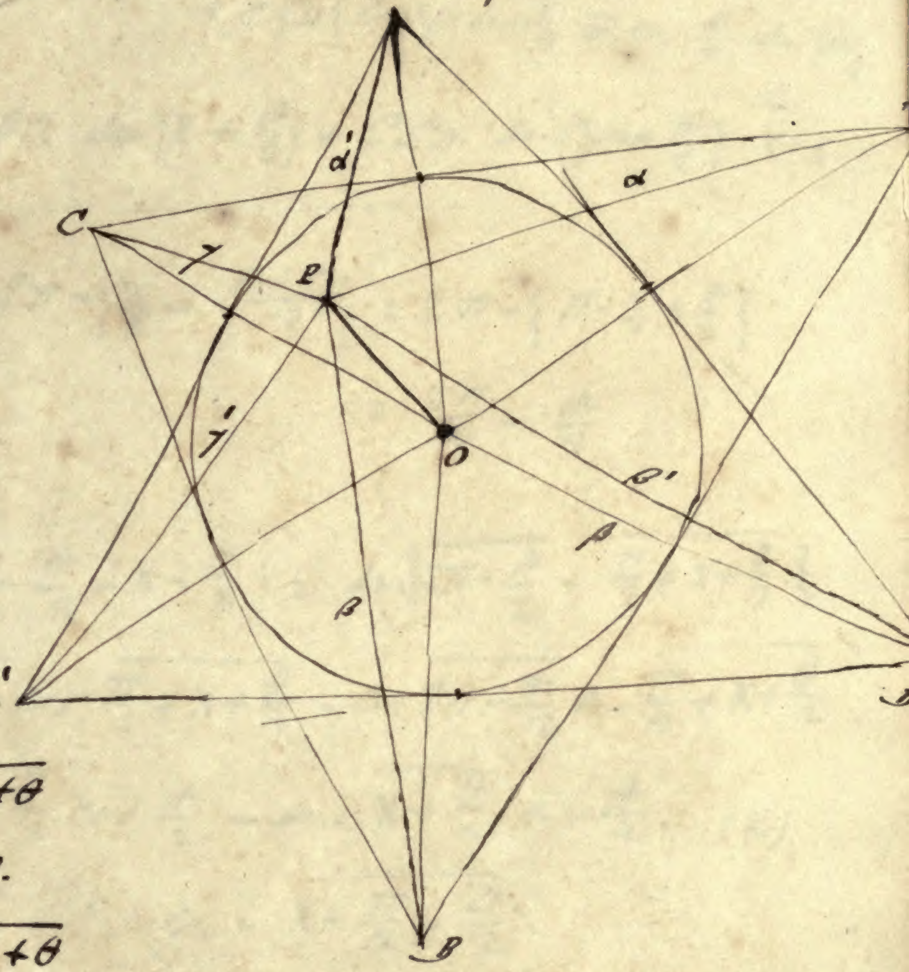
[Mathematical problems
Vol. 2]

Trigonometrical Problems

If the centre of the inscribed ⊙ of a triangle be fixed and α. β. γ represent the distances of its angles from any other fixed point then will $a\alpha^2 + b\beta^2 + c\gamma^2$ be constant for all positions of the Δ.



ABC the initial points of the Δ.
O the cent. of the inscribed ⊙
P any fixed point
A'B'C' the points of the Δ after it has revolved thro' any angle θ

$\angle POA' = \varphi \therefore \angle POA = \overline{\varphi+\theta}$

$\angle POC' = \psi \therefore \angle POC = \overline{\psi-\theta}$

$\angle POB' = \chi \therefore \angle POB = \overline{\chi+\theta}$

now.

$\alpha^2 = AO^2 + PO^2 - 2AO.PO.\cos\overline{\varphi+\theta}$

$\alpha'^2 = A'O^2 + PO^2 - 2A'O.PO.\cos\varphi.$

$\beta^2 = BO^2 + PO^2 - 2BO.PO.\cos\overline{\chi+\theta}$

$\beta'^2 = B'O^2 + PO^2 - 2B'O.PO.\cos\chi.$

$\gamma^2 = CO^2 + PO^2 - 2CO.PO.\cos\overline{\psi-\theta}$

$\gamma'^2 = C'O^2 + PO^2 - 2C'O.PO.\cos\psi.$

$\therefore a\alpha^2 + b\beta^2 + c\gamma^2 \gtreqless a\alpha'^2 + b\beta'^2 + c\gamma'^2$

$a\,AO\cos\overline{\varphi+\theta} + b.BO\cos\overline{\chi+\theta} + cCO\cos\overline{\psi-\theta} \gtreqless a\,A'O\cos\varphi + b\,B'O\cos\chi + cC'O\cos\psi$

$a\,AO\{\cos\varphi\cos\theta - \sin\varphi\sin\theta - \cos\varphi\} \gtreqless \left\{\begin{array}{l} b.BO.\{\cos\chi - \cos\chi\cos\theta + \sin\chi\sin\theta\} \\ +cCO\{\cos\psi - \cos\psi\cos\theta - \sin\psi\sin\theta\} \end{array}\right.$

$-a\,AO\{\cos\varphi(1-\cos\theta) - \sin\varphi\sin\theta\} \gtreqless \left\{\begin{array}{l} b.BO.\{\cos\chi(1-\cos\theta) + \sin\chi\sin\theta\} \\ +c.CO\{\cos\psi(1-\cos\theta) - \sin\psi\sin\theta\} \end{array}\right.$

$$-a.AO.\left\{\cos\frac{\theta}{2}\sin\phi+\sin\frac{\theta}{2}\cos\phi\right\} > = < \left\{ \begin{array}{l} b.Bo\left(\cos\chi\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\sin\chi\right) \\ +c.Co\left\{\cos\psi\sin\frac{\theta}{2}\mp\cos\frac{\theta}{2}\sin\psi\right\}. \end{array}\right.$$

$$-a\,AO\sin\left(\phi+\frac{\theta}{2}\right) > = < b.BO.\sin\left(\chi+\frac{\theta}{2}\right)+c.Co.\sin\left(\psi\mp\frac{\theta}{2}\right) \quad (1)$$

Now $\chi+\psi = 2\pi - C'OB' = 2\pi - \left\{\pi - \frac{B+C}{2}\right\} = 2\pi - \left\{\pi - \frac{\pi}{2} + \frac{A}{2}\right\}$

$$= \frac{3\pi}{2} - \frac{A}{2}.$$

$$\therefore \psi = \frac{3\pi}{2} - \frac{A}{2} - \chi.$$

$$\therefore \sin\left(\psi-\frac{\theta}{2}\right) = \sin\left(\frac{3\pi}{2}-\frac{A}{2}-\chi-\frac{\theta}{2}\right) = \sin\left\{\overline{2\pi-\frac{A}{2}} - \overline{\frac{\pi}{2}+\chi+\frac{\theta}{2}}\right\}$$

$$= \sin\overline{2\pi-\frac{A}{2}}\cos\overline{\frac{\pi}{2}+\chi+\frac{\theta}{2}} - \cos\overline{2\pi-\frac{A}{2}}\sin\overline{\frac{\pi}{2}+\chi+\frac{\theta}{2}}$$

$$= \cos\overline{\chi+\frac{\theta}{2}}\cos\frac{A}{2} - \sin\overline{\chi+\frac{\theta}{2}}\sin\frac{A}{2}. \quad (2).$$

Also. $\overline{\chi-\phi} = \frac{\pi}{2}+\frac{C}{2} \quad \therefore \phi = \overline{\chi-\frac{\pi}{2}-\frac{C}{2}}$

$$\therefore \sin\overline{\phi+\frac{\theta}{2}} = \sin\left\{\overline{\chi+\frac{\theta}{2}} - \overline{\frac{\pi}{2}+\frac{C}{2}}\right\}$$

$$= \sin\overline{\chi+\frac{\theta}{2}}\cos\overline{\frac{\pi}{2}+\frac{C}{2}} - \cos\overline{\chi+\frac{\theta}{2}}\sin\overline{\frac{\pi}{2}+\frac{C}{2}}$$

$$= \sin\overline{\chi+\frac{\theta}{2}}\sin\frac{C}{2} - \cos\overline{\chi+\frac{\theta}{2}}\cos\frac{C}{2}. \quad (3).$$

Substituting in (1) the values of $\sin\overline{\phi+\frac{\theta}{2}}$, $\sin\overline{\psi-\frac{\theta}{2}}$ found in (2) and (3) we have.

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(1) [illegible]

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[illegible]

(2) [illegible]

[illegible]

[illegible]

[illegible]

(3) [illegible]

[illegible]

[illegible]

$$a\alpha^2+b\beta^2+c\gamma^2 >=< a\alpha'^2+b\beta'^2+c\gamma'^2$$

$$a.AO\left\{\begin{matrix}\cos\overline{X+\frac{\theta}{2}}\cos\frac{C}{2}\\+\sin\overline{X+\frac{\theta}{2}}\sin\frac{C}{2}\end{matrix}\right\}^2 >=< \left\{\begin{matrix}b.BO\sin\overline{X+\frac{\theta}{2}}\\+cCO\left\{\cos\overline{X+\frac{\theta}{2}}\cos\frac{A}{2}-\sin\overline{X+\frac{\theta}{2}}\sin\frac{A}{2}\right\}\end{matrix}\right.$$

Again we have $AO = r\operatorname{cosec}\frac{A}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\cdot\sqrt{\frac{bc}{(s-b)(s-c)}}$

$$= \sqrt{\frac{(s-a)\cdot bc}{s}}$$

also $\cos\frac{C}{2} = \sqrt{\frac{s.(s-c)}{ab}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

$$\therefore aAO\cos\frac{C}{2} = a\sqrt{\frac{(s-a).bc}{s}}\cdot\sqrt{\frac{(s-c)s}{ab}} = \sqrt{(s-a)(s-c)ac} \quad (5)$$

$$a\,AO.\sin\frac{C}{2} = a\sqrt{\frac{(s-a)bc}{s}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}} = (s-a)\sqrt{\frac{s-b}{s}.ac}. \quad (6)$$

Similarly $CO = r\operatorname{cosec}\frac{A}{2} = \sqrt{\frac{(s-c)ab}{s}}$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \;;\; \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$\therefore c.CO.\cos\frac{A}{2} = c\sqrt{\frac{(s-c)ab}{s}}\sqrt{\frac{s(s-a)}{bc}} = \sqrt{(s-a)(s-c)ac}. \quad (7)$$

$$c.CO\sin\frac{A}{2} = c\sqrt{\frac{(s-c)ab}{s}}\sqrt{\frac{(s-b)(s-c)}{bc}} = (s-c)\sqrt{\frac{s-b}{s}.ac}. \quad (8).$$

Hence collecting all the terms and substituting the values of $a.AO\cos\frac{C}{2}$; $a\,AO.\sin\frac{C}{2}$; $c.CO\cos\frac{A}{2}$; $cCO\sin\frac{A}{2}$. found in (5).(6).(7)(8) in $=^n$ (5) we have.

$$a\alpha^2 + b\beta^2 + c\gamma^2 >=< a\alpha'^2 + b\beta'^2 + c\gamma'^2$$

$$\left.\begin{array}{l}\sqrt{(s-a)(s-c).ac}\cos\overline{X+\frac{\theta}{2}} \\ +(s-a)\sqrt{\frac{(s-b)ac}{s}}\sin\overline{X+\frac{\theta}{2}}\end{array}\right\} >=< \left\{\begin{array}{l} b\sqrt{\frac{s-b}{s}ac}\sin\overline{X+\frac{\theta}{2}} \\ +\sqrt{(s-a)(s-c)ac}\cos\overline{X+\frac{\theta}{2}} \\ -(s-c)\sqrt{\frac{(s-b)ac}{s}}\sin\overline{X+\frac{\theta}{2}}\end{array}\right.$$

$$(s-a)\sqrt{\frac{(s-b)ac}{s}}\sin\overline{X+\frac{\theta}{2}} >=< (b-s+c)\sqrt{\frac{(s-b)ac}{s}}.\sin\overline{X+\frac{\theta}{2}}$$

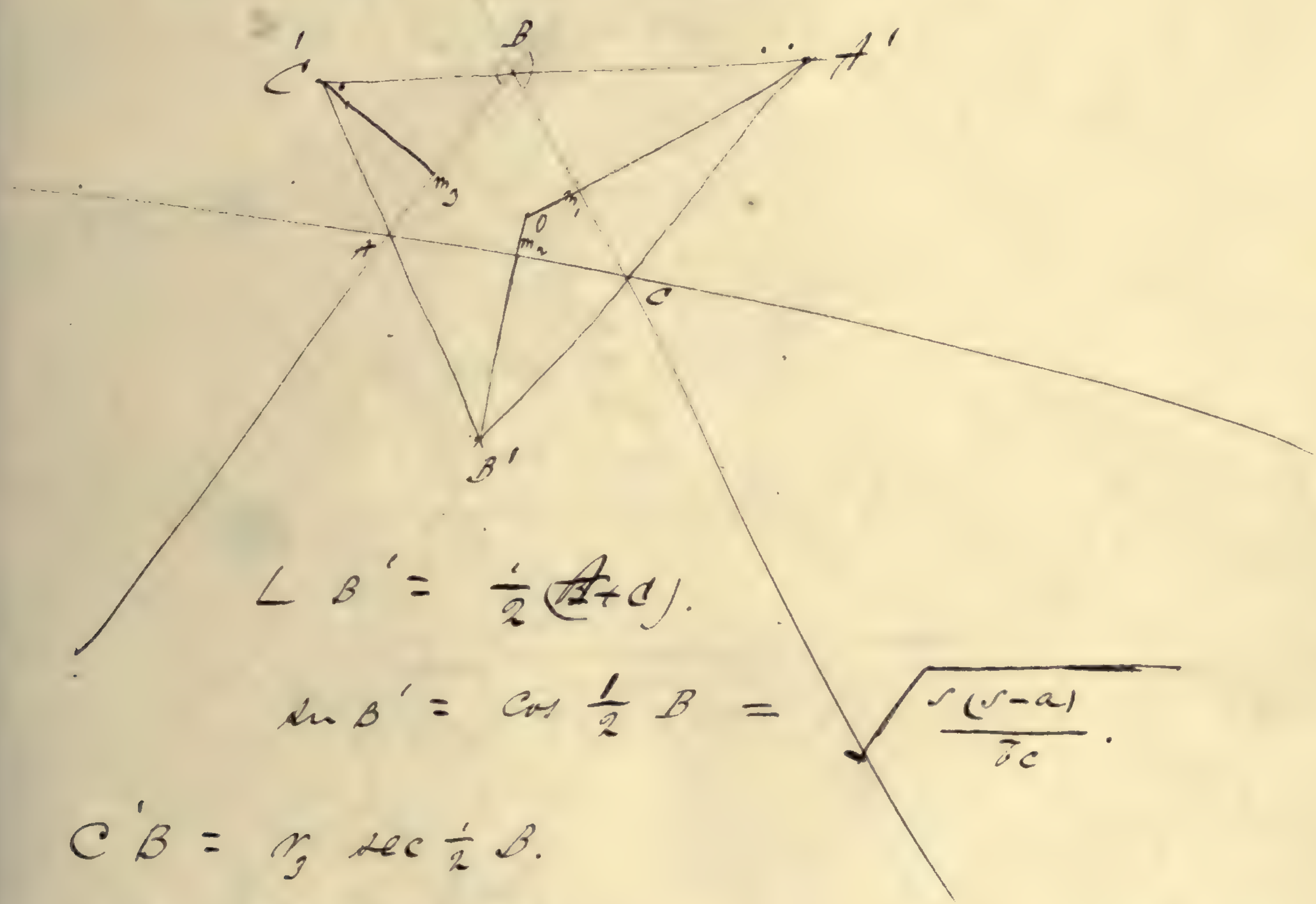
$$s-a >=< b-s+c.$$

$$2s >=< a+b+c.$$

But $2s = a+b+c \therefore a\alpha^2 + b\beta^2 + c\gamma^2 = a\alpha'^2 + b\beta'^2 + c\gamma'^2.$

and as the result is independent both of θ and of the distance OP. it is evident the result holds whatever be the value of θ or whatever be the distance OP.

The centres of the escribed ⊙s of a triangle are joined forming another triangle: this is treated similarly and so on; shew that the radii of the circumscribing ⊙s form a geometric progn whose common ratio is 2.



$$\angle B' = \frac{1}{2}(A+C).$$

$$\sin B' = \cos\frac{1}{2}B = \sqrt{\frac{s(s-a)}{bc}}.$$

$$C'B = r_3 \sec\tfrac{1}{2}B.$$

$$A'B = r_1 \sec\tfrac{1}{2}B.$$

$$\therefore A'C' = (r_3 + r_1)\sec\tfrac{1}{2}B.$$

$$R_2 = A'O = \tfrac{1}{2}A'C' \sec\tfrac{1}{2}B$$

$$= \tfrac{1}{2}(r_3 + r_1)\sec^2\tfrac{1}{2}B.$$

$$r_3 = \frac{S}{s-c} = \frac{\sqrt{s(s-a)(s-b)}}{\sqrt{s-c}}; \quad r_1 = \frac{\sqrt{s(s-b)(s-c)}}{\sqrt{s-a}}$$

$$\therefore R_2 = \frac{1}{2}\sqrt{s(s-b)}\left\{\frac{\sqrt{s-a}}{\sqrt{s-c}} + \frac{\sqrt{s-c}}{\sqrt{s-a}}\right\}\frac{ac}{s(s-b)}$$

$$= \frac{ac}{2}\,\frac{s-a+s-c}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{acb}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

Also $R_1 = \dfrac{b}{2\sin B}$.

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2}$$

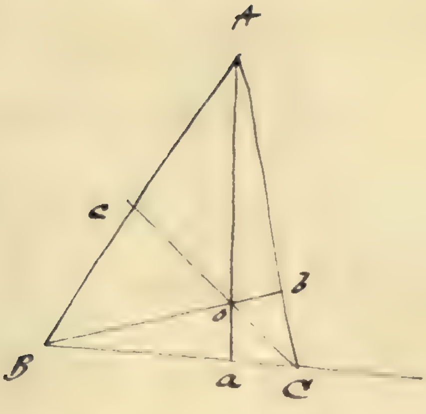
$$= \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore R_1 = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

$$\therefore R_2 = 2R_1$$

wherefore the successive radii form a geometrical progressn. with common ratio 2.

⊥ from the ∠s upon the opposite side meet in a point.



Eqn to AC is. (take a for origin. & aA, aC axes of y, x.)

$$\frac{y}{Aa} + \frac{x}{aC} = 1$$

$$y = -\frac{Aa}{aC}x + Aa.$$

Eqn to the ⊥ Bb is $y = \frac{Ca}{Aa}(x - Ba)$.

~~similarly eqn to ⊥ Cc is~~

$$y = Ba$$

Eqn to AB is $\frac{y}{Aa} - \frac{x}{Ba} = 1$

$$y = \frac{Aa}{Ba}x + Aa.$$

Eqn to Cc is $y = -\frac{Ba}{Aa}(x + aC)$

at the point of intersectn of these lines the values of y are equal.

$$\therefore \frac{Ca}{Aa}(x - Ba) = -\frac{Ba}{Aa}(x + aC)$$

$$\frac{Ca}{Aa}x = -\frac{Ba}{Aa}x.$$

or $x = 0$.

∴ the point of intersectn is in Aa.

$$\frac{y}{C'C} - \frac{x}{AC} = 1$$

$$y = \frac{C'C}{AC} x + C'C$$

$$y = - \frac{AC}{C'C}(x - CB)$$

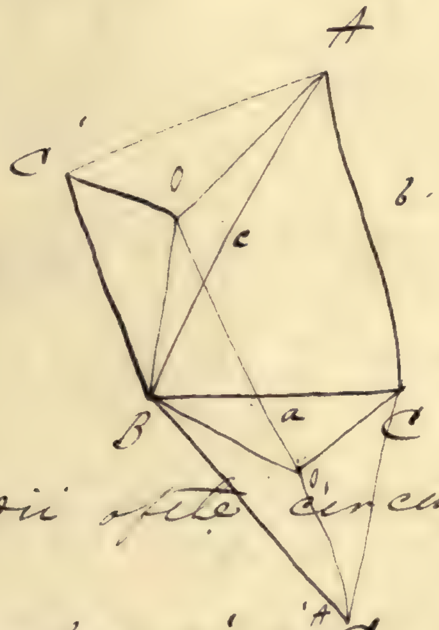
$$= - \frac{AC}{C'C}$$

$$y = \frac{CB}{C'C} \cdot (x - C)$$

$$C'C\,y = - AC\,x + AC \cdot CB$$

$$C'C\,y = CB\,x - CB \cdot CB.$$

If on the sides of a $\triangle$ are constructed 3 $\triangle$s whose vertical angles are equal to those of the $\triangle$ respectively opposite shew that the $\triangle$ formed by joining the centres of the $\odot$'s circumscribed about them is in every respect equal to the given one.



Let $r.\ r_1\ r_2$ be the radii of the circumscribed $\odot$'s

$\therefore r = \frac{c}{\sin C'} = \frac{c}{\sin C}$ for $C' = C$.

$r_1 = \frac{a}{\sin A'} = \frac{a}{\sin A}$ for $A = A' = \frac{c}{\sin C} = r$

$\therefore r = r_1 = r^2$.

$\therefore AO.\ BO = BO.\ BO_1$

$\angle OBO = \angle B$ for each $= 2C'$.

$\therefore OO' = AB$

similarly for the other sides.

$$\cos 7\theta = \cos^7\theta - \frac{7.6}{1.2}\cos^5\theta(1-\cos^2\theta) + \frac{7.6.5.4}{1.2.3.4}\cos^3\theta\{1-2\cos^2\theta+\cos^4\theta\}$$

$$- \frac{7.6.5.4.3.2}{1.2.3.4.5.6}\cos\theta.\{1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\}.$$

$$=(1+21+35+7)\cos^7\theta - \{21+70+21)\cos^5\theta$$

$$+(35+21)\cos^3\theta - 7\cos\theta.$$

$\therefore$ $\cos 7\theta + 7\cos\theta = 0$ becomes

$$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta = 0.$$

$$\therefore \cos^3\theta = 0 \quad \therefore \theta = \frac{\pi}{2} = \frac{3\pi}{2} = \frac{5\pi}{2}.$$

$$8\cos^4\theta - 14\cos^2\theta + 7 = 0.$$

from which the other values of θ are easily deduced.

Solvelle $=$ⁿ. $\cos 7\theta + 7\cos\theta = 0.$

Shew that $\theta = \sin\theta \sec\frac{\theta}{2} \sec\frac{\theta}{2^2} \sec\frac{\theta}{2^3}$ &c to ∞.

we have $\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$.

$$= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}.$$

$$= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

$$= 2^4\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^4}\sin\frac{\theta}{2^4}$$

$$= 2^\infty\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \text{ \&c to infy. } \sin\frac{\theta}{2^\infty}.$$

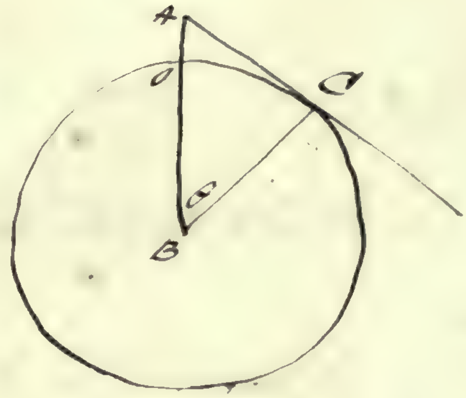
$\therefore 2^\infty \sin\frac{\theta}{2^\infty} = \sin\theta \sec\frac{\theta}{2}\sec\frac{\theta}{2^2}\sec\frac{\theta}{2^3}$ &c to infinity.

But $\sin\frac{\theta}{2^\infty} = \frac{\theta}{2^\infty}$.

$\therefore 2^\infty \sin\frac{\theta}{2^\infty} = \theta$.

or $\theta = \sin\theta \sec\frac{\theta}{2}\sec\frac{\theta}{2^2}\sec\frac{\theta}{2^3}$ &c to infinity.

If the height of a station in feet be increased by $\frac{1}{2}$ and the square root be taken, the result is the distance of the horizon in miles nearly:



$AC = r.\tan\theta.$

$\cos\theta = \frac{r}{r+a} \quad \therefore \tan^2\theta = \sec^2\theta - 1$

$= \left(\frac{r+a}{r}\right)^2 - 1$

$\therefore \frac{AC^2}{r^2} = \frac{r^2 + 2ar + a^2 - r^2}{r^2}$

$AC = \sqrt{2ar + a^2} = \sqrt{2ar}$ nearly.

now $r = 5280 \times 3960$ feet nearly.

$= \frac{3}{4}.(5280)^2.$

$\therefore AC = 5280\sqrt{\frac{3}{2}a}$ feet nearly

$= \sqrt{\frac{3}{2}a}$ miles nearly.

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$$\frac{\cos A - \sin B \sin C}{\cos B \cos C} = \frac{\cos D - \sin E \sin F}{\cos E \cos F}$$

$$\frac{\cos A - \sin B \sin C}{\cos B \cos C} \mp 1 = \frac{\cos D - \sin E \sin F}{\cos E \cos F} \mp 1$$

$$\frac{\cos A - \sin B \sin C \mp \cos B \cdot \cos C}{\cos B \cos C} = \frac{\cos D - \sin E \sin F \mp \cos E \cos F}{\cos E \cos F}$$

$$\frac{\cos A \mp \cos(B \mp C)}{\cos B \cos C} = \frac{\cos D - \cos \overline{E - F}}{\cos E \cos F}$$

$$\frac{\cos B \cos C}{\cos E \cos F} = \frac{-\cos A + \cos \overline{B - C}}{-\cos D + \cos \overline{E - F}}$$

$$= \frac{1 - \cos A - (1 - \cos \overline{B - C})}{1 - \cos D - (1 - \cos \overline{E - F})}$$

$$= \frac{\operatorname{vers} A - \operatorname{vers} \overline{B - C}}{\operatorname{vers} D - \operatorname{vers} \overline{E - F}}$$

$$\ldots{}^{2}\log.\ \frac{a+\sqrt{a^2-x^2}}{x} = \frac{a^3}{3}\log.\ \frac{a+\sqrt{a^2-x^2}}{x} - \frac{1}{3}\int_{/x} x^3.$$

(1) $\sin a + \sin\beta + \sin\gamma = 4\cos\frac{1}{2}a\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma$.

$(a+\beta) = 180^\circ - \gamma$.

$\frac{1}{2}(a+\beta) = 90^\circ - \frac{1}{2}\gamma \quad \sin\frac{1}{2}\overline{a+\beta} = \sin 90^\circ - \frac{1}{2}\gamma = \cos\frac{1}{2}\gamma$.

$\sin a + \sin\beta = 2\sin\frac{1}{2}\overline{a+\beta}.\cos\frac{1}{2}\overline{a-\beta}$.

$= 2\cos\frac{1}{2}\gamma\cos\frac{1}{2}\overline{a-\beta}$.

$\sin\gamma = 2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma$.

$\therefore \sin a + \sin\beta + \sin\gamma = 2\cos\frac{1}{2}\gamma(\cos\frac{1}{2}\overline{a-\beta} + \sin\frac{1}{2}\gamma)$.

$= 2\cos\frac{1}{2}\gamma(\cos\frac{1}{2}\overline{a-\beta}. + \cos\frac{1}{2}\overline{a+\beta}.$

$= 4\cos\frac{1}{2}a\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma$.

(2) $\sin a - \sin\beta + \sin\gamma = 4\sin\frac{1}{2}a\cos\frac{1}{2}\beta\sin\frac{1}{2}\gamma$.

$\sin a - \sin\beta = 2\cos\frac{1}{2}\overline{a+\beta}\sin\frac{1}{2}\overline{a-\beta}$.

$= 2\sin\frac{1}{2}\gamma\sin\frac{1}{2}\overline{a-\beta}$.

$\sin\gamma = 2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma$.

$\therefore \sin a - \sin\beta + \sin\gamma = 2\sin\frac{1}{2}\gamma\sin\frac{1}{2}\overline{a-\beta}. + 2\cos\frac{1}{2}\gamma\sin\frac{1}{2}\gamma$.

$= 2\sin\frac{1}{2}\gamma(\sin\frac{1}{2}\overline{a-\beta}. + \sin\overline{a+\beta}.)$.

$= 4\sin\frac{1}{2}\gamma\sin\frac{1}{2}a\cos\frac{1}{2}\beta$.

$\sin 2a + \sin 2\beta + \sin 2\gamma = 4\sin a\sin\beta\sin\gamma$.

$\sin 2a + \sin 2\beta = 2\sin\overline{a+\beta}\cos\overline{a-\beta}. = 2\sin\gamma\cos\overline{a+\beta}$.

$\sin 2\gamma = 2\cos\gamma\sin\gamma$.

$\sin 2a + \sin 2\beta + \sin 2\gamma. = 2\sin\gamma(\cos\overline{a+\beta}. + \cos\overline{a-\beta}.)$.

$= 4\sin\gamma(\sin a\sin\beta)$.

The angles of a spherical Δ are each $= 120°$. If s be the area of the Δ, s' the area of the polar Δ.

Then $\tan\frac{s}{3} : \tan\frac{s'}{3} :: 6\sqrt{2}+\sqrt{3} : 2\sqrt{2}-\sqrt{3}$.

Since each ~~side of~~ of the $\angle$s of the Δ cont. $120°$ we have $A + a' = 180°$ $\therefore a' = 60° = b' = c'$.

$$\therefore \frac{a'+b'+c'}{2} = 90°$$

$$s-a = s-b = s-c = 30°$$

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \cdot \sin(s-a)}} = \frac{1}{\sqrt{2}} \text{ or } \tan A' = 2\sqrt{2}.$$

$$\therefore s' = 3\tan^{-1} 2\sqrt{2} - 180°$$

$$\frac{s'}{3} = \tan^{-1} 2\sqrt{2} - 60° = \tan^{-1} \frac{2\sqrt{2}-\sqrt{3}}{1+2\sqrt{6}}$$

$$\text{also } \frac{s}{3} = 60°.$$

$$\therefore \tan\frac{s}{3} : \tan\frac{s'}{3} :: \sqrt{3} : \frac{2\sqrt{2}-\sqrt{3}}{1+2\sqrt{6}}.$$

$$:: 6\sqrt{2}+\sqrt{3} : 2\sqrt{2}-\sqrt{3}.$$

Prove that any small error of the Solar Tables in AR. (R). & North Polar distance (Δ), & obliquity (I). are connected by the =n.

$$\delta\Delta + \sin\Delta \sin\lambda \sec I \,\delta I + \sin\Delta \cos\lambda \tan I.\, \delta R = 0.$$

we have

$$\cos\Delta = \sin I.\sin\lambda$$

$$\tan R = \cot\Delta \cot I$$

$$\cos\lambda = \cos R \sin\Delta.$$

$$\cos R\,\delta a + \operatorname{cosec}\Delta \cot I\,\delta\Delta + \operatorname{cosec}^2 I \cot\Delta\,\delta I = 0.$$

$$\cos\lambda\,\delta a + \operatorname{cosec}\Delta \cot I\,\delta\Delta + \operatorname{cosec}^2 I \cos\Delta.\,\delta I = 0.$$

$$\text{or}\quad \cos\lambda \sin\Delta \tan I\,\delta R + \delta\Delta + \frac{\cos\Delta \sin\Delta}{\cos I \sin I}\,\delta I = 0$$

$$\text{or}\quad \delta\Delta + \cos\lambda \sin\Delta \tan I\,\delta R + \sin\lambda \sec I \sin\Delta\,\delta I = 0.$$

The angles of a plane triangle are in geometrical prog. and the common ratio is $\frac{1}{2}$. shew that the greater side =

$$\text{2 perimeter} \sin 12^\circ.\tfrac{6}{7}.$$

Let x. y. z. be the sides. a. 2a. 4a the angles.

$$\therefore \frac{x}{y} = \frac{\sin a}{\sin 2a} = \frac{1}{2\cos a} \quad \therefore y = 2x\cos a.$$

$$\frac{x}{z} = \frac{\sin a}{\sin 4a} = \frac{1}{4\cos a\cos 2A} \quad \therefore z = 4x\cos a\cos 2A.$$

$\therefore$ if z is the 7th side we have.

$$4\cos a\cos 2A\, 7 = \angle\, 2\left(1 + 2\cos a + 4\cos a\cos 2a\right)\sin\frac{a}{2}.$$

$$4\cos\frac{a}{2}\cos a\cos 2A\, 7 = \angle\left(1 + 2\cos a + 4\cos a\cos 2a\right)2\sin\frac{a}{2}\cos\frac{a}{2}.$$

$$2\cos\frac{a}{2}\left(\cos 3a + \cos a\right)7 = \angle\, \sin a + 2\sin a\cos a + 4\cos a\sin a\cos 2a.$$

$$\left.\begin{array}{l}\cos\frac{7a}{2} + \cos\frac{5a}{2}\\ + \cos\frac{3a}{2} + \cos\frac{a}{2}\end{array}\right\} 7 = \angle\, \sin a + \sin 2a + \sin 4a.$$

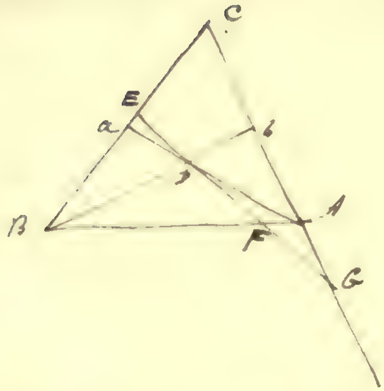
Now since $a + 2a + 4a = 7a = \pi$, $\frac{7a}{2} = \frac{\pi}{2} \therefore \cos\frac{7a}{2} = 0.$

also $\cos\frac{5a}{2} = \sin a$, $\cos\frac{3a}{2} = \sin 2a$. $\cos\frac{a}{2} = \sin 4a$.

$$\therefore \left.\begin{array}{l}\cos\frac{7a}{2} + \cos\frac{5a}{2}\\ + \cos\frac{3a}{2} + \cos\frac{a}{2}\end{array}\right\} = \sin a + \sin 2A + \sin 4a.$$

$\therefore$ the greatest side $= 2\,\text{perim} \sin 12^\circ\frac{6}{7}.$

$$\frac{Da}{DE} = \frac{3\,Da}{DE} = 3 \cdot \frac{\sin CEG}{\sin AaC}$$



$$\therefore \frac{1}{DE} = 3 \cdot \frac{\sin CEG}{AC \cdot \sin C} = \frac{3}{AC} \cdot \frac{CG}{EG}$$

$$\therefore \frac{GE}{DE} = \frac{3 \cdot CG}{AC}$$

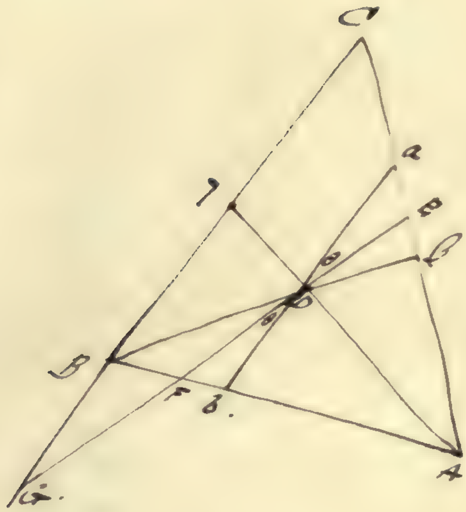
Sim^{ly} $$\frac{1}{DF} = \frac{\sin DFA}{DA \cdot \sin aAB} = 3 \cdot \frac{\sin DFA}{DC \cdot \sin B} = \frac{3}{AC} \cdot \frac{\sin DFA}{\sin A} = \frac{3}{AC} \cdot \frac{AG}{FG}$$

$$\therefore \frac{FG}{DF} = \frac{3 \cdot AG}{AC}$$

$$\therefore \frac{GE}{DE} - \frac{FG}{DF} = 3$$

or $$\frac{DG}{DE} - \frac{DG}{DF} = 1$$

$$\frac{1}{DE} - \frac{1}{DF} = \frac{1}{DG}$$



$$\frac{Da}{DE} = \frac{\sin(C+\theta)}{\sin C} \quad \therefore \quad \frac{1}{DE} = \frac{3}{a}\left\{\frac{\sin(C+\theta)}{\sin C}\right\}.$$

$$\frac{Db}{DF} = \frac{\sin(B-\theta)}{\sin B} \quad \therefore \quad \frac{1}{DF} = \frac{3}{a}\left\{\frac{\sin(B-\theta)}{\sin B}\right\}.$$

$$\therefore \quad \frac{1}{DE} - \frac{1}{DF} = \frac{3}{a}\left\{\cos\theta + \cot C \sin\theta - \cos\theta + \cot B \sin\theta\right\}.$$

$$= \frac{3}{a}\cdot\left\{\frac{\sin(B+C)\sin\theta}{\sin C \sin B}\right\} = \frac{3}{a}\cdot\frac{\sin A \sin\theta}{\sin C \sin B}.$$

$$\frac{Fb}{Db} = \frac{\sin\theta}{\sin(B-\theta)} \quad \therefore \quad Fb = \frac{a}{3}\cdot\frac{\sin\theta}{\sin(B-\theta)} = \frac{b}{3}\cdot\frac{\sin A \sin\theta}{\sin B \sin(B-\theta)} = \frac{c}{3}\cdot\frac{\sin A \sin\theta}{\sin C \sin(B-\theta)}$$

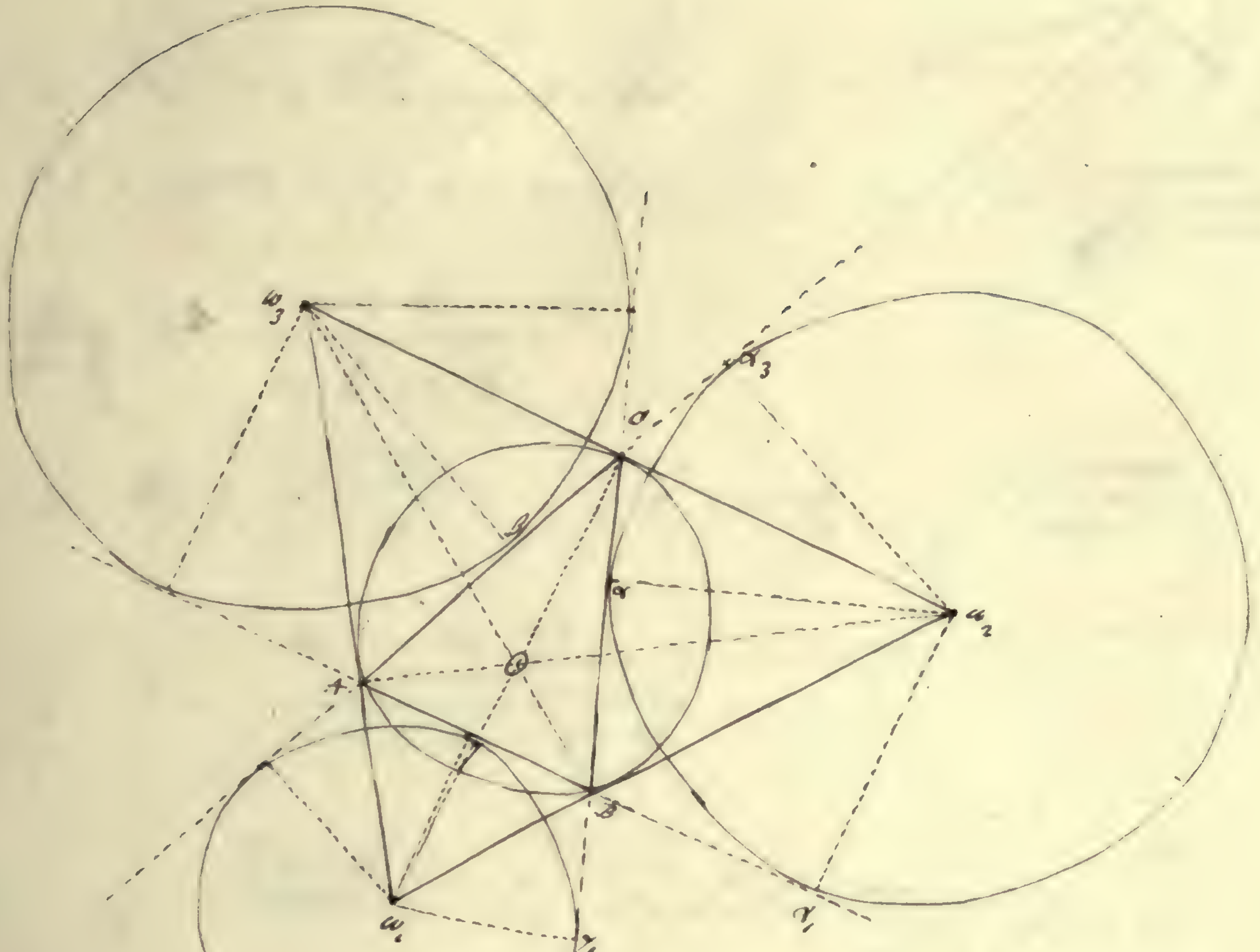
also by similar Δs $bF : bB :: DE : DG$

$$\therefore \quad \frac{1}{DG} = \frac{1}{DF} \times \frac{\sin A \sin\theta}{\sin C \sin(B-\theta)} \quad \text{for } Bb = \frac{c}{3}.$$

$$= \frac{3}{a}\cdot\frac{\sin(B-\theta)}{\sin B} \times \frac{\sin A \sin\theta}{\sin C \sin(B-\theta)}$$

$$= \frac{3}{a}\,\frac{\sin A \sin\theta}{\sin C \sin B}$$

$$= \frac{1}{DE} - \frac{1}{DF}.$$



(1) $\angle \gamma B \gamma_1 = \angle \alpha B \alpha_1 = \pi - C$ Also $(B\omega_2)^2 = (\omega_2 \alpha_2)^2 + (\alpha B)^2 = (\omega_2 \gamma_1)^2 + (B\gamma_1)^2$

$\therefore \alpha B = B\alpha_1$

or $B\omega_2$ bisects the angle $\alpha B \gamma_1$

$\omega_1 B \omega_2$ is a straight line.

(2). $B\alpha_1 = B\alpha, \; C\alpha_3 = C\alpha$

$\therefore A a_3 + A\alpha_1 = a + b + c = 2s$

and $(Aa_3)^2 + (a_3\alpha_1)^2 = (Aa_1)^2 + (a_2 a_1)^2$

$\therefore Aa_3 = Aa_1 = s$

(3). $\angle \gamma A \omega_1 = \frac{1}{2}(\pi - A)$

$\angle \gamma B \omega_1 = \frac{1}{2}(\pi - B)$

$\therefore \angle A \omega_1 B = \pi - \frac{1}{2}(2\pi - A - B)$

$= \frac{1}{2}(A + B) = \frac{\pi}{2} - \frac{C}{2}$

$\therefore \angle A \omega_1 \gamma = \frac{B}{2}$

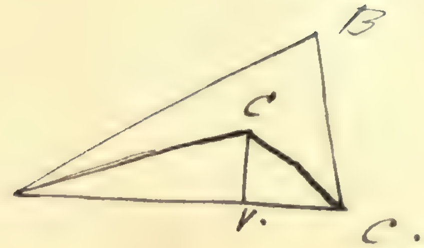
(4). $\angle \omega_1 A \gamma = \frac{\pi}{2} - \frac{A}{2}$

$\angle OA\gamma = \frac{A}{2}$

$\therefore \omega_1 A \omega_2 = \frac{\pi}{2}$.

(5). $\angle \omega_1 A \gamma = \frac{\pi}{2} - \frac{\pi}{2} \; \therefore \angle A \omega_1 = \frac{A}{2}$.

Let $r_1 = AC = r_1$ the reqd. dist.



Then CV = radius of inscribed $\odot$ $= \sqrt{\dfrac{s(s-a)(s-b)(s-c)}{s}} = r$

$$AC = \frac{r}{\sin\frac{A}{2}} = \frac{r\cos\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}$$

Now $\sin\frac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, and $\cos\frac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

$$\therefore\ r_1 = \frac{r\cos\frac{A}{2}\,.\,bc.}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{bc\cos\frac{A}{2}}{s} \text{ by substituting for } r.$$

$$= \frac{2bc.\cos\frac{A}{2}}{a+b+c}.$$

C is the centre of the inscribed $\odot$. find AC.

In any Δ.

$$\cos C = \frac{a^2+b^2-c^2}{2ab}.$$

$$2ab\cos C = a^2+b^2-c^2.$$

If a is constant while $b \cdot c$ vary.

$$2a\cos C\,\delta b = 2b\delta b - 2c\delta c.$$

Similarly we have.

$$2ac\cos B = a^2+c^2-b^2.$$

$$2a\cos B\,\delta c = 2c\delta c - 2b\delta b.$$

$$\therefore \cos C\,\delta b + \cos B\,\delta c = 0.$$

In a plane Δ. given $a.\ A.\ \overline{b+c}$; or $c.\ A.\ \overline{a+b}$.
Solve the Δ by the aid of log.s.

In the first case $a+b+c = 2s$ is known.

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\therefore\ bc = \frac{\cos^2\frac{A}{2}}{s(s-a)}.$$

or $\log b + \log c = 2\,L\cos\frac{A}{2} - \log s - \log (s-a) - 20.$

gives $bc = \beta$ suppose

also $b+c = \alpha$ suppose.

$$\therefore\ b^2+2cb+c^2-4bc = \alpha^2-4\beta.$$

$$\therefore\ b-c = \sqrt{\alpha^2-4\beta}.$$

$$\therefore\ b = \frac{1}{2}\left\{\alpha+\sqrt{\alpha^2-4\beta}\right\}.\quad c = \frac{1}{2}\left\{\alpha-\sqrt{\alpha^2-4\beta}\right\}.$$

also $L\sin B = L\sin A - \log a + \log b.$

$L\sin C = L\sin A - \log a + \log c.$

In the 2nd case $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$$= \frac{(b+a)(b-a)+c^2}{2bc} = \frac{\beta.(2b-\beta)+c^2}{2bc} \text{ if } \beta = a+b.$$

$$\therefore\ (2c\cos A-2\beta).b = c^2-\beta^2.$$

$$b = \frac{c^2-\beta^2}{2\{c\cos A-\beta\}};\quad a = \beta-b.$$

and the Δ may be solved as before.

$$\tan^{-1} \frac{4(x-x^3)}{1+x^2} = 2\pi - 2\,\mathrm{chord}^{-1} \frac{2}{\sqrt{1+x^2}}.$$

$$\mathrm{chord}^{-1} \frac{2}{\sqrt{1+x^2}} = 2\sin^{-1} \frac{1}{\sqrt{1+x^2}}.$$

$$= 2\tan^{-1} \left\{ \frac{\frac{1}{\sqrt{1+x^2}}}{\sqrt{\frac{1+x^2-1}{\sqrt{1+x^2}}}} \right. = 2\tan^{-1} \frac{1}{x}$$

$$\therefore \tan^{-1} \frac{4(x-x^3)}{1+x^2} + 4\tan^{-1} \frac{1}{x} = 2\pi.$$

$$\tan^{-1} \frac{4(x-x^3)}{1+x^2} + 2\tan^{-1} \frac{\frac{2}{x}}{1-\frac{1}{x^2}} = 2\pi.$$

$$2\tan^{-1} \frac{2x}{x^2-1} = \tan^{-1} \frac{\frac{4x}{x^2-1}}{1-\frac{4x^2}{x^4-2x^2+1}}$$

$$= \tan^{-1} \frac{4x(x^2-1)}{x^4-6x^2+1}$$

$$\tan^{-1} \frac{4(x-x^3)}{1+x^2} + \tan^{-1} \frac{4x(x^2-1)}{x^4-6x^2+1}$$

$$= \tan^{-1} \frac{4(x-x^3)\{x^4-6x^2+1-1-x^2\}}{16x^2(x^2-1)^2+(1+x^2)(x^4-6x^2+1)} = 2\pi.$$

[illegible]

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$$\therefore \tan 2\pi = 0 = \frac{4(x-x^3)(x^4-7x^2)}{16x^2(x^2-1)^2+(1+x^2)(x^4-6x^2+1)}$$

$\therefore$ the numerator of this fraction $= 0$.

or $4x^3 \cdot (1-x^2)(x^2-7) = 0$.

$$\therefore \quad x = 0.$$

$$x^2 = 1$$

$$x^2 = 7$$

$\log_a m = \log_a b \cdot \log_b c \cdot \log_c d \ldots \log_l m.$

Let $m = a^x = b^y = c^z = d^{z'}$ &c $= l^{x'}$

$\therefore\ x = \log_a m\ ;\ y = \log_b m,\ z = \log_c m$ &c.

$a^x = b^y \ \therefore\ a = b^{\frac{y}{x}}$

$\therefore\ \frac{y}{x} = \log_b a.$

or $\dfrac{\log_b m}{\log_a m} = \log_b a \quad \therefore\ \log_a m = \log_b m \log_a b.$

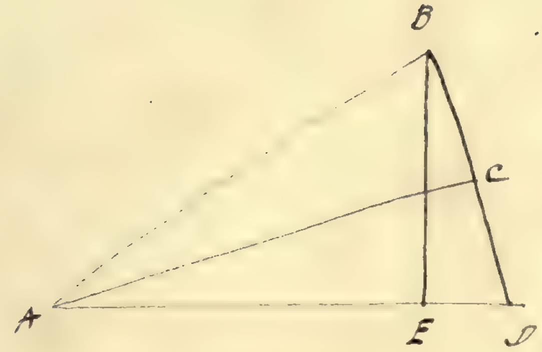
similarly $\log_b m = \log_c m \log_b c.$

$\log_c m = \log_d m \log_c d$

— — —

$\log_k m = \log_l m \log_k l.$

$\therefore\ \log_a m = \log_a b \log_b c \log_c d \ldots \log_l m.$



$$\sin\theta = \frac{BC}{AB}.$$

$$\therefore \frac{BD^2}{AB^2} = 4\sin^2\theta = \frac{BE^2 + ED^2}{AD^2}$$

$$= \frac{BE^2 + (AD - AE)^2}{AD^2}$$

$$= \frac{BE^2 + AD^2 - 2\,AD.AE + AE^2}{AD^2}$$

$$= \frac{2\,AD^2 - 2\,AD.AE}{AD^2} = 2\left\{1 - \frac{AE}{AD}\right\}$$

$$\therefore 2\sin^2\theta = 1 - \cos 2\theta.$$

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}.$$

The double value indicated by the radical point out the 2 values of BC, and DC.

Geometry

Find the =n of the conoidal surface generated by a horizontal line which meets the helix and the axis of the vertical cylinder on which the helix is traced.

Take the axis of the cylinder for the axis of z.

Then $z = \beta$. $y = \alpha x + \gamma$ are the =ns of the line.

and since it always meets the axis of z.

$$0 = 0 + \gamma \quad \therefore \gamma = 0.$$

$\therefore$ the =ns of the generating line are

$$z = \beta. \quad y = \alpha x.$$

And the generating line meets the helix whose =ns are. $y = \sqrt{a^2 - x^2}$. $z = na \cos^{-1}\frac{x}{a}$.

$$\therefore \alpha x = \sqrt{a^2 - x^2}. \quad (\alpha^2 + 1)x^2 = a^2$$

$$x = \frac{a}{\sqrt{\alpha^2 + 1}}$$

$$\therefore \beta = na \cos^{-1}\frac{1}{\sqrt{\alpha^2 + 1}}.$$

$$\text{or } z = na \cos^{-1}\frac{x}{\sqrt{x^2 + y^2}}.$$

is the req^d =n.

Find the area of the section of an ellipsoid cut off by a plane passing thro' the centre, and inclined at ∠s $\alpha . \beta . \gamma$ to the axes of $x . y . z$.

The equation to the plane is $x\cos\alpha + y\cos\beta + z\cos\gamma = 0$.

Hence. $$\left(\frac{x'\cos\beta - y'\cos\alpha\cos\gamma}{a\sin\gamma}\right)^2 + \left(\frac{x'\cos\alpha + y'\cos\beta\cos\gamma}{b\sin\gamma}\right)^2 + y'^2\frac{\sin^2\gamma}{c^2} = 1$$

Now when the equation to an ⊙ is $Ax^2 + 2Bxy + Cy^2 = d$, its area is

$$\frac{\pi d}{\sqrt{AC - B^2}}.$$

In this case $AC - B^2$ becomes.

$$\frac{\sin^2\gamma}{\sin^4\gamma}\left(\frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2}\right)\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right) + \frac{1}{c^2}\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right)$$

$$-\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2\left(\frac{\cos^2\alpha\cos^2\beta\cos^2\gamma}{\sin^4\gamma}\right).$$

$$\frac{\cos^2\gamma}{\sin^4\gamma} \cdot \frac{\cos^4\alpha + \cos^4\beta}{a^2b^2} + \frac{1}{c^2}\left(\frac{\cos^2\beta}{a^2} + \frac{\cos^2\alpha}{b^2}\right)$$

$$+ \frac{2}{a^2b^2}\left(\frac{\cos^2\alpha\cos^2\beta\cos^2\gamma}{\sin^4\gamma}\right\}.$$

$$= \frac{1}{a^2b^2c^2}\left\{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma\right\}.$$

$$\text{area} = \frac{\pi d}{\sqrt{AC - B^2}}$$

$$= \frac{\pi abc}{\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}}.$$

Find =ⁿ of the conical surface whose vertex is a point in the axis of z, and directrix a ⊙ whose =ⁿ is $x^2+y^2+z^2=r^2$; $z=ax+b$.

The =ⁿˢ of the generating line in any point are.

$x = \alpha.(z-c)$. (1) c = vertical coord of the given point.

$y = \beta(z-c)$ (2)

and we have to eliminate $x.y.z.$ between these =ˢ

and $x^2+y^2+z^2=r^2$; (3) $z = ax+b$. (4).

from (1) and (2) $\frac{x}{\alpha} = \frac{y}{\beta}$ $\therefore \beta x = \alpha y$.

from (1) and (4) $x = \alpha\{ax+b-c\} = \frac{\alpha(b-c)}{1-a\alpha}$.

$$\therefore y = \frac{\beta.(b-c)}{1-a\alpha}.$$

$$z = \frac{\alpha a(b-c)+b-ba\alpha}{1-a\alpha}$$

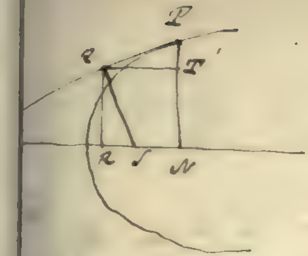
$$= \frac{b-ac\alpha}{1-a\alpha}.$$

substitute in (3).

$$(\beta^2+\alpha^2)(b-c)^2+(b-ac\alpha)^2 = (1-\alpha a)^2r^2.$$

or $$\frac{x^2+y^2}{(z-c)^2}.(b-c)^2+\left(b-\frac{acx}{z-c}\right)^2 = \left(1-\frac{ax}{z-c}\right)^2r^2.$$

$$(x^2+y^2)(b-c)^2+\{b(z-c)-acx\}^2 = (z-c-ax)^2r^2.$$



Let h k be the coord^s of S, and let SN = x, NP = y be the coord^s of P. PT a tangent at that point SQ a ⊥ upon the tangent from S. SR = x', SQ = y' the coord^s of Q.

Then from the similar Δs SQR, QT'P

$$\frac{SR}{QR} = \frac{PT'}{QT'}, \text{ or } \frac{x'}{y'} = \frac{y-y'}{x+x'} \text{ or } x'^2 + y'^2 = yy' - xx'$$

also if α be the ∠ at which the tangent at P cuts the axis of x we have

$$\tan\alpha = \frac{2a}{y} = \sqrt{\frac{a}{x}} = \frac{x'}{y'}$$

$$\therefore \frac{a}{x} = \frac{x'^2}{y'^2} \text{ or } x = a\frac{y'^2}{x'^2}$$

$$y^2 = 4ax = 4a^2\frac{y'^2}{x'^2} \therefore y = 2a\frac{y'}{x'}$$

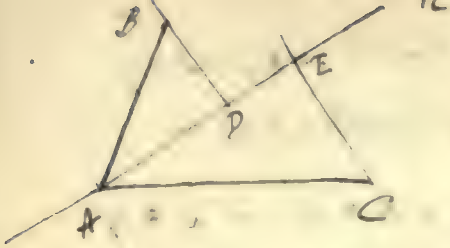
But $x'^2 + y'^2 = yy' - xx'$

$$= 2a\frac{y'^2}{x'} - a\frac{y'^2}{x'}$$

$$= a\frac{y'^2}{x'}$$

$$\therefore x'^2 + y'^2 = a\frac{y'^2}{x'}$$ the eq^n req^d.

To find the locus of the intersection of the tangent to the parabola, and the ⊥ upon it from a point (h, k).



Let the forces a & b act in the directions AB & AC inclined at ∠ 60°

Let AR bisect the angle ~~between~~ BAC

Now if AB is resolved in the directions AR. DB we have

$AD = a \cos 30° = \frac{a}{2}\sqrt{3}$

$BD = a \sin 30° = \frac{a}{2}$

Similarly AC ~~is~~ is equivalent to AE. EC

$AE = b \cos 30° = \frac{b}{2}\sqrt{3}$

$CE = b \sin 30° = \frac{b}{2}$

∴ the resultant in the direction AR =

$\frac{1}{2}(a + b)\sqrt{3}$.

Lemma: forces acting in directions ~~inclined~~ including 60° find the composition of force in the line bisect the angle.

$$\frac{x}{\cos w + a} + \frac{y}{a \cos w + b} = 1 \quad \therefore x(a\cos w + b) + y(b\cos w + a) = (b\cos w + a)(a\cos w + b)$$

$$\text{or } y = -\frac{a\cos w + b}{b\cos w + a}\, x + a\cos w + b$$

$$\therefore m = -\frac{a\cos w + b}{b\cos w + a}$$

Let m' = the tangent of the angle at which the line perpendicular to the given line cuts the axis of x

$$\therefore m' = -\frac{1 + m\cos w}{m + \cos w}$$

$$= -\left\{ \frac{1 + \left\{-\frac{a\cos w + b}{b\cos w + a}\cos w\right\}}{-\frac{a\cos w + b}{b\cos w + a} + \cos w} \right\}$$

$$= -\left\{ \frac{b\cos w + a - a\cos^2 w - b\cos w}{b\cos^2 w + a\cos w - a\cos w - b} \right\}$$

$$= -\left\{ \frac{a(1 - \cos^2 w)}{b(\cos^2 w - 1)} \right\} = \frac{a}{b}$$

$\therefore y = \frac{a}{b} x$ is the equation to the line through the origin parallel to the required line

and since a and b are points in the line we have

$$b = \frac{a}{b} a.$$

Subtracting the latter eqn from the former we have

$$y - b = \frac{a}{b}(x - a) \quad \text{or } (y - b)b = (x - a)a$$


Capella
Cor. Caroli
Corona borealis
Arcturus

Let p' be the length of the perp — substituting in eq.

$$p' = \frac{y' - mx' - c}{\sqrt{1 + 2m\cos w + m^2}}$$, recollecting that $y' = b$, $x' = a$, $m = -\frac{a\cos w + b}{b\cos w + a}$

$c = a\cos w + b$

$$p' = \frac{\left\{b + \frac{a^2\cos w + ab}{b\cos w + a} - a\cos w - b\right\}\sin w}{\sqrt{1 - \frac{2a\cos w + 2b}{b\cos w + a}\cos w + \left(\frac{a\cos w + b}{b\cos w + a}\right)^2}} = \frac{(a^2\cos w + ab - ab\cos^2 w - a^2\cos w)\sin w}{\sqrt{(a\cos w + b)^2 - 2\cos w\{(a\cos w + b)(b\cos w + a)\} + (a\cos w + b)^2}}$$

$$= \frac{ab(1 - \cos^2 w)\sin w}{\sqrt{a^2 + b^2 + 2ab\,(1 - \cos^2 w)}} = \frac{ab\sin^2 w}{a + b}$$

Sub. $S = \dfrac{a}{\sqrt{a^2-y^2-x^2}}$.

$$\int_y = a \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} + C$$

from $x=0$ to $x = \sqrt{a^2\sin^2\alpha - y^2}$

$$\int_y = a \sin^{-1} \sqrt{\frac{a^2\sin^2\alpha - y^2}{a^2-y^2}}$$

$$\int = a \int_y^{0 \ldots a\sin\alpha} \sin^{-1} \sqrt{\frac{a\sin^2\alpha - y^2}{a^2-y^2}}$$

$$= \frac{a\pi}{2} \cdot (a - a\cos\alpha).$$

Find the position of =m of a sphere floating in a fluid.

We have $aX = bY = cZ$.

The =n to the sphere $z = \sqrt{a^2 - x^2 - y^2}$.

=n to the surf. $z = c\{1 - \frac{x}{a} - \frac{y}{b}\}$.

$$V = \int_x\int_y\int_z 1 = \int_x\int_y \{2\sqrt{a^2-x^2-y^2} - c(1 - \frac{x}{a} - \frac{y}{b})\}$$

$$= \int_x \{(a^2 - x^2)\pi - \frac{cb}{2}(1 - \frac{x}{a})^2\}$$

$$= \frac{4}{3}\pi a^3 - \frac{cb}{a^2}\{a'' - a' + \frac{a}{3}\} = \frac{1}{6}\{8\pi a^3 - cba'\}$$

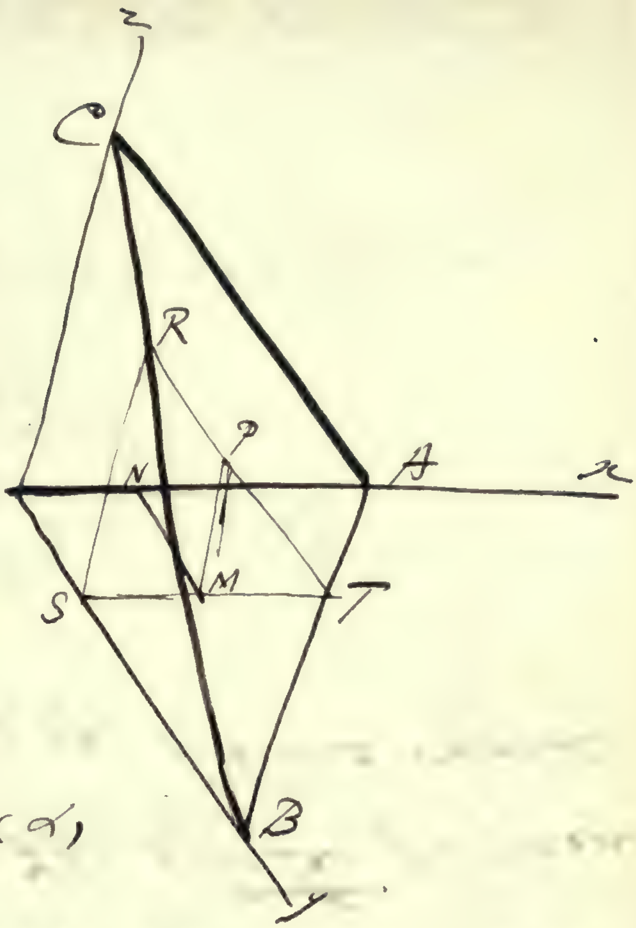
$$= \frac{4}{3}\pi a^3 - \frac{cba'}{6}$$

$$VX = \int_x\int_y\int_z x = \int_x \{(a^2x - x^3)\pi - \frac{cb}{2}\int_a^{a'}\{x - \frac{2x^2}{a_1} + \frac{x^3}{a^2}\}$$

$$= a^4\pi - \frac{cba'}{24} = \frac{1}{24}\{24a^4\pi - cba'\}$$

$$VY = \int_x\int_y\int_z y = \int_x\int_y \{2y\sqrt{a^2-x^2-y^2} - cy(1 - \frac{x}{a} - \frac{y}{b})\}$$

$$\frac{8}{3}(a^2 - x^2 - y^2)^{\frac{3}{2}} - c(\frac{y^2}{2} - \frac{xy^2}{a} - \frac{y^3}{b})$$



$ON = x \quad NM = y \quad MP = z$

$OA = a \quad OB = b \quad OC = c$

$$\frac{SM}{ST} + \frac{PM}{RS} = 1$$

$$\frac{x}{ST} + \frac{z}{RS} = 1 \qquad (\alpha)$$

$$\frac{ST}{OA} + \frac{OS}{OB} = 1$$

or $$\frac{ST}{a} + \frac{y}{b} = 1$$

Again $$\frac{RS}{CO} + \frac{OS}{OB} = 1$$

or $$\frac{RS}{c} + \frac{y}{b} = 1$$

$$\frac{x}{a\left(1-\frac{y}{b}\right)} + \frac{z}{c\left(1-\frac{y}{b}\right)} = 1$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

If a curve in space is a plane curve its plane must coincide with some plane

$$z = Ax + By + C.$$

$$\therefore d_x z = A + B d_x y.$$

$$d_x^2 z = B d_x^2 y.$$

$$d_x \left(\frac{d_x^2 z}{d_x^2 y} \right) = 0.$$

$$\text{or } d_x^2 y \, d_x^3 z - d_x^2 z \, d_x^3 y = 0.$$

and this condit.n must be identically satisfied by the coords of the proposed curve if the curve is a plane curve.

Prove that the sum of the squares of the projection of an ⊙ on 3 planes at right angles to each other is constant —

no. ellipsoid

$u = ax^2 + by^2 + cz^2 + 2a'zy + 2b'xz + 2c'xy + d = 0.$

the eqn to the tangent plane is

$(x'-x)\frac{du}{x} + (y'-y)\frac{du}{y} + (z'-z)\frac{du}{z} = 0.$

$\frac{du}{x} = 2ax + 2b'z + 2c'y.$

$\frac{du}{y} = 2by + 2a'z + 2c'x.$

$\frac{du}{z} = 2cz + 2a'y + 2b'x.$

$\therefore (x'-x)\{(ax+b'z+c'y) + (y'-y)(by+a'z+c'x)$

$+(z'-z)(cz+a'y+b'x) = 0.$

$x'(ax+b'z+c'y) + y'(by+a'z+c'x) + z'(cz+a'y+b'x)$

$- ax^2 - by^2 - cz^2 - b'zx - c'xy - a'zy - c'xy$

$-a'zy - b'xz \quad = 0.$

or $x'(ax+b'z+c'y) + y'(by+a'z+c'x)$

$+ z'(cz+a'y+b'x) + d = 0.$

$\frac{x}{\;} + \frac{y}{\;} = 1$

the tangent plane is ∥ to the axis of z we have.

$$c'z + a'y + b'x = 0$$

$$\therefore z = -\frac{(a'y + b'x)}{c}$$

substitute in the eqn for the surface

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy + d = 0$$

$$\therefore ax^2 + by^2 + \frac{(a'y + b'x)^2}{c} - 2\frac{(a'y + b'x)^2}{c} + d + 2c'xy = 0$$

$$cax^2 + cby^2 - a'^2y^2 - 2a'b'xy - b'^2x^2 + cd + 2cc'xy = 0.$$

$$(cb - a'^2)y^2 + (ca - b'^2)x^2 + (2cc' - 2a'b')xy + cd = 0$$

$$\text{or } Ay^2 + Bxy + Cx^2 + cd = 0.$$

The area of this is $\dfrac{\pi C'}{\sqrt{A'B'}}$

where $C' = cd$.

$$A' = \tfrac{1}{2}\{A + C + \sqrt{A + C + (B^2 - 4AC)}\}.$$

$$B' = \tfrac{1}{2}(\overline{A + C} - \sqrt{A + C + B^2 - 4AC}$$

$$\therefore \text{area of } \odot = \frac{2\pi cd}{\sqrt{-B^2 + 4AC}}$$

$$= \frac{\pi cd}{\sqrt{(cb - a'^2)(ca - b'^2) - (cc' - a'b')^2}}$$

If the tangent plane is // to the axis of x we have

$$ax + b'z + c'y = 0.$$

$$x = \frac{-(b'z + c'y)}{a}.$$

∴ the eqn to the projectn of the envelope on the plane of xy is

$$\frac{(b'z + c'y)^2}{a} + by^2 + cz^2 + 2a'zy - \frac{2(b'z + c'y)^2}{a} + d = 0.$$

$$aby^2 + acz^2 + 2aa'zy - \{b'^2z^2 + 2b'c'zy + c'^2y^2\} + ad = 0.$$

$$(ab - c'^2)y^2 + (ac - b'^2)z^2 + 2(aa' - b'c')zy + ad = 0.$$

The area of this ⊙ is $\dfrac{\pi a d}{\sqrt{(ab - c'^2)(ac - b'^2) - (aa' - b'c')^2}}$

$$\therefore A_x^2 = \frac{\pi^2 a d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}.$$

similarly $$A_z^2 = \frac{\pi^2 c d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}$$

$$A_y^2 = \frac{\pi^2 b d^2}{abc - bb'^2 - cc'^2 - aa'^2 + 2a'b'c'}.$$

$$\therefore A_x^2 + A_y^2 + A_z^2 = \frac{\pi^2 d^2 \{a + b + c\}}{abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c'}.$$

If a. b. c. be semi axes of the ⊙oid.

$a + b + c = \{a' + b' + c'\}$ $\quad a'b'c' = abc - aa'^2 - bb'^2 - cc'^2 + 2a'b'c'.$

Let $u = f(x, y) = 0$ be the eq^n to the curve.

$\therefore (x'-x)\frac{du}{dx} + (y'-y)\frac{du}{dy} = 0$ = eq^n to the tangent. (1)

The eq^n to the $\perp$ from the origin upon this plane

is $y'\frac{du}{dx} = x'\frac{du}{dy}$. or $y' = \dfrac{\frac{du}{dy}}{\frac{du}{dx}}\, x'$.

$\therefore$ putting eq^n (1) under the form

$$x'\frac{du}{dx} + y'\frac{du}{dy} = x\frac{du}{dx} + y\frac{du}{dz}.$$

and substituting for y'

$$x'\left\{\frac{du}{dx} + \frac{\left(\frac{du}{dy}\right)^2}{\frac{du}{dx}}\right\} = x\frac{du}{dx} + y\frac{du}{dz}.$$

$$\text{or } x' = \frac{\left(x\frac{du}{dx} + y\frac{du}{dz}\right)\left(\frac{du}{dx}\right)}{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}.$$

$$\therefore y' = \frac{\left(x\frac{du}{dx} + y\frac{du}{dz}\right)\left(\frac{du}{dy}\right)}{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}.$$

$$\therefore p^2 = x'^2 + y'^2 = \frac{\left\{x\frac{du}{dx} + y\frac{du}{dy}\right\}^2}{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}.$$

$$p = \frac{x\frac{du}{dx} + y\frac{du}{dy}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}}.$$

Find the area of the ⊙ $ay^2 + bxy + cx^2 + d = 0$. shewing the necessary condition that it may represent a ⊙

Change the direct^n of the coord^t axes. by changing

$$x = x'\cos\theta - y'\sin\theta.$$
$$y = x'\sin\theta + y'\cos\theta$$

∴ The transformed =^n becomes.

$$a\{x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta\}$$
$$+ b\{(x'^2 - y'^2)\sin\theta\cos\theta + x'y'(\cos^2\theta - \sin^2\theta)$$
$$+ c\{x'^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta\}.$$

or $(a\cos^2\theta - b\sin\theta\cos\theta + c\sin^2\theta)y'^2 + \{(a-c)2\sin\theta\cos\theta + b\cos^2\theta - \sin^2\theta\}x'y'$
$+ (a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta)x'^2 + d = 0.$

If we make the coeff^t of $x'y' = 0$ we have. $\tan 2\theta = \frac{-b}{a-c}$.
and the =^n becomes of the form $Ay'^2 + Bx'^2 = -d$.

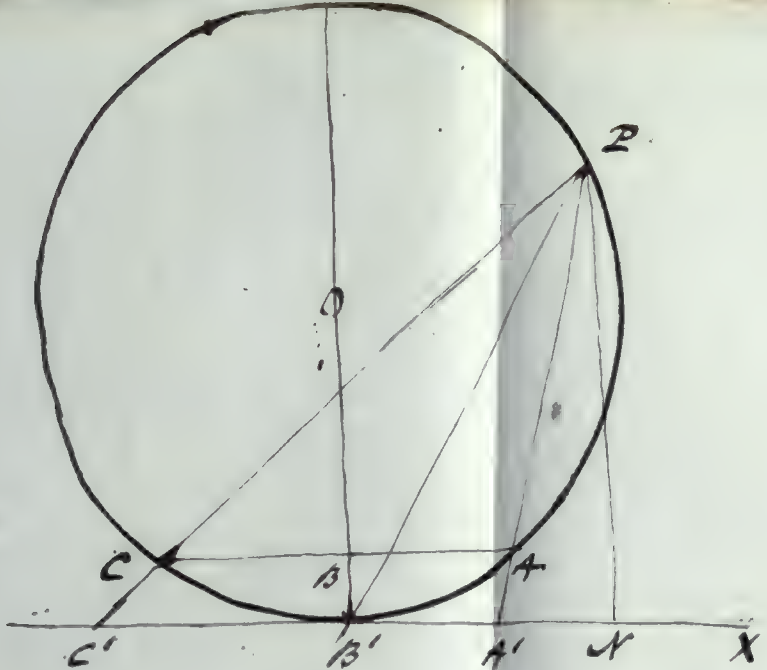
where $A = \frac{1}{2}\{a + c + \sqrt{(a-c)^2 - b^2}\}$ $B = \frac{1}{2}\{a + c - \sqrt{(a-c)^2 - b^2}\}$.

now area of ⊙ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

∴ area of ⊙ $\frac{Ay^2}{-d} + \frac{Bx'^2}{-d} = 1$ is $\frac{-\pi d}{\sqrt{AB}}$.

$$= \frac{-2\pi d}{\sqrt{4ac - b^2}}$$

∴ The necessary conditions that it may be an ⊙ are that d must be negative, and $4ac > b^2$.



C'PA any segment containing angle CPA standing on arc CBA; CB' = B'A – then if B'P [be joined] the joining should bisect the angle C'PA.

Join CA; and let BC = BA = a. BB' = c, radius of Circle = r, BO, BX rectangular axes – B' origin – $x' = B'N$, $y' = PN$ co-ordinates of P.

The eqn to the ⊙ is $x^2 = 2ry - y^2$.

eqn to C'P is $\frac{y-y'}{x-x'} = \frac{y'-c}{x'+a}$ ∴ $\tan PC'N = \frac{y'-c}{x'+a}$ ∴ $\operatorname{cosec} PC'N = \frac{\sqrt{(y'-c)^2+(x'+a)^2}}{y'-c} = \frac{\sqrt{2r(c+y')-cy'+ax'}}{y'-c}$

or $y(a+x') = x(y'-c) + ay' + cx'$

Let $y=0$ ∴ $x = B'C' = -\frac{cx'+ay'}{y'-c}$.

eqn to A'P is $\frac{y-y'}{x-x'} = \frac{y'-c}{x'-a}$ ∴ $\tan PA'N = \frac{y'-c}{x'-a}$ ∴ $\operatorname{cosec} PA'N = \frac{\sqrt{2r(c+y')-cy'-ax'}}{y'-c}$

or $y(x'-a) = x(y'-c) - ay' + cx'$

Let $y=0$ ∴ $x = BA' = \frac{cx'-ay'}{y'-c}$.

Now $C'P = y' \operatorname{cosec} PC'N = \frac{y'}{y'-c}\sqrt{2r(c+y')-cy'+ax'}$, and $PA' = \frac{y'}{y'-c}\sqrt{2r(c+y')-cy'-ax'}$

$$\therefore \frac{C'P^2}{PA'^2} = \frac{2r(c+y')-cy'+ax'}{2r(c+y')-cy'-ax'} = \frac{2c^2ry' - 2c^2y'^2 + 2cry'^2 + 2acx'y'}{2c^2ry' - 2c^2y'^2 + 2cry'^2 - 2acx'y'}$$

$$= \frac{c^2(2ry'-y'^2) + y'^2(2rc-c^2) + 2acx'y'}{c^2(2ry'-y'^2) + y'^2(2rc-c^2) - 2acx'y'} = \frac{c^2x'^2 + y'^2a^2 + 2acx'y'}{c^2x'^2 + y'^2a^2 - 2acx'y'}$$

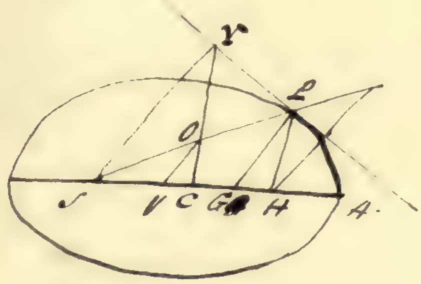
$$= \frac{B'C'^2}{B'A'^2}$$

or $\frac{C'P}{PA'} = \frac{B'C'}{B'A'}$ ∴ C'PA is bisected by the line B'P.

[illegible]

P is a point of an ⊙, S a focus, C the centre, PY a tangent at P. SY ⊥ to PY. prove that if CY intersect SP in O

$SO = OP = OY$.



Since SY is ⊥ to the tangent at P

$CY = CA$.

Let PG be a normal at G. OV ∥ to PG.

then $CG = e^2x \therefore SG = ae + e^2x$

$\therefore SV = VG = \frac{1}{2}ae + \frac{1}{2}e^2x$.

$\therefore VC = \frac{1}{2}ae - \frac{1}{2}e^2x$.

$\therefore SV : VC :: a+ex : a-ex$

$\therefore CO : OY :: a-ex : a+ex$.

$CO+OY : OY :: a-ex+a+ex : a+ex$.

$CY : OY :: 2a : a+ex$.

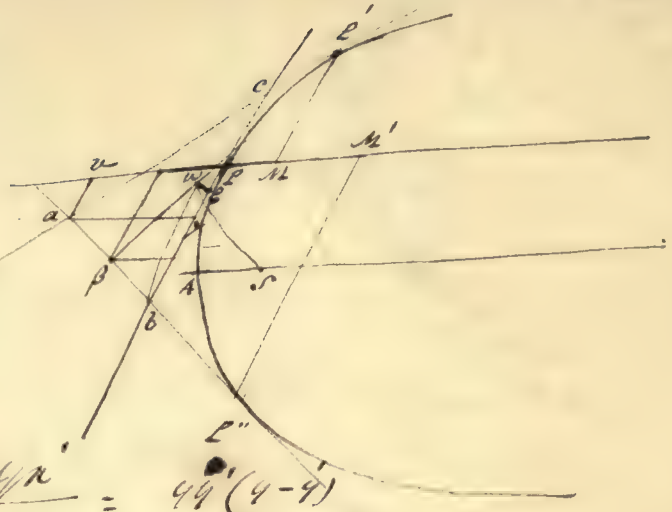
But $CY = a \therefore OY = \frac{1}{2}(a+ex)$.

also $SP = a+ex \therefore SO = OY = \frac{1}{2}(a+ex)$

$\therefore SO = OP = OY$.

Hymers Conic Sections
P. 52. Ex. 2.

Let the ⊙ be referred to the [tan]gent cb and diam. ℓM' as axes.
[ℓa] = h. va = k. the points of intersect[ion of] the diameters. aℓ'. aℓ". ℓM = x. Mℓ' = y
[ℓ]M' = x', M'ℓ" = y' coords. of ℓ'. ℓ"



$\therefore ky = 2a.(x+h),\ ky' = 2a.(x'+h).$

$\therefore x+h = \frac{y}{y'}.(x'+h)$; or $h = \frac{xy' - yx'}{\frac{y}{y'} - y'} = \frac{yy'(y-y')}{4a.\ y-y'}$

$\ell v = \frac{yy'}{4a} = \frac{yy'}{4a} = \sqrt{xx'}.$

$k = \frac{2a\sqrt{x}}{2\sqrt{a}.\sqrt{x}}\{\pm\sqrt{x} + \sqrt{x'}\} = \frac{1}{2}\{\pm 2\sqrt{ax} + 2\sqrt{ax'}\} = \frac{1}{2}(y'-y) = av.$

$\therefore cb = \frac{1}{2}(y+y')$; $\gamma b = \frac{1}{2}y' - \ell\gamma = \frac{1}{2}y = \ell c$

$\gamma c = \ell c + \ell\gamma = \frac{1}{2}y' = \ell b.$

Let β be the middle point of ab.

the coord[s] of β are $x'' = \frac{1}{2}v\ell = \frac{1}{2}\sqrt{xx'}$

$y'' = \frac{1}{2}\{av + \ell b\} = \frac{1}{4}\{2y' - y\}$

Also since cℓ = γb. the coords of δ. the middle point of cb

are $x''' = 0,\ y''' = \frac{1}{2}\ell\gamma = \frac{1}{4}(y'-y).$

Let βw, δw, be drawn ⊥ to ab. cb.

The eq.n to βw is $y'' - Y = m.(x''-X) = \frac{\sqrt{x'}}{\sqrt{a}}.(x''-X)\frac{\sin\alpha}{\sin\overline{\omega-\alpha}}$

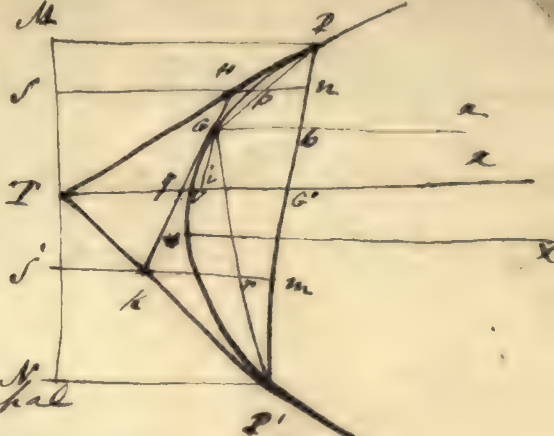
eq.n to δw is $y'''-Y = n'.X = \frac{\sin\beta}{\sin\overline{\omega-\beta}}.X.$

where $\cot\alpha = \frac{dy'}{dx'}$, $\cot\beta = -\frac{dy}{dx}$.

