MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XL.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1884.

*** Of this series forty volumes have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

LIST OF CONTRIBUTORS.

ALDIS, J. S., M.A.; H.M. Inspector of Schools.
ALLEN, Rev. A. J. C., M.A.; St. Peter's Coll., Camb.
ALLMAN, Professor GEO. J., LL.D.; Galway.
ANDERSON, ALEX., B.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor; Pesaro.
BALL, ROBT. STAWELL, LL.D., F.R.S.; Professor of Astronomy in the University of Dublin.
BASU, SATISH CHANDRA; Presid. Coll., Calcutta.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BELTRAMI, Professor; University of Pisa.
BERG, F. J. VAN DEN; Professor of Mathematics in the Polytechnic School, Delft.
BESANT, W. H., M.A.; Cambridge.
BHUT, ATH BIGAH, M.A.; Delhi.
BICKERDIKE, C.; Allerton Bywater.
BICKMORE, C. E.; New College, Oxford.
BIDDLE, D.; Gough H., Kingston-on-Thames.
BIRCH, Rev. J. G., M.A.; London.
BLACKWOOD, ELIZABETH; Boulogne.
BLYTHE, W. H., B.A.; Egham.
BORCHARDT, Dr. C. W.; Victoria Strasse, Berlin.
BOSANQUET, R. H. M., M.A.; Fellow of St. John's College, Oxford.
BOURNE, C. W., M.A.; Bedford County School.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof. COLIN; Andersonian Univ. Glasgow.
BUCHHEIM, A., Ph.D.; Schol. New College, Oxford.
BUCK, E., B.A.; Oakhill, Bath.
BURNSIDE, W. S., M.A.; Professor of Mathematics in the University of Dublin.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., B.A.; 14, Grafton Sq., Clapham.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge, Member of the Institute of France, &c.
CHAKRAVARTI, BYOMAKESA, M.A.; Professor of Mathematics, Calcutta.
CHASE, PLINY EARLE, LL.D.; Professor of Philosophy in Haverford College.
CLARKE, Colonel A. R., C.B., F.R.S.; Hastings.
COATES, W. M., B.A.; Trinity College, Dublin.
COCHEZ, Professor; Paris.
COCKLE, Sir JAMES, M.A., F.R.S.; Ealing.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrew's.
CONSTABLE, S.; Swinford Rectory, Mayo.
COTTERILL, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
CROCKER, J., M.A.; Plymouth.
CROFTON, M. W., B.A., F.R.S.; Prof. of Math. and Mech. in the R. M. Acad., Woolwich.
CULVERWELL, E. P., B.A.; Sch. of Trin. Coll., Dubl.
CURTIS, ARTHUR HILL, LL.D., D.Sc.; Dublin.
DARBOUX, Professor; Paris.
DAVIS, J. G., M.A.; Abingdon.
DAVIS, R. F., B.A.; Wandsworth Common.
DAWSON, H. G., B.A.; Baymount, Dublin.
DAY, Rev. H. G., M.A.; Richmond Terr., Brighton.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DICK, G. R., M.A.; Fellow of Caius Coll., Camb.
DOBSON, T., B.A.; Hexham Grammar School.
DROZ, Prof. ARNOLD, M.A.; Porrentruy, Berne.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTWOOD, G., M.A.; Saxonville, Massachusetts.
EASTON, BELLE; Lockport, New York.
EDWARD, J., M.A.; Head Master of Aberdeen Collegiate School.
EDWARDES, DAVID; Erith Villas, Erith, Kent.
ELLIOTT, E. B., M.A.; Fell. Queen's Coll., Oxon.
ELLIS, ALEXANDER J., F.R.S.; Kensington.
EMTAGE, W. T. A.; Pembroke Coll., Oxford.
ESCOTT, *ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.*
ESSENNELL, EMMA; Coventry.
EVANS, Professor, M.A.; Lockport, New York.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in Queen's College, Belfast.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FINCH, T. H., B.A.; Trinity College, Dublin.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FUORTES, E.; University of Naples.
GALBRAITH, Rev. J., M.A.; Fell. Trin. Coll., Dublin.
GALE, KATE K.; Worcester Park, Surrey.
GALLATLY, W., B.A.; Earl's Court, London.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., B.A.; Stud. of Ch. Ch., Oxford.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GOLDENBERG, Professor, M.A.; Moscow.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Harold, B.A.; King's Coll., Cambridge.
HARLEY, Rev. R., F.R.S.; Huddersfield College.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HARRIS, J. R., M.A.; Clare College, Cambridge.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; Hammersmith.
HERBERT, A., M.A.; King Alfred's Sch., Wantage.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
INGLEBY, C. M., M.A., LL.D.; London.
JELLY, J. O., B.A.; Magdalen College, Oxford.
JENKINS, MORGAN, M.A.; London.
JOHNSON, J. M., B.A.; Radley College, Abingdon.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, SWIFT; Trin. Coll., Dublin.
JONES, L. W., B.A.; Merton College, Oxford.
KEALY, J. A., M.A.; Wilmington, Delaware.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rect.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
KNOWLES, R., B.A., L.C.P.; Tottenham.
LACHLAN, R., B.A.; Lewisham.
LADD, CHRISTINE; Professor of Natural Sciences and Mathematics, Union Springs, New York.
LARMOR, J., M.A.; Queen's College, Galway.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAX, W. G., B.A.; Trinity College, Cambridge.
LEIDHOLD, R., M.A.; Finsbury Park.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, A., D.Sc., F.R.S.E.; Examiner in Mathematics in the University of Edinburgh.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MACMAHON, Capt. P. A.; R. M. Academy.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.

McAlister, Donald, M.A., D.Sc.; Cambridge.
McCay, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
McClelland, W. J., B.A.; Prin. of Santry School.
McColl, H., B.A.; 73, Rue Sibliquin, Boulogne.
McDowell, J., M.A.; Pembroke Coll., Camb.
McIntosh, Alex., B.A.; Bedford Row, London.
McLeod, J., M.A.; R.M. Academy, Woolwich.
McVicker, C. E., B.A.; Trinity Coll., Dublin.
Malet, Prof., M.A.; Queen's Coll., Cork.
Mannheim, M.; Prof. à l'Ecole Polytech., Paris.
Marks, Sarah; Cambridge Street, Hyde Park.
Martin, Artemas, M.A., Ph.D.; Editor & Printer of *Math. Visitor* & *Math. Mag.*, Erie, Pa.
Martin, Rev. H., D.D., M.A.; Edinburgh.
Mathews, G. B., M.A.; Leominster.
Matz, Prof., M.A.; King's Mountain, Carolina.
Mee, W. M., B.A.; Belturbet.
Merrifield, J., LL.D., F.R.A.S.; Plymouth.
Merriman, Mansfield, M.A.; Yale College.
Meyer, Mary S.; Girton College, Cambridge.
Miller, W. J. C., B.A., (Editor); The Paragon, Richmond-on-Thames.
Minchin, G.M., M.A.; Prof. in Cooper's Hill Coll.
Mitcheson, T., B.A., L.C.P.; City of London Sch.
Monck, Henry Stanley, M.A.; Prof. of Moral Philosophy in the University of Dublin.
Moncourt, Professor; Paris.
Moon, Robert, M.A.; Ex-Fell. Qu. Coll., Camb.
Moore, H. K., B.A.; Trin. Coll., Dublin.
Morel, Professor; Paris.
Morgan, C., B.A.; Salisbury School.
Morley, T., L.C.P.; Bromley, Kent.
Morley, F., B.A.; Woodbridge, Suffolk.
Morrice, G. G., B.A.; The Hall, Salisbury.
Moulton, J. F., M.A.; Fell. of Ch. Coll., Camb.
Muir, Thomas, M.A., F.R.S.E.; Bishopton.
Mukhopadhyay, Asutosh, M.A.; Bhowanipore.
Nash, A. M., M.A.; Prof. in Pres. Coll., Calcutta.
Nelson, R. J., M.A.; Naval School, London.
Newcomb, Prof. Simon, M.A.; Washington.
Nicolls, W., B.A.; St. Peter's Coll., Camb.
Openshaw, Rev. T. W., M.A.; Clifton.
O'Regan, John; New Street, Limerick.
Orchard, H. L., M.A., L.C.P.; Hampstead.
Owen, J. A., B.Sc.; Tennyson St., Liverpool.
Panton, A. W., M.A.; Fell. of Trin. Coll., Dublin.
Pendlebury, Rev. C., M A.; London.
Perryman, W.; Carbrook, Sheffield.
Phillips, F. B. W.; Balliol College, Oxford.
Pillai, C. K., M.A.; Trichy, Madras.
Pirie, A., M.A.; University of St. Andrew's.
Polignac, Prince Camille de; Paris.
Pollexfen, H., B.A.; Windermere College.
Prudden, Frances E.; Lockport, New York.
Purser, Prof. F., M.A.; Queen's College, Belfast.
Putnam, K. S., M.A.; Rome, New York.
Ran, B. Hanumanta, M.A.; Head Master, Normal School, Madras.
Rawson, Robert; Havant, Hants.
Reeves, G. M., M.A.; Lee, Kent.
Renshaw, S. A.; Nottingham.
Reynolds, B., M.A.; Notting Hill, London.
Richardson, Rev. G., M.A.; Winchester.
Rippin, Charles R., M.A.; Woolwich Common.
Roberts, R. A., M.A.; Schol. of Trin. Coll., Dublin.
Roberts, S., M.A., F.R.S.; Tufnell Park, London.
Roberts, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
Roberts, W. R., M.A.; Ex-Sch. of Trin. Coll., Dub.
Robson, H. C., B.A.; Sidney Sussex Coll., Cam.
Rosenthal, L. H.; Scholar of Trin. Coll., Dublin.
Roy, Kaliprasanna, M.A.; Professor in St. John's College, Agra.
Royds, J., L.C.P.; Kiveton Park, Sheffield.
Rücker, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
Ruggero, Simonelli; Università di Roma.
Russell, J. W., M.A.; Merton Coll., Oxford.
Russell, R., B.A.; Trinity College, Dublin.
Rutter, Edward; Sunderland.
Salmon, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
...mpson, C. H., M.A.; Balliol College, Oxford.
Sanders, J. B.; Bloomington, Indiana.
Sanderson, Rev. T. J., M.A.; Royston, Cambs.
Sarkar, Nilkantha, M.A.; Calcutta.
Savage, Thomas, M.A.; Fell. Pemb. Coll., Camb.
Scheffer, Professor; Mercersbury Coll., Pa.
Scott, A. W., M.A.; St. David's Coll., Lampeter.
Scott, Charlotte A.; Girton College, Camb.
Scott, R. F., M.A.; Fell. St. John's Coll., Camb.
Serret, Professor; Paris.
Sharp, W. J. C., M.A.; Hill Street, London.
Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham.
Shepherd, Rev. A. J. P., B.A.; Fellow of Queen's College, Oxford.
Simmons, Rev. T.C., M.A.; Christ's Coll., Brecon.
Siverly, Walter; Oil City, Pennsylvania.
Smith, C., M.A.; Sidney Sussex Coll., Camb.
Stabenow, H., M.A.; New York.
Steggall, J. E. A., B.A.; Clifton.
Stein, A.; Venice.
Stephen, St. John, B.A.; Caius Coll., Cambridge.
Stewart, H., M.A.; Framlingham, Suffolk.
Swift, C. A., B.A.; Grammar Sch., Weybridge.
Sylvester, J. J., D.C.L., F.R.S.; Professor of Mathematics in the University of Oxford, Member of the Institute of France, &c.
Symons, E. W., M.A.; Fell. St. John's Coll., Oxon.
Tait, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
Tanner, Prof. H. W. L., M.A.; Bristol.
Tarleton, F. A., M.A.; Fell. Trin. Coll., Dub.
Taylor, Rev. C., D.D.; Master of St. John's College, Cambridge.
Taylor, H. M., M.A.; Fell. Trin. Coll., Camb.
Taylor, W. W., M.A.; Ripon Grammar School.
Tebay, Septimus, B.A.; Farnworth, Bolton.
Terry, Rev. T. R., M.A., Fell. Magd. Coll., Oxon.
Thomas, Rev. D., M.A.; Garsington Rect., Oxford.
Thomson, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Camb.; Brinkley Rectory, Newmarket.
Tirelli, Dr. Francesco; Univ. di Roma.
Todhunter, Isaac, F.R.S.; Cambridge.
Torelli, Gabriel; University of Naples.
Torry, Rev. A. F., M.A.; St. John's Coll., Camb.
Townsend, Rev. R., M.A., F.R.S.; Professor of Nat. Phil. in the University of Dublin, &c.
Traill, Anthony, M.A., M.D.; Fellow and Tutor of Trinity College, Dublin.
Trowbridge, David; Waterburgh, New York.
Tucker, R., M.A.; Mathematical Master in University College School, London.
Turrell, I. H.; Cumminsville, Ohio.
Turriff, George, M.A.; Aberdeen.
Vincenzo, Jacobini; Università di Roma.
Vose, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
Walenn, W. H.; Mem. Phys. Society, London.
Walker, G. F., M.A.; Queen's Coll., Camb.
Walker, J. J., M.A., F.R.S.; Hampstead.
Walmsley, J., B.A.; Eccles, Manchester.
Warburton-White, R., B.A., Salisbury.
Warren, R., M.A.; Trinity College, Dublin.
Watherston, Rev. A. L., M.A.; Bowdon.
Watson, Rev. H. W.; Ex-Fell. Trin. Coll., Camb.
Wertsch, Fr.; Weimar.
White, J. R., B.A.; Worcester Coll., Oxford.
White, Rev. J., M.A.; Cowley College, Oxford.
Whiteside, G., M.A.; Eccleston, Lancashire.
Whitworth, Rev. W. A., M.A.; Fellow of St. John's Coll., Camb.; Hammersmith.
Wickersham, D.; Clinton Co., Ohio.
Wilkins, W.; Scholar of Trin. Coll., Dublin.
Williamson, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
Wilson, J. M., M.A.; Head-master, Clifton Coll.
Wilson, Rev. J., M.A.; Rect. Bannockburn Acad.
Wilson, Rev. J. R., M.A.; Royston, Cambs.
Woodcock, T., B.A.; Wellington, Salop.
Wolstenholme, Rev. J., M.A., Sc.D.; Professor of Mathematics in Cooper's Hill College.
Woolhouse, W. S. B., F.R.A.S., &c.; London
Wright, Dr. S. H., M.A.; Penn Yan, New York.
Wright, W. E., B.A.; Herne Hill.
Young, John, B.A.; Academy, Londonderry.

CONTENTS.

Mathematical Papers, &c.

Solved Questions.

[illegible]

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4641. (By the late Professor CLIFFORD, F.R.S.)—If a circular cubic with a double point O be cut by a circle in four points, A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

Solution by W. J. C. SHARP, M.A.

The inverse of the cubic about the node is a conic, the asymptotes of which are parallel to the nodal tangents, and the axes to the bisectors of the angles between these.

Now, if a, b, c, d, e, f, g, h be the points inverse to A, B, C ... H, the line joining ab will be parallel to that joining EF, since $OA \cdot OE = OB \cdot OF$ and $Oa \cdot OA = Ob \cdot OB$, and therefore $\dfrac{OE}{OF} = \dfrac{OB}{OA} = \dfrac{Oa}{Ob}$.

Similarly all the connectors of a, b, c, d are parallel to the corresponding connectors of E, F, G, H. And each pair of lines connecting a, b, c, d are equally inclined to the axis of the conic. And therefore, &c.

[The property might be more generally enunciated, as it is true for the inverse of a conic, that is to say, for any bicircular quartic or circular cubic with a double point.]

7143. (By Professor SYLVESTER, F.R.S.)—If

$$\log F(x, y) = \Sigma \log \left(x - 2 \cos \frac{2\lambda\pi}{\kappa} y \right),$$

where λ is to assume all values prime to κ and not exceeding $\tfrac{1}{2}(\kappa - 1)$; prove that, when x, y are relative primes, $F(x, y)$ can have no *prime* factors other than divisors of κ or of the form $\kappa i \pm 1$.

Note by the PROPOSER.

If κ be any given number, $\phi\kappa$ the number of numbers λ less than κ and prime to it, and Fx the product of the $\phi\kappa$ factors $\left(x-\cos\frac{2\lambda\pi}{\kappa}\right)$, it is, in this Question, required to prove that, for all integer values of x, the factors of Fx prime to κ are of the form $\kappa i \pm 1$.

Ex. 1.—Let $\kappa = 8,\ \phi\kappa = 4,\ Fx = (x^2-2)^2$,
and the odd factors of x^2-2 are of the form $8i\pm1$.

Ex. 2.—Let $\kappa = 12,\ \phi\kappa = 4,\ Fx = (x^2-3)^2$,
and the odd factors of x^2-3 not containing 3 are of the form $12i\pm1$.

Ex. 3.—Let $\kappa = 18,\ \phi\kappa = 6,\ Fx = (x^3-3x+1)(x^3-3x-1)$,
and the factors of $x^3-3x\pm1$ not containing 3 are of the form $18i\pm1$.

[Prof. SYLVESTER states that this last example is of paramount importance in his new theory concerning the resolution of integers into the sum or difference of two rational cubes. With the exception of the two numbers 66 and 74, all the numbers up to 100 inclusive can now either be resolved into two cubes or proved to be irresolvable, and it is likely that with further trial these two exceptions can be made to disappear.]

7227. (By the EDITOR.)—Show that (1) two sets of n things, whereof the individuals are marked 1, 2, 3, ... n, can be permuted so that no two individuals marked with the same number shall occupy the same place in each set, in $(n!)^2\left\{\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right\}$ ways; and therefrom (2), if two examiners, working simultaneously, examine a class of 12 boys, the one in Classics and the other in Mathematics, so that the boys are examined individually, for 5 minutes each, in each subject, a suitable arrangement, such that no boy shall be wanted by both examiners at once, can be made in 84407190782745600 ways.

Solution by W. J. C. SHARP, M.A.; D. BIDDLE; *and others.*

1. The first set may be permuted in $n!$ ways; and with any particular arrangement, *ex. gr.*, 1, 2, 3, ... n, the following numbers of arrangements of the second set will be excluded by the conditions:—

$(n-1)!$	in which 1 stands first;
$(n-1)!-(n-2)!$	,, 2 ,, 2nd, and 1 is not first;
$(n-1)!-2(n-2)!+(n-3)!$	,, 3 ,, 3rd, and 1 is not first;
&c. &c.	&c. or 2 second;

$$(n-1)!-(n-1)(n-2)!+\frac{(n-1)(n-2)}{1.2}(n-3)!-\&c.,$$

in which n is last, and no other in the same place as in the chosen arrangement of the first set. These exclusions amount to

$$n(n-1)!-\frac{n(n-1)}{1.2}(n-2)!+\frac{n(n-1)(n-2)}{1.2.3}(n-3)!-\&c.$$
$$=n!\left\{1-\frac{1}{1.2}+\frac{1}{1.2.3}-...+(-1)^{n-1}\frac{1}{n}\right\}$$

and therefore the number of included sets is

$$n!\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\&c.\quad+(-1)^n\frac{1}{n!}\right\}.$$

Hence the number of ways is that given in the question.

2. Applying the formula in (1) to the particular case, the number of ways is $(12!)^2\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-...+\frac{1}{12!}\right)$, which gives the stated result.

II. *Solution by the* Rev. T. P. Kirkman, M.A., F.R.S.

1. The solution of the general question is readily seen by the following discussion of the particular case thereof, which forms parts (2):—

2. If the number of the boys were three, the solution would be, giving to each examiner every possible roll-call,

231 312 123 213 132 321
123 231 312 132 321 213

312 123 231 321 213 132
123 231 312 132 321 213,

which is $S_3 . 3!$, $S_3 = 2$ being the number of the all-disturbed permutations of 123; $(3! = 3.2.1)$.

For 12 boys the solution is $S_{12} . (12)!$, S_{12} being the number of the all-disturbed permutations of 123 ... 9*abc*. Each permutation is distinguished by its *circles*. Thus 23164589*abc*7 is one of the number $B_{6,3,3}$, having one circle of six and two circles of three. S_{12} is the sum of such B_p.

In my memoir *On the Theory of Groups and many-valued Functions* (Manchester Memoirs, 1862), at p. 284, there is a demonstration that the number of all-disturbed arrangements of N elements for the partition

$$Aa+Bb+...+Jj=N,\ (J>1),\ (p),$$

showing *a* circles of A, *b* circles of B, &c., A > B, B > C, &c., is

$$\frac{N!}{a!\,b!...j!\,A^aB^b...J^j}=B_p.$$

The number of skeleton groups there spoken of, is that of the different arrangements of N so partitioned that can be written under 12 ... 9*abc* ..., the unity of N elements.

The partitions *p* before us of N = 12 are 12·1, 10·1 + 2·1, 9·1 + 3·1, 8·1 + 2·2, &c.; say, (12), (10) 2, 93, 822, 84, 75, 732, 66, 642, 633, 6222, 543, 5322, 552, 444, 4422, 4332, 42222, 3333, 33222, 222222. The above formula gives all the B_p for N = 12, as follows:—

$$B_{12}=11!;\ B_{10,2}=\frac{12!}{10.2};\ B_{9,3}=\frac{12!}{9.3};\ B_{8,22}=\frac{12!}{2!\,8.2^2};$$
$$B_{8,4}=\frac{12!}{8.4};\ B_{7,5}=\frac{12!}{7.5};\ B_{7,3,2}=\frac{12!}{7.3.2};\ B_{6,6}=\frac{12!}{2!\,6^2};$$

$$B_{6,4,2} = \frac{12!}{6.4.2}; \quad B_{6,3,3} = \frac{12!}{2!\,6.3^2}; \quad B_{6,2,2,2} = \frac{12!}{3!\,6.2^3};$$

$$B_{5,4,3} = \frac{12!}{5.4.3}; \quad B_{5,3,2,2} = \frac{12!}{2!\,5.3.2^2} \quad B_{5,5,2} = \frac{12!}{2!\,5^2.2};$$

$$B_{4,4,4} = \frac{12!}{3!\,4^3}; \quad B_{4,4,2,2} = \frac{12!}{2!\,2!.4^2.2^2}; \quad B_{4,3,3,2} = \frac{12!}{2!\,4.3^2.2};$$

$$B_{4,2,2,2,2} = \frac{12!}{4!\,4.2^4}; \quad B_{3,3,3,3} = \frac{12!}{4!\,3^4};$$

$$B_{3,3,2,2,2} = \frac{12!}{2!\,3!\,3^2.2^3}; \quad B_{2,2,2,2,2,2} = \frac{12!}{6!\,2^6}.$$

We can find $S_1, S_2, \ldots S_N$ very easily from the formula $N! = (1+S)^N$, where for S^e we have to write S_e; $N>1$ and $S_1 = 0$; or from

$$S_n - nS_{n-1} = (-1)^n, \; n>1, \text{ and } S_{2-1} = 0.$$

This gives $S_{12} = 176214841$; thus the number required for N_{12} is $N!\,176214841$, which gives the stated result.

7268. (By W. S. M'Cay, M.A.)—Prove that two equal non-intersecting circles are polar reciprocals to an imaginary parabola.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

Let the equation of one circle be $(x-a)^2+(y-b)^2-r^2 = 0$, ...(1), and that of the parabola be $\quad y^2 - px = 0$ .... (2); then the equation of a tangent to the circle at the point (x', y') will be

$$(x-a)(x'-a)+(y-b)(y'-b)-r^2 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (3),$$

and if (x_1, y_1) be the pole of this tangent with regard to the parabola, (3) must be identical with $2y_1y - p\,(x_1+x) = 0$; hence the conditions

$$\frac{y'-b}{x'-a} = -\frac{2y_1}{p}, \quad \frac{a\,(x'-a)+b\,(y'-b)+r^2}{x'-a} = -x_1, \; (x'-a)^2+(y'-b)^2-r^2=0;$$

from which equations, on eliminating $x'y'$, we obtain

$$\{2by_1 - p\,(a+x_1)\}^2 - r^2\{p^2+4y_1^2\} = 0,$$

or, transferring the origin of coordinates to the centre of the circle (1), we obtain, as the equation of the polar reciprocal required,

$$\{2b\,(y+b) - p\;x+2a)\}^2 - r^2\{p^2 + 4\,(y+b)^2\} = 0 \;\ldots\ldots\ldots$$

Now, that this should be a circle, we must have $b = 0$, and p^2 [illegible]

which reduces (4) to $\quad (x+2a)^2+y^2-r^2 = 0$ [illegible] ....

while (1) and (2) are transformed to

$$x^2+y^2-r^2 = 0 \text{ and } y^2-2\ldots$$ [illegible]

Of these (5) and (6) represent two circles [illegible]

or are wholly external to each other a[illegible]

sents an imaginary parabola, whose [illegible]

distance between the centres of the [illegible]

7331. (By Professor Malet, F.R.S.)—If $\Delta \equiv 1-kx^2$, prove that

$$14\int_0^1 \Delta^{\frac{3}{2}} \log\Delta\, dx - 8\int_0^1 \frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx = 9\int_0^1 \Delta^{\frac{3}{2}}\, dx + 3\,(1-k)^{\frac{3}{2}}\,[2\log(1-k)-3].$$

Solution by Hanumanta Ran; Prof. Nash, M.A.; *and others.*

Since
$$\int \Delta^{\frac{3}{2}} \log\Delta\, dx = x\,\Delta^{\frac{3}{2}} \log\Delta + \int\left\{\tfrac{3}{2}\,.\,2kx\,\frac{\log\Delta}{\Delta^{\frac{1}{2}}} + \frac{2kx}{\Delta^{\frac{1}{2}}}\right\} x\, dx$$
$$= x\,\Delta^{\frac{3}{2}}\log\Delta + \tfrac{3}{2}\int (1-\Delta)\,\frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx - \tfrac{2}{3}\int x d\,\Delta^{\frac{3}{2}},$$

we have, therefore, $\quad 14\int \Delta^{\frac{3}{2}}\log\Delta\, dx - 8\int \frac{\log\Delta}{\Delta^{\frac{1}{2}}}\, dx$

$$= 6x\,\Delta^{\frac{3}{2}}\log\Delta - 9\int x\, d\,\Delta^{\frac{3}{2}} = 6x\,\Delta^{\frac{3}{2}}\log\Delta - 9x\,\Delta^{\frac{3}{2}} + 9\int \Delta^{\frac{3}{2}}\, dx.$$

Integrating between the limits 1 and 0, we obtain the required result.

5850. (By Professor Sylvester, F.R.S.)—1. Suppose an arborescence subject to the law that at every joint each stem or branch splits up into m, the main stem being reckoned as a free branch. Prove that, if n is the number of such joints, $(m-1)\,n+2$ will be the number of free branches.

2. If $m = 2$, *i.e.* for the case of dichotomous ramification, it will be found that, making as above no distinction between the main stem and any free branch, the number of *distinct forms of arborescence*, when there are 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. joints, will be respectively 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, &c. Let such number be called N. Required to express N generally in terms of n, when the arborescence is dichotomous.

Solution by W. J. C. Sharp, M.A.

The total number of branches including the stem is $mn+1$, and of these $n-1$ are not free; therefore, the number of free branches

$$= mn+1-(n-1) = n\,(m-1)+2.$$

Hence, if $m = 2$, the number of free branches is $n+2$.

If now it be assumed that all systems which have the same number of joints with two free branches (free joints, say) belong to the same species; then, denoting by p the number of free joints, and by q the number of joints with one free branch, $2p+q = n+2$ the number of free branches. Also q cannot exceed $n-2$.

Then N is the number of integer solutions of the above equation which satisfies the condition. If $n = 1$, the equation, is not applicable as there is one joint with three free branches.

For $n = 2, 3, 4, 5, 6, 7$, &c., the values of N are 1, 1, 2, 2, 3, 3, &c., N being $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

7344. (By T. Muir, M.A., F.R.S.E.)—Prove the theorem of continuants which for the case of the 4th order is

$$K(a-1, a, a, a+1) = aK(a, a, a).$$

Solution by the Proposer.

