' to a, and b' to b) at angles α, β; shew that

$$\frac{a'^2 - b'^2}{a^2 - b^2} = \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)}.$$

98. Of two conjugate diameters of an hyperbola one only meets the curve. If one be drawn through a given point of the curve, find where the other meets the conjugate hyperbola.

99. If PQ be a chord of a parabola, normal at P, and T the point of intersection of the tangents at P and Q; prove that PT is bisected by the directrix.

100. If two chords of a conic section be drawn, of which one bisects the other, and the straight lines joining their extremities be produced to intersect, the line joining the two points of intersection shall be parallel to the chord bisected.

101. If a right cylinder of circular base be cut by a plane, the section will be an ellipse whose eccentricity is equal to the cosine of the inclination of the cutting plane to the axis of the cylinder.

102. From the general equation to a curve of the second order deduce the equation to that diameter, which bisects all chords parallel to one of the coordinate axes; and state the successive transformations by which it may be reduced to the form $y^2 = mx + nx^2$.

103. If a paraboloid of revolution be cut by a plane parallel to its axis, the section is the same as the generating parabola.

104. The chord of curvature, through the focus, at any point P of a parabola = 4SP.

105. In an ellipse, SY . HZ = BC^2. Prove this, and deduce the relation between SY and SP.

106. If the tangent at any point of an hyperbola be produced to meet the asymptotes, the area of the triangle cut off is constant.

SECTION VIII.

QUESTIONS ON DIFFERENTIAL CALCULUS.

1. Find $d\left\{\frac{1}{\sqrt{-1}}\cdot(x\sqrt{-1}+\sqrt{1-x^2})\right\}$. 1821

2. Of all triangles upon equal bases and with equal vertical angles, the isosceles has the greatest perimeter.

3. Investigate a differential expression for the radius of curvature of a curve, referred to rectangular coordinates.

4. Investigate the differential expression for the subtangent of a curve; and mention the analytical characters of a double, triple, and conjugate point.

5. If a be the arc of a circle whose radius is unity, a', a'', a''', ... the arcs of its successive involutes, then

$$a + a' + a'' + a''' + \dots \textit{in infinitum} = e^a - 1.$$

6. Investigate the differential expression for the length of a curve.

7. Trace the curve whose equation is

$$x^4 - a^2 x^2 + a^3 y = 0,$$

and find its points of contrary flexure.

8. Trace the curve, the equation to which is

$$xy + ay + bx = c.$$

9. Trace the curve, the equation to which is 1822

$$y^2 - ax - ab = 0.$$

Draw a tangent to it at any point, and determine the angle at which the curve cuts the axis.

10. Shew how the true value of a fraction may be found, the numerator and denominator of which both vanish upon assigning a particular value to the variable quantity, and find the value of

$$\frac{\tan x + \sec x - 1}{1 + \tan x - \sec x} \text{ when } x = 0.$$

11. Find the radius of curvature at any point of the catenary.

12. Prove *Newton's* fourth Lemma (Sect. I.), and by means of it find the content of an oblate spheroid.

13. Differentiate

$$(1)\ \log \frac{\sqrt{4ax + x^2} + x}{\sqrt{4ax + x^2} - x}:$$

$$(2)\ \frac{e^x \sqrt{e + x}}{e^{-x}\sqrt{e - x}}.$$

14. Required the radius of curvature of the curve whose equation is $\frac{x}{y} = \frac{a + y}{a - x}$, and determine the coordinates of the centre.

15. Find the n^{th} differential of $\sqrt{1 - x^2}$:

$$\text{find also } \int \frac{dx}{\sqrt{1 - x^2}} \int \frac{dx}{\sqrt{1 - x^2}};$$

and $\int dx\,(a^2 + b^2 - 2ab \cos x)^n$ from $x = 0$ to $x = 180°$; and determine the relation between x and y when

$$\frac{d^2y}{dx^2} - \mathrm{A}\,\frac{dy^2}{dx^2} + \mathrm{B}\,\frac{dy^3}{dx^3} = 0.$$

16. Given the magnitude of a spherical surface, find the radius of the sphere so that the corresponding spherical segment may be the greatest possible.

17. Differentiate

$$(1)\ \log (2x + 1 + 2\sqrt{1 + x + x^2}):$$

$$(2)\ \cos\theta + \sec\theta;$$

and find the values of

$$(1)\ \frac{a^{\log x} - 1}{\log x} \text{ when } x = 1: \qquad (2)\ \frac{\log \tan x}{\log \tan \frac{x}{2}} \text{ when } x = 0.$$

18. Shew that cones and cylinders upon equal bases are to one another as their altitudes.

19. Three points being given in position; it is required to draw a straight line through one of them, so that the rectangle of the perpendiculars let fall upon it from the other two may be the greatest possible. 1823

20. Find the differential of an arc the tangent of whose half is x.

21. If in the spherical triangle ABC, c and C be constant, and the other angles and sides variable, then will AC and BC be the corresponding values of ϕ and θ in the differential equation.

$$\frac{d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)}} + \frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = 0 \text{ where } e = \frac{\sin C}{\sin c}.$$

22. Determine that point of the cubical parabola where the curvature is the greatest.

23. Find the greatest triangle that can be inscribed in a given circle.

24. Two given spheres are situated at the extremities of the diameter of a given circle: determine the position of an eye in the circumference, where the surface seen is the greatest possible.

25. Trace the curve whose equation is 1824

$$(y - c)^2 = (x - a)^4 (x - b),$$

and determine the position of its tangent at the point where $x = a$.

26. In *Newton's* second Lemma, if the ordinate vary as the m^{th} power of the abscissa, find the limit of the sum of the areas of the circumscribing parallelograms.

27. Find x so that $\dfrac{\tan^3 x}{\tan 3x}$ may be a maximum.

28. If u be a function of x, and in the equation $\dfrac{du}{dx} = 0$, there be m roots equal to a and n roots equal to b, then there will be one minimum value of u for each of the roots a and b if m and n be odd, and neither maxima nor minima values when they are even.

1825 29. Find the point of contrary flexure of a spiral, where the angle varies inversely as the n^{th} power of the radius vector.

30. Explain the method of drawing asymptotes to spirals, and apply it to the hyperbola considered as a spiral having the pole in the focus.

31. Investigate the differential expression for the radius of curvature, and apply it to find the radius of curvature of the logarithmic curve.

32. Compare the curvatures of an ellipse at the extremities of the major and minor axes.

1826 33. Trace the curve whose equation is $y^2 = \dfrac{x^3 - bx^2}{x + c}$, and determine the nature of its singular points.

34. Find the shortest line which can be drawn touching a given ellipse, and intercepted by the tangents drawn at the extremities of the axes of the ellipse.

35. Find the value of $\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{1 - \cos x}$, when $x = 0$.

36. Trace the curve whose equation is $a^2 y = \dfrac{x^5}{x^2 - b^2}$.

37. Find all the angles in which the curve whose equation is

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 = 2\,a^2\left(\frac{1}{y^2} - \frac{1}{x^2}\right)$$

cuts the axis, and determine the value of its greatest ordinate.

38. Three given points are taken in the circumference of a given circle; find its vertical position on a horizontal plane, that the sum of their altitudes may be the greatest or least possible.

39. Shew that the evolute of a cycloid is an equal cycloid, and find its position.

40. The vertex of a parabola is A and the axis AN, and in the ordinate NP a point Q is taken always equidistant from A and P; find the equation to the curve which is the locus of Q; trace it, and determine the angles in which it cuts the axis and the arcs of the parabola.

41. Trace the curve whose equation is $\sqrt{y} = \frac{a}{\sqrt{x}} + \sqrt{x}$, and determine the positions of its asymptotes.

42. Differentiate

$$\frac{a}{a+x} + \frac{b}{b+x} - \frac{a+b}{a-b} \log\left(\frac{a+x}{b+x}\right);$$

and find the value of

$$\frac{a^x \sin ax - b^x \sin bx}{c^x \operatorname{vers} cx - e^x \tan ex}, \text{ when } x = 0.$$

43. Given one of the angles and the perimeter of a plane triangle, to find the sides, when the area is the greatest possible.

44. A parabola and hyperbola have the same vertex and the same axis; draw a tangent to the former which shall cut the latter in a given angle.

45. If u be a homogeneous function of $x, y, z, \ldots$ of n 1827
dimensions, and $p, q, r \ldots$ the values of $\frac{du}{dx}, \frac{du}{dy}, \ldots$

$$(n-1)\, du = xdp + ydq + \ldots$$

46. Find the differentials of $\log(\sin x)$ and $\frac{\sqrt[m]{(1+x^2)^2}}{\sqrt[n]{(1-x^2)^2}}$.

47. If $y = x^3 - 2x^2 + x + 4$, find the maximum and minimum values of y, distinguish them from each other, and shew that they are not the greatest and least values that y admits of.

48. Differentiate the continued fraction $\cfrac{x^2}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \ldots}}}$

49. Shew that in general a parabola may be found which shall have a much more intimate contact with a given curve than any circle whatever.

50. Given $z = x + e^z$. Required z in terms of x.

51. A circle being described on the axis-major of an ellipse, and a tangent drawn to each curve at the points where an ordinate to the axis meets them, find where the angle between these tangents is greatest; and shew what is the ultimate point of contact in this case, when the eccentricity of the ellipse is diminished *sine limite*.

52. Find the value of the fraction $\dfrac{3.5.9.17\ldots.}{2.4.8.16\ldots.}$ when its numerator and denominator are continued *sine limite*.

53. Trace the curve whose equation is $a^{\frac{3}{2}}\,y = (x-a)^2\sqrt{x}$.

54. If ordinates $y_1, y_2 \ldots y_n$ be drawn, at equal intervals beginning from the origin, to the catenary whose equation is

$$\frac{y}{a} = \tfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right);$$

prove that

$$(y_1)^n = \left(\frac{a}{2}\right)^{n-1}\left\{y_n + ny_{n-2} + \frac{n(n-1)}{1.2}\,y_{n-4} + \ldots.\right\},$$

and write down the last term of the series.

55. Determine the points of a given ellipse in which the sum of the conjugate diameters is the greatest or least possible, and distinguish the maximum from the minimum.

56. At points of a curve where the curvature is a maximum or a minimum, the circle of curvature has a contact of a higher order than the second.

57. Find the equation to the curve cutting at right angles all equal parabolas having their axes in the same line.

58. Draw all the rectilineal asymptotes to the curve whose equation is $y = \dfrac{x^3 + ax^2 + a^3}{x^2 - a^2}$; and trace the curve whose equation is $x^4 + y^4 = 2axy^2$.

59. Find the equation to the curve cutting off equal arcs from all circles which have their centres in the same line, and their circumferences passing through a given point in that line; and prove that the distances from this point at which the curve cuts the line are as the numbers $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \ldots$ and the tangents of the angles at which it cuts it as $1, 3, 5, \ldots.$

1828 60. Define the radius of curvature, and prove that in an ellipse it $= \dfrac{CD^2}{PF}$.

61. Find the algebraic equation to the cissoid of *Diocles*, trace the curve, and deduce the polar equation, the cusp being the pole.

62. Prove without the use of the integral calculus that the solid content of a cone is one-third that of a cylinder of the same base and altitude.

63. Define continuous curvature, and shew that the arc, chord, and tangent of any curve of continuous curvature are ultimately equal.

64. Draw the curve whose equation is $y = \sin x + 2 \sin 2x$. Find all its points of maximum, flexure, and intersection; and shew after what values of x its form will recur.

65. On a given triangle a pyramid is to be constituted of a given content. Determine it so that its surface may be the least possible.

66. Define the differential coefficient of any function, and from that definition find the differential coefficient of $\dfrac{a^2+x^2}{a-x}$, and of $\tan x$.

67. Investigate *Maclaurin's* theorem independently of *Taylor's* theorem, and apply it to find the series for a circular arc in terms of its sine.

68. If y be a function of x, determine the conditions requisite for y to be a maximum or minimum; and exemplify the theory when $y = (x-a)^n$, both when n is even, and when it is odd.

69. Shew how to determine when a curve is concave, and when convex to the axis. Trace the curve whose equation is

$$a^3 y = x^4 - bx^3 - b^2x^2,$$

and determine the number and nature of its singular points.

70. Find the m^{th} differential coefficient of $\sqrt{\cos x}$.

71. In an ellipse in which the semi-axes are CA, CB, and the abscissa and ordinate CM and MP, in MP take

$$\mathrm{MQ} = \frac{\mathrm{CB}^2}{\mathrm{CM} + \mathrm{MP}},$$

trace the curve which is the locus of Q; find its maximum and minimum ordinates, and the angles made by its two extremities with the axes.

72. Shew generally how to find the evolute of any curve whose equation is given, and find that of the common parabola.

73. In the expression $y = 2x^3 - 15x^2 + 36x$, find for what values of x, y is a maximum or minimum, and in each case which.

74. From the equation $z = f\left(\dfrac{y^2 - x^2}{x}\right)$ to eliminate by differentiation the quantity $f\left(\dfrac{y^2 - x^2}{x}\right)$.

75. $y^2 = \dfrac{x^3 - a^3}{x + a}$; trace the curve and draw its asymptotes.

76. At points of greatest and least curvature, the osculating circle will have with the curve a contact of a higher than the second order.

77. Shew how to determine the value of a vanishing fraction in all cases; and find the value of

$$\frac{1 - \dfrac{2x}{\pi}}{\text{cotan } x}, \text{ when } x = \frac{\pi}{2},$$

$$\text{and of } \frac{\sqrt{a} - \sqrt{x} + \sqrt{a - x}}{a - \sqrt{2ax - x^2}}, \text{ when } x = a.$$

1829 78. Trace the curve, of which the equation is $y^5 = ax^4 + mx^5$; draw its asymptote, and determine its singular points.

79. Find the equation between the angle and radius vector in a spiral, in which the radius vector is always equal to n times the chord of curvature drawn through the pole. Find also the value of the radius of curvature in such a spiral.

80. To inscribe the greatest ellipse in a given semi-circle, one axis of the ellipse being parallel to the diameter of the semi-circle.

81. Find the limiting ratio of the corresponding *increments* of $\sqrt{a^2 + x^2}$ and x; and those of $\dfrac{e^x + 1}{e^x - 1}$ and x.

82. Expand $\sin x$ and $\cos x$ by *Taylor's* theorem; and find limits of the value of the terms after the n^{th} term in each case.

83. Trace the curve of which the equation is

$$y = x \cdot \frac{x-a}{x-2a};$$

and find its maximum and minimum ordinates.

84. The chord of curvature at any point (x, y) of a curve, drawn through a point whose coordinates are α, β,

$$= \frac{2(1+p^2)\{y-\beta-p(x-\alpha)\}}{q\sqrt{(x-\alpha)^2+(y-\beta)^2}}, \text{ where } p = \frac{dy}{dx},\ q = \frac{d^2y}{dx^2}.$$

85. If b_n represent the coefficient of x^n in the expansion of any function of x by *Maclaurin's* theorem, and a_n in a similar expansion of the hyperbolic logarithm of that function, prove that

$$a_n = \frac{1}{nb_0}\{-(n-1)b_1a_{n-1}-(n-2)b_2a_{n-2}- \ldots -b_{n-1}a_1+nb_n\},$$

and apply this theorem to determine the relation between the coefficients in the expansion of hyp. log $\cos x$.

86. Trace the curve of which the equation is $y^2(x-a) = x^3 - b^3$, when $a > b$ and when $a < b$. Find its asymptotes and singular points.

87. Apply *Lagrange's* theorem to determine the least root of the equation

$$x^3 - 5x + 7 = 0.$$

88. Investigate the conditions requisite in order that a function of two variables x, y may be a maximum or minimum. Apply them to find when

$$u = x^4 + y^4 - 4axy^2$$

is a maximum or minimum.

89. Trace the curve whose equation is $y = \dfrac{a^2x}{a^2+x^2}$, and find the number and nature of its singular points.

90. Eliminate by differentiation the constants from the equation $y^2 = ax + bx^2$, and shew how many derivatives of the m^{th} order there are to an equation containing n arbitrary constants.

1830 91. Define the differential coefficient of any function, and from that definition find the differential coefficient of uv, u and v being functions of x.

92. Find the differential of the surface of a solid of revolution.

93. Prove *Maclaurin's* theorem, and thence expand $\tan^{-1} x$ to 5 terms.

94. Define the radius of curvature, and shew that in curves referred to rectangular coordinates it $= \dfrac{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$: shew also that in general the circle of curvature at once touches and cuts the curve.

95. Trace the curve whose equation is

$$a^2y = x^3 - \frac{b^4}{x},$$

and determine the number and nature of its singular points.

96. Eliminate by differentiation $f\left(\dfrac{y}{x}\right)$ and $\phi\ (xy)$ from the equation

$$z = xf\left(\frac{y}{x}\right) + \phi\,(yx).$$

97. Of all spherical triangles which have the same base and equal perpendiculars from the vertex to the base, shew that the isosceles has the greatest vertical angle; and from the result prove that the same is true in plane triangles.

98. Expand $f(x + h,\ y + k)$ in a series ascending by powers of h and k.

99. Prove *Lagrange's* theorem.

100. In each of the conic sections, the radius of curvature

$$= \frac{(\text{normal})^3}{(\frac{1}{2}\text{ lat. rect.})^2}.$$

101. Explain the method of finding whether a curve has multiple points, and find the number and nature of the multiple points of the curve the equation of which is

$$y^4 - 2a^2y^2 - 2ax^3 - 3a^2x^2 + a^4 = 0.$$

102. Explain the transformation of the independent variable, and transform the equation

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0,$$

where x is the independent variable, into one where θ is the independent variable, θ being equal to $\cos^{-1}x$.

103. The radius of curvature of a spiral $= \frac{rdr}{dp}$.

104. Investigate the differential coefficients of 1831

$$\sqrt{a^2+x^2}, \log x, \text{ and } \sin x.$$

105. If u be a function of y, and y a function of x, then

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

106. Find the equation to a straight line touching a given curve at a given point; and apply it to draw a tangent to the ellipse at the extremity of the latus rectum.

107. If AP be any curve referred to a pole S, and if u be the solid generated by the revolution of the area ASP about AS, SP $= r$, and the angle ASP $= \theta$,

$$\text{shew that } \frac{du}{d\theta} = \frac{2}{3}\pi r^3 \,.\, \sin\theta.$$

108. Trace the curve of which $a^m y = x(x-a)^m$ is the equation, m being an even number; find its maximum and minimum ordinates, point of contrary flexure, and the angle at which it cuts the axis of x.

109. Prove *Maclaurin's* theorem, and apply it to expand u as far as x^3 when $u^3 - 6ux - 8 = 0$.

110. Find generally an expression for the radius of curvature of a spiral curve, and apply it to determine the radius of curvature of the reciprocal spiral, the equation to which is

$$r = \frac{m}{\theta}.$$

111. Find the equation to a curve of the parabolic kind, that will pass through four given points.

112. Investigate *Lagrange's* theorem.

113. Expand $f(\overline{x+h}, \overline{y+k})$, and if $n = f(x, y)$, shew that each term involving differential coefficients of the p^{th} order will be of the form

$$\frac{1}{1.2.3\,..\,p}\cdot\frac{d^p n}{dx^{p-n}\,dy^n}\,(n+1)^{th} \text{ term of expansion of } (h+k)^p.$$

114. Trace the curve whose equation is $ay^2 = x^3 - bx^2$.

115. Let x and y be functions of a third variable t; it is required to determine what substitutions must be made for

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},$$

in an expression in which x is the independent variable, to change it to one in which f is the independent variable.

1832 116. Trace the curve whose equation is $y = \dfrac{x-3}{(x-1)(x-2)}$, determine its greatest and least ordinates, and shew that it has a point of contrary flexure corresponding to an abscissa between 5 and 6.

117. Investigate the differential coefficients of $\sqrt{2ax + x^2}$, a^x, and $\sin mx$.

118. Prove *Taylor's* theorem.

119. In determining trigonometrically the height of an object standing on a horizontal plane, where should the angle of elevation be observed, that for a given error in the observation the corresponding error in the height may be a minimum?

120. Prove that the sectorial differential of the area of any curve is $\dfrac{r^2 d\theta}{2}$ in polar coordinates, and $\dfrac{xdy - ydx}{2}$ in rectangular coordinates.

121. Obtain a general expression for the radius of curvature of a plane curve referred to rectangular coordinates, and apply it to find the radius of curvature at the extremity of the latus rectum of a parabola.

122. Find the value of $\dfrac{\sin . m\theta - \cos . (m . \frac{\pi}{2} - n\theta)}{\sin . n\theta + \cos . (n . \frac{\pi}{2} + m\theta)}$, when $\theta = 0$, m being a whole number of the form $4p + 1$, and n of the form $4p + 3$.

123. If x and y be coordinates to any point P in a plane curve APQ, $x + h$ the abscissa belonging to any other point Q in the curve, and $AP = s$; prove that the surface of the solid generated by the revolution of the arc PQ about the axis of x, is equal to

$$2\pi h \left\{u + \frac{du}{dx} h + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} + \dots \right\}$$

$$\text{where } u = y \frac{ds}{dx}.$$

124. If ρ and ρ' be the radii of curvature of a curve and its evolute at corresponding points, prove that $\dfrac{\rho'}{\rho} = \dfrac{3pq^2 - r(1+p^2)}{q^2}$ where p, q, r are the first, second, and third differential co-efficients at the point in the curve.

125. If, when a particular value is given to x, the expansion of $f(x + h)$ contain a fractional power of h between the numbers n and $n + 1$, shew that all the differential coefficients of $f(x)$ higher than the n^{th} will be infinite for that value of x.

126. At points where the radius of curvature is a maximum or a minimum, prove that the circle of curvature and the curve have a degree of contact higher than the second.

Will the circle and curve cut each other at such points?

127. Define the maxima and minima values of a function, and shew how to determine those of a function of one variable, distinguishing the maxima and minima. Determine the greatest rectangle which can be inscribed in a given portion of a parabola.

128. Determine *all* the maximum or minimum values of the 1833
function

$$x^{1+p} (a - x)^{1-p},$$

(p being a proper fraction) which correspond to different forms of p.

129. If three equations be assigned connecting one system of variables x, y, z with another of r, θ, and ϕ, shew how to transform an expression involving x, y, z, and the partial differential coefficients of z with regard to x and y, into an equivalent one involving only r, θ, ϕ, and the partial differential coefficients of r with regard to θ and ϕ. Apply the method to the expression

$$1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2,$$

when $z = r \,.\, \sin\phi$, $y = r \cos\phi \,.\, \sin\theta$, $x = r \,.\, \cos\phi \,.\, \cos\theta$.

130. Define a differential coefficient, and investigate those of $\tan x$ and $\dfrac{x}{\sqrt{x^2 + a^2}}$.

131. Find an expression for the subtangent at a given point of a curve, and shew that it is the same whether the coordinate axes be rectangular or oblique.

132. Express (p) the perpendicular from the pole upon the tangent at any point of a spiral curve in terms of the polar coordinates r and θ. If p remain finite while r is continually increased, prove that its ultimate value is the same as that of $\dfrac{r^2 d\theta}{dr}$.

133. Explain the principle of changing the independent variable: apply it to find $\dfrac{ds}{dz}$ when $\dfrac{ds}{dx} = \dfrac{\sqrt{1 - e^2x^2}}{1 - x^2} = \mathrm{R}$, and $z = \dfrac{x}{\mathrm{R}}$.

134. Trace the curve of which the equation is $x^3 + x^2y + y^3 = ax^2$, determining the positions of its rectilinear asymptote and point of contrary flexure.

135. If when $f(x + h)$ is expanded by *Taylor's* theorem, $f_m(z)$ denote generally the value which the m^{th} differential coefficient of $f(x)$ assumes when z is put for x, prove that

$$f(x + h) = f(x) + f_1(x) \,.\, h + f_2(x) \,.\, \frac{h^2}{1.2} + \dots$$

$$+ f_{n-1}(x) \,.\, \frac{h^{n-1}}{1.2.3 \dots (n-1)} + f_n(x + \lambda h) \,.\, \frac{h^n}{1.2.3 \dots n}$$

where λ is always $= 0$, or 1, or some intermediate quantity.

136. Find the conditions that a function of two independent variables may be a *maximum* or *minimum*, and deduce the equation for determining x when the function is

$$ab \sin x + bc \sin y - ac \sin (x + y).$$

137. Shew how the points of contrary flexure of a curve may be determined. Apply the method to the curve whose equation is

$$y = x (x - 1) (x - 2),$$

and trace the curve.

138. With what radius must a circle be described about the 1834
centre of a given ellipse, so that the parallelogram circumscribing the two figures may have the least possible area?

139. If the chord of a conic section, whose eccentricity is (e), subtend at its focus a constant angle $(2a)$; prove that it will always touch a conic section having the same focus, whose eccentricity is $e \cos a$.

140. If in any segment of a parabola a polygon be inscribed, having the same base as the segment, the sum of the cube roots of the areas of all the partial segments standing upon the sides of the polygon is equal to the cube root of the area of the whole segment.

141. Define a differential coefficient; and investigate those of $a + bx^n$, and of $\tan^{-1} u$, u being a function of x. Also, find the limiting ratio of the corresponding increments of the area of a spiral and the angle at the pole.

142. Investigate the equation $\dfrac{d^2 f(x, y)}{dx\, dy} = \dfrac{d^2 f(x, y)}{dy\, dx}$, and shew its truth when $f(x, y) = x^y$.

143. Adapt the common expression for the radius of curvature, to the case where x and y are functions of t; and thence find, in terms of t, the radius of curvature at any point of the curve whose equations are $x = a \cos t$, $y = b \sin t$.

144. Shew that a curve is convex or concave towards the axis of x at any point, according as the values of y and $\dfrac{d^2 y}{dx^2}$ at that point have the same or different signs. In finding points of inflexion, why is it necessary to examine the roots of $\dfrac{d^2 y}{dx^2} = \infty$?

145. The tax on a given quantity of a certain article was raised from fourpence to sixpence, and the revenue which it produced was thereby increased in the proportion of 7 to 4; at a subsequent period, the tax being further raised to ninepence, its produce was diminished, yet so as to bear to its original amount the proportion of 3 to 2; determine what amount of tax is most likely to produce the greatest revenue.

146. If the equation $f(x) = 0$ has one root only between a and b, prove that, in order to approximate to it successfully by *Newton's* method, we must be certain that neither of the equations $f'(x) = 0$, $f''(x) = 0$, has a real root between these limits, and must begin the approximation from that limit which, when substituted for x in $f(x)$ and $f''(x)$, gives results with the same sign. Shew also how to find the number of correct figures in the approximate value of the root after each process.

147. Shew how to distinguish the maximum from the minimum values of a function of one variable; determine those of y when

$$y^4 - 4a^2xy + x^4 = 0.$$

148. Define the evolute of a curve. State the reason for the name, and investigate the property on which it depends.

149. The sum of the reciprocals of the n^{th} powers of the values of y in the equation $z - y + \phi y = 0$, is equal to the sum of all the terms of the developement of y^{-n} which involve negative powers of z.

150. Eliminate by differentiation the arbitrary function from the equation $z = x\phi(x + my)$; and having given the equations $\rho = a(1 - e \cos u)$ and $u = m + e \sin u$, express ρ in terms of m in a series ascending by powers of e, to three terms.

151. Find the angle which the radius vector makes with the tangent at any point of a spiral; and shew that it is constant in the spiral whose equation is $\rho = ae^{\theta}$.

152. Apply *Maclaurin's* theorem to find the expansion of y to three terms from the equation $y^2(1 + x) = 1 - nx$.

1835 153. Express $f(x)$ in a series ascending by powers of x, when the expansion is possible. Apply the method to $\log(a + bx)$.

154. Determine the values of x which make $(x-a)(b-x)^2$ a maximum or minimum, distinguishing between them.

155. Explain on what the contact between two curves at an assigned point depends. Find the radius of curvature at a given point of the curve determined by the equations

$$x = a(1 - \cos\theta),\ y = a(\theta + \sin\theta).$$

156. Having given the equation to a spiral between r and θ, find the equation between p and r. Ex. $r = a \sec n\theta$.

157. Draw the asymptotes to the curve whose equation is $y^2 x = (x + a)(x - b)^2$; trace it, find its minimum ordinates and points of contrary flexure.

158. If the equation $\dfrac{du}{dx} = 0$ contain several sets of equal roots, investigate the circumstances under which u will have maximum or minimum values corresponding to them.

159. Expand $\tan(x+h)$ to four terms by *Taylor's* theorem. Shew that $f(a + h) - f(a)$ is positive, if $\dfrac{df(x)}{dx}$ is positive for all values of x between a and $a + h$.

160. Give a method of approximating to the greatest root of an equation without the aid of *Lagrange's* theorem. Apply it to the equation

$$x^3 + qx^2 + r = 0.$$

161. Find the radius and coordinates of the centre of the circle of absolute curvature; and transform the expressions into others, in which the arc is the independent variable.

162. Eliminate the arbitrary function from the equation

$$y - bz = \phi(x - az).$$

163. Give a definition of a differential coefficient; and in- 1836
vestigate that of $\dfrac{x^2 - 1}{x}$.

Shew that if $u = \phi(y)$, and $y = f(x)$, $\dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx}$; and hence find $\dfrac{d}{dx}\left(\dfrac{x^2 - 1}{x}\right)^3$.

164. Find the value of $\sqrt{a^2 - x^2} \cdot \cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$ when $x = a$. Determine the maximum or minimum values of $\frac{(a + ex)^2}{ae - x}$.

165. The normal of a curve touches its evolute. Determine the whole length of the evolute of an ellipse.

166. Investigate the differential coefficient of a surface of revolution. What is the surface generated by the revolution about the axis of x of the curve whose equations are

$$x = a \operatorname{vers} \theta, \quad y = a\theta + a \sin\theta\,?$$

167. Shew that the polar subtangent $= \frac{r^2\, d\theta}{dr}$. Find that of the hyperbola referred to its focus as pole; and thence determine the position of its asymptote.

168. If x, y be rectangular coordinates of any point in a curve, and r its distance from the origin, prove that the coordinates of the corresponding point in the evolute are $\frac{\left(\frac{d^2(r^2)}{dy^2}\right)}{2\left(\frac{d^2x}{dy^2}\right)}$ and $\frac{\left(\frac{d^2(r^2)}{dx^2}\right)}{2\left(\frac{d^2y}{dx^2}\right)}$: apply these formulæ to determine the evolute of an ellipse.

169. Having given $cx(b - y) = ay(c - z) = bz(a - x)$, find the maximum value of each of these expressions.

170. The area of a polygon of a given number of sides, circumscribing a given oval figure, will be the least possible when each side is bisected in the point of contact.

171. Determine the multiple point in the curve whose equation is

$$ay^3 - 2ax^2y - x^4 = 0.$$

Find also the points where it is parallel to the coordinate axes.

172. If $z = f(x, y)$, where x and y are both functions of two other variables r and θ, express $\frac{dz}{dx}$ and $\frac{dz}{dy}$, in terms of r,

θ, $\frac{dz}{dr}$ and $\frac{dz}{d\theta}$. Prove that $\frac{xdz}{dy} - \frac{ydz}{dx} = \frac{dz}{d\theta}$, where $x = r \cos \theta$ and $y = r \sin \theta$.

173. Apply *Lagrange's* theorem to find x^n from the equation $x^2 + 2ax + 1 = 0$.

174. What are the analytical characteristics of a point of osculation, and of a cusp? Ex. $y = a \operatorname{vers}^{-1} \frac{x}{a} + \sqrt{2ax - x^2}$.

175. Trace the curve whose equation is $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 1$; determine its asymptote, points of inflexion, and the angles at which it cuts the axes.

176. Eliminate by differentiation the constant from the equation

$$\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y).$$

SECTION IX.

QUESTIONS IN INTEGRAL CALCULUS.

1821 1. FIND the area of a curve in which the abscissa $= \theta$, and the ordinate $= \dfrac{\cos^2 \theta}{r}$, between the values of $\theta = 0$, and $\theta = 90°$.

Integrate
$$\int \frac{dx \sqrt{2ax + x^2}}{x}.$$
$$\int \frac{dx}{(1 + x^2)^3}.$$

2. Find the integrals of
$$\frac{dx}{\sqrt{(a - bx^2)}}, \quad \frac{x\,dx}{(x + a)(x + b)},$$
$$d\theta\,(\sin \theta)^3, \qquad a^x x^3\,dx.$$

1822 3. Integrate the quantities
$$\frac{dx}{x^n \sqrt{1 + x^2}}, \text{ n being an even number}; \quad \frac{dx}{x^2 \sqrt{a + bx + cx^2}}.$$

4. Integrate the quantities
$$\frac{dx}{(1 - x^2)^{\frac{3}{2}}}, \quad \frac{dx}{a - bx^2}, \quad \frac{dx}{x^3 - 3x^2 + 2x}.$$

5. How much of the Earth's surface may be seen by a person raised n radii above it?

6. Integrate
$$(1) \quad \frac{dx}{\sqrt{a + b\sqrt{x}}}:$$

(2) $\dfrac{dx\sqrt{a^2+x^2}}{x^8}$:

(3) $e^x \cos x dx$.

7. Find the area included between any two radii of a spiral, where the angle contained between them is the measure of their ratio.

8. Find the integral of

$$x^m dx\,(\log x)^m \text{ from } x = 0 \text{ to } x = 1.$$

9. Shew that the content of a sphere : the content of the greatest cone that can be inscribed in it :: $3^3 : 2^3$.

10. The equation to a curve is $y = x^3 - 9x^2 + 24x + 16$; determine the values of the abscissa when the ordinate is a maximum, and when a minimum; and find the area included between those ordinates.

11. Integrate

(1) $\dfrac{x^2dx}{(1+x^2)^2}$: (2) $\dfrac{x^{\frac{1}{2}}dx}{\sqrt{a+b\sqrt{x}}}$: (3) $adx \int bdx \int cdx$.

12. Integrate

$$\frac{dx}{\sqrt{x^2+y^2}} - \frac{xdy}{y\sqrt{x^2+y^2}} = 0;$$

13. Integrate $\dfrac{dx}{\sqrt[m]{1-x^m}}$. 1823

14. Integrate

$$\frac{xdx}{\sqrt{a^4x^{-4}+1}},\quad \frac{x^{\frac{1}{2}}dx}{a^3-5x^3} \text{ and } \frac{d\theta}{(\tan\theta)^2}.$$

15. If there be taken the evolute of a logarithmic spiral, the evolute of that evolute, and so on *ad infinitum*, find the sum of the arcs of all the successive evolutes.

16. Integrate

(1) $\dfrac{x^4\,dx}{\sqrt{(2ax-x^2)}}$.

(2) $\dfrac{dx}{\sqrt{(1-x^2)}} \cdot \log x$ between $x = 0$, and $x = 1$.

17. Find the integrals of $\frac{xdx}{\sqrt{(a^4 + x^4)}}$, and of v^2dx, where $v = \log x$.

18. Trace the curve whose equation is $y = \sec x$, draw a tangent to it, and find its area.

19. Integrate

$$\frac{dx}{(a + b \sin x)^2} \text{ and } \frac{x^m\, dx}{(x + \sqrt{1 + x^2})^2}.$$

1824 20. Integrate

$$\frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}, \quad \frac{dx}{a + bx + cx^2}, \quad a^x x^3\, dx.$$

21. The sum of the squares of the coefficients of the expansion of $a^x = \frac{1}{\pi} \int e^{2k \cos x}\, dx$ taken between $x = 0$, and $x = \pi$; k being $= \log a$.

22. Find the integrals of

$$\frac{(x^3 + x^2 + 2)\, dx}{x^5 - 2x^3 + x}; \quad a^x \sin^3 x dx.$$

23. In a parabola find the area included between the curve, its evolute, and its radius of curvature.

24. Find the integrals of

$$\frac{x^6\, dx}{1 + x^2}; \quad \frac{x^{\frac{1}{2}}\, dx}{\sqrt{(2a - x)}}; \quad \frac{dx}{\sin x}.$$

25. Integrate

$$x^2\, dx\, (\log x)^2, \quad \frac{x^3\, dx}{\sqrt{(x^2 + 1)}} \text{ and } dy + \frac{nydx}{\sqrt{1 + x^2}} = adx.$$

26. ACB is a quadrant of a circle whose centre is C, CA, CB its radii, AD, BE equal arcs, DE the chord of the arc DE; shew that the solid generated by the revolution of the circular segment DE, about either radius is equal to twice the sphere whose diameter is $\sqrt{2} \sin \frac{1}{2}$ DE.

27. Integrate

$$\frac{dx}{(1 - x)(1 - 2x)^{\frac{1}{2}}}, \quad \frac{dx}{(x - a)\, x^{\frac{2n+1}{2}}} \text{ and } x^2 d^2y = xdx\, dy + nydx^2.$$

28. If A_1 is the area of a given logarithmic spiral from a 1825
distance r to the centre, A_2 the corresponding area of the curve traced out by the perpendicular upon its tangent, A_3 that of the curve traced out by the perpendicular on *its* tangent, and so on continually; find the value of

$$A_1 + A_2 + A_3 + \dots .$$

29. Trace the spiral in which $r = a \cos \theta$, and find its area between the values $\theta = 0$ and $\theta = \frac{\pi}{2}$.

30. Find the area of the least parabola, which can circumscribe a given circle.

31. Shew that the length of a parabola, whose equation is $ay^{2n} = x^{2n+1}$, where n is any whole positive number, may be found in terms of the abscissa, in a finite algebraical form.

32. The whole surface of a cone is three times the area of the base. Find its vertical angle.

33. If an ordinate be drawn to the axis of a cycloid from any point P, cutting the circle described upon the axis in D; and from P and D there be drawn tangents to the cycloid and circle intersecting one another in T. Required the curve which is the locus of T, and the area which is contained between it and the cycloid.

34. Integrate the following differentials: 1826

$$\frac{dx}{x^4 + 1}; \; e^x \sin mx dx; \; dy - (x + y)^n dx = a^n dx.$$

35. Integrate the following differentials:

$$\frac{(x + 4)\,dx}{x^3 - x^2}; \; \frac{x^4 dx}{\sqrt{1 - x^2}}; \; \sqrt{x}dx + \sqrt{y}dx = \sqrt{y}dy.$$

36. Two equal parabolas have a common axis prove that the area included between one of them and a straight line touching the other is a constant magnitude.

37. Integrate $\frac{\tan \theta \, d\theta}{1 - \tan^2 \theta}$.

38. An ellipse whose semi-axes are a and b and eccentricity e, revolves about its major axis; shew that the surface of the solid thus generated is

$$= 2\pi b\left\{b + \frac{a}{e}\,\mathrm{A}\right\};$$

A being a circular arc whose sine $= e$ to a radius $= 1$.

39. Integrate

$$\frac{(\sec\theta - \tan\theta)\,d\theta}{\sec\theta + \tan\theta} \text{ and } \frac{e^{\theta}\sin m\theta d\theta}{1 - 2e^{\theta}\cos m\theta + e^{2\theta}}.$$

40. Given a the area corresponding to any rectangular coordinates in the figure of secants; to find the content of the solid generated by its revolution round the axis.

41. Integrate $\sin\theta\cos\theta\operatorname{vers}\theta d\theta$.

42. Trace the curve whose equation is

$$x^4y^4 - a^4y^4 + b^8 = 0, \text{ and find its area.}$$

43. Integrate the following differentials:

$$\frac{dx\sqrt{a^2 + x^2}}{x}, \quad \frac{(\sin x)^5\,dx}{(\cos x)^3}.$$

44. Integrate the following differentials:

$$\frac{x^2dx}{(a + x)(x^2 + b^2)}, \quad \frac{dx}{x^5 + 2x^3 + 3x^2}.$$

1827 45. A cord, the ends of which are joined, is suspended freely over two pegs in the same horizontal line, so as to form two catenaries, of which the arcs are $2s$ and $2s'$, and the tensions at the lowest points a and a'; prove that

$$s - s' : a' - a :: a' + a : s + s'.$$

46. Find the integrals of

$$\frac{dx}{\sqrt{ax - bx^2}}, \quad \frac{x^2\,dx}{x^2 - 1} \text{ and } e^{\log x}\,dx.$$

47. Trace the curve whose equation is

$$y = \sqrt[3]{\frac{a^4}{x}} - x,$$

and find the area, and the solid formed by its revolution from $x = 0$ to $x = a$.

48. Compare the volume of a sphere with that of the least cone that can be described about it.

49. In the ordinate PN of a parabola πN is taken proportional to the curvature of the parabola at P; find the area of the curve which is the locus of π.

50. A box is full of small spherical shot: what portion of 1828
the space is empty?

51. Integrate

$$x^3dx\sqrt{\frac{1+x^2}{1-x^2}}; \quad x^{nx}dx\begin{Bmatrix}x=0\\ x=1\end{Bmatrix};$$

52. Find the differential of the arc of any curve, and apply it to determine the length of the common parabola.

53. Explain the method of resolving any rational fraction into its simple or quadratic factors, and shew the use of such resolution in the integration of

$$\frac{dx}{(x^3-x^2)(x^2+x+1)}.$$

54. Integrate the differentials:

$$\frac{x^4dx}{\sqrt{2ax-x^2}}, \quad \frac{\sin^2 xdx}{\cos^4 x}, \quad e^{az}\cos^m zdz.$$

55. Integrate $\dfrac{dx}{\sqrt{1+x+x^2}}$, and $\dfrac{x^{2n}\,dx}{1+x^2}$.

56. Integrate the following differentials:

$$\frac{dx}{(x+a)(x^2+a^2)}, \quad \frac{x^5\,dx}{\sqrt{1-x^2}}.$$

57. Integrate the following differentials: 1829

$$\frac{dx}{\sqrt[3]{1+x}+\sqrt[5]{1+x}}, \quad \frac{dx}{(x^2+1)\sqrt{x^2-2}},$$

$x^6 \sin xdx$ from $x=0$ to $x=\dfrac{\pi}{2}$.

58. Define a differential; and hence find the differential of a solid of revolution. Also, apply this to prove that the sphere is $\frac{2}{3}$ of the circumscribing cylinder.

59. Integrate $\dfrac{dx}{x^3+1}$; and shew that between the limits $x=0$ and $x=1$, it is

$$\frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}.$$

60. Prove that the integral of $\dfrac{x^{m-1}\,dx}{x^n+1}$ between the limits $x=0$ and $x=\infty$, is equal to $\dfrac{\pi}{n\sin\dfrac{m}{n}\pi}$, n being greater than m.

61. Integrate

$$\frac{dx}{(a^2+x^2)^4},\quad \frac{dx}{a+b\sin x+c\cos x}.$$

62. Integrate

$$\frac{dx}{\sqrt[3]{1+x^3}},\quad \frac{x\,dx}{\sqrt{a+bx+cx^2}},\quad \frac{dx}{\cos^3 x}.$$

63. Integrate

$$dx\,(a^2+x^2)^{\frac{5}{2}} \text{ and } d\theta\sin m\theta\cos n\theta\cos p\theta.$$

1830 64. Trace the curve, the equation to which is $y=e^{\sin x}$, and express in a series the area which recurs.

65. Integrate the following differentials:

$$\frac{dx}{x^3(x^2+4)},\quad \frac{x^5\,dx}{\sqrt{1-x^2}},\quad \sqrt{x}\,dx\,\sqrt{1-x^3},\quad \frac{dx}{\cos x},\quad e^x x^m\,dx.$$

66. Find the locus of the intersections of the tangents of an hyperbola with the perpendiculars upon them from the centre: determine its maximum ordinate, its area, and the angles at which it intersects the axis.

67. Find the differential of a solid of revolution, and apply it to find the content of a segment of a paraboloid, the radii of the greater and smaller ends of which are a and b respectively, and the distance between them c.

68. Find $\displaystyle\int\frac{dx}{\sqrt{x(1-x)}}$, $\displaystyle\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$,

$$\int\frac{A(\sin\theta)^n+B(\cos\theta)^n}{(\cos\theta)^{n+2}}\,d\theta,\quad \int\cos(a+b\theta)\cos(A+B\theta)\,d\theta.$$

69. Trace the curve whose equation is $ay^2 = \dfrac{x^4}{x - a}$, and find its area.

70. Find the content of the solid generated by the revolution of the curve, the equation to which is 1831

$$(a^2 + x^2)\, y^2 - x^2\,(a^2 - x^2) = 0,$$

about the axis of x.

71. Find $\int (a + bx)^3 . x^2 dx$, $\int \dfrac{x^5 dx}{1 + x^2}$, $\int x^2 dx \sqrt{1 - x^2}$,

$$\int \frac{dx}{x\sqrt{ax^2 - b}},\ \int d\theta . \cos\theta . \cos m\theta.$$

72. Shew generally how to resolve a rational fraction into factors, and apply your method in the integration of

$$\frac{dx}{(x^3 - 4x^2)\,(x^2 + 1)}.$$

73. Integrate $\dfrac{dx}{(1 + x^2)^3}$, $e^{mx}\, x^3 dx$.

74. Find the polar equation to the rectangular hyperbola, the centre of the hyperbola being the pole, and thence determine the area of the sector.

75. Integrate $\dfrac{dx}{\sqrt{a + bx + cx^2}}$, $\dfrac{dx}{\cos x}$.

76. Find the length of the arc of a cycloid. 1832

77. Find $\int (a + bx^2)\, dx$; $\int \dfrac{x^3 dx}{\sqrt{1 - x^2}} \begin{Bmatrix} x = 0 \\ x = 1 \end{Bmatrix}$;

$$\int \frac{\sin 2\theta . d\theta}{a + b\,(\cos\theta)^2};\ \text{and}\ \int \frac{y dx - x dy}{x^2 + y^2}.$$

78. Shew that $\int \dfrac{dx}{(1 + x^2)^n}$ between the limits $x = 0$, and $x = \infty$, is

$$\frac{1.3.5\,...\,(2n - 3)}{2.4.6\,...\,(2n - 2)} . \frac{\pi}{2}.$$

79. Express $(\cos\theta)^n$ in terms of the cosines of the multiples of θ; and find $\int (\cos\theta)^n\, d\theta \begin{Bmatrix} \theta = 0 \\ \theta = \pi \end{Bmatrix}$.

1833 80. Integrate the following functions:

$$\frac{x+3}{(x^2+1)\,.\,(x-1)^2},\quad \frac{1}{(a^2+x^2)^{\frac{5}{2}}},$$

and find the length of the curve, the equation to which is $ay^2 = x^3$.

81. If $ax^2 + 2bxy + cy^2 = d$ be the equation to an ellipse, prove that its area $= \pi \dfrac{d}{\sqrt{ac-b^2}}$.

82. Find the length of the curve, the equation to which is $8a^3y = x^4 + 6a^2x^2$, and the equation to its involute.

83. Find $\displaystyle\int \frac{x^2dx}{\sqrt{2ax-x^2}}$ from $x = 0$, to $x = 2a$.

84. Prove that between the limits $x = a$, $x = b$,

$$\int udx = h\,(u_{x=a} + u_{x=a+h} + \ldots u_{x=b-h})$$

$$+ \frac{h^2}{1.2}\left(\frac{du}{dx_{x=a}} + \frac{du}{dx_{x=a+h}} + \ldots + \frac{du}{dx_{x=b-h}}\right)$$

$$+ \frac{h^3}{1.2.3}\left(\frac{d^2u}{dx^2_{x=a}} + \frac{d^2u}{dx^2_{x=a+h}} + \ldots + \frac{d^2u}{dx^2_{x=b-h}}\right) + \ldots$$

$$\text{where } h = \frac{1}{n}(b-a).$$

85. Prove that $\int x^m\,dx\,(a+bx^n)^{\frac{p}{q}}$ may be obtained in finite terms when $\dfrac{m}{n}$ or $\dfrac{m}{n} + \dfrac{p}{q}$ is an integer.

86. Integrate $\dfrac{1}{x^4+1}$, and reduce $\displaystyle\int \frac{dx}{\sqrt{a+bx+cx^2}}$ to the form $\int Zdz$, where Z is a rational function of z.

87. Integrate from $x = 0$ to $x = \infty$, $e^{-a^2p^2}\cos rx$,

$$\text{and } \frac{\cos(m+n)x}{\sin x}\cdot\frac{x}{1+x^2},$$

m being an odd integer, and n an indefinitely small quantity; prove that the latter integral is discontinuous.

1834 88. Integrate the following equations:

$$\frac{du}{dx} = \frac{1}{x^4+4x+3},\ \frac{du}{dx} = x\,\text{versin}^{-1},\ \frac{du}{dx} = (x+c)\sqrt{(x-b)(a-b)};$$

89. $\int \frac{a + b \sin^2 \phi d\phi}{\sqrt{1 - c^2 \sin^2 \phi}} = \frac{\pi}{2}(1 + c_1)(1 + c_2) \ldots\ldots (1 + c_n)$ $\{a + \frac{1}{2} b (1 + \frac{1}{2} c_1 + \frac{1}{4} c_1 c_2 \ldots \frac{1}{2^{n-1}} c_1 c_2 \ldots c_{n-1})\}$, from $\phi = 0$, to $\phi = \frac{\pi}{2}$ where $c_1, c_2, \ldots$ are derived from c, and from one another, by the law $c_{r+1} = (1 - \sqrt{1 - c_r^2}) \div (1 + \sqrt{1 - c_r^2})$; and the process is carried on till c_n does not differ sensibly from zero. Prove this, and thence find the first two terms of the developement, in a series of cosines of multiple angles, of $(a^2 - 2aa' \cos \omega + a'^2)^{-\frac{1}{2}}$ in the case where a' is not much less than a.

90. Integrate $\frac{1}{bx + cx^2}$, $\sqrt{e^2x^2 - ax}$; and find the area of the curve $a^2y = x^2\sqrt{a^2 - x^2}$, between the limits $x = 0$, $x = a$.

91. A plane curve, referred to polar coordinates, is defined by the equation $r = (a^2 - b^2) \cdot \frac{\sin \theta \cdot \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$; prove that its area $= \frac{\pi}{2}(a - b)^2$.

Shew that the curve represented by the equations $\frac{a}{x} + \frac{c}{z} = 1$, $\frac{b}{y} + \frac{c}{z} = 1$, is an hyperbola; when the coordinate axes are inclined to each other at given oblique angles, determine the position of its centre, the magnitude and direction of its axes.

92. Find the value of

$$\int \frac{d\theta}{(\sin \theta)^6 \cdot (\cos \theta)^4}, \text{ and } \int \frac{(x^{\frac{1}{3}} + 1)^{\frac{5}{2}}\, dx}{x^{\frac{1}{6}}}.$$

93. Prove that every rational proper fraction can be resolved into the sum of a series of fractions of all or some of the forms $\frac{A}{x - a}$, $\frac{B}{(x - b)^n}$, $\frac{Cx + D}{x^2 - 2ax + a^2 + \beta^2}$, and $\frac{Ex + D}{(x^2 - 2a'x + a'^2 + \beta'^2)^m}$, and shew how to integrate each of them.

94. Explain what is meant by the process of integration; and find the integrals of $\sqrt{\frac{a + x}{a - x}}$, $x^2 a^x$, $\cos mx \cdot \cos nx$.

1835 95. In the spiral, whose equation is $r = a \sec \frac{\theta}{2}$, the area included by the curve, the asymptotes and the tangent at the apse $= 4a^2$.

96. Find the differential coefficient of the volume of a solid of revolution, and determine the volume generated by the revolution round the axis of x of the area of the curve whose equation is $ay = x\sqrt{a^2 - x^2}$.