Eliminate Y and by substitut.g for x''. y'' y''' their values we get Y, and X. the coord.s of w. and it will be found that

$w\delta^2 = w\beta^2$

Let the tangents at P & P' intersect in T.
& let HK be a tangent at G, join PG,
P', PP' — the Δ PGP' shall be
double of the Δ HTK.



Through T draw Ta ∥ to AX the principal
axis of the parabola. then PG' is an ordinate to the diam. Ta
∴ VT = VG'. Similarly if through G the diam. Gx be drawn,
and Gi ∥ to PP' ; vi = qv ∴ Tq = iG' = Gb

Also if Hn, km be drawn ∥ to the axis AX
they will bisect GP, GP' in p and r

$$\text{and } nm = \tfrac{1}{2} PP' = PG'$$

Now if MP, NP', Ss' be drawn ∥ to Tq

area of Δ ~~H~~Tk = $\frac{1}{2}$ Tq × SS'

area of Δ PGP' = $\frac{1}{2}$ Gb × MN

and Tq has been shewn to be = Gb

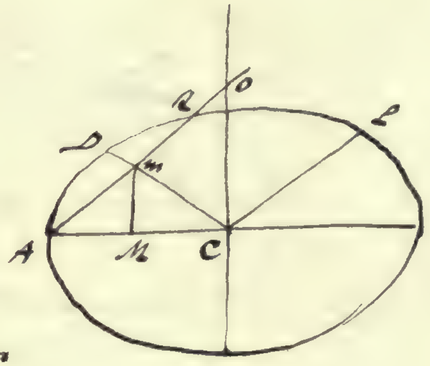
and MN is double of SS'

∴ Δ ~~H~~Tk is $\frac{1}{2}$ Δ PGP'.

The triangle contained by 3 tangents to a P. is double of the Δ
contd. by the 3 Chds. joining the points of contact.

If CL be any ½ diameter of an ⊙ & AQO be diameter from extr. of the major axis ∥ CL, and meets the curve Q. and the minor axis produced in O, shew that $2CL^2 = AO.AQ$

Let x'y' be the coord^s of L.



$\therefore y = \frac{y'}{x'}x$ is the $=^n$ to CL.

$y = \frac{y'}{x'}(x+a)$ is the $=^n$ to AO.

$y = -\frac{b^2x'}{a^2y'}x$ is the $=^n$ to CD, the semi conjugate axis, which bisects AQ in m

If X.Y. be coord^s of m. we have.

$$\frac{y'}{x'}(X+a) = -\frac{b^2x'}{a^2y'}X \text{ , or. } \frac{a^2y'^2+b^2x'^2}{a^2y'x'}X = \frac{b^2}{y'x'}X = -\frac{y'}{x'}a.$$

$$\therefore X = CM = -y'^2\frac{a}{b^2}.$$

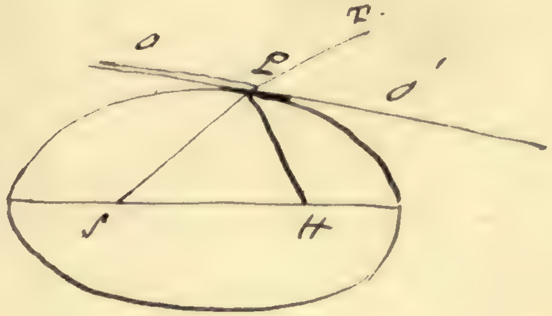
$$\text{or } AM = \frac{a}{b^2}\{b^2-y'^2\} = \frac{a}{b^2}.\frac{b^2}{a^2}x'^2 = \frac{x'^2}{a}.$$

$$\therefore AQ = 2AM \sec QAC = \frac{2x'^2}{a}\frac{\sqrt{x'^2+y'^2}}{x'} = \frac{2x'}{a}\sqrt{x'^2+y'^2}.$$

$$AO = AC \sec QAC = \frac{a}{x'}\sqrt{x'^2+y'^2}.$$

$$\therefore AQ.AO = 2(x'^2+y'^2) = 2CL^2$$

Qⁿ In an elliptic orbit of small eccentricity the ∠r velocity is nearly constant about that focus in which the sun is not.



vel in $PO = \dfrac{h}{SP \sin SPO}$.

∴ PT is ⊥ to HP.

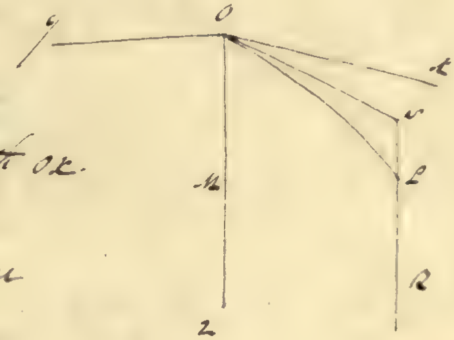
vel in $PT = \dfrac{h}{SP}$.

∠r vel about $H = \dfrac{h}{SP.PH}$

which is nearly constant.

Hydrostatical Problems

orce $\propto \frac{1}{u^4}$. To find the attraction on a particle of fluid in the interior of the fluid when each particle attracts with a force which vanishes when the dist. of the particle from the attracted point is finite.



yOx the tangent plane at O.
yOz. xOz. planes of greatest and least curv^{re} thro O.
OSz a plane passing thro z making $\angle\theta$ with Ox.
Oz a normal at O, which take for axis of z.
$OS = \rho$. $SL = z$. $SQ = z_1$ $OM = r$, $ML = u$
$MQ = u_1$

The eq^n to the surface is $z = \frac{\rho^2}{2}\left\{\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right\}$.

$$u_1^2 = \rho^2 + (z_1 - r)^2 \quad ; \quad u^2 = \rho^2 + (z - r)^2.$$

$$u_1 \frac{du_1}{dz_1} = z_1 - r \quad ; \quad u\frac{du}{d\rho} = \rho - r\frac{dz}{d\rho} \text{ nearly}.$$

$$= \rho\left\{1 - r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}.$$

$$\therefore \rho = u\frac{du}{d\rho}\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}.$$

$$\text{Attr} = \int_\theta^{0\,2\pi}\int_\rho^{0\,\infty}\int_{z_1}^{z\,\infty} \rho\,\frac{z_1 - r}{u_1}\cdot\frac{1}{u_1^4}.$$

$$\int_{z_1} \frac{z_1 - r}{u_1}\,\frac{1}{u_1^4} = \int_{z_1} \frac{1}{u_1^4}\frac{du_1}{dz_1} = \int_{u_1} \frac{1}{u_1^4} = -\frac{1}{3u_1^3}$$

$z_1 = \infty$ gives $u_1 = \infty$. $z_1 = z$ gives $u_1 = u$.

$$\therefore \int_{z_1}^{z\,\infty} \frac{z_1 - r}{u_1}\,\frac{1}{u_1^4} = +\frac{1}{3u^3}.$$

$$\text{Attr} = \int_\theta^{0\,2\pi}\int_\rho^{0\,\infty}\frac{\rho}{3u^3} = \int_\theta^{0\,2\pi}\int_\rho^{0\,\infty}\frac{1}{3u^2}\frac{du}{d\rho}\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}$$

$$= \int_\theta^{0\,2\pi}\left\{1 + r\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}\int_u^{r\,\infty}\frac{1}{3u^2} = \int_\theta^{0\,2\pi}\left\{\frac{1}{3r} + \frac{1}{3}\left(\frac{\cos^2\theta}{R} + \frac{\sin^2\theta}{S}\right)\right\}$$

$$= \frac{1}{3}\int_\theta^{0\,2\pi}\left\{\frac{1}{r} + \frac{1}{2}\left\{\frac{1+\cos 2\theta}{R} + \frac{1-\cos 2\theta}{S}\right\}\right\} = \frac{2\pi}{3}\left\{\frac{1}{r} + \frac{1}{2}\left\{\frac{1}{R} + \frac{1}{S}\right\}\right\}$$

$$\therefore \text{Attract}^n \text{ at } M = \frac{2\pi}{3}\left\{\frac{1}{r} + \frac{1}{2}\left(\frac{1}{R} + \frac{1}{S}\right)\right\}$$ which when r is infinitely small $= \infty$

for the completion of the calculation when the surface of the fluid varies; we have in general.

$$K\cdot\frac{du}{dt}\left(\int_{z=c}-\int_{z=x}\right)\frac{1}{Z}+\frac{1}{2}\left(1-\frac{K^2}{X^2}\right)u^2=g\,(c-x).$$

now $X\frac{du}{dt} = Ku \quad \therefore \quad \frac{du}{dt} = u\cdot\frac{K}{X}.$

$$\frac{du}{dt}=\frac{du}{dx}\frac{dx}{dt}=u\frac{du}{dx}\cdot\frac{K}{X}.$$

$$\therefore \frac{K^2}{X}\left(u\frac{du}{dx}\right)N+\frac{1}{2}\left(1-\frac{K^2}{X^2}\right)u^2=g\cdot(c-x).$$

$$\left(u\cdot\frac{du}{dx}\right)+\frac{1}{2}u^2\,\frac{X}{N}\cdot\left(\frac{1}{K^2}-\frac{1}{X^2}\right)=\frac{g}{K^2}\,\frac{X}{N}\cdot(c-x).$$

$$e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}\left\{2u\frac{du}{dx}+u^2\,\frac{X}{N}\cdot\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\right\}=\frac{2g}{K^2}\cdot\frac{X}{N}(c-x)\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}$$

or $$d_x\cdot\left\{u^2\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}\right\}=\frac{2g}{K^2}\,\frac{X}{N}(c-x)\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}.$$

Integrating we have.

$$u^2\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}=\frac{2g}{K^2}\int_x\frac{X}{N}(c-x)\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}.$$

$$\therefore \frac{K^2}{2}u^2=e^{-\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}\left\{g\int_x\frac{X}{N}(c-x)\,e^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}\right.$$

Miller Hydrostatics. Art 76.

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

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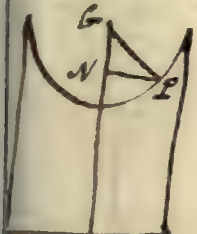
A cylindrical vessel conts. fluid is whirled round its axis ⊥ to the horizon with a given angular velocity. find the form which the surface assumes.



Any point in the surf. is acted upon by 3 forces gravity in direct. GN — the centrifugal force in direct. NP, and the resultn of the fluid in directn PG.

Now the centrifugal force $= \frac{v^2}{y} = \frac{a^2y^2}{y} = a^2y$ where a = the angular velocity,

$\therefore NP : GN :: a^2y : g$

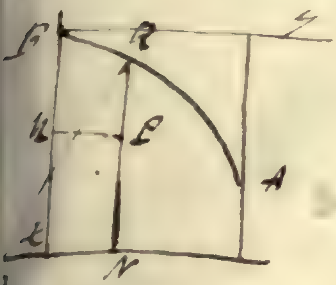
$y : GN :: a^2y : g$

$GN = \frac{g}{a^2}$

∴ GN the subnormal is constt.

or the curve a Pa whose latus rectum is $\frac{2g}{a^2}$.

A fluid is acted on by gravity in direction of.
and by another in direction PM which is $\propto$ to PM.
Find the eq^n of the surf.



The forces are g in direct. AP
and μy in direct. MP.
Let QE be a normal to the surface
at Q. then QE represents the compound
force.

Take B for the origin By. Bx rectangular axes.

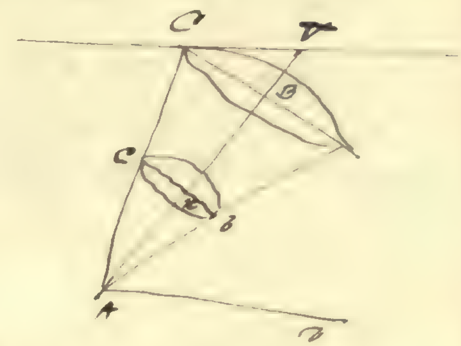
$$\therefore \frac{dx}{y} = \frac{g}{\mu y} \quad \therefore x = \frac{g}{\mu}(\log y - \log c)$$

$$\frac{\mu x}{g} = \log \frac{y}{c}.$$

$$\therefore \frac{y}{c} = e^{\frac{\mu x}{g}}.$$

$$y = c\, e^{\frac{\mu x}{g}}.$$

given cone filled with fluid is supported with its axis inclined to the horizon at a given ∠. on what sectn. of the base is this a max.?



Let ∠ $B + V = \theta$.

cb the regd. sectn.

$Ax = x$. $y = mx$ the eqn. to the generatg. line by the revolution of which the cone is generated ∴ $AB = a$ ∴ $BC = ma$.

the depth of the c. of g. of the regd. sectn. below the surf. of the fluid is $ma\cos\theta + (a-x)\sin\theta$.

the area of the sectn. is $\pi m^2 x^2$

∴ the pressure $= \mu\pi m^2 x^2 \{ma\cos\theta + (a-x)\sin\theta\}$ a maximum.

$$\therefore\ 2x(ma\cos\theta + (a-x)\sin\theta) - x^2\sin\theta = 0$$

$$2a(m\cos\theta + \sin\theta) - 3x\sin\theta = 0.$$

$$x = \frac{2}{3}\left(\frac{am\cos\theta + a\sin\theta}{\sin\theta}\right)$$

$$= \frac{2}{3}AV.$$

A cube is filled with fluid and made to revolve about an axis thro' its cent. of g. with an angular velocity α. Compare the pressure on its sides with the weight of the fluid neglect. grav.

Supp. the pressure on any pt. of the sides we have.

$$p = \rho\left\{\frac{1}{2}\alpha^2(x^2+y^2)\right\}.$$

Integrate with respect to x from $x = \frac{1}{2}a$ to $x = -\frac{1}{2}a$.

$$p' = \rho\left\{\frac{1}{2}\alpha^2\left(\frac{a^3}{3} + ay^2\right)\right\}.$$

Integrate from $y = \frac{1}{2}a$ to $y = -\frac{1}{2}a$.

$$p'' = \rho\left\{\frac{a^4\alpha^2}{3}\right\}.$$

$\therefore$ the pressure on the four sides $= \dfrac{4\rho a^4\alpha^2}{3}$.

and the weight $= g\rho a^3$.

$$\therefore p : w :: \frac{4\rho a^4\alpha^2}{3} : g\rho a^3$$

$$:: 4\alpha^2 a : 3g.$$

A conical vessel is filled with fluid whose density ∝ depth; the pressure on the base being equal to that on the sides find the vertical ∠.

Let θ = vertical ∠.

a = length of the axis of the cone.

$\therefore \pi a^2 \tan^2\theta$ = area of the base.

$\therefore \mu\pi a^4 \tan^2\theta$ = press on the base.

also $2\pi x\,\delta x \tan\theta \sec\theta$ = area of small annulus at depth x measured along the axis

$$\therefore 2\pi\mu x^3 \delta x \tan\theta \sec\theta = \text{press on } \delta s.$$

$$\therefore \frac{\pi \mu a^4}{2} \tan\theta \sec\theta = \text{whole pressure}$$

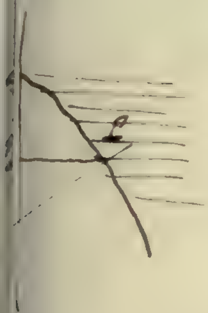
on the sides

$$\therefore \frac{\sec\theta}{2} = \tan\theta \quad \text{by question.}$$

$$\tan^2\theta + 1 = 4\tan^2\theta$$

$$\frac{1}{\sqrt{3}} = \tan\theta$$

or $\theta = 30^\circ$.



If P be a pt. in a dyke. the normal pressure at $P = ds . x + h \times \rho$.

where ds is an element of the curve

∴ the horizontal pressure $= \rho x \dfrac{dx}{ds} . \dfrac{ds}{dy}$ if $x + h = x$.

$= \rho x dx$.

similarly the ⊥ pressure $= \rho x dy$.

∴ the whole horizontal press $= \frac{1}{2} \rho (x^2 + C)$.

——— vertical ——— $= \rho x (y + C)$.

[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

To determine the nature of the stem of an hydrometer that the part (x) of the stem above the surface of the fluid may be $\because^l$ to the specific grav. of the fluid in which it is immersed.

Let a = value of x when specif. gr. $= 1$

$$\therefore \frac{x}{a} = s = \frac{w}{M}.$$

$$\frac{dx}{a} = -\frac{w\,dM}{M^2}.$$

Let B = whole magnitude of the instrument

$$\therefore M = B - \pi\int_x y^2 \quad \text{(if the stem is a solid of revolution)}$$

$$\therefore \frac{1}{a} = -\frac{w\pi y^2}{(B - \pi\int_x y^2)^2}.$$

~~$B^2 - 2\pi B\int_x$~~

$$B - \pi\int_x y^2 = y\sqrt{w\pi a}.$$

$$-\pi y^2 = dy_x \cdot \sqrt{w\pi a}.$$

$$\frac{-\pi}{\sqrt{w.\pi a}} = \frac{dy_x}{y^2}.$$

$$x\sqrt{\frac{\pi}{wa}} = \frac{1}{y} + C$$

$$\text{or } (x + C')y = \sqrt{\frac{wa}{\pi}}.$$

the $= ^n$ to a $\frac{1}{st}$ $\frac{?}{?}$ referred to its asymptotes.

[An] irregular body is slightly elevated or depressed i[n a] small vessel reqd. the time of an oscillatn.

Let a. b. be sectns of the surface of the fluid and of the plane of flotation.

x = depressn. or elevatn. of the solid

y = elevatn. or depressn. of the water.

Then $y(a-b) = b.x$

$$\therefore y = \frac{b x}{a-b}.$$

$$\therefore y + x = \frac{a x}{a-b}.$$

$$\therefore \text{the water displaced} = \frac{ab x}{a-b}.$$

[Let] ρ = density of the water. σ = density of the solid. V = volume of the solid g = force of grav.

$$\text{moving force on the solid} = g\rho.\frac{ab.x}{a-b}.$$

$$\therefore \text{acceleratg. force} = g\rho\frac{ab x}{\sigma V}.$$

$$\therefore \frac{d^2x}{dt^2} - \frac{g\rho ab}{V\sigma}x = 0$$

$$x = A\cos\left\{\sqrt{\frac{g\rho ab}{V\sigma}}.t + B\right\}.$$

x is a max when $\cos\sqrt{\frac{g\rho ab}{V\sigma}}t = 1$

$$\therefore t = 0,\ 2\pi.\ 4\pi\ \&c.$$

wherefore the body will oscillate continually the time of an oscillatn. being $2\pi\sqrt{\frac{V\sigma}{g\rho ab}}$.

A cylinder which when left to itself floats vertically in the surface of a fluid be elevated through a given depth determine the motion when left go.

Let h = height of the cylinder.

ρ = sp. grav. σ = sp. grav. of the fluid

a = height of elevation.

x = dist of the top of the cylinder from the position of rest at time t.

Then the moving force on the cylinder is

$\pi b^2 g \sigma x$ if b = rad of the base.

∴ accelerating force = $\pi b^2 h \sigma$

$$\therefore \frac{d^2x}{dt^2} - \frac{g\sigma}{h\rho} x = 0$$

$$x = A \cos\left(\sqrt{\frac{g\sigma}{h\rho}}\, t + B\right)$$

$$\frac{dx}{dt} = -\sqrt{\frac{g\sigma}{h\rho}} \cdot A \sin\left(\sqrt{\frac{g\sigma}{h\rho}} \cdot t + B\right).$$

$x = a$ when $t = 0$.

$$\therefore a = A \cos\left(\sqrt{\frac{g\sigma}{h\rho}} \cdot t + B\right) = A$$

$\frac{dx}{dt} = 0$ when $t = 0$. $\therefore 0 = -\sqrt{\frac{g\sigma}{h\rho}}\, A \sin B$ $\therefore B = 0$.

$$x = a \cos \sqrt{\frac{g\sigma}{h\rho}} \cdot t$$

x reaches its greatest value when $\cos \sqrt{\frac{g\sigma}{h\rho}} \cdot t = 1$

$$\therefore \sqrt{\frac{g\sigma}{h\rho}} \cdot t = 2\pi \quad \text{or} \quad t = 2\pi \cdot \sqrt{\frac{h\rho}{g\sigma}}.$$

which is the time of a complete oscillation

If the body be slightly disturbed, and the $=^m$ be stable it will continue to oscillate.

and if the vertical distance of the metacentre of the body from its c. of grav. $= D$. and $k^2 =$ rad² of gyration, the length of the isochronous pendulum $= \dfrac{k^2}{D}$.

Ex. a cone floating with its axis vertical.

$h =$ height of the cone $\therefore \frac{3h}{4} =$ height of its c. of grav.

$\rho =$ density of the fluid $\sigma =$ density of the cone. $x =$ part of its axis immersed $\therefore \rho x^3 = \sigma h^3 \therefore x = h\sqrt[3]{\frac{\sigma}{\rho}}$

$\therefore mh\sqrt[3]{\frac{\sigma}{\rho}} =$ rad of plane of flotat. $\therefore \frac{\pi}{4}.m^4h^4\left(\frac{\sigma}{\rho}\right)^{\frac{4}{3}} = Ak^2$.

$\therefore \pi m^2h^2.\left(\frac{\sigma}{\rho}\right)^{\frac{2}{3}}.\frac{h}{3}\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}}.HM = \frac{\pi}{4}.m^4h^4\left(\frac{\sigma}{\rho}\right)^{\frac{4}{3}}$.

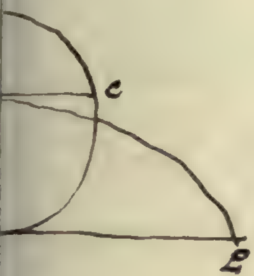
$\therefore HM = \frac{3}{4}.m^2h\left(\frac{\sigma}{\rho}\right)^{\frac{1}{3}} = \frac{3m^2h}{4}\sqrt[3]{\frac{\sigma}{\rho}}$

$\therefore$ dist. of the metacentre from the bottom of the cone is $\frac{3}{4}\left(h\sqrt[3]{\frac{\sigma}{\rho}} + m^2h\sqrt[3]{\frac{\sigma}{\rho}}\right) = \frac{3}{4}(m^2+1)h\sqrt[3]{\frac{\sigma}{\rho}}$.

$\therefore$ dist of met. from the c. of g. of the solid $=$

$\frac{3h}{4}\left\{1-(m^2+1)\sqrt[3]{\frac{\sigma}{\rho}}\right\}$.

$\therefore$ length of the isoch. pend $= \dfrac{4k^2}{3h\left(1-(m^2+1)\sqrt[3]{\frac{\sigma}{\rho}}\right)}$



AB the vertical side of a vessel filled with fluid. D an orifice. DE the course of the issuing stream. BCA a semicircle.

the velocity of the issuing stream = that acquired from B ∴ B is a point in the directrix

$\therefore AE^2 = 4 BD \cdot DA$

$= 4 DC^2$

$AE = 2 DC = 2 \sin BC.$

The times in which two hemispheres are emptied the one by an orifice in the vertex, and the other by an equal orifice in the base are as 3 : 5.

Req. the proport. of the radii.

Let a = rad of the one which is emptied through an orifice in the base.

Then for the mot. of the fluid we have.

$$S\frac{dx}{t} = k\sqrt{2gx}. \quad \text{or } \pi(a^2 - x^2) = k\sqrt{2gx}.$$

$$\frac{\pi}{k\sqrt{2g}}\left\{a^2x^{-\frac{1}{2}} - x^{\frac{3}{2}}\right\} = -\frac{dt}{dx} \therefore t = \frac{\pi}{k\sqrt{2g}}\left\{2a^2x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right\} + C.$$

Take integral from $x = 0$ to $x = a$.

$$t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{8}{5}a^{\frac{5}{2}}\right\}.$$

Let a' = rad of the other

$$\therefore \frac{\pi}{k\sqrt{2g}}\left(2a'x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) = \frac{dt}{dx} \therefore t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{4}{3}a'x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right\}$$

Take the limits as before $t = \frac{\pi}{k\sqrt{2g}}\left\{\frac{14}{15}a'^{\frac{5}{2}}\right\}$.

$\therefore$ By the question

$$3 : 5 :: \frac{14}{15}a'^{\frac{5}{2}} : \frac{8}{5}a^{\frac{5}{2}} :: \frac{7}{3}a'^{\frac{5}{2}} : 4a^{\frac{5}{2}}$$

$$12a^{\frac{5}{2}} = \frac{35}{3}a'^{\frac{5}{2}}.$$

$$a = a'\left(\frac{35}{36}\right)^{\frac{2}{5}}.$$

Determine the nature of an orifice to be made in the side of a vessel which is always filled to the same altitude, so that the quantity discharged ∝ as height of the portion, and when the whole orifice is open, the quantity discharged = C.

The area of any portⁿ of the orifice $= \int y\,dx$.

$\therefore \int y\,dx\,\sqrt{2gx} = C\int dx$.

$y\sqrt{2gx} = C = m\sqrt{2gb}$ {if m = length of the ordinate at depth b}.

$y^2 x = m^2 b$.

also the whole quantity discharged =

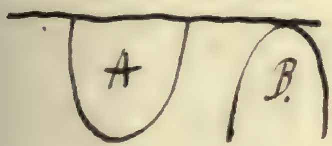
$2m(b-a)\sqrt{2gb}$, where b = depth of the lowest, and a that of the highest part of the orifice below the surf. of the fluid

$$\therefore 2m(b-a)\sqrt{2gb} = C$$

$$m = \frac{C}{2(b-a)\sqrt{2gb}}$$

$$\therefore y^2 x = \frac{C^2}{8 \cdot g \cdot (b-a)^2}$$

Compare the quantities of fluid discharged by two equal ⌓'s, in the side of a vessel kept constantly full. one of them having its base, and the other its vertex downwards, and the summits of both coinciding with the surface of the fluid.



Let $y = \sqrt{mx}$ be the eqn to B.

$\therefore y = \sqrt{-(mx - ma)} = y = \sqrt{m(a-x)}$ is the eqn to A.

Now the quantities discharged at distance x from the surfaces are

$2\sqrt{2gx}\sqrt{mx}$, and $2\sqrt{2gx}\cdot\sqrt{m(a-x)}$ respectively.

$\therefore$ they vary as the whole quantities discharged are

$2\sqrt{2gm}\cdot a^2$ and $2\sqrt{2gm}\cdot\int_x \sqrt{ax - x^2}$ or $2\sqrt{2gm}\cdot\frac{a^2\pi}{4}$.

wherefore $\frac{\text{quantity discharged by A}}{\text{quantity discharged by B}} = \frac{\pi}{4}$.

If a stream of fluid fall ⊥ on a circular plane and at an angle 30° on an elliptic plane whose minor axis equals the diameter of the circle and impel these planes with equal forces; to determine the proportion of the major and minor axis.

$R \propto A v^2 \sin^2 \theta$.

In the ⊙ $A = \pi a^2$

$\therefore R \propto v^2 \pi a^2$.

Let $2k$ = major axis of the ellipse

Then $A = \pi a k$.

$$R \propto v^2 \pi a k \sin^2 30^\circ \propto \frac{\pi a k}{4} v^2$$

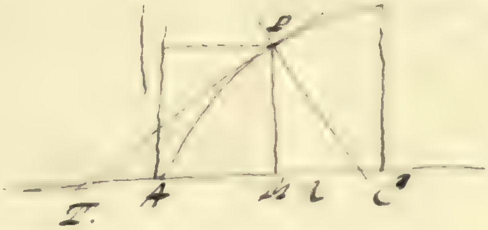
$$\therefore \pi a^2 = \frac{\pi a k}{4}$$

$$k = 4a.$$

or the major axis is to the minor as 4 : 1

Find the resistance to the motion of a solid through a fluid whose velocity is v.

Let R = reqd resistance.



$x \cdot y \cdot$ the coords of P.

$2\pi y\, ds$ = area of the element described by the revolution of P about AM.

also $\sin^3$ of inclinat. $= \dfrac{\left(\frac{dy}{dx}\right)^3}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \dfrac{\left(\frac{\sqrt{m}}{\sqrt{x}}\right)^3}{\left(\frac{m+x}{x}\right)^{\frac{3}{2}}} = \dfrac{m^{\frac{3}{2}}}{(m+x)^{\frac{3}{2}}}$.

$$\therefore \frac{dR}{v} = \frac{2\pi \sqrt{mx}\, m^{\frac{3}{2}}}{(m+x)^{\frac{3}{2}}} ds = \pi \cdot 2^{\frac{3}{2}} m^2 \frac{\sqrt{x}}{(m+x)^{\frac{3}{2}}} ds = \pi \cdot 2^{\frac{3}{2}} m^2 \frac{x}{(m+x)^2}$$

$$R = \pi 2^{\frac{3}{2}} m^2 \left\{ \int_x \frac{m+x}{(m+x)^2} - \int_x \frac{m}{(m+x)^2} \right\} = \pi 2^{\frac{3}{2}} m^2 \left\{ \int_x \frac{1}{m+x} - m \int_x \frac{1}{(m+x)^2} \right\}$$

$$= \pi 2^{\frac{3}{2}} m^2 \left\{ \log(m+x) + m \frac{1}{m+x} \right\} + C.$$

$R = 0 \quad x = 0$

$$0 = \pi 2^{\frac{3}{2}} m^2 \log m + 1 + C$$

$$R = 2^{\frac{3}{2}} m^2 \pi \left\{ \log\left(\frac{m+x}{m}\right) + \frac{m}{m+x} - 1 \right\}.$$

At C $x = b$ suppose.

$$\therefore R = 2^{\frac{3}{2}} m^2 \pi \left\{ \log\left(\frac{m+b}{m}\right) + \frac{m}{m+b} - 1 \right\}.$$

Compare the resistance on a cube moving in the direction of its diagonal with that on a cube moving in a direction ⊥ to one of its sides.

Let v = velocity. a = side of the stream

the sine of the ∠ of inclination of each side of the fluid cube is $\frac{1}{\sqrt{3}}$.

∴ if R = resist. on the cube mov. diagonally.

r = ——— ⊥ to a side

$$R : r :: \frac{1}{3\sqrt{3}} \times 3 : 1$$

$$\therefore R = \frac{r}{\sqrt{3}}$$

$$r = R\sqrt{3}.$$

Find the Resistance of a stream to the motion thro' it of the solid generated by the revolution of cycloid about its axis.

Let R = Resistance

$$\therefore \frac{dR}{dx} = 2\pi y \cdot \frac{\left(\frac{dy}{dx}\right)^3}{1+\left(\frac{dy}{dx}\right)^2} \times \frac{1}{2}\rho v^2$$

$$= 2\pi\left\{a\,\text{vers}^{-1}\frac{x}{a} + \sqrt{2ax-x^2}\right\}\cdot\frac{(2a-x)^{\frac{3}{2}}}{2a\cdot\sqrt{x}}\cdot\frac{1}{2}\rho v^2$$

$$\therefore R = \frac{\pi\rho v^2}{2a}\int_x \left(a\,\text{vers}^{-1}\frac{x}{a}\right)\left\{\frac{(2a-x)^{\frac{3}{2}}}{\sqrt{x}}\right\} + \frac{\pi\rho v^2}{2a}\int_x (2a-x)^2.$$

Now $$\int_x \frac{(2a-x)^{\frac{3}{2}}}{\sqrt{x}} = \int_x \frac{(2a-x)^2}{\sqrt{2ax-x^2}} = \int_x \frac{2a^2}{\sqrt{2ax-x^2}} + 2a\int_x \frac{a-x}{\sqrt{2ax-x^2}} - \int_x \sqrt{2ax-x^2}$$

$$= 2a^2\text{vers}^{-1}\frac{x}{a} + 2a\sqrt{2ax-x^2} - \frac{x-a}{2}\sqrt{2ax-x^2} - \frac{a^2}{2}\text{vers}^{-1}\frac{x}{a}.$$

$$\int_x \frac{(2a-x)^{\frac{3}{2}}}{\sqrt{x}}\,a\,\text{vers}^{-1}\frac{x}{a} = a\,\text{vers}^{-1}\frac{x}{a}\left\{\frac{3a^2}{2}\text{vers}^{-1}\frac{x}{a} + \frac{5a-x}{2}\sqrt{2ax-x^2}\right\}.$$

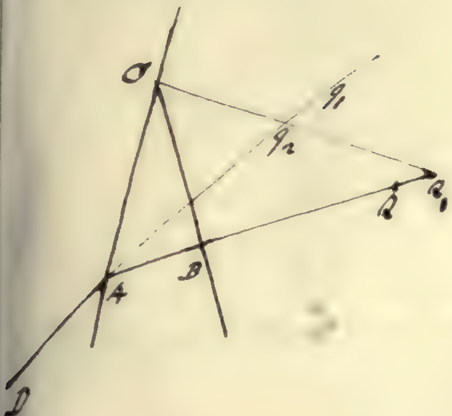
$$-a\int_x\left\{\frac{3a^2}{2}\text{vers}^{-1}\frac{x}{a}\frac{1}{\sqrt{2ax-x^2}} + \frac{5a-x}{2}\right\}.$$

$$= \frac{3a^3}{2}\left(\text{vers}^{-1}\frac{x}{a}\right)^2 + a\,\text{vers}^{-1}\frac{x}{a}\left(\frac{5a-x}{2}\right)\sqrt{2ax-x^2}$$

$$-\frac{3a^3}{4}\left(\text{vers}^{-1}\frac{x}{a}\right)^2 - \frac{1}{2}\left\{5a^2x - \frac{ax^2}{2}\right\} + C.$$

$$= \frac{3a^3\pi^2}{4} - 4a^3 = \frac{1}{4}\left\{3a^3\pi^2 - 16a^3\right\}$$ from $x=2a$ to $x=0$.

Optical Problems



QBAD is the course of the axis of a pencil which is incident directly at B on the first surface of a prism. find its primary and secondary foci after refractn.

Let $BQ = u$ $BC = a$. ι = inclination of the prism $Aq_2 = v_2$ $Aq_1 = v_1$ the secondary and primary foci. Q_1 the geometrical focus of the pencil after refractn at the first surface.

Then $AB = a \sin \iota = z$

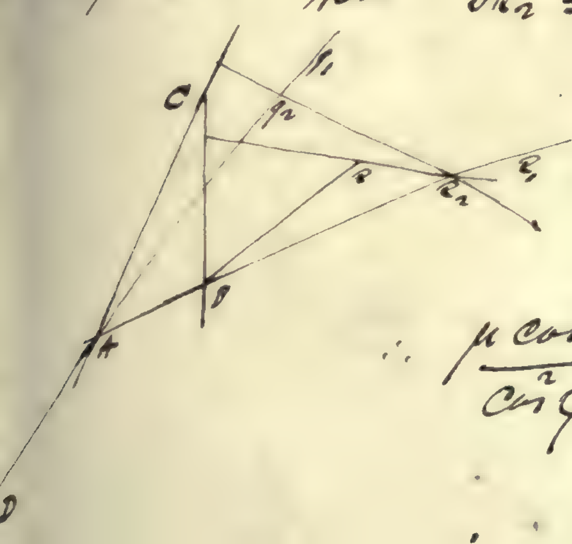
$BQ_1 = \mu u \;\therefore\; AQ_1 = \mu u + a \sin\iota$

If ϕ, ϕ' be angles of incidence and emergence at A

$$Aq_1 = v_1 = \frac{\mu \cos^2\phi'}{\cos^2\phi}\left(\mu u + a \sin\iota\right)$$

$$Aq_2 = v_2 = \mu\left(\mu u + a \sin\iota\right)$$

If the incidence is not direct at B, but ψ, ψ' ∠s of incidn. & reflectn. then $BQ_2 = \mu u$, $BQ_1 = \frac{\mu \cos^2\psi'}{\cos^2\psi} u$



$$\therefore AQ_2 = a \frac{\sin\iota}{\cos(\psi'+\iota)} + \mu u.$$

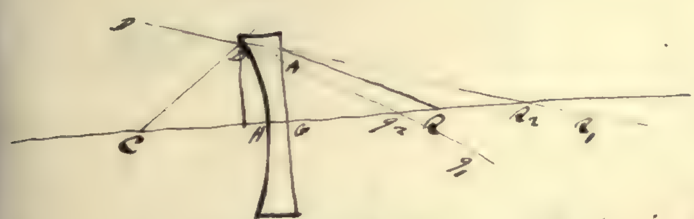
$$AQ_1 = a.\frac{\sin\iota}{\cos(\psi'+\iota)} + \frac{\mu\cos^2\psi'}{\cos^2\psi} u$$

$$\therefore \frac{\mu\cos^2\phi'}{\cos^2\phi} v_1 = \frac{a \sin\iota}{\cos(\psi'+\iota)} + \frac{\mu\cos^2\psi'}{\cos^2\psi} u$$

$$v_1 = a \frac{\cos^2\phi}{\text{[cut off]}} \frac{\sin\iota}{\text{[cut off]}} + \frac{\cos^2\psi'\cos^2\phi}{\text{[cut off]}} u$$

[illegible]

A pencil is incident obliquely on a plano concave lens. find the foci of the emergent pencil



Q the origin of the pencil. Q_2, Q_1 the foci after refr. at the first surf.
q_2 q_1 ——— second surf.

$HQ = u$ $Dq_2 = v_2$; $Dq_1 = v_1$ $GH = t$. $CH = r$ the radius of the ⊙ of which DH is an arc. φ φ' ∠s of incidence and refract. at A.
ψ ψ' ∠s of emergence and inciden. at D.

To find AD. let $CH = x$. $DH = y$.

$$y = \sqrt{r^2 - x^2} = (r + t + GQ_2 - x)\tan\varphi.$$

$$r^2 - x^2 = (r+t+GQ_2)^2 - 2(r+t+GQ)x + x^2.$$

$$\therefore x = \frac{(r+t+GQ_2)^2 - r^2}{2(r+t+GQ)}$$

$$AQ_2 = \mu u \quad \therefore GQ_2 = \mu u \cos\varphi \quad \therefore x = \frac{(r+t+\mu u\cos\varphi)^2 - r^2}{2(r+t+\mu u\cos\varphi)}$$

$$AQ_1 = \frac{\mu\cos^2\varphi'}{\cos^2\varphi}u$$

$$\therefore DQ_2 = \mu u + \frac{(r+t+\mu u\cos\varphi)^2 - r^2}{2(r+t+\mu u\cos\varphi)}$$

$$DQ_1 = \frac{\mu\cos^2\varphi'}{\cos^2\varphi}u + \frac{(r+t+\mu u\cos\varphi)^2 - r^2}{2(r+t+\mu u\cos\varphi)}$$

now
$$\frac{\mu\cos^2\psi'}{v_1} - \frac{\cos^2\psi}{DQ_1} = -\frac{\mu\cos\psi' - \cos\psi}{r}$$

and
$$\frac{\mu}{v_2} - \frac{1}{DQ_2} = -\frac{\mu\cos\psi' - \cos\psi}{r}.$$

Calculate the position and dimensions of the least ○ of aberration of a direct pencil refracted at a plane surface. the origin of the pencil being at a dist. 20 from the surface, and the distance of the extreme ray from the axis 5.

O the origin E the geometrical focus. Hs. hs. the directions of the extreme rays after refract. hr. Ir the direct. of any other rays. meeting Hs in t.

mt the rad of the least ○ of aberrat. $AK = y'$ $AH = y$. $\mu = \frac{4}{3}$.

$\frac{mr}{mt} = \frac{Ar}{Ak} = \frac{Ar}{y'}$ $\therefore mr = mt.Ar.\frac{1}{y'}$

similarly ; — $ms = mt.Ar.\frac{1}{y}$. $\Big\} \therefore sr = mt.Ar.\left(\frac{y+y'}{yy'}\right)$.

But $Er : Es :: y'^2 : y^2$ $\therefore rs : Es :: y^2 - y'^2 : y^2$

$\therefore rs = Es.\frac{y^2-y'^2}{y^2} = mt.Ar.\frac{y+y'}{yy'}$ or $mt = \frac{Es}{Ar}.\frac{(y-y')y'}{y}$.

& for the least that mt is a maximum. $y = 2y'$ $\therefore y' = \frac{y}{2}$.

$mt = \frac{y}{4}\frac{Es}{Ar}$. $mr = \frac{1}{4}Es$

Now $Ar = \mu u + \frac{\mu^2-1}{\mu}.\frac{y'^2}{2u} = \frac{4.20}{3} + \frac{7.3}{9.4}.\frac{25}{4.2.20}$

$= \frac{80}{3} + \frac{35}{384} = \frac{10275}{384}$.

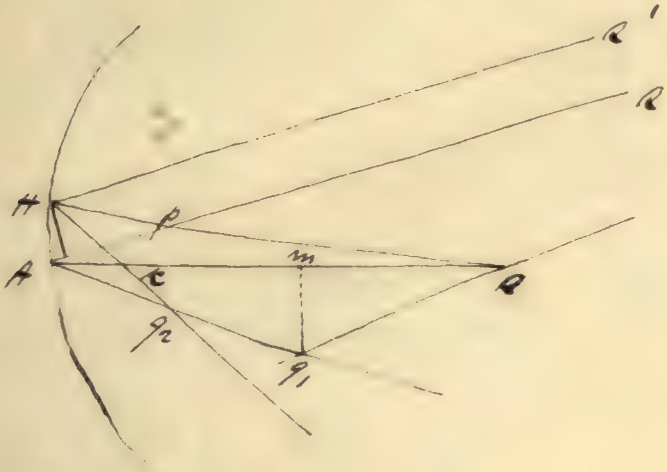
$Es = \frac{\mu^2-1}{\mu^2}.\frac{y^2}{2u}$

$= \frac{7.3}{9.4}.\frac{25}{40} = \frac{7.5}{12.8} = \frac{35}{96}$.

$\therefore Am = \frac{10275+140}{384} = \frac{10415}{384} =$ [illegible]

$mt = \frac{5}{4}.\frac{35}{96}.\frac{384}{10275} = \frac{35}{2055} = \frac{7}{411}$

A small pencil of ∥ rays is incident obliquely at an ∠ of 30° on a spherical reflecting surface whose radius is 60 inches. find the focal lines of the emergent pencil



RA the axis of the incident pencil. Aq_1 the axis of the reflected pencil – since the pencil consists of ∥ rays. the line thro' O. from the origin may be considered ∥ to RA.

$$\therefore \angle q_1OA = \angle q_1AO = 30°.$$

$$\therefore Aq_1 = \frac{r \sec 30°}{2} = \frac{30.2}{\sqrt{3}} = \frac{40}{\sqrt{3}}.$$

also. if φ. be the ∠ of incidence of RA. R'H.

φ + δφ — ∠ of reflect. of R'H.

$$\delta\phi = \angle R'HO - RAO = apO - aAO = HOA = \frac{AH}{r} \text{ nearly}.$$

$$\delta\phi = q_2HO - q_2AO = k - HRA - k + Hq_2A$$

$$= \frac{AH \cos\phi}{q_2} - \frac{AH}{r} = \frac{AH}{r}$$

$$\therefore \frac{1}{q_2} = \frac{2}{r\cos\phi} = \frac{2}{20.\sqrt{3}} = \frac{1}{10\sqrt{3}}$$

$$\therefore q_2 = 10\sqrt{3}. = \frac{30}{\sqrt{3}}.$$

A small pencil is incident obliquely on a plane refracting surface. the distance of the origin from the point of incidence is 20 inches, and the refractive index $\frac{4}{3}$. find the position of the focal lines. the angle of incidence being 30°.

Let $AQ = 20$.

$Hq_2 = v_2$ } the secondary
$Hq_1 = v_1$ } and primary foci.

then $\frac{\mu \cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = 0$.

$$\therefore v_1 = \frac{\mu u \cos^2\phi'}{\cos^2\phi} = \frac{u}{\cos^2\phi}\left\{\mu(1 - \sin^2\phi')\right\}$$

$$= \frac{u}{\cos^2\phi}\left\{\mu - \sin^2\phi\right\}.$$

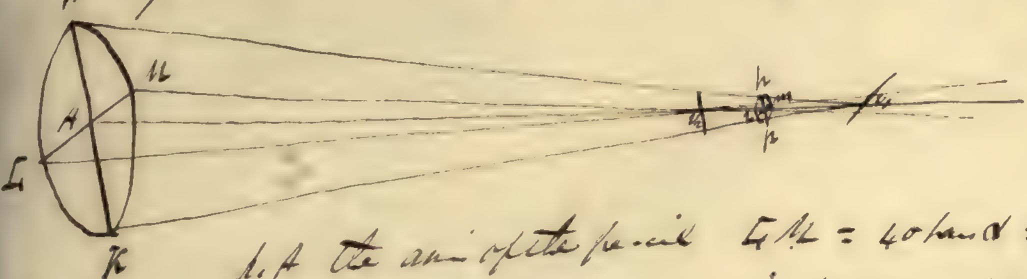
$$= \frac{20.4}{3}\left\{\frac{4}{3} - \frac{1}{4}\right\}$$

$$= \frac{20.4.13}{3.3.4} = \frac{260}{9}$$

also $v_2 = \mu u = \frac{4.20}{3} = \frac{80}{3} = \frac{240}{9}$.

$$\therefore v_1 - v_2 = \frac{20}{9}.$$

[C]alculate the posn. and dimensions of the ⊙ of least confusion of an oblique pencil incident on a plane surface at a ∠ of 30°. after refraction. the refractive index being $\frac{4}{3}$, and the distance of the origin of the pencil from the point of incidence 20 inches, semi vertical ∠ α.



p.A the axis of the pencil $LM = 40\tan\alpha = c$ suppose

$\therefore HK = c\sec 30° = \frac{2c}{\sqrt{3}}$.

Now. $\frac{hk}{mp_1} = \frac{HK}{Ap_1} = \frac{2c}{\sqrt{3}.v_1}$ or $hk = \left(\frac{v_1 - Ao}{v_1}\right)\frac{2c}{\sqrt{3}}$

$\frac{lm}{ov_2} = \frac{LM}{Av_2} = \frac{c}{v_2} \therefore lm = \left(\frac{Ao - v_2}{v_2}\right)c.$

But $lm = hk \therefore \frac{Ao - v_2}{v_2} = \frac{2(v_1 - Ao)}{\sqrt{3}.v_1}$ or $Ao(\sqrt{3}.v_1 - 2v_2) = v_1v_2(\sqrt{3}+2)$

$Ao = \frac{v_1v_2(\sqrt{3}+2)}{v_1\sqrt{3} - 2v_2}.$

$\therefore lm = \left(\frac{v_1(\sqrt{3}+2)}{v_1\sqrt{3} - 2v_2} - 1\right)c$

$= \frac{2(v_1 + v_2)c}{v_1\sqrt{3} - 2v_2}.$

But $\mu\frac{\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = 0 \quad \frac{1}{v_1} = \frac{\cos^2\phi}{\mu u\cos^2\phi'} = \frac{3.3}{4.4.20.(1 - \frac{9}{4.16})}$

$= \frac{3.3.4.16}{4.4.20.55} = \frac{9}{275}$

$\therefore v_1 = \frac{275}{9} = 30\frac{5}{9}.$

$v_2 = \mu u = \frac{4.20}{3} = \frac{80}{3} = 26\frac{2}{3}.$

$\therefore lm. = \frac{2\{.\frac{515}{9}\}c}{30\frac{5\sqrt{3}}{9} - \frac{480}{9}} = \frac{1030c}{150\sqrt{3} - 480}$

A spherical surface

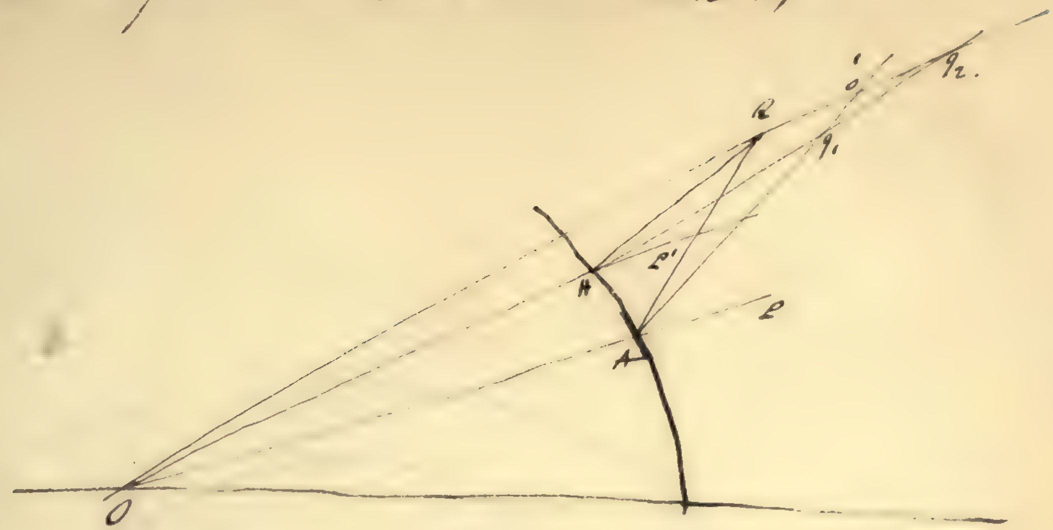
Find the position of the focal lines of an oblique pencil reflected at a convex spherical surface.

Take origin O the centre of the surface. R the axis of the pencil incident & produced backward to meet RO in q_2 the secondary focus, RH any ray which after reflect. is produced backward to meet Aq_2 in q_1 the primary focus. ∴ if lines be considered +ve when measured in direct. more nearly opposite to that of the incident pencil, both v and Aq_1, Aq_2 are −ve.

$$\angle HRA = \frac{HN}{HR} = \frac{AH \sin HAR}{HR} = \frac{AH\cos\phi}{u} \text{ nearly.}$$ {φ, φ' = opposite ∠s of incid. of AR, HR.

$$\text{Now } \phi = \angle RAE - \angle RHE'$$
$$= \angle ROA + \angle ORA - \angle ROH - \angle ORH$$
$$= \angle HOA + \angle HRA = \frac{AH}{r} + \frac{AH\cos\phi}{u}.$$
$$\text{Also } \phi = q_1AO - q_1HO = \pi - HOA - \pi + Hq_1A$$
$$= \frac{AH\cos\phi}{v_1} - \frac{AH}{r}.$$
$$\therefore \frac{\cos\phi}{v_1} - \frac{1}{r} = \frac{\cos\phi}{u} + \frac{1}{r}.$$
$$\frac{\cos\phi}{-v_1} + \frac{1}{u} = -\frac{2}{r\cos\phi}$$
$$\text{Let } \angle AOq_2 = \theta \quad \therefore \frac{AO}{-Aq_2} = \frac{-\sin(\phi+\theta)}{\sin\theta} ; \quad \frac{AO}{AR} = \frac{\sin(\phi-\theta)}{\sin\theta} : -u$$
$$\therefore \frac{1}{-v_2} + \frac{1}{u} = -\frac{2\cos\phi}{r}.$$

Oblique refraction at a convex spherical surface.



$$\delta\phi = \angle RAe - \angle RHe = \frac{AH}{r} + \frac{AH\cos\phi}{u}.$$

$$\delta\phi' = \angle q_1Ae - \angle q_1He'$$

$$= \{AO'O + O'OA\} - (Hq_2O + q_2OH).$$

$$= \{\overline{Hq_2O} + O'q_1q_2 + O'OA\} - (\overline{Hq_2O} + q_2OH)$$

$$= Hq_1A + HOA = \frac{AH\cos\phi'}{v} + \frac{AH}{r}.$$

$$\therefore \frac{\delta\phi}{\delta\phi_1} = \frac{\mu\cos\phi'}{\cos\phi} = \frac{\frac{1}{r} + \frac{\cos\phi}{u}}{\frac{1}{r} + \frac{\cos\phi'}{v}}$$

$$\therefore \frac{\mu\cos^2\phi}{v} - \frac{\cos^2\phi}{u} = -\left\{\frac{\mu\cos\phi' - \cos\phi}{r}\right\}.$$

$$\frac{-r}{AR} = \frac{\sin(\phi-\theta)}{\sin\theta} = -\sin\phi\cot\theta + \cos\phi$$

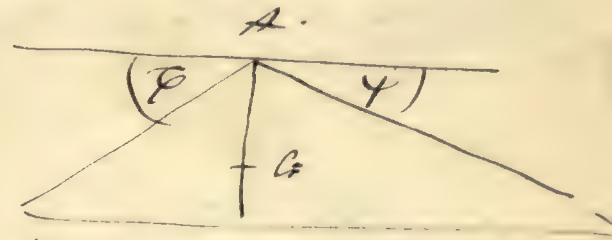
$$-\frac{r}{AO} = \frac{\sin(\phi'-\theta)}{\sin\theta} = -\sin\phi'\cot\theta + \cos\phi'$$

$$\therefore \frac{\mu}{v_2} - \frac{1}{u} = -\frac{1}{r}\left\{\mu\cos\phi' - \cos\phi\right\}.$$

Mechanical Problems

bent lever. often. the 2 arms are a & b. and the angle θ makes small oscillations in one plane. the length of the isochronous pendulum is

$$\frac{2}{3} \cdot \frac{a^3+b^3}{\sqrt{a^4+2a^2b^2\cos\theta+b^4}}.$$



$AG = h.$

we have $M(k^2+h^2) = \frac{1}{3}(a^3+b^3).$

$2Mh = a^2\sin\varphi + b^2\sin\psi$. (1)

$0 = a^2\cos\varphi - b^2\sin\psi$ (2)

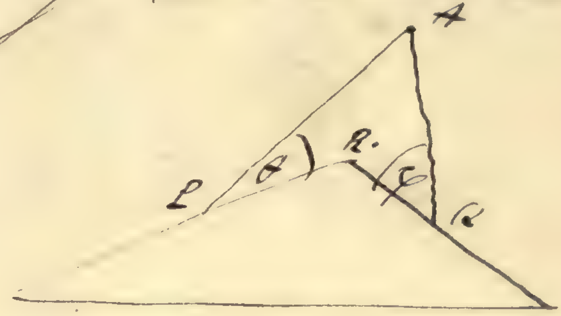
square (1) and (2), and add.

$(2Mh)^2 = a^4 + 2a^2b^2\cos\theta + b^4$

$\therefore$ length of isochronous pendulum =

$$\frac{2}{3} \cdot \frac{a^3+b^3}{\sqrt{a^4+2a^2b^2\cos\theta+b^4}}.$$

Two weights are connected by a string passing round a fixed pulley, and slide along two inclined planes. Shew that the velocities of the two weights at any time are inversely ∝ to the cosines of the angles which their strings make with the planes.



$RP = s. \quad RQ = s'$

Now $\delta AQ = \delta s' \cos\phi$

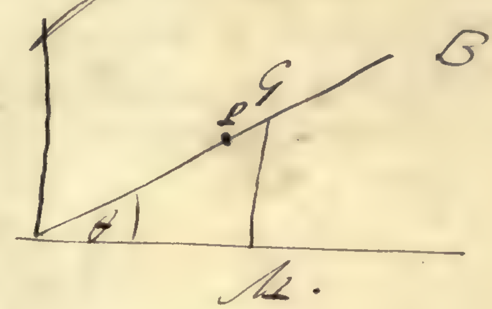
$\delta AP = \delta s \cos\theta$ } ultly.

$\delta AQ = \delta AP.$

$\therefore \delta s' \cos\phi = \delta s . \cos\theta.$

$$\frac{\frac{ds}{t}}{\frac{ds'}{t}} = \frac{\cos\phi}{\cos\theta}.$$

A rod revolves about one extremity in a horizontal plane: find the diffl-=n of the path of a ring which slides along it.



AM = x. MG = y.

AP = r. AG = a.

M = mass of rod. m = mass of ring.

Conservat. of areas.

$$M\left((k^2+a^2)\frac{d\theta}{t} + mr^2\frac{d\theta}{t}\right) = C \quad (1)$$

Vis Viva

$$M.(k^2+a^2)\left(\frac{d\theta}{t}\right)^2 + m\left\{\left(\frac{dr}{t}\right)^2 + r^2\left(\frac{d\theta}{t}\right)^2\right\} = C_1$$

$$\text{Now } \frac{d\theta}{t} = \frac{C}{M(k^2+a^2)+mr^2}.$$

$$\frac{\left\{M.(k^2+a^2) + m.\overline{r^2 + \left(\frac{dr}{\theta}\right)^2}\right\}}{M.(k^2+a^2) + mr^2} = k^2$$

is the diffl-=n reqd.

[illegible] the [illegible] for long tabular [illegible] [illegible] [illegible] long .

Nicholson's hydrometer whose weight is w, is slightly depressed till the weight of the fluid displaced becomes $w+w'$: when the hydrometer in rising reaches its posit^n of $=^m$ a small weight w'' is placed in the upper dish: det. the motion, and shew that if an additional weight $= \sqrt{w'^2 + w''^2 + \frac{w''}{w} w'^2}$. be inserted at a proper time, the instrument may be reduced to a state of $=^m$

let a be the depth of the c of gr. of the instrument from its posit^n of rest at the time the weight of the fluid displaced is $w+w'$

x = depth of the same at a time t.

$\therefore$ for the mot^n. $\frac{d^2x}{dt^2} = -\frac{x}{a} \cdot g \cdot \frac{w'}{w} = -\frac{g}{a} \cdot \frac{w'}{w} x$.

$$\therefore \frac{dx^2}{dt^2} = -\frac{g}{a} \cdot \frac{w'}{w} x^2 + C.$$

but $\frac{dx}{dt} = 0$ when $x = a$.

$$\therefore \frac{dx^2}{dt^2} = \frac{g}{a} \frac{w'}{w} (a^2 - x^2).$$

At the posit^n of $=^m$. $x = 0$ $\therefore \frac{dx}{dt} = a\sqrt{\frac{g}{a} \cdot \frac{w'}{w}}$.

Let the weight w'' be now placed in the upper dish.

$\therefore \frac{d^2x'}{dt^2} = -\frac{g}{a} \cdot \frac{w'}{w+w_{''}} \cdot x$ is the $=^n$ for the subsequent mot^n.

$$\therefore \left(\frac{dx'}{dt}\right)^2 = -\frac{g}{a} \cdot \frac{w'}{w+w_{''}} x'^2 + C$$

when $x' = 0$. $\left(\frac{dx'}{dt}\right)^2 = \frac{g}{a} \cdot \frac{w'}{w} a^2$

$$\therefore \left(\frac{dx'}{dt}\right)^2 = \frac{g}{a} \cdot \frac{w'}{w+w_{''}} \left\{ \frac{w+w_{''}}{w} \cdot a^2 - x'^2 \right\}.$$

$$\frac{dt}{dx'} = \sqrt{\frac{(w+w_{''})a}{w'g}} \cdot \frac{1}{\sqrt{\frac{w+w_{''}}{w} a^2 - x'^2}}$$

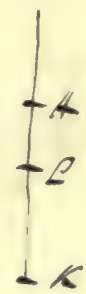
$$\therefore t = \sqrt{\frac{(w+w_{''})a}{w'g}} \cdot \sin^{-1} \frac{x'\sqrt{w}}{a\sqrt{w+w_{''}}} + C$$

$$t = 0 \quad x' = a. \quad \therefore C = 0.$$

$$t = \sqrt{\frac{(w+w_{''})a}{w'g}} \cdot \sin^{-1} x' \sqrt{\frac{w}{w+w_{''}}}.$$

Nicholsons hydrometer.

A the surface of the fluid.
C the plane of flotation.



A C = x. then for the motion at first if
C = d.

$$\frac{dv}{t} = -\frac{gx}{d}.$$

$$\left(\frac{dx}{t}\right)^2 = C - \frac{gx^2}{d} \text{ when } x = d\frac{w'}{w}, \frac{dx}{t} = 0.$$

$$\overline{\frac{dx}{t}}\Big|^2 = \frac{g}{d}\left\{d^2\left(\frac{w'}{w}\right)^2 - x^2\right\}$$

and vel, when it passes thro' its position of = m

$$= g d.\left(\frac{w'}{w}\right)^2.$$

II. To find subsequent motion, measuring x from the new plane of flotation which is below the former by a quantity $= d.\frac{w''}{w}$, we have

$$\frac{w + w''}{g}\left(\frac{dx}{t}\right)^2 = C - \frac{wx^2}{d}.$$

and when $x = d.\frac{w''}{w}$ we have

$$\overline{vel}\,\Big|^2 = g d.\left(\frac{w'}{w}\right)^2.$$

[illegible]

Nicholsons hydrometer.

$$\frac{w+w''}{g}\cdot\left(\frac{dx}{t}\right)^2 = (w+w'')\,d\left(\frac{w'}{w}\right)^2 + wd.\left(\frac{w'}{w}\right)^2 - \frac{w\,x^2}{d}.$$

and when $\left(\frac{dx}{t}\right) = 0$.

$$\frac{wx}{d} = (w+w'')\,d\left(\frac{w'}{w}\right)^2 + wd\left(\frac{w''}{w}\right)^2$$

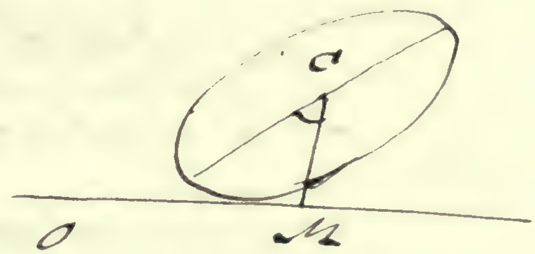
and additional weight to be placed to sustain it at the additional depth $= w.\frac{x}{d}$

$$\therefore \frac{w^2x^2}{d^2} = (w+w'')\,\frac{w'^2}{w} + w''^2.$$

$\therefore$ Additional weight $= \sqrt{w'^2 + w''^2 + \frac{w''}{w}.\,w'^2}$.

A rotatory motion is given to an elliptic cylinder which is then suddenly laid lengthwise upon a smooth horizontal plane: det. the subsequent mot? What must be the original ~~motion~~ ∠r velocity that the body may assume a posit^n of perpetual rest?

Let ω be the initial ∠r velocity, immediately after the cylinder is placed on the plane.



$CM = y$; M = its mass.

Then $M\left\{\left(\frac{dy}{dt}\right)^2 + k^2\left(\frac{d\phi}{dt}\right)^2\right\} = 2Mg\{h - y\}$.

$$y = a\sqrt{1 - e^2\sin^2\phi} = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}.$$

$$\frac{dy}{dt} = -\frac{a^2\cos\phi\sin\phi + b^2\sin\phi\cos\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\cdot\frac{d\phi}{dt}$$

$$= -\frac{(a^2 - b^2)\sin\phi\cos\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\frac{d\phi}{dt}$$

also $\left(\frac{d\varphi}{dt}\right)^2 \left\{ k^2 + \frac{(a^2-b^2)\sin^2\varphi\cos^2\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi} \right\}$

$$= C - 2g\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}.$$

Let α be the original value of φ.

$$\therefore \omega^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right. = C - 2g.\sqrt{a^2\cos^2\alpha - b^2\sin^2\alpha}$$

$$\therefore \left(\frac{d\varphi}{dt}\right)^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\varphi\cos^2\varphi}{a^2\cos^2\varphi + b^2\sin^2\varphi} \right\}$$

$$- \omega^2 \left\{ \frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right.$$

$$= 2g \left\{ \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi} + \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \right\}.$$

If it assume a positn of permanent rest then

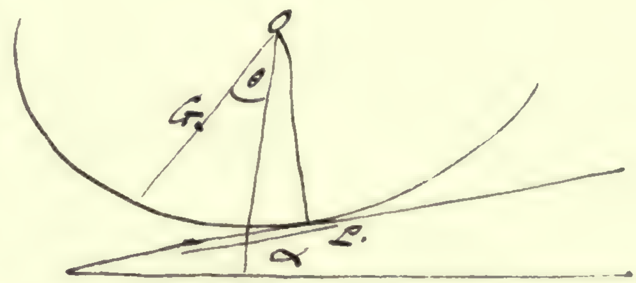
when $\varphi = 0$ $\frac{d\varphi}{dt} = 0$.

$$\therefore \omega^2 = \frac{2g\left\{ -\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + a \right\}}{\frac{a^2+b^2}{4} + \frac{(a^2-b^2)\sin^2\alpha\cos^2\alpha}{a^2\cos^2\alpha + b^2\sin^2\alpha}}.$$

[illegible]

A cylindrical body oscillates on a plane, inclined at a very small ∠ α to the horizon, the friction being such as to prevent all sliding. Its radius $= a$, and the distance of the cent. of grav from the axis being $= c$. Shew that the time of oscillation is very little different from that on a horizontal plane, but that the extent of oscillation is > by $\frac{2a\alpha}{c}$

O the cent. of the cylinder. G: c of grav



α = inclin: of the plane.
θ = inclin: of GO to the vertical at time t.

$$\therefore M\{k^2 + CG^2\}\frac{d^2\theta}{t} = -Mg\{GO\sin\theta + CO\sin\alpha\}$$

$$\frac{d^2\theta}{t} + \frac{g.GO}{k^2+Ag^2}\theta = -g.\frac{CO\sin\alpha}{k^2+AG^2}.$$

$$\theta = A\cos\overline{nt+B} - g.\frac{CO\sin\alpha}{k^2+AG^2}\ \frac{k^2+AG^2}{g.GO}.$$

$$= A\cos nt+B - g.\frac{CO.\sin\alpha}{g.GO}.$$

Let θ_1, θ_2 be the max and min values of θ.

$$\therefore \theta_1 = A - \frac{CO. \sin\alpha}{GO.}$$

$$-\theta_2 = -A - \frac{CO \sin\alpha}{GO.}$$

$$\therefore \theta_1 - \theta_2 = -\frac{2\,CO. \sin\alpha}{GO}$$

$$= -\frac{2ad}{c}.$$

An inelastic homogeneous cylinder rolls down a perfectly rough inclined plane which terminates in a perfectly rough horizontal plane. Find the velocity of the cylinder along the horizontal plane.

Let u = velocity of cylinder before impact

$= u\cos\alpha \parallel$ to Ax. a = rad of cylinder

$u\sin\alpha \perp$ to Ax.

v = velocity of cylinder after impact.

$\therefore m'k^2\left\{\frac{u}{a} - \frac{v}{a}\right\} = R'a.$ (1)

$m'(u\cos\alpha - v) = -R' = -\frac{m'k^2}{a^2}\cdot(u-v) = -\frac{m}{2}\cdot(u-v)$

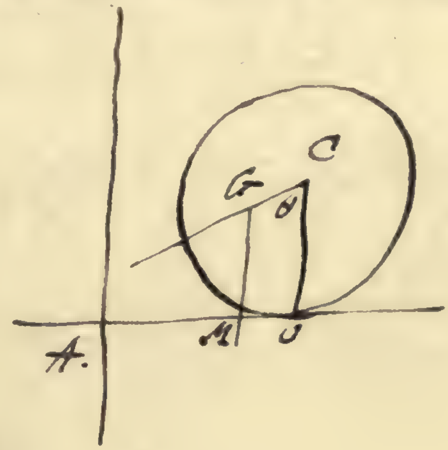
$\therefore u(2\cos\alpha - 1) = v$ $u\{2\cos\alpha + 1\} = 3v.$

$\therefore v = \frac{1}{3}u(2\cos\alpha + 1).$

$\therefore R' = m(v - u\cos\alpha) = \frac{1}{3}u(1 - \cos\alpha)$

also $R = mu\sin\alpha$. since the velocity $\perp$ to the plane after impact is zero.

An angular velocity having been impressed upon a heterogeneous sphere about an axis perpendicular to the vertical plane which contains its c of grav. G and its geometrical cent. C. and passing thro G. It is then placed upon a smooth horizontal plane: to det. the magnitude of the impressed angular velocity. that G may rise into a point in the vertical sck and there rest.



Let AM = x. MG = y be coords of G the centre of grav. R = pressure on the plane at O.

$$\therefore\ m k^2 \frac{d^2\theta}{dt^2} = -Rc\sin\theta$$

$$m\frac{d^2y}{dt^2} = R - mg$$

$$\therefore\ k^2\frac{d^2\theta}{dt^2} = -c\sin\theta\left(\frac{d^2y}{dt^2}+g\right)$$

but $y = a - c\cos\theta$

$$\frac{dy}{dt} = c\sin\theta\cdot\frac{d\theta}{dt}$$

$$\frac{d^2y}{dt^2} = c\cos\theta\cdot\left(\frac{d\theta}{dt}\right)^2 + c\sin\theta\cdot\frac{d^2\theta}{dt^2}$$

$$\therefore \left(k^2 + c^2 \sin^2\theta\right) \frac{d^2\theta}{dt^2} + c^2 \sin\theta \cos\theta \left(\frac{d\theta}{dt}\right)^2 = -cg \sin\theta.$$

$$k^2 \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + 2c^2 \sin^2\theta \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} + 2c^2 \sin\theta \cos\theta \cdot \frac{d\theta}{dt} \left(\frac{d\theta}{dt}\right)^2 = -2cg \sin\theta \cdot \frac{d\theta}{dt}.$$

$$\int_t \sin^2\theta \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} = \sin^2\theta \cdot$$

$$\int_t \sin^2\theta \cdot \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} = \sin^2\theta \cdot \frac{d\theta}{dt} \left(\frac{d\theta}{dt}\right) - \int_t 2\sin\theta\cos\theta \left(\frac{d\theta}{dt}\right)^2 \left(\frac{d\theta}{dt}\right) - \int_t \sin^2\theta \cdot \frac{d^2\theta}{dt^2} \cdot \left(\frac{d\theta}{dt}\right).$$

$$= \frac{1}{2} \sin^2\theta \left(\frac{d\theta}{dt}\right)^2 - \int_t \sin\theta \cos\theta \left(\frac{d\theta}{dt}\right)^3.$$

$$\therefore \int_t \sin^2\theta \cdot \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + \int_t \sin\theta \cos\theta \left(\frac{d\theta}{dt}\right)^3 = \frac{1}{2} \sin^2\theta \left(\frac{d\theta}{dt}\right)^2.$$

$$\therefore \left(k^2 + c^2 \sin^2\theta\right) \left(\frac{d\theta}{dt}\right)^2 = 2cg \cos\theta + C.$$

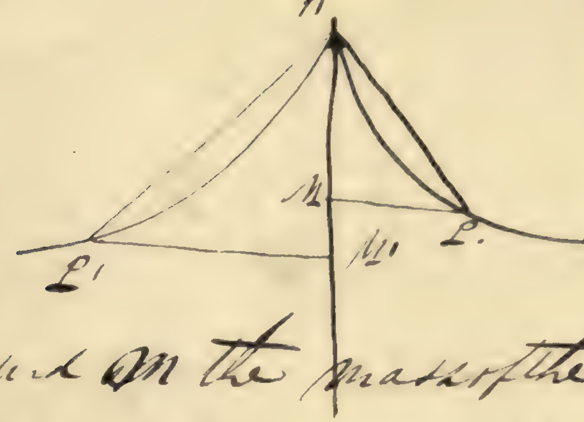
when $\theta = \alpha$. $\frac{d\theta}{dt} = \omega$.

$$\therefore \left(k^2 + c^2 \sin^2\alpha\right) \omega^2 = 2cg \cos\alpha + C.$$

$$\therefore k^2 \left(\left(\frac{d\theta}{dt}\right)^2 - \omega^2\right) + c^2 \left(\sin^2\theta \cdot \left(\frac{d\theta}{dt}\right)^2 - \sin^2\alpha \cdot \omega^2\right) = 2cg \left(\cos\theta - \cos\alpha\right).$$

when $\theta = \pi$. $\frac{d\theta}{dt} = 0$.

$$\therefore k^2 \omega^2 + c^2 \sin^2\alpha\, \omega^2 = 2cg \left(1 + \cos\alpha\right.$$



[...]ichever of m' = the mass of the body whose co-ords are AM' M'P'; and m the mass of the body whose co-ords are AM. MP.

$$m'\left(\frac{ds'}{dt}\right)^2 + m\left(\frac{ds}{dt}\right)^2 = 2g(my - m'y')$$

$$AM = y. \quad AM' = y'$$

or if $v.\ v'$ be the velocities.

$$m'v'^2 + mv^2 = 2g.(my - m'y')$$

But $$\left(\frac{v}{v'}\right)^2 = \left(\frac{ds}{ds'}\right)^2.$$

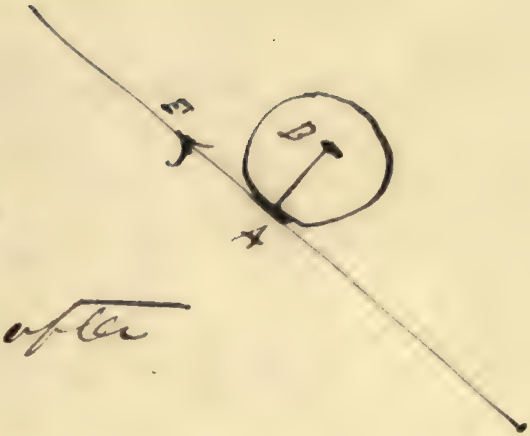
$$\therefore v^2 = v'^2 \frac{ds^2}{(ds')^2}.$$

$$\therefore \left(m' + m.\frac{ds^2}{ds'^2}\right)v'^2 = 2g(my - m'y')$$

$$v'^2 = 2g(my - m'y') \cdot \frac{ds'^2}{m'(ds')^2 + m\,ds^2}.$$

$$v^2 = 2g(my - m'y')\ \frac{ds^2}{m'ds'^2 + m\,ds^2}.$$

Let u be the velocity the instant before sliding



u' = velocity the instant after sliding.

$$\therefore m(u - u') = +E.$$

also if ω = ∠r velocity communicated.

$$mk^2\omega = Ea = mk^2\frac{u'}{a}.$$

$$\therefore E = \frac{mk^2}{a^2}u' = \frac{m}{2}u'$$

$$\therefore mu - mu' = +\frac{m}{2}u'$$

$$\therefore u = \frac{3}{2}u' \quad \therefore u' = \frac{2}{3}u$$

$$E = \frac{m}{2}u' = \frac{m}{2}\cdot\frac{2}{3}u = \frac{m}{3}u.$$

A homogeneous cylinder slides without rolling down an inclined plane which is for a certain space quite smooth, and after acquiring a given velocity is suddenly caused by the roughness of the surface to roll without sliding. Find the vel. of the axis of the cylinder the instant

let the axis major of the ellipse be the axis of x.

then the eqns of motion are

$$\frac{d^2x}{dt^2} = 0 \quad (1) \qquad \frac{d^2y}{dt^2} = -E. \quad (2)$$

now $y^2 = \frac{b^2}{a^2}\sqrt{a^2-x^2}$ $\therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{\sqrt{a^2-x^2}}$

from (1) $\frac{dx}{dt} = \beta$.

also $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{b^2\beta}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}$.

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left\{-\frac{b^2\beta}{a^2}\,\frac{x}{\sqrt{a^2-x^2}}\right\}$$

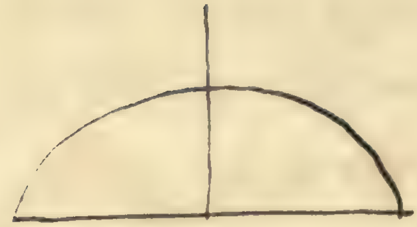
$$= \frac{d}{dx}\left\{-\frac{b^2\beta}{a^2}\cdot\frac{x}{\sqrt{a^2-x^2}}\right\}\frac{dx}{dt}.$$

$$= -\frac{b^2\beta^2}{a^2}\cdot\frac{d}{dx}\frac{x}{\sqrt{a^2-x^2}}$$

$$= -\frac{b^2\beta^2}{a^2}\cdot\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$$

$$= -\frac{b^4\beta^2}{a^2y^3}. \quad = E.$$

A particle describes an ellipse under the action of a force ⊥ to its major axis. find the law of force.



Let the base of the cycloid be the axis of x.

Then for the motion we have

$$\frac{d^2x}{dt^2} = 0 \quad (1) \qquad \frac{d^2y}{dt^2} = F.$$

now $x = a(1-\cos\theta)$ $\qquad y = a(\theta + \sin\theta)$.

$$\frac{dx}{d\theta} = a\sin\theta \qquad \frac{dy}{d\theta} = a(1+\cos\theta)$$

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx} = \frac{1+\cos\theta}{\sin\theta}.$$

now $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \beta . \frac{1+\cos\theta}{\sin\theta}.$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left\{\beta . \frac{1+\cos\theta}{\sin\theta}\right\} = \frac{d}{d\theta}\left\{\beta \frac{1+\cos\theta}{\sin\theta}\right\}\frac{d\theta}{dt}.$$

now $\frac{dx}{dt} = \beta = \frac{dx}{d\theta}\frac{d\theta}{dt} = a\sin\theta . \frac{d\theta}{dt}.$

$$\therefore \frac{d\theta}{dt} = \frac{\beta}{a\sin\theta}$$

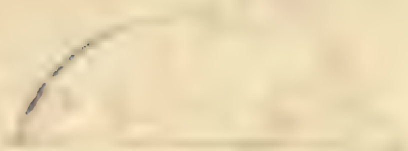
$$\therefore \frac{d^2y}{dt^2} = \frac{\beta^2}{a\sin^3\theta}\left\{-\sin^2\theta + (1+\cos\theta)\cos\theta\right\}$$

$$= \frac{2\beta^2\cos^2\theta}{a\sin^3\theta}.$$

$$\frac{1}{F} = \frac{a}{2\beta^2}\left(\tan^2\theta\sin\theta\right)$$

$$= \frac{a}{2\beta^2}\left\{2\sin\theta - \sin 2\theta\right\}.$$

A particle describes an [arc of a cycloid] under the action of a force at right angles to the major axis × find the force at any point of the orbit.
× || to its base.



II. If the angular velocity be variable then $\frac{dw}{t}$ is the ∠r accelerat. fce: and the linear acc. force in direct. of motion is $c.\frac{dw}{t}$. the acc. force ⊥ to the direct. of motn is as before cw^2 ∴ the resultant of these two is $c\sqrt{w^4+(\frac{dw}{t})^2}$ applied at G. and moment of this about A is

$$cGM.\sqrt{w^4+(\frac{dw}{t})^2}$$ which must be > g AM.

$$\therefore \frac{AM}{GM} > \frac{c}{g}.\sqrt{w^4+(\frac{dw}{t})^2}.$$

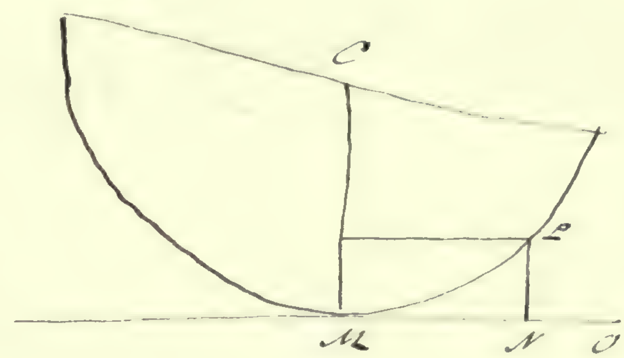
$$\frac{b + cx}{x^3 + a^3} = \frac{A}{x + a} \cdot \frac{P}{Q}.$$

$$b + cx = A\cdot(x^2 - ax + a^2) + P\cdot(x + a).$$

$$\text{Let } x = -a.$$

$$b - ac = A.$$

A thin hemispherical shell whose rad = a rests on a smooth horizontal plane, a particle of the same mass as the shell is placed at the lowest point of the internal surface. Shew that if the shell be projected with a horizontal velocity = $2\sqrt{ga}$, the particle will ascend just as high as the vertex of the shell and then descend.



$OM = z$. $ON = x$. $NP = y$

vel of the particle

by conservat of the motion of the cent of gravity.

$$M\frac{dz}{dt} + m\frac{dx}{dt} = Mu \quad (1) \quad = M \cdot 2\sqrt{ga} \quad (1)$$

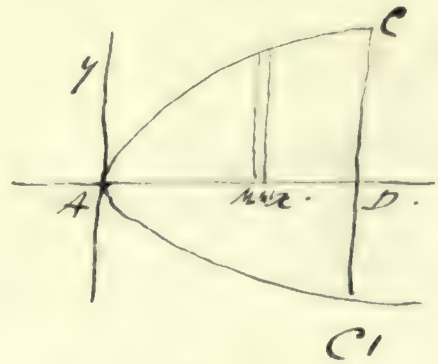
Vis Viva

$$M\left(\frac{dz}{dt}\right)^2 + m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = m(C - 2gy).$$

$$M\,4ga = m\,C$$

$$\therefore \left(\frac{dz}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2g(2a - y) \quad (2).$$

To find the rad. of gyr. of a port. of a parabola bounded by a double ordinate to the axis about a ⊥ line thro' the vertex.



A the vertex. Ay. Ax the axes of y & x.
the rad of gyr. about the perpendicular axis = rad of gy. about Ay. + rad of gy. about Ax.

To find k^2 for the axis of y. we have

$$mk^2 = \rho \int_0^{x'} x^2 y \, dx = \rho \int_0^{x'} 2\sqrt{m} \cdot x^{\frac{5}{2}} = \rho \cdot \frac{2\sqrt{m} \cdot x'^{\frac{7}{2}} \cdot 2}{7} = \frac{2x'^2}{7} \rho x' y'$$

$$m = \frac{2}{3} \rho x' y' \quad \therefore k^2 = \frac{3}{7} x'^2.$$

To find k' for the axis of x.

$$mk'^2 = \rho \int_0^{y'} y^2 (x' - x) = \rho \int_0^{y'} \left\{ y^2 x' - \frac{y^4}{4a} \right\}.$$

$$= \rho \left\{ \frac{y'^3}{3} x' - \frac{y'^5}{20a} \right\} = \rho x' y' \left\{ \frac{y'^2}{3} - \frac{y'^2}{5} \right\}.$$

$$= \rho x' y' \cdot \frac{2}{15} y'^2$$

$$m = \frac{2}{3} \rho x' y' \quad \therefore k'^2 = \frac{1}{5} y'^2.$$

$\therefore$ the required rad of gyration $= k^2 + k'^2 =$

$$= \frac{3}{7} x'^2 + \frac{1}{5} y'^2.$$

To find the radius of gyrat. of a regular polygon, about an axis ⊥ to it thro' its centre.