Making use twice of the identity

$$K(a+\omega, b, c, d, \ldots) = K(a, b, c, d, \ldots) + \omega K(b, c, d, \ldots),$$

we have $\quad K(a-1, a, a, a+1) = K(a, a, a, a+1) - K(a, a, a+1)$

$$= K(a, a, a, a+1) - K(a, a, a) - K(a, a).$$

But the first term of this expansion equals $(a+1)K(a, a, a) + K(a, a)$. Hence the theorem is established. As an easy deduction from this, we have

$$\frac{a}{a+1+}\,\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a-1} + \frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a} = 1.$$

7334. (By C. E. McVicker, M.A.)—Adopting the usual notation for the radii of the circles connected with a plane triangle, prove that

$$r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 - 6R(a^2+b^2+c^2) \ldots\ldots\ldots\ldots\ldots (1);$$

$$32R^3 - 6R(r_1^2 + r_2^2 + r_3^2 + r_2) + (r_1^3 + r_2^3 + r_3^3 - r^3) = 0 \ldots\ldots\ldots (2).$$

Solution by Hanumanta Ran; G. Heppel, M.A.; *and others.*

1. Let $\quad r_1 + r_2 + r_3 = u, \;\; r_1r_2 + r_2r_3 + r_3r_1 = v, \;\; r_1r_2r_3 = w,$

then $\;u = 4R + r, \; v = s^2, \; w = rs^2,$ and $4Rr + r^2 - s^2 = -\frac{1}{2}(a^2+b^2+c^2).$

Now $\;r_1^3 + r_2^3 + r_3^3 = u^3 - 3uv + 3w = 64R^3 + 48R^2r + 12Rr^2 + r^3 - 12Rs^2,$

$\therefore\; r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 + 12R(4Rr + r^2 - s^2) = 64R^3 - 6R(a^2+b^2+c^2).$

2. Again $\;r_1^2 + r_2^2 + r_3^2 = u^2 - 2v = 16R^2 + 8Rr + r^2 - 2s^2,$

$\therefore\; r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 + 2\,.\,(4Rr + r^2 - s^2) = 16R^2 - (a^2+b^2+c^2),$

therefore $\;32R^3 - 6R(r_1^2 + r_2^2 + r_3^2 + r^2) + r_1^3 + r_2^3 + r_3^3 - r^3 = 0.$

7341. (By A. Martin, B.A.)—Solve the equations

$$yz(y+z-x) = a, \;\; zx(z+x-y) = b, \;\; xy(x+y-z) = c.$$

Solution by W. W. Taylor, M.A.; Hanumanta Ran; *and others.*

If we put $\quad p = x+y+z, \;\; q = xy+yz+zx, \;\; r = xyz,$

the given equations may be written

$$yz\,p - 2r = a, \;\; z\,xp - 2r = b, \;\; xy\,p - 2r = c \ldots\ldots\ldots (1, 2, 3).$$

By addition, and from (1) and (2), we obtain, respectively,

$$qp - 6r = a + b + c, \quad zp^2r = (a+2r)(b+2r) \text{ .... } (4, 5).$$

By adding (5) and two similar equations, and from (1), (2), (3), we have

$$p^3r = (ab + bc + ca) + 4(a+b+c)r + 12r^2 \text{ .... } (6),$$

$$(a+2r)(b+2r)(c+2r) = x^2y^2z^2p^3 = p^3r^2 = r\,[ab + bc + ca + 4(a+b+c)r + 12r^2].$$

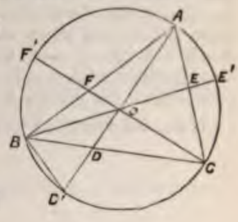
On simplification, this equation becomes $4r^3 - (ab + bc + ca)\,r = abc$, a cubic in r; and when r is found, p can be found from (6), and q from (4).

The values of x, y, z are then the roots of the cubic $x^3 - px^2 + qx - r = 0$.

7335. (By W. J. C. Sharp, M.A.)—If O be the centre of the circle drawn round the triangle ABC, and AO, BO, CO be produced to meet the opposite sides in D, E, F, and the circle in D′, E′, F′ respectively; prove that

$$\frac{DD'}{AD} + \frac{EE'}{BE} + \frac{FF'}{CF} = 1, \; BD.DC + OD^2 = AE.EC + OE^2 = AF.FB + OF^2.$$

Solution by Hanumanta Ran; R. Knowles, B.A.; *and others.*



$$AD = 2R.\frac{\sin B \sin C}{\cos(B-C)}; \quad DD' = 2R.\frac{\cos B \cos C}{\cos(B-C)};$$

therefore $\dfrac{DD'}{AD} = \cot B \cot C$;

therefore, since $A + B + C = 180°$,

$$\frac{DD'}{AD} + \frac{EE'}{BE} + \frac{FF'}{CF} = 1;$$

the second part of the question follows at once from Euclid III. 35.

7349. (By Sarah Marks.)—If a number consist of 7 digits, whose sum is 59, show that the probability that it will be exactly divisible by 11 is $\frac{4}{21}$.

Solution by Professor Roy; G. Heppel, M.A.; *and others.*

In order that the number may be divisible by 11, the digits in the even places must together be equal to 24.

Now set	8888999	gives	4	such arrangements	out of	35
	7889999	,,	24	,,	,,	105
	7799999	,,	0	,,	,,	21
	6899999	,,	12	,,	,,	42
	5999999	,,	0	,,	,,	7,

so that there are 40 suitable arrangements out of 210, and the chance is $\frac{4}{21}$.

4513 & 5691. (By the late Professor CLIFFORD, M.A.)—If the intersections of two circles $A = 0$, $B = 0$ are concentric with the antifoci of the intersections of $C = 0$, $D = 0$, then *vice versâ*; and if this property hold for the pairs AB, CD, and also for the pairs AC, DB, prove that it will likewise hold for the pairs AD, CB.

Solution by W. J. C. SHARP, M.A.

If (h, k) and (h', k') be two points, the antipoints are

$$\left(\frac{h+h'+(-1)^{\frac{1}{2}}(k-k')}{2},\ \frac{k+k'-(-1)^{\frac{1}{2}}(h-h')}{2}\right)$$

and
$$\left(\frac{h+h'-(-1)^{\frac{1}{2}}(k-k')}{2},\ \frac{k+k'+(-1)^{\frac{1}{2}}h-h')}{2}\right),$$

so that the points and antipoints are concentric, and the first proposition follows at once, since the common chords bisect each other.

Again, if a, b, c, d be the centres of the circles, then, if the property hold for AB, CD, the perpendiculars from a and b upon $A-B=0$, and those from c and d upon $C-D=0$, meet these lines at their intersection; and if the property hold for AC, BD, the perpendiculars from a and c upon $A-C=0$, and from b and d upon $B-D=0$, meet these lines at the same point, and these conditions involve the identity of the four lines $A-B=0$, $A-C=0$, $C-D=0$, and $B-D=0$, which are, therefore, also identical with $A-D=0$ and $B-C=0$, which proves the second proposition, the circles A, B, C, D forming a system through two points.

[If the intersections be imaginary, the antipoints are real, and *vice versâ*.]

7154. (By the Rev. G. RICHARDSON, M.A.)—If three circles whose centres are O_1, O_2, O_3, and radii r_1, r_2, r_3 respectively, be coaxal, prove that

$$r_1^2 \cdot O_2O_3 + r_2^2 \cdot O_3O_1 + r_3^2 \cdot O_1O_2 + O_2O_3 \cdot O_3O_1 \cdot O_1O_2 = 0.$$

Solution by J. P. JOHNSTON, B.A.; J. O'REGAN; *and others.*

Take the line on which the centres lie and the radical axis as axes of coordinates, then the equations of the circles are

$$x^2+y^2-2k_1x+\delta^2=0,\quad x^2+y^2-2k_2x+\delta^2=0,\quad x^2+y^2-2k_3x+\delta^2=0,$$

where k_1, k_2, k_3 are the distances of centres from the radical axes. Then

$$r_1^2 = k_1^2-\delta^2,\quad r_2^2=k_2^2-\delta^2,\quad r_3^2=k_3^2-\delta^2;$$

therefore

$$r_1^2(k_2-k_3)+r_2^2(k_3-k_1)+r_3^2(k_1-k_2) = k_1^2(k_2-k_3)+k_2^2(k_3-k_1)+k_3^2(k_1-k_2)$$
$$= -(k_2-k_3)(k_3-k_1)(k_1-k_2).$$

But $k_2-k_3 = O_2O_3$, &c.; therefore the stated result follows.

7346. (By D. EDWARDES.)—Prove that, whatever be the value of n,

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(1-\sin\theta\cos\phi)^{\frac{1}{2}n}\sin\theta\,d\theta\,d\phi=\frac{\pi}{n+2}.$$

Solution by DR. CURTIS; BELLE EASTON; *and others.*

The limits of integration show that this integral is extended to the surface of a quadrantal triangle on the surface of a sphere of radius unity, or, taking as axes of coordinates x, y, z, the radii of the sphere drawn to the three angular points of the triangle, and supposing the radius vector to any assumed point within the area to make with x, y, z, angles α, β, γ, we have $\alpha=\theta$, $\cos\beta=\sin\alpha\cos\phi$, $\cos\gamma=\sin\alpha\sin\phi$; therefore, if $d\Omega$ denote the element of the area, we have

$$\int_0^{\frac{1}{2}\pi}\int_0^{\pi}(1-\sin\theta\cos\phi)^{n}\sin\theta\,d\theta\,d\phi=\iint(1-\cos\beta)^{\frac{1}{2}n}\,d\Omega$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}2^{\frac{1}{2}n}\sin^n\tfrac{1}{2}\beta\sin\beta\,d\beta\,d\psi=2^{\frac{1}{2}n+1}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^{n+1}\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\,d\beta\,d\psi$$

$$=2^{\frac{1}{2}n}\pi\int_0^{\frac{1}{2}\pi}\sin^{n+1}\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\,d\beta=\frac{2^{\frac{1}{2}n+1}\pi}{n+2}\left(_0^{\frac{1}{2}\pi}\sin^{n+2}\tfrac{1}{2}\beta\right)=\frac{\pi}{n+2}\,\frac{2^{\frac{1}{2}n+1}}{2^{\frac{1}{2}(n+2)}}=\frac{\pi}{n+2}.$$

7192. (By Professor MATZ, M.A.)—Show that the sum of the series for

$$I=\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{\theta}\,d\theta\text{ is }1{\cdot}3749833960=\frac{11}{8}\text{ nearly.}$$

Solution by W. H. BLYTHE, M.A.; Professor NASH, M.A.; *and others.*

The series required is

$$a-\frac{a^3}{3\,.\,3\,!}+\frac{a^5}{5\,.\,5\,!}-\frac{a^7}{7\,.\,7\,!}+\text{\&c., where }a=\frac{\pi}{2};$$

and the annexed computation shows the result to be that stated in the Question.

$\log a=\log\pi-\log 2$
$\quad=\cdot1961199$
$\log a^3=\cdot5783597$
$\log a^5=\cdot9605995$
$\log a^7=1{\cdot}3428383$
$\log a^9=1{\cdot}7250781$
$\log a^{11}=2{\cdot}1073179$
$\log a^{13}=2{\cdot}4895577$
$\log a^{15}=2{\cdot}8717975$
$\log\frac{1}{3!}=\bar{1}{\cdot}2218487$

$\log\frac{1}{5!}=\bar{3}{\cdot}9208187$
$\log\frac{1}{7!}=\bar{4}{\cdot}2975694$
$\log\frac{1}{9!}=\bar{6}{\cdot}2402369$
$\log\frac{1}{11!}=\bar{8}{\cdot}1988442$
$\log\frac{1}{13!}=\overline{10}{\cdot}0057296$
$\log\frac{1}{15!}=\overline{13}{\cdot}6835103,$

$$\log(\text{1st term}) = \log a = \cdot 1961199$$

$$\log(\text{2nd term}) = \log a^3 + \log\frac{1}{3\,!} - \log\ 3 = \bar{1}\cdot 3230871$$

$$\log(\text{3rd term}) = \log a^5 + \log\frac{1}{5\,!} - \log\ 5 = \bar{2}\cdot 1824482$$

$$\log(\text{4th term}) = \log a^7 + \log\frac{1}{7\,!} - \log\ 7 = \bar{4}\cdot 7953097$$

$$\log(\text{5th term}) = \log a^9 + \log\frac{1}{9\,!} - \log\ 9 = \bar{5}\cdot 0110725$$

$$\log(\text{6th term}) = \log a^{11} + \log\frac{1}{11\,!} - \log 11 = \bar{7}\cdot 2647694$$

$$\log(\text{7th term}) = \log a^{13} + \log\frac{1}{13\,!} - \log 13 = \bar{9}\cdot 3813439$$

$$\log(\text{8th term}) = \log a^{15} + \log\frac{1}{15\,!} - \log 15 = \overline{11}\cdot 3792165.$$

[The series is evidently convergent, for each alternate is positive and negative, and each successive term causes the sum to be either greater or less than some fixed limit, the terms also diminishing by a ratio less than unity.]

1st term =	1·5707963268
2nd ,, =	·2104200000
difference =	1·3603763268
3rd term =	·0152211724
sum =	1·3755974992
4th term =	·0006241800
diff. =	1·3749733192
5th term =	·0000102583
sum =	1·3749835775
6th term =	·0000001839
diff. =	1·3749833936
7th term =	·0000000024
sum =	1·3749833960
8th term =	·0000000000
Result =	1·3749833960.

7317. (By Asutosh Mukhopâdhyây.) — Any number (m) of tangents are drawn to a parabola, such that the arcs between the points of contact subtend equal angles at the focus. If 2α be the angle which the axis of the parabola makes with the radius vector drawn to the adjacent point of contact, prove that the product of the perpendiculars from the focus on the tangents varies inversely as $\sin m\alpha$.

Solution by A. H. CURTIS, LL.D, D.Sc.

The solution of this question may be obtained by reciprocation from a particular case of a theorem (an extension of COTES' theorem), due to the late Prof. MACCULLAGH, which, so far as it concerns the present question, may be enunciated as follows:—If any point O, taken on the circumference of a circle of radius a, be joined to all the angular points of a regular polygon of m sides inscribed in the circle, and the angle subtended at the centre by the point O and any angular point of the polygon, *e.g.* the adjacent one, be denoted by 2α, the equation whose roots are the squares of these joining lines, $r_1, r_2, \dots r_m$, is of the form

$$z^m - Az^{m-1} + Bz^{m-2} - \&c. \; + (-1)^m 4a^{2m} \sin^2 m\alpha = 0,$$

where the coefficients A, B, &c., are functions of a, and therefore

$$(r_1 . r_2 \dots r\;)^2 = 4a^{2m} \sin^2 m\alpha, \text{ or } r_1 . r_2 \dots r_m = 2a^m \sin m\alpha.$$

If, taking O as origin, we reciprocate this theorem, we obtain the one proposed; for 2α is double of the angle between the diameter through O (the axis of the parabola), and a perpendicular to r_1 (co-directional with the *adjacent* tangent to the parabola into which the circle reciprocates), and is consequently identical with the angle expressed by 2α in the question.

The proof of MACCULLAGH's theorem adapted to the special case is as follows:—Let r_k denote the line joining O to the k^{th} angular point, then

$$z = r_k^2 = 4a^2 \sin^2 \tfrac{1}{2}\left(2\alpha + \frac{k2\pi}{m}\right),$$

or $$\frac{z}{2a^2} = 2\sin^2 \tfrac{1}{2}\left(2\alpha + \frac{k2\pi}{m}\right) = 1 - \cos\left(2\alpha + \frac{k2\pi}{m}\right),$$

therefore $$\cos\left(2\alpha + \frac{k2\pi}{m}\right) = \left(1 - \frac{z}{2a^2}\right).$$

Now, if $\cos\phi = u$, $2\cos m\phi = (2u)^m - m(2u)^{m-2} + \dfrac{m \,.\, m-3}{1 \,.\, 2}(2u)^{m-4} - \&c.$,

therefore $$2\cos(2m\alpha) = 2\cos(2m\alpha + 2k\pi) = 2\cos m\left(2\alpha + \frac{k2\pi}{m}\right)$$

$$= \left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^m - m\left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^{m-2} + \frac{m \,.\, m-3}{1 \,.\, 2}\left\{2\left(1 - \frac{z}{2a^2}\right)\right\}^{m-4}$$

— &c. (MURPHY's *Theory of Equations*, pp. 32 and 75);

the term independent of z in this equation, which will result on putting

$z = 0$, is $$2^m - m2^{m-2} + \frac{m \,.\, m-3}{1 \,.\, 2} 2^{m-4} - \&c. \; - 2\cos(2m\alpha),$$

or $$2\cos m(\cos^{-1} 1) - 2\cos(2m\alpha), \text{ or } 2(1 - \cos 2m\alpha), \text{ or } 4\sin^2 m\alpha,$$

therefore $$\frac{r_1^2 . r_2^2 \dots r_m^2}{(a^2)^m} = 4\sin^2 m\alpha,$$

and therefore $$r_1^2 . r_2^2 \dots r_m^2 = 2a^m \sin m\alpha.$$

7189. (By Professor SYLVESTER, F.R.S.)—Sum the series

$$1+(x-i)+\frac{(x-2i)(x-2i-1)}{1\,.\,2}+\frac{(x-3i)(x-3i-1)(x-3i-2)}{1\,.\,2\,.\,3}+\dots\dots$$

Solution by the PROPOSER.

The answer may be got by calling the sum u_x, and satisfying the equations $u_x-u_{x-1}-u_{x-i-1}=0$, $u_0=1$, $u_1=1$, $u_2=1$, ... $u_{i-1}=1$.
Thus *ex. gr.*, we have

$$1+(x-1)+\frac{(x-2)(x-3)}{1\,.\,2}+\frac{(x-3)(x-4)(x-5)}{1\,.\,2\,.\,3}$$
$$+\dots=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^x-\left(\frac{1-\sqrt{5}}{2}\right)^x\right\}.$$

6938. (By C. MORGAN, B.A.)—If ABCDEF be a rectilineal figure, prove that the sum of the tangents of its interior angles is equal to the difference between the sums of the products of the tangents taken 3 and 5 together.

Solution by the PROPOSER; Professor NASH, M.A.; *and others.*

$$\tan[(A+B)+(C+D)+(E+F)]=$$
$$\frac{\tan(A+B)+\tan(C+D)+\tan(E+F)-\tan(A+B)\,.\tan(C+D)\,.\tan(E+F)}{\text{some denominator.}}$$

But $A+B+C+D+E+F=4\pi$; therefore the numerator of the above fraction is zero; or, putting a, b, c ... for tan A, tan B ..., we have

$$\frac{a+b}{1-ab}+\frac{c+d}{1-cd}+\frac{e+f}{1-ef}-\frac{(a+b)(c+d)(e+f)}{(1-ab)(-1cd)(1-ef)}=0,$$

or $$a-acd-aef+acdef+b-bcd-bef+bcdef+c-cab-cef+cabef$$
$$+d-dab-def+dabef+e-eab-ecd+eabcd+f-fab-fcd+fabcd$$
$$-ace-bce-ade-bde-acf-bcf-adf-bdf=0,$$

or $$a+b+c+d+e+f=\left(\begin{matrix}acd+aef+\dots\\ \text{20 terms}\end{matrix}\right)-\left(\begin{matrix}acdef+bcdef+\dots\\ \text{6 terms}\end{matrix}\right).$$

7376. (By Professor CAYLEY, F.R.S.)—Show how the construction of a regular heptagon may be made to depend on the trisection of the angle $\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)$.

Solution by R. RAWSON; Professor MATZ, M.A.; *and others.*

Putting θ for the angle subtended by the side of the heptagon, we have

$$7\theta = 2\pi, \text{ and } \sin 7\theta = \sin\theta\,(64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1) = 0;$$

hence, if $4\cos^2\theta = z$, we have $z^3 - 5z^2 + 6z - 1 = 0$ .... (1);
and, by the relation $z = v + \frac{5}{3}$, equation (1) is transformed into

$$v^3 - \tfrac{7}{3}v - \tfrac{7}{27} = 0 \qquad\qquad (2).$$

The trigonometrical solution of (2) is well known to be $v = \frac{2}{3}\sqrt{7}\cos\alpha$,

where $$\cos 3\alpha = \frac{7}{2\,.\,27}\left(\frac{9}{7}\right) = \frac{1}{2\sqrt{7}};$$

when therefore 3α, or $\cos^{-1}\dfrac{1}{2\sqrt{7}}$, is trisected,

then $4\cos^2\theta = 2\cos 2\theta + 2 = \frac{1}{3}(5 + 2\sqrt{7}\cos\alpha)$ is readily constructed.

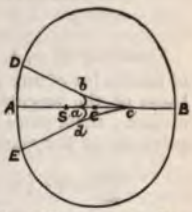
[It may be remarked that for $z = 2\cos\theta + \frac{1}{3}$, we have also the equation $z^3 - \frac{7}{3}z - \frac{7}{27} = 0$, which is identical with (2) above.]

7379. (By Professor WOLSTENHOLME, M.A.) — In the limaçon $r = a + b\cos\theta$, if $a > 2b$, prove that the length of the whole arc of the evolute, and the whole area of the evolute, will be, respectively,

$$4\left\{\frac{a(a^2-3b^2)}{a^2-4b^2} - (a^2-b^2)^{\frac{1}{2}}\right\}; \quad \frac{\pi}{9}\left\{\frac{(a^2-b^2)^{\frac{3}{2}}}{(a^2-4b^2)^{\frac{1}{2}}} - a^2 - \tfrac{1}{2}b^2\right\}.$$

Solution by D. EDWARDES; BELLE EASTON; *and others.*

We have $$\rho = \frac{(a^2+b^2+2ab\cos\theta)^{\frac{3}{2}}}{a^2+2b^2+3ab\cos\theta}.$$



When $\cos\theta = -\dfrac{b}{a}$, there is a cusp on the evolute, and at this point $\rho = (a^2-b^2)^{\frac{1}{2}}$. The radii of curvature at the vertices are $\dfrac{(a+b)^2}{a+2b}$ and $\dfrac{(a-b)^2}{a-2b}$.

Hence evidently the length

$$= 2\left[\frac{(a+b)^2}{a+2b} + \frac{(a-b)^2}{a-2b} - 2(a^2-b^2)^{\frac{1}{2}}\right] = 4\left[\frac{a(a^2-3b^2)}{a^2-9b^2} - (a^2-b^2)^{\frac{1}{2}}\right].$$

Let E be the area of the evolute and A that of the limaçon, and let s be the arc of the limaçon measured from the farther apse. Then $\mathrm{E} + \mathrm{A} = \int\rho\,ds$, between the limits 0 and π of θ, that is, $\mathrm{E} + \mathrm{A} = \displaystyle\int_0^\pi \frac{(a^2+b^2+2ab\cos\theta)^2}{a^2+2b^2+3ab\cos\theta}\,d\theta.$

The result follows from observing that $a > 2b$, $\mathrm{A} = \pi(a^2 + \frac{1}{2}b^2)$, and

$$\int_0^\pi \frac{\cos^2\theta\,d\theta}{\mathrm{A}+\mathrm{B}\cos\theta} = \mathrm{A}\int_0^\pi \frac{\cos^2\theta\,d\theta}{\mathrm{A}^2 - \mathrm{B}^2\cos^2\theta}.$$

5330. (By the late Professor CLIFFORD, F.R.S.)—Show that

$$\int_0^{\frac{1}{2}\pi} \cos(\alpha \tan x)\, e^{\beta \tan x}\, dx = \tfrac{1}{2}\pi e^{-\alpha}(\cos\beta + \sin\beta).$$

Note by J. J. WALKER, M.A., F.R.S.

It may be of interest to show that this Question, proposed as far back as 1863, is erroneous; nor is it likely that such an integral could be evaluated in a finite form. Assume $\tan x = y$, so that the given integral becomes

$$u = \int_0^\infty \cos(\alpha y)\, e^{\beta y} \div (1+y^2)\,.\,dy. \quad \text{Then } u - \frac{d^2u}{d\alpha^2} = \int_0^\infty \cos \alpha y \,.\, e^{\beta y}\, dy.$$

For the value of u as given in the Question, the sinister of this equality is zero; but the dexter, a well-known integral, is infinite for β positive, and equal to $\beta \div (\alpha^2+\beta^2)$ for β negative.

7076. (By Professor TOWNSEND, F.R.S.)—Two circular cylinders round axes passing through the point of no linear acceleration O of a rigid body in motion, in directions parallel to those of the angular velocity and of the angular acceleration, at any instant of the motion, being supposed described through any arbitrary point P of the body; show that the entire linear acceleration of P, at the instant, consists of two distinct components, due respectively to angular velocity and to angular acceleration, the former normal to the first and the latter tangential to the second of the two aforesaid cylinders, and each directly proportional to the radius of its cylinder.

Solution by the PROPOSER.

Denoting by u, v, w the components of the linear velocity of the centre of inertia (or of any other definite point) $\bar{x}\,\bar{y}\,\bar{z}$ of the body, and by p, q, r those of the angular velocity of its entire mass, at any instant of the motion; then, since for the components of the linear velocity of any other point xyz of the mass at the instant,

$$\frac{dx}{dt} = u + q(z-\bar{z}) - r(y-\bar{y}), \quad \frac{dy}{dt} = v + \&c., \quad \frac{dz}{dt} = w + \&c.;$$

therefore, for those of the linear acceleration of xyz at the instant,

$$\frac{d^2x}{dt^2} = \frac{du}{dt} - (q^2+r^2)(x-\bar{x}) + pq(y-\bar{y}) + pr(z-\bar{z}) + \frac{dq}{dt}(z-\bar{z}) - \frac{dr}{dt}(y-\bar{y}),$$

with similar values for $\dfrac{d^2y}{dt^2}$ and for $\dfrac{d^2z}{dt^2}$, the dexters of which equated to 0 give at once three independent linear equations for the determination of $(x_0-\bar{x})$, $(y_0-\bar{y})$, $(z_0-\bar{z})$, and therefore of x_0, y_0, z_0, for the point O of no linear acceleration at the instant; and show that for every instant there is always one, and generally but one, such point in the space of the motion, not necessarily included in, but definitely connected with, the body's mass.

Denoting now by ξ, η, ζ the relative coordinates $(x-x_0)$, $(y-y_0)$, $(z-z_0)$ with respect to O of any other point P of the mass at the instant, and substituting in the dexters of the preceding equations for x, y, z their equivalents $(x_0+\xi)$, $(y_0+\eta)$, $(z_0+\zeta)$, we get from them immediately, by virtue of the three aforesaid evanescences characteristic of the position of O, for the components of the linear acceleration of P at the instant,

$$\frac{d^2x}{dt^2} = -(q^2+r^2)\,\xi + pq\eta + rp\zeta + \frac{dq}{dt}\zeta - \frac{dr}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = -(r^2+p^2)\,\eta + qr\zeta + pq\xi + \frac{dr}{dt}\xi - \frac{dp}{dt}\zeta,$$

$$\frac{d^2z}{dt^2} = -(p^2+q^2)\,\zeta + rp\xi + qr\eta + \frac{dp}{dt}\eta - \frac{dq}{dt}\xi,$$

which show that, as stated in the question, the entire linear acceleration of P at the instant consists of two distinct parts, one depending on p, q, r, and directed normally to the cylinder of radius ρ through x, y, z, whose equation to origin O is

$$(q\zeta-r\eta)^2+(r\xi-p\zeta)^2+(p\eta-q\xi)^2=(p^2+q^2+r^2)\,\rho^2,$$

and the other depending on $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$, or p', q', r', and directed tangentially to the cylinder of radius ρ' whose equation to same origin is

$$(q'\zeta-r'\eta)^2+(r'\xi-p'\zeta)^2+(p'\eta-q'\xi)^2=(p'^2+q'^2+r'^2)\,\rho'^2,$$

and each component acting perpendicularly to the axis, and varying directly as the radius of its cylinder.

Expressed in terms of the radii ρ and ρ' of the two cylinders, and of the components pqr and $p'q'r'$ of the angular velocity and angular acceleration of the mass at the instant, the values of the two parts of the linear acceleration at xyz are respectively

$$(p^2+q^2+r^2)\,\rho \text{ and } \pm\,(p'^2+q'^2+r'^2)^{\frac{1}{2}}\,\rho\ ;$$

as is evident from the above equations.