97. Integrate the differential coefficient $\dfrac{1}{x^2(x^2 - x - 2)}$,

and make $\displaystyle\int \frac{x^n dx}{\sqrt{2ax - x^2}}$ depend on $\displaystyle\int \frac{x^{n-1}}{\sqrt{2ax - x^2}}$.

98. The area of the segment included between the arc PP′ and chord of an ellipse or hyperbola, is equal to the area of the segment included between the conjugate arc DD′, and chord of the ellipse or opposite hyperbola.

99. The frustum of a cone, the radii of whose ends are R, r, and altitude h, is cut by a plane, which touches the circumference of each end; find the axes of the section, and shew that the frustum is divided into parts, which are to each other as $R^{\frac{3}{2}} : r^{\frac{3}{2}}$.

100. Express $\cos^6\theta$ in terms of the cosines of multiples of θ, and find $\int \cos^6\theta d\theta$ between the limits $\theta = 0$ and $\theta = \frac{\pi}{2}$.

1836 101. Integrate the following differential coefficients:

$$\frac{1}{x\sqrt{x^2 \pm a^2}}, \quad \frac{3x - 2}{(x - 1)^2(x^2 + 1)}, \quad \frac{ax^6 - x^7}{\sqrt{2ax - x^2}}.$$

102. Make $\int dx\, x^m (a + bx^n)^p$ depend upon $\int dx\, x^{m+n} (a + bx)^p$. Shew for what relation of the coefficients the method fails, and how in that case the integration is to be effected.

Find $\int dx e^{-ax} \sin mx$ between the limits $x = 0$ and $x = \infty$.

103. Integrate the expressions

$$u^3 \left(\frac{a + x}{a - x}\right)^{\frac{2}{3}}, \quad \left(\sec \frac{x}{2}\right)^6 . \cos x.$$

SECTION X.

QUESTIONS IN DIFFERENTIAL EQUATIONS.

1. Integrate the equations 1821

(1) $$\frac{dx}{1+x+x^2}+\frac{dy}{1+y+y^2}=0.$$

(2) $$y=\frac{xdy^2}{dx^2}+\frac{dy}{dx}.$$

(3) $$\frac{d^2z}{dx^2}+\frac{3d^2z}{dxdy}+\frac{2d^2z}{dy^2}=x+y.$$

2. The curve which is expressed by the particular solution of a differential equation of the first order, is the locus of the intersections of the curves which arise from giving every possible value to the constant in the general solution.

3. Find the equation of the curve traced out by the extremities of the perpendiculars upon the tangents of a circle, drawn from a point in its circumference; and find its greatest ordinate.

4. The integration of the partial differential equation

$$Pp+Qq=R,$$

where $p=\frac{dz}{dx}$ and $q=\frac{dz}{dy}$, and P, Q and R are functions of the variables x, y and z, is reduced to the integration of equations of two variables, when any one of the equations

$$\mathrm{P}dy - \mathrm{Q}dx = 0,$$
$$\mathrm{P}dz - \mathrm{R}dx = 0,$$
$$\mathrm{Q}dz - \mathrm{R}dy = 0,$$

involves two variables only.

5. If a ladder slides down a perpendicular wall, shew that each stave describes a quadrant of an ellipse, except the middle one, which describes a quadrant of a circle.

6. Given, in the equation $\frac{d^2u}{dv^2} + u = 0$, $u = a \sin v + b$. $\cos v$, to solve the equation $\frac{d^2u}{dv^2} + u + \Pi = 0$, by the *Variation of the Parameters.*

7. Integrate

$$\frac{dx}{\sqrt{1+x^2}} + adx + 2bydy = 0, \quad \frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}.$$

1822 8. Investigate the equation to the curve, in which the area has the same ratio to the square of the ordinate, that the ordinate has to the abscissa.

9. Integrate

$$dy + ydx = ax^3\, dx\,;$$
$$dx\, dy - (x + a)\, d^2y - \frac{xdy^2}{b} = 0.$$

1823 10. $aydy - by^2dx + cxdx = 0.$

11. Required the curve, which within its own arc, its evolute and radius of curvature shall contain the least area.

12. Find the relation of x and y in the equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (y - a)\, x^2 = 0.$$

1824 13. Integrate $ydydx = (x + a)\, dy^2 + adx^2.$

14. Find the relation between x and y in the equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

15. Integrate 1825

$$\frac{dx}{x^3\sqrt{(1-x^2)}}; \quad \frac{dx}{x^2-5x+6}.$$

16. Shew how the complete integral of the differential equation

$$Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + D\frac{d^3y}{dx^3} + \ldots\ldots = X,$$

is to be obtained, when A, B, C, D, ... are constant quantities, and X a function of x.

17. Integrate the equations,

$$(x^2 + yx)\,dy = (x-y)\,dx.$$

$$(2y^2x + 3y^3)\,dx + (2x^2y + 9xy^2 + 8y^3)\,dy = 0.$$

$$x^3\,dx^3 + dy^3 = axdx^2\,dy.$$

18. Required the nature of the curve which cuts perpendicularly a series of similar concentric ellipses.

19. Find the relation of x and y in the equation 1826

$$\frac{d^2y}{dx^2} - a\frac{dy^2}{dx^2} + bx\frac{dy^3}{dx^3} = 0.$$

20. Integrate

$$(a^2y + x^3)\,dx + (b^3 + a^2x)\,dy = 0.$$

21. Integrate

$$x^2ydx - y^3dy = x^3dy, \quad x^3\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0,$$

$$z - px - qy = 3ap^{\frac{1}{3}}\,q^{\frac{1}{3}},$$

where $p = \dfrac{dz}{dx}$ and $q = \dfrac{dz}{dy}$.

22. What curve is that in which the perpendicular from the origin on the tangent is always equal to the abscissa? 1827

23. If $\dfrac{dx}{1+x^2} + \dfrac{dy}{1+y^2} = 0$, find an algebraical value of y in terms of x.

24. Integrate the differentials and differential equations

$$\frac{x\log x dx}{\sqrt{1-x^2}}, \quad \frac{dx}{\sqrt[3]{1+x} - \sqrt{1+x}}, \quad \frac{d^ny}{dx^n} = y,$$

$$dx + 2xdy = x^2y^2dy.$$

1828 25. Integrate

$$\frac{d^2x}{dt^2} = 2x - 5y, \ \frac{d^2y}{dt^2} = x + 2y;$$

26. Integrate the differential equations:

$$\sqrt{x}dx + \sqrt{y}dy = \sqrt[4]{xy}dy, \ \frac{d^2y}{dx^2} + \frac{y}{x^2} = x^3 + ax^2.$$

27. Explain the relation which exists between the curves, which the complete integral of any differential equation of the first order represents, and that which is defined by a particular solution of that equation; and shew how the particular solution may be deduced from the complete integral. Exemplify in the case where $y = x \tan a - \dfrac{x^2}{4h\cos^2 a}$ is the complete integral, a being the arbitrary constant.

28. Solve the differential equation

$$(a + bx + cy)\,dx = (a' + b'x + c'y)\,dy.$$

29. Given a solution of a differential equation of the first order, find whether it is included in the complete integral.

30. Given the solution of the equation $\dfrac{d^2u}{dv^2} + u = 0$, solve

$$\frac{d^2u}{dv^2} + u + a\cos v = 0,$$

by the method of the variation of parameters.

31. Shew the method of integrating $Pp + Qq = R$, when neither $Pdy - Qdx = 0$, nor $Pdz - Rdx = 0$, are integrable separately and independently, and explain the process fully.

32. If u be an homogeneous function of x and y of m dimensions, shew that $mu = \dfrac{du}{dx}x + \dfrac{du}{dy}y$; and hence find a factor which renders $Mdx + Ndy = 0$ integrable, M and N being homogeneous functions.

1829 33. If tangents be drawn from a given point to each of a given system of curves, shew generally how to determine the curve, which is the locus of all the points of contact; and apply the method when tangents are drawn from a given point, to a system of concentric similar ellipses.

34. Integrate $\dfrac{dx}{dy} = xy + x^2y^3$,

$$2x\frac{dz}{dx} + y\frac{dz}{dy} = 2xy,$$

and $\dfrac{d^4y}{dx^4} - \dfrac{d^2y}{dx^2} + y = 0$, in a rational form.

35. Integrate $\dfrac{dy}{dx} = \dfrac{10 + 6y - 4x}{6x - 9y + 3}$.

36. Integrate

$$\left.\begin{aligned} \frac{d^2y}{dt^2} + 2\frac{d^2x}{dt^2} &= 2y - 3x - 1 \\ \frac{d^2x}{dt^2} - 2\frac{d^2y}{dt^2} &= x + 16y - 3 \end{aligned}\right\}.$$

37. If u_1, u_2 be values of u which satisfy the differential equation

$$\frac{d^2u}{dt^2} + M\frac{du}{dt} + Nu = 0,$$

M and N functions of t, shew that the integral of

$$\frac{d^2u}{dt^2} + M\frac{du}{dt} + Nu + \Pi = 0 \text{ is } c_1u_1 + c_2u_2 - u_1\int\frac{u_2}{u_1}\int\frac{\Pi dt^3}{u_1 d\frac{u_2}{u_1}}.$$

38. Integrate

$$\frac{dx}{(x+2)^2(1+x^2)},\quad \frac{dx}{x^3\sqrt[3]{a^3 - x^3}};\ \text{and if } \frac{dy}{dx} = 1 + xy,$$

shew that

$$y = 1 + \frac{1}{1.3} + \frac{1}{1.3.5} + \ldots \text{ between } x = 0, \text{ and } x = 1:$$

also, integrate

$$\frac{z+y}{y}dx - \frac{x+y}{z}dz - \frac{y^2 - xz}{y^2}dy = 0,$$

by applying the criterion of integrability.

39. Integrate the differentials $e^x \sin^m x dx$, and the differential equation

$$x^2dy = (x^2 - ay^2)\,dx.$$

40. Integrate $\frac{(x+2)\,dx}{(x^2+1)(x-3)^2}$, and the following differential equation $dy + y\,dx = ax^3dx$.

41. Shew under what condition the equation

$$P\,dx + Q\,dy + R\,dz = 0$$

is integrable. Find also the factor which will render it integrable when homogeneous.

42. Explain the method of integrating the partial differential equation $Pp + Qq = R$, when $P\,dy - Q\,dx = 0$ and $P\,dz - R\,dx = 0$ are integrable separately. Apply the method to the equation $py + qx = z$.

1830 43. Find the volume of a solid the equation to which is

$$z = e^{-\frac{y}{a}(a^2+x^2)}$$

$$\text{between} \begin{Bmatrix} x = 0 \\ x = \infty \end{Bmatrix} \text{ and } \begin{Bmatrix} y = 0 \\ y = \infty \end{Bmatrix},$$

$$\text{and integrate } \frac{dx}{(1-x^{\frac{2}{3}})^{\frac{3}{2}}},\quad d\theta\,\frac{a + b\tan\theta}{A + B\tan\theta},$$

$$\frac{dy}{dx}\frac{d^2y}{dx^2} + (2x+a)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4x + 2a = 0.$$

44. Integrate

$$\frac{d^2y}{dx^2} + a^2y = e^x \cos ax,$$

$$z - px - qy = m(x + y + z).$$

45. Integrate $\frac{x^4dx}{\sqrt{(2ax - x^2)}}$ between the limits $x = 0$ and $x = a$;

$$\frac{dx}{x^3+1},\quad (7x + 5y + 3)\frac{dy}{dx} + 28x + 20y - 7 = 0.$$

46. If from a point two straight lines be drawn and their extremities be joined by a curve; find its nature when the length is a maximum, the area contained by the two lines and the curve being given.

47. Explain the method of integrating the partial differential equation

$$\frac{d^2z}{dx^2} + \mathrm{P}\frac{d^2z}{dxdy} + \mathrm{Q}\frac{d^2z}{dy^2} = \mathrm{R},$$

where P, Q, R are functions of x, y, z, $\frac{dz}{dx}$, $\frac{dz}{dy}$.

48. Integrate the following differential equations:

$$\frac{dy}{dx} + y = xy^3, \quad \frac{d^2y}{dx^2} - n^2y = \cos mx, \quad m^2\left(\frac{dy}{dx}\right)^3 = \left(y - x\frac{dy}{dx}\right)^2:$$

explain also the relation which subsists between the particular solution and complete integral of a differential equation.

49. Approximate to the following integrals:

$$dx \log(1 + e\cos x), \text{ and } \frac{d^2y}{dx^2} - ax^m\, y = 0;$$

and solve the following partial differential equation

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}.$$

50. Integrate the differential equations, 1831

$$\frac{d^2y}{dx^2} + \mathrm{M}\frac{dy}{dx} + \mathrm{N}^2y = \mathrm{A}\sin nx, \text{ and } \frac{z - b'}{x - a'} = \frac{dz}{dy} + \frac{dz}{dx}\cdot\frac{y-b}{x-a};$$

and obtain a particular integral of

$$\frac{d^2z}{dy^2} + \frac{2dz}{dx} + \left(\frac{dz^2}{dx^2} - a^2\right)\frac{d^2z}{dx^2} = 0.$$

51. Eliminate the arbitrary function by differentiation from the equation $\frac{z}{x} = \phi\left(\frac{y}{x}, \frac{x}{t}\right)^t$; and prove that the integral of every differential equation of the first order and degree between four variables, is of the form $\mathrm{P} = f(\mathrm{Q}, \mathrm{R})$.

52. Shew that every differential equation of the n^{th} order has n first integrals.

Integrate $\dfrac{dx}{x^2\sqrt{2ax - x^2}}$, $\dfrac{d\theta}{1 - e^2 . \cos^2\theta}$,

$$xy^2\,dy + y^3\,dx = a^3\,dx,$$

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= a - b\frac{dx^2}{dt^2} \\ \frac{d^2y}{dt^2} &= b\left(c - \frac{dy}{dt}\right)^2 \end{aligned}\right\} \begin{array}{l} \text{eliminate } t \text{ supposing } x,\ y,\ t,\ \dfrac{dx}{dt},\ \dfrac{dy}{dt} \\ \text{to vanish together.} \end{array}$$

$$\text{and } \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = a^2.$$

53. Explain fully the mode of solving the equation

$$\frac{d^ny}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + \ldots . + My = 0,$$

where the coefficients are all constant, and where the equation

$$u^n + Au^{n-1} + \ldots . + M = 0$$

contain p equal roots.

54. Integrate

$$\frac{dy}{dx} = x + y, \text{ and } \frac{dy}{dx} = 1 + \frac{y^2}{x^2}.$$

55. Integrate

$$\frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}} + \frac{dy}{\sqrt{A + By + Cy^2 + Dy^3 + Ey^4}} = 0.$$

56. Find a multiplier which will render the homogeneous equation $Mdx + Ndy = 0$ an exact differential.

Integrate the equation $\dfrac{d^2u}{d\theta^2} + u = k\cos\theta.$

57. Having given a value which satisfies a differential equation of the first order, find whether it is included in the complete integral.

58. Prove that either of the two first integrals of a differential equation of the second order will give the same particular solution.

59. Integrate $dy + Pydx = Qdx$.

1832 60. Integrate $\left.\begin{aligned} \frac{d^2u}{d\theta^2} + u + \frac{a^2b^2}{u^3} &= 0 \\ \text{and } \frac{d\theta}{dt} &= hu^2 \end{aligned}\right\},$

θ and t commencing when u has its greatest value a.

61. If any number of circles be described having different radii, and touching a given straight line at a given point, find the equation to the curve which shall cut off equal arcs of the circles measured from that point.

62. Integrate

$$\frac{dx}{x^2\sqrt{2ax - x^2}}, \quad y\frac{dy}{dx} - \frac{ay^2}{x^2} = \frac{b}{x^3}.$$

63. Integrate the differentials $\dfrac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$, $\dfrac{(x + 2)\,dx}{x^4 + x^2}$, and transform the equation $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\cdot\dfrac{dy}{dx} + \dfrac{2y}{x^2} = ax^m$, to one in which $\log x$ shall be the independent variable, and integrate it by the variation of parameters.

64. Integrate

$$\frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + 4y = 0, \text{ and } x\frac{dz}{dx} + y\frac{dz}{dy} = \frac{xy}{z}.$$

65. If a series of curves be described after a given law, prove that the curve which is the locus of their successive intersections touches them all.

66. A series of straight lines having their extremities in two lines Ax, Ay at right angles to each other, are subject to the condition that the rectangle contained by the segments which they cut off from Ax, Ay is constant, prove that the locus of their successive intersections is an hyperbola.

67. Prove that Mdx + Ndy is an exact differential when

$$\frac{d\text{M}}{dy} = \frac{d\text{N}}{dx};$$

and shew how to integrate it in this case.

68. Investigate the nature of a curve which shall cut at right angles all the tangents which can be drawn to a given circular arc. 1833

69. If $\dfrac{d^ny}{dx^n} = z$ where z is a known function of x, prove that

$$y = \frac{1}{1.2.3\ldots n}\int dx\,(a - x)^n + \text{C}_1x^{n-1} + \text{C}_2x^{n-2} \ldots .$$
$$+ \text{C}_{n-1}\,x + \text{C}_n,$$

where $C_1, C_2 \ldots\ldots C_n$ are arbitrary constants, and in the first term x is to be substituted for a after integration.

70. Solve the following equations:

$$\frac{dy}{dx} = x \,.\, e^x \,.\, \cos x, \quad \frac{dy}{dx} = \sin (mx + ny),$$

$$\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{4x^2} = Ax^2 + Bx + C,$$

$$(1 - x^2) \,.\, \left(\frac{d^2y}{dx^2}\right)^2 + 2x \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0,$$

$$f(x) + f(y) = a + f(x + y);$$

and express the sum of the series

$$1 - \frac{e^2}{1.2} \,.\, \cos 2\theta + \frac{e^4}{1.2.3.4} \,.\, \cos 4\theta - \ldots .$$

continued *ad infinitum.*

71. Integrate the following equations:

$$\frac{du}{dx} = \frac{a}{bx + cx^2}, \quad \frac{du}{dx} = (x^2 + 1)\sqrt{x^2 + 4},$$

$$y^3 \frac{dy}{dx} + 3y^2 x + 2x^3 = 0, \quad x \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} - n^2 xy = 0.$$

Find the value of $\int dx \left(\frac{1 - n \tan^2 x}{1 + n \tan^2 x}\right)^m$ between the limits $x = 0$, $x = \frac{\pi}{2}$; and shew that when c is nearly $= 1$, the values of $\int dx\, (1 - c^2 \sin^2 x)^{-\frac{1}{2}}$ and $\int dx\, (1 - c^2 \sin^2 x)_{\frac{1}{2}}$ between the same limits, are respectively

$$\log_e \frac{4}{b} + \frac{b^2}{4}\left(\log_e \frac{4}{b} - 1\right), \text{ and } 1 + \frac{b^2}{2}\left(\log_e \frac{4}{b} - \frac{1}{2}\right),$$

very nearly, where $b = \sqrt{1 - c^2}$.

72. Integrate $\frac{d^2u}{d\theta^2} + n^2 u = A \cos n\theta$.

73. Prove that every differential equation of the n^{th} order between two variables has n integrals of the $(n - 1)^{th}$ order.

74. If M and N be homogeneous functions of x and y in the equation $M + N \frac{dy}{dx} = 0$, then is $\frac{1}{Mx + Ny}$ a factor, which will render the equation integrable.

75. Integrate 1834

$$ny\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+1=0,\ (x-mz)\frac{dz}{dx}+(y-nz)\frac{dz}{dy}=0,$$

$$f(x+f'x)=fx+\tfrac{1}{2}(f'x)^2.$$

76. Find the curve which, of all those that can be drawn between two given points, contains between itself, the curve constantly touched by a line drawn from any point in it perpendicular to the radius vector, and the two perpendiculars to the radii vectores at its extremities, the greatest area.

77. The centre of each of a series of circles lies on the axis of a plane curve, and the radius of each is the corresponding perpendicular ordinate; determine the curve, when the locus of the points of ultimate intersection of the circles is a given parabola, whose axis coincides with the axis of the curve.

78. Solve the following equations:

$$\frac{du}{dx}=\frac{\sqrt{1+\sin x}}{\sin x},\quad x^4\cdot\frac{d^2y}{dx^2}+n^2y=0,$$

$$y\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=a^2-2y^2.$$

79. If μ and ν be the amplitudes of two elliptic functions of the first order, which are respectively equal to the sum and difference of two functions whose amplitudes are ϕ and ψ, prove that

$$\sin\mu\,.\,\sin\nu=\frac{\sin^2\phi\sin^2\psi}{1-c^2\,.\,\sin^2\phi\,.\,\sin^2\psi}.$$

Integrate $x\dfrac{dy}{dx}=y+\sqrt{x^2+y^2}$, and shew how we may find a factor which shall render integrable any homogeneous equation of the first order, and that the number of such factors is infinite.

80. Shew how to find the equation to the curve which cuts at a given angle each of a series of curves described after a given law. When the angle vanishes, prove that the singular solution of the differential equation belongs to the curve required.

81. Shew how the equation

$$a\frac{d^2z}{dx^2}+b\frac{d^2z}{dx\,dy}+c\frac{d^2z}{dy^2}=f(x,y)$$

may be integrated. Apply the method to obtain the integral of $\frac{d^2z}{dx^2}-4n\frac{d^2z}{dx\,dy}+3n^2\frac{d^2z}{dy^2}z=0$ under such form, that the surface represented by it may intersect two of the coordinate planes in a straight line and circle respectively.

1835 82. Integrate the equations

$$\frac{d^2u}{dx^2}=\sqrt{a^2-x^2}\begin{pmatrix}x=0\\x=a\end{pmatrix},\text{ and }\frac{dy}{dx}=\frac{ay-\sqrt{x^2+y^2}}{ax}.$$

83. Having given a solution of a differential equation, shew whether it is a singular solution or particular integral. Are $y^2=2x+1$, and $y^2+x^2=0$ singular solutions or particular integrals of the equation $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$?

84. If $\frac{d^{2n}y}{dx^{2n}}+p_1\frac{d^{2n-1}y}{dx^{2n-1}}+\ldots\ldots+p_{2n}=\mathrm{X}$,

where $k^{2n}+p_1k^{2n-1}+\ldots+p_{2n}=\mathrm{K}=\{(k+a)^2+\beta^2\}^n$; prove that

$$e^{ax}y=\mathrm{C}_0+\mathrm{C}_1x+\ldots+\mathrm{C}_{n-1}x^{n-1})\cos\beta x+(c_0+c_1x+\ldots$$
$$+c_{n-1}x^{n-1})\sin\beta x$$
$$+\cos\beta x\,.\int dx\,\{\sec^2\beta x\,.\int dx\,(\cos^2\beta x\,.\int dx\,(\sec^2\beta x\ldots\ldots$$
$$\int dx)\sec^2\beta x\,.\int dx\,(\cos\beta x\,.\,e^{ax}\mathrm{X})\},$$

where the integral sign occurs $2n$ times; aud thence exhibit the form of the complete solution of a linear equation, whatever be the nature of the roots of $\mathrm{K}=0$.

1836 85. When $n=\frac{3a^2+2}{a^3+4a}$, the expression $\frac{1+n\cos x}{(1+a\cos x)^4}$ can be integrated in finite terms.

Integrate $\left(1-y^2-\frac{y^4}{x^2}\right)\left(\frac{dy}{dx}\right)^2-\frac{2y}{x}\cdot\frac{dy}{dx}+\frac{y^2}{x^2}=0$.

An integral of the equation

$$\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}+x=0\text{ is }y^2+(x-1)^2=0,$$

ascertain whether it is a singular solution.

86. Shew that the singular solution of the partial differential equation

$$(z - px - qy)^m = Ap^a q^b \text{ is } x^a y^b z^c = A \cdot \frac{a^a b^b c^c}{m^m};$$

$$m \text{ being } = a + b + c.$$

87. Shew that the equation $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x\frac{dy}{dz} - y}{y\sqrt{x^2 + y^2}}$ satisfies the criterion of integrability, and integrate it.

88. If u be a homogeneous function of $x, y, z \ldots.$ of n dimensions, shew that $nu = x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} + \ldots..$ Deduce a factor which renders integrable a homogeneous differential equation of the first order and degree.

89. Investigate the condition that $Ldx + Mdy + Ndz = 0$ may be integrable by a factor. Apply it to integrate the equation

$$(cy - bz)\,dx + (az - cx)\,dy + (bx - ay)\,dz = 0.$$

Integrate the equation $x^2\frac{d^2z}{dx^2} - y^2\frac{d^2z}{dy^2} = 0$

90. If the integral of the equation

$$\frac{d\phi}{\sqrt{1 - c^2\sin^2\phi}} + \frac{d\psi}{\sqrt{1 - c^2\sin^2\psi}} = 0.$$

be represented by $F_c\phi + F_c\psi = F_c\mu$, shew that

$$\cos\mu = \cos\phi\cos\psi - \sin\phi\sin\psi\sqrt{1 - c^2\sin^2\mu}.$$

91. Integrate the equation

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = 0,$$

and thence, by the variation of parameters,

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = \cos x.$$

SECTION XI.

QUESTIONS IN ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

1821 1. If, through a given point within a sphere, three planes pass, each of which is at right angles to the other two, the sum of the areas of the sections of the sphere is a given quantity.

2. Shew that all the sections of an ellipsoid made by parallel planes are similar ellipses.

3. Find the solid content of a sphere by referring it to three rectangular coordinates.

1822 4. Investigate a differential expression for the surface of a solid of revolution; and apply it to find the surface of the solid generated by the figure of tangents revolving round its axis.

5. Shew that the stereographic projection of a great circle of a sphere is a circle; and find the radius.

1823 6. Find the equation to a curve surface in which the normals from every point meet a given plane in a given straight line.

7. If three planes be at right angles to each other, find the equation to the surface to which if a tangent plane be drawn, the content of the solid formed by this and the other three planes will be constant.

1824 8. The equation of the curve made by the intersection of a plane with a surface of the second degree is a quadratic equation.

9. If δ = distance of a plane from the origin of the three rectangular coordinates AX, AY, AZ; and if α, β, γ be the angles which δ makes with these three respectively, then will the equation to the plane be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \delta.$$

10. Draw a line perpendicular to two straight lines not in the same plane.

11. Every surface of the second degree may be generated by a circle of variable radius moving parallel to itself, the centre moving along a diameter of the surface. Prove it in the case of the ellipsoid.

12. Explain the nature of the stereographic projection of 1825
the sphere, and shew that the projections of all circles, the planes of which do not pass through the eye, are circles.

13. Find the equation to the curve surface in which the tangent plane at any point intersects the axis of z at a distance from the origin equal to m times the corresponding value of z.

14. Draw a perpendicular to two given straight lines not in the same plane.

15. Find the equations to a straight line drawn from a given 1826
point perpendicular to a given plane.

16. Find the equation to the curve surface in which the tangent plane at any point cuts off from the axis of z a portion equal to the distance of the point of contact from the origin of the coordinates.

17. Find the sum of the projections of the three sides of a plane triangle upon three planes at right angles to each other.

18. Find that point in the surface of a given paraboloid through which if two planes be drawn, one perpendicular and the other parallel to the axis, the sum of the areas of the sections shall be a maximum

19. Three points move with equal velocities in three rectangular axes; one of them commences its motion from the origin, the other two from two given points equally distant

from the origin; find the equation to the surface, to which the plane, passing through the contemporaneous positions of the points, shall always be a tangent.

1827 20. If all possible ellipsoids be described of which the axes a, b, c are subject to the condition $c : a :: a : b :: b :$ a given line; shew how the equation to the surface they all touch may be found, and find the equations to the curve in which it is touched by any one of the ellipsoids, and of the curve which is the locus of the intersections of all such curves. The centre and the directions of the axes being the same in all the ellipsoids.

21. In a stereographic projection of the sphere it is required to draw a great circle through two given points.

22. Find the equation to a conical surface in general, and deduce from it that of a common right cone.

23. A plane passing through a given point, and always touching a surface of the second order, traces out a plane curve on the surface.

24. If x and y be coordinates of any point of the shortest line drawn between two given points on a surface formed by the revolution of a plane curve round the axis of z, and ds the differential of the line, prove that $xdy - ydx = cds$, c being a constant; and find the equations to the line when the surface is a paraboloid.

1828 25. Perpendiculars are drawn from a given point upon an infinite number of planes all passing through another given point; find the locus of the intersections of the perpendiculars with the planes.

26. Apply the differential expression for the volume of any solid referred to three rectangular coordinates to find the volume of a portion of a paraboloid whose equation is

$$x^2 + y^2 = 2az,$$

cut off by a plane whose equation is

$$\mathrm{B}y - \mathrm{C}z = 0.$$

27. A hexagonal pyramid whose sides are isosceles triangles is placed with its base on the plane of x and y; find the sum of the projections of its sides on the three coordinate planes.

28. If an ellipsoid, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, be cut by a plane passing through the origin perpendicular to the plane of xy, prove that the normals to the surface drawn from every point of the intersection of the plane with the ellipsoid will cut the plane of xy in a straight line, and find the equation to that line.

29. Explain the method of drawing a tangent plane at any proposed point of a given curve surface, and find the equation to that plane in the case of an ellipsoid.

30. Shew that in a curve surface the sections of the greatest and least curvature are at right angles to each other.

31. Define conical surfaces, and investigate their general equation.

32. If the equations to two planes be $\left\{\begin{matrix} ax+by+cz=d \\ \alpha x+\beta y+\gamma z=\delta \end{matrix}\right\}$, find the angle made by the planes.

33. AM, MN, NP being the coordinates to the point P 1829
in a curve surface, find the equation to the surface when the foot of the normal lies always in the centre of gravity of the triangle AMN.

34. Prove that the stereographic projection of any circle on the sphere is a circle; and find the centre and radius of the circle.

35. If x_1, y_1, z_1 be the distances from the origin of the coordinates at which a tangent plane to a curve surface cuts the axes x, y, z, and if $x_1^n+y_1^n+z_1^n=a^n$, prove that the equation to the surface touched is

$$x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}+z^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}.$$

36. State the general nature of developable surfaces. Investigate the partial differential equation of the second order which belongs to them, and integrate for the partial differentials of the first order. Shew also how such surfaces may be obtained from given curves of double curvature.

37. Draw a tangent plane to any curve surface, and determine the angle it makes with the plane of xy.

38. Having given the equation to a plane, find the equations to a straight line perpendicular to it, and passing through a given point.

39. Investigate the general form of the equation to cylindrical surfaces, and apply it when the directrix is a curve of which the equations are

$$x^2 + y^2 = r^2 \text{ and } z = cx.$$

40. In any surface of the second order which has a centre, the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.

1830 41. If $Ay^3 + Bxy^2 + Cxy + Dx^2 + Ey + Fx = 0$ be the equation to a curve; $By^2 + Cy + 2Dx + F = 0$ is the equation to a parabola which bisects all the chords parallel to the axis of x.

$2By^2 + Cy + 2Dx + F = 0$ and $Cy + 2Dx + F = 0$ are equations to a parabola and straight line which are asymptotes to the curve.

The two parabolas and the straight line have a common point of contact in the bisection of that chord which passes through the origin.

42. If two straight lines meeting one another, be parallel to two straight lines which meet one another but are not in the same plane with the first two, the plane which passes through these is parallel to the plane passing through the others.

43. Having given the equation to a plane, and the coordinates of a given point without it, find the equations to a straight line drawn from the point perpendicular to the plane; and determine its length.

44. A vertical prismatic column, the horizontal section of which is an equilateral and equiangular pentagon, is cut by a given plane; find the sides and angles of the section.

45. Prove that if the tangent plane to any curve surface make with the three coordinate planes the least possible volume,

the distance of the intersections of the plane and axes from the origin are respectively $3x$, $3y$ and $3z$, x, y and z being the coordinates of the point of contact.

46. A plane is so moved as always to cut off from a given paraboloid of revolution equal volumes; determine the equation to the surface to which it is always a tangent.

47. Find the surface in which the tangent plane always cuts the axis of z at distances from the origin proportional to $\frac{1}{z^n}$; and when $n = 1$ give to the arbitrary function that particular form which will produce the equation to the ellipsoid.

48. $$\left.\begin{aligned} x &= az \\ y &= bz \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x^2 + y^2 &= 2cx \\ x^2 + y^2 &= m^2z^2 \end{aligned}\right\}$$

are the equations to a straight line and curve of double curvature; find the equation to the surface generated by a straight line moving always parallel to the plane of xy, and passing through the straight line and the curve.

49. State the general nature of developable surfaces. Investigate the partial differential equation of the second order which belongs to them.

50. Find the equation to a plane, and determine the constants when the distances of the intersections of the plane with the coordinate axes from the origin are given.

51. Find the general equation to conical surfaces; and if a conical surface be described about a surface of the second order, shew that the curve of contact will be in one plane.

52. In the surface whose equation is

$$Az^2 + By^2 + Cx^2 + Kx = 0,$$

shew in what cases the surface will be respectively an ellipsoid, hyperboloid, elliptic paraboloid, hyperbolic paraboloid, and a paraboloid of revolution.

53. At any point in any curve surface, the sections of greatest and least curvature are at right angles to each other.

1831 54. The straight line joining any points P and Q of the surface of an ellipsoid is bisected by a plane passing through the centre of the ellipsoid, and through the line of intersection of the tangent planes at P and Q.

55. A curve surface is described by a straight line always passing through two straight lines, the equations to which are $x = a,\ y = b$; and $x = a',\ z = b'$; and through a curve, $z = f(y)$, in the plane zy; shew that the equation to the surface is $\dfrac{xb' - az}{x - a'} = f\left(\dfrac{xb - ay}{x - a}\right)$.

56. Find the equation to a plane, having given the distances of its intersections with the three coordinate axes from the origin.

57. Find the differential of a surface of revolution, and apply it to find the portion of the surface of the sphere included between two given parallel planes.

58. Required the equation of the plane in which two given straight lines lie, which intersect each other in space.

59. Define points of simple and double inflexion in a curve of double curvature, and shew how to determine them.

60. Find the equation to a surface which envelopes a series of surfaces described after a given law.

61. The equations to a straight line and plane being given, find the conditions that they may be at right angles to one another.

1832 62. The common section of two spherical surfaces is a circle of which the radius is

$$\frac{2}{a}\sqrt{\mathrm{S}\,.\,(\mathrm{S} - r)(\mathrm{S} - r')(\mathrm{S} - a)},$$

where r and r' are the radii of the spheres, (a) the distance of their centres, and $\mathrm{S} = \dfrac{r + r' + a}{2}$.

63. Perpendiculars are let fall from a given point in the axis of z on tangent planes to the surface whose equation is $x^2 + y^2 = az$, determine the locus of their extremities; shew that

when the given point is the origin of coordinates, the intersection of the locus with the planes zy or zx is a cissoid.

64. A given square is placed in a spherical surface, and four planes are drawn passing through the centre of the sphere and the sides of the square; determine the surface cut off.

65. Having given the equations to two straight lines in space which do not meet, find the equation to a plane which coincides with one of them and is parallel to the other.

66. A straight line CD is always perpendicular to a fixed straight line AB which is inclined at a given angle to the plane of xy; find the surface traced out by CD, the part of it between AB, and the plane of xy being always of the same length.

Shew that the intersection of this surface with the plane of xy is an ellipse.

67. Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ the equation to a plane, find its equation when the coordinates are so transformed that the axis of x is parallel to the plane, the origin and the axis of z remaining unaltered.

68. Every surface of the second order may be generated by the motion of a circle of variable radius parallel to itself, the centre of the circle moving along a diameter of the surface.

69. Find the differential of the volume of a solid, and apply it to find the content of an ellipsoid.

70. Obtain the general equation to the tangent plane at any point of a curve surface; and find the points where the tangent plane to the surface whose equation is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$, cuts the axes of coordinates.

71. Find the form of the surface, the equation to which is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$$

and shew what the principal sections are.

72. The equations of a line are $x=az$, and $y=bz$; determine the angles which it makes with the plane xy, and with the axis of x.

73. Compare the volumes of an oblate and prolate spheroid generated by the same ellipse.

1833 74. A curve being defined by the equations to its projections upon the coordinate planes of xy and xz; prove that it is a plane curve if

$$\frac{d^3y}{dx^3}\,\frac{d^2z}{dx^2} - \frac{d^3z}{dx^3}\cdot\frac{d^2y}{dx^2} = 0,$$

and shew how to determine the equation to the plane in which it lies.

75. Having given the angles which a straight line makes with the axes, oblique or rectangular, and the coordinates of a point in it, find the equations to its three projections on the coordinate planes.

76. If a cone envelope an elliptic paraboloid, the line joining its vertex and the centre of the curve of contact is parallel to the axis of the surface, and bisected by the surface; and the volume of the cone is $\frac{4}{3}$rds of that of the enveloped segment of the paraboloid.

77. If a curve traced on a surface be of such a nature that its osculating plane at every point contains the normal to the surface at that point, shew that its equations are

$$\frac{d^2x}{ds^2} + p\cdot\frac{d^2z}{ds^2} = 0, \quad \frac{d^2y}{ds^2} + q\cdot\frac{d^2x}{ds^2} = 0,$$

x, y, z being the coordinates of the extremity of an arc whose length is s, and p, q the partial differential coefficients of z derived from the equation to the surface. Integrate these equations when the surface is a sphere.

78. Investigate formulæ for the transformation of coordinates from one system of three rectangular axes (x, y, z), to another (x', y', z'); the position of the latter being determined by the angle which the axis of x' makes with the plane of xy, and that which its projection on the plane of xy makes with the axis of x, the axis of y' being in the plane of xy.

79. Investigate an equation for determining the lines of curvature of a given surface; apply it to the case of an ellipsoid.

80. If S be a portion of a curve surface referred to three rectangular axes, prove that $\frac{d^2S}{dx\,dy} = \sqrt{1 + p^2 + q^2}$, and prove that if a right cone be cut by a cylinder having a base of given area with its axis parallel to that of the cone, the intercepted portion of the conical surface is constant.

81. Shew that 1834

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = d$$

(where d denotes a positive quantity) is the equation to an ellipsoid, hyperboloid of one, or hyperboloid of two sheets, according as the cubic $(s - a)(s - b)(s - c) - a'^2(s - a) - b'^2(s - b) - c'^2(s - c) = 2a'b'c'$, gives for s three positive values, two or one. When the surface is an ellipsoid, shew also that its volume

$$= \frac{4\pi d^{\frac{3}{2}}}{3\sqrt{abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c'}}.$$

82. Having given that

$$(y^2 - Cx^2)\{\beta(a - \gamma) + Ca(\beta - \gamma)\} + Ca\beta(a - \beta) = 0$$

is the equation to the projection of the lines of curvature of the ellipsoid whose equation is $\beta\gamma x^2 + a\gamma y^2 + a\beta z^2 = a\beta\gamma$; prove that according as the plane of xy does or does not contain the mean axis, the arbitrary constant C admits, for each point of the ellipsoid, two values of contrary signs, or two negative values; so as to give an ellipse and hyperbola in the former, and two ellipses in the latter case.

83. Find the equation to a plane considered as generated by a straight line which moves in a direction parallel to itself along a straight line given in position. Express the result in terms of the perpendicular from the origin, and the angles which it makes with the axes of the coordinates.

84. Three chords of an ellipsoid are drawn through a given point, each at right angles to the plane containing the other two; prove that if a rectangle be formed by the segments into which each chord is divided at the given point, the sum of the reciprocals of these rectangles is constant.

85. Determine the line of intersection of the tangent planes drawn at two consecutive points of the surface of an ellipsoid

If a series of such points be taken in one plane, prove that the corresponding lines of intersection will all pass through one point; and determine the coordinates of that point when the equation to the plane is $z = Ax + By + C$.

86. Through every point of the surface whose equation is $n^2z^2 = y^2 - 2ax$, two straight lines can be drawn coinciding with it in all their points.

87. Determine the surface, every point of which is the intersection of three normals to an oblate spheroid.

88. Shew how to find the equation to the section of a surface made by a plane perpendicular to one of the coordinate planes, and deduce the equation to a plane which, passing through the mean axis of an ellipsoid, cuts it in a circle.

89. Define a diametral surface relative to a given surface, and shew that for a surface of the second order it is a plane. If a surface be defined by an equation of the n^{th} order, of what order is the equation to its diametral surface?

90. Find the position of a plane on which the sum of the projections of any number of plane areas is a maximum.

91. Mention the different ways in which developable surfaces may be supposed to be generated, and find their differential equation. Shew that surfaces generated by the motion of a straight line may be distinguished into two classes, in one of which the tangent planes at points in the same generating line are all coincident, and in the other all distinct.

92. Find the equation to the normal plane at any point of a curve of double curvature; and shew how the equation to the surface formed by the continual intersection of such planes may be determined.

1835 93. The area of a section of an ellipsoid, made by a plane passing through the centre and inclined at angles α, β, γ to the principal axes,

$$= \frac{\pi abc}{\{a^2 \sin^2 \alpha + b^2 \sin^2 \beta + c^2 \sin^2 \gamma\}^{\frac{1}{2}}}.$$

94. Find the equation to the surface, in which lie all the evolutes to the curve formed by the intersection of the surfaces, $y^2 = 4a(x+z)$, $z^2 = 4a(x+y)$; and determine the equations to that evolute which cuts the axis of x at a distance $7a$ from the origin.

95. Having given the equation to a plane, determine the constants in terms of the distances from the origin of the intersections of the plane with the coordinate axes, rectangular or oblique. In the former case, what are the equations to a line perpendicular to the plane, and passing through the origin?

96. Let *any* two chords AA′, BB′ in a surface of the second order be drawn through a fixed point; the locus of the points of intersection of AB, A′B′ is a plane.

97. Let SY be drawn from the origin S perpendicular to the tangent plane at any point P of a surface; then will $\frac{SY^2}{SP}$ be the perpendicular on the tangent plane at the corresponding point of the surface which is the locus of Y.

98. Find the equation to a plane which passes through a given straight line, and through the shortest distance between the line and the axis of x.

99. Investigate formulæ for the transformation of coordinates from one system of three rectangular axes (x, y, z) to another (x', y', z'), the position of the latter being determined by the angle which the axis of x' makes with the plane xy, and that which its projection on the plane xy makes with the axis of x, the axis of y' being in the plane xy.

100. Investigate the form of the surface whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

and find the equation to the asymptotic surface.

101. Investigate the differential expression for the area of a curve surface referred to three rectangular axes; and apply it to find that portion of the surface of a cone which is included between two planes perpendicular to the axis, and at a given distance from each other.

1836 102. Shew that the equation of the surface swept out by a line cutting the directrix of a parabola at right angles, and touching the paraboloid generated by revolution of the parabola about its axis, is

$$\frac{x^2}{z^2} - \frac{y^2}{l^2} = 1;$$

the axis of revolution, directrix, and a perpendicular to them being taken for the coordinate axes; and l being the latus rectum.

103. The orthogonal projection of a straight line, whose length is L, upon any straight line in space $=$ L . cosine of their mutual inclination. Prove this, and apply it in finding the equation of a sphere referred to oblique coordinates.

104. Determine the equation to the surface traced out by the circumference of a circle, whose centre is a fixed point, and which always passes through each of two given straight lines at right angles to each other, and not in the same plane.

105. A quadrilateral figure is inscribed in a small circle of a sphere, determine the position of an eye on its surface when the stereographic projection of the quadrilateral is a rectangle.

106. Find the angle between a straight line and a plane whose equations are given, and thence the conditions that they may be at right angles to each other.

107. Find the general equation to conical surfaces. Apply it to prove that the stereographic projection of any plane section of a paraboloid of revolution upon a plane perpendicular to its axis is a circle, the eye being placed in the vertex.

108. Determine the radii of curvature at any point of a surface in terms of the coordinates of that point; and shew how to find those points of a surface at which the radii of curvature are equal and have the same sign.

109. Determine the volume of the solid, the equation of whose surface is $\frac{x^2}{z^2} - \frac{y^2}{c^2} = 1$, contained between the coordinate planes and that whose equation is $\frac{x}{a} + \frac{y}{b} = 1$.

SECTION XII.

QUESTIONS IN FINITE DIFFERENCES AND SERIES.

1. Sum the series 1821

$$1 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{5}{16} + \frac{3}{16} + \dots \text{ to } n \text{ terms},$$

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots \text{ to } n \text{ terms},$$

by the method of increments.

2. The series

$$1 - \frac{3}{x} + \frac{5}{x^2} - \frac{7}{x^3} + \frac{9}{x^4} - \dots$$

is a recurring series: find its scale of relation and its sum *in infinitum.*

3. A recurring series may generally be resolved into two or more geometric series.

4. Sum the series

$$1^2 - 2^2 + 3^2 - 4^2 + \dots \pm x^2.$$

5. Find $\Delta^3 . u_x$, $\Sigma . x^3$, and $\Sigma . \sin x\theta$.

6. Sum the series 1822

$$\frac{1}{2.4.6} + \frac{1}{4.6.8} + \dots \text{ to } n \text{ terms and } in\ infinitum.$$

7. Given

$$\left.\begin{array}{l}\log 6753 = 3.8294967 \\ \log 6754 = 3.8295611\end{array}\right\} \text{ to find } \log .67532.$$

8. Given the logarithms of $1 + x$ and $1 + 2x$, shew how the logarithm of $1 + 3x$ may be computed.

9. Sum the following series:

(1) $\frac{2x}{1.3} + \frac{3x^2}{3.5} + \frac{4x^3}{5.7} + \ldots$ *in infinitum.*

(2) $\frac{\cos\theta}{1^2} - \frac{\cos 2\theta}{2^2} + \frac{\cos 3\theta}{3^2} - \ldots$ *in infinitum.*

(3) $\tan\theta\left(\tan\frac{\theta}{2}\right)^2 + 2\tan\frac{\theta}{2}\left(\tan\frac{\theta}{4}\right)^2 + 4\tan\frac{\theta}{4}\left(\tan\frac{\theta}{8}\right)^2$
$+ \ldots$ to (n) terms.

10. Sum the series

$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots$ to n terms and *in infinitum.*

$1^2 + 3^2 + 5^2 + \ldots$ to n terms.

$1 + 2x + 11x^2 + 43x^3 + \ldots$ *in infinitum.*

1823 11. Sum the series

$$\frac{1}{m} + \frac{2}{m(m+a)} + \frac{2(a+2)}{m(m+a)(m+2a)} + \ldots \textit{ in inf.}$$

when m is greater than 2.

12. Sum the series:

(1) $\frac{1}{3.5.7} + \frac{2}{4.6.8} + \frac{3}{5.7.9} + \ldots$ to n terms.

(2) $\frac{1}{1.4.7} + \frac{1}{2.6.9} + \frac{1}{3.8.11} + \ldots$ *in inf.*

(3) $1 + n\cos\theta + \frac{n(n-1)}{1.2}\cos 2\theta$
$+ \frac{n(n-1)(n-2)}{1.2.3}\cos 3\theta + \ldots$

13. Sum the following series:

(1) $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots$ *ad inf.* when n is infinite.

(2) $\frac{x^2}{1.2} + \frac{x^5}{1.2.3.4.5} + \frac{x^8}{1.2.3.4.5.6.7.8} + \ldots$ *ad inf.*

(3) $\frac{1}{2.1^2} + \frac{1}{2^2.2^2} + \frac{1}{2^3.3^2} + \ldots$ *ad inf.*

14. Sum the series:

(1) $1 + 3x + 5x^2 + 7x^3 + \ldots \ldots$ *ad inf.*

(2) $\frac{1}{1.2.4} + \frac{1}{2.3.5} + \frac{1}{3.4.6} + \ldots \ldots$ to n terms by increments.

15. Prove that $\sin A + \sin 2A + \sin 3A + \ldots$ is a recurring series; find the scale of relation, and by means of it the sum of n terms.

16. Sum the series: 1824

$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \ldots \ldots \text{ to } n \text{ terms;}$$

$$\frac{1}{1.2} - \frac{3}{4.5} + \frac{5}{7.8} - \ldots \ldots \text{ to infinity;}$$

$$\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \ldots \ldots \text{ to infinity.}$$

17. Required the sum of 1825

$$\frac{1^2}{2.3.4.6} + \frac{2^2}{3.4.5.7} + \ldots \ldots \text{ to } n \text{ terms.}$$

18. Sum the series

$\sin a + \sin(a + b) + \sin(a + 2b) + \ldots \ldots$ to n terms.

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \ldots \text{ to infinity.}$$

19. Sum the series

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots \ldots \text{ to } n \text{ terms.}$$

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \ldots \ldots \text{ to infinity.}$$

20. Sum the series

$$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \ldots \ldots \text{ to } n \text{ terms.}$$

$$\frac{5}{1.2.3.4} + \frac{7}{2.3.4.5} + \frac{9}{3.4.5.6} + \ldots \ldots \text{ to } n \text{ terms}$$

$$1^3 + 2^3 + 3^3 + \ldots \ldots + x^3.$$

1826 21. Sum the following series:

$$\frac{\sin\theta}{1}\frac{1}{2} - \frac{(\sin\theta)^3}{1.2.3}\frac{1}{3} + \frac{(\sin\theta)^5}{1.2.3.4.5}\frac{1}{4} - \dots \text{ ad inf.}$$

$$2 + 5 + 24 + 83 + 334 + \dots \text{ to } n \text{ terms.}$$

22. Sum the following series:

$$1.2^2 + 2.3^2 + 3.4^2 + \dots \text{ to } n \text{ terms.}$$

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \dots \text{ to infinity.}$$

23. The quantities $a, b, c, d \dots$ are in arithmetical progression; prove that the terms of any order of the differences of the quantities $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \dots$ increase or decrease according as the progression decreases or increases.

24. Sum the series

$$\sin^2 a + \sin^2(a+b) + \sin^2(a+2b) + \dots \text{ to } n \text{ terms.}$$

25. Find the sum of the series

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots \text{ in inf.}$$

and of $\frac{nx^3}{1.2.3} + \frac{n(n-1)x^4}{1.2.3.4} + \frac{n(n-1)(n-2)x^5}{1.2.3.4.5}$

$$+ \dots \text{ to } n \text{ terms.}$$

26. Sum the following series:

$$\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \dots$$

to n terms and to infinity.

$$\frac{1^2}{1^2 - k^2}\cos\theta + \frac{2^2}{2^2 - k^2}\cos 2\theta + \frac{3^2}{3^2 - k^2}\cos 3\theta + \dots$$

27. Integrate the following equation of differences:

$$u_{x+1} - au_x = x^2.$$

1827 28. Sum the series

$$\cos x - \cos 2x + \cos 3x - \dots \text{ ad inf.}$$

$$\frac{1}{2.3.5} + \frac{2}{3.5.9} + \frac{4}{5.9.17} + \frac{8}{9.17.33} + \dots$$

to n terms; and find the integral of $a^x(1 + x)^2$.