Let $CC' = c$. and the polygon be of n sides $\therefore \angle CAD = \frac{\pi}{n}$.

The moment of inertia of an element pq about the proposed axis is $m.r^3 \delta\theta . \delta r$.

$\therefore m\int_r r^3 \delta\theta = \frac{m}{4}. r^4 . \delta\theta + C$

take the integral from $r = 0$ & $r = AB \sec\theta$.

$\therefore \frac{m}{4}. AB^4 . \sec^4\theta\,\delta\theta$ = moment of inertia of the slice $A\,o\,v$.

$\therefore \frac{m}{4}. AB^4 . \int_{\tan\theta} (1+\tan^2\theta) = \frac{m}{4} AB^4 . \left\{ \tan\theta + \frac{1}{3}\tan^3\theta \right\} + C$

take the integral from $\theta = \frac{\pi}{n}$ to $\theta = -\frac{\pi}{n}$. and we have for the moment of inertia of the whole $\triangle ACC'$

$$\frac{m}{2}. AB^4 . \left(\tan\frac{\pi}{n} + \frac{1}{3}\tan^3\frac{\pi}{n}\right) = \frac{m}{2}.\tan^3\frac{\pi}{n}.AB^4 \left\{ \frac{1}{\tan^2\frac{\pi}{n}} + \frac{1}{3} \right\}.$$

$\therefore$ moment of inertia of the whole polygon =

$$\frac{mn}{2}. AB^4 . \tan^3\frac{\pi}{n} \left\{ \frac{3 + \tan^2\frac{\pi}{n}}{3\tan^2\frac{\pi}{n}} \right\}.$$

also mass of the polygon $= n(AB.BC)m$.

$$\therefore r^2 = \frac{AB^3 . \tan^3\frac{\pi}{n}}{6.BC} \left\{ \frac{3\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}} \right\}$$

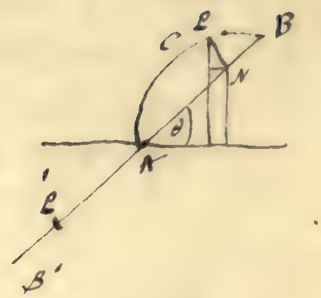
$$= \frac{BC^2}{3}. \left\{ \frac{2\cos^2\frac{\pi}{n} + 1}{2\sin^2\frac{\pi}{n}} \right\} \text{ for } BC = AB\tan\frac{\pi}{n}.$$

$$= \frac{CC'^2}{12} \left\{ \frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}} \right\} = \frac{c^2}{12} \left\{ \frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}} \right\}$$

$$\frac{\frac{1}{h}}{1+\frac{r^2}{h^2}} = \frac{h}{h^2+r^2}$$

$$\frac{\frac{1}{h}}{\sqrt{1-\frac{r^2}{h^2}}} = \frac{1}{\sqrt{h^2-r^2}}$$

$B'B$ is an inflexible rod, to one end of which is attached a semi-elliptical tube ACB. $B'B$ is moveable about A in a vertical plane, which also coincides with the plane of the B. Two particles a, b are placed one at B capable of moving along BCA, the other at B' capable of moving along AB'. Determine the motion of each particle.



Let P be the place of a at the time t. $AN = x$. $NP = y$.

P' ——— b ——— $AP' = r$.

by the principle of Vis Viva.

$$a\left(x^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) + br^2\left(\frac{d\theta}{dt}\right)^2 = C'.$$

by the principle of conservation of areas.

$$\left(a \cdot x^2 + br^2\right)\left(\frac{d\theta}{dt}\right) = C''.$$

now $\left(\frac{dx}{dt}\right)^2 = \left(\frac{dx}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2$; $\left(\frac{dy}{dt}\right)^2 = \left(\frac{dy}{d\theta}\right)^2\left(\frac{d\theta}{dt}\right)^2$.

$$\therefore a\left(x^2 + \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right) + br^2 = \frac{C'}{\left(\frac{d\theta}{dt}\right)^2}$$

$$= \frac{C'}{C''^2}\cdot\left(ax^2 + br^2\right)^2$$

$$a\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\} = \left\{\frac{C'}{C''^2}\left(ax^2 + br^2\right) - 1\right\}\left(ax^2 + br^2\right).$$

or $$\frac{a'^2 - e^2x^2}{a'^2 - x^2}\cdot\left(\frac{dx}{d\theta}\right)^2 = \left\{\frac{C'}{C''^2}\left(ax^2 + br^2\right) - 1\right\}\left\{x^2 + \frac{b}{a}r^2\right\}. \quad (A).$$

also $k^2\frac{d^2\theta}{dt^2}(a+b) = a(x\cos\theta - y\sin\theta)g + b\cos\theta g$

$= a\left(x\cos\theta - \frac{b}{a}\sqrt{a^2-x^2}\cdot\sin\theta\right)g \cdot (a+b)\cos\theta g.$

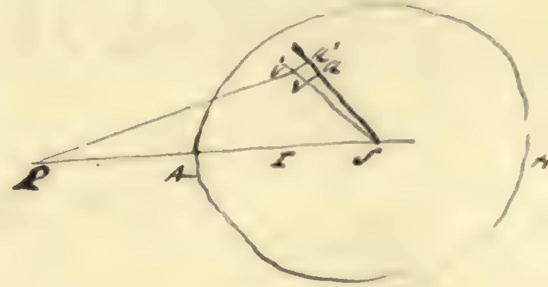
$\therefore (a+b)k^2\left(\frac{d\theta}{dt}\right)^2 = 2a\left(x\sin\theta + \frac{b}{a}\sqrt{a^2-x^2}\cdot\cos\theta\right)g + b\sin\theta g$. $\therefore \frac{d\theta}{dt} = f(x\cdot\theta) = \frac{C''}{ax^2+br^2}$

substitute in A and we have the diff: eq. of the path of P.

$$\frac{a'^2 - e^2x^2}{a'^2 - x^2}\cdot\left(\frac{dx}{d\theta}\right)^2 = \frac{C'}{C''}\left(\frac{1}{f(x\cdot\theta)} - \frac{1}{C''}\right)\left\{\frac{1}{f(x\cdot\theta)}\right\} \text{ or } \left(\frac{dx}{d\theta}\right)^2 = \frac{C'' - f(x\theta)}{f(x\cdot\theta)^2}\cdot\frac{C'}{C''}\cdot\frac{a'^2 - x^2}{a'^2 - e^2x^2}.$$

S is the centre. SA is the radius of a sphere. each of whose particles exerts an attractive force varying as $\frac{1}{u^n}$. Having assumed in SA produced any point E and having taken SE : SA :: SA : SI. find the ratio of the attractions which the whole sphere exerts on equal corpuscles placed at A and I.

Let a sect. of the sph. made by a plane passing thro' SE, $SE = b$. $SA = a$ ∴ $SI = \frac{a^2}{b}$



$SV = SQ = r$. $SV' = SQ' = r+\delta r$. $\angle ESV = \theta$.

$\angle ESQ = \theta + \delta\theta$

Then the force of attract. at a distance $u = \mu u^n$.

Also we have $u^2 = b^2 + r^2 - 2br\cos\theta$

The mass of the element $VQ = \rho r\delta r . \delta\theta$.

mass of annulus generated by the revolut. of this element about the axis of a.

$= 2\rho\pi . r^2\delta r . \delta\theta . \sin\theta = -2\pi\rho . r^2\delta r . \partial(\cos\theta) = \frac{2\pi\rho}{b} . r\delta r . u . du$.

The attract. of this annulus in direct. $ES = \frac{2\pi\rho}{b} . r\delta r . u\,du \left\{ \frac{u^2 + b^2 - r^2}{2bu} \right\}$.

∴ attract. of the sphere in the same direct. on the particle $= \left\{ . \frac{\pi\rho}{b^2} \int_r \int_u r \left\{ u^{n+2} + (b^2 - r^2) u^n \right\} \right.$.

$$= -\frac{\pi\rho}{b^2} \int_r . r \left\{ \frac{(b+x)^{n+3} - (b-x)^{n+3}}{n+3} + (b^2 - r^2) \frac{(b+x)^{n+1} - (b-x)^{n+1}}{n+1} \right\}.$$

$$= -\frac{\pi\rho}{b^2} \left\{ \begin{array}{l} \int_r \dfrac{(b+r)^{n+4} - b(b+r)^{n+3} + (b-r)^{n+4} - (b-r)^{n+3}}{n+3} \\ + \int_r r \left\{ \dfrac{b-r . (b+r)^{n+2} - (b+r)(b-r)^{n+2}}{n+1} \right\} \end{array} \right\}.$$

$$= -\frac{\pi\rho}{b^2} \int_r \left\{ \frac{(b+r)^{n+4} - b . (b+r)^{n+3} + (b-r)^{n+4} - (b-r)^{n+3}}{n+3} - \frac{1}{n+1} \left(\begin{array}{l} (b+r)^{n+4} - 3b(b+r)^{n+3} + 2b^2(b+r)^{n+2} \\ (b-r)^{n+4} - 3b . (b-r)^{n+3} + 2b^2 . (b+r)^{n+2} \end{array} \right. \right.$$

$$= -\frac{\pi\rho}{b^2} \left(\frac{1}{n+3} \left\{ \frac{(b+a)^{n+5}}{n+5} - \frac{b . (b+a)^{n+4}}{n+4} - \frac{(b-a)^{n+5}}{n+5} + \frac{b(b-a)^{n+4}}{n+4} \right\} - \frac{1}{n+1} \left\{ \begin{array}{l} \dfrac{(b+a)^{n+5} - (b-a)^{n+5}}{n+5} \\ - \dfrac{3b}{n+4}\left\{(b+a)^{n+4} - (b-a)^{n+4}\right\} \\ + \dfrac{2b^2}{n+3}\left\{(b+a)^{n+3} - (b-a)^{n+3}\right\} \end{array} \right. \right.$$

$$\frac{\pi\rho}{b^2} \left\{ \frac{(b+a)^{n+5} - (b-a)^{n+5}}{(n+5)(n+3)(n+1)} - \frac{(3n+8) b \left\{(b+a)^{n+4} - (b-a)^{n+4}\right\}}{(n+5)(n+3)(n+1)} + \frac{2b^2}{(n+1)(n+3)} . (b+a)^{n+3} - (b-a)^{n+3} \right\}.$$

$$\frac{\pi\rho}{(n+5)b^2(n+3)(n+1)} \left\{ 2\left\{(b+a)^{n+5} - (b-a)^{n+5}\right\} - (3n+8)\left\{b . (b+a)^{n+4} - (b-a)^{n+4}\right\} + 2(n+5)b^2\left\{(b+a)^{n+3} - (b-a)^{n+3}\right\} \right\}$$

for the attract: of the particle at I. we have also.

$$A = \frac{\pi f}{b^2} \cdot \int r \int u \cdot r \cdot \left(u^{n+2} + (b'^2 - r^2) u^n\right)$$

$$= \frac{\pi f}{b^2} \int r \cdot r \left\{ \frac{(r-b')^{n+3} - (r+b')^{n+3}}{n+3} + \frac{(b'^2 - r^2)}{n+1} \cdot (r-b')^{n+1} - (r+b')^{n+1} \right.$$

where $b' = \frac{a^2}{b}$.

$$\frac{\pi f \cdot b^2}{a^4} \int r \left\{ \frac{\left(r - \frac{a^2}{b}\right)^{n+4} + \frac{a^2}{b} \cdot \left(r - \frac{a^2}{b}\right)^{n+3} - \left(r + \frac{a^2}{b}\right)^{n+4} + \frac{a^2}{b}\left(r + \frac{a^2}{b}\right)^{n+3}}{n+3} \right.$$

$$\frac{\pi f b^2}{a^4} \int r \cdot r \left\{ -\left(\frac{a^2}{b} + r\right)\left(r - \frac{a^2}{b}\right)^{n+2} - \left(\frac{a^2}{b} - r\right)\left(r + \frac{a^2}{b}\right)^{n+2} \right\}.$$

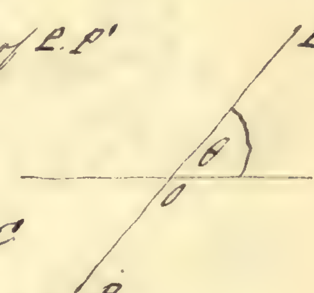
$$\frac{\pi f b^2}{a^4} \left\{ \frac{\left(\frac{ba - a^2}{b}\right)^{n+5}}{n+5} + \frac{a^2}{b} \cdot \frac{\left(\frac{ba - a^2}{b}\right)^{n+4} - \left(\frac{ba + a^2}{b}\right)^{n+4}}{n+4} - \left(\frac{ab + a^2}{b}\right)^{n+5} \frac{1}{n+5} \right\} \cdot \frac{1}{n+3}$$

$$+ \frac{\pi f b^4}{(n+1) a^4} \left\{ \frac{\left(\frac{ab - a^2}{b}\right)^{n+5} - \left(\frac{ab + a^2}{b}\right)^{n+5}}{n+5} + \frac{3a^2}{b} \cdot \frac{(ab - a^2)^{n+4} - (ab + a^2)^{n+4}}{n+4} + \frac{2a^4}{b^2} \cdot \frac{(ab - a^2)^{n+3} - (ab + a^2)^{n+3}}{n+3} \right.$$

$$\frac{\pi f b^4}{(n+5) a^2 (n+3)(n+1)} \left\{ \begin{array}{l} 2\left(\frac{ba + a^2}{b}\right)^{n+5} - \left(\frac{ba - a^2}{b}\right)^{n+5} - (3n+8)\left(\frac{ba + a^2}{b}\right)^{n+4} - \left(\frac{ba - a^2}{b}\right)^{n+4} \\ + 2(n+5)\left\{\left(\frac{ba + a^2}{b}\right)^{n+3} - \left(\frac{ba - a^2}{b}\right)^{n+3}\right\} \frac{b^4}{a^4} \end{array} \right\} \cdot B.$$

∴ the attractions are as $A : B$.

Two particles P P' are connected together by a rigid rod without inertia which passes thro' a small ring at O; the rod rests upon a horizontal plane. Suppose any impulse communicated to the particles find the paths they will describe.



Let OP = r OP' = l − r ∴ PP' = l. m m' masses of P P'

Then by principle of Vis Viva

$$m\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} + m'\left\{\left(\frac{d}{dt}\overline{l-r}\right)^2 + (l-r)^2\left(\frac{d\theta}{dt}\right)^2\right\} = C$$

when $t = 0$ let $\frac{dr}{dt} = \beta$, $r = a$, $l - r = a'$, $\frac{d\theta}{dt} = \omega$.

$$\therefore (m+m')\beta^2 + (ma^2 + m'a'^2)\omega^2 = C$$

$$\therefore (m+m')\left(\frac{dr}{dt}\right)^2 + \{mr^2 + m'(l-r)^2\}\left(\frac{d\theta}{dt}\right)^2 = (m+m')\beta^2 + (ma^2+m'a'^2)\omega^2$$ (

by principle of Conservat: of Areas.

$$\{mr^2 + m'(l-r)^2\}\frac{d\theta}{dt} = C' = (ma^2 + m'a'^2)\omega.$$

Now since $\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$ we have

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cdot \frac{(ma^2+m'a'^2)^2\omega^2}{\{mr^2+m'(l-r)^2\}^2}$$

$$\therefore (m+m')\left(\frac{dr}{d\theta}\right)^2 + mr^2 + m'(l-r)^2 = \frac{C\{mr^2+m'(l-r)^2\}^2}{(ma^2+m'a'^2)^2\omega^2}$$

$$\text{or } (m+m')\left(\frac{dr}{d\theta}\right)^2 = \left\{A.[mr^2+m'(l-r)^2] - 1\right\}\left\{mr^2+m'(l-r)^2\right\}$$

$$\text{where } A = \frac{C}{(ma^2+m'a'^2)^2\omega^2} = \frac{(m+m')\beta^2 + (ma^2+m'a'^2)\omega^2}{(ma^2+m'a'^2)^2\omega^2}$$

Ex. $\frac{\mu}{r^4}$ a body is projected from an apse with a velocity which is to velocity in a ⊙ at the same distance as $\sqrt{2} : \sqrt{3}$. find the orbit described and the time before the body falls to the centre.

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \mu u^4.$$

$$\frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2}u^2$$

$$= \frac{3a}{2}.u^2.$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = a u^3 + C.$$

when $\frac{du}{d\theta}=0$ $u = \frac{1}{a}$

$$\therefore \frac{1}{a^2} = \frac{1}{a^2} + C \therefore C = 0.$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2.(au-1) \text{ or } \frac{d\theta}{du} = \frac{1}{u\sqrt{au-1}}.$$

{ In a ⊙ $v = \sqrt{E.R} = \sqrt{\frac{\mu}{r^3}}$ initially

$$\therefore v' = \sqrt{\frac{2\mu}{3r^3}}.$$

at the apse dist $= a$.

$$\therefore h^2 = v'^2 a^2 = \frac{2\mu}{3a}.$$

$$\therefore \theta = \int \frac{du}{u\sqrt{au-1}}$$

let $\sqrt{u-1} = z \therefore \frac{du}{\sqrt{u-1}} = 2dz.$

$$u = z^2+1$$

$$\therefore \frac{du}{u\sqrt{u-1}} = \frac{2dz}{z^2+1}.$$

$$\therefore \theta = 2\int_z \frac{1}{z^2+1} = 2\tan^{-1}z. \quad c=0.$$

$$\tan\frac{\theta}{2} = z = \sqrt{au-1}.$$

$$\sec^2\frac{\theta}{2} = \frac{a}{r}.$$

$$\therefore r = a\cos^2\frac{\theta}{2}.$$

$$= \frac{a}{2}(\cos\theta-1)$$

again $\left(\left(\frac{dr}{d\theta}\right)^2+r^2\right)\left(\frac{d\theta}{dt}\right)^2 = v'^2 - 2\int_r \frac{\mu}{r^4}$

$$\frac{a^2}{4}\left\{\sin^2\theta+\cos^2\theta-2\cos\theta+1\right\}\left(\frac{d\theta}{dt}\right)^2 = \ldots$$

$$\left\{1-\cos\theta\right\}\left(\frac{d\theta}{dt}\right)^2 = v'^2 + \frac{2}{3}\frac{\mu}{r^3}$$

$$= v'^2 + \frac{2}{3}\frac{\mu}{a^3}\sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)^2.$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{v'^2}{a^2\cos^4\frac{\theta}{2}} + \frac{2}{3}\frac{\mu}{a^3}.\sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)^2.$$

$$= \frac{\sec^2\frac{\theta}{2}}{a^2}\left\{v'^2 + \frac{2}{3}\frac{\mu}{a^3}.\sec^6\frac{\theta}{2}\right\}.$$

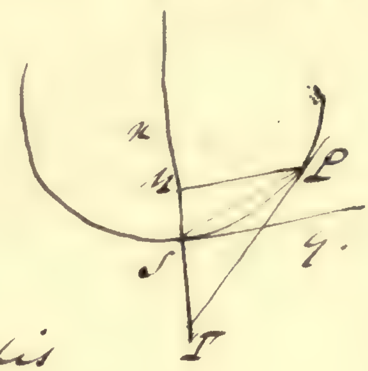
$$\therefore \frac{dt}{d\theta} = \frac{a}{\sec\frac{\theta}{2}}.\frac{1}{\sqrt{v'^2+\frac{2}{3}\frac{\mu}{a^3}\sec^6\frac{\theta}{2}}}.$$

$$\therefore t = a\int_\theta \frac{\cos\frac{\theta}{2}}{\sqrt{v'^2+\frac{2}{3}\frac{\mu}{a^3}\sec^6\frac{\theta}{2}}}$$

when $\cos\theta = 1$ or $\theta = 0$.

[illegible]

A centre of force which is repulsive and varies as (dist)^n is situated at S the lowest point of a vertical ⊙. how far will a body placed at S and acted on by gravity ascend up the curve.



P the posit.^{n} of the body at the time t. $SP = r$. $SM = x$. $MP = y$.

The force estimated along the tangent is

$$v\frac{dv}{ds} = -\frac{\mu}{r^n}\frac{dr}{ds} + g\frac{dx}{ds}$$

$$\therefore \frac{1}{2}v^2 = \frac{\mu}{n-1}\cdot\frac{1}{r^{(n-1)}} + gx + C.$$

when $r = 0$ $v = 0$ $x = 0$ $\therefore C = 0$

$$\frac{1}{2}v^2 = \frac{\mu}{n-1}\cdot\frac{1}{r^{(n-1)}} + gx.$$

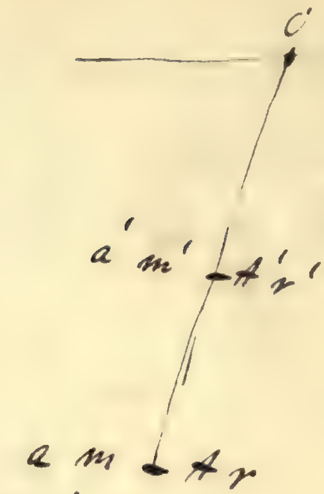
$$= \frac{\mu}{n-1}\cdot\frac{1}{(2ax)^{\frac{n-1}{2}}} + gx \quad \text{for } r^2 = x^2 + y^2 = 2ax.$$

when $v = 0$ the body arrives at the highest point

$$\therefore 0 = \frac{\mu}{n-1}\cdot\frac{1}{(2a)^{\frac{n-1}{2}}} + g.x.^{\frac{n+1}{2}}.$$

$$\therefore x = \sqrt[n+1]{\left\{\frac{\mu}{(n-1)g}\frac{1}{(2a)^{\frac{n-1}{2}}}\right\}^2}.$$

A pendulum consists of an indefinitely thin rod OA, and a globe, of which the centre is A; to determine the point A' in the line OA at which the centre of another globe must be fixed in order that the oscillations of the 2 globes may be executed in the shortest time.



let r' = rad of A' $\therefore \frac{2}{5}r'^2$ = rad of gyr. about its diam.

$$\frac{2}{5}r'^2 + a'^2 = \text{———— about the axis thro O}$$

$$\text{do.}$$

r = rad of A $\therefore \frac{2}{5}r^2 + a^2 =$ ————

$$\therefore k^2 = a^2 + a'^2 + \tfrac{2}{5}(r'^2 + r^2).$$

also $h(m+m') = m'a' + ma \quad \therefore h = \dfrac{m'a' + ma}{m+m'}$

$\therefore$ length of isochronous pendulum $= \dfrac{k^2 + h^2}{h}$

$$= \frac{(m+m')\{a^2 + a'^2 + \tfrac{2}{5}(r^2 + r'^2)\}}{m'a' + ma}.$$

for the shortest oscillations this length must be a max.

$$\therefore d_{a'}\left\{\frac{(m+m')\{a^2+a'^2+\tfrac{2}{5}(r'^2+r^2)\}}{m'a'+ma}\right\} = 0 \text{ gives}$$

the reqd lengths of a'.

A body is projected in the plane (xy), with a velocity (v) in a direction making an angle (α) with the axis of (x) and is acted on by a force μx^n // to the axis of y. Find the $=^n$ to the orbit described.

we have. $\frac{d^2y}{dt^2} = \mu x^n$. (A)

$\frac{d^2x}{dt^2} = 0 \quad \therefore \frac{dx}{dt} = v\cos\alpha.$

$x = vt\cos\alpha.$ (1)

substitute in A and we have.

$$\frac{d^2y}{dt^2} = -\mu v^n t^n \cos^n\alpha.$$

$$\therefore \frac{dy}{dt} = -\frac{\mu}{n+1}.v^n\cos^n\alpha.t^{n+1} + v\sin\alpha.$$

$$y = -\frac{\mu}{(n+1)(n+2)}.v^n\cos^n\alpha.t^{n+2} + vt\sin\alpha.$$

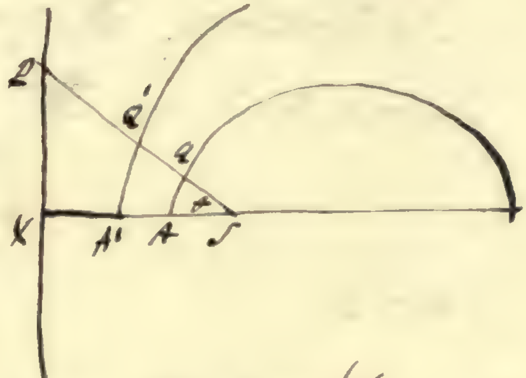
$$= vt\sin\alpha - \frac{\mu.(v\cos\alpha.t)^{n+2}}{(n+1)(n+2)v^2\cos^2\alpha}$$

but $v\cos\alpha.t = x \quad \therefore t = \frac{x}{v\cos\alpha}.$

$$\therefore y = x\tan\alpha - \frac{\mu x^{n+2}}{(n+1)(n+2)v^2\cos^2\alpha}.$$

A particle moves from any point in the directrix of an ⊙ in a straight line towards a centre of force $\propto \frac{1}{(\text{dist})^2}$ in the nearer focus. find the velocity when it arrives at the curve.

S the focus. Q the pt. in the directrix



$AS = a(1-e)$. $AX = \frac{a(1-e)}{e}$ ∴ $SX = \frac{a(1-e^2)}{e}$

$SP = \frac{a(1-e^2)}{e\cos\theta}$.

for the mot: along SP. $\frac{d^2r}{dt^2} = -\frac{\mu}{r^2}$ ∴ $\left(\frac{dr}{dt}\right)^2 = 2\left(\frac{\mu}{r} - \frac{\mu}{a}\right)$

at the commencement of mot. $r = SP$. and at Q $r = \frac{a(1-e^2)}{1+e\cos\theta}$

∴ if V = vel. at Q $V^2 = 2\mu\left\{\frac{1+e\cos\theta}{a(1-e^2)} - \frac{e\cos\theta}{a(1-e^2)}\right\}$.

$$= \frac{\mu}{\frac{a}{2}(1-e^2)} = \frac{\mu}{\frac{1}{4}\text{ latus rectum}}.$$

If A'Q' be a part of a ⊙ic orbit $SA' = A'X = a'$ suppose

$$\therefore v^2 = 2\mu\left\{\frac{1}{r} - \frac{1}{a}\right\}.$$

$$V^2 = 2\mu\left\{\frac{1+\cos\theta}{2a} - \frac{\cos\theta}{2a}\right\} = \frac{\mu}{a} = \frac{\mu}{a}$$

$$= \frac{\mu}{\frac{1}{4}\text{ latus rectum}}.$$

At one end C of a straight tube CA is a centre of force $= \mu \cdot (\text{dist})$. The tube revolves in one plane round C with an angular velocity $= \sqrt{\frac{\mu}{2}}$. and the body descends down the tube under the actⁿ of the force in C. Then that its path will be a ⊙ on the initial position of the tube.

CA the initial posⁿ of the tube

CA' any other posⁿ.

P the place of the particle at the time t. $\angle A'CA = \theta$. $CP = r$. $CA = a$.

now the velocity of $P = r\sqrt{\frac{\mu}{2}}$.

the centripetal force $= \frac{v^2}{r} = r\frac{\mu}{2}$

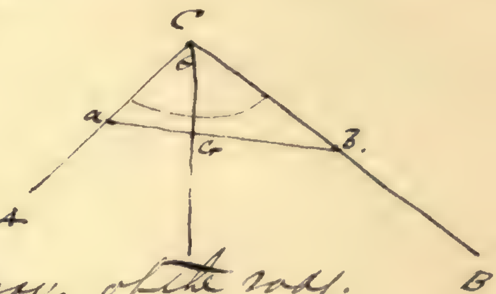
∴ the effective force on the particle in directⁿ $PC = \frac{\mu r}{2}$.

wherefore $\frac{d^2r}{t} = -\frac{\mu}{2} r$. $\therefore \left(\frac{dr}{t}\right)^2 = \frac{\mu}{2}\{a^2 - r^2\}$

$$\therefore \frac{dt}{r} = \sqrt{\frac{2}{\mu}} \cdot \frac{-1}{\sqrt{a^2-r^2}} \quad \therefore t = \sqrt{\frac{2}{\mu}} \cdot \cos^{-1}\frac{r}{a}.$$

or $r = a\cos t\sqrt{\frac{\mu}{2}}$. which is the polar eq. to the ⊙ on CA.

A bent lever of which the arms are of length a, b, and the angle between them θ, makes small oscillations in its own plane about the angular point. Find the length of the isochronous simple pendulum.



Here $l = \dfrac{k^2+h^2}{h}$.

Let a, b, be the middle points of CA, CB. Join ab. & take $aG = \dfrac{b}{a+b} \cdot ab$. then G is the cent. of grav. of the rods.

Now $ab = \sqrt{\dfrac{a^2}{4} + \dfrac{b^2}{4} - \dfrac{ab}{2}\cos\theta}$.

$h^2 = CG^2 = (Ca)^2 + (aG)^2 - 2\,Ca \cdot aG \cdot \cos Cab$.

where $\cos Cab = \dfrac{\dfrac{a^2}{4} + (ab)^2 - \dfrac{b^2}{4}}{a \cdot (ab)}$.

∴ after reduction we have.

$h^2 = CG^2 = \dfrac{1}{4(a+b)^2} \cdot \left(a^4 + b^4 + 2a^2b^2\cos\theta\right)$

or $h = \dfrac{1}{2(a+b)} \cdot \sqrt{a^4+b^4+2a^2b^2\cos\theta}$.

also rad of gyr^n. of CA about the axis of rotation $= \frac{1}{3}a^2$.

——— CB ——————————— $= \frac{1}{3}b^2$.

∴ $h^2 + k^2 = k_1^2 = \frac{1}{3}(a^2+b^2)$.

$$l = \frac{h^2+k^2}{h} = \frac{\frac{2}{3}\cdot(a^2+b^2)(a+b)}{\sqrt{a^4+b^4+2a^2b^2\cos\theta}}.$$

$$v^2 = (1+\varepsilon)\sqrt{\frac{\mu . f(a) -}{f'a}}$$

v is the given vel. ε is the eccent.

V = vel.^{y} in ⊙
v = vel.^{y} at y^e apse

$v : V : \sigma : r$

$$V = \frac{r}{\sigma} . v$$

$$\frac{r^2 . v^2}{\sigma^2} = V^2 = \frac{2\mu}{r} - \frac{\mu}{a_1}$$

$$= \mu . \left(\frac{2}{a_1(1+\varepsilon)} - \frac{1}{a_1} \right)$$

$$= \frac{\mu . (1-\varepsilon)}{a_1(1+\varepsilon)} = \frac{\mu (1-\varepsilon)}{a}$$

$$⊙ \quad \frac{\sigma}{r} = \sqrt{\frac{f(a)}{a f'(a)}}$$

$$\therefore \frac{\mu . (1-\varepsilon)}{a} = v^2 \frac{f(a)}{a . f'(a)}$$

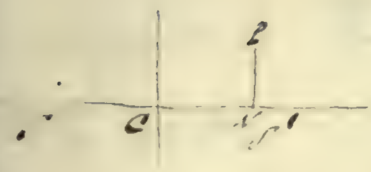
$$v^2 = \mu \frac{f'(a)}{f(a)} . (1-\varepsilon)$$

The velocity of the given body at the apse. (Newton Prop 45) being known. find the eccent. of the ⊙ employed in the proposit^n.

$\frac{(3-1)x}{2}$ [illegible]

[illegible] $= \frac{[illegible]}{8}$

$(3-1)\cdot \frac{[illegible]}{[illegible]}$ [illegible]



let S, the centre of force. ∴ $S_1 = 2c$

let C the middle point of S S be taken for the origin of coords. & the position of the point in the space. x, y, the coords of P.

Then $\frac{d^2x}{dt^2} = \mu\left\{\frac{a+x}{(a+x)^2+y^2} - \frac{a-x}{(a-x)^2+y^2}\right\}$.

$2\frac{dx}{dt}\frac{d^2x}{dt^2} = \mu \cdot \left\{\frac{\frac{d}{dt}\{(a+x)^2+y^2\}}{(a+x)^2+y^2} + \frac{\frac{d}{dt}\{(a-x)^2+y^2\}}{(a-x)^2+y^2}\right\}$

∴ $\left(\frac{dx}{dt}\right)^2 + \mu \log_e\{(a+x)^2+y^2\}\{(a-x)^2+y^2\}$.

but this by the question is constant.

∴ $\{(a+x)^2+y^2\}\{(a-x)^2+y^2\} = c^4$

which is the eqn to the lemniscate.

Determine the value of $\frac{d^2u}{d\theta^2}+u$ at an apse in terms of the radius of curvature.

Let ρ be the radius of curv.

$$\therefore \rho = \frac{1}{u^3}\cdot\frac{\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}^{\frac{3}{2}}}{\left(\frac{d^2u}{d\theta^2}+u\right)}$$

therefore $\frac{d^2u}{d\theta^2}+u = \frac{1}{u^3}\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)^{\frac{3}{2}}\frac{1}{\rho}$.

Now at an apse $\frac{du}{d\theta}=0$.

$$\therefore \frac{d^2u}{d\theta^2}+u = \frac{1}{\rho}$$

A body revolves in an ellipse the centre of force being the focus. shew that $h = \sqrt{\mu a (1-e^2)}$.

$$P = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

at the further apse. the direction of the body is $\perp$ to the radius vector. and the radius vector $= a(1+e)$. also the velocity at this point $= \sqrt{\frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right)}$

$$= \sqrt{\frac{\mu}{a} \cdot \frac{1-e}{1+e}}.$$

$$\therefore h = vp = a(1+e)\sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

$$= \sqrt{\mu \cdot a(1-e^2)}.$$

also $P = \frac{2 \text{ area of ellipse}}{h} = \frac{\pi ab}{\sqrt{\mu \cdot a(1-e^2)}}$.

Now $b = a\sqrt{1-e^2}$.

$$\therefore P = \frac{\pi \cdot a^{\frac{3}{2}}\sqrt{a(1-e^2)}}{\sqrt{\mu}\sqrt{a(1-e^2)}} = \frac{\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

Divide a vertical space into 3 parts such that the time of descent down each may be the same.

$$s = \frac{1}{2} f t^2$$

Changing t into 1. 2. 3 successively, and denoting by $s.\ s_1\ s_2$ the spaces through which the body has fallen at the end of these periods we have $s = \frac{1}{2} f$, $s' = \frac{4}{2} f$, $s'' = \frac{9}{2} f$

∴ the spaces descended are as 1 : 4 : 9 —

wherefore if a = the whole vertical space which a body descends in a given space of time, and it be divided into the spaces

$\frac{a}{9}$, $\frac{3a}{9}$, $\frac{5a}{9}$, the times of descent down each will be the same.

A body is projected upwards with a given velocity, find the space which it has described after a given time.

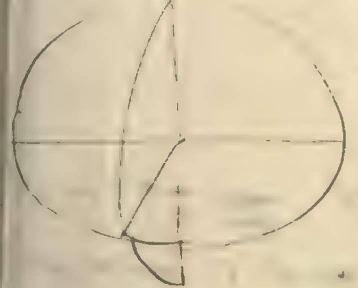
let s = space described, t the time &c —

Then $s = tv - \frac{1}{2} f t^2$ =

solving the eq. with respect to t

$$t = \frac{v}{f}\left\{ 1 \pm v\sqrt{v^2 - 2sf} \right\}$$

The double result corresponding to the two values of the signs before radical refers to the time of passing the same point of its course, ~~in the~~ when moving upwards, and when falling.



$$Ax^2 + By^2 + Cz^2 = 1.$$

Let $x = mz$ $y = nz$ (the line $\perp$ to a diameter

$$A(mz+h)^2 + B(nz+k) + C(z+l)^2 = 1$$

must give $=$ values of z with different signs.

or $\quad 2Amh + 2Bnk + 2Cl = 0$

the $\perp$ to the diametral plane.

or $\quad Amx + Bny + Cz = 0$ suppose . (1)

$A m_1 x + B n_1 y + Cz = 0$ (2) is the $\perp$ to the diametral plane of $\left.\begin{matrix} x = m_1 z \\ y = n_1 z \end{matrix}\right\}$ (β)

$A m_2 x + B n_2 y + Cz = 0$ (3) $\quad$ of $\left.\begin{matrix} x = m_2 z \\ y = n_2 z \end{matrix}\right\}$ (γ)

But the intersections of (1) & (2) must coincide with (γ) in order that (α),(β),(γ) may be conjugate diameters

$$A(mn_1 - m_1 n)x + C(n_1 - n) = 0$$

$$\therefore x = \frac{C(n - n_1)}{A(mn_1 - m_1 n)} z = m_2 z$$

$$y = \frac{C(m - m_1)}{A(n_1 m - m_1 n)} z = n_2 z$$

$$\therefore \frac{C(n - n_2)}{A(mn_2 - m_2 n)} = m_1$$

$$\frac{C(m - m_2)}{A(n_2 m - m_2 n)} = n_1$$

A body moves in a curve acted upon by a force ∥ to the axis of y. Prove that the force $= u^2 \frac{d^2y}{dx^2}$ where u is the velocity ∥ to the axis of x. and apply this result to determine the path of a projectile upon the earths surface.

$$\frac{d^2y}{dt^2} = F. \qquad \frac{d^2x}{dt^2} = 0.$$

$$\frac{dx}{dt} = C. = u.$$

Now $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = u.\frac{dy}{dx}.$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left\{u.\frac{dy}{dx}\right\} = \frac{d}{dx}\left\{u.\frac{dy}{dx}\right\}\frac{dx}{dt}$$

$$= u.\frac{d^2y}{dx^2}.u$$

$$= u^2.\frac{d^2y}{dx^2}.$$

$\therefore F = \frac{d^2y}{dt^2} = u^2.\frac{d^2y}{dx^2}.$

For the case of a projectile

$$u^2\frac{d^2y}{dx^2} = -g.$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{g\sec^2\alpha}{v^2}. \quad \text{where } \alpha = \angle \text{ of proj}^n.$$

$$\therefore \frac{dy}{dx} = -\frac{gx.\sec^2\alpha}{v^2} + C.$$

$$x = 0 \quad \frac{dy}{dx} = \tan\alpha.$$

$$\therefore \frac{dy}{dx} = \tan\alpha - \frac{gx.\sec^2\alpha}{v^2}$$

$$y = x\tan\alpha - \frac{gx^2}{2v^2}\sec^2\alpha.$$

A heavy body is projected along the interior surface of a cylinder the axis of which is vertical determine the path.

Let the point of projection be taken for the origin of coords.

Let the centre of the horizontal circular section of the ~~which the~~ cylinder which passes through the point of projection be taken for the origin of coords.

Let α be the direction of projection with respect to the horizon. then $v\cos\alpha$ is the constant horizontal velocity ∴

$\frac{v\cos\alpha}{r}$ = the constant angular velocity estimated horizontally.

But since the body is subject to the action of gravitation if z be the vertical coord. of the body at the time t. $z =$ ~~$gt - \frac{1}{2}$~~ $vt\sin\alpha - \frac{1}{2}gt^2$.

wherefore for the coord. at any time

$$x^2+y^2 = r^2.$$

$$z = vt\sin\alpha - \frac{1}{2}gt^2.$$

or if θ be the angle described by the radius vector estimated horizontally

$$\theta = \frac{t\,v\cos\alpha}{r}.$$

$$z = vt\sin\alpha - \frac{1}{2}gt^2$$

Find the attrn. of a uniform cylindrical rod, the base of which is very small, in the direction of its length upon a point any where without it.

AB the cylindrical rod. k the area of a vertical section of the rod. P the point. a. −c the coords of P. Ay. Ax. axis of x.

Av = x then the element at v = k δx. the attr of this element on the pt = $\frac{k\rho\,\delta x}{Pv^2} = \frac{k\rho\,\delta x}{a^2+(c+x)^2}$ in direction Pv.

∴ the attn in direction of the length of the rod = $\frac{k\rho a.\,\delta x}{\{a^2+(c+x)^2\}^{\frac{3}{2}}}$

∴ if A = whole attn.

$$A = \int_{x}^{0.B.} \frac{k\rho a.}{(a^2+(c+x)^2)^{\frac{3}{2}}}$$

$$= \frac{k\rho}{a}.\frac{c+x}{\{a^2+(c+x)^2\}^{\frac{1}{2}}} + C'$$

$$0 = \frac{k\rho}{a}.\ \frac{c}{a} + C'.$$

$$\therefore A = \frac{k\rho}{a}\left\{\frac{c+b}{\{a^2+(c+b)^2\}^{\frac{1}{2}}} - \frac{c}{a}\right\}.$$

The resistance of the air varying as the square of the velocity determine the motion of a body falling from rest, and show that after a certain finite time the velocity becomes nearly uniform

$$\frac{d^2x}{dt^2} = g - kv^2 \quad \text{or} \quad \frac{dv}{dt} = g - kv^2$$

$$\therefore \frac{dt}{dv} = \frac{1}{g - kv^2} = \frac{1}{k} \cdot \frac{1}{\frac{g}{k} - v^2}$$

$$\therefore t = \frac{1}{2\sqrt{kg}} \cdot \log\left(\frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}\right)$$

$$\therefore \varepsilon^{2t\sqrt{kg}} = \frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} - v\sqrt{k}}$$

$$(\sqrt{g} - v\sqrt{k})\,\varepsilon^{2t\sqrt{kg}} = \sqrt{g} + v\sqrt{k}$$

$$\sqrt{g}\{\varepsilon^{2t\sqrt{kg}} - 1\} = v\sqrt{k}\{1 + \varepsilon^{2t\sqrt{kg}}\}$$

$$\therefore v = \sqrt{\frac{g}{k}} \cdot \left(\frac{\varepsilon^{2t\sqrt{kg}} - 1}{\varepsilon^{2t\sqrt{kg}} + 1}\right)$$

which is nearly uniform after a certain finite time t.

A body attracted to a constant centre of force moves directly towards it from rest through a medium of which the resistance $= \frac{n}{2}\frac{(vel)^2}{dist}$: find the velocity acquired through a given space and determine the dist. from the cent. when the velocity is a maximum.

A the point from which the body starts. $AS = a$; E the posit. of the body at a time t. $ES = x$.

For the motion

$$\frac{dv}{t} = C - \frac{nv^2}{2x} \quad \text{or} \quad \frac{dv}{x} = C\frac{dt}{x} - \frac{nv}{2x} \quad \text{for } v = \frac{dx}{t}$$

$$\therefore \frac{d_x v}{v} = \frac{C d_x t}{d_t x} - \frac{n}{2x}$$

$$d_x(\log v) = C$$

For the motion $v\frac{dv}{x} = C - \frac{n}{2}\frac{v^2}{x}$

$$v\frac{dv}{x} + \frac{n}{2}\frac{v^2}{x} = C$$

$$d_x\left(\varepsilon^{n\log x} \cdot v^2\right) = 2C\varepsilon^{n\log x}$$

$$\varepsilon^{n\log x} v^2 = 2C\int_x \varepsilon^{n\log x}$$

$$= 2C\cdot x\varepsilon^{n\log x} - 2Cn\int_x \varepsilon^{n\log x}$$

$$= \frac{2Cx}{1+n}\cdot\varepsilon^{n\log x} + C$$

$$0 = \frac{a}{1+n}\cdot\varepsilon^{n\log a} + C$$

$$\therefore \varepsilon^{n\log x}\cdot v^2 = \frac{2C}{1+n}\left\{x\varepsilon^{n\log x} - a\varepsilon^{n\log a}\right\}$$

$$v^2 = \frac{2C}{1+n}\left(x - a\varepsilon^{n\log\frac{a}{x}}\right)$$

Find the attraction of a sphere on a particle without it, the force varying inversely as the square of the dist.

E the centre of the sphere.
a the point at which the particle is placed. $Ea = b$. c = rad of the sphere.

$EC = EC' = r$. $CEa = \theta$. $C'Ea = \theta + \delta\theta$. $aC = u$.

$u^2 = b^2 + r^2 - 2br\cos\theta$ $\quad\therefore\ 2u\frac{du}{d\theta} = 2br\sin\theta$ (1)

area of the element $C'C = r\,\delta\theta\cdot\delta r$.

vol of the annulus generated by the revolution of this annulus about the line aE $= 2\pi r^2\sin\theta\,\delta\theta\,\delta r$

$= \frac{2\pi u r}{b}\,\delta u\,\delta r$ from (1)

$\therefore$ attr. of this annulus in direct. $Ea = \frac{2\pi\rho r}{b\,u}\,\delta u\,\delta\theta\sin\theta$

$$= \frac{\pi\rho}{b^2}\int_r\int_u\left(\frac{u^2+b^2-r^2}{u^2}\right)r$$

$$\int_{b-r}^{b+r} u\cdot r\frac{(b^2-r^2)}{u^2} + r = -r(b^2-r^2)\left(\frac{1}{b+r}-\frac{1}{b-r}\right) + r\{b+r-b+r\}$$

$$= -r\{b-r-b-r\} + r(b+r-b+r)$$

$$= +4r^2$$

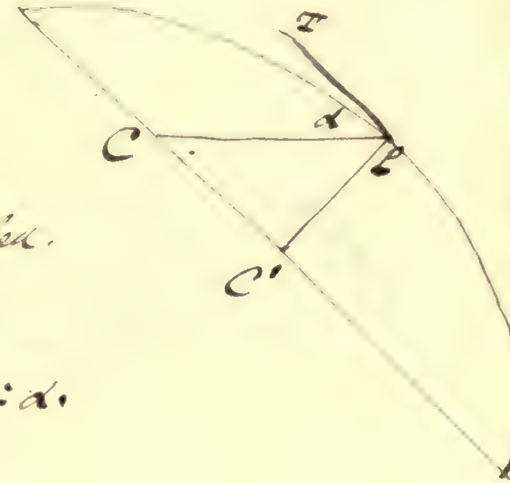
and $\int_0^c 4r^2\,dr = \frac{4}{3}c^3$.

$\therefore$ attr. of the sphere $= \frac{4}{3}\frac{\pi\rho c^3}{b^2}$

$$= \frac{\text{mass of the sphere}}{b^2}$$

[illegible]

Force $= \frac{\mu}{SP^2}$. At a distance SP from the centre bodies are projected in all directions with a velocity $= \sqrt{\frac{\mu}{SP}}$. shew that the axis major of any of the ellipses described will be parallel to the direction of proj? and their centres will all lie in the surface of a sphere of which SP is a diameter.



C the cent. of force. P the pt. & PT the direct. of either of the bodies.

then $\rho = \frac{v^2}{2F} = \frac{SP}{2}$.

let a. b be semi axes of the orbit described.

$2a = \frac{\rho^2}{r-\rho} = 2 \cdot SP \therefore a = SP$.

$b = \frac{\rho}{\sqrt{r-\rho}} \cdot r \sin\alpha = r\sin\alpha = SP \sin\alpha$.

$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \sin^2\alpha = \cos^2\alpha$.

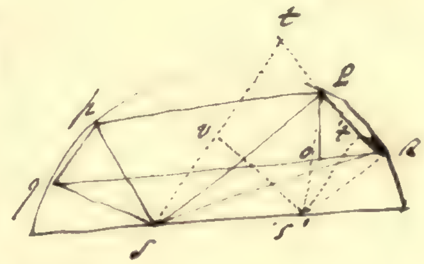
If θ = in clinat. of PT to the axis ~~major~~ minor

$\sin\theta = \frac{1}{e}\cos\alpha = 1 \therefore$ the axis minor is $\perp$ to the direction of projn. $\therefore$ axis major is $\parallel$ to the direction of projn.

Also $CC' = CP\cos\alpha$. and as $CC'P$ is a right angle C' is a point in the sphere of which CP is the diameter, and the same is true of either of the other orbits $\therefore$ all the orbits have their cent. in that sphere, and their axes major $\parallel$ to the direct. of proj?

In an ⊙ described round a force in one of the foci two chords Qp, Rq are drawn ∥ to the axis major, shew that the difference of the times in which the arcs QR, pq are described varies as the distance between the chords.

Let S be the centre of force.
S' the other focus of the ⊙
St, S't' ⊥ on the chord QR.



Now the times of describing the arcs will be ∝ to the areas swept out by the radii vectores.

also area Spq = area $S'QR$.

∴ the difference of the areas = area QSR − area $QS'R$

$$= \tfrac{1}{2} QR.(St - S't')$$

$$= \tfrac{1}{2} QR . Sv.$$

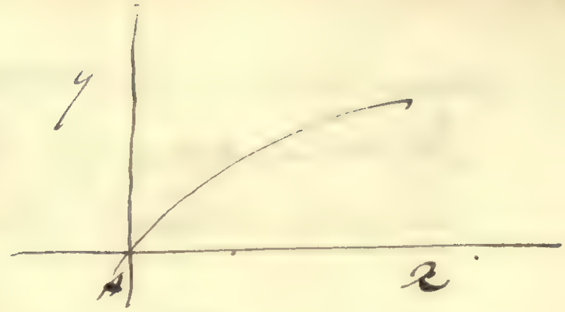
Now by similar △s $SS' : Sv :: QR : QO$

or $QO \times SS' = QR . Sv$

∴ the difference between the areas varies as $QO . SS'$
∝ QO for SS' is constant.

but the difference of the areas is ∝ to the difference of the times of describing the arcs

∴ the diff of the time reqd. to describe the arcs is ∝ to QO the dist. between the chords.



$$d_t^2 x = d_s x \cdot 2g \cdot \sqrt{\frac{v}{k}}.$$

$$= d_s x \cdot 2g \cdot \sqrt{\frac{d_t s}{k}}.$$

$$\therefore (d_t^2 x)^2 = (d_s x)^2 \frac{4g^2}{k} d_t s = \frac{4g^2}{k} d_t x.$$

Then $d_t x = w$.

$$\therefore d_t w = \frac{2g}{\sqrt{k}} \sqrt{w}, \quad d_w t = \frac{\sqrt{k}}{2g} \frac{-1}{\sqrt{w}} \quad \therefore t = \frac{\sqrt{k}}{g}(\sqrt{w} + \sqrt{a})$$

$$\therefore \sqrt{w} = \left(\sqrt{a'} - \frac{gt}{\sqrt{k}}\right). \quad d_t x = \left(\sqrt{a'} - \frac{gt}{\sqrt{k}}\right)^2.$$

Similarly $d_t y = \left(\sqrt{a_1'} - \frac{gt}{\sqrt{k}}\right)^2$.

But $v^2 = (d_t x)^2 + (d_t y)^2 = \left(\sqrt{a} - \frac{gt}{\sqrt{k}}\right)^4$.

where a is some constant.

$$\therefore v = \left(\sqrt{a} - \frac{gt}{\sqrt{k}}\right)^2.$$

A body is projected in a medium where the resistance $= 2g \cdot \sqrt{\frac{v}{k}}$. Shew that the velocity after any time $= \left(\sqrt{a} - \frac{gt}{\sqrt{k}}\right)^2$.

$F = \mu\{r + \frac{2a^3}{r^2}\}$ find with what velocity a body would revolve in a ⊙ at distance a, and if the velocity were suddenly doubled shew that the new orbit would have an apse at distance $3a$.

If $r = a$ $F = 3\mu a$. shew in a ⊙ $v = \sqrt{F.R}$

∴ In this case the velocity $= a\sqrt{3\mu}$.

also $h^2u^2(\frac{d^2u}{d\theta^2} + u) = \frac{\mu}{u} + 2a^3\mu . u^2$

or $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\frac{1}{u^3} + \frac{2a^3 u}{h^2}$.

But $h^2 = a^2v'^2 = 3\mu a^4 \times 4 = 12\mu a^4$.

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{12a^4}\frac{1}{u^3} + \frac{2u}{12a}$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = -\frac{1}{12a^4}\frac{1}{u^2} + \frac{u}{3a} + C.$$

at the time the velocity was doubled

$$\frac{du}{d\theta} = 0 \quad u = \frac{1}{a}.$$

$$\therefore \frac{1}{a^2} = -\frac{1}{12a^2} + \frac{1}{3a^2} + C$$

$$\frac{12 + 1 - 4}{12a^2} = \frac{3}{4a^2} = C.$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = -u^2 - \frac{1}{12a^4}\frac{1}{u^2} + \frac{u}{3a} + \frac{3}{4a^2}$$

at an apse $\frac{du}{d\theta} = 0$.

$$\therefore u^2 + \frac{1}{12a^4}\frac{1}{u^2} = \frac{u}{3a} + \frac{3}{4a^2}.$$

which is satisfied if $u = \frac{1}{3a}$.

∴ at the apse $r = 3a$.

[illegible]

[illegible]

[illegible]

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$$[illegible]$$

When a body describes an ellipse about a centre of force in the centre, it moves between the extremity of any pair of conjugate diameters in the same time.

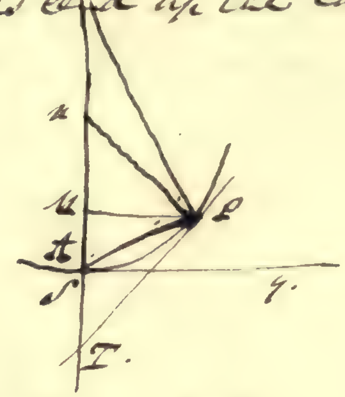
For the eqn to the ⊙ referred to any conjugate diameters as axes is $y^2 = \frac{b'^2}{a'^2}(a'^2 - x^2)$.

∴ the area contained between any semi conjugate axis = $\frac{1}{4}$ area of the ⊙

wherefore since the radius vector sweeps out ~~the~~ equal areas in equal times, and the centre of force is at the origin of coord^s, the radius vector will sweep out the space between any semi conjugate diameters in $\frac{1}{4}$ of the whole periodic time

∴ the body moves between the extremities of any two conjugate diameters in the same time

A centre of force wh. is repulsive and varies as dist.$^{-n}$ is situated at S the lowest point of a vertical ⊙; how far will a body placed at S a.d acted on by gravity ascend up the curve.



Let P be the posit. of the body at the time t

SM = x. MP = y coords of P.

the force of S upon P $= \frac{\mu}{SP^n} = \frac{\mu}{r^n}$ suppose in direction SP $= \frac{\mu x}{SP^{n+1}}$, in directn SM.

$\tan PMT = \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax-x^2}}$.

$\tan MSP = \frac{y}{x} = \frac{\sqrt{2ax-x^2}}{x}$ $\therefore \tan SPT = \dfrac{\frac{\sqrt{2ax-x^2}}{x} - \frac{a-x}{\sqrt{2ax-x^2}}}{1+\frac{a-x}{x}}$

$= \frac{(2ax-x^2)-ax+x^2}{a\sqrt{2ax-x^2}}$

$= \frac{x}{\sqrt{2ax-x^2}}$

$\therefore \cos SPT = \frac{\sqrt{2ax-x^2}}{\sqrt{2ax}}$

$= \sqrt{\frac{2a-x}{2a}}$.

the force exerted by S along the tangent $=$

$\frac{\mu}{r^n} \cdot \frac{\sqrt{2a-x}}{\sqrt{2a}}$.

the whole force along the tangent $=$

$$g\,\frac{\sqrt{2ax-x^2}}{a} - \frac{\mu}{(2ax)^{\frac{n}{2}}} \cdot \frac{\sqrt{2a-x}}{\sqrt{2a}}.$$

$$v\frac{dv}{ds} = -\frac{\mu}{r^n}\frac{dr}{s} + g\frac{dx}{s}$$

$$\frac{1}{2}v^2 = \frac{\mu}{n-1}\,\frac{\mu}{r^{n-1}} + gx + C$$

$x = 0 \quad r = 0 \quad v = 0 \quad \therefore C = 0$.

$$\frac{1}{2}v^2 = \frac{\mu}{n-1}\,\frac{1}{(2ax)^{\frac{n-1}{2}}} + gx.$$

when $v = 0$ we have.

$$0 = \frac{\mu}{n-1} + (2a)^{\frac{n-1}{2}}\,g\,x^{\frac{n+1}{2}}$$

$\therefore x =$

Cf. $P = \mu u + \mu' u^3$ the apsidal angle $= \dfrac{\pi\left(1 + \frac{\mu'}{h^2}\right)}{\sqrt{1+\frac{\mu'}{h^2}}} = \pi\sqrt{1+\frac{\mu'}{h^2}}$

From this shew that the above value of P is coincident with that given by Newton for a body moving in a revolving orbit

$$\frac{8A^2}{G \cdot Ch^2} + \frac{G^2 - E^2}{E^2} \cdot \frac{4A^2}{Ch^3}$$

we have the apsidal angle $= \dfrac{G}{E} \pi$.

$$\therefore \frac{G}{E} = \sqrt{1 + \frac{\mu'}{h^2}} \qquad \frac{G^2 - E^2}{E} = + \frac{\mu'}{h^2}$$

also $8A^2 = 2h^2$ and $4A^2 = h^2$.

$\therefore$ the expression becomes $\left(Ch = r = \dfrac{1}{u}\right)$

$$\frac{2h^2}{G} \cdot u^2 + \frac{\mu'}{h^2} \cdot \frac{h^2}{r^3}$$

$$= \mu u^2 + \mu' u^3 \quad \left(\text{for } \frac{2h^2}{G} = \mu\right.$$

Shew that Newton's expression for the force

$$= \frac{2h^2}{SY^2 \cdot PV}$$

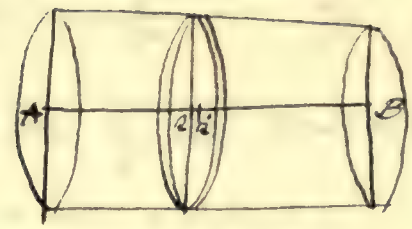
agrees with the analytical expression $P = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$

we have $SY^2 = \frac{1}{u^2+\left(\frac{du}{d\theta}\right)^2}$.

also $PV = \frac{1}{2u^2} \cdot \frac{u^2+\left(\frac{du}{d\theta}\right)^2}{u+\frac{d^2u}{d\theta^2}}$.

$$\therefore \frac{2h^2}{SY^2 \cdot PV} = h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right).$$

-body attached to a certain



Let a = rad of the base of the cylinder. b = its height.
Take AB the axis of the cylinder for the axis of x. then the mass of a section contained between two planes $\perp$ to the axis of x, and at distances of x. $x+\delta x$ from A will be.

$$\pi\rho a^2\ \delta x.$$

$\therefore$ if M = mass of the cylinder and k rad of gyr.

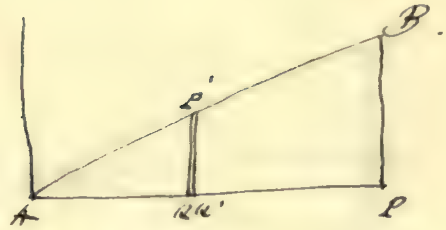
$$Mk^2 = \frac{\pi\rho a^4}{2}\int_x 1 \quad \text{from } x = 0 \text{ to } x = b$$

$$= \frac{\pi\rho a^4 b}{2}$$

$$\text{But } M = \pi\rho a^2 b$$

$$\therefore k^2 = \frac{a^2}{2}.$$

[illegible]



Let AC the axis of the cone be the axis of x. $y = mx$ the eqn to the generating line AB.

Then the mass of a thin circular sect. generated by the revolut. of Paa' about the axis of x $(Aa = x \quad Aa' = x + \delta x) = \rho \delta x . \pi y^2$

and the moment of inertia about the axis of x =

$\frac{1}{2} \pi \rho y^4 \delta x.$

∴ the moment of inertia of the whole cone

$= \frac{1}{2} \pi \rho \int_0^{AC} dx . y^4 \quad = \frac{1}{2} \pi \rho m^4 \int_0^b dx\, x^4 \quad$ if $AC = b.$

$= \frac{1}{2} \frac{\pi \rho m^4 b^5}{5}.$

∴ if k = rad of gyr we have

$M k_2 = \frac{\pi \rho m^4 b^5}{2.5.}$

But mass of the cone $= \frac{m^2 b^3 \pi \rho}{3.}$

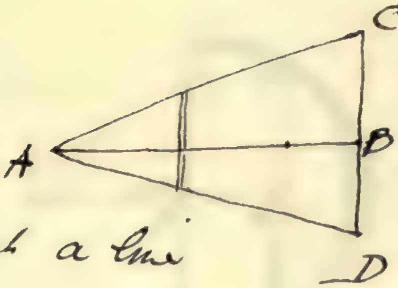
$\therefore k^2 = \frac{3}{2.5} . m^2 b^2$

$= \frac{3}{2.5} BC^2 = \frac{3}{10} BC^2.$

[illegible]

AB the axis of the cone $= b$

a = rad of the base.



(1) To find the moment of inertia about a line thro' A. ⊥ to the axis of the cone.

The mass of a slice of the cone thickness δx cut off by planes ⊥ to the axis at distances x, $x+\delta x$ from A will be

$$\frac{\pi\rho a^2}{b^2} \cdot x^2 \delta x.$$

and the moment of inertia of this slice about any diam will be

$$\pi\rho \frac{a^4 x^4}{4b^4}\delta x \quad \therefore \quad \frac{(\pi\rho a^2)(a^2+4b^2)}{4b^4} x^4 \delta x = \text{moment of inertia}$$

about an axis thro' A ⊥ to AB.

∴ if M = mass of the cone and k rad of gyr. about this axis

$$Mk^2 = \frac{\pi\rho a^2}{4b^4} \cdot (a^2+4b^2) \int_0^b x^4$$

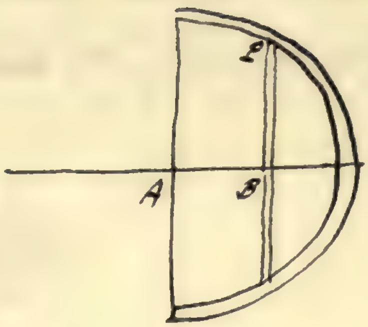
$$= \frac{\pi\rho a^2 b}{20} (a^2+4b^2).$$

But $M = \frac{a^2 \pi . b\rho}{3}$ $\therefore k^2 = \frac{3}{20} \cdot (a^2+4b^2)$ —

wherefore if k_1 = rad of gyr. about an axis thro' the cent. of grav. ⊥ to the axis of x we have

$$k_1^2 = \frac{3a^2}{20} + \frac{3.4}{20} b^2 - \frac{4}{9} b^2 = \frac{27a^2 - 28b^2}{180}.$$

A the cent. of the sphere.
a. b. the radii of the outer and
inner spheres.



The moment of inertia of the annulus contained between planes ⊥ to the axis at a distance x and $x+\delta x$ from A will be.

$$\pi\rho\frac{(a'^4-b'^4)\,\delta x}{2} = \frac{\pi\rho}{2}.\left(a^2-x^2\right)^2-\left(b^2-x^2\right)^2\delta x.$$

∴ Moment of inertia of the hollow sphere =

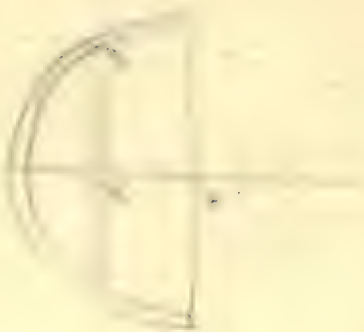
$$\pi\rho\left\{\int_x^a a^4-2ax^2+x^4-\int_x^b b^4-2bx^2+x^4\right)$$

or $$Mk_1^2=\pi\rho\left\{.\ a^5-\frac{2}{3}a^5+\frac{1}{5}a^5-b^5+\frac{2}{3}b^5-\frac{1}{5}b^5\right.$$

$$=\frac{\pi\rho 8}{15}\left(a^5-b^5\right)$$

But $M=\frac{4}{3}\pi\rho(a^3-b^3)$

$$\therefore k^2=\frac{2}{5}.\left(\frac{a^5-b^5}{a^3-b^3}\right)$$



Find the velocity with which a body must move near the earth's surface in order to continue revolving about it in a circular orbit.

$$P = 2\pi \sqrt{\frac{R}{E}} = 2\pi \sqrt{\frac{3960 \times 5280}{32.2}}.$$

$$= 2\pi \sqrt{\frac{660 \times 6 \times 660 \times 8}{32.2}} = 660 \times 8 \times \pi \sqrt{\frac{3}{32.2}}$$

Now $\sqrt{\frac{3}{32.2}} = \sqrt{.0931677} = .30523.$

$$\therefore \quad P = \pi \times 5280 \times .30523$$

$$= 3.1416 \times 5280 \times .30523$$

$$= 50623 \text{ seconds nearly}$$

$$= 1^{hr}\ 24'\ 23''$$

The force tending to the centre of a ⊙ whose radius is c, being $\mu\left(r + \frac{2c^4}{r^3}\right)$ find the velocity with which a body will describe the ⊙ and shew that if the velocity be suddenly doubled it will come to an apse at a distance c.

1st. to find the velocity.

$$v = \sqrt{F.R} = \sqrt{\mu r^2 3} = r\sqrt{3\mu} \text{ . for } r = c.$$

let h = 2 area described in 1 second, in the curve after the velocity is doubled.

$$h^2 = 4v^2.r^2 = 12c^4\mu.$$

∴ since the force is central

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu\left(\frac{1}{u} + 2c^4u^3\right)$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{1}{12c^4}\left\{\frac{1}{u^3} + 2c^4u\right)$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{12c^4}\left\{\frac{1}{u^3} - 10c^4u\right\}.$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{12c^4}\left\{-\frac{1}{u^2} - 10c^4u^2\right) + C.$$

at the point of proj. $\frac{du}{d\theta} = 0$ $u = \frac{1}{c}$.

$$0 = \frac{1}{12c^4}\left\{-11c^2\right) + C.$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{1}{12c^4}\left\{-\frac{1}{u^2} - 10c^4u^2 + 11c^2\right\}.$$

or $2a = \frac{r^2}{r-s}$

$s = \frac{v^2}{2F} = \frac{12\mu c^2}{6\mu c} = 2c.$

$\therefore 2a = \frac{4c^2}{c-2c} = -4c$

A spherical surface is described in space, having in its centre a force $\propto \frac{1}{r^2}$. shew that if a particle be let fall from this surface, and be projected in any direction at any moment of its descent with the velocity acquired it will move in an $\odot$ the major axis of which = the radius of a sphere.

To find the velocity acquired.

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad \therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{\mu}{a} = \frac{2\mu}{x}\frac{(a-x)}{a}$$

$$= \frac{\mu}{x}\left(2 - \frac{x}{\frac{1}{2}a}\right). \quad (1)$$

Now in an $\odot$ $F = \frac{\mu}{x^2}$ we have

$$v^2 = \frac{\mu}{x}\left\{2 - \frac{x}{AC}\right\}. \quad (2)$$

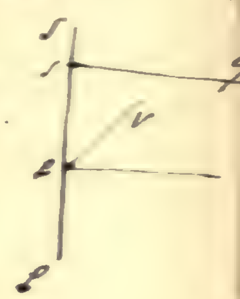
Compare (2) with (1) and we see they are similar, and that $AC = \frac{1}{2}a$, or AA' the major axis of the $\odot = a$

Two masses S. E. S being = 2E attract each other with forces ∝ $\frac{1}{(\text{dist})^2}$, and descend towards each other in a straight line from rest. When they have approached by $\frac{1}{2}$ their original distance E receives a new velocity ⊥ to its motion, and equal to that which S has acquired — what curve will afterwards be described by each about the other.

Since $S = 2E$, and the mutual attraction operates the same on each, the velocity of E = twice the velocity of S. Also S will evidently have moved through $\frac{1}{6}$ of the whole space.

$$\therefore\ v^2 = \left(\frac{dx}{dt}\right)^2 = 2\mu\left\{\frac{1}{x} - \frac{1}{a}\right\} = \frac{2\mu}{5a} = \text{velocity of } S$$

$$v_1^2 = \qquad\qquad \frac{4\mu}{5a} = \text{velocity of } E$$



Let a force equal to that which E exerts upon S be applied to both E and S, in an opposite direction to that in which S had a tendency to move. This will not alter the relative position of S and E. and S will be at rest.

Let SE be the axis of x. Sy ⊥ to SE the axis of y. also let x, y be the coords of E at any time t.

then $\frac{d^2x}{dt^2} = -\frac{(S+E)\,x}{r^3}$. $\frac{d^2y}{dt^2} = -\frac{(S+E)}{r^3}\cdot y$.

Or using polar coords as the force is central.

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = (S+E)\,u^2 = \mu . u^2 \text{ suppose.} \quad (3)$$

Now if v'^2 = velocity at E $v'^2 = \frac{6\mu}{5a}$ $(= v^2 + v_1^2)$.

also $\angle SEv = \tan^{-1}\frac{1}{2}$ $\therefore \sin SEv = \frac{1}{\sqrt{5}}$

$$\therefore h^2 = SE^2 \sin^2 SEv . v_1^2 = \frac{a^2}{4}\cdot\frac{6\mu}{25a} = \frac{3a\mu}{50}$$

∴ eqn (3) becomes $\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} = \frac{50}{3a}$.

$$\therefore\ u = \frac{50}{3a} + A\cos(\theta - B)$$ which compared with $u = \frac{1}{a(1-e^2)}\{1 + e\cos\ldots\}$

since $\frac{50}{3a} = \frac{1}{a(1-e^2)}$. $A = \frac{e}{a(1-e^2)} = \frac{50}{3a}e$ ∴ A = and the angle on axis may be found.

wherefore E describes an ⬭ about S.

Similarly it may be shown that S describes an ⬭ about E, and that the two bodies describe ellipses about each other.

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \mu u^3 + \mu' u^2$$

$$\frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2} + \frac{\mu' u}{h^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1-\frac{\mu'}{h^2}\right)u - \frac{\mu}{h^2} = 0$$

$$\therefore u = \frac{\mu}{h^2}\cdot\frac{h^2}{h^2-\mu'} + A\cos\left(\frac{h^2}{h^2-\mu'}\,\theta - B\right)$$

$$= \frac{\mu}{h^2-\mu'} + A\cos.\frac{h^2}{h^2-\mu'}(\theta - B)\}$$

Now in an ⊙ $u = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)}\cos(\theta - B')$

∴ the proposed orbit will be an ellipse, the velocity of the plane of the ⊙ being to that of the radius vector as $\frac{\mu'}{h^2-\mu'}$ is to unity.

2. The central force consists of two parts, of which one varies as $\frac{1}{(\text{dist.})^2}$ and the other as $\frac{1}{(\text{dist.})^3}$. shew that the orbit described will coincide with the path of the body revolving in an ⊙ round a centre of force in a focus, while the plane of the ⊙ revolves at the same time about that centre.

A heavy ball attached by a string to a fixed point is held in such a position that the string is horizontal, find the tension of the string at the lowest point.

For the tension of the string we have. $T - R = \frac{mv^2}{\rho} + mg$.
where m = mass of the body. R = centrif. force.

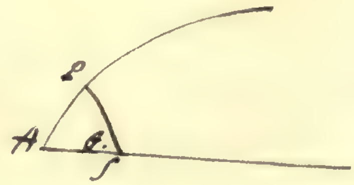
At the lowest point $v^2 = 2ga$, if a = rad of the ⊙ or the length of the string. also $\rho = r = a$.

∴ at the lowest point we have

$$T = 2mg + mg = 3mg.$$

= 3 times the weight of the body.

Find the time of describing a given angle of a [pa]bolic orbit round the focus.



we have $r^2 \frac{d\theta}{dt} = h.$

$$\therefore \frac{dt}{d\theta} = \frac{r^2}{h} = \frac{m^2}{h}.\sec^4\frac{\theta}{2}.$$

$$\text{or } \frac{dt}{d\theta} = \frac{2m^2}{h}.\left(1+\tan^2\frac{\theta}{2}\right)\frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore \frac{dt}{dx} = \frac{2m^2}{h}.\{1 + x^2) \quad \therefore t = \frac{2m^2}{h}\left(x + \frac{1}{3}x^3\right) + C.$$

$$\therefore t = \frac{2m^2}{h}\left\{\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right\} + C.$$

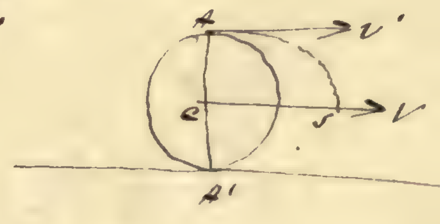
when $t = 0$ $\theta = 0$ $\therefore C = 0$.

$$T = \frac{2m^2}{h}\left\{\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right\}.$$

A molecule is projected from the wheel of a coach in rapid motion. given the velocity of the coach, find the greatest distance before the wheel to which the molecule can be thrown.

CV the direction of the motion of the coach.

v = velocity of the coach.



at A' a molecule on the surface of the wheel would be at rest, at A its velocity would be in direct: Av' ∥ to the horizontal plane, and $= 2v$.

let As represent the path of the molecule after being projected at A. where its vel. would evidently be a maximum. because the whole angular velocity at that point is in the direction of the motion of the coach.

let r = rad of the wheel.

Then if t be the time the molecule takes to arrive at s

$t = \sqrt{\frac{2r}{g}}$ ∴ $cs = 2v.\sqrt{\frac{2r}{g}}$.

But during this interval the axis of the wheel would have advanced thro' a space. $v\sqrt{\frac{2r}{g}}$.

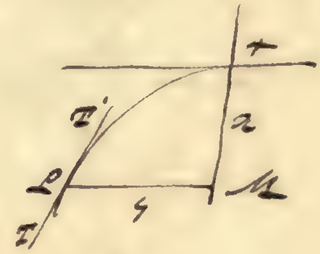
∴ the dist. of the particle before the axle of the wheel $= v\sqrt{\frac{2r}{g}}$. and before the point in nearest point in the circumf. of the wheel $= v\sqrt{\frac{2r}{g}} - r$.

when the molecule reaches the ground its distance before the axis of the wheel $= 2v\sqrt{\frac{r}{g}}$. which is the reqd. distance.

A body acted on by gravity moves on the convex surface of the cycloid whose vertex is the highest point the velocity at the highest point being given, det. the point where it will leave the cycloid and the latus rectum of the parabola afterwards described

x.y. coord^s. of the reqd. pt. L.

Let R = resistance on the curve outwards.



$R = mg\frac{dy}{ds} - m\frac{v^2}{\rho}$.

$\rho = -\frac{1}{\frac{d^2y}{dx^2}}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = \frac{x^{\frac{3}{2}}\sqrt{2a-x}}{a}\cdot\left(\frac{2a}{x}\right)^{\frac{3}{2}} = 2\sqrt{2a}\cdot\sqrt{2a-x}$

$\therefore \frac{dy}{ds} = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{\sqrt{2a-x}}{\sqrt{2a}}$

$\therefore R = mg\left\{\frac{\sqrt{2a-x}}{\sqrt{2a}} - \frac{(h+x)}{\sqrt{2a}\sqrt{2a-x}}\right\}$ if h = space due to velocity at A.

at the reqd point R = 0

$\therefore \frac{mg}{\sqrt{2a}}\left\{\frac{2a-x-h-x}{\sqrt{2a-x}}\right\} = 0$ or $x = \frac{1}{2}(2a-h)$.

Again at L if LT be a tangent. $\tan T'Ly = \frac{1}{\frac{dy}{dx}} = \sqrt{\frac{x}{2a-x}}$

$= \sqrt{\frac{\frac{1}{2}(2a-h)}{2a-a+\frac{h}{2}}} = \sqrt{\frac{a-\frac{h}{2}}{a+\frac{h}{2}}}$

$= \sqrt{\frac{2a-h}{2a+h}}$.

$\therefore$ vel $\parallel$ to $LM = \sqrt{g(2a+h)}\cdot\cos TLy$

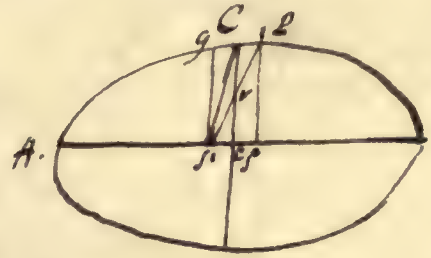
$= \sqrt{g(2a+h)}\cdot\frac{\sqrt{2a+h}}{\sqrt{4a}} = \frac{2a+h}{2}\sqrt{\frac{g}{a}}$.

$\therefore$ the latus rectum $= 4\left\{\frac{(2a+h)^2}{4}\cdot\frac{g}{2ag}\right\}$

$= \frac{(2a+h)^2}{2a}$.

[illegible]

A body moves in an $\odot$ of small eccentricity and is acted on by a central force tending to the focus; shew that its angular distance from the extremity of the nearer apse will be determined nearly by the eq.: $\theta = \frac{\pi}{2} + 2\tan^{-1} e$, at the end of $\frac{1}{4}$ of the periodic time.



S' the focus of the $\odot$ S'P nearly bisects CC' in v since the $\odot$ is small. Hence area CvP = area S'vC' nearly or S'P is nearly the position of the radius vector at the end of ¼ of the periodic time

now $gSA = \frac{\pi}{2}$.

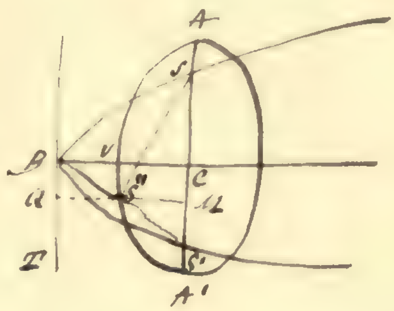
$$\tan gS'C = \tan CS'P \text{ nearly} = \frac{S'C}{CC'} = \frac{ae}{b} = \frac{ae}{a\sqrt{1-e^2}}$$

$$= \frac{ae}{a} = e \text{ nearly}$$

$\therefore gS'P = 2\tan^{-1} e$ nearly.

and the whole $\angle ASP = \frac{\pi}{2} + 2\tan^{-1} e$ nearly.

A body describes a ⊙ about a centre of force in a point in the circumference of an ellipse & the foci of parabola are in the circumf of the ⊙ the force varying inversely as (dist)², shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ⊙ the centre of force is placed.



$y^2 = 4mx$ the eqn to the ⊙

$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ eqn to the ⊙

S'' the centre of force $CM = x$. $MS = y$ coords of S

Then the whole area traced out by the radius vector $= \frac{2}{3} SS' \cdot BC - \frac{1}{2} SS' \cdot SM$

$= SS' \left\{ \frac{2}{3} BC - \frac{1}{2} \frac{b}{a} \sqrt{a^2 - x^2} \right\}$

$= \frac{2}{3} SS' \left\{ BC - \frac{3}{4} \frac{b}{a} \sqrt{a^2 - x^2} \right\}$

Let v' be the velocity at B.

$\therefore h = v\,SB \sin SBT + v' = v\,Sa + v'$

$= v \cdot \left(BC - \frac{b}{a} \sqrt{a^2 - x^2} \right) + v'$

$= v'' \left\{ BC - \frac{3}{4} \frac{b}{a} \sqrt{a^2 - x^2} \right\}.$

if $v' = \frac{v}{4} \frac{b}{a} \sqrt{a^2 - x^2}$

Then $T = \frac{2}{3} \frac{SS'}{v''}$ which is constant.

Given the pressure on the sides of a wedge, the coefficient of fricts. the weight of the wedge, and that of a hammer which strikes it, and the modulus of elasticity between their surfaces, to find how far one blow of the hammer with given velocity will drive the wedge.

P = pressure on the wedge.

$\therefore \mu P$ = friction

m = mass of wedge.

m' = mass of hammer.

$v' v$. velocity of hammer immediately before and immediately after impact

$$m'(v' - v)^{-\mu P} = mv.$$

for since gravity ~~and friction~~ requires time to act, the weight of the wedge will not alter the initial velocity.

$$m'v' - \mu P = (m + m')v \qquad \therefore v = \frac{m'v' - \mu P}{m + m'}.$$

$\therefore$ for the motn of the wedge.

$$\frac{d^2x}{t} = \frac{\mu P}{m} - g.$$

$$\frac{dx}{t} = \left(\frac{\mu P}{m} - g\right)t + \frac{m'v' - \mu P}{m + m'} \qquad (1)$$

$$x = \frac{1}{2}\left(\frac{\mu P}{m} - g\right)t^2 + \frac{m'v' - \mu P}{m + m'}t.$$

which gives the reqd space x in terms of t.

But from (1), if $\frac{dx}{t} = 0$ $\quad t = \frac{(m'v' - \mu P)m}{(m + m')(\mu P - mg)}$.

$$\therefore x = \frac{1}{2} \cdot \frac{\mu P - mg}{m}\left\{\frac{(m'v' - \mu P)^2 m^2}{(m + m')^2(\mu P - mg)^2}\right\} + \frac{(m'v' - \mu P)^2 m}{(m + m')^2(\mu P - mg)}$$

$$= \frac{3}{2} \cdot \left(\frac{m'v' - \mu P}{m + m'}\right)^2 \left(\frac{m}{\mu P - mg}\right).$$

Camb. Problems.

let V = moment of the impulse communicated.

x = distance from the centre of the point of its applicatn. and suppose it to act at right angles to that distance.

a = rad of the earth $\therefore \frac{4}{3}\pi a^3$ = its volume.

Now $\because$ V acting at x is equivalent to V acting at the earths centre, and a couple whose moment is Vx.

If V were applied at the earths centre it would produce a velocity $= \frac{3V}{4\rho\pi a^3}$ in the direction of the impulse communicated.

let r = distance of the earth from the Sun at the time the impulse was communicated.

$$\therefore P = \frac{2\pi . a'^{\frac{3}{2}}}{\sqrt{\mu}} \quad \text{where } a' = \text{semi major axis of the earths orbit}$$

$$\text{But } \frac{3V}{4\rho\pi a^3} = \sqrt{\frac{\mu}{r}\left(\frac{2a'-r}{a'}\right)} \quad \therefore \sqrt{\mu} = \sqrt{\frac{a'r}{2a'-r}} \cdot \frac{3V}{4\rho\pi a^3}$$

$$\therefore P = \frac{8}{3} \cdot \frac{\pi^2 a' \rho a^3}{V} \sqrt{\frac{2a'-r}{r}}.$$

Let α be the angular velocity communicated to the earth by the couple whose moment is Vx.