If at any instant of the motion the axes of angular velocity and of angular acceleration of the mass happen to coincide in the space of the motion, that is, if at any instant $p':q':r'=p:q:r$, the two aforesaid cylinders corresponding to them coincide also completely at the instant, for every point of the body; and the linear acceleration from point to point of the mass follows consequently the same simple law as for the motion of a lamina in its plane,—there being then, in fact, an *axis*, in place of, as usual, only a *centre*, of no linear acceleration of the motion.

[Professor MINCHIN remarks that the theorem here enunciated was published by him in *Nature* (for November 18, 1880), and that he had arrived at the result for uniplanar motion, and mentioned it to Professor WOLSTENHOLME, who communicated to him the extension to three-dimensional motion. The general result was known previously, and is to be found in the works of some foreign authors. In MINCHIN's *Uniplanar Kinematics*, the theorem has been made use of for the solution of several problems.]

7332. (By the EDITOR.)—If p_1, p_2, p_3 be the perpendiculars from the vertices of a triangle on the opposite sides; d_1, d_2, d_3 the distances from the vertices to the points of contact of the escribed circles with the opposite sides; and $l_1^2 = d_1 + r_1^2$, $l_2^2 =$ &c., $l_3^2 =$ &c.; prove that

$$\frac{l_1^2}{p_1r_1} = \frac{l_2^2}{p_2r_2} = \frac{l_3^2}{p_3r_3} = \frac{2}{r}(\mathrm{R}-r), \quad \frac{l_1^2}{bcr_1} = \frac{l_2^2}{car_2} = \frac{l_3^2}{abr_3} = \frac{1}{r} - \frac{1}{\mathrm{R}}.$$

Solution by G. HEPPEL, M.A.; Prof. MATZ, M.A.; *and others.*

Let D be the foot of the perpendicular from A (the figure may be easily imagined); E the point of contact with BC. Then $\mathrm{CE} = s-b$, $\mathrm{CD} = b\cos \mathrm{C}$;

therefore $$\mathrm{DE} = \frac{a^2+b^2-c^2}{2a} - \frac{a+c-b}{2} = \frac{(b-c)\,s}{2},$$

$$\mathrm{DE}^2 = \frac{(b-c)^2 s^2}{a^2} = \frac{[a^2-4\,(s-b)(s-c)]\,s^2}{a^2} = s^2 - \frac{4\Delta^2 . s}{a^2\,(s-a)} = s^2 - p_1^2 - \frac{4\Delta^2}{a\,(s-a)}$$

hence we have $$l_1^2 = \mathrm{DE}^2 + p_1^2 + r_1^2 = s^2 - \frac{4\Delta}{a\,(s-a)} + \frac{\Delta^2}{(s-a)^2},$$

therefore $$\frac{l_1^2}{p_1r_1} = \frac{s^2\,a\,(s-a)}{2\Delta^2} - 2 + \frac{a}{2\,(s-a)} = \frac{as\,[s\,(s-a)+(s-b)(s-c)]}{2\Delta^2} - 2$$

$$= \frac{abcs}{2\Delta^2} - 2 = \frac{2\mathrm{R}}{r} - 2 = \frac{2}{r}(\mathrm{R}-r),$$

therefore $$\frac{l_1^2}{bcr_1} = \frac{p_1}{bc} \cdot \frac{2}{r} . (\mathrm{R}-r) = \frac{2\sin \mathrm{C}}{rc} . (\mathrm{R}-r) = \frac{1}{r} - \frac{1}{\mathrm{R}}.$$

[If O_1 be the centre of the circle escribed to the side a, we have

$$\mathrm{AO}_1 = s\sec \tfrac{1}{2}\mathrm{A}, \quad r_1 = s\tan \tfrac{1}{2}\mathrm{A}, \quad p_1 = c\sin \mathrm{B} = \frac{bc}{2\mathrm{R}};$$

$$\frac{\mathrm{AO}_1^2}{p_1r_1} = \frac{2s}{p_1 \sin \mathrm{A}} = \frac{2\Delta}{p_1 r \sin \mathrm{A}} = \frac{4\mathrm{R}\Delta}{p_1 ar} = \frac{2\mathrm{R}}{r}; \quad l_1^2 = \mathrm{AO}_1^2 - 2p_1r_1;$$

$$\frac{l_1^2}{p_1r_1} = \frac{2\mathrm{R}}{r} - 2 = \frac{2}{r}(\mathrm{R}-r); \quad \frac{l_1^2}{bcr_1} = \frac{l_1^2}{2\mathrm{R}p_1r_1} = \frac{1}{r} - \frac{1}{\mathrm{R}}.]$$

7309 & 7362. (By W. J. C. SHARP, M.A., and G. HEPPEL, M.A.)—If S_r denote the sum of the r^{th} homogeneous products of any quantities; s_r the sum of the r^{th} powers of the same quantities; and p_r the sum of the combinations taken r together; prove that

$$\left.\begin{array}{c} \mathrm{S}_1 = s_1, \; 2\mathrm{S}_2 = \mathrm{S}_1 s_1 + s_2, \; 3\mathrm{S}_3 = \mathrm{S}_2 s_1 + \mathrm{S}_1 s_2 + s_3 \\ r\mathrm{S}_r = \mathrm{S}_{r-1}s_1 + \mathrm{S}_{r-2}s_2 + \mathrm{S}_{r-3}s_3 + \ldots + s_r \end{array}\right\} \ldots\ldots(7309),$$

$$\mathrm{S}_r = \mathrm{S}_{r-1} . p_1 - \mathrm{S}_{r-2}p_2 + \mathrm{S}_{r-3} . p_3 \;\ldots\ldots\; \pm p_r \ldots\ldots\ldots\ldots(7362).$$

Solution by G. HEPPEL, M.A.

Let $a, b, c, d \ldots k, l$ be the quantities; S_r the sum of r^{th} homogeneous products; s_r the sum of r^{th} powers; p_r the sum of combinations taken r together. Let $u = (1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)\ldots$ or its equivalent $\frac{1}{1-p_1x+p_2x^2-\ldots}$; $v = p_1x-p_2x^2+\ldots$; so that $u = \frac{1}{1-v}$ and $u = uv+1$; u_1, v_1, u_2, v_2, &c., be the successive differential coefficients of u and v with respect to x; and U_1V_1, U_2V_2, &c. be the values of these when $x=0$. Then S_r is the coefficient of x^r in u; therefore $S_r = U_r \div r!$

Also $\quad u_r = u_r v + r u_{r-1} v_1 + \frac{1}{2} r (r-1) u_{r-2} \cdot v_2 + \ldots\ldots + u v_r.$

Put $x = 0$, and remember that $V_r = \pm r! \cdot p_r$,

then $\quad U_r = rU_{r-1} \cdot p_1 - r(r-1) U_{r-2} \cdot p_2 + \ldots \pm r! \, p_r,$

therefore $\quad S_r = S_{r-1} p_1 - S_{r-2} \cdot p_2 + S_{r-3} \cdot p_3 - \ldots \pm p_r.$

In TODHUNTER's *Theory of Equations* and elsewhere, it is proved that, $s_r = s_{r-1} p_1 - s_{r-2} p_2 + s_{r-3} p_3 - \ldots \pm r p_r$; but the following new and shorter proof depends only on first principles:—

$$s_r = p_1 s_{r-1} - \Sigma (a^{r-1} . b), \quad -\Sigma (a^{r-1} . b) = -p_2 s_{r-2} + \Sigma (a^{r-2} . bc),$$
$$\Sigma (a^{r-2} . bc) = p_3 s_{r-3} - \Sigma (a^{r-3} . bcd), \text{ \&c., \&c.,}$$
$$\mp \Sigma (a^2 bcd \ldots k) = \mp p_{r-1} s_1 \pm r p_r;$$

therefore, adding the equations, $s_r = s_{r-1} p_1 - s_{r-2} \cdot p_2 + s_{r-3} p_3 - \ldots \pm r p_r$.
The series in 7309 is derived from these, as follows:—

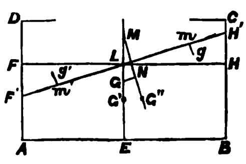
It is evidently true for the first few terms. Assume it to be true as far as $(r-1)$ terms; so that $(r-1) S_{r-1} - s_{r-1} = S_{r-2} s_1 + S_{r-3} s_2 + \ldots + S_1 s_{r-2}$. Then, from the series found above, we obtain

$$rS_r - s_r = (rS_{r-1} - s_{r-1}) p_1 - (rS_{r-2} - s_{r-2}) p_2 + \ldots \pm (rS_1 - s_1) p_{r-1}$$
$$= S_{r-1} \cdot p_1 - 2S_{r-2} \cdot p_2 + 3S_{r-3} \cdot p_3 - 4 S_{r-4} \cdot p_4 + \ldots \pm (r-1) S_1 p_{r-1}$$
$$+ (S_{r-2} \cdot s_1 + S_{r-3} \cdot s_2 + S_{r-4} s_3 + \ldots + S_1 \cdot s_{r-2}) p_1$$
$$- (S_{r-3} \cdot s_1 + S_{r-4} \cdot s_2 + S_{r-5} \cdot s_3 + \ldots + S_1 \cdot s_{r-3}) p_2 + \ldots \pm S_1 p_{r-1}.$$

Hence, collecting coefficients of S_{r-1}, S_{r-2}, &c., and using the second series, $\quad rS_r - s_r = S_{r-1} \cdot s_1 + S_{r-2} \cdot s_2 + S_{r-3} \cdot s_3 + \ldots + S_1 s_{r-1}.$

7378. (By Professor HAUGHTON, F.R.S.)—A homogeneous rectangular parallelepiped, the edges of which are a, b, c, floats in a liquid whose density is ρ; and is turned through an angle θ (the top remaining above the surface of the liquid), so that the plane of a, b remains parallel to itself; find the limiting value of θ, when the solid will cease to right itself.

Solution by R. RAWSON; SARAH MARKS; *and others.*



Let ABCD be a vertical section through the centre of gravity (G) of the parallelepiped. The liquid lines FH, F′H′ before and after the solid is turned through an angle θ, are determined—(1) by the equality of the weights of the displaced liquid and parallelepiped; (2) by the equality of the weights of the *in* and *out*. Hence EL $= aw : w'$, where w, w' are the weights of the cubic units of the solid and liquid respectively. Bisect EL in G′, which is the centre of gravity of the displaced liquid. Let G″ be the centre of gravity of the displaced liquid, after the solid has been turned through the angle θ.

Draw G″M perpendicular to F′H′, meeting EL produced in M, which is called by Atwood and others the metacentre corresponding to the definite angle θ. When, however, the angle θ is indefinitely diminished, the point M becomes the metacentre first assigned by Bouguer. Let g, g' be the centre of gravity of the *in* and *out*, respectively; draw gm, $g'm'$ perpendicular to the liquid line F′H′ and GN perpendicular to G″M.

Now, $abcw$. GN = the righting moment, or the force by which the solid endeavours to gain its upright position.

Atwood has shown that $\mathrm{GN} = \dfrac{mm'.v}{\mathrm{V}} - \mathrm{GG'}\sin\theta$ ....(1),

where V, v are the volumes of immersion and *in* and *out* respectively. (See Fincham's *Outlines of Shipbuilding*, p. 152); and it is readily seen that

$$\mathrm{V} = \frac{abcw}{w'},\quad \text{and}\quad v = \frac{b^2c\tan\theta}{8},\quad \mathrm{GG'} = \frac{a}{2} - \frac{aw}{2w'} = \frac{a\,(w'-w)}{2w'},$$

therefore $$\frac{v}{\mathrm{V}} = \frac{b^2c\tan\theta\times w'}{8\,.\,abcw} = \frac{bw'\tan\theta}{8aw};$$

$$mm' = 2\mathrm{L}m = 2\left[\tfrac{1}{2}\mathrm{LH'} + \tfrac{1}{3}\left(\mathrm{LH}\cos\theta - \tfrac{1}{2}\mathrm{LH'}\right)\right] = \tfrac{2}{3}\mathrm{LH'} + \mathrm{LH}\cos\theta);$$

but $2\mathrm{LH'} = b\sec\theta$, and $2\mathrm{LH} = b$, $\therefore\ mm' = \tfrac{1}{3}b\,(\sec\theta + \cos\theta)$.

Substituting in (1),

$$\mathrm{GN} = \frac{\sin\theta}{2}\left\{\frac{b^2w'}{12aw}\left(1 + \frac{1}{\cos^2\theta}\right) - \frac{a\,(w'-w)}{w'}\right\}\ \ldots\ldots\ldots\ldots(2)$$

and $$\mathrm{GM} = \tfrac{1}{2}\left\{\frac{b^2w'}{12aw}\left(1 + \frac{1}{\cos^2\theta}\right) - \frac{a\,(w'-w)}{w'}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The solid will cease to right itself when GN equals zero,

or $$\cos\theta = \frac{bw'}{(12a^2ww' - 12a^2w^2 - b^2w'^2)^{\frac{1}{2}}},$$

a value which is possible when $\left(\dfrac{w}{w'}\right)^2 - \dfrac{w}{w'} + \dfrac{b^2}{12a^2} < 0$.

7397. (By C. Bickerdike.)—A point moves with constant velocity in a straight line, prove that its angular velocity about any fixed point is inversely as the square of the distance from this point.

Solution by G. S. Carr, B.A.

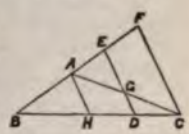
Let p = perpendicular on the line; then, geometrically, we have

$$\frac{r\,d\theta}{dx} = \frac{p}{r};\ \text{therefore}\ \frac{d\theta}{dt} = \frac{p}{r^2}\frac{dx}{dt} = \frac{pv}{r^2}.$$

[For another proof, see also Thomson and Tait's *Nat. Phil.*, p. 30.]

7398. (By R. KNOWLES, B.A.)—In the side BC (> AB) of a triangle ABC, BD is taken equal to one-half of AB + BC; and in BA produced, BE is taken equal to BD; prove that DE bisects AC at G.

Solution by KATE GALE; S. GREENIDGE; *and others.*



Make EF = AE and join FC; then EF = AE = BD − AB = $\frac{1}{2}$(AB + BC) − AB = $\frac{1}{2}$(BC − AB); DC = BC − $\frac{1}{2}$(AB + BC) = $\frac{1}{2}$(BC − AB); therefore AE = EF = DC; hence CF is parallel to GE. and as AE = EF, therefore AG = GC.

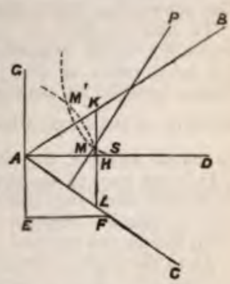
[If AH be drawn parallel to DGE, we have

$$BD = \tfrac{1}{2}(BH + BC) = BH + \tfrac{1}{2}HC;$$

therefore HD = DC, and consequently AG = GC.]

7336 & 7369. (By W. H. BLYTHE, M.A., and A. H. CURTIS, LL.D.) —Through a given point to draw a straight line which shall (7336) bisect a given triangle, (7369) form with two given straight lines a given area.

Solution by G. HEPPEL, M.A.; Professor MATZ, M.A.; *and others.*



(7369).—Let AB, AC be the given lines, and first let the given point P be outside BAC. Let AD be the bisector; AE, perpendicular to AD, a side of the square equal to the given area. Draw EF parallel to AD. Make AG = EF. Let the circle with diameter EG cut AD in H. Draw KHL perpendicular to AD, and in AD make AS = AK. Let the circle whose radius is AH cut the circle whose diameter is SP in M and M′. Then one of the lines PM, PM′ will be the line required.

For $AH^2 = AE \,.\, AG = AE \,.\, EF$,

and $HK : AH = AE : EF$;

$\therefore$ AH . HK or Δ KAL = AE^2 = given area.

Now S is the focus of a hyperbola, whose major axis is twice AH, and whose asymptotes are AB and AC; and the latter part of the construction is merely the ordinary method of drawing a tangent from P to such a hyperbola. This must cut off from the angle BAC the area required. If P is inside the angle, the construction is the same, but both lines PM, PM′, will give a solution.

(7336).—If A be one angle of the triangle ABC, we have merely to take AK a mean proportional to AB and $\frac{1}{2}$AC; and proceed as before.

angles acute. Hence we have

$$p = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\phi_1}^{\phi_2} d\theta \,.\, 2\pi \sin\mu\, d\mu \sin\phi\, d\phi$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta} \operatorname{cosec}^2\theta \sin\psi\, d\psi - \int_0^{\theta} \cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi} \tan^2\tfrac{1}{2}\theta\, d\theta = \frac{1}{2\pi} - \frac{1}{8}.$$

2. If $\theta < \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AD or EA′; or if $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AE or DA′; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AE; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AD; the triangle will have one obtuse angle. Hence we have

$$p_1 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\phi_1} \sin\phi\, d\phi + \int_{\phi_2}^{\pi} \sin\phi\, d\phi\right\} d\theta \,.\, 2\pi \sin\mu\, d\mu$$

$$+ \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\phi_1} \sin\phi\, d\phi + \int_{\phi_1}^{\pi} \sin\phi\, d\phi\right\} d\theta \,.\, 2\pi \sin\mu\, d\mu$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_0^{\phi_2} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\phi_1} 2\pi\sin\mu\, d\mu \sin\phi\, d\phi\right\} d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\pi} \sin\mu\, d\mu - \int_0^{\theta} \operatorname{cosec}^2\theta \sin\psi\, d\psi + \int_0^{\theta} \cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$+ \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\pi} \sin\mu\, d\mu - \int_{\theta}^{\pi} \operatorname{cosec}^2\theta \sin\psi\, d\psi + \int_{\theta}^{\pi} \cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi} (3 - \sec^2\tfrac{1}{2}\theta)\, d\theta + \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi} (3 - \operatorname{cosec}^2\tfrac{1}{2}\theta)\, d\theta = \frac{9}{8} - \frac{3}{2\pi}.$$

3. If $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in DE; the triangle will have two obtuse angles. Hence we have

$$p_2 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_2}^{\phi_1} d\theta \,.\, 2\pi \sin\mu\, d\mu \sin\phi\, d\phi$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_{\phi_2}^{\phi_1} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_1}^{\phi_2} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta} \operatorname{cosec}^2\theta \sin\psi\, d\psi - \int_0^{\theta} \cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$+ \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_{\theta}^{\pi} \operatorname{cosec}^2\theta \sin\psi\, d\psi - \int_{\theta}^{\pi} \cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi} \tan^2\tfrac{1}{2}\theta\, d\theta + \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi} \cot^2\tfrac{1}{2}\theta\, d\theta = \frac{3}{2\pi} - \frac{3}{8}.$$

6670. (By Belle Easton.)—Through a given point P, between two given lines AB, AC, draw a straight line BPC meeting the given lines in B and C, so that BPC may be a minimum.

Solution by G. Heppel, M.A.; J. O'Regan; *and others.*

Take AB and AC as axes. Let $AB = h$, $AC = k$, and let the point P be (a, b); then, since P is on BC, $\frac{a}{h} + \frac{b}{k} = 1$, therefore $k = \frac{bh}{h-a}$, and if l be the length of BC, $l^2 = h^2 + \frac{b^2h^2}{(h-a)^2} - \frac{2bh^2}{h-a} . \cos A$. Put $x = h - a$

and $c = b \cos A$, then $l^2 = (x+a)^2 + \frac{b^2(x+a)^2}{x^2} - \frac{2c(x+a)^2}{x}$;

therefore $$l^2 = (x+a)^2 \left(1 + \frac{b^2}{x^2} - \frac{2c}{x}\right);$$

therefore, if l is a minimum, we have

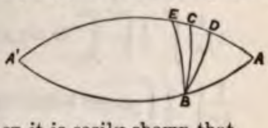
$$(x+a)\left(\frac{c}{x^2} - \frac{b^2}{x^3}\right) + 1 + \frac{b^2}{x^2} - \frac{2c}{x} = 0, \text{ or } x^3 - cx^2 + acx - ab^2 = 0,$$

a cubic equation which determines x. If the lines are at right angles, $c = 0$ and $x = \sqrt[3]{ab^2}$, whence by substitution $l^2 = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^3$.

5980. (By Professor Seitz, M.A.)—Three points, taken at random in the surface of a sphere, are joined by arcs of great circles; show that the chance (1) that the triangle formed has all its angles acute, is $\frac{1}{2\pi} - \frac{1}{8}$; (2) that it has one obtuse angle, is $\frac{9}{8} - \frac{3}{2\pi}$; (3) that it has two obtuse angles, is $\frac{3}{2\pi} - \frac{3}{8}$; and (4) that it has all its angles obtuse, is $\frac{3}{8} - \frac{1}{2\pi}$.

Solution by the Proposer.

Let A, B, C be the random points. We may suppose one of the points, as A, fixed. Produce the arcs AB and AC till they meet at A'; draw the arcs BD and BE perpendicular, respectively, to ACA' and ABA'.



Let $\angle BAC = \theta$, arc $AB = \mu$, $AC = \phi$, $AD = \phi_1$, $AE = \phi_2$, $BD = \psi$, $BE = \omega$, and take unity for the radius of the sphere. Then it is easily shown that $\cos\phi_1 \sin\mu \, d\mu = \text{cosec}^2\theta \sin\psi \, d\psi$, and $\cos\phi_2 \sin\mu \, d\mu = \cot^2\theta \sin\omega \sec^2\omega \, d\omega$.

An element of surface at B is $2\pi \sin\mu \, d\mu$, and at C it is $d\theta \sin\phi \, d\phi$.

If $\theta < \frac{1}{2}\pi$ and $\phi < \frac{1}{2}\pi$, or if $\theta > \frac{1}{2}\pi$ and $\phi > \frac{1}{2}\pi$, E will lie in DA'; but if $\theta < \frac{1}{2}\pi$ and $\phi > \frac{1}{2}\pi$, or if $\theta > \frac{1}{2}\pi$ and $\phi < \frac{1}{2}\pi$, E will lie in AD.

1. If $\theta < \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in DE, the triangle will have all its

angles acute. Hence we have

$$p = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\phi_1}^{\phi_2} d\theta \,.\, 2\pi \sin\mu \, d\mu \sin\phi \, d\phi$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta} \operatorname{cosec}^2\theta \sin\psi \, d\psi - \int_0^{\theta} \cot^2\theta \sin\omega \sec^2\omega \, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi} \tan^2 \tfrac{1}{2}\theta \, d\theta = \frac{1}{2\pi} - \frac{1}{8}.$$

2. If $\theta < \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AD or EA′; or if $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AE or DA′; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in AE; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in AD; the triangle will have one obtuse angle. Hence we have

$$p_1 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\phi_1}\sin\phi \, d\phi + \int_{\phi_2}^{\pi}\sin\phi \, d\phi\right\} d\theta \,.\, 2\pi\sin\mu \, d\mu$$

$$+ \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\phi_1}\sin\phi \, d\phi + \int_{\phi_1}^{\pi}\sin\phi \, d\phi\right\} d\theta \,.\, 2\pi\sin\mu \, d\mu$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_0^{\phi_2} 2\pi\sin\mu \, d\mu \sin\phi \, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_0^{\phi_1} 2\pi\sin\mu \, d\mu\sin\phi \, d\phi\right\} d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\pi}\sin\mu \, d\mu - \int_0^{\theta}\operatorname{cosec}^2\theta\sin\psi \, d\psi + \int_0^{\theta}\cot^2\theta\sin\omega\sec^2\omega \, d\omega\right\} d\theta$$

$$+ \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\pi}\sin\mu \, d\mu - \int_{\theta}^{\pi}\operatorname{cosec}^2\theta\sin\psi \, d\psi + \int_{\theta}^{\pi}\cot^2\theta\sin\omega\sec^2\omega \, d\omega\right\} d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}(3 - \sec^2\tfrac{1}{2}\theta) \, d\theta + \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}(3 - \operatorname{cosec}^2\tfrac{1}{2}\theta) \, d\theta = \frac{9}{8} - \frac{3}{2\pi}.$$

3. If $\theta < \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in ED; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in DE; the triangle will have two obtuse angles. Hence we have

$$p_2 = \frac{1}{8\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_2}^{\phi_1} d\theta \,.\, 2\pi\sin\mu \, d\mu\sin\phi \, d\phi$$

$$+ \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_{\phi_2}^{\phi_1} 2\pi\sin\mu \, d\mu\sin\phi \, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_1}^{\phi_2} 2\pi\sin\mu \, d\mu\sin\phi \, d\phi\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\operatorname{cosec}^2\theta\sin\psi \, d\psi - \int_0^{\theta}\cot^2\theta\sin\omega\sec^2\omega \, d\omega\right\} d\theta$$

$$+ \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_{\theta}^{\pi}\operatorname{cosec}^2\theta\sin\psi \, d\psi - \int_{\theta}^{\pi}\cot^2\theta\sin\omega\sec^2\omega \, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_0^{\frac{1}{2}\pi}\tan^2\tfrac{1}{2}\theta \, d\theta + \frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\cot^2\tfrac{1}{2}\theta \, d\theta = \frac{3}{2\pi} - \frac{3}{8}.$$

4. If $\theta > \frac{1}{2}\pi$, $\phi < \frac{1}{2}\pi$, and C lies in DA′; or if $\theta > \frac{1}{2}\pi$, $\phi > \frac{1}{2}\pi$, and C lies in EA′; the triangle will have all its angles obtuse. Hence we have

$$p_3 = \frac{1}{8\pi^2}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\frac{1}{2}\pi}\int_{\phi_1}^{\pi} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi + \int_{\frac{1}{2}\pi}^{\pi}\int_{\phi_2}^{\pi} 2\pi \sin\mu\, d\mu \sin\phi\, d\phi\right\} d\theta$$

$$= \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left\{\int_0^{\pi}\sin\mu\, d\mu - \int_{\theta}^{\pi}\operatorname{cosec}^2\theta \sin\psi\, d\psi + \int_{\theta}^{\pi}\cot^2\theta \sin\omega \sec^2\omega\, d\omega\right\} d\theta$$

$$= \frac{1}{4\pi}\int_{\frac{1}{2}\pi}^{\pi}\left(3 - \operatorname{cosec}^2 \tfrac{1}{2}\theta\right) d\theta = \frac{3}{8} - \frac{1}{2\pi}.$$

6833. (By the EDITOR.)—Show that the volume between $x = 0$ and $x = 2l$ of the solid bounded by the surface whose equation is

$$a(y^4 - z^4) - z^2(x^3 - 2ay^2 + 2c^3) - y^2(bx^2 + c^2x + c^3) = 0,$$

is

$$\frac{\pi}{3a}\left(6l^4 + 4bl^3 + 3c^2l^2 + 9c^3l\right).$$

Solution by D. EDWARDES; G. EASTWOOD, M.A.; *and others.*

While x is constant, the equation of the section is

$$r^2 = A\cos^2\theta + B\sin^2\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $aA = bx^2 + c^2x + c^3$, and $aB = x^3 + 2c^3$. The area of (1) is $\frac{1}{2}\pi\,(A + B)$; therefore the required volume is

$$\frac{\pi}{2a}\int_0^{2l}\left(x^3 + bx^2 + c^2x + 3c^3\right) dx = \frac{\pi}{6a}\left(12l^4 + 8bl^3 + 6c^2l^2 + 18c^3l\right) = \&c.$$

6739. (By Professor WOLSTENHOLME, M.A.)—If $u^2 = 0$ be the rational equation of the second degree of a conic referred to Cartesian coordinates inclined at an angle ω, prove that the equations giving (1) the foci, (2) the director circle, (3) all four directrices, are

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} = \frac{d^2u}{dx\,dy}\sec\omega, \quad \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 2\frac{d^2u}{dx\,dy}\cos\omega \quad\ldots\ldots (1, 2);$$

$$\left\{\frac{du}{dx}\frac{du}{dy}\cos 2\omega \left[\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right]\cos\omega - u\frac{d^2u}{dx\,dy}\right\}^2$$

$$= \left\{2\frac{du}{dx}\frac{du}{dy}\cos\omega + \left(\frac{du}{dx}\right)^2 - u\frac{d^2u}{dx^2}\right\}\left\{2\frac{du}{dx}\frac{du}{dy}\cos\omega + \left(\frac{du}{dx}\right)^2 - u\frac{d^2u}{dy^2}\right\} \ldots (3).$$

Solution by the PROPOSER.