29. Sum the series 1828

$$1 + \frac{1}{2}x + \frac{1.3}{2.3}x^2 + \frac{1.3.5}{2.3.4}x^3 + \ldots . \textit{in inf.}$$

30. Solve the following equation of differences,

$$u_{x+2} + 2au_{x+1} + a^2u_x = \mathrm{A}x^3 + \mathrm{B}x^2 + \mathrm{C}x.$$

31. Sum the series

$$\left.\begin{array}{l} \frac{1}{1.3}\frac{1}{2^3} - \frac{1}{2.4}\frac{1}{2^4} + \frac{1}{3.5}\frac{1}{2^5} - \ldots . \\ \cos\theta - \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta - \ldots \end{array}\right\}, \textit{ad inf.}$$

32. Prove the following theorem in finite differences,

$$\Sigma u_x v_x = u_x \Sigma v_x - \Delta u_x \, \Sigma^2 v_{x+1} + \Delta^2 u_x \, \Sigma^3 v_{x+2} - \ldots .$$

and apply it to find the sum of x terms of the series

$$1.2.3a - 2.3.4a^2 + 3.4.5a^3 - \ldots .$$

33. Determine $\Delta^n u_x$ in a series involving

$$u_{x+n},\ u_{x+n-1},\ u_{x+n-2},\ \ldots .$$

34. Shew that every recurring series may be resolved into a certain number of geometric series, and exemplify the method by resolving the following series, and finding the sum of any number of its terms

$$1 + 4 + 18 + 80 + 356 + \ldots .$$

35. Sum the following series:

$$\left.\begin{array}{l} \frac{1}{1.2} + \frac{3}{2.3} + \frac{5}{3.4} + \frac{7}{4.5} + \ldots . \\ 1^3 + 2^3 + 3^3 + 4^3 + \ldots . \end{array}\right\} \text{ to } n \text{ terms.}$$

36. Sum the series 1829

$$\frac{16}{2.3.4} - \frac{2.21}{3.4.5.3} + \frac{2^2.26}{4.5.6.3^2} - \ldots .$$

to n terms and to infinity.

37. Find the sum of the series

$$1.3.4 + 2.4.5 + 3.5.6 + \ldots . \text{ to } n \text{ terms,}$$

$$\text{and of } \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots . \textit{in inf.}$$

38. Sum the series

$$\frac{1}{2^2-3^2}+\frac{1}{4^2-3^2}+\frac{1}{6^2-3^2}+\ldots \text{ to } n \text{ terms, and to infinity};$$

$$1.3\sin\theta + 3.5\sin 3\theta + 5.7\sin 5\theta + \ldots \text{ to } n \text{ terms};$$

$$\frac{1}{1^3.3^3}+\frac{1}{1^3.5^3}+\ldots+\frac{1}{3^3.5^3}+\ldots \textit{ ad inf.}$$

39. $u_x u_{x+\pi} = a(u_x + u_{x+\pi})$. Give a complete solution of this equation, and determine the particular values of the constants when it is the equation to a conic section about the focus.

40. Find Σx^n in a series proceeding according to the descending powers of x.

41. Explain the method of summing any recurring series, and sum the series 2, 5, 13, 35, ... to n terms.

42. Shew that the solution of the equation of differences

$$u_{x+n} + A_x u_{x+n-1} + \ldots + P_x u_x = Q_x,$$

may be made to depend upon the solution of the equation

$$u_{x+n} + A_x u_{x+n-1} + \ldots + P_x u_x = 0.$$

43. Prove that

$$\Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n 0^{n+1}}{1.2\ldots(n+1)}\frac{d^{n+1}u_x}{dx^{n+1}} + \frac{\Delta^n 0^{n+2}}{1.2\ldots(n+2)}\frac{d^{n+2}u_x}{dx^{n+2}} + \ldots$$

1830 44. Having given

$$\log 8801 = 3.9445320,$$
$$\log 8802 = 3.9445814,$$
$$\log 8804 = 3.9446800,$$
$$\log 8805 = 3.9447294;$$

find log 8803.

45. Solve the following equations of differences:

$$\Delta^3 x + \Delta^2 x + \Delta x = x^3,$$

$$u_x u_{x+1} + u_x u_{x+2} + u_{x+1} u_{x+2} = m^2.$$

46. Sum the following series:

$$\frac{1}{1.3}+\frac{1}{4.6}+\frac{1}{7.9}+\frac{1}{10.12}+\ldots \textit{ ad inf.}$$

$$\sec\theta\cos\theta + 4(\sec\theta)^2\cos 2\theta + 13(\sec\theta)^3\cos 3\theta$$
$$+ 40(\sec\theta)^4\cos 4\theta + \ldots \text{ to } n \text{ terms}.$$

47. Reduce $\dfrac{u_{x+1}}{u_x u_{x+2} u_{x+3}}$ to an integrable form when $u_x = a + bx$: and sum the series

$$\frac{2^2}{1.3.4.5} + \frac{3^2}{2.4.5.6} + \frac{4^2}{3.5.6.7} + \ldots \text{ to } n \text{ terms.}$$

48. Prove that

$$\Sigma u_x = \int u_x dx - \frac{u_x}{2} + \frac{1}{2.1.2.3}\frac{du_x}{dx} - \frac{1}{6.1.2.3.4.5}\frac{d^3 u_x}{dx^3} + \ldots$$

the coefficients being the same as those of t in the expansion of $\dfrac{1}{e^t - 1}$.

49. Prove that

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} \ldots$$

and thence shew that

$$1.2.3 \ldots n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots$$

50. Prove that the integral of 1831

$$\Delta^3 u_x + a\Delta^2 u_x + b\Delta u_x + c u_x = 0,$$

is $u_x = C\alpha^x + C'\beta^x + C''\gamma^x$, if $\alpha - 1$, $\beta - 1$, $\gamma - 1$, be the roots of $z^3 + az^2 + bz + c = 0$. Also, by varying the parameters, obtain the integral of

$$\Delta^3 u_x + a\Delta^2 u_x + b\Delta u_x + c u_x = A_x.$$

51. Prove that $\dfrac{d^m u_x}{dx^m} = \{\log(1+\Delta)\}^m . u_x$, where any power of Δ as Δ^p, in the expansion of $\{\log(1+\Delta)\}^m$, when u_x is joined to it, denotes the p^{th} difference of u_x.

And if $a_1, a_2, a_3, \ldots a_n$ be the coefficients of

$$\Delta^m u_x, \ \Delta^{m+1} u_x, \ \ldots \ \Delta^{m+n} . u_x$$

in the expansion of $\{\log(1+\Delta)\}^m . u_x$; shew that a_n is found in terms $a_{n-1} \ldots$ from the following equation,

$$\begin{aligned} n a_n = {} & \tfrac{1}{2}\{n - (m+1)\} a_{n-1} - \tfrac{1}{3}\{n - 2.(m+1)\} a_{n-2} \\ & + \tfrac{1}{4}.\{n - 3.(m+1)\} a_{n-3} - \ldots \\ & \pm \frac{1}{n}.\{n - (n-1).(m+1)\} a_1 \mp \frac{nm}{n+1}. \end{aligned}$$

52. Sum the following series:

$$(1)\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ ad\ inf.$$

$$(2)\ 1+5+21+85+\ldots\ldots \text{ to } n \text{ terms.}$$

53. Explain the connexion between the differential calculus, as depending on the theory of limits, and the calculus of differences; and deduce *Taylor's* theorem from the equation

$$u_{x+n}=u_x+\frac{n}{1}\Delta u_x+\frac{n.(n-1)}{1.2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x$$
$$+\ldots\ldots$$

54. Determine the five first terms of Σx^n.

1832 55. Sum the series

$$\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\ldots\ldots \text{ to } n \text{ terms.}$$

$$\sin\theta+e^2\sin(\theta-a)+e^4.\sin(\theta-2a)+e^6.\sin(\theta-3a)\ldots\ ad\ inf.$$

56. Obtain the complete integral of the equation of differences

$$u_{x+n}+au_{x+n-1}+bu_{x+n-2}+\ldots\ldots+ku_x=l.$$

1833 57. Integrate the equations of differences

$$\Delta u_x = x^4-3x^2,\ (u_x+1)^2+u_x=2.$$

If $u_x=\dfrac{1}{a+}\ \dfrac{1}{-b+}\ \dfrac{1}{a+}\ \dfrac{1}{-b+\ldots\ldots}$ to x fractional terms, prove that

$$au_x=1-\frac{\sin(x-1)\theta}{\sin(x+1)\theta},\ \text{or}=1-\frac{\cos(x-1)\theta}{\cos(x+1)\theta},$$
$$(\text{where } 2\sin\theta=\sqrt{ab}),$$

according as x is odd or even.

Prove that $m^n=(1+\Delta)^m 0^n$, and apply the result to shew that

$$1^n-n.2^n+\frac{n(n-1)}{1.2}\cdot 3^n-\ldots\pm(n+1)^n=(-1)^n 1.2.3..n;$$

and thence deduce the theorems, that if $n+1$ be a prime number,

$$1.2.3\ldots n+1,\ \text{and}\ \left(1.2.3\ldots.\frac{n}{2}\right)^2+(-1)\frac{n}{2},$$

are each exactly divisible by it.

58. Shew that

$$\Sigma u_x v_x = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} + \ldots .$$

and apply it to sum $1^2 - 2^2 + 3^3 - \ldots$ to x terms.

59. Find the sums of the following series:

$$\sin a - \sin (a + b) + \sin (a + 2b) - \ldots \text{ to } n \text{ terms,}$$

$$\left.\begin{array}{l} \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots \\[2ex] \dfrac{1}{1^2.2} + \dfrac{1}{2^2.3} + \dfrac{1}{3^2.4} + \ldots\ldots \end{array}\right\} \textit{ad inf.}$$

60. If $a_0, a_1, a_2 \ldots . a_n$ denote the coefficients of an expanded binomial, prove that 1834

$$a_0 a_r + a_1 a_{r+1} + a_2 a_{r+2} + \ldots + a_{n-r} a_n = \frac{2n(2n-1)\ldots(n-r+1)}{1.2.3\ldots(n+r)}.$$

Express the general term of the expansion of $\log_e (\cos x)$ by *Bernouilli's* numbers; and find the sum of the following infinite series, where $u = x^2 + px + q$,

$$1 + \frac{t}{1}\frac{du}{dx} + \frac{t^2}{1.2}\frac{d^2u^2}{dx^2} + \frac{t^3}{1.2.3}\frac{d^3u^3}{dx^3} + \ldots .$$

61. Integrate

$$\Delta u_x = \frac{1}{x^2 + 7x + 10}; \quad u_x + 2 + u_x = \text{A} . \cos (mx + \text{B});$$

$$f(a + x) . f(a - x) = a^2 - x^2.$$

62. Prove that

$$\Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n o^{n+1}}{1.2.3\ldots(n+1)}\frac{d^{n+1} u_x}{dx^{n+1}}$$

$$+ \frac{\Delta^n o^{n+2}}{1.2.3\ldots(n+2)}\frac{d^{n+2} u_x}{dx^{n+2}} + \ldots .$$

Shew how the numbers $\Delta^n o^n$, $\Delta^n o^{n+1}$, ... may be calculated; and find the numerical value of $\Delta^3 . o^5$.

63. Shew in what respect singular solutions of equations of differences differ from those of differential equations; illustrate by the example

$$u_x = x(u_{x+1} - u_x) + (1 - u_{x+1} + u_x)^2,$$

a complete integral of which is $u_x = x(1 - a) + a^2$.

64. Shew in what case a recurring series can be resolved into geometrical progressions, and in what case it cannot.

Resolve $1 + 5 + 17 + 53 + \ldots\ldots$ into its component series.

65. Sum the series

$$.5 + .25 + .125 + \ldots\ldots \textit{ in inf.}$$

$$\left.\begin{array}{l}\cos a + \cos 3a + \cos 5a + \ldots\ldots \\ \frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \frac{18}{3.4.5.6} + \ldots\ldots\end{array}\right\} \text{to } n \text{ terms.}$$

1835 66. Prove that

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots\ldots$$

and determine from three observed values of a quantity near its maximum or minimum made at given equidistant times, when the maximum or minimum took place.

67. Find the generating function of $\Delta^m u_{x-mr}$, and prove that

$$u_{x+n} = u_x + n\Delta u_{x-r} + \frac{n.(n+2r-1)}{1.2}\Delta^2 u_{x-2r} + \ldots\ldots$$

68. Integrate $u^2_{x+1} + au_{x+1}u_x + bu^2_x = cb^x$.

Is $u^2_x + \sqrt{-1}.(-1)^x = 0$ a singular solution of

$$u^2_{x+1} + u_{x+1}u_x + u^2_x = (-1)^x\,?$$

Apply the calculus of finite differences to solve the equation $(a-b+c-d+\ldots \text{ to } n \text{ terms})^2 = a+b+c+d+\ldots \text{ to } n$ terms.

What solid of revolution is that, in which the sum of the areas of any two sections, equi-distant from the extremities of its axis and perpendicular thereto, is invariable?

69. Prove that

$$\Sigma u_x = C + xu_x - \frac{x(x+1)}{1.2}\Delta u_x + \frac{x(x+1)(x+2)}{1.2.3}\Delta^2 u_x - \ldots$$

hence find the sum of the series $1^3 + 2^3 + 3^2 \ldots + x^3$.

Sum also the series $1 + 2x + 3x^2 + 4x^3 \ldots + nx^{n-1}$.

70. Shew that $\Delta^n \sin x = \left(2\sin\frac{h}{2}\right)^n \sin\left\{x + \frac{n}{2}(\pi+h)\right\}$, where h is the increment of x. Also prove that $h^n.\frac{d^n u}{dx^n} = \Delta^n u + A_1\Delta^{n+1}u + A_2\Delta^{n+2}u + \ldots$ where $A_1\, A_2 \ldots$ are the coefficients of t^{n+1}, t^{n+2} .... respectively in the development of

$$\{\log_e(1+t)\}^n.$$

SECTION XIII.

QUESTIONS IN THE CALCULUS OF VARIATIONS.

1. Find the variation of $\int Vdx$, and explain the use of that 1828
part of the result which is without the integral sign, and exemplify it by finding the shortest distance between two given straight lines not in the same plane.

2. If V be any function of $x, y, p, q, \ldots$ prove that when 1829
$\int Vdx$ is a maximum or minimum,

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \ldots = 0.$$

3. Explain fully the mode of applying the Calculus of Vari- 1830
ations to cases wherein it is required to determine one function u a maximum or minimum, the value of another function v being given; and exemplify it by finding the curve of quickest descent from one given point to another, the length of the curve being given: shew also how the constants introduced by integration may be determined in this case.

4. The path of a ship in a horizontal plane is referred to 1831
rectangular axes, that of x being in the direction of the wind, and its velocity in sailing from one given point to another is assumed to be $f\left(\frac{dy^2}{dx^2}\right)$; required the nature of the brachystochronous path between the given points.

5. Find the variation of $\int Vdx$, where V is a function of $x, y, p, q \ldots$ and the conditions requisite for a maximum or minimum value.

Apply the two resulting equations to find the shortest line that can be drawn from one given curve to another given curve.

1832 6. Shew that the cycloid is the curve of quickest descent from one given curve to another, the motion being supposed to commence from the first, and that at the points where it meets the curves their tangents are parallel.

1833 7. If V be a function of x, y, and the differential coefficients of y with respect to x, find the variation δV.

1834 8. Find the shortest line that can be drawn on a surface of revolution from a given point to a given curve.

1836 9. Assuming the expression for $\delta\int V dx$, shew that the condition of $\int V dx$, between given limits, being a maximum or minimum, gives rise to two independent equations. Prove that the shortest line between two curves of double curvature is a straight line perpendicular to both.

SECTION XIV.

QUESTIONS IN STATICS.

1. Given a and b the arms of a straight lever which turns on an axis, the radius of which is r. P would maintain the equilibrium acting perpendicularly at the distance (a), if there were no friction: but a weight p must be added to it in order to overcome the friction. Find the proportion of the friction to the pressure. 1821

2. A beam 30 feet long, balances itself upon a point at $\frac{1}{3}$rd of its length from the thicker end. But when a weight of 10 lbs. is suspended at the other end, the prop must be moved two feet towards it to maintain the equilibrium. What is the weight of the beam?

3. Explain how wheels assist the motion of a carriage.

4. Explain *Archimedes'* screw.

5. Investigate expressions for determining the position of the centre of gravity of a plane surface, bounded by a curve whose equation is given.

6. If a body be balanced upon a horizontal plane, and a slight motion be given to it, its centre of gravity will move horizontally; prove this, and shew in what cases the equilibrium is stable.

7. Shew, when P sustains W upon a screw, if a slight motion be given to the machine, that P's velocity : W's velocity :: W : P.

1822 8. If a triangular prismatic beam is supported at both ends, shew that it is twice as strong when the edge is uppermost, as when the base is.

9. Find the position of the centre of gravity of the quadrant of a circular area.

10. When the same string passes over any number of pullies, and the parts of the string supporting any pulley at the lower block are not parallel to one another, find the proportion between P and W in equilibrio.

11. Two weights sustain each other on two inclined planes, having a common altitude, by means of a string parallel to the planes; compare the pressures.

12. A parallelogram and a triangle upon the same base and between the same parallels revolve round the base as an axis; prove that the solid generated by the triangle equals one third of that generated by the parallelogram.

13. State generally the principle of virtual velocities; and from it deduce the position of equilibrium of a straight rod of uniform density placed on two inclined planes.

14. The distance of the centre of gravity of a cycloid from the vertex $= \frac{7}{12}$ths of the axis; compare, from this, the contents of the solids generated by its revolution round the base and a tangent at the vertex.

15. P supports W upon an axle, by means of a perpetual screw acting upon the circumference of the wheel. Required their proportion.

16. A plane of given form and area is supported in the air as a kite, the wind acting in a direction parallel to the horizon; the weight of the string and materials being (w), and the horizontal pressure of the wind equivalent to a weight (p) upon each square foot; required the angle made by the plane with the horizon, and the greatest weight it can support.

1823 17. A chain of uniform density is suspended at its extremities by means of two tacks in the same horizontal line at a given distance from each other; find the length of the chain so that the stress upon either tack may be equal to the chain's weight.

18. A beam of given length and weight is placed with one end on a vertical, and the other on a horizontal plane: find the force necessary to keep it at rest, and the pressures on the two planes.

19. On a lever of uniform density, every inch weighing w oz. a weight of W oz. is suspended at a given distance from the fulcrum which is placed at one extremity. What must be the length of the lever, so that the whole may be supported by the least possible power acting in an opposite direction at the other extremity?

20. The beam of a false balance being of uniform density and thickness, it is required to shew that the lengths of the arms are respectively proportional to the differences between the true and apparent weights.

21. Determine the length of a straight line drawn through the centre of gravity of a given isosceles triangle, making a given angle with the base, and terminated by the sides.

22. Determine the conditions of equilibrium of a material point situated in a canal of indefinitely small dimensions and acted upon by any number of forces.

23. If a pole rests with one end on the ground against a 1824
wall, and the other attached to a string fixed in the wall, find the tension of the string.

24. If α, β, γ be the angles which the resultant of three forces acting at right angles to one another makes with each of them respectively, then will

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1.$$

25. Deduce the equation to the catenary,

$$2y = a(e^{\frac{x}{a}} + e^{-\frac{x}{a}}),$$

where $a =$ tension at the lowest point; and prove that its radius of curvature is equal to its normal.

26. If the sides of a triangle ABC be bisected in the points D, E, F; then the centre of the circle inscribed in the triangle DEF is the centre of gravity of the perimeter of the triangle ABC.

27. A and B are two given points in a horizontal line, to which are fastened two strings of given lengths; the string BC passes through a ring at C, and is fastened to a given weight W; find the position in which the weight will rest.

28. If any number of forces p, q, r .... in different planes, acting on a point, make the angles α, β, γ .... with the resultant, then will

$$p \cos \alpha + q \cos \beta + r \cos \gamma + \ldots . \text{ be a maximum.}$$

1825 29. Find the position of the centre of gravity of the area of a semi-parabola.

30. Determine the position of equilibrium of a uniform rod, one end of which rests against a plane perpendicular to the horizon, and the other on the interior surface of a given hemisphere.

31. An arch, where the equilibrium is preserved by the weights of the voussoirs, is so constructed that the centres of gravity of the voussoirs are in a catenary curve and the joints perpendicular to the curve. Required the equation from which the length of the voussoirs may be obtained.

32. When a chain fixed at two points is acted upon by a central attractive or repulsive force, the tension at any point is inversely as the perpendicular let fall from the centre of force on the tangent at that point. Required proof.

33. Determine the point in the curve surface on which a semi-paraboloid will rest on a horizontal plane.

34. A given weight is to be supported at a given point upon a straight lever of uniform density by a power acting at its extremity on the same side of the fulcrum. Required the least power which will support the system, and the corresponding length of the lever.

35. A circular hoop is supported in a horizontal position, and three weights of 4, 5, and 6 pounds respectively are suspended over its circumference by three strings meeting in the centre; what must be their positions so that they may sustain one another?

36. The resultant and sum of two forces are given, and also the angle which one of them makes with the resultant; it is required to determine the forces and the angle at which they act. 1826

37. If t be the length of a part of the catenary, the weight of which is equal to the tension at a point whose abscissa is x and corresponding arc s, prove that

$$s + x : s - x :: \sqrt{t + s} : \sqrt{t - s}.$$

38. A chain suspended at its extremities from two tacks in the same horizontal line forms itself into a cycloid; prove that the density at any point $\propto \sec^3 (\frac{1}{2}\theta)$, and the weight of the corresponding arc $\propto \tan (\frac{1}{2}\theta)$, θ being the arc of the generating circle measured from the vertex.

39. One end of a beam is connected to a horizontal plane by a hinge, about which the beam is suffered to revolve in a vertical plane; the other end is attached to a weight by means of a string passing over a pulley in the same vertical plane; find the position of equilibrium.

40. Find the pressure which a given power exerts by means of a common vice, the dimensions of which are given. 1827

41. The voussoirs of a bridge being very small, and the equilibrium maintained by the vertical pressure of the masonry above them, of what portion of a circle must the intrados consist, that the ascent of the bridge from a level road may be continuous, supposing the thickness of the arch at its summit to be to the radius of the intrados as 1 to $5\frac{1}{3}$?

42. An uniform elastic string being of such a length that when it hangs vertically, if an equal quantity were appended to the lowest point it would stretch it to twice that length, what weight must be appended at the middle point that the increase of length may be three quarters of the original?

43. Two given weights being attached to given points in the circumference of a wheel, find the position in which the greatest weight will be supported on the axle.

44. If two given equal weights sustain each other by a string passing over a smooth curve, the plane of which is vertical, the sum of the pressures on any arc depends only on the directions of its extremities.

45. If the particles of a hollow elastic cylinder be so arranged, that on its being subjected to a given internal pressure they may all be in the same given degree of dilatation, find how the thickness must be altered, in order that the strength of the cylinder may increase in arithmetical progression; the internal radius of the cylinder being supposed to remain constant.

1828 46. If a point be kept at rest by three forces acting upon it at the same time, any three lines which are in the direction of those forces and form a triangle will represent them.

47. A body is placed on a horizontal plane; find when it will be supported.

48. A ladder of uniform thickness rests with its lower end on a horizontal plane, and its upper end on a slope inclined 60° to the horizon: the ladder makes an angle of 30° with the horizon; find the force which must act horizontally at the foot to prevent sliding.

49. In a given sphere rests a given plane triangle of uniform thickness; find the angle which it makes with the horizon.

50. Find the form of a uniform chain suspended from any two points on the surface of an upright cone, and resting on the curve surface. Find the tension when it becomes a horizontal circle.

51. The equilibrium of the screw will take place when the power is to the weight as the distance of two contiguous threads to the whole circle described by the point where the force is applied.

52. If any number of forces act in the same plane upon a rigid body, determine their resultant, and the equation of the straight line in which the resultant acts.

53. Prove the formula for the place of the centre of gravity of any body, viz. $h = \frac{\int x dm}{m}$, and apply it to find the centre of gravity of a common parabola.

54. State the most recent and approved experiments, whereby it is ascertained that the decrement of velocity arising from friction is the same for all velocities.

55. When any number of forces act on a body, shew that the plane on which the sum of the projections of the *moments* is a maximum, is perpendicular to the planes with respect to which this sum is 0.

56. If two weights acting perpendicularly upon a straight lever on opposite sides of the fulcrum, or two forces in opposite directions on the same side of it, are inversely as their distances from the fulcrum, they will balance each other. 1829

57. If on an isosceles wedge, of which the angle is $2a$, a power P acting perpendicular to the base, balance a resistance W acting on each of the sides in a direction making an angle ι with a perpendicular to the side,

$$P : W :: \sin a : \cos \iota.$$

58. When a system is in equilibrium, if a small motion be given to its parts, the centre of gravity will neither ascend nor descend.

59. A cone and sphere of given weights support each other between two given inclined planes, the cone resting on its base. Determine what must be the vertical angle of the cone, that the equilibrium may subsist.

60. Find the ratio of the power to the weight in that system where each pulley hangs by a separate string; first, when the strings are parallel; secondly, when they are not.

61. Find the resultant of any number of parallel forces acting on a rigid body, and shew that they cannot in all cases be reduced to a single force which shall have the same effect.

62. Determine the equation to the catenary, the force of gravity being supposed constant.

63. A weight W is suspended from a point P of an uniform catenary APA′. O and O′ are the lowest points of two uniform catenaries, of which AP and A′P are parts. Shew that W is equal to the *difference* or *sum* of the weights of the portions OP, O′P of the catenaries, according as AP and A′P are *one* or *both* less than a semi-catenary.

64. The content of any segment of a right or oblique prismatic solid is equal to the area of one end of the segment, multiplied into the perpendicular let fall upon it from the centre of gravity of the area of the other end.

65. If three parallel forces acting at the angular points A, B, C of a plane triangle are respectively proportional to the opposite sides a, b, c; prove that the distance of the centre of parallel forces from A

$$= \frac{2bc}{a+b+c} \cos \frac{A}{2}.$$

66. A body being acted upon by any number of forces in the same plane; find the equations of equilibrium.

1830 67. A cord passing round a fixed point is drawn in different directions by two equal forces acting at a given angle; find the pressure on the point.

68. Explain the method of graduating the common steelyard.

69. Find the relation between the power and the weight when there is an equilibrium on the inclined plane.

70. Find the centre of gravity of any system of points whatever.

71. Find the resultant of any number of forces acting in the same plane upon a rigid body, and the equation to the line in which it acts.

72. Investigate the equation to the catenary between the arc and abscissa; and shew that the tension at the vertex is equal to the weight of a portion of the catenary of the same length as the radius of curvature at the vertex.

73. A uniform rod rests with one of its extremities in a semi-circle whose axis is vertical, find the nature of the line supporting its other extremity so that it may rest in every position.

74. If a hemisphere and paraboloid of equal bases and similar materials have their bases cemented together, the whole solid will rest on a horizontal plane on any point of the spherical

surface if the altitude of the paraboloid $= a\sqrt{\frac{3}{2}}$, a being the radius of the hemisphere.

75. State the principle of virtual velocities, and prove it when two bodies are in equilibrium on a bent lever.

76. When a chain fixed at two points is acted upon by a central attractive or repulsive force, the tension at any point is inversely as the perpendicular from the centre of force upon the tangent at that point.

77. If two equal weights act perpendicularly on a straight 1831
lever, they may be kept in equilibrium round any fulcrum by the same force as if they were collected at the middle point between them.

78. Define the centre of gravity, and find it in a plane triangle.

79. Two weights keep each other in equilibrium on a bent lever:

(1) Compare them.

(2) Prove that if an indefinitely small motion be given, the centre of gravity will neither ascend or descend.

80. Assuming that if δp, δq, δr be the virtual velocities of three forces P, Q, R which keep a point at rest,

$$P\delta p + Q\delta q + R\delta r = 0,$$

in whatever direction the virtual motion of the point takes place; prove that the forces are proportional to the sides of a triangle drawn in their directions.

81. A ladder rests with its foot on a horizontal plane, and its upper extremity against a vertical wall; having given its length, the place of its centre of gravity, and the ratios of the friction to the pressure both on the plane and on the wall; find its position when in a state bordering upon motion.

82. Find the relation of the power to the weight when there is equilibrium on the screw.

83. A body is supported on a plane curve by forces X and Y acting in the directions of the rectangular axes of x and y; prove that

$$Xdx + Ydy = 0.$$

84. If a uniform chain be suspended from two piers, the points of suspension being in the same horizontal line; shew that when the chain is nearly horizontal, the tension is nearly equal to the weight of a length $\frac{s^2}{4(s-b)}$ of the same chain, where s = length of the chain, and b = the distance between the points of suspension. Shew also that such a length may be given to the chain as to render the tension at either pier a minimum; and investigate an equation for determining the minimum tension.

85. Three uniform beams AB, BC, CD, of the same thickness, and of lengths l, $2l$, l respectively, are connected by hinges at B and C, and rest on a perfectly smooth sphere, the radius of which $= 2l$, so that the middle point of BC, and the extremities of A, D are in contact with the sphere; shew that the pressure at the middle point of BC $= \frac{91}{100}$ of the weight of the beams.

86. Graduate the common steelyard.

87. Find the centre of gravity of a semi-parabola.

1832 88. If two weights acting perpendicularly on a straight lever on opposite sides of the fulcrum balance each other, they are to one another inversely as their distances from the fulcrum.

89. Find the force requisite to draw a carriage wheel over an obstacle, supposing the weight of the carriage collected at the axis of the wheel.

90. Find the relation between P and W in equilibrium, on a system of pulleys where each string is attached to the weight, supposing the weights of the pulleys to be equal.

91. State generally the principle of virtual velocities. If any point of the system in equilibrio press against a resisting surface, prove that the virtual velocity of the normal force is nothing. Apply the principle to the following problem. Two bodies P and Q are connected by a rigid rod, one of which rests on a given plane; determine the nature of the curve on

which the other must rest that they may be in equilibrio in all positions.

92. Find the centre of gravity of the surface of a hemisphere, and $\int \frac{d\theta}{\cos^6\theta} \begin{pmatrix} \theta = 0 \\ \theta = \frac{\pi}{4} \end{pmatrix}$.

93. Find the forces parallel to the axes of an ellipsoid that keep a particle at rest on every point of its surface; prove that the resultant of all the forces varies inversely as the perpendicular from the centre on the tangent plane.

94. Explain the action of toothed wheels on each other, and find the ratio of P to W in the case of equilibrium.

95. Find the magnitude and point of application of the resultant of any number of parallel forces acting on a rigid body. Explain the meaning of the result when there are only two equal and opposite forces. What is their moment round any point in the plane in which they act?

96. A uniform catenary of given length is suspended from two given points at the same height, and is nearly horizontal; in consequence of an expansion of its materials the vertex of the catenary is observed to have descended through a small given altitude; find the increase of the length of the catenary, supposing its expansion to have been uniform throughout.

97. CA and CB are the arms of a uniform bent lever; determine the distance of its centre of gravity from C, having given the lengths of the arms and the angle ACB.

98. A uniform beam rests with one end against a smooth vertical wall, and with the other on a horizontal plane, the friction on which is $\frac{1}{n}$th of the pressure; determine the inclination of the beam to the horizon when just supported by the friction.

99. Find the centre of gravity of a triangular pyramid, and prove that its distance from the base is $\frac{1}{4}$th of the altitude.

100. Find the proportion of the power to the weight when there is equilibrium on the inclined plane, the power acting in any given direction.

1833 101. If any number of forces P, Q, . . . and P', Q', ... acting upon the arms of a lever to turn it in opposite ways round a fixed point C, be such that $P \cdot CM + Q \cdot CN + \ldots = P' \cdot CM' + Q' \cdot CN' + \ldots$ where CM, CN, . . . are the perpendiculars from C on the directions of P, Q, . . . there will be an equilibrium.

102. Having given the coordinates of any number of bodies, considered as points in the same plane; determine those of their centre of gravity.

103. Find the ratio of the power to the weight on a system of pullies, in which each pulley hangs by a separate string, and all the strings are parallel.

104. Having given that the resultant of two forces, applied at a point, is in the direction of the diagonal of the parallelogram, whose sides represent the forces in magnitude and direction; shew that it is represented in magnitude by the diagonal.

105. The pressure on the fulcrum of a lever, acted on by any number of forces in the same plane, is equal to the resultant of all the forces, supposing them applied at that point, retaining their directions.

106. An ellipsoid rests on a horizontal plane on the extremity of its mean axis; shew how to estimate the stability with regard to a slight displacement in any direction. Define the direction which distinguishes between stable and unstable equilibrium.

107. Apply the principle of virtual velocities to shew, that when three forces, acting perpendicularly upon the sides of a scalene wedge, keep each other in equilibrio, they are proportional to those sides.

108. If a rope applied to the arc of any curve be drawn by two forces acting at its extremities, and one of them be on the point of preponderating; prove that it is greater than the other

in the ratio of $\epsilon^{na} : 1$, where n is the ratio of friction to pressure at every point of the arc, a the angle between the normals at the points where the rope leaves the curve, and $\epsilon = 2.7182818$.

109. Let p, p' be two forces into which a given system acting upon a rigid body may be resolved; a, θ, the least distance and inclination of their directions; prove that $pp'a \sin\theta$ is invariable; also, if the same system of forces be resolved into a single resultant force and a single couple; prove that the moment of the couple multiplied by the sine of the angle which its plane makes with the resultant, is invariable.

110. In a plane triangle, if the line joining the centre of the circumscribed circle, and the point of intersection of the perpendiculars be trisected, the point of division which is nearest to the centre of the circumscribed circle is the centre of gravity of the area of the triangle.

111. The centre of gravity of three weights $a.(w-\alpha)^2$, $b.(w-\beta)^2$, $c.(w-\gamma)^2$, whatever be the value of w, will be situated in a line of the second order to which the lines joining the centres of gravity of the weights are tangents.

112. Required the equation to the common catenary sus- 1834
pended from two points in the same horizontal line. Shew how the constant may be determined when the length of the chain and the distance between the points of suspension are given.

113. In a system of parallel forces, having given the distances of the points of application from one another, and from a fixed point, find the distance of the centre of the system from that point. Apply the result to find the distance of the centre of gravity of a triangular pyramid from one of the angular points; assuming that it coincides with the centre of gravity of four equal bodies placed in the angular points.

114. Find the necessary equations of equilibrium of a rigid body acted on by any forces, and state the modifications of those equations, when, instead of being free, the body has one point immoveable, or two points immoveable.

115. Define the terms *couple* and *axis of a couple*; if the magnitudes of the moments and directions of the axes of two couples be represented by adjacent sides of a parallelogram,

prove that its diagonal will represent in moment and axis the resultant couple.

116. Explain the effect of friction in supporting an arch. How is the true theory of the arch connected with the theory of roofs?

117. If A, B, C represent the moments of a force round each of three rectangular axes which meet in a point, and α, β, γ be the angles which a straight line through the point of intersection makes with each axis, the moment of the force round this line is $A \cos \alpha + B \cos \beta + C \cos \gamma$.

118. Enunciate *Guldinus*' properties of the centre of gravity, and apply one of them to find the distance of the centre of gravity of the area of a semi-parabola from its axis.

119. State the principle of virtual velocities, and thence deduce the relation of the power to the resistance on the screw.

1834 120. If any number of forces P, Q, . . . and P′, Q′, ... in the same plane acting upon the arms of a lever to turn it in opposite ways round a fixed point C, be such that

$$P . CM + Q . CN + \dots = P' . CM' + Q' . CN' + \dots$$

where CM, CN, ... are the perpendiculars from C on the directions of P, Q, . . ., there will be an equilibrium. State the axioms and propositions which you assume in this proof.

121. If three forces act on a point in the directions of the sides of a triangle taken in order, and keep it at rest, they are represented in magnitude by the sides of the triangle. Is the same true of the sides of a polygon?

122. When P begins to move vertically from the state in which it balances W on the single moveable pulley with strings not parallel, P's actual velocity : W's actual velocity : : W : P. Does the proposition thus enunciated hold for all the mechanical powers?

123. In the combination of levers used in the Stanhope printing-press, find the relation between the pressure exerted and the force applied; and make the advantages of such a combination appear.

124. If a heavy cord passing over two pullies, fixed in a horizontal line, be kept at rest by equal weights attached to its extremities, prove that no possible increase of the weights can stretch it so as to become horizontal.

125. A number of forces in one plane act upon a point in given directions; determine the magnitude and direction of their resultant. How may both be expressed by one algebraical formula?

126. Define the centre of gravity, and shew how to find its position for two bodies considered as points. If a spherical surface be cut by two parallel planes, prove that the centre of gravity of the intercepted portion bisects the line joining the centres of the circular ends.

127. If a lever, kept at rest by weights P, Q, suspended from its arms a, b, so that they make angles α, β, with the horizon, be turned about its fulcrum through an angle 2θ, prove that the vertical spaces described by P and Q, are to one another as $a \cos(\alpha + \theta) : b \cos(\beta - \theta)$; and thence deduce the equation of virtual velocities.

128. If S and D represent respectively the semi-sum and semi-difference of the greatest and least angles, which the direction of a power supporting a weight on a rough inclined plane may make with the plane, and ϕ be the least elevation of the plane when a body would slide down it; prove that the cosine of the angle, at which the same power being inclined to a smooth plane of the same elevation would support the same weight, $= \dfrac{\cos S}{\cos \phi} \cdot \cos(D + \phi)$.

129. Explain the weighing machine for turnpike roads.

130. If a right-angled triangle be supported in a horizontal position by vertical threads fastened to its angular points, each of which can just bear an additional tension of 1 lb.; determine within what portion of the area a weight less than 3 lbs. may be placed without destroying the equilibrium.

131. Two forces balance each other on a lever moveable about a cylindrical axis; shew that the centre of the axis must

lie in the direction of their resultant. Find also how far it may be removed from this line, when friction is considered, before motion will ensue.

132. Find the form of an elastic lamina of uniform breadth and thickness, fixed at one end, and acted upon at the other by a given force, mentioning the hypotheses on which the investigation proceeds.

133. Find the conditions of equilibrium of any number of forces acting in the same plane upon a rigid body, and apply them to determine the position of a beam resting upon a point with one end against a vertical plane.

134. A combination of mechanical powers consists of a cylinder (turned by a winch) on which is the thread of a screw working in the teeth of a wheel; and round the axle of the wheel passes a cord, drawing a weight up an inclined plane to which its direction is parallel. Compare the force turning the winch with the weight drawn up the plane.

135. If the inclination to the horizon of a plane on which a body is placed, be slowly increased till the body begins to move; shew that its tangent, at the instant motion begins, expresses the ratio of friction to pressure.

1835 136. Explain the construction and graduation of the common steelyard. On what account is the common balance preferable in determining small weights?

137. Find the relation of P to W in the isosceles wedge.

138. Shew in what case a body will remain at rest when placed on a horizontal plane.

139. If three forces keep a point in equilibrium, and three lines be drawn making with the directions of the forces three equal angles towards the same parts; these three lines will form a triangle whose sides will represent the three forces respectively.

140. By what experiments is it proved that the friction between the same substances depends only upon the pressure? Shew how the coefficient of friction may be practically determined.

141. If in a system consisting of any number of particles a point be taken, and if each particle be multiplied by the square of its distance from the point, the sum of these products will be the least when the point is the centre of gravity.

142. Find the centre of gravity of the surface of an equilateral spherical triangle.

143. The centre of gravity of a system of bodies, *placed* in any given position, and acted upon only by the force of gravity and the reactions of the surfaces upon which the particles move, will *continually* descend *until* it becomes the lowest possible.

144. Explain the action of an oar when used in rowing; and determine the effect produced, having given the distances of the fulcrum and the hand of the rower from the side of the boat.

145. If any number of forces act in the same plane on a rigid body, determine the condition that they may have a single resultant.

146. A roof ACB, consisting of beams which form an isosceles triangle with its base AB horizontal, supports a given weight at C; find the horizontal force at A. Why must a pointed arch carry a heavy weight at its vertex?

147. How is a notion of force acquired? Give a definition 1836
of it. What is a resultant? Assuming the resultant of two forces, determine that of any number of forces acting on a point.

148. Find the distance of the centre of gravity of any number of bodies, in the same straight line, from a point lying between two of the bodies.

149. Determine the relation of P to W in a system of pullies, where the strings are parallel and each attached to the weight; the weights of the pullies being taken into account.

150. If two forces, acting perpendicularly on a straight lever on the same side of the fulcrum, are inversely as their distances from the fulcrum, they will balance each other.

A uniform heavy rod, at a given point of which a given weight is attached, is sustained at one end; determine its length when the force, which applied at the other end will keep it horizontal, is a minimum.

151. Explain and illustrate the proposition "in any machine, what is gained in power is lost in time, and conversely." Point out the principal advantages gained by the use of machines.

152. A body, the lower surface of which is spherical, rests upon a sphere; find in what case the equilibrium is stable.

153. Three forces act on a point in directions respectively perpendicular to three rectangular coordinate planes, and each varying as the coordinate to which it is parallel; shew that there are two planes, in either of which if the point be situated the resolved part of the whole force, which is parallel to the plane, tends to the origin and varies as the distance of the point from it.

154. If the vertical angle of a right cone of circular base be greater than $\sin^{-1}\frac{1}{4}$, the frustum cut off by any plane will be supported with its base on a horizontal plane.

If the vertical angle be less than $\sin^{-1}\frac{1}{4}$, determine the limits for the inclination of the cutting plane to the axis that the frustum may stand.

155. The extremities of a given flexible and uniform heavy chain are attached to unequal arms of a straight lever; investigate an equation to find the position of equilibrium of the lever, neglecting its weight.

156. Find the resultant of two parallel forces acting on a rigid body, and thence that of any number.

157. State briefly the results of experiments on friction.

Investigate the condition of equilibrium when two forces act at the extremities of a rope coiled about a rough cylindrical axle; and thence explain the effect of coiling the rope about the axle in working a capstan.

158. State and prove *Guldin's* property of the centre of gravity, by which the volume of a solid of revolution may be determined. Apply it to find that of the frustum of a right cone in terms of its altitude and the radii of its ends.

159. A rigid body is acted on by a system of forces which have not a single resultant; shew that they may be reduced to a single force and a couple whose plane is perpendicular to the direction of the force. Find the equations of the line in which the force acts, referred to any origin and rectangular axes, and the moment of the couple.

160. In the single moveable pulley, the strings not parallel, shew that P . P's velocity = W . W's velocity. Determine whether the equilibrium is stable or unstable.

SECTION XV.

QUESTIONS IN PURE DYNAMICAL SCIENCE, NOT INCLUDING CENTRAL FORCES.

1821 1. Find the law of force by which a body may describe a rectangular hyperbola, the force acting in parallel lines perpendicular to one of its asymptotes.

2. Two pendulums, the lengths of which are L and l, begin to oscillate together, and are again coincident after n oscillations of the first pendulum. Given L to find l.

3. In what direction must a body be projected with a given velocity from a point in a given inclined plane, that the range may be the greatest possible?

4. Two bodies are projected from two given points in given directions and with given velocities; find their distance at the end of t''.

5. A rod is placed in an inclined position, with one end upon a perfectly smooth horizontal plane; find the equation of the curve described by the other extremity whilst it falls.

6. Explain what is meant by *accelerating force;* and when P draws up Q in a machine, find the force accelerating Q's ascent, both when P and Q move with the same and with different velocities, the inertia and friction of the parts of the machine not being considered.

7. Two equal bodies A and B are connected by a string of given length: A is placed in a horizontal groove, and B

hangs freely down, the string passing through an aperture which is continued along the bottom of the groove: a given velocity is given to A; find the position of B at the end of t''.

8. A body descends by gravity, and describes in the n^{th} second of its fall a space $= p$ times the space described in the last but n; required the whole space.

9. Mention some of the experiments and observations, from which we may infer the truth of the second law of motion.

10. A given globe rolls down a given inclined plane in a medium resisting as the square of the velocity; to find the time of describing a given space.

11. Two bodies A and B are placed upon a horizontal plane, and connected by a rigid rod without weight: a body C impinges upon a given point of the rod, in a given direction and with a given velocity; define the motions of A and B, the body C not being connected with the system after the impact.

12. When a body is uniformly accelerated from rest, to find the space described in a given time.

13. Two equal weights are fixed, one at the middle point, and the other at the extremity of an inflexible and imponderable rod, which is suspended at the other extremity; if this compound pendulum be made to vibrate through small arcs, to find the time of its vibrations.

14. Apply *D'Alembert's* principle to find the velocity and time when p draws q over a fixed pulley.

15. A sphere of given radius is suspended by a point at a distance from its centre equal to its diameter. Find the time of its oscillation, and the point within the sphere at which it must be suspended so as to oscillate in the same time.

16. A body oscillates in a cycloid; compare the whole tension of the string at any point with the weight of the body.

17. A body oscillates in a cycloid, in a medium the resistance of which $\propto$ (vel.)^2; construct for the resistance at any point.

18. What must be the solid of revolution, so that when 1822
suspended by its vertex the centre of oscillation may be in its base?

19. Two equal heavy balls are suspended, by wires of the same given length, from the vertical axis of a machine, and are just in contact. How far will they separate from one another when a given angular velocity is communicated to the system?

20. A body falls down a given inclined plane, and, at the instant that it begins to fall, another is projected upwards from the bottom of the plane with a velocity equal to that acquired in falling through n times its length. Where will they meet?

21. A bucket descends into a well, unwinding a string from a cylinder of given weight and radius. What is the velocity acquired in falling through a given space, and the time of descent, the weight of the string being neglected?

22. Three equal weights are placed at the angles of an equilateral triangle without weight, which is suspended by an axis perpendicular to its plane bisecting one of its sides; find the centre of oscillation.

23. A ball whose elasticity : perfect elasticity :: $n : 1$, is projected with a given velocity in a direction making an angle of 60° with the horizon, and when at its greatest height is reflected by a vertical plane; determine where the ball will again strike the horizon and the whole time of flight.

24. Define the centre of spontaneous rotation; shew generally how it may be found, and determine it when a straight rod of uniform density and given length is struck perpendicularly at a given point.

25. Find the whole times of ascent and descent of a body urged by the force of gravity in a medium whereof the resistance varies as the square of the velocity, and give *Newton's* constructions.

26. Two straight rods equal in length are suspended by their extremities, one being of uniform density, and the density of the other varying as the n^{th} power of the distance from the point of suspension; and they make small oscillations in times which are as $\sqrt{5} : \sqrt{6}$. Required the value of n.

27. Compare the momentum of a paraboloid with that of its inscribed cone having the same base and vertex, when they both revolve round their common axis.

28. A string wrapped round a cylindrical annulus of uniform density whose radii are R and r, passes over a fixed pulley, and has a weight attached to it; find the space descended by the annulus in a given time.

29. The magnitudes of three perfectly elastic bodies are in harmonical progression; prove that the momentum communicated to either of the extremes by the impact of the other equals the momentum of the mean moving with the velocity of the impinging body before impact.

30. If a body describe the arc of a cycloid by a force acting parallel to its base; prove that the force varies inversely as $2 \sin \theta - \sin 2\theta$; θ being the corresponding arc of the generating circle reckoned from the vertex.

31. Explain the different uses of a fly-wheel in machinery.

32. P descends vertically, drawing Q over a fixed pulley; find the pressure upon the axis of the pulley, and its value when P is indefinitely increased.

33. A pendulum is composed of two thin wires of equal length, at right angles to each other at the point of suspension, and vibrating in their own plane. Find the time of a small oscillation; and the angle at which they must be inclined to each other so that the time of oscillation may be doubled.

34. A weight W is raised upon a moveable pulley. The two extremities of the cord are wound in different directions about two cylinders, which have a common axis but different radii; and the power P descends unwinding a string from a wheel of given radius upon the same axis. What is the force which accelerates P's descent, when the strings are parallel to each other?

35. Explain *Atwood's* machine; and mention some of the facts which it establishes in the theory of motion uniformly accelerated or retarded.

36. When the force by which a watch-balance is actuated varies as the n^{th} power of the distance from the point of the spiral spring's quiescence; find the alteration in the daily rate, in consequence of a given change in the arc of vibration.

37. A second's pendulum of given length, in the form of a thin rectangular bar, suspended at the middle of its extremity by an axis perpendicular to its plane, is carried to the top of a mountain. The length of the bar is diminished by a given quantity in consequence of a change of temperature, the breadth remaining the same, and it loses t'' in a day. What is the height of the mountain?

38. A cannon ball weighing 24 lbs strikes a wall with a velocity of 1700 feet. Find the weight of a beam, terminated by a hemisphere of the same diameter as that of the ball, which, when moved with a velocity of 10 feet, may penetrate to the same depth; and the weight of a similar beam, which may have the same effect in shaking the wall.

39. A body not affected by gravity falls down the axis of a thin cylindrical tube infinite in length, the particles of which attract with a force which varies inversely as the square of the distance. Find the velocity acquired in falling through a given space.

40. Shew that if a body oscillates in a cycloid, in a medium the resistance of which is constant, the successive altitudes to which it will rise are in arithmetical progression.

41. If a body is acted upon by any forces which would, if separately communicated, cause it to revolve about given axes with given angular velocities; find its axis of rotation, and its angular velocity; and apply the conclusion where there are three axes at right angles to each other.

1823 42. Two bodies A and B descend from the same extremity of the vertical diameter of a circle, one down the diameter, and the other down the chord of 30°. Find the ratio of A to B when their centre of gravity moves along the chord of 120°.

43. A semi-circular area is placed with its vertex upon a horizontal plane; find the time of one of its small oscillations.

44. A body projected in a direction parallel to the horizon, and acted upon by the force of gravity, describes a common cycloid; shew that the resistance of the medium, and the velocity at any point, vary respectively as $\sin\frac{\theta}{2}$, and $\cos\frac{\theta}{2}$, θ being the corresponding arc of the generating circle.

45. Divide the length of a given inclined plane into three parts, so that the times of descent down them may be equal.

46. Two bodies, A and B, whose elasticity is m, moving in opposite directions with velocities a and b, impinge directly upon each other; find the distance between them when t'' from the moment of impact have elapsed.

47. Determine geometrically that point in the hypothenuse of a given right-angled triangle whose base is parallel to the horizon, from which the time of a body's descent to the right angle may be the least possible.

48. Divide the arc of a cycloid into two parts, so that the times of a body's oscillating through them may be in the ratio of 1 : 5.

49. If the velocities of two balls A and B, whose elasticity is e, be a and b before impact, and u and v after; also if α and β be the velocities lost and gained, then will

$$\mathrm{A}a^2 + \mathrm{B}b^2 = \mathrm{A}u^2 + \mathrm{B}v^2 + \frac{1-e}{1+e}\{\mathrm{A}\alpha^2 + \mathrm{B}\beta^2\}.$$

50. A body descends down the convex side of a logarithmic curve placed with its asymptote parallel to the horizon; find where it leaves the curve.

51. If S be the momentum of inertia of a system, in respect of an axis which passes through its centre of gravity, S′ the momentum of inertia of the same system, in respect of an axis parallel to the first and distant from it by the space k, and M the mass of the system, then

$$\mathrm{S} = \mathrm{S} + \mathrm{M}k^2.$$

52. If from any point in a rectangular hyperbola whose axis is vertical, two lines be drawn to the extremities of the axis major, the times of descent down them will be equal.

53. Amongst all the axes passing through the centre of gravity of a triangle in its own plane, find that for which the momentum of inertia is a maximum or a minimum.

54. If two bodies be projected at equal angles of elevation, and with equal velocities, one in a non-resisting medium, and the other in a medium whose resistance corresponding to the

velocity $v = nv^2$; also, if s' be the arc of the parabola, and s the arc of the other curve described by the bodies when they are moving in directions making equal angles with the horizon, then will $e^{2ns} = 1 + 2ns'$.

55. The initial force accelerating a body down a circular arc is to the force accelerating it down the chord as 2 to 1 ultimately.