Take an elementary section of the earth $\perp$ to the axis of revolution, ~~at a distance~~ contained between planes at a distance x and $x + \delta x$ from the centre.

The product of each particle of this elementary $\odot$ into its velocity about the axis $= \int_0^{2\pi}\int_0^y r^2 \delta\theta\delta r . r\alpha$

$$= \frac{2}{4}\pi\alpha y^4 = \frac{1}{2}\pi\alpha . \{a^2 - x^2)^{\frac{3}{2}}$$

$\therefore$ for the whole sphere we have $\frac{2}{3}\pi\alpha \int_{x}^{a} \left(a^2 - x^2\right)^{\frac{3}{2}}$

$\frac{1}{2}\pi\alpha . \int_{-a}^{a} (a^2 - x^2)^2 .$

$(a^2+x^2)^2 = a^4+2a^2x^2+x^4.$

$\therefore \int_x (a^2+x^2)^2 = a^4x+\frac{2}{3}a^2x^3+\frac{1}{5}x^5.$

$\therefore {}^{a}\!\!\int^{-a} (a^2+x^2)^2 = 2a^5+\frac{4}{3}a^5+\frac{2}{5}a^5 = \frac{a^5}{15}\{30+20+6\}$

$= \frac{56}{15}a^5$

$\therefore$ the moments of the sphere about its axis $= \frac{28}{15}a^5\pi\alpha$

$\therefore Vx = \rho\alpha.\frac{28}{15}a^5\pi$ or $\alpha = \frac{15Vx}{28\rho\alpha\pi a^5}$

but time of diurnal rotation $= \frac{2\pi}{\alpha}$

$= \frac{56\pi^2\rho a^5}{15Vx}$

suppose in round numbers the earth makes 365 daily rotations, during the period of an annual revolutn.

$\therefore \frac{365\times 56\pi^2\rho a^5}{15Vx} = \frac{8}{3}.\frac{\pi^2 a'\rho a^3}{V}\sqrt{\frac{2a'-r'}{r'}}$

$\frac{73\times 7a^2}{x} = a'\sqrt{\frac{2a'-r'}{r'}}$

$\therefore x = \frac{511a^2\sqrt{r'}}{a'\sqrt{2a'-r'}}$

1)

Let $x^2+y^2+z^2=c^2$ be the $=^n$ to the sphere.

then $\delta x\delta y\delta z$ = ~~vol~~ volume of element.

$\therefore V=\int_x\int_y\int_z 1.$

$=\int_x\int_y z = \int_x zy + C.$

$=\int_x\int_y \sqrt{c^2-(x^2+y^2)}.$

$=\int_x \left\{ \frac{y}{2}\sqrt{c^2-x^2-y^2}+\frac{c^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{c^2-x^2}} \right.$

the limits of y are $y=0$ and $y=\sqrt{c^2-x^2}$.

$\therefore V={}^0\!\int_x^c \frac{c^2-x^2}{2}\cdot\frac{\pi}{2}.$

$=\left(\frac{c^3}{2}-\frac{c^3}{6}\right)\frac{\pi}{2}=\frac{\pi c^3}{6}$ = ~~vol~~ vol of the eighth part of the sphere.

$\therefore \frac{4}{3}\pi c^3$ = vol of the whole sphere.

p15

Let a be the depth of the lowest part of the ⌓ from the surface of the fluid.

Let x be the depth of any horizontal section of the fluid below the surface.

Then the (velocity)^2 of the fluid at this depth $= 2gx$.

$\therefore$ $2y\sqrt{2gx} \cdot \delta x$ = volume of the fluid discharged in a unit of time, contained between horizontal sections of the fluid at distances x and $x + \delta x$ from the surface.

$\therefore \int_x^{a} \cdot (2y\sqrt{2gx}) = 4\sqrt{2gm} \cdot \int_x^{a} \cdot x = 2\sqrt{2gm} \cdot a^2$. is the quantity of fluid discharged during a unit of time.

also the area of the ⌓ $= \frac{4}{3} xy = \frac{8}{3} a\sqrt{ma}$.

$\therefore$ the mean velocity $= \dfrac{3 \cdot 2a^2\sqrt{2gm}}{8a\sqrt{ma}} = \frac{3}{4} \cdot \sqrt{2ga}$.

And the horizontal section which has this velocity is situated at a depth $= \frac{9a}{16}$ from the top.

Let $\log \frac{c}{r} = \frac{\theta}{a}$ be the eqn to the spiral

then $v^2 = C - 2\int_r \frac{\mu}{r^2} = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)$ if a = original distance of the particle

$$\therefore \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right).$$

but $d\theta = \frac{a}{r}\,dr$. $\therefore r^2\left(\frac{d\theta}{dt}\right)^2 = a^2\left(\frac{dr}{dt}\right)^2$.

$$\therefore (1+a^2)\left(\frac{dr}{dt}\right)^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right).$$

$$\frac{dr}{dt} = \frac{1}{\sqrt{1+a^2}}\cdot\sqrt{\frac{2\mu}{a}}\cdot\sqrt{\frac{a-r}{r}}.$$

$$\frac{dt}{dr} = \sqrt{1+a^2}\sqrt{\frac{a}{2\mu}}\left\{\frac{\frac{a}{2}-r}{\sqrt{ar-r^2}} - \frac{a}{2}\cdot\frac{1}{\sqrt{ar-r^2}}\right\}.$$

$$\therefore t = \sqrt{\frac{a(1+a^2)}{2\mu}}\left\{\sqrt{ar-r^2} - \frac{a}{a}\text{vers}^{-1}\frac{2r}{a}\right) + C.$$

at the commencement of motion $t = 0$. $r = 2a$.

$$0 = \sqrt{\frac{a(1+a^2)}{2\mu}}\left\{-\frac{a}{2}\pi\right) + C.$$

$$\therefore t = \sqrt{\frac{a(1+a^2)}{2\mu}}\left\{\frac{a}{a}\left(\pi - \text{vers}^{-1}\frac{2r}{a}\right) + \sqrt{ar-r}\right).$$

after one revolution $r = c\varepsilon^{-\left(\frac{\theta}{a}-2\pi\right)}$.

$$\therefore t' = \sqrt{\frac{a(1+a^2)}{2\mu}}\left\{\frac{a}{2}\left(\pi - \text{vers}^{-1}\frac{2c\varepsilon^{-\left(\frac{\theta}{a}-2\pi\right)}}{a}\right) + \sqrt{ac\varepsilon^{-\left(\frac{\theta}{a}-2\pi\right)} - c^2\varepsilon^{-\left(\frac{2\theta}{a}-4\pi\right)}}\right\}$$

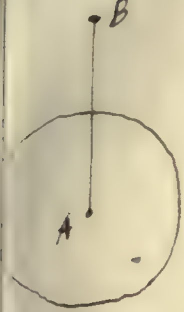
when the particle arrives at the centre $r = 0$.

$$\therefore t'' = \frac{a^{\frac{3}{2}}\,2\pi}{2^{\frac{3}{2}}}\sqrt{\frac{1+a^2}{\mu}}\Big\}.$$

$$\therefore \frac{t'}{t''} = \frac{\frac{a}{2}\left(\pi - \text{vers}^{-1}\frac{2c\varepsilon^{-\left(\frac{\theta}{a}-2\pi\right)}}{a}\right\} + 2\sqrt{ac\varepsilon^{-\left(\frac{\theta}{a}-2\pi\right)} - c^2\varepsilon^{-\left(\frac{2\theta}{a}-4\pi\right)}}}{a\pi}$$

p. 161

[illegible]



Let the lowest point in the circular arc be taken for the origin of coord^s. and let x, y be the coord^s of the centre of the sphere at a time t.

$\therefore \frac{d^2s}{dt^2} = -g\frac{dx}{ds} \quad \therefore \left(\frac{ds}{dt}\right)^2 = -2gx + C$

at the commencement of the motion let x = h.

$\therefore \frac{ds}{dt} = \sqrt{2g(h-x)} \quad$ or $\frac{dt}{ds} = \frac{1}{\sqrt{2g}\sqrt{h-x}}$

$\therefore \frac{dt}{dx} = \frac{\frac{ds}{dx}}{\sqrt{2g}\cdot\sqrt{h-x}} \qquad$ In the ⊙ $\frac{ds}{dx} = \frac{a}{\sqrt{2ax-x^2}}$

$\therefore \frac{dt}{dx} = \frac{a}{\sqrt{2g}}\cdot\frac{1}{\sqrt{hx-x^2}}\cdot\frac{1}{\sqrt{2a-x}} = \frac{1}{2}\sqrt{\frac{a}{g}}\cdot\frac{1}{\sqrt{hx-x^2}}\left\{1+\frac{1}{2}\left(\frac{x}{2a}\right)+\frac{1\cdot 3}{2\cdot 4}\left(\frac{x}{2a}\right)^2+\dots\right.$

$\therefore t = -\frac{1}{2}\sqrt{\frac{a}{g}}\left\{\text{vers}^{-1}\frac{2x}{h} + \frac{1}{4a}\left(\sqrt{hx-x^2}+\frac{h}{2}\text{vers}^{-1}\frac{2x}{h}\right)+\&c.\right\}+C.$

when t = 0 x = h.

$\therefore 0 = \frac{1}{2}\sqrt{\frac{a}{g}}\left\{\pi + \frac{h}{8a}\pi + \&c.\right\} + C$

$\therefore t = -\frac{1}{2}\sqrt{\frac{a}{g}}\left(\text{vers}^{-1}\frac{2x}{h} + \frac{1}{4a}\left(\sqrt{hx-x^2}+\frac{h}{2}\text{vers}^{-1}\frac{2x}{h}\right)+\dots\right)$

at the time of a complete oscillation h = −x.

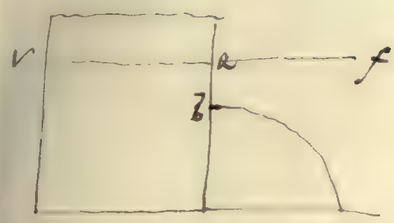
$\therefore T = \sqrt{\frac{a}{g}}\cdot\frac{5\pi}{4} = \frac{\pi}{2}\sqrt{\frac{a}{g}}\left\{1+\frac{h}{8a}\right.$

$\therefore T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$ nearly.

Let l be the distance from B measured in the direction BA of the centre of oscillation

$\therefore l = \frac{k^2+h^2}{h} = \frac{\frac{2}{5}\cdot\frac{a^2}{4}+4a^2}{2a} = \frac{22}{20}a = \frac{11}{10}a.$

p13)



ar the section of the surf a a of the fluid made by a vertical plane passing thro' the orifice $ab = c$. πe^2 = area of section of the cylinder. Then the (velocity)^2 of the issuing stream is $2gc$. also af is the direction of the $\therefore$ the latus rectum $= 4c$. $= L$ suppose.

Let x = depth of the surface below the horizontal line va at a time t.

$$\therefore \frac{dx}{dt} \cdot \pi e^2 = a\sqrt{2g(c-x)}.$$

$$\frac{dt}{dx} = \frac{\pi e^2}{a\sqrt{2g}} \cdot \frac{1}{\sqrt{c-x}}.$$

$$t = -\frac{\pi e^2}{a\sqrt{2g}} \cdot \frac{1}{2}\sqrt{c-x} + C.$$

at the commencement of motion $x = 0$.

$$\therefore t = \frac{\pi e^2}{a\sqrt{2g}} \cdot \frac{1}{2}\left\{\sqrt{c} - \sqrt{c-x}\right\}.$$

when the fluid reaches b $x = c$.

$\therefore$ if T = the time reqd.

$$T = \frac{\pi e^2}{2a} \cdot \sqrt{\frac{c}{2g}} \cdot \Big\}.$$

Let l be the length of a pendulum which vibrates once during this period.

$$\therefore T = \pi\sqrt{\frac{l}{g}}.$$

$$\therefore \frac{e^2}{2a}\sqrt{\frac{c}{2}} = \sqrt{l} \quad \therefore l = \frac{ce^4}{8a^2}.$$

$$\therefore \frac{L}{l} = \frac{4c}{\frac{ce^4}{8a^2}} = \frac{32a^2}{e^4}$$

P. 13

1 h.g.

Let α = inclination of the plane to the horizon. take BA for the axis of x.

$$\therefore \frac{dv}{dt} = g \sin\alpha - \mu v^2$$

$$\therefore \frac{dv}{g \sin\alpha - \mu v^2} = dt \text{ or } \frac{dt}{dv} = \frac{1}{\mu} \cdot \frac{1}{\frac{g}{\mu}\sin\alpha - v^2}.$$

$$\therefore t = \frac{1}{2\sqrt{\mu g}} \cdot \log\left\{\frac{\sqrt{g\sin\alpha} - \sqrt{\mu}\,v}{\sqrt{g\sin\alpha} + \sqrt{\mu}\,v}\right\}. \quad (1)$$

Again $v\frac{dv}{dx} = g\sin\alpha - \mu v^2$.

$$\frac{v\,dv}{g\sin\alpha - \mu v^2} = dx.$$

$$-\frac{1}{2\mu} \cdot \frac{d(g\sin\alpha - \mu v^2)}{g\sin\alpha - \mu v^2} = dx.$$

$$\therefore -\frac{1}{2\mu} \cdot \log(g\sin\alpha - \mu v^2) = x + C.$$

$$-\frac{1}{2\mu} \cdot \log(g\sin\alpha) = C.$$

$$\therefore \frac{1}{2\mu} \cdot \log \frac{g\sin\alpha}{g\sin\alpha - \mu v^2} = x.$$

$$\varepsilon^{2\mu x} = \frac{g\sin\alpha}{g\sin\alpha - \mu v^2}$$

$$\therefore \mu v^2 \cdot \varepsilon^{2\mu x} = g\sin\alpha(1 - \varepsilon^{2\mu x})$$

$$\therefore v = \sqrt{\frac{g\sin\alpha}{\mu}} \cdot \left(\sqrt{\varepsilon^{-2\mu x} - 1}\right)$$

$$\therefore t = \frac{1}{2\sqrt{\mu g}} \cdot \log\left\{\frac{1 - \sqrt{\varepsilon^{-2\mu x} - 1}}{1 + \sqrt{\varepsilon^{-2\mu x} - 1}}\right\}.$$

$$r = \frac{1}{d^2_x y}\left(1+(d_x y)^2\right)^{\frac{3}{2}}$$

$$d_x y = \frac{-x}{\sqrt{a^2-x^2}} \quad \therefore \quad 1+(d_x y)^2 = \frac{a}{\sqrt{a^2-x^2}}$$

$$d^2_x y = \frac{-\sqrt{a^2-x^2} \mp x^2}{(a^2-x^2)^{\frac{3}{2}}} = \frac{-a^2}{(a^2-x^2)^{\frac{3}{2}}}$$

$$\therefore \rho = \frac{(a^2-x^2)^{\frac{3}{2}}}{a^2}\left\{\frac{-a^{\frac{3}{2}}}{(a^2-x^2)^{\frac{3}{2}}}\right.$$

$$\therefore X - a = +\frac{d_x y}{d^2_x y}\left(1+(d_x y)^2\right) \qquad \text{for } x^2+y^2$$

for $(x-a)^2+(y-b)^2 = c^2$.

$$y - b = -\frac{1}{d^2_x y}\left(1+(d_x y)^2\right) \qquad \therefore x-a+(y-b)\,d_x y = 0$$

$$X - a = \frac{+x}{\sqrt{a^2-x^2}}\,\frac{(a^2-x^2)^{\frac{3}{2}}}{a^2}\left(\frac{a^2}{(a^2-x^2)}\right)^{\frac{3}{2}}$$

$$= \frac{x}{a}\sqrt{a^2-x^2}, \quad = x$$

$$\therefore X = x + a.$$

$$Y = \frac{(a^2-x^2)^{\frac{3}{2}}}{a^2}\cdot\frac{a^2}{\sqrt{a^2-x^2}}$$

$$= \sqrt{a^2-x^2}$$

$$Y - b = \sqrt{a^2-(X-a)^2} = \sqrt{2aX - X^2}$$

$$Y = b + \sqrt{2aX - X^2}$$

$$d^2_x z + 3\,d_x d_y z + 2\,d^2_y z = x + y.$$

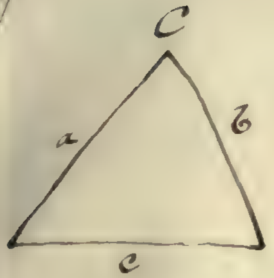
$$(d_x z + 2\,d_y z)(d_x z + d_y z)$$

$$\text{Let } z = e^{m(x+y)}$$

$$\therefore d^2_x z = (m^2 + 3m^2 + 2m^2)\,e^{m(x+y)} = (x+y)\,e$$

$$6m^2\,e^{m(x+y)} = (x+y).$$

3)



c the base. C the constant vertical angle.

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \quad \therefore \quad 1+\cos C = \frac{(a+b)^2-c^2}{2ab}$$

$$\text{or } 2\cos^2\frac{C}{2} = \frac{(a+b+c)(a+b-c)}{2ab} = \frac{s(s-2c)}{2ab}$$

where s = perimeter

$$\therefore \; s^2 - 2cs = A \cdot ab$$

$$s^2 - 2cs + c^2 = A \cdot ab + c^2$$

$$s - c = \sqrt{A \cdot ab + c^2}$$

$$s = c + \sqrt{A ab + c^2}$$

$$ds = \frac{A \cdot (a\,db + b\,da)}{\sqrt{A \cdot ab + c^2}} = 0.$$

$$\therefore \; a\,db + b\,da = 0.$$

$$2ab\cos C = a^2 + b^2 - c^2.$$

$$2\cos C(a\,db + b\,da) = 2a\,da + 2b\,db.$$

$$\therefore \; \mu \; 2\cos C\,(a\,db - b\,da) = 2\mu a\,da + 2\mu b\,db.$$

$$(2\cos C \cdot a - 2b)\mu - a = 0$$

$$\therefore \; \mu = \frac{a}{2(a\cos C - b)}$$

$$(2\cos C\, b - 2a)\mu - b = 0$$

$$\therefore \; \mu = \frac{b}{2(b\cos C - a)} = \frac{a}{2(a\cos C - b)}$$

which can only be satisfied by $a = b$.

for $ab\cos C - b^2 = ab\cos C - a^2$.

$b^2 = a^2$ or $b = a$.

(05)

Let AB be the initial position of the rod $A'B'$ its position at the expiration of a time t

$AM' = x$. $M'B' = y$; $AM = b$. $MB = c$

Then if $R =$ ~~resistance~~ thrust down the rod at time t we have.

$$\frac{R}{g} = \sin B'A'M' \quad \therefore \quad R = g \cdot \frac{y}{a}$$

$\therefore$ for the force acting $\parallel$ to AM we have.

$$\frac{d^2x}{dt^2} = -g \cdot \frac{y}{a} \cdot \cos B'A'M' = -\frac{g}{a^2} \cdot y\sqrt{a^2-y^2}.$$

$$\frac{d^2y}{dt^2} = -g\,\frac{y}{a}\sin B'A'M' = g(\cos^2 B'A'M') = -\frac{g}{a^2}(a^2-y^2)$$

$$\therefore \left(\frac{dy}{dt}\right)^2 = -\frac{2g}{a^2}\left\{a^2y - \tfrac{1}{3}y^3\right\} + C.$$

at the commencement of motion $\left(\frac{dy}{dt}\right)^2 = 0$. $y = c$

$$0 = -\frac{2g}{a^2}\left(a^2c - \tfrac{1}{3}c^3\right) + C.$$

$$\therefore \frac{dy}{dt} = \frac{\sqrt{2g}}{a}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}.$$

Now $$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt} = \frac{dx}{dy}\frac{dy}{dt} = \frac{\sqrt{2g}}{a}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\,\frac{dx}{dy}$$

$$\therefore \frac{d^2x}{dt^2} = d_t\left\{\frac{dx}{dy}\,\frac{\sqrt{2g}}{a}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right)$$

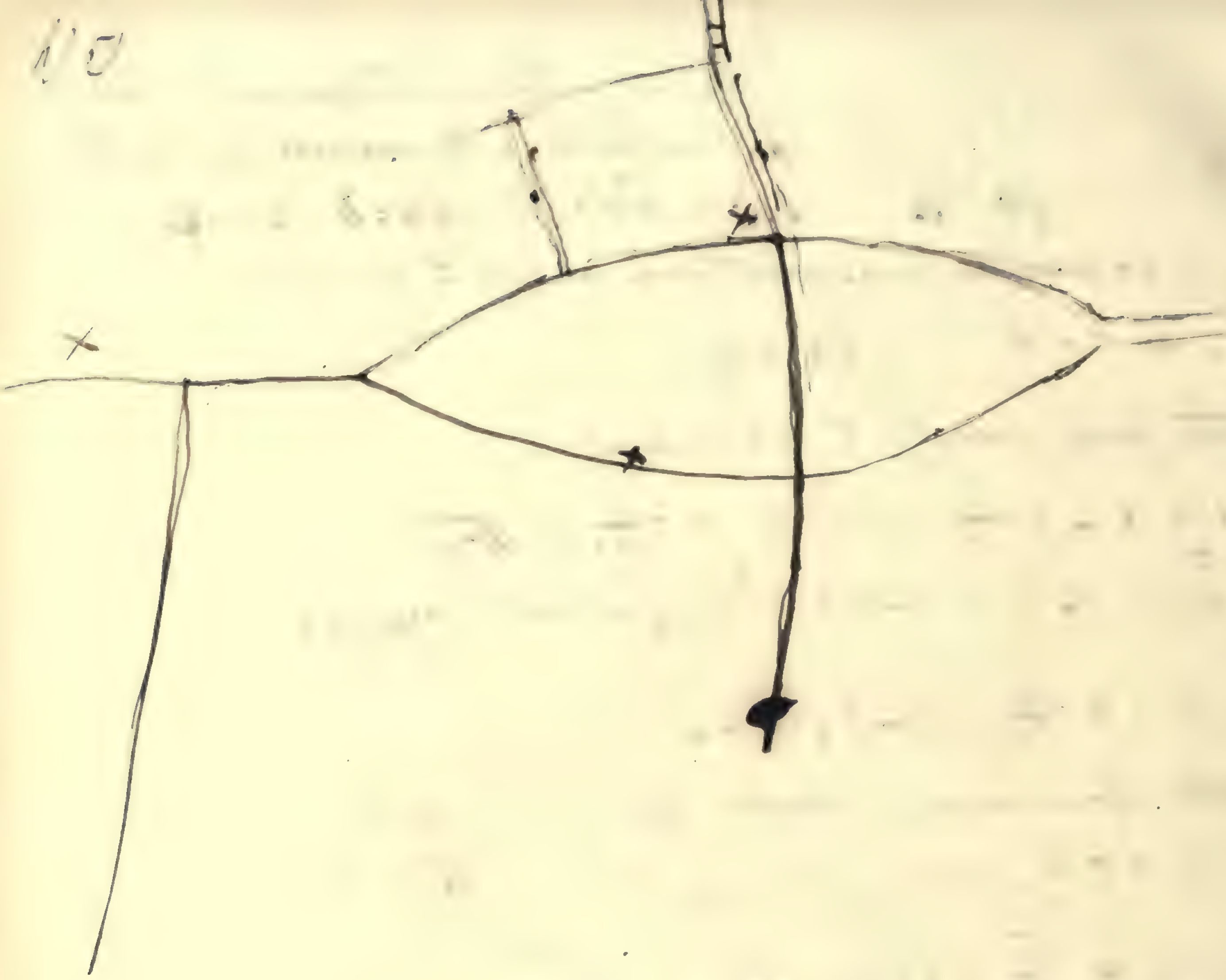
$$= d_y\left\{\frac{dx}{dy}\,\frac{\sqrt{2g}}{a}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\} d_t y)$$

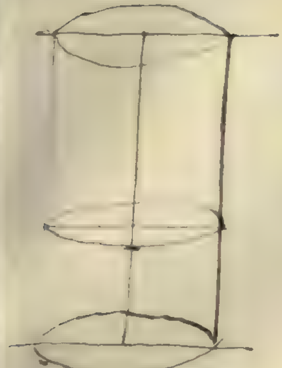
$$= d_y\left\{\frac{dx}{dy}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\}\cdot\left\{\frac{2g}{a^2}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\right\}.$$

$$\therefore y\sqrt{a^2-y^2} = 2\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}\left\{\frac{d^2x}{dy^2}\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)} + \frac{dx}{dy}\,\frac{-a^2+3y^2}{\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}}\right\}.$$

$$= 2\frac{d^2x}{dy^2}\left\{2(a^2(c-y) - \tfrac{1}{3}(c^3-y^3))\right\} + \frac{(3y^2-a^2)\frac{dx}{dy}}{\sqrt{a^2(c-y) - \tfrac{1}{3}(c^3-y^3)}}$$

N° 5





Let x = depth of the base of the cylinder below the surface of the fluid at a time t.

$$\therefore \frac{d^2x}{dt^2} = \rho(x-a) \qquad \begin{array}{l}\rho = \text{density of the fluid} \\ 2a = \text{height of the cylinder}\end{array}$$

$$\therefore 2\frac{dx}{dt}\frac{d^2x}{dt^2} = 2\rho x\frac{dx}{dt} - 2a\rho\cdot\frac{dx}{dt}$$

$$\left(\frac{dx}{dt}\right)^2 = \rho(x^2-2ax) + C.$$

At the commencement of motion $\frac{dx}{dt} = 0$. $x = 2a$.

$$\therefore 0 = \rho \times 0 + C \quad \therefore C = 0$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \rho(x^2-2ax).$$

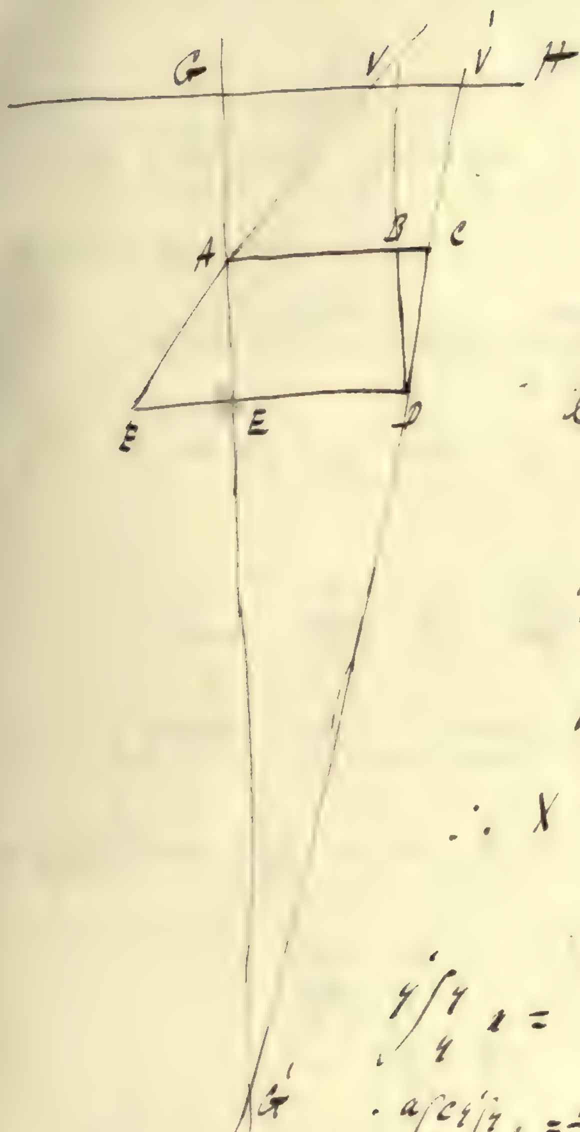
when the cylinder reaches its highest position

$$\frac{dx}{dt} = 0 \qquad \therefore 0 = x^2 - 2ax$$

$$\therefore x = 0 \text{ or } x = 2a.$$

wherefore in its highest position the base of the cylinder will be level with the surface of the fluid —

l. 81



Take GH for the axis of y.
GE vertical for the axis of x.
Divide the trapezoid into the □ AD
whose sides are ∥ to the axis of x and y.
and the △'s AE'E. BCD.

~~let y = mx be eqn to AE.~~

~~$y = -\frac{b}{a}x + b$ = eqn to CD~~

$$y = -\frac{GV}{GA}x + GV = -\frac{b}{a}x + b = \text{eqn to } AE$$

$$y' = -\frac{GV'}{GG'}x + GV' = -\frac{b'}{a'}x + b' = \text{eqn to } CD.$$

$$\therefore X = \frac{\int_a^c \int_y^{y'} x^2}{\int_a^c \int_y^{y'} x} \cdot \quad Y = \frac{\int_a^c \int_y^{y'} xy}{\int_a^c \int_y^{y'} x}$$

$$\int_y^{y'} x = x(y'-y) = x^2\left(\frac{b}{a} - \frac{b'}{a'}\right) + x(b'-b).$$

$$\therefore \int_a^c \int_y^{y'} x = \frac{1}{3}(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + \frac{1}{2}(c^2-a^2)(b'-b)$$

Again $\int_y^{y'} x^2 = x^3\left(\frac{b}{a} - \frac{b'}{a'}\right) + x^2(b'-b).$

$$\therefore \int_a^c \int_y^{y'} x^2 = \frac{1}{4}(c^4-a^4)\left(\frac{b}{a} - \frac{b'}{a'}\right) + \frac{1}{3}(c^3-a^3)(b'-b).$$

$$\int_y^{y'} xy = \frac{1}{2}\left\{x^3\left(\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + 2x^2\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + x(b'^2 - b^2)\right\}.$$

$$\therefore \int_a^c \int_y^{y'} xy = \frac{1}{6}(c^4-a^4)\left(\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + \frac{1}{3}(c^3-a^3)\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + \frac{1}{2}(c^2-a^2)(b'^2-b^2)$$

$$\therefore \bar{X} = \frac{1}{2}\left\{\frac{3(c^4-a^4)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 4(c^3-a^3)(b'-b)}{2(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 3(c^2-a^2)(b'-b)}\right\}.$$

$$\bar{Y} = \frac{1}{2}\left\{\frac{2(c^4-a^4)\left(\frac{b'^2}{a'^2} - \frac{b^2}{a^2}\right) + 4(c^3-a^3)\left(\frac{b^2}{a} - \frac{b'^2}{a'}\right) + 6(c^2-a^2)(b'^2-b^2)}{2(c^3-a^3)\left(\frac{b}{a} - \frac{b'}{a'}\right) + 3(c^2-a^2)(b'-b)}\right\}.$$

Let $\frac{\mu}{r^2}$ = force acting upon Moon at a distance r.

∴ when the mass of the earth is doubled.

$$F = -\frac{h^2}{2} \cdot d_r\left(\frac{1}{p^2}\right) = \frac{2\mu}{r^2}$$

$$\therefore \frac{h^2}{2}\frac{1}{p^2} = \frac{2\mu}{r} + C$$

at the time the mass of the earth was doubled $p = r = \rho$

$$\therefore \frac{h^2}{2\rho^2} = \frac{2\mu}{\rho} + C$$

$$\therefore C = \frac{h^2\rho - 4\mu\rho^2}{2r^2\rho} = \frac{h^2 - 4\mu\rho}{2\rho^2}$$

$$\times \quad \therefore \frac{h^2}{2}\frac{1}{p^2} = \frac{2\mu}{r} + \frac{h^2\rho - 4\mu\rho^2}{2\rho^3\rho}$$

$$\frac{r^2\rho h^2}{h^2\rho - 4\mu r^2}\frac{1}{p^2} = 4\mu \quad \times$$

$$\times \therefore \frac{h^2}{2} \cdot \frac{1}{p^2} = \frac{2\mu}{r} + \frac{h^2 - 4\mu\rho}{2\rho^2}.$$

$$\therefore \frac{\rho^2 h^2}{h^2 - 4\mu\rho}\frac{1}{p^2} = \frac{2\rho}{h^2 - 4\mu\rho} \cdot \frac{2\mu}{r} + 1.$$

$$\therefore \frac{\rho^2 h^2}{4\mu\rho - h^2}\frac{1}{p^2} = \frac{4\mu\rho}{4\mu\rho - h^2} \cdot \frac{1}{r} - 1$$

$$\therefore 2a = \frac{4\mu\rho}{4\mu\rho - h^2} \qquad b^2 = \frac{4\mu\rho - h^2}{\rho^2 h^2}$$

∴ & can be det'd

Let T. t. be the times of oscillations

$\therefore T = \pi\sqrt{\frac{L}{g}}, \quad t = \pi\sqrt{\frac{l}{g}}.$

also $nT = (n \pm 1)t$.

$\therefore n^2 L = (n \pm 1)^2 . l$

$\therefore l = \frac{L . n^2}{(n \pm 1)^2}.$

The eq^n to the hyp referred to its asymptotes will be

$x^2 - y^2 = a^2$ (1)

also suppose the force to act ∥ to the axis of x.

$\therefore \frac{d^2y}{dt^2} = 0$ (2) $\frac{d^2x}{dt^2} = -P$ (3)

From (1) $x\frac{dx}{dt} - y\frac{dy}{dt} = 0$ (4)

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 - y\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)^2 = 0.$$

or $Px + \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 = 0$ for $\frac{d^2y}{dt^2} = 0$.

From (4) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \frac{x^2}{y^2}\cdot\left(\frac{dx}{dt}\right)^2.$$

$$\text{or } \left(\frac{dx}{dt}\right)^2 = \frac{y^2}{x^2}\cdot\left(\frac{dy}{dt}\right)^2.$$

From (2) $\frac{d^2y}{dt^2} = 0 \quad \therefore \frac{dy}{dt} = c$

$$\therefore \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 = \frac{y^2 - x^2}{x^2}\left(\frac{dy}{dt}\right)^2$$

$$= (y^2 - x^2)\frac{c^2}{x^2}.$$

$$\therefore Px + \frac{(y^2 - x^2)c^2}{x^2} = 0.$$

$$\text{or } Px^3 = a^2c^2.$$

$$P = \frac{a^2c^2}{x^3} \therefore = \frac{\mu}{x^3}.$$

Diff^l Calculus.

Transform $\int V.\,dx_1\,dx_2\,dx_3\,dx_4$ to polar coords.

$$x_1 = r\sin\theta\cos\phi = s\cos\phi$$

$$x_2 = r\sin\theta.\sin\phi = s\sin\phi.$$

$$x_3 = r\cos\theta\cos\psi = u\cos\psi$$

$$x_4 = r\cos\theta\sin\psi = u\sin\psi.$$

$$dx_1\,dx_2 = \left(\frac{dx_1}{s}\frac{dx_2}{\phi} - \frac{dx_2}{s}\frac{dx_1}{\phi}\right) ds\,d\phi$$

$$= s\,ds.\,d\phi.$$

$$dx_3\,dx_4 = \frac{dx_3}{u}\frac{dx_4}{\psi} - \frac{dx_4}{u}\frac{dx_3}{\psi}$$

$$= u\,du\,d\psi.$$

$$\therefore\ dx_1\,dx_2\,dx_3\,dx_4 = su.\,du\,ds\,d\phi\,d\psi.$$

But $u = r\cos\theta$

$s = r\sin\theta$.

$$du\,ds = \frac{du}{r}\frac{ds}{\theta} - \frac{du}{\theta}\frac{ds}{r}$$

$$= r\,d\theta\,dr.$$

$$\therefore\ dx_1\,dx_2\,dx_3\,dx_4 = sur.\,dr\,d\theta\,d\phi\,d\psi$$

$$= r^3\sin\theta\cos\theta.\,dr\,d\theta\,d\phi\,d\psi.$$

$$\therefore \iiiint V.\,dx_1\,dx_2\,dx_3\,dx_4 = \iiiint V r^3\sin\theta\cos\theta.\,dr\,d\theta\,d\phi\,d\psi.$$

Trace and find the area of the curve

$r = a\cos\theta + b.$

$r^2 = a^2\cos^2\theta + 2ab\cos\theta + b^2.$

$A = \frac{1}{2}\int_\theta r^2$

$= \frac{1}{2}\int_\theta \left\{ \frac{a^2}{2}(1+\cos 2\theta) + 2ab\cos\theta + b^2 \right\}.$

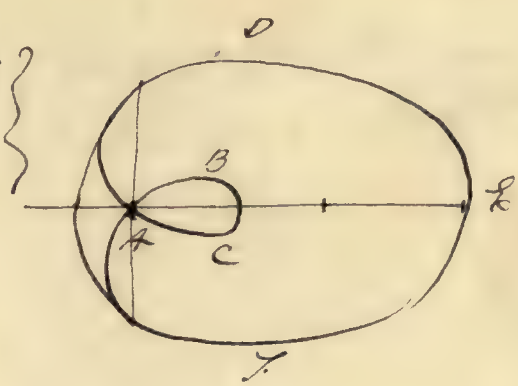
$= \frac{1}{2}\left\{ \left(\frac{a^2}{2}+b^2\right)\theta + \frac{a^2}{4}\sin 2\theta + 2ab\sin\theta \right\} + C$

Take double of the value of this integral from $\theta = 0$ to $\theta = \cos^{-1} - \frac{b}{a}$.

$= \frac{1}{2}\left\{ (a^2+2b^2)\cos^{-1}\frac{b}{a} - \frac{a^2}{a}\cdot\frac{b}{a}\sqrt{1-\frac{b^2}{a^2}} + 4ab\cdot\sqrt{1-\frac{b^2}{a^2}} \right\}$

$= \frac{1}{2}\left\{ (a^2+2b^2)\cos^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} + 4b\sqrt{a^2-b^2} \right\}$

$= \frac{1}{2}\left\{ (a^2+2b^2)\cos^{-1}\frac{b}{a} + 3b\sqrt{a^2-b^2} \right\}$



which = area of ADEF.

area of ABC =

$\frac{1}{2}\left\{ (a^2+2b^2)\left(\pi - \cos^{-1}\frac{b}{a}\right) - 3b\cdot\sqrt{a^2-b^2} \right\}.$

Determine whether the =ⁿ $4y^2+5xy+x^2+2y+3x+4=0$ has asymptotes, and find the =ⁿ to them.

$$4y^2+5xy+x^2+2y+3x+4=0.$$

$$y^2+\frac{1}{4}(2+5x)y+\frac{1}{64}\{2+5x\}^2 = \frac{1}{64}\{2+5x\}^2-\frac{1}{64}\{64+48x+16x^2\}$$

$$= \frac{1}{64}\{4+20x+25x^2-64-48x-16x^2\}.$$

$$y+\frac{1}{8}\cdot(2+5x) = \frac{1}{8}\{9x^2-28x-60\}^{\frac{1}{2}}$$

$$=\frac{3x}{8}\left\{1-\frac{28}{9}\frac{1}{x}-\frac{60}{9x^2}\right\}^{\frac{1}{2}}.$$

$$=\frac{3x}{8}\left\{1-\frac{1.28}{2.9}\cdot\frac{1}{x}+\alpha\right\}.$$

$$=\frac{3x}{8}-\frac{7}{12}+\alpha.$$

$$y=\pm\left\{\frac{1}{4}x-\frac{29}{24}\right\}.$$ is the =ⁿ to the asymptote.

The fact of the given curve having asymptotes might have been deduced from $b^2-4ac = 25-16 = 9$ a positive integer or from the curve's depending on x.

Trace the curve whose eqn is $5y^2+3xy+2x^2+5y+6x+7=0$.

Here $b^2-4ac = 9-40 = -31$.

$$f(h,k) = f + \frac{ae^2-bed+cd^2}{-31} = \frac{5.36-3.6.5+2.25}{-31}+7 = -\frac{357}{31}.$$

∴ A and C are of the same sign and diff. from $f(h,k)$

or the curve is an ellipse

$$h = \frac{2ae-bd}{-31} = \frac{2.5.6-3.5}{-31} = -\frac{15}{31}$$

$$k = \frac{2cd-be}{-31} = \frac{2.2.5-3.6}{-31} = \frac{16}{31}.$$

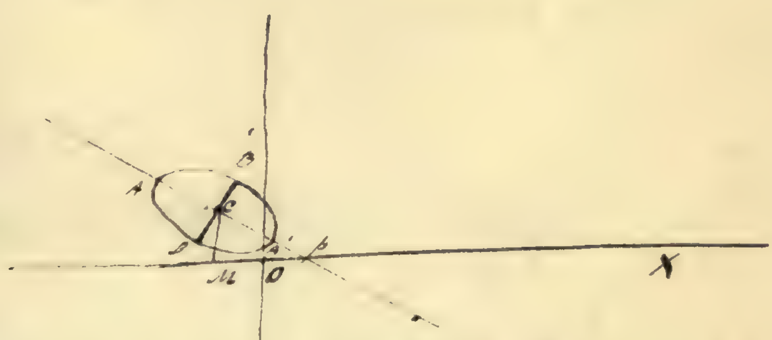
$$\tan 2\theta = \frac{-b}{a-c} = \frac{-3}{3} = -1 \quad \text{or } \theta = 157\tfrac{1}{2}.$$

$$A = \tfrac{1}{2}\left\{a+c+\sqrt{(a-c)^2+m}\right\} = \tfrac{1}{2}\left\{7+\sqrt{68}\right\}.$$

$$B = \tfrac{1}{2}\left\{a+c\mp\sqrt{(a-c)^2+m}\right\} = \tfrac{1}{2}\left\{7-\sqrt{68}\right\}.$$

∴ $Ay^2+Bx^2 = -f(h,k)$ becomes

$$\frac{31(7+\sqrt{68})}{714}y^2 + \frac{31(7-\sqrt{68})}{714}x^2 = 1$$



The curve is represented by the fig. where O is the original, C the new origin of co-ord^s. $OM = -\frac{15}{31}$, $MC = \frac{16}{31}$. $\angle CPX = 157\tfrac{1}{2}$.

$CA' = \left\{\frac{31(7+\sqrt{68})}{714}\right\}^{-\frac{1}{2}}$. $CB' = \left\{\frac{714}{31(7-\sqrt{68})}\right\}^{\frac{1}{2}}$. the semi-axes.

$u = (a+bx^n)^m = y^m$ suppose.

$\therefore \frac{du}{x} = my^{m-1}\frac{dy}{x}$.

$= m.(a+bx^n)^{m-1}.\, nbx^{n-1}$

$= mnb.\, x^{n-1}.(a+bx^n)^{m-1}$.

$\frac{du^2}{x} = mnb\{(n-1)x^{n-2}.(a+bx^n)^{m-1} + (m-1)x^{2n-2}.bn.(a+bx^n)^{m-2}$

$= mnb.\, x^{n-2}.(a+bx^n)^{m-2}\{(n-1)(a+bx^n) + (m-1)bnx^n$.

$= \text{———— ————} \{a(n-1) - bx^n + mn.bx^n\}$.

$= \text{— —} \qquad \{(n-1)a + (mn-1)bx^n.\}$

$d^n_x.(a+bx^n)^m = mnb.\, d^{n-1}_x\{x^{n-1}.(a+bx^n)^{m-1}\}$.

$= mnb\{(n-1)(n-2) — 3.2.1.(a+bx^n)^{m-1}$

$+ \quad n(n-1) — 3.2.1.(m-1)nbx^n.(a+bx^n)^{m-2}\}$.

$+\{.\frac{n.(n-1)^2}{12.}(n-2) - 3.(m-1)(m-2)n^2b^2.x^{2n}.(a+bx^n)^{m-3}\}$

$\therefore$ &c.

$\cdot)$.

$$u = \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} = y^{\frac{1}{2}}.$$

$$\frac{du}{x} = \frac{1}{2} y \cdot \frac{dy}{x} = \frac{1}{2}\{x + (1+x^2)^{\frac{1}{2}}\}^{-\frac{1}{2}}\{1 + \frac{x}{(1+x^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\left\{ \frac{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}}} \right..$$

$$d^r_x u = \frac{1}{2} d^{r-1}_x \left\{ x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} . (1+x^2)^{-\frac{1}{2}} \right\}.$$

$$= \frac{1}{2} . d^{r-1}_x . (1+x^2)^{-\frac{1}{2}} . (x + (1+x^2)^{\frac{1}{2}})^{\frac{1}{2}} + \alpha.$$

$$u' = (1+x^2)^{-\frac{1}{2}}.$$

$$du'_x = -x . (1+x^2)^{-\frac{3}{2}}.$$

$$d^2_x u' = -(1+x^2)^{-\frac{3}{2}} + 3x^2 . (1+x^2)^{-\frac{5}{2}}.$$

$$= -(1+x^2)^{-\frac{3}{2}} + 3.(1+x^2)^{-\frac{3}{2}} - 3.(1+x^2)^{-\frac{5}{2}}.$$

$$= 2.(1+x^2)^{-\frac{3}{2}} - 3.(1+x^2)^{-\frac{5}{2}}.$$

$$d^3_x u' = -2.3.x.(1+x^2)^{-\frac{5}{2}} + 3.5.x.(1+x^2)^{-\frac{7}{2}}.$$

$$d^4_x u' = -2.3.(1+x^2)^{-\frac{5}{2}} + 2.3.5.(1+x^2)^{-\frac{5}{2}} - 2.3.5.(1+x^2)^{-\frac{7}{2}}$$

$$+ 3.5.(1+x^2)^{-\frac{7}{2}} - 3.5.7.(1+x^2)^{-\frac{7}{2}} + 3.5.7.(1+x^2)^{-\frac{9}{2}}.$$

$$= 2.3.4.(1+x^2)^{-\frac{5}{2}} - 3.5.8.(1+x^2)^{-\frac{7}{2}} + 3.5.7.(1+x^2)^{-\frac{9}{2}}.$$

$$d^5_x u' = -3.5.x\{2.4.(1+x^2)^{-\frac{7}{2}} - 8.7.(1+x^2)^{-\frac{9}{2}} + 7.9.(1+x^2)^{-\frac{11}{2}}.$$

$$u = \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

$$\frac{du}{dx} = \frac{1}{2}\{x + (1+x^2)\}^{-\frac{1}{2}} \cdot \left\{1 + \frac{x}{(1+x^2)^{\frac{1}{2}}}\right\} = \frac{1}{2} \cdot \frac{(x + (1+x^2)^{\frac{1}{2}})^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}}}$$

$$\frac{d^2u}{dx^2} = \frac{1}{2}\frac{d}{dx} \cdot (x + (1+x^2)^{\frac{1}{2}})^{\frac{1}{2}} \cdot (1+x^2)^{-\frac{1}{2}}.$$

$$= \frac{1}{2} \cdot \{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}\left\{\frac{(1+x^2)^{-1}}{2} - \frac{3x}{(1+x^2)^{\frac{3}{2}}}\right.$$

$$= \frac{1}{2} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}} \left\{\frac{(1+x^2)^{\frac{1}{2}}}{2} - 3x\right\}.$$

$$= \frac{1}{2} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}}{(1+x^2)^{\frac{3}{2}}} \left\{\frac{(1+x^2)^{\frac{1}{2}} + x - 7x}{2}\right\}$$

$$= \frac{1}{4} \cdot \frac{\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{3}{2}}}{(1+x^2)^{\frac{3}{2}}} - \frac{7}{4} x \frac{(x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}.$$

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$$u = \epsilon^{x^n}.$$

$$d_x u = \epsilon^{x^n}\left\{ n x^{n-1} \right\}.$$

$$d_x^2 u = \epsilon^{x^n}\left\{ n.(n-1)x^{n-2} + n^2.x^{2(n-1)} \right\}.$$

$$d_x^3 u = \epsilon^{x^n}\left\{ n.(n-1)(n-2)x^{n-3} + 2.n^2.(n-1)x^{2n-3} \right.$$
$$\left. + n^2.(n-1)x^{2n-3} + n^3.x^{3.(n-1)} \right.$$

$$= \epsilon^{x^n}\left\{ n.(n-1)(n-2)x^{n-3} + 3.n^2.(n-1)x^{2n-3} + n^3.x^{3(n-1)}. \right\}.$$

$$d_x^4 u = \epsilon^{x^n}\left\{ n.(n-1)(n-2)(n-3)x^{n-4} + 3.n^2.(n-1)(2n-3)x^{2n-4} + 3n^3.(n-1)x^{3n-4} \right.$$
$$\left. + n^2.(n-1)(n-2)x^{2n-4} + 3n^3.(n-1)x^{3n-4} + n^4.x^{4(n-1)}. \right.$$

$$= \epsilon^{x^n}\left\{ n.(n-1)(n-2)(n-3)x^{n-4} + n^2.(n-1)(7n-11).x^{2n-4} \right.$$
$$\left. + 6.n^3.(n-1)x^{3n-4} + n^4.x^{4(n-1)}. \right.$$

$$d_x^r u = \epsilon^{x^n}\left\{ n.(n-1) - (n-r+1)x^{n-r} + \&c \text{ ——— } \right.$$
$$\left. \text{—— –} \quad + n^r.x^{r(n-1)} \right..$$

$$u = \varepsilon^{\sin x}$$

$$\frac{du}{x} = \varepsilon^{\sin x}\{\cos x\}.$$

$$\frac{d^2u}{x} = \varepsilon^{\sin x}\{-\sin x + \cos^2 x\} = \varepsilon^{\sin x}.\{-\sin x + \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\}.$$

$$\frac{du^2}{x} = \varepsilon^{\sin x}\{-\cos x - \sin 2x - \tfrac{1}{2}\sin 2x + \tfrac{1}{2}\cos 2x \cos x + \tfrac{1}{2}\cos x\}$$

$$= \varepsilon^{\sin x}\{-\tfrac{1}{2}\cos x + \tfrac{3}{2}\sin 2x + \tfrac{1}{2}\cos 2x \cos x\}.$$

$$u = \log.\{x + (1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

$$\frac{du}{x} = \frac{\left(x + (1+x^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}}{2\,(1+x^2)^{\frac{1}{2}}} \div \left(x + (1+x^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

$$= \frac{1}{2}.\ \frac{1}{(1+x^2)^{\frac{1}{2}}}.$$

$u = \log.(\log x). = \log y$ suppose.

$$\therefore \frac{du}{x} = \frac{\frac{dy}{x}}{y} = \frac{1}{x \log x}.$$

$$\therefore \int_x \frac{1}{x}.\log x^{-1} = \int_x (\log x)^{-1}.d_x \log x = \frac{d_x.\log_\varepsilon x}{\frac{\log x}{\varepsilon}} = \log\{\tfrac{\log x}{\varepsilon}\}.$$

$$= \frac{\log x}{\log x} - \int_x x \log x.$$

$$u = \log^n x = \log.\log^{n-1} x = \log y.$$

$$\therefore\ d_x u = \frac{d_x y}{y} = \frac{d_x . \log^{n-1} x}{\log^{n-1} x} = \frac{1}{x \log x \log^2 x \ldots \log^{n-1} x}.$$

$$\therefore \int_x \frac{1}{x \log x \log^2 x \log^3 x} = \int_x \frac{d_x \log x}{\log x} \cdot \frac{1}{\log^2 x \log^3 x}.$$

$$= \int_x \frac{d_x(\log^3 x)}{\log^3 x} = \log.(\log^3 x) = \log^4(x).$$

$$u = \log.(\sin x) \quad \therefore\ d_x u = \frac{d_x \sin x}{\sin x} = \cot x.$$

$$\therefore \int_x \cot x = \int_x \frac{\cos x}{\sin x} = \int_x \frac{d_x \sin x}{\sin x} = \log.(\sin x).$$

$$u = \log \cos x \quad \therefore\ d_x u = -\frac{\sin x}{\cos x} = -\tan x.$$

$$\therefore \int_x \tan x = \int_x \frac{\sin x}{\cos x} = -\int_x \frac{d_x \cos x}{\cos x} = -\log.(\tan x).$$

$$\int_x \frac{m}{\sin mx} = \int_x \frac{m}{2\sin\frac{1}{2}mx\cos\frac{1}{2}mx}$$

$$= \int_x \frac{m\sec^2\frac{1}{2}mx}{2\tan\frac{1}{2}mx} = \int_{\tan\frac{1}{2}mx} \frac{1}{\tan\frac{1}{2}mx}$$

$$= \log_c \tan\frac{1}{2}mx = \log_c\left\{\frac{\sin\frac{1}{2}mx}{\cos\frac{1}{2}mx}\right\}.$$

$$= \log_c\left(\frac{1-\cos mx}{1+\cos mx}\right)^{\frac{1}{2}}.$$

V. $$u = \log_c\left(\frac{1-\cos mx}{1+\cos mx}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_c(1-\cos mx) - \frac{1}{2}\log_c(1+\cos mx)$$

$$= \frac{m}{2}\left\{\frac{\sin mx}{1-\cos mx} + \frac{\sin mx}{1+\cos mx}\right\}.$$

$$= \frac{m\sin mx}{1-\cos^2 mx} = \frac{m}{\sin mx}.$$

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$u = \log(\tan x)$

$$\frac{du}{dx} = \frac{\sec^2 x}{\tan x} = \frac{1}{\cos^2 x}\cdot\frac{\cos x}{\sin x} = \frac{1}{\sin x \cos x}.$$

$$\therefore \int_x \frac{1}{\sin x\cos x} = \int_x \frac{\sec^2 x}{\tan x} = \int_{\tan x} \frac{1}{\tan x} = \log.(\tan x).$$

$u = \text{Cos.}(\sin x). \quad = \cos y$

$$\frac{du}{dx} = -\sin y \frac{dy}{dx} = -\sin.(\sin x)\cos x.$$

$$\int_x -\sin(\sin x)\cos x = \int_x -\sin(\sin x)\, d_x \sin x.$$

$$= \text{Cos.}(\sin x).$$

$u = \sin.(\log x).$

$$\frac{du}{dx} = \frac{\text{Cos.}(\log x)}{x}.$$

$$\int_x \text{Cos.}(\log x)\frac{1}{x} = \int_x \text{Cos.}(\log x).\, d_x \log x$$

$$= \int_{\log x} \text{Cos.}(\log x) = \sin.(\log x).$$

$$u = \sin^{-1}\frac{x}{(1+x^2)^{\frac{1}{2}}} = \sin^{-1} y.$$

$$\frac{du}{dx} = \frac{\frac{dy}{dx}}{\sqrt{1-y^2}} = +\frac{1}{(1+x^2)^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}$$

$$= +\frac{1}{1+x^2}$$

$$\int_x \frac{1}{1+x^2} = \tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}.$$

(7). $u = \sin^{-1}\frac{1-x^2}{1+x^2}.$

$$\frac{du}{dx} = \frac{\frac{-2x(1+x^2)-(1-x^2)2x}{(1+x^2)^2}}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{1+x^2}}} = \frac{\frac{-4x}{(1+x^2)^2}}{\frac{2x}{1+x^2}}$$

$$= \frac{-2}{1+x^2}.$$

$$\therefore \int_x \frac{-2}{1+x^2} = 2\tan^{-1} x$$

$$= \quad 1+$$

$\therefore d^r w =$

$m(m-1) - (m-r+1)\cdot(a+x)^{m-r}\cdot(c+x)^{-n}\Big\{1 - \frac{r.n.}{m-r+1}\ \frac{a+x}{c+x}$

$d^n \log x.$

$d^r_x w = n.(n-1)\cdot(n-r+1)x^{n-r}.\log x\Big\{.1 + \frac{r}{n-r+1}$

$= n.(n-1) - (n-r+1)x^{n-r}\Big\}\log x + \frac{r}{n-r+1}$

$\Big\{\log x + \frac{r}{n-r+1} - \frac{r.(r-1)}{1\cdot 2.}\ \frac{1}{(n-r+1)(n-r+2)}$

$w = \varepsilon^{ax}.\ x^{n.}$

$d^r_x w = \varepsilon^{ax}\Big\{a^r.x^n + rn.a^{r-1}x^{n-1}. + \frac{r(r-1)}{1\,2.}n(n-1).a^{r-2}x^{n-2} \propto\Big\}.$

$d^n_x w.\ \varepsilon^{ax} x^r = \varepsilon^{ax}\Big\{a^n x^r + rn.a^{n-1}x^{r-1} + \frac{r(r-1)}{1\,2.}n(n-1)a^{r-2}x^{n-2}.$

$= a^{n-r}.x^{r-n}$

$\therefore d^n_x\, \varepsilon^{ax} x^r = a^{n-r}\cdot x^{r-n}\cdot d^r_x.\varepsilon^{ax}x^n.$

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[illegible] $=$ [illegible] $+ \frac{[illegible]}{\sqrt{(n-1)}}$ [illegible]

[illegible] $=$ [illegible] $+ \frac{[illegible]}{\sqrt{n-1}}$ [illegible]

[illegible] $=$ [illegible]

[illegible] $+ \frac{n-k+1}{k} - \frac{[illegible]}{\sqrt{n-1}}$ [illegible]

$= n(n-1)$ [illegible] $- (n-k+1)$ [illegible] $+ \frac{n-k+1}{k}$

$=$ [illegible] $+ \frac{n-k+1}{k}$

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[illegible] $= \frac{n-k+1}{k} \cdot \frac{[illegible]}{[illegible]}$

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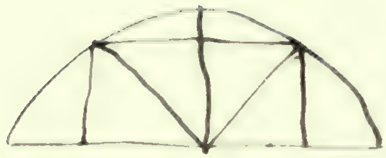
$w = \varepsilon^{ax} \cdot \cos nx \cdot x^{m}$

$d^{r}_{x} w = (a^2+n^2)^{\frac{r}{2}} \cdot \cos\left(nx + \frac{r\varphi}{2}\right) x^{m}$



$d = m(m-1) \ldots (m-r+1) x^{m-r} \cdot \varepsilon^{ax} \left\{ \cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos\left(nx + \frac{\varphi}{2}\right) \right.$

$\cos\left(nx + \varphi + \frac{\pi}{2}\right)$

$\cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos\left(nx + \varphi + \frac{\pi}{2}\right) + \frac{r(r-1)}{1 \cdot 2} \frac{a^2+n^2}{(m-r+1)(m-r+2)}$

$\cos nx +$

$\cos nx + \frac{r}{1} \cdot \frac{(a^2+n^2)^{\frac{1}{2}}}{m-r+1} \cdot \cos(nx+\varphi) + \cdot$

$= (a^2+n^2)^{\frac{r}{2}} \cdot \cos(nx + r\varphi) x^{m}$

$w = \varepsilon^{ax} \cdot \cos nx \cdot x^{m}$.

$d_x w = \varepsilon^{ax} \cdot (a^2+n^2)^{\frac{r}{2}} \cdot x^{m} \left\{ \cos(nx + r\varphi) + \frac{r}{x} \cdot \frac{m}{(a^2+n^2)^{\frac{1}{2}}} \cos \frac{\quad}{x} \right)$

$\cdot d_r \cdot \varepsilon^{ax} \cdot \cos nx \cdot \sin x$

$= \varepsilon^{ax} \cdot (a^2+n^2)^{\frac{r}{2}} \left\{ \cos(nx + r\varphi) \cdot \sin x + \frac{r}{\sqrt{a^2+n^2}} \cos\{nx + (r-1)\varphi\} \cos x \right.$

$- \frac{r \cdot (r-1)}{1 \cdot 2 (a^2+n^2)} \cdot \cos\{nx + (r-2)\varphi\} \sin x$

$u = \sin nx.$

$\frac{du}{dx} = n\cos nx. = n\sin\left(nx + \frac{\pi}{2}\right).$

$\frac{d^2u}{dx^2} = n^2\cos\left(nx + \frac{\pi}{2}\right) = n^2\sin\left(nx + \pi\right).$

$\therefore \frac{d^ru}{dx^r} = n^r \sin \frac{r\pi}{2}$

$= n^r \sin\left(nx + \frac{r\pi}{2}\right).$

$u = \cos nx.$

$\frac{du}{dx} = -n\sin(nx) = n\cos\left(nx + \frac{\pi}{2}\right)$

$\frac{d^2u}{dx^2} = -n^2\sin\left(nx + \frac{\pi}{2}\right) = n^2\cos\left(nx + \pi\right).$

$\therefore \frac{d^ru}{dx^r} = n^r. \cos\left(nx + r\frac{\pi}{2}\right).$

(21).

$$u = \frac{x}{a^2+x^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x}{x+a\sqrt{-1}} - \frac{x}{x-a\sqrt{-1}}\right\}.$$

$$= -\frac{1}{2a\sqrt{-1}}\left\{1 - \frac{a\sqrt{-1}}{x+a\sqrt{-1}} - 1 - \frac{a\sqrt{-1}}{x-a\sqrt{-1}}\right\}.$$

$$= +\frac{1}{2}\left\{\frac{1}{x+a\sqrt{-1}} + \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$\therefore d^r_x u = \frac{r(r-1).\ 3.2.1}{2}\left\{\frac{(x+a\sqrt{-1})^{r+1} + (x-a\sqrt{-1})^{r+1}}{(a^2+x^2)^{r+1}}\right\}.$$

Let $\theta = \tan^{-1}\frac{a}{x}$ $\therefore x = \sqrt{a^2+x^2}\cos\theta$. $a = \sqrt{a^2+x^2}.\sin\theta$

$$\therefore d^r_x u = \frac{(-1)^r r(r-1).\ 3.2.1}{2}.\frac{(a^2+x^2)^{\frac{r+1}{2}}}{(a^2+x^2)^{r+1}}.\cos(r+1)\theta.$$

$$= (-1)^r.\ \frac{r.(r-1).\ 3.2.1}{2}.\frac{\cos(r+1)\theta}{(a^2+x^2)^{\frac{r+1}{2}}}.$$

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$$u = \frac{x}{a^2+x^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x}{x+a\sqrt{-1}} - \frac{x}{x-a\sqrt{-1}}\right\}.$$

$$d_x u = -\frac{1}{2a\sqrt{-1}}\left\{\frac{x+a\sqrt{-1}}{x+a\sqrt{-1}} \cdot \frac{-a\sqrt{-1}}{x+a\sqrt{-1}} - \frac{x-a\sqrt{-1}}{x-a\sqrt{-1}} \cdot \frac{-a\sqrt{-1}}{x-a\sqrt{-1}}\right\}.$$

$$= -\frac{1}{2a\sqrt{-1}}\left(1 - \frac{a\sqrt{-1}}{x+a\sqrt{-1}} - 1 - \frac{a\sqrt{-1}}{x-a\sqrt{-1}}\right).$$

$$= +\frac{1}{2}\cdot\left(\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right).$$

$$\therefore d_x^r u = \frac{(-1)^{r+1} \cdot r(r-1) \ldots 3.2.1 \sin (r+1)\theta}{(a^2+x^2)^{\frac{r+1}{2}}}.$$

$$d_x^r . \frac{1}{a^2+x^2} = (-1)^r . \frac{r(r-1) \ldots 3.2.1 \sin (r+1)\theta}{a.(a^2+x^2)^{\frac{r+1}{2}}}.$$

$$d_x^r . \frac{x}{a^2+x^2} = (-1)^r . \frac{r(r-1) \ldots 3.2.1. \cos (r+1)\theta}{(a^2+x^2)^{\frac{r+1}{2}}}.$$

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$\frac{1}{[illegible]} - \frac{1}{[illegible]} \Big) \cdot \frac{1}{5} + =$

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$$u^n = \frac{1}{(a^2+x^2)} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$d^r_x u^n = ~~\text{[struck out]}~~$$

$$= (-1)^{r+1}\cdot\frac{r\cdot(r-1)\cdots 3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{1}{(x+a\sqrt{-1})^{r+1}} - \frac{1}{(x-a\sqrt{-1})^{r+1}}\right\}$$

$$= (-1)^{r+1}\cdot\frac{r(r-1)\cdots 3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{(x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1}}{(x^2+a^2)^{r+1}}\right\}$$

Let $\tan\theta = \dfrac{a}{x}$.

$$\therefore \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{x}{\sqrt{a^2+x^2}}$$

$$\therefore x = \cos\theta\sqrt{a^2+x^2}.$$

$$\therefore a = \sin\theta\sqrt{a^2+x^2}.$$

$$\therefore (x-a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}\left\{\cos(r+1)\theta - \sqrt{-1}\sin(r+1)\theta\right\}.$$

$$(x+a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}\cdot\left\{\cos(r+1)\theta + \sqrt{-1}\sin\cdot(r+1)\theta\right\}.$$

$$\therefore (x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1} =$$

$$-2\sqrt{-1}(a^2+x^2)^{\frac{r+1}{2}}.\sin(r+1)\theta.$$

$$\therefore d^r_x u^n = \frac{r\cdot(r-1)\cdots 3\cdot 2\cdot 1}{a(a^2+x^2)^{\frac{r+1}{2}}}(-1)^r\cdot\sin\cdot(r+1)\theta.$$

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$$u^n = (a^2+x^2)^{-n}.$$

$$\therefore d_x^r u^n = n(n+1) - (n+r-1)(a^2+x^2)^{n+r}(2x)^{-r}\Big\}.$$

$$1 + \frac{r(r+1)}{n+r-1}\frac{(2x)^2}{(a^2+x^2)}$$

$$= n.(n+1) - (n+r-1).(a^2+x^2)^{n+r}.(2x)^{r}.$$

$$-\frac{1}{2a\sqrt{-1}}\Big\{\frac{1}{u+a\sqrt{-1}} - \frac{1}{u-a\sqrt{-1}}.$$

$$= \frac{u-a\sqrt{-1} - u - a\sqrt{-1}}{u^2+a^2}$$

$$\therefore \frac{1}{u^2+a^2} = -\frac{1}{2a\sqrt{-1}}\Big\{\frac{1}{u+a\sqrt{-1}} - \frac{1}{u-a\sqrt{-1}}.\Big\}.$$

$$\frac{1}{u+a\sqrt{-1}} = (u+a\sqrt{-1})^{-1}$$

$$d_u = -(u+a\sqrt{-1})^{-2}.$$

$$d_u^2 = 1\cdot 2.\,(u+a\sqrt{-1})^{-3}.$$

$$\therefore d_u^r = r(r-1)\ldots 3\cdot 2\cdot 1.\,(u+a\sqrt{-1})^{-r}.$$

$$\therefore\ u = \frac{1}{x^2+a^2} = -\frac{1}{2a\sqrt{-1}}\left\{\frac{1}{x+a\sqrt{-1}} - \frac{1}{x-a\sqrt{-1}}\right\}.$$

$$\therefore\ d^r u = (-1)^{r+1} \cdot \frac{r(r-1)\ldots 3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{1}{(x+a\sqrt{-1})^{r+1}} - \frac{1}{(x-a\sqrt{-1})^{r+1}}\right\}$$

$$= (-1)^{r+1} \cdot \frac{r(r-1)\ldots 3\cdot 2\cdot 1}{2a\sqrt{-1}}\left\{\frac{(x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1}}{(x^2+a^2)^{\frac{r+1}{2}}}\right\}$$

$$\frac{a}{x} = \tan\theta.$$

$$\therefore \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{x}{\sqrt{a^2+x^2}}.$$

$$\therefore\ x = \sqrt{a^2+x^2}\,.\cos\theta.$$

$$a = \sqrt{a^2+x^2}\,\sin\theta.$$

$$\therefore (x\mp a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}.\{\cos(r+1)\theta \mp \sqrt{-1}\sin(r+1)\theta\}.$$

$$(x+a\sqrt{-1})^{r+1} = (a^2+x^2)^{\frac{r+1}{2}}\{\cos(r+1)\theta + \sqrt{-1}\sin(r+1)\theta\}.$$

$$\therefore (x-a\sqrt{-1})^{r+1} - (x+a\sqrt{-1})^{r+1} =$$

$$(a^2+x^2)^{\frac{r+1}{2}}\ 2\sin(r+1)\theta$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

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[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$u^n = (a^2 + x^2)^n$ Here $u = a^2 + x^2$.

$u' = 2x.$ $c = 1.$

$\therefore\ d^r u^n = n(n-1) - (n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1} \cdots\right.$

$= n(n-1) - (n-r+1)(a^2+x^2)^{n-r}.(2x)^r$

If $n = r$ we have

$d^r u^n = n(n-1) - 3.2.1.\ x^n.\ 2^n.$

$d^r u^n = n(n-1) - (n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1}\cdot\frac{cu}{u'^2} + \frac{r(r-1)(r-2)(r-3)}{(n-r+1)(n-r+2)}\cdot\frac{c^2u^2}{u'^4}\right.$

$\therefore\ (a^2 + 4x + 9x^2)^n.$

Here $u' = 4 + 18x.$

$c = 9.$

$n(n-1)(n-2) \ldots (n-r+1)\left\{1 + \frac{r(r-1)}{n-r+1}\cdot\frac{(a^2+4x+9x^2)9}{(4+18x)^2}\right.$

If $n = r$ we have.

$n(n-1) - 3.2.1\left\{1 + \frac{n(n-1)}{1}\cdot\frac{(a^2+4x+9x^2)9}{4(2+9x)^2}\right.$

$f(u) = \phi(a + bx + cx^2)^n$

$f(u+h) = f(u) + f'(u)h + ch^2)^n$

$= (u + u'h + ch^2)^n$

$= (u+u'h)^n + n(u+u'h)^{n-1} c h^2 + \frac{n(n-1)}{1 \cdot 2}(u+u'h)^{n-2} c^2 h^4 +$

$\therefore$ coeff. of $\frac{h^r}{\lfloor r}$ =

$2 \cdot$

$\frac{n-1}{1} \cdot u^{n-2}\, u'c\, h^3$

$n-5+3$

$n-2 \cdot u^{n-3} u' c^2 h^5$

$\frac{n(n-1) \cdot \cdot (n-r+1)}{\lfloor r} \cdot u^{n-r} \cdot u'^{r} h^r$.

$+ \frac{n \cdot (n-1) - (n-r+2)}{1 \cdot \lfloor r-2} \cdot u^{n-r+1} u'^{r-2} \cdot c \cdot h^r$

$+ \frac{n \cdot (n-1)}{1 \cdot 2} \cdot \frac{(n-2)(n-3) \cdot (n-r+3)}{\lfloor r-4} u^{n-r+2} u'^{r-4} \cdot c^2 \cdot h^r$.

$\therefore$ coeff. of $\frac{h^r}{\lfloor r}$ =

$n(n-1) - (n-r+1) u^{n-r} \cdot u'^{r} \left\{ 1 + \frac{(r-1)r}{n-r+1} \cdot \frac{uc}{u'^2} + \frac{(r-3)(r-2)(r-1)r}{1 \cdot 2(n-r+1)(n-r+2)} \frac{u^2c^2}{u'^4} \right.$

$\left(1 + \frac{u'}{2u}h\right)^2 = 1 + \frac{u'}{u}h + \frac{u'^2}{4u^2}h^2$.

$d_h^r u = n(n-1) - (n-r+1) \cdot u^{n-r} u'^{r} \left\{ 1 + \frac{r \cdot (r-1)}{n-r+1} \frac{cu}{u'^2} \right.$

$+ \frac{r(r-1)(r-2)(r-3)}{(n-r+1)(n-r+2) \cdot 1 \cdot 2} \cdot \frac{c^2u^2}{u'^4}$

$\therefore\ n = \sqrt{a^2+n^2}\,.\sin\varphi\ ;\quad a = \sqrt{a^2+n^2}\,.\cos\varphi\,.$

$\therefore\ d_x u = \varepsilon^{ax}\,.\sqrt{a^2+n^2}\,(\cos\varphi\cos nx - \sin\varphi\sin nx)\,.$

$\therefore\ d_x u = \sqrt{a^2+n^2}\,\{\varepsilon^{ax}\,.\ \cos(nx+\varphi)\}\,.$

$d_x^2 u = a^2+n^2\,.\ \varepsilon^{ax}\,.\ \cos.(nx+2\varphi)\,.$

$d_x^r u = (a^2+n^2)^{\frac{r}{2}}\,.\ \varepsilon^{ax}\,.\ \cos(nx+r\varphi)\,.$

$u = \log x\,.$

$d_x u = \frac{1}{x} = x^{-1}\,.$

$d_x^2 u = -x^{-2}\,.\quad d_x^3 u = 2\,.\,x^{-3}\,.$

$\therefore\ d_x^r u = (-1)^{r-1}\,.\,\underline{(r-1)}\ x^{-r}\,.$

$$u = e^{x\cos\theta}\cos.(x\sin\theta).$$

$$\frac{du}{dx} = e^{x\cos\theta}\{\cos(x\sin\theta)\cos\theta - \sin(x\sin\theta)\sin\theta\}.$$

$$= e^{x\cos\theta}.\cos(\theta + x\sin\theta).$$

$$\frac{d^2u}{dx^2} = e^{x\cos\theta}\{\cos(\theta + x\sin\theta)\cos\theta - \sin(\theta + x\sin\theta)\sin\theta\}.$$

$$= e^{x\cos\theta}.\cos(2\theta + x\sin\theta).$$

$$\therefore \frac{d^r u}{dx^r} = e^{x\cos\theta}.\cos(r\theta + x\sin\theta).$$

$$u = e^{ax}\cos nx.$$

$$\frac{du}{dx} = e^{ax}\{-n\sin nx + a\cos nx\}$$

$$= e^{ax}\{a\cos nx - n\sin nx\}.$$

Let $\frac{n}{a} = \tan\varphi \therefore \sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}} = \frac{n}{\sqrt{a^2+n^2}}$

$$\cos\varphi = \frac{1}{\sqrt{1+\tan^2\varphi}} = \frac{a}{\sqrt{a^2+n^2}}$$

12. $u = \frac{1+x}{1-x} = \frac{1-x}{1-x} + \frac{2x}{1-x} = 1 + \frac{2x}{1-x}$.

$$\frac{du}{x} = \frac{(1-x)2 + 2x}{(1-x)^2} = \frac{2}{(1-x)^2}.$$

$$\therefore d^2_x u = +\frac{2.2}{(1-x)^3} = 2.2.(1-x)^{-3}.$$

$$d^3_x u = 3.2.2.(1-x)^{-4}.$$

$$\therefore d^r_x u = r.(r-1) \ldots 3.2. \; 2.(1-x)^{-(r+1)}$$

$$\frac{d^r(uv)}{x} = v.\frac{d^r u}{x} + \frac{r}{1}\frac{dv}{x}.\frac{d^{r-1}(u)}{x} + \frac{r.(r-1)}{1.2.} d^2_x v . d^{r-2}_x (u).$$

$$d^r_x(uv) = v.d^r_x u + r.\frac{dv}{x}.d^{r-1}_x(u) + \ldots$$

$$\frac{r.(r-1)}{1.2.} d^2_x v . d^{r-2}_x . (u).$$

137. $uv = x^n \cdot (1-x)^n.$

$$\therefore d^r(uv) = n\cdot(n-1)\cdots(n-r+1)x^{n-r}\cdot(1-x)^n$$

$$- n^2\cdot(n-1)\cdots(n-r+2)x^{n-r+1}\cdot r\cdot(1-x)^{n-1}$$

$$+ n^2\cdot(n-1)^2\cdots(n-r+3)^{n-r+2}\cdot\frac{r\cdot(r-1)}{1\cdot 2}(1-x)^{n-2}.$$

&c. &c. —

$u = x^n.$

$\frac{du}{dx} = nx^{n-1}.$

$\frac{d^2u}{dx^2} = n(n-1)x^{n-2}.$

$\frac{d^3u}{dx^3} = n(n-1)(n-2)x^{n-3}.$

$\frac{d^ru}{dx^r} = n(n-1)(n-2) \text{———} (n-r+1)x^{n-r}.$

$u = (a+bx)^n.$

$\frac{du}{dx} = n(a+bx)^{n-1}.b.$

$= nb(a+bx)^{n-1}$

$\frac{d^2u}{dx^2} = n(n-1)(a+bx)^{n-2}.b^2$

$= n(n-1)b^2(a+bx)^{n-2}.$

$\therefore \frac{d^ru}{dx^r} = n(n-1) \text{———} (n-r+1)(a+bx)^{n-r}.b^r.$

$$u = \frac{1}{x^n} = x^{-n}.$$

$$\therefore\ d_x u = -n \cdot x^{-(n+1)}$$

$$d_x^2 u = n \cdot (n+1)\, x^{-(n+2)}.$$

$$d_x^3 u = -n \cdot (n+1)(n+2)\, x^{-(n+3)}.$$

$$\therefore\ d_x^r u = (-1)^r \cdot n \cdot (n+1)(n+2) \ldots (n+r-1) \cdot x^{-(n+r)}.$$

$$u = a^x.$$

$$d_x u = a^x \cdot \log_\varepsilon a.$$

$$d_x^2 u = a^x \cdot (\log_\varepsilon a)^2.$$

$$\therefore\ d_x^r u = a^x \cdot (\log_\varepsilon a)^r.$$

$$u = \varepsilon^{nx}.$$

$$d_x u = n \cdot \varepsilon^{nx}. \qquad d_x^2 u = n^2 \cdot \varepsilon^{nx}.$$

$$\therefore\ d_x^r u = n^r \cdot \varepsilon^{nx}.$$

Find the eq^n to the asymptote of the curve $y^4+2axy^2-x^4=0$ and trace the curve.

$$y^4+2axy^2-x^4=0$$

$$y^4+2axy^2+a^2x^2 = x^4+a^2x^2 = x^2(x^2+a^2)$$

$$y^2+ax = x^2\sqrt{1+\frac{a^2}{x^2}} = x^2\left(1+\frac{a^2}{2x^2}\right) \text{ if } x=\infty$$

$$\therefore y = \pm\sqrt{x^2-\frac{ax}{2}+\frac{a^2}{2}} \qquad = \pm\left(x^2+\frac{a^2}{2}\right)$$

$$= \pm\left(x-\frac{a}{4}\right). \text{ the eq}^n \text{ to a pair of asymptotes.}$$

Also $x=0$ gives $y=0$ and the origin is a point in the curve.

$$(4y^3+4axy)\frac{dy}{dx} = 4x^3-2ay^2 \text{ or } \frac{dy}{dx} = \frac{2x^3-ay^2}{2y^3+2axy} = \frac{0}{0} \text{ at origin}$$

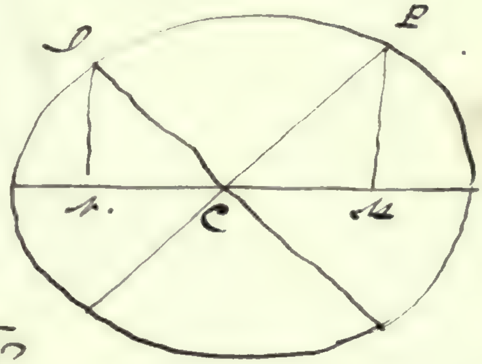
$$p = \frac{d_x\{2x^3-ay^2\}}{d_x(2y^3+2axy)} = \frac{6x^2-2ayp}{6y^2p+2ay+2axp} = \frac{0}{0}$$

$$= \frac{12x-2ap^2}{12p^2y+4ap} = -\frac{p^2}{2p} = -\frac{p}{2}.$$

$$\therefore p = -\frac{p}{2} \text{ or } p=0, p=\infty$$

or the curve at the origin cuts the axes of x & y at right ∠'s

Find those conjugate diameters of an $\odot$ whose sum is a max, and whose sum is a min.



Let CP, CD be semi conjugate diameters.

x, y coords of P ∴ $\frac{ay}{b}$, $\frac{bx}{a}$, are coords of D.

∴ $u = CP + CD = \sqrt{x^2+y^2} + \frac{1}{ab}\sqrt{a^4y^2+b^4x^2}$

$$= \sqrt{b^2+e^2x^2} + \sqrt{a^2-e^2x^2}$$

$$\frac{du}{x} = e^2x\left\{\frac{1}{\sqrt{b^2+e^2x^2}} - \frac{1}{\sqrt{a^2-e^2x^2}}\right\} = 0$$

∴ $e^2x^2 = 0$ and $\frac{1}{\sqrt{b^2+e^2x^2}} - \frac{1}{\sqrt{a^2-e^2x^2}} = 0$

The former gives $x = 0$ and $u = CP + CD = a + b$ a max.

The latter gives $b^2 + e^2x^2 = a^2 - e^2x^2$

$$2e^2x^2 = a^2 - b^2$$

$$(a^2-b^2)\frac{x^2}{a^2} = \frac{a^2-b^2}{2}$$

$$x = \frac{a}{\sqrt{2}}. \quad y^2 = \frac{b^2}{a^2}\left(a^2 - \frac{a^2}{2}\right) = \frac{b^2}{2}.$$

∴ $CP = CD = \frac{\sqrt{a^2+b^2}}{\sqrt{2}}$. or the conjugate diameters are equal to each other for a minimum.

Prove that at the points of greatest and least curv. the contact of the ⊙ of curv. is of the 3rd order.

we have $\rho = -\frac{1}{\frac{d^2y}{dx^2}}\cdot\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$.

$$\therefore \frac{d\rho}{dx} = -\left\{\frac{\left(\frac{d^2y}{dx^2}\right)^2 3.\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}\left(\frac{dy}{dx}\right) - \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}.\frac{d^3y}{dx^3}}{\left(\frac{d^2y}{dx^2}\right)^2}\right\}$$

∴ for a max or min we have.

$$3\frac{dy}{dx}.\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3}.$$

$$\text{or } \frac{d^3y}{dx^3} = \frac{3.\left(\frac{dy}{dx}\right).\left(\frac{d^2y}{dx^2}\right)^2}{1+\left(\frac{dy}{dx}\right)^2}.$$

let $(Y-b)^2+(X-c)^2=c^2$ be the eq^n to the ⊙ of curv.

$$(Y-b)\frac{dY}{dX}+X-c=0.$$

$$(Y-b)\frac{d^2Y}{dX^2}+\left(\frac{dY}{dX}\right)^2+1=0. \quad \therefore Y-b = -\frac{1+\left(\frac{dY}{dX}\right)^2}{\frac{d^2Y}{dX^2}}$$

$$(Y-b)\frac{d^3Y}{dX^3}+3\frac{dY}{dX}.\frac{d^2Y}{dX^2}=0.$$

$$\therefore \frac{d^3Y}{dX^3} = -\frac{3\frac{dY}{dX}.\frac{d^2Y}{dX^2}}{Y-b.} = \frac{3\frac{dY}{dX}\left(\frac{d^2Y}{dX^2}\right)^2}{1+\left(\frac{dY}{dX}\right)^2}.$$

For a contact of the 3rd order. $\frac{dY}{dX}=\frac{dy}{dx}$. $\frac{d^2Y}{dX^2}=\frac{d^2y}{dx^2}$. $\frac{d^3Y}{dX^3}=\frac{d^3y}{dx^3}$.

$$\therefore \frac{d^3y}{dx^3} = \frac{3.\frac{dy}{dx}.\left(\frac{d^2y}{dx^2}\right)^2}{1+\left(\frac{dy}{dx}\right)^2},$$ which is also the condit^n of maximum or minimum curvature.