If $u^2 = 0$ be the rational equation of a conic referred to Cartesian coordinates inclined at an angle ω, and we move the origin to (x, y) a focus, using (X, Y) for current coordinates, the equation will become of the form $k\,(X^2+Y^2+2XY\cos\omega) = (pX+qY+r)^2$;

but u^2 will become

$$u^2 + X\,.\,2u\frac{du}{dx} + Y\,.\,2u\frac{du}{dy} + X^2\left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2}\right\} + Y^2\left\{\left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2}\right\}$$
$$+ 2XY\left(\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy}\right) = 0\,;$$

and, making this coincide with the former,

$$u^2 = \lambda r^2,\quad u\frac{du}{dx} = \lambda pr,\quad u\frac{du}{dy} = \lambda qr,$$
$$\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2} = \lambda\,(p^2-k),\quad \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2} = \lambda\,(q^2-k),$$
$$\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy} = \lambda\,(pq-k\cos\omega)\,;$$

whence $\lambda p^2 = \left(\frac{du}{dx}\right)^2,\ u\frac{d^2u}{dx^2} = -\lambda k;\ \lambda q^2 = \left(\frac{du}{dy}\right)^2,\ u\frac{d^2u}{dy^2} = -\lambda k;$

$$\lambda pq = \frac{du}{dx}\frac{du}{dy},\quad u\frac{d^2u}{dx\,dy} = -\lambda k\cos\omega\,;$$

or the equations for the foci will be

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2} = \sec\omega\,\frac{d^2u}{dx\,dy}.$$

The equation of two tangents drawn from a point (xy) is

$$4u^2U^2 = \left(X\frac{d\,(u^2)}{dx} + Y\frac{d\,(u^2)}{dy} + \ldots\right),$$

and the condition for these being at right angles is, if

$$u^2 \equiv (a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2,$$
$$4u^2\,(a+b-2h\cos\omega) = \left(\frac{du^2}{dx}\right)^2 + \left(\frac{du^2}{dy}\right)^2 - 2\left(\frac{du^2}{dx}\right)\left(\frac{du^2}{dy}\right)\cos\omega.$$

But $$a = \tfrac{1}{2}\frac{d^2\,(u^2)}{dx^2} = \left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2}\right\};$$

$$b = \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dx\,dy},\quad h = \frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy};$$

so that the equation becomes

$$4u^2\left\{\left(\frac{du}{dx}\right)^2 + u\frac{d^2u}{dx^2} + \left(\frac{du}{dy}\right)^2 + u\frac{d^2u}{dy^2} - 2\cos\omega\left(\frac{du}{dx}\frac{du}{dy} + u\frac{d^2u}{dx\,dy}\right)\right\}$$
$$= 4u^2\left(\frac{du}{dx}\right)^2 + 4u^2\left(\frac{du}{dy}\right)^2 - 8\cos\omega\,u^2\frac{d^2u}{dx\,dy},$$

or $$4u^3\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - 2\cos\omega\frac{d^2u}{dx\,dy}\right) = 0\,;$$

that is, the equation of the director circle is

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - 2\frac{d^2u}{dx\,dy}\cos\omega = 0.$$

A pair of directrices, the director circle, and the conic, have four common points; hence the equation of a pair of parallel directrices will be

$$\lambda = u\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right),$$

where λ is some constant; and the two values of λ can be most readily determined by making this equation represent two parallel straight lines.

But $$u\frac{du}{dx} = ax+hy+g,\quad \left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2} = a;$$

$$\left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2} = b;\quad \frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy} = h,$$

or the equation for the pair of directrices is

$$\lambda\,(ax^2+by^2+c+2fy+2gx+2hxy) = u^2\,(a+b-h\cos\omega)-(ax+hy+g)^2$$
$$-(hx+by+f)^2+2\,(ax+hy+g)(hx+by+f)\cos\omega;$$

the coefficient of x^2 is

$$a\lambda-a\,(a+b-2h\cos\omega)+a^2+h^2-2ah\cos\omega,\ \text{ or }\ a\lambda+h^2-ab.$$

Similarly the coefficient of y^2 is $b\lambda+h^2-ab$; and the coefficient of $2xy$ is

$$\lambda h-h\,(a+b-h\cos\omega)+ah+bh-(ab+h^2)\cos\omega,\ \text{ or }\ \lambda h+(h^2-ab)\cos\omega$$

so that the equation for λ is

$$(a\lambda-\mathrm{P})^2\,(b\lambda-\mathrm{P}) = (h\lambda-\mathrm{P}\cos\omega)^2,\quad \text{where } \mathrm{P} = ab-h^2,$$

or $$\lambda^2-\lambda\,(a+b-2h\cos\omega)+(ab-h^2)\sin^2\omega = 0.$$

Hence, to get the equation of all four directrices, we have to eliminate λ between the equations $\lambda = u\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right)$,

$$\lambda^2-\lambda\,(a+b-2h\cos\omega)+(ab-h^2)\sin^2\omega = 0;$$

where $a = \left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2}$, $b = \left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2}$, $h = \frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy}$.

Hence $$a+b-2h\cos\omega-\lambda = \left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2-2\frac{du}{dx}\frac{du}{dy}\cos\omega,$$

and the final equation is

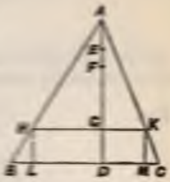
$$u\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2-2\frac{du}{dx}\frac{du}{dy}\cos\omega\right\}\left\{\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}-2\frac{d^2u}{dx\,dy}\cos\omega\right\}$$
$$=\left\{\left[\left(\frac{du}{dx}\right)^2+u\frac{d^2u}{dx^2}\right]\left[\left(\frac{du}{dy}\right)^2+u\frac{d^2u}{dy^2}\right]-\left(\frac{du}{dx}\frac{du}{dy}+u\frac{d^2u}{dx\,dy}\right)^2\right\}\sin^2\omega.$$

And this will be found (I hope) to coincide with the given equation, which I must have obtained in some different way, long forgotten.

7290. (By S. Tebay, B.A.)—In a given triangle inscribe a rectangle having one side parallel to the base, and the perimeter equal to given straight line.

Solution by R. Knowles, B.A.; Belle Easton; *and others.*

Let ABC be the triangle, AD perpendicular to BC. Make DE = BC, and let DF be half the given straight line. Find AG a fourth proportional to AE, AF, AD, and through G draw HK parallel to BC, and HL, KM perpendicular to BC. Then LK is the rectangle required.



By similar triangles,

$$AD : BC \text{ (or } DE) = AG : HK \text{ (or } FG);$$

whence $$AE : DE = AF : FG,$$

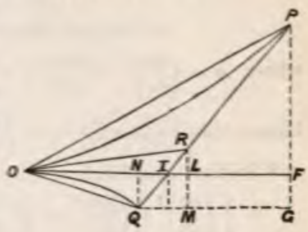
and $$AE : AF = DE : FG = AD : AG,$$

which proves the construction.

7205. (By C. Leudesdorf, M.A.)—The tangent at any point P of the cissoid $y^2(a-x) = x^3$ cuts the curve again at Q, and R is a point on PQ such that $RP = 2RQ$. Show that, if the straight lines joining R, P, Q to the origin make angles θ, α, β respectively with the axis of x, then $\cot\theta = \tan\alpha - \cot\beta$.

Solution by J. S. Jenkins; Kate Gale; *and others.*

Let (x', y') be the coordinates of P; then the equation to the tangent at P is $y - y' = \dfrac{x'^{\frac{1}{2}}(3a-2x')}{2(a-x')^{\frac{3}{2}}}(x-x')$, therefore $OI = \dfrac{ax'}{3a-2x'}$; and combining the equations of tangent and curve, we find the tangent cuts the curve as represented by



$$(x-x')^2\left[(4a-3x')\,a^2x - a^3x'\right] = 0,$$

whence, for point Q, we have

$$x = \frac{ax''}{4a-3x} = ON, \quad y = \frac{-ax'^{\frac{3}{2}}}{2(a-x')^{\frac{1}{2}}(4a-3x')} = NQ.$$

By similar triangles, $OL = \dfrac{x'(2a-x')}{(4a-3x')}$, $\quad RL = \dfrac{x'^{\frac{3}{2}}(a-x')^{\frac{1}{2}}}{(4a-3x')}$;

whence $\quad \cot\theta = \dfrac{OL}{RL} = \dfrac{2a-x'}{x'^{\frac{1}{2}}(a-x')^{\frac{1}{2}}}$, and $\cot\beta = \dfrac{ON}{NQ} = -\dfrac{2(a-x')^{\frac{1}{2}}}{x'^{\frac{1}{2}}}$,

therefore $\quad \cot\theta + \cot\beta = \dfrac{2a-x'-2(a-x')}{x'^{\frac{1}{2}}(a-x')^{\frac{1}{2}}} = \dfrac{x'^{\frac{1}{2}}}{(a-x')^{\frac{1}{2}}} = \tan\alpha,$

therefore $$\cot\theta = \tan\alpha - \cot\beta.$$

6941. (By the Rev. T. W. OPENSHAW, M.A.)—Find the equation to the circle circumscribing the triangle formed by two tangents to the ellipse $a^2y^2+b^2x^2=a^2b^2$ and their chord of contact.

Solution by the PROPOSER; BELLE EASTON; *and others.*

If (h, k) be the point of intersection of the tangents, the equation to a circle through the points of contact will be of the form

$$(a^2yk+b^2xh-a^2b^2)(a^2yk-b^2xh-d)=\lambda(a^2y^2+b^2x^2-a^2b^2).$$

The conditions for a circle, and that the circle shall pass through (h, k), give

$$\lambda=\frac{a^4k^2+b^4h^2}{a^2-b^2},\quad d=-\frac{a^2b^2(h^2+k^2)}{a^2-b^2}.$$

By substitution, the equation to the circle is

$$(a^2k^2+b^2h^2)(x^2+y^2)-b^2h(h^2+k^2+a^2-b^2)x-a^2k(h^2+k^2-a^2+b^2)y$$
$$-(a^2k^2-b^2h^2)(a^2-b^2)=0.$$

If (h, k) is on the ellipse, this will give the osculating circle.

3269. (By the EDITOR.)—Prove that the chord that joins the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $l\alpha^{\frac{1}{2}}+m\beta^{\frac{1}{2}}+n\gamma^{\frac{1}{2}}=0$ is parallel to

$$\frac{l\alpha}{\alpha_1^{\frac{1}{2}}+\alpha_2^{\frac{1}{2}}}+\frac{m\beta}{\beta_1^{\frac{1}{2}}+\beta_2^{\frac{1}{2}}}+\frac{n\gamma}{\gamma_1^{\frac{1}{2}}+\gamma_2^{\frac{1}{2}}}=0.$$

Solution by the Rev. J. L. KITCHIN, M.A.; N. SARKAR, B.A.; *and others.*

Let $\quad l_1\alpha+m_1\beta+n_1\gamma=0$ be the equation to the chord;

then $\quad l_1\alpha_1+m_1\beta_1+n_1\gamma_1=0,\ l_1\alpha_2+m_1\beta_2+n_1\gamma_2=0$;

therefore $\quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}=0$, the equation to the chord,

which is $\quad \alpha(\beta_1\gamma_2-\beta_2\gamma_1)+\beta(\alpha_2\gamma_1-\alpha_1\gamma_2)+\gamma(\alpha_1\beta_2-\alpha_2\beta_1)=0$;

but $\quad l\alpha_1^{\frac{1}{2}}+m\beta_1^{\frac{1}{2}}+n\gamma_1^{\frac{1}{2}}=0,\ l\alpha_2^{\frac{1}{2}}+m\beta_2^{\frac{1}{2}}+n\gamma_2^{\frac{1}{2}}=0$;

therefore $\quad \dfrac{l}{(\beta_1\gamma_2)^{\frac{1}{2}}-(\beta_2\gamma_1)^{\frac{1}{2}}}=\dfrac{m}{(\alpha_2\gamma_1)^{\frac{1}{2}}-(\alpha_1\gamma_2)^{\frac{1}{2}}}=\dfrac{n}{(\alpha_1\beta_2)^{\frac{1}{2}}-(\alpha_2\beta_1)^{\frac{1}{2}}}$;

$\therefore\ \alpha l[(\beta_1\gamma_2)^{\frac{1}{2}}+(\beta_2\gamma_1)]^{\frac{1}{2}}+m\beta[(\alpha_2\gamma_1)^{\frac{1}{2}}+(\alpha_1\gamma_2)]^{\frac{1}{2}}+n\gamma[(\alpha_1\beta_2)^{\frac{1}{2}}+(\alpha_2\beta_1)]=0.$

Now, if this line move parallel to itself up to $(\alpha', \beta', \gamma')$, it becomes

$$2\alpha l(\beta'\gamma')^{\frac{1}{2}}+2m\beta(\alpha'\gamma')^{\frac{1}{2}}+2n\gamma(\alpha'\beta')^{\frac{1}{2}}=0,$$

or $\quad \dfrac{l\alpha}{\alpha'^{\frac{1}{2}}}+\dfrac{m\beta}{\beta'^{\frac{1}{2}}}+\dfrac{n\gamma}{\gamma'^{\frac{1}{2}}}=0$, the tangent.

But if the line given in the question move in the same way, we get

$$\frac{l\alpha}{2\alpha'^{\frac{1}{2}}}+\frac{m\beta}{2\beta'^{\frac{1}{2}}}+\frac{n\gamma}{2\gamma'^{\frac{1}{2}}}=0,\quad\text{or}\quad \frac{l\alpha}{\alpha'^{\frac{1}{2}}}+\frac{m\beta}{\beta'^{\frac{1}{2}}}+\frac{n\gamma}{\gamma'^{\frac{1}{2}}}=0.$$

Hence both lines are parallel to the tangent, and are therefore parallel.

7201. (By R. F. Scott, M.A.)—Prove that

$$\int_0^\pi x \log(1-\sin^2\alpha\sin^2 x)\,dx = 2\pi^2\log\cos\tfrac{1}{2}\alpha.$$

Solution by D. Edwardes; Prof. Matz, M.A.; *and others.*

Writing $\pi - x$ for x, the integral is $\frac{1}{2}\pi\int_0^\pi \log(1-\sin^2\alpha\sin^2 x)\,dx$.

If $$u = \int_0^\pi \log(1-\sin^2\alpha\sin^2 x)\,dx,$$

$$\frac{du}{d\alpha} = -\sin 2\alpha\int_0^\pi \frac{\sin^2 x}{1-\sin^2\alpha\sin^2 x}\,dx = -2\pi\tan\tfrac{1}{2}\alpha;$$

therefore $u = 4\pi\log\cos\frac{1}{2}\alpha + C$. But, when $\alpha = 0$, $u = 0$, therefore $C = 0$; hence $u = 4\pi\log\cos\frac{1}{2}\alpha$; therefore $I = \frac{1}{2}\pi u = 2\pi^2\log\cos\frac{1}{2}\alpha$.

7072. (By Ath Bigah Bhut.)—If a^3, b^3, c^3, d^3 denote

$$\begin{vmatrix} x, & y, & z \\ u, & x, & y \\ z, & u, & x \end{vmatrix}, \quad \begin{vmatrix} y, & z, & u \\ x, & y, & z \\ u, & x, & y \end{vmatrix}, \quad \begin{vmatrix} z, & u, & x \\ y, & z, & u \\ x, & y, & z \end{vmatrix}, \quad \begin{vmatrix} u, & x, & y \\ z, & u, & x \\ y, & z, & u \end{vmatrix};$$

exhibit the values (severally) of x, y, z, u, in terms of a, b, c, d.

Solution by W. H. Blythe, M.A.; Belle Easton; *and others.*

The following are evident identities, reducing to determinants with two identical rows

$$d^3x + a^3y + b^3y + c^3u = 0,\quad c^3x + d^3y + a^3y + b^3u = 0,\quad b^3x + c^3y + d^3y + a^3u = 0;$$

hence $$\frac{x}{\begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} b^3, & c^3, & d^3 \\ a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \end{vmatrix}}$$ = two similar expressions in z, and u, a^3, b^3, c^3, d^3.

If we write this $\frac{x}{\lambda_1} = \frac{y}{\lambda_2} = \frac{z}{\lambda_3} = \frac{u}{\lambda_4} = \mu$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ being known in terms of a, b, c, d, substitute $x = \lambda_1\mu$, &c., for x, y, z, u in the first determinant, we obtain

$$a^3 = \mu^3\begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \lambda_4, & \lambda_1, & \lambda_2 \\ \lambda_3, & \lambda_4, & \lambda_1 \end{vmatrix}, \quad x = a\begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix} \div \begin{vmatrix} \lambda_1, & \lambda_2, & \lambda_3 \\ \lambda_4, & \lambda_1, & \lambda_2 \\ \lambda_3, & \lambda_4, & \lambda_1 \end{vmatrix}^{\frac{1}{3}},$$

where $$\lambda_1 = \begin{vmatrix} a^3, & b^3, & c^3 \\ d^3, & a^3, & b^3 \\ c^3, & d^3, & a^3 \end{vmatrix}$$

with similar expression for λ_2, λ_3, and λ_4.

6832. (By Professor MATZ, M.A.)—Find the values of X and x from the equations
$$\sin X^{\log x} = \frac{1}{\pi}, \quad \log (Xx) = \frac{2}{\pi}.$$

Solution by G. M. REEVES, M.A. ; J. O'REGAN ; *and others.*

Let $u = \log X$ and $v = \log x$; then the equations are equivalent to
$$uv = \log \sin^{-1} \frac{1}{\pi}, \quad u + v = \frac{2}{\pi}; \text{ whence we obtain}$$
$$u = \frac{1}{\pi} \pm \left(\frac{1}{\pi^2} - \log \sin^{-1} \frac{1}{\pi}\right)^{\frac{1}{2}} = \log X,$$
$$v = \frac{1}{\pi} \mp \left(\frac{1}{\pi^2} - \log \sin^{-1} \frac{1}{\pi}\right)^{\frac{1}{2}} = \log x.$$

7138. (By G. G. MORRICE, B.A.)—A triangle Δ is formed by the straight lines $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$, $a_3x + b_3y = c_3$; another triangle Δ_1 is formed by the external bisectors of its angles Δ_2 by the external bisectors of Δ_1; show that, if
$$s_1 = a_1 + a_2 + a_3, \quad s_2 = b_1 + b_2 + b_3, \quad s_3 = c_1 + c_2 + c_3,$$
Area of Δ_{2r} is
$$\frac{\frac{1}{2} \left| \frac{1}{3}(4^r - 1)s_1 + a_1, \; \frac{1}{3}(4^r - 1)s_2 + b_2, \; \frac{1}{3}(4^r - 1)s_3 + c_3 \right|^2}{\left| \frac{1}{3}(4^r - 1)s_1 + a_1, \; \frac{1}{3}(4^r - 1)s_2 + b_2 \right| \times \left| \frac{1}{3}(4^r - 1)s_1 + a_2, \; \frac{1}{3}(4^r - 1)s_2 + b_3 \right| \times \left| \frac{1}{3}(4^r - 1)s_1 + a_3, \; \frac{1}{3}(4^r - 1)s_2 + b_1 \right|}.$$

Solution by the PROPOSER ; Prof. NASH, M.A. ; *and others.*

$$\text{The area of } \Delta_0 = \frac{\frac{1}{2}\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}^2}{\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \times \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} \times \begin{vmatrix} a_3, & b_3 \\ a_1, & b_1 \end{vmatrix}}.$$

The equation of the external bisector of the angle between $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ is $(a_1 + a_2)x + (b_1 + b_2)y = c_1 + c_2$.

$$\text{Hence} \quad \Delta_1 = \frac{\frac{1}{2}\begin{vmatrix} s_1 - a_1, & s_2 - b_1, & s_3 - c_1 \\ s_1 - a_2, & s_2 - b_2, & s_3 - c_2 \\ s_1 - a_3, & s_2 - b_3, & s_3 - c_3 \end{vmatrix}^2}{\begin{vmatrix} s_1 - a_1, & s_2 - b_1 \\ \cdots & \cdots \quad \cdots \\ \cdots & \cdots \quad \cdots \end{vmatrix} \times \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} \times \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix}}.$$

Proceeding in the same way by substituting for any element the sum of the remaining elements in the same column, we have successively corresponding to a_1 the elements $s_1 + a_1$, $3s_1 - a_1$, $5s_1 + a_1$, $11s_1 - a_1$, $21s_1 + c_1$,

$43s_1 - a_1$, $85s_1 + e_1$. Now the series $1+3+5+11+21+43+\ldots$ may be written $1+(2^1+1)+(2^2+1)+(2^3+3)+(2^4+5)+\ldots$ So that any term $T_r = T_{r-2}+2^r$, $T_{r-2} = T_{r-4}+2^{r-2}$, $\therefore\ T_r = T_0+2^r+2^{r-2}+2^2 = \dfrac{4^r-1}{3}$.

$$\text{Hence}\qquad \Delta_{2r} = \frac{\tfrac{1}{2}\begin{vmatrix} \dfrac{4^r-1}{3}s_1+a_1, & \begin{matrix}\cdots\\ \cdots\end{matrix} \\ \cdots\quad\cdots & \cdots \end{vmatrix}^2}{\begin{vmatrix}\cdots\cdots\\ \cdots\cdots\end{vmatrix} \times \begin{vmatrix}\cdots\cdots\\ \cdots\cdots\end{vmatrix} \times \begin{vmatrix}\cdots\cdots\\ \cdots\cdots\end{vmatrix}}.$$

6953 & 7126. (By Professor Wolstenholme, M.A., D.Sc.)—(6953.) A circle is drawn with its centre O on the parabola $y^2 = 4ax$, and such that triangles can be inscribed in the parabola whose sides touch the circle: prove that (1) the radius of the circle is twice the normal to the parabola at O cut off by the axis; (2) the envelope of these circles consists of two distinct curves, one of which is the parabola $y^2 = 4a(x+4a)$, and the other is a quartic of the fourth class, whose equation may be written

$$2y^2+x^2-38ax-239a^2 = (x+21a)^{\frac{3}{2}}(x+5a)^{\frac{1}{2}};$$

(3) if the circle touch these curves in the points P, Q, the tangents at O, P, Q to their respective loci concur in a point which is the polar with respect to the parabola of the normal at O; and (4) if θ be the angle which the tangent at O makes with the axis, the tangents at P, Q will make angles $-\theta$, 3θ (or $3\theta \pm \pi$) with the axis. (The quartic envelope is the first negative pedal of the curve whose equation referred to the focus as pole is $r = 3a\sec\frac{1}{3}\theta$.)

(7126.) With a point O on the ellipse $a^2y^2+b^2x^2 = a^2b^2$ as centre is described a circle such that triangles can be circumscribed to the circle and inscribed in the ellipse; prove that (1) the envelope of such circles consists of two distinct curves, of which one is the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = \left(\dfrac{a^2+b^2}{a^2-b^2}\right)^2$, and the other is a curve of order 6, class 6, having 6 nodes (2 or 4 real), 6 bitangents (6 or 4 real), 4 cusps, and 4 inflexions (probably all impossible), so that its reciprocal has the same Plückerian numbers, and osculating the elliptic envelope in four points (where the normal and tangent are equally inclined to the axes). Also (2), if P, Q be the points of contact of the circle with these two curves, the tangents at O, P, Q will concur in one point, which is the polar with respect to the given ellipse of the normal at O; and if θ be the angle which the tangent at O makes with the axis, the tangents at P, Q will make angles $\pi-\theta$, 3θ with the axis. If $a^2 > 3b^2$, the maximum distance of Q from the centre is $(a^2+b^2)^{\frac{3}{2}} \div (a^2-b^2)$; (3) the radius of curvature at Q of the locus of Q is $\dfrac{a^2b^2}{p^3}\cdot\dfrac{4p^2+a^2+b^2}{3(a^2-b^2)}$, where p is the perpendicular from the centre on the tangent at O; and (4) trace the locus of Q when $b^2 = 2a^2, 3a^2, 4a^2, -2a^2, -3a^2, -5a^2$.

Solution by the Proposer.

(6953.) If we take the coordinates of O to be am^2, $2am$, then $m = \cot\theta$, and the equation of a circle with centre O is $(x-am^2)^2+(y-2am)^2 = r^2$;

and forming the discriminant of $k(y^2-4ax)+(x-am^2)^2+(y-2am)^2-r^2$,
we find $$\Delta = 4a^2,\ \Theta = 4a^2(1+m^2),\ \Theta' = r^2,\ \Delta' = r^2,$$
and the condition that triangles can be inscribed in the parabola whose sides touch the circle is $\Theta'^2 = 4\Theta\Delta'$, *i.e.*, $r^2 = 0$ or $r^2 = 16a^2(1+m^2)$. For real triangles the second is the only available value, and we see that, if OG be the normal at O terminated by the axis, the radius of the circle will be 2OG. But, if P be a point on the parabola $y^2 = 4a(x+4a)$ such that $x+4a = am^2$, $y = -2am$, and OM, PN be perpendicular to the axis; then, if OP meet the axis in T, we have $\text{MN} = 4a$, $\text{NT} = 2a$, or PO is the normal at P. Thus the parabola $y^2 = 4a(x+4a)$ touches every such circle and is part of the envelope. The other point of contact of the circle with its envelope will then be found by drawing PQ perpendicular to the tangent at O, and bisected by this tangent (for this tangent is the line joining consecutive centres of circles). We get then, for the coordinates of Q,

$$x+4a-am^2 = \frac{y+2am}{-m},\quad x-4a+am^2-m(y-2am)+2am^2 = 0;$$

whence $x+5a = \dfrac{a(3-m^2)^2}{1+m^2}$, $y = 2am\,\dfrac{3m^2-5}{1+m^2}$, also $x+21a = \dfrac{a(5+m^2)^2}{1+m^2}$,

from which equations may be obtained

$$2y^2+x^2-38ax-239a^2 = (x+21a)^{\frac{3}{2}}(x+5a)^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots\ldots (2).$$

The tangents to the circle at P, Q will intersect on the tangent at O, since this is the join of two consecutive centres; whence (3) the tangents at O, P, Q concur in one point, and this point (U) is given by the equations $$x-my+am^2 = 0,\quad x+4a+my+am^2 = 0,$$
that is, $$x+2a+am^2 = 0,\quad my+2a = 0,$$
whence the locus of U is $y^2(x+2a)+4a^3 = 0$. Also, (4) the tangents at O, P being equally inclined to the axis, that at P makes an angle $\pi-\theta$ with the axis, and the angle $\text{UQP} = \text{UPQ} = \text{POU} = \frac{1}{2}\pi-2\theta$, whence the angle which UQ, the tangent at Q, makes with the axis is $\frac{1}{2}\pi+\theta-(\frac{1}{2}\pi-2\theta)$ or 3θ. (Here θ has been assumed $<\frac{1}{4}\pi$; the angle may be 3θ or $3\theta\pm\pi$.)

The equation of the tangent at Q will then be

$$y+2a\tan\theta = \tan 3\theta\,(x+2a+a\cot^2\theta);$$

and the perpendicular on it from S, the focus of the given parabola, will be $$a(3+\cot^2\theta)\sin 3\theta-2a\tan\theta\cos 3\theta$$

$$\equiv \frac{2a}{\cos\theta}(\sin 3\theta\cos\theta-\cos 3\theta\sin\theta)+\frac{a\sin 3\theta}{\sin^2\theta} \equiv 4a\sin\theta+\frac{a\sin 3\theta}{\sin^2\theta} \equiv \frac{3a}{\sin\theta}.$$

Hence the equation of the pedal of the locus of Q with respect to S is $r = 3a\sec\frac{1}{3}\theta$; or the locus of Q is the first negative pedal of this curve, when S is pole.

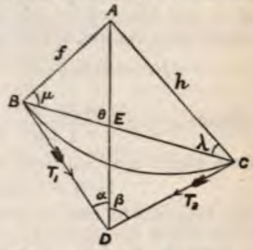
[The quartic has two cusps (impossible) when $m^2+5=0$, $x+21a=0$, $y^2+500a^2=0$; one crunode when $m^2=\frac{5}{3}$, $y=0$, $x=-\frac{13}{3}a$; one bitangent when $m^2=3$, $x=-5a$, $y^2=12a^2$. PLÜCKER's equations then prove that the class of the curve is 4, and that there are two inflexions which I suppose are impossible from tracing the curve. The curve and its reciprocal will therefore have all PLÜCKER's numbers the same for the two curves ($m=n=4$, $\delta=\tau=1$, $\kappa=\iota=2$). It would be interesting to receive an investigation as to whether any projection of this curve could be its own reciprocal.