56. If the force vary inversely as the square of the distance, and m = force at a distance 1, and l = latus rectum of the conic section, then the area described in $1'' = \sqrt{\frac{lm}{8}}$.

57. P descending vertically draws Q up an inclined plane by means of a string passing over a pulley above it; find the velocity acquired by P in describing a given space.

58. Determine the pressure upon the axis of any vibrating body in any given position.

59. If the force of gravity be uniform and act perpendicularly to the plane of the horizon; it is required to determine the motion of a projectile in a medium whose resistance is proportional to the velocity. (*Newton*, Prop. 4, Book II.)

1824 60. Given the direction and velocity of projection; find the direction and velocity of a projectile at the end of t'', and its height above the horizontal plane passing through the point of projection.

61. A and B are two material points connected by an inflexible rod without weight; B moves on the horizontal plane CB, A descends along the inclined plane AC, the motion of the rod being in a vertical plane. Compare the velocity which A has at the point C with the velocity which it would have, were it to descend freely down AC.

62. A body is projected upwards from the lower extremity of a given vertical line with a given velocity; after what time must another be projected downwards from the upper extremity with the same velocity, so as to meet the former in the middle point of the line?

63. A body, which is half elastic, descends along the arc of an inverted cycloid, and is reflected by the axis which is vertical; find the space described in the time of n oscillations of a pendulum vibrating in the same cycloid.

64. To make a body oscillate in a given hypocycloid.

65. A and B are two balls whose elasticity $= e$, and A strikes B at rest; prove that if B be infinitely greater than A, A's momentum before impact : momentum communicated to B :: $1 : 1 + e$.

66. If a body, whose elasticity $= m$, descend from rest through an altitude h, by the uniform force of gravity in a medium whose resistance to a velocity $v = gkv^2$, and impinging on a horizontal plane, rise and fall alternately; prove that the whole space described by the body

$$= \frac{1}{gk^2} \log \left\{ \frac{m^2 e^{-2gk^2 h} - 1}{m^2 - 1} \right\}.$$

67. A given weight P draws another given weight W up an inclined plane of given height and length, by means of a string parallel to the plane; when and where must P cease to act that W may just reach the top?

68. Find the centre of oscillation of an isosceles triangle, vibrating about an axis which is perpendicular to its plane, and passes through the angle contained by the equal sides of the triangle.

69. Find the velocity acquired by an inelastic body descending through a system of three planes, the first being vertical, the second inclined at 45°, and the third at 15° to the horizon, respectively.

70. Find the isosceles triangle of a given area, which, vibrating about an axis passing through its vertex perpendicular to its plane, shall oscillate in the least time possible.

71. Having given the centre of gyration of a circle revolving about a diameter, find the centre of gyration of an ellipse revolving about either axis.

72. A body half elastic moving along a horizontal plane is reflected by a hard plane inclined at an obtuse angle to its

course; prove that the time of flight on the inclined plane : time of acquiring the velocity before impact by a body descending vertically :: tangent of plane's inclination : radius.

73. A body acted on by gravity ascends along the concave part of a vertical semi-circle from the extremity of the horizontal rudius; find its initial velocity, so that after quitting the circumference it may pass through the centre.

74. Find the centre of gyration of a right cone revolving about an axis passing through its centre of gravity parallel to one of its sides.

75. Find the correction due to the length of a pendulum for the thickness of its axis.

76. P and Q are two material points connected by an inflexible line PQ; P moves along a groove and PQ on a smooth horizontal plane. Having given the initial position of the rod and the quantity and direction of the motion communicated to P, find the angular velocity of the rod when it coincides with the grove.

77. Resistance varies partly as the velocity and partly as the velocity squared; construct for the time when a body is projected downwards with a velocity greater than the greatest it can acquire from rest.

1825 78. A weight P is supported by a string which passes over a fixed pulley, is wound several times round a cylinder Q, and attached to a pin on the other side, the strings being all parallel.

79. If the string be now cut between the cylinder and this point of support, required the motions of P and Q.

80. Prove that in the direct impact of perfectly hard bodies, the difference of the vires vivæ before and after impact is equal to the sum of the vires vivæ of the bodies moving with the velocities lost and gained, respectively.

81. If P and Q are two weights connected by a string passing over a fixed pulley, and acting in directions parallel to two given inclined planes having a common altitude, and P descends; find the space described in t'' and the velocity acquired.

82. Find the time in which a straight line of given length will oscillate when suspended at its extremity.

83. If a body oscillate in a cycloid beginning at the highest point, the tension of the string at any point arising from the centrifugal force equals the tension arising from gravity.

84. Find the centre of gyration of an hyperboloid revolving about its axis.

85. A body, whose elasticity : perfect elasticity :: e : 1, projected from a horizontal plane, at a given elevation, with a given velocity, impinges against a perfectly hard vertical plane, whose distance from the point of projection is given. Required the horizontal range of the body; the vertical plane being supposed perpendicular to the plane of the body's motion.

86. A body describing a parabola, by a uniform force acting in parallel lines, receives an impulse. Given the line which the body would describe by the impulse alone in a given time T; and the chord of the parabolic arc, which it would have described in the same time T if no impulse had been communicated to it; to find the chord of the arc which it actually does describe.

87. A weight P raises up another weight Q by means of a string passing over a fixed cylindrical pulley; given P, Q and the weight of the pulley, to compare the tensions of the two parts of the string; the weight of the string and the friction at the axis of the pulley being neglected.

88. A body oscillates in a cycloid in a resisting medium, where the resistance is as the square of the velocity; find the resistance at any point of the body's motion. (*Newton*, Prop. 29, Book II.)

89. Prove, from the preceding proposition, if a space S be taken so as to be represented by the rectangle of the hyperbola in *Newton's* figure on the same scale that the cycloidal arc is represented by the hyperbolic area, and V be the velocity acquired in falling through S by the force of gravity in medio non resistente; that V is the limit of the velocity which can be acquired in the resisting medium by the force of gravity.

90. Compare the time in which a sphere slides down an inclined plane with the time in which it rolls down the same plane.

91. A given uniform rod moves in the same plane in a hemisphere. Determine its motion.

92. Several bodies are projected from a given point, with the same velocity, in different directions, being acted upon by the force of gravity. Find the locus of them all at the end of a given time.

93. Find the elasticity of two bodies A and B, and their proportion to each other, so that when A impinges upon B at rest, A may remain at rest after impact, and B move on with an n^{th} part of A's velocity.

94. If a solid cylinder and a thin hollow cylinder of the same weight and radius roll together from rest down a given inclined plane, how far will they be separated after a given time?

95. Two bodies are projected from the same point in a horizontal plane with equal velocities and have the same horizontal range. Required the directions of projection so that the area included between the parabolas described may be the greatest possible.

96. Find the time in which a pendulum would oscillate in a hypocycloid within the earth, the diameter of the wheel being half the earth's radius.

97. P descends drawing Q over a fixed pulley. Find the space described in a given time, the string being conceived to have weight.

98. A straight bar of given length is made to oscillate in its own plane about an axis situated in a line which bisects it at right angles. Required the point of suspension, so that the time of oscillation may be the least possible.

99. Find the curve described by a body projected in a medium, the resistance of which varies as the velocity, and acted upon by gravity.

100. Prove that an arc of a circle, which does not exceed 60°, is a curve of quicker descent than any other curve which can be drawn within the same arc: and that the arc of 90° is a curve of quicker descent than any other curve which can be drawn without the same arc.

101. Find the time of a small oscillation of an oblique-angled parallelogram vibrating in its own plane about an axis passing through one of its angular points. 1826

102. If from a point in a horizontal plane, any number of bodies be projected in the same vertical plane with such velocities and in such directions, that the areas of the parabolas described shall be equal to one another; find the curve which shall touch them all.

103. Two circles are situated in the same vertical plane; determine analytically and geometrically the straight line of swiftest descent from one to the other; and shew that the two results agree.

104. Find the moment of inertia of a rectangle revolving in its own plane, round an axis passing through its centre of gravity.

105. From the top of a tower two bodies are projected with the same given velocity at different given angles of elevation, and they strike the horizon at the same place. Find the height of the tower.

106. Two spheres of given magnitudes and elasticity, not affected by gravity, are projected at the same time from given points with given velocities in opposite directions in the same straight line; find when and where their impact takes place, and their positions at the end of any assigned time after impact.

107. If the force be repulsive and vary as the distance from the centre of the globe; prove that the oscillations in an epicycloid are isochronous; and having given the radii of the globe and wheel, find the velocity at any point and the actual time of an oscillation.

108. Find the motions of two equal balls connected by an inflexible rod without weight, one of them being attached to a

given weight by means of a string passing over a fixed pulley, and the other moving on a perfectly smooth horizontal plane.

109. A straight rod which is always parallel to the horizon descends freely by the force of gravity, and at the same time revolves uniformly about one of its extremities; required the equations to the surface traced out by it, and to the tangent plane at any point.

110. A body is projected perpendicularly upwards, and the time between its leaving a given point and returning to it again is given; find the velocity of projection and the whole time of motion.

111. A globe of given weight and radius rolls down the surface of a hemispherical bowl from rest; find the velocity acquired at any point of its descent.

112. The bias and velocity of projection of a bowl are such as to cause it to describe a given logarithmic spiral; find the direction of projection, so that, after having described a path of given length, it may impel the jack in a given direction.

113. A body projected in the direction of the action of a constant force, describes P and Q feet in the p^{th} and q^{th} seconds; find the magnitude of the force and the velocity of projection.

114. Two bodies begin to descend at the same time down two given inclined planes from given points in the same vertical line; find their distance from each other at the end of any assigned time.

115. An uniform rod is made to vibrate about a point, so that the time of its oscillation is a minimum; find the force exerted on the point of suspension in any given position.

116. A semi-circle, the plane of which is vertical and base horizontal, has an uniform chain of given length, placed in a given position upon its circumference; find the velocity of the chain at the end of a given time.

117. Two given weights are connected by a string passing through a hole in a horizontal plane: one of them is projected in any direction in the horizontal plane, the other descends vertically by the action of gravity; find the motion of the bodies, and the curve described on the plane.

118. If a body be projected in a medium, the resistance of which varies as the velocity, and be acted on by gravity, and another be projected *in vacuo* at the same angle, and with the same velocity, and acted upon by the same constant force, and if t_1 and t_2 be the times of describing two arcs in the medium, and *in vacuo*, so related to each other that the tangents at their extremities shall cut the axis at the same angle; the $e^{kt_1} - 1 = kt_2$; k being the resistance to velocity 1.

119. If a body be projected downwards with a given velo- 1827
city, what is its velocity after describing a given space?

120. Prove, that when a mass entirely free is struck at any point, the motion of translation is the same as if the direction of the impact passed through the centre of gravity, and that of rotation as if the centre were fixed by an axis. By this proposition find the distance of the centre of percussion from a fixed axis.

121. Find, geometrically, the inclination of the path of a projectile to the horizon at a given time after the beginning of its motion, and prove by that means that the trajectory is a parabola.

122. Explain the division of a string which produces the several notes of the diatonic scale. What alteration would be made in the general pitch by assuming 256 instead of 240 for the number of vibrations constituting the tenor C?

123. If two elastic balls in the ratio of 1 to 3 meet directly with equal velocities, the larger one will remain at rest.

124. Shew that a tennis ball projected along an inclined roof, but not in the direction in which it would naturally fall, describes a parabola, and find its latus rectum, having given the inclination of the roof, and the velocity and direction of projection.

125. If a circle of given radius oscillate flatways through a small angle, determine the content of the solid which it traces out, having given the time of the oscillation and the whole angle through which it oscillates.

126. Find how long a given sphere, suspended by a twisted string which is suffered to untwist, will continue to turn in the same direction.

1828 127. By what experiments is the third law of motion established?

128. Two bodies of given magnitudes and elasticity impinge directly upon each other with given velocities; find the velocity of each after impact.

129. A body is projected from a given point in a given direction with a given velocity, and acted upon by gravity; find where it will strike a given plane.

130. Find the velocity and direction of projection of a ball, that it may be 100 feet above the ground at one mile distance, and may strike the ground at three miles.

131. A straight rod moves on a smooth horizontal plane, subject to the condition of always passing through a given point: determine its motion. Prove that the varying centre of gyration of the rod with respect to the fixed point will describe areas uniformly about that point.

132. Find the time of a body's descent down any arc of a cycloid, and shew that the times of the whole oscillations are as the square roots of the lengths of the strings.

133. If a rigid body oscillate about a horizontal axis, find the length of a simple pendulum which shall oscillate in the same time.

134. A uniform rod is at liberty to move freely in a vertical plane about a horizontal axis; find the nature of the circumference of a wheel which, revolving uniformly about a given horizontal axis, shall cause the rod to revolve uniformly also: the point of contact of the wheel and the rod being always at the same distance from the point of suspension.

135. A ring slides down a perfectly smooth rod revolving uniformly in a vertical plane; find the motion of the ring.

136. An oblique parallelopiped oscillates about one of its edges which is in a horizontal position; determine its motion and the pressure it exerts against the axis of suspension in any position.

137. If a body oscillate in a cycloid, in a medium the resistance of which varies as the velocity, and s be the first arc

of descent, prove that the whole space described by the body before the motion ceases

$$= s \cdot \frac{e^{\frac{k\pi}{\sqrt{\left(\frac{g}{4a} - k^2\right)}}} + 1}{e^{\frac{k\pi}{\sqrt{\left(\frac{g}{4a} - k^2\right)}}} - 1},$$

where $2k$ = resistance to velocity 1, g = gravity, $2a$ = diameter of the generating circle, and π = the semi-circumference of a circle the radius of which is 1.

138. A bent lever, of which the arms are a and b, and the angle θ, makes small oscillations in its own plane; the length of the isochronous simple pendulum is

$$\frac{2}{3} \frac{a^3 + b^3}{\sqrt{a^4 + b^4 + 2a^2 b^2 \cos\theta}}.$$

139. State and explain *D'Alembert's* principle, and apply it to determine the pressure on the axis about which a body revolves when acted on by a single force in a plane perpendicular to the axis.

140. Give *Newton's* construction for determining the path of a projectile acted upon by gravity in a medium whose resistance $= \propto$ velocity, and apply the differential equations of motion to determine the actual equation.

141. When any number of bodies move uniformly in straight lines in different planes, their centre of gravity also moves uniformly and in a straight line.

142. Having given the moment of inertia round any axis passing through the centre of gravity of a body, to determine that round any axis parallel to the former.

143. A globe is projected vertically upwards with a given velocity c, in a medium where the resistance is $= k \angle$ (vel.)^2, and is acted on by gravity; determine the relation between the time, space, and velocity.

144. A pendulum is taken to the top of a hill; how many seconds a day does it lose?

145. Find the general equation for the motion of a vibrating cord.

146. Find the velocity acquired by a cylinder unrolling and descending vertically through a given space.

1829 147. If two imperfectly elastic bodies impinge obliquely on each other with given velocities and directions, find the velocities and directions of their motions after impact.

148. If a body acted on by gravity be projected from a given point A with a given velocity so as to strike a given point Q, find the direction of projection; and if AI bisects the angle which AQ makes with the vertical, shew that there are two such directions equally inclined to AI.

149. A body urged towards a plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity. Find what force must act at the same time on the body parallel to the plane, that it may move in a given parabola having its axis in the plane; and determine the circumstances of the motion.

150. A uniform rod is oscillating about one extremity; find the tendency of the *vis inertiæ* in any given position to bend the rod at any point, and determine at what point that tendency is the greatest.

151. A body is oscillating in a cycloid, in a medium where the resistance varies as the $(\text{vel.})^2$, and the density varies inversely as the arc measured from the lowest point; prove by a method similar to *Newton's* in Book II. Prop. 26, that the time of descent to the lowest point will be the same from all altitudes; and apply the integral calculus to find the whole time, supposing the resistance at the highest point corresponding to any velocity v, to be less than $\frac{v^2}{l}$, where l is the length of the pendulum.

152. Find the limit of the velocity communicated by a body A to C, through an indefinite number of mean proportionals between A and C, the bodies being supposed perfectly elastic.

153. Explain *D'Alembert's* principle, and apply it to determine the motion of two bodies connected together by a wheel and axle; the inertia of the machine being taken into account.

154. The heights of the ridge and eaves of a house are H and h, and the roof is inclined at 30° to the horizon. Find where a sphere rolling down the roof from the ridge will strike the ground, and also the time of descent from the eaves.

155. A river, of which the breadth is a, flows with a velocity u, and a swimmer, whose velocity is nu, always aims at a mark on the farther bank directly opposite to the place where he entered the river. Find the curve in which he swims, and shew that the time of his arriving at the mark is equal to

$$\frac{na}{(n^2-1)\,u}.$$

156. A uniform rod, of which the elasticity is e, falls upon a smooth horizontal plane: given the altitude from which it falls, and its inclination to the horizon; find its motion after rebounding.

157. An imperfectly elastic body slides down a smooth plane of given length, and is reflected from the horizontal plane. Find the inclination of the plane that the range may be a maximum, and find the range.

158. Find the ratio of the height to the diameter of the base of a cylinder, that the moment of inertia may be the same about any axis whatever passing through its centre of gravity.

159. If a body oscillate in a medium in which the resistance varies as the square of the velocity; the differences between the times of oscillation in the medium and in vacuo are proportional to the arcs nearly. (*Newton*, Book II. Prop. 27.)

160. When a body is acted upon by forces X and Y in the directions of the coordinates x and y,

$$\text{prove } \frac{d^2x}{dt^2} = \text{X and } \frac{d^2y}{dt^2} = \text{Y}.$$

161. Assuming the ordinary expansion of $\delta\int \mathrm{V}dx$, determine the requisite addition to be made to it when V involves the limiting values of x, y, p, ...; and apply the method to find

the position of the curve of quickest descent from one curve to another, when the motion commences from the first curve.

162. A body moveable about a fixed axis is acted upon by a single force in a plane perpendicular to the axis; find the pressure on the axis arising from that force, and thence determine fully the coordinates of the centre of percussion.

163. When a body moves upon a surface of revolution, find the re-action of the surface.

164. If a body be acted upon by any forces, the motion of the centre of gravity will be the same as if all the forces were applied at that point; and the motion of rotation will be the same as if the centre of gravity were fixed and the same forces applied.

165. Define the radius of gyration of any body or system of bodies moveable about a fixed axis, and investigate an expression for determining its magnitude: apply also this expression to a sphere revolving about a diameter.

166. Find the angles which the axis of instantaneous rotation makes with the coordinates x, y, z.

167. If a point move through one or more spaces bounded by parallel planes, and be acted upon by a force which is perpendicular to the planes, and which is the same at the same distance from them, the angle of incidence is to the angle of emergence in a given ratio. (*Newton*, Book I. Prop. 94.)

168. Explain the nature and use of the ballistic pendulum, and perform the requisite calculations in the experiment.

169. If a body move in a surface of revolution acted upon by a centre of force situated in the axis, the areas described, projected on a plane perpendicular to the axis, are proportional to the times. (*Newton*, Book I. Prop. 55.)

170. Shew that when a body moves in an inverted cycloid, the force by which it is urged along the curve varies as the arc to the lowest point; and hence shew that the oscillations are isochronous.

1830 171. Two bodies whose common elasticity is e, moving with given velocities, impinge directly upon each other; it is required to determine their velocities after impact.

172. A perfectly smooth rod in a vertical plane revolves uniformly round a vertical axis, and a ring placed on it is attracted to a horizontal plane by a force varying as the distance in addition to the uniform force of gravity; required the form of the rod that the ring may remain on whatever point it is placed.

173. The equation to the path of a projectile is

$$y = ax + \frac{g}{k^2}\text{hyp. log}\,(1 - bx),$$

gravity ($= g$) acting parallel to the axis of y; shew that the resistance $= k$ velocity.

174. A circular sector revolves through any angle round one of its extreme radii; find the centre of gravity of the solid generated, its density varying as the n^{th} power of the distance from the centre of the circle.

175. In the above case, supposing the angle of the sector and the angle through which it has revolved to remain the same, prove that as the radius varies, the motion of the centre of gravity will be in a plane passing through the centre of the circle; find the line of motion and the equation to the plane.

176. A body acted on by gravity oscillates in a curve, and a chain of given length, suspended from the horizontal ordinate where the motion commences, is divided by the ordinate at each point in two parts proportional to the two parts of the tension at that point arising from the centrifugal force and from gravity. What is the curve?

177. A paraboloid revolving round its axis strikes a body P in a direction perpendicular to the radius, and P, being attracted to the intersection of the radius and axis by a force varying as $\frac{1}{D^2}$, after impact describes a parabola of the same dimensions as the generating one. Determine the velocity of rotation and the point of impact.

178. If a pendulum of length l vibrate in a small circular arc in a medium of which the resistance $= kv^2$ to velocity v, and if s be the arc described from the commencement of a vibration to the point where the velocity is greatest when the

friction at the axis of suspension is taken into account, and s' the corresponding arc when the friction is neglected, prove that

$$e^{2ks'} - e^{2ks} = f\,\frac{2kl}{g},$$

f being the constant effect of friction, and g gravity.

179. If two bodies A and B of elasticity e impinge directly on each other with velocities a and b respectively, and if u and v be their respective velocities after impact, and p and q the velocities lost and gained respectively,

$$\text{then } \mathrm{A}a^2 + \mathrm{B}b^2 = \mathrm{A}u^2 + \mathrm{B}v^2 + \frac{1-e}{1+e}(\mathrm{A}p^2 + \mathrm{B}q^2).$$

180. Shew that the centres of oscillation and suspension are reciprocal, and explain the use of this property in finding practically the length of a pendulum.

181. A body attracting with a force varying directly as the distance moves uniformly in a straight line; determine the motion of another body situated in the same plane and subject to its influence.

182. A body descends down the arc of a vertical catenary having its vertex at the lowest point; find the curve of ascent when the oscillations are isochronous, the two curves being so united at the lowest point as to have a common tangent.

183. A corpuscle is attracted by two straight lines at right angles to each other, the particles of which attract with forces varying as $\frac{1}{\mathrm{D}^2}$: having given the position of the corpuscle and the length of one of the lines, find the length of the other when the direction in which the corpuscle begins to move is towards their common intersection.

184. A body descends in a straight line in a medium whereof the density varies as the square root of the distance from a given point, and is urged by a constant force tending to that point; find the velocity and time corresponding to a given space, supposing the resistance to vary as the density and velocity jointly.

185. Two balls connected together by an inflexible and inextensible line are constrained to move, the one on a horizontal

plane, the other on an inclined plane which is at liberty to move freely on the horizontal plane; find the motions of the balls and of the plane, supposing the motion of the rod to be in a vertical plane.

186. A right-angled triangle vibrates in its own plane about an axis passing through its vertex, find the length of the isochronous simple pendulum; and if one of the sides be slightly diminished and the other as much increased, determine the variation of the pendulum.

187. A body falls towards a centre of force which varies as $\frac{1}{D^3}$, in a medium of which the density varies as $\frac{1}{D^3}$, and the resistance varies as (velocity)^2. Prove that at any distance r from the centre,

$$(\text{velocity})^2 = \frac{m}{h}\left\{1 - e^{-h\left(\frac{1}{r^2}-\frac{1}{a^2}\right)}\right\},$$

where m = force at distance 1, h = density at distance 1, and a = distance from centre at the beginning of the motion.

188. A uniform rod vibrates in a medium the resistance of which varies as the velocity; find the time of one of its small oscillations.

189. A body moving on a curve is acted on by forces X and Y parallel to the axes of the curve; find the reaction, and apply it to find the tension of a string, at the lowest point, when a body oscillates in a circle through an arc of 120°.

190. A ladder rests with one end on a smooth horizontal plane, and the other against a smooth vertical wall; find the horizontal force at its foot which will keep it at rest; and when the force is removed determine its motion.

191. A pile of weight w is driven by a hammer H impinging with a velocity v, the friction being represented by F; find the motion; and when the velocity given to the pile is small, approximate to the whole space through which it is driven.

192. A body is acted on by gravity; find the tautochronous curve in a medium in which the resistance varies partly as the velocity and partly as the square of the velocity; and from the result prove that it is a cycloid when the resistance vanishes, or varies as the velocity.

193. Find the moment of inertia of an ellipse revolving in its own plane about any axis.

194. If a system move in any manner whatever, prove that $\int \Sigma mvds$ is a minimum.

195. Find the time of an oscillation in a hypocycloid, the body being acted upon by a force varying as the distance from the centre of the globe.

196. Find the moment of inertia of a system about any axis passing through the origin of the coordinates, and the moment of inertia about any axis in terms of the moments about the principal axes.

197. If a body acted upon by gravity be projected in a medium the resistance of which varies as the square of the velocity, find the equation to the curve described; and when the resistance vanishes, shew that it is the equation to a parabola.

1831 198. Define *inertia, mass, weight, moving force, accelerating force.*

199. A body acted on by gravity descends in a straight line; find the space described in a given time, and prove that it is equal to half the space which would be described in the same time with the last acquired velocity continued uniform.

200. A body descends on an inclined plane, find the accelerating force; and prove that at any point in the descent, the velocity is equal to that which would be acquired down the perpendicular height.

201. If the material particles $m, m', \ldots$ of any system in motion, pass from the positions $a, a', \ldots$ to $b, b', \ldots$ during the very small time δt, and $c, c', \ldots$ be the positions they would have had, if during δt only the impressed forces had acted; then forces proportional to and in the directions of $\overline{bc}$, $\overline{b'c'}, \ldots$ will produce equilibrium with the pressures on the fixed points and axes of the system.

202. Also, if $e, e' \ldots$ be the places $m, m' \ldots$ would have had, if the impressed motions had been compounded with uniform motions during δt along the actual paths with the

velocities at a, a' . . .; then forces proportional to and in the directions of $\overline{be}$, $\overline{b'e'}$, . . . will keep the system at rest; and if γ, γ', . . . be any other positions compatible with the conditions of the system $\Sigma \, . \, m\overline{eb}^2 \angle \Sigma \, . \, m\overline{e\gamma}^2$.

203. An elastic chord AabcB is stretched between two fixed points, A, B; the portion abc is made to assume the form of two straight lines ab, bc, the points a and c being in the straight line joining A, B, and b at a small distance from it: when the chord is suddenly left to itself, what motion will take place, and what will happen when the motion reaches the fixed points?

204. Having given the direction and velocity of projection of a body acted upon by gravity, find the equation of the trajectory.

205. Find the time of oscillation in a cycloidal arc.

206 A system of material points moveable about a horizontal axis, has all its parts acted on by gravity; it is required to determine the accelerative force, and to find a point of the system which shall be accelerated exactly as much as a single point in the same position.

207. Shew that if a material particle, moving in any manner in space, be solicited by the forces X, Y, Z, in the directions of three rectangular axes, and x, y, z be the coordinates of its place at the time t,

$$X = \frac{d^2x}{dt^2}, \quad Y = \frac{d^2y}{dt^2}, \quad Z = \frac{d^2z}{dt^2}.$$

208. If two equal bodies which attract each other with forces varying as $\frac{1}{(\text{dist.})^2}$ are constrained to move in two straight lines at right angles to one another, shew that they will arrive together at the point of intersection of the lines, from whatever points their motions commence. And having given their distance at the beginning of the motion, find the time to the point of intersection.

209. A given hemisphere rests with its base upon a horizontal plane, and a given uniform rod, one end of which is moveable about a horizontal axis fixed in the plane, is palced against the

hemisphere so as to be a tangent to a great circle of it; and the rod by its pressure puts the hemisphere in motion; find the equation for determining the motion of the rod when the friction of the plane varies as the pressure of the hemisphere upon it. And when there is no friction, find the angular velocity of the rod when it comes to the plane.

210. If a body be attracted by a force which varies as $\frac{1}{(\text{dist.})^n}$, find the value of n when the velocity acquired from an infinite distance to a distance r from the centre is equal to the velocity that would be acquired from r to $\frac{r}{4}$.

211. A body being acted upon by any forces, prove that the motion of the centre of gravity will be the same as if all those forces acted at that centre, and that the motion of rotation will be affected as if the centre of gravity were fixed, and the same forces applied.

212. The moment of inertia of any system, with respect to any given axis, is equal to the moment about an axis parallel to this passing through the centre of gravity, together with the moment of the whole body collected in its centre of gravity about the given axis.

213. If any number of bodies be acted upon by their mutual attractions, their centre of gravity will either be at rest, or move uniformly in a straight line.

214. A body which is symmetrical with respect to a vertical plane passing through the centre of gravity revolves about a horizontal axis, the body being acted on by gravity only; find the pressure on the axis.

215. State and prove the principle of the conservation of *vis viva;* and when the *vis viva* is a maximum, shew that the body passes through a position of stable equilibrium, and when a minimum through a position of unstable equilibrium.

216. A cylindrical body unrolls itself from a vertical string, the other end of which passes over a fixed pulley and supports a weight; it is required to determine the motion.

217. A second's pendulum is carried to the top of a mountain, and there loses 48″ 6 in a day; determine the height of the mountain, supposing the earth's radius to be 4000 miles.

218. What is the numerical value of the measure of the force of gravity? and by what experiments is it determined? 1832

219. When a body is accelerated in a straight line by a uniform force, and sets off with a given velocity, find the velocity acquired and the space described in a given time.

220. Two weights are connected by a string passing over a fixed pulley, and slide along two inclined planes; shew that the velocities of the weights at any time are inversely as the cosines of the angles which their strings make with the planes.

221. A body acted on by gravity moves on the convex surface of a cycloid, the velocity at the highest point being $\sqrt{2gh}$; determine the point where it will leave the curve, and the latus rectum of the parabola afterwards described.

222. A and B are two points on opposite sides of a plane curve, and a body goes from A to B in the least time possible, its velocity on the two sides of the curve being in the ratio of $1 : n$. Shew that the path of the body cuts the curve in angles whose cosines are in the same ratio.

223. A uniform rectangular parallelopiped is supported on two hinges placed on one of its edges which is vertical; determine the magnitude and direction of the pressure on the hinges. Also if the body be struck by a given horizontal force perpendicularly to one of its faces at a given point, determine the angular velocity communicated to it, and the consequent pressure on the hinges.

224. State the second law of motion, and the experiments by which it is established.

225. What is the meaning of the terms velocity and accelerating force in variable motion? Investigate the equation $f = \frac{dv}{dt}$.

226. In a system consisting of any number of points in the same plane moveable about an axis perpendicular to that plane, a force P acts to turn the system; apply *D'Alembert's* principle to find the effective accelerating force on any point.

227. If a pendulum oscillating in a small circular arc be acted upon, in addition to the force of gravity, by a small horizontal force (as the attraction of a mountain) in the plane in which it oscillates, having given the number of seconds gained in a day, find the horizontal force.

228. A cylinder descends vertically by unrolling itself from a string the end of which is fixed; and at any point of its descent a weight suspended by a string is attached to the cylinder at the point where the former string is a tangent; shew that this weight will not alter the motion of the cylinder.

229. If a uniform straight rod oscillate about one extremity through a small angle, in a medium of very small density, of which the resistance varies as the square of the velocity, find the difference between the angles described on opposite sides of the vertical; and shew that it varies as the length of the rod when its thickness and density are given.

230. A door opened through a given angle is to be shut by means of a weight attached to a string passing over a pulley and acting horizontally on the door, the pulley being in the post against which the door shuts; to determine the motion.

231. Of all right cones having the same volume, determine that which will oscillate about an axis through its vertex and perpendicular to the axis of the figure in the least time possible.

232. Find the time of an oscillation in the cycloid, when the resistance varies as the velocity.

233. Define the moment of inertia, and that of a globe revolving round its axis.

234. State the properties of the principal axes of rotation, and find the moment of inertia about any axis in terms of the moments about the principal axes.

235. A small weight is attached to an uniform rod which oscillates about a given point in it; determine the simple pendulum, and shew that there are two points, one above, another below the point of suspension, where a small change in the position of the weight does not affect the length of the simple pendulum.

236. Find the equation to the curve described by a projectile acted on by gravity; and determine its greatest distance from a plane passing through the point of projection.

What laws of motion are employed in this investigation, and in what manner?

237. Find the centre of oscillation of any body.

238. Two bodies whose common elasticity is e, moving with given velocities, impinge directly on each other; determine their velocities after impact. 1833

239. Compare the space described by a projectile in the direction of projection, with its vertical fall in the same time. If the velocity be given, determine the angle of projection, that the focus may lie in the horizontal line through the point of projection.

240. Define accelerating force, and state how it is measured. Describe *Atwood's* machine, with the nature and objects of the experiments made by it.

241. Find the time of oscillation in a small circular arc.

242. Having given the moment of inertia of any body about an axis through its centre of gravity, find the moment of inertia about an axis parallel to the former, at a known distance from it. Also having given the moment of inertia of a plane figure about each of two axes in it, at right angles to one another, find its moment about an axis through their intersection perpendicular to them.

243. A given plane area revolves with an uniform angular velocity about a horizontal axis fixed in its own plane and at a given altitude above a horizontal plane. Determine the point at which a sphere of given mass should be opposed to its impact that it may be projected to the greatest possible distance on the horizontal plane.

244. A pendulum is observed to make n vibrations in a certain time at a place of known latitude, and, by calculation from an assumed approximate value of the earth's ellipticity, the number of vibrations in the same time performed by the same pendulum, at the place of observation and the equator,

ought to be n' and N respectively. Shew that a nearer value of the ellipticity is obtained by multiplying the assumed one by

$$1 + \frac{n' - n}{n' - N}.$$

245. An inflexible straight rod is set in motion round a vertical axis passing through one extremity, about which it is capable of revolving freely in a horizontal plane. Determine the motion of a ring sliding freely along it; prove that the whole *vis viva* of the system is constant.

246. A groove in form of a cycloid, with its vertex downwards and base horizontal, is cut in a solid vertical plane; determine the time of oscillation of a heavy body moving along it, while the vertical plane itself is capable of moving freely along a smooth horizontal plane, and the curve which the body describes in space.

247. From two points in the same vertical line, and at given distances from a fixed horizontal plane, two equal elastic balls are dropped; determine the successive points of meeting, and the times in which each will return to its original position.

248. If on a rough horizontal plane revolving uniformly about a vertical axis a rough sphere be placed; determine its initial motions, and shew that its path in space will be a circle.

249. A particle attracted to two centres of force varying inversely as the square of the distance, will oscillate in the arc of an hyperbola of which they are the foci, supposing it to have been originally at rest in such a position as to be attracted equally by each.

250. A semi-cycloid is placed with its axis vertical and vertex downwards, and from different points in it a number of heavy bodies are let fall at the same instant, each moving down the tangent at the point from which it sets out; prove that they will reach the involute all at the same instant.

251. A rocket ascends vertically in a medium of which the resistance varies as the velocity; the inflammable composition contained in it being supposed to produce a constant moving force, and to be exhausted in n'' at an uniform rate of consumption; determine the height to which the rocket will ascend.

252. Determine the motion of two heavy particles connected by an inflexible rod without weight, one of which moves on a surface of revolution, and the other is constrained to move in the axis of the surface which is vertical. Find the velocity of the particle on the surface, when the other continues stationary.

253. A body acted on by gravity descends from rest down a given circular arc, the tangent to which at the lowest point is horizontal; compare the initial accelerating force with that down the chord.

254. Find the centre of spontaneous rotation when a body acted on by no forces is struck by another, and determine its path, neglecting the inertia of the striking body.

255. State the principle of least action, and apply it to find the path described by a projectile when acted on by gravity.

256. Having given the centres of gravity and oscillation of any number of bodies revolving round a common axis, determine the centre of oscillation of the system.

257. When a body is moveable about a fixed axis, define the centre of percussion, and investigate the coordinates which determine its position. Shew that it coincides with the centre of oscillation, when the axis of rotation is parallel to one of the principal axes through the centre of gravity.

258. When a vibration is propagated along a cord fixed at one end, prove that the reflected wave returns with a velocity equal to that of the incident, but at the opposite side of the axis.

259. Determine the time in which a cylindrical body rolls down a given inclined plane.

260. A body moves on any curved surface acted on by any forces; to find the pressure.

261. Explain how velocity and accelerating force are mea- 1834
sured. If a heavy body, in vacuo, fall from rest through $16\frac{1}{10}$ feet in 1″, determine the force of gravity.

262. Determine the path of a heavy body projected obliquely, and shew distinctly at what points of the investigation the 1st and 2nd laws of motion are applied.

263. Describe the experiments from which it appears that, in the direct impact of elastic bodies of the same kind, the force of restitution bears a constant ratio to the force of compression.

Prove that the direction and velocity of the motion of the common centre of gravity are not altered by the impact of two bodies.

264. State and explain *D'Alembert's* principle; and apply it to determine the motion of two weights, when one draws up the other by a wheel and axle, neglecting the inertia of the machine.

265. Of an hyperbola whose major axis is horizontal, determine the diameters down which a heavy body will descend in a given time; and that down which it will descend in the shortest time.

266. A hemisphere rests on a horizontal plane with a string fastened to its edge, which, passing over a pulley, supports a weight; determine the position of equilibrium, and, if the string be cut, the motion of the hemisphere.

267. Having given that the principal axes of a body are determined by a cubic of the form

$$(s-a)(s-b)(s-c)-a'^2(s-a)-b'^2(s-b)-c'^2(s-c) = 2a'b'c';$$

prove that it has for a limiting equation, the quadratic to which it is reduced by making any two of the quantities a', b', c', vanish; and thence, that all its roots are real.

268. From a fixed point S the straight line Sp is drawn continually representing the velocity of a body moving freely in one plane, and continually parallel to the tangent at the corresponding point of the body's path; prove that the force which acts upon the body will be continually represented by the velocity of the point p along the curve which is its locus. If the motion of the body be that of a projectile in vacuo, prove that the locus of p is a straight line.

269. If two of the principal axes, drawn through any point of a body, lie in a plane which passes through the centre of gravity, prove that every point of the body, situated in that plane, will also have two of its principal axes lying in the same place.

If one of these two principal axes be always parallel to a given straight line, determine the locus of the corresponding points of the body.

270. A perfectly elastic solid of revolution, turning about its axis at a given rate, impinges on a hard smooth plane. If before impact the centre of gravity moves perpendicular to the plane with a velocity v, determine the motion of rotation after impact, and prove that the centre of gravity will move in the same direction with a velocity $v \cdot \frac{p^2 - k^2}{p^2 + k^2}$, where p is the perpendicular from the centre of gravity on the normal at the point of impact, and k is the radius of gyration round an axis through the centre of gravity perpendicular to the axis of the solid.

271. Determine the apparent path of a projectile to a person, who advances uniformly in a straight line towards the point of projection.

272. For what axes of suspension is the time of a small oscillation of a solid body an absolute minimum? Take the case of an ellipsoid.

273. A heavy body, symmetrical with respect to a vertical plane passing through its centre of gravity, revolves about a horizontal axis perpendicular to that plane; determine the pressure on the axis. Apply the result to the case of a cylinder whose axis is bisected by the axis of rotation.

274. Find the differential equation of the first order to the path of a projectile in a medium where the resistance $\propto$ (vel)^2; determine the velocity at any point, and shew that it is least after the body has passed the highest point.

275. Investigate equations for finding the angular velocities of a rigid body acted upon by any forces, about three principal axes intersecting in its centre of gravity which is supposed to be fixed. When the body is acted upon by no forces, what is meant by the invariable plane?

276. Employ the equations of motion to find the path of a particle, upon a smooth horizontal plane, fastened by a thread to a point whose motion is uniform and rectilinear in that plane.

277. Find the range of a projectile on a horizontal plane, not passing through the point of projection, and the direction of projection when the range is the greatest.

1835 278. Define *mass* and *weight*; and state the measures of moving and accelerating force. Describe some simple experiments by which it appears, that when pressure communicates motion directly, the momentum communicated in a given time is proportional to the pressure.

279. A body is projected vertically downwards with a given velocity, find the time of describing a given space; and explain the two roots of the quadratic equation.

280. A body is projected from the top of a tower with a given velocity and in a given direction; find the range on the horizontal plane passing through the foot of the tower, and the time of flight.

281. In the direct impact of two perfectly elastic bodies, the sum of each body multiplied by the square of its velocity is the same before and after impact. Shew that the same is true when the bodies impinge obliquely.

282. A heavy body oscillates in a circular arc, find the tension of the string in any position of the body.

283. If a rigid body move in any manner whatever, its *vis viva* at any instant is equal to the *vis viva* of the whole mass collected at its centre of gravity, together with the *vis viva* round its centre of gravity.

284 A billiard ball A in motion is struck by an equal one B moving with the same velocity, and in a direction making an angle of α° with that in which A is moving, in such a manner that the line joining their centers at the time of impact is in the direction of B's motion; find the velocities of the bodies after impact, and shew that that of A will be a maximum, when $\tan\frac{\alpha}{2} = \left(\frac{e}{1+e-e^2}\right)^{\frac{1}{2}}$, e being the elasticity of the bodies.

285. Three equal rods AW, BW, CW without weight support a weight W; their lower ends rest on a smooth horizontal plane and are connected by strings in such a manner that the rods and strings form the edges of a regular tetrahedron; if the

string BC be cut, and the point A be prevented from sliding, compare the velocities of the ends of the rod BW, and shew that they are equal, when the string AB has revolved through 15°.

286. A cone of given form, and supported at G its centre of gravity, has a motion communicated to it, round an axis through G, perpendicular to the line joining G with a point in the circumference of the base, and in a plane passing through this point and the axis of the cone. Determine the position of the invariable plane; and explain clearly the motion of the cone's vertex.

287. A circular arc without weight, whose length is l subtending an angle a at the centre, is placed upon a plane and acted upon at every point by a constant repulsive force f tending from one extremity of the arc; shew that it will rest with its chord c parallel to the plane if friction $\div$ pressure $>$ $\cot \frac{a}{4}$.

If the position of equilibrium be slightly disturbed and friction sufficient to prevent all sliding, the time of a small oscillation

$$= \pi \operatorname{cosec} \frac{a}{4} \sqrt{\frac{l-c}{2f}}.$$

288. A sphere will *roll from rest* between two given curves in the shortest time possible down the involute of a cycloid. The curve will have its axis vertical, and cusp at the point from which the sphere begins to fall. It will cut the lower curve at right angles, and the two curves in points at which the tangents are parallel.

289. An uniform slender rod acted upon by gravity g is placed between two planes (one horizontal and the other vertical) having at a point in their common intersection an attractive force $\propto \dfrac{1}{(\text{dist.})^2}$ which at the centre of gravity of the rod *always* $= \dfrac{g}{2}$. Shew that it will rest in a position of *unstable* equilibrium at an inclination of $\dfrac{3\pi}{8}$ to the horizon.

If the rod be originally placed in any given position, determine the motion.

290. A shot is fired at random with a given velocity towards a tower, whose horizontal distance from the cannon $= \frac{1}{2}$ the greatest range, and whose altitude subtends an angle $= \tan^{-1}\frac{3}{4}$ at the point of projection; supposing the cannon is capable of being elevated from a horizontal position through an angle of 80°, shew that the odds in favour of the balls striking the tower are 3 : 1.

291. Explain the application and use of the pendulum in the common clock. Describe the dead beat escapement, and shew in what its superiority over the recoil escapement consists.

292. Form a given mass into a cone, such that the moments of inertia round all its principal axes may be equal.

293. Determine the moment of inertia of a cylinder round its axis, and the time of its rolling down a given inclined plane.

294. A particle oscillates in a small circular arc acted on by gravity and by a small disturbing force; determine the effect of the disturbing force on the time and arc of vibration.

295. Explain the mechanical effect of a locomotive engine.

296. When a body is acted on by a central force, the velocity at any point of the brachystochronous path between two given curves varies as the perpendicular on the tangent from the centre of force. Shew also that the path cuts the curves at right angles.

297. The time of an oscillation being $\pi\sqrt{\frac{l}{g}}$, find the alteration in the time corresponding to given small and contemporaneous alterations in l and g.

298. A body acted on by gravity is projected vertically upwards with a given velocity in a fluid in which the resistance varies as the square of the velocity; determine the accelerating force and the whole time of the ascent.

299. A semi-circular area acted on by gravity revolves round a horizontal axis which coincides with its diameter; find the pressure on the axis in any position.

300. Having given the moments of inertia of a body round three rectangular axes, find its moments of inertia round any other axis passing through the origin; and shew that the sum of the moments round any three rectangular axes passing through the same point, is constant.

301. Determine the centre of oscillation of a pendulum consisting of a uniform rod with a weight attached to it, and the effect produced on the time of an oscillation by a small change in the position of the weight.

302. A body descending vertically draws an equal body 25 feet in $2\frac{1}{2}$ seconds up a plane inclined at 30° to the horizon, by means of a string passing over a pulley at the top of the plane; determine the force of gravity.

303. State the distinction between *accelerating force* and *moving force*. 1836

If a body be projected vertically upwards or downwards, shew that, when it has described a given space, its velocity $=$ the velocity of projection $\mp$ that which gravity would generate in the *time* the body has been in motion.

Why may not the last term of this equation be the velocity which gravity would generate in a body falling from rest through the same *space*?

304. Two bodies whose common elasticity is e, moving with given velocities, impinge directly on each other; determine their actual and relative velocities after impact.

305. The path of a projectile in vacuo is a parabola with its axis vertical; and the velocity at any point is that acquired in falling from the directrix.

306. If a body oscillate in a circular arc, the accelerating force varies as the sine of its angular distance from the lowest point; prove this, and find the time of a small oscillation.

307. Define the centre of oscillation of a rigid system, and find its position.

Two heavy particles connected by a rod without weight are suspended at a given point in the rod; find the time of a small oscillation.

308. A weight P descending vertically draws Q up an inclined plane by means of a string passing over a pulley fixed

above the plane; and it is observed, that at the instant when the two parts of the string between the pulley and the weights become parallel, Q rises off the plane. Compare P and Q, having given the positions of the pulley and plane, and the length of the string.

309. A rotatory motion about its axis is communicated to an elliptic cylinder, which is then suddenly laid lengthwise upon a perfectly smooth horizontal plane; determine the subsequent angular motion. Does your solution hold good for all degrees of original velocity? What must be the original angular velocity that the body may assume a position of permanent rest?

310. A uniform chain hangs vertically, its lower end just resting on the earth's surface at the equator; determine its length that it may hang in equilibrium without any fixed support. What is the greatest tension it has to sustain? If the chain were removed so that its lower end should rest on the earth in a given latitude, what would be the form of equilibrium? While hanging in the first position, if an indefinitely small downward motion be given to it, required the velocity with which the last link will strike the ground. (In this and the following question the earth's ellipticity to be considered insensible.)

311. The vertical at any point being defined to coincide with the direction in which a heavy body would begin to descend if let fall; required the nature of the curve which is vertical at all points through which it passes.

312. A body in motion is deflected at equal intervals of time by equal impulses in parallel directions; determine its velocity and direction of motion after any time. Apply the result to find the motion of a body continually deflected by a constant force acting in parallel lines.

313. A uniform heavy rod, moveable freely about a fixed fulcrum, is connected by a hinge at the end of its shorter arm with another, the farther extremity of which slides along a smooth vertical plane; determine the motion, when it takes place in a vertical plane perpendicular to the given one.

314. Shew that the motion of a body, acted on by no forces, about its centre of gravity may be imitated mechanically in the

following manner: Describe an ellipsoid, rigidly connected with the body whose equation is $Ax^2 + By^2 + Cz^2 = h^2$, the origin being the centre of gravity, and A, B, C the moments of inertia about the principal axes of the body which are also the coordinate axes, and h a quantity determinable from the initial motion; at the point where this ellipsoid was intersected by the axis of rotation at the commencement of the motion draw a tangent plane, which let remain fixed in space; and upon this plane let the ellipsoid be rolled, so that the angular velocity of the body about a line from the origin perpendicular to the plane may be uniform.

315. Compare the times of titubation of an elliptical cylinder on each of two horizontal planes, of which one is smooth and the other rough. Explain the result when the body degenerates into a circular cylinder.

316. Two bodies are let fall from the same point at an interval of one second, how many feet are they apart at the end of one minute from the fall of the first?

317. When a body moves on a surface of revolution, the force at any point acting in a plane through the axis, its projection on a plane perpendicular to the axis describes areas proportional to the times about the intersection of the axis with the same plane. Shew also how to find the path upon the surface.

318. Enunciate and prove the principle of the conservation of the motions of translation and rotation.

319. Given one principal axis of a body, find the other two.

320. A heavy body is projected vertically upwards in a medium of which the resistance $= kv^2$; determine its velocity when it again arrives at the point of projection.

321. The moment of inertia of a body about an axis through its centre of gravity is less than that about any parallel axis. Find the moment of inertia of a triangle about an axis perpendicular to its plane through its centre of gravity.

322. Find the direction in which an imperfectly elastic ball must be projected from a given point, so that after reflection at a given plane, it may strike another given point.

SECTION XVI.

QUESTIONS ON PLANE ASTRONOMY.

1821 1. Why does the apparent distance of two fixed stars increase as they approach the horizon?

2. Shew that the hour of sunrise and sunset together = 12 hours nearly; and find the correction necessary if the sun's declination should have changed by a given quantity.

3. In latitude 45°, find the time of sunrise on the longest day.

4. Explain the method of determining accurately the obliquity of the ecliptic: to what inequalities is it subject? and from what causes do they arise?

5. Given the focal length and aperture of a *Herschelian* telescope; during what time will the image of a given star be visible in the tube?

6. Explain fully what is meant by the term *mean* in Astronomy.

7. Explain what is meant by the equation of time, and from what causes it arises: at what times of the year is it nothing, and at what times is its negative and positive value respectively a maximum?

8. Supposing γ Draconis to be affected by a sensible annual parallax, in what manner will its apparent place be affected both by aberration and parallax on March 20, June 21, September 23, and December 23, its right ascension being 18 hours nearly?

9. The level of a transit instrument is suspended by hooks from its axis; in what manner are the errors which arise from the axis not being horizontal, or from the level not being parallel to the axis, distinguished from each other, and by what adjustments are they corrected?

10. Explain the method of determining accurately, when the first point of Aries is upon the meridian.

11. Explain the method of determining the sun's meridian altitude by means of a sextant: 1st, on the open sea; 2nd, when the sun's altitude is not less than 60°, but a neighbouring coast is on the same side of the ship with the sun; 3dly, on land.

12. A degree of latitude in latitude 45°, is nearly an arithmetic mean between a degree at the equator and the pole.

13. What are the principal phenomena presented by the sun and earth in the course of a month, to a lunar observer?

14. Explain the mode of constructing a catalogue of the fixed stars.

15. If two straight lines intersect each other in a circle, the sum of the arcs cut off between their extremities is the same as that cut off by any two lines respectively parallel to them, and intersecting each other within the circle. Prove this property, and shew its use in correcting observations made with circular instruments inaccurately centered.

16. When is the planet Venus stationary, and when retrograde?

17. Explain the phases of the earth as they would be seen from the moon.

18. Explain fully the method of finding the longitude at sea by the observed distance between the moon and a star and their altitudes.

19. On board a ship in north latitude, Jupiter is observed on the meridian at $3^h. 4^m. 56^s.$ and his corrected altitude is $29° 6' 42''$. One of his satellites is at the same instant eclipsed. His tabulated declination is $5° 4' 35''$ north, and the tabulated time of the eclipse $7^h. 0^m. 32^s.$ Required the latitude and longitude of the ship.

20. Explain *Borda's* circle of repetition; and the method of finding the latitude by the zenith distances of stars near the meridian.