[illegible]

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Shew that a conic section may be det^d. which shall have a contact of the 4th order with a curve at any point; and if at any point it be a P, at the next preceding and succeeding points it will be an E and H.

Let $y^2 + bx^2 + cxy + dy + ex + f = 0$ be the equation to the section. Since this contains 5 arbitrary constants, we may assign to them such values that at a proposed point $Y = y$; $d_x Y = d_x y$, $d_x^2 Y = d_x^2 y$; $d_x^3 Y = d_x^3 y$. $d_x^4 Y = d_x^4 y$; $Y = f(X)$ being the equation to the proposed curve ∴ the section may have a contact of the 4th order with the curve.

$$\underline{b \quad a \quad c}$$

$$b^2 - 4ac = 0$$

$$\frac{b}{a} \cdot \frac{b}{c} - 4 = 0 \qquad d_x^2 y \quad d_x^3 y$$

Body X

Prove that all ⊙s whose axes vary subject to the cond^n $a^2+mb^2=c^2$ may be touched by 4 straight lines.

Let the =^n to an ⊙ be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad a^2+mb^2=c^2$$

Then $\dfrac{x^2}{a^3}+\dfrac{y^2}{b^3}\dfrac{db}{da}=0$ & $a+m\dfrac{db}{da}=0$

$$\frac{b^3}{a^3}\cdot\frac{x^2}{y^2}=\frac{db}{da}=\frac{a}{mb}.$$

$$mb^4x^2=a^4y^2.$$

$$\frac{b^2}{a^2}=\pm\frac{y}{x}\frac{1}{\sqrt{m}}.$$

$$\therefore a^2\left(1\pm\frac{b^2}{a^2}m\right)=c^2.$$

$$a^2\left(1\mp\frac{y\sqrt{m}}{x}\right)=c^2.$$

$$c^2=\left(x\pm y\sqrt{m}\right)^2.$$

$$x\pm y\sqrt{m}=\pm c.$$

the =^ns to 4 straight lines

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~~$u = 1\cdot 3 \cdot 5 \ldots 2n$~~

Let $\quad x = \cos 2z$

$$\therefore 1-x = 2\sin^2 z \qquad 1+x = 2\cos^2 z$$

$$\therefore u = 2\cdot(1\cdot 3\cdot 5 \ldots \overline{2n-1})\left\{(\sin z)^{2n-1}\cdot(\cos z) - \frac{\overline{2n-1}\cdot\overline{2n-2}}{1\cdot 2\cdot 3}\cdot(\sin z)^{2n-3}(\cos z)^3 + \ldots\right.$$

$$= \frac{1}{n}\cdot \lfloor\overline{2n-1}\,!\left\{2n(\sin z)^{2n-1}\cos z - \frac{2n\cdot 2n-1\cdot 2n-2}{\lfloor 3}\sin z^{2n-3}\cos z^3 + \ldots\right.$$

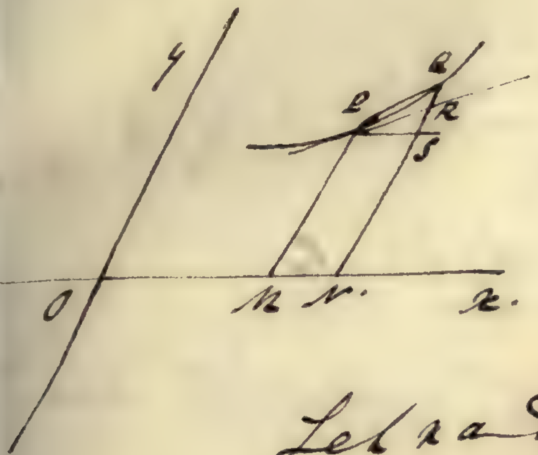
$$= \frac{\frac{1}{n}\,\lfloor 2n-1\,!}{2\sqrt{-1}}\left\{(\sqrt{-1}\sin z + \cos z)^{2n} - (\sqrt{-1}\sin z - \cos z)^{2n}\right\}$$

$$= \frac{1}{n}\cdot\frac{\lfloor 2n-1\,!}{2\sqrt{-1}}\left\{(\cos 2nz + \sqrt{-1}\sin 2nz) - (\cos 2nz - \sqrt{-1}\sin 2nz)\right\}$$

$$= \frac{1}{n}\cdot 1\cdot 3\cdot 5 \ldots 2n-1\ \sin 2nz$$

$$= 1\cdot 3\cdot 5\ldots \overline{2n-1}\cdot\frac{\sin(n\cos^{-1} z)}{n}$$

Find the =n to the tangent at any point of a curve, the axes being oblique.



Let x and y be coords of P.

$MN = h$.

$\therefore \overline{x+h}$ and $y + d_x y . h + d_x^2 y . \frac{h^2}{1.2}$ = the coords of Q.

then whether the axes be rectangular or oblique the =n to PQ is

$$Y - y = \frac{SQ}{PS} . \overline{X - x} \qquad (PS \text{ is } \parallel \text{ to } ON).$$

$$Y - y = \frac{d_x y . h + d_x^2 y . \frac{h^2}{1.2} ...}{h} . \overline{X - x}.$$

$$= \left(d_x y . + d_x^2 y . \frac{h}{1.2} + \ldots\right) \overline{X - x}$$

$$= d_x y . \overline{X - x}$$ if h is indefinitely diminished

Find the value of $\dfrac{x^3\cos^2 x + \sin x}{x}$ when $x = 0$.

$$u = \frac{x^3\cos^2 x + \sin x}{x} = \frac{0}{0} \text{ if } x = 0.$$

$$= \frac{3x^2\cos^2 x - 2x^3\cos^2 x + \cos x}{1} = 1 \text{ if } x = 0.$$

Find the value of $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$ when $x = a$.

Let $x = a + h$. $\therefore \sqrt{x^2 - a^2} = \sqrt{2ah + h^2} = \sqrt{2ah} + \frac{1}{2}\cdot\frac{h^2}{\sqrt{2ah}} + \text{\&c.}$

$$\sqrt{x-a} = \sqrt{h}.$$

$$\sqrt{x} = \sqrt{a+h} = \sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a}} + -$$

$\therefore \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$ becomes $\dfrac{\sqrt{a} + \frac{1}{2}\frac{h}{\sqrt{a}} - \sqrt{a} + \sqrt{h}}{\sqrt{2ah} + \frac{1}{2}\frac{h^2}{\sqrt{2ah}}}$

$$= \frac{\sqrt{h}\left\{1 + \frac{1}{2}\sqrt{\frac{h}{a}} + \text{\&c}\right\}}{\sqrt{h}\left(\sqrt{2a} + \frac{1}{2}\frac{h}{\sqrt{2a}} + -\right)}$$

$$= \frac{1 + \frac{1}{2}\sqrt{\frac{h}{a}} + \text{\&c}}{\sqrt{2a} + \frac{1}{2}\frac{h}{\sqrt{2a}}} = \frac{1}{\sqrt{2a}} \text{ if } h = 0.$$

Given the bases of 2 Δ's and the sum of their ~~altitudes~~ perimeters, shew that in order that their area may be a max, the Δ's must be isosceles; and the sines of the Δ's of the bases must be as the bases.

Let $p \cdot p'$ be the perimeters.

$b \cdot b'$ the bases.

$x \cdot y;\ x'\ y'$ the other sides of the Δ's

c = sum of the areas.

$$\therefore c = \sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)} + \sqrt{\tfrac{p'}{2}\left(\tfrac{p'}{2}-b'\right)\left(\tfrac{p'}{2}-x'\right)\left(\tfrac{p'}{2}-y'\right)} \qquad (1)$$

$$p+p' = b+b'+x+y+x'+y' \quad - - - - - - \qquad (2).$$

From the first $=^n$.

$$-\tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)} \cdot \frac{dx+dy}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}} - \tfrac{1}{2}\sqrt{\tfrac{p'}{2}\left(\tfrac{p'}{2}-b'\right)} \cdot \frac{dx'+dy'}{\sqrt{\left(\tfrac{p'}{2}-x'\right)\left(\tfrac{p'}{2}-y'\right)}} = 0.$$

From the $2^{nd} =^n$

$$dx+dy+dx'+dy' = 0.$$

$$\lambda\, dx+\lambda\, dy+\lambda\, dx'+\lambda\, dy' = 0.$$

Collect the coefft of $dx \cdot dy$. &c.

$$\left\{\lambda - \tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)}\right\} \frac{dx}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}} = 0 = \left(\lambda - \tfrac{1}{2}\sqrt{\tfrac{p}{2}\left(\tfrac{p}{2}-b\right)}\right) \frac{dy}{\sqrt{\left(\tfrac{p}{2}-x\right)\left(\tfrac{p}{2}-y\right)}}$$

$$\therefore dy = dx \quad \text{or} \quad x = y.$$

similarly $x' = y'$ and the Δ's are isosceles.

$a^3 y = x^4.$

$a^3 \frac{dy}{dx} = 4x^3.$

$a^3 \frac{d^2y}{dx^2} = 12x^2$

$a^3 \frac{d^3y}{dx^3} = 24x.$

$a^3 \frac{d^4y}{dx^4} = 24.$

$\therefore$ the first diff[l]. coeff[t]. which does not vanish when $x=0$ is of an even order $\therefore$ there is no p[t]. of inflex[n]. at the origin

$y = x^2 \pm x^{\frac{5}{2}}$.

$d_x y = 2x + \frac{5}{2} x^{\frac{3}{2}}$. $= 0$ if $x = 0$.

$d_x^2 y = 2 + \frac{5}{2} x^{\frac{1}{2}}$. $= 2$ if $x = 0$.

also if $x = 0 - h$ $y = h^2 \pm (-h)^{\frac{5}{2}}$. which is impossible

∴ since $d_x^2 y = 2$. whether $x = 0+h$ or $0-h$. and y is impossible. there is a cusp of the 2nd. ~~[illegible]~~. at the origin

three quantities $\overline{a+z}$, $\overline{a+z}+h$, $\overline{a+z}+h'$; approximation to the root (a) of an $=^n$ being substituted for the unknown quantity give results h. $h+\delta$, $h+\delta'$; show that z will be very nearly found from the $=^n$.

$$z^2(h\delta'-h'\delta)+z(h^2\delta'-h'^2\delta)+h.hh'(h'-h)=0.$$

---

let $f(x)=0$ be the proposed $=^n$.

$$\therefore f\,\overline{a+z}=f(a)+f'(a)z+f''(a)\frac{z^2}{1\cdot2}+\&c=h.$$

But $f(a)=0$ for a is a root of the $=^n$.

$$\therefore f''\frac{(a)}{1\cdot2}=\frac{h-f'(a)z}{z^2}\text{ very nearly.}$$

$$f(a+z+h)=f(a)+f'(a)(z+h)+f''(a)\frac{(z+h)^2}{1\cdot2}+\&c.=h+\delta$$

or substituting for $f''\frac{(a)}{1\cdot2}$ we have.

$$f'(a)(z+h)+\frac{h-f'(a)z}{z^2}.(z+h)^2=h+\delta$$

$$f'(a).\left\{z+h-\frac{(z+h)^2}{z}\right\}=h+\delta-h.\left(\frac{z+h}{z}\right)^2$$

$$-zf'(a)=\frac{z^2(h+\delta)-h(z+h)^2}{(z+h)h}$$

[illegible]

[illegible]

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$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

similarly. changing h into h', δ into δ'

$$-2f'(a) = \frac{z^2(h+\delta') - h(z+h')^2}{(z+h')h'}.$$

Equate the two values of $-2f'(a)$.

$$h'z^2(h+\delta)(z+h') - (z+h')h'h\{z^2+2hz+h^2\} =$$

$$hz^2(h+\delta')(z+h) - (z+h)hh\{z^2+2h'z+h'^2\}$$

which by reduction assumes the form.

$$z^2(h'\delta - h\delta') + (h'^2\delta - h^2\delta')z + 2hh'(h'-h) = 0.$$

Body

If $p = \dfrac{ay - 2bx}{3y - ax}$, find the value of p when $x = y = 0$.

$$p = \frac{ay - 2bx}{3y - ax} = \frac{0}{0}$$

$$= \frac{ap - 2b}{3p - a}$$

$$3p^2 - 2ap = -2b$$

$$p^2 - \frac{2}{3}ap + \frac{a^2}{9} = \frac{1}{9}\{a^2 - 6b\}$$

$$p - \frac{a}{3} = \frac{1}{3}\sqrt{a^2 - 6b}$$

$$\therefore\ p = \frac{1}{3}\{a \pm \sqrt{a^2 - 6b}\}$$

Trace the curve whose =n is

$$x^4 + 2ax^2y - ay^3 = 0$$

we have $\frac{dy}{dx} = \frac{4x^3 + 4axy}{3ay^2 - 2ax^2} = \frac{0}{0}$ at the origin

$$= \frac{12x^2 + 4axp + 4ay}{6ayp - 4ax} = \frac{0}{0}$$

$$= \frac{24x + 8ap}{6ap^2 - 4a} = \frac{4ap}{3ap^2 - 2a}.$$

$\therefore 3ap^3 - 6ap = 0$ or $p = 0$.

$p^2 = 2$ or $p = \pm\sqrt{2}$.

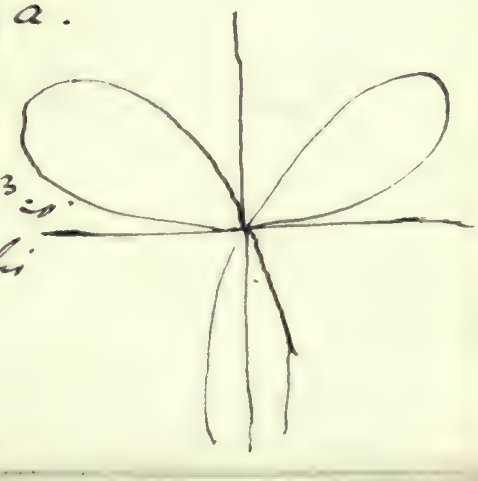
$x = 0$ gives $y = 0$. $\therefore$ the origin is a point in the curve. Also $x = \infty$ gives $\frac{dy}{dx} = \infty$ or the tangent at an infinite distance is ∥ to the axis of y, also for $\frac{dy}{dx} = \infty$ $3ay^2 - 2ax^2 = 0$ or $\frac{y^2}{x^2} = \frac{2}{3}$ $\therefore \frac{y}{x} = \sqrt{\frac{2}{3}}$.

which value of y substituted in the =n gives

$$x^4 + 2a.x^2\sqrt{\frac{2}{3}}. - \frac{2}{3}x^3.a\sqrt{\frac{2}{3}} = 0.$$

$$x + \frac{4}{3}a\sqrt{\frac{2}{3}} = 0. \text{ or } x = \pm\sqrt{\frac{2}{3}}.\frac{4}{3}a.$$

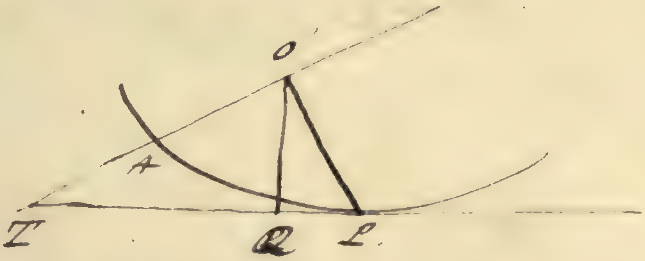
The curve is represented by the fig.



To find the max. value of y. we have. $u = x^4 + 2ax^2y - ay^3 = 0$

$\frac{du}{dx} = 4x^3 + 4axy = 0$. $x = 0$ or $y = -\frac{x^2}{a}$. substitute this value of y in the =n and $x = \pm a\sqrt{2}$, $\therefore y = 2a$.

Find the curve whose pole shall trace out a catenary when it rolls along a st line.



O the pole. AP the curve.

OP = r. OQ = p the ⊥ upon the tangent.

$OP = OQ \sec QOP = OQ \frac{ds}{dx}$. for PT is a tangent at P.

$= p \cdot \frac{ds}{dx}$.

$\therefore$ if $p = f(r)$ be the eq^n to AP, $y = f(y \cdot \frac{ds}{dx})$ will be eq^n to the locus of the pole.

the eq^n to the catenary is $y = \frac{a}{2}\left\{ e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right\}$.

$$\frac{dy}{dx} = \frac{1}{2}\left\{ e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right\}.$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\cdot\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = \frac{y}{a}.$$

$$\therefore y\frac{ds}{dx} = \frac{y^2}{a}$$

but $y = f(y\frac{ds}{dx}) = \sqrt{ay\frac{ds}{dx}}$

$$\therefore p = \sqrt{ar}.$$

$p^2 = ar$ is the eq^n to AP which is the common (·).

The locus of the center of an ellipse rolling along a st. line is $\frac{dx}{dy} = \frac{y^2}{\sqrt{a^2-y^2}\sqrt{y^2-b^2}}$.

The $\perp$ from the centre on the tangent is

$$p^2 = \frac{a^2b^2}{a^2+b^2-r^2}.$$

$\therefore$ the eqn to the reqd locus is

$$y^2 = \frac{a^2b^2}{a^2+b^2-y^2\left(\frac{ds}{dx}\right)^2}$$

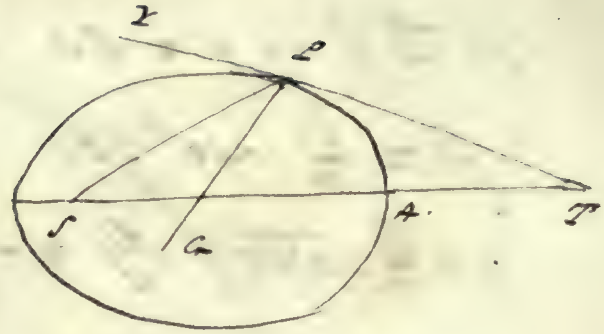
$$a^2+b^2-y^2\left(\frac{ds}{dx}\right)^2 = \frac{a^2b^2}{y^2}.$$

$$\frac{(a^2+b^2)y^2-a^2b^2}{y^2} = y^2\left(1+\left(\frac{dy}{dx}\right)^2\right).$$

$$\frac{-y^4+(a^2+b^2)y^2-a^2b^2}{y^4} = \left(\frac{dy}{dx}\right)^2.$$

$$\therefore \frac{dx}{dy} = \frac{y^2}{\sqrt{a^2-y^2}\sqrt{y^2-b^2}}.$$

The same otherwise.



$$\cos SPZ = \sin SPG = e \cos STZ$$

$$\sec SPZ = \frac{1}{e}\sec STZ = \frac{1}{e}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \sec^3 SPZ = \frac{1}{e^3}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$$

$$= \rho \cdot \frac{\frac{d^2y}{dx^2}}{e^3}$$

$$\therefore \rho = \left(1-\frac{x^2}{a^2}\right)$$

$$1-\sin^2 SPG = \cos^2 SPG = \left\{1-\frac{e^2}{1+\left(\frac{dy}{dx}\right)^2}\right\}$$

$$\sec^2 SPG = \frac{1+\left(\frac{dy}{dx}\right)^2}{1-e^2+\left(\frac{dy}{dx}\right)^2} = \left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{b^4}{a^2y^2}$$

$$\sec^3 SPG = \frac{b^6}{a^3y^3}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$$

$$\frac{a^3y^3}{b^6}\sec^3 SPG = \rho\frac{d^2y}{dx^2}$$

$$= \frac{b^4}{}$$

$$\therefore \frac{b^6}{a^3y^3}\sec^3 SPG = \frac{b^4}{a^2y^3}\rho$$

$$\frac{b^2}{a}\sec^3 SPG = \rho.$$

In the ⊙ if $\rho = LO$. then

$$\rho = \frac{b^2}{a} \cdot \sec^3 SLO.$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}; \quad \frac{dy}{x} = -\frac{b^2 x}{a^2 y}. \quad \frac{d^2y}{x} = -\frac{b^4}{a^2y^3}.$$

$$\therefore \rho = -\frac{1}{\frac{d^2y}{x}}\left(1+\left(\frac{dy}{x}\right)^2\right)^{\frac{3}{2}} = \frac{a^2y^3}{b^4}\left\{\frac{a^4y^2+b^4x^2}{a^4y^2}\right\}^{\frac{3}{2}}$$

$$= \frac{1}{ab^4}\cdot\left(b^4+(a^2-b^2)y^2\right)^{\frac{3}{2}}.$$

also $\tan SLO = \frac{\frac{du}{\theta}}{u} = \frac{e\sin\theta}{1-e\cos\theta}.$

$$\therefore \sec^2 SLO = \frac{1-2e\cos\theta+e^2}{1-2e\cos\theta+e^2\cos^2\theta} = \frac{1-\frac{2\sqrt{a^2-b^2}}{a}\cdot\frac{ae+x}{a+ex}+\frac{a^2-b^2}{a^2}}{1-\frac{2\sqrt{a^2-b^2}}{a}\frac{ae+x}{a+ex}+\frac{a^2-b^2}{a^2}\left(\frac{ae+x}{a+ex}\right)^2}$$

$$= \frac{a^4-\left(\frac{a^2-b^2}{a}\right)x^2-2(a^2-b^2)a^2-2\cdot(a^2-b^2)^{\frac{3}{2}}x+(a^2-b^2)\{a^2+2\sqrt{a^2-b^2}\cdot x+\frac{a^2}{a}\ldots}{a^4-(a^2-b^2)x^2-2(a^2-b^2)a^2-2\cdot(a^2-b^2)^{\frac{3}{2}}x+(a^2-b^2)\{a^2-b^2+2\sqrt{a^2-b^2}\cdot x\ldots}$$

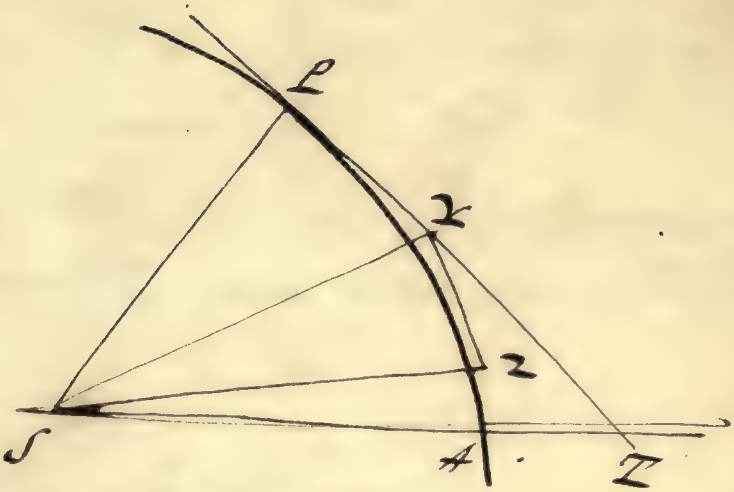
$$= \frac{(a^2-b^2)b^2x^2-a^4b^2}{(a^2-b^2)(a^4+b^2a^2)-a^6} = \frac{(a^2-b^2)(b^2y^2)-a^2b^2}{(a^2-b^2)(a^2+b^2)-a^4}$$

$$= \frac{b^4+(a^2-b^2)y^2}{b^4}.$$

$$\text{Also } \rho = \frac{1}{ab^4}\left\{b^4+(a^2-b^2)y^2\right\}^{\frac{3}{2}} = \frac{b^2}{a}\left\{\frac{\sqrt{b^4+(a^2-b^2)y^2}}{b^2}\right\}^{\frac{3}{2}}.$$

$$= \frac{b^2}{a}\sec^3 SLO.$$

If $f(\rho, p) = 0$ be the eqn to a spiral, then $f\left(\rho', \frac{\rho'^2}{p'}\right) = 0$ is the eqn to the locus of Y.



we have $SY^2 = SP \cdot SZ$.

or $\rho p' = p^2 = \rho'^2$

$$\left.\begin{aligned} \therefore \rho &= \frac{\rho'^2}{p'} \\ p &= \rho' \end{aligned}\right\}$$

or $f\left(\rho', \frac{\rho'^2}{p'}\right) = 0$ is the eqn to the locus of Y.

Trace the curve whose $=^n$ is $\frac{y}{a} = \sqrt{\frac{a}{x} - 1}$, shewing there is a point of contrary flexure if $x = \frac{3a}{4}$.

since $x = 0$ makes $y = \infty$. the axis of y is an asymptote also if x be a negative quantity, or a +ve quantity $> a$ y is impossible.

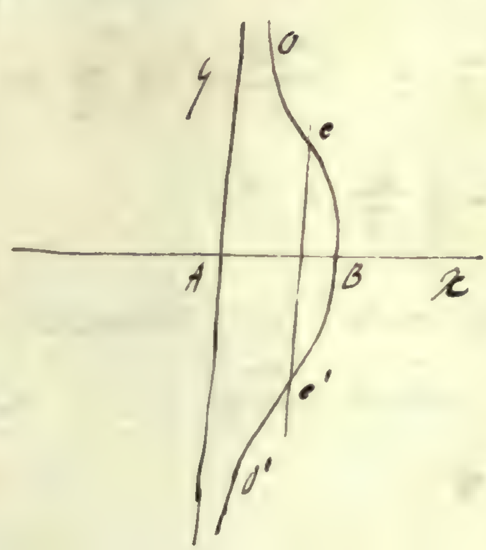
$$\frac{y}{a} = \sqrt{\frac{a-x}{x}} \text{ gives } \frac{d_x y}{a} = -\frac{1}{2}\,\frac{a}{x^{\frac{3}{2}}\sqrt{a-x}}.$$

∴ if $x = a$. $d_x y = \infty$, or the curve cuts the axis of x at right angles. similarly the curve is ∥ to the axis of y if $x = 0$.

$$\frac{d_x^2 y}{a} = + \frac{a}{4}\left\{\frac{3ax^2 - 4x^3}{(ax^3 - x^4)^{\frac{3}{2}}}\right\} = 0 \text{ if } x = \frac{3a}{4}.$$

Now $d_x^2 y$ is +ve if $4x < 3a$ ∴ the curve has its convexity turned to the axis of x. if $x = \frac{3a}{4}$. there is a point of inflexn, and if $x > \frac{3a}{4}$. $d_x^2 y$ is −ve, and the curve is convex towards the axis of x.

The shape of the curve is that annexed



If $y^4 - x^5 + x^4 + 3x^2y^2 = 0$ shew that the origin is a conjugate point.

$$y^4 - x^5 + x^4 + 3x^2y^2 = 0$$

$$(4y^3 + 6x^2y)\frac{dy}{dx} = 5x^4 - 4x^3 - 6xy^2$$

$$\frac{dy}{dx} = \frac{5x^4 - 4x^3 - 6xy^2}{4y^3 + 6x^2y} = \frac{0}{0} \text{ if } x=0,\ y=0$$

$$p = \frac{20x^3 - 12x^2 - 12xyp}{12y^2p + 12xy + 6x^2p} = \frac{0}{0}.$$

$$= \frac{60x^2 - 24x - 12yp - 12xp^2}{24yp^2 + 12xp + 12y + 12xp} = \frac{0}{0}$$

$$= \frac{120x - 24 - 24p^2}{24p^3 + 36p} = \frac{-2 - 2p^2}{2p^3 + 3p}$$

$$2p^4 + 3p^2 + 2p^2 = -2$$

$$p^4 + \frac{5}{2}p^2 + \frac{25}{16} = -1 + \frac{25}{16} = -\frac{9}{16}.$$

$$\therefore p^2 = \frac{1}{4}\left\{-5 \pm \sqrt{-9}\right\}.$$

$\therefore$ the value of p^2 and therefore of p is impossible when $x=0$, $y=0$ $\therefore$ the origin is a conjugate point

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Integral Calculus.

Boby

Find the area of the evolute of a ⊙ (Brady)

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1$$

$$\frac{y}{\beta} = \left(1-\left(\frac{x}{\alpha}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \therefore\ y = \frac{\beta}{\alpha}\left(\alpha^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Let $x^{\frac{2}{3}} = z^2 \quad \therefore\ \frac{2}{3}\log x = 2\log z$

$$dx = 3\frac{x}{z}dz = 3z^2dz.$$

$$\therefore\ y\,dx = \frac{3\beta}{\alpha}\left\{\alpha^{\frac{2}{3}} - z^2\right\}^{\frac{3}{2}} z^2\,dz.$$

$$A = \frac{3\beta}{\alpha}\int_z (a^{\frac{2}{3}} - z^2)^{\frac{3}{2}} z^2 = \frac{3\beta}{\alpha}\left\{-\int_z (a^{\frac{2}{3}}-z^2)^{\frac{5}{2}} + a^{\frac{2}{3}}\int_z (a^{\frac{2}{3}}-z^2)^{\frac{3}{2}}\right.$$

Now $\int_z (a^{\frac{2}{3}}-z^2)^{\frac{5}{2}} = z(a^{\frac{2}{3}}-z^2)^{\frac{5}{2}} + 5\int_z z^2(a^{\frac{2}{3}}-z^2)^{\frac{3}{2}}$

$$= \frac{z(a^{\frac{2}{3}}-z^2)^{\frac{5}{2}}}{6} + \frac{5a^{\frac{2}{3}}}{6}\int_z (a^{\frac{2}{3}}-z^2)^{\frac{3}{2}}$$

$$\therefore\ -\int_z (a^{\frac{2}{3}}-z^2)^{\frac{5}{2}} + a^{\frac{2}{3}}\int_z (a^{\frac{2}{3}}-z^2) = \frac{a^{\frac{2}{3}}}{6}\int_z (a^{\frac{2}{3}}-z^2)^{\frac{3}{2}}$$

$$= \frac{a^{\frac{2}{3}}}{6}\left\{\frac{z(a^{\frac{2}{3}}-z^2)^{\frac{3}{2}}}{4} + \frac{3a^{\frac{2}{3}}}{4}\int_z (a^{\frac{2}{3}}-z^2)^{\frac{1}{2}}\right\}$$

$$= \frac{a^{\frac{4}{3}}}{8}\int_z (a^{\frac{2}{3}}-z^2)^{\frac{1}{2}} \text{ from } z = a^{\frac{1}{3}} \text{ to } z = -a^{\frac{1}{3}}$$

$$= \frac{a^{\frac{4}{3}}}{8}\left\{\frac{z}{2}(a^{\frac{2}{3}}-z^2)^{\frac{1}{2}} + \frac{a^{\frac{2}{3}}}{2}\sin^{-1}\frac{z}{a^{\frac{1}{3}}}\right\}$$

$$= \frac{a^2}{8}\frac{\pi}{2} \text{ from } z = a^{\frac{1}{3}} \text{ to } z = -a^{\frac{1}{3}}$$

Take twice of this as the curve is symmetrical with respect to the axis and multiply by $\frac{3\beta}{a}$ and we have the whole area of the curve

$$= 2\cdot\frac{3\beta}{\alpha}\cdot\frac{\alpha^2\pi}{8\cdot 2} = \frac{3\alpha\beta\pi}{8}.$$

If $f(m) = \int_0^\pi \frac{\cos m\theta}{1 - e\cos\theta}\, d\theta$ where m is an integer.

$f(m+2) = \frac{2}{e} \cdot f(m+1) - f(m)$.

Let $\cos\theta = \frac{1}{2}(x + \frac{1}{x})$

$$\frac{\cos m\theta}{1 - e\cos\theta} = \frac{\cos m\theta}{1 - \frac{e}{2}(x + \frac{1}{x})} = \cos m\theta \left\{ 1 + \frac{e}{2}(x + \tfrac{1}{x}) + \frac{e^2}{4}(x^2 + \tfrac{1}{x^2} + 2) + \frac{e^3}{8}\left\{ x^3 + \tfrac{1}{x^3} + 3(x + \tfrac{1}{x}) \right\} \right\} \&c.$$

$$= \cos m\theta \left\{ 1 + e\cos\theta + \frac{e^2}{2}\{\cos 2\theta + 1\} + \frac{e^3}{4}\{\cos 3\theta + 3\cos\theta\} + \&c. \right.$$

$$f(m) = \int_0^\pi \left\{ \cos m\theta + \frac{e}{2}\{\cos(m+1)\theta + \cos(m-1)\theta\} + \frac{e^2}{2}\left\{\cos m\theta + \frac{1}{2}\{\cos(m+2)\theta + \cos(m-2)\theta\}\right\} + \frac{e^3}{8}\{\cos(m+3)\theta + \cos(m-3)\theta + 3\cos(m+1)\theta + 3\cos(m-1)\theta\} + \&c. \right\}$$

$$f(m+2) = \int_0^\pi \left\{ \cos(m+2)\theta + \frac{e}{2}\{\cos(m+3)\theta + \cos(m+1)\theta\} + \frac{e^2}{2}\left\{\cos(m+2)\theta + \frac{1}{2}[\cos(m+4)\theta + \cos m\theta]\right\} \&c. \right.$$

$$f(m+1) = \int_0^\pi \left\{ \frac{2}{e}\cos \overset{+1}{m}\theta + \cos(m+2)\theta + \cos m\theta + e\left\{\cos \overset{+1}{m}\theta + \frac{1}{2}[\cos(m+3)\theta + \cos(m-1)\theta]\right\} \right.$$

$\frac{2}{e} \cdot f(m+1) - f(m) =$

$$\int_0^\pi \left\{ \frac{2}{e}\cos \overset{+1}{m}\theta + \cos(m+2)\theta + \frac{e}{2}(\cos(m+1)\theta + \cos(m+3)\theta\} ~~- \cos(m-1)\theta~~ \right.$$

$$= \int_0^\pi \left\{ \cos(m+2)\theta + \frac{e}{2}\{\cos(m+1)\theta + \cos(m+3)\theta\} + \&c \right.$$

for $\int_0^\pi \cos m\theta = 0$.

$= f(m+2)\theta$.

$$\int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} = -\frac{1}{n}\int_x \frac{\sin(n\cot^{-1}x)-n}{1+x^2}\cdot\frac{1}{(1+x^2)^{\frac{n-2}{2}}}$$

$$= -\frac{1}{n}\cdot\cos(n\cot^{-1}x)\frac{1}{(1+x^2)^{\frac{n-2}{2}}} \quad \frac{n-2}{n}\int_x \cos(n\cot^{-1}x)\frac{x}{(1+x^2)^{\frac{n}{2}}}.$$

$$\int_x \frac{n-2}{n}\cos(n\cot^{-1}x)\frac{x}{(1+x^2)^{\frac{n}{2}}} = -\frac{n-2}{n^2}\int_x \frac{\cos(n\cot^{-1}x)-n}{(1+x^2)}\cdot\frac{x}{(1+x^2)^{\frac{n-2}{2}}}$$

$$= -\frac{n-2}{n^2}\cdot\sin(n\cot^{-1}x)\frac{x}{1+x^2} - \frac{(n-2)^2}{n^2}\cdot\int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}}.$$

$$\therefore \int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} = +\left(\frac{n-2}{n}\right)^2\int_x \frac{\sin n(\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}}.$$

$$-\frac{1}{n}\frac{\cos.(n\cot^{-1}x)}{(1+x^2)^{\frac{n-2}{2}}} + \frac{n-2}{n^2}\cdot\frac{\sin(n\cot^{-1}x)x}{(1+x^2)^{\frac{n-2}{2}}}.$$

$$\text{or } \int_x \frac{\sin(n\cot^{-1}x)}{(1+x^2)^{\frac{n}{2}}} =$$

$$\frac{n^2}{n^2-(n-2)^2}\left\{\left(\frac{n-2}{n}\right)^2 \frac{\sin(n\cot^{-1}x)x}{(1+x^2)^{\frac{n-2}{2}}} - \frac{1}{n}\frac{\cos.(n\cot^{-1}x)}{(1+x^2)^{\frac{n-2}{2}}}\right\}$$

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[illegible]

$= $ [illegible]

J — [illegible]

$$\int_x (\log x)^n (x^n + p x^{n-1} \cdots + p_n) = \int_x f(x) \cdot (\log x)^n$$

$$= Fx \cdot (\log x)^n - n \int_x \frac{F(x)}{x} \cdot (\log x)^{n-1}.$$

continuing the process we should at last come to

$$\int_x \varphi(x) \log x = \psi(x) \cdot \log x - \int_x \frac{\psi(x)}{x}.$$

$$= \psi x \log(x) - \chi(x).$$

$$\int_x x^2 \cdot \log \frac{a+\sqrt{a^2-x^2}}{x} = \frac{x^3}{3} \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) - \frac{1}{3}\int_x x^3 \left\{\frac{\frac{-x^2}{\sqrt{a^2-x^2}} - (a+\sqrt{a^2-x^2})}{x(a+\sqrt{a^2-x^2})}\right\}$$

$$= \frac{x^3}{3} \log \frac{a+\sqrt{a^2-x^2}}{x} + \frac{a}{3}\int_x \frac{x^2}{\sqrt{a^2-x^2}}$$

$$= \frac{x^3}{3} \log \frac{a+\sqrt{a^2-x^2}}{x} - \frac{a}{3}\int_x \sqrt{a^2-x^2} + \frac{a^3}{3}\int_x \frac{1}{\sqrt{a^2-x^2}}$$

$$= \frac{x^3}{3} \log \frac{a+\sqrt{a^2-x^2}}{x} - \frac{a x \sqrt{a^2-x^2}}{6} + \frac{a^3}{6} \sin^{-1}\frac{x}{a}.$$

$$\int_x \frac{1}{1+\tan x} = \frac{1}{2}\int_x \frac{2\cos x}{\cos x + \sin x} = \frac{1}{2}\int_x \left\{\frac{-\sin x + \cos x + \cos x + \sin x}{\cos x + \sin x}\right\}$$

$$= \frac{1}{2} \log(\cos x + \sin x) + \frac{1}{2} x.$$

$$\int_x \frac{1+\sin x}{1+\cos x} = \frac{1}{2}\int_x \sec^2 \frac{x}{2} + \int_x \frac{\sin x}{1+\cos x}.$$

$$= \tan \frac{x}{2} - \log_\varepsilon (1+\cos x).$$

$$\int_x \frac{x^m}{\sqrt{2ax-x^2}} = \int_x \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} = -2x^{m-\frac{1}{2}}.\sqrt{2a-x} + (2m-1)\int_x . x^{m-\frac{3}{2}}\sqrt{2a-x}.$$

$$\int_x x^{m-\frac{3}{2}}\sqrt{2a-x} = -\int_x \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} + 2a\int_x \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}$$

$$\therefore \int_x \frac{x^m}{\sqrt{2ax-x^2}} = -2x^{m-\frac{1}{2}}.\sqrt{2a-x} - (2m-1)\int_x \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} + (2m-1)2a\int_x \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}$$

or $$\int_x \frac{x^{m-\frac{1}{2}}}{\sqrt{2a-x}} = -\frac{x^{m-\frac{1}{2}}\sqrt{2a-x}}{m} + \frac{(2m-1)a}{m}.\int_x \frac{x^{m-\frac{3}{2}}}{\sqrt{2a-x}}.$$

$$\int_x \frac{x^m}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}.\sqrt{2ax-x^2}}{m} + \frac{(2m-1)a}{m}.\int_x \frac{x^{m-1}}{\sqrt{2ax-x^2}}.$$

$$\int_x \varepsilon^{ax}(\cos x)^n = \frac{1}{a}\varepsilon^{ax}\cdot\cos^n x + \frac{n}{a}\int_x \varepsilon^{ax}\cdot\cos^{n-1}x\sin x\,.$$

$$\int_x \varepsilon^{ax}\cos^{n-1}x\sin x = \frac{1}{a}\varepsilon^{ax}\cdot\cos^{n-1}x\sin x - \frac{1}{a}\int_x \varepsilon^{ax}\left\{(n-1)\cos^{n-2}x\cdot\sin^2 x + \cos^n x\right\}$$

$$= \frac{1}{a}\varepsilon^{ax}\cos^{n-1}x\sin x - \frac{1}{a}\int_x \varepsilon^{ax}\left\{-(n-1)\cos^{n-2}x(1-\sin^2 x) - (n-1)\cos^{n-2}x + \cos^n x\right\}$$

$$= \frac{1}{a}\varepsilon^{ax}\cos^{n-1}x\sin x - \frac{n}{a}\int_x \varepsilon^{ax}\cos^n x + \frac{n-1}{a}\int_x \varepsilon^{ax}\cos^{n-2}x$$

$$\therefore \frac{a^2+n^2}{a^2}\int_x \varepsilon^{ax}\cos^n x = \frac{\varepsilon^{ax}}{a^2}\left\{a\cos^n x + n\cos^{n-1}x\sin x\right\} + \frac{n(n-1)}{a^2}\int_x \varepsilon^{ax}\cos^{n-2}x$$

$$\int_x \varepsilon^{ax}\cos^n x = \frac{1}{a^2+n^2}\varepsilon^{ax}\cdot\left(a\cos^n x + n\cos^{n-1}x\sin x\right) + \frac{n(n-1)}{a^2+n^2}\int_x \varepsilon^{ax}\cos^{n-2}x\,.$$

$$\int_x \sqrt{a+b\tan^2 x} = \int_x \frac{a+b\sec^2 x - b}{\sqrt{a+b\tan^2 x}}$$

$$= \sqrt{b}\cdot\int_{\tan x}\frac{1}{\sqrt{\frac{a}{b}+\tan^2 x}} + (a-b)\cdot\int_x \frac{\cos x}{\sqrt{a(1-\sin^2 x)+b\sin^2 x}}$$

$$= \sqrt{b}\cdot\log\left\{\tan x + \sqrt{\tfrac{a}{b}+\tan^2 x}\right\} + \sqrt{b-a}\cdot\int_{\sin x}\frac{1}{\sqrt{\frac{a}{b-a}+\sin^2 x}}$$

$$= \sqrt{b}\cdot\log\left\{\tan x + \sqrt{\tfrac{a}{b}\cdot\tan^2 x}\right\} + \sqrt{b-a}\,\log\left\{\sqrt{\tfrac{a}{b-a}} + \sqrt{\tfrac{a}{b-a}+\sin^2 x}\right\}$$

$$\frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{A}{1+x^2} + \frac{B}{a^2+b^2x^2}.$$

$$1 = A(a^2+b^2x^2) + B.(1+x^2).$$

$$\therefore Aa^2 + B = 1. \quad Ab^2 + B = 0.$$

$$\therefore A = \frac{1}{a^2-b^2} ; B = \frac{-b^2}{a^2-b^2}.$$

$$\int_x \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left\{\int_x \frac{1}{1+x^2} - b^2\int_x \frac{1}{a^2+b^2x^2}\right)$$

$$= \frac{1}{a^2-b^2}\left(\tan^{-1} x - \frac{b}{a} \tan^{-1} \frac{xb}{a}\right) + C.$$

but $x=0$ gives $C=0$.

$$\therefore \int_x \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left\{\tan^{-1} x - \frac{b}{a} \tan^{-1} \frac{xb}{a}\right\}.$$

If $x = \infty$ we have.

$$\int_0^\infty \frac{1}{(1+x^2)(a^2+b^2x^2)} = \frac{1}{a^2-b^2}\left(\frac{\pi}{2} - \frac{b}{a}\frac{\pi}{2}\right).$$

$$= \frac{a-b}{a(a^2-b^2)}\frac{\pi}{2} = \frac{\pi}{2}\cdot\frac{1}{a(a+b)}.$$

$$\int_x \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = x\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} + \sqrt{b^2-a^2}\int_x \frac{x^2}{\sqrt{a^2-x^2}\cdot(b^2-x^2)} + C.$$

$$\int_0^a \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = \sqrt{b^2-a^2}\int_0^a \frac{x^2}{\sqrt{a^2-x^2}\,(b^2-x^2)}.$$

$$\int_x \frac{x^2}{\sqrt{a^2-x^2}\,(b^2-x^2)} = -\int_x \frac{b^2-x^2-b^2}{\sqrt{a^2-x^2}\,(b^2-x^2)} = -\int_x \frac{1}{\sqrt{a^2-x^2}} + b^2\int_x \frac{1}{\sqrt{a^2-x^2}\,(b^2-x^2)}$$

$$= -\sin^{-1}\frac{x}{a} + \frac{b}{\sqrt{b^2-a^2}}\tan^{-1}\frac{b\sqrt{a^2-x^2}}{x\sqrt{b^2-a^2}} + C.$$

let $x=0$ and the integral becomes

$$0 + \frac{b}{\sqrt{b^2-a^2}}\cdot\frac{\pi}{2} + C. \quad \therefore \; = -C = \frac{b}{\sqrt{b^2-a^2}}\,\frac{\pi}{2}.$$

let $x=a$ and the integral becomes

$$-\frac{\pi}{2} + C = 0 \quad \therefore \quad C = \frac{\pi}{2} =$$

$$\therefore \sqrt{b^2-a^2}\int_0^a \frac{x^2}{\sqrt{a^2-x^2}\cdot(b^2-x^2)} = \frac{\pi}{2}\left(b - \sqrt{b^2-a^2}\right)$$

$$\int_x \frac{\alpha x^2+\beta}{x^4+2px^2+q^2} = \alpha\int_x \frac{x^2}{x^4+2px^2+q^2} + \beta\int_x \frac{1}{x^4+2px^2+q^2}.$$

$$\int_x \frac{1}{x^4+2px^2+q^2} = \frac{1}{q^2}\int_x \frac{1}{\frac{x^4}{q^2}+\frac{2p}{q^2}x^2+1} = \frac{1}{q^{\frac{3}{2}}}\int_z \frac{1}{z^4+\frac{2p}{q}z^2+1}.$$

If $p > q$; $z^4+\frac{2p}{q}z^2+1$ may be reduced to two real factors $(z^2+a^2)(z^2+b^2)$.

$$\therefore \int_z \frac{1}{z^4+\frac{2p}{q}z^2+1} = \int_z \frac{A}{z^2+a} + \int_z \frac{B}{z^2+b}.$$

If $p < q$. $z^4+\frac{2p}{q}z^2+1 = \left(z^2-2z\cos\frac{\theta}{2}+1\right)\left(z^2-2z\cos\frac{3\theta}{2}+1\right)$.

Let $\left(z^2-2z\cos\frac{\theta}{2}+1\right) = (z-a)(z-b)$

$\therefore$ The partial fractions are $\frac{A}{z-a}+\frac{B}{z-b}$.

$$A = \frac{1}{4\left(a^2-\cos\frac{\theta}{2}\right)a} = \frac{1}{4\sqrt{-1}.\sin\theta.a} = \frac{b}{4\sqrt{-1}.\sin\theta}.$$

$$B = \frac{-a}{4\sqrt{-1}\sin\theta}.$$

$\therefore$ The partial fraction is

$$\frac{1}{2\sin\theta}.\frac{x\sin\frac{\theta}{2}}{x^2-2x\cos\frac{\theta}{2}+1} = \frac{\sin\frac{\theta}{2}}{2\sin\theta}.\frac{x-\cos\frac{\theta}{2}+\cos\frac{\theta}{2}}{x^2-2x\cos\frac{\theta}{2}+1}$$

$\therefore$ The whole integral is

$$\frac{\sin\frac{\theta}{2}}{\sin\theta}.\log\left(x^2-2x\cos\frac{\theta}{2}+1\right)-\frac{1}{4}\tan^{-1}\left(\frac{x-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)$$

$$+\frac{\sin\frac{3\theta}{2}}{\sin\theta}\log\left(x^2-2x\cos\frac{3\theta}{2}+1\right)-\frac{1}{4}\tan^{-1}\frac{x-\cos\frac{3\theta}{2}}{\sin\frac{\theta}{2}}.$$

If $p = q$. we have:

$$\int_x \frac{\alpha x^2+\beta}{x^4+2px^2+q^2} = \alpha\int_x \frac{x^2+q-q+\frac{\beta}{\alpha}}{(x^2+q)^2} = \alpha\int_x \frac{1}{x^2+q} + \beta-\alpha q\int_x\frac{1}{(x^2+q)^2}$$

$$\int_x \frac{1}{(x^2+q)^2} = \frac{x}{2q(x^2+q)} + \frac{1}{2q}\int_x \frac{1}{x^2+q}.$$

$$\therefore \int_x \frac{\alpha x^2+\beta}{x^4+2px^2+q^2} = \beta-\alpha q\cdot\left\{\frac{x}{2q(x^2-q)}\right\} + \frac{\beta+\alpha q}{2q^{\frac{3}{2}}}\cdot \tan^{-1}\frac{x}{\sqrt{q}}.$$

$$\int_x \frac{\alpha+\beta x}{(c^2-x^2)\sqrt{a+bx}} = \frac{\alpha-c\beta}{2c}\int_x \frac{1}{(c+x)\sqrt{a+bx}} + \frac{\alpha+c\beta}{2c}\int_x\frac{1}{(c-x)\sqrt{a+bx}}.$$

$$= \frac{\alpha-c\beta}{c\sqrt{bc-a}}\cdot\tan^{-1}\sqrt{\frac{a+bx}{bc-a}}$$

$$+ \frac{\alpha+c\beta}{c\sqrt{bc+a}}\cdot\tan^{-1}\frac{\sqrt{a+bx}}{\sqrt{bc+a}}.$$

$$\int_x \sqrt{(a-x)(x-c)} = \int_x \sqrt{-ac+(a+c)x-x^2} = \int_x\sqrt{\left(\frac{a-c}{2}\right)^2-\left\{\frac{a+c}{2}-x\right\}^2} = \int_x\sqrt{\left(\frac{a-c}{2}\right)^2-\left\{x-\frac{a+c}{2}\right\}^2}$$

$$= \frac{1}{2}\left(x-\frac{a+c}{2}\right)\sqrt{(a-x)(x-c)} + \frac{1}{2}\left\{\frac{a-c}{2}\right\}^2\cdot \sin^{-1}\left(\frac{2x-(a+c)}{a-c}\right)$$

$$\int_x \sin x\sqrt{1-c^2\sin^2 x} = -\cos x\sqrt{1-c^2\sin^2 x} + \int_x \frac{-c^2\sin x\cos^2 x}{\sqrt{1-c^2\sin^2 x}}$$

$$= -\cos x\sqrt{1-c^2\sin^2 x} - \int_x \sin x\left\{\frac{c^2-1+1-c^2\sin^2 x}{\sqrt{1-c^2\sin^2 x}}\right.$$

$$= -\frac{1}{2}\cos x\sqrt{1-c^2\sin^2 x} + \frac{1}{2}(1-c^2)\int_x \frac{\sin x}{\sqrt{1-c^2\sin^2 x}}.$$

$$\int_x \frac{\sin x}{\sqrt{1-c^2\sin^2 x}} = \frac{1}{c^2}\int_x \frac{\sin x}{\sqrt{\frac{1}{c^2}-1+(\cos x)^2}} = -\frac{1}{c^2}\cdot\log.\left\{\cos x+\sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\}+C$$

$$\therefore \int_x \sin x\sqrt{1-c^2\sin x} = -\frac{1}{2}\cos x\sqrt{1-c^2\sin^2 x} - \frac{1}{2}\frac{1-c^2}{c^2}\log\left\{\cos x+\sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\}$$

$$= -\frac{1}{2}\left\{\cos x\sqrt{1-c^2\sin^2 x} + \frac{1-c^2}{c^2}\log.\left\{\cos x+\sqrt{\frac{1-c^2}{c^2}+\cos^2 x}\right\}\right\}$$

$$\int_t e^{-a^2t^2} t^{2n} = \int_t t\, e^{-a^2t^2}\cdot t^{2n-1} = -\frac{1}{2a^2}e^{-a^2t^2}\cdot t^{2n-1} + \frac{2n-1}{2a^2}\int_t e^{-a^2t^2}\cdot t^{2n-2} + C.$$

&c &c &c.

$$\therefore \int_t e^{-a^2t^2} t^{2n} = -\frac{1}{2a^2}\cdot e^{-a^2t^2}\cdot t^{2n-1} - \frac{2n-1}{2^2 a^4}e^{-a^2t^2}t^{2n-3} - \&c.$$

$$= -e^{-a^2t^2}\left\{\frac{t^{2n-1}}{2a^2} + \frac{(2n-1)t^{2n-3}}{2^2a^4} + \frac{(2n-1)(2n-3)}{2^3a^6}\cdot t^{2n-5}\cdots\cdots + \frac{(2n-1)\cdots 3.1}{2^n a^{2n}}t\right\} + \frac{(2n-1)\cdots 3.1}{2^n a^{2n}}\int_t e^{-a^2t^2}.$$

$$\therefore \int_{t\,-\infty}^{+\infty} e^{-a^2t^2} t^{2n} = \frac{(2n-1)(2n-3)\cdots 3.1}{2^n a^{2n}}\int_{t\,-\infty}^{+\infty} e^{-a^2t^2}.$$

$$= \left\{\frac{(2n-1)(2n-3)\cdots 3.1}{2^n\cdot a^{2n+2}}\right\}\sqrt{\pi}.$$

$$\frac{1}{(x+a)^m (x+b)^n} = \frac{1}{(x+a)^{m+n}\left(1+\frac{b-a}{x+a}\right)^n}$$

$$= \frac{1}{(b-a)^{m+n-2}} \frac{\left(\frac{1}{x+a}\right)^2 \left(\frac{b-a}{x+a}\right)^{-m-n+2}}{\left(1+\frac{b-a}{x+a}\right)^n}$$

$$= \frac{1}{(b-a)^{m+n-2}} \frac{\left(\frac{1}{x+a}\right)^2 \left(1+\frac{b-a}{x+a}-1\right)^{-m-n+2}}{\left(1+\frac{b-a}{x+a}\right)^n}$$

$$= -\frac{1}{(b-a)^{m+n-1}} \frac{d_x u\,(u-1)^{-m-n+2}}{u^n}$$

$$= -\frac{1}{(b-a)^{m+n-1}} \cdot \frac{d_x u}{u^n} \left\{ u^{-m-n+2} - (m+n-2)\,u^{-m-n+1} + \&c. \right.$$

$$= -\frac{1}{(b-a)^{m+n-1}} \left\{ \frac{d_x u}{u^{m+2n-2}} - m+n-2\,\frac{d_x u}{u^{m+2n-1}} + \&c \right)$$

$$= -\frac{1}{(b-a)^{m+n-1}} \left\{ -\frac{1}{m+2n-1}\,\frac{1}{(u^{m+2n-1})} + \frac{m+n-2}{m+2n} \cdot \frac{1}{u^{m+2n}} + \&c_2 \right)$$

$$\int \frac{1}{(x+a)^m} \cdot \frac{1}{(x+b)^n} = -\frac{1}{m-1} \cdot \frac{1}{(x+a)^{m-1}} \frac{1}{(x+b)^n}$$

$$+ \frac{n}{m-1} \cdot \frac{1}{(x+a)^{m-1}} \frac{1}{(x+b)^{n+1}}$$

$$\int \frac{1}{x+c} = \int \frac{1}{\sqrt{a-x} + \sqrt{b+x}}$$

$$\frac{\sqrt{a-x} + \sqrt{b+x}}{\sqrt{a-x} - \sqrt{b+x}} \quad a-x$$

$$\sqrt{a-x} - \sqrt{b+x} = Z$$

$$\frac{-dx}{\sqrt{a-x}}$$

$$\frac{\alpha x+\beta}{(x+p)(x+q)} = \frac{A}{x+p} + \frac{B}{x+q}$$

$$\alpha x+\beta = A(x+q) + B(x+p)$$

$$\alpha = (A+B) \quad \therefore \quad \alpha - A = B$$

$$\beta = Aq + Bp = Aq + \alpha p - Ap$$

$$= \alpha p + A(q-p)$$

$$\beta - \alpha p = A(q-p)$$

$$A = \frac{\beta-\alpha p}{q-p}$$

$$B = \alpha - \frac{\beta-\alpha p}{q-p}$$

$$= \frac{\alpha q - \alpha p - \beta + \alpha p}{q-p}$$

$$\therefore \int_x \frac{\alpha x+\beta}{(x+p)(x+q)\sqrt{a+bx+cx^2}}$$

$$\frac{\beta-\alpha p}{q-p}\int_x \frac{1}{(x+p)\sqrt{a+bx+cx^2}}$$

$$+\frac{\alpha q-\beta}{q-p}\int_x \frac{1}{(x+q)\sqrt{a+bx+cx^2}}$$

$$\int_x \frac{1}{x^2+c} = \frac{1}{\sqrt{c}}\tan^{-1}x + C$$

$x=0$ gives $C=0$ $\therefore \int_x^{\infty 0} \frac{1}{x^2+c} = \frac{1}{\sqrt{c}}\frac{\pi}{2}$.

$$\int_x \frac{1}{x^4-4x+3} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+2x+3}.$$

$$\therefore\quad 1 = A.(x^2+2x+3) + B.(x-1)(x^2+2x+3) + \overline{Cx+D}.(x-1)^2.$$

$$x = 1.$$

$$1 = A.6 \quad\therefore\quad A = \tfrac{1}{6}.$$

$$-\frac{x^2}{6} - \frac{x}{3} - \frac{1}{2} = -\frac{\left(\frac{x^2}{6}+\frac{x}{3}-\frac{1}{2}\right)}{x-1} = -\left(\frac{x}{6}+\frac{1}{2}\right) = B(x^2+2x+3) + \overline{Cx+D}.x-1$$

$$\text{Let } x = 1$$

$$\therefore -\frac{2}{3} = 6B \quad\therefore\quad B = -\frac{1}{9}.$$

$$\frac{x}{6} - \frac{1}{2} + \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{3} = \frac{1}{18}\left(\frac{2x^2+x-3}{x-1}\right) = \frac{2x+3}{18} = Cx+D.$$

$$\therefore \int_x \frac{1}{x^4-4x+3} = \frac{1}{6}\int_x \frac{1}{(x-1)^2} - \frac{1}{9}\int_x \frac{1}{x-1} + \frac{1}{18}\int_x \frac{2x+3}{x^2+2x+3}.$$

$$\int_x \frac{\;}{\sqrt[4]{a+bx^4}} = \frac{1}{3b}\int_x \frac{1}{x^3}\cdot\frac{3bx^3}{\sqrt[4]{a+bx^4}} = \frac{1}{3b}\cdot\frac{(a+bx^4)^{\frac{3}{4}}}{x^3} + \frac{3}{3b}\int_x \frac{(a+bx^4)^{\frac{3}{4}}}{x^4}.$$

$$a + 2bx^{n} = z^{-2n}$$

$$\therefore\ -2n \cdot z^{-2n-1}\, dz = 2n\, b x^{n-1}\, dx$$

$$dx = -\frac{z^{-2n-1}}{b x^{n-1}}$$

$$\therefore\ x^{n-1}\, dx = -\frac{1}{b} \cdot \frac{1}{z^{2n+1}}$$

$$a + bx^{n} = \frac{z^{-2n} + a}{2}$$

$$\therefore\ \frac{x^{n-1}\, dx}{(a+bx^{n})(a+2bx^{n})^{\frac{1}{2n}}} = -\frac{1}{b} \cdot \frac{z^{-2n-1}\, dz}{z^{-1} \cdot \frac{z^{-2n} + a}{2}} =$$

$$= -\frac{2}{b} \cdot \frac{dz}{1 + a z^{2n}}$$

$$\therefore\ \int_x \frac{x^{n-1}}{(a+bx^{n})(a+2bx^{n})^{\frac{1}{2n}}} = -\frac{2}{b} \int_z \frac{1}{1 + a z^{2n}}$$

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$$\frac{b+cx}{x^3+a^3} = \frac{A}{x+a} + \frac{Bx+C}{x^2-ax+a^2}$$

$$b+cx = A.(x^2-ax+a^2) + (Bx+C)(x+a).$$

let $x = -a$.

$$b-ac = A.\{3a^2\} \quad \therefore \quad A = \frac{b-ac}{3a^2}$$

let $x^2 = ax-a^2$.

$$b+cx \quad \therefore \; B.ax - Ba^2 + Bax + Cx + Ca.$$

$$2B.a + C = c \quad \therefore C = c - 2Ba.$$

$$b = Ca - Ba^2$$

$$= ac - 3Ba^2 \quad \therefore + \frac{b-ac}{3a^2} = -B.$$

$$C = \frac{3a^2c + 2ba - 2a^2c}{3a^2} = \frac{ac+2b}{3a}$$

$$\therefore \int_x \frac{b+cx}{x^3+a^3} = \frac{b-ac}{3a^2}\int_x \frac{1}{x+a} + \frac{1}{3a^2}.\frac{(ac-b)x + (ac+2b)}{x^2-ax+b}..$$

$$\int_x \frac{1+c^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} = \frac{1}{2}\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{1}{2}\int_x \frac{1-cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}$$

$$= \frac{1}{2}\left\{ \frac{1}{\sqrt{2c+p}}\cdot lg.\ \frac{1-cx^2}{x\sqrt{2c+p}+\sqrt{1+px^2+c^2x^4}} + \frac{1}{\sqrt{p-2c}}\cdot lg.\ \frac{1+cx^2}{x\sqrt{p-2c}+\sqrt{1+px^2+c^2x^4}} \right\}.$$

$$\int_x \frac{x^2}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} = \frac{1}{4c}\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} - \frac{1}{4c}\int_x \frac{1-cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} \Big\}.$$

$$\int_x \frac{\sqrt{1+px^2+c^2x^4}}{1-c^2x^4} = \int_x \frac{1+px^2+c^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}$$

$$= \frac{2c+p}{4c}\cdot\int_x \frac{1+cx^2}{1-cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{2c-p}{4c}\cdot\int_x \frac{1+cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}.$$

$$\frac{p + qx^2 + pc^2x^4}{1 - c^2x^4} = A.\,\frac{1+cx^2}{1-cx^2} + B.\,\frac{1-cx^2}{1+cx^2}.$$

$$\therefore\; p + qx^2 + pc^2x^4 = A.(1+cx^2)^2 + B.(1-cx^2)^2.$$

$$\therefore\; p = A + B. \qquad \therefore\; p + \frac{q}{2c} = 2A.\quad A = \frac{2pc+q}{4c}$$

$$\frac{q}{2c} = A - B.$$

$$B = \frac{4pc - 2pc - q}{4c} = \frac{2pc-q}{4c}$$

$$\therefore \int_x \frac{p+qx^2+pc^2x^4}{1-c^2x^4}X = \frac{2pc+q}{4c}\int_x \frac{1+cx^2}{1-cx^2}\,\frac{1}{\sqrt{1+px^2+c^2x^4}} + \frac{2pc-q}{4c}\int_x \frac{1-cx^2}{1+cx^2}\cdot\frac{1}{\sqrt{1+px^2+c^2x^4}}.$$

$$\int_x \frac{1+cx^2}{1-cx^2}\,\frac{1}{\sqrt{1+px^2+c^2x^4}} = \int_x \frac{x^{-2}+c}{x^{-1}-cx}\cdot\frac{1}{\sqrt{x^{-2}+p+c^2x^2}} = -\int_x \frac{d_x(x^{-1}-cx)}{x^{-1}-cx}\cdot\frac{1}{\sqrt{(x^{-1}-cx)^2+2c+p}}.$$

$$= \frac{1}{\sqrt{2c+p}}\cdot \log.\,\frac{x^{-1}-cx}{\sqrt{2c+p}+\sqrt{x^{-2}+p+c^2x^2}}.$$

Similarly, we have.

$$\frac{1-cx^2}{1+cx^2}\,\frac{1}{\sqrt{1+px^2+c^2x^4}} = \int_x \frac{x^{-2}-1}{x^{-1}+cx}\cdot\frac{1}{\sqrt{x^{-2}+p+c^2x^2}} = -\int_x \frac{d_x(x^{-1}+cx)}{x^{-1}+cx}\cdot\frac{1}{\sqrt{(x^{-1}+cx)^2+p-2c}}$$

$$= \frac{1}{\sqrt{p-2c}}\,\log.\,\frac{x^{-1}+cx}{\sqrt{p-2c}+\sqrt{x^{-2}+p+c^2x^2}}.$$

$$\therefore \int_x \frac{p+qx^2+pc^2x^4}{1-c^2x^4}\cdot\frac{1}{\sqrt{1+px^2+cx^4}} =$$

$$\frac{2pc+q}{4c\sqrt{2c+p}}\,\log.\left\{\frac{1-cx^2}{x\sqrt{2c+p}+\sqrt{1+px^2+c^2x^4}}\right\} + \frac{2pc-q}{4c\sqrt{p-2c}}\,\log.\left\{\frac{1+cx^2}{x\sqrt{p-2c}+\sqrt{1+px^2+c^2x^4}}\right\}.$$

$$\int_x \frac{\sqrt{a+bx^2}}{x^4} = \int_x x^{-3}\sqrt{ax^{-2}+b}. = \frac{2}{3}\frac{\left(ax^{-2}+b\right)^{\frac{3}{2}}}{-2a} = -\frac{1}{3a}\cdot\left(ax^{-2}+b\right)^{\frac{3}{2}}$$

$$= -\frac{1}{3a}\cdot\frac{\{a+bx^2\}^{\frac{3}{2}}}{x^3}.$$

$$\int_x \frac{1}{(1+x)\sqrt{1-x^2}} = \int_x \frac{1}{(1+x)\sqrt{2.(1+x)-(1+x)^2}} = \frac{1}{(1+x)^2\sqrt{\frac{2}{1+x}-1}}$$

$$= -\sqrt{\frac{2}{1+x}-1} = -\sqrt{\frac{1-x}{1+x}}.$$

$$\int_x \frac{1}{x\sqrt{a+bx^n}} = \int_x \frac{x^{\frac{n}{2}-1}}{x^{\frac{n}{2}}\sqrt{a+bx^n}} = \frac{1}{\sqrt{b}}\int_{x^{\frac{n}{2}}} \frac{1}{x^{\frac{n}{2}}\sqrt{\frac{a}{b}+x^n}}$$

$$= \frac{1}{\sqrt{a}}\log.\left\{\frac{x^{\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}}+\sqrt{\frac{a}{b}+x^n}}\right\}.$$

$$\int_x \frac{1+cx^2}{1-cx^2}.\frac{1}{\sqrt{1+bx^2+c^2x^4}}. = \int_x \frac{x^{-2}+c}{x^{-1}-cx}.\frac{1}{\sqrt{x^{-2}+b+c^2x^2}}$$

$$= -\int_x. \frac{d_x.\left(x^{-1}-cx\right)}{x^{-1}-cx}\frac{1}{\sqrt{\left(x^{-1}-cx\right)^2+b+2c}}$$

$$= \frac{1}{\sqrt{b+2c}}.\log\left\{\frac{x^{-1}-cx}{b+2c+\sqrt{x^{-2}+b+c^2x^2}}\right\}.$$

$$\int_x \frac{1}{x\sqrt{a+bx+cx^2}} = \int_x \frac{x^{-2}}{\sqrt{ax^{-2}+bx^{-1}+c}} = -\frac{1}{2\sqrt{a}} \cdot \int_x \frac{-2ax^{-2}}{\sqrt{\left(ax^{-1}+\frac{b}{2}\right)^2+\frac{4ac-b^2}{4}}}$$

$$= -\frac{1}{2\sqrt{a}} \log. \left(\left(ax^{-1}+\frac{b}{2}\right) + \sqrt{a}\sqrt{ax^{-1}+bx^{-1}+c} \right)$$

$$\int_x \frac{x}{\sqrt{a+bx+cx^2}} = \frac{1}{2c} \cdot \int_x \frac{2cx+b}{\sqrt{a+bx+cx^2}} - \frac{b}{2c}\int_x \frac{1}{\sqrt{a+bx+cx^2}}$$

$$\int_x \frac{x^2}{\sqrt{a+bx+cx^2}} = \frac{1}{2c} \int_x x \cdot \frac{2cx+b}{\sqrt{a+bx+cx^2}} - \frac{b}{2c} \cdot \int_x \frac{x}{\sqrt{a+bx+cx^2}}.$$

$$= \frac{1}{c} \cdot x\sqrt{a+bx+cx^2} - \frac{1}{c} \cdot \int_x \sqrt{a+bx+cx^2}$$
$$- \frac{b}{2c^2}\sqrt{a+bx+cx^2} + \frac{b^2}{4c}\int_x \frac{1}{\sqrt{a+bx+cx^2}}.$$

$$\int_x x\sqrt{a^2+x^2}\cdot\log_e\left(\frac{a^2-b^2}{b^2}\right) = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}}\cdot\log_e\left(\frac{a^2-b^2}{b^2}\right)$$

$$-\frac{2}{3}\int_x \frac{(a^2+x^2)^{\frac{3}{2}}x}{x^2-b^2}.$$

Now $$\frac{(a^2+x^2)^{\frac{3}{2}}x}{x^2-b^2} = x\sqrt{a^2+x^2}\left\{\frac{a^2+x^2-b^2+b^2}{(x^2-b^2)}\right\}$$

$$= x\sqrt{a^2+x^2} + c^2\,\frac{x\sqrt{a^2+x^2}}{x^2-b^2}$$

$$= x\sqrt{a^2+x^2} + c^2\left\{\frac{a^2x+x^3-b^2x+b^2x}{(x^2-b^2)\sqrt{a^2+x^2}}\right\}$$

$$= x\sqrt{a^2+x^2} + c^2\,\frac{x}{\sqrt{a^2+x^2}} + c^4\,\frac{x}{(x^2-b^2)\sqrt{a^2+x^2}}.$$

$$\int_x x\sqrt{a^2+x^2} = \frac{1}{3}\cdot(a^2+x^2)^{\frac{3}{2}}.$$

$$c^2\int_x \frac{x}{\sqrt{a^2+x^2}} = c^2\sqrt{a^2+x^2}.$$

$$c^4\int_x \frac{x}{(x^2-b^2)\sqrt{a^2+x^2}} = c^4\int_z \frac{1}{z^2-(a^2+b^2)} \quad \text{where } z=\sqrt{a^2+x^2}.$$

$$= \frac{c^4}{2\sqrt{a^2+b^2}}\cdot\log\left(\frac{z-c}{z+c}\right).$$

$$= \frac{c^3}{2}\log\left\{\frac{\sqrt{a^2+x^2}-c}{\sqrt{a^2+x^2}+c}\right\}$$

$$= \frac{c^3}{2}\cdot\log\left\{\frac{a^2+x^2-c^2}{\{\sqrt{a^2+x^2}+c\}^2}\right\} = \frac{c^3}{2}\cdot\log\frac{x^2-b^2}{(\sqrt{a^2+x^2}+c)^2}$$

$$= c^3\log\frac{\sqrt{x^2-b^2}}{\sqrt{a^2+x^2}+c}.$$

$$\therefore \int_x x\sqrt{a^2+x^2}\log_e\left(\frac{x^2-b^2}{b^2}\right) =$$

$$(a^2+x^2)^{\frac{3}{2}}\log_e\left(\frac{x^2-b^2}{\ldots}\right) - \frac{2}{3}\left\{c^2\sqrt{a^2+x^2}+\frac{1}{3}(a^2+x^2)^{\frac{3}{2}}+c^3\log\left(\frac{\sqrt{x^2-b^2}}{\ldots}\right)\right\}$$ [illegible]

$$2\cos(mx^2+\alpha)\cos rx = e^{(mx^2+\alpha)\sqrt{-1}}\cos rx + e^{-(mx^2+\alpha)\sqrt{-1}}\cos rx$$

$$= e^{\alpha\sqrt{-1}}\, e^{mx^2\sqrt{-1}}\cos rx$$

$${}^{0}\!\int_x^{\infty} e^{-a^2x^2}\cos rx = \frac{\sqrt{\pi}}{2a}\, e^{-\frac{r^2}{4a^2}}$$

let $-a^2 = m\sqrt{-1}$

$$\therefore e^{\alpha\sqrt{-1}}\,{}^{0}\!\int_x^{\infty} e^{m\sqrt{-1}\,x^2}\cos rx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{-m\sqrt{-1}}}\, e^{-\frac{r^2}{4m\sqrt{-1}}} \times e^{\alpha\sqrt{-1}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}}\;\frac{e^{+\frac{r^2}{4m}\sqrt{-1}}}{e^{-\frac{\pi}{4}\sqrt{-1}}}\; e^{\alpha\sqrt{-1}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}}\; e^{\left(\alpha+\frac{\pi}{4}+\frac{r^2}{4m}\right)\sqrt{-1}}.$$

$$e^{\alpha\sqrt{-1}}\cdot{}^{0}\!\int_x^{\infty} e^{-m\sqrt{-1}\,x^2}\cos rx = \frac{1}{2}\sqrt{\frac{\pi}{m}}\cdot e^{-\left(a+\frac{\pi}{4}+\frac{r^2}{4m}\right)\sqrt{-1}} \Big\}$$

$$\therefore {}^{0}\!\int_x^{\infty} \cos(mx^2+\alpha)\cos rx = \frac{1}{4}\sqrt{\frac{\pi}{m}}\left\{ e^{\left(a+\frac{\pi}{4}+\frac{r^2}{4m}\right)\sqrt{-1}} + e^{-\left(a+\frac{\pi}{4}+\frac{r^2}{4m}\right)} \right\}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{m}}\cdot\cos\left(a+\frac{\pi}{4}-\frac{r^2}{4m}\right)$$

$$\int_x^{0\,\infty} \frac{\sin x \cos nx}{x} = \frac{1}{2} \int_x^{0\,\infty} \left(\frac{\sin(n+1)x}{x} + \frac{\sin(1-n)x}{x} \right).$$

$$\int_x^{0\,\infty} \varepsilon^{-ax} \cos nx = \frac{a}{n^2+a^2}.$$

$$\int_x^{0\,\infty} \varepsilon^{-ax} \frac{\sin nx}{x} = \tan^{-1} \frac{n}{a}.$$

let $a = 0$

$$\therefore \int_x^{0\,\infty} \frac{\sin nx}{x} = \frac{\pi}{2}.$$

$$\therefore \int_x^{0\,\infty} \frac{\sin(n+1)x}{x} = \frac{\pi}{2}.$$

$\int_x^{0\,\infty} \frac{\sin(1-n)x}{x} = \frac{\pi}{2}$ if n is less than 1.

$= -\frac{\pi}{2}$ if n is greater than 1.

$$\therefore \int_x^{0\,\infty} \frac{\sin x \cos nx}{x} = \frac{1}{2} \left\{ \frac{\pi}{2} + \frac{\pi}{2} \right) \text{ or } \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{2} \right)$$

$= \frac{\pi}{2}$ or zero according as

n is less or greater than unity

$$\int_x \varepsilon^{ax} \sin \overline{mx+\alpha} = \frac{1}{a}\varepsilon^{ax} \sin \overline{mx+\alpha} - \frac{m}{a}\int_x \varepsilon^{ax} \cos \overline{mx+\alpha}.$$

$$= \frac{1}{a}\varepsilon^{ax} \sin \overline{mx+\alpha} - \frac{m}{a^2}\varepsilon^{ax} \cos \overline{mx+\alpha} - \frac{m^2}{a^2}\int_x \varepsilon^{ax} \sin \overline{mx+\alpha}.$$

$$= \frac{\varepsilon^{ax}}{m^2+a^2}\left\{ a \sin \overline{mx+\alpha} - m \cos \overline{mx+\alpha} \right\}.$$

Let $\tan\theta = \frac{m}{a}$ ∴ $\sin\theta = \frac{m}{\sqrt{a^2+m^2}}$. $\cos\theta = \frac{a}{\sqrt{a^2+m^2}}$.

$$\therefore \int_x \varepsilon^{ax} \sin \overline{mx+\alpha} = \frac{\varepsilon^{ax}}{\sqrt{a^2+m^2}} \sin \overline{mx+\alpha-\theta}.$$

$$= \frac{\varepsilon^{ax}}{a\sec\theta} \sin \overline{mx+\alpha-\theta}$$

$$\therefore \int_x^r \varepsilon^{ax} \sin \overline{mx+\alpha} = \frac{\varepsilon^{ax}}{(a\sec\theta)^r} \sin \overline{mx+\alpha-r\theta}.$$

$$\int_x x^n \varepsilon^{ax} \sin \overline{mx+\alpha} = x^n \int_x \varepsilon^{ax} \sin \overline{mx+\alpha} - n \int_x \left\{ x^{n-1} \int_x \varepsilon^{ax} \sin \overline{mx+\alpha} \right\}$$

$$= \frac{x^n \varepsilon^{ax}}{a\sec\theta} \sin \overline{mx+\alpha-\theta} - \frac{n}{a\sec\theta}\int_x x^{n-1} \varepsilon^{ax} \sin \overline{mx+\alpha-\theta}.$$

$$\int_x x^{n-1} \varepsilon^{ax} \sin \overline{mx+\alpha-\theta} = \frac{x^{n-1}\varepsilon^{ax}}{a\sec\theta} \sin \overline{mx+\alpha-2\theta} - \frac{n(n-1)}{(a\sec\theta)^2} x^{n-2} \varepsilon^{ax} \sin \overline{mx+\alpha-2\theta}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\therefore \int_x x^n e^{ax} \sin \overline{mx+\alpha} = \frac{x^n e^{ax}}{a \sec\theta} \sin \overline{mx+\alpha-\theta}$$

$$- \frac{n x^{n-1} e^{ax}}{(a\sec\theta)^2} \sin \overline{mx+\alpha-2\theta}$$

$$+ \frac{n(n-1)\, e^{ax}}{(a\sec\theta)^3} x^{n-2} \sin \overline{mx+\alpha-3\theta} \; \&c.$$

the last term being $\dfrac{n(n-1)\ldots 2.1}{(a\sec\theta)^{n+1}} \sin \overline{mx+\alpha-(n+1)\theta}$.