(7126.) This is the corresponding Question for the *ellipse*, and an analogous investigation would prove the theorems in this Question.

7321. (By D. Edwardes.)—The extremities of a heavy uniform string are attached to the ends of a weightless bent lever, whose arms are at right angles to one another and of lengths f, h. If α, β, θ are the inclinations to the vertical, in the position of equilibrium, of the tangents to the string at its extremities and of the line joining its extremities, prove that
$$\cot\theta = \frac{f^2\cot\alpha - h^2\cot\beta}{h^2+f^2-hf(\cot\alpha+\cot\beta)}.$$

Solution by Dr. Curtis; W. H. Blythe, M.A.; *and others.*

If λ, μ, denote the angles opposite to f and h, respectively, in the triangle, whose vertex is at the fulcrum A, and whose sides are f and h, or AB and AC, while T_1, T_2 denote the tensions along BD, CD, the tangents at the extremities of the catenary in which the string hangs; then, for the equilibrium of the lever,



$$T_1\sin\alpha - T_2\sin\beta = 0,$$
and, taking moments round A,
$$T_1 f\sin(\theta-\alpha+\mu) = T_2 h\sin(\theta+\beta-\lambda),$$
or
$$f\sin\beta\sin(\theta-\alpha+\mu) = h\sin\alpha\sin(\theta+\beta-\lambda);$$
expanding $\sin(\theta-\alpha+\mu)$ and $\sin(\theta+\beta-\lambda)$, and remembering that
$$\sin\lambda = \cos\mu = \frac{f}{\sqrt{(f^2+h^2)}},\quad \cos\lambda = \sin\mu = \frac{h}{\sqrt{(f^2+h)^2}},$$
we easily obtain the result,
$$\cot\theta = \frac{f^2\cot\alpha - h^2\cot\beta}{f^2+h^2-fh(\cot\alpha+\cot\beta)} \quad\text{.......................}\quad (1),$$
It is plain, however, that this equation is insufficient to solve the problem, which involves *three* unknown quantities, α, β, and θ. Two other equations must consequently be obtained. Now, as the weight of the lever is neglected, it is necessary for equilibrium that the vertical through D, the line along which the weight of the catenary acts, should pass through the fulcrum A, therefore (*geometrically*)
$$\frac{f\sin(\theta+\mu)}{h\sin(\theta-\lambda)} = \frac{\text{BE}}{\text{CE}} = \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta},$$
or
$$\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{f}{h}\cot(\theta-\lambda) \quad\text{......................}\quad (2).$$

Again, if k be the parameter of the catenary, and y_1, y_2 the vertical ordinates of B and C,
$$\sqrt{(f^2+h^2)}\sin\theta = y_2-y_1 = k\left(\frac{1}{\sin\beta}-\frac{1}{\sin\alpha}\right) = k\,\frac{\sin\alpha-\sin\beta}{\sin\alpha\sin\beta},$$
while l, the given length of the string,
$$= k(\cot\alpha+\cot\beta) = k\,\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta},$$
therefore
$$\frac{\sqrt{(f^2+h^2)}\sin\theta}{l} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)} \quad\text{.........................}\quad (3).$$

Equations (1), (2), and (3) solve the problem. It is not difficult to discuss in a similar manner the question in which the weight of the lever (homogeneous or not) is considered, and the arms are inclined at any angle.

6925. (By Professor MATZ, M.A.)—Solve the equation

$$2\left[\log\left(1+\sin^2\theta\right)\right]^{\frac12} = \frac{a\left[2-\log\left(2-\cos^2\theta\right)\right]}{\left[1-\log\left(2-\cos^2\theta\right)\right]^{\frac12}}.$$

Solution by J. HAMMOND, M.A.; Professor COCHEZ; *and others.*

Writing $2-\log\left(2-\cos^2\theta\right) = 2x$, the equation is

$$(2-2x)^{\frac12} = \frac{ax}{(2x-1)^{\frac12}}, \quad \text{or} \quad a^4x^4-8x^3+20x^2-16x+4 = 0.$$

The completion of the square gives $(x^2-4x+2)^2 = (1-a^4)\,x^4$.........(1),

whence $x^2\left[1-(1-a^4)^{\frac12}\right]-4x+2 = 0$, or $x = \dfrac{2\pm\left[2+2\,(1-a^4)^{\frac12}\right]^{\frac12}}{1-(1-a^4)^{\frac12}}$.

But $\quad 2+2\,(1-a^4)^{\frac12} = \left[(1+a^2)^{\frac12}+(1-a^2)^{\frac12}\right]^2 = \alpha^2$, suppose.

Then $\quad x = \dfrac{2\,(2\pm\alpha)}{4-\alpha^2} = \dfrac{2}{2\pm\alpha} = \dfrac{2}{2\pm(1+a^2)^{\frac12}\pm(1-a^2)^{\frac12}}$.

Neglecting the ambiguities, which can be introduced again at pleasure,

$$x = \frac{2}{2+\alpha}, \quad \log\left(2-\cos^2\theta\right) = \frac{2\alpha}{2+\alpha}, \quad 2-\cos^2\theta = e^{\frac{2\alpha}{2+\alpha}}.$$

The simplest form of the final result is $\sin\theta = \left(e^{\frac{2\alpha}{2+\alpha}}-1\right)^{\frac12}$.

7395. (By R. TUCKER, M.A.)—If we have

$$\Delta \equiv \begin{vmatrix} ac^2, & ba^2, & cb^2 \\ ab^2, & bc^2, & ca^2 \\ \cos A, & \cos B, & \cos C \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} ac, & a^2, & bc \\ ab, & bc, & a^2 \\ \frac12, & \cos B, & \cos C \end{vmatrix},$$

where the elements involved are those of a plane triangle, prove that

$$2\Delta = (a^2+b^2+c^2)\,\Delta'.$$

Solution by G. G. MORRICE, B.A.; J. O'REGAN; *and others.*

Putting, in the last row of Δ, $\cos A = \dfrac{a\,(b^2+c^2-a^2)}{2abc}$, with two similar substitutions, and then subtracting the first two elements in each column from the third, we get

$$-\Delta = \begin{vmatrix} c^2, & b^2, & a^2 \\ a^2, & c^2, & b^2 \\ b^2, & a^2, & c^2 \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1, & b^2, & a^2 \\ 1, & c^2, & b^2 \\ 1, & a^2, & c^2 \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)\,;$$

$$2\Delta' = a^4-b^2c^2+2a^3(b\cos C+c\cos B)+2abc\,(b\cos B+c\cos C)\,;$$

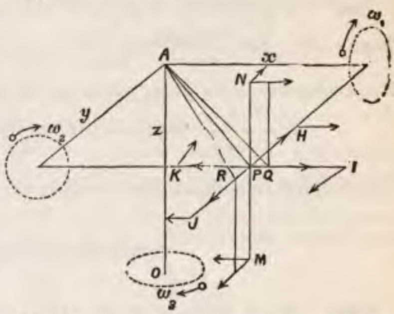
but $\quad 2abc\,.\,b\cos B = b^2(c^2+a^2-b^2),\quad 2abc\,.\,c\cos C = c^2(a^2+b^2-c^2)\,;$

therefore $2\Delta' = a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2$, therefore, &c.

7403. (By Professor Sylvester, F.R.S.)—From the principle of conservation of areas, deduce geometrically Euler's equations for the motion of a body revolving about a fixed point.

I. *Solution by* G. S. Carr, M.A.

Let P be an element of the body, O the fixed point and origin of coordinates, ω_1, ω_2, ω_3 the resolved angular velocities.



The impressed couple about the z axis of rotation is proportional to the acceleration in the description of areas about that axis by the point P. ω_3 contributes directly to this the area

$$\tfrac{1}{2}AP^2\frac{d\omega_3}{dt} = \tfrac{1}{2}(x^2+y^2)\frac{d\omega_3}{dt}$$

(not shown in the figure).

Let the velocity of P parallel to z and due to ω_1 be represented by $PN = y\omega_1$, and that due to ω_2 by $PM = x\omega_2$. The acceleration parallel to x, caused by PN and ω_2, is $y\omega_1\omega^2$ at N, which may be superposed at P and represented by PQ. Similarly the acceleration parallel to y, caused by PM and ω_1, is $x\omega_2\omega_1$ at M, represented by PR. The areas due to these accelerations are $+$APR and $-$APQ, that is, $\frac{1}{2}x^2\omega_2\omega_1$ and $-\frac{1}{2}y^2\omega_1\omega_2$.

Thus for the whole acceleration of the momentum round the z axis (referred to principal axes of the body), we have

$$\Sigma(x^2+y^2)\frac{d\omega_3}{dt}-\Sigma(y^2-x^2)\,\omega_1\omega_2,\quad\text{or}\quad C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2 = N,$$

the impressed couple; and this is Euler's third equation.

There are eight areas in addition to the foregoing due to accelerations indicated by the arrows. But two of these arising from accelerations at J and K are always equal and of opposite sign, while the rest disappear in the summation when principal axes are taken.

The entire twelve areas are as follows. The first factor of each is the perpendicular from A upon the base of the triangle, which base represents an acceleration of velocity. Four areas due to direct accelerations at P, parallel to the x and y axes (the double area is written in each case):—

$$xx\frac{d\omega_3}{dt},\quad yy\frac{d\omega_3}{dt},\quad -yz\frac{d\omega_2}{dt},\quad -xz\frac{d\omega_1}{dt}.$$

And eight areas due to accelerations at right angles to velocities, viz.:—

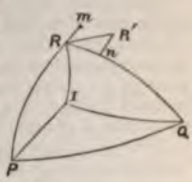
at M, $yx\omega_2^2$ and $xx\omega_2\omega_1$; at N, $-yy\omega_1\omega_2$ and $-xy\omega_1^2$;

at I, $xz\omega_2\omega_3$; at J, $yx\omega_3^2$; at K, $-xy\omega_3^2$; at H, $-yz\omega_1\omega_3$.

Putting A, B, C, F, G, H for the moments and products of inertia about the axes (when not principal ones), these areas give the known result

$$-G\frac{d\omega_1}{dt}-F\frac{d\omega_2}{dt}+C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2-H\,(\omega_1^2-\omega_2^2)-F\omega_3\omega_1+G\omega_2\omega_3=N.$$

II. *Solution by the* PROPOSER.



1. Suppose no forces acting on the body, then there will be an invariable line perpendicular to the resultant instantaneous axes. Of a sphere described about the fixed point, let P, Q, R be the intersections with the principal axes, and I with the invariable line at the time t.

Let IP $= \lambda$, IQ $= \mu$, IR $= \nu$, and let p, q, r be the angular velocities, and A, B, C the moments of inertia in respect to P, Q, R. Take R$m = qdt$, R$n = pdt$, and in the time dt let R get to R′ and call IR′ $= \nu'$; then

$$\frac{Ap}{\cos\lambda}=\frac{Bq}{\cos\mu}=\frac{Cr}{\cos\nu}\quad\text{and}\quad\frac{Cr'}{\cos\nu'}=\frac{Cr}{\cos\nu}=G,$$

the constant angular momentum; hence we have

$$\delta r = 0-\frac{r\sin\nu}{\cos\nu}\,\delta\nu=-\frac{r\sin\nu}{\cos\nu}\,(\mathrm{R}m\cos\mathrm{IRP}-\mathrm{R}n\cos\mathrm{IRQ})$$

$$=-dt\,\frac{r\sin\nu}{\cos\nu}\left(q\,\frac{\cos\lambda}{\sin\nu}-p\,\frac{\cos\mu}{\sin\nu}\right)=dt\left(p\,\frac{Bq}{C}-q\,\frac{Ap}{C}\right).$$

Thus we have $\qquad Cdr=(B-A)\,pqdt;$

and, similarly, $\quad Bdq=(A-C)\,rpdt,\quad Adp=(C-B)\,qrdt.$

2. When external forces are expressed whose moments about P, Q, R are L, M, N, the angular motions of P, Q, R in the time dt will be those due to the motion of the body in the preceding instant of time acted on by no forces during the time dt, *plus* those due to the actions of the forces in the time dt on the body at rest before they begin to act, and consequently the equations become

$$Cdr=(B-A)\,pqdt+N,\quad Bdq=(A-C)\,rpdt+M,\quad Adp=(C-B)\,qrdt+L.$$

6897. (By Professor TOWNSEND, F.R.S.)—An equiangular spiral or spherical surface being supposed the frictionless catenary of a uniform cord, or the frictionless trajectory of a material particle, constrained to rest or move on the surface, under the action of a force passing perpendicularly in every position through the axis of the spiral; show that the force varies, for the catenary inversely as the square, and for the trajectory inversely as the cube, of the distance from the axis.

Solution by the Proposer.

Denoting by α the constant angle of intersection of the spiral with all the meridians of the surface that pass through its poles, by T the current tension of maintenance at any point of the catenary, by V the current velocity of description at any point of the trajectory, by ρ the current distance of the point from the axis of the spiral, and by F the current force per unit of length or of mass at the point; then since, on elementary dynamical principles, for the catenary $dT = -F\,d\rho$ and $T\rho\sin\alpha = \text{const.} = h$, and for the trajectory $VdV = Fd\rho$ and $V\rho\sin\alpha = \text{const.} = k$, therefore at once $F = h\operatorname{cosec}\alpha\,.\,\rho^{-2}$ in the former case, and $= -k^2\operatorname{cosec}^2\alpha\,.\,\rho^{-3}$ in the latter case; and therefore, &c.

Since manifestly, as in the corresponding well-known cases of equilibrium and of motion in a plane, the tension of maintenance T in the former case is that to infinity, and the velocity of description V in the latter case is that from infinity, under the action of the force; hence, conversely, a uniform cord or material particle, constrained to rest or move without friction on a spherical surface with the tension of maintenance or the velocity of description to or from infinity, under the action of a repulsive force varying inversely as the square, or of an attractive force varying inversely as the cube of the distance from a fixed diameter of the sphere, through which it passes perpendicularly in every position, will have for its catenary or trajectory an equiangular spiral of the surface of which the diameter is the axis.

Denoting by μ the absolute force of the repulsion or attraction, it appears at once from the above that $\mu = h\operatorname{cosec}\alpha$ for the catenary, and $= k^2\operatorname{cosec}^2\alpha$ for the trajectory. Hence, for different equiangular spirals about the same axis of force, whether belonging or not to the same spherical surface having its centre on the axis, the constants h and k, as for the corresponding cases of equilibrium and motion in a plane, vary as $\sin\alpha$ in both cases alike.

7036. (By R. Tucker, M.A.)—Find (1) the maximum triangle, inscribed in an ellipse, two of whose sides pass through the foci; and show (2) that in this case when the excentricity equals $\frac{1}{3}\sqrt{5}$, the angle between the focal chord is 60°.

Solution by J. S. Jenkins, M.A.; Sarah Marks; *and others.*

Let ABC be the triangle, the two sides of which pass through the foci, and let the coordinates of C be (x', y'); then, since the sides pass through the foci, we readily obtain the following equations:—

$$AC = \frac{2(a+\epsilon x')^2}{(a+a\epsilon^2)^2+2\epsilon x'},\quad BC = \frac{2(a-\epsilon x')^2}{(a+a\epsilon^2)^2-2\epsilon x'},\quad \sin ACB = \frac{2b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}}{a^2-\epsilon^2x'^2};$$

whence Area $\triangle ABC = \dfrac{4b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}\,(a^2-\epsilon^2x'^2)}{(a+a\epsilon^2)^2-4\epsilon x'^2}$, and by differentiating we find this expression to be a maximum when $x'^2 = \dfrac{3a^2\epsilon^4+6a^2\epsilon^2-a^2}{8\epsilon^2}$, therefore the maximum area of the triangle ABC is $\dfrac{3b^2}{2\sqrt{2}\,.\,a}\,(4a^2-3b^2)^{\frac{1}{2}}$.

Again, $\sin ACB = \dfrac{2b\epsilon\,(a^2-x'^2)^{\frac{1}{2}}}{a^2-\epsilon^2x'^2} = \dfrac{\sqrt{3}}{2}$ (when $\epsilon = \frac{1}{5}\sqrt{5}$) $= \sin 60^\circ$.

7411. (By C. Leudesdorf, M.A.)—S is the focus, A the vertex, of the parabola $y^2 = 4ax$. A conic has double contact with the parabola and also with the circle on SA as diameter; prove that its director circle will envelope the curve $y^2(16x+25a) = 4(x+a)(a^2+4ax-4x^2)$.

Solution by D. Edwardes; R. Knowles, B.A.; *and others.*

The equation of a conic having double contact with the curves in question will be $\mu^2x^2-2\mu(2y^2+x^2-5ax)+(x+3a)^2 = 0$. Writing down the equation of a pair of tangents from x, y: equating to zero the sum of the coefficients of x^2 and y^2, the equation of the director circle becomes

$$(\mu-1)^2(x^2+y^2)+(\mu-1)(10ax+4a^2)+16ax+25a^2 = 0,$$

whose envelope is $a(5x+2a)^2 = (x^2+y^2)(16x+25a)$, which reduces to the stated form.

7409. (By W. S. M'Cay, M.A.)—Two circles A, B are inverted from an origin O into two circles A′, B′; if O be on a polar with respect to A or B of either of their centres of similitude, prove that after inversion O will still be on a polar with respect to B′ or A′ of one of their centres of similitude.

Solution by G. B. Mathews, B.A.; Kate Gale; *and others.*

Let the circles be $(x-d)^2+y^2 = r^2$, $(x-d')^2+y^2 = r'^2$............(A, B). The polar of the origin with regard to (A) is $dx = d^2-r^2$; and if this goes through the centre of similitude

$$d\,.\,\frac{r'd+rd'}{r+r'} = d^2-r^2,\quad rdd' = rd^2-(r+r')\,r^2;$$

therefore $r(r+r') = d(d-d')$ or $d^2-r^2 = rr'+dd'$......................(1).

Now, if $\delta\rho$, $\delta'\rho'$ refer to the inverse circles, it is easily seen, geometrically,

that $\delta = \frac{1}{2}\left(\dfrac{k^2}{d-r}+\dfrac{k^2}{d+r}\right) = \dfrac{k^2d}{d^2-r^2}$, $\rho = \frac{1}{2}\left(\dfrac{k^2}{d-r}-\dfrac{k^2}{d+r}\right) = \dfrac{k^2r}{d^2-r^2}$;

and similarly, $\delta' = \dfrac{k^2 d'}{d'^2 - r'^2},\quad \rho' = \dfrac{k^2 r'}{d'^2 - r'^2};$

therefore $\rho\rho' + \delta\delta' = \dfrac{k^4 (rr' + dd')}{(d^2 - r^2)(d'^2 - r'^2)} = \dfrac{k^4}{d'^2 - r'^2}$ by (1) $= \delta'^2 - \rho'^2$,

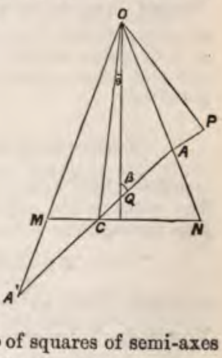
an equation of the same form as (1) with accented letters for unaccented, and *vice versâ*: hence the theorem.

[The Proposer remarks that the theorem *must* be true if A, B are orthogonal, for the tangents from O are then harmonic (Townsend's *Modern Geometry*, Vol. I, p. 287, and *Educational Times* for June, 1878), and, being unchanged by inversion, O must also be on the locus of points from which tangents to A′, B′ are harmonic. In looking for a geometric proof (which is not difficult) of the theorem, it turned out to be true for any two circles. It appears also, more generally, that for any position of O, the figure formed by O, the two centres of similitude S, S′ of A, B, and the two inverses of S, S′ to A and B, preserves its species (but in opposite cyclical order) after inversion.]

7412. (By J. J. Walker, M.A., F.R.S.)—The sides of a right cone make an angle α with the axis; prove that the locus of centres of sections by planes making with the axis an angle β is a coaxial right cone generated by a line through the vertex, and inclined to the axis at an angle equal to $\tan^{-1}\tan^2\alpha\cot\!an\,\beta$; also that the ratio of the axes of such a section is $[\sin(\alpha+\beta)\sin(\alpha-\beta)]^{\frac{1}{2}}\sec\alpha$; and that, if p is the perpendicular distance of the plane of the section from the vertex of the cone, then the distance of the centre from the foot of p is equal to

$$p\sin\beta\cos\beta \,/\, \sin(\alpha+\beta)\sin(\alpha-\beta).$$

Solution by the Rev. T. C. Simmons, M.A.; Prof. Nash, M.A.; *and others.*



1. Let C be the centre of the section by a plane through AA′; then we have

$$\frac{2\,CQ}{OQ} = \frac{A'Q}{OQ} - \frac{AQ}{OQ},$$

$$\frac{2\sin\theta}{\sin(\beta-\theta)} = \frac{\sin\alpha}{\sin(\beta-\alpha)} - \frac{\sin\alpha}{\sin(\beta+\alpha)}$$

$$= \frac{2\sin^2\alpha\cos\beta}{\sin(\beta-\alpha)\sin(\beta+\alpha)},$$

$$\frac{\sin(\beta-\theta)}{\sin\theta} = \frac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha\cos\beta};$$

from which we obtain

$$\beta\cot\theta = \frac{\sin^2\beta\cos^2\alpha}{\cos\beta\sin^2\alpha} \text{ or } \tan\theta = \tan^2\alpha\cot\beta.$$

2. Draw MCN perpendicular to axis; then ratio of squares of semi-axes

$$= \frac{MC}{A'C}\cdot\frac{CN}{AC} = \frac{\sin(\beta-\alpha)}{\cos\alpha}\cdot\frac{\sin(\beta+\alpha)}{\cos\alpha}.$$

3. Draw OP perpendicular to A′CA; then we have

$$=\mathrm{CP} = \mathrm{A'P} - \frac{\mathrm{AA'}}{2} = p\cot(\beta-\alpha) - \frac{\mathrm{OA}\sin 2\alpha}{2\sin(\beta-\alpha)}$$

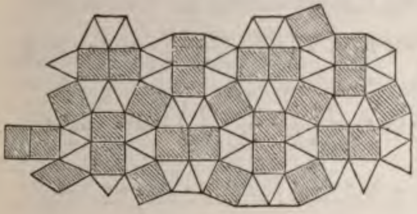
$$= p\cot(\beta-\alpha) - \frac{p}{2}\cdot\frac{\sin 2\alpha}{\sin(\beta+\alpha)\sin(\beta-\alpha)}$$

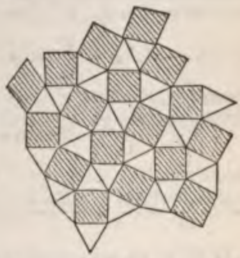
$$= \frac{p}{2}\cdot\frac{2\cos(\beta-\alpha)\sin(\beta+\alpha)-\sin 2\alpha}{\sin(\beta+\alpha)\sin(\beta-\alpha)} = \frac{p\sin\beta\cos\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}.$$

[The above is the correct formula for determining the figure and position of the "Iris seen in water" (*Phil. Mag.*, 1854).]

6904. (By the Rev. W. A. WHITWORTH, M.A.)—Required patterns to cover an area with black square tiles, and white equilateral triangular tiles, the side of the square, and the side of the triangle being equal, and the pattern regular; (1) using 2 triangles to 1 square, (2) using 7 triangles to 3 squares.

Solution by the PROPOSER.





The required patterns are as clearly shown in the annexed figures, which need no explanation.

7406. (By Professor HUDSON, M.A.)—Two inclined planes, of the same altitude and inclinations α, β, are placed back to back with an interstice between them. Two weights P, Q are placed one on each plane at the bottom, and connected by a string which passes over two small smooth pulleys at the top and under a movable pulley, weight W, which hangs between the two planes, the free portion of the string being parallel. Find the least value of W, in order that both weights may be drawn up; and, if they arrive at the top at the same time, prove that

$$\frac{4(\sin^2\alpha-\sin^2\beta)}{\mathrm{W}} = \frac{2\sin\alpha+\sin\alpha\sin\beta+\sin^2\beta}{\mathrm{P}} = \frac{2\sin\beta+\sin\alpha\sin\beta+\sin^2\alpha}{\mathrm{Q}}.$$

I. *Solution by* R. RAWSON; G. B. MATHEWS, B.A.; *and others.*

Let T = the tension of the string, then the moving forces of W, P, Q are $W-2T, \quad T-P\sin\alpha, \quad T-Q\sin\beta.$

Put x, y for the spaces moved through by P, Q respectively in (t) seconds; then $\frac{1}{2}(x+y)$ will evidently be the space moved through by W in the same time. The differential equations of motion are, therefore, as follows:

$$\frac{d^2x}{dt^2} = g\left(\frac{T}{P} - \sin\alpha\right), \quad \frac{d^2y}{dt^2} = g\left(\frac{T}{Q} - \sin\beta\right) \text{ ....(1, 2)},$$

$$\frac{d^2(x+y)}{dt^2} = 2g\left(1-\frac{2T}{W}\right) \text{ ....(3)}.$$

From these we obtain $$T = \frac{2+\sin\alpha+\sin\beta}{\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}} \text{ ....(4)}.$$

Integrating (1), (2), (3), we obtain

$$x = \frac{g}{2}\left(\frac{T}{P}-\sin\alpha\right)t^2, \quad y = \frac{g}{2}\left(\frac{T}{Q}-\sin\beta\right)t^2 \text{.........(5, 6)},$$

$$x+y = g\left(1-\frac{2T}{W}\right)t^2 \text{..........................(7)}.$$

The constants of integration are zero, since the motions of P, Q, W are zero when $t=0$. If, therefore, P, Q arrive at the top at the same time,

$$\frac{T}{P}\sin\alpha - \sin^2\alpha = \frac{T}{Q}\sin\beta - \sin^2\beta, \text{ or } T\left(\frac{\sin\alpha}{P} - \frac{\sin\beta}{Q}\right) = \sin^2\alpha - \sin^2\beta.$$

From (4),

$$\left(\frac{\sin\alpha}{P} - \frac{\sin\beta}{Q}\right)(2+\sin\alpha+\sin\beta) = \left(\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}\right)(\sin^2\alpha - \sin^2\beta);$$

therefore

$$\frac{4(\sin^2\alpha-\sin^2\beta)}{W} = \frac{2\sin\alpha+\sin\alpha\sin\beta+\sin^2\beta}{P} - \frac{2\sin\beta+\sin\alpha\sin\beta+\sin^2\alpha}{Q}.$$

If W be strong enough to draw P, Q up the inclined plane, we have

$$W > 2T, \text{ or } W > \frac{2PQ}{P+Q}(\sin\alpha+\sin\beta).$$

II. *Solution by* D. EDWARDES.

Since, in any the same interval, the space passed over by W is half the sum of the spaces described by P and Q (the free portions being parallel), therefore at any instant the velocity of W is half the sum of the velocities of P and Q. Hence the acceleration of W is half the sum of the accelerations of P and Q, that is,

$$2\frac{W-2T}{W} = \frac{T-P\sin\alpha}{P} + \frac{T-Q\sin\beta}{Q};$$

whence $T = \dfrac{2+\sin\alpha+\sin\beta}{\frac{1}{P}+\frac{1}{Q}+\frac{4}{W}}$, subject to the condition that T is greater

than the greatest of the quantities P sin α and Q sin β, or W greater than the greatest of the values $\frac{4PQ\sin\alpha}{(2+\sin\beta)Q-P\sin\alpha}$, $\frac{4PQ\sin\beta}{(2+\sin\alpha)P-Q\sin\beta}$.
Let t be the whole time of motion. Then $\frac{2s}{f}=\frac{2s'}{f'}=t^2$, or $f\sin\alpha=f'\sin\beta$; and, substituting for f and f' their values, viz.,

$$g\,\frac{T-P\sin\alpha}{P} \text{ and } g\,\frac{T-Q\sin\beta}{Q},$$

we have the required equation.