1822 21. The autumnal equinox takes place at 6 in the evening, the moon being full at the same instant, and in her ascending node; the next night the moon rises at the *same hour*. Required the north latitude of the place.

22. By what methods may the variation of the compass be determined; and to what point does the true north correspond when the variation is 22° 30′ west?

23. Find the latitude of the place in which the longest day contains 16 hours.

24. The plane of a vertical dial is inclined at an angle of 45° to the plane of the meridian in a latitude whose sine $= \frac{1}{\sqrt{3}}$; find the position of the substile, the altitude of the stile, and the hour lines.

25. The earth being considered a perfect sphere, prove that at any place the length of a degree of latitude : the length of a degree of longitude :: radius : cosine of latitude.

26. Given the latitudes and longitudes of two places on the earth's surface, to find their distance.

27. Given the latitude of the place and the length of the day, to find the time of the year.

28. What effects are produced by aberration in the apparent places of the moon and the planets?

29. Given the place of a planet at noon, on March 20th, in Libra 3° 4′ 30″: on the 21st in 8° 7′ 7″: on the 22nd in 13° 19′ 30″: and on the 23rd in 18° 41′ 44″; to find its place on the 22nd at 6^h.

30. If Jupiter and Saturn are in conjunction with one another, and in opposition to the sun, on a given day; and their periodic times are 12 years and 29.5 years respectively; when will they again be in the same position?

31. Given the altitude of the sun, and the breadth of the penumbra which the top of a mountain throws upon a horizontal plane; to find the height of the mountain.

32. Explain the method of deducing the sun's parallax from the transit of an inferior planet.

33. Two places in the same latitude whose difference of longitude is l, are distant a miles from each other; find their latitude. 1823

34. If z = true zenith distance of a planet, p = its parallax at that distance, and P = horizontal parallax; then

$$\tan\left(\frac{z}{2}+p\right) = \tan\frac{z}{2}\tan^2\left(45^\circ+\frac{P}{2}\right).$$

35. The sine of half the angle that measures the duration of the shortest twilight $= \dfrac{\sin 9^\circ}{\cos \text{lat}}$.

36. If a comet move in a hyperbola whose semi-axis $= a$, and eccentricity $= ae$, its place at the end of t'' after leaving the perihelion may be determined from the equations,

$$(1)\ t = \frac{Pa^{\frac{3}{2}}}{2\pi}\left\{e\tan\theta - \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\},$$

$$(2)\ \tan\frac{v}{2} = \sqrt{\left(\frac{e+1}{e-1}\right)}\tan\frac{\theta}{2},$$

where v = true anomaly reckoned from the perihelion, and P = earth's periodic time, her mean distance being 1.

37. If the longitudes of a planet in three different points of its orbit be denoted by a, b, c, and its latitudes at those points by α, β, γ; then will

$$\tan\beta . \sin(c-a) = \tan\alpha . \sin(c-b) + \tan\gamma . \sin(c-a).$$

38. If the sun's longitude $= c$, and the obliquity of the ecliptic $= \phi$, then will the equation of time arising from the obliquity of the ecliptic

$$= \tan^2\frac{\phi}{2}\sin 2c - \frac{1}{2}\tan^4\frac{\phi}{2}\sin 4c + \frac{1}{3}\tan^6\frac{\phi}{2}\sin 6c - \ldots.$$

ad infinitum converted into time.

39. Find the perihelion distance of the comet, moving in the plane of the ecliptic, that stays the longest time within the earth's orbit.

40. If S = surface of a portion of the earth ABCD, AB being an arc of the equator, and AC, BD two arcs of circles of latitude; also if AB = c, AC = a, BD = b, then will

$$\tan\frac{S}{2} = \frac{\sin\frac{a+b}{2}}{\cos\frac{a-b}{2}}\tan\frac{c}{2}.$$

41. In consequence of the aberration of light, every star appears to describe an ellipse in the heavens, of which the true place of the star is the centre. Prove this, and find the axes of the ellipse.

42. Given the latitude of a place, find the time of the year when a given star rises at a given hour.

43. Explain the construction, adjustment, and use of the zenith sector.

1824 44. Given the precession in right ascension of a star, find the corresponding change in the angle of position.

45. Find the height of a lunar mountain.

46. Find the time of the sun's passing the vertical wire of a telescope.

47. In what latitude is the angle between 3 and 4 o'clock hour lines, on a horizontal dial, a maximum?

48. In an ellipse find the locus of the intersections of the lines SP and CQ, which cut off the true and eccentric anomalies.

49. If the radius vector and perpendicular on the tangent in any curve described by a revolving body be denoted by r and p; and in the curve of a star's apparent aberration, as seen from this body, by r' and p', then will $r : p :: r' : p'$.

50. On a given day, in a given latitude, the sun being on the meridian, determine geometrically the angle at which a rod of given length must be inclined to the horizon, that its shadow may be the greatest possible.

51. The altitudes of two stars as they cross the prime vertical are observed, and the difference of their right ascensions is known; find the latitude of the place.

52. State how the sun, planets, and fixed stars are affected by aberration; and shew that the part of the aberration arising from the motion of the planet varies as $\frac{\cos \text{SPT}}{\sqrt{\text{SP}}}$, S being the sun, T the earth, and P the planet.

53. If I be the obliquity of the ecliptic, L the sun's longitude, A his right ascension, then

$$A = L - \left\{ \tan^2 \frac{I}{2} \cdot \frac{\sin 2L}{\sin 1''} - \tan^4 \frac{I}{2} \cdot \frac{\sin 4L}{\sin 2''} + \tan^6 \frac{I}{2} \cdot \frac{\sin 6L}{\sin 3''} - \ldots \right\}.$$

54. If the latitude of a place be determined by observing 1825
the altitude of the sun at 6 o'clock, and the tabulated declination be affected by a small error, find the corresponding error in latitude.

55. Given the distance of Jupiter from the sun, his radius, and the times of his diurnal and annual revolutions, to compare the aberration of a given star when it passes the meridian of an observer in his equator at mid-day and at mid-night.

56. In a given latitude, find the altitude of the sun on the day of the equinox at 9 in the morning.

57. If v represent the true anomaly of a planet, reckoning from the perihelion, u the eccentric anomaly and e the eccentricity; when e is small, $v - u = e \sin u$, nearly. Required proof.

58. Explain the cause of aberration of light; shew how it is to be measured, and distinguish accurately between the aberration of the fixed stars and the aberration of the planets.

59. Given the latitude of the place and the declination of the sun. Find the time that the sun is above the horizon.

60. Given the time of sunrise and the altitude of the sun when due east on the same day, to find the latitude of the place and the declination of the sun.

61. The N.P.D. of α Aquilæ being 81° 38′ 25″, and its observed zenith distance when on the meridian 43° 50′ 45″, find the latitude of the place; and state the several corrections which must be applied to obtain an accurate result.

62. Shew that a horizontal dial, constructed for north latitude l, will be a vertical meridional dial for south latitude $90 - l$.

1826 63. If P be the pole of the heavens, Z the zenith, and S a given star, find when the angle ZSP increases fastest.

64. At what hour on a given night, in a given latitude, will the vertical circle passing through a known star cut the equator in a given angle?

65. Enumerate the arguments by which the diurnal rotation of the earth round its axis and its annual motion round the sun, are established.

66. In any latitude find when the time of the rising of the sun's disk bears the least ratio to the time of its crossing the meridian.

67. If λ be the true latitude of a place, and θ the latitude on *Mercator's* chart constructed to a radius $= 1$; prove that $2 \tan \lambda = e^{\theta} - e^{-\theta}$.

68. Find the interval between the heliacal rising and setting of a given star, to a spectator in a given latitude.

69. Prove that the semi-axes major and minor and the semi-latus rectum of an elliptic orbit are respectively an arithmetic, geometric, and harmonic mean, between the aphelion and perihelion distances.

70. Having given the contemporaneous altitudes of the sun and a known star, on a given day, and also the angular distance between them; find the latitude of the place and the hour of the day.

71. Determine when the sum of the zenith distances of two known stars in a given latitude is a maximum.

72. It is required to graduate a horizontal dial, the style of which is in the meridian, and inclined to the horizon at a given angle, so that on a given day it shall shew the apparent time in a given latitude.

1827 73. There were five Sundays in February 1824; in what year will this happen again?

74. Find when the inclination of the ecliptic to the horizon increases fastest.

75. What probable and adequate cause has been assigned for the secondary planets always turning the same face towards their primaries?

76. If a star whose right ascension is $19° 25'$ pass over the meridian $2^h 18'$ of sidereal time before the sun, what is the sun's right ascension when on the meridian?

77. Explain the moon's phases, and why part of the disk is always visible.

78. The times of the sun's rising and setting being calculated for a certain place, what correction is necessary to make them serve for another place not far distant from it?

79. The style of a horizontal dial being bent down, its edge coincides with the 9 o'clock hour-line. For what latitude was it constructed?

80. Supposing the sun to remain above the horizon a given number of days, find the latitude.

81. If L be the length in miles of an arc of a great circle of the earth, D the depression in feet of one extremity of it below a tangent drawn at the other, $D = \frac{2}{3} L^2$ nearly.

82. Prove that *Brinkley's* formula for the mean refraction is reducible to the same form as *Bradley's*.

83. Given the position of the moon's nodes, and the inclination of her orbit to the ecliptic, to find when her latitude and declination are equal.

84. In a chart on *Mercator's* projection the length of the meridian from the radius of $30°$ to that of $60°$ is to the radius of the sphere as the natural logarithm of $\frac{\sqrt{3}+1}{3-\sqrt{3}}$ to 1.

85. Supposing the latitude of a star to be $60°$, its longitude $95°$, and that of the sun $65°$, what is the aberration in longitude? In what sense is $20'' 25$ the maximum of aberration?

86. On the supposition of a homogeneous atmosphere, the refraction may be expressed by the formula

$$r = \frac{m-1}{\sin 1''} \tan \left(Z - \frac{\delta}{(m-1)(1+\delta) r}\right),$$

δ being the ratio of the height of the homogeneous atmosphere to the radius of the earth.

87. Given the altitudes of two known stars at the same instant of time; required the latitude of the place. How may this problem be solved geometrically on a sphere?

88. Convert 17° 25′ 8″ into time at the rate of 15° to one hour.

89. Find the azimuth of two known stars which are seen at the same instant in one vertical plane.

90. A known circumpolar star reaches its maximum azimuth at two different places at the same instant; having given the values ofthe maximum azimuth at the two places, find their latitudes and the difference of longitude.

1828 91. State the principal arguments for the diurnal rotation of the earth round an axis, and its annual motion round the sun.

92. Having given the right ascension and declination of a star, find its latitude and longitude, and adapt the formulæ to logarithmic computation.

93. What must be the relation of the distances from the sun, of a superior and inferior planet, that their synodical revolutions may be equal?

94. Three stars A, B, C are very nearly in a great circle, the angle made by A and C at B being $180 - \beta$, where β is small. Shew that if t be the time which elapses between AB and BC being vertical,

$$t = \frac{\sin z}{\cos a \cos l} \cdot \frac{\beta}{15},$$

where z is the zenith distance and a the azimuth of B, and l the latitude of the place.

95. A style projects from the vertex of an upright cone; trace the hour lines on the surface of the cone; and find the time in each day during which the dial will serve.

96. If a body fall to the earth in the time t'', the deviation to the east of the point from which it fell will be $\frac{1}{3} g a t^3 \cos l$; where l is the latitude, and a the angle described by the earth in $1''$.

97. Given three altitudes of a known star observed very near the meridian, and the differences of the times of observation; determine the latitude of the place.

98. Describe the manner in which *Bradley* discovered aberration, and in which he distinguished nutation from it.

99. If z be the true zenith distance, P the horizontal parallax, p the parallax in seconds,

$$p = \sin \mathrm{P} \frac{\sin z}{\sin 1''} + \sin^2 \mathrm{P} \frac{\sin 2z}{\sin 2''} + \sin^3 \mathrm{P} \frac{\sin 3z}{\sin 3''} + \&c.$$

100. Having given the latitude of the place, find the day of the year on which the shadow of a given ellipse placed perpendicular to the meridian with its major axis vertical will be an ellipse of half the eccentricity.

101. The interval between the passages of a known circumpolar star through the plane of a vertical instrument with a given azimuth is observed; find the latitude of the place and the area of the spherical surface contained between the vertical circle and the apparent path of the star.

102. Explain the use of observations made by reflexion on the polar star in adjusting a transit instrument.

103. Compare the portion of the surface of the earth illuminated by the sun in perigee with that illuminated in apogee, taking into account the magnitude of the sun.

104. Apply *Lagrange's* theorem to the determination of the first three terms of the series expressing the true anomaly in terms of the mean, the series ascending by powers of ϵ and the anomaly being measured from the perihelion.

105. Given three distances of a planet from the sun, and the corresponding arguments of latitude, to find the place of the perihelion, and the true anomaly at the first observation.

106. Construct a vertical south dial for a given latitude.

107. Shew how a planet, superior or inferior, may have its motion direct or retrograde, or may be stationary.

108. Explain the method of determining the sun's parallax by observations made on the transit of Venus over the sun's disk.

109. Find the aberration of a given star in R.A., in terms of the R. A., declination, obliquity, and sun's longitude.

110. Find the precession in north polar distance.

1829 111. Explain the causes of change of seasons, and of the different lengths of day and night. Shew that the full moon in winter is longer above the horizon than in summer.

112. Explain by what observations the path of the sun among the fixed stars is determined.

113. Shew that the aberration of a star takes place towards a point of the ecliptic 90° before the earth's place, and that it varies as the sine of the angle of the earth's way.

114. Explain the cause of twilight, and shew how its duration may be found from the declination of the sun and the latitude of the place. Find also on what day it is shortest at a given place.

115. The style of a horizontal dial, which is accurately graduated for a given place, is bent through a small given angle δ; if a be the hour angle at which there is no error in the time denoted by the dial, find the error in the time denoted at any hour angle h, on a given day; and shew that at six o'clock the error is independent of the latitude of the place, and vanishes at sunset.

116. Explain the use of the astronomical clock, and shew how we may determine whether it goes uniformly at all hours. Also, explain clearly why the mean daily rate determined by the assistance of tables, differs from that determined by direct observations on the transits of stars, and state for what stars this disagreement is perceptible.

117. Explain *Flamstead's* method of determining the right ascension of the sun, by observations made near the equinoxes; and shew that it will serve to determine the place of the equinox.

118. On a given day and hour, find the azimuth of the ascending point of the ecliptic, and the longitude and altitude of the nonagesimal degree.

119. Find the times of the beginning and end of a lunar eclipse, and the number of digits eclipsed.

120. The first of February of the present year (1829) falls a Sunday. Find generally when this will happen again; and write down all the years in which it will occur during the present century.

121. If at a place between the tropics, c be the zenith distance of a known star, when the ecliptic comes upon the zenith of the place of observation, and θ the longitude of the earth, when the corresponding aberration in zenith distance vanishes, prove that θ is determined by the equation

$$\cot(\theta - \mathrm{L}) = \frac{\sin^2\lambda \cos c}{\sqrt{\sin(c+\lambda)\sin(c-\lambda)}},$$

where L and λ represent the longitude and latitude of the star.

122. If the observed angular distance of two points be a, their observed elevations above the horizon being H and h, and θ the angular distance reduced to the horizon, shew that

$$\sin^2\frac{\theta}{2} = \frac{\sin\frac{1}{2}(a + \mathrm{H} - h)\sin\frac{1}{2}(a + h - \mathrm{H})}{\cos \mathrm{H} \cos h}.$$

123. Find when the altitude of a known star increases fastest.

124. Explain fully the method of determining the sun's parallax by the transit of Venus over the sun's disk.

125. Explain accurately what is meant by the *equation of time;* and shew that it vanishes four times in a year.

126. Construct a vertical south dial, and determine how much of it must be graduated.

127. Having given three geocentric places of a comet, find the corresponding heliocentric and geocentric distances.

128. Find the moon's parallax by observations made out of the plane of the meridian, and shew how the effect of refraction may be avoided.

129. Find the length of the longest day in latitude 52° 13′, the obliquity of the ecliptic being 23° 28′.

Having given 10.1105786 = log tan 52° 13′,
9.6376106 = log tan 23° 28′,
9.7481230 = log cos 55° 57′,
9.7483099 = log cos 55° 56′.

1830 130. Explain what is meant by the *error of the line of Collimation*, and shew how it may be avoided.

131. Having given the sidereal time of any phenomenon, find the corresponding mean solar time.

132. Find the latitude of the place of observation, from two equal altitudes of the sun before and after noon, and the time between.

133. Determine the precession in north polar distance and right ascension, and shew when it is additive and when subtractive.

134. An ellipse may be constructed so that if any abscissa be taken to represent the aberration in longitude of a given star, the corresponding ordinate will represent the aberration in latitude, coordinates being measured from the centre along the axes; prove this and determine the axes.

135. Two planets P_1, P_2 revolve in circular orbits at the distances r_1, r_2 from the sun, and when they appear stationary to one another cot P_2's elongation seen from $P_1 = \frac{1}{2}\tan\theta$; shew that $\frac{r_1}{r_2} = \frac{1}{2}\tan\frac{\theta}{2}\tan\theta$.

136. $\left.\begin{array}{r} Ax + By + Cz = 0 \\ A'x + B'y + C'z = 0 \end{array}\right\}$ are the equations to the planes in which two planets move. Apply them to find the inclination of the orbits to one another, in terms of their inclinations to the ecliptic and of the longitudes of their ascending nodes, the ecliptic being in the plane of x and y.

137. Explain fully the *equation of time*, and shew at what seasons that part of it arising from the obliquity of the ecliptic is positive, and at what seasons negative.

138. Shew that the inclination of the ecliptic to the horizon is a minimum when Aries rises, and a maximum when it sets; and explain the phenomenon of the harvest moon.

139. Having given the variation of the obliquity of the ecliptic, find the corresponding variations in right ascension and declination.

140. Determine the latitude of the place of observation from observing the times of the rising of two known stars.

141. Find $\sin x$ from the equation

$$\sin x \cos x + a \sin^2 x = b,$$

and shew its use in the solution of the following problem: to determine how much the azimuth of a known star on the horizon is affected by refraction.

142. Find the longitude of the perihelion and the time of the earth's passing through it.

143. Find the sun's right ascension by *Flamstead's* method. Why must the observations be made near an equinox?

144. Explain *Mercator's* projection of the sphere, and find the length of the projection of an arc of the meridian included between the latitudes of 30° and 60°.

145. Find the two parts of solar nutation, and prove that they are connected by the equation to an ellipse, the axes of which are in the ratio of $\cos I : 1$, where I is the obliquity of the ecliptic.

146. Having given the length of a degree of latitude, and also the length of a degree in a direction perpendicular to the meridian, in a given latitude; find the ellipticity of the earth.

147. Shew how to determine whether a planet is a superior or an inferior one; and having given the synodic period of a planet and the length of a year, find the planet's period.

148. Construct a horizontal dial, and find the limits beyond which it is unnecessary to graduate it.

149. Determine the circular orbit of a planet from two observations.

150. In a given latitude find the sun's azimuth, his declination and the time of the day being given; and adapt the trigonometrical formula to logarithmic computation.

1831 151. State the arguments from which it is inferred that the earth revolves round its axis and round the sun.

152. Shew how to draw a meridian line by observing the shadow of a vertical gnomon on a horizontal plane, and correct for the change in the sun's declination between the observations.

153. Explain the nature of the five astronomical corrections: refraction, parallax, aberration, precession, and nutation.

154. The longitude of α Arietis is $35^\circ\ 4'\ 41''$, its north polar distance is $67^\circ\ 25'\ 1''.7$, and the obliquity of the ecliptic is $23^\circ\ 27'\ 46''.3$; find its angle of position.

$$\begin{aligned}
\log \cos 35^\circ\ 4'\ 41'' &= 9.9129496,\\
\log \sin\ 23\ 27\ 46.3 &= 9.6000517,\\
\log \sin\ 67\ 25\ \ 1.7 &= 9.9653546,\\
\log \sin\ 20\ 39\ 52.3 &= 9.5476467.
\end{aligned}$$

155. When the vertical plane in which a transit instrument moves, nearly coincides with the meridian, to find the deviation.

156. Investigate the two equations

$$nt = u - e \sin u,$$

$$\text{and } \tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2},$$

the former expressing the relation between the eccentric and mean anomaly, and the latter that between the eccentric and true anomaly.

157. What are the sidereal, the tropical, and the anomalistic years? Which is longest, and which shortest?

158. Determine the relative positions of the earth and an inferior planet, when the latter appears stationary.

159. In a given latitude a vertical rod is placed at a given distance from an east and west wall, so as to cast a portion of

its shadow upon it; find the equation to the extremity of the shadow traced upon the wall on a given day. Shew what the equation becomes when the sun is in the equator, and the latitude of the place 45°.

160. Having given u_0, u_a, u_{2a}, three values of a function near its maximum, observed at times 0, a, $2a$; find the time when the function will be a maximum. And if the declinations of the sun at noon on three successive days were 23° . 27′, 23° . 27′ . 9, 23° . 27′ . 6, find when the declination was greatest.

161. If the earth be an oblate spheroid, and from any point Q above it perpendiculars QM, QN be drawn to the axis and equator respectively, intersecting a meridian in P and p; and if tangents PT, pt to this meridian meet the axis and equator in T and t, and if the straight line which joins T and t cuts the meridian in E and F, E and F are the extreme points of the meridian visible from Q.

162. If normals be drawn at every point of the rhumb line, find the locus of their intersection with the equator, the earth being considered an oblate spheroid.

163. Having given the latitudes of two places on the earth's surface, one of which is N. E. of the other, find the difference of their longitudes and their distance from each other, considering the earth a sphere.

164. Express the radius of curvature of the meridian in a spheroid of small ellipticity in terms of the latitude.

If λ, λ' be the latitudes of two stations on the same meridian, prove that the length of the arc included between them is

$$b(\lambda'-\lambda)\left\{1+\frac{e}{2}-\frac{3e}{2}\cos(\lambda'+\lambda)\,.\,\frac{\sin(\lambda'-\lambda)}{(\lambda'-\lambda)}\right\} \text{ where}$$

$$e=\frac{a-b}{a}.$$

165. Investigate the precession in N . P . D and R . A ., and shew when they are additive and when subtractive.

166. Explain the method of finding the longitude of a place by the observed distance of the moon from the sun or a fixed star, and deduce formulæ adapted to logarithmic computation.

167. Shew how the time, duration, and magnitude of a lunar eclipse may be computed.

168. If λ be the angle which the normal to any point in an ellipse makes with the axis major, the length of the normal

$$= \frac{b^2}{a\,(1 - e^2 \sin^2 \lambda)^{\frac{1}{2}}}.$$

169. Distinguish between a sidereal, a solar, and a mean solar day; and define the equation of time, stating the two causes from which it arises.

170. Construct a horizontal dial for a given latitude, and determine the limits beyond which it is unnecessary to graduate it.

1832 171. Explain the causes of different lengths of days at different seasons of the year. Shew that the duration of the longest day is greater, and that of the shortest day less as the latitude increases.

172. Explain the effects of parallax; and obtain the value of parallax at a given altitude in terms of the horizontal parallax.

173. Given the latitudes of two places, and their difference of longitude; determine the inclination of their horizons, and the day of the year on which the sun sets to both places at the same instant.

174. Explain the *Gregorian* correction of the calendar. The length of the mean tropical year being 365.242264 days, in how many years will the error of this correction amount to a day?

175. Determine the relation between the right ascension a, and the longitude l of the sun. Shew also that

$$l - a = \tan^2 \frac{w}{2} . \sin 2l - \tfrac{1}{2} \tan^4 \frac{w}{2} \sin 4l + \ldots,$$

where w is the obliquity.

176. At what time of the day will a star, of which the right ascension is 270°, pass the meridian at the time of the equinoxes? How will its place be affected by aberration? How was it shewn that the effects of aberration did not arise from a change in the position of the earth's axis?

177. From an equinox to a solstice the daily increase of the sun's declination is continually diminished.

178. If the earth's orbit were a circle, shew that the whole duration of daylight at every place on the earth's surface in the course of a year would be the same.

Taking into account the eccentricity and position of the earth's orbit, is there more daylight on the north or south side of the equator?

179. Supposing the eccentricity of the earth's orbit to be equal to cos 89° 2′, and the perihelion at the winter solstice; shew that the summer half of the year is longer than the winter half by 7 days 20 hours nearly.

180. Feb. 6, 1824. Moon's A.R. at 0^h was 23° 21′ 55″,
Feb. 6, ... 12 . . 29 37 9,
Feb. 7, ... 0 . . 36 5 56.

Determine her A.R. on Feb. 6, at 15^h.

181. Having given the error in the observed azimuth of a heavenly body, find the corresponding error in its declination. In measuring an arc of the meridian, shew that the chain of triangles should be near the meridian.

182. Having given the equation between the mean and eccentric anomalies $nt = u - e \sin u$, and that between the true and eccentric

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2},$$

apply *Lagrange's* theorem to find a series for v in terms of nt as far as e^3.

183. Explain what is meant by the equation of time; and find when that part of it which is caused by the obliquity is additive or subtractive, and when it is greatest.

184. Investigate the nutation in right ascension of a given star: also express the nutation in terms of the maximum nutation, the longitude of the moon's ascending node, and the longitude of the moon's ascending node corresponding to the maximum nutation.

185. Having given the lengths of two arcs of the meridian, and the latitude of their extremities, find the compression of

the earth. If the arcs be on the same side of the meridian, the operation is more accurate in proportion as they are more distant from each other.

186. In orbits of small eccentricity, the greatest equation of the centre is nearly twice the eccentricity.

187. If R be the refraction of a given apparent zenith distance, when the temperature is 50°, and the barometer 29.6 inches, and r the refraction for the same zenith distance, when the temperature is $50° + t°$, and the barometer b inches; then $r = \frac{b}{29.6} \cdot \frac{1 - \beta t}{1 + at} \cdot R$ nearly; the elastic force of the air increasing a parts of the whole, and the mercury expanding β parts of the whole, for each degree of temperature above 50°.

188. Having given the perihelion distance D of a comet in a parabolic orbit, find an expression for the time through any angle v of true anomaly; and having given corresponding tabulated values of v and t in an orbit whose perihelion distance is 1, find corresponding values of v and t in any other orbit.

189. Enumerate the elements of a planet's orbit. What does each determine?

Find the node of a planet's orbit from observations made on the planet near its node?

190. Explain *Mercator's* projection of the sphere, and find the length of an arc of the meridian on this projection, supposing the earth spherical.

191. Explain the cause of twilight; and determine its duration at a given place on a given day.

192. Find the relation between the mean and eccentric anomalies, and that between the eccentric and true anomalies.

1833 193. Explain the moon's phases, and state the cause which renders the dark part visible near new-moon.

194. State the three laws of planetary motion discovered by *Kepler*, and the nature of the observations by which they were established. Shew that in the conjunction of Jupiter and Saturn the disturbance which the former produces on the orbit

of the latter is greater than the effect of the latter on that of the former.

195. By what observations is it shewn that the sun's apparent annual motion is in a great circle? Shew how the obliquity of the ecliptic is determined.

196. Explain the construction of the common horizontal dial, and draw the last hour lines in a given latitude.

197. Determine the position of the ecliptic with respect to the meridian and horizon of a given place, at any time. In what astronomical investigations are the quantities determined above required?

198. Determine the longitude by observing the increase of the moon's right ascension in the interval between passing the meridian of Greenwich and that of the place of observation.

199. Find the moon's parallax by observations made not in the plane of the meridian; and shew how the correction for refraction may be avoided by observing the zenith distance of a star nearly in contact with the moon.

200. Express the latitude and longitude of a star in terms of its observed altitude and azimuth, the observations being made at a given place when the first point of Aries was on the meridian.

201. Determine the points of its orbit between which the earth must be situated when a given superior planet appears stationary; the orbits of both earth and planet being supposed circular, and in planes inclined to each other.

202. Find the latitude from observing the angular distance of the extreme points of the horizon in which the sun appears at rising in the course of a year; and if α, β denote the distances of those points from the point in which the sun rises when the declination is δ, prove that sine of the obliquity

$$= \sin\delta \cdot \frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}.$$

203. Let two heliocentric distances of a comet revolving in a hyperbolic orbit be denoted by ρ, ρ'; and the length of the

line joining the two positions of the comet by c; and let ϕ, ϕ' be two angles, such that

$$\rho + \rho' + c = 2a(\sec\phi' - 1),$$

$\rho + \rho' - c = 2a(\sec\phi - 1)$, $2a$ being the major axis; prove that (taking the earth's mean distance and period for the units of distance and time) the time of moving between the two positions

$$\frac{a^{\frac{3}{2}}}{2\pi}\{\tan\phi + \tan\phi' - \log_\varepsilon(\tan\phi + \tan\phi')(\sec\phi + \sec\phi')\}.$$

204 If D be the apparent diameter of the sun, θ the altitude of its centre, and s_1, s_2 the respective lengths of the pure shadow and penumbra cast by a vertical rod upon a horizontal plane, prove that

$$\frac{s_1}{s_2} = \frac{1}{2}\left\{\frac{\sin 2\theta}{\sin \mathrm{D}} - 1\right\}.$$

205. Find the aberration of a given star in right ascension. In what respect was γ Draconis peculiarly fitted for discovering the effects of aberration, at Greenwich? How did *Bradley* separate the effects of aberration from those of nutation and precession?

206. Shew how to find the inclination of the sun's equator to the ecliptic, and the time of the sun's rotation. On what grounds is it probable that the sun, in addition to his rotatory motion, has a motion of translation?

207. To determine the latitude from zenith distances of a known star observed very near the meridian.

208. Point out all the circumstances in which solar and lunar eclipses differ; and calculate the circumstances of a solar eclipse at a given place.

209. To find Jupiter's distance from the sun by an observed eclipse of a satellite. Explain the method of determining the longitude by observations of those eclipses.

210. Having given the inclination to the primitive of a great circle of the sphere, find the radius and position of the centre of its stereographic projection; and shew that the centres of the projections of the meridians upon the horizon of any place lie in a straight line.

211. Express the radius of curvature at any point of the elliptic meridian in terms of the latitude. Mention the various methods that have been employed to determine the figure and dimensions of the earth. If the length of a degree be $69\frac{1}{2}$ miles, what will be the length of the earth's radius supposing it a sphere?

212. By what observations does it appear that the stars 1834
describe parallel circles with a uniform angular motion, all completing their revolutions in the same time, at a distance compared with which the dimensions of the earth are evanescent?

213. Describe the construction and adjustments of the mural circle; and the nature and objects of the observations made with it.

214. Find the latitude and hour angle from two altitudes of the sun and the time between, neglecting the change of declination in the interval. At what time of the year will the result thus obtained be the most accurate?

215. Explain the principle of *Mercator's* method of projection; and find in it the length of an arc of the meridian, supposing the earth to be a sphere.

216. Determine the precession in declination of a given star whose right ascension is less than 90°. How is the mean value of the whole precession determined from observation? If its amount be determined from the observed annual precession of a star in declination, why is the result different from that which would be obtained from the observed annual precession in right ascension?

217. Explain the phases of an opaque heavenly body illuminated by the sun's rays. What is the cause of the occasional *gibbous* appearance of Mars? and why is not the same perceived in the other superior planets? At what point of his orbit is this appearance most remarkable?

218. Why are the sun's transit and his culmination not contemporaneous events? Prove that in the interval between them, his centre describes an hour angle whose sine equals

$\frac{m \sin(l - \delta)}{\cos l \cos \delta}$, very nearly; l being the latitude of the place, δ the declination of the sun's centre, and m the ratio of the apparent motions of the centre in declination and right ascension.

219. At a given hour on a given night the moon is observed to rise in the east point; determine the longitude of the node of her orbit, supposing its inclination to the ecliptic known.

220. If during one revolution of a planet the times when it crosses the ecliptic and attains its greatest latitudes be observed, shew how to determine whether its orbit is eccentric, and what is approximately the inclination of its axis major to the line of nodes.—If the axis major be perpendicular to the line of nodes, prove that an approximate value of the eccentricity is obtained from the expression $\frac{\pi}{8} \cdot \frac{T - t}{t}$ where T and t represent the respective times of motion on different sides of the ecliptic.

221. Shew that the equation of time attains two maximum and two minimum values in a year, and determine approximately the corresponding positions of the sun.

Prove that the maximum value, which occurs next after the summer solstice, is much less than that which occurs next before the vernal equinox.

222. A transit instrument is placed with its axis N and S in a horizontal position; find the latitude by observations upon a known star.

223. Shew how the longitude of a place on the earth's surface may be found by observing the distance of the moon from the sun or a fixed star. What other methods are used to determine the longitude, and why is it more difficult to ascertain the longitude than the latitude of a ship?

224. Construct a horizontal dial. Why is the style parallel to the earth's axis? If it deviate from this position by a small given angle, and the time be known when the error is nothing, find the correction for any other time.

225. Find the error with which the hour angle of a heavenly body is affected, when computed from an observed zenith

distance, as far as it arises from parallax. Apply it to find the moon's parallax, and thence her distance from the earth.

226. Find the aberration perpendicular to any plane, taking into account the elliptic form of the earth's orbit.

227. State the theorem by which the computation of a triangle on the terrestrial sphere is reduced to that of a plane triangle having sides of the same length, and apply it to solve a geodesic triangle where two sides and the included angle are given. What advantage does the approximate possess over the exact method?

228. Explain the correction of annual parallax for a fixed star, and investigate its effect on the declination. Shew that its present amount in any direction may be determined from the corresponding formula for aberration three months hence.

229. Find the distance of any point on the earth's surface from its centre in terms of the latitude of that point; and shew that the number of seconds in the angle of the vertical is $\frac{180 \, . \, 60 \, . \, 60}{\pi} \cdot \epsilon \sin 2l$ nearly. What is the use of this expression?

230. Investigate formulæ for changing a catalogue, in which the stars are registered according to their right ascensions and declinations, into one where they are registered according to their longitudes and latitudes.

231. State the observations by which the changes in the 1835
sun's right ascension and declination are determined. How does it appear that the section of his apparent orbit with the celestial sphere is a great circle?

232. Compare the lengths of the longest day at two given places. What is the least latitude, in which the sun does not set for 24 hours?

233. Explain the nature and cause of aberration. In what plane does it take place? Supposing the right ascension of γ Draconis to be 270°, describe the effects of aberration on this star at the equinoxes and solstices, which led *Bradley* to the discovery of aberration; and illustrate your explanation by a figure.

234. Determine the coefficient of refraction from observations of circumpolar stars. If the declination of any one star be known correctly, shew how a table of refraction may be formed.

235. Shew from observation that the curve which the earth describes round the sun is an ellipse, and determine its eccentricity.

236. Supposing an approximate value of the longitude of any place on the earth's surface to be known, shew how a more correct value may be found from an observed occultation of a fixed star by the moon. How may the possible errors of the lunar tables be obviated?

237. The earth is touched by two equal conical surfaces, the planes of whose bases coincide with that of the equator; and its surface appears projected upon them to an eye placed in its centre. Shew that by a proper assumption of the form of the cones, the earth's surface may be thus projected on a plane circle; and illustrate the formation of the chart by laying down the position of a place of given latitude and longitude.

238. Determine the position of the place nearest to the north pole, at which the sun rises on a given day at the same instant as at Greenwich. If y, z be the zeniths of the place and Greenwich, and yzx a quadrant of a great circle on the celestial sphere, the projection of the locus of x on the horizon of z is an arc of an ellipse, whose eccentricity $=$ cos. lat. of Greenwich.

239. Explain the method by which the parallax of a fixed star has been attempted to be found by observations on the fixed double stars; and investigate an equation for determining the times of the year at which the observations on a given star for this purpose may be most advantageously made.

If at one of these times u be the difference of the longitudes of the sun and star, λ the latitude of the latter, δ the angular distance between the two stars, and a the change of position of the line joining them in the course of half a year, shew that the parallax of the greater star $= a \sin \delta \sqrt{1 + \cos^2 u \, . \cot^2 \lambda}$.

240. If l be the latitude of a place between the tropics, a and δ the sun's right ascension and declination; the times when the ecliptic is vertical are determined from the equation $h = a + \sin^{-1} (\sin a \tan l \cot \delta)$.

241. The extremity of the shadow intercepted by a line drawn in the dial plane perpendicular to the substyle of a horizontal or vertical south dial, will move uniformly round some fixed point A in the substyle.

If the hour lines be graduated from this property, and a small error be committed in assuming the position of the point A; the error of the time indicated by the dial at the time t will be $2a \sin 30t$, where a denotes the error at one o'clock.

242. At midnight, two known stars appear in their real verticals, one at the vernal, the other at the autumnal equinox; shew that their azimuths are supplementary; determine also the latitude of the place and the obliquity of the ecliptic.

243. The altitude of a star on the prime vertical is observed with a sextant whose arc is not exactly 60°, but which is accurately *subdivided*; find the error in the arc from observing the difference between the sidereal time deduced from the altitude and the real sidereal time.

244. Having given the heliocentric, find the geocentric place of a planet at any time. How is the time in which a planet is in its node determined?

245. Having given the lengths of two degrees in the same meridian at different given latitudes, shew how the earth's form and magnitude may be deduced. By what observations is it known (1) that the arc measured is exactly one degree; (2) that it lies in the same meridian?

246. Explain the method of setting the sidereal clock.

A star rises at $5^h. 47^m. 56^s.$ sidereal time, find the mean solar time, supposing the mean sun's right ascension when last on the meridian to have been $1^h. 47^m. 6^s.$

247. Find the apparent differences of the latitudes and longitudes of the sun and moon at any assigned time during a solar eclipse.

248. If i be the inclination of a plane to the horizon, and a the inclination of a line in it to the intersection of the plane with the horizontal plane; shew that the inclination θ of the line to the horizon is found from the equation $\sin\theta = \sin i . \sin a$.

249. Explain the use and construction of a vernier; and if n divisions of the instrument are equal to $n + 1$ of the vernier, and the m^{th} divisions coincide, determine the true reading off.

250. Determine the time corresponding to a given true anomaly in a very eccentric ellipse; and having given the anomaly in a parabolic orbit, find the anomaly in a very eccentric ellipse, the time and perihelion distance being the same in both orbits.

251. Find the duration of a lunar eclipse. On what account are eclipses of the moon more frequent at a given place than those of the sun.

1836 252. What evidence have we that the earth revolves about the sun, and rotates about its own axis?

253. Describe and explain the phenomena of day and night throughout the year in our latitude. How must the explanation be modified for places near the poles?

254. How was the motion of light discovered, and by what remarkable astronomical phenomenon was the discovery confirmed? What is parallax? Having given the horizontal parallax of a heavenly body at the equator, find the parallax of the same body at any other place.

255. Find the latitude of a place by observing three altitudes of an unknown star near the meridian. What are the chief advantages of this method?

256. Shew how to find the time by an observed altitude of a known star, when east of the meridian. Prove that the error in time from a given error in altitude is less the nearer the star is to the prime vertical at the instant of observation.

257. Explain the terms "solar and lunar ecliptic limits." Shew that there may be seven eclipses in the course of a year. What effect is produced by the earth's atmosphere in lunar eclipses?

258. When that part (E) of the equation of time, which arises from the obliquity (ω) of the ecliptic, is a maximum, $\sin E = \tan^2 \frac{\omega}{2}$.

If l, δ be a star's latitude and declination, its distance (d) from the sun, at the moment of his crossing the equator, may be found from

$$\sin^2 d \,.\, \sin^2 \omega = \sin^2 l - 2 \cos \omega \sin l \sin \delta + \sin^2 \delta;$$

prove this equation, and prepare it for calculation by a table of logarithms.

259. In 365^d, 5^h, 48^m, the sun's longitude is increased by $360°$; what is his mean daily motion?

260. Having given the sun's altitude and azimuth, determine the points where an arch of given colour in the primary rainbow meets the horizon.

At a place of given latitude determine the season of the year, during which it is impossible to see a primary rainbow at mid-day; and on a given day, during that period, for what time before and after noon this is the case.

261. At a place of given latitude determine the nearest approach of the ascending point of the ecliptic to the meridian, and at what hour on a given day it is attained.

262. Determine the positions of all stars such that, when the aberration either in right ascension or declination vanishes, the other shall be a maximum.

263. Investigate a method of determining the longitude, by observing the distance of the moon from the sun or a fixed star. Why is the use of this method particularly convenient at sea?

264. Explain the method of determining the place of the node of a planet's orbit from observations made on the planet near its node.

265. Construct a vertical south-east dial. What is the greatest inclination of a vertical dial to the meridian, that it may shew the time of sunrise throughout the year?

266. Find when that part of the equation of time which is caused by the obliquity is additive or subtractive.

Shew how to correct a watch by a sun-dial.

SECTION XVII.

QUESTIONS IN CENTRAL FORCES AND PHYSICAL ASTRONOMY.

1821 1. The moon revolves in a circle about the earth, and the quantity of matter in the earth is suddenly doubled. Compare the eccentricity of the orbit now described with its axis-major, and with the original radius of the moon's orbit.

2. Find the force of the sun to disturb the motions of the moon.

3. A pendulum vibrating seconds at the earth's surface is carried to a distance from the centre of the earth equal to that of the moon. What is the time of its vibration?

4. Find the time in which the moon would fall to the earth, if suddenly deprived of its angular motion.

5. In the 11th section of *Newton*, the mean motion of the nodes or apsides varies as the periodic time of P directly, and as the square of the periodic time of T inversely.

6. Find an expression for that part of the force of Jupiter to disturb Saturn, which acts in the direction of a tangent to Saturn's orbit.

7. If the accelerating gravity of a satellite of Jupiter to the sun was greater than that of Jupiter at the same distance, in the ratio of $e : 1$, where e differs little from unity, then the distance of the centres of the sun and of the orbit of the satellite, would be greater than the distance of the centres of the sun and Jupiter in the ratio of $\sqrt{e} : 1$.

8. If the sun and moon be both supposed to be in the equator; to compare the lengths of a lunar and a tide day, for a given position of the luminaries.

9. In elliptical orbits of small eccentricity, the diminution of angular velocity in moving from the lower apse to the higher, is nearly proportional to the increase of distance.

10. A body describes an ellipse, the force being in the centre; given the force at a given distance, to find the actual periodic time.

11. A body is projected from a given point, in a given direction with a given velocity, when the force varies inversely as $(\text{dist.})^2$; find the latus-rectum of the orbit described.

12. The earth being a sphere, and its radius 4000 miles, what must be its diurnal rotation that a body at the equator may lose half its weight?

13. A body describes a logarithmic spiral, and approaches the centre by a space which is small compared with the whole distance. Compare the time of one revolution with the time to the centre.

14. At what distance from its centre must the earth, considered as a sphere, receive a single impulse, so as to produce its diurnal and annual rotation?

15. A comet describes 90° from the perihelion in 100 days. Compare its perihelion distance with the radius of a planet's circular orbit which revolves about the sun in 942 days.

16. If a body is projected in any direction, and acted upon continually by two forces tending to fixed centres, not both in the same plane with the direction of projection, it will describe by lines drawn from the two fixed points equal solids in equal times about the line joining the two points.

17. Find the mean horary motion of the moon's nodes, when 1822
the line of the nodes is in octants.

18. A body is projected in a given direction with a given velocity from a given point, and is acted upon by a repulsive force which varies as the distance from another given point. Required the curve which it will describe.

19. A body describes a circle about a fixed point, the force varying inversely as the square of the distance; another body, the attractive force of which varies in the same law, is introduced into the system; how will this affect the velocity of the body, the form of the orbit, and the periodic time?

20. Given the velocity of projection equal to the velocity in a circle at the same distance, $\left(\text{force} \propto \frac{1}{D^2}\right)$; required the direction in which a body must be projected at a given distance, that the focus of the conic section described may bisect the semi-axis major, and determine the magnitudes and positions of the axes.

21. Investigate the apsidal equation, and shew what number of possible positive roots it can have, when the velocity is acquired from a finite distance and the force varies as D^{n-1}.

22. Find the horary motion of the moon's nodes in a circular orbit. (*Newton*, Book III. Prop. 30.)

23. The sum of the areas described by any number of bodies round a given point, multiplied by the respective masses of the bodies, is proportional to the time, if they be supposed to be acted upon only by their mutual attractions, and by the force tending to the given point.

24. Given the time, construct for the inclination of the lunar orbit to the plane of the ecliptic. (*Newton*, Book III. Prop. 35.)

25. Determine the point in P's orbit, (Sect. 11,) where the tangential ablatitious force is a mean proportional between the addititious and central ablatitious forces.

26. The velocity in an ellipse at the greatest distance is half that with which a body would move in a parabola at the same distance. What is the eccentricity of the ellipse?

27. Shew that the inclination of the moon's orbit is the greatest, when the line of the nodes is in syzygy; and the least, when the nodes are in quadrature and the moon in syzygy.

28. Two material points S and P, the mass of the first being twice that of the second, attract each other with a force which varies inversely as the square of the distance. When they have

approached each other by half their original distance, P receives a new perpendicular impulse, which communicates to it a velocity equal to that which S has acquired. What curve is now described by each about the other?

29. Find the whole variation in the inclination of the moon's orbit, as the moon moves from quadrature to syzygy; the line of the nodes lying in quadrature. (*Newton*, Book III. Prop. 34, Cor. 4.)

30. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic. (*Newton*, Book III. Prop. 34.) 1823

31. Find the angular distance of a body from the vertex of a common parabola where the velocity is equal to half the greatest velocity.

32. The difference of the forces on P and p (*Newton*, Sect. 9.) $\propto \frac{1}{\mathrm{CP}^3}$; required a proof.

33. If the force $\propto \frac{1}{\text{dist.}^4}$, and a body be projected at an apse with the velocity acquired in descending from an infinite distance to that point, construct the curve described, and find the time of descent to the centre.

34. Prove that the periodic time of a body revolving in an ellipse round the focus $= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m}}$, where $a =$ semi-axis, and $m =$ force at a distance 1; and apply this result to deduce the actual time of falling down AC, in Prop. 32, Sect. 7.

35. Shew that in consequence of the mean disturbing force of the sun in the direction of the radius vector, the distance of the moon from the earth is increased by a 358th part, and her angular velocity diminished by a 179th part.

36. If the force $\propto \frac{1}{\text{dist.}}$, then the time of descent to the centre $= a\sqrt{\frac{\pi}{2m}}$, where $a =$ whole distance, and $m =$ force at a distance 1 from the centre.

37. Explain fully the motion of the apsides, and the variation of the eccentricity of P's orbit, Cor^s. 8 and 9, Prop. 66.

38. If a body describing a spiral in a medium whose density $\propto \frac{1}{\text{dist.}}$, cut the radius vector at B in the same angle as at A, and with a velocity which is to that at A :: $\sqrt{SA} : \sqrt{SB}$, then will the distances at which the body cuts the radius vector, and the times of successive revolutions, be in geometric progression. (*Newton*, Book II. Cor. 7, Prop. 15.)

39. Determine the angular distance of a body from the vertex of an ellipse whose eccentricity $= \frac{1}{2}$, at which the velocity : greatest velocity :: $1 : \sqrt{3}$.

40. Shew that the velocity acquired by a body in falling from infinity to the earth's centre, is to the velocity of a secondary at the earth's surface :: $\sqrt{3} : 1$.

41. Find where an inferior planet will appear stationary, supposing the force of gravity to vary inversely as the cube of the distance, and the orbits of the earth and planet to be circular.

42. In *Cotes's* first spiral, it is required to shew that the successive distances at which the curve cuts the apsidal line may be represented by a series of the form,

$$\frac{c}{c^2+1}, \quad \frac{c^2}{c^4+1}, \quad \frac{c^3}{c^6+1}, \quad \ldots \quad \frac{c^n}{c^{2n}+1}.$$

43. To determine the mean motion of the moon's nodes. (*Newton*, Book III. Prop. 32.)

1824 44. Force varies as $\frac{bA^m + cA^n}{A^3}$, find the angle between the apsides.

45. Explain the nature of centrifugal force, and shew that in all curves it $= \frac{h^2}{\text{dist.}^3}$, where $h =$ twice the area described in one second of time.

46. Let a be the distance from the centre of force, from which a body must fall externally to acquire the velocity in a circle whose radius $= r$ when the force $\propto \frac{1}{\text{dist.}}$; and let b be

the distance to which it must fall internally to acquire the same velocity; then will a, r, b be in geometric progression.

47. If at the distance a from the centre of force, a body be projected at an angle of 45° with the distance, with a velocity which is to that in a circle at the same distance as $\sqrt{2}$ to $\sqrt{3}$; and the force $= \frac{2a^2 n}{\text{dist.}^5} + \frac{n}{\text{dist.}^3}$; required the curve described.

48. A body acted on by gravity moves on the surface of a cone whose axis is vertical; find the law of force tending to the vertex, by which the projection of its path on a horizontal plane passing through the vertex may be described; and hence deduce the angle between the apsides when the orbit is nearly circular.

49. If a body describe the spiral of *Archimedes*, the force being in the pole, and its motion beginning from that point; then will the times of the successive revolutions be as the differences of the cubes of the natural numbers; and the excess of the time of the $\overline{n+1}^{th}$ revolution above that of the $n^{th} = n \times$ excess of the 2^{nd} above the 1^{st}.

50. If S be the sun, and A, B two planets that appear stationary to one another; then tan SBA : tan SAB :: periodic time of A : periodic time of B.

51. If the periodic times of a body revolving in a circle round the centres of force S and R be the same, compare the force tending to S with that tending to R.

52. If P′ be a point so taken in the radius vector SP of a parabola, that SP′ = SY the perpendicular on the tangent, then will the locus of the point P′ be the elliptic spiral; prove this, and compare the times of two bodies describing AP and AP′, the absolute forces being the same in both cases.

53. If a body describe an oval round a centre of force, the distance at which the angular velocity is equal to the mean angular velocity is $\sqrt{\frac{A}{\pi}}$, where A is the area of the figure, and $\pi = 3.14159$.

54. An imperfectly elastic body revolving in an ellipse whose eccentricity is $\frac{1}{2}$, is reflected at the mean distance by a plane coincident with the distance, so as to move after impact in the direction of the axis minor; find the degree of elasticity, and compare the periodic times in the two ellipses $\left(F \propto \frac{1}{D^2}\right)$.

55. Find the horary motion of the moon's nodes in a circular orbit.

1825 56. Find the space through which a body must fall externally that it may acquire the velocity with which it moves in an ellipse about the centre.

57. If the mass of a planet is four times that of the earth, and the distance of its satellite 16 times that of the moon from the earth, in how many months will the satellite revolve?

58. When the centripetal force varies inversely as the n^{th} power of the distance, n being greater than 3; find the equation to the spiral, which a body, projected with a velocity equal to the velocity acquired by falling from an infinite distance, describes; and determine the number of revolutions which it makes, before it falls into the centre.

59. Compare the mean addititious force with the force by which P is retained in its orbit round T. (*Newton*, Prop. 66, Cor. 17.)

60. State the phenomena, from which it appears that the force by which the moon is retained in its orbit, tends to the earth, and that this force varies inversely as the square of the distance.

61. In the 10th Lemma of the 1st Section of *Newton*, where the abscissa AD represents the time, the ordinate DB the velocity, and the area ABD the space described; if a straight line be drawn touching the curve AB in B the extremity of the ordinate, the tangent of the angle which this line makes with the axis will represent the force. Required proof.