$\cos mx = \frac{1}{2} \varepsilon^{mx\sqrt{-1}}$

$$\cos mx = \frac{1}{2}\left(\varepsilon^{mx\sqrt{-1}} + \varepsilon^{-mx\sqrt{-1}}\right).$$

$$= \frac{1}{2}\varepsilon^{\sqrt{-1}}\left\{\varepsilon^{mx} + \varepsilon^{-mx}\right\}.$$

$$\therefore \varepsilon^{-ax^2}\left\{\varepsilon^{mx} + \varepsilon^{-mx}\right\} = 2\varepsilon^{\sqrt{-1}} \cdot \varepsilon^{-ax^2} \cdot \cos mx.$$

$$= 2\varepsilon^{-ax^2\sqrt{-1}} \cos mx.$$

But. $\int_x^{0/\infty} \varepsilon^{-ax^2} \cos cx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot \varepsilon^{-\frac{c^2}{4a}}.$

$$\therefore \int_x^{0/\infty} \varepsilon^{-ax^2}\left\{\varepsilon^{mx} + \varepsilon^{-mx}\right\} = \sqrt{\frac{\pi}{a}}\, \varepsilon^{\frac{m^2}{4a}}.$$

$$\int_x^{0/\infty} e^{-ax^2} \frac{1}{2}\left(e^{cx\sqrt{-1}} + e^{-cx\sqrt{-1}}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{c^2}{4a}}$$

let $c = m\sqrt{-1}$

$\therefore c^2 = -m^2$

$$\int_x^{0/\infty} e^{-ax^2} \frac{1}{2}\left(e^{-mx} + e^{mx}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot e^{\frac{m^2}{4a}}$$

$$\varepsilon \left\{ \varepsilon^{mx} + \varepsilon^{-mx} \right\} = 2\varepsilon \cos mx.$$

but $\int_0^\infty \varepsilon^{-ax^2} \cos cx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot \varepsilon^{-\frac{c^2}{4a}}$;

$$\therefore \int_0^\infty \varepsilon^{-ax^2} \left\{ \varepsilon^{mx} + \varepsilon^{-mx} \right\} = 2\int_0^\infty \varepsilon^{-ax^2} \cos mx.$$

$$= \sqrt{\frac{\pi}{a}} \cdot \varepsilon^{-\frac{m^2}{4a}}$$

If $x = \tan^2 z$. $\therefore$ $1+x = \sec^2 z$.

$$d_z x = 2 \cdot \tan z \cdot \sec^2 z.$$

$$\frac{x^{m-1}}{1+x} = \frac{\tan^{2m-2} z}{\sec^2 z} \qquad \text{when } x = 0 \quad z = 0 \cdot$$
$$x = \infty \quad z = \frac{\pi}{2}.$$

$$^{0}\!\int_x^{\infty} \frac{x^{m-1}}{1+x} = {}^{0}\!\int_z^{\frac{\pi}{2}} \frac{x^{m-1}}{1+x} \, d_z x.$$

$$= 2\, {}^{0}\!\int_z^{\frac{\pi}{2}} \cdot \frac{\tan^{2m-2}}{\sec^2 z} \cdot \tan z \sec^2 z \cdot$$

$$= 2\, {}^{0}\!\int_z^{\frac{\pi}{2}} \cdot \tan^{2m-1} z.$$

But $^{0}\!\int^{\infty} \frac{x^{m-1}}{1+x} = \frac{\pi}{\sin m\pi}$

$$\therefore \ {}^{0}\!\int_z^{\frac{\pi}{2}} \tan^{2m-1} z = \frac{\pi}{2 \sin m\pi}.$$

$$\int_t t^{2n+1} \varepsilon^{-at^2} = \int_t t^{2n} \cdot t\varepsilon^{-at^2} = -\frac{1}{2a}\varepsilon^{-at^2} \cdot t^{2n} + \frac{n}{a} \cdot \int_t t^{2n-1} \cdot \varepsilon^{-at^2}$$

Now the integrated part of this expression vanishes when $t = 0$ or when $t = \infty$.

$$\therefore {}^0\!\int^\infty_t t^{2n+1} \cdot \varepsilon^{-at^2} = \frac{n}{a} \cdot {}^0\!\int^\infty_t t^{2n-1} \varepsilon^{-at^2}.$$

$$= \frac{n(n-1)}{a^2} \cdot {}^0\!\int^\infty_t \cdot t^{2n-3} \varepsilon^{-at^2} \quad \&c.$$

$$= \frac{n(n-1) \ldots 2.1}{a^{n+1}} \cdot {}^0\!\int^\infty_t t \varepsilon^{-at^2}.$$

~~But ${}^0\!\int^\infty_t \varepsilon^{-at^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.~~

~~$\therefore {}^0\!\int^\infty_t t^{2n+1} \varepsilon^{-at^2} = \frac{n(n-1) \ldots 2.1}{a^{n+1}}$~~

$$\int_t \cdot t\varepsilon^{-at^2} = -\frac{1}{2}\varepsilon^{-at^2} + C.$$

∴ ~~$\int_t$~~ ~~$t = \infty$~~ when $t = \infty$ the integral vanishes

$t = 0 \quad C = \frac{1}{2}$.

$$\therefore {}^0\!\int^\infty_t \cdot t\varepsilon^{-at^2} = \frac{1}{2}$$

$$\therefore {}^0\!\int^\infty t^{2n+1} \varepsilon^{-at^2} = \frac{n(n-1) \ldots 2.1}{\ldots}$$

$$\int_t t^{2n} \cdot \varepsilon^{-at^2} = \int_t t^{2n-1} \cdot t\,\varepsilon^{-at^2}$$

$$= -\frac{1}{2a} \cdot t^{2n-1} \cdot \varepsilon^{-at^2} + \frac{2n-1}{2a} \cdot \int_t t^{2n-2} \cdot \varepsilon^{-at^2}$$

$$\therefore \int_{t}{}_{0}^{\infty} t^{2n}\,\varepsilon^{-at^2} = \frac{2n-1}{2a} \cdot \int_t t^{2n-2} \cdot \varepsilon^{-at^2}$$

$$= \frac{(2n-1)(2n-3) \cdots 5.3.1}{(2a)^n} \cdot \int_{t}{}_{0}^{\infty} \varepsilon^{-at^2}$$

But $\displaystyle \int_{t}{}_{0}^{\infty} \varepsilon^{-at^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$$\therefore \int_{t}{}_{0}^{\infty} t^{2n}\,\varepsilon^{-at^2} = \frac{(2n-1)(2n-3) \cdots 5.3.1}{(2a)^n} \cdot \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\tan\theta = \frac{a\sin cx}{1+a\cos cx} \quad \therefore \cos\theta = \frac{1+a\cos cx}{\sqrt{1+2a\cos cx+a^2}}$$

$$= \frac{1+a\cos cx}{r}.$$

$$\text{or } r = \frac{1+a\cos cx}{\cos\theta}.$$

$$\therefore r^n \cos^n\theta = 1 + na\cos cx + \frac{n(n-1)}{1\cdot2}a^2\cos^2 cx - \cdots \left\{1 - \frac{n(n-1)}{1\cdot2}\tan^2\theta\right.$$

$$\text{But } \tan^2\theta = \frac{a^2\sin^2 cx}{(1+a\cos cx)^2}.$$

$$\therefore r^n \cos n\theta = 1 + na\cos cx + \frac{n(n-1)}{1\cdot2}\cos 2cx + \&c.$$

$$\int_0^\infty \frac{1}{1+x^2} = \tan^{-1}x + C = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{\cos cx}{1+x^2} = \frac{\pi}{2}\varepsilon^{-c}.$$

$$\int_0^\infty \frac{\cos 2cx}{1+x^2} = \frac{\pi}{2}\varepsilon^{-2c} \quad \&c \&c.$$

$$\therefore \int_0^\infty \frac{r^n\cos n\theta}{1+x^2} = \frac{\pi}{2}\left\{1 + n\varepsilon^{-c} + \frac{n(n-1)}{1\cdot2}\varepsilon^{-2c} + \&c.\right\}$$

$$= \frac{\pi}{2}\left(1+\varepsilon^{-c}\right)^n.$$

$$\tan^{-1}\frac{m\sin cx}{1-m\cos cx} = \frac{m\sin cx}{1-m\cos cx} - \frac{1}{3}\left\{\frac{m\sin cx}{1-m\cos cx}\right\}^3 + \frac{1}{5}\left\{\frac{m\sin cx}{1-m\cos cx}\right\}^5 - \&c$$

$$= m\sin cx + \frac{m^2}{2}\left\{2\sin cx\cos cx\right\} + \frac{m^3}{3}\left\{3\sin cx\cos^2 cx - \sin^3 cx\right\}$$

$$+ \frac{m^4}{4}\left\{4\sin cx\cos^3 cx + 4\sin^3 cx\cos cx\right\} + \&c.$$

$$= m\sin cx + \frac{m^2}{2}\sin 2cx + \frac{m^3}{3}\sin 3cx + \frac{m^4}{4}\sin 4cx + \&c.$$

$$\therefore \frac{x}{1+x^2}\tan^{-1}\left(\frac{m\sin cx}{1-m\cos cx}\right) = \frac{mx\sin cx}{1+x^2} + \frac{m^2}{2}\cdot\frac{x\sin 2cx}{1+x^2} + \&c.$$

$$\text{And } \int_0^\infty \frac{x\sin cx}{1+x^2}\,dx = \frac{\pi}{2}\varepsilon^{-c};\quad \int_0^\infty \frac{x\sin 2cx}{1+x^2}\,dx = \frac{\pi}{2}\varepsilon^{-2c}\ \&c.$$

$$\therefore \int_0^\infty \frac{x}{1+x^2}\tan^{-1}\frac{m\sin cx}{1-m\cos cx}\,dx = \frac{\pi}{2}\left\{m\varepsilon^{-c} + \frac{m^2}{2}\varepsilon^{-2c} + \frac{m^3}{3}\varepsilon^{-3c} + \&c\right\}$$

$$= \frac{\pi}{2}\log\left(\frac{1}{1-m\varepsilon^{-c}}\right).$$

Diff.[l] Equations.

$$d_x^2 y + n^2 y - A\cos\overline{mx+a} - B\cos\overline{nx+\beta} = 0.$$

multiply by $\sin\overline{nx+\beta}$.

$$d_x^2 y.\sin\overline{nx+\beta} + n^2 y.\sin\overline{nx+\beta} - A\cos\overline{mx+a}\,\sin\overline{nx+\beta} - B\cos\overline{nx+\beta}\,\sin\overline{nx+\beta}$$

$$\int_x d_x^2 y\,\sin\overline{nx+\beta} = d_x y.\sin\overline{nx+\beta} - n\int_x d_x y\cos\overline{nx+\beta}$$

$$= d_x y\,\sin\overline{nx+\beta} - n y\cos\overline{nx+\beta} - \int_x n^2 y\,\sin\overline{nx+\beta}$$

$$\therefore \int_x d_x^2 y\,\sin\overline{nx+\beta} + \int_x n^2 y\,\sin\overline{nx+\beta} = d_x y\,\sin\overline{nx+\beta} - ny\cos\overline{nx+\beta}.$$

$$\int_x \cos\overline{mx+a}\,\sin\overline{nx+\beta} = \frac{1}{m}\sin\overline{mx+a}\,\sin\overline{nx+\beta} - \frac{n}{m}\int_x \sin\overline{mx+a}\,\cos\overline{nx+\beta}$$

$$= \frac{1}{m}\sin\overline{mx+a}\,\sin\overline{nx+\beta} - \frac{n}{m}\left\{-\frac{1}{m}\cos\overline{mx+a}\,\cos\overline{nx+\beta} - \frac{n}{m}\int_x \cos\overline{mx+a}\,\sin\overline{nx+\beta}\right\}$$

$$= \frac{m\sin\overline{mx+a}\,\sin\overline{nx+\beta} + n\cos\overline{mx+a}\,\cos\overline{nx+\beta}}{m^2-n^2}.$$

$\therefore$ the integral of the whole line is dividing the whole by $\sin^2\overline{nx+\beta}$.

$$\frac{d_x y}{\sin\overline{nx+\beta}} - \frac{ny\cos\overline{nx+\beta}}{\sin^2\overline{nx+\beta}} - \frac{A}{n^2-m^2}\left\{-\frac{m\sin\overline{mx+a}}{\sin\overline{nx+\beta}} - \frac{n\cos\overline{mx+a}\,\cos\overline{nx+\beta}}{\sin^2\overline{nx+\beta}}\right\}$$

$$-\frac{B}{4n}\,\frac{\cos\overline{2nx+2\beta}}{\sin^2\overline{nx+\beta}} + \frac{C}{\sin^2\overline{nx+\beta}} = 0.$$

Integrate again and we have.

$$\frac{y}{\sin\overline{nx+\beta}} - \frac{A}{n^2-m^2}\cdot\frac{\cos\overline{mx+a}}{\sin\overline{nx+\beta}} + \frac{B}{2n}\int_x 1 - \left(C-\frac{B}{4n}\right)\int_x d_x\frac{\cot\overline{nx+\beta}}{n}$$

$$+\frac{B}{2n}x + \left(\frac{B}{4n^2}-\frac{C}{n}\right)\cot\overline{nx+\beta} + C_2.$$

$$\therefore\quad y = \frac{A}{n^2-m^2}\cos\overline{mx+a} - \frac{B}{2n}x\sin\overline{nx+\beta} + \left(\frac{C}{n}-\frac{B}{4n^2}\right)\cos\overline{nx+\beta}$$

$$-C_2\sin\overline{nx+\beta}$$

$$= \frac{A}{n^2-m^2}\cos\overline{mx+a} - \frac{B}{2n}x\sin\overline{nx+\beta} + C_1\cos\overline{nx+\beta_1}.$$

(1) $d^2_\theta u \mp n^2 u = 0$.

Let $u = \varepsilon^m$ ∴ by substitution $m^2 - n^2 = 0$ or $m = n$ or $-n$.

$$\therefore\ u = C\varepsilon^{n\theta} + C_1\varepsilon^{-n\theta}$$

(2). $d^2_\theta u - n^2 u - a = 0$.

Let $u = u' - \frac{a}{n^2}$ or $u' = u + \frac{a}{n^2}$.

$$\therefore\ d^2_\theta . u' - n^2 u' = 0$$

$$\therefore\ u' = C\varepsilon^{n\theta} + C_1\varepsilon^{-n\theta}$$

$$\text{or } u = -\frac{a}{n^2}.\ C\varepsilon^{n\theta} + C_1\varepsilon^{-n\theta}.$$

(3). $d^2_\theta u - n^2 u - \theta^{n} = 0$.

Let u' be such a value of u that $d^2_\theta u \mp n^2 u = 0$

$$\therefore\ u' = C\varepsilon^{n\theta} + C_1\varepsilon^{-n\theta}.$$

$$\varepsilon^{-n\theta} u' = C + C_1\varepsilon^{-2n\theta}.$$

$$d_\theta . \varepsilon^{-n\theta} u' = -2n . \varepsilon^{-2n\theta}.$$

$$d_\theta \varepsilon^{2n\theta} d_\theta \varepsilon^{-n\theta} u' = 0. \text{ or } \varepsilon^{-n\theta} . d_\theta \varepsilon^{2n\theta} . d_\theta \varepsilon^{-n\theta} u = \theta^{n}$$

$$\varepsilon^{2n\theta} d_\theta . \varepsilon^{-n\theta} u = \int_\theta . \varepsilon^{n\theta} . \theta^{n} = \frac{1}{n}\varepsilon^{n\theta} . \theta^n - \int \varepsilon^{n\theta} . \theta^{n-1}$$

$$= \varepsilon^{n\theta}\left\{\frac{\theta^n}{n} - \frac{\theta^{n-1}}{n} + \frac{n-1}{n^2}\theta^{n-2} - \frac{(n-1)(n-2)}{n^3}\theta^{n-3} + \ldots (-1)^{n-1}\frac{(n-1)(n-2)\ldots 1.2}{n^n}.\theta\ (-1)^n\frac{(n-1)\ldots 3.2.1}{n^{n+1}}\right\} + C$$

$$\therefore\ \varepsilon^{-n\theta} u = \int_\theta \varepsilon^{-n\theta}\left\{\frac{\theta^n}{n} - \frac{\theta^{n-1}}{n} + \frac{n-1}{n^2}\theta^{n-2} + \ldots \ \ldots \&c \ \ (-1)^{n-1}\frac{(n-1)(n-2)\ldots 3.2.}{n^n}\theta + (-1)^n\frac{(n-1)(n-2)\ldots 3.2.1}{n^{n+1}}\right\} + C\varepsilon^{-2n\theta}.$$

see next page.

$$\varepsilon^{-n\theta} u = \int_\theta \varepsilon^{-n\theta} \left\{ \frac{\theta^n}{n} - \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^2}\theta^{n-2} + \dots - (-1)^{n}\frac{(n-1).(n-2)\dots 3.2}{n^n}.\theta + (-1)^n\frac{(n-1)\dots 3.2.1}{n^{n+1}} \right\} \pm \int_\theta \varepsilon^{-2n\theta}.$$

Now $\int_\theta \varepsilon^{-n\theta}.\frac{\theta^n}{n} = -\frac{\varepsilon^{-n\theta}}{n}\left\{\frac{\theta^n}{n} + \frac{\theta^{n-1}}{n^2} + \frac{n-1}{n^2}\theta^{n-2} + \dots \text{———} (+1)^n.\frac{(n-1)(n-2)\dots 3.2.1}{n^{n+1}}\right\}.$

$$-\int_\theta \varepsilon^{-n\theta}\frac{\theta^{n-1}}{n^2} = \frac{\varepsilon^{-n\theta}}{n^2}\left\{\frac{\theta^{n-1}}{n} + \frac{(n-1)\theta^{n-2}}{n^2} + \&c \text{———} (+1)^n\frac{(n-1)(n-2)\dots 3.2.1}{n^{n+1}}\right\}.$$

$$\therefore \frac{1}{n}\left\{\int_\theta \varepsilon^{-n\theta}\theta^n - \int_\theta \varepsilon^{-n\theta}.\theta^{n-1}\right\} = -\frac{1}{n^2}.\varepsilon^{-n\theta}.\theta^n.$$

Again $\frac{n-1}{n^2}.\int_\theta \varepsilon^{-n\theta}.\theta^{n-2} = -\frac{n-1}{n^3}.\varepsilon^{-n\theta}.\theta^{n-2} + \frac{(n-1)(n-2)}{n^3}\int_\theta \varepsilon^{-n\theta}.\theta^{n-3}.$

$$\therefore \frac{n-1}{n^2}\int_\theta \varepsilon^{-n\theta}\theta^{n-2} - \frac{(n-1)(n-2)}{n^3}.\int_\theta \varepsilon^{-n\theta}\theta^{n-3} = -\frac{n-1}{n^3}.\varepsilon^{-n\theta}.\theta^{n-2}.$$

wherefore we have at last.

$$\varepsilon^{-n\theta} u = -\frac{1}{n^2}\varepsilon^{-n\theta}.\left\{\frac{n\theta^n}{n^2} + \frac{n(n-1)\theta^{n-2}}{n^2} + \frac{n.(n-1)(n-2)(n-3)}{n^4}\theta^{n-4} + \&c\right\} \quad \mathcal{B} + \&c$$

Now in the series A. the number of terms is $n+1$. if n be odd. the last term is negative and there will be $\frac{1}{2}(n+1)$ pairs of terms. and $\therefore$ B will become.

$$\varepsilon^{-n\theta} u = -\frac{1}{n^2}\varepsilon^{-n\theta}\left\{\frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \dots (+1)^{n-1}.\frac{n(n-1)\dots 3.2}{n^{n-1}}\theta\right\} - \frac{1}{2n}\varepsilon^{-2n\theta} + C_1$$

If n be even it becomes.

$$\varepsilon^{-n\theta} u = -\frac{1}{n^2}\varepsilon^{-n\theta}\left\{\frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \dots (+1)^{n-2}.\frac{n(n-1)\dots 4.3}{n^{n-2}}\theta^2 + \frac{n(n-1)\dots 3.2.1}{n^n}\right\} - \frac{1}{2n}\varepsilon^{-2n\theta} + C_1$$

$$\therefore u = -\frac{1}{n^2}\left\{\frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2} + \frac{n(n-1)(n-2)(n-3)\theta^{n-4}}{n^4}\dots\dots + \frac{n(n-1)(n-2)\dots 3.2}{n^{n-1}}\theta\right\} + C_1\varepsilon^{n\theta} + C_2\varepsilon^{-n\theta}.$$

or $= -\frac{1}{n^2}\left\{\frac{n\theta^n}{n} + \frac{n(n-1)\theta^{n-2}}{n^2}\dots\dots + \frac{n(n-1)\dots 3.2.1}{n^n}\right\} + C_1\varepsilon^{n\theta} + C_2\varepsilon^{-n\theta}$. according as n is odd or even.

The last example may be more conveniently performed thus.

$$d^2_\theta u - n^2 u - \theta^n = 0.$$

$$(d^2_\theta - n^2)u = \theta^n$$

$$+ n^2\left\{\frac{1}{n^2}d^2_\theta - 1\right\}u = \theta^n \text{ or } u = -\frac{1}{n^2}\cdot\frac{\theta^n}{\left(1-\frac{1}{n^2}d^2_\theta\right)}$$

$$= -\frac{1}{n^2}\left\{\theta^n + \frac{n(n-1)}{n^2}\theta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{n^4} + \&c. \right\} + C_1 \varepsilon^{n\theta} + C_2 \varepsilon^{-n\theta}.$$

which is the same result as before.

4). $$d^2_\theta u = n^2 u + A \cos \overline{n\theta + b}.$$

As in example 3 we have.

$$\varepsilon^{2n\theta} d_\theta \varepsilon^{-n\theta} u = A\int_\theta \varepsilon^{n\theta}.\cos\overline{n\theta+b}. = -\frac{A}{n}\varepsilon^{n\theta}\cos\overline{n\theta+b} - \frac{An}{n}\int \varepsilon^{n\theta}\sin\overline{n\theta+b}$$

$$= -\frac{A}{n}\varepsilon^{n\theta}\cos\overline{n\theta+b} - \frac{A}{n}\varepsilon^{n\theta}\sin\overline{n\theta+b} + A\int_\theta \varepsilon^{n\theta}\cos\overline{n\theta+b}$$

$$= -\frac{A}{2n}\varepsilon^{n\theta}\left\{\cos\overline{n\theta+b} + \sin\overline{n\theta+b}\right\} + C_1$$

$$\therefore \varepsilon^{-n\theta} u = -\frac{A}{2n}\int_\theta \varepsilon^{-n\theta}.\cos\overline{n\theta+b} + \sin\overline{n\theta+b} + C_1\int_\theta \varepsilon^{-2n\theta}.$$

$$\int_\theta \varepsilon^{-n\theta}\cos\overline{n\theta+b} = -\frac{1}{n}\varepsilon^{-n\theta}\cos\overline{n\theta+b} - \int_\theta \varepsilon^{-n\theta}.\sin\overline{n\theta+b}$$

$$\therefore \int_\theta \varepsilon^{-n\theta}\left(\cos\overline{n\theta+b} + \sin\overline{n\theta+b}\right) = -\frac{1}{n}\varepsilon^{-n\theta}\cos\overline{n\theta+b}. + C_2$$

$$\varepsilon^{-n\theta} u = \frac{A}{2n^2}\varepsilon^{-n\theta}.\cos\overline{n\theta+b} - \frac{C_1}{2n}\varepsilon^{-2n\theta} + C_2.$$

$$u = \frac{A}{2n^2}\cdot\cos\overline{n\theta+b} + C_2\varepsilon^{n\theta} + C_3\varepsilon^{-n\theta}.$$

The last example may be more shortly performed thus.

$$d^2_\theta\left(u-\frac{A}{2n^2}\cos\overline{m\theta+b}\right)-n^2\left(u-\frac{A}{2n^2}\cos\overline{m\theta+b}\right)=0.$$

$$\therefore u-\frac{A}{2n^2}\cos\overline{m\theta+b}=C_1\varepsilon^{n\theta}+C_2\varepsilon^{-n\theta}.$$

5). $d^2_\theta u - n^2u + A\sin\overline{m\theta+b}+B\cos\overline{p\theta+c}=0.$

As before we have.

$$\varepsilon^{2n\theta}d_\theta\,\varepsilon^{-n\theta}u = -A\int_\theta \varepsilon^{n\theta}\sin\overline{m\theta+b} - B\int_\theta\varepsilon^{n\theta}\cos\overline{p\theta+c}$$

$$-A\int_\theta\varepsilon^{n\theta}\sin\overline{m\theta+b} = -\frac{A}{n}\varepsilon^{n\theta}\sin\overline{m\theta+b}+\frac{Am}{n}\int_\theta\varepsilon^{n\theta}\cos\overline{m\theta+b}.$$

$$+\frac{Am}{n}\left\{\frac{1}{n}\varepsilon^{n\theta}\cos\overline{m\theta+b}+\frac{m}{n}\int_\theta\varepsilon^{n\theta}\sin\overline{m\theta+b}\right\}$$

$$=-A\left\{\frac{n\varepsilon^{n\theta}\sin\overline{m\theta+b}-m\varepsilon^{n\theta}\cos\overline{m\theta+b}}{m^2+n^2}\right\}+C$$

$$-B\int_\theta\varepsilon^{n\theta}\cos\overline{p\theta+c} = -B\left\{\frac{n\varepsilon^{n\theta}\cos\overline{p\theta+c}+p\varepsilon^{n\theta}\sin\overline{p\theta+c}}{p^2+n^2}\right\}+C_1$$

$$\therefore \varepsilon^{2n\theta}d_\theta\,\varepsilon^{-n\theta}u = -\frac{A\varepsilon^{n\theta}}{m^2+n^2}\left\{n\sin\overline{m\theta+b}-m\cos\overline{m\theta+b}\right\}-\frac{B\varepsilon^{n\theta}}{p^2+n^2}\left\{n\cos\overline{p\theta+c}+p\sin\overline{p\theta+c}\right\}+C.$$

$$\therefore\varepsilon^{-n\theta}u = -\frac{A}{m^2+n^2}\int_\theta\left\{\varepsilon^{-n\theta}n\sin\overline{m\theta+b}-m\varepsilon^{-n\theta}\cos\overline{m\theta+b}\right\}-\frac{B}{p^2+n^2}\int_\theta\varepsilon^{-n\theta}\left\{n\cos\overline{p\theta+c}+p\sin\overline{p\theta+c}\right\}+C\varepsilon^{-2n\theta}.$$

$$=+\frac{A}{m^2+n^2}\varepsilon^{-n\theta}\sin\overline{m\theta+b}+\frac{B}{p^2+n^2}\varepsilon^{-n\theta}\cos\overline{p\theta+c}-\frac{1}{2n}C_1\varepsilon^{-2n\theta}+C_2.$$

$$\therefore u=\frac{A}{m^2+n^2}\sin\overline{m\theta+b}+\frac{B}{p^2+n^2}\cos\overline{p\theta+c}+C_2\varepsilon^{n\theta}+C_3\varepsilon^{-n\theta}.$$

$u = \theta^{n-1}\cos\theta \cdot \frac{du}{d\theta} = (n-1)\theta^{n-2}\cdot\cos\theta \mp \theta^{n-1}\sin\theta.$

$d^2_\theta u = (n-1)(n-2)\theta^{n-3}\cos\theta \mp 2(n-1)\theta^{n-2}\sin\theta - \theta^{n-1}\cos\theta.$

$d^4_\theta u = \cos\theta\cdot\theta^{n-1} + 4(n-1)\sin\theta\cdot\theta^{n-2} - 6\cos\theta(n-1)(n-2)\theta^{n-3} - 4\sin\theta(n-1)(n-2)(n-3)\theta^{n-4} + \&c$

&c &c

$u = \theta^{n-2}\cos^2\theta = \frac{1}{2}\cdot\theta^{n-2}\cdot\cos 2\theta + 1$

$\therefore d^2_\theta \cdot \frac{1}{2}\cdot\theta^{n-2}\cdot(\cos 2\theta + 1) = \frac{1}{2}\cdot(n-2)(n-3)\theta^{n-4} +$

$\frac{1}{2}\left(-2^2\cos 2\theta\cdot\theta^{n-2} - 2^2\cdot\sin 2\theta\cdot(n-2)\theta^{n-3} + \cos 2\theta\cdot(n-2)(n-3)\theta^{n-4}\right.$

- -

$\frac{1}{n^2}\cdot\theta^n - \frac{a}{n}\cos\theta + \frac{a}{n^3}\cos\theta - \frac{a}{n^5}\cos\theta \;\&c \;\Big|\; \theta^{n-1} + \frac{n(n-1)}{n^2}\Big|\; 1 - \frac{2a}{n^2}\sin\theta + \frac{4a}{n^4}\sin\theta \;\&c + \frac{a^2}{2}\cos^2\theta - \frac{a^2}{n^2}\cos 2\theta \;\&c \Big\}\; \theta^{n-2} + \frac{n(n-1)(n-2)}{n^3}\Big|\; \frac{a\cos\theta}{n} - \frac{6a\cos\theta}{n^2} \;\&c - \frac{2a^2\sin 2\theta}{n} \;\&c \Big\}\; \theta^{n-3}$

6.

As an extension of Ex. 3 we may take the following.

$$d^2_\theta u - n^2 u + (\theta - a\cos\theta)^n = 0.$$

$$\therefore (d^2_\theta - n^2)u = -(\theta - a\cos\theta)^n$$

$$-n^2\left(1 - \frac{1}{n^2}d^2_\theta\right)u = -(\theta - a\cos\theta)^n$$

$$u = +\frac{1}{n^2}\left\{\frac{\theta^n - n\theta^{n-1}a\cos\theta + \frac{n(n-1)}{1\cdot 2}\theta^{n-2}a^2\cos^2\theta + \&c}{\left(1 - \frac{1}{n^2}d^2_\theta\right)}\right\}.$$

$$= \frac{1}{n^2}\left\{\theta^n + \frac{n(n-1)}{n^2}\theta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{n^4}\theta^{n-4} + \&c\right.$$

$$-\frac{a}{n}\left\{\theta^{n-1}\cos\theta + \frac{1}{n^2}d^2_\theta\left(\theta^{n-1}\cos\theta\right\} + \frac{1}{n^4}d^4_\theta\left\{\theta^{n-1}\cos\theta\right\}\&c.\right.$$

$$+\frac{(n-1)a^2}{1\cdot 2n}\cdot\left\{\theta^{n-2}\cos^2\theta + \frac{1}{n^2}\cdot d^2_\theta\left\{\theta^{n-2}\cos^2\theta\right\} + \frac{1}{n^4}d^4_\theta\left\{\theta^{n-2}\cos^2\theta\right\} + \&c.\right.$$

$$+ \&c \quad \&c.$$

$$= \frac{1}{n^2}\left\{\theta^n + \frac{n(n-1)}{n^2}\cdot\theta^{n-2} + \frac{n\ldots(n-3)}{n^4}\theta^{n-4} + \&c.\right.$$

$$-\frac{na}{n^2}\left\{\cos\theta.\theta^{n-1} - \frac{\cos\theta.\theta^{n-1}}{n^2} - \frac{2(n-1)}{n^2}\sin\theta\,\theta^{n-2} + \frac{(n-1)(n-2)\cos\theta}{n^2}.\theta^{n-3}\right.$$

$$+\frac{1}{n^4}\cos\theta.\theta^{n-1} + \frac{4(n-1)}{n^4}\sin\theta.\theta^{n-2} - \frac{6(n-1)(n-2)}{n^4}\cos\theta.\theta^{n-3} - 4.\frac{n\ldots\overline{n-3}}{n^4}\theta^{n-4} + \&c.$$

$$+\frac{n(n-1)a^2}{1\cdot 2n^2}\left\{\ldots\ldots\ldots\ldots\ldots\quad \cos^2\theta.\theta^{n-2}\ldots\ldots\ldots\ldots\right.$$

$$\ldots\ldots\ldots -\frac{2}{n^2}\cos 2\theta.\theta^{n-2} - \frac{2(n-2)}{n^2}\sin 2\theta\,\theta^{n-3} + \frac{(n-2)(n-3)}{2n^2}\cos 2\theta.\theta^{n-4} + \&c.$$

$$\&c \quad \sim\sim$$

Collect the results and we have. $\ldots + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\, n^2}.a^4\cos^4\theta.\theta^n$

see opposite.

$$d_x^2 y - n^2 y - A \cos \overline{mx+a} - B \cos \overline{nx+\beta} = 0 .$$

$p = d_x y = f(x y)$ the proposed diffl.

$y = \phi(x, c)$ the complete integral (1)

Now let $y = u$ be a solution suppose it derived from
assigning c the value c'

$\therefore \quad \phi(x,c) - u =$ ~~...~~ a series of ascending powers of $c - c'$

$= az$ suppose where a is the lowest
power of $c-c'$ & z a function
wh. does not vanish when $c = c'$

$\therefore \quad y = u + az$ is the complete value of y

$\therefore d_x y = d_x u + a d_x z = f(x, u+az)$,

$= f(x,u) + az . d_{y=u} f(xy) + \dots$

but $\because u$ is a solution

$\therefore d_x u = f(x, u)$

$\therefore a d_x z = \quad az . d_{y=u} f(xy) + \dots \qquad (2)$

Now if $y = u$ be a particular integral obtained
by making $c = c'$ i.e. $a = 0$ & $\therefore$ (2) is satisfied
but if $y = u$ be a singular solution then (2) must
be satisfied by the vanishing of z independent
of the value of a ...

but ∵ $d_y z$ does not necessarily vanish when $z = 0$

in here d_y &c... $= \frac{\text{finite}}{0} = \infty$ when $y = 0$

Which cond[n] is the same as $d_{p_1} v = 0$ whilst

$d_{y_1} v$ remains finite ∵ $d_y v = -\frac{d_{y_1} v}{d_{p_1} v}$

when $v = 0 = + \&c\, y\, p_1$ is the form of the ~~equation~~;

& these solutions deduced by this condition are singular solutions, none else.

$d_y v = 0 = d_{y_1} v + d_{p_1} v \cdot d_y p =$

$\therefore d_y p = -\frac{d_{y_1} v}{d_{p_1} v}$

---

but if p be given by $x = ^2 + \&c\, y\, p_1 = 0 = v$ suppose

then $d_y v = 0 = d_{y_1} v + d_{p_1} v \cdot d_y p$

$\therefore d_y p = -\frac{d_{y_1} v}{d_{p_1} v}$

∴ the cond[n] $d_y p = \infty$ is equivalent to

$d_{p_1} v = 0$ whilst at same time $d_{y_1} v$ is finite

and the solutions deduced from these conditions are singular solutions, but if $y = u$ do not satisfy both these cond[ns] it is a particular integral

$$\frac{x^{p-1}}{(1+ax)(1-x)^p} = \frac{x^{p-1}}{(1-x)^p}\left\{1 - ax + a^2x^2 - \&c\right\}.$$

Now $\int_x^{0/1} \frac{x^{p-1}}{(1-x)^p} = \frac{\pi}{\sin p\pi}$.

$\therefore -a \int_x^{0/1} \frac{x^p}{(1-x)^p}$

$$\int_x^{0/1} \frac{x^{m+p-1}}{(1-x)^p} = \frac{m+p-1}{m}\int_x^{0/1}\frac{x^{m+p-2}}{(1-x)^p}$$

$$= \frac{m+p-1}{m}\cdot\frac{m+p-2}{m-1}\cdots \frac{p}{1}\int_x^{0/1}\frac{x^{p-1}}{(1-x)^p}$$

$$= \frac{(m+p-1)(m+p-2)\ldots p}{\lfloor m}\cdot\frac{\pi}{\sin p\pi}$$

$$\int_x^{0/1} \frac{x^{m+p-1}}{(1-x)^p} = \frac{(m+p-1)(m+p-2)\cdots p}{\lfloor m.}\,\frac{\pi}{\sin p\pi}.$$

$$\therefore -a\int_x^{0/1}\frac{x^p}{(1-x)^p} = \frac{-ap}{1}\,\frac{\pi}{\sin p\pi}.$$

$$a^2\int_x^{0/1}\frac{x^{p+1}}{(1-x)^p} = \frac{a^2(p+1)p.}{1.2}\,\frac{\pi}{\sin p\pi}.$$

$$-a^3\int_x^{0/1}\frac{x^{p+2}}{(1-x)^p} = \frac{-a^3(p+2)(p+1)p}{1.2.3.}\,\frac{\pi}{\sin p\pi}.$$

[illegible]

∴ By addition

$$\int_0^1 \frac{x^{p-1}}{(1+ax)(1-x)^p} = \left(1 - pa + \frac{p(p+1)}{1.2.}a^2 - \frac{p(p+1)(p+2)}{\lfloor 3}a^3 \ldots\right)\frac{\pi}{\sin p\pi}$$

$$= \frac{1}{(1+a)^p}\cdot\frac{\pi}{\sin p\pi}.$$

$$(1-x^2)\frac{dy}{dx} + 2xy = ax^2+b.$$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\,y = ax^2+b.$$

Integrat^g. factor is $\varepsilon^{\int_x \frac{2x}{1-x^2}}$

$$= \frac{1}{1-x^2}.$$

$$\therefore \frac{d}{dx}\left\{\frac{y}{(1-x^2)}\right\} = \frac{ax^2+b}{(1-x^2)^2}$$

$$= -\frac{a}{1-x^2} + \frac{a+b}{(1-x^2)^2}.$$

$$\therefore \frac{y}{1-x^2} = -\frac{a}{2}\cdot\log_\varepsilon\left(\frac{1+x}{1-x}\right) + (a+b)\left\{\frac{x(1-x^2)}{2} + \frac{1}{2}\int_x \frac{1}{1-x^2}\right.$$

$$= -\frac{a}{2}\cdot\log_\varepsilon\cdot\left(\frac{1+x}{1-x}\right) + (a+b)\left\{\frac{x\cdot(1-x^2)}{2} + \frac{1}{4}\log_\varepsilon\left(\frac{1+x}{1-x}\right)\right.$$

$$= \frac{b-a}{4}\cdot\log_\varepsilon\cdot\left(\frac{1+x}{1-x}\right) + \frac{a+b}{2}\cdot x\cdot(1-x^2)$$

$$(1+x^2)\, d_x y + xy + 1 = 0.$$

$$d_x y + \frac{x}{1+x^2}\, y + \frac{1}{1+x^2} = 0.$$

Integrat[g] factor $= \sqrt{1+x^2}$.

$$\sqrt{1+x^2}.\, d_x y + \frac{xy}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} = 0.$$

$$d_x \left\{ y\sqrt{1+x^2} \right) + \frac{1}{\sqrt{1+x^2}} = 0.$$

$$y\sqrt{1+x^2} = \log c - \log.\left(x + \sqrt{1-x^2}\right)$$

$$= \log.\left(\frac{c}{x+\sqrt{1-x^2}}\right).$$

$$\therefore\ c = \left(x + \sqrt{1-x^2}\right) \varepsilon^{y\sqrt{1+x^2}}. -$$

$$d_x y - \frac{2}{\sqrt{x}} y = a \frac{\sqrt{y}}{\sqrt{x}}.$$

$$\frac{d y_x}{2\sqrt{y}} - \frac{1}{\sqrt{x}} \sqrt{y} = \frac{a}{2\sqrt{x}}$$

$$d_x \sqrt{y} - \frac{1}{\sqrt{x}} \cdot \sqrt{y} = \frac{a}{2\sqrt{x}}.$$

Integrat^g. factor $\varepsilon^{-2\sqrt{x}}$.

$$\therefore d_x\left(\varepsilon^{-2\sqrt{x}} \cdot \sqrt{y}\right) = a \cdot \frac{\varepsilon^{-2\sqrt{x}}}{2\sqrt{x}}.$$

$$\therefore \sqrt{y} = -\frac{a\,\varepsilon^{2\sqrt{x}}}{2} \cdot \int_x - \frac{\varepsilon^{-2\sqrt{x}}}{\sqrt{x}}.$$

$$= -\frac{a}{2} \varepsilon^{2\sqrt{x}} \left\{ \varepsilon^{-2\sqrt{x}} + C_1 \right\}.$$

$$= -\frac{a}{2} \left\{ 1 + C_1 \varepsilon^{2\sqrt{x}} \right\}.$$

$$\frac{dz}{dx} = \frac{a}{\sqrt{a^2-y^2-x^2}}.$$

$$\therefore z = a\left\{\sin^{-1}\frac{x}{\sqrt{a^2-y^2}} - \sin^{-1}\varphi(y)\right\}.$$

$$px + qy + z = 0.$$

$$P = x. \quad Q = y. \quad R = -z.$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{x} = \frac{dy}{y} \quad \therefore \log\left(\frac{x}{y}\right) = \log c.$$

$$\frac{x}{y} = c$$

$$\frac{dx}{x} = -\frac{dz}{z}.$$

$$\therefore \log x + \log z = \log c_2.$$

$$\frac{x}{z} = c_2. \qquad xz = f\left(\frac{y}{x}\right)$$

$y = bx$
$z = ax$

$$x \cdot ax = f(bx) = \frac{a}{b^2}(bx)^2$$

$$\therefore \frac{x}{y} = f\left(\frac{x}{z}\right). \qquad xz = \frac{a}{b^2}\left(\frac{y}{x}\right)^2.$$

when $y = ax^2$, $z = bx$.

$$\therefore \frac{1}{ax} = f\left(\frac{x}{bx}\right) = f\left(\frac{1}{b}\right).$$

$$d_x^2 z - d_y^2 z = ax + by.$$

$$(d_x^2 - d_y^2) z = ax + by.$$

$$\therefore\ z = (d_x^2 - d_y^2)^{-1} \cdot (ax + by)$$

$$= (d_x^2)^{-1} \cdot (ax + by) + \varphi(y + x) + \psi(y - x)$$

$$= \int_x^2 \cdot (ax + by) + \varphi(y + x) + \psi\,\overline{y - x}$$

$$= \frac{ax^3}{6} + \frac{bx^2 y}{2} + \varphi \cdot \overline{y + x} + \psi\,\overline{y - x}$$

$$x^2 p + y^2 q = z^2.$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}.$$

$$\therefore -\frac{1}{x} + \frac{1}{y} = c.$$

$$-\frac{1}{x} + \frac{1}{z} = c_2.$$

$$\therefore \frac{1}{z} - \frac{1}{x} = f.\left(\frac{1}{y} - \frac{1}{x}\right)$$

$$p\cdot(z+y) + q\cdot(z+x) = (x+y).$$

$$\frac{dx}{z+y} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

$$\therefore \frac{dx+dy+dz}{2(x+y+z)} = -\frac{dx-dy}{x-y} = -\frac{dx-dz}{x-z}.$$

$$= -\frac{dy-dz}{y-z}$$

$$\therefore \tfrac{1}{2}\log(x+y+z) + \log(y-z) = \log c_1$$

$$(x+y+z)(y-z)^2 = c_1$$

$$(x+y+z)(x-z)^2 = c_2.$$

$$\therefore f\left\{(x+y+z)(x-z)^2, (x+y+z)(y-z)^2\right\} = 0$$

[illegible]

[illegible]

[illegible]

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$$d_x^2 y + \frac{1}{x} \cdot d_x y - \frac{1}{x^2} \cdot y = 1$$

$$x \cdot d_x^2 y + d_x y - \frac{y}{x} = x.$$

$$d_x^2(xy) - d_x y - \frac{y}{x} = x.$$

$$d_x xy =$$

$$\frac{y}{x} = z.$$

$$y = xz.$$

$$\therefore d_x y = z + x d_x z.$$

$$d_x^2 y = 2 d_x z + x d_x^2 z.$$

$$\text{Let } x = e^t. \quad d_t x = e^t = x.$$

$$d_x y = d_t y \, d_x t = \frac{d_t y}{d_t x}.$$

$$= \frac{1}{x} \cdot d_t y$$

$$d_x^2 y = d_x \left(d_t y \cdot d_x t \right) = d_t \left(d_t y \, d_x t \right) d_x t.$$

$$= d_t^2 y \left(d_x t \right)^2 + d_t y \cdot d_x^2 t$$

$$d_x y = \frac{d_t y}{d_t x}. \quad d_x^2 y = \frac{d_t x \, d_t^2 y - d_t y \, d_t^2 x}{\left(d_t x \right)^3}.$$

$$\frac{dz}{t} + \frac{dx}{t} = 2\sqrt{ga}.$$

Also $(z-x)^2 + (a-y)^2 = a^2.$

$$\left(\frac{dz}{t} - \frac{dx}{t}\right)(z-x) = (a-y)\frac{dy}{t}.$$

$$2.\left(\sqrt{ga} - \frac{dx}{t}\right)(z-x) = (a-y)\frac{dy}{t}.$$

or $\frac{dx}{t} = \sqrt{ga} - \frac{(a-y)\frac{dy}{t}}{2(z-x)}$

$$\frac{dz}{t} = \sqrt{ga} + \frac{(a-y)\frac{dy}{t}}{2(z-x)}.$$

$$\therefore 2ga + \left(\frac{a-y}{z-x}\right)^2 \frac{\left(\frac{dy}{t}\right)^2}{2} + \left(\frac{dy}{t}\right)^2 = 2g(2a-y).$$

Let $\frac{dy}{t} = 0$ then

$$a = 2a-y. \therefore y = a.$$

$y^2 - 2xyp + (1+x^2)p^2 - 1 = 0 = V.$

for a singular solut. $d_p V = 0$.

$\therefore -2xy + 2(1+x^2)p = 0.$

$$p = \frac{xy}{1+x^2}$$

$\therefore$ V becomes

$$y^2 - \frac{2x^2y^2}{(1+x^2)} + \frac{x^2y^2}{1+x^2} - 1 = 0.$$

$$y^2 - \frac{x^2y^2}{1+x^2} - 1 = 0.$$

$y^2 - 1 - x^2 = 0 \qquad y^2 = 1 + x^2.$

$$p = \frac{y - p}{x - p} \quad \text{(Find a singular solution.)}$$

$$\therefore \frac{px - p^2 + p - y}{x - p}$$

$$px - p^2 + p - y = 0$$

$$x - 2p + 1 = 0$$

$$p = \frac{x+1}{2}$$

$$\frac{2x^2 + 2x}{4} - \frac{x^2 + 2x + 1}{4} + \frac{2x + 2}{4} - y = 0$$

$$\frac{x^2}{4} + \frac{2x}{4} + \frac{1}{4} - y = 0$$

$$\frac{1}{4}(x+1)^2 - y = 0$$

$$\left(y - x\cdot\frac{dy}{dx}\right)^2 + \left(x - \frac{y}{\frac{dy}{dx}}\right)^2 = a^2$$

$$y^2 - 2xy\cdot p + x^2p^2 + x^2 - \frac{2xy}{p} + \frac{y^2}{p^2} = a^2$$

$$\left(\frac{y^2}{p^2} + x^2 - \frac{2xy}{p}\right)(p^2+1) = a^2 \;\therefore\; \left(\frac{y}{p} - x\right) = \frac{a}{\sqrt{p^2+1}}$$

$$\therefore\; \frac{y}{p^2} = \frac{ap}{(p^2+1)^{\frac{3}{2}}}$$

$$y^2(p^2+1)^3 = a^2p^6$$

$$y^{\frac{2}{3}}(p^2+1) = a^{\frac{2}{3}}p^2 \;\therefore\; \left(y^{\frac{2}{3}} - a^{\frac{2}{3}}\right)p^2 = -y^{\frac{2}{3}}$$

$$p^2 = \frac{-y^{\frac{2}{3}}}{y^{\frac{2}{3}} - a^{\frac{2}{3}}} = \frac{y^{\frac{2}{3}}}{a^{\frac{2}{3}} - y^{\frac{2}{3}}}$$

$$p^2+1 = \frac{-a^{\frac{2}{3}}}{y^{\frac{2}{3}} - a^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{a^{\frac{2}{3}} - y^{\frac{2}{3}}}$$

And $\frac{y^2}{p^2} + x^2 - \frac{2xy}{p} = \frac{a^2}{p^2+1}$

$$y^{\frac{4}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right) + x^2 - 2xy^{\frac{2}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}} = a^{\frac{4}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)$$

$$-y^{\frac{2}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}} + x = a^{\frac{2}{3}}\left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$x = \left(a^{\frac{2}{3}} - y^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$x^{\frac{2}{3}} = a^{\frac{2}{3}} - y^{\frac{2}{3}}$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$y \cdot d_x^2 y + \sqrt{1+(d_x y)^2} = 0.$$

$$\frac{d_x^2 y}{\sqrt{1+(d_x y)^2}} + \frac{1}{y} = 0.$$

$$\frac{d_x p}{\sqrt{1+p^2}} + \frac{1}{y} = 0.$$

$$\frac{p \cdot d_y p}{\sqrt{1+p^2}} + \frac{1}{y} = 0.$$

$$\sqrt{1+p^2} + \log y = \log c$$

$$\sqrt{1+p^2} = \log \cdot \frac{c}{y}.$$

$$y \cdot d_x^2 y + (d_x y)^2 = 0$$

$$d_x \cdot (y \cdot d_x y) = 0$$

$$\therefore \; y \cdot d_x y = C$$

$$d_x y = \frac{C}{y}$$

$$d_y x = \frac{y}{C}$$

$$\therefore \; x = \frac{1}{2}\frac{y^2}{C} + C'$$

$$\frac{dy}{dx} + \frac{a}{1-x^2} \cdot y = \frac{1+x}{(1-x)^3}$$

$$\int_x P = \frac{a}{2} \cdot ~~\text{[illegible]}\ \frac{1+x}{1+x}~~ = \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}}$$

$$\therefore\ d_x\left(y \cdot \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}}\right) = ~~\int_x~~ \frac{(1+x)^{\frac{a}{2}+1}}{(1-x)^{\frac{a}{2}+3}}$$

$$y = \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a}{2}} \frac{1+x}{(1-x)^3}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a+2}{2}} \frac{1}{(1-x)^2}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \left\{\frac{1}{2} \cdot \int_x \left(\frac{1+x}{1-x}\right)^{\frac{a+2}{2}} d_x\left(\frac{1+x}{1-x}\right)\right\}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}} \left\{\frac{1}{a+4} \cdot \left(\frac{1+x}{1-x}\right)^{\frac{a+4}{2}} + C\right.$$

$$= \frac{1}{a+4} \cdot \left(\frac{1+x}{1-x}\right) + C \cdot \left(\frac{1-x}{1+x}\right)^{\frac{a}{2}}.$$

$$(1-x^2)^{\frac{1}{2}}\frac{dy}{dx} - ny = x(1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} - \frac{n}{(1-x^2)^{\frac{1}{2}}}y = x.$$

Integrating factor $= \varepsilon^{\int_x \frac{-n}{(1-x^2)^{\frac{1}{2}}}} = \varepsilon^{-n\sin^{-1}x}.$

$$\therefore d_x\left\{y\,\varepsilon^{-n\sin^{-1}x}\right\} = x\,\varepsilon^{-n\sin^{-1}x}.$$

$$y = \varepsilon^{n\sin^{-1}x}\cdot\int_x x\,\varepsilon^{-n\sin^{-1}x}.$$

$$\int_x x\,\varepsilon^{-n\sin^{-1}x} = -\frac{x\sqrt{1-x^2}}{n}\cdot\varepsilon^{-n\sin^{-1}x} + \frac{1}{n}\int_x \sqrt{1-x^2}\,\varepsilon^{-n\sin^{-1}x}.$$

$$= -\frac{x\sqrt{1-x^2}}{n}\cdot\varepsilon^{-n\sin^{-1}x} - \frac{1}{n^2}(1-x^2)\,\varepsilon^{-n\sin^{-1}x} - \frac{2}{n^2}\int_x \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,x\,\varepsilon^{-n\sin^{-1}x}$$

$$= -\left\{\frac{nx\sqrt{1-x^2}+1-x^2}{n^2+2}\right\}\varepsilon^{-n\sin^{-1}x} + C$$

$$\therefore y = -\left\{\frac{nx\sqrt{1-x^2}+1-x^2}{n^2+2}\right\} + C\,\varepsilon^{n\sin^{-1}x}.$$

$$d_x y - \frac{n}{\sqrt{1-x^2}} \cdot y = x$$

$$\int_x \frac{1}{\sqrt{1-x^2}} \cdot \varepsilon^{-n\sin^{-1}x}$$

$$\int \mathcal{L} = -n \sin^{-1} x$$

$$d_x (y \cdot \varepsilon^{-n\sin^{-1}x}) = x \cdot \varepsilon^{-n\sin^{-1}x}$$

$$y \cdot \varepsilon^{-n\sin^{-1}x} = \frac{x^2}{2} \cdot \varepsilon^{-n\sin^{-1}x} + \frac{n}{2} \int_x \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{-x\sqrt{1-x^2}}{n} \cdot \varepsilon^{-n\sin^{-1}x} + \frac{1}{n} \int_x \varepsilon^{-n\sin^{-1}x} \cdot \sqrt{1-x^2} - \sigma$$

$$= x\sqrt{1-x^2} \left(\frac{1}{\sqrt{1-x^2}} \cdot \varepsilon^{-n\sin^{-1}x}\right)$$

$$= \frac{x}{2} \cdot \varepsilon^{-n\sin^{-1}x} + \frac{n}{2} \int_x \frac{x^2}{\sqrt{1-x^2}} \cdot \varepsilon^{-n\sin^{-1}x}$$

$$= \frac{x^2}{2} \cdot \varepsilon^{-\varphi} + \frac{n}{2} \cdot \int_x \frac{1}{\sqrt{1-x^2}} \cdot \varepsilon^{-\varphi} - \sqrt{1-x^2} \cdot \varepsilon^{-\varphi}$$

$$= \frac{x^2}{2} \cdot \varepsilon^{-\varphi} + \frac{n}{2} \cdot \left(-\frac{1}{n} \varepsilon^{-\varphi}\right) - \frac{n}{2} \cdot x^2$$

$$\sin^{-1} x = \theta$$

$$x = \sin\theta$$

$$d_\theta x = \cos\theta$$

$$y \cdot \varepsilon^{-n\theta} = \int_x x \cdot \varepsilon^{-n\theta} = \int_\theta \sin\theta \cos\theta \cdot \varepsilon^{-n\theta}$$

$$= \frac{1}{2} \cdot \int_\theta \sin 2\theta \cdot \varepsilon^{-n\theta}$$

$$= -\frac{1}{2} \cdot \varepsilon^{-n\theta} \cdot \frac{n \sin 2\theta + 2\cos 2\theta}{n^2+4}$$

$$= -\frac{1}{2} \cdot \varepsilon^{-n\theta} \cdot \frac{2 \cdot (1-2x^2) + 2n \cdot x\sqrt{1-x^2}}{n^2+4}$$

$$= -\varepsilon^{-n\theta} \cdot \frac{n x\sqrt{1-x^2} + 1 - 2x^2}{n^2+4}$$

$$d_x^2 y = f(y).$$

$$2 \cdot d_x y \, d_x^2 y = 2 \cdot f(y) \, d_x y.$$

$$(d_x y)^2 = \phi(y) + C$$

$$d_x y = \sqrt{\phi(y) + C} \, .$$

$$d_y x = \frac{1}{\sqrt{\phi(y) + C}}$$

$$x = \int_y \frac{1}{\sqrt{\phi(y) + C}}$$

$$E.\left(d_x^2 y \cdot d_x y \cdot y\right) = 0$$

$$d_x^2 y = d_x p = d_y p \cdot d_x y = p \cdot d_y p.$$

$\therefore$ $E.\left(d_x^2 y \cdot d_x y \cdot y\right)$ becomes.

$E.\left(p \cdot d_y p \cdot p \cdot y\right)$ which can

be reduced to an =n of the first order.

$$d_x^2 y + n^2 y = 0$$

$$(d_x^2 + n^2) y = 0$$

$$y = A \cos \overline{nx + B}$$

$$d_x^2 y = ax + by$$

$$d_x^2 (ax + by) - b (ax + by) = 0$$

$$\therefore \quad ax + by = C e^{\sqrt{b} \cdot x} + C_1 e^{-\sqrt{b} \cdot x}$$

$$(1 + x^2) d_x^2 y + 1 + (d_x y)^2 = 0$$

$$\frac{d_x^2 y}{1 + (d_x y)^2} + \frac{1}{1 + x^2} = 0$$

$$\therefore \tan^{-1} (d_x y) + \tan^{-1} x = \tan^{-1} C$$

$$\tan^{-1} \frac{d_x y + x}{1 - x d_x y} = \tan^{-1} C$$

[illegible]

$$\therefore \frac{\frac{dy}{dx} + x}{1 - x\frac{dy}{dx}} = C$$

$$\frac{dy}{dx} + x = C - Cx\frac{dy}{dx}$$

$$(Cx+1)\frac{dy}{dx} = C - x$$

$$\frac{dy}{dx} = \frac{C - x}{Cx + 1} = \frac{1}{C}\cdot\frac{C^2 - Cx - 1 + 1}{Cx + 1}$$

$$= \frac{C^2+1}{C^2}\,\frac{1}{x + \frac{1}{C}} - \frac{1}{C}$$

$$\therefore y = \frac{C^2+1}{C^2}\log\left(x + \frac{1}{C}\right) - \frac{x}{C} + c$$

$$P \cdot d_x^2 y + \frac{1}{2} d_x P \cdot d_x y + C y = 0 \cdot \{ P = f(x)$$

$$P\, 2 d_x y \cdot d_x^2 y + d_x P \cdot (d_x y)^2 + 2 C y\, d_x y = 0 \cdot$$

$$d_x \cdot \{ P \cdot (d_x y)^2 \} + 2 C y \cdot d_x y = 0$$

$$P \cdot (d_x y)^2 + C y^2 + C' = 0 \cdot$$

$$\frac{y.dx - x dy}{y\sqrt{y^2 - x^2}} = \frac{y.dx - x dy}{y^2} \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}$$

$$= \frac{d.\left(\frac{x}{y}\right)}{\sqrt{1 - \frac{x^2}{y^2}}} = 0.$$

$$\therefore \sin^{-1}\frac{x}{y} = C.$$

$$\frac{x^{n-1}}{y^{n+1}}\left\{ny\,dx - nx\,dy\right) = 0.$$

$$d\left(\frac{x^n}{y^n}\right) = 0.$$

$$\therefore \frac{x^n}{y^n} = C.$$

Reduce $(a+bx)^2 d_x^2 y + A(a+bx) d_x y + By = f(x)$
to a linear $=^n$ with constant coeffts.

let $a+bx = z$.

$\therefore b^2 z^2 . d_z^2 y + Ab.z. d_z y + By = f(x)$.

let $z = \varepsilon^{-t}$.

Then $d_z y = \dfrac{d_t y}{d_t z} = -\dfrac{1}{z} d_t y$.

$d_z^2 y = \dfrac{d_t^2 y}{(d_t z)^2} - \dfrac{d_t y \; d_t^2 z}{(d_t z)^3}$

$= \dfrac{d_t^2 y}{z^2} - \dfrac{d_t y}{z^2}$.

$\therefore z^2 . d_z^2 y = d_t^2 y - d_t y$.

$z . d_z y = - d_t y$.

$\therefore$ the $=^n$ becomes

$b^2 . (d_t^2 y - d_t y) - Ab . d_t y + By = f(x)$

$d_t^2 y - \left(\dfrac{A}{b} + 1\right) d_t y + \dfrac{B}{b^2} y = \dfrac{1}{b^2} f(x)$.

in wh. the coeffts are constant.

$$x\cdot(a+bx)\, d_x^2 y + (c+ex)\, d_x y = 0.$$

$$\frac{d_x^2 y}{d_x y} + \frac{c+ex}{x(a+bx)} = 0$$

$$\frac{d_x^2 y}{d_x y} + \frac{e}{2b}\left\{\frac{2bx+a}{ax+bx^2}\right\} + \frac{c-\frac{ae}{2b}}{b\left\{\frac{a^2}{4b^2}+\frac{ax}{b}+x^2-\frac{a^2}{4b^2}\right\}}$$

$$\therefore \log. d_x y + \frac{e}{2b}\log.(ax+bx^2) + \frac{c-\frac{ae}{2b}}{b}\int_x \frac{1}{\left(x+\frac{a}{2b}\right)^2-\left(\frac{a}{2b}\right)^2}$$

$$\log. d_x y + \frac{e}{2b}\log.(ax+bx^2) + \frac{c-\frac{ae}{2b}}{a}\log.\left(\frac{2bx+2a}{2bx}\right) = \log C.$$

$$\frac{Cx^{-2}}{ax^{-1}+b} + \frac{e}{a+bx}$$

$$d_x^2 y \, d_x^3 y = a.$$

$$q \, d_x q = a.$$

$$q^2 = 2ax + C$$

$$d_x p = \sqrt{2ax + C}.$$

$$p = d_x y = \frac{1}{3a}\left(2ax + C\right)^{\frac{3}{2}} + C'$$

$$y = \frac{1}{3a} \cdot \frac{1}{5a}\left(2ax + C\right)^{\frac{5}{2}} + C'x + C''$$

$$y^3 \cdot d_x^2 y = c$$

$$d_x^2 y = \frac{c}{y^3}.$$

$$2 d_x y \cdot d_x^2 y = \frac{2 c\, d_x y}{y^3}.$$

$$(d_x y)^2 = -\frac{c}{y^2} + c'$$

$$= \frac{c' y^2 - c}{y^2}.$$

$$d_y x = \frac{y}{\sqrt{c' y^2 - c}}.$$

$$\therefore x = \frac{1}{c'} \sqrt{c' y^2 - c} + c''.$$

$d_x^2 y + n^2 y = 0$

$y = A \cos nx + B \sin nx$

when $x = 0$ $y = 1$ $\therefore B = 0$

$1 = A \cos. \text{zero}$.

but cos zero $= 1$ $\therefore A = 1$

$y = \cos nx$

$d_x y = -n \sin nx = 0$ if $x = 0$.

$\therefore y = \cos nx$ satisfies the reqd condition

$$\Delta.\ \frac{\cot \frac{\theta}{2^x}}{2^x} = \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}}.$$

Express $\Delta^n u_x$ in terms of $d_x^n u_x$ and successive diff. coeffts.

Prove. $d_x^n u_x = \log.(1+\Delta)^n u_x$.

$$\Delta \frac{\cot \frac{\theta}{2^x}}{2^x} = \frac{\cot \frac{\theta}{2^{x+1}}}{2^{x+1}} - \frac{\cot \frac{\theta}{2^x}}{2^x} = \frac{2^x}{2^x . 2^{x+1}} \left\{ \cot \frac{\theta}{2^{x+1}} - 2 \cot \frac{2\theta}{2^{x+1}} \right\}.$$

$$= \frac{1}{2^{x+1}} \left\{ \cot \frac{\theta}{2^{x+1}} - \left\{ \frac{1 - \tan^2 \frac{\theta}{2^{x+1}}}{\tan \frac{\theta}{2^{x+1}}} \right\} \right\}$$

$$= \frac{1}{2^{x+1} \tan \frac{\theta}{2^{x+1}}} \left\{ 1 - 1 + \tan^2 \frac{\theta}{2^{x+1}} \right\} = \frac{\tan \frac{\theta}{2^{x+1}}}{2^{x+1}}.$$

$$E^n u_x = u_x + n . \Delta u_x + \frac{n(n-1)}{1 . 2} . \Delta^2 u_x + \&c.$$

$$= u_x + nh . \frac{\Delta u_x}{h} + \frac{nh(nh-h)}{1 . 2} . \frac{\Delta^2 u_x}{h^2} + \&c.$$

Let h decrease indefinitely while $nh = t$ remains constant.

then $\frac{\Delta u_x}{h}$ the limiting ratio of the diff. of u_x and the increment $= d_x u_x$

$$\frac{\Delta^2 u_x}{h^2} = d_x^2 u_x \quad \&c.$$

$$\therefore E^n u_x = u_x + t d_x u_x + \frac{t^2}{1 . 2} . d_x^2 u_x \quad - - + \frac{t^n}{\lfloor n} . d_x^n u_x.$$

$$= \left\{ 1 + t d_x + \frac{t^2 d_x^2}{1 . 2} + \text{---} \right\} u_x = \varepsilon^{t d_x} u_x.$$

$$(xp - y)(xp - 2y) + x^3 = 0.$$

$$x^2p^2 - 3xyp + 2y^2 + x^3 = 0$$

∴ for a singular solut.

$$2x^2p - 3xy = 0.$$

$$p = \frac{3y}{2x}.$$

$$\therefore \frac{9y^2}{4} - \frac{9}{2}y^2 + 2y^2 + x^3 = 0.$$

$$-\frac{y^2}{4} + x^3 = 0.$$

$$y^2 - 4x^3 = 0$$

$$\left(y - x \cdot d_x y\right)^2 = b^2 + a^2 \left(\frac{dy}{x}\right)^2$$

$$y^2 - 2xy\, d_x y + x^2 \left(\frac{dy}{x}\right)^2 = b^2 + a^2 \left(\frac{dy}{x}\right)^2$$

$$V = y^2 - 2xyp + (x^2 - a^2)p^2 - b^2 = 0$$

$$d_p V = -2xy + 2(x^2 - a^2)p = 0$$

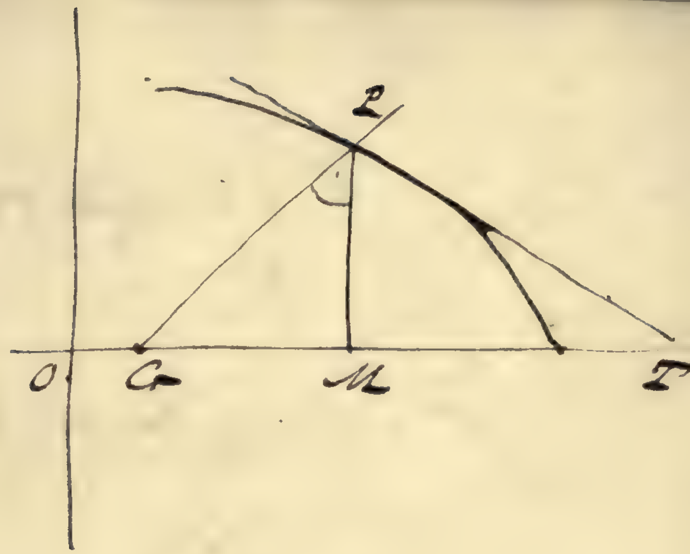
$$\therefore\ p = \frac{xy}{x^2 - a^2}$$

$$\therefore\ y^2 - \frac{2x^2y^2}{x^2 - a^2} + \frac{x^2y^2}{(x^2 - a^2)} - b^2 = 0$$

$$y^2 - \frac{x^2y^2}{x^2 - a^2} - b^2 = 0$$

$$-a^2y^2 = b^2(x^2 - a^2)$$

$$a^2(b^2 - y^2) = b^2x^2.$$



PG the normal =

$y\sqrt{1+(d_x y)^2}$.

$$\rho = -\frac{(1+(d_x y)^2)^{\frac{3}{2}}}{d_x^2 y}$$

If the normal = rad of curve we have.

$$y\sqrt{1+(d_x y)^2} + \frac{(1+(d_x y)^2)^{\frac{3}{2}}}{d_x^2 y} = 0$$

$$y\, d_x^2 y + 1 + (d_x y)^2 = 0.$$

$$\frac{d_x^2 y}{1+(d_x y)^2} + \frac{1}{y} = 0.$$

$$d_x(y \cdot d_x y) + 1 = 0.$$

$$y \cdot d_x y + x = C.$$

$$\frac{y^2}{2} + \frac{x^2}{2} = Cx + C_1$$

[illegible]

$$(\text{normal})^3 = y^3 \cdot \{1 + (d_x y)^2\}^{\frac{3}{2}}$$

$$\text{rad of curv} = \mp \frac{\{1 + (d_x y)^2\}^{\frac{3}{2}}}{d_x^2 y}$$

If $(\text{normal})^3 \propto \rho$ we have

$$C y^3 = \mp \frac{1}{d_x^2 y}$$

$$\therefore d_x^2 y \pm \frac{1}{C y^3} = 0$$

$$2 d_x y \, d_x^2 y \pm \frac{2 d_x y}{C y^3} = 0$$

$$(d_x y)^2 \mp \frac{1}{C y^2} = C'$$

$$d_x y = \frac{1}{\sqrt{C}} \cdot \frac{1}{y}$$

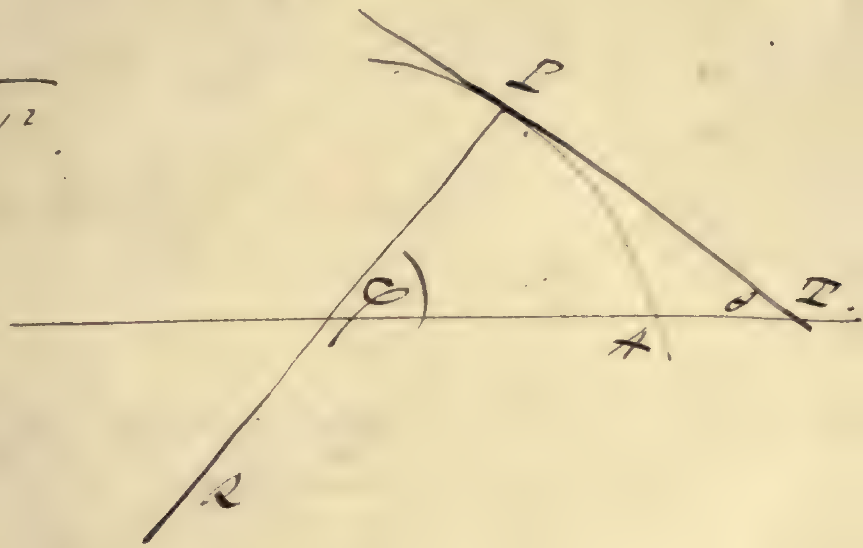
$$d_y x = \sqrt{C} \cdot y$$

$$x = \sqrt{C} \cdot \frac{y^2}{2} + C'$$

$$= C \pm \frac{1}{Cy^2}$$

$$= \frac{C'Cy^2 \pm 1}{Cy^2}$$

Find a curve in wh. radius of curv. ∝ $\operatorname{cosec}^3\phi$.



$\operatorname{cosec}\phi = \sec\theta = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

$\operatorname{cosec}^3\phi = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$.

$\rho = -\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

$\therefore$ diff $= \mu \operatorname{cosec}^3\phi$

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \mu\cdot\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{\mu}.$$

$$y = \frac{x^2}{2\mu} + C_1x + C_2.$$

$$z = f\left(\frac{x-z}{y-z}\right) = f(v) \text{ suppose.}$$

$$d_x z = d_v \cdot f(v) \; d_x v.$$

$$d_y z = d_v f(v) \; d_y v.$$

$$v = \frac{x-z}{y-z}.$$

$$d_x v = \frac{(y-z)(1-d_x z) + (x-z)\, d_x z}{(y-z)^2}.$$

$$= \frac{y-z + (x-y)\, d_x z}{(y-z)^2}.$$

$$d_y v = \frac{(x-y)\, d_y z - x - z}{(y-z)^2}.$$

$$\therefore \quad \frac{d_x z}{y-z + (x-y)\, d_x z} = \frac{d_y z}{(x-y)\, d_y z - (x-z)}$$

$$(x-y)\, d_x z\, d_y z - (x-z)\, d_x z =$$

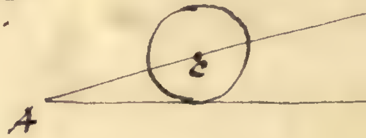
$$(y-z)\, d_y z + (x-y)\, d_x z \cdot d_y z.$$

$$(y-z)\, d_y z + (x-z)\, d_x z = 0.$$

The centre of a ⊙ moves along a line $y = mx + c$ its rad. increases so as always to be ∝ to the abscissa of its centre; find the curve which envelopes the ⊙s.

Let x. y. be the coords of the cen. of the ⊙

∴ the eq^n to the ⊙ is $(X-x)^2+(Y-y)^2 = c'^2$



but $c' = \mu x$; $y = mx + c$

$$\therefore \left(X - \frac{c'}{\mu}\right)^2 + \left(Y - \frac{mc'}{\mu} - c\right)^2 = c'^2. \qquad (A)$$

Differentiate with respect to c'

$$\frac{1}{\mu}\cdot\left(X - \frac{c'}{\mu}\right) + \frac{m}{\mu}\left\{Y - \frac{mc'}{\mu} - c\right) + c' = 0.$$

$$\left(1 - \frac{1}{\mu^2} - \frac{m^2}{\mu^2}\right)c' + \frac{m}{\mu}\{Y - c) + \frac{X}{\mu} = 0.$$

$$c' = \frac{m\mu\{Y + X - c)}{m^2+1-\mu^2}.$$

Substitute in A and we have.

$$\left(X - \frac{m\{Y+X-c)}{m^2+1-\mu^2}\right)^2 + \left(Y - \frac{m^2\{Y+X-c)}{m^2+1-\mu^2}\right)^2 = \left(\frac{m\mu.\overline{Y+X-c}}{m^2+1-\mu^2}\right)$$

$$\{(m^2-\mu^2)X - mY + c\}^2 + \{(1-\mu^2)Y - m^2X + m^2c\}^2 = m^2\mu^2\{Y + X - c\}$$

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

In a conic section $\rho = \dfrac{(\text{normal})^3}{(\text{semi lat. rect})^2}$

let $y^2 = nx + mx^2$ be the eqn to a conic secn.

$\therefore \rho = -\dfrac{1}{\frac{d^2y}{x}}\left(1+\left(\frac{dy}{x}\right)^2\right)^{\frac{3}{2}}$.

also the normal $= y.\left(1+\left(\frac{dy}{x}\right)^2\right)^{\frac{1}{2}}$.

$\therefore \rho = -\dfrac{1}{\frac{d^2y}{x}} \left(\dfrac{\text{normal}}{y^3}\right)^3$.

$2y\dfrac{dy}{x} = n + 2mx. \quad \therefore \dfrac{dy}{x} = \dfrac{n+2mx}{2y}.$

$2y\dfrac{d^2y}{x} + 2\left(\dfrac{dy}{x}\right)^2 = 2m. \quad \therefore \dfrac{d^2y}{x} = \dfrac{m-\left(\frac{dy}{x}\right)^2}{y}.$

$\dfrac{d^2y}{x} = \dfrac{4my^2 - \{n^2+4mnx+4m^2x^2\}}{4y^3} = -\dfrac{n^2}{4y^3} + \dfrac{4my^2-4mnx-4m^2x^2}{4y^3}$

$= -\dfrac{n^2}{4y^3}$ for $4m(y^2 - nx - mx^2) = 0$

$\therefore \rho = \dfrac{4y^3}{n^2}.\dfrac{(\text{normal})^3}{y^3} = \dfrac{(\text{normal})^3}{\left(\frac{n}{2}\right)^2}$

and $\frac{n}{2}$ is the semi-latus rectum.

$$u = \int_x \sqrt{1+p^2}$$

$$\delta u = \int_x \left\{ \left(\sqrt{1+p^2} + \frac{p\cdot\delta p}{\sqrt{1+p^2}}\right)(1 + d_x \cdot \delta x) - \sqrt{1+p^2} \right\}$$

$$= \int_x \frac{p\cdot \delta p}{\sqrt{1+p^2}} + \sqrt{1+p^2}\cdot d_x(\delta x)$$

$$= \int_x p\cdot\frac{(d_x \delta y - p\, d_x \delta x)}{\sqrt{1+p^2}} + \sqrt{1+p^2}\cdot d_x \cdot \delta x$$

$$= \int_x p\cdot\frac{(d_x\cdot \overline{\delta y - p\,\delta x} + d_x p\cdot \delta x)}{\sqrt{1+p^2}} + \sqrt{1+p^2}\cdot d_x \delta x$$

$$= \int_x \left\{ p\cdot\frac{d_x(\delta y - p\,\delta x)}{\sqrt{1+p^2}} + d_x(\delta x\cdot\sqrt{1+p^2}) \right\}$$

$$= \delta x\cdot\sqrt{1+p^2} + (\delta y - p\,\delta x)\frac{p}{\sqrt{1+p^2}} - \int_x (\delta y - p\,\delta x)\, d_x\left(\frac{p}{\sqrt{1+p^2}}\right)$$

2. $$= -\int_x (\delta y - p\,\delta x)\; d_x \frac{p}{\sqrt{1+p^2}}$$

since at limits δy δx are each zero

& $\therefore\ d_x\left(\frac{p}{\sqrt{1+p^2}}\right) = 0$

$$\frac{p}{\sqrt{1+p^2}} = c$$

$$\therefore\ p = c_1$$

$$y = c_1 x + c_2$$

Calculus of Variations
find the shortest curve joining two points.

tion

$$u = \int_x \sqrt{1+p^2}$$

$$\delta u = \int_x \left(\sqrt{1+d_x y'^2} \cdot d_x x' - \sqrt{1+p^2} \right)$$

$$= \int_x \left\{ \sqrt{(d_x x')^2 + (d_x y')^2} - \sqrt{1+p^2} \right\}$$

$$= \int_x \left\{ \sqrt{(1+d_x \delta x)^2 + (p + d_x \delta y)^2} - \sqrt{1+p^2} \right\}$$

$$= \int_x \left\{ \sqrt{1+p^2} + \frac{d_x \delta x + p \cdot d_x \delta y}{\sqrt{1+p^2}} \right.$$

Find the equation to the rectangular trajectory of a system of lemniscates comprised in the equation $r_1 r_2 = c^2$; $r_1 r_2$ being radii vectores drawn from the poles to any point in the curve, and c being the variable parameter.

$2a$ = distance between the poles. Take the origin at the bisection of this distance. Then

$$r_1^2 = r^2 + a^2 + 2ar\cos\theta.$$

$$r_2^2 = r^2 + a^2 - 2ar\cos\theta.$$

$$\therefore r_1^2 r_2^2 = c^4 = (r^2+a^2)^2 - 4a^2r^2\cos^2\theta.$$

$$= r^4 + a^4 - 2a^2r^2\cos 2\theta.$$

$$0 = r^3 - a^2 r\cos 2\theta + a^2 r^2 \sin 2\theta \frac{d\theta}{dr}.$$

If then at any point of the curve $\frac{r\,d\theta}{dr} = \varphi(r\,\theta)$

$$0 = r^2 - a^2\cos 2\theta + a^2\sin 2\theta \cdot \varphi(r\,\theta)$$

but $r\,\theta$ being a point in the trajectory we have

$$0 = 1 + \frac{r\,d\theta}{dr}\varphi(r\,\theta)$$

$$0 = r^3 - a^2\left(r\cos 2\theta + \sin 2\theta \frac{dr}{d\theta}\right)$$

$$0 = r^3 d\theta - a^2 dr \sin 2\theta.$$

$$\therefore \frac{a^2 dr \sin 2\theta}{r^3 \sin^3 2\theta} = \frac{d\theta}{\sin^3 2\theta} \quad \therefore \frac{a^2}{2r^2\sin 2\theta} = -\int \frac{1}{\sin^3 2\theta} = -\int_\theta \frac{\sin^2 2\theta + \cos^2 2\theta}{\sin^3 2\theta}$$

$$\therefore \frac{a^2}{r^2} = 2\cos 2\theta - \sin 2\theta \log_\epsilon (\tan\theta)^{\frac{3}{4}} + c\sin 2\theta.$$

$$\left(\frac{d\theta}{dt}\right)^2 = 3g\left\{\frac{1}{a\cos\alpha} - \frac{1}{b\,d\alpha}\right) - \frac{3}{2}g\left(\frac{1}{a}\right.$$

$$\int \frac{d\theta}{(\sin\frac{\theta}{2})\cos\theta} = -\frac{b}{2}\log\tan\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\cdot\log\tan\left\{\left(\frac{\theta - 45^\circ}{2}\right)\right\}$$

$$\left(\frac{d\theta}{dt}\right)^2 = 3g\left\{\frac{1}{a\cos\theta} - \frac{1}{b\,d\alpha}\right) - \frac{3}{8}g\cdot\frac{1}{a}\cdot\log\tan\left(\frac{\theta-45^\circ}{2}\right) - \frac{1}{b}\tan\frac{\theta}{2}$$

$$= \frac{3g}{2a}$$

$$gm\frac{a}{2}\cdot(\cos\alpha - \cos\theta)\cos\theta$$

$$d^2\theta \qquad gm\frac{a}{2}\cdot(d\alpha - d\theta)\,d\theta$$

$$gm\frac{a}{2}\left\{\cos\alpha\cos\theta - \sin\alpha\sin\theta\right) + C$$

$$gm\frac{a}{2}\cdot\cos(\theta+\alpha) + C$$

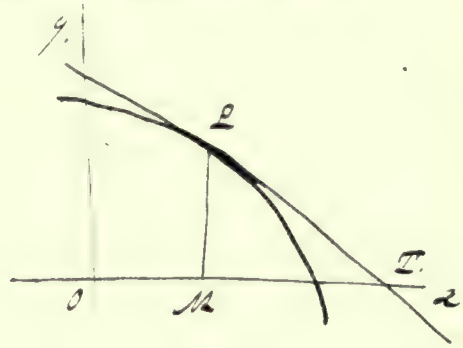
$$v\frac{dv}{ds} = gm\frac{a}{2}\cos(\theta-\alpha) + C$$

$$v^2 = c^2 - 2gx$$

$$\frac{d^2x}{dt^2} = -g\cos\alpha\cos\theta$$

Diffl = n. Brady

Find a curve in wh. the subtangent is a multiple of the abscissa.



$y = f(x)$ the reqd curve.

O the origin. $OM = x$. $ML = y$. coords of L.

Then subtangent $MT = -y \cot LTM$.

$$= \frac{-y}{\frac{dy}{dx}}.$$

Absolute abscissa $= OM = x$.

$$\therefore mx = -\frac{y}{\frac{dy}{dx}}.$$

$$\therefore \frac{\frac{dy}{dx}}{y} = -\frac{1}{mx}.$$

$$\log y = -\frac{1}{m}\log x = \log\left(\frac{1}{x}\right)^{\frac{1}{m}} + \log C.$$

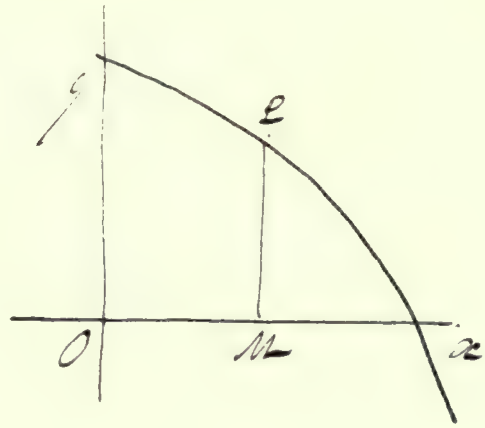
$$\text{or } y = \left(\frac{C}{x}\right)^{\frac{1}{m}}.$$

$$y^m = \frac{C}{x}.$$

Body

Greg. P. 486. Ex. 2.

Let $yOML$ be the given area.



$\therefore A = \int_x y.$

Also the rectangle contd. by the ordinate and the abscissa is xy.

$$\therefore \int_x y = m \cdot xy.$$

$$\frac{1}{mx} = \frac{y}{\int_x y}.$$

$$\frac{1}{m}\log x + \log C = \log_c \int_x y.$$

$$\therefore (Cx)^{\frac{1}{m}} = \int_x y.$$

$$\therefore y = \frac{m C^{\frac{1}{m}}}{m+1} x^{\frac{m+1}{m}}$$

$$y = C' x^{\frac{m+1}{m}}.$$

$$y = C^{\frac{1}{m}} \frac{1}{m} x^{\frac{1-m}{m}}.$$

$$= C' x^{\frac{1-m}{m}}.$$

Greg. p. 437. Ex. 3. Booly

$$p = \frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\therefore x^2 \left\{1+\left(\frac{dy}{dx}\right)^2\right\} = y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2.$$

$$x^2 = y^2 - 2xy\frac{dy}{dx}.$$

Put $y^2 = vz$.

$$\therefore x^2 = vz - x\frac{d}{dx}(vz).$$

$$= vz - xv\frac{dz}{dx} - xz\frac{dv}{dx}.$$

$$vz - xv\frac{dz}{dx} = 0, \quad \frac{\frac{dz}{dx}}{z} = \frac{1}{x}.$$

$$\log z = \log x - \log C.$$

$$z = \frac{x}{C}.$$

But $z\frac{dv}{dx} + x = 0.$

$$\frac{x}{C}\frac{dv}{dx} + x = 0.$$

$$\frac{dv}{dx} = -C$$

$$v = -C_1 x + C_2.$$

$$\therefore y^2 = zv = -C_1 x^2 + C'x.$$

$$\text{or } y^2 + x^2 = C'x.$$

put of 3 white and two black balls, two are placed at random in one bag, and 3 in the other. after which a white ball is accidentally lost: find the probability that a person going to either bag will draw a white ball.

There are 10 combinations of 5 things taken 3 together. The probability of there being 3 white balls in the large bag is $\frac{1}{10}$

——————— 2 white ~~and one~~ black. —— $\frac{6}{10}$

1 white and 2 black —— $\frac{3}{10}$.

∴ if the white ball had not been lost, the probability of drawing a white ball would have been. $\frac{3+2.6+3}{30} = \frac{3}{5}$

Also there are 10 combinations of 5 things taken 2 together. The probability of there being 2 white balls in the small bag is $\frac{3}{10}$

——————— 1 white and 1 black —— $\frac{6}{10}$

2 black ——————— $\frac{1}{10}$.