7417. (By R. Russell, B.A.)—Show that $A_1, A_2 \dots A_{2n}$ can be found such that, if a certain invariant relation holds between $a_1, a_2 \dots a_{2n}$,

$$A_1(x-a_1)^{2n}+A_2(x-a_2)^{2n}+\dots+A_{2n}(x-a_{2n})^{2n} \equiv P(x-a_1)(x-a_2)\dots(x-a_{2n}).$$

Solution by G. B. Mathews, B.A.; Sarah Marks; *and others.*

Since the left-hand side is rational and homogeneous of degree $2n$ in x, it is sufficient to make it vanish when $x = a_1\, a_2 \dots a_{2n}$ respectively: hence

$$A_2(a_1-a_2)^{2n}+A_3(a_2-a_3)^{2n}+\dots=0,$$
$$A_1(a_2-a_1)^{n}+\dots+A_5(a_2-a_5)^{2n}+\dots=0;$$

whence the invariant relation

$$\begin{vmatrix} 0 & (a_1-a_2)^{2n} \dots (a_1-a_{2n})^{2n} \\ (a_2-a_1)^{2n} & 0 \quad \dots \quad \dots \quad \dots \\ (a_3-a_1)^{2n} & \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ 1 & \end{vmatrix} = 0,$$

and the A's are proportional to first minors of this determinant.

7407. (By Professor Wolstenholme, M.A.)—Prove that the three conics $x^2+ay=a^2$, $x^2-y^2=ax$, $y^2-xy=a^2$ have three common points

$$\frac{x}{\sin\frac{2}{7}\pi}=\frac{y}{\sin\frac{3}{7}\pi}=\frac{-a}{\sin\frac{1}{7}\pi},\quad \frac{-x}{\sin\frac{1}{7}\pi}=\frac{y}{\sin\frac{2}{7}\pi}=\frac{a}{\sin\frac{3}{7}\pi},\quad \frac{x}{\sin\frac{3}{7}\pi}=\frac{-y}{\sin\frac{1}{7}\pi}=\frac{a}{\sin\frac{2}{7}\pi};$$

the other common points of them, taken two and two, being $x=y=\infty$; $x=0,\ y=a$; $y=0,\ x=a$.

Solution by R. Rawson; G. B. Mathews, B.A.; *and others.*

Put $x = ax_1$ and $y = ay_1$ in each conic; then we have

$$x_1^2+y_1=1,\quad x_1^2-y_1^2=x_1,\quad y_1^2-x_1y_1=1 \quad\dots\dots\dots\dots\dots(1).$$

The elimination of x_1, y_1 respectively from the 1st and 2nd, 1st and 3rd, and 2nd and 3rd conics gives, respectively,

$$(x_1-1)(x_1^3+x_1^2-2x_1-1)=0,\quad y_1(y_1^3+2y_1^2-y_1-1)=0 \dots\dots(2,3),$$
$$x_1(x_1^3+x_1^2-2x_1-1)=0,\quad (y_1-1)(y_1^3+2y_1^2-y_1-1)=0 \dots\dots(4,5),$$
$$(x_1^3+x_1^2-2x_1-1)=0,\quad (y_1^3+2y_1^2-y_1-1)=0 \dots\dots\dots\dots(6,7).$$

From these equations it is readily inferred that the conics have three common points which are determined by the three roots of the cubics (6) and (7); the other common points, taken two and two, being obvious from the same equations.

From (6) and (7) it follows that $y_1 x_1 = 1$. It now remains to find the roots of the cubic (6). In order to find the roots as given by the learned proposer, it will be necessary to deviate slightly from the ordinary method of solving cubic equations by trigonometrical tables.

If $7\theta = \pi$, then $\sin\theta\left\{64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1\right\} = 0$ ....(8), the roots of which are $\pi, \frac{1}{7}\pi, \frac{2}{7}\pi, \frac{3}{7}\pi, \ldots \frac{6}{7}\pi$ (see HYMERS' *Trig.*, p. 89). Equation (8) is by the relation $\cos 2\theta = 2\cos^2\theta - 1$ readily transformed into $(2\cos 2\theta)^3 + (2\cos 2\theta)^2 - 2(2\cos 2\theta) - 1 = 0$ ....(9).
Compare (9) with (6), the three roots of which are, therefore,

$$2\cos 2\theta, \quad 2\cos 4\theta, \quad 2\cos 6\theta.$$

Hence $$x = 2a\cos\tfrac{2}{7}\pi = a\,\frac{2\sin\frac{2}{7}\pi\cos\frac{2}{7}\pi}{\sin\frac{2}{7}\pi} = a\,\frac{\sin\frac{4}{7}\pi}{\sin\frac{2}{7}\pi} = \frac{a\sin\frac{3}{7}\pi}{\sin\frac{2}{7}\pi}.$$

The corresponding value of y is obtained as follows:—

$$y = a(1-x_1)(1+x_1) = -a\left(\frac{\sin\frac{4}{7}\pi - \sin\frac{2}{7}\pi}{\sin\frac{2}{7}\pi}\right)\left(\frac{\sin\frac{4}{7}\pi + \sin\frac{2}{7}\pi}{\sin\frac{2}{7}\pi}\right)$$
$$= -a\,\frac{2\cos\frac{3}{7}\pi\sin\frac{1}{7}\pi\; 2\sin\frac{3}{7}\pi\cos\frac{1}{7}\pi}{\sin^2\frac{2}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi}{\sin\frac{2}{7}\pi} = -\frac{a\sin\frac{1}{7}\pi}{\sin\frac{2}{7}\pi}.$$

Again, $$x = 2a\cos\tfrac{4}{7}\pi = -2a\cos\tfrac{3}{7}\pi$$
$$= -a\,\frac{2\sin\frac{3}{7}\pi\cos\frac{3}{7}\pi}{\sin\frac{3}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi}{\sin\frac{3}{7}\pi} = -\frac{a\sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}.$$

The corresponding value of y is

$$y = a(1-x_1)(1+x_1) = a\left(\frac{\sin\frac{3}{7}\pi + \sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)\left(\frac{\sin\frac{3}{7}\pi - \sin\frac{1}{7}\pi}{\sin\frac{3}{7}\pi}\right)$$
$$= a\,\frac{2\sin\frac{2}{7}\pi\cos\frac{1}{7}\pi\; 2\cos\frac{2}{7}\pi\sin\frac{1}{7}\pi}{\sin^2\frac{3}{7}\pi} = a\,\frac{\sin\frac{2}{7}\pi}{\sin\frac{3}{7}\pi}.$$

Again, $$x = 2a\cos\tfrac{6}{7}\pi = -2a\cos\tfrac{1}{7}\pi = -a\,\frac{2\sin\frac{1}{7}\pi\cos\frac{1}{7}\pi}{\sin\frac{1}{7}\pi} = -\frac{a\sin\frac{2}{7}\pi}{\sin\frac{1}{7}\pi}.$$

The corresponding value of y is

$$y = a(1-x_1)(1+x_1) = -a\left(\frac{\sin\frac{5}{7}\pi + \sin\frac{1}{7}\pi}{\sin\frac{1}{7}\pi}\right)\left(\frac{\sin\frac{5}{7}\pi - \sin\frac{1}{7}\pi}{\sin\frac{1}{7}\pi}\right)$$
$$= -a\,\frac{2\sin\frac{3}{7}\pi\cos\frac{2}{7}\pi\; 2\cos\frac{3}{7}\pi\sin\frac{2}{7}\pi}{\sin^2\frac{1}{7}\pi} = -a\,\frac{\sin\frac{6}{7}\pi\sin\frac{4}{7}\pi}{\sin^2\frac{1}{7}\pi} = -a\,\frac{\sin\frac{3}{7}\pi}{\sin\frac{1}{7}\pi},$$

which proves the beautiful property stated in the question.

7368. (By S. TEBAY, B.A.)—Prove the following formula for finding the Dominical or Sunday letter for any given year (given in WOOLHOUSE's excellent little manual on the weights and measures of all nations, in *Weale's Series*)—$L = 2(\frac{1}{4}c)_r + 2(\frac{1}{4}y)_r + 4(\frac{1}{7}y)_r + 1$ (rejecting sevens); where c is the number of completed centuries, and y the years of the current century; the suffix r indicating *remainder* after each division.

Solution by the PROPOSER.

The Julian year contains 365·25 days, and the Gregorian year 365·2425 days; the difference is ·0075 $= \frac{3}{400}$ day, which will amount to 3 days in 400 years. Hence in any proposed year it is only necessary to consider the remainder after division by 400.

The year $Y = 100c + y$, divided by 400, leaves remainder $100\left(\frac{c}{4}\right)_r + y$.

Since $365 = 7 \times 52 + 1$, omitting 7's, we have $2\left(\frac{c}{4}\right)_r + \left(\frac{y}{7}\right)_r$ days.

The number of Julian leap-years is $25\left(\frac{c}{4}\right)_r + \frac{1}{4}\left\{y - \left(\frac{y}{4}\right)_r\right\}$; but the centuries represented by $\left(\frac{c}{4}\right)_r$ are not leap-years in the Gregorian calendar. Therefore the number of leap-years is

$$24\left(\frac{c}{4}\right)_r + \frac{1}{4}\left\{y - \left(\frac{y}{4}\right)_r\right\};$$

whence, adding this to the above, and omitting sevens, the remaining days are

$$5\left(\frac{c}{4}\right)_r + \left(\frac{y}{7}\right)_r + \frac{1}{4}\left\{\left(\frac{y}{7}\right)_r - \left(\frac{y}{4}\right)_r\right\}.$$

Since the first day of the year 1 is Sunday, if we deduct 1 from the above expression, and cast out sevens, the remainder is the number of days after the last Sunday in the year Y. This remainder, taken from 7, gives the Dominical number. Hence, subtract 1, and to avoid fractions multiply by 8, and cast out sevens; the remaining days are

$$5\left(\frac{c}{4}\right)_r - 2\left(\frac{y}{4}\right)_r + 3\left(\frac{y}{7}\right)_r - 1,$$

which, being taken from $7 + 7\left(\frac{c}{4}\right)_r + 7\left(\frac{y}{7}\right)_r$,

gives $\quad L = 2\left(\frac{c}{4}\right)_r + 2\left(\frac{y}{4}\right)_r + 4\left(\frac{y}{7}\right)_r + 1$ (omitting sevens).

7291. (By D. EDWARDES.)—If the radii of the escribed circles of a triangle are the roots of $x^3 - px^2 + qx - t = 0$, prove that the radii of the escribed circles of its orthocentric triangle are the roots of

$$(pq-t)^3x^3 - 2(pq-t)^2(q^2-pt)x^2 + 16qt^2(pq-t)x - 8t^2[4q^3 - (pq+t)^2] = 0.$$

Solution by the PROPOSER; R. KNOWLES, B.A.; *and others.*

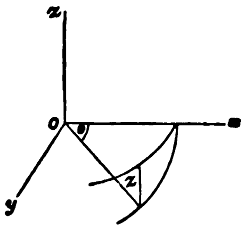
We have $\quad 4R = \frac{pq-t}{q}, \quad \Delta = tq^{-\frac{1}{2}}, \quad$ and since $\Delta = R\sigma$,

therefore $$\Sigma\rho_1\rho_2 = \sigma^2 = \frac{16qt^2}{(pq-t)^2}.$$

6870. (By D. EDWARDES.)—A particle under no forces is projected with velocity V along a rough helix; prove that it makes the first n complete revolutions in the time $\dfrac{a}{\mu V \cos^2 a}\left(e^{2\mu n\pi \cos a}-1\right)$, a being the pitch of the screw, and a radius of cylinder upon which the helix could be drawn.

Solution by G. M. REEVES, M.A.; SARAH MARKS; *and others.*

If k be normal pressure and μk the friction, we get (mass of particle being unity)



$$a\frac{d^2\theta}{dt^2}\cdot\tan a=-\mu k\sin a,\ \text{and}\ a\left(\frac{d\theta}{dt}\right)^2=k;$$

therefore $$\frac{d^2\theta}{dt^2}+\left(\frac{d\theta}{dt}\right)^2=-\mu\cos a;$$

therefore $$-\frac{1}{\dfrac{d\theta}{dt}}=C-\mu\cos a\,.\,t,$$

$$C=-\frac{a}{V\cos a}\quad\because\ \frac{d\theta}{dt}=\frac{V\cos a}{a}\ \text{initially};$$

therefore $$\frac{1}{\dfrac{d\theta}{dt}}=\frac{a}{V\cos a}+\mu\cos a\,.\,t,\quad \frac{dt}{d\theta}=\frac{a+\mu V\cos^2 a\,.\,t}{V\cos a};$$

therefore $$\frac{1}{\mu\cos a}\cdot\log\left(a+\mu V\cos^2 a\,.\,t\right)=\theta+C.$$

Taking $\theta=0$, when $t=0$, $C=\dfrac{\log a}{\mu\cos a}$,

$$\log\left(a+\mu V\cos^2 a\,.\,t\right)=\mu\cos a\,.\,\theta+\log a,\quad 1+\frac{\mu V\cos^2 a}{a}\,.\,t=e^{\mu\cos a\,.\,\theta};$$

therefore $$t=\frac{a}{\mu V\cos^2 a}\cdot\left(e^{\mu\theta\cos a}-1\right),$$

and time of describing the angle $2n\pi$ is

$$\frac{a}{\mu V\cos^2 a}\cdot\left(e^{2n\pi\cos a}-1\right).$$

7276. (By S. TEBAY, B.A.)—A substance P, suspended from one extremity of a lever, is balanced by a weight Q at the other end, or by a weight Q′ from a second fulcrum: find P, and show that there are two values (P, P′) such that PP′ = QQ′; also, if a be the length of the lever, and a/p the distance between the two fulcrums,

$$Q=(p-1)^2m^2-n^2,\quad Q'=(p+1)^2m^2-n^2,\quad P=(pm\pm n)^2-m^2;$$

m, n being any integers prime to one another.

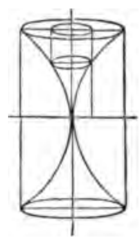
Solution by R. KNOWLES, B.A.; E. J. HENCHIE; *and others.*

The points A′, B′, C′ are evidently the mid-points of EF, FD, DE; therefore $\Delta A'B'C' = \frac{1}{4}\Delta DFE = \frac{1}{4}(ODE + ODF + OFE)$

$$= \tfrac{1}{8}r^2 (\sin A + \sin B + \sin C) = \tfrac{1}{2}r^2 \cos\tfrac{1}{2}A \cos\tfrac{1}{2}B \cos\tfrac{1}{2}C.$$

7408. (By the EDITOR.)—If a portion of the parabola $y^2 = 4ax$ cut off by the terminal ordinate c, revolve around the tangent at the vertex, show that the volumes of (1) the solid thus generated, and (2) the greatest cylinder that can be cut therefrom, are $\frac{\pi}{40}\frac{c^5}{a^2}$, $\frac{16\pi}{3125}\frac{c^5}{a^2}$.

Solution by G. B. MATHEWS, M.A.; Prof. MATZ, M.A.; *and others.*



1. Volume $= 2\int_0^c \pi x^2 dy = \frac{\pi}{8a^2}\int_0^c y^4\, dy = \frac{\pi c^5}{40a^2}$.

2. Constructing a cylinder as in figure, its volume

$$= \pi x^2 (c-y) = \frac{\pi}{16a^2} y^4 (c-y);$$

making this a maximum, we have $y^3(4c-5y) = 0$, whence $y = \frac{4}{5}c$,

and volume $= \frac{\pi}{16a^2} \cdot \frac{4^4}{5^5} \cdot c^5 = \frac{16\pi}{3125}\frac{c^5}{a^2}$.

5682. (By E. W. SYMONS.)—A series of triangles are inscribed in an ellipse so that their orthocentres coincide with the centre of the ellipse; find (1) the locus of their centroids; and (2) prove that the normals at the vertices generally meet in a point.

Solution by D. EDWARDES; J. A. KEALY, M.A.; *and others.*

Let α, β, γ be the excentric angles of the corners of one such triangle; then, since the perpendicular through α to the chord through β, γ passes through the centre, we have $b^2 \sin\alpha = a^2 \tan\frac{1}{2}(\beta+\gamma)\cos\alpha$,

similarly $b^2 \sin\beta = a^2 \tan\frac{1}{2}(\gamma+\alpha)\cos\beta$ .... (A);

whence, dividing and reducing,

$$\sin\tfrac{1}{2}(\alpha-\beta)\,[\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)] = 0;$$

hence, generally, the normals at α, β, γ meet in a point. Again the equation A gives $(a^4+a^2b^2)\cos\alpha\cos\beta+(b^4+a^2b^2)\sin\alpha\sin\beta+a^2b^2=0$.

In the same way, by eliminating β between two such equations,

$$(a^4+a^2b^2)\cos\alpha\cos\gamma+(b^4+a^2b^2)\sin\alpha\sin\gamma+a^2b^2=0.$$

Hence β, γ are the roots of

$$(a^4+a^2b^2)\cos\alpha\cos\theta+(b^4+a^2b^2)\sin\alpha\sin\theta+a^2b^2=0;$$

therefore $$\frac{\cos\frac{1}{2}(\beta+\gamma)}{(a^4+a^2b^2)\cos\alpha}=\frac{\sin\frac{1}{2}(\beta+\gamma)}{(b^4+a^2b^2)\sin\alpha}=\frac{\cos\frac{1}{2}(\beta-\gamma)}{-a^2b^2}\;\ldots\ldots\ldots\ldots(\text{B}).$$

Now if (x, y) are the coordinates of the centroid, by (B),

$$\frac{3x}{a}=\cos\alpha+\cos\beta+\cos\gamma=\cos\alpha-\frac{2a^4b^2\cos\alpha}{(a^2+b^2)(a^4\cos^2\alpha+b^4\sin^2\alpha)};$$

$$\frac{3y}{b}=\sin\alpha+\sin\beta+\sin\gamma=\sin\alpha-\frac{2a^2b^4\sin\alpha}{(a^2+b^2)(a^4\cos^2\alpha+b^4\sin^2\alpha)};$$

hence, squaring and adding, we have, for the locus,

$$\text{I}\left(\frac{x^2}{a^2}=\frac{y^2}{b^2}\right)+\left(\frac{a^2-b^2}{a^2+b^2}\right)^2,$$

an ellipse concentric with and similar to the original.

7231. (By the Editor.)—If A, B, C are candidates for an office, the election to which is in the hands of $8m+1$ electors; and $3m$ votes, together with the casting vote if necessary, are promised to A, and $2m$ votes to B; show that the remaining votes be given so that A may be successful, in $\frac{1}{2}(7m^2+11m+2)$ ways.

Solution by the Rev. J. L. Kitchin, M.A.; E. Rutter; *and others.*

$3m+1$ things can be put into parcels of p, q, r in $(3m+1)!\div(p!\,q!\,r!)$ ways; here subject to the conditions that $p+q+r=3m+1$, and that p, q, r are so chosen that when placed with $3m$, $2m$, 0, for A, B, C respectively, $3m+p$, $2m+q$, r may be such that $3m+p$ may be $>2m+q$ or r; and that, whenever $3m+p$ equals either $2m+q$ or r, the casting vote goes to A; or this case is reckoned also in A's favour. This is the same as $m+p>q$, or $>r-2m$. The total number of ways will equal the sum of the values of $(3m+1)!\div(p!\,q!\,r!)$ with all possible values of p, q, r, taken subject to these conditions, plus the cases in which $m+p=q$ or $3m+p=r$.

The solution is for A, $3m$, $3m+1$, &c.; for B, $2m$, &c.; for C, 0, &c.; and, when counted up, this gives rise to the series

$$m+2+m+3+\ldots+2m+2+2m+1+2m+\ldots+3+2,$$

as is plain on examination; of which the first part

$$=\tfrac{1}{2}(m+1)(3m+2)=\tfrac{1}{2}(3m^2+5m+2),$$

and the second part $=\frac{1}{2}(2m+1)(2+2m)-1=2m^2+3m$; hence the sum is $\frac{1}{2}(7m^2+11m+2)$ ways, as stated.

7405. (By Professor TOWNSEND, F.R.S.) — **The rectangular coordinates (x_1, y_1) of a variable point P_1, in a fixed plane, being supposed connected with those (x_2, y_2) of another point P_2, in the same or in another plane, by a relation of the form $f(x_1+iy_1, x_2+iy_2) = 0$, where f is the representative of any function.**

(1) If P_1 describe a curve of small magnitude in its plane, show that P_2 will describe a curve of similar form in its plane.

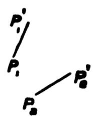
(2) If P_1 and P_2 be the stereographic projections, of a variable point P on a fixed sphere, upon the planes of the great circles of which any two arbitrary centres of projection O_1 and O_2 on the sphere are the poles: show that (x_1, y_1) and (x_2, y_2) are connected as in (1), and determine the form of f corresponding to the case.

Solution by G. B. MATHEWS, B.A.; Professor NASH, M.A.; *and others.*

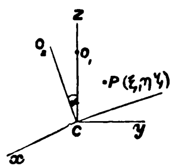
1. If $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, we have

$$\frac{\delta f}{\delta z_1}\cdot dz_1 + \frac{\delta f}{\delta z_2}\cdot dz_2 = 0, \text{ for small variations.}$$

Hence $dz_1 : dz_2 = -\dfrac{\delta f}{\delta z_2} \div \dfrac{\delta f}{\delta z_1}$, a ratio depending only on the values of z_1, z_2; therefore, supposing P_1, P_2 in the same plane, and P_1', P_2' any consecutive positions, the complex (quaternion) ratio $\dfrac{P_1P_1'}{P_2P_2'}$ = constant for small displacements; hence P_1', P_2' describe similar small curves.



2. Take CO_1 for axis of z, O_1CO_2 for plane of yz, and let ξ, η, ζ be the coordinates of P where $\xi^2+\eta^2+\zeta^2 = 1$; then the line O_1P is



$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z-1}{\zeta-1};$$

hence, putting $z = 0$, we get

$$x_1 = \frac{\xi}{1-\zeta}, \quad y_1 = \frac{\eta}{1-\zeta}.$$

Hence we have $x_1^2+y_1^2 = \dfrac{\xi^2+\eta^2}{(1-\zeta)^2} = \dfrac{1-\zeta^2}{(1-\zeta)^2} = \dfrac{1+\zeta}{1-\zeta}$,

whence $$\zeta = \frac{x_1^2+y_1^2-1}{x_1^2+y_1^2+1}, \quad 1-\zeta = \frac{2}{x_1^2+y_1^2+1},$$

also $$\xi = \frac{2x_1}{x_1^2+y_1^2+1}, \quad \eta = \frac{2y_1}{x_1^2+y_1^2+1}.$$

If $\angle O_1CO_2 = \alpha$, we get

$$x_2 = \frac{\xi}{1-(\zeta\cos\alpha-\eta\sin\alpha)} = \frac{2x_1}{(x_1^2+y_1^2+1)+2y_1\sin\alpha-(x_1^2+y_1^2-1)\cos\alpha}$$

$$= \frac{x_1}{(x_1^2+y_1^2)\sin^2\tfrac{1}{2}\alpha+2y_1\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha+\cos^2\tfrac{1}{2}\alpha}$$

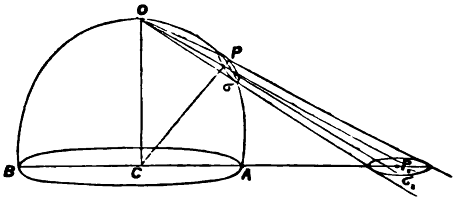
$$= \frac{x_1}{x_1^2\sin^2\tfrac{1}{2}\alpha+(y_1\sin\tfrac{1}{2}\alpha+\cos\tfrac{1}{2}\alpha)^2},$$

$$y_2' = \frac{\zeta \sin \alpha + \eta \cos \alpha}{1 - (\zeta \cos \alpha - \eta \sin \alpha)} = \frac{(x_1^2 + y_1^2 - 1) \sin \alpha + 2y_1 \cos \alpha}{2 x_1^2 \sin^2 \tfrac{1}{2}\alpha + 2 (y_1 \sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha)^2}$$

$$= \frac{(x_1^2 + y_1^2 - 1) \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha + y_1 (\cos^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}\alpha)}{x_1^2 \sin^2 \tfrac{1}{2}\alpha + (y_1 \sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha)^2};$$

$$x_2 + iy_2 = \frac{x_1(\cos^2 \tfrac{1}{2}\alpha + \sin^2 \tfrac{1}{2}\alpha) + iy_1(\cos^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}\alpha) + i(x_1^2 + y_1^2 - 1) \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha}{[(x_1 + iy_1) \sin \tfrac{1}{2}\alpha + i \cos \tfrac{1}{2}\alpha] [(x_1 - iy_1) \sin \tfrac{1}{2}\alpha - i \cos \tfrac{1}{2}\alpha]}$$

$$= \frac{i(x_1 + iy_1) \cos \tfrac{1}{2}\alpha + \sin \tfrac{1}{2}\alpha}{(x_1 + iy_1) \sin \tfrac{1}{2}\alpha + i \cos \tfrac{1}{2}\alpha} = f(x_1 + iy_1).$$



This may be confirmed by geometry; thus, let a small curve σ be described about P on the sphere, and let σ_1 be its projection on the plane AB.

Then the planes of σ, σ_1 are ultimately perpendicular to CP, CO, and are therefore equally inclined to OPP_1, the axis of the slender cone $O(\sigma)$; hence the curves σ, σ_1 are ultimately similar; so for the curves σ, σ_2; therefore σ_1, σ_2 are ultimately similar, so that, if $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, z_2 varies continuously with z_1, and the limit of $\frac{\delta z_2}{\delta z_1}$ is independent of the direction of dz_1; therefore &c.

7440. (By W. J. C. Sharp, M.A.)—Prove that (1) the tangents to the nine-point circle of a triangle, at the points where it meets either side, make angles with that side equal to the difference of the angles adjacent to the side; and (2) the tangent at the middle point makes angles with the other sides which are equal to the opposite angles of the triangle.

Solution by G. Heppel, M.A.; Sarah Marks; *and others.*

1. If AD be the perpendicular from A on BC, then, with D as origin and BC as axis of x, the equation to the nine-point circle is

$$x^2 + y^2 - \text{R} \sin (\text{B} - \text{C})\, x - \text{R} \cos (\text{B} - \text{C})\, y = 0;$$

hence the tangent at the origin is $\text{R} \sin (\text{B} - \text{C})\, x + \text{R} \cos (\text{B} - \text{C})\, y = 0$.

2. The tangent at the middle point also makes an angle equal to (B − C) with the axis of x, and therefore an angle C with AB.

[If O be the centre of the nine-point circle, and M the middle point of BC, the angle made with BC by the tangent at either D or M is $\sin^{-1} \frac{a\text{D}}{2 \text{ radius}} = \sin^{-1} \frac{b \cos \text{C} \sim \cos \text{B}}{2\text{R}} = \text{B} \sim \text{C}$, which proves (1), and (2) follows as above.]

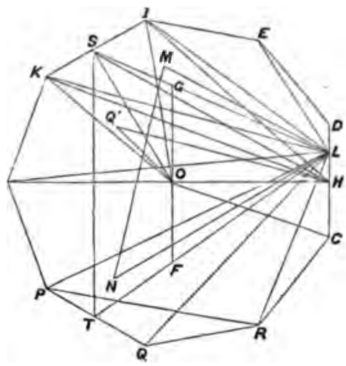
7355. (By Professor SEITZ, M.A.)—If P, Q, R be three consecutive vertices of a regular polygon of n sides and area Δ, and AB the diameter of the circumscribing circle, and if a triangle be formed by joining three random points on the surface of the polygon: prove that the respective averages of the (1) area and (2) square of area of the triangle are

$$\frac{\Delta}{36n^2}\left\{26\left(\frac{AB}{PQ}\right)^2+\left(\frac{AB}{PR}\right)^2-9\right\},\quad \frac{\Delta^2}{24n^2}\left\{2\left(\frac{AB}{PQ}\right)^2+\left(\frac{AB}{PR}\right)^2-1\right\}.$$

6348 & 6985. (By W. S. B. WOOLHOUSE, F.R.A.S.)—If five points be taken at random on the surface of a regular polygon of n sides, prove (1) that the probabilities that they will be the corners of a (1) convex, (2) regular pentagon, are respectively

$$p_1 = 1-\frac{5}{36n^2}\left\{46\left(\frac{AB}{PQ}\right)^2-\left(\frac{AB}{PR}\right)^2-15\right\},\quad p_2 = \tfrac{7}{18}-\cdot 10\bar{4}\sqrt{5}.$$

Solution by Professor E. B. SEITZ, M.A.