62. Investigate the formula of lunar nutation in right ascension; find the longitude of the moon's node when it is

a maximum, and thence its maximum value; and by means of these values express the nutation in right ascension in a more simple formula.

63. Determine generally the resistance of a medium, so that a body acted upon by a centripetal force, whose law is known, may move in a given curve; and thence find the resistance when the force is uniform and acts in parallel lines, so that the curve may be a circular arc.

64. Find the law of the force tending to the pole of the logarithmic spiral.

65. When the force varies inversely as the n^{th} power of the distance, compare the velocity acquired by falling from an infinite distance with the velocity of a body revolving in a circle.

66. Find the point in a given hyperbola where the velocity of a body acted on by a force tending to its focus is twice as great as the velocity in a parabola at the same distance.

67. If the earth's motion about its axis were to cease, how much would a clock keeping true time in a given latitude gain in 24 hours?

68. There is a fixed centre of force which varies inversely as the square of the distance; and about this as a focus an ellipse is described, the axes of which are to one another in the proportion of $\sqrt{2} : 1$. A perfectly elastic body falls from a distance equal to the axis-major in the direction of the radius vector passing through the extremity of the axis-minor and impinges on the ellipse. Required the motion after impact.

69. The earth being supposed a sphere revolving about its axis with a given angular velocity, find the curve, in a meridional plane, which is the locus of a body, the centrifugal force of which opposed to gravity is every where equal to the force of gravity acting upon it.

70. State the methods by which the masses of all the planets and of their satellites may be compared with that of the sun.

71. If a body be revolving in an ellipse about the focus, and the force be suddenly made to vary inversely as the cube of the distance, the actual force at the mean distance being unaltered, what will be the curve described?

1826 72. Investigate the equation to the orbit in which the centripetal force is always n times as great as the centrifugal, and find the time of one revolution.

73. Determine that point in an ellipse described round a centre of force situated in the focus, where the linear velocity is n times as great as the paracentric.

74. If a body describe a logarithmic curve by a force acting perpendicularly to its axis; prove that the force at any point varies as the body's distance from the axis, and the velocity as the square root of the chord of curvature parallel to it.

75. If a body describe a circle by a force in the circumference, and at the same time the circle revolve about the centre of force in its own plane with an angular velocity varying inversely as the square of the body's distance; prove that the path traced out in fixed space is the spiral of *Archimedes*, and find the law of force by which it may be described.

76. Required the law of force when the space due externally to the velocity in a circle : space due internally :: $\sqrt{e} : 1$; e being the base of the hyperbolic system of logarithms.

77. If any number of hyperbolas have a common centre, and at distances proportional to their major axes double ordinates be drawn; shew that bodies acted upon by the same absolute force situated in the centre will describe any of the arcs thus cut off in equal times.

78. Prove that the force by which a body may describe any of the conic sections, round a centre of force in the vertex, varies inversely as the square of the distance, and directly as the cube of the secant of the angle which it makes with the axis.

79. A body projected in a given direction with a given velocity and attracted towards a given centre of force, has its velocity at every point : the velocity in a circle at the same distance :: $1 : \sqrt{2}$; find the orbit described, the position of its apse, the magnitude of its axis and the law of force.

80. If a body describe an equilateral hyperbola, round

a centre of force situated in the centre, and if θ be the angle described by the body from an apse in time t, prove that

$$\sin 2\theta = \frac{e^{4\sqrt{kt}} - 1}{e^{4\sqrt{kt}} + 1},$$

the force at distance 1 being represented by k.

81. Apply the differential equations of motion to determine the density of the medium, so that a body may describe a given curve about a given centre of force; and find the law of the density, when the curve is a circle and a force is situated in its circumference varying as $\frac{1}{D^n}$.

82. If a body describe an ellipse uniformly, round two centres of force situated in the foci; prove that the forces at any point of the ellipse are equal, and inversely proportional to the square of the corresponding conjugate diameter.

83. A body projected from a given point in a plane is 1827
attracted by forces $\frac{\alpha}{x^3}$ in the direction of x, and $\frac{\beta}{y^3}$ in the direction of y; prove that if the velocity and direction of projection be rightly assumed, it will describe a circle round the origin as a centre, and find how the velocity varies in different parts of the orbit.

84. Compare the difference of the forces in the fixed and moveable orbits, with the force in a circle at the same distance described with the same angular velocity.

85. According to what law must the centripetal force vary, that the areas *dato tempore* in all circles uniformly described about the centre may be equal?

86. Find the disturbing forces of Venus on the earth when their heliocentric longitudes differ by 45°.

87. If two bodies S and P attracting each other with forces varying inversely as the square of the distance, revolve about each other, S being much greater than P, the actual time of one revolution will be less than if S were immoveable in the ratio of 1 to $1 + \frac{P}{2S}$.

88. Supposing the orbit of a comet to lie in one plane, if the force attracting it towards the sun vary as that power of the distance whose index is $2 - \delta$, (δ being a small fraction,) the heliocentric angle between two successive perihelia will be $360° \dfrac{\delta}{1 + \delta}$.

89. Prove geometrically that $\frac{1}{2}(ydx - xdy)$ is the element of a sectorial area about the origin of the coordinates.

90. Compare the force at a given point of an ellipse described about the focus, with that in a circle at the same distance described with the velocity in the ellipse at that point.

91. Find where the space due externally to the velocity in an ellipse, force in focus, is thrice the space due internally.

92. If the force vary inversely as the 7^{th} power of the distance, and a body be projected from an apse with a velocity which is to the velocity in a circle at the same distance :: $1 : \sqrt{3}$; find the polar equation to the curve described, and transform it to rectangular coordinates.

93. Force varying inversely as the square of the distance, if a body be projected with n times the velocity in a circle at the same distance, and in a direction making an angle a with the distance; the angle θ between the axis-major and the distance may be determined from the equation

$$\tan(\theta - a) = (1 \sim n^2) \tan a.$$

1828 94. If equal areas be described by a body in equal times about a given point in a given plane, the body is urged by a force tending to that point. (*Newton*, Book I. Prop. 2.)

95. If a body be projected from a given point in a given direction with a given velocity about a centre of force varying as the distance, shew that it will describe an ellipse having its centre in the centre of force, and find the magnitudes and positions of the axes. (*Newton*, Book I. Prop. 10. Cor. 1.)

96. Find the law of the force tending to the focus of an hyperbola. (*Newton*, Book I. Prop. 12.)

97. If the mass of the earth increase slowly and uniformly, find the resulting equation of the moon's place at any given time, the orbit being nearly circular.

98. Mention the steps of the proof by which *Newton* shewed that every particle of matter gravitates to every other particle with a force which is inversely as the square of the distance.

99. A body falls towards a centre of force which varies as some power of the distance, determine the cases in which we can integrate so as to find the time of descent.

100. Knowing the force which varies as $\frac{1}{D^2}$, and the velocity of projection from a given point, to find the path described. (*Newton*, Book I. Prop. 17.)

101. State and prove *Newton's* construction for the path of a body projected from an apse with a velocity less than that acquired by falling from an infinite distance, and acted upon by a force varying inversely as the cube of the distance. (*Newton*, Book I. Prop. 41. Cor. 3.)

102. If the force vary as $\frac{1}{A^2} + \frac{1}{A^3}$, find the angle between the apsides by *Newton's* method. (Prop. 45.)

103. Describe the variations which take place in the inclination of P's orbit during one revolution of the line of nodes. (*Newton*, Book I. Prop. 66. Cor. 10.)

104. Shew how very small secular inequalities in the mean motions of two planets may be introduced when their mean motions are nearly commensurable.

105. Find the actual velocity of the point P, (*Newton*, Book I. Sect. vii. Prop. 39.) the force tending to C being supposed to vary directly as the distance.

106. The force in an orbit $\propto \frac{m}{\sqrt{a^2 + r^2}}$, where r is the radius vector; find the angle between the apsides when the orbit is nearly circular.

107. Shew from *Newton's* construction for determining the horary increment of the area described by the moon in a circular orbit round the earth at rest, that the velocity generated by the tangential ablatitious force between quadrature and syzygy is to that which would be generated in the same time by the mean addititions :: $3 : \pi$.

108. To determine the horary motion of the moon's nodes in a circular orbit. (*Newton*, Book III. Prop. 30.)

109. Explain fully the principles on which *Newton* calculates the correction in the motion of the nodes due to the unequable description of areas, and shew that the mean decrement when the nodes are in quadratures is equal $\frac{1}{4}$ decrement in syzygy.

110. A body may revolve in the equiangular spiral in a medium of which the density is inversely as the distance from the centre by a force varying inversely as the square of the distance from the centre. (*Newton*, Book II. Prop. 15.) Prove this geometrically and analytically.

111. If a uniform force act upon a body tending to give it a motion of rotation about an axis always perpendicular to the axis about which it is at each instant revolving, and always in the same plane, the angular velocity is unaltered. Shew hence that the angular velocity of the earth is not affected by the action of the sun and moon.

112. If gravity act upon a system of bodies m', m'', . . . and h', h'', ... be the vertical spaces described, and v', v'', ... be the actual velocities of the bodies, prove that

$$m'v'^2 + m''v''^2 + \ldots . = 2g\,(m'h' + m''h'' + \ldots .).$$

113. Find the place of a body in a parabolic orbit at any assigned time. (*Newton*, Book I. Prop. 30.)

114. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic. (*Newton*, Book III. Prop. 34.)

115. The centre of gravity of the earth and moon describes an orbit round the sun much more nearly elliptical than that described by the earth or moon.

116. Compare the times of bodies oscillating in a hypocycloid, revolving about the centre, and falling to the same centre, the force varying as the distance. (*Newton*, Book I. Prop. 52, Cor. 3.)

117. A body descends in a straight line towards a centre of force varying as the distance, find the velocity acquired in descending through a given space, and the time of descent (*Newton*, Book I. Prop. 38.)

118. Find the difference of the forces in the fixed and moveable orbits. (*Newton*, Book I. Prop. 44.)

119. Prove analytically that the areas described by a body about any centre of force are in the same plane, and proportional to the time.

120. Find the law of force acting in parallel lines by which 1829
a body will be made to describe a portion of the circle. (*Newton*, Book I. Prop. 8.)

121. Find the law of force tending to the focus, by which a body may be made to describe an ellipse. (*Newton*, Book I. Prop. 11.)

122. Explain the nature of centrifugal force, and shew that in a body revolving about a centre it varies inversely as the cube of the distance.

123. If a body be projected from a given point with a given velocity in a given direction, and acted on by a force which varies inversely as the square of the distance; find the conic section described.

124. Prove that the motion of a body P round T, when disturbed by a body S, is determined by the equation

$$\frac{d^2x}{dt^2} + \frac{\text{P} + \text{T}}{r^3}\,x + \frac{d\text{R}}{dx} = 0,$$

together with two similar equations in y and z, where $r = \text{PT}$,

$$\text{and R} = \frac{\text{S}\,.\,r}{\text{ST}^2}\cos\angle\,\text{STP} - \frac{\text{S}}{\text{SP}};$$

and by means of them exemplify fully the method of determining the variation of the elements of P's orbit arising from the disturbance, when that disturbance is small.

125. At similar points in similar curves described round centres of force similarly situated, the forces are as the squares of the velocities directly, and the distances inversely.

126. Find the place of a body in an elliptic trajectory after a given time. (*Newton*, Book I. Prop. 31.)

127. State and prove the proportion which *Newton* has given in Book I. Prop. 44, Cor. 1, for determining the actual value of

the difference of the forces in the fixed and moveable orbits, and apply it to determine the whole force on p, when the fixed orbit is an ellipse, and the force in the focus.

128. If P and S attract each other, the curve which P describes relatively to S, may be described by P round S fixed. (*Newton*, Book I. Prop. 58.)

129. A body revolving in a given ellipse, force in focus, leaves the higher apse; and when it arrives at the lower apse A, the absolute force is suddenly altered, so that the body describes a similar ellipse, of which A is the higher apse; and a like alteration takes place when the body arrives at the lower apse of the new ellipse, and so on at each successive apse. Find the time to the centre of force.

130. The parallax of the moon

$$= \mathrm{P}\left\{1 + e\cos(c\theta - \alpha) + m^2\cos(\overline{2 - 2m\theta} + 2\beta) + \frac{15}{8}me\cos(\overline{2 - 2m - c\theta} + 2\beta + \alpha)\right\},$$

where P $=$ mean parallax, $m = \dfrac{\text{sun's mean motion}}{\text{moon's mean motion}}$, $\theta =$ longitude of moon, $\alpha =$ longitude of perigee, $-\beta =$ sun's mean longitude when $\theta = 0$. Explain fully the effects of the several terms in the above expression on the moon's orbit.

131. A body acted on by a centripetal force varying partly as $\dfrac{1}{\mathrm{D}^3}$ and partly as $\dfrac{1}{\mathrm{D}^5}$, is projected with the velocity which would be acquired in falling from infinity, at an angle with the distance whose tangent $= \sqrt{2}$, the forces being equal at the point of projection. Required the orbit described, and the time of descent to the centre.

132. In *Newton*, Book I. Prop. 66, explain the effect of the disturbing force of S in producing a motion of the nodes of P's orbit, and a variation of the inclination.

133. Find the horary motion of the nodes in an elliptic orbit. (*Newton*, Book III. Prop. 31.)

134. Find the horary increment of the area described by the moon, and compare its values at quadrature and syzygy. (*Newton*, Book III. Prop. 26.)

135. Determine by the principles of *Newton's* seventh section the spaces due to the velocity externally and internally in an ellipse, the force being in the centre.

136. Find the effect of the disturbing force of the sun on the apsides of the lunar orbit during one revolution of the moon. (*Newton*, Book I. Prop. 66, Cor. 7.)

137. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic, and thence by *Newton's* construction determine its mean monthly value.

138. If the velocities of two bodies at any equal distances from the centre of force be the same, and if one body move in a straight line to or from the centre and the other in a curve, their velocities will be the same at all other equal distances. (*Newton*, Book I. Prop. 40.)

139. If S and P revolve round their common centre of gravity by their mutual attraction, shew that each will move in the same manner as if a body were placed in the centre of gravity exerting a force varying according to the same law. What is the magnitude of this body for each of the two S and P, the force varying inversely as the square of the distance?

140. A body is projected round a centre of force varying as 1830
the distance, with a given velocity, in a given direction; find the magnitudes and positions of the axes of the orbit described, and also the periodic time. (*Newton*, Book I. Prop. 10, Cors. 1 and 2.)

141. A body revolves in a parabola, find the law of the force tending to the focus. (*Newton*, Book I. Prop. 13.)

142. In different conic sections described round the same centre of force, situated in the focus, the latera recta are as the squares of the areas described in a given time. (*Newton*, Book I. Prop. 14.)

143. When a body descends from a point A towards a centre of force S, the force varying as the distance, shew that the space described, the velocity, and the time of motion, are respectively proportional to the versed sine, sine and arc of the circle of radius SA; and find the time to the centre.

144. Explain *Newton's* method of finding the angle between the apsides in orbits nearly circular, and apply it when the force $= mA + \frac{m_1}{A^2}$.

145. *Newton*, Section XI. Prop. 66.

146. In the case of the sun, moon, and earth, find the whole force on the moon in the direction of the radius vector, the orbits being considered in the same plane.

147. Determine the orbit described and the time of describing any angle when a body is projected round a centre of force varying as $\frac{1}{D^7}$, at an angle whose tangent $= \frac{3^{\frac{1}{2}}}{2^{\frac{5}{6}}}$, and with a velocity which is to the velocity in a circle at the same distance $:: \sqrt{2} : \sqrt{3}$.

148. A body describes a circle of given radius uniformly, acted upon by two forces each varying as the distance and without the plane of the circle; find the velocity of the body and the position of the plane of its orbit.

149. If a body revolve in an ellipse round the focus, prove that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius vector during the progress of the body from the higher to the lower apse, and point out the effect on the eccentricity.

150. A body is acted on by two forces, the one repulsive and varying as the distance from a given point, the other constant and acting in parallel lines. Determine the motion of the body.

151. Prove that the centre of gravity of the earth and moon describes about the sun very nearly an ellipse in one plane, and that the area described by its radius vector is very nearly proportional to the time.

152. Find the horary variation of the inclination of the moon's orbit. (*Newton*, Book III. Prop. 34.)

153. As the line of nodes of the moon's orbit moves from syzygy to quadrature, the inclination of the orbit to the ecliptic is diminished; and as the line of nodes moves from quadrature

to syzygy, the inclination is increased. (*Newton*, Book I. Prop. 66, Cor. 10.)

154. Find the horary increment of the area which the moon describes about the earth in a circular orbit. (*Newton*, Book III.)

155. $s = k\,\{\sin(g\theta - \gamma) + \frac{3m}{8}\sin(2-2m-g)\theta + 2\beta + \gamma\}$,

where s = tangent of moon's latitude, γ = longitude of node, $-\beta$ = the sun's mean longitude when $\theta = 0$. Explain the effect of these terms, and thence shew that the inclination of the orbit is greatest when the line of nodes is in syzygies, and least when it is in quadratures.

156. Find the ratio of the diameters of the lunar orbit, supposing it to have been originally without eccentricity. (*Newton*, Book III. Prop. 28.)

157. If two bodies S and P attract each other mutually, the orbit which P appears to describe about S in motion may be described about S fixed, by the action of the same force. (*Newton*, Book I. Prop. 58.)

158. Find the mean horary motion of the nodes of the lunar orbit supposed to be elliptical. (*Newton*, Book III. Prop. 31.)

159. Eliminate t from the differential equations:

$$\frac{d\left(\rho^2 \frac{d\theta}{dt}\right)}{dt} = \mathrm{T}\rho,$$

$$\text{and } \frac{d^2(\rho s)}{dt^2} = -\mathrm{S}.$$

160. The difference of the forces at corresponding points in the fixed and revolving orbits varies inversely as the cube of the distance. (*Newton*, Book I. Prop. 44.)

161. Compare the axis-major of the ellipse apparently described by P round T, with that of the ellipse described by P round T fixed in the same periodic time. (*Newton*, Book I. Prop. 60.)

162. A body being acted upon by a force tending to a centre, 1831
the areas described are in one plane, and proportional to the times of description. (*Newton*, Prop. 1.)

163. Investigate an expression for the force by which a body may be made to describe any orbit whatever round a fixed centre in the same plane with it; and apply it to find the law of force tending to the focus of an ellipse. (*Newton*, Props. 6 and 11.)

164. If a body by the action of any centripetal force move in any manner, and another body ascend or descend in a straight line, and their velocities in any case of equal distances be equal, their velocities at all equal distances will be equal. (*Newton*, Prop. 40.)

165. Prove that the difference of the forces in the fixed and moveable orbits varies inversely as the cube of the distance. (*Newton*, Prop. 44.)

166. Required the part of the sun's disturbing force perpendicular to the plane of the moon's orbit.

167. Find the quantities of matter of planets which have satellites. And these being given, shew how to determine the quantity of matter of those which have not satellites.

168. If x, y, z, r, and x', y', z', r' be the coordinates and distances of two planets m and m' from the centre of the sun supposed at rest, and if $\lambda =$ the distance of m from m_1,

$$\text{and } Q = \frac{xx' + yy' + zz'}{r'^3} - \frac{1}{\lambda},$$

prove that the axis-major of the ellipse of curvature at the point (x, y, z,) of m's orbit is

$$= \frac{1 + m}{2m_1 \int \left(\frac{dQ}{dx} dx + \frac{dQ}{dy} dy + \frac{dQ}{dz} dz \right)},$$

where $1 =$ mass of the sun.

169. How does it appear that the mean motions of the planets are subject to no *secular* variations?

170. If a be the mean distance of a planet from the sun, and $l =$ the length of the line of nodes, then the time of the planet's passage from node to node through the perihelion is

$$= \frac{a^{\frac{3}{2}} p}{\pi} \left\{ \tan^{-1} \sqrt{\frac{l}{2a - l}} - \frac{l}{2a} \cdot \frac{\sqrt{2a - l}}{l} \right\},$$

where $p =$ periodic time of the earth about the sun, and $1 =$ its mean distance from it.

171. A body drawn towards the origin of rectangular co-ordinates by a force mr, and from the plane yz by a force nx, is projected perpendicularly to the plane xz, from a point in it distant by D from each of the axes of x and z, with the velocity $\sqrt{m}\,.\,D$; required the equations of the orbit, and the positions of the apses.

172. Find the horary motion of the moon's nodes in a circular orbit. (*Newton*, Vol. III. Prop. 30.)

173. If the revolving orbit (*Newton*, Sect. 9,) be an ellipse, force in the focus, and $=\dfrac{F^2}{A^2}$ at distance A, prove that the force on the body in fixed space

$$=\frac{F^2}{A^2}+\frac{RG^2-RF^2}{A^3}.$$

If the orbit be nearly circular, shew that this expression approximates to $G^2A^{\frac{F^2}{G^2}-3}$ as its limit.

174. In what positions of the node and moon is the inclination of the lunar orbit to the ecliptic respectively a maximum and minimum? (*Newton*, Prop. 66, Cor. 10.)

175. The mean horary variation of the inclination of the lunar orbit to the plane of the ecliptic, for a given position of the line of nodes, varies nearly as the sine of twice the angular distance of the nodes from syzygy. (*Newton*, Book III. Prop. 34, and Cors. 1, 2, 3.)

176. Find the variation of the moon. (*Newton*, Book III. Prop. 29.)

177. Having given

$$nt=\theta-2e\sin\theta+\frac{3e^2}{4}\sin 2\theta-\frac{e^3}{3}\sin 3\theta,$$

apply *Lagrange's* theorem to prove that

$$\theta=nt+\left(2e-\frac{e^3}{4}\right)\sin nt+\frac{5e^2}{4}\sin 2nt+\frac{13e^3}{12}\sin 3nt,$$

neglecting higher powers of e than the cube.

178. The centre of gravity of the earth and moon describes about the sun, very nearly, an ellipse in one plane, and the area passed over by its radius vector is very nearly proportional to the time.

179. The resistance of a medium being $Q.v^2$, and the force tending to a fixed centre being P, the differential equation of the trajectory is

$$\frac{d^2u}{dv^2} + u - \frac{\mathrm{P}e^{2\int \mathrm{Q}ds}}{h^2u^2} = 0.$$

180. Find the mean annual motion of the moon's nodes. (*Newton*, Book III. Prop. 32.)

181. In what positions of the apse of the lunar orbit is the eccentricity respectively a maximum and minimum? (*Newton*, Prop. 66. Cor. 9.)

182. A body descends in a straight line from a given point towards a centre of force, the force varying as $\frac{1}{\mathrm{D}^2}$; find the time of describing any space, and the time to the centre.

183. The mean disturbing force of the sun on the moon in the direction of the radius vector is ablatitious and equal to half the mean addititious force, supposing the orbits to be in the same plane.

1832 184. When a body moves in a curve in one plane, if equal areas are described in equal times about a fixed point in the plane, the force acting on the body tends to that point. (*Newton*, Prop. 2.)

185. A body revolves in an ellipse, required to find the law of force tending to the centre of the ellipse. (*Newton*, Prop. 10.)

Having given the force at a given distance, determine the periodic time.

186. A body describes a circle radius (radius r) in time T, and is accelerated by a tangential force $k \sin 2\theta$, where θ is the angle described from the beginning of the motion, and k a small quantity whose square may be neglected; shew that the velocity at any point

$$= \frac{2\pi r}{\mathrm{T}}\left(1 - \frac{k\mathrm{T}^2 \cos 2\theta}{8\pi^2 r}\right).$$

187. Two material points P and Q are connected by a rigid rod, and attracted towards a fixed centre of force; shew, from the equations of motion, that the principle of *vis viva*

obtains. Also, if the force vary directly as the distance, each body will describe round the other areas proportional to the times.

188. A body descends in a straight line towards a centre of force, the force varying inversely as the square of the distance, determine the time of describing a given space from rest, and also the time to the centre.

189. In the 9th Section of *Newton*, if the revolving orbit be an ellipse, force in the focus, prove that the force on the body in fixed space varies as

$$\frac{F^2}{A^2} + \frac{R(G^2 - F^2)}{A^3}.$$

190. Explain generally how the motion of the nodes and the variation of the inclination of P's orbit round T are necessary consequences of the disturbing force of S. If the nodes be in the octants after quadratures, how will the inclination of P's orbit, and the motion of the nodes be affected in a whole revolution of P? (*Newton*, Sect. 11.)

191. If x be the distance from the axis of the earth at which it must be struck when at its nearer apse, so as to have its rotatory motion and motion of translation communicated to it by the same force, and x_1 the corresponding distance when the earth is at its farther apse, prove that

$$x : x_1 :: 1 - e : 1 + e,$$

where e is the eccentricity of the earth's orbit, and the earth a solid of revolution.

192. If v and v_1 be the true anomalies described in the same time t in a very eccentric ellipse and parabola of the same perihelion distance D.

$$\text{Having given } t = \frac{D^{\frac{3}{2}}\sqrt{2}}{\sqrt{\mu}}\left\{\tan\frac{1}{2}v + \frac{1}{3}\tan^3\cdot\frac{1}{2}v \right.$$
$$\left. + (1-e)\tan\frac{1}{2}v\left(\frac{1}{4} - \tan^2\frac{1}{2}v - \frac{1}{5}\tan^4\cdot\frac{1}{2}v\right)\right\}$$

prove that

$$\tan\frac{x}{2} = \frac{1}{20}\cdot(1-e)\tan\cdot\frac{1}{2}v_1\left(4 - 3\cos^2\cdot\frac{1}{2}v_1 - 6\cos^4\cdot\frac{1}{2}v\right)$$

nearly, where $v_1 + x = v$.

193. Find the mean motion of the moon's nodes. (*Newton*, Book III. Prop. 32.)

194. A body is projected from a given point, in a given direction, with a velocity acquired from infinity, and is acted on by a central force varying as $\frac{1}{D^6}$; find the equation to the curve described, and trace it.

195. Required the mean value of the central disturbing force acting on the moon, and the quantity by which it increases her periodic time.

196. Explain the physical cause of solar precession and solar nutation.

197. State the steps by which it is proved that if m and m' be the masses of two spheres, and r the distance of their centres, the moving force with which they attract each other will be as $\frac{mm'}{r^2}$.

198. In the lunar theory, having given

$$\frac{P}{h^2u^2} = a\left\{1 - \frac{3k^2}{4} + \frac{3k^2}{4}\cos 2(\theta - \gamma)\right\}$$
$$- \frac{m'a'^3}{h^2a^3}\left\{\frac{1}{2} + \frac{3}{2}\cos\{(2 - 2m)\theta - 2\beta\}\right\}$$
$$\frac{T}{u^3} = -\frac{3}{2}\cdot\frac{m'a'^3}{a^4}\sin\{(2 - 2m)\theta - 2\beta\},$$
$$\text{and} \quad \frac{d^2u}{d\theta^2} + u + 2a\int\frac{T\,d\theta}{h^2u^3} - \frac{P}{h^2u^2} = 0,$$

where m and k are small quantities of the first order, and $\frac{m'a'^3}{h^2a^4}$ of the second order; it is required to integrate the last equation as far as terms of the second order.

199. Compare the disturbing forces of the sun on the moon to gravity at the earth's surface. (*Newton*, Book III. Prop. 25.)

200. The ratio of the angular motion of the node to the angular motion of the apse is the same in different systems. (*Newton*, Prop. 66, Cor. 16.)

How is it shewn in the lunar theory that this ratio is much greater for the moon than for one of the satellites of Jupiter or Saturn?

201. What are the principal differences in the investigation of the disturbance of the moon by the sun, and that of one planet by another?

202. Deduce a numerical comparison between the centrifugal force at the equator, and the force of gravity, the radius of the earth being 4000 miles, and gravity at the equator $32\frac{1}{6}$ feet.

203. Compare the axis-major of the ellipse described by P round S in motion, with that of the ellipse described by P round S fixed, in the same periodic time. (*Newton*, Sect. 11.)

204. Explain the nature of centrifugal force; and when a body revolves in a plane curve about a centre of force, shew that it varies inversely as the cube of the distance.

205. If a body be projected from a given point in a given direction with a given velocity, find the orbit described, the force varying inversely as the square of the distance; and when the orbit is an ellipse, shew that the axis-major is independent of the direction of projection.

206. Explain *Newton's* method of finding the angle between the apsides in orbits nearly circular; and when the force is equal to $\frac{b}{A^2} \pm cA$, shew that the apses regress or progress according as the $+$ or $-$ sign is taken.

207. Prove that the force in any given orbit round a fixed 1833
centre varies as the limit of $\frac{QR}{SP^2 \cdot QT^2}$, (*Newton*, Prop. 6) and make this expression general for different orbits.

208. A body revolves in an ellipse, required to find the force tending to the focus. (*Newton*, Prop. 40.)

209. Apply the equations of motion to prove that when a body describes a curve by the action of a centripetal force, the velocity at any point is independent of the nature of the path described. Is this true when the body is acted on by any number of centripetal forces?

210. In the system of three bodies S, T, P, (*Newton*, Sect. 11.) in the same plane, compare the addititious and ablatitious force of S on P; and shew that the addititious force is to the force of T on P :: (period of P)^2 : (period of T)^2 nearly.

Why are the principles of the 9th section applicable to the determination of the apsidal motion in the case of a satellite of Jupiter with greater accuracy than in the case of the moon?

211. A body is projected from a point near a centre of force which varies inversely as the square of the distance, in a direction perpendicular to the line joining the point of projection with the centre of force, and so as to describe an ellipse about that centre. Shew that the point of projection will coincide with the nearer or further apse, according as the velocity of projection is greater or less than that with which a circle might be described at the same distance.

212. A body considered as a point is attracted towards each of two fixed centres by a force which varies inversely as the square of the distance. Determine the equation to the surface on which it may remain at rest in every position.

213. If v be the velocity of a body moving freely in space corresponding to any time t, ρ the radius of the circle of absolute curvature at the corresponding point of its path, and P the whole force by which it is acted upon; apply the differential equations of motion referred to three rectangular axes to prove that

$$p^2 = \left(\frac{dv}{dx}\right)^2 + \left(\frac{v^2}{\rho}\right)^2.$$

214. The motion of the moon's nodes, while the moon describes a small angle $\delta\theta$, being $3m^2 \cos \alpha \sin \theta \sin (\alpha + \theta)\, \delta\theta$, where θ is the moon's distance from quadratures, and α the node's distance from quadratures; to determine the whole motion in a revolution of the moon.

215. The differential equation for determining the inclination of the radius vector of the moon to the plane of the ecliptic (second approximation) being

$$\frac{d^2s}{\delta\theta^2} + \frac{3}{2} m^2k \sin(g\theta - \gamma) - \frac{3}{2} m^2k \sin \{(2 - 2m - g)\,\theta - 2\beta + \gamma\}$$
$$= 0,$$

where g is a quantity nearly $= 1$; determine the value of s, and explain the effect of the several terms.

216. A body moves in a parabola urged by a force tending to the focus; determine its position at any time. (*Newton*, Prop. 30.)

217. Investigate the alterations of the semi-axis-major, and of the eccentricity of the disturbed orbit of a planet; the equations for determining the motion of a disturbed planet being

$$C = \left(\frac{dr_1}{dt}\right)^2 + r_1^2\left(\frac{d\theta_1}{dt}\right)^2 - \frac{2\mu}{r_1} + 2\int dt\,\frac{dR}{dt},$$

$$0 = \frac{d}{dt}\left(r_1^2 \cdot \frac{d\theta_1}{dt}\right) + \frac{dR}{dt_1};$$

and the radius vector and longitude in the undisturbed orbit being expressed by the series

$$r = A + B\cos(nt + \varepsilon - \omega) + C\cos 2(nt + \varepsilon - \omega) + \ldots\ldots$$

$$\theta = nt + \varepsilon + M\sin(nt + \varepsilon - \omega) + N\sin 2(nt + \varepsilon - \omega) + \ldots$$

218. To find the diameters of the orbit, into which the disturbing forces would convert the orbit of the moon, if circular when undisturbed.

219. Explain *Newton's* method of accounting for the motions of the apsides of an orbit, and shew that the disturbing centripetal force which causes it, varies inversely as the cube of the distance from the centre.

220. When bodies describe different circles with uniform 1834
motions, the forces tend to the centres of the circles, and are as the squares of the velocities divided by the radii of the circles. Prove that this is also true of similar portions of similar curves having centres of force similarly situated.

221. When several bodies revolve in ellipses about the same centre of force varying inversely as the square of the distance, the latus rectum in each orbit is as the square of the area described by the radius vector in a unit of time.

222. Enunciate the propositions in which *Newton* proves that a body attracted towards a centre of force, varying inversely as the square of the distance, obeys in its motion the laws established by *Kepler's* observations for the planets.

223. A body acted upon by a force varying as $(\text{distance})^{-n}$, falls in a straight line to the centre of force from a distance (a); prove that for different values of a the time varies as $a^{\frac{n+1}{2}}$.

224. Explain the effect of the disturbing force on the inclination of P's orbit to a given fixed plane passing through S. For what positions of the line of nodes is the inclination greatest and least? (*Newton*, Prop. 66, Cor. 10.)

225. Several bodies move about a centre of force, varying inversely as the square of the distance, and all pass through a given point with the velocity in a circle at that distance; shew that they all return to the point after the same interval, and that the centres of their orbits lie in a sphere.

226. If the moon moved in the ecliptic, shew that the force of the earth to produce rotation about her axis perpendicular to that plane would nearly $= \dfrac{3\mu \sin 2\theta}{2r^3} \dfrac{A - B}{C}$; μ, r being the earth's mass and distance from the moon; A, B, C the principal momenta of inertia of the lunar spheroid; and θ the angular distance, at the moon's centre, of the earth from one of the principal axes which lie in the ecliptic.

227. Assuming the expression for the force in the preceding problem, find the moon's libration in longitude; and by comparing it with the result of observation, draw inferences as to the moon's form, and the initial relation between her motions of rotation and revolution.

228. Investigate the deviation from the principle of conservation of areas when the point about which the areas are described is the centre of one body in the system much larger than the others.

229. If $cz + c'y + c''x = 0$ be the equation to the invariable plane, when the common centre of gravity is supposed to coincide with the centre of the large body, prove that the equation to the corresponding plane in the case mentioned will be

$$(c - c_z)\, z + (c' - c_y)\, y + (c'' - c_x) = 0,$$

$$\text{where } c_z = \frac{1}{\mu t} \cdot \int dt\, \Sigma mm' \left\{ y^2 \frac{d}{dt} \frac{x'}{y} \frac{y^2 d}{dt} \frac{x'}{y} + y'^2 \cdot \frac{d}{dt} \frac{x}{y'} \right\},$$

c_y, and c_z are similar functions of (z, x) and (z, y) respectively, and μ denotes the sum of the mass of the large body and either one of the others.

230. Explain fully the physical cause of the motion of the moon's nodes, and the connexion between it and the phenomenon of nutation.

231. Find the whole force upon the moon parallel to the projection of her radius vector, expanded in a series of cosines of the difference of longitude of the sun and moon. What order of small quantities does the result include, and to what standard are they referred?

232. The equation for determining the orbit of a body round a fixed centre of force being $\dfrac{d^2y}{d\theta^2} + u - \dfrac{P}{h^2u^2} = 0$; determine its integral when $P = mu^2 + m'u^3$. Shew how to determine the arbitrary constants when the point, the direction, and the velocity of projection, are given.

233. A planet m is disturbed by another m', and one term in R which produces the most important term in the perturbation in longitude is

$$m' \mathrm{P} \cos\{(pn - qn')\,t + p\epsilon - q\epsilon' + \mathrm{Q}\},$$

where P and Q are constant, p and q integers, and $pn - qn'$ very small; prove that the ratio between the coefficients of the long inequalities produced by it in the longitudes of m and m' respectively is

$$= -\frac{m'}{m}\sqrt{\frac{a'}{a}} \text{ nearly.}$$

234. Find the horary increment of the area which the moon describes in a circular orbit about the earth. (*Newton*, Book III. Prop. 26.)

235. Explain *Newton's* method of finding the angle between the apsides, in an orbit nearly circular. Apply it to the case where the central force is the same at all distances.

236. Enunciate and prove *Newton's* eleventh lemma. What is meant by saying that every curve of finite curvature is ultimately a parabola? How is it proved? 1835

237. Prove that $F = 2 \text{ limit } \frac{QR}{T^2}$, and find the space due to the velocity at any point by the action of the force at that point continued uniform.

238. A body is projected with a given velocity at a given distance from a centre of force varying inversely as the square of the distance; shew that the axis-major of the orbit described is independent of the direction of projection.

239. Whilst the earth moves from perihelion to aphelion, the mean motion of the moon is retarded, and conversely. (*Newton*, Prop. 66, Cor. 6.)

240. S is a centre of force, which varies as $(\text{dist.})^{-2}$. From different points in the straight line SA, bodies are projected at right angles to SA with velocities which are inversely as their distances from S. Shew that the orbits described will all pass through the circumference of a fixed circle, and that the periodic times will be as the cubes of the minor axes.

241. Force $\propto \frac{1}{(\text{dist.})}$, shew that the attraction of an ellipse in a direction parallel to its axis-major, on a point whose coordinates are f, g, is equal to

$$(1 \pm 1)\frac{\pi k b f}{a+b} + (1 \mp 1)\frac{\pi k b}{ae^2} \times$$

$$\left\{f - \sqrt{\frac{\sqrt{(g^2 + a^2e^2 - f^2)^2 + 4f^2g^2} - (g^2 + a^2e^2 - f^2)}{2}}\right\},$$

the upper or lower signs being used according as the attracted point is within or without the ellipse. Find also the value of the expression when g and e vanish.

242. Shew that when the moon's apse line is in syzygies it will progress from the effect both of the central and tangential disturbing forces. Supposing it to regress when in quadratures, by what considerations does it appear that upon the whole the line of apses progresses considerably?

243. Find the horary motion of the moon's nodes in a circular orbit. By what observations does it appear that the force by which the moon is retained in her orbit tends to the earth,

and varies nearly inversely as the square of her distance from the earth's centre?

244. The moon's parallax is

$$P\left\{1 + e\cos(c\theta - \alpha) + m^2\cos(\overline{2 - 2m}\,.\,\theta - 2\beta)\right.$$

$$\left. + \frac{15}{8}\,me\cos(\overline{2 - 2m - c}\,.\,\theta - 2\beta + \alpha)\right\}.$$

Explain the effect of the last term of this expression on the form of the moon's orbit.

245. Having given the equations

$$\left(\frac{dr_1}{dt}\right)^2 + r_1^2\left(\frac{d\theta_1}{dt}\right)^2 - \frac{2\mu}{r} + 2\int dt\frac{dR}{dt} = C,$$

$$\frac{d^2(r_1^2)}{dt^2} = 2C + \frac{2\mu}{r_1} - 4\int dt\,\frac{dR}{dt} - 2r_1\frac{dR}{dr_1},$$

investigate the equation for the perturbation in longitude. What terms in the resulting equation are most important?

246. Apply the equations of motion to shew that a body acted upon by a central force will describe a curve lying in one plane, and that the areas described about the centre of force are proportional to the times.

247. A body describes an ellipse round a centre of force in the centre, find the periodic time and also the time of describing a given angle after leaving an apse.

248. Explain *Newton's* method of finding the angle between the apsides in orbits nearly circular. Ex. Force $\propto$ (dist.)^n.

249. The velocities of a body at different points of a curve, 1836
described about a centre of force, are inversely as the perpendiculars from the centre upon the tangents at those points.

250. Find the law of force and the periodic time in an ellipse about the focus. Hence, assuming the law of gravitation, deduce *Kepler's* third law.

251. Explain *Newton's* method of finding the angle between the apsides in an orbit nearly circular.

Ex. Force varying partly as the distance and partly inversely as the square of the distance.

252. When the sun is moving from apogee to perigee, shew that, in consequence of his disturbing force, the moon is always in advance of her mean place.

253. The radius vector of a planet's orbit is affected with a small periodical inequality; shew that its effect may be represented by continued and periodical alterations of the eccentricity and longitude of the perihelion; the period of either being $\frac{P \cdot T}{P \sim T}$ where P is the period of the planet, and T that of the inequality.

254. A body describes an ellipse about a centre of force in one of its foci, and its velocity is slightly increased in the direction of its motion; shew that the eccentricity will be thereby diminished or increased, as the distance of the body from the focus is greater or less than its mean distance; and the apse will advance or recede, as the body is moving from the lower to the higher apse, or from the higher to the lower.

255. Find the horary motion of the moon's nodes in a circular orbit; and shew that the mean horary motion of the nodes is half the horary motion when the moon is in syzygy.

256. If a body attracted to a centre of force be projected with a given velocity from a given point, its velocity, when at a given distance from the centre, is independent of the direction of projection.

257. In the expression for the moon's longitude there occurs the term $-\frac{k^2}{4}\sin(2gpt - 2\gamma)$; shew that it is nearly the difference between her longitudes measured on her orbit and on the ecliptic.

Explain the effect of the term $k \cdot \frac{3m}{8}\sin\{(2-2m-g)\theta - 2\beta + \gamma\}$ in the expression for the tangent of the moon's latitude. What do the several letters in this term represent?

258. The equation for determining the perturbation of the radius vector of a planet being

$$0 = \frac{d^2(r\delta r)}{dt^2} + n^2\left(\frac{a^3}{r^3}\right) r\delta r + 2n\int dt\,\frac{dR}{d\epsilon} + r\,\frac{dR}{dr},$$

explain the method of solving it by approximation. Shew that a force, which goes through all its values nearly in the time of a revolution, will produce a considerable inequality in the radius vector.

259. The apparent orbit of P to a spectator at S in motion may be described round S fixed by the action of the same force, if P be projected with a proper velocity and in a proper direction. (*Newton*, Prop. 58.)

SECTION XVIII.

QUESTIONS IN HYDROSTATICS AND THE THEORY OF SOUND.

1821 1. A clepsydra is constructed to mark equal portions of time, in the form of a paraboloid having its vertex downwards, the equation to the generating curve being $y^4 = ax$. How must the scale on the axis be graduated?

2. Explain the construction of the steam engine; and having given the weight upon the piston, the quantity of steam admitted, and the content of the cylinder, find the velocity of the piston at any point, and the time of describing the cylinder.

3. Compare the pressure on the surface of a sphere filled with water, with the weight of a sphere of mercury of the same magnitude.

4. Find the centre of pressure of a trapezoidal plane surface immersed vertically in a fluid, two of whose sides are parallel to each other, and to the surface of the fluid.

5. A cylinder of given length is pressed down in a vertical position into a fluid, so that its upper end is on a level with the surface, the specific gravity of the cylinder being one half that of the fluid: the pressure being removed, to find the greatest height to which the upper end of the cylinder will rise above the surface of the fluid.

6. Distinguish between the centre of gravity and centre of pressure, and shew that the former is always nearer to the surface of the fluid than the latter.

7. Graduate a thermometer according to *Fahrenheit's* scale.

8. A small aperture (a) is made in the vertical side of a cylindrical vessel filled with a fluid; the area of its horizontal section being A. Compare the latus rectum of the parabola first described by the spouting fluid with the length of a pendulum vibrating once while the surface of the fluid descends to the orifice.

9. A vessel is kept filled with a fluid; and an aperture is made in its perpendicular side in the form of a parabola, the vertex of which coincides with the surface of the fluid. Find the depth of a horizontal section such that if the whole fluid issued with its velocity, the quantity discharged in a given time would be the same as when each horizontal section flows with its own velocity.

10. An isosceles triangle is immersed perpendicularly in a 1822
fluid with its vertex coincident with the surface and its base parallel to it. How must it be divided by a line parallel to the base, so that the pressure upon the upper and lower parts respectively may be in the ratio of 1 : 7?

11. How may the phenomena of the trade winds be explained?

12. A vessel of given altitude empties itself through an orifice of given dimensions in its lowest point, and the upper surface descends with a given uniform velocity; find the content of the vessel.

13. Find the time of emptying a sphere filled with fluid through an orifice in its lowest point.

14. An upright cylindrical vessel empties itself through an orifice in the base; compare the pressures upon the concave surface at first, and when half the time of emptying has elapsed.

15. The orifices in the equal bases of two upright prismatic vessels are in the ratio of 2 : 1, and the vessels are emptied in equal times; compare their altitudes.

16. A life-boat contains 100 cubic feet of wood, specific gravity .8; and 50 feet of air, specific gravity .0012. When filled with fresh water, what weight of iron ballast, specific gravity 7.645, must be thrown in before it will begin to sink?

17. In the scale of *Reaumur's* thermometer, the freezing point of water is 0, and the boiling point 80°. In the centigrade thermometer, those points are 0 and 100° respectively. What will be the degree of heat marked by each, when *Fahrenheit's* thermometer stands at 59° ?

18. The axis of *Archimedes*' screw is inclined at a given angle to the horizontal section of the water. Find the highest and lowest points of the spiral tube, its point of inflection, and the quantity of water which can be raised in a given time by a given power.

1823 19. A given paraboloid filled with fluid is placed with its vertex downwards and its axis vertical; determine the time of emptying one half of its content through a given orifice in the vertex.

20. Find the centre of resistance of a semi-circle revolving round its diameter in a medium whose resistance $\propto$ (velocity)^2.

21. Two cylindrical vessels of given dimensions, containing given quantities of water, are made to communicate with each other by means of a small orifice at their bases; find the time elapsed before the water stands at the same altitude in both vessels.

22. Find the different positions of equilibrium of a parabola floating in a fluid with its vertex immersed.

23. A sphere of given weight and dimensions descends in a fluid; find the pressure on the bottom of the vessel that contains it, arising from the action of the sphere on the fluid.

24. Compare the resistances on a globe and the circumscribing cylinder moving in a fluid with equal velocities in the direction of the axis of the cylinder.

25. A given weight is suspended by a string and immersed in a river whose inclination to the horizon is known. Having observed the angle which the string makes with a vertical line, it is required to determine the velocity of the stream.

26. A cylinder of given length is just immersed vertically in two fluids whose specific gravities are as 1 : 2, and the pressures of the fluids upon the convex surface are as 2 : 3; find the length of the cylinder immersed in each fluid.

27. A given cylinder filled with fluid revolves round its vertical axis with a given uniform velocity; find what quantity of fluid will escape if a small orifice be made in the centre of its base.

28. Two equal cylinders balance at the extremities of equal 1824
arms of a straight lever, when immersed in fluids, whose densities vary as the depth. The surface of one coincides with that of the fluid, and the depth of the upper surface of the other is equal to n times its altitude. Compare the densities of the fluids at equal depths.

29. A cylinder placed with its axis vertical in a fluid rests with an m^{th} part immersed; when placed in another fluid it rests with an n^{th} part immersed; to what depth would it sink in a mixture composed of equal quantities of these fluids?

30. The altitude of the mercury in a barometer placed in a given cylindrical diving bell is observed at the beginning and end of a descent; find the depth descended.

31. The lower end of a barometer is immersed in a bason of mercury, and the upper is suspended from the extremity of the beam of a common balance; what weight suspended at the other extremity of the beam will keep it in equilibrium?

32. If when the mercury in a true barometer stands at an altitude a, the mercury in an imperfect one of given length stands at the altitude b; what will be the height of the mercury in the true barometer, when it stands at an altitude c in the imperfect one?

33. A rod of given length and weight, and of uniform density, rests with one end in water, and the other on the edge of the vessel which contains it; find the magnitude of the part immersed, and the pressure on the side of the vessel.

34. A wine-glass in the form of a paraboloid is partly filled with water, and then inverted on a table; given the weight of the glass, required the greatest quantity of water that can be contained without running out.

35. A circle whose plane is vertical, is just immersed in a fluid; divide it by a horizontal line into two such parts, that the pressures on them may be equal.

36. Compare the pressures on the upper and lower halves of a hemispherical vessel filled with fluid.

37. A hemispherical vessel rests with its base on a horizontal plane; having given the weight and inner radius of the basin, find the specific gravity of a fluid which when just filling it shall begin to run out at the bottom.

38. A vertical cylindrical tube is connected by a horizontal branch with a cubical vessel of water, and the water is made to ascend in the tube by a condensing syringe applied to the top of the vessel. Having given the dimensions of the vessel, tube, and syringe, and the elevation of the water in the tube and vessel; find the number of descents of the piston.

39. A cylindrical tube is filled with fluid and closed at both ends; compare the pressures on its sides at the earth's surface and at a given altitude above it, supposing the bulk of the fluid from change of temperature to be diminished an n^{th} part, and the axis to be vertical in both positions.

1825 40. Given the radius of the moon and her mass compared with that of the earth, to find the density of the atmosphere at the moon's surface, supposing it to be similar to our own.

41. Shew that a sphere is emptied through a small orifice at the lowest point in less time than any other spherical segment of the same capacity.

42. A flood-gate moves upon a vertical axis, the area on one side of the axis being the quadrant of a circle, and on the other side a parallelogram of the same altitude. Required the width of the parallelogram so that the gate may just open by the pressure of the water when it has risen to the top.

43. Explain the construction of a fire-engine.

44. Compare the resistance upon the surface of a cone moving in a fluid with a given velocity in the direction of its axis, with the resistance upon its base.

45. Find the time in which a vessel formed by the revolution of a cycloid about its axis, placed with its axis vertical and its vertex downwards, will empty itself through a small orifice at the vertex.

46. A solid of revolution, whose axis is perpendicular to the horizon, empties itself through a small given orifice. Required its nature when the velocity of the descending surface is uniform.

47. A paraboloid, generated by the revolution of a parabola, whose equation is $y^n = a^{n-1} x$, placed with its vertex downwards and its axis vertical, empties itself through a small orifice at the vertex; and the value of n is such, that if a sphere empty itself in the same time that the paraboloid does, half the sphere will empty itself in the same time that half the paraboloid does; compare the distance of the descending surface from the vertex when half the paraboloid is emptied with the distance at first.

48. Supposing the force of gravity to vary inversely as the n^{th} power of the distance; by what law does the density of the atmosphere vary?

49. Determine at what angle the wind must strike against the sails of a mill, so that the effect to put them in motion may be the greatest possible.

50. A spherical bubble composed of matter the specific gravity of which is S, and filled with gas of the specific gravity s, just floats in air, specific gravity σ. Required the thickness of the bubble.

51. A cylinder of given altitude has its lower half filled with mercury and the rest with water. Find the time in which it will empty itself through a small orifice in the base.

52. A weight being suddenly removed from the deck of a vessel, the area of which at the surface of the water is given, she is observed to make a small vertical oscillation in t''. Required the weight of the vessel.

53. A rod of given length and uniform density is supported 1826
in a fluid, the density of which varies as the depth, by a string attached to it at a given point; find the position of equilibrium, supposing one extremity of the rod coincident with the surface.

54. Find the resistance to a cycloid moving in a fluid in the direction of its base.

55. Determine the magnitude of a sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

56. Find the time of emptying the frustum of a cone, the radii of whose ends are R and r and altitude h, through a small orifice in its less base.

57. A globe of given weight and magnitude, after descending by the force of gravity a feet in air, passes into another medium whose density is n times as great; required the relations between the space described, the velocity acquired, and the time of motion.

58. A small orifice is made in the side of an upright cylindrical vessel, and the vessel revolves about its axis with a given uniform velocity; find the path traced out by the fluid on the horizontal plane.

59. The velocities of the different parts of a river vary as their distances from the bank; required the path described by a boat moving in a course inclined to the stream at an angle of 45°; also, the velocity at any place, and the time of reaching a point at a given distance from the bank.