∴ under the same circumstances as before, the probability of drawing a white ball would be. $\frac{2\times3+6}{20} = \frac{12}{20} = \frac{3}{5}$. as in the case of the large bag, and as would also have been the case if the balls had not been separated.

∴ The probability of drawing a white ball from either bag after one white one was lost, would be either if the balls all remained together, or if separated into 2 bags. $\frac{2}{4}$ or $\frac{1}{2}$.

$$\sqrt{a+\sqrt{b}} = \sqrt[4]{\alpha}+\sqrt[4]{\beta}.$$

$$\sqrt{a-\sqrt{b}} = \sqrt[4]{\alpha}-\sqrt[4]{\beta}.$$

$$\sqrt{a^2-b} = \sqrt{\alpha}-\sqrt{\beta}.$$

$$\therefore\ a^2-b = \alpha - 2\sqrt{\alpha\beta}+\beta.$$

$$a^2+2a\sqrt{b}+b = \alpha + 4\alpha^{\frac{3}{4}}\beta^{\frac{1}{4}} + 6\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} + 4\alpha^{\frac{1}{4}}\beta^{\frac{3}{4}} + \beta.$$

$$\therefore\ 2(a+\sqrt{b})\sqrt{b} = 4\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\left(\alpha^{\frac{1}{2}}+2\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}+\beta^{\frac{1}{2}}\right)$$

$$(a+\sqrt{b}.)\sqrt{b} = 2.\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}.\left(\sqrt[4]{\alpha}+\sqrt[4]{\beta}\right)^2$$

$$\therefore\ b = 4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}.$$

$$a^2-b = \alpha-2\sqrt{\alpha\beta}+\beta$$

$$\therefore\ a^2 = \alpha+2\sqrt{\alpha\beta}+\beta.$$

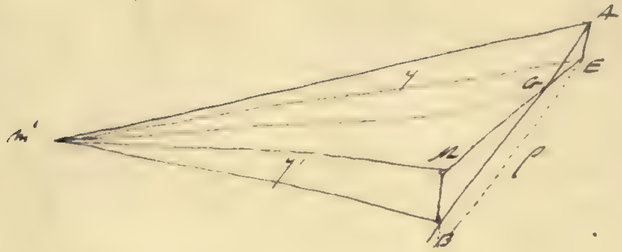
$$\therefore\ 1-\frac{a^2}{b} = \frac{b-a^2}{b} = \frac{4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}-\alpha-2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}-\beta}{4\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}.$$

$$= -\left(\frac{\sqrt{\alpha}+\sqrt{\beta}}{2\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}}\right)^2.$$

$\therefore\ 1-\frac{a^2}{b}$ is a perfect square.

if $\sqrt{a\pm\sqrt{b}} = \sqrt[4]{\alpha}\pm\sqrt[4]{\beta}.$

Lunar Theory.



$$y^2 = (m'A)^2 + AE^2$$
$$= r'^2 + 2r'GA\cos\overline{\theta-\theta'} + GA^2 + s^2 GA^2$$

$$\therefore \frac{1}{y^3} = \frac{1}{r'^3}\left\{1 - \frac{3GA}{r'}\cos\overline{\theta-\theta'}\right\} \text{ nearly.}$$

Similarly $\frac{1}{y'^3} = \frac{1}{r'^3}\left\{1 + \frac{3GB}{r'}\cos\overline{\theta-\theta'}\right\}$ nearly.

$$\therefore \frac{1}{y'^3} - \frac{1}{y^3} = \frac{3}{r'^4}\left\{GA + GB\cos\overline{\theta-\theta'}\right\} = \frac{3\rho}{r'^4}\cos\overline{\theta-\theta'}.$$

$$\frac{1}{\sqrt{1+s^2}}\left\{\frac{MG}{y'^3} + \frac{EG}{y^3}\right\} = \frac{AG}{y'^3} + \frac{EG}{y^3} = \frac{M\rho}{\mu r'^3}\left\{1 - \frac{3M\rho}{\mu r'}\cos\overline{\theta-\theta'}\right\} + \frac{E\rho}{\mu r'^3}\left\{1 + \frac{3E\rho}{\mu r'}\cos\overline{\theta-\theta'}\right\}$$
$$= \frac{\rho}{r'^3}\left\{1 + \frac{3(E-M)\rho}{\mu\, r'}\cos\overline{\theta-\theta'}\right\} = \frac{\rho}{r'^3} \text{ nearly.}$$

$$P = \frac{\mu}{\rho^2(1+s^2)^{\frac{3}{2}}} + m'\left\{\frac{MG}{\sqrt{1+s^2}\,y'^3} + \frac{EG}{\sqrt{1+s^2}\,y^3} - r'\cos\overline{\theta-\theta'}\left(\frac{1}{y'^3} - \frac{1}{y^3}\right)\right\}.$$
$$= \frac{\mu}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m'\rho}{r'^3} - \frac{3m'\rho}{r'^3}\cos^2\overline{\theta-\theta'}$$
$$= \frac{\mu}{\rho^2}\frac{1}{(1+s^2)^{\frac{3}{2}}} + \frac{m'\rho}{r'^3} - \frac{3m'\rho}{2r'^3}\left\{1 + \cos 2\overline{\theta-\theta'}\right\}.$$
$$= \frac{\mu}{\rho^2}\left(1 - \frac{3}{2}s^2\right) - \frac{m'\rho}{r'^3}\left\{\frac{1}{2} + \frac{3}{2}\cos 2\overline{\theta-\theta'}\right\}. \text{ nearly.}$$

$$S = \frac{\mu s}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m's}{\sqrt{1+s^2}}\left\{\frac{MG}{y'^3} + \frac{EG}{y^3}\right\}$$
$$= \frac{\mu s}{\rho^2(1+s^2)^{\frac{3}{2}}} + \frac{m's\rho}{r'^3} \text{ nearly}$$
$$\therefore\; S - Ps = \frac{3m'\rho}{2r'^3}\left\{1 + \cos 2\overline{\theta-\theta'}\right\}.$$

$$T = -m'r'\sin\overline{\theta-\theta'}\left(\frac{1}{y'^3} - \frac{1}{y^3}\right) = -m'r'\sin\overline{\theta-\theta'}\left\{\frac{3\rho}{r'^4}\cos\overline{\theta-\theta'}\right\}.$$
$$= -\frac{3m'\rho}{2r'^3}\sin 2\overline{\theta-\theta'} \text{ nearly.}$$

let $R = \dfrac{1}{\sqrt{1-2ph+h^2}}$

$$\frac{dR}{dp} = \frac{h}{(1-2ph+h^2)^{\frac{3}{2}}}$$

$$\therefore (1-p^2)\frac{dR}{dp} = \frac{h(1-p^2)}{(1-2ph+h^2)^{\frac{3}{2}}}$$

$$\frac{d}{dp}\left\{(1-p^2)\frac{dR}{dp}\right\} = \frac{-(1-2ph+h^2)\,2ph+3h(1-p^2)h}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= h\left\{\frac{3h-3hp^2-2p+4p^2h-2ph^2}{(1-2ph+h^2)^{\frac{5}{2}}}\right\}$$

$$= h\left\{\frac{3h+p^2h-2p-2ph^2}{(1-2ph+h^2)^{\frac{5}{2}}}\right\}$$

also $\dfrac{dR}{dh} = \dfrac{p-h}{(1-2ph+h^2)^{\frac{3}{2}}}$

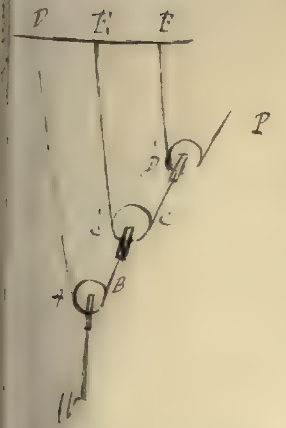
$$h^2\frac{dR}{dh} = \frac{ph^2-h^3}{(1-2ph+h^2)^{\frac{3}{2}}}$$

$$\frac{d}{dh}\left(h^2\frac{dR}{dh}\right) = \frac{(1-2ph+h^2)(2ph-3h^2)+(ph^2-h^3)\,3(p-h)}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= \frac{2ph-4p^2h^2+2ph^3-3h^2+\cancel{6ph^3}-\cancel{3h^4}+3p^2h^2-\cancel{6ph^3}+\cancel{3h^4}}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$$= \frac{2ph-p^2h^2+2ph^3-3h^2}{(1-2ph+h^2)^{\frac{5}{2}}}$$

$\therefore \dfrac{d}{dp}\left\{(1-p^2)\dfrac{dR}{dp}\right\} + \dfrac{d}{dh}\left\{h^2\dfrac{dR}{dh}\right\} = 0$. $\quad\therefore \dfrac{d}{dp}(1-p^2)\dfrac{dL_n}{dp} + n(n+1)L_n = 0$



in the annexed system supposing AD to be the angle, EC ... C' BD' ... θ''

The ~~pressure~~ pressure at B = $\frac{w}{2\cos\theta}$

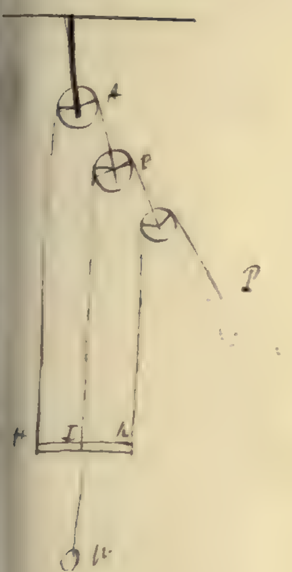
—— —— $C = \frac{\text{press. B}}{2\cos\theta'} = \frac{w}{2^2\cos\theta\cos\theta'}$

—— —— —— $P = \frac{\text{press. at C}}{2\cos\theta''} = \frac{w}{2^3\cos\theta\cos\theta'\cos\theta''}$

and in the same manner if n be the number of moveable pullies we have

$$P = \frac{w}{2^n\cos\theta\cos\theta'\cos\theta'' \text{ — } \cos\theta^{n}}$$

$$\left.P = \frac{w}{2^n\cos^n\theta}\right\} \text{ if } \theta = \theta' = \theta'' \text{ — }$$



$$\left.\begin{array}{l} P = \text{pressure at C.} = \text{pressure at K} \\ 2P\cos\theta = \text{pressure at B.} = \text{———— I} \\ 2^2P\cos\theta\cos\theta' = \text{pressure at A} = \text{———— H} \end{array}\right\}$$

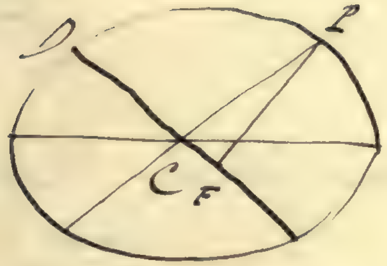
Similarly if n be the number of moveable pullies the pressure on the last string =

$$\left.2^nP\cos\theta\cos\theta' \text{ — — } \cos\theta^{(n)}\right\}$$

now if θ = θ' = θ'' ———— each

$$W = 2P\left\{\cos\theta + 2\cos^2\theta + 2^2\cos^3\theta + \text{ — } 2^{n-1}\cos^n\theta\right\}$$

$$\therefore P = \frac{W}{\left\{2\cos\theta + 2^2\cos^2\theta + 2^3\cos^3\theta \text{ —— } 2^n\cos^n\theta\right\}}$$



Let CP. CD be = semi conjugate diameters.

Chd of Curv. at P. through C = $\frac{2CD^2}{CP}$

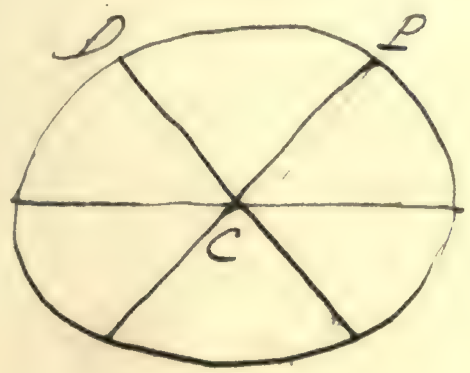
Diam of Curv at P $\frac{2CD^2}{PF}$

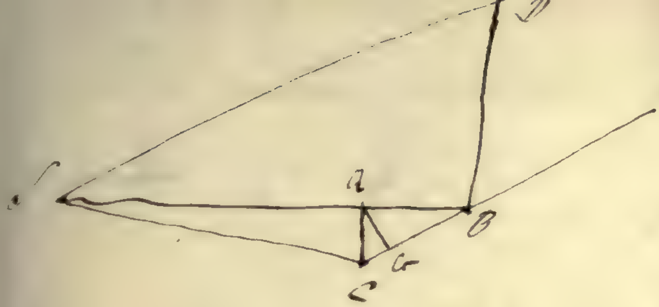
∴ Chd of Curv × Diam of Curv = $\frac{4CD^4}{CP.PF}$

$$= \frac{CD^4 + 2CD^2.CP^2 + PC^4}{CD.PF} \quad \text{if } CD = PC$$

$$= \frac{(CD^2 + CP^2)^2}{CD.PF}$$

$$= \frac{(a^2+b^2)^2}{ab}$$


D
P
C



Let CB be the direc.ⁿ of mot.ⁿ before the change of mot.ⁿ
BD the direc.ⁿ after the mot. BD = 2CB. — SB = a

Then area of △ SCB $= \frac{SB.BC}{2} \sin 30 = \frac{SB.BC}{4}$.

area of △ SBD = ½ SB.BD = $\frac{2}{2}$ SB.BC.

∴ the .

CA : AB.

CB + CA : AB :: AB : CB − CA

$CB^2 - CA^2 = AB^2$

Let the body uninfluenced by S move in the straight line CB. Join SB. this is the direc.ⁿ of the rad. vector. Resolve the velocity in the direc.ⁿ AB and CA. these are ∥ᴸ to the sides of the △ CBA.

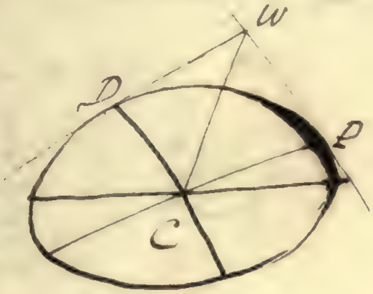
now $BA^2 = CB^2 - AC^2 = (CB + AC)(CB - AC)$

∴ (CB + AC) : BA :: BA : (CB − AC)

and this being true will be true in the limit when the force wh.ᶜʰ acts on the body ~~holding~~ causing it to move in the curve

∴ the velocity in the direc.ⁿ of the rad. vector is a mean ∥ᴸ. ~~[illegible]~~ between the sum and diff. of the

If c, c' are the curvatures at points of an ⊙ where the tangents are ⊥, $c^{\frac{2}{3}} + c'^{\frac{2}{3}}$ is constant.



Let the tangents at D and P meet at right angles in W.

C the centre of the ⊙. CD and CP the semi conjugate diams at D and P are always ∥ to the tangents at the extremity ∴ CP is ⊥ to CD.

Let c = curvat at P. c' curvat at D

$$c = \frac{2CD^2}{CP}, \qquad c' = \frac{2CP^2}{CD}.$$

$$\therefore c^{\frac{2}{3}} + c'^{\frac{2}{3}} = 2^{\frac{2}{3}}\left\{\frac{CD^{\frac{4}{3}}}{CP^{\frac{2}{3}}} + \frac{CP^{\frac{4}{3}}}{CD^{\frac{2}{3}}}\right\}$$

$$= 2^{\frac{2}{3}}\left\{\frac{CD^2 + CP^2}{CP.CD^{\frac{2}{3}}}\right\}$$

$$= 2^{\frac{2}{3}}\,\frac{a^2+b^2}{(ab)^{\frac{2}{3}}}$$ which is constant.

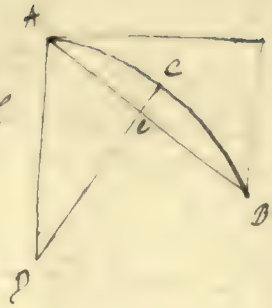
$$c \propto \frac{PF}{CD^2} \propto \frac{AC.BC}{CD^3}$$

$$c^{\frac{2}{3}} \propto \frac{1}{CD^2}$$

$$c'^{\frac{2}{3}} \propto \frac{1}{CP^2} \quad \therefore c^{\frac{2}{3}} + c'^{\frac{2}{3}} \propto \frac{1}{CD^2} + \frac{1}{CP^2}$$

$$\propto \frac{CD^2 + CP^2}{CD^2.CP^2}$$

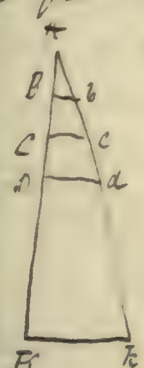
Let ACB be part of a circle. Centre D. bisect ACB in C and join DC. this will evidently bisect AB. wherefore since AB ultimately = ACB, Ac ultimately = AC. Let ∠ ADC = θ.



Then $\frac{\sin\theta}{\theta} = \frac{\frac{Ac}{AD}}{\frac{AC}{AD}} = \frac{Ac}{AC} = 1$

when θ equals 0.

1. At the begining of the eleventh page where it is stated area ABD : area Abd :: $AD^2 : Ad^2$ it is necessary that the ordinates should be ∥ to each other ∴ here the limitation is introduced that the ordinates should be drawn making any other given angle with AE also unless the angle at which the curve and the straight line met were finite the angle at the vertex of the Δ's here compared would not be finite



Let Ak be a straight line. AB . BC &c = intervals. Bb . Cc &c the velocities acquired at the end of the first, second &c intervals Then by similar Δs AB : Bb :: AC : Cc.

F = $\frac{\text{velocity acquired in any time}}{\text{time}}$ ∴ v = f t ∝ t if F is constant

F = $\frac{Bb}{AB}$

F = $\frac{Bb}{AB} = \frac{Cc}{AC}$ = f

F ∝ $\frac{\text{space}}{\text{time}^2}$. $\frac{\text{now space described in time AB}}{\text{space — AC}} = \frac{AB . Bb}{AC . Cc}$

But Bb : Cc :: AB : AC by similar Δ. $= \frac{AB^2}{AC^2}$

F ∝ $\frac{\text{space}}{\text{time}^2} = \frac{AB^2}{AC^2} \cdot \frac{AC^2}{AB^2} = 1$ which is constant.

Mechanical Problems. St Johns. May 1836.

…[illegible] given the ratios of two moving forces acting upon two unequal masses, also the ratio of the spaces described, and of the velocities acquired find the proportion of the masses.

A bent lever has arms whose lengths are in the proportion m : n, and whose densities are in the like ratio. find the ratio of m : n when the lever rests with its lower arm inclined 60° to the horizon. the angle between the arms is 135°.

A weight in the form of an isosceles Δ is to be supported by means of two unequal rods of given length and weight.

A uniform ladder rests with one end on a rough horizontal plane, and the other against a vertical wall, making given angle of elevation. supposing friction to be just sufficient to maintain the equilibrium in that position find what addition of force will be requisite to maintain the equilibrium, a man of given weight being supposed to ascend.

There are two balls whose elasticity is e. A impinges obliquely on B with a given velocity. find the angles made by A and B after impact with A's original direction. prove when A's deviation is the greatest possible $\tan\theta = \frac{1}{2}\left\{\frac{1+e}{1-e}\right\}\sqrt{\frac{1-e}{2}}$, θ being the deviation. It appears from the investigation that the deviation is independent of the original velocity. In practice however this is not the case, on ordinary billiard tables for instance. account for this difference.

A cylinder whose base is an isosceles Δ is laid with its axis ∥ to the intersection of two given inclined planes at right angles to each other. the points of contact with the planes being the extremities of the unequal side. find the position of equilibrium and shew that if the equilibrium be stable b is less than $a\frac{\sqrt{5}}{2}$, a being the base of the triangle and b the other side.

A horizontal shaft is to be uniformly and alternately raised and depressed through a given space by a wheel revolving uniformly acting underneath required the form of the wheel. — An ellipse. the pivot being placed in one of the foci.

Required the size of the wheel which a horse of given size acting at a given distance from it will draw most easily over a given obstacle, the weight of the wheel being given.

A body P is drawn upwards by another weight Q acting over a fixed pulley. A small oscillation is given to P. required the number of small oscillations made by P during the ascent.

Two bodies of masses m, m′ are projected in vacuo with velocities v, v′ at different angles of elevation in different vertical planes. shew that the path of the centre of gravity is a ⌒ and find its latus rectum.

Find the centre of gravity of a segment of an ellipsoid contained between two ∥ planes, the planes being ⊥ to two principal axes.

1 If a pendulum be slightly altered in length, find the number of oscillations gained or lost in a day. Ex. the seconds pendulum length 39.13 inches altered by the .01 inches.

2. AB is a uniform heavy beam moveable about A in a vertical plane. D is a fixed pulley vertically over A. W a weight fastened to a string which passes over D. and is fastened to B. find the position of equilibrium.

3. Find the conditions of equilibrium of any number of forces acting in any direction upon a rigid body. State the modifications which they undergo when the body is moveable (1) round a fixed axis, (2) round an axis along which it can slide.

4. A body acted on by any forces, rests on the plane of xy upon its base, the eq. of boundary of which is $y = fx$. find the conditions of equilibrium.

5. A body consisting of two straight arms AC. BC, making an angle of 150° with each other balances itself on a rough horizontal plane, when AC lies upon the plane. If a weight W be placed at the extremity of CA and the arm CB be struck at a given point by a weight P which has fallen through 5 feet find how high it will ascend.

6. If a force p act in the plane xy at a point x. y. upon a rigid body moveable round the axis of z and inclined at an angle α to the axis of x. shew that $p(x \sin\alpha - y \cos\alpha)$ represents the moment of p to turn the body round the axis.

7. Find the eq. to the catenary between horizontal and vertical co-ords. If points of suspension be in a horizontal line and their distance from each other nearly equal to the length of the chain find the depression of the lowest point.

8. If a flexible thread be acted upon by forces which are not in the same plane, and f be the resultant of all the forces acting at a point P making an angle θ with the tangent, and an angle φ with the radius of absolute curvature and if t be the tension of the string at that point shew that

$$\frac{dt}{ds} = f \cos\theta, \text{ and } t = f\rho \cos\phi$$

9. Investigate the formulæ for finding the centre of gravity of any curve figure, and apply it to find that of a hemisphere.

10. Find the eq.^s of motion of a body moving in a plane, and acted on by any forces in that plane. Hence shew that a body describing the arc of a curve uniformly is acted on at every point by a force which varies inversely as the radius of curvature.

A body is acted on by a central force, find the polar eq. to the curve described viz $\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0$ and shew that the velocity at any given distance is independent of the curve described.

When a body descends along the arc (s) of a curve by the action of gravity shew that the pressure at the point $x, y = g\frac{dx}{ds} + \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$, ρ being the radius of curvature at the point. Ex. when a body oscillates in a whole cycloid find where the tension of the string = the weight of the body.

State the principle of virtual velocities — Apply it to determine the position of a weight P sustained upon a sphere by another weight Q with which it is connected by a string passing over a pulley in the vertical diameter of the sphere produced.

Shew that the moment of inertia of a body round an axis through its centre of gravity is less than that round any ‖ axis. Find the moment of inertia of a body in the form of an ellipse about an axis through its centre & perpendicular to its plane.

P and P′ are the masses of two bodies connected by a string which passes over a fixed pulley C; P is raised until PC is horizontal and is then suffered [illegible]; determine the motion of P.

Diff Cal. St John. June 1835.

Define a multiple point, and shew how to find the angles which the different branches of a curve make with the axis of x at a multiple point. Ex. $y^6 - a^2x^2y^2 + x^6 = 0$

Investigate Lagrange's theorem for expressing $f(y)$ in a series of powers of x when $y = z + x\cdot\varphi(y)$. Ex $(1+xy^r)^n$.

Integrate $\frac{1-x^2}{(1+x)^2(x^2+a^2)}$; $\frac{1}{(a+b\cos x)^2}$; $(\cos mx . \cos nx)^2$

Integrate $\frac{1}{\sqrt{(x^2-y^2)(x^2-z^2)}}$ by an elliptic function, and prove that $\int_z^\infty{}_0 \varepsilon^{-z^2} = \frac{1}{2}\sqrt{\pi}$

Shew that the area bounded by the curve whose eq. is $xy = \frac{a^2}{2}\log_\varepsilon \frac{a^2}{a^2-x^2}$ and by its asymptote, and by the axis of x is $\frac{\pi^2 a^2}{12}$

Prove that there are n arbitrary constants in the complete integral of a diff! eq. of the nth order, and n arbitrary functions in the complete integral of a partial diff! equation of the nth order.

Integrate the eqns.

$d_x y + Py = Qy^n$; $y = x f d_x y + \varphi\, d_x y$; $P d_x z + Q d_y z = R$.

Prove that $\int_{x=\frac{\pi}{2}} \int_{x=0} \frac{1}{\sqrt{1-c^2\sin^2 x}} = \frac{\pi}{2}(1+c_1)(1+c_2)\dots(1+c_n)$

where $c_1, c_2 \dots$ are derived from c and from one another by the law $c_{r+1} = (1-\sqrt{1-c_r^2}) \div (1+\sqrt{1-c_r^2})$, and the process is continued till $c_n = 0$ nearly.

Let $d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \dots p_n y = X$ be a linear eq. of the nth order with constant coeffts. and let $-a_1, -a_2, \dots -a_n$ be the roots of its auxiliary eq. so that

$$m^n + p_1 m^{n-1} + p_2 m^{n-2} + \dots p_n = (m+a_1)(m+a_2)\dots(m+a_n)$$

then if we multiply both sides of the proposed eq. by $\varepsilon^{a_n x}$, and integrate, the result will be a linear eq. of the $(n-1)$th order with the same auxiliary eq. except that it wants the factor $(m+a_n)$.

Apply the above theorem to shew that the complete solution of the eq. is $y = \varepsilon^{-a_1 x}\int_x \varepsilon^{(a_1-a_2)x}\int_x \varepsilon^{(a_2-a_3)x}\dots\int_x \varepsilon^{(a_{n-1}-a_n)x}\int_x \varepsilon^{a_n x} X$; and integrate the equation $d_x^4 y - y = \cos x$.

Find the complete solution of the eq. $d_x^2 z - v^2 d_y^2 z = 2v^2\left\{\frac{d_y z}{y} - \frac{z}{y^2}\right\}$ by assuming $z = d_y\left(\frac{u}{y}\right)$

12. In the wheel and axle suppose the axle to be rough, and that a perfectly flexible rope is wound round it n times. To the end of which rope is attached a given weight which is allowed to descend freely till there are m revolutions of the rope left round the axle. Find the velocity acquired and then what a resistance or force is necessary in order that the weight may descend uniformly after this time, the inertia of the machine being neglected, and the rope being supposed without weight.

13. A uniform flexible rope is suspended from two points in the same horizontal line, find the form in which it will hang if a given weight be suspended at a point of the rope at a given distance from the extremity.

14. State and prove D'Alembert's principle, and ~~find~~ apply it to find the angular accelerating force on any body round a fixed horizontal axis when acted upon by gravity.

15. Find the proportion of P to W in the first system of pullies, when the strings are not parallel.

Define force, mass, weight, reaction, tension.
Component, resultant.

find the result. of two forces acting on a particle.

3 forces act on a particle in direct. ⊥ to each other find the magnitude of result.

If the forces acting on a pt. x. y. z. in direct. ∥ to the co-ord. axes be $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$, shew that the eq.n to the direct. of the result. $\frac{x'-x}{\Sigma X} = \frac{y'-y}{\Sigma(Y)} = \frac{z'-z}{\Sigma(Z)}$

If a point rest on a plane curve under forces X. Y in the plane of the curve ∥ to co-ord. axes then shew that cond. of eq.m is $0 = X + Y\frac{dy}{dx}$

find the corresponding condit. when a particle rests on a curve in space under forces X. Y. Z. ∥ to coord. axes.

The axis of a par. is vertical, a given particle rests upon it by a str. passing over a pulley at the foc. and supporting another given weight. find the condit. that there may be eq.m and also the condit. of the particle resting at the end of the latus rectum.

given impulsive force, and state how it is measured
Two bodies of given elasticity moving in opposite direct. in the same straight line impinge, find their subsequent motion
When two bodies impinge, the motion of the centre of gravity is the same before & after impact
Two balls lie in contact on a table. Shew how one of them may be struck, that the other may move off in a given direct.
A ball dropped vertically on two other equal balls lying in contact on a horizontal plane it strikes both at once in a vertical plane through their centres — determine the subsequent motion.
Two balls on a table are connected by a loose string — if one is projected in a given manner, find their motion after string becomes tight.
Given the measure of force when a yard is unit of space, find its measure when a foot is the unit

State and prove Guldin's properties.

Find c.g. approx. of the vol. and surf. of a hemisp.

Find the vol. and area of the surface of a cattle-ring [illegible].

= in to the common catenary.

If a [illegible] roll along a straight line then the focus traces out a caten.

The c.g. approx. of any portion of the cat. will lie in the vertical passing thro' the int. of the tangents at its extrem.

A uniform chain of length l hangs from 2 pts in a horizontal line distance c, and α is the angle which the tangents at the pts of suspension make with the vertical.

$$\tan\alpha \cdot \log\tan\frac{\alpha}{2} = -\frac{c}{l}$$ find then that the posit. of the chain is known.

When the force tends to a cent. the cat. assumes such a form that the tension at any point varies as the ⊥ from the c. upon the tangt. at that point.

Length of an elastic string hanging vertically and supporting a given wt.

A uniform triangular lamina is placed in a smooth hemisph.

then that $\tan\theta = \frac{c}{\sqrt{R}}$

R = rad

r = rad of ⊙ circumscribing triangle

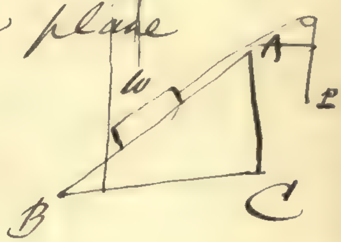
c = distance of c.g. approx. from circum.

[illegible]

If a point be acted on at the same instant by 3 pressures each of wh. varies as the sine of the angle cont.d between the lines in wh. the other two act it will rem. at rest.

) Given a pressure. shew how to resolve it into 2 pressures at right angles to each other.

). a weight W is ~~placed~~ supported upon an inclined plane by a power P which acts over a pulley.



find the pressure on the plane, and the relat. betw. P and W.

Find the pressure on the fulcrum of a lever kept at rest by any two pressures.

If 3 pres. act in a plane, and keep a lever steady, the I. If their dir. are ∥ one of the pressures = the other two II. If their directions in not ∥ the lines of act. of every pressure pass through same one point.

State the principle of virtual velocities and apply it to find the cond. of equil. of a system of smooth bodies lies in general the highest or lowest point. the parts of the system being in contact or fixed smooth surfaces.

Find the posit. of $=^m$ of a uniform rod resting between ~~inclined plane~~ ⊥ inclined plane.

Of a rod of ... the lower end resting on a smooth ... on the rim of ... it lies.

The lower end of a rod is supported on a curve. the rod resting also on a given smooth peg. find the nature of the curve that any posit. may be one of $=^m$.

Define stable and unstable $=^m$. when a body whose surface is spherical rests on upon a sphere. find the condt of stable $=^m$.

Find cent of grav. of a circular arc of 60°, of a paraboloid of revolut. and of the surf of a right cone.

Methods. 1 the $=^{ce}$ of $=^m$.

2. the use of virtual velocities

3 cert. cases in wh. the $=^m$ of a body is reducible to the $=^m$ of a pt.

Define the moment and axis of a couple and show that the axis is a proper representation of the couple

Show that the arm of a couple may be turned about any point in its own plane without altering its efficiency

State & prove the ▱ of couples.

Forces 1-3-5 & 7 act at equal distances in // directions on a rigid rod — find the single resultant couple.

Find the magnitude and resultant of a system of forces acting in one plane on a rigid body.

Show in what case the forces will not be reducible to a single resultant.

The vertex of a cone rests on a horizontal table. Find the moment of a couple which will keep it with its slant side vertical.

A rectangular ▱pped is placed on an inclined plane with its length horizontal: as the inclination of the plane is increased determine whether it will roll first or slide first

-ions in act on a rigid body in space. find the condn. that they may admit a single resultant.

Forces (2–2, 3–3) act at the opposite corners of a rectangle find the couple to which they are = ent.

Find the condns. of = m of any rigid body acted on by ‖ forces.

— = m ——— when the origin is a fixed pt. and the body is capable of sliding along the axis of z fixed.

When a body capable of turning about an axis is in = m. the press on the axis is the same as if the forces were directly applied at it in ‖ directions.

Forces act on a rectangular ▱ piped. represented in magnitude and direction by 3 edges which neither meet nor are ‖. find the = nt. force and couple. shewing whether the forces admit of a single resultant or not.

Two uniform beams of given lengths rest pressing against each other in a horizontal plane and against 2 vertical planes. find the posn. and press on the planes.

Define

Two forces act in one plane on a rigid body - find their result. (1) when their directions meet. (2) when the directions are ∥ and opposite. In the latter case show where the proof will fail if the forces are equal.

Define the moment of a force with regard to an axis and a plane. Define the centre of a system of ∥ forces, and find the coord^s of its position.

A ◠ has its axis vertical. find how far from the vertex a particle will rest - the coeff. of friction being given.

A particle rests on a sphere attached to the vertex by a string. how does the tension vary with the length of the string, also how does the pressure on the curve vary

A particle rests on a rough inclined plane under the action of a given force. find the locus of the limiting direction of force.

$\frac{a}{\sqrt{n}}$

$2\sqrt{a}\cdot\sqrt{n}$

$\sqrt{\frac{a}{n}}$

Show that any system of forces can always be reduced to 2 single forces - and in the case when it can be reduced to one find the Eq. to the line in wh: it acts

A body acted on by given forces has 2 points fixed show how to find the press: on these points ~~Ex:~~

Ex: a rect: ABCD is supported by hinges at A & B AB being vertical find the press. on the hinges

A convex surface rests on a hor: plane show how to determine whether the =m is stable or unstable.

Ex: A ⅁loid the length of whose axis = ¾ its lat: rect.

State & prove Guldinus 1st prop: & state the 2 purposes to wh: it may be applied. Ex: (1) find c. of gr. of the quad: of a ⊙cle (2) find the vol: of a right cone (3) find ~~the~~ convex surface of a right cone

5) Find =n to catenary in terms of x & y & show that diff: bet: the Tension at any 2 points ∝ the vertical dist: bet: them.

) Find condns of =m of a cord (1) on rough ⊙cle (2) on any rough curve & show that if 2 given = wts sustain ea: other by a string passing over a smooth curve the plane of wh: is vertical the sum of the press: on any arc depends only on the posns of its extremities

Prove that $v = d_t s$ & assuming that $f = d_t v$ show that $v d_s v$ & in =ns $v = \pm d_t s$ $f = \pm d_t v$ $f = \pm d_t^2 s$ $f = \pm v d_s v$ state when the +ve & -ve signs are to be used

) Prove the =ns of motion on a pt. referred to 3 rect: axes

3) Supposing a ~~shaft~~ hole bored to the center of the earth the rad: of earth ~~Ex~~ = a & a body be let fall from a height a above the surface find time of reaching the center & the vel: there acquired. assuming force of gr: outside the earth to ∝ ~~inversely as the sq:~~ $\frac{1}{dist^2}$ & inside directly as the distance

$\frac{22}{4}$

(1) Under the action of a central force the area described is prop. ... Hyp. (?)

(2) Find the eqn to the orbit described ...

(3) Force $= \frac{\mu}{(dist)^2}$. A body is projected towards the centre S from a pt. A such that $SA = a(1+e)$ & with a vel: $= \sqrt{\frac{\mu}{a}\frac{(1-e)}{1+e}}$. Find the time of reaching a pt. at a distance $a(1-e)$ from S & the vel: there. Find also how far the body will go on the other side of S & the time of reaching that extreme point.

(4) The force is attractive & $\propto$ [illegible]. A body is projected from A with a vel. $k\sqrt{\mu}$ & in a dirn ⊥ to SA. Apply the eqns of motion to find the vel: at any point & the eqn to the orbit. $SA = h$

(5) Suppose a body to move uniformly with a vel: c in the curve $y^2 = 4ax$. Find $d_t^2 x$, $d_t^2 y$ & show that the whole force at any pt. is in the dirn of the normal at that pt.

$t^{-2} d_t t = \ldots$

$a + ae \quad a - ae$

$SA - SP = 2ae$

$d_t A = d\theta \ldots d_t\theta$

$d_t r = d\theta r$

$r^2 d_t\theta = h^2$: $d_t\theta = \frac{h^2}{r^2}$

[If] three forces keep a point at rest, any one of them [is] proportional to the sine of the angle between the direc[tio]ns of the two others.

A stone dropped from a bridge, strikes the water in [...] what is the height of the bridge? Also, if the stone [be] projected downwards, with a velocity of 3 feet per second, [in w]hat time will it strike the water?

Two weights sustain each other upon two inclined planes [hav]ing a common altitude, by means of a string which is [para]llel to the planes; find their position, taking into [acco]unt the weight of the string which is supposed to be of [uni]form thickness.

[If] a body, whose elasticity is (e), impinge on a plane [wit]h a velocity (v), and be reflected with a velocity (v'); and [a], a', be the angles which the directions of its motion, in [the] two cases, make with a perpendicular to the plane, prove [that] $\tan a = e \tan a'$; $v \sin a = v' \sin a'$.

[T]he locus of the centres of gravity of the areas of all [righ]t-angled triangles on the same hypothenuse ($2a$), is a [circl]e whose radius $= \frac{a}{3}$. The locus of the centres of gravity [of t]heir perimeters is a spiral, whose equation is [...] $\sin\theta - \sin\frac{\pi}{4}$); the pole being in the middle point [of t]he hypothenuse, and θ measured from that line.

[A] hemispherical bowl is terminated by a cylindrical [rim], having the same external and internal radius; [wha]t must be its breadth, in order that the vessel may [rest] upon any point of its spherical base?

7. A perfectly elastic ball is projected from the middle point of the base of a vertical square towards one of the angles, and after having been reflected at the sides containing that angle, falls at the opposite angle; find the velocity of projection.

8. Two weights P and Q, connected by a string of given length, are placed upon an inclined plane of the same length, P being at the highest point, and Q at the lowest; after what time will their distance be a minimum?

9. A uniform rod whose length equals twice the diameter, passes through a hole in a spherical shell, and rests with one end against the internal surface; shew that if a be its inclination to the vertical, when in its position of stable equilibrium, $\frac{1}{3}(\pi+a)$ and $\frac{1}{3}(\pi-a)$ will be its inclinations, in its positions of unstable equilibrium.

10. A uniform rod of given length (a) is bent into the form of a cycloid, and oscillates about a horizontal line joining its extremities; prove that the length of the isochronous pendulum $= \frac{a}{5}$.

11. A convoy moving uniformly along a road which runs east and west, is perceived at the instant it is due south of a battery; at what elevation, and towards what point of the compass, must a cannon, loaded with a given charge, be fired at the same instant, so as just to hit it?

Deduce the coeff^t of refract^n from the greatest and least difference

~~Describe~~ the style of the horizontal dial which is accurately graduated for a given place is bent thro' a small ∠ δ. α is the hour ∠ and there is no error allowed in the time of the dial shew that h = the hour ∠ at any time = $\delta \sin \overline{h-\alpha} \cdot \{\cot \ell + \cot h \tan 2\ell\}$

Gregory or Cass. Tel.
focal length of mm. 20. 1
foc length of eye glass $\frac{1}{3}$ of an inch.
magnifying power 800.

---

focal length of mirrors 30. 2 inch
foc length of eye glass 1 inch.
magnifying 300

---

with Ramsden eye place double the magnifying power, and the same detail will result.

$m = \frac{2}{3} \frac{f_0^2}{f_{0,1}^2}$

Explain the defects of the astronomical telescope in its simplest form considering light homogeneous.
Examine the effect on the magnifying power, the field of view, and the brightness if a portion of the object glass or eye glass be stopped up.
Explain the compensation which makes Huyghens eye piece achromatic.
Shew why Huyghens eye piece cannot be used with a telescope intended for measuring.
Describe the Astronomical telescope with Ramsdens eye piece & find the distance of the field glass from the focus of the object glass.
If Cassegrains telescope is used with Huyghens eye piece shew that the magnifying power equals $\frac{2}{3} \cdot \frac{f_0^2}{f_e \cdot f_m}$ f_e = focal length of eye glass.
A Gregorian telescope has Ramsdens eyepiece the focal length of large mirror is 12 inches. the lenses of the eye piece are $\frac{2}{3}$ of an inch apart and it magnifies 600 times. find the focal length of the small mirror. $\frac{16}{450}$ inch
The eye glass of an astronomical has a focal length of 2 inch – find in wh. direction & how far it must be moved for any eye which sees at a dist. of 8 inches and compare the magnifying power to such an eye with that of an ordinary case

Describe Microscope of Doublets Lentant —

hence the sagitta of an arc as shown which it is ultimately = 1/4 of ... subtense.

Shew that every curve of finite curvature has a common ... as its limiting form. If the curve be a circle, what is the latus rectum of the ...

Shew according to what power of the arc subtense ultimately vary in order that the curvature may be finite

State and prove Prop I. A body moving in a curve whose angle of 30° to the distance is suddenly made to move ⊥ to the distance with double that velocity. find the change in the rate of describing areas about the centre of force.

The velocity in any curve in the direction of the radius vector is a mean ... between the sum and diff of the whole velocity and the part ⊥ to the radius vector.

State and prove Prop II.

$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$

$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{dy}{dx}$ and prove Lemma 7. and thence shew that $\frac{\sin \theta}{\theta} = 1$ when $\theta = 0$

State and prove Lemma 9. and shew in what part of the proof each limitation of the statement is taken into acc[oun]t.

State and prove Lemma 10. If the curve AK be a straight line shew that the force is constant. If it be a [illegible] with given latus rectum touching AK. find the relation between the space described from rest and time.

Shew how the curvature of two curves which have a common tangent at the point where they meet may be compared.

Of the two curves $y = a^{\frac{2}{3}}x^{\frac{1}{3}}$ and $y = \sqrt{bx}$. shew which has the greater curvature at the origin

Define the circle of curvature at any point of a plane curve, and prove that its diam[eter]

$$= \lim \frac{(\text{arc})^2}{\text{subtense} \perp \text{to tangent}}$$

Hence find the rad[ius] of curv[ature] at the vertex of a [parabola]. & also at the vertex of a curve

$$y = \sqrt{\frac{a^2x + x^3}{b}}$$

into ran angular velocity is incrassed. which shall about the cen: of force it vanish $\frac{1}{dist^2}$

A body is projected and acted upon by a force varying as the dist: the space due to velocity is ½ dist. the angle of project. is 60°. find the axis of the orbit with position.

A body is projected and acted upon by a force varying as $dist^{-2}$ & the space due to velocity is ½ dist. the angle of proj. 30°. find the axes of the orbit with posit.

A body is describing an ⊙ about the force. which suppose that the absolute force μ is changed into $\mu + d\mu$. find the consequent changes in the major axis and eccent.

Force var d^{-2}. a body falls in a st. line to the cen: of force. find the time of motion & the velocity acquired. Deduce the velocity acquired from inf. and space due.

Of two bodies one falls directly to the cen: of force. the other moves in an orbit which at apsis the velocities are equal and at equal dis. shew it be equal and also = distances.

When a body describes an ⊙ the angular velocity of the focal dist: is double that of the tangent.

A body describes a circle about an interior pt. find when the angular velocity = the mean angular velocity.

Condit. of =m of a solid floating freely in a fluid.
The press of air under a given temp varies inversely as the (?) ... occurs
when the =m of a solid is slightly disturbed from the vertical thro the cen. of grav of the fluid displaced.
Define the metacent. of a floating body and show how its posit. affects the stability of equilib.
Find the met. of a triangular prism floating with axis horizontal and one edge immersed.
Find the met. of a cone floating with its axis vertical
~~Find~~ Investigate the relation betw the density press and temp of air

$$\Pi = \mu \frac{\rho}{\tau} \{ 1 + \cdot 00366 t \}$$

+ find the diff. ... 2 stations by means of barom. in both cases.

Find the force with which a stream impels a plane. the ~~thing~~ stream impinging (1) directly (2) obliquely on the plane.
Find ——— ⊥ to the generating line of a cylinder impels it (1) in that direct: (2) in direct: at right angles to the same.

——— ratio of resolution in direct: of its axis impels it in that direct:.

A sphere moves thro' a fluid find retarding force arising from the resistance of the fluid.

Describe the constr. and use of Sykes Hydrom.

Hydraulic Ram.

Find the posit: of the rudder when the effect of the rudder in turning the ship is a max.

From a given cone cut off a portion by a plane ∥ to the base, so that the whole resist. of the remaining frustum moving in direct: of axis shall be the min. — and the middle pt of the axis of the frustum must be equidistant from the vertex and from the ~~edge~~ base of the base

Smeaton's exp: and find the density of the strokes of the piston.

Compare the sp: gravs of two fluids by weighing the same solid in each.

Desc. the cons. and use of the atmos. Engine

1 Explain fully how the pressure at any point in a mass of fluid is measured; and find the pressure at any point in a mass of fluid at rest acted upon by gravity

2 Find the pressure of a heavy incompressible fluid upon any surface.

3 Describe the experiments by which it appears that at a given temperature the density of air is proportional to its pressure.

4 Compare the specific gravities of two fluids by means of a common hydrometer.

5 Describe Watt's double acting steam engine and the high pressure steam engine.

6 Describe the forcing pump, and shew how it is employed in a fire engine.

7 Compare the specific gravities of air and water, and state in round numbers the value of the ratio for some given temperature and pressure.

8 A cylindrical vessel containing fluid is inclined till the surface of the fluid passes through the highest point of its base; find the magnitude and point of application of the resultant of the pressure upon the base.

[illegible] may?

9 State the law which connects the density, temperature, and pressure of atmospheric air; and if a heavy piston confine in a cylindrical vessel a given quantity of air at different temperatures, compare its distances from the base in the two cases.

10 Shew that the principle of virtual velocities obtains in the equilibrium of forces applied to pistons in the sides of a vessel filled with an incompressible fluid.

11 Express the equations of the motion of a fluid in terms of rectangular co-ordinates.

12 Explain what is meant by the metacentre of a floating body, and its position determines the stability or instability of the equilibrium. Find the metacentre of a right-angled cone floating in a fluid of twice its specific gravity

13 Find the pressure at any point in a mass of fluid at rest under the action of any forces; and apply the result to determine the pressure at any point of a fluid contained in an open vessel which revolves uniformly about a vertical axis.

14 Describe the Hydraulic Ram.

1 Define the following terms: "specific gravity", "density", "volume", and "mass". Give the principal equations which connect them.

2 (1) Shew how to find the specific gravity of a body by weighing it in a fluid heavier than itself.

(2) The specific gravities of oil and sea water are respectively 913 and 1030 and that of air 1.2, find the specific gravity of a cone which rests with one-third of its axis in each of the fluids, the vertex being downwards

3 (1) There is a vessel, the external and internal figures of which are similar cones on the same axis; the space between the cones is filled with materials whose density varies as the n^{th} power of their distance from the vertex, measuring along the axis. At what depth will it float in a fluid whose density varies as the m^{th} power of their depth?

(2) The nature of the fluid being the same as in the last question, find the centre of pressure, of a rhombus which hangs vertically; having its lesser angle at the surface

4 A hollow semi-paraboloid of given size and weight, is suspended by 3 strings in such a position, that the plane of section coincides with the surface of the fluid. Find the tensions of the strings

5 The transverse section of the bank of a canal, is a portion of a circle, included between the arc, abscissa, and ordinate. Find the conditions that it may not be overturned.

6 (1) Define the Metacentre, and shew how it assists us in discovering the stability or unstability of a floating body. If the specific gravity of a cone be to that of a fluid :: 1 : 2; shew that in order that the equilibrium may be stable $a > 2\sec^{-1}\sqrt{\sqrt{2}-1}$, where a is the vertical angle of the cone.

(2) In the above problem, the cone is supposed to float with its axis vertical. Suppose that not to be the case, find the position of equilibrium

7 Describe accurately the high-pressure steam engine, distinguishing between the high and low pressure.

8 The tube of a common barometer being supposed to be of equal bore throughout, suppose the air at the sealed end to have been imperfectly exhausted, and that under a given pressure with temperature 50°, the difference of the columns of mercury at the two ends was $h_{,}$; a given quantity of mercury being poured into the open end, the difference of altitude was observed to be $h_{,,}$, and afterwards $h_{,,,}$; $h_{,,}$ being the altitude for a moment, in consequence of the air in the tube being heated to 50° + t by compression and $h_{,,,}$ the altitude, when the air has cooled down to the temperature of the surrounding atmosphere. Find the quantity of air in the tube, and the atmospheric pressure

When a fluid acted on by gravity is at rest, the pressure is the same at all points in the same horizontal plane, whatever be the form of the vessel containing it.

Find the sum of the perpendicular pressures of a fluid on any surface, and shew in what case it will be the same as their resultant. Ex. A cone with its base in the surface of a fluid.

Shew that the resultant of the pressure of a fluid on the surface of a solid immersed in it is equal to the weight of the fluid displaced. Hence shew that if a hemispherical bowl of given weight (W) float upon a fluid with $\frac{1}{3}$ of its axis below the surface a weight ($5W$) put into it will make it float with $\frac{2}{3}$ of its axis below the surface.

Shew how to measure the pressure of the atmosphere and correct for the temp of the mercury.

State accurately how the boiling and freezing points of a thermometer are determined, and what is the temperature on Reaumur, which is denoted by the same number according to Fahrenheit and the Centigrade scale.

In the common pump, find the tension of the rod, the height through which the water rises at each stroke, and the least value of the range of the piston.

Describe Smeaton's air pump, and the barometer gauge.

Find the specific gravity of a solid whose weight is less than the weight of the fluid displaced by it by weighing it in air and in fluid. Ex.

Compare the specific gravities of two fluids by means of Nicholson's hydrometer, and shew how the instrument may be employed to determine the weight of a small solid.

Describe the hydraulic ram.

Find the difference in altitude of two stations by means of the barometer, taking into account the variation of gravity in the same vertical.

Find the metacentre of a cube of wood floating in water with one face horizontal when specific gravity of water = $\frac{3}{2}$ specific gravity of wood.

Find the pressure at any point of a mass of fluid acted upon by given forces.

Determine the form of the surface of a heavy fluid in a cylindrical vessel, revolving uniformly about its axis which is vertical, and if there be a solid cylinder floating on the fluid and having its axis coincident with that of the vessel find how much it will be depressed.

Find the relation between the pressure and velocity in fluid motion when the velocity at any point is independent of the time; and thence deduce the velocity with which an incompressible fluid acted on by gravity, issues through a very small orifice in the vessel containing it.

Investigate the four equations of the motion of a fluid in terms of rectangular co-ordinates, and express them in terms of φ.

Hydrostatics. S^t Johns. May. 1842.

Shew how to find the value of a $\int\int z$ of a surface made by a plane this theorem

The sect. of the Poid which ~~passes thr.~~ the the mean axis 2b — and makes with the plane of xy an ∠ whose tan is $\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}$ is a ⊙. $d:=^n$ is $x^2+y^2=b^2$.

Find the =ns to the tangent plane and normal at a point x. y. z. at the surface $u = f(x. y. z) = 0$.

the number of normals to a surface of the n^{th} order that may pass thr. the same given point cannot exceed n^3-n^2+n.

Investigate the formulas $d_x d_y V = z$. & $d_x d_y S = z\sqrt{1+p^2+q^2}$ and explain how the limits of the integrat. are determ.

Find Vol of Poid.

Given $\int_x^{0/a} \sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{b^2-x^2}} = \frac{\pi}{2}\left(b-\sqrt{b^2-a^2}\right)$ find area of spherical surface intercepted by a right cone whose vertex is centre of a sphere.

Find the =ns to the common helix and its length.

$$\int_x^{0/\pi} \log\left(1+2m\cos x+m^2\right) = 0. \text{ when } m<1$$
$$= 2\pi\log m \quad m>1.$$

$$\int_x^{0/\infty} \frac{\sin x\cos nx}{x} = \frac{\pi}{2} \text{ when } n \text{ lies between } 1 \text{ and } -1$$
and zero in all other cases.

$$\int_x^{0/\infty} \frac{x\sin cx}{1+x^2} = \frac{\pi}{2}\varepsilon^{-c}$$ Prove $$\int_x^{0/\infty} \frac{x}{1+x^2}\tan^{-1}\left(\frac{m\sin cx}{1-m\cos cx}\right) = \frac{\pi}{2}\log\frac{1}{1-m\varepsilon^{-c}}$$

Define a diametral surface. Shew that for surfaces of the 2nd order it is a plane and find its surface.
Every surface of the 2nd order has at least one diametral plane. ⊥ to the —
Every variety of surfaces of the 2nd order is comprehended in the =n. $Ax^2 + By^2 + Cz^2 + 2Ax + 2By + 2Cz + D = 0$.
Investigate the condit: of the surface represented by the preceding =n being an ellipsoid or H —
Prove that x^2 represents a contin [illegible] generated by revolution about the axis $x = y = z$. and length of real axis $\sqrt{2}$

$$x^2 + 2y^2 + 3z^2 + 4xy + 4xz + 6yz + 2x + 2y + 2z$$

represents a contin H od. the coords of whose centre are $h = k = l$

Prove $1 + \frac{n}{1} a \cos cx + \frac{n(n-1)}{1 \cdot 2} a^2 \cos 2cx = r^n \cos n\theta$.
$r^2 = 1 + 2a \cos cx + a^2$. $\tan\theta = \frac{a \sin cx}{1 + a \cos cx}$
$\int_0^\infty \frac{r^n \cos n\theta}{1 + x^2} = \frac{\pi}{2}(1 + a e^{-c})^n$

...force velocity, and state how it is measured.

State and prove the first law of motion.

Show that if velocities impressed at once on a particle be represented by 2 straight lines forming sides of a parallelogram the resulting velocity will be represented by one of the diagonals of the ▭.

If a particle have velocities v. v. in direction of the axes of x and y, find the eq. to the direction of its motion from ~~the point~~

Also find its velocity in the direction of a straight line whose eq. is $\frac{x}{a}+\frac{y}{b}=1$

State and prove the 3rd law of motion.

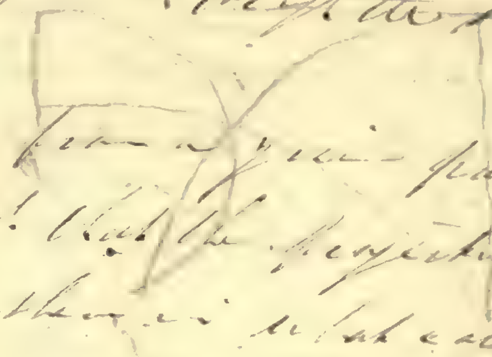
A body urged from rest by a constant force during two seconds has a velocity with which it will describe 100 yards in 3 seconds. Find f.

A force represented to a second as a unit, after [illegible] [illegible] unity for its measure; find its measure referred to a minute.

The velocity generated by a gun in a bullet of [...]ince weight is 1000 ft. per second, and the barrel has been described in $\frac{1}{10}$ of a second. Find the moving force, supposed uniform, which has acted on the bullet.

Given the velocity and direction of a projectile, find its eq. on a horizontal plane through the point of projection. also on an inclined plane through the point.

Given the velocity of proj. etc. from a given point, find the direction of proj. that the projectile shall hit a given point, and show in what case the question is impossible.



Find the eq. referred to rectangular axes to the path of a projectile.

Show from this eq. what is the curve of the path, and that the space due to the velocity at any point is the distance of that point from the directrix.

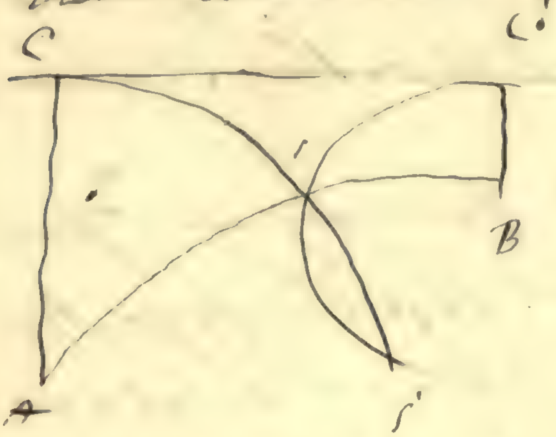
In any chord of the path of a projectile the velocity of a body at the two ends of it resolved ⊥ to the chord is the same.

A ball is projected in a vertical plane and strikes upon a vertical wall. find the distance from the wall where it again strikes the horizontal plane through the point of proj.

Two balls are projected simultaneously from the same point with velocities v, v' at inclinations α, α', show that the time elapsed between their passages through the point common to their paths

$$= \frac{2}{g}\{v v' \sin(\alpha-\alpha')\} \div v\cos\alpha + v'\cos\alpha'$$

the construction in the second case is



Let A be the point of projection. B the point through which it shall pass.

then since the velocity is given AC the distance from directrix is known

with centre A & distance AC describe circle, & centre B & rad = BC' draw

SS' are the points of intersection

2 press: acting ⊥ on a str: lever on the same side of the Fulcrum will bal: when they are inversely as their distances from the Fulcrum

If 2 wts bal: in one posn on a str: lever they will balance in any posn – in the true. also when the lever is heavy

A press: P acts at the pt A in a direction making an ∠ α with Ax show that the resolved pts of P in Ax & Ay ⊥ to Ax are $P\cos\alpha$ & $P\sin\alpha$ (1st) when α lies bet: 0 & 90° (2d) when bet: 180° & 270°

Prove the truth of the principle viz: that when a body is kept in = by pressures in one plane the sum of the Moments of the pressures one way round any pt whatever in the plane = the sum of the moments of the press: the other way round the same pt in the case of question (1) the given pt being ½ way bet: the pts of application of P & Q

Find rel: of P to W on the Inclined plane P acting H.l to the plane & show that in this case $\dfrac{P\text{'s } V.V}{W\text{'s } V.V} = \dfrac{W}{P}$

Find rel: of P to W when there is = in the combination of the Wheel & Axle with the single moveable pulley. & show that in this case $\dfrac{P\text{'s } V.V}{W\text{'s } V.V} = \dfrac{W}{P}$

Define the centre of gravity & find that of a pyramid

Define the terms Acc. force Moving force & Vel: & give their measures assuming the Acc. force of gr: to be 32.2 when a second is the unit of time find it when a minute is the unit

State the 3 laws of motion & the ▭m of Vel's point out the ~~[illegible]~~ diff: bet: the 2d law of mot & the ▭m of Vel. & the purposes to wh. they are respectively applied

Show that the motion of the center of gravity of 2 bodies moving in 1 plane is not affected by Impact

A body is projected with a vel: of 161 ft: in a direction making an ∠ sin $\frac{4}{5}$ with the horizon find how high it will rise & how long. find also its posn directn & vel: after 3" & also after 6"

When a pressure P produces motion in a body whose wt is W find the Acc. force & the space it will drag it [illegible] in t"

Show that the space due to the Vel: at any pt of the path of a projectile is the distance of that pt from the directrix

(14) Find the greatest height of the projectile above a given incline passing through the point of projection

A body is projected upwards or downwards with a velocity u. if it after describing a space s has a velocity v. shew that $v^2 = u^2 \pm 2gs$.

Since $\sqrt{2gs}$ is the velocity which gravity would generate from rest through a space s explain why it is not true that $v = u \pm \sqrt{2gs}$

whereas the eq. $v = u \pm ft$ is true

The height of a room is 10 feet. a body of elasticity $\frac{1}{2}$ is projected upwards, and after striking the ceiling and floor just rises to the ceiling again find the velocity of project.

Two given bodies connected by a string move over a fixed pulley. find the space which either describes in t" from rest.

If the string which connects them be now cut. find their position after t" more.

A body slides down an inclined plane. shew that the velocity which it acquires is the same as if it fell down freely the same height.

Divide an inclined plane into n parts, so that the time of descent down each of them may be the same.

P is a body lying on a table, and Q slides down an inclined plane. when the string becomes ... determine the subsequent motion of P and Q

[illegible] the hyperbola [illegible] that $HP - SP = 2a$.
Shew how a point of the curve may be described by a con[illegible]
Find the polar eq. to the hyperbola from focus the ex[illegible]
trace the curve
Find the eq. to the tangent and its position when the
point of contact is above [illegible] the direction.
Eq. to normal. [illegible] that it bisects the external
angle between the focal distances.
Draw a tangent to an hyperbola from a given
external point.
Find the [illegible] conjugate to a given [illegible] and
find its eq. referred to the same axes.
Of a pair of conjugate diam. only one
can meet the hyperbola.

$$\frac{SP}{4} \quad CP^2 - CD^2$$

$$SP \cdot HP = CD^2$$

All [illegible] whose ~~[illegible]~~ sides [illegible] and
conjugate at the extremity of a pair of conjugate
diam. are equal in area and have their diagonal
coincident with asymptotes.
The locus of the vertex of a $\triangle$ on a given
base, and having the line bisecting the vertical
angle always [illegible] itself is an hyperbola
If an ellipse and hyperbola have the same
foci, they cut each other at $\perp$ [illegible]

[illegible] [illegible] ab ⊥ is a ○ passing thro'
the point of intersection.

Define the eql. of a curve.

and that the co-ords of the curve rep. by the genl.
=" of the second order.

Find the eql. of the curve

$y = bx + \frac{c}{x}$ of the eql. of the curve

when the exp. for the co-ords become $\frac{0}{0}$

then that the =" rep. two ‖ st. lines.

If x & y be rect. coord. then what is meant by
$Ax^2 + Bxy + Cy^2 = 0$. $xy = bx^2 + c$

Transform $Ay^2 + Bxy + Cx^2 + f(hk) = 0$

into another of the form $Ay^2 + Bx^2 + f(hk) = 0$.

and then there is only one syst. of rect. axes by wh. this transf. can be effected.

Find the axes of the curve $ay^2 + bxy$ when $b^2 - 4ac$

is diff from Zero.

Prove that $y^2 - mxy + x^2 = A$ rep. an

ellipse when $a(\sqrt{2} \mp 1)$

~~The~~ =" being given $x^2 + y^2 = (~~A~~ mx + ny)^2$

find the eccel. of the, locus of the axes.

Shew that the degree of the $\pm n$ is not altered by change in the ~~sect~~ axis of coor

A st. line cannot meet a curve of the n^{th} order in more than n points

A curve of the n^{th} order may be made to fulfil $n(n+3)$ conditions

Shew how to find the point of the centre of a curve

Ex. $x^2 + y^2 = (a + mx + ny)^2$.

Define a diam of a curve, and shew that it must be of the $\frac{1}{2}n(n-1)$ order if the proposed curve be of the n^{th} order

Shew how to find the locus of the middle pts of a system of || chords and apply the method to curves of the 2

Ex. = n to 2 curves by $x^3 + y^3 = 3axy$

and $(x^3 + y^3)a = x^2y^2$

Shew that a line inclined at 45° to the axis of x is an axis of each of the

Find the distance bet. 2 pts, and the angle wh. the line joining them forms with the axes.

Find the =n to a plane supposing it to be generated by a straight line wh. moves so as to slide along a given fixed line.

Find =ns to a st. line in space.

Find condit. that 2 planes may be // to one another.

When a st. line is // to a plane, the project. of the line and the trace of the plane upon any plane whatever are at right angles to each other.

Find condit. that a st. line and plane may be at right angles to each other.

Given =ns to a st. line find angles which it makes with axes.

Find =n to a st. line wh.

Find angles between 2 lines having given with their =ns (2) the angles wh. they respectively make with the axes.

Find =n to a plane thro' the origin, and // to 2 given lines.

Find =n to a plane wh. shall contain a given line and be ⊥ to a given plane.

Find =n to surf. generated by ~~[illegible]~~ a line moving along a line whose =ns are $x = mz + a$. $y = nz + b$. and // to a plane

[illegible]

Find the area of the surf of a segment of a sphere.
— volume of a segment of a sphere.
Spheroid

area of the surf of an oblate spheroid. deduce the area of the surf of a sphere.
length of an arc of a common (?).
volume of a figure generated by a plane moving ∥ to itself and increasing or diminishing in magnitude. ✓

volume of a pyramid.
area of curve $ay^2 = a^2\sqrt{a^2-x^2}$.
length of curve $a^3y = x^4 + 6a^2x^2$.
area of curve $y^2 = \dfrac{x^3}{2a-x}$.

Locus of the foot of a ⊥ dropped from a point α.β. upon the tangent to an ⊙ has for its eq^n ~~$\rho + \alpha + \beta + \beta\sin\theta = \sqrt{a^2 s +}$~~ $\rho + \alpha\cos\theta + \beta\sin\theta = 0$
area $= \frac{\pi}{2}\left(a^2+b^2+\alpha^2+\beta^2\right)$

arc of a cycloid rolls along a st. line touching its axis ∥ at the beginning and end of the motion. the area contd. betw the line and the curve which is the locus of the vertex $= \pi a^2\left(1+\frac{\pi^2}{6}\right)$

Find the condit. wh. the 3 angles made by a line thro' the orig. must satisfy.

Find eqns to a st. line passing thro' a given pt. and ⊥ to a given plane.

Shew that the angle between two planes = angle between 2 st. lines ⊥ to them

Investigate directly the condit. in order that two st. lines whose eqns are given may be ⊥ to each other.

The eqn $mm' + nn' + 1 =$

by $m = m'$ $n = n'$

Find the angle of inclinat. of two planes whose eqns are given

Find the angle of inclinat. of a line and plane.

Find distance of a given point from a given plane.

Find co-ord of the foot of the ⊥ dropped from a given pt. upon a given plane.

Find the distance of a given point from a given line

Find the co-ord of the foot of the perpendicular dropped from a given pt. on a given line

Prove that the shortest dist. between 2 lines meets them both at right angles

Find the eqn to a plane wh. contains one given line and be ∥ to another.

Find eqn to a plane wh. passes thro' a given pt. and [illegible]

[illegible]

. " of the discontinuous Hoid and shew that it has an catenary conical asymptote.
. =" of the elliptic Hoid and determine its form.
. =" of the Hi.i oid and determine the nature of the traces on co ord planes
find the project. of a limited line - " on any plane.
D10. a line.
. find area of the project. of any plane surface on a pl..
. Express the distance of a point from the origin and of 2 points from one another by oblique co ords.
the locus of a point whose distance from a fixed point ~~is~~ to equals n times its distance from a fixed line is a spheroid or a continuous Hoid whose semi-axes $\frac{nc}{1-n^2}$ and $\frac{nc}{\sqrt{1-n^2}}$ c being the perpendic. from the point upon the given line.
If an ⊙ roll on a straight line. its centre will describe a curve whose = " ..

$$\frac{dy}{dx} = \frac{y^2}{\sqrt{(a^2-y^2)(y^2-b^2)}}$$

the tangent & normal ~~each~~ make equal angles with the focal [in a parab.]
distance, and a line drawn through parallel to the axis
making the inclination with the axes when the two lines inferred
$y^2 - x^2 = 0$ by the eq. are ⊥ to each

~~and~~ define a diameter, and shew that all diameters
of a parabola are straight lines // to axis
find the eq. to a parab. referred to any diamet.
and the tangent to the extremity of diam as axis
the double ordinate of any diameter ~~which~~ [through the focus]
is = 4 times the focal distance of the extr.
: the tangents at the extremity of any chord
will intersect in the diam. of which the chord is
an ordinate
Any parabolic segment is $\frac{2}{3}$ of the □ whose
sides are its abscissa & ordinate.
find the eq. to ellipse & reduce it to
the form $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$
Shew that the ellipse has two foci whose
distance from centre = ae, and the distance of directrix
$\frac{a}{e}$. latus rectum $= 2a(1 - e^2)$
the △ contained by 3 tangents to a parab. is
double of △ contained by the 3 chords joining
the points of contact.

Define the centre and axes of a curve, and show that the origin is the centre and co-ord. axes are conjugate axes of the curve.

$Ax^2 + By^2 = c$

Find a conic section, and then in which case the curve ... the focus may be at the centre and the directrix.

Find the eq. to parab. and trace the curve.

Then that $SP = x + a$, and find the polar eq.

Find the eq. to tangent and normal at a given point.

and shew that they tend to become parallel to the axis.

Prove the subtangent = 2 abscissa, and subnormal = ½ latus rect.

$SP = ST = SG$.

The perp. from the focus upon the tangent to the parabola intersects the tangent in the line touching the parabola.

$SY^2 = SP \times SA$.

Find eq. to tangent to a parab. in

$$y = mx + \frac{a}{m}$$

then find the locus of the intersection of the tangents to parab. and the [illegible] from a point (h, k).

Find eq. to normal under the form

$$y = mx - 2am - am^3$$

and shew the locus of the intersection of two normals which are at right angles to each other.

[illegible] to a parab. find the locus of the intersection of two tangents to a parab. including a constant angle.

[illegible]

$$\int_x \frac{1}{(1-x^2)^{\frac{3}{2}}} = (1-x^2)^{-\frac{3}{2}}$$

$$\int_x (1-x^2)^{-\frac{3}{2}} \cdot \int_x (1-x^2)^{-\frac{1}{2}}$$

$$P = x(1-x^2)^{-\frac{1}{2}} \therefore \frac{dP}{x} = (1-x^2)^{-\frac{3}{2}}\left\{(1-x^2) + x^2\right.$$

$$= (1-x^2)^{-\frac{3}{2}}$$

$$\therefore \int_x \frac{1}{(1-x^2)^{-\frac{3}{2}}} = \frac{x}{(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \int_x \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} = \frac{x\sin^{-1}x}{(1-x^2)^{\frac{1}{2}}} - \int_x x$$

$$= \frac{x\sin^{-1}x}{(1-x^2)^{\frac{1}{2}}} - \frac{1}{2}x^2$$

Eliminate a & b from $(x-a)^2+(y-b)^2=c^2$.

$y = b\cos\overline{mx+a}$.

c from $\sqrt{x^2+y^2} = c\,\varepsilon^{\tan^{-1}\frac{x}{y}}$.

a & b from $2xy + ay^2 - bx^2 = 0$.

Explain why it does not admit of a derivative of the 2nd order.

Integrate: $2axy + ax^2 d_x y = y^3 + 3xy^2 \cdot d_x y$.

× $x + y \cdot d_x y = x d_x y - y$. $\sqrt{x^2+y^2} = c\,\varepsilon^{\tan^{-1}\frac{y}{x}}$.

$(1-x^2) d_x y + 2xy = ax^2 + b$.

× $(1+x^2) d_x y + xy + 1 = 0$.

× $\sqrt{x} \cdot d_x y - 2y = a\sqrt{y}$.

$d_x y = \sin(mx + ny)$.

× $d_x y + L + y\, d_x\left(\frac{1}{2}\log\frac{R}{L}\right) + y^2 R = 0$.

∴ $y = -\sqrt{\frac{L}{R}} \tan$

$y = -\sqrt{\frac{L}{R}} \tan \int_x \sqrt{RL}$.

L & R are fns of x.

~~$\frac{d^2y}{dx^2} + \frac{1}{2}\frac{dy}{dx} \cdot \frac{d^4}{dx^2} = 0$~~

~~$x^2 \frac{d^2y}{dx^2} + 5 + 2x^3$~~

~~$\frac{\quad}{x + x^4} \cdot \frac{dy}{dx} - \frac{2 + x^3}{x^2 + x^5} \cdot 6y = 0$~~

$y = x^2$ is a solut. find ~~complete integral~~.

Point out the diff. between a total and partial diff. eq.

Integrate. $y \cdot \frac{dx - x\,dy}{y\sqrt{y^2 - x^2}}$. $\frac{2ay \cdot (y\,dx - x\,dy)}{(x^2+y^2)^{\frac{3}{2}}\sqrt{x^2 - y^2}}$.

$\frac{x^{n-1}}{y^{m+1}} \cdot (my\,dx - nx\,dy)$

Find the condit. for $P\,dx + Q\,dy + R\,dz$ admits of a solut. of the form $f(x \cdot y \cdot z \cdot c) = 0$

$(y+z)\,dx + (x+z)\,dy + (x+y)\,dz = 0$.

$2y \cdot dx + 2x\,dy$. + ~~[illegible]~~ $(ay + az^2)\,dz$.

$(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy + (x^2 + y^2 + xz)\,dz = 0$

Find eq.n to the surface which has the property that the normal at any point passes thro' the center of f. of the 4 whose sides are the coords of $x \cdot y$ of the point.

Eq.n to a surface belonging to conical surfaces and a surface of revolut. about the axe

$\{x(x-a) + y \cdot (y-b)\}\,dz = (x\,dx + y \cdot dy)(z - c)$ and that

this not integrable unless $a = b = c$. Integrate this the general

$\frac{\sqrt{a+cx^2}}{x^4}$. $\frac{1}{(1+x)\sqrt{1-x^2}}$. $\frac{1}{x\sqrt{a+bx^n}}$. $\frac{1}{\sqrt[4]{a+bx^4}}$. $\frac{1+cx^2}{(1-cx^2)} \times \frac{1}{\sqrt{1+bx^2+c^2x^4}}$.

$\frac{x^5}{a^2+x^2}$. $\frac{2x+3}{x^3+x^2-2x}$; $\frac{x^2-2}{x^3+4x^2+4x}$; $\frac{x^2-x+1}{x^3+x^2+x+1}$.

Resolve $\frac{1}{1-x^n}$ into partl. fractn. n by odd.

& ~~x^m~~ [illegible] the general $\frac{x^m}{1+x^n}$, $\frac{x^m}{x^{2n}-2x^n\cos\theta+1}$.

If $\frac{A}{x-a}$ is a partial fracti in the resolution of $\frac{U}{V}$

. $A = \frac{U_{x=a}}{dV_{x=a}}$ x.

In the resolutn of $\frac{U}{Q(x-a)^n}$ the general term of the group of partial fractns corresponds to $(x-a)^r$ is $\frac{1}{\lfloor r} \cdot d^r_x\left(\frac{U}{Q}\right)(x-$

Hence resolve. $\frac{1}{(x-a)^n(x-b)^n}$

I showed him thought it would him confident on

the cent. of a ⊙ moves along a line $y = mx + c$ its rad increases so as always to be :: l to the abscissa of its cent. f d the curve which envelopes the ⊙s

the curve touched by the chord joining foci
is an ⊙ is another ⊙ whose axes are $\frac{a}{\sqrt{2}}$ $a\sqrt{2}$. $b\sqrt{2}$.

the curve touched by the diams of a semi ⊙ that rolls along a st. line is a cycloid.

The curve touched by the base of a cycloid which rolls along a st. line is

$$\frac{x}{a} = \left\{\left(\frac{y}{2}\right)^{\frac{2}{3}} + 2\right\}\sqrt{1 - \left(\frac{y}{2}\right)^{\frac{2}{3}}}.$$

Find the = n to the tangent of a curve – the coords being oblique.
Find the cond. of a point of contrary flexure and prove that the curve is convex or concave &c as y and $\frac{d^2y}{dx^2}$ have the same or diff. sign
Prove that 2 curves which have a contact of the nth order will touch or cut accord. as n is odd or even.

Prove $\frac{1}{\rho^2} = (d_s^2 x)^2 + (d_s^2 y)^2$.

In a conic sect. $\rho = \frac{\text{normal}^3 \text{ cube}}{(\text{semi latus rect.})^2}$.

ρ in an ellipse $= a - 2c + 3c \sin^2 \frac{\pi}{4}$. where $c = a - b$ and is small

Prove that at the points of greatest and least curv. the contact of the circle of curv. is of the 3rd order.

Shew that a conic sect. may be det. which shall have contact of the 4th order with a curve at any point, and if at any point it be a parabola at the preceding and succeeding.
Shew that it will be ellipse or hyperbola.

Find the coords of the cent. of curv. at a given pt. of ellipse and hence the = n to the evolute.

Find the = n to the asymptote to the curve. $y^4 + 2ax^2y^2 - x^4 = 0$.

If the point O in a Δ be such that $AO + BO + CO$ is a min each of the angles at O is $120°$.

$u = \frac{x}{a^2} + x^2$. then $d_x^n u = \{(-1)^n \frac{1 \cdot 2 \ldots n}{a^{n+1}}\} \cos(n+1)\theta \sin^{n+1}\theta$.

when $\cot\theta = \frac{x}{a}$.

If an ellipse be inscribed in a semicircle, with major axis ∥ to diam. the locus of the cent. of curv. of the ellipse at the point of ~~contact~~ contact is $\rho = a \cos^2\theta$, a = rad. of semi circle

$y^2 = \frac{b^2}{a} \cdot x - a$.

Explain in what cases the true value of a vanishing fraction cannot be found by successive diff: of its numerator and denominator and shew how to find the true value in those cases.

Explain in what cases $d_x y$ derived from $u = f(x, y) = 0$ will assume the form $\frac{0}{0}$, and how its values are to be detd.

$d_x y = \dfrac{4y - 2bx}{3y - ax}$. find its value when $x = y = 0$.

Shew that $f(x)$ may be a max. or a min. for values of x not furnished by the $=^n$. $d_x f(x) = 0$.

Investigate the condit^n. that $f(x, y)$ may be a max. or min. where x & y are independent.

$x^4 + 2ax^2y - ay^3 = 0$. shew that at the origin the values of $d_x y$ are 0 and $\pm\sqrt{2}$.

And find the max and min values of x and y.

Given the bases of 2 △'s and the sum of their perimeters, shew that in order that the sum of their areas may be a max. the △s must be isosceles, and the sines of the ∠s opp the bases similar as the bases.

In the $=^n$: $p^2 = \dfrac{b^2 r}{2a - r}$, find the max and min values of p and explain the result.

Find the conjugate diam. of an ⊙ whose sum is a max. and sum a minimum.

Find the value of $\dfrac{a^3 \cos^2 x + \sin x}{x}$ when $x = 0$.

—— $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$ when $x = a$.

If $a^3y = x^4$. det. if the origin is a pt of inf.

$y = x^2 \pm x^{\frac{5}{2}}$. there is a cusp of the 2nd kind at the origin

Trace $\frac{y}{a} = \sqrt{\frac{a}{x} - 1}$. shew that there are pts of contrflex. when $x = \frac{3a}{4}$.

If O be the cent. of curv. at P in a C, and OQ ⊥ to OP meets CP in Q. prove that the rad of curv. of the C at P $= \frac{b^2}{a} \sec^3 SPO$, and ρ of the evolute at O $= 3QO$.

If $f(p \cdot \rho) = 0$ be the = n to a spiral. then $f\left(p', \frac{p'}{\rho'}\right) = 0$ is the = n to the locus of ρ.

Shew how to draw a tangent to a curve whose relat. is defined by a relat. betn. SP and HP.

Ex. $SP + nHP = c$.

Shew how to det. multiple and conjugate points.

$y^4 + 2ay^2x + x^4 - 2ax^3 = 2ax^3$ the origin is a double point

$y^4 - x^5 + x^4 + 3x^2y^2 = 0$. shew that the origin is a conjugate point.

$y^2 = \frac{x^3}{2a - x}$. there is a cusp of the first kind at the origin

nonetheless $\tan SPY = \frac{r}{\text{diff.}}$ $\qquad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

$$\rho = r = \frac{u^3 \left(\frac{ds}{d\theta}\right)^3}{u + \frac{d^2u}{d\theta^2}} \qquad \frac{d\theta}{ds} =$$

of $ASP = \frac{1}{2}\rho^2 = \frac{1}{2}p\,ds$

to the polar eqn. to the hyperbola.

$$\frac{p^2}{c^2} = \frac{a^2 - \rho^2}{a^2 - c^2}$$

Det. the asymptotes and points of inflexion of $\frac{r}{a} = \frac{\theta^2}{\theta^2 - 1}$.

In the curve $\rho = \frac{b^2}{a + \sqrt{a^2 - b^2}\cdot\cos\theta}$. find the relation between p & ρ.

In the involute of a ⊙ find the eqn. to p at ρ, and show that the locus of p is the spiral of Archimedes.

$au = \cos^3\frac{\theta}{3}$. find the relation between p and ρ.

and show that the locus of p is a ⊙.

Find the eqn. to a curve wh. touches a series of curves described after a given law.

The centre of curv. is the intersect. of consecutive normals at a point

the ends of a straight line of given length move in rectangular axes of x and y. find the curve wh. it is always a tangent.

the evolute of an hypocycloid is another hypocycloid the arc of a great ○ intercepted between the equator, and the zenith of a place measured on a ~~secondary~~ to the

Reduce to an integrable form the following.

$$\frac{1}{(x+a)^m(x+b^n)^n} \quad \frac{1}{(x^5-x)^2} \quad \frac{1}{(x^2+a)^2(x^2+b)^2}$$

$$\frac{\alpha x+\beta}{(x+p)(x+q)} \quad \frac{1}{\sqrt{a+bx+cx^2}} \quad \frac{1}{\sqrt{a-x}-\sqrt{b+x}} \quad \text{dem} = 2$$

$$\int_x \frac{1}{(x^{2n}+1)\sqrt{(x^{2n}+1)^{\frac{1}{n}}-x^2}} = \tan^{-1}.\ (x^{-2n}+1)^{-\frac{1}{2n}}$$

$$\int_x^{\pi/0} \frac{\cos rx}{1+e\cos x} = \frac{\pi}{\sqrt{1-e^2}}\left\{\frac{\sqrt{1-e^2}-1}{e}\right\}^r.$$

Hence the $\int_x^{\pi/0}$ of $\frac{\cos rx}{(a+b\cos x)^n}$. r and n integers

$$\int_x^{\infty/0} \frac{1}{x^2+c} = \frac{\pi}{2\sqrt{c}}.\ \text{Hence deduce} \int_x^{\infty/0} \frac{1}{(x^2+c)^n}$$

$$\int_t^{\infty/0} e^{-t^2} = \frac{1}{2}\sqrt{\pi}.$$

$$\int_x^{\pi/0} \sin mx = \frac{1}{m}.$$

Shew how to integrate $px^n + p_1 x^{n-1} + p_2 x^{n-2} + \text{—}$

Ex. (x^2+1).