(7355). 1. Let

CDE ... IK ... PQR ...

be the regular polygon, O its centre, and OH its apothegm.

Let Δ_3 be the average area of the triangle when L, one of the points, is taken at random in the perimeter of the polygon, and M, N, the other two points, are taken at random on the surface.

Let OH $= a$, CD $= s$, and $x =$ the apothegm of a regular polygon of n sides, whose centre is O, and whose sides are parallel to those of the given polygon. Then, if L be taken in the perimeter of this polygon, and M, N on its surface, the average area of LMN will be $\Delta_3 x^2 \div a^2$, and we have

$$\Delta_1 = \int_0^a \frac{\Delta_3 x^2}{a^2}\left(\frac{\Delta x^2}{a^2}\right)^2\left(\frac{nsx}{a}\right)dx \div \int_0^a\left(\frac{\Delta x^2}{a^2}\right)^2\left(\frac{nsx}{a}\right)dx = \tfrac{3}{4}\Delta_3.$$

Let the polygon be divided into triangles by drawing lines from L to all the vertices; and let LIK and LPQ be the t^{th} and $(r+t)^{\text{th}}$ triangles respectively; and let S, T be the middle points of IK, PQ; G, F the centres of gravity of LIK, LPQ; HL $= y$, and $\angle$COH $= \theta = \pi \div n$; then

$$\angle \text{HSI} = \angle \text{SHD} = t\theta,\quad \text{HS} = 2a\sin t\theta,$$

$$\text{area HIK} = \tfrac{1}{2}\text{HS}\,.\,\text{IK}\sin\text{HSI} = as\sin^2 t\theta,$$

$$\text{area LIK} = \text{HIK}\,(\text{HS}-2\text{LH}\cos\text{SHD}) \div \text{HS} = s\sin t\theta\,(a\sin t\theta - y\cos t\theta),$$

$$\text{area LPQ} = s\sin(r+t)\,\theta\,[a\sin(r+t)\,\theta - y\cos(r+t)\,\theta],$$

$$\text{area LGF} = \tfrac{4}{9}\text{LST} = \tfrac{4}{9}a\sin r\theta\,[2a\sin t\theta\sin(r+t)\,\theta - y\sin(r+2t)\,\theta].$$

If M, N be taken in the triangle LIK, the average area of LMN will be $\frac{4}{27}$LIK (see WILLIAMSON'S *Integral Calculus*, p. 321), and while L ranges and the moment [illegible]

$$= \tfrac{1}{4} [illegible]$$

$$= \tfrac{1}{4} [illegible]$$

hence [illegible]

Therefore [illegible]

and [illegible]

[illegible]

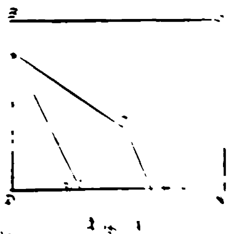
[illegible]

which is [illegible]

[illegible]

[illegible]

[illegible]



[illegible]

hence we have [illegible]

[illegible]

therefore [illegible]

And since, for a general solution, [illegible]

equations [illegible] and [illegible] give

[illegible]

an equation which, for $r = 2$, gives the stated equation to the locus

2. At O the curve is approximately $2y^3 = \pm c^2x$, and at A, $y^3 = \pm 2c^2x$;

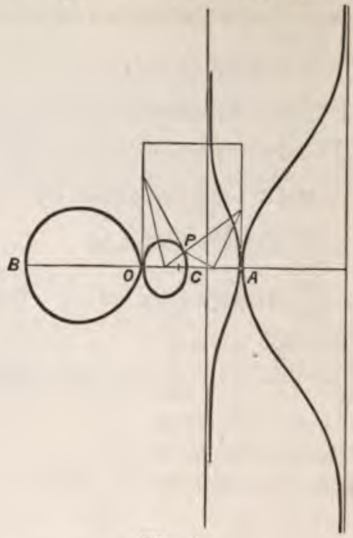


Fig. 2.

thus each branch has a point of inflexion. The asymptotes are $x = \frac{2}{3}c$, $x = 2c$. The sign of $\frac{2^{\frac{1}{3}}x^{\frac{2}{3}} - (c-x)^{\frac{2}{3}}}{x^{\frac{2}{3}} - 2^{\frac{2}{3}}(c-x)^{\frac{2}{3}}}$ determines the sign of y^2, and between $x = c(\sqrt{2}-1)$ and $x = \frac{2}{3}$ the curve is imaginary. In the figure, we have $OC = c(\sqrt{2}-1)$, and $OB = c(\sqrt{2}+1)$. The curve is symmetrical with respect to the axis of x.

5426. (By Professor Wolstenholme, M.A.)—Prove that (1) the two points whose distances from A, B, C, the angular points of a triangle, are as sin A, sin B, sin C, and the two whose distances are as cos A, cos B, cos C (one of which is the orthocentre), lie on the straight line joining the centre (O) of the circumscribed circle and the orthocentre (L); (2) the two former points Q, Q′ are real for any acute-angled triangle, and lie in LO produced, their positions being determined by

$$\frac{QL}{OL} = \frac{2k+2}{3k+1}, \quad \frac{Q'L}{OL} = \frac{2-2k}{1-3k},$$

where $k^2 = \frac{\cos A \cos B \cos C}{1 + \cos A \cos B \cos C}$; (3) P is always real, and lies in OL produced, so that OL . OP = square on the radius of the circumscribed circle, and $\frac{AP}{AL} = \frac{BP}{BL} = \frac{CP}{CL} = \frac{OP}{R} = \frac{R}{OL} = \frac{1}{(1-8\cos A\cos B\cos C)^{\frac{1}{2}}}$.

Hence the points will be fixed for all triangles inscribed in the same circle and having the same centroid.

Solution by the Proposer.

1. The second point (P) whose distances from A, B, C are as cos A : cos B : cos C (one such point being L, the centre of perpendiculars) lies in GL produced (G being the centroid), so that

$$GP : GL = 1+4\cos A\cos B\cos C : 1-8\cos A\cos B\cos C.$$

If (x, y, z) be areal coordinates of P, we have

$$PA^2 = -a^2yz + b^2z(y+z) + c^2y(y+z) \equiv b^2z^2 + c^2y^2 + 2bc\cos A\, yz,$$

$$\frac{b^2z^2+c^2y^2+2bc\cos A\, yz}{\cos^2 A} = \frac{c^2x^2+a^2z^2+2ca\cos B\, zx}{\cos^2 B} = \frac{a^2y^2+b^2x^2+2ab\cos C\, xy}{\cos^2 C}.$$

But $$b^2z^2+c^2y^2+2bc\cos A\, yz$$
$$= (x+y+z)(b^2z+c^2y) - (a^2yz+b^2zx+c^2xy) \equiv b^2z+c^2y-S,$$

or $$\frac{b^2z+c^2y-S}{\cos^2 A} = \frac{c^2x+a^2z-S}{\cos^2 B} = \frac{a^2y+b^2x-S}{\cos^2 C},$$

whence $$(\cos^2 B-\cos^2 C)(b^2z+c^2y) + (\cos^2 C-\cos^2 A)(c^2x+a^2z)$$
$$+ (\cos^2 A-\cos^2 B)(a^2y+b^2x) = 0,$$

or the point lies on G'L. Let then $x = \lambda+\mu\tan A$, $y = \lambda+\mu\tan B$, $z = \lambda+\mu\tan C$, which for proper values of λ, μ represent any point on GL.

Substituting these in any single equation, the term involving μ^2 disappears, and (taking the first equation) we get

$$\lambda\,[(2b^2+2c^2-a^2)\cos^2 B - (2c^2+2a^2-b^2)\cos^2 A]$$
$$+ 2\mu\,\{[b^2\tan C + c^2\tan B + bc\cos A(\tan B+\tan C)]\cos^2 B$$
$$-[c^2\tan A + a^2\tan C + ca\sin^2 B(\tan C+\tan A)]\cos^2 A\} = 0,$$

or $$\lambda\,(2\sin^2 C - \cos^2 B - \cos^2 A - \cos^2 B + \sin^2 A)$$
$$+ 2\mu\,[\sin C\cos C - \tan C(\cos^2 B - \sin^2 A)] = 0,$$

or $\lambda\,[3-2(\cos^2 A+\cos^2 B+\cos^2 C)] + 2\mu\tan C(\cos^2 C - \cos^2 B + \sin^2 A) = 0,$

or $$\lambda(1+4\cos A\cos B\cos C) + 2\mu\tan C\,.\,2\sin A\sin B\cos C = 0,$$

or $$\lambda(1+4\cos A\cos B\cos C) + 4\mu\sin A\sin B\sin C = 0.$$

But the point divides GL in the ratio $\mu\tan A\tan B\tan C : 3\lambda$,

or $$GP : PL = 1+4\cos A\cos B\cos C : -12\cos A\cos B\cos C,$$

whence $$GP : GL = 1+4\cos A\cos B\cos C : 1-8\cos A\cos B\cos C,$$

or, if O be the centre of the circumscribed circle,

$$OL = OP(1-8\cos A\cos B\cos C).$$

But $OL^2 = R^2(1-8\cos A\cos B\cos C)$, whence $OP\,.\,OL = R^2$, or P, L are reciprocal points with respect to the circumscribed circle (which suggests that there must be a much simpler proof from some other point of view).

Again, if p, q, r be the distances of P from A, B, C,

$$a^2(p^2-q^2)(p^2-r^2)+\ldots-b^2c^2(q^2+r^2)-\ldots+a^2b^2c^2=0,$$

or, if $p=k\cos A,\quad q=k\cos B,\quad r=k\cos C,$

$$k^4[a^2(\cos^2 A-\cos^2 B)(\cos^2 A-\cos^2 C)+\ldots]-k^2[b^2c^2(\cos^2 B+\cos^2 C)+\ldots]$$
$$+a^2b^2c^2=0,$$

and one value of k^2 is $4R^2$, whence we get for the other

$$4k^2 . R^2=\frac{a^2b^2c^2}{a^2(\sin^2 A-\sin^2 B)(\sin^2 A-\sin^2 C)+\ldots+\ldots},$$

or
$$k^2=\frac{4R^2\sin^2 A\sin^2 B\sin^2 C}{\sin^2 A(\sin^2 A-\sin^2 B)(\sin^2 A-\sin^2 C)+\ldots+\ldots}$$
$$=\frac{4R^2\sin A\sin B\sin C}{\sin A\sin(A-B)\sin(A-C)+\ldots+\ldots},$$

and denominator
$$=\tfrac{1}{2}\sin A[\cos(B-C)-\cos(B+C-2A)+\ldots+\ldots$$
$$=\tfrac{1}{4}(\sin 2B+\sin 2C+\sin 4A-\sin 2A+\ldots]$$
$$=\tfrac{1}{4}[\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C]$$
$$=\sin A\sin B\sin C-\sin 2A\sin 2B\sin 2C$$
$$=\sin A\sin B\sin C(1-8\cos A\cos B\cos C),$$

or
$$k^2=\frac{4R^2}{1-8\cos A\cos B\cos C},$$

whence, finally, we obtain

$$\frac{AP}{AL}=\frac{BP}{BL}=\frac{CP}{CL}=\frac{OP}{R}=\frac{R}{OL}=\left(\frac{1}{1-8\cos A\cos B\cos C}\right)^{\frac{1}{2}}.$$

2. The two points Q, Q′, whose distances from A, B, C are as sin A, sin B, sin C, will be similarly determined by the equations

$$\frac{b^2z+c^2y-S}{\sin^2 A}=\frac{c^2x+a^2z-S}{\sin^2 B}=\frac{a^2y+b^2x-S}{\sin^2 C},$$

whence they both satisfy the equation

$$(b^2z+c^2y)(\sin^2 B-\sin^2 C)+\ldots+\ldots=0,$$

the same as before, or both points lie upon GL. Taking $x=\lambda+\mu\tan A$, &c., as before, we get the quadratic for $\lambda:\mu$,

$$\lambda^2[(2b^2+2c^2-a^2)\sin^2 B-(2a^2+2c^2-b^2)\sin^2 A]$$
$$+2\lambda\mu[b^2\tan C+c^2\tan B+bc\cos A(\tan B+\tan C)\sin^2 B$$
$$-c^2\tan A-a^2\tan C-ca\cos B(\tan C+\tan A)\sin^2 A]$$
$$+\mu^2[(b^2\tan^2 C+c^2\tan^2 B+2bc\tan B\tan C\cos A)\sin^2 B$$
$$-(c^2\tan^2 A+a^2\tan^2 C+2ca\tan C\tan A\cos B)\sin^2 A]=0,$$

which, when divided by b^2-a^2, gives

$$2\lambda^2(\sin^2 A+\sin^2 B+\sin^2 C)+2\lambda\mu[2\tan A\tan B\tan C+2\sin A\sin B\sin C)$$
$$+\mu^2\tan^2 A\tan^2 B\tan^2 C=0,$$

or $$4\lambda^2(1+\cos A\cos B\cos C)+4\lambda\mu\tan A\tan B\tan C(1+\cos A\cos B\cos C)$$
$$+\mu^2\tan^2 A\tan^2 B\tan^2 C=0,$$

or $$\frac{2\lambda}{\mu\tan A\tan B\tan C}=-1\pm\left(1-\frac{1}{1+\cos A\cos B\cos C}\right)^{\frac{1}{2}}\equiv-1\pm k,$$

whence $\quad GQ:QL=2:3(-1+k),\quad GQ':Q'L=2:-3(1+k).$

Hence the two points Q, Q′ are real when cos A cos B cos C is positive, or for any acute-angled triangle. Measuring distances from O instead of G,

we get $\frac{OQ-\frac{1}{3}OL}{OL-OQ}=\frac{-2}{3(1-k)}$, $\frac{3OQ-OL}{OL-OQ}=\frac{-2}{1-k}$, $\frac{2OQ}{2OL}=\frac{-2}{1-3k}$,

or $OQ:OL=2:3k-1$, and, since k is less than $\frac{1}{3}$, OQ, OL are of opposite signs, or Q lies in LO produced. Similarly $\frac{OQ'}{OL}=\frac{-2}{1+3k}$, or Q′ also lies in LO produced. Moreover

$$\frac{1}{OQ}+\frac{1}{OQ'}=-\frac{1}{OL}=\frac{1}{LO}=\frac{2}{2LO},$$

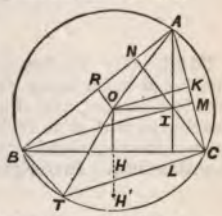
or, if LO be produced to a point R so that OR is twice LO, Q, Q′ will divide OR harmonically. Also $LQ \, , \, LQ' = 9R^2$.

If the equation for the distances from A, B, C be formed as before, the coefficient of k^4 will be the same as before (since $\sin^2 B - \sin^2 C = \cos^2 C - \cos^2 B$, &c.),

whence $\frac{AQ \, . \, AQ'}{\sin^2 A}=\frac{AP \; AL}{\cos^2 A}=\frac{4R^2}{1-8\cos A\cos B\cos C}$, &c.

7428. (By Professor Sylvester, F.R.S.)—If O is the centre of the circle circumscribed about the triangle ABC, and I the intersection of the three perpendiculars from the angles upon the opposite sides of the triangle; prove (1) that the distance of O from any side is half the distance of I from the opposite angle; and hence (2) that OI is the resultant of the three equal forces OA, OB, OC.

Solution by W. G. Lax, B.A.; Margaret T. Meyer; *and others.*



1. Produce AO to cut the circumscribing circle in T, and join TB, TC. Then, since AT is a diameter, ACT is a right angle; hence TC is parallel to OK, and TC = 2OK.

Also CT is parallel to BI, and, since similarly ABT is a right angle, BT is parallel to CI; hence BTCI is a parallelogram; therefore CT = BI and CT = 2OK, therefore BI = 2OK; similarly CI = 2OR, and AI = 2OH.

2. Produce OH to H′ so that OH′ = 2OH; then the resultant of two equal forces OB, OC is 2OH = OH′; hence the resultant of OB, OC, OA is that of OH′ and OA. Now OH′ is parallel and equal to AI, therefore OH′IA is a parallelogram; therefore OI is the resultant of OH′ and OA, that is, of OA, OB, OC.

[If through A, B, C parallels be drawn to BC, CA, AB, the triangle thus formed is similar to ABC and double its linear dimensions; and I being the centre of the circle inscribed in this triangle, we have (1) AI = 2OH, &c.; again, if H, the mid-point of BC, be joined to A, it will cut OI in G, in such wise that, AG = 2DG, and G thus being the centroid, OI = 3OG, and therefore (2) is true.

Since O is the centre of the nine-point circle of the larger triangle, the line joining the centre of this circle to that of the circumscribed circle is the resultant of the forces represented by the three radii of the nine-point circle to the middle points of the sides.

It may be shown that the line joining the centres of the inscribed and nine-point circles passes through the centroid, so that the nine-point centre is in OI and indeed bisects it, for $OG = \frac{1}{3}OI = \frac{2}{3}OO'$.

The trilinear equation to this line is

$$\alpha \sin 2A \sin (B-C) + \beta \sin 2B \sin (C-A) + \gamma \sin 2C \sin (A-B) = 0.]$$

7437. (By J. J. Walker, M.A., F.R.S.) — Prove the following formula of reduction for the parts of any spherical triangle ABC :—

$$(\sec a \sin b \cos A - \sin c)^2 + (\sec a \cos b - \cos c)^2 (1 - \operatorname{cosec}^2 a \sin^2 A) = \tan^2 a \cos^2 B \cos^2 C.$$

[The formula is employed, without proof, on p. 69 of Vol. 37 of *Reprints*.]

Solution by D. Edwardes ; A. Martin, M.A. ; *and others.*

Substituting for cos A its value in terms of the sides, the first term becomes $\cos^2 c \tan^2 a \cos^2 B$, and the second $\sin^2 c \tan^2 a \cos^2 B \left(1 - \frac{\sin^2 A}{\sin^2 a}\right)$,

$$\therefore \text{ left side } = \tan^2 a \cos^2 B \left(1 - \frac{\sin^2 c \sin^2 A}{\sin^2 a}\right) = \tan^2 a \cos^2 B (1 - \sin^2 c)$$
$$= \tan^2 a \cos^2 B \cos^2 C.$$

7426. (By Professor Haughton, F.R.S.)—In a work erroneously attributed to Sir Isaac Newton, it is stated, that if two spheres, each one foot in diameter, and of a like nature to the Earth, were distant by but the fourth part of an inch, they would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time. Investigate the truth of this statement.

Solution by R. Rawson ; Professor Matz, M.A. ; *and others.*

Let D, D_1, D_2 be the diameters of Earth (= 7912·41 × 5280 feet) and of (Earth$_1$), (Earth$_2$) similar in material to the Earth ; O the common centre of the three spheres ; Q another point such that $OQ = a$; and A, A_1 the attractive forces of the (Earth) and (Earth$_1$) at the point Q ; then (Earnshaw's *Dynamics*, p. 318) if ρ is the density of the (Earth) we have

$$A = \frac{\pi\rho D^3}{6a^2}, \quad A_1 = \frac{\pi\rho D_1^{\,3}}{6a^2}, \quad \text{whence } \frac{A_1}{A} = \frac{D_1^{\,3}}{D^3} \quad \text{......(1, 2, 3).}$$

Let g, g_1 be the accelerating forces at Q produced by A, A_1; then, since accelerating forces are proportional to the moving forces,

$$\frac{g_1}{g} = \frac{A_1}{A} = \frac{D_1^3}{D^3} \text{ from (3), therefore } g_1 = \frac{gD_1^3}{D^3} \quad \text{............(4).}$$

If μ, μ_1 be the accelerating forces of A_1, A_2 at a unit distance from O, where A_2 is the attractive force of the ($Earth_2$), we have

$$\mu = g_1 a^2 = \frac{gD_1^3 a^2}{D^3}, \quad \mu_1 = \frac{gD_2^3 a^2}{D^3} \quad \text{.................(5, 6).}$$

If, however, Q be on the surface of the (Earth), then $2a = D$, and $g = 32\frac{1}{18}$; hence (5) and (6) become

$$\mu = \frac{gD_1^3}{4D}, \quad \mu_1 = \frac{gD_2^3}{4D} \quad \text{........................(7, 8).}$$

Again, if O, P be the initial positions of ($Earth_1$), ($Earth_2$) respectively, and O′, P′ their positions at the end of (t) seconds; and we put $OP = \beta$, $OO' = x$, $OP' = y$, the dynamical equations of motion are

$$\frac{d^2x}{dt^2} = \frac{\mu_1}{(y-x)^2}, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{(y-x)^2}, \text{ whence } \frac{d^2(y-x)}{dt^2} = -\frac{\mu+\mu_1}{(y-x)^2} \text{...(9,10,11).}$$

This equation, as integrated by EARNSHAW, pp. 73, 74, is as follows:—

$$t = \left(\frac{\beta}{2(\mu+\mu_1)}\right)^{\frac{1}{2}} \left\{ \tfrac{1}{2}\beta\pi + [(y-x)(\beta-y+x)]^{\frac{1}{2}} - \tfrac{1}{2}\beta \operatorname{vers}^{-1} \frac{2(y-x)}{\beta} \right\} \text{...(12).}$$

If we say that the spheres come together when they touch, then $y-x = \frac{1}{2}(D_1 + D_2)$, and if we further take $D_1 = D_2 = 1$, then (12) becomes

$$t = \left(\frac{D\beta}{g}\right)^{\frac{1}{2}} \left\{ \tfrac{1}{2}(\beta\pi) + (\beta-1)^{\frac{1}{2}} - \tfrac{1}{2}\beta \operatorname{vers}^{-1} \frac{2}{\beta} \right\} \quad \text{.......... (13).}$$

When $\beta = \frac{49}{18}$ feet, this formula gives for (t) the value 150·91 seconds. This result is so wide apart from a month that I am afraid of having made a slip.

By integrating (11) the velocity of approach is readily seen to be

$$[2(\mu+\mu_1)]^{\frac{1}{2}} \left\{ \frac{1}{y-x} - \frac{1}{\beta} \right\} = \left\{ \frac{32\frac{1}{18}}{7912{\cdot}41 \times 5280 \times 49} \right\}^{\frac{1}{2}} = {\cdot}000125 \text{ nearly,}$$

in the case here considered.

7448. (By D. EDWARDES.)—If a rectangular hyperbola pass through the centre of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, touch it at a point P, whose eccentric angle is α, and intersect it in Q, R; prove that tangents to the ellipse at Q, R intersect on the straight line

$$b^3x \cos\alpha + a^3y \sin\alpha + ab(a^2+b^2) = 0.$$

Solution by R. E. SPREAD, M.A.; MARGARET T. MEYER; *and others.*

Let the tangents at Q, R intersect at T (ξ, η); then the equation to any conic passing through Q, R, and touching the ellipse at the point (α), is $a^2y^2 + b^2x^2 - a^2b^2 + \lambda(a^2y\eta + b^2x\xi - a^2b)^2 (ay \sin\alpha + bx\cos\alpha - ab) = 0$;

and the condition of this going through the centre is

$$-a^2b^2+\lambda \, . \, a^3b^3=0, \text{ hence } \lambda=(ab)^{-1}.$$

Substituting for λ, and putting $co \, . \, x^2+co \, . \, y^2=0$ for rectangular hyperbola, we get $(ab^3+b^3\xi\cos\alpha)+(a^3b+a^3\eta\sin\alpha)=0$; hence the locus is

$$b^3x\cos\alpha+a^3y\sin\alpha+ab\,(a^2+b^2)=0.$$

7450. (By R. Tucker, M.A.)—If a circle passing through the focus of a given conic intersects the conic in points $(\theta_1, \theta_2, \theta_3, \theta_4)$, prove that (1) $\Sigma\cos\theta$ is dependent upon the excentricity only; and (2) if the diameter of the circle be inclined to the axis of the conic at an angle $\sin^{-1} l/d$, where $2l$ is the latus rectum and d the diameter of the circle, then one of the angles (θ) is a right angle.

Solution by J. Hammond, M.A.; Margaret T. Meyer; *and others.*

The intersections of the conic and circle are given by the equations $r=\dfrac{l}{1-e\cos\theta}=d\cos(\theta-\alpha)$; hence, writing x for $\cos\theta$ and m for $\dfrac{l}{d}$, we have

$$\frac{m}{1-ex}=x\cos\alpha+\sin\alpha\,(1-x^2)^{\frac{1}{2}};$$

whence

$$\frac{m^2}{(1-ex)^2}-\frac{2m\,x\cos\alpha}{1-ex}+x^2=\sin^2\alpha,$$

or

$$e^2x^4-2ex^3+(1-e^2\sin^2\alpha+2m\,e\cos\alpha)\,x^2$$
$$+\,2x\,(e\sin^2\alpha-m\cos\alpha)+m^2-\sin^2\alpha=0.$$

Therefore $\Sigma\cos\theta$ (or Σx) $=\dfrac{2}{e}$, and if $\sin\alpha=m=\dfrac{l}{d}$, one value of x is zero, or one of the angles is a right-angle [which is also obvious geometrically].

6820. (By H. G. Dawson.)—If $\alpha, \beta, \gamma, \delta$ be the roots of

$$(abcde)\,(x1)^4=0,$$

prove that the equation whose roots are $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, &c.,

$$\begin{vmatrix} 3, & -z, & -\left(\frac{1}{4}z^2-\frac{4Hz}{a^2}+\frac{4I}{a^2}\right) \\ z, & \frac{1}{4}z^2-\frac{4Hz}{a^2}+\frac{I}{a^2}, & \frac{6J}{a^3} \\ \frac{1}{4}z^2-\frac{4Hz}{a^2}+\frac{I}{a^2}, & \frac{I}{a^2}z+\frac{6J}{a^3}, & -\frac{2J}{a^3}z \end{vmatrix}=0,$$

where $H=b^2-ac$, $I=ae-4bd+3c^2$, and $J=ace+2bcd-eb^2-ad^2-c^3$.

Solution by the PROPOSER; Prof. NASH, M.A.; *and others.*

Since $(\alpha-\beta)^2+(\gamma-\delta)^2 = \frac{16H}{a^2} - 4\lambda$ and $(\alpha-\beta)^2(\gamma-\delta^2) = \frac{4I}{a^2}+\frac{24J}{a^3}$,

where λ is a root of $(az)^3 - I(az) + 2J$,

the roots of $y^2 - y\left(\frac{16H}{a^2} - 4\lambda\right) + \frac{4I}{a^2} + \frac{24J}{a^3\lambda} = 0$ are $(\alpha-\beta)^2$, $(\gamma-\delta)^2$.

Hence the result of eliminating λ between this equation and

$$(a\lambda)^3 - I(a\lambda) + 2J = 0 \quad\text{.............................(1)}$$

must be the equation of differences. Now we may write the above equation

$$\lambda^2 y + \left(\frac{y^2}{4} - \frac{4Hy}{a^2} + \frac{I}{a^2}\right)\lambda + \frac{6J}{a^3} = 0.$$

Conbining this with (2), we get

$$3\lambda^2 - \lambda y - \left(\frac{y^2}{4} - \frac{4Hy}{a^2} + \frac{4I}{a^2}\right) = 0,$$

and
$$\lambda^2\left(\frac{y^2}{4} - \frac{4Hy}{a^2} + \frac{I}{a^2}\right) + \lambda\left(\frac{I}{a^2}y + \frac{6J}{a^3}\right) - \frac{2J}{a^3}y = 0.$$

Eliminate λ and λ^2, and we get the required result.

6990. (By J. HAMMOND, M.A.)—Referring to Professor CAYLEY's Question 5244, prove that the 16 nodes lie by sixes on sixteen conics, that six of these conics intersect at each node, and that four conicoids may be found, each of which passes through four of the conics and twelve of the nodes, the tetrahedron of reference being self-conjugate with respect to all four of the conicoids.