60. A parabola with its axis vertical, has its vertex coincident with the surface of a fluid in which it is immersed; divide it by lines parallel to the surface into four parts, so that the pressures upon them may be equal.

61. How far must a given frustum of a sphere be immersed in a fluid, with its axis vertical, that the pressure on its two ends may be equal to n times the pressure on its curve surface?

62. A cone containing a given quantity of fluid, has its axis inclined to the horizon at a given angle; find the time of emptying through a small orifice in its vertex.

63. Find the centre of pressure of the sector of a circle, the axis of the sector being supposed to be vertical.

1827 64. The curve surface of a conical vessel of water, placed with its axis vertical and base uppermost, is composed of an infinite number of triangular sectors, admitting of revolution round hinges at the vertex of the cone, and confined by a string passing round the base; find the tension of the string.

65. A body weighs four ounces *in vacuo*, and if another body which weighs three ounces in water be attached to it, the whole in water weighs two ounces and a quarter; find the specific gravity of the former body.

66. There are two air-pumps, one with a receiver A and barrel B, the other with a receiver B and barrel A; compare the quantities of air exhausted by them in t turns.

67. To what depth will a given paraboloid placed with its axis vertical sink in a fluid of three times the specific gravity of itself.

68. What must be the form of a surface of revolution in which, when filled with water which runs out by a small orifice at the lowest point, the surface descends from its greatest altitude with an uniformly accelerated motion?

69. If a cylinder (weight w), attached by a string passing over a pulley to a weight $= \frac{1}{2} w$, be just immersed vertically in a fluid of the same specific gravity as itself; find the greatest velocity acquired by the cylinder, and the time of its ascending to its greatest height.

70. What part of its bulk at 60° does a body expand for each additional degree of temperature, supposing it to expand .05 parts of the magnitude which is at 32° for each degree above 32°?

71. Find the density of the air and the altitude of the mercury in a barometer at a given depth within the earth; gravity being supposed to vary as the distance from the earth's centre, and the temperature of the air, from the surface to where the barometer stands, to remain constant.

72. A cone of given specific gravity rests in a given fluid with its vertex immersed and axis vertical, shew that the nature of the equilibrium will not be affected by altering the altitude of the cone; and find the vertical angle when the equilibrium is indifferent.

73. How is it shewn that fluids press equally in all directions? 1828
Apply this principle to the explanation of the hydrostatical paradox and *Bramah's* press.

74. Find the density of the air in a common condenser after t descents of the piston.

75. When different planes move in directions perpendicular to their surfaces in different fluids and with different velocities, the resistances will be as the squares of their velocities $\times$ densities of the fluids $\times$ areas of the planes.

76. In a steam engine working expansively, the influx of steam is stopped when it has filled $\frac{1}{m}$ of the cylinder, and the piston is afterwards driven by the expansion of the steam. Compare the effect of a given quantity of steam so employed with its effect when it is not stopped; the effect being measured by the force $\times$ space moved through.

77. The height of a homogeneous atmosphere is the same for whatever distance above the earth's surface we find it.

78. Compare the resistance on the arc of a plane curve moving in a fluid in the direction of its axis, with the resistance on the base; and apply the formula to the case of a semi-circle.

79. A heavy piston descends by its own weight in a close cylinder filled with atmospheric air; find the velocity at any point of its descent, and shew how to approximate to the whole length of the oscillation.

80. The particles of a fluid mass are attracted to two equal constant centres of force, and a uniform motion of rotation is given to the mass about the line joining those centres; find the equation to the surface which the fluid will assume.

81. A given hemispherical vessel filled with fluid is whirled round its vertical axis, so that the surface of the fluid which remains, touches the lowest point of the hemisphere; find the angular velocity and the quantity of fluid remaining.

82. Investigate the general equation of equilibrium of any fluid; and shew from the equation that the resultant of the forces at any point in the surface of a fluid incompressible and perfectly free, is a normal to the surface.

83. Shew how to determine the altitudes of mountains by the barometer, and explain the corrections to be applied in consequence of a change in temperature.

84. Construct the common pump, and find the height which the water rises at each stroke.

85. If a body float on a fluid, determine its stability at a small angle of inclination from a given position of equilibrium; and the time of one of its small oscillations.

86. Divide a cylinder filled with fluid into two such parts, that the times of emptying the fluid contained in each, through a small orifice in the base, may be the same.

87. Explain how fluids press equally in all directions; and 1829
from this shew that in all tubes communicating with each other, a fluid will stand at the same altitude.

88. Shew that when a body floats in a fluid, the weight of the body is equal to that of the fluid displaced, and that their centres of gravity are in the same vertical line; and hence explain the construction and use of the hydrometer.

89. If a given quantity of air be left in the tube of a barometer, find the depression below the standard altitude.

90. A hemisphere with its base downwards is filled with fluid, and inclined at a given angle to the horizon; the base being moveable about a tangent at its upper extremity, find the force which applied at the centre will keep it at rest.

91. If a solid hemisphere rests on its base wholly immersed in a fluid of less specific gravity than itself, whilst the fluid flows horizontally against it with a given velocity, find the whole pressure of the hemisphere against the bottom of the vessel, the solid being kept at rest by the friction.

92. Prove that the pressure upon any portion of a vessel filled with a fluid of uniform density is equal to the weight of a column of fluid whose base is the area of the surface pressed, and altitude the perpendicular depth of its centre of gravity below the surface of the fluid; and find the whole pressure on the surface of a spherical segment filled with fluid.

93. If a floating body revolve round a horizontal axis, and so pass through all its positions of equilibrium, they will be alternately stable and unstable.

94. If the distances above the surface of the earth increase in arithmetical progression, the corresponding densities of the air will decrease in geometrical progression; the force of gravity being invariable.

95. A hemispherical vessel filled with fluid revolves round its vertical diameter with such an angular velocity that half the fluid is thrown out; required the pressure on the surface.

96. What is the least depth of fluid, in which a given cone can rest permanently with its axis vertical, the vertex of the cone resting on the base of the vessel, and the specific gravities of the cone and fluid being given.

97. In a common pump find the height through which the water ascends at any stroke.

98. Find the specific gravity of a body lighter than the fluid in which it is weighed.

99. Define the metacentre of any floating body, and investigate a formula for its position.

100. Explain the construction of the fire-engine, and shew the use of the air-vessel.

101. If an incompressible fluid contained in any vessel flow through an orifice k with a velocity u, z' being the vertical coordinate of the upper surface, and y' its area, z, y the same quantities for any other horizontal section, p the pressure on a unit of this section,

$$p = \Pi + g(z - z') - k\frac{du}{dt}\int\frac{dz}{y} - \frac{k^2u^2}{2}\left(\frac{1}{y^2} - \frac{1}{y'^2}\right),$$

where Π is the atmospheric pressure on the surface. Also, from this expression find the velocity of the issuing fluid when the orifice is small.

102. Construct the air-pump, and shew that the quantities of air expelled by successive strokes are in geometrical progression.

103 Define specific gravity; what is the relation between the weight, volume, and specific gravity of any substance? Explain the meaning of the numbers given in tables of specific gravity. 1830

104. Describe *Nicholson's* hydrometer, and shew how the specific gravity of a fluid may be found by it.

105. Explain the experiment by which it is ascertained that the density of air is proportional to the force which compresses it.

106. Describe the common pump, and find the least play of the piston which will enable the pump to work, the lower valve being at the surface of the water.

107. Water issues from the horizontal surface of a fountain, at an angle a, with a velocity due to h, through a circular annulus of which the radius is r; V is the volume contained by the surface of the fountain, the ascending and the descending stream; and V′ by the surface, the ascending stream, and a plane touching it at the highest point; prove that $\frac{V}{V'}$ is constant when $\sin 2a \propto \frac{r}{h}$.

108. Define the centre of pressure, and find it in the case of a semi-parabola immersed in a fluid with its base contiguous to the surface.

109. Determine all the positions of equilibrium of an equilateral triangle floating on a fluid with one angle immersed.

110. The compressing force of the air varying as the density, and the force of gravity varying inversely as the square of the distance from the centre of the earth, find the relation between the density of the air at any altitude and the density at the earth's surface.

111. The axis of a given cone filled with fluid is inclined at a given angle to the horizon; find how much of the fluid will flow out, and determine the pressure exercised by the remainder upon the conical surface.

112 Explain the method of determining altitudes above the earth's surface by the barometer, account being taken of the variation of temperature.

113. Find the specific gravity of a body lighter than the fluid in which it is weighed.

114. Find the conditions of the stable and unstable equilibrium of a floating body; and if it revolve about a horizontal axis, shew that it passes alternately through positions of stable and unstable equilibrium.

115. Having given the quantity of air in the tube of a barometer, determine the depression below the standard altitude.

1831 116. If given volumes of two fluids of known specific gravities be mixed together, find the specific gravity of the compound, supposing its volume to be equal to the sum of the volumes of the parts.

117. If the particles of an elastic fluid repel each other with forces varying inversely as the n^{th} power of their distances; prove that the compressing force varies as $(\text{density})^{\frac{n+2}{3}}$.

118. Find the density of air in the air-pump after n turns.

119. Determine the pressure exercised by an incompressible fluid of uniform density against the surface of the vessel containing it.

120. If A and a be the areas of horizontal sections of a waterfall at heights H and h above the horizontal, find the height of the fall, the $(\text{velocity})^2$ of the water at any point being as the height from which it has fallen.

121. Find the times of emptying a given prismatic vessel filled with water, by a cycloidal syphon of small bore placed with its base horizontal; the vertex of the syphon resting on the edge of the vessel.

122. Water retained at a constant elevation, issues from one vessel into another through a cylindrical pipe of radius r, and thence into the air through another cylindrical pipe of radius r'; having given the depth H of the mouth of the latter below the constant surface, find the velocity and pressure at a given depth h in the other pipe.

123. If the particles of a fluid mass, revolving with a given angular velocity, be attracted to a fixed centre by a force,

which is any function of the distance from the centre, the ellipticity, supposed small, will be half the ratio of the centrifugal force to the attraction at the equator.

124. Find the general equation of the equilibrium of incompressible fluids. If the fluid be heterogeneous, shew that all surfaces of equal pressure are of uniform density.

125. Let any particle of a mass of incompressible fluid, the coordinates of which at any time t are x, y, z, be acted upon by the forces X, Y, Z, in the directions of the axes of coordinates; and let u, v, w, be its velocities in the same directions; then if $udx + vdy + wdz = d\phi$, a complete differential, the pressure and motion are determined by the equations

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

$$p = \int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dx) - \frac{d\phi}{dt} - \frac{1}{2}\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right).$$

126. If the force of gravity be considered constant, and altitudes from the earth's surface be taken in arithmetical progression, the corresponding densities of the air will decrease in geometrical progression.

127. Explain *Bramah's* hydrostatic press. What is the limit to its practical application?

128. When a body floats on a fluid, the weight of the fluid 1832
displaced is equal to the weight of the body; and the centres of gravity of the body and the fluid displaced are in the same vertical line.

129. There is a constant east wind under the equator, and a constant north-east wind near it on the north side, and a constant south-east wind near it on the south side; explain the causes of these phenomena.

130. Describe the barometer, and mention the purposes to which it is applied. Also calculate the correction to be made in any observed altitude of the mercury when the change of level of the mercury in the basin is taken into account.

131. There are three bodies of equal bulk, but of different given specific gravities; it is required to suspend them from

three points in a straight lever, so that in whatever fluid they be wholly immersed, they may balance about its fulcrum.

132. A cubical vessel is filled with fluid and covered with a lid moveable about one of its edges as a hinge; if the vessel be moved into such a position that three edges meeting in one point are inclined at given angles to the vertical, determine the force which must be applied at a given point of the lid to keep it at rest.

133. Shew that the vertical pressure against the sides and bottom of any vessel is the weight of the fluid contained in it. How does it appear that the horizontal pressure will have no tendency to communicate any lateral motion to the vessel?

134. If the pressure of the air vary as the density, and the force of gravity vary as $\frac{1}{(\text{dist.})^2}$, find the density at a given height; and shew that it is finite at an infinite height.

135. If a body be let fall into a fluid of the same specific gravity as itself, it will sink to the bottom whatever be the depth of the fluid, the resistance varying as the square of the velocity.

136. Find the velocity of issuing through a small orifice in a vessel in which the water is retained at a given altitude.

137. State the sole condition necessary for the equilibrium of any fluid homogeneous or heterogeneous; and prove that when it is fulfilled, the force at every point of the external surface is perpendicular to the surface.

138. A fluid mass revolves round an axis, and is attracted towards a point in that axis by a force varying as the distance, required its form; and having given the mass of the fluid, determine the axes.

139. A body weighs (a) lbs. in vacuo, and (b) lbs. in water; another body weighs (a') lbs. in vacuo, and (b') lbs. in water; compare the specific gravities of the two bodies.

1833 140. Find the pressure on a plane immersed in a heavy fluid.

141. Explain the process of filling the tube of a mercurial

thermometer, and find what degree on *Reaumur's* thermometer will correspond to $(n)^\circ$ on *Fahrenheit's.*

142. Shew how to determine the specific gravities of bodies insoluble in water, by their weights in air, and in water.

143. From the principle that when a mass of fluid is in equilibrium, the state of rest is not altered by supposing any portion of the mass to become solid, deduce the equality of the pressure of fluid in all directions.

144. Explain the action of *Watts's* steam engine. On what principle is the efficiency of the engine, when employed in raising water, determined from the work done? What is the distinction between *high pressure* and *low pressure* engines, and in what cases are the former requisite?

145. The axis of a cylindrical vessel containing a known quantity of fluid is inclined at a given angle to the horizon; determine the centre of pressure of its base.

146. Between two planes which meet the horizontal plane in the same line, and each of which is inclined to it at an angle whose tangent is $\frac{3}{2}$, a hollow sphere first filled with a homogeneous fluid is placed, shew that the hemispheres on each side of a plane through their intersection may be supposed disunited without disturbing the equilibrium.

147. When a sphere acted on by gravity falls from rest in a medium resisting as the square of the velocity, find the velocity and resistance in terms of the space described. Hence explain the fall of the barometer in rainy weather.

148. State the experiments by which it appears that in all cases the pressure of air at a given temperature varies inversely as the space occupied.

149. Find the resultant of the pressure of a fluid on the surface of a solid, wholly or partly immersed in it. When the body floats in equilibrium, compare its volume with the volume of the part immersed. 1834

150. Having given the form of the diving bell, find the space occupied by the air in it at any depth below the surface. If the bell be prismatic, solve the resulting equation, and explain the roots.

151. Compare the specific gravities of a solid and a fluid, by weighing the solid in air and in the fluid.

152. How is the pressure at a given point of a mass of fluid measured? Prove that, when an elastic fluid of uniform temperature acted on by gravity is at rest, its density is the same at all points in the same horizontal plane. What considerations lead us to the conclusion that the height of our atmosphere is finite?

153. Describe *Smeaton's* air-pump; find the density of the air in the receiver after any number of strokes of the piston, and explain a contrivance by which it may be practically measured.

154. A barometer, consisting of two equal tubes connected by another of larger section into the ends of which they are inserted, has a quantity of lighter fluid above the mercury; shew the greater sensibility of this instrument, by comparing the variation of altitude of the compound column, owing to a given change of atmospheric pressure, with that of the mercurial column in a common barometer, the common surface of the two fluids always falling within the larger tube.

155. A solid is generated by the revolution, through a given angle, of a right-angled triangle about one of the sides containing the right angle; determine the moment of a couple which, acting in the plane of symmetry, will support the solid with its axis vertical in a given fluid.

156. A given quantity of incompressible fluid, each particle of which is attracted towards a fixed centre by a force which varies as the distance, is separated from it by a given fixed plane; determine the pressure which the fluid exerts upon the plane.

157. Describe and explain the hydraulic ram. Account fully for the downward motion of the valve through which the waste takes place.

158. Find the relation between the density, temperature, and pressure of an elastic fluid, referring to the necessary experiments and assuming the results.

159. The attraction of the particles of a fluid to each other and to glass being supposed sensible only at very small distances, find the relation between the diameter of a capillary tube, and the height to which the fluid will rise.

160. When a fluid moves, acted on by any forces, determine the effective accelerating force in the direction of its motion at any point. Employ the result to find the velocity with which an incompressible fluid issues through an indefinitely small orifice in a vessel containing it.

161. Define the centre of pressure of a plane surface immersed in a fluid. Why is its position independent of the inclination, and lower than the centre of gravity of the surface?

162. What is the nature of the transmission of fluid pres- 1835
sure? Apply the principle to explain *Bramah's* press.

163. Compare the specific gravities of a solid and fluid, the specific gravity of the solid being less than that of the fluid.

164. Having given the graduations of two thermometers for freezing and boiling water, determine the graduation of one, when the other marks t^0.

165. Find the magnitude and direction of the resultant of the pressure of a fluid on the surface of a solid immersed in it.

166. Describe the common barometer, and find the correction to be applied when a small quantity of air remains in the upper part of the tube. What other circumstances affect the accuracy of the apparent altitude?

167. Compare the duties of an atmospheric steam engine for one oscillation of the piston, (1) when the cylinder as usual is open at the top, (2) when hermetically closed.

168. The pressure on a quarter of the surface of a hemispheroidal bowl filled with fluid (bounded by two planes passing through the axis c which is vertical) will be equivalent to a single force acting in the straight line whose equations are

$$x = y = \frac{cz}{\pi a} + \frac{3}{16} \cdot \frac{a^2 - c^2}{a}.$$

169. Assuming the differential expression for the pressure

at any point of a fluid mass in equilibrium when acted on by known forces, shew that the resultant of the forces at a point in the surface is in the direction of the normal; and that the equilibrium is possible when the forces arise from the attraction to a fixed centre, or to every particle of a solid or fluid mass.

170. Find the centre of pressure of the area of a quadrant of a circle, one side of the quadrant coinciding with the surface of the fluid.

1836 171. In what respect does the effect produced by the application of pressure to a fluid differ from that produced by the application of the same pressure to a solid body? Explain what is termed the hydrostatical paradox.

172. Investigate the conditions to be fulfilled that a body may float in equilibrium on a fluid.

173. Explain the principle and action of the syphon; and thence the phenomena of tide-wells or reciprocating springs.

174. Compare the specific gravities of two fluids by weighing them in the same vessel.

175. Describe the construction of the common mercurial thermometer. Why can it not be depended on for temperatures above 212°? What methods are used to measure intense heat and cold? Explain the construction and use of the register thermometer.

176. Investigate the position of the metacentre of a floating vessel, partly filled with fluid. What effect is produced upon the stability of a ship by throwing out the ballast and leaking an equal weight of sea water?

177. A cubical vessel, full of fluid, is held in a given position, compare the pressures on its several sides.

178. Determine the velocity of the wind, blowing horizontally, when it is just able to overturn a given cylinder standing on a horizontal plane, and prevented from sliding.

179. The times of emptying a segment of a sphere through equal small orifices in its vertex and base are as $1 : n$, the base being horizontal in both cases; compare the volume of the segment with that of the whole sphere.

180. When a fluid of uniform density is acted on by any forces, state the conditions under which there will be equilibrium; and shew that they are satisfied, if the forces tend to fixed centres and are functions of the respective distances. When the fluid is elastic, shew how the pressure and density at any point are determined. Why cannot the atmosphere of the earth continue in equilibrium?

181. Determine the velocity with which fluid issues through a small orifice in a vessel kept constantly full.

182. Determine the difference of altitude of two stations by means of a barometer and thermometer, the force of gravity being constant.

183. Describe the diving bell, and find the tension of the rope by which it is suspended.

THEORY OF SOUND.

1. Prove that a column of air in a cylindrical pipe of length 1831
l, closed at one end, may be made to vibrate so that at distances $2\lambda, 4\lambda \ldots . 2k\lambda$ from the closed end, the air will be stationary, and at distances $\lambda, 3\lambda, 5\lambda, \ldots . (2k+1)\lambda$, the last of which $= l$, the density will be constant.

2. Prove that the vibrations propagated from any point of disturbance in a cylindrical column of air, are such that the velocity of the particles is proportional to the condensation, and the condensed particles move in the direction of propagation, the rarefied in the contrary direction.

3. Determine the velocity of sound, and shew that it is 1832
independent of its loudness.

How is the difference between the theoretical and experimental determinations accounted for?

4. In consequence of a small disturbance communicated to the particles of an elastic fluid mass at rest, a point whose coordinates are x, y, z is transferred to a position whose coordinates are $x + \lambda u$, $y + \lambda v$, $z + \lambda w$, where λ is a small

constant positive quantity; prove that the density at that point is increased or diminished according as

$$\left(\frac{du}{dx}\right) + \left(\frac{dv}{dy}\right) + \left(\frac{dw}{dz}\right) \text{ is } - \text{ or } +.$$

1834 5. Explain the phenomenon of musical beats and the formation of harmonics. If the original notes make m and n vibrations in a given time, find the number of vibrations of the resultant note.

1835 6. The error arising in the computation of the velocity of sound in any gas from neglecting the change of temperature, is corrected by the introduction of a factor $\sqrt{K}$, explain the physical meaning of K; and shew that if its value for atmospheric air be known, its corresponding value for any known gas may be found by comparing with a monochord the fundamental notes in two equal tubes of the air and gas under the same pressure and at the same temperature.

1836 7. Investigate the condition for the single propagation of a pulse of sound in a thin straight tube filled with uniform elastic fluid. If the pulse meet a second medium, shew that it will be divided into two, running in opposite directions.

SECTION XIX.

QUESTIONS IN OPTICS AND THE THEORY OF LIGHT.

1. Shew that the image of a straight line placed between the centre and principal focus of a concave mirror is a hyperbola. 1821

2. Shew how to find the focal length of a double concave lens by experiment.

3. If the air near the surface of the ground be less dense than at small altitudes above it, there will be observed inverted images of distant horizontal objects.

4. An object appears brighter, *cæteris paribus,* when seen through a convex lens, than when seen through a concave lens.

5. Prove that objects appear erect in *Galileo's* telescope.

6. By what experiments is it proved, that light consists of rays differing in colour and refrangibility?

7. Shew that the image of an indefinite straight line perpendicular to the axis of a convex lens, and nearer to its centre than the principal focus of parallel rays incident in the opposite direction, forms the arcs of two opposite hyperbolas; and find the semi-axes.

8. A person wishes to see distinctly when under water. What kind of glasses must he use, and of what focal length? 1822

9. A hollow cylinder is viewed by an eye placed in its axis produced. Compare its apparent capacity when empty and when filled with water.

10. A ray of light issuing from a point in the extreme ordinate of a parabola is incident in a direction parallel to the axis, and after two reflections at the curve meets the ordinate again; prove that the length of the path described will be the same from whatever point in the ordinate the ray proceeds.

11. A ray of homogeneal light is refracted through a double convex lens; compare the densities in the different parts of the circle of chromatic dispersion.

12. Find the point at which a ray of light parallel to the axis must be incident upon a concave spherical reflector, that after two reflections it may cut the axis in a given angle.

13. A sphere of glass and another of water being placed in air, what must be the proportion of their radii, that their magnifying powers may be the same.

1823 14. A ray of light issues from the extremity of the diameter of a semi-circle, and is reflected by the circumference; determine the point of incidence, so that after reflection the ray may pass through a given point in the diameter produced.

15. Where must a ray of light parallel to the axis of a concave spherical reflector be incident, that after reflection it may divide the radius in the ratio of $\sqrt{3} - 1 : 1$?

16. If a ray of light refracted into a sphere, emerge from it after any given number of reflections, determine the distance of the incident ray from the axis, when the arc of the circle intercepted between the axis and the point of emergence is a minimum.

17. Find the equation of the caustic of the parabola, when the rays are incident perpendicular to the axis. Trace the caustic, and find the angles at which it cuts the axis, its maximum ordinate, its area, and its length.

18. If a ray of light QACS be refracted through a prism IKL in a plane perpendicular to its axis, and if the vertical

angle KIL $= \alpha$, QAK $= \theta$, ACL $= \phi$, and the whole deviation of the ray $= \delta$, then will

$$\tan\left(\phi - \frac{\alpha}{2}\right) = \frac{\tan\left(\theta + \frac{\delta + \alpha}{2}\right)\tan\frac{\delta + \alpha}{2}}{\tan\frac{\alpha}{2}}.$$

19. If f, f', f'' . . . . be the focal lengths of any number of contiguous lenses, the thickness of each being very small, and F be the focal length of the compound lens, then

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} + \frac{1}{f''} + \ldots .$$

20. If the focus of incident rays be placed any where in the axis of an elliptic reflector, find the geometrical focus of reflected rays.

21. If A, A′ denote the intensities of two lights, and n, n' the number of laminæ of any substance, through which when the lights are viewed they appear equally obscure; also the light lost in passing through each lamina being supposed to be $\frac{1}{m^{th}}$ of that which entered it; then will

$$A : A' :: \left(\frac{m-1}{m}\right)^{n'} : \left(\frac{m-1}{m}\right)^{n}.$$

22. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third, and reflected successively by each of them; it is required to shew that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

23. Investigate the nature of the surface, such that if two lights of given intensities be placed in two given points, every point in it may be proportionally illuminated by each.

24. If a prism be turned round its axis which is perpendicular to the incident rays of the sun, till the spectrum neither ascend nor descend, shew that the refractions at the points of incidence and emergence are equal.

25. A ray of light proceeding from a moveable point is 1824
reflected by a fixed plane mirror to a given point; find the locus of the first point when the path of the ray is of given length.

26. Define the centre of a lens, and prove it to be a fixed point.

27. If a prism be laid on its base in the open air, and the eye be placed in a proper position; the base will appear to be divided into two parts, the one much brighter than the other, and separated from one another by a coloured bow, concave towards the eye. Give *Newton's* explanation of the cause of this phenomenon.

28. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness $(c) = a$; then if the radius of the second surface $= a + \frac{13c}{9}$, the image of a distant object formed by reflection at the first surface will coincide with the image formed by reflection at the second surface, and by refraction at the first.

29. Describe the experiment by which *Newton* shewed that the more refrangible rays are the more reflexible.

30. In a given latitude, on a given day of the year, determine the time during which a primary, and secondary rainbow may be seen.

31. The image of a circular arc, concentric with the reflector, subtends the same angle as the object both from the surface and the centre.

32. If (i) be the angle of incidence of a ray passing through a prism in a plane perpendicular to its axis, (e) the angle of emergence, (a) the vertical angle of the prism, and $1 : n ::$ sin I : sin R out of the ambient medium into the prism, then

$$\sin e = \frac{1}{n} \sin a \sqrt{(1 - n^2 \sin^2 i)} - \cos a \sin i.$$

33. If an object be placed between two plane reflectors inclined at any angle; find the locus of an eye such that the length of the ray by which any given image is seen may be equal to a given line.

34. Q is the focus of incidence of a pencil of rays which passes nearly perpendicularly through the sides of a prism whose vertical angle A is small, q the focus of emergent rays,

and QC, qc perpendicular to the first and second surfaces respectively. Having given the ratio of sin I : sin R out of the ambient medium into the prism, and also QC, CA and the angle A, find qc and Ac.

35. Trace the image of an indefinite straight line in contact with a spherical reflector. 1825

36. Shew how the polar equation to caustics formed by reflection may be found generally; and apply the method when the reflecting curve is a circular arc, and the radiating point in the circumference of the circle.

37. Having given the refracting powers of two mediums, find the ratio of the focal lengths of a convex and concave lens formed of these substances, which when united produce images nearly free from colour.

38. Find the geometrical focus of a small pencil of rays diverging from a given point in the axis and incident nearly perpendicularly on an elliptic reflector.

39. State what advantage is obtained by substituting a glass rectangular prism for a plane reflector, in the construction of *Newton's* telescope; and trace the course of a pencil of rays on that supposition.

40. Find the focal length of a double convex glass lens of inconsiderable thickness when the radii of the surfaces are equal, and sin I : sin R :: 3 : 2.

41. Two straight rods inclined to each other at a given angle are immersed vertically in a fluid in a given position; find the angle formed by their images. 1826

42. Find the principal focus of a reflector generated by the revolution of a cycloid about its axis, and determine the relation between the distances of the conjugate foci from the vertex.

43. A pencil of parallel rays incident upon a transparent medium bounded by parallel plane surfaces, is partly reflected at the upper surface, partly refracted by the medium, and afterwards reflected at the lower surface; prove that all the rays of the emergent pencil are parallel, and find the angle of incidence that its breadth may be the greatest possible.

44. Prove that the image of a straight line formed by rays refracted through a sphere is a conic section; and find the magnitudes and positions of its axes and the latus rectum.

45. An eye is situated in a given point of the straight line joining the centres of two given spheres, but not between them; find the visible surface of each.

46. Two rods, of given lengths, are erected perpendicular to a given plane; find the locus of an eye in that plane, to which the sum of their apparent magnitudes will always be the same.

1827 47. A ray of light which passes through two media, bounded by parallel plane surfaces, will emerge parallel to its first direction, if the deviation in passing out of one medium into another under a given angle of incidence be supposed proportional to the difference of the densities of the media.

48. Prove that an object seen through the astronomical telescope appears inverted, but may be made to appear erect by two additional glasses, and find the magnifying power of the telescope thus formed.

49. At what distance from a luminous sphere must a point be situated so as to receive the greatest quantity of light from it?

50. Prove that a spheroidal mirror may be made to reflect diverging rays accurately to one point; and by the help of this proposition find the common formula for a spherical mirror.

51. How must the equidistant seats of a lecture-room be constructed, so that persons of equal height sitting on them, and directing their eyes to the same given point, may be equally able to see over each other's heads?

52. Explain the method of finding the refracting power of a soft substance by placing it between glass lenses.

53. A small pencil of rays, parallel to the axis of a hemisphere of denser medium, is incident nearly perpendicularly on the convex surface; find the geometrical focus of emergent rays.

54. Prove that the image of a straight line, seen through a prism, the angle of which is small, is a straight line; and

compare the angle between the object and image with the angle of the prism when the object is parallel to one side of the prism, and in a plane which is perpendicular to the two sides.

55. Explain what is meant by the polarization of light, and prove experimentally that it may be produced by reflexion from transparent media.

56. If parallel rays be incident nearly perpendicularly upon 1828
a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface, is to its distance from the centre, as the sine of incidence to the sine of refraction.

57. Prove that the image of a straight line formed by a plane refracting surface is a straight line, and having given the inclination of the line to the refracting surface, find the inclination of the image.

58. Construct the solar microscope.

59. Construct *Gregory's* telescope; find its magnifying power and its greatest field of view.

60. Explain the theory of the aberration of light, and define clearly the plane in which it takes place.

61. A telescope consists of three convex lenses whose focal lengths are $30m$, $3m$, m, the two latter being at a distance $2m$. Find its magnifying power, and the distance of the first two lenses. Trace the course of the rays.

62. When a ray of light passes through a prism in a plane perpendicular to its axis, the deviation is a minimum when the incident and emergent rays make equal angles with the sides.

63. Find the field of view in *Galileo's* telescope.

64. Find the longitudinal aberration of parallel rays refracted by a spherical surface.

65. Why do objects appear further off and smaller, when viewed through the wrong end of a telescope?

66. If the object placed before a spherical reflector be a straight line, the image will be a conic section. Prove this,

and shew how the different parts of the image are formed, when the object is placed between the principal focus and the surface of the reflector.

67. Required the geometrical focus of a thin pencil of rays after being refracted at a curved surface.

68. If diverging rays fall upon a concave spherical surface of a rarer medium, to find the geometrical focus of refracted rays.

69. If a small pencil of parallel homogeneal rays be refracted into a sphere, and the ratio of the sine of incidence to the sine of refraction be known, to find at what angle the rays must be incident, that they may emerge parallel after any given number of reflections within the sphere.

1829 70. If Q and q be the conjugate foci of rays incident nearly perpendicular on a spherical reflector; E the centre and F the principal focus,

$$FE^2 = FQ \, . \, Fq.$$

71. A convex spherical refracting surface of a denser, and a concave of a rarer medium, diminish the divergency and increase the convergency of all pencils of rays incident nearly perpendicularly, unless the focus of incident rays be between the surface and centre of the refractor.

72. Construct *Gregory's* telescope. Draw accurately the course of the extreme ray. Express the magnifying power in terms of the focal lengths and the distance between the principal foci of the reflectors.

73. If the object placed before a lens, or a spherical reflecting or refracting surface, be a conic section, of which the focus and axis coincide respectively with the centre and axis of the reflector or refractor; the image will be a conic section. Prove it in each case, and find the eccentricity and axes of the image.

74. Find the whole longitudinal aberration when a pencil of parallel rays passes through a plano-convex lens of inconsiderable thickness, the rays being incident on the convex side in a direction parallel to the axis of the lens.

75. Explain the formation of the primary and secondary rainbows, and why the red arc is exterior in the former case and interior in the latter.

76. Converging rays are incident upon a concave spherical reflector; find the longitudinal and lateral aberrations.

77. All objects from the zenith to the horizon are visible to an eye under water, and appear to be bounded by a conical surface of which the eye is the vertex. Explain this, and having given the refracting power of water, find the vertical angle of the conical surface. State also what is seen beyond the limits of the conical surface.

78. If rays are incident upon a common looking-glass in an oblique direction from a candle, one faint image is observed before the principal image, and a row of them behind it. Explain this; and find the caustic formed by the rays emergent from the glass after reflection at the quicksilver, the thickness of the glass and its *refracting* power being given.

79. The shadow cast by an oblate spheroid resting on its vertex in the sun, on a horizontal plane, is an ellipse; and the spheroid stands in its focus.

80. If a small pencil of parallel homogeneal rays be refracted into a sphere, find at what angle the rays must be incident that they may emerge parallel after any given number of reflections within the sphere.

81. Find the caustic when the reflecting curve is a logarithmic spiral, and the luminous point in the pole.

82. A person can see distinctly at the distance of four inches; find the focal length and nature of a lens which will enable him to see distinctly at the distance of sixteen inches.

83. Having given the position of an object placed between two plane mirrors inclined to each other at a given angle, find the number and positions of the images. 1830

84. Diverging rays are incident nearly perpendicularly upon a given spherical refracting surface; having given the focus of incidence, find the focus after refraction, and prove that the conjugate foci move in the same direction upon the axis of the surface.

85. Explain the construction of the divided object glass micrometer, and shew its use.

86. A ray of light is refracted through a prism, the angle of which is 60° and index of refraction $\sqrt{2}$, so as to undergo the least possible deviation; determine that deviation. Shew also that no ray can be directly transmitted through a prism of the same refracting power when the angle exceeds 90°.

87. A given opaque sphere and a given luminous paraboloid of revolution have their axes in the same line; the distance between them being known, deduce the equation to the surface of the shadow, and find the form of the shadow thrown on a given plane.

88. Determine the form of a surface which shall refract a pencil of rays proceeding from a given point accurately to another given point.

89. Explain the formation of the primary rainbow, and shew that the breadth of the bow = radius of the red arc − radius of the violet arc + the apparent diameter of the sun's disk.

90. Construct the astronomical telescope, and having given the focal lengths of the object and eye-glasses, find the position of the eye when the field of view is the greatest.

91. Prove that the eye cannot be achromatic for objects at all distances.

92. Find experimentally the refracting power of any transparent substance.

93. Determine the centre and diameter of the least circle of chromatic aberration, in a given lens.

94. Prove that the image of a straight line placed before a spherical reflector is a conic section, and determine in what cases it is each particular conic section.

95. When a small pencil of diverging rays is incident obliquely on a concave refracting surface, the ultimate intersection of two refracted rays in a normal plane is determined from the equation

$$\frac{(\cos\phi_1)^2}{v}\frac{1}{v} - \frac{(\cos\phi)^2}{\mu}\frac{1}{u} = \left(\cos\phi_1 - \frac{\cos\phi}{\mu}\right)\frac{1}{r}.$$

96. Having given the radius of the arc of any colour in the primary rainbow, find the index of refraction, for that colour, out of air into water.

97. Rays fall upon a plane reflecting surface; prove that the foci of incident and reflected rays are on opposite sides of the reflector, and equidistant from it. 1831

98. A small pencil of rays diverging from a point in the axis of a spherical reflector is incident nearly perpendicularly upon it; find the focus of reflected rays. When the foci are on opposite sides of the reflector, the divergence of the reflected is less than that of the incident pencil by a constant quantity.

99. Find the principal focus of a double convex lens, the thickness of which is inconsiderable.

100. When a very small conical pencil of rays is reflected at a spherical surface, the transverse section of the reflected pencil will be a circle, at a distance from the point of reflection, which is an harmonic mean between the distances of the focal lines in the primary and secondary planes from the same point.

101. If any number of rays be incident parallel to the axis on the surface of an elliptic paraboloid, the reflected rays will all pass through each of two parabolas lying in the principal planes of the paraboloid.

102. Find the angle at which a small pencil of parallel rays must be incident on a sphere of water, that the rays after one reflexion within the sphere may emerge parallel in the plane of refraction.

103. The distance of the least circle of aberration from the approximate focus, is three-fourths of the longitudinal aberration of the extreme ray, and its diameter is half the lateral aberration.

104. Construct the astronomical telescope, and find its magnifying power.

105. A small pencil of diverging rays is incident obliquely on a reflecting surface at its centre, and the section of the surface in the primary plane is a circle of given curvature; find the curvature of the section in the secondary plane, that the foci of rays reflected in the two planes may coincide.

106. A reflector is formed of an indefinite number of faces, which are tangents to a parabola, and rays diverge from a given point in the axis of the parabola; find the locus of the images produced by reflection.

107. A small pencil of rays being refracted obliquely at a convex spherical surface, determine the foci in the primary and secondary planes.

108. Determine the aberration of a pencil of parallel rays refracted directly through a double convex lens; and if the index of refraction be $\frac{3}{2}$, find the ratio of the radii when the aberration is a minimum.

109. Determine the relation between the focal length of two lenses which shall achromatize each other when separated by a given interval.

110. A ray of light, which passes through a prism in a plane perpendicular to the axis, is turned towards the thicker part of the prism, if it be denser than the surrounding medium.

111. If a luminous point be placed between two plane mirrors inclined to each other, prove that there will be a series of images which lie in the circumference of a circle: determine also the number of images.

1832 112. How does it appear that light emanates in straight lines? and how has its velocity been determined?

113. When a small pencil of diverging rays is incident on a concave spherical reflector, prove that FQ : FA :: FA : Fq. Shew that the foci Q and q are on the same side of F, and move in opposite directions.

114. Describe the astronomical telescope in its simplest form, and determine its magnifying power.

What are the obstacles to the perfection of refracting telescopes? and what means have been taken to obviate them?

115. If a small cylindrical pencil of rays fall obliquely on the centre of a spherical mirror, the lengths of the focal lines in the primary and secondary planes are to one another as $1 : \cos^3\phi$, where ϕ is the inclination of the axis of the pencil

to the axis of the mirror. Also the diameter of the *circle of least confusion* is $\frac{\lambda(1-\cos^3\phi)}{1+\cos^3\phi}$, where λ is the diameter of the circular section of the incident pencil.

116. In the astronomical telescope state fully the defects of the simple eye-glass. How are they remedied by the achromatic eye-piece?

117. A small pencil of rays not parallel is incident obliquely on a plane refracting surface; determine the foci in the primary and secondary planes.

118. Explain the formation of the image of a portion of a straight line placed before a convex spherical reflector, compare the linear magnitudes of the object and image, and draw the course of the extreme rays from the object to an eye in a given position.

119. If a straight rod be immersed in water in a horizontal position, shew that it will appear to be lengthened; and if it be immersed in a vertical position, that it will appear to be shortened.

120. A slender rod partly immersed in water contained in a vessel, casts a shadow on the bottom of the vessel from a light placed near it; it is observed that the shadow of the part of the rod immersed is separated from that of the other part by a bright interval; shew that this is a consequence of capillary attraction.

121. At a given hour on a given day find the inclination of a plane reflector to the horizon and meridian, that it may reflect the sun's rays in a given horizontal direction.

122. A small pencil of parallel rays falls directly on a double concave glass lens of equal radii, and after refraction at the first surface is reflected at the second; shew that it will diverge from a point at a distance from the surface equal to $\frac{3}{7}$ of the radius, neglecting the thickness of the lens.

123. A pencil of diverging rays is incident directly on a double convex lens; find the focus of refracted rays to the second approximation.

124. Determine the exterior colour of the primary rainbow, and shew that the order of colours in the secondary bow is the reverse of that in the primary.

125. The deviation of a ray passing through a prism is a minimum when the angles of incidence and emergence are equal. Shew how this proposition is used to find the refractive power of any substance.

126. A mixed pencil of parallel rays passes eccentrically through two lenses of given focal lengths; determine the interval between the lenses that the directions of all the partial emergent pencils may be parallel.

127. A pencil of parallel rays passes directly through a lens; determine the diameter of the least circle of chromatic dispersion.

128. Explain the method by which *Newton* proved the sun's light to consist of rays which differ in refrangibility and colour; and that each ray of the solar spectrum, when once separated by refraction, does not admit of farther separation.

129. Find the focal length of a lens of inconsiderable thickness; and shew that a short-sighted person requires concave glasses, and a long-sighted person convex ones.

1833 130. A small pencil of rays is incident nearly perpendicularly on a convex spherical mirror, determine the geometrical focus of reflected rays; and shew that the increase of divergency, or decrease of convergency, is the same for all pencils.

131. Describe the eye; why do objects viewed at near distances appear enlarged and distorted?

132. Parallel rays refracted at a plane surface continue parallel.

133. Define a ray of light, and state the experiment which proves that if the refracted ray becomes the incident, the incident becomes the refracted ray.

134. Describe the astronomical telescope in its simplest form, tracing the course of the extreme rays. How is a uniformity of brightness secured in the field of view? What are the additions requisite to make the telescope a good one?

135. When a small pencil of diverging rays is incident obliquely on a plane refracting surface, determine the foci after refraction in the primary and secondary planes; and shew that

after refraction through a prism, these foci will coincide, if the angles of incidence and emergence are equal.

136. A given small luminous object is placed at a given point in the line passing through the centre and perpendicular to the plane of a circular area. Compare the illumination of the whole area with that which it would receive if the light thrown on it was uniformly the same as that at the centre.

137. Parallel rays are incident upon a cylinder in a direction perpendicular to its axis; shew that the equation to a section of the caustic surface is

$$(\mu^2 - 1)\, y = \{(\mu^2 a)^{\frac{2}{3}} - x^{\frac{2}{3}}\}^{\frac{3}{2}} + \{a^{\frac{2}{3}} - (\mu^2 x)^{\frac{2}{3}}\}^{\frac{3}{2}},$$

the centre of the corresponding section of the cylinder, whose radius is a, being the origin, and its diameter, perpendicular to the incident rays, the axis of x, and μ the index of refraction.

138. A small plane reflector stands upon a horizontal plane and inclined at a given angle to it. Determine how great a length of his own person a man standing before it at a given distance from it can see, and where his position must be that he may see the greatest possible length.

139. A small pencil of diverging rays is incident directly on a concave spherical refractor; to investigate a formula for determining the focus of refracted rays, giving the first and second approximations.

140. When a ray of solar light is refracted by passing through a prism, describe the appearance of the spectrum when seen in its state of greatest purity. How do different media affect the positions of the fixed lines in the spectrum? Are the phenomena the same for all kinds of light?

141. A small pencil of diverging rays passes directly through a lens, required the diameter of the least circle of chromatic dispersion.

142. The general formula for direct refraction through a double convex lens being

$$\frac{1}{u} + \frac{1}{v} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) + \frac{\mu - 1}{2\mu^2}\left\{\left(\frac{1}{r} + \frac{\mu + 1}{u}\right)\left(\frac{1}{r} + \frac{1}{u}\right)^2 + \left(\frac{1}{s} + \frac{\mu + 1}{v}\right)\left(\frac{1}{s} + \frac{1}{v}\right)^2\right\} y^2,$$

to determine the ratio of $r : s$, when the aberration is a minimum.

143. The distances of the conjugate foci from the first and second focal centres of a lens are connected by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left\{\frac{1}{r} - \frac{1}{s} - \frac{\mu - 1}{\mu}\frac{t}{rs}\right\}.$$

144. Rays tending to form an image in the axis of a concave lens are refracted to an eye situate in that axis, find the visual angle; and determine the position of the image when the visual angle is the same for all distances of the eye.

1834 145. Explain the experiments by which it appears that when a ray of light passes out of one medium into another, the sine of incidence is to the sine of refraction in a given ratio; and that if the refracted ray becomes the incident, the incident becomes the refracted ray.

146. Investigate, geometrically or algebraically, the relation between the distances, from the surface of a spherical reflector, of the conjugate foci of a pencil incident nearly perpendicularly upon it. Adapt the formula to diverging or converging rays, drawing the requisite figures.

147. Describe the construction and use of *Hadley's* quadrant, and state why the reflectors are at right angles to its plane. What is meant by the index error?

148. Explain the construction of *Newton's* telescope, tracing the course of a ray. What are the defects attending the use of a simple eye-glass in a telescope?

149. A given small pencil of diverging rays is reflected obliquely at the centre of a spherical mirror; shew how to determine the form of the reflected pencil. Explain the term focal lines; and, having given the foci of reflected rays in the primary and secondary planes, determine the position and magnitude of the circle of least confusion.

150. A small pencil of parallel rays passes directly into a sphere of water and is reflected at the second surface, determine the geometrical focus of emergent rays.

151. The ends of a glass cylinder are worked into portions of a convex and concave spherical surface, radii r, s, respec-

tively, having their centres in the axis of the cylinder; shew that the distance of these surfaces, in order that an eye placed at the concave surface may see the image of a distant object distinctly, must $= \frac{\mu (r - s)}{\mu - 1}$; and that the magnifying power will $= \frac{r}{s}$.

152. A small pencil of diverging rays, whose axis lies in a given plane, is reflected obliquely at the centre of a concave spherical mirror; determine the spaces, within which if the point of divergence be situated, the reflected rays in the primary and secondary planes will both diverge or both converge; and shew that they are separated by a space, within which if the point be situated, the reflected rays in the primary plane will converge, and those in the secondary diverge.

When the rays reflected in the primary plane are parallel, determine the locus of the geometrical focus of the rays reflected in the secondary plane.

153. If a pencil of parallel rays pass directly through a lens, the distance of the geometrical focus of emergent rays from the nearer focal centre in a convex lens, and from the farther focal centre in a concave lens, is the same on which ever surface the rays are incident.

154. If an eye be placed in air close to the surface of a clear stagnant fluid, prove that the apparent form of a circular arc in the fluid, whose centre coincides with the place of the eye, and whose plane is perpendicular to the surface, is defined by the equation

$$r = a \cdot \frac{m \sin^2 \theta}{m^2 - \cos^2 \theta},$$

where a is the radius of the circle, m the index of refraction, and the radius vector r is drawn from the place of the eye making the angle θ with the surface.

155. If the thickness of a concavo-convex lens be equal to $(\mu + 1)$ times the distance between the centres of its spherical surfaces, shew that a point may be found in its axis, from which if rays diverge and fall upon the concave surface, they will diverge accurately from a point after emergence.

156. Determine the form of a small pencil of rays after passing obliquely through the centre of a lens. What are the advantages and disadvantages of limiting the aperture of a lens, and why is the object-glass of an astronomical telescope a lens of considerable aperture and small magnifying power?

157. A ray of light passes through a double convex lens of small thickness; shew that if ϵ and η be the angles at which the incident and emergent rays are respectively inclined to the axis,

$$\frac{c \tan \eta}{b \tan \epsilon} = 1 + \frac{\mu - 1}{2\mu} \left\{ \frac{1}{r}\left(\frac{1}{r} + \frac{1}{b}\right) - \frac{1}{s}\left(\frac{1}{s} + \frac{1}{c}\right) \right\} y^2.$$

158. Explain the terms refractive and dispersive powers of a medium. Shew that they may be measured by the quantities $\mu - 1$ and $\frac{\Delta\mu}{\mu - 1}$, where μ = the constant ratio of the sine of incidence to the sine of refraction. State also the nature and cause of secondary spectra.

159. The principal focus of a sphere bisects the distance between the focus after the first refraction, and the extremity of the diameter in the direction of which the rays are incident.

1835 160. An object is placed between two plane mirrors inclined at an angle of 20°, find the locus of the images and their number.

161. A small pencil of rays is incident directly on a convex spherical refractor of a denser medium, find the focus of emergent rays. When the incident pencil is converging, determine for what positions of its focus the convergency is increased by the refraction.

162. Explain the construction and use of *Hadley's* sextant; when applied on land to determine the altitude of a star, how is the want of an accurate horizon supplied?

163. Define the term critical angle; and prove that a pencil of rays cannot pass through a prism whose refracting angle is greater than twice the critical angle of the substance of which it is composed.

164. Find the magnitude and position of the least circle of aberration when a pencil of diverging rays is refracted directly at a plane surface.

165. In the astronomical telescope, whether with a simple or compound eye-piece, determine the magnifying power; and shew that it is equal to the $\dfrac{\text{breadth of the object glass}}{\text{breadth of the emergent pencil}}$. How does this enable us to determine practically the magnifying power?

166. Determine the distance at which a short-sighted person can see distinctly, from observing the angle through which a glass prism is turned from its position of minimum deviation, in order that he may see distinctly through it a line of homogeneous light.

Where must an eye be situated, in order that when a luminous point is placed in the axis of a convex lens, the lens may appear wholly illuminated?

167. To an eye placed at the aperture of the large mirror in *Gregory's* telescope there will appear an inverted image of both mirrors near the smaller; and if the axis of the smaller be slightly disturbed, the images will be shifted towards that part of it which is most inclined from the larger. Prove this property, and explain its use in the practical adjustment of the telescope.

168. If b, b' be the breadths of the p^{th} and q^{th} rainbows respectively, and δ the sun's apparent diameter; shew that

$$b' = b + \left(\sqrt{\frac{(q+1)^2 - \mu^2}{(p+1)^2 - \mu^2}} - 1\right)(b - \delta) \text{ nearly.}$$

169. If N, n, F, be the focal centres and principal focus of a lens, the distances of the conjugate foci measured from two points N$'$, n', so situated in the axis that $nn' = p \,.\, \mathrm{NN}' = (1 - p)\, n\mathrm{F}$, are connected by the equation $\dfrac{p}{v} - \dfrac{1}{pn} = \dfrac{1}{n\mathrm{F}}$; p being any constant quantity.

170. If a small pencil of rays pass directly through a medium bounded by concentric spherical surfaces whose radii are r, s; the equation between the distances of the conjugate foci measured from the centre is

$$\frac{1}{v} - \frac{1}{u} = \frac{\mu - 1}{\mu}\left(\frac{1}{r} - \frac{1}{s}\right).$$

171. Having given the focal lengths of two lenses, find that of the system formed by their combination when the lenses are placed (1) in contact, (2) at a given interval. Explain briefly the principle of the achromatic eye-piece.

172. A pencil of diverging rays is refracted obliquely at the centre of a thin lens; find the distances from the centre of the foci of refracted rays in the primary and secondary planes.

173. Why is the form of the rainbow circular? State the cause of the partial brightness of the sky within the bow.

174. Two lenses of equal focal length $3l$ are placed at a distance $2l$ from each other; required the forms of the lenses so as to throw the best possible image of a plane object on a distant plane surface, the pencils being defined by a diaphragm placed in the focus of the compound lens. Having given

$$\gamma = \frac{1}{3(\alpha - \beta)^2}\{28x^2 + 20(\alpha + \beta)x + 10\alpha\beta + 3\beta^2 + 27\}.$$

175. Having given the focus of a pencil of rays incident directly on a spherical reflector, find the focus of reflected rays to the first and second approximations.