2. Shew how to integrate the product of consecutive terms of an arithmetic progn.

Also a fraction where Den. is a prod.t of consecutive terms.

Integrate a^x. $\log x$. $\cos r\theta$. $\dfrac{1}{\cos r\theta \cos(r+1)\theta}$. $a^x \sin r\theta$.

Shew how to resolve $x^n + p_1 x^{n-1} + \ldots$ into factorials.

Resolve $x^4 - 3x^2$ into factorials and integrate it

Also. $\overline{3x+1}\cdot\overline{3x+4}$. ; $x\cdot(x+1)(x+2)(x+3)$

$$\Sigma.\ \frac{x^2+6x+12}{x(x+1)(x+2)\,2^x} = -\frac{x+3}{x\cdot(x+1)\,2^{x-1}}.$$

$$\Sigma.\ \frac{1}{4x^2-9} = -\frac{1}{6}\cdot\frac{12x^2-12x-1}{(4x^2-1)(2x-3)}.$$

$z = f\left(\frac{x-z}{y-z}\right)$. form the partial diffl = n of the proposed of wh. it is the integral.

$p(x-z) + q(y-z) = 0$

$\frac{dz}{dx} = \frac{a}{\sqrt{a^2-y^2-x^2}}$.

$px + qy + z = 0$, & det. the arbitrary fn. so that when $y = ax^2$ $z = bx$.

$p(z+y) + q(z+x) = x+y$ if $\left\{(x+y+z)(z-x)^2, (x+y+z)(z-y)^2\right\} = 0$.

$(x+y)p + (y-x)q = z$. $z = (x^2+y^2)^{\frac{1}{2}} f\left\{\tan^{-1}\frac{x}{y} - \log\sqrt{x^2+y^2}\right\}$

$x^2p + y^2q = z^2$.

$(2az + cx)p + (2bz - cy)q = (ay + bx)c$

Ansr. $z + ay - bx = f(z^2 - cxy)$

Find a surface in wh. the portion of the axis of y. z between the origin and tangent plane. ∝ dist. of the pt. from the origin. $\left\{z + \sqrt{x^2+y^2+z^2}\right\} x^{n-1} = f\left(\frac{y}{x}\right)$

Find a surface wh. cuts at right ∠s all surfaces comprised in the =n. $(ax^2 + by^2 + cz^2) = D$. when D assumes all possible values. $\frac{z^b}{y^c} = f\left(\frac{z^a}{x^c}\right)$.

To integrate $d_x^2 z - d_y^2 z = (ax + by)$. $z = \frac{ax^3}{6} + \frac{bx^2y}{2} + \varphi(y+x) + \psi(y-x)$.

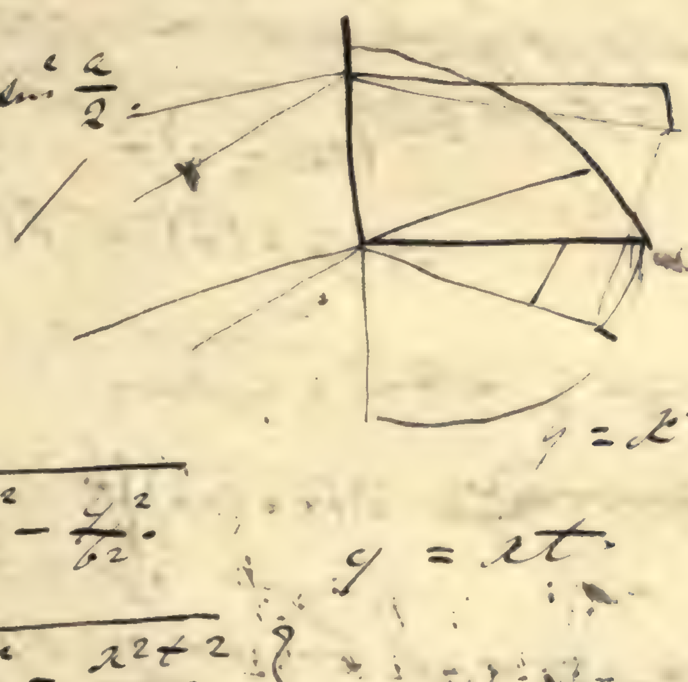
$d_x^2 z - d_x d_y z + d_y^2 z = \varepsilon^{(ax+by)}$

$z = \frac{\varepsilon^{(ax+by)}}{(a-b)^2} + x\,\varphi(y+x) + \psi(y-x)$.

Find a surface in wh. the area of any portion bears a constant ratio to the area of its project: on the plane of xy. A right cone about the axis of z is a particular case.

$\sqrt{1+p^2+q^2} = \text{constant}$.

$$\frac{4\pi a^3}{3} \cdot \sin^2 \frac{a}{2}.$$



$$d_t \, T = $$

$$d_x \cdot d_y \, T = z.$$

$$d_x \cdot d_t \, T = xz.$$

$$y = xt.$$

$$= x\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

$$y = xt.$$

$$= x\sqrt{1 - \frac{x^2}{a^2} - \frac{x^2 t^2}{b^2}}$$

$$= \frac{x^2}{b}\sqrt{\frac{b^2}{x^2} - \frac{b^2}{a^2} - t^2}.$$

$$= \frac{x^2}{b}\left\{\frac{t}{2}\sqrt{\quad} + \frac{1}{2}\left(\frac{b^2}{x^2} - \frac{b^2}{a^2}\right) \sin^{-1} \frac{t}{\sqrt{\frac{b^2}{x^2} - \frac{b^2}{a^2}}}.\right.$$

$d_x y = f(ax+by+c)$. Separate the variables.

$d_x y = \dfrac{3xy^2-x^2}{1+6y^2-3x^2y}$. is exact.

$d_x y = xy^2+x$.

$x d_x y - y = \sqrt{x^2+y^2}$.

$\dfrac{x^2+y^2}{a+x} + y d_x y = 0$. $d_x y + y = a$

$+ d_x y + by^2 = ax^m$. separate the variables if $m = -2$, -4

$\dfrac{x d_x y - y}{\sqrt{1-(d_x y)^2}} = f(x^2 \cdot y^2)$ separate the variables

$M + N d_x y = 0$ be homogeneous. $\dfrac{1}{Mx+Ny}$. renders it exact. Hence integrate. $\dfrac{xy+y^2}{xy-x^2} + d_x y = 0$.

Investigate a factor which will make $d_x y + P y - R = 0$ integrable.

$y d_x y + y + \dfrac{1}{cx} = 0$. becomes integrable by the factor $\dfrac{1}{x}\{\varepsilon^{-\frac{c}{2}(x+y)^2}\}$

$d_x y = \dfrac{y(m+cx)}{y+a+bx+cx^2}$ separate the variables by assuming the second side $= z$.

The trajectory of a st. line $y = cx$ has for its eq. $= a$

$a \log. \dfrac{\sqrt{x^2+y^2}}{C} = \tan^{-1}\dfrac{y}{x}$.

Find the diffl. eq. to the trajectory of a curve of the 2nd order.

Find the curve the locus of the ext. of whose subtangent is a curve of the 2nd order.

$u = l_y.(a^2+b^2) - l_y b^2$.

$\frac{du}{dx} = e^u = \frac{a^2+b^2}{b^2}$

$\therefore\ du\, e^u = \frac{2x}{b^2}$

$\therefore\ \frac{du}{dx} = \frac{2x}{b^2} \cdot \frac{b^2}{x^2+a^2}$

$= \frac{2x}{x^2+a^2}$

$d_x^2 y + a^2 y = 0$.

$d_x^2 y = ax + by$.

$L.\, d_x^2 y + \frac{1}{2} d_x L\, d_x y + cy = c$ L a fn of x

$L.\, d_x^2 y + d_x y = \frac{y}{x}$

$d_x y - (x+a) d_x^2 y - \frac{x}{p}.\, (d_x y)^2 = 0$.

$(1+x^2) d_x^2 y + (d_x y)^2 + 1 = 0$

$x.\, d_x^2 y - d_x y = x^2$.

$d_x^2 y + (\varepsilon^x - 1) d_x y = \varepsilon^{2x}$.

Find the curve in which ρ is a fn of the abscissa.

——— ρ = normal on the curve concave or convex.

——— ρ varies as the cube of the normal.

——— ρ varies as the sec φ where φ is the $\angle$ which the rad. vector makes with the curve

Reduce. $E(d_x^2 y.\, d_x y.\, y) = 0$ to an = n of the first order.

$d_x^2 y\, d_x^3 y = a$

$x(a+bx) d_x^2 y + (c+ex) d_x y = 0$

$y^3.\, d_x^2 y = c$

Integrate $y\, d_x^2 y + (d_x y)^2 = 0$.

$y\, d_x^2 y + \sqrt{1 + (d_x y)^2} = 0$.

$d_x^2 y = f(y)$.

If u be a particular integral of the = n. $d_x^2 y + P d_x y + Q = 0$ the complete integral u. $\int_x \frac{\varepsilon^{-\int P}}{u^2}$.

Explain the term singular solut: of a diffl = n.
Shew how to obt: all the singular solutions from the general integral
Having given a solut of a diffl = n. shew whether it is included in the complete integral or not.
Shew how to find the singular solut: from the diffl = n without knowing the complete integral.
Every factor proper to make a proposed = n integrable is made infinite by the singular solut:

$y^2 - 2xyp + (1+x^2)p^2 = 1.$ $y^2 = 1 + x^2$ is the sing. solut:

$(xp - y)(xp - 2y) + x^3 = 0.$ $y^2 - 4x^3 = 0$ is the solut.

diffl = n. $yp^2 - 2xp - y = 0.$ $x^2 + y^2 = 0$ is a singular solut: $y^2 = 2x + 1$ a particular integral.

Find the curve in wh. the length of tangent terminated by the axis of x is invariable.

The product of the ⊥s upon the tangent from 2 given pts is invariable

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such that by drawing from a point 2 lines to 2 fixed pts and a ∥ line joining the fixed pts cutting the curve wh ∠ is γ. α. we may always have $\tan^2\beta = \tan\alpha \tan\gamma$

Find singular solutions of
$d_x y = \dfrac{y - d_x y}{x - d_x y}$; $(y - x d_x y)^2 = b^2 + a^2 (d_x y)^2$;
$(x+1)^2 = 4y.$

$x^2 \cdot d_x^2 y - x d_x y + y = a \log x$

$x^2 \cdot d_x^2 y - 2x \cdot d_x y + 2y = 0.$

Express x. y. in terms of t from the eqns.

$$d_t x + y + 5x = e^t.$$

$$d_t y - x + 3y = e^{2t}.$$

Solve. $d_t^2 x + d_t^2 y + x + 2y = t.$

$$4 d_t^2 x + d_t^2 y - 3x + y = e^t.$$

Apply $=^n \cdot y = y_{x=a} + \frac{x-a}{1} \cdot d_{x=a} y + \&c$ to.

integrate $d_x^2 y = f \cdot (xy \cdot d_x y).$

Ex. $y - x d_x y + \frac{1}{2} x^2 \cdot d_x^2 y = 0.$

$d_x^2 y + \frac{5+2x^3}{x+x^4} d_x y - \frac{2+x^3}{x^2+x^5} \, 6y = 0$

$y = x^2$ is a solution find the complete integral.

Integrate the linear eq^ns of the nth order & the coeffts being const

— $d_x^2 y - 3 d_x y + 2y = 0$

Integrate the linear eq^n of the nth order with variable coeffts.

— $d_x^2 y + 2 d_x y + y = x^3$.

— $d_x^2 y + n^2 y = 0$ so that when $x = 0$ we may have $y = 1$ $d_x y = 0$.

— $d_x^2 y + \frac{1}{x} d_x y - \frac{1}{x^2} y = 1$ so that when $x = 1$ we may have $y = a \cdot d_x y = b$.

— $d_x^2 y + n^2 y + ax + b = 0$.

— $d_x^2 y + 2m d_x y + n^2 y = 0$

— $a^2 d_x^4 y - d_x^2 y = 0$.

— $d_x^4 y - 4 d_x^3 y + 6 d_x^2 y - 4 d_x y + y = 0$

— $d_x^4 y + 2n^2 d_x^2 y + n^4 y = 0$

— $d_x^2 y + n^2 y + b \cos \overline{mx + a} = 0$.

— $d_x^2 y + n^2 y = a \cos \overline{nx + a}$.

— $d_x^2 y - a^2 y = \varepsilon^x$

— $(a + bx)^2 d_x^2 y + A(a + bx) d_x y + By = f(X)$ reduce to linear eq^n with constant coeffts.

$x^3 d_x^2 y = (y - x d_x y)^2 \quad y = x \log \left(\frac{c + c_1 x}{x} \right)$.

$d_x^4 y - a^4 y = x^3; \quad y = -\frac{x^3}{a^4} + C\varepsilon^{ax} + C_1 \varepsilon^{-ax} + C_2 \cos \overline{ax + C_3}$.

$$\left(d_x - a\right)^r y = \varepsilon^{ax}.$$

$$\therefore y = \frac{\varepsilon^{ax}}{(a-a)^r} + \left(C_1 + C_2 x + C_3 x^2 \cdots\right)\varepsilon^{ax}.$$

$$\left(d_x - a\right)^r y = \varepsilon^{mx}.$$

$$\therefore y = \frac{\varepsilon^{mx}}{(m-a)^r} + \left(C_1 + \text{———}\right)\varepsilon^{ax}$$

$$= \frac{\varepsilon^{mx} - \varepsilon^{ax}}{(m-a)^r} + C' \cdots$$

$$= \frac{x\cdot\varepsilon^{mx}}{r\,(m-a)^{r-1}}$$

$$= \frac{x^r\cdot\varepsilon^{ax}}{\lfloor r} + \left(C' + C_1 x \cdots\right)\varepsilon^{ax}.$$

$$d_x^2 y + n^2 y = \cos nx.$$

$$\left(d_x^2 + n^2\right) y = \cos nx$$

$$\therefore y = \frac{\cos nx}{-n^2 + n^2} + A\cos nx + B\sin nx.$$

$$\left(d_x^2 + n^2\right) y = \cos mx.$$

$$\therefore y = \frac{\cos mx - \cos nx}{n^2 - m^2} + A_1 \cos nx$$

$$= \frac{-x\sin mx}{-2n} = \frac{x\sin nx}{2n} + \text{＋}.$$

1) If $p = f(\varphi)$ be the relation between the rad-vector & ⊥ on the tangent, then locus of the pole when the curve rolls along a st. line in the plane is $y = f(y \cdot d_x s)$.
1) If the curve be a ⊙ the locus of the focus is the common Catenary.
The locus of Cⁿ of a ⊙ rolling along a st line is $d_x y^2 = \frac{y^2}{\sqrt{a^2-y^2}\sqrt{y^2-b^2}}$
Integrate $y = xp + f(p)$. Ex. $\frac{x + y d_x y - c}{x - c} = \frac{(x + y d_x y)^2}{x^2 + y^2}$.
Integrate $y = x f(p) + \varphi(p)$.
Find a curve whose subnormal is constant.
or ∝ a:

(normal)² ∝ abscissa + subnormal

Find a curve in wh. the normal bisects the ∠ between the focal distances. $y^2(1+p^2) = a^2(x+py)$.
Let 2c be the distance of the 2 foci along with the middle point the = n to the curves.

$$(1-p^2)xy = p(x^2+y^2-c^2).$$

Integrate. $(axy)(d_x y)^2 + (x^2 - ay^2 - b) d_x y - xy = 0$.

Assume $d_x y = \frac{x}{y} \cdot a$.

$$x_x y + \frac{1}{P} d_x^2 P = y^2:$$

$$y + \frac{d_x P}{P} + \frac{1}{P^2} \cdot \frac{1}{\int_x \frac{1}{P^2}} = 0. \text{ by } + \frac{d_x P}{P} = \frac{1}{2}.$$

$\times \dfrac{1}{x(a+bx)^{\frac{2}{3}}}$ $\dfrac{x}{(x^3+a^3)^{\frac{2}{3}}}$ $\times \dfrac{1}{\sqrt{a\varepsilon^{x}+b\varepsilon^{2x}}}$.

$\int_0^\infty \dfrac{\cos rx}{\varepsilon^{qx}} = \dfrac{q}{q^2+r^2}$. thence deduce $\int_0^\infty \times \dfrac{x^p \cos rx}{\varepsilon^{qx}}$.

If p be the $\perp$ upon the tangent from an assumed point in the axis and ϕ the $\angle$ which it makes with the axis. & t the length of the tangent which it intercepts. and s the corresponding arc of curve then $d_\phi(s-t) = p$.

Hence shew that the lengths of 2 arcs of an $\odot$ the diff of whose lengths can be expressed algebraically.

Find a series for the length of an elliptic quad. when the ecc. is not greater than $\frac{1}{\sqrt{2}}$.

Find the area of the whole surface of a spheroid.

Find the vol. of any segment of a [illegible].

Shew how to approx. to the area of a curve by the method of equidistant ordinates.

Find the area of the surf. of the ungula of a right cone cut off by a plane thro' the cent. of its base

Find the area of the nodes of the curve

$$y^3 + 3axy + x^3 = 0.$$

Find the area of a sector of a spiral whose eq^n is

$\rho = a\sec\theta + \theta$.

If the eq^n to a spiral be $\rho = \dfrac{a}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

the area $= \dfrac{\pi}{4}\left\{\sqrt{a-b}\right\}^2$

Prove that terms of 3rd order must be retain the coeff^{t}=s [illegible] to be 2nd order.

$d_\theta^2 s + s + \frac{3}{2}m^2k \sin \overline{g\theta-\gamma} - \frac{3}{2}mk \sin \overline{2-2m-g\cdot\theta-2\gamma-g}$.

find the value of s. and interpret the result.

Explain why the motion of the node is obtained by the 2nd approx^{n}. better than the motion of the apse.

Given the value of θ the mean long^{n}.

$pt - 2e\sin(cpt-\alpha) + \frac{5e^2}{4}\sin 2cpt - 2\alpha + \frac{15}{4}me \sin \overline{2-2m-c}\,pt - 2\beta + \alpha$

$- 3me' \sin(mpt+\beta-\zeta)$ find from this the mean velocity of the apse in any revolut. shew that the apse line progresses in syzygy. and regresses in quad. Find also the periodic time of the Moon in any revolut. and shew how it depends on the position of the sun.

$\times \int \frac{p+qx^2+pc^2x^4}{1-c^2x^4} \cdot \frac{1}{\sqrt{1+px^2+c^2x^4}}$.

Assume the rational part $= A. \frac{1+cx^2}{1-cx^2} + B. \frac{1-cx^2}{1+cx^2}$.

Det. A & B.

(1) $\times \frac{1+c^2x^4}{1-c^2x^4} \cdot \frac{1}{\sqrt{1+px^2+c^2x^4}}$ (2) $\times \frac{x^2}{1-c^2x^4} \cdot \frac{\times 1}{\sqrt{1+px^2+c^2x^4}}$. (3) $\times \frac{\sqrt{1+px^2+c^2x^4}}{1-c^2x^4}$

$\times \frac{b+cx}{x^3+a^3}$; $\frac{\times 1}{x^4-4x+3}$.

Integrate without substitut. $x^{m-1}(a+bx^n)^{\frac{p}{q}}$. $\frac{m}{n} = +ve$ integer. $\frac{m}{n}+\frac{p}{q} = -ve$ integer

$\frac{1}{(a+bx^n)(a+2bx^n)^{\frac{1}{2n}}}$. Let $a+2bx^n = \left(\frac{x}{z}\right)^{2n}$. integral $= \int_z \frac{1}{a+b^2z^{2n}}$.

If $a+2bx^n = z^{-2n}$. $\int_x \frac{x^{n-1}}{(a+bx^n)(a+2bx^n)^{\frac{1}{2n}}} \times = -\frac{2}{b} \int_z \frac{1}{1+az^{2n}}$.

Rationalize. $\frac{\alpha+\beta x+\gamma x^2}{p+qx+rx^2} \cdot \frac{1}{\sqrt{a+bx+cx^2}}$

$\frac{x^{m-1}}{(a+bx^n)^{\frac{m}{n}}}$. rationalize and integrate. $\frac{x}{(1+x^3)^{\frac{2}{3}}}$.

$d_x^2 y = a \cos mx + b \cos nx$

$d_x^2 y + n^2 y = a \cos mx + b \cos nx$

$d_x^2 y + 4y = 1 + \sin 2x + \cos 5x$

write down the proper assumptn for obtaing a solutn

State how the forces on the moon and find the value of T. Shew that this force always tends towards the nearer syzygy.

Investigate the diffl =n for determing s. the tangent of the Moons latitude. in solving this =n state what difficulty arises unless the result of its first approxn is used in its obvious form and how the difficulty may be evaded.

Explain why it is necessary to retn in the diffl. =n some terms of the 3rd order to obtain a solutn which is correct to the 2nd order.

Supposing the tangentl force the only disturbg force on the Moon find the value of u to the 2nd order and interpret it. Supposing e of the 2nd order.

Given $\int_0^{\infty} dx\, e^{-ax^2} \cos cx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{c^2}{4a}}$.

Prove $\int_0^{\infty} dx\, e^{-ax^2}\left(e^{mx} + e^{-mx}\right) = \sqrt{\frac{\pi}{a}}\, e^{\frac{m^2}{4a}}$.

Given $\int_0^{\infty} dx\, \frac{x^{m-1}}{1+x} = \frac{\pi}{\sin m\pi}$ where $m > 0 < 1$

then $\int_0^{\frac{\pi}{2}} dx\, (\tan x)^{2m-1} = \frac{\pi}{2\sin m\pi}$.

Given $\int_0^1 dx\, \frac{x^{p-1}}{(1-x)^p} = \frac{\pi}{\sin p\pi}$ Prove $\int_0^1 dx\, \frac{x^{p-1}}{(1+ax)(1-x)^p} = \frac{\pi}{(1+a)^p \sin p\pi}$.

Define a recurring series and prove that it may be generally resolved into 2 or more geometl. Progs.

Find general term of $1 + 4 + 18 + 80 + 356$ &c.

Find n terms of the series of which every term beginning with the 3rd = sum of the 2 preceeding.

Explain when a series is said to be convergent or divergent. $a + ax + ax^2 + \ldots$ is convergent or divergent according as x is $<$ or $>$ 1.

If the limit $\left(\frac{u_{n+1}}{u_n}\right)$ when $n = \infty$ be less than 1. the series $u_1 + u_2 + \ldots$ &c is convergent and divergent in the contrary case.

$$\int_0^\infty dt\, e^{-at^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}. \quad \text{Prove} \int_0^\infty dt\, t^{2n} e^{-at^2} = \frac{1\cdot 3\cdot 5\cdots 2n-1}{(2a)^n}\cdot\frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty dt\, t^{2n+1} e^{-at^2} = \frac{1\cdot 2\cdots n}{2a^{n+1}}$$

$$\int dx\, e^{ax}\sin(mx+\alpha) = \left(\frac{e^{ax}}{a\sec\theta}\right)\sin\overline{mx+\alpha-\theta}. \qquad \tan\theta = \frac{m}{a}.$$

$$\int^r dx\, e^{ax}\sin\overline{mx+\alpha} \quad \frac{e^{ax}}{(a\sec\theta)^r}\sin\overline{mx+\alpha-r\theta}$$

$$\int dx\, e^{ax} x^n \sin\overline{mx+\alpha} = \left(\frac{e^{ax}x^n}{a\sec\theta}\right)\sin\overline{mx+\alpha-\theta}$$

$$-\frac{e^{ax}\, n x^{n-1}}{(a\sec\theta)^2}\sin\overline{mx+\alpha-2\theta} \quad \text{\&c to infin.}$$

Find eqns of condit. that 2 lines may be at right ∠s to one another & that a line and plane may be at right angles to each other. the axes oblique or rectangular.

Given eqns to 3 planes passing thro' the same point find the condit. that they may be in one plane.

Det. the point of intersect. of a plane and ⊥ dropped upon it from a given pt. & the length of ⊥n.

Det. the point of intersect. of a line and a ⊥ let fall upon it from a given pt. and the length of ⊥n.

Find the ∠s of inclinat. of 2 planes whose eqns are given and also of a line and plane.

[illegible]

$$\int_0^1 \frac{x^{m-1}+x^{n-m-1}}{(1+x^n)} = \frac{\pi}{n}\,\mathrm{cosec}\,\frac{m\pi}{n}.$$

Prove $\displaystyle\int_0^1 \frac{x^{m-1}-x^{n-m-1}}{(1+x^n)\log x} = \log\tan\frac{m\pi}{2n}.$

$$\int_0^\infty \frac{\epsilon^{ax}+\epsilon^{-ax}}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\sin mx = \frac{1}{2}\cdot\frac{\epsilon^{m}-\epsilon^{-m}}{\epsilon^{m}+2\cos\alpha+\epsilon^{-m}}.$$

Prove $\displaystyle\int_0^\infty \frac{\epsilon^{ax}-\epsilon^{-ax}}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\,\frac{\sin mx}{x} = \tan^{-1}\left(\frac{\epsilon^{m}-1}{\epsilon^{m}+1}\right)\tan\frac{\alpha}{2}$

$$\int_0^\infty \frac{\epsilon^{ax}-\epsilon^{-ax}}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\cos mx = \frac{\sin\alpha\ [illegible]}{\epsilon^{m}+2\cos\alpha+\epsilon^{-m}}.$$

Eqn = n cont. only 2 variables will represent a cylindrical
surface // to that axis whose coord. it does not involve
Find = n to contains Hord., and the value of the chord //
to plane of yz. Prove that it can have an infinite no. of st. lines
entirely coinciding with the surface.
Find = n to Hei Cord. and then that it has plane asymptotes
[illegible] of st. lines coinciding with the surface.
and can have an infinite no. of st. lines [illegible] upon another line.
Find the length of project. of a limited line upon another line.
Find dist of pt. from the origin and of 2 pts from one another
in terms of oblique coords.
Find = n to the surface generated by a st. line which constantly passes
thro' 3 fixed lines.
Find the locus of a pt. whose dist. from origin [illegible] ⊥ dist. measured
|| to the plane of xy. from a given line
[illegible] from one system of coords. to another one ~~oblique~~ rectangular
& one oblique.

$$\int_0^1 dx \; \frac{x^{m-1}+x^{n-m-1}}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}$$

Integrate with respect to m.

$$\int_0^1 dx \; \frac{x\left(x^{m}+x^{n-m}\right)}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{m\pi}{n}.$$

$$\therefore \int_0^1 dx \; \frac{x\left(x^{m}+x^{n-m}\right)}{(1+x^n)\log x} = \frac{\pi}{n}\int_m \frac{1}{\sin \frac{m\pi}{n}}$$

$$= \frac{\pi}{n} \cdot \frac{1}{\frac{\pi}{n}} \log \tan \frac{m\pi}{2n}.$$

$$= \log\left(\tan \frac{m\pi}{2n}\right)$$

there is proveth following.

$\tan SPL = \frac{\rho}{d_\theta \rho}$

$\frac{1}{p^2} = u^2 + (d_\theta u)^2$

$(d_\theta s)^2 = \rho^2 + (d_\theta \rho)^2$

$$r = \frac{u^3 (d_\theta s)^3}{u + d_\theta^2 u}$$

$d_\theta ASP = \frac{1}{2}\rho^2 = \frac{1}{2} p \, d_s \theta$

express the rad of curv. in terms of p and $d_s p$.

Let the asymp. ie the spiral of the spiral

$$\frac{\rho}{a} = \frac{\theta^2}{\theta^2 - 1}$$

obt. the polar eq. of the cycloid

$$\frac{p^2}{c^2} = \frac{a^2 - \rho^2}{a^2 - c^2}$$ hence show that the earlier is a semi epicycloid.

In the curve $\rho = \frac{b^2}{a + \sqrt{a^2 - b^2}\cos\theta}$

then that $p^2 = \frac{b^2 \rho}{2a - \rho}$

$\frac{\rho}{a} = \left(\sec\frac{\theta}{3}\right)^3$ find the relat. bet. p and ρ $p^3 = \rho^2 a$.

and prove the locus of y. is the common [illegible]

In general bet. p and ρ in the involute of a ⊙ and show that [illegible] of Archimedes.

$u = x^4 - 8x^3 + 24x^2 - 32x$ find wh. max or min if $x = 2$.
If the sum of the edges of a rect. parallelopiped is a and one edge is double of the other, shew that the remaining edge must $= \frac{a}{3}$ in order that the volume may be a max.
Define wh. 2 curves are said to have a contact of the nth order. Prove that two curves wh. have a contact of the nth order will touch or cut each other as n is odd or even.
Prove Def. the circle of curv. and find its rad. — the co ord. of its cent.
In the con. ... rad of curv. at a pt. $P = 2(SP)^{\frac{3}{2}} \cdot \sqrt{SA}$.
In any con. sec. rad of curv $= \dfrac{(\text{normal})^3}{\left(\frac{L}{2}\right)^2}$ $\left(\frac{L}{2}\right)$

In the ellipse rad of curv $= \dfrac{b^2}{a(1-e^2 \sin^2\alpha)^{\frac{3}{2}}}$
where α is the angle wh. normal makes with major axis
If $(a-b) = c$ $\quad r = (a - 2c + 3c \sin^2\alpha)$ nearly.
In the cycloid the rad of curv $= 2$ normal.
catenary —— $\frac{y}{a} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$
rad of curv = normal.
Prove that rad of curv $= \dfrac{1}{\sqrt{(d_s^2 x)^2 + (d_s^2 y)^2}}$

the length of the least cone that can circums. a given sphere is 2 diam.

Which sector must be taken out of a given ~~semicircle~~ circle. in order that the rem. may form a cone of the gr. volume.

∴ the angle of the sec. is $2\pi\left(1-\sqrt{\tfrac{2}{3}}\right)$

$u = 3x^5 - 125x^3 + 2160x$ find its great. maxi.

when $x = 3$.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\frac{1}{\sqrt{(1-x^2)(x+2)}} = 2\sin^{-1}\sqrt{\frac{x+2}{3}}$$

$$\frac{1}{(1+x)\sqrt{1+x-x^2}} = \sin^{-1}\frac{3x+1}{(x+1)\sqrt{5}}$$

$$\frac{\sqrt{ax+bx^3}}{x} = -\frac{2\sqrt{3}}{3}\left\{\log\left\{\sqrt{\frac{a}{x^3}}+\sqrt{b+\frac{a}{x^3}}\right\}\right\} + \frac{2}{3}\left(a+bx^3\right)$$

$$\left(x^2+a\right)\sqrt{x^2+b} = \frac{x}{4}\left(x^2+b\right)^{\frac{3}{2}} + \left(a-\frac{1}{4}b\right)\left\{\frac{x}{2}\sqrt{x^2+b}+\frac{b}{2}\log\left(x+\sqrt{x^2+b}\right)\right\}$$

$$\frac{1}{(x+a)\sqrt{x^2-a^2}} = \left\{-\frac{2}{\sqrt{a^2+a}}\right\}\tan^{-1}\sqrt{\frac{a-ax}{x+ax}}$$

$$\frac{2x+3}{x^3+x^2-2x} \quad ; \quad \frac{x^2-2}{x^3+4x^2+4x} \quad ; \quad \frac{x^2-x+1}{x^3+x^2+x+1}$$

$$\frac{x^3+a^3}{(x+a)^3} \quad ; \quad \frac{x^2+1}{(x^2+3)(x^2+5)} \quad ; \quad \frac{1-x\cos\alpha}{1-2x\cos\alpha+x^2}$$

$$\frac{1}{x\sqrt{a+bx^3}}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$(a-x)(b-x)^{\frac{m}{n}}$

$\int \frac{cx+ex^3}{(a+bx^2)^n}$. $\int \frac{1}{\sqrt{a-x}}$. $\int \frac{x^{n-1}}{\sqrt{a+bx^n}}$.

$\int \frac{1}{(x^2+a^2)^{\frac{3}{2}}}$

When the exp. $(a+bx^n)^{\frac{p}{q}} x^{m-1}$ is such a form that unless $\frac{m}{n}$ is a pos. int. or $\frac{m}{n}+\frac{p}{q}$ a neg. it is integrable

i. the integ. of the to elem. log. forms.

— circular forms.

$\sqrt{u^2 \pm a^2}$. $d_x u$ $\sqrt{2au \pm u^2}$. $d_x u$. $\sqrt{a^2-u^2}$

$\int \frac{1}{a+bx \pm cx^2}$; $\int \frac{1}{x\sqrt{a+bx+cx^2}}$; $\int \frac{1}{x^2\sqrt{a+bx+cx^2}}$.

Log.

$\int \frac{1}{(c+ex)\sqrt{a+bx}}$; $\int \frac{1}{(c+ex)\sqrt{a+bx^2}}$; $\int \frac{1}{(c+ex^2)\sqrt{a+bx^2}}$.

$\int \frac{1}{\sqrt{(a+bx)(c+ex)}}$. $\int \frac{1}{(c+ex)\sqrt{a+bx^2}}$. $\int \frac{1}{(p+qx)\sqrt{a+bx+cx^2}}$. $\int \frac{x}{(c+ex^2)\sqrt{a+bx^2}}$.

$\frac{p+qx^2}{(c^4-x^4)\sqrt{a+bx^2}}$. $\int \frac{(p+qx)\sqrt{a+bx^2}}{c+ex^2}$ $\frac{1}{(px+qx^2)\sqrt{a+bx+cx^2}}$.

$\frac{x^2+ac}{x\sqrt{(a^2-x^2)(x^2-c^2)}}$:

every integ. an arbitrary const. must be added. the value of wh. will be known if the value of the integ. corresp. to a given value of x be known.

Investigate the intg. of x^{m-1} and det. the const. suppose the value of the integ 0 when $x = a$.

1 ———— $a u^{m-1} d_x u$. u being a fn of x.

$\frac{d_x u}{\sqrt{u}}$ ————

Having given $\int_x x^m = \frac{x^{m+1}}{m+1} + C$ deduce the result $\int_x \frac{1}{x} = \log_\varepsilon x + C$.

Investigate $\frac{n d_x u}{u}$. Shew how to intg. all rational & algeb. fns of x.

$\int_x u = \int_z (u . d_x z)$. ———— u a fn. of x and u a fn of z.

Shew how to intg. $(a+bx^n)^{\frac{p}{q}} x^{m-1}$. wh. $\frac{m}{n}$ a posit. or $\frac{m}{n} + \frac{p}{q}$ a neg. int.

Intty..

$u = (2x^3+3x^2)^4(x^2+x)$, $\frac{b+2cx}{\sqrt{a+bx+cx^2}}$ $\frac{1}{\sqrt[3]{a-x}}$, $\frac{1}{a+bx}$

$\frac{1}{bx+cx^2}$, $\frac{2-4x}{x^2-x-2}$, $\frac{5+7x^3}{x}$, $x^2\sqrt{a+x}$

$\frac{1}{x^3\sqrt{2ax-x^2}}$, $(d)\ (a-x)\sqrt{b-x}$. $(x^2+7)\sqrt{x^2+4}$.

$\frac{1}{x+x^{\frac{2}{3}}}$. $x^3(a+bx^2)^{\frac{2}{3}}$.. $\frac{m+n\sqrt{x^{-n}}}{x+x^{-n+1}}$

$$\frac{6x^6-3x^5+2x^3+4x-2}{7x^4-x^3}.$$

(d) $= (a-b+b-x)\sqrt{b-x} = (a-b)\sqrt{b-x} + (b-x)^{\frac{3}{2}}$.

Example what is meant by taking an integral between limits.

$\sqrt{a^2-x^2}$; $\int_x^{0}\int^{1} x^{n-1}$; $\int_x^{0}\int^{\infty} e^{-ax} \sin mx$; $\int_x^{0}\int^{\infty} \sin mx = \frac{1}{m}$.

$\frac{\pi}{2}$ $(\sin x)^n$; $\int_x^{0}\int^{\infty} \frac{x^n}{e^{ax}} = \frac{1.2.\ldots n}{a^{n-1}}$. $\int_x^{0}\int^{\infty} \left(1+\frac{x^2}{n}\right)^{-n} = \frac{\pi\sqrt{n}}{2} \frac{1.3.(2n\ldots}{2.4.6.(2n\ldots}$

show what it becomes when n is infinite.

$u = C + xu + \frac{x^2}{1.2} \frac{du}{dx} + \frac{x^3}{1.2.3} d$

$v_x = v_{x=a} (x-a) + dv_{x=a} \frac{(x-a)^2}{1.2} + d^2v_{x=a} \frac{(x-a)^3}{1.2.3}$ &c.

$\frac{1}{\sqrt{a+bx}} = \frac{2}{b}\left\{\sqrt{a+b} - \sqrt{a}\right\}$. $\int_x^{0}\int^{1} \frac{1}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}} \cdot \log \left\{\frac{2c+b+2\sqrt{c}\sqrt{a\ldots}}{b+2\sqrt{ac}}\right.$

and shew how the former results contain

$\frac{1}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$ $\int_x^{0}\int^{\infty} \frac{1}{(a+b\cos x)^2} = \left(\frac{\pi a}{(b^2)}\right)^{\frac{3}{2}}$.

$\int^{\pi}$ $\frac{\cos rx}{1+e\cos x} = \frac{\pi}{\sqrt{1-e^2}} \left\{\frac{\sqrt{1-e^2}-1}{e}\right\}^r$

$\int^{\infty}$ $\frac{1}{x^2+c}$. $\int_x \frac{1}{\sqrt{2ax-x^2}}$.

Expand $\log_{b}(1+x)$ $\tan^{-1} x$. $\sin^{-1} x$.

$\therefore$ $f(x)$ will have equal roots or not according as $f(x)$ and $f'(x)$ have a com. meas.

then how to find the commensurable root of $f = 0$.

The root of a reciprocal = ^{n} of an even no. of dimensions may be found by the solut. of an = ^{n} of $\frac{1}{2}$ the no. of dimensions.

The imaginary roots of $x^n - 1 = 0$ n being a prime no. are the n powers of α from 1 to $n-1$. α being one of the imag. roots

any radical has as many values as there are units in its index

Find the condit. $ax^2 + bx + c = 0$ being resolvable into 2 simple factors and for being a perfect sq.

Solve a Cubic by Cardan's method, and shew that the solut. extends only to the cases in wh. 2 of the roots are impossible.

$x^4 - \frac{x^2}{2} + \frac{3}{16} = 0$ has $= ^n$ $(x - \frac{1}{2})(x - \frac{1}{2})(x^2 + x + \frac{3}{4})$

$x^3 - 4x^2 - 6x + 12$ has a commensurable root.

$(x+2)(x^2 - 6x + 6)$

$x^5 + 2x^4 - 6x^3 - 4x^2 + 13x - 6 = 0$ find how oft the root unity is repeated - 3 times.

$x^4 + 2x^3 - 9x^2 - 22x - 22 = 0$ has 2 roots such that $\alpha + \beta + 2 = 0$ $\therefore d = -(\beta + 2)$

$x^4 - 14x^3 + 61x^2 - 84x + 36 = 0$ has 2 pairs of = roots.

$(x-1)^2 (x-6)^2$.

x^3 9) $x^3 - x^2 - 9x + 9$ has roots in harmonic proport.

$= (x^2 - 9)(x-1)$

$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$

has roots of the form $\alpha . \alpha . \alpha . \beta . \beta$. $(x-3)^3 (x-2)^2$.

if the roots of $x^3 - px^2 + qx - r = 0$ be in arithmetic prog. $9r - pq \div 6q - 2p^2$ is one of them.

$$2PR\,dy + 2P^2R + y\{P\,dR - R\,dP\} + 2y^2PR^2 = 0$$

$$d.\left(y\sqrt{\frac{R}{P}}\right) = \sqrt{\frac{R}{P}}\cdot y$$

$$= \sqrt{\frac{R}{P}}\,dy + \frac{1}{2}y\cdot\frac{P\,dR - R\,dP}{P^2\sqrt{\frac{R}{P}}}$$

$$= \frac{PR\,dy + \frac{1}{2}y\,(P\,dR - R\,dP)}{P\sqrt{PR}}$$

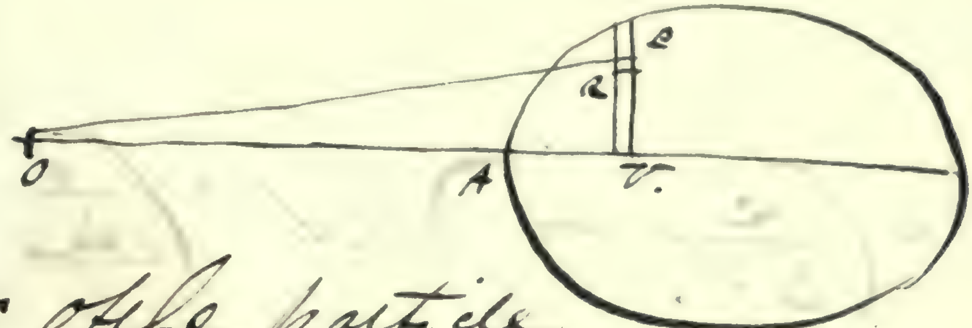
$$PR\,dy + \frac{1}{2}y\,(P\,dR - R\,dP)$$

$$+ PR.\,(P + y^2R) = 0$$

$$\sqrt{PR}\,d\left(y\sqrt{\frac{R}{P}}\right) + PR\{P + y^2R\} = 0$$

$$\sqrt{PR}\,d\left(y\sqrt{\frac{R}{P}}\right) + P^2R\left(1 + \frac{y^2R}{P}\right) = 0$$

1)



O the positⁿ of the particle

OA = a. $\delta x \delta y$ an element of the ellipse.

attraction of this element upon O in direction OA is $\mu . \delta x \delta y . \dfrac{(a+x)}{\{(a+x)^2+y^2\}^{\frac{3}{2}}}$.

∴ whole attraction of ellipse =

$$\mu . \int_x^{a+c} \int_y \frac{a+x}{((a+x)^2+y^2)^{\frac{3}{2}}}.$$

$$\int_y \frac{a+x}{((a+x^2)+y^2)^{\frac{3}{2}}} = \frac{(a+x)y}{\{(a+x)^2+y^2\}^{\frac{1}{2}}}.$$

Limy $= \frac{b}{c}\sqrt{2cx-x^2}$.

∴ attract. of $\odot$ =

$$\frac{\mu b}{c} \int_x^{a+c} \frac{(a+x)\sqrt{2cx-x^2}}{\sqrt{a^2+2.(a-\frac{b^2}{c})x+(1-\frac{b^2}{c^2})x^2}}.$$

$$\frac{P_1 \, d_1\left(y\sqrt{\frac{R}{P}}\right)}{1+\left(y\sqrt{\frac{R}{P}}\right)^2} + P_1\sqrt{PR} = 0$$

$$d_1 \tan^{-1}\left(y\sqrt{\frac{R}{P}}\right) + \sqrt{PR} = 0$$

$$\tan^{-1}\left(y\sqrt{\frac{R}{P}}\right) + \int_1 \sqrt{PR} = 0$$

$$\therefore y = -\sqrt{\frac{P}{R}}\tan\int_1\sqrt{PR}$$

Let h.g.f. h'g'f' be the coords of 2 particles the former, on an ellipsoid whose =ⁿ is.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

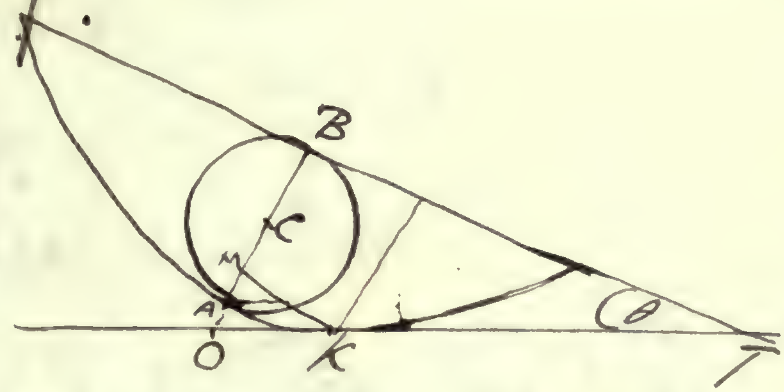
the latter on an ellipsoid whose =ⁿ is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$$

Then the attraction of the first particle upon the second ellipsoid is to the attraction of the second particle on the first ellipsoid // to the axis of z as ab : αβ.

Similar expressions hold for the attractions // to the axes of x & y.

The use of this theorem is to make the attraction of an ellipsoid upon an internal partic. depend upon the attraction of another ellipsoid upon an external particle.



The eqn to BT is

$$y = (X - OT)\tan\theta$$

Now $OT = OK + KT$

Let $AM = x$, $MK = y$

Then $\cot\theta = \sqrt{\dfrac{2a-x}{x}}$

Now $OK = AK = 2\sqrt{2ax}$

$$KT^2 = (2a-x)^2 + \frac{(2a-x)^3}{x}$$

$$= (2a-x)^2\,\frac{2a}{x}$$

$$\therefore OT = 2\sqrt{2ax} + \sqrt{2a}\,.\,\frac{2a-x}{\sqrt{x}}$$

$$= \sqrt{\frac{2a}{x}}\,.\,(2x + 2a - x)$$

$$= \sqrt{\frac{2a}{x}}\,.\,(x + 2a)$$

$x + y = 8$

$\frac{x-y}{2} + \frac{2x-y}{3} + \frac{4}{3} = 0$

$3x - 3y + 4x - 2y = - 8$

$\therefore 7x - 5y = - 8$

$7x + 7y = 56$

$\therefore 12y = 48$

$\therefore y = 4$

and $x + y = 8$

$\therefore x = 4$

$\frac{x+3}{x} + 4x = 14$

$\therefore x + 3 + 4x^2 = 14x$

$\therefore 4x^2 - 13x = - 3$

$\therefore x^2 - \frac{13}{4}x + \frac{169}{64} = - 3 + \frac{169}{64} = \frac{-192}{64} + \frac{169}{64} = -\frac{23}{64}$

$\therefore x - \frac{13}{8} = \pm \frac{\sqrt{23}}{8}$

$\therefore x = \frac{\sqrt{23}}{8} + \frac{13}{8}$

Now $x \cot^2\theta = 2a - x$

$$\therefore\ 2a = x \operatorname{cosec}^2\theta$$

$$x = 2a \sin^2\theta$$

$$\therefore\ OT = \frac{1}{\sin\theta}\left(2a\sin^2\theta + 2a\right)$$

$$= 2a.\ \frac{\sin^2\theta + 1}{\sin\theta}$$

(2) $$\therefore\ Y = 2a.\frac{1+\sin^2\theta}{\sin\theta}.\tan\theta - X\tan\theta$$

$$= 2a\,\frac{1+\sin^2\theta}{\cos\theta} - X\tan\theta$$

$$\text{or } Y\cos\theta + X\sin\theta = 2a\left(1+\sin^2\theta\right)$$

$$\therefore\ -Y\sin\theta + X\cos\theta = 4a\sin\theta\cos\theta$$

$$\therefore\ Y^2 + X^2 = 4a^2\left\{1 + \sin^4\theta + 2\sin^2\theta\cos^2\theta\right\}$$

$$= 4a^2\left\{1 - \sin^4\theta + 2\sin^2\theta\right\}$$

Gen $S = \left\{2a + (n-1)b\right\}\frac{n}{2}$

Here $a = 4$, $b = -7$, $n = 20$

$\therefore S = \left\{8 + (20-1)\,{-7}\right\}\frac{20}{2}$

$= (8 + 19 \times -7)\,10$

$= (8 - 133)\,10$.

$= -125 \times 10 = -1250$

G $S = \left\{2a + (n-1)b\right\}\frac{n}{2}$

Here $a = \frac{n-1}{n}$, $b = -\frac{1}{n}$, $n = n$ ..

$\therefore S = \left\{\frac{2n-2}{n} + (n-1)\,{-\frac{1}{n}}\right\}\frac{n}{2}$.

$\therefore S = \left\{\frac{2n-2}{n} + \frac{-n+1}{n}\right\}\frac{n}{2}$

$\therefore S = \frac{n-1}{n} \times \frac{n}{2}$

$\therefore S = \frac{n-1}{2}$

We have then

$$y = 2a\{\cos\theta + \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta\}$$

$$= 2a\{2\cos\theta - \cos^3\theta - 2\cos\theta + 2\cos^3\theta\}$$

$$\therefore \cos^3\theta = \frac{y}{2a}$$

$$\cos^2\theta = \left\{\frac{y}{2a}\right\}^{\frac{2}{3}}$$

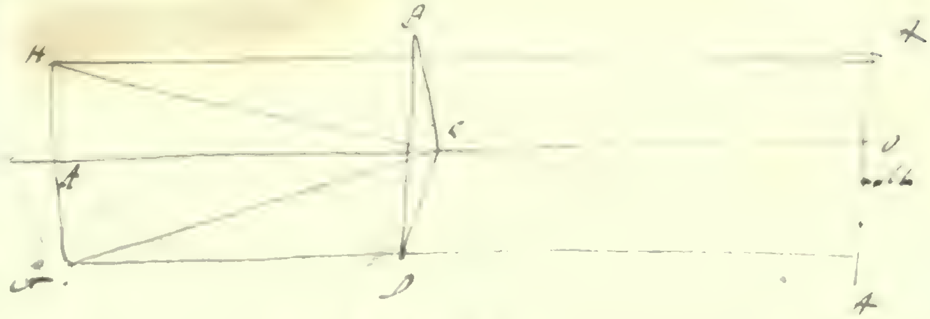
$$\therefore y\cdot\left\{\frac{y}{2a}\right\}^{\frac{1}{3}} + x\left\{1 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\} =$$

$$2a\left\{2 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\}$$

$$y\left(\frac{y}{2a}\right)^{\frac{1}{3}} + (x - 2a)\left\{1 - \left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\} = 2a$$

$$y^{\frac{4}{3}} + (x - 2a)\left\{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}\right\} = (2a)^{\frac{5}{3}}$$

$$(2a)^{\frac{1}{3}}\cdot\left\{(2a)^{\frac{4}{3}} - y^{\frac{4}{3}}\right\} = (x - 2a)\left\{(2a)^{\frac{2}{3}} - y^{\frac{2}{3}}\right\}$$



Let H be the source of light from which a wave diverging is refracted thro' the prism whose sides are inclined to each other at a small angle ι. Let H' be the point from which, if a wave diverged without being refracted, it would [illegible] proceed in the same manner as the wave which proceeds from H is refracted: and H' a point from which a wave ~~would proceed~~ without refraction would proceed as the wave from H after refraction at C.

Then $\angle HCH' = \angle ACB = (\mu-1)\iota$ also $HH' = (\mu-1)\sin\iota$. Let $HC = a$, $CO = b$, and [illegible] a point on the screen upon which the waves from H and H' fall. The undulation from H may be represented by

$$\frac{c}{HO}\sin\frac{2\pi}{\lambda}\left(vt - HO + \ldots\right)$$

a ditto from H' may be represented by

$$\frac{c}{H'O}\sin\frac{2\pi}{\lambda}\left(vt - H'O + B\right).$$

Since the waves from H and H' [illegible]

$$\therefore (2a)^{\frac{1}{3}}\left\{(2a)^{\frac{2}{3}} + y^{\frac{2}{3}}\right\} = X - 2a$$

$$\therefore 2a + (2a)^{\frac{1}{3}} y^{\frac{2}{3}} = X - 2a$$

$$\therefore (2a)^{\frac{1}{3}} y^{\frac{2}{3}} = X - 4a$$

... in the [illegible] phase $A = B$.

Now in the coeffts. we may replace HM and GM by $AC = a+b$ nearly.

∴ the whole disturbance at M is represented by

$$\frac{c}{a+b}\left\{\sin\frac{2\pi}{\lambda}(vt - HM + A) + \sin\frac{2\pi}{\lambda}(vt - GM + A)\right\}$$

$$= \frac{2c}{a+b}\cos\frac{\pi}{\lambda}(HM - GM)\sin\frac{2\pi}{\lambda}\left(vt - \frac{HM+GM}{2} + A\right).$$

∴ the intensity of the light is represented by

$$\frac{4c^2}{(a+b)^2}\cos^2\frac{\pi}{\lambda}(HM - GM).$$

Also $HM^2 = ((\mu-1)a\sin\alpha + bx)^2 +$

$$HM^2 = (a+b)^2 + ((\mu-1)a\sin\alpha + x)^2 \quad \text{writg. } a+b \text{ for } AC.$$

$$\therefore HM = a+b + \frac{1}{2}\frac{((\mu-1)a\sin\alpha + x)^2}{(a+b)}$$

$$GM = a+b + \frac{1}{2}\frac{((\mu-1)a\sin\alpha - x)^2}{a+b}$$

$$\therefore HM - GM = \frac{2(\mu-1)a\sin\alpha}{a+b}\cdot x$$

∴ intensity of light =

$$\frac{4c^2}{(a+b)^2}\cos^2\frac{2\pi}{\lambda}\left(\frac{a(\mu-1)\sin\alpha}{a+b}\,x\right)$$

A perfectly flexible chain of variable thickness hangs in the form of a curve whose equation taken from the lowest point is $\frac{y}{b} = \log \sec \frac{x}{b}$. Find the law of variation of thickness, and shew that the chain offers an equal resistance to rupture at every point.

If m = unit of mass. T = tension of the chain at the lowest point. $m = \frac{T}{g} \cdot \frac{d^2_x y}{d s_x}$.

Now $\frac{y}{b} = \log \sec \frac{x}{b}$ or $e^{\frac{y}{b}} = \sec \frac{x}{b}$.

$\frac{d y_x}{b} e^{\frac{y}{b}} = \frac{1}{b} \sec \frac{x}{b} \tan \frac{x}{b}$ or $d y_x = \tan \frac{x}{b}$.

$d^2_x y = \frac{1}{b} \cdot \sec^2 \frac{x}{b}$.

$d s_x = \sqrt{1 + (d_x y)^2} = \sqrt{1 + \tan^2 \frac{x}{b}} = \sec \frac{x}{b}$.

$\therefore\ m = \frac{T}{bg} \cdot \sec \frac{x}{b} = \frac{T}{bg} \cdot e^{\frac{y}{b}}$.

Tension at $P = T \sec \theta$

$= T \sec \frac{x}{b}$

$\frac{\text{Tension at } P}{m} = \text{Constant}$

$$d_s y = \frac{d_x y}{\sqrt{1+(d_x y)^2}}$$

$$d_s x = \frac{1}{\sqrt{1+(d_x y)^2}}$$

$$\frac{x\,d_x y + y}{\sqrt{1+(d_x y)^2}} = \sqrt{x^2+y^2}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

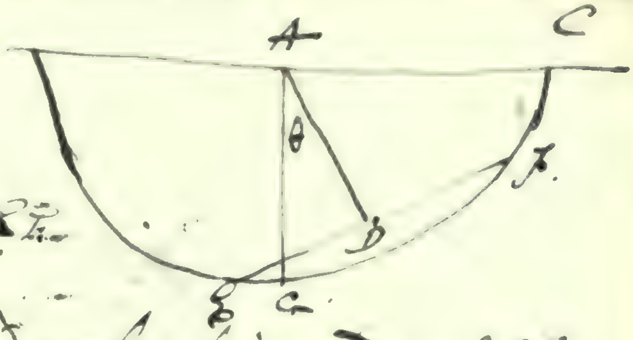
$$\frac{x\,d_\theta y + y\,d_\theta x}{\sqrt{(d_\theta x)^2+(d_\theta y)^2}} = r$$

$$\left.\begin{aligned} d_\theta y &= r\cos\theta + \sin\theta\, d_\theta r \\ d_\theta x &= -r\sin\theta + \cos\theta\, d_\theta r \end{aligned}\right\}$$

$$x\,d_\theta y + y\,d_\theta x = r^2\cos 2\theta + d_\theta r \sin 2\theta . r$$

$$(d_\theta x)^2 + (d_\theta y)^2 = r^2 + (d_\theta r)^2$$

$$r\cos 2\theta + d_\theta r . \sin 2\theta = \sqrt{r^2 + (d_\theta r)^2}$$



B E C a vertical section of the bowl: passing thro' the rod E F

let D be the c.e.n.t. of grav. of the rod which is in the middle
point AD = c. ∠ GAD = θ (GA a vertical line)

∴ for the motion of the rod we have

$$m \cdot k^2 \cdot \frac{d^2\theta}{dt^2} = -gm \cdot c \sin\theta \qquad k^2 = \text{rad of gyr. about } A$$

$$k^2 \cdot \frac{d^2\theta}{dt^2} + gc \sin\theta = 0$$

If θ is small we may assume $\sin\theta = \theta$.

$$\therefore k^2 \cdot \frac{d^2\theta}{dt^2} + gc\,\theta = 0$$

$$\frac{d^2\theta}{dt^2} + g\frac{c}{k^2}\,\theta = 0 \qquad \frac{d^2\theta}{dt^2} + \frac{gc}{k^2}\theta = 0 \quad \therefore\ \frac{d^2\theta}{dt^2} + m\theta = 0$$

$$\theta = A\sin\left\{\sqrt{\frac{gc}{k^2}} \cdot t + B\right\} \qquad \frac{\pi}{2} \qquad \frac{\pi k}{\sqrt{gc}}$$

If 2a = length of the rod. Then rad of gyration about an axis thro' D ∥ to the axis of rotation $= \frac{a^2}{3}$.

$$\therefore k^2 = c^2 + \frac{a^2}{3} = \frac{3c^2 + a^2}{3}$$

$$\therefore \theta = A\sin\left\{\sqrt{\frac{3gc}{3c^2+a^2}} \cdot t + B\right\}$$

$$\frac{r^2 \operatorname{Cos} 2\theta + 2r\, d_\theta r \operatorname{Sin} 2\theta + \operatorname{Sin}^2 2\theta\, (d_\theta r)^2}{\operatorname{Cos} 2\theta}$$

$$= r^2 + (d_\theta r)^2$$

$$r^2 \operatorname{Sin}^2 2\theta + (d_\theta r)^2 \operatorname{Cos}^2 2\theta = 2r\, d_\theta r \operatorname{Sin} 2\theta \operatorname{Cos} 2\theta$$

$$(r \operatorname{Sin} 2\theta - d_\theta r \operatorname{Cos} 2\theta)^2 = 0$$

$$r \operatorname{Sin} 2\theta = d_\theta r \operatorname{Cos} 2\theta$$

$$\frac{d_\theta r}{r} = -\frac{1}{2}\, \frac{d_\theta \operatorname{Cos} 2\theta}{\operatorname{Cos} 2\theta}$$

$$\log_e r = \log_e C - \frac{1}{2} \log_e \operatorname{Cos} 2\theta$$

$$r = \frac{C}{\sqrt{\operatorname{Cos} 2\theta}}$$

$$r^2 \operatorname{Cos} 2\theta = C$$

Let m, m' be the masses of P & Q.

∴ ~~force on P = g(m-m')~~

∴ force on P $= mg - T$.

acc. force on P $= g - \frac{T}{m}$

acc. force on Q $= \frac{T}{m'} - g$.

$\therefore g - \frac{T}{m} = \frac{T}{m'} - g$.

$2g = T \cdot \left\{ \frac{1}{m'} + \frac{1}{m} \right\} = \frac{m+m'}{mm'} T$.

$\therefore T = \frac{2g\, m m'}{m+m'}$.

which gives T.

also acc. force on P $= g - \frac{T}{m} = g - \frac{2gm'}{m+m'}$

$= \frac{(m-m')g}{m+m'}$

~~acc. force on Q~~ $= \frac{T}{m'} - g$ = acc. force on Q.

$= g\left\{\frac{2m}{m+m'} - 1\right\} = g\left\{\frac{m-m'}{\ldots}\right\}$

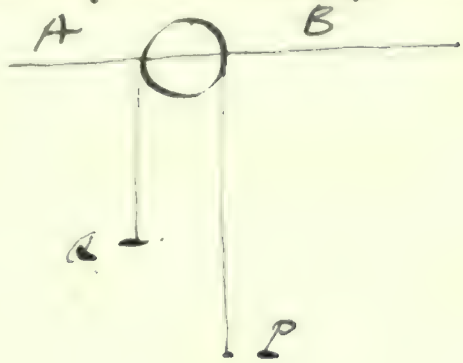
Let v be the velocities of P and Q at the time the string was cut.

For the mot. of P: $\frac{d^2x}{dt^2} = +g$ — x = dist. from plane A' B' at any time

$\therefore \frac{dx}{dt} = +gt + v$ — a = dist. at the time the string was cut.

$x = a + vt + \frac{1}{2}gt^2$.

Similarly for the mot. of Q. $\frac{dx'}{dt} = gt - v$ $\therefore x' = b - vt + \frac{1}{2}gt^2$



2. $x^3 d_x^2 y = (y - x d_x y)^2$

~~$d_x\left(\frac{x}{y}\right)$~~

$$d_x\left(\frac{y}{x}\right) = \frac{y - x d_x y}{x^2}$$

$$\therefore x^3 d_x^2 y = x^4 \left(d_x \frac{y}{x}\right)^2$$

$$x d_x^2 y = x^2 \left(d_x \frac{y}{x}\right)^2$$

$$d_x\left(x^2 d_x \frac{y}{x}\right) = x^2 \left(d_x \frac{y}{x}\right)^2$$

$$x^2 d_x\left(\frac{y}{x}\right) = z$$

$$d_x z = x^2 \frac{z^2}{x^4} = \frac{z^2}{x^2}$$

$$\frac{1}{z} = \frac{1}{x} + C$$

$$x = z + Cxz$$

$$= x^2 d_x\left(\frac{y}{x}\right)(1 + Cx)$$

$$d_x\left(\frac{y}{x}\right) = \frac{1}{x(1 + Cx)} = \frac{x^{-2}}{x^{-1} + C}$$

$$\frac{y}{x} = -\log_e\left(x^{-1} + C\right) = \log_e \frac{x}{Cx + 1}$$

4) $x^9+x^8-9x^7+3x^6-8x^5-8x^4+3x^3-9x^2+x+1=$

divide by $x+1$

$1+1-9+3-8-8+3-9+1+1$

$1 \quad \frac{-1}{0} \quad \frac{0}{-9} \quad \frac{9}{12} \quad \frac{-12}{-20} \quad \frac{20}{12} \quad \frac{-12}{-9} \quad \frac{9}{0} \quad \frac{0}{1} \quad \frac{-1}{0}$

and the reduced =tion is

$x^8-9x^6+12x^5-20x^4+12x^3-9x^2+1=0$

$x^4-9x^2+12x-20+\frac{12}{x}-\frac{9}{x^2}+\frac{1}{x^4}=$

or $\left(x^4+\frac{1}{x^4}\right)-9\left(x^2+\frac{1}{x^2}\right)+12\left(x+\frac{1}{x}\right)-20=$

put $x+\frac{1}{x}=z$

$\therefore x^2+\frac{1}{x^2}=z^2-2$

$x^3+\frac{1}{x^3}=z^3-2z-z$

$=z^3-3z$

$x^4+\frac{1}{x^4}$

$=z^4-3z^2-z^3+3z$

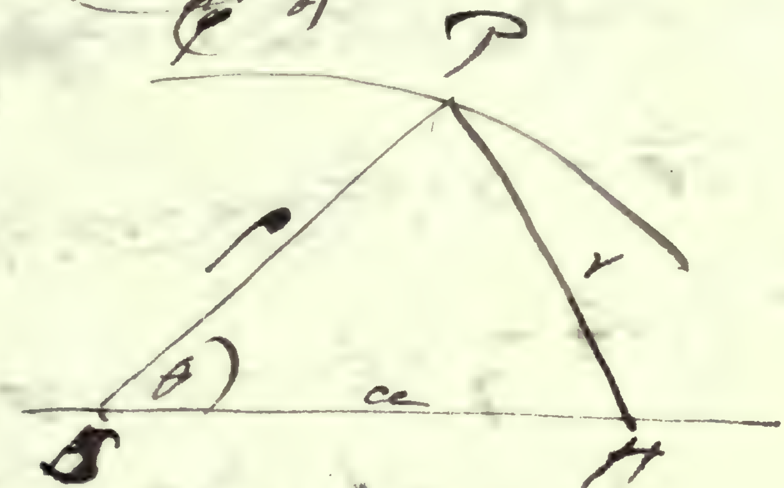
$=z^4-z^3-3z^2+3z$

$\therefore z^4-z^3-3z^2+3z-9(z^2-2)+12z-20=$

$z^4-z^3-12z^2+15z-2=0$

$$\tan(\rho' - \rho) = \frac{\frac{r}{dr/d\theta} - \frac{\rho}{d\rho/d\theta}}{1 + \frac{\rho r}{dr\, d\rho}}$$

&c tan



$$f(r, \rho) = 0$$

$$r^2 = \rho^2 + a^2 - 2\rho a \cos\theta$$

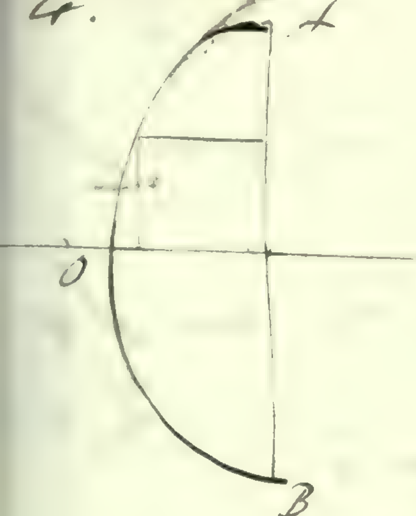
$$f\{\rho, \sqrt{\rho^2 + a^2 - 2\rho a \cos\theta}\} = 0$$

$$r + n\rho = c$$

$$n\rho + \sqrt{\rho^2 + a^2 - 2\rho a \cos\theta} = c$$

$$\rho^2 + a^2 - 2\rho a \cos\theta = c^2 - 2cn\rho + n^2\rho^2$$

4.



k, the rad. of gyration about AB

$$\Sigma m(2a-x)^2 = \int_s (2a-x)^2$$

$$= \int_x (2a-x)^2 \, d_x s$$

$s^2 = 8ax$ $\quad a = $ rad. of generating ⊙

$s \, d_x s = 4a$

$d_x s = \frac{4a}{2\sqrt{2ax}} = \sqrt{\frac{2a}{x}}$

$\therefore$ mom. of inertia $= \int_x \sqrt{2a} \frac{(2a-x)^2}{\sqrt{x}}$

$$= \sqrt{2a} \int_x \left(\frac{4a^2}{\sqrt{x}} - 4a\sqrt{x} + x^{\frac{3}{2}} \right)$$

$$= \sqrt{2a} \left\{ 8a^2 x^{\frac{1}{2}} - \frac{8a}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} \right\} + C$$

$$= \sqrt{2a} \left\{ 8a^2\sqrt{2a} - \frac{16a^2}{3}\sqrt{2a} + \frac{8a^2}{5}\sqrt{2a} \right\}$$

~~$= 16a^2($~~

$$= 2a \left(8a^2 - \frac{16a^2}{3} + \frac{8a^2}{5} \right)$$

$$= 2a^3 \frac{(120 - 80 + 24)}{15}$$

$$= \frac{328 a^3}{15}$$

$\therefore (h^2 - 1)\rho^2 - 2.(hc + a\cos\theta)\rho + c^2 - a^2 = 0$

find $d\rho$ in terms of θ & ρ

& $\tan SPy = \frac{\rho}{d\rho}$

$h = 1$

& $\rho = \frac{c^2 - a^2}{2(hc + a\cos\theta)}$

= to Conic Section.

$mk = \Sigma \delta s . x = \int_x \frac{ax}{\sqrt{2ax - x^2}}$

$= -a\sqrt{2ax - x^2} + a^2 \text{vers}^{-1}\frac{x}{a}$

from 0 to x

$\therefore\ l = \frac{mk_1^2}{mk} = 2a$ wh. is independ^t

of the length of the arc.

$$x = \frac{\pi}{2} + \theta$$

$$\int_x \frac{1}{\sin x \cdot \sqrt{1+\cos x}} = -\int_\theta \frac{1}{\sin\theta \cdot \sqrt{2} \cdot \cos\frac{\theta}{2}}$$

$$= -\frac{1}{\sqrt{2}}\int_\theta \frac{1}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}$$

$$= -\frac{1}{2\sqrt{2}}\int_\theta \frac{\sec^2\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= -\frac{1}{2\sqrt{2}}\int_\theta \left\{\frac{1}{\sin\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right\}$$

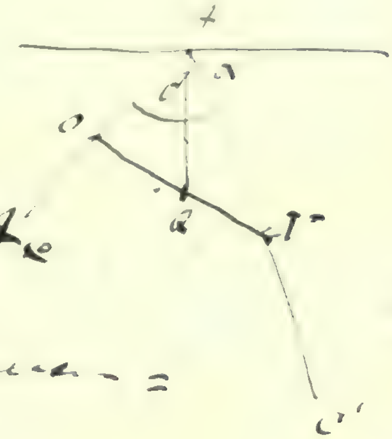
$$= -\frac{1}{\sqrt{2}}\int_{\frac{\theta}{2}} \left\{\frac{1}{\sin\frac{\theta}{2}} - \frac{2\,d_\theta \cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right\}$$

$$= -\frac{1}{\sqrt{2}}\left\{\log_\varepsilon \tan\frac{\theta}{4} + \frac{2}{\cos\frac{\theta}{2}}\right\}$$

$$= -\frac{1}{\sqrt{2}}\left\{\log_\varepsilon \tan\left(\frac{x}{4} - \frac{\pi}{8}\right) + 2\sec\left(\frac{x}{2} - \frac{\pi}{4}\right)\right\}$$

$AB = a$. $AC' = b$ the arms.

k = dist. of centre of grav. of AB & AC' from A. k'^2 = rad. of gyr. [illegible] [illegible] about a horizontal axis thro' A



∴ the length of the [illegible] [illegible] pendulum = $\frac{k^2}{k}$.

Now $k^2 = \frac{1}{3}(a^3 + b^3)$.

also if O. V. be the middle points of AB, AC', & a be such that

$Oa = \frac{a}{a+b} \cdot OV$. a is the centre of grav. of the rods. and Aa is vertical in position of equilm.

$OV = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta}$.

$Oa = \frac{a}{a+b} \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta}$.

$Aa^2 = \frac{b^2}{4} + Oa^2 + \frac{b}{2} \cdot Oa . \cos AOV$.

$= \frac{b^2}{4} + aO^2 + \frac{b}{2} \cdot \frac{a}{a+b} \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - \frac{ab}{2}\cos\theta} \left\{ \frac{\frac{b^2}{4} + OV^2 - \frac{a^2}{4}}{b . OV} \right\}$.

$= \frac{b^2}{4} + aO^2 + \frac{b}{2} \cdot \frac{a}{a+b} \left\{ \frac{b^2}{4} + OV^2 - \frac{a^2}{4} \right\}$.

which by reduction becomes $\frac{1}{2} \sqrt{a^4 + 2a^2b^2\cos\theta + b^4}$.

$\therefore \frac{k^2}{k} = \frac{2}{3} \cdot \frac{a^3 + b^3}{\sqrt{a^4 + 2a^2b^2\cos\theta + b^4}}$

$$d_x y = \sin(mx + ny)$$

Let $mx + ny = z$

Then $m + n\,d_x y = d_x z$

$$\therefore \frac{d_x z - m}{n} = \sin z$$

$$d_x z = m + n \sin z$$

$$x = \int_z \frac{1}{m + n \sin z}$$

$$= \frac{1}{n} \int_z \frac{1}{\frac{m}{n} + \sin z}$$

$$= \frac{1}{n} \int_z \frac{\sec^2 \frac{z}{2}}{\frac{m}{n}\left(\tan^2 \frac{z}{2} + 1\right) + 2 \tan \frac{z}{2}}$$

$$= \frac{1}{m} \int_z \frac{\sec^2 \frac{z}{2}}{\tan^2 \frac{z}{2} + \frac{2n}{m} \tan \frac{z}{2} + 1}$$

$$= \frac{2}{m} \int \frac{d_z \tan\frac{z}{2}}{\left(\tan\frac{z}{2} + \frac{n}{m}\right)^2 + 1 - \frac{n^2}{m^2}}$$

$$x = \frac{2}{m} \cdot \frac{m}{\sqrt{m^2 - n^2}} \cdot \tan^{-1} \frac{m \tan\frac{z}{2} + n}{\sqrt{m^2 - n^2}}$$

$$= \frac{2}{\sqrt{m^2 - n^2}} \tan^{-1} \frac{m \tan\left(\frac{nx + my}{2}\right) + n}{\sqrt{m^2 - n^2}}$$

The $=^n$ to the H.ic paraboloid is

$$x = \frac{y^2}{l} - \frac{z^2}{l'}$$

for a plane ⊥ to the axis x is some constant $= h$ suppose.

∴ the $=^n$ to the $\int\int^n$ is $\frac{z^2}{hl'} - \frac{y^2}{hl} = 1$

which represents an H. with major axis $= \sqrt{hl'}$, minor axis $= \sqrt{hl}$. the minor axis being ∥ to the axis of y.

From the form of the $=^n$. which contains only even powers of z and y the origin is the centre of the curve. ~~The axes are the same as the axes of z. and y of the paraboloid.~~

The trace on yz is. $y = \pm\sqrt{\frac{l}{l'}} \cdot z$.

which represents 2 straight lines.

$$P d_x^2 y + d_x y = \frac{y}{x}$$

$$P d_x \{ x^2 d_x (\tfrac{y}{x}) \} + x d_x y - y = 0$$

Now $x^2 d_x (\tfrac{y}{x}) = x d_x y - y = z$

$$P d_x z + z = 0$$

$$d_x z + \frac{1}{P} z = 0$$

Integrating factor $= \varepsilon^{+\int_x \frac{1}{P}}$

$$d_x \left(z \varepsilon^{+\int_x \frac{1}{P}} \right) = 0$$

$$z = C \varepsilon^{-\int_x \frac{1}{P}} = x^2 d_x (\tfrac{y}{x})$$

$$\therefore y = Cx \int_x \frac{1}{x^2} \varepsilon^{-\int_x \frac{1}{P}}$$

which is the solution of the Equation

Let θ θ' be the ∠s wh. the normal to the front makes with the optic axes.

Then the normal makes ∠s cosines $l\ m\ n$ with the axes & the optic axes make ∠s cosines

$\pm\sqrt{\frac{a^2-b^2}{a^2-c^2}}$ $\quad 0 \quad$ $\pm\sqrt{\frac{b^2-c^2}{a^2-c^2}}$ with the axes

$$\therefore \cos\theta = l\sqrt{\frac{a^2-b^2}{a^2-c^2}} + n\sqrt{\frac{b^2-c^2}{a^2-c^2}}$$

$$\text{or } (a^2-c^2)\cos\theta = l\sqrt{(a^2-b^2)(a^2-c^2)} + n\sqrt{(b^2-c^2)(a^2-c^2)}$$

$$\text{Now } \frac{l^2}{v^2-a^2} + \frac{m^2}{v^2-b^2} + \frac{n^2}{v^2-c^2} = 0$$

$$\text{or } l^2(v^2-b^2)(v^2-c^2) + m^2(v^2-a^2)(v^2-c^2) + n^2(v^2-a^2)(v^2-b^2) = 0$$

$$\text{or } l^2(v^2-a^2+a^2-b^2)(v^2-a^2+a^2-c^2) + m^2(v^2-a^2)(v^2-a^2+a^2-c^2) + n^2(v^2-a^2)(v^2-a^2+a^2-b^2) = 0$$

$$\text{or } (v^2-a^2)^2 + A(v^2-a^2) + l^2(a^2-b^2)(a^2-c^2) = 0$$

∴ if v_1^2 v_2^2 be the 2 values ~~roots~~ of ~~this~~ v^2

$$d_x p - \frac{1}{x} p = x$$

$$\frac{d_x p}{x} - \frac{1}{x^2} p = 1$$

$$d_x \left(\frac{p}{x}\right) = 1$$

$$\frac{p}{x} = x + C$$

$$d_x y = x^2 + Cx$$

$$y = \frac{x^3}{3} + \frac{Cx^2}{2} + C_1$$

The eq for the vel y is

$$\frac{v^2}{v^2 - a^2} + \frac{w^2}{v^2 - b^2} + \frac{w^2}{v^2 - c^2} = \ldots$$

4/2

$$(v_1^2 - a^2)(v_2^2 - a^2) = b^2(a^2 - b^2)(a^2 - c^2)$$

Simly $$(v_1^2 - c^2)(v_2^2 - c^2) = a^2(b^2 - c^2)(a^2 - c^2)$$

$$\therefore (a^2 - c^2)\cos\theta = \sqrt{(v_1^2 - a^2)(v_2^2 - a^2)} + \sqrt{(v_1^2 - c^2)(v_2^2 - c^2)}$$

Simly $$(a^2 - c^2)\cos\theta' = \sqrt{(v_1^2 - a^2)(v_2^2 - a^2)} - \sqrt{(v_1^2 - c^2)(v_2^2 - c^2)}$$

Now $$\sin^2\theta \sin^2\theta' = 1 - (\cos^2\theta + \cos^2\theta') + \cos^2\theta\cos^2\theta'$$

$$\therefore (a^2 - c^2)^2 \sin^2\theta \sin^2\theta' = (a^2 - c^2)^2 - (a^2 - c^2)^2(\cos^2\theta + \cos^2\theta') + (a^2 - c^2)^2 \cos^2\theta\cos^2\theta'$$

$$(a^2 - c^2)^2(\cos^2\theta + \cos^2\theta') = 2(v_1^2 - a^2)(v_2^2 - a^2) + 2(v_1^2 - c^2)(v_2^2 - c^2)$$

$$= 4v_1^2 v_2^2 - 2a^2(v_1^2 + v_2^2) - 2c^2(v_1^2 + v_2^2) + 2(a$$

$$= 4v_1^2 v_2^2 - 2(a^2 + c^2)(v_1^2 + v_2^2) + 2(a^4 + c^4)$$

$$d_x y = x(y^2+1)$$

$$\frac{d_x y}{y^2+1} = x$$

$$\tan^{-1} y = \frac{x^2}{2} + C$$

$$x^2 + y^2 + (a+x)\, y\, d_x y = 0$$

$$x^2 + y^2 + \frac{1}{2}(a+x)\, d_x y^2 = 0$$

$$d_x(y^2) + \frac{2}{a+x}\cdot y^2 = -\frac{2x^2}{a+x}$$

Integrating factor $= \varepsilon^{\int_x \frac{2}{a+x}}$

$$= (a+x)^2$$

$$(a+x)^2 d_x(y^2) + 2y^2(a+x) = -2x^2(a+x)$$

$$d_x\{(a+x)^2 y^2\} = -2x^2(a+x)$$

$$(a+x)^2 y^2 = C - \frac{2ax^3}{3} - \frac{x^4}{2}$$

31

Let Y_m. Z_n be any 2 Laplaces coefft. of the mth. and nth. orders respectively.

Since Z_n is a Laplaces coefft of the nth order it satisfies the = n.

$$n(n+1).\int_0^{2\pi}\int_{-1}^{+1} Y_m . Z_n . d\mu' d\phi' =$$

$$-\int_0^{2\pi}\int_{-1}^{+1} Y_m \left\{ d_\mu . (1-\mu^2) d_\mu Z^n + \frac{1}{1-\mu^2} . d^2_\phi . Z_n \right\} d\mu' d\phi'.$$

Now integrating by parts observing that $1-\mu^2=0$ for both limits we have.

$$\int_{\mu'}^{-1\,+1} Y_m . d_\mu (1-\mu^2) d_\mu Z^n) = -\int_{\mu'}^{-1\,+1} d_\mu Y_m (1-\mu^2) . d_\mu Z^n$$

Similarly. we have.

$$\int_{\phi'}^{0\,2\pi} Y_m \frac{1}{1-\mu^2} . d^2_\phi Z_n = -\int_{\phi'}^{0\,2\pi} d_\phi Y_m \frac{1}{(1-\mu^2)} d_\phi Z^n$$

Similar expressions occur for the other quantity. Y_m. By subtraction we have.

$$\therefore \{n(n+1) - m.(m+1)\} \int_\phi^{0\,2\pi} \int_\mu^{+1} Y_m Z_n \} = 0.$$

and if n. does not = m. this can only be satisfied by

$$\int_\phi^{0\,2\pi} \int_\mu^{+1} Y_m Z_n = 0.$$

$$f(p, \rho) = 0$$

$$p = SY \quad \rho = SP$$

Now $SP . SZ = SY^2$

$$\rho . p_1 = p^2 = \rho_1^2$$

$$\therefore \quad p = \rho_1$$

$$\therefore \quad \rho = \frac{\rho_1^2}{p_1}$$

$$\therefore f\left\{\rho_1 \frac{\rho_1^2}{p_1}\right\} = 0$$

if $f(p, \rho) = 0$ be the eqn. to the curve, then $f\left(\rho', \frac{\rho'^2}{p'}\right)$ is the eqn. to the locus of Z.

4/2

$$mk = \int_s (2a - x) = \int_x (2a - x)\, d_x S$$

$$= \sqrt{2a}\left(4a\sqrt{x} - \frac{2}{3}x^{\frac{3}{2}}\right) + C$$

$$= 8a^2 - \frac{8a^2}{3} = \frac{16a^2}{3}$$

$$l = \frac{mk_1^2}{mk} = \frac{32\,8 \cdot a}{15 \times \frac{16}{3}} = \frac{32\,8}{80} \cdot a$$

$$= 2\cdot 1\, a$$

now α = length of Cycloid

$$= 4\sqrt{2ax} = 8a \qquad a = \frac{\alpha}{8}$$

$$l = \frac{2\cdot 1}{8} \cdot \alpha$$