Solution by the PROPOSER; SARAH MARKS; *and others.*

The sixteen conics are the conics of contact of the sixteen singular tangents, and the four conicoids are

$$Ax^2+By^2+Cz^2+Dw^2 = 0,\quad Ay^2+Bx^2+Cw^2+Dz^2 = 0 \text{......(1, 2)},$$
$$Az^2+Bw^2+Cx^2+Dy^2 = 0,\quad Aw^2+Bz^2+Cy^2+Dx^2 = 0 \text{......(3, 4)}.$$

For, by the solution to Question 5244,

$$A\beta^2+B\alpha^2+C\delta^2+D\gamma^2 = 0,\quad A\gamma^2+B\delta^2+C\alpha^2+D\beta^2 = 0,$$
$$A\delta^2+B\gamma^2+C\beta^2+D\alpha^2 = 0.$$

If now we denote the nodes by $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$, the order in Quest. 5244 being preserved, and the conics of contact of the singular tangents by A_1, A_2, &c, then it is evident, from the reduced form of the equation to the surface, that

A_1, A_2, A_3, A_4	and the twelve nodes	b, c, d	lie on	(1),
B_1, B_2, B_3, B_4	„ „	c, d, a	„	(2),
C_1, C_2, C_3, C_4	„ „	d, a, b	„	(3),
D_1, D_2, D_3, D_4	„ „	a, b, c	„	(4);

and it may be noticed that the eight intersections of (2), (3), and (4) are the four nodes $\dot{a}$, and the four points $(\alpha, -\beta, -\gamma, -\delta)$, $(\alpha, -\beta, \gamma, \delta)$, $(\alpha, \beta, -\gamma, \delta)$, $(\alpha, \beta, \gamma, -\delta)$, which are nodes of the quartic surface, whose equation only differs from that in question by the sign of its last term.

The twelve nodes b, c, d, on (1), are arranged by sixes on the four conics A_1, A_2, A_3, A_4; for the equation to the plane of A_1 being

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

is satisfied by the coordinates of six nodes, viz.,

$$(\beta, -\alpha, -\delta, \gamma), \quad (\beta, -\alpha, \delta, -\gamma), \quad (\gamma, -\delta, -\alpha, \beta), \quad (\gamma, \delta, -\alpha, -\beta),$$
$$(\delta, -\gamma, \beta, -\alpha), \quad (\delta, \gamma, -\beta, -\alpha);$$

i.e., the six nodes b_2, b_3, c_2, c_4, d_3, d_4 lie on the conic A_1, and the positions of all the nodes may be tabulated as follows:

$$\begin{array}{c|cccccc|} A_1 & b_2, & b_3, & c_2, & c_4, & d_3, & d_4 \\ A_2 & b_1, & b_4, & c_1, & c_3, & d_3, & d_4 \\ A_3 & b_1, & b_4, & c_2, & c_4, & d_1, & d_2 \\ A_4 & b_2, & b_3, & c_1, & c_3, & d_1, & d_2 \end{array}, \qquad \begin{array}{c|cccccc|} B_1 & c_3, & c_4, & d_2, & d_4, & a_2, & a_3 \\ B_2 & c_3, & c_4, & d_1, & d_3, & a_1, & a_4 \\ B_3 & c_1, & c_2, & d_2, & d_4, & a_1, & a_4 \\ B_4 & c_1, & c_2 & d_1, & d_3, & a_2, & a_3 \end{array},$$

$$\begin{array}{c|cccccc|} C_1 & d_2, & d_3, & a_2, & a_4, & b_3, & b_4 \\ C_2 & d_1, & d_4, & a_1, & a_3, & b_3, & b_4 \\ C_3 & d_1, & d_4, & a_2, & a_4, & b_1, & b_2 \\ C_4 & d_2, & d_3, & a_1, & a_3, & b_1, & b_2 \end{array}, \qquad \begin{array}{c|cccccc|} D_1 & a_3, & a_4, & b_2, & b_4, & c_2, & c_3 \\ D_2 & a_3, & a_4, & b_1, & b_3, & c_1, & c_4 \\ D_3 & a_1, & a_2, & b_2, & b_4, & c_1, & c_4 \\ D_4 & a_1, & a_2, & b_1, & b_3, & c_2, & c_3 \end{array}.$$

Also the node a_1 is the intersection of the six conics B_2, B_3, C_2, C_4, D_3, D_4, and so for the others. This is most easily seen from the fact that the node $(\alpha, \beta, \gamma, \delta)$ and the plane $(\alpha, \beta, \gamma, \delta)$ are pole and polar with respect to $x^2+y^2+z^2+w^2 = 0$, so that our tables give by a simple interchange of large and small letters the six singular tangent planes, or, what is the same, the six conics, which intersect at each particular node.

6699. (By Professor Townsend, F.R.S.)—A circular plate of invariable form being supposed, by a small movement of translation in the direction of any diameter, to put in continuous irrotational strain, in the plane of its mass, a surrounding lamina of any incompressible substance extending radially in all directions from its circumference to a fixed boundary at infinity; show that the potential and displacement line-systems of the strain are two systems of circles, passing both through the centre of the plate, and touching respectively its diameters perpendicular and parallel to the direction of its movement.

Solution by the Proposer.

Taking for axes of coordinates any pair of rectangular diameters of the plate in its original position, and denoting by r its radius, by l and m the components of its small movement of translation, by ξ and η those of the resulting small displacement of any point xy of the lamina, and by u and v the potential and displacement functions respectively of the strain; then since, by hypothesis, $\xi = 0$ and $\eta = 0$ for all positions of xy at infinity, while for all points on the circumference of the plate in its original position, that is, for all points satisfying the equation $x^2+y^2=r^2$, they are connected with x and y by the relation $(x+\xi-l)^2+(y+\eta-m)^2=r^2$, or, neglecting small quantities of the second order, by the relation $x\xi+y\eta=lx+my$; we have accordingly, for the solution of the entire problem (including the above particulars) of the strain, first to find, if possible, a potential function u which shall satisfy, at once, the general equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ throughout the entire extent of the lamina, and the aforesaid particular conditions at its outer and inner boundaries, and then derive from it the corresponding displacement function v in the usual manner. Both functions, in the present case, are found readily as follows. As the function $lx+my$ is homogeneous in x and y, and satisfies for all values of them the equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, therefore the function $k\,.\,\frac{lx+my}{x^2+y^2}$, where k is any constant, satisfies, for all values of k, the general equation $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$ throughout the entire extent of the lamina, and the particular conditions $\xi=0$ and $\eta=0$ at its outer boundary, and for the particular value $k=r^2$ that for its inner boundary also. It follows therefore from the aforesaid considerations that $u=r^2\,.\,\frac{lx+my}{x^2+y^2}$, and, by derivation from it in the usual manner, that $v=r^2\,.\,\frac{ly-mx}{x^2+y^2}$; which accordingly are the potential and displacement functions, respectively, of the strain.

Putting $u=c$ and $v=c$, which represent, for different values of c, the potential and displacement line-systems, respectively, of the strain; we get, for the two systems of lines respectively, the equations

$$(x^2+y^2)=\frac{r^2}{c}(lx+my) \quad\text{and}\quad (x^2+y^2)=\frac{r}{c}(ly-mx),$$

which manifestly establish the two particulars of the question.

To find the principal displacement of the strain at any point xy of the lamina. From the above value of u, by a first differentiation with respect to x and to y, we get at once

$$\left[\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right]^{\frac{1}{2}}=(l^2+m^2)^{\frac{1}{2}}\,.\,\frac{r^2}{x^2+y^2};$$

from which we see that, throughout the entire extent of the strained mass, the principal displacement depends only on the distance from the centre of the plate, and varies from point to point inversely as the square of that distance.

To find the principal dilatation of the strain at any point xy of the lamina. From the same, by a second differentiation with respect to x and to y, we get again at once

$$\left[\left(\frac{d^2u}{dx\,dy}\right)^2-\frac{d^2u}{dx^2}\,.\,\frac{d^2u}{dy^2}\right]^{\frac{1}{2}}=2\,(l^2+m^2)^{\frac{1}{2}}\,.\,\frac{r^3}{(x^2+y^2)^{\frac{3}{2}}};$$

from which we see that, throughout the entire extent of the strained mass, the principal dilatation depends only on the distance from the centre of the plate, and varies from point to point inversely as the cube of that distance.

To find the principal axes of the strain at any point xy of the lamina. From the general equation for the determination of their directions in any case of strain in two dimensions, viz.,

$$(\mu^2-1)\frac{d^2u}{dx\,dy}+\mu\left(\frac{d^2u}{dx^2}-\frac{d^2u}{dy^2}\right)=0,$$

referred in the present case for convenience, to the diameters coinciding with and orthogonal to the movement of the plate, as those of x and y respectively, we have, for the determination of μ at any point xy of the strained mass, the equation $y(3x^2-y^2)(\mu^2-1)+x(3y^2-x^2)\,2\mu=0$; from which it appears that $\mu=0$ or ∞ when $y=0$ and when $\frac{y}{x}=\pm\sqrt{3}$, and that $\mu=\pm 1$ when $x\equiv 0$ and when $\frac{x}{y}=\pm\sqrt{3}$; and therefore that the principal axes of the strain are parallel and perpendicular to the movement of the plate for all points on its diameter of displacement or on either of the two inclined at angles of 60° to it; and are inclined at angles of 45° to the direction of its movement for all points on its perpendicular diameter, or on either of the two inclined at angles of 60° to it, and therefore at angles of 30° to the direction of its movement.

To find the principal dilatation line-systems of the strain throughout the entire extent of the lamina. Solving for μ from the preceding equation, and substituting for it $\frac{dy}{dx}$ in the result, we get at once, for the differential equations of the two orthogonal systems in question,

$$\frac{d^2u}{dx\,dy}\cdot dy+\frac{d^2u}{dx^2}\cdot dx=+\left[\left(\frac{d^2u}{dx\,dy}\right)^2+\left(\frac{d^2u}{dx^2}\right)\right]^{\frac{1}{2}}\cdot dx,$$

which, by transformation to polar coordinates for which $u=r^{-1}\cos\theta$, become respectively, after the usual reductions,

$$\frac{dr}{r}+\frac{\sin\frac{1}{2}\theta\,.\,d\theta}{\cos\frac{1}{2}\theta}=0,\quad\text{and}\quad\frac{dr}{r}-\frac{\cos\frac{1}{2}\theta\,.\,d\theta}{\sin\frac{1}{2}\theta}=0;$$

the integrals of which, viz., $r^{-\frac{1}{2}}\cos\frac{1}{2}\theta=a^{-\frac{1}{2}}$ and $r^{-\frac{1}{2}}\sin\frac{1}{2}\theta=b^{-\frac{1}{2}}$, where a and b are the two constants of integration, represent, for different values of a and b, two systems of cardioids oppositely situated with respect to each other, having a common cusp at the origin, and a common axis of figure in the displacement diameter of the plate; which accordingly are the line-systems of greatest dilatation and of greatest condensation of the strain.

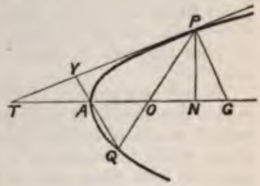
7380. (By Professor HUDSON, M.A.)—If from the vertex A of a parabola, AY be drawn perpendicular to the tangent at P, and YA produced meet the curve again in Q; prove that PQ cuts the axis in a fixed point.

I. *Solution by* A. MARTIN, M.A.; R. KNOWLES, M.A.; *and others.*

The equation to AY is $y=-\frac{y_1}{2a}x$; and this cuts the curve also in the points $y = -8a^2$, $x = \frac{16a^3}{y_1^2}$; hence the equation to PQ is

$$y - y_1 = \frac{y_1 + 8a^2}{x_1 - \frac{16a^3}{y_1}}(x - x'), \text{ and when } y = 0 \text{ we have } x = 2a.$$

II. *Solution by the* Proposer.



Let the tangent at P and PQ meet the axis in T and O; and draw the normal PG and the ordinate PN; then we have

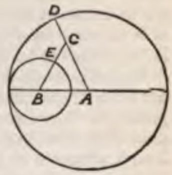
TA : AO = PO : OQ by property of the parabola,

= OG : AO by similar triangles AQO, GPO.

∴ OG = AT = AN, ∴ AO = NG = 2AS.

7449. (By C. Bickerdike.)—If a circle A is touched internally by a circle B, and a circle C touches both A and B; show that the locus of the centre of C is an ellipse round the centres of A and B.

Solution by Kate Gale; G. Bayliss, B.A.; *and others.*



Let A, B, C be the centres of the circles; then, since CD = CE, we have

$$AC + BC = AD + BE = \text{constant};$$

hence the centre of the third circle describes an ellipse round A, B.

On the Relative Values of the Chessmen. *By* D. Biddle.

The difficulties surrounding this inquiry are numerous and great; and, since many attempts have already been made to arrive at a decisive estimate as to the average value attaching to the several pieces employed in Chess, it might seem reasonable to refer in detail to these previous efforts. But mathematical questions are not to be decided by authority. It may suffice, therefore, to say that some of the ablest papers on the subject have been written by Mr. W. J. C. Miller, and appeared in the *Huddersfield College Magazine*, out of which has grown the *British Chess Magazine*. The pity is, that Mr. Miller had not more scope to carry his investigations to a complete issue.

In estimating the value of the several chessmen, we must duly regard the objects of the game, so far as each player is concerned; viz., to checkmate the adverse king, and to avoid the same on the part of his own king. The capture of adverse pieces is quite of secondary importance, to be engaged in only as tending to accomplish the ultimate object ever to be held in view.

Again, we must bear in mind that the relative value of the several pieces is in great measure a matter of range, the meanest piece being as powerful within its sphere as the highest. Therefore, supposing each piece to be the centre of forces which issue from it in various definite directions, and each square to be the focus of forces which impinge upon or pass through it, these forces must be regarded as equal, except in range.

Except as a passive obstruction, no piece has power over the square on which it stands. A piece standing on a given square obstructs the forces which would otherwise pass through, and receives both these, and also those which terminate at the given square, into itself. The forces thus received by a piece are of two kinds—those coming from friendly, and those from adverse, powers. That which gives intensity to a force coming from an adverse power is the possibility that capture is intended. That which gives virtue to a force coming from a friendly power is the security it affords, that, if such capture take place, reprisals will follow: a piece is thus said to be "guarded." It is of importance, however, to remember that the king can neither be taken nor guarded in this way.

Now, when exchanges take place, it is of moment to know the mean value of the pieces involved. When two pieces stand in a relation of mutual antagonism, the effect of each upon the other is inversely as their respective mean values, other things being equal. In fact, it is a law in Chess, that where the capabilities of two or more pieces are, in respect of any given mode of movement, identical, that which has the least mean value is really the most potent for the time being. Within its sphere, no piece is so powerful as the pawn, simply because no piece can incur such risks.

In estimating the mean value of the several chessmen, we must first consider their range on a clear board, and how this is affected by their position. For there are three ways in which the full powers of the various pieces may be curtailed; viz., (1) by the piece having a position on the board where it has less scope, (2) by the presence of obstructions to its progress in the shape of other pieces, and (3) by its being captured by the enemy and removed from the board altogether.

As to the effect of *position* on the range of a piece, it may be remarked that the castle is the only piece which in any position, on a clear board, commands the same number of squares. Whether in the middle of the board, or at the sides, or even in the corners, the castle, when the board is clear, commands 14 squares, not including the one it occupies. But every other piece varies in range, according to its position, considerably. For the purpose of elucidating this variation in range, it is advantageous to divide the chessboard into a series of concentric square plots. Thus, the whole board consists of 8^2 squares; within that is a set of 6^2 squares; within that, again, a set of 4^2 squares; and finally a central set of 2^2 squares. Having done this, we can assert that only the castles have full power in any part of the 8^2 set; that the king and pawns (as pawns) have their full power only inside the 6^2 set; the knights only inside the 4^2 set; and the queen and bishops only within the very central set of all, the 2^2 set. The bishop of each colour is therefore limited to two squares when desirous of exhibiting the full extent of his powers; and there are only four squares whence even the queen can take her full command of the board. The following is a tabular statement of the maximum and minimum range on a clear board of each piece:—

	Max.	Min.		Max.	Min.
King	8	3	Bishop	13	7
Queen	27	21	Knight	8	2
Castle	14	14	Pawn	2	1

To find the *average* range of a piece on a clear board, it is necessary to take the sum of its several ranges in every possible position, and thence deduce the mean. We obtain the following results:—

	Average.		Average.
King	$6\frac{9}{16}$	Knight	$5\frac{1}{4}$
Queen	$22\frac{3}{4}$	Pawn, for taking	$1\frac{3}{4}$
Castle	14	Pawn, moving only	$1\frac{1}{8}$
Bishop	$8\frac{3}{4}$		

And, taking the range of the pawn (for capture) as unit, this gives—pawn 1, knight 3, bishop 5, castle 8, queen 13; which is precisely the result arrived at, though in a somewhat different manner, by Mr. Miller.

It must not be supposed, however, that this table gives a strictly true account of the comparative value of the several pieces. There are many ingredients in the character of a piece, and even in the mode of its movement, besides the range, which greatly affect its value. This will appear as we consider the subject of *Obstructions*, which we have classed next to Position, as a main cause of varying powers in any piece.

Under the head of Obstructions is to be classed whatever interferes with the movement of a piece, in any of its characteristic directions, over and beyond mere faults of position which arise from the constitution of the chessboard and the arrangement of the squares. The cause of obstruction is almost invariably some other piece, the only exceptions arising from certain laws of the game, such as that in regard to "castling," in which it is laid down that castling cannot be accomplished if either the king or the particular castle have previously moved, nor if the king be in check, nor if the enemy command either of the two squares involved in the king's movement; viz., that on to, and that over, which he must move. But, although the cause of obstruction is almost invariably some other piece, the piece immediately obstructing is not necessarily an enemy. On the contrary, the most troublesome obstructions are, as a rule, pieces of the same side. This is especially the case at the beginning of the game; and, if such obstructions be traced to their origin, they are found to arise very generally from the peculiarity of the pawn, which, in capturing, takes a different direction from that in which it otherwise moves, and which accordingly cannot itself remove from its onward path that which obstructs it, but is bound to remain where it is so long as the obstacle is before it, unless the enemy place some piece in diagonal juxtaposition with it, or some other piece of its own side can capture the obstacle. It is easy to see how a pawn thus circumstanced will interfere with the free movement of its superiors, and how moves are inevitably lost in making the attempt either to get it out of the way, or to take a circuitous course round it. The pawns which, as a rule, open the game on each side (the kings' pawns) are thus situated from the first. But it is of importance to remember that the advance of the enemy is in this instance retarded no less than our own.

Obstructions may arise when the very square to which we are desirous of moving some piece is (1) already occupied, or (2) commanded by the enemy, or (3) when one or more intervening squares are occupied.

The king, knight, and pawn are never affected by (3); no piece but the king by (2) absolutely,—though other pieces, of course, incur the risk of capture when placed on a square which the enemy commands,—and, except in the case of the pawn, even (1) is an absolute bar only when the piece occupying the square is of the same side as that which it is sought to move thereto.

An adverse piece, even when unguarded, if it come under any but the first category, is an obstruction of a serious nature. For, if it command the desired square, that it is itself unguarded is a matter of very little moment; and if it intervene between the said square and our piece, it necessitates for its capture at least a distinct move, and this may even terminate the game, as where it prevents our at once sheltering our king from check, and where the loss of a move is the loss of everything. But even this is not so galling as to suffer defeat through the stupid obstructiveness of some piece of our own, which, though itself free to move, is in the way of some other piece, and requires time for its removal. Nor is it irrelevant to our subject to remark here how important a factor in the game of Chess is *time*, and what endless complications arise from the fact that each player is rigidly restricted, whenever it becomes his turn to play, to a single distinct act. "Castling" and giving "check by discovery" are no real exceptions to this law.

But it is necessary to observe that obstructions are not only such as prevent the movement of a piece from the square it stands on to some position which is peculiarly valuable. It may so happen that a piece is attacked, either by one of less value or by one like the knight, whose assault it is utterly unable to parry, and where flight is absolutely essential to safety. Then, if there are obstacles in the way to prevent its retreat, its case is hopeless, and the loss may be considerable, unless the enemy's attention can be diverted for a time to some other part of the board, and the play be gradually brought round so as to interfere with those obstacles or the attacking piece. In cases of this kind, the position which the piece occupies on the board is of considerable importance. For instance, the queen, if in the middle of the board, requires eight distinct obstacles to prevent her moving; whereas, at the boundaries, five will suffice, and, in the corners, three. And much the same may be said of the other pieces; but we need speak of no other except the king, who, after all, is chiefly concerned in this matter, since if he be attacked, and there be no retreat open to him, nor any piece to interpose in his defence, he is checkmated, and the game is lost.

The king, if at perfect liberty, has the same number of directions to choose from as the queen, though his move, except in castling, extends to the next square only. But he is differently affected by obstructions; for whereas, in the case of the queen, a distinct obstacle in each direction is requisite to prevent movement absolutely, the same adverse piece will often suffice to prevent movement in several directions in the case of the king, owing to the fact that he moves but one square at a time, and cannot move even to a square that is vacant, if an adverse piece *commands* it.

For the sake of convenience, we may regard the squares surrounding the king, and to any one of which he may move when perfectly free, as forming, together with his own, a set consisting of three rows of three squares each. An adverse piece, whether standing at a distance or near, may obstruct the king's passage to a variable number of those adjacent squares. Thus, even the pawn, if placed on the square immediately in front of the king, or on that next but one in front, prevents the king from moving to two squares. In the former case, indeed, it appears to obstruct movement in *three* directions; that is, to its own square as well as to two others. But this is an appearance only; for, where the king is concerned, actual obstacles arise only from the presence of pieces of his own side. We have already observed that an adversary may intercept the passage of other pieces, even when itself unguarded, and thus cause the loss of a move; but this cannot happen to the king. Nothing on the part of an adverse piece can obstruct the king's movements but the fact of its *commanding* certain of the squares around him; for he may capture an unguarded adversary that stands on a square adjacent to him, and in the case of one that is guarded the entire obstruction to his appropriating its square lies in the piece or pieces guarding. But it is of importance to remember that to the king an adverse piece is equally obstructive, whether guarded or unguarded, so far as concerns the squares which it commands.

We may next remark that it often becomes necessary to obstruct the movement of the adversary's king before directly "checking" him. If we can hold him to his present position, though he be not in check, or if we can so obstruct his movements as to drive him at any moment wheresoever we will, the most difficult part of the process of checkmating is already accomplished.

We have already given the mean *range* of each piece on a clear board: excluding the king, the same table will serve to indicate the average number of squares from which the adverse king can be "checked" by the several pieces mentioned; since it is obvious, that whatever be the number of squares included in the range of any piece, such also will be the number of distinct positions from which, on a clear board, the same piece could command the said square, or check the king standing on it. It is owing to this fact that Mr. Miller arrived at the same result as given in the Table of Mean Ranges, he having taken the power of checking as his guide.

But we have shown that to check the king is not always the most serviceable act which a piece can perform. Obstruction of his movements, with or without checking, is equally necessary. Consequently, it is of importance that we discover what power the several pieces have of affecting the adverse king's movements in the various positions in which he may be placed, and what is the average of each piece for the whole board.

Then, since there is an advantage in being able to obstruct as regards several directions by means of one piece,—the queen, for instance, when favourably placed, can check the adverse king, and at the same time prevent his moving in as many as five directions,—it is of importance to calculate the comparative value of each piece in this respect also. But, although it may be doubtful whether a piece which is able to obstruct as regards two directions from three distinct positions, is on the whole of the same value as one that is able to obstruct as regards three directions from two positions, they will in the present paper be held as equivalent.

Then, another fact to be taken into consideration, and which Mr. Miller has partially taken into account under the title of "Safe-checking," is, that some pieces can affect the king from various positions without placing themselves in danger of being captured by him, whilst others incur this danger more frequently, and require to be guarded. There is, on the contrary, the comparative possibility of interception to be taken into account; for the farther from the king the position whence a piece might affect him, the greater the opening for interposition, except in the case of the knight.

Now, our calculations can be greatly facilitated by the recollection that the chessboard can be divided into similar and equivalent blocks of squares; so that, if we suppose the king to stand in succession upon the several squares of any one such block, and examine the powers of the various pieces in reference to him there, we are in a position to state without further trouble what are their powers in reference to him, when he is placed on the similar squares in the other blocks. Thus, by dividing the board into four equal blocks by two lines drawn through the centre, at right angles to each other, and parallel to the sides, it is not difficult to perceive that the squares of the several blocks are similarly placed as regards the whole board, and consequently that whichever block the king be placed in, if he stand on similar squares, the total power over him of any piece, except the bishops (which can only command squares of the colour they stand on) and the pawns (which have no command at any time over the first two rows of squares), will be the same on a clear board.

The annexed diagram shows which are the similar squares in the several blocks:—

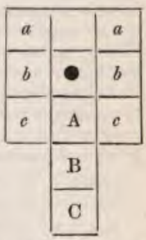
Q	M	H	D	N	O	P	Q
P	L	G	C	I	K	L	M
O	K	F	B	E	F	G	H
N	I	E	A	A	B	C	D
D	C	B	A	A	E	I	N
H	G	F	E	B	F	K	O
M	L	K	I	C	G	L	P
Q	P	O	N	D	H	M	Q

But the effect of similarity in lightening our labours does not end here; for in each block there are squares which, as regards the whole board, are, to all intents and purposes, similarly situated. Thus B = E, C = I, D = N, G = K, H = O, and M = P. The only squares that have none corresponding in the same block are A, F, L, Q. We have therefore only to make our computations in regard to ten squares, to be furnished with data for calculating the powers of the various pieces, so far as their influence upon the movements of the adverse king is concerned, in respect of the whole board. The process is very simple; for, given the power of a piece as regards each of the aforesaid ten squares, we double it in the case of the six that have corresponding squares in the same block; then, adding these several sums to the other four, derived from our examination regarding the single squares A, F, L, Q, we obtain the sum total for the block; and, dividing this by 16, we obtain the mean power of the piece for the whole board.

We have, however, excepted the bishops and the pawns. The bishops, being each restricted to one colour, meet with strictly similar squares in two only out of the four blocks. The two bishops are equal in power as regards the whole board, though very different as regards any particular square. The plan to adopt, therefore, in regard to them is to examine them together as regards each of the aforesaid ten squares, to add together the twin results in each case, and, treating the several sums as above, to halve the final result. The mean power of each bishop is thus obtained.

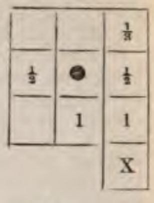
To find the mean power of the pawns, our plan is to take the board by columns. Over two squares in each column the pawns have no command. Moreover, when once they have moved forward, they cannot, as the other pieces can, return to their former position. Our calculations as to the possible effect of the pawns upon the king must necessarily be alike for the four central columns. But the border columns differ considerably, inasmuch as the king when standing on one of their squares can be checked by a pawn on one side only, and the columns next to these are also different, in that the king, when in them, cannot be obstructed by pawns from so many positions. However, the results for the eight columns being added together, the sum must be divided by 64, and again by 8, to give the mean power of the single pawn. In the following calculations the capability of *obstructing* the adverse king, though in regard to one direction only, is held as of equal value with *checking;* and to obstruct in regard to two, three, four, or five directions, as of equal value with the capability of checking from two, three, four, or five distinct positions.

A piece placed in immediate proximity to the king, and therefore in danger, unless supported, of being captured by him, is deemed to be only half as potent as one having equal command from a distance. To take an example, there are three positions on each side of the king (when in the middle of the board), from which the knight can, without checking, obstruct his movements in regard to two directions. The positions of the knight are represented in the accompanying diagram by the squares A, B, C, and the squares commanded are *a, a*; *b, b*; *c, c*, respectively. It is manifest that, in estimating the value of the knight, we must not regard him as equally valuable at all three. At either B or C, he is of double the value that he is at A, because he obstructs to the same extent, and does not require the aid of another piece to prevent his immediate capture.



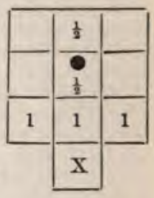
The liability to interposition from which every piece operating from a distance suffers, except the knight, it is somewhat difficult