1836 176. State the laws of reflection. Find the image of any straight line placed before a plane reflector.

177. Find the relation between the conjugate focal distances, when a small pencil of rays is refracted directly at a spherical surface. Apply the formula to find the focal length of a thin meniscus lens of glass, the radii of whose surfaces are 9 and 12 inches.

178. Explain the general principle of telescopes. What is the field of view of the astronomical telescope? Why are large instruments necessary for some observations?

179. Determine the form of a surface on which, when rays diverging from a given point are incident, they may diverge after reflection from another given point.

180. Explain the term "dispersive power" of a medium; and shew how it is measured. Find the conditions under which two prisms with small refracting angles will achromatize each other.

181. Rays issuing from a luminous point are incident upon a thin lens. A portion of those that enter the lens is allowed to proceed at once through the second surface; a second portion, however, does not escape till it has been twice internally reflected; a third portion four times reflected; a fourth portion six times, and so on. Shew that a row of images will be formed at distances from the lens which are in harmonic progression.

182. A ray of light falls on the convex surface of a hemispherical lens in a direction parallel to its axis, is reflected at the plane surface, and emerges through the convex; shew that the angles of incidence and emergence are equal, and that the distance between the points of incidence and emergence is equal to twice the deviation of the ray at either refraction.

183. A concave lens of glass is placed behind a given crossed lens of the same material, and joins closely with it; determine the radius of the outer surface that a pencil of parallel rays may pass directly through both lenses without aberration.*

184. Light passing through a lens of small aperture is received upon a screen placed at the geometrical focus; determine the law of brightness of the image. What is the peculiar difficulty of solving this question, when the screen is placed in any other position? for instance, what is the law of brightness of the least circle of aberration?

185. If r and s be respectively the radii of the first and second surfaces of a thin lens of given focal length, the longitudinal aberration will be the least possible when

$$\frac{1}{r}+\frac{1}{s}=\frac{2(\mu+1)}{\mu+2}\left(\frac{1}{v}+\frac{1}{u}\right), \text{ and } \frac{1}{r}-\frac{1}{s}=\frac{1}{\mu-1}\left(\frac{1}{v}-\frac{1}{u}\right).$$

186. The image of each point of a luminous object placed before a reflecting or refracting surface being a caustic, the whole image will consist of an indefinite number of caustics;

* The aberration for a pencil of diverging rays through a glass convex lens is $-v^2 \cdot \frac{y^2}{6f^3}\left\{7x^2+10ax+\frac{13}{4}a^2+\frac{27}{4}\right\}$, where $\frac{1}{r}=\frac{1+x}{f}$ and $\frac{1}{u}=\frac{1+a}{2f}$.

X

reconcile this fact with the common theory of images, and explain the use of a diaphragm.

A small curvilinear object is placed before a lens in a plane perpendicular to its axis, so as to form an image by excentrical pencils limited before incidence by a diaphragm placed on the axis; determine the curvature of the image where it is distinct.*

187. Determine the index of refraction between two media, having given the indices between each of them and a third. Given the refracting and critical angles of a prism, find the greatest inclination at which a ray may enter it so as to emerge.

188. A small pencil of parallel rays passes directly through a plano-convex lens, determine the longitudinal aberration of the extreme rays; shew that it is greater when the rays are incident on the plane than when on the convex surface.

189. A small pencil of diverging rays is incident directly on a plane refracting surface, determine the longitudinal aberration.

THEORY OF LIGHT.

1821 1. A ray of light, after being refracted through media possessing equal dispersive powers, will always appear coloured at its emergence, unless the incident and emergent rays are parallel.

1827 2. If the refraction of light be the effect of any causes which act throughout a certain distance from the surface of a medium, and the intensity of which depends solely on the distance from that surface, the ratio of the sines of incidence and refraction must be constant.

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* The formulæ for the primary and secondary focal lines are respectively,

$$\frac{1}{k} = \frac{1}{f} - \frac{1}{h} + \frac{1}{k^2}\left(3V + \frac{1}{\mu}\right)\frac{z^2}{2f},$$

$$\frac{1}{k} = \frac{1}{f} - \frac{1}{h} + \frac{1}{k^2}\left(V + \frac{1}{\mu}\right)\frac{z^2}{2f}.$$

3. Explain briefly the optical experiments and theories to which the following terms refer: *Fits of easy transmission and reflection; Polarization; Plane of Polarization; Depolarization; Depolarizing Axis; Fringes; Interferences.* 1828

4. Mention the facts from which it appears that the phenomena of the extraordinary ray in a double refracting crystal can be accounted for on the supposition of a repulsive force emanating from the axis.

5. Explain the theory of the interferences of light, and determine the colour, origin, and intensity of a ray resulting from the interference of two similar rays, differing in origin and intensity. 1830

6. What properties of a medium in which the density varies as the pressure, correspond to and serve to explain the following observed properties of light: its rectilinear and uniform transmission; the different intensity of different rays; the difference of intensity of the same ray at different distances from its origin; the difference which the eye distinguishes in rays by colour; their crossing in all possible ways without mutual disturbance; the interference of two rays, the paths of which nearly coincide in direction and differ in length by a multiple of a certain interval? 1831

7. Demonstrate the laws of reflection and refraction on the undulatory theory of light.

8. In the undulatory theory of optics, define the terms wave, length of a wave, front of a wave, phase of a wave. How is the effect of any wave in disturbing a given point found? If the maximum vibration produced at a point by a succession of waves be $\frac{c\lambda}{\pi r \cos\theta} \sin \frac{2\pi b \cos\theta}{\lambda}$, it is required to compare the values of this expression for different values of θ according as λ is much greater or much less than b. 1832

9. Shew how to express algebraically the transmission of an undulation.

10. In the undulatory theory of light, explain the principle of interferences. By what experiments is the existence of such 1833

interference proved? Give the mathematical investigation of some experiment for determining the effect of the interference of rays.

1834 11. A prism is cut out of a biaxal crystal having its edge parallel to one of the axes of elasticity; describe the nature of the refractions which take place in a plane perpendicular to the edge.

12. State the two theories which have been imagined to account for the phenomena of light. Point out the principal objection which has been made to each, and describe the corresponding experiment.

13. Prove, from the principles of the undulatory theory, that light proceeding from a bright point by a small hole into a room, ought not to spread through the room in the same manner as sound coming in the same direction and through the same hole.

14. A plate of Iceland spar, bounded by planes perpendicular to its axis, is interposed between a polarizing and an analyzing plate, the latter being so placed that no light is reflected without the interposition of the crystal; investigate the intensity of the light in various parts of the image seen after reflection at the analyzing plate.

The difference of retardations of the ordinary and extraordinary rays produced by passing through the crystal for a small angle of incidence, is

$$\mathrm{T}.\ \frac{c^2 - a^2}{2av}\ .\ \sin^2 i.$$

1835 15. A luminous point is placed at a given distance from a plate of glass bounded by two parallel planes; required the form of the emergent wave of light at a given time.

16. Define the axes of elasticity of a crystal; and supposing the displacement of a molecule to be exceedingly small compared with the distances between the molecules, and proportional to the force resulting from it, prove that there are three axes of elasticity in every crystal at right angles to each other.

17. A ray of light is incident on a plate of Iceland spar, which is bounded by planes perpendicular to the axis of the

crystal; shew that if ϕ' and ϕ_1 be the angles of refraction for the ordinary and extraordinary rays, and e the eccentricity of the spheroid of refraction,

$$\frac{1}{\sin^2\phi_1} - \frac{(1-e^2)^2}{\sin^2\phi'} = e^2.$$

18. Light diverging from a point falls on one side of a prism which has a very obtuse angle opposite to this side; state the phenomena observed, and explain them on the theory of undulations.

19. Two convex lenses of small curvature are placed in contact; having given the intensities of the reflected (1) and transmitted (2) light, trace fully the phenomena of *Newton's* rings. By what probable analogy is the circumstance, that the rings formed by the reflected and transmitted light are complementary to each other, illustrated?

$$(1)\quad \frac{4a^2e^2\sin^2\frac{\pi}{\lambda}V}{(1-e^2)^2+4e^2\sin^2\frac{\pi}{\lambda}V}. \qquad (2)\quad \frac{a^2(1-e^2)^2}{(1-e^2)^2+4e^2\sin^2\frac{\pi}{\lambda}V}.$$

20. In a dark room two bright points of light are placed 1836
very near to each other; describe the phenomena exhibited upon a screen placed in any position.

21. A succession of waves of light, whose fronts are parallel to a plane screen which has a small opening in it, is moving towards the screen, and the magnitude of a vibration on any point of a semicircle behind the screen, and of which the centre lies in the opening, $= \frac{c\lambda}{\pi r\cos\theta} . \sin\frac{2\pi b\cos\theta}{\lambda} . \sin\frac{2\pi}{\lambda}(vt-r)$; investigate the appearance on the semicircle, and shew that it obviates a material objection against the undulatory theory.

22. What is meant by polarized light? Account for the separation of common light into two pencils by a doubly refracting crystal, and for their consequent polarization. If the crystal be uniaxal, shew that an extraordinary wave will diverge from a point in the form of a spheroid of revolution.

SECTION XX.

QUESTIONS ON ATTRACTIONS, TIDES, FIGURE OF THE EARTH, &c.

1821 1. The attraction of a spherical shell upon a particle placed without it, is the same as if the whole matter in the shell were placed in its centre.

2. In a revolving spheroid of small eccentricity, if polar gravity : equatoreal sensible gravity : radius of equator : semi-axis, gravity is everywhere perpendicular to the spheroidal surface.

3. Shew, by measuring the area ANB in the 12th section, that if a sphere be composed of particles, the attraction of which $\propto \frac{1}{(\text{dist.})^2}$, the attraction of the whole sphere on an external particle varies in the same law.

1822 4. Shew that if M and S represent the height of the tide produced by the moon and sun respectively; retardation of tide at new and full moon : retardation in quadratures :: $M-S : M+S$.

5. The density of different parts of a circle varies as the square of the distance from the centre; find the velocity ac-

quired by a corpuscle attracted towards this circle in a line passing through its centre and perpendicular to its plane, the attractive force of each particle varying as $\frac{1}{D^2}$.

6. Find the attraction of a rectangle on a corpuscle situated in one of its sides produced, in a direction perpendicular to the other side; the force tending to each particle of the rectangle varying inversely as the square of the distance.

7. If two given spheres touch each other internally, and 1824
the interior be taken away, find a point within the remainder such that a particle being placed there shall remain at rest. $\left(\text{Attraction of each particle} \propto \frac{1}{\text{dist.}^2}\right)$.

8. If the particles of two spheres attract with forces varying as the distance, the force with which the spheres attract each other is as the distance between their centres.

9. Let Q be a point in a semicircle whose diameter is AB, join AQ and in AQ produced, take AP a mean proportional between AB and AQ; find the equation of the curve which is the locus of the point P; its area; the content of the solid generated by its revolution, and the radius of curvature at its vertex.

10. Prove that the curve in the last problem possesses these properties.

(1.) That a particle placed anywhere in its perimeter as at P, will attract a particle at A, in the direction AB, with a constant force; supposing the force of attraction to vary as $\frac{1}{\text{dist.}^2}$.

(2.) That if it be made the revolving orbit in the ninth section of *Newton*, and $G = \frac{F}{2}$, the orbit traced out in fixed space will be the lemniscate of *Bernoulli.*

11. In what latitude will a ring surrounding the earth, and parallel to the equator, attract a particle placed in the earth's centre with the greatest force possible?

12. If a circle whose diameter is equal to the whole tide in any given latitude be placed vertically, and so as to have the lower extremity of its diameter coincident with the level of low water, prove that the tide will rise or fall over equal arcs in equal times.

13. Supposing a comet of the same magnitude and density as the moon, on its nearest approach to the earth, to be distant thirty radii from the earth's centre; required the magnitude of the tide raised by the comet.

14. If particles of a spherical shell attract with forces varying as $\frac{1}{D^2}$, and a cylindrical rod of uniform density whose length equals n times the radius of the sphere pass through the shell; find the pressure on the shell when the rod is at rest, the part of it within the shell being equal to the radius of the sphere.

15. If the inscribed sphere be taken away from the earth, find the time in which a particle situated in the plane of the earth's equator within the space occupied by the inscribed sphere will reach the inner surface of the remaining meniscus; the earth being supposed an oblate spheroid of small ellipticity.

1825 16. A second's pendulum is carried to the height of one radius above the earth, and another is sunk to the depth of half a radius. Compare the times of their oscillations.

17. A corpuscle placed within a circle is attracted to every particle in the circumference with a force that varies inversely as the square of the distance; prove, when the distance of the corpuscle from the centre is small, that the attraction on the corpuscle varies nearly as its distance from the centre, and draws it from the centre.

18. Given the ratio of the periodic time of the moon to the time of the earth's revolution about its axis, and the ratio of the mean distance of the moon to the mean semi-diameter of the earth, to find the ratio of the polar and equatorial diameters of the earth nearly.

19. Given the heights of the spring and neap tides, to compare the densities of the sun and moon; their apparent diameters being considered as equal.

20. Given the declination of the moon, to find the duration of the superior or inferior tide occasioned by her action alone.

21. If $\cos(\lambda + \delta) : \cos(\lambda - \delta) :: 1 : 3$, where λ is the lati- 1826
tude of the place and δ the declination of the moon; prove that the time of the ebbing or flowing of the superior tide : the time of the ebbing or flowing of the inferior tide :: 2 : 1.

22. Find that section of a sphere which attracts a corpuscle placed at a given point in the axis produced with the greatest possible force, the force of each particle $\propto \frac{1}{(\text{dist})^2}$.

23. Shew that Saturn's ring cannot be a homogeneous and 1827
regular solid of revolution.

24. Of all conical surfaces of equal altitudes, determine that which exerts the greatest attraction on a particle at its vertex.

25. Explain clearly, from elementary principles, why the moon's attraction causes a tide on the opposite side of the earth.

26. Prove that there are generally either two homogeneous fluid spheroids of equilibrium or none, for the same time of rotation; and supposing the eccentricity of the one spheroid very small, find the ratio of the axes in the other.

27. A particle is placed anywhere within a triangle, the 1828
sides of which are composed of particles attracting with forces varying as $\frac{1}{D^2}$; find the direction in which it will begin to move.

28. A given quantity of matter is to be formed into a cone; find its form, that its attraction on a particle at its vertex may be a maximum, the attraction of each particle varying as $\frac{1}{D^2}$.

29. If the whole force at the pole of an oblate spheroid be to that at the equator as the equatorial radius to the polar, and to any point within the spheroid canals of any form be drawn, the pressure on that point will be the same whatever be the form or direction of the canal.

30. Find the attraction of a spheroid of finite eccentricity on a particle in its equator.

31. Construct for the place of high water in a given position of the sun and moon, and find an expression for the actual height of the compound tide.

32. Give an analysis of the reasoning by which *Newton* explains the theory of the tides, and deduce a numerical comparison between the force of the sun on the tides, and the force of gravity.

33. Find the attraction of a spherical shell in which the attraction of each particle $\propto \frac{1}{(\text{dist.})^2}$ according to *Newton's* method, and analytically.

1829 34. Find the attraction on a particle placed within a heterogeneous spheroidal shell of small eccentricity, the density being the same throughout concentric spheroidal surfaces of different eccentricities, and the internal and external surfaces being of given eccentricity, and the density uniform throughout them.

35. Find the length of the tide-day, the sun and moon being in the equator, and shew how the densities of the sun and moon may be compared, by observing the lengths of the greatest and least tide-days.

36. A spherical surface being constituted of particles the forces of which vary as $\frac{1}{D^2}$; shew that the attraction of the whole surface on a particle without it, varies inversely as the square of the distance of the particle from the centre. (*Newton*, Book I. Prop. 71.)

37. The elevation of the summit of the spheroid produced by the attraction of the sun and moon on a fluid sphere is double of the depression of the equator below the sphere.

38. Investigate the motion of the pole of the earth produced by the moon in one sidereal revolution.

39. Determine the attraction of an oblate spheroid on a particle situated in its equator.

40. In a homogeneous spheroid attracting a point on the surface, the effect of the force parallel to the equator is as the distance from the axis.

41. Find the attraction of a homogeneous spheroid of small eccentricity on a particle situated in its pole.

42. If the earth be an oblate spheroid of small ellipticity 1830
with semi-axes a and b, the ratio of the mean density to that at the surface is

$$\frac{3}{k^2a^2}\left(1 - \frac{k\,(4a - b)}{3 \tan kb}\right) \text{ very nearly,}$$

assuming the density to be uniform throughout each spheroidal stratum at the same distance from the earth's surface, and to vary as $\dfrac{\sin kr}{r}$ at different distances, where k is a constant quantity and r the polar semi-axis of the surface of equal density.

43. Find the attraction of an oblate spheroid on a particle in its equator.

44. Having given the declination of the moon, compare the magnitudes and durations of the superior and inferior tides in any latitude, the effect of the sun on the tide being neglected.

45. When the force at the pole of a revolving fluid spheroid is to the force at the equator as the equatorial radius is to the polar radius, any two canals drawn from any points in the surface and meeting within it, will balance each other.

46. Supposing the earth to be spherical, and the matter in 1831
its interior to be compressed according to the law $p = k\,(\rho^2 - \delta^2)$, p being the pressure and ρ the density at any distance r from the centre, and δ the density at the surface; shew that $\rho \propto \dfrac{\sin qr}{r}$, q being a certain constant.

47. If a plumb-line be drawn from the vertical by a small quantity of matter, at a small elevation m above the earth's surface; shew that the deviation will be a maximum, when $\sin\theta = \dfrac{m}{r\sqrt{2}}$, where θ is the angular distance at the earth's centre of the plumb-line and attracting point, and r the radius of the earth supposed spherical.

48. Having given the positions of the sun and moon supposed to be in the equator, find the interval between high water at any place and the passage of the moon over the meridian.

49. Investigate the change in the inclination of the ecliptic produced by the sun's action on the earth.

50. Find the proportion of the axis of a homogeneous revolving fluid spheroid; shew that there are two forms of equilibrium, and find the eccentricity in each case when the centrifugal force is small.

51. If several angular velocities be impressed on a body at the same time, the resulting axis of rotation, and the angular velocity about it, will be found by finding the direction and magnitude of the resultant of forces in the directions of the several axes of rotation, and proportional to the angular velocities.

52. Find the attraction of a spherical shell on a particle without it, the law of attraction being $\frac{1}{(\text{dist.})^2}$.

1832 53. If the earth be a solid of revolution, such that r is the radius of curvature of its meridian at the equator, and $r + \mu \sin^4 . l$ the radius of curvature of its meridian in any latitude l, prove that the polar and equatorial axes are respectively $r + \frac{1}{5}\mu$, and $r + \frac{8}{15}\mu$.

54. Determine the height to which the disturbing force of the sun would raise the waters of the earth above the equicapacious sphere, supposing the earth to be at rest and to be covered with fluid.

55. Find the attraction of a spherical shell on any point without it, the force of attraction of each particle varying as $\frac{1}{(\text{dist.})^2}$, and hence determine the attraction of a homogeneous sphere.

1833 56. If CA be the radius of a sphere, the centre of which is C, and Q, P two points in the same diameter, Q within and

P without the sphere, so situated that $CQ \cdot CP = CA^2$, and if the attractive force of each particle of the sphere $\propto$ (distance)$^{-n}$, prove that the attraction on Q : attraction on P :: $\sqrt{QC.PC^n} : \sqrt{PC.QC^n}$.

57. Prove that the attraction of an oblate spheroid on a particle at any point in its surface in a direction perpendicular to the axis, is proportional to the distance of the particle from the axis, and in a direction perpendicular to the equator to its distance from the equator.

58. The tide produced at the equator by the joint action of the sun and moon, being

$$\tfrac{1}{2}\{h \sin^2\delta \cos 2(\theta - \lambda) + h' \sin^2\delta' \cos 2(\theta' - \lambda')\},$$

where θ, θ' are the respective hour angles of the sun and moon, determine the time and height of high water. On what suppositions is the above formular obtained?

59. The earth being supposed a homogeneous oblate spheroid, find the angular velocity generated in an indefinitely small time round an equatorial axis by the action of the sun.

60. Prove that the attraction of a homogeneous sphere on a particle without it, is the same as if all the mass were collected in its centre; force varying inversely as the square of the distance. Is this result true for any other laws of force?

61. What is the practical method of observing the number 1834
of vibrations made by a pendulum in 24 hours? Shew how the mass of the earth may be determined by a comparison of the rates of oscillation of a pendulum at the surface, and at a point below the surface.

62. Shew that velocity of the earth's rotation is unaltered by the action of the sun and moon, first proving that the points of intersection of the axis of instantaneous rotation with a sphere described about the earth's centre lie nearly in a small circle.

63. By what experiments is it shewn that terrestrial bodies are attracted by the earth and by each other proportionably to their quantities of matter?

64. Compare the retardation of the tide at syzygy and quadrature. What is meant by the establishment of a port?

65. Having given the attractions of an oblate spheroid of small eccentricity upon particles at its pole and equator, express the ellipticity of a spheroid of equilibrium revolving slowly, in terms of the ratio of the centrifugal force at its equator to gravity. How may we compare the ellipticities of two planets which have satellites, supposing them homogeneous?

1835 66. The pole of the earth, affected only by the sun's attraction, traces out a curve on the celestial sphere, whose equation is

$$\frac{\tan i}{n} \cdot \theta = \cos^{-1} \frac{\phi - 1}{n} - \left\{1 - \left(\frac{\phi - i}{n}\right)^2\right\}^{\frac{1}{2}},$$

where $\phi = \text{KP}$, θ equal the inclination of KP to an arc drawn through K and the first point of ♈, n a constant quantity, and i the mean value of ϕ.

67. A particle placed on the inner surface of a spheroidal shell, bounded by similar concentric spheroidal surfaces, will remain at rest, supposing the force of attraction of each particle to vary inversely as the square of the distance.

68. Compare the retardation of the spring and neap tides; and find the time of high water when the moon is syzygy.

69. Find the attraction of an oblate spheroid on a particle situated at its pole.

70. Find the law of attraction, that a sphere may attract a particle without it in the same manner as if the whole mass were collected in the centre.

71. Assuming the attraction of a homogeneous spheroid on any point to be known, enunciate all the propositions necessary for finding the attraction of a heterogeneous spheroid on a point within it; and supposing the latter given, and the density of equally dense spheroidal surfaces a known function of the polar distance, give the train of reasoning from which the ellipticity of the earth may be determined.

72. If a uniform force acting upon a body tend to give it a motion of rotation round an axis which is always perpen-

dicular to the axis round which it is at each instant revolving, and the axes be always in the same plane, the angular velocity will be unaltered.

73. State the theory of universal gravitation, and point out generally the evidence on which it has been received. How is it shewn that bodies are attracted towards the earth by forces tending to each part of the mass?

74. A sphere, whose surface is perfectly smooth, by its 1836
attraction keeps attached to it the point of a needle, the other end of which rests upon a perfectly smooth horizontal plane, situated below the sphere at a distance less than the length of the needle. Suppose the needle originally not placed in a position of equilibrium, determine the nature of its motion.

75. Determine the attraction of a right prism of square base on a particle in its axis, the force to each particle varying inversely as the square of the distance.

76. Find the attraction of a sphere upon a point, and thence that of one sphere upon another; the force varying inversely as the square of the distance.

77. Supposing the earth to have been originally a homogeneous fluid mass revolving uniformly about its axis, shew that there will be equilibrium when the force at the pole is to that at the equator as the radius of the equator is to that of the pole.

78. Investigate the motion of the pole of the earth produced by the moon in one sidereal revolution.

79. The equation for determining the ellipticities of the spheroids of equal density in the earth is

$$\frac{e}{c^3}\int dc\,\rho c^2 - \frac{\int dc\,\rho\,\frac{d\,(c^5 e)}{dc}}{5c^5} - \frac{1}{5}\left\{\int_{c=c} - \int_{c=c}\right\}\rho\,d_c e - \frac{\pi}{2\mathrm{T}^2} = 0;$$

shew that when $\rho = \mathrm{A}\,\dfrac{\sin qc}{c}$, its solution may be made to depend on that of the equation $\dfrac{d^2v}{dc^2} - \dfrac{6v}{c^2} + q^2 v = 0.$

Having given the complete solution of this

$$v = C\left\{\left(1 - \frac{3}{q^2c^2}\right)\sin(qc + C') + \frac{3}{qc}\cos(qc + C')\right\},$$

determine the earth's ellipticity at the surface.

80. When electricity is latent in any body, the quantity of positive electricity contained in it is equal to that of negative.

81. Shew that the action of the moon produces contemporaneous tides on opposite sides of the earth. At a given place find the elevation of the tide caused by the joint action of the sun and moon, both supposed to be in the equator.

SECTION XXI.

MISCELLANEOUS QUESTIONS.

1. If P be any rational function of x, in which the highest power of x is less than n; and if A, B, C, D, . . . K be the values of

$$\mathrm{P}, \frac{d\mathrm{P}}{dx}, \frac{d^2\mathrm{P}}{dx^2}, \frac{d^3\mathrm{P}}{dx^3} \cdots \cdots \frac{d^{n-1}\mathrm{P}}{dx^{n-1}},$$

when $x = a$, then

$$\frac{\mathrm{P}}{(x-a)^n} = \frac{\mathrm{A}}{(x-a)^n} + \frac{\mathrm{B}}{(x-a)^{n-1}} + \frac{\mathrm{C}}{1.2.(x-a)^{n-2}}$$
$$+ \frac{\mathrm{D}}{1.2.3.(x-a)^{n-3}} + \cdots\cdots + \frac{\mathrm{K}}{1.2\ldots(n-1)(x-a)}.$$

2. Divide a given paraboloid into two parts in the ratio of $m : n$, by a plane inclined at a given angle to the axis.

3. Prove that the chord of curvature $= \dfrac{\mathrm{PQ}^2}{\mathrm{QR}}$.

4. Solve the equation $\phi x^2 = \phi(2x) + 2$.

5. Prove the following formula for small arcs,

$$l \sin x = lx + \tfrac{1}{3} l \cos x.$$

6. Find the part of a sphere cut out by three planes passing through its centre, and inclined to each other at angles of 120°.

7. Two straight lines are inclined to each other at a given angle, find the area of all the circles which can be described

touching each other and the two given lines, the position of the centre of the last circle being given.

8. Find that point in the surface of a spherical triangle from which, if straight lines be drawn to the angular points, the pyramid thus formed shall be a maximum.

9. Prove that $1 \, . \, 2 \, . \, 3 \, \ldots \, x = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$ very nearly, when x is large, and shew the utility of this formula in the solution of the following problem.

10. In a pack of fifty-two cards, containing an equal number of red and black cards, determine the probability that in drawing any even number of cards, there shall be an equal number of red and black cards; supposing that the probability of drawing any even number is the same. A numerical result is required.

11. In the series of quantities $A_1, A_2, A_3, \ldots$

$$\text{if } A_1 = r \tan\left(\sin\frac{2\pi}{3} + a\right), \; A_2 = r \tan\left(\sin\frac{4\pi}{3} + a\right),$$

and the remaining ones be derived according to the following law:

$$A_1 \, . \, A_2 \, . \, A_3 = r^2 (A_1 + A_2 + A_3),$$
$$A_2 \, . \, A_3 \, . \, A_4 = r^2 (A_2 + A_3 + A_4), \ldots$$

$$\text{prove that } A_n = r \tan\left(\sin\frac{2n\pi}{3} + a\right).$$

12. If $x = m \tan(z - nx)$ where x is small compared with z,

$$\text{prove that } x = \frac{m}{2} \frac{\sin 2z}{mn + \cos^2 z} \text{ very nearly.}$$

13. If $-P_{m-p} x^{m-p}, \; -P_{m-q} x^{m-q}, \; -P_{m-r} x^{m-r}, \; - \ldots$ be the negative terms of an equation of m dimensions, then will the greatest root of this equation be less than the sum of the two greatest of the quantities $P_{m-p}^{\frac{1}{p}}, P_{m-q}^{\frac{1}{q}} \ldots$

14. Having given the first two terms of the expansion of $(a^2 + b^2 + 2ab \cos\theta)^{-\frac{1}{2}}$ in a series of the form

$$A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \ldots$$

shew how from them the first two terms of the expansion of $(a^2 + b^2 + 2ab \cos\theta)^{-\frac{3}{2}}$ may be determined.

15. If S_1 represent the sum of the ordinates in the quadrant of a circle whose radius is 1,

S_2 represent the sum of their squares,

S_3 . . . . cubes,

. . . .

prove that $S_{n-1} S_n = \dfrac{3}{n+1} S_1 S_2$.

16. Integrate the following differentials and differential equations:

$$\frac{dx}{x^4+1}, \quad \frac{dx}{x^4\sqrt{1-x^2}}, \quad \frac{dx}{\sqrt{a-x}-\sqrt{x}},$$

$$(x^2+y^2)\,dx + x^2y\,dy = 0, \quad (1+x)\frac{d^2y}{dx^2} + a\frac{dy}{dx} = 0,$$

$$1 + p^2 + q^2 = m^2;$$

and also the following equations of differences:

$$f(x^2) - f(x) = m \text{ and } u_x\, u_{\pi+x} = k^2.$$

17. There are two urns A and B, the former containing three white and the latter three black balls; a ball is taken from each at the same time and put into the other, and this operation is repeated three times; what is the probability that A will contain three black and B three white balls?

18. Develope $\sin(\alpha + \beta x + \gamma x^2)$ in a series of the form $A + Bx + Cx^2 + Dx^3 + \ldots$.

19. If the true centre of the moon's orbit move uniformly in a circle about the mean centre, the result is a change of the moon's place of the form $m \sin 2\{(☽ - \odot) - A\}$, where A is the moon's anomaly.

20. If m be the maximum lunar nutation in N. P. D., l the longitude of the moon's ascending node, shew that when the longitude $= l'$, the nutation $= m\cos(l' - l)$.

21. The coefficient of x^n in the expansion of

$(1 + x + 2x^2 + 3x^3 + \ldots \textit{ad inf.})^2$ is equal to $\dfrac{n^3 + 11n}{6}$.

22. If $A_1, A_2 \ldots \ldots A_n$ and $B_1, B_2 \ldots \ldots B_n$ be two series of positive numbers arranged in order of magnitude, of which A_1 and B_1 are respectively the greatest, shew that $\frac{A_1}{B_1} + \frac{A_2}{B_2} + \ldots \ldots + \frac{A_n}{B_n}$ is less, and $\frac{A_1}{B_n} + \frac{A_2}{B_{n-1}} + \ldots \ldots + \frac{A_n}{B_1}$ greater than if the denominators $B_1, B_2 \ldots \ldots B_n$ be arranged in any other order under $A_1, A_2 \ldots \ldots A_n$.

23. Two vessels, of which the capacities are a and b, are filled, the one with wine and the other with water; equal quantities c are taken from each and poured into the other, and this operation is repeated n times. Find the quantities of wine and water remaining in each vessel.

24. Expand $\frac{1}{1 + e\cos x}$ into a series of the form $A + B\cos x + C\cos 2x + \ldots$, and explain the law of the coefficients.

25. If a circle be described on AM the axis-major of an ellipse, and if an ordinate to the axis meet the ellipse in P and the circle in Q, S being a point in the major-axis, the areas ASP, ASQ are in a constant ratio.

26. If $\left.\begin{array}{l} aX + bY + cZ = 0 \\ a_1X + b_1Y + c_1Z = 0 \end{array}\right\}$ where $\begin{array}{l} X = ax + a_1x_1 + a_2, \\ Y = bx + b_1x_1 + b_2, \\ Z = cx + c_1x_1 + c_2, \end{array}$

then $X^2 + Y^2 + Z^2$

$$= \frac{\{a_2(bc_1 - b_1c) + b_2(a_1c - ac_1) + c_2(ab_1 - a_1b)\}^2}{(bc_1 - b_1c)^2 + (a_1c - ac_1)^2 + (ab_1 - a_1b)^2}.$$

27. Solve the functional equation

$$(\psi a^x)^2 \psi\left(\frac{1 - a^x}{1 + a^x}\right) = b^2 a^x.$$

28. If AB be an elliptic quadrant, CP, CD semi-conjugate diameters, PF perpendicular to CD, K the point, in which CD produced meets the circumscribing circle, and KMQ a line perpendicular to the major-axis meeting the ellipse in Q, then will arc BP − arc AQ = CF.

29. The square of the area of any one of the faces of a triangular pyramid, is equal to the sum of the squares of the other three, minus twice the rectangle contained by the product of every two and the cosine of their inclination.

30. A mortgage is taken on an estate worth N acres of it; land rises n per cent. in price, and in consequence the mortgage is only worth N_1 acres, and it is then paid off. During the continuance of high prices another mortgage is taken, which is worth N acres as before; prices return to their former level, and the mortgage is worth N_2 acres;

$$\text{shew that } N - N_1 : N_2 - N = 1 : 1 + \frac{n}{100}.$$

31. The ages of a man and his wife are respectively $85 - m$ and $86 - n$; find the present worth of an annuity A to be paid to the wife after the death of her husband, supposing one male out of every 85, and one female out of every 86, to die annually; and n greater than m.

32. If a straight line PSP_1 revolve about a fixed point S; and if perpendiculars PM, P_1M_1 be drawn upon a fixed straight line passing through S, find the equation to the curve in which $\frac{1}{PM^n} + \frac{1}{P_1M_1{}^n} = \frac{2}{a^n}$.

33. If P, Q be any two points in a curve referred to a pole S, and PM, QN perpendiculars on a fixed straight line TMN, find the equation of the curve, when the area SPQ is to the area PQMN as 1 to n, and construct the curve when $n = 1$.

34. Three given quantities, $(a+z)$, $(a+z)+h$, $(a+z)+h'$, approximations to the root a of an equation, being substituted for the unknown quantity, give results n, $n + \delta$, $n + \delta'$; shew that z will be very nearly found from the equation,

$$z^2 (h\delta' - h'\delta) + z (h^2\delta' - h'^2\delta) + nhh' (h' - h) = 0.$$

35. If each of the angles of a spherical triangle, the sides of which are very small compared to the radius of the sphere, be diminished by a third part of the spherical excess, the angles so diminished may be taken as the angles of a plane triangle, the sides of which are equal in length to those of the spherical triangle.

36. Having given the three edges of a parallelopipedon, and the angles they make with each other, find its solidity.

37. Having given the length of a string, the density of which at every point is a given function of the distance from one extremity, determine the form which when suspended at two given points it must assume, in order that its centre of gravity may be the lowest possible.

38. The three edges of a triangular pyramid which meet, and the angles which they make with each other, being given, find its content.

39. The ratio of the apparent axes of Saturn's ring being known from observation, investigate an equation for determining its inclination to the ecliptic. Since Saturn's centre does not coincide with that of the ring, what probable supposition will account for the stability of the equilibrium?

40. The greatest probability of correctness in solving a small spherical triangle from three observed angles and a measured side, is when the angle opposite to the known side is less than a right angle, and the other two sides are nearly equal. If two angles only be observed, for what value of the third angle will the errors probably be smallest?

41. Out of three white balls and two black ones, two are placed at random in one bag and three in another, after which a white ball is accidentally lost; find the probability that a person going to either bag will draw a white ball.

42. To an eye placed at the centre, every section of an ellipsoid whose projection on a tangent plane at the extremity of one axis is a circle, passes through a fixed point in the less of the remaining axis.

43. If $u = f(x, y) = F(r, z)$,

and $r = \phi(ax + cz) = \psi(ax - by)$,

shew that $\dfrac{1}{a}\dfrac{du}{dx} + \dfrac{1}{b}\dfrac{du}{dy} + \dfrac{1}{c}\dfrac{du}{dz} = 0$.

44. Integrate the following equations:

$$\sqrt{x}\,\frac{dy}{dx} - 2y = a\sqrt{y},\quad x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = a\log_e x,$$

$$u_{x+2} - 2au_{x+1} + (a^2 + b^2)\,u_x = x,$$

and shew that if

$$\Delta = \sqrt{1 - c^2 (\sin\theta)^2},$$

$$\int \frac{d\theta}{\Delta^{2m+1}} \text{ between the limits } \theta = \frac{\pi}{2} \text{ and } \theta = 0,$$

$$= \frac{1}{(1-c^2)^m} \int d\theta \Delta^{2m-1}.$$

45. If a string without weight be stretched between two points on any surface, the pressure at any point varies inversely as the radius of absolute curvature.

46. A cask full of wine contains (a) gallons, (b) gallons are drawn from it, and the void filled with a mixture of wine and water in a given ratio; (b) gallons are again taken from this and replaced as before. How much wine will remain after this process has been repeated (n) times. Solve the problem also when the strength of the mixture, which replaces the void in each process, increases as the numbers 1, 2, 3 . . . n.

47. If the base of a right pyramid be a regular polygon of n sides, having given the length of one of these sides and the length of one of the edges of the pyramid, find the inclination of any two of its adjacent faces, and the solid angle at the vertex.

48. In *instrumental* observations, explain the advantage of taking the mean of several observations.

If the only errors of observation that can arise are a, 0, and $-a$, and if it is equally probable that any one of those shall arise, what is the probability that one-third of three observations will be the true value of the quantity sought?

49. Having given the edges of a parallelopiped and the angles they make with each other, find the diagonal.

50. If H and h be the small altitudes of two objects near the horizon, and a the angle which these two objects subtend at the station of the observer, and $a + x$ the value of a reduced to the horizon, prove that

$$\frac{x}{\sin 1''} = \left(\frac{H + h}{2}\right)^2 \tan\frac{a}{2} - \left(\frac{H - h}{2}\right)^2 \cot\frac{a}{2}.$$

51. Each of a series of parabolas is described to pass through two given points, and have its axis parallel to a given fixed line. Prove that the locus of their foci is an hyperbola, and determine the ratio of its axes.

52. From each point of a parabola a straight line is drawn perpendicular to the chord joining it with the vertex. Determine the locus of the ultimate intersections of all such lines.

53. Which is greater, n^3+1, or $n+n^2$; n, or $\log_\varepsilon(1+n)$; chord $108°$, or chord $36°$ + chord $60°$?

In the equations $\left.\begin{aligned} 2(ab+xy)+(a+b)(x+y)=0 \\ 2(cd+xy)+(c+d)(x+y)=0 \end{aligned}\right\}$ shew that the values of x and y are

$$\frac{cd-ab\pm\{(a-c)(a-d)(b-c)(b-d)\}^{\frac{1}{2}}}{(a+b)-(c+d)};$$

also shew that if a, b, c, d, are the roots of *any* biquadratic equation, the values of x and y are real.

54. Shew that a square greater in area than the face of the cube in the proportion of 9 to 8, can be just placed within the cube.

55. If two circles touch one another internally, and any circle be described touching both, prove that the sum of the distances of its centre from the centres of the two given circles will be invariable. Also, if a series of such circles be described touching each of the given circles and one another, and a, b, r, be respectively the radii of the given circles, and of the first circle in the series, prove that the radius of the $(n+1)^{th}$ circle will be

$$=\frac{ab(a-b)r}{abr+\{n(a-b)\sqrt{r}\pm\sqrt{ab(a-b-r)}\}^2}.$$

56. If the three sides of a triangle be tangents to a parabola, its area will be half of that of the triangle whose angular points are the points of contact.

57. From the middle point of the hypothenuse of a right-angled triangle draw a line so as to be equally inclined to it and the base, and terminated in the base produced, and from the middle point of this line draw a second equally inclined to

it and the base produced, and terminated in the base produced, and so on; prove that the ultimate value of the length of the last line is $a \div \alpha$, a being the altitude of the triangle, and α the opposite angle.

58. Having given that in the year 1600, the 1st of March fell on the fourth day of the week (Wednesday), shew that in the year $s.\,10^2 + m$ it falls on the x^{th} day, x being the remainder after dividing by 7 the quantity $4 + m + \frac{1}{4}m + 5s + \frac{1}{4}s$, where the integral parts only of $\frac{1}{4}m$, $\frac{1}{4}s$ are to be taken. Enumerate the remaining steps for finding when Easter falls in the same year.

59. The arcs of great circles joining the angular points of a spherical triangle and the poles of the opposite sides, meet one another in the same point, and by their intersection with the sides determine the angular points of a spherical triangle whose perimeter is less than that of any other which can be inscribed in the first.

60. Find the locus of the extremity of a line drawn from the centre of a conic section such that the rectangle contained by it, and the diameter perpendicular to it, is equal to the rectangle under the axes.

61. Find the curve which touches all circles whose centres are in a given curve, and peripheries pass through a given point. Apply the method to shew that when the curve and point are a conic section and its centre, the touching curve is that constructed in the preceding problem.

62. Find the permanent temperature at any point of a fine metallic wire of indefinite length, exposed to a uniform current of air of given temperature, and having one of its extremities subjected to a constant source of heat.

63. If there be n quantities forming a geometric progression whose common ratio is r, and S_m denote the sum of the m first terms of such a series, prove that the sum of their products taken two and two together $= \dfrac{r}{r+1} \cdot S_n \cdot S_{n-1}$.

64. Straight lines are drawn from a fixed point to the several points of a straight line given in position, and on each as base is described an equilateral triangle. Determine the locus of the vertices.

65. Find the volume of any tetrahedron of which the lengths, the inclination, and the least distance of two opposite edges are given.

66. If in any one trial an event may happen m ways and fail in n ways, prove that in $r(m+n)$ trials it is most likely to happen rm times and fail rn times. Find the probability that in the same number of trials it will happen not fewer than $r(m-1)$ times, nor more than $r(m+1)$ times; and shew that this probability continually approximates to certainty, as the number of trials increases.

67. Expand $(\sin^{-1}x)^2$ in a series of ascending powers of x by *Maclaurin's* theorem, and prove that

$$(u)_{x=0} + \frac{x}{1}\cdot\left(\frac{du}{dx}\right)_{x=0} + \frac{x^2}{1.2}\left(\frac{d^2u}{dx^2}\right)_{x=0} + \ldots + \frac{x^n}{1.2.3\ldots n}\left(\frac{d^nu}{dx^n}\right)_{x=0}$$
$$= \frac{x^{n+1}}{1.2\ldots n}\cdot\left\{\frac{d^n}{dz^n}\left(\frac{u_z}{x-z}\right)_{z=0}\right\},$$

where u_x and u_z denote the same functions of the independent variables x and z.

68. Give a construction for drawing normals to a parabola from a given point. Shew that in certain situations of the point it will be possible to draw three, and in others one; and that these situations are separated by a curve from any point of which two normals can be drawn.

69. By what experiments is it shewn that terrestrial bodies are attracted by the earth and by each other, proportionally to their quantities of matter.

70. Having given the number of years which an individual wants of 86, find the present worth of an annuity to be paid during his life; it being supposed that out of 86 persons born, one dies every year till they are all extinct.

71. When m is of the form $\alpha + \beta\sqrt{-1}$ find the relation between a, b, α, β, in order that the expression $(a + b\sqrt{-1})^m$ may be exhibited in a possible form.

72. The sides of a plane triangle are arcs of equal circles, radius $= 1$, prove that the algebraical sum of its sides and angles is always equal to $180°$, those sides being considered negative which are concave to the interior of the triangle. If the three

angles of such a triangle be given, investigate a formula for the determination of a side, and adapt it to logarithmic calculations.

73. Integrate the equations

$$(1 + x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0, \quad \Delta u_x = \frac{x^2}{a^x}, \quad u_{x+1} + u_x = a^x,$$

and shew that if $u = x^2 + px + q$,

$$u^r \cdot \frac{d^{n+r}u^n}{dx^{n+r}} = (n - r + 1) . (n - r + 2) \ldots (n + r) \cdot \frac{d^{n-r}u^n}{dx^{n-r}}.$$

74. Find the equation to a straight line passing through a given point and perpendicular to a given straight line. Shew that the equation

$$y^2 - xy - 2x^2 + 5x - y - 2 = 0$$

represents two straight lines.

75. If S be the surface of a spherical triangle, each of whose angles is 120°, and S′ that of its supplemental triangle,

$$\tan\frac{S}{3} : \tan\frac{S'}{3} = 6\sqrt{2} + \sqrt{3} : 2\sqrt{2} - \sqrt{3}.$$

76. When the price of barley is p shillings a quarter, a farmer gains r per cent. on the capital he expends on a farm of n acres, in which the produce of each acre increases in an arithmetic progression, and the worst acre repays only the cost of its cultivation. When the price falls to $(p - q)$ shillings, shew that in order that each acre may at least repay the capital expended upon it, which is the same for all, *not fewer* than $\dfrac{50(n-1)q}{r(p-q)}$ acres must be thrown out of cultivation.

77. If in a given right-angled triangle elliptic quadrants be inscribed, of which the axes coincide with the sides of the triangle, the locus of the points of intersection of their chords is the evolute of a quadrant of an ellipse.

78. Integrate the following equations:

$$\frac{du}{dx} = \left(\frac{a}{x^2} - \frac{x}{a^2}\right)^{\frac{1}{2}}, \quad \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2 - \frac{d^2y}{dt^2} = y \text{ where } x = e^t + e^{-t},$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{3}{4x^2}, \quad x^2\frac{d^2z}{dx^2} - y^2\frac{d^2z}{dy^2} = xy,$$

and prove that

$$\int e^{-x^2} dx \text{ between the limits } x = \infty,\ x = 0;$$
$$= \frac{e^{-a^2}}{2a} \cdot \cfrac{1}{1 + \cfrac{\frac{1}{2a^2}}{1 + \cfrac{\frac{2}{2a^2}}{1 + \cfrac{\frac{3}{2a^2}}{1 + \dots}}}}$$

79. The equation to that logarithmic spiral which is *its own* evolute, is $\log_\varepsilon \left(\frac{r}{c}\right)^{\frac{1}{\theta}} = \left(\frac{c}{r}\right)^{\frac{(4n-1)\pi}{2\theta}}$, where c is arbitrary, and n any whole number.

80. Prove that

$$\int dx e^{-n \cot 2\beta x^2} \sin (nx^2 + a) = \sqrt{\frac{\pi \sin 2\beta \{\sin (a + \beta)\}^2}{4n}},$$

between the limits $x = \infty$, $x = 0$.

81. The talent of each of two individuals A and B being known to be at least $\frac{1}{m}$th of that of his opponent, the chance that each of them will have been successful in one of the two first trials of ability is $\frac{1 + 4m + m^2}{3(1 + m)^2}$.

Also, having given A's talent at least $\frac{1}{m}$th of B's, and B's talent at least $\frac{1}{n}$th of A's; find their respective chances of success in both trials.

82. The equation to a curve so drawn that every chord passing through the pole shall subtend a right angle from a fixed point is

$$r \cos \left(\frac{2\lambda \pm 1}{2} \theta + f(\theta)\right) = a \cos \left(\frac{2\lambda \mp 1}{2} \theta + f(\theta)\right)$$

where λ is any whole number, and $f(\theta + \pi) - f(\theta) = 0$; give also a complete solution of the latter equation.

83. The lines drawn from the angles of a spherical triangle to the middle points of the opposite sides, meet in one point;

and if E be the spherical excess, a' the arc joining the middle points of the sides b and c, $\cos a' = \cos\frac{a}{2}\cdot\cos\frac{E}{2}$.

84. Let four points be taken at random in a plane, join them two and two in every possible way, the joining lines being produced, if necessary, to intersect. Join these points of intersection two and two, in every possible way, producing as before the joining lines. Every line in the figure so formed is divided harmonically.

85. The constitution of a medium is such that if through any particle a line be drawn, all the particles through which it passes are symmetrically arranged upon it. Shew that the disturbance of a particle by the transmission of a plane wave through such a medium is expressed by the following equations:

$$\frac{d^2\xi}{dt^2} = \frac{d^2}{dx^2}(A\xi + D\eta + E\zeta),$$

$$\frac{d^2\eta}{dt^2} = \frac{d^2}{dx^2}(D\xi + B\eta + F\zeta),$$

$$\frac{d^2\zeta}{dt^2} = \frac{d^2}{dx^2}(E\xi + F\eta + C\zeta),$$

the coefficients of ξ, η, ζ being certain constants dependent on the nature of the medium; x, y, z coordinates of the place of rest of the particle, of which ξ, η, ζ are the disturbances at the time t parallel to the coordinates; and the plane yz is supposed parallel to the front of the wave.

From the three equations in the last question find by integration the values of ξ, η, and ζ; and deduce an expression for the velocity with which the wave is transmitted through the medium.

86. Find the volume of any tetrahedron of which the lengths, the inclination, and the least distance of two opposite edges are given.

87. A certain effect, the measure of which is z, is observed to be periodic and continuous; and its periodicity is found to depend upon two quantities, whose measures are x, y, in such a manner that while x gradually increases to $x + h$, or while

y gradually increases to $y + k$, z goes through one period. Shew that the dependence of z on x and y may be expressed by the equation

$$z = \Sigma \left\{ a_i \sin 2i\pi \left(\frac{x}{h} \pm \frac{y}{k} + C_i \right) \right\},$$

i being any integer. Shew also that any periodic and continuous effect may be considered as resulting from the superposition of a number of symmetrical effects which differ only in their intensity and period.

88. If n be a prime number, the expression

$$A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots + A_m$$

cannot admit of more than m different integral values of x, less than n, which will render it a multiple of n; the coefficients being integers, and A_0 prime to n. Also when it does admit of exactly m such different values of x, the quantities

$$A_0 S_1 + A_1;\ A_0 S_2 - A_2;\ A_0 S_3 + A_3;\ \ldots\ A_0 S_m - (-1)^m A_m$$

are multiples of n: S_r denoting the sum of the products of those values of x taken r at a time. Prove these properties, and thence shew that

$$(1 . 2 . 3 \ldots . n - 1) + 1;$$

and $(1 . 2 . 3 \ldots . \overline{n-1}) \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n-1}\right)$

are multiples of n.

89. If the vertical angle of a cone be less than 90°, it may be cut by two planes at right angles to each other so that the sections are parabolas.

Two planes are drawn cutting a cone in parabolas, so that the principal sections perpendicular to them are at right angles to each other, and the line joining the vertices of the parabolas is projected on a plane through the vertex, and perpendicular to the axis of the cone; shew that this projection is bisected by the plane which passes through the vertex and the line of intersection of the cutting planes.

90. How many terms are there in a symmetrical function of m letters, each term containing n of them, and each letter in the term raised to some given power, supposing several of the indices in each term equal to each other?

A number of letters are combined together in homogeneous products, the sum of the indices in each being n, shew that the sum of all such combinations when any letter (a) is omitted minus the sum when any other (b) is omitted is equal to $(b-a)$ times the sum when all the letters are taken, and the sum of the exponents in each term is $n-1$.

91. Let τ be the mean interval between successive passages of the moon over the meridian of a place in lat. λ, and δ be her declination at a time t, reckoned from the high-water of a known spring-tide; let τ', δ' be corresponding quantities for the sun: prove, on the supposition that each luminary causes vertical isochronous oscillations of the ocean, which vary slowly in extent, that the height of the tide, at that time above its mean height, is nearly

$$M\cos^2(\lambda-\delta)\cos\frac{4\pi t}{\tau}+S\cos^2(\lambda-\delta')\cos\frac{4\pi t}{\tau'}.$$

THE END.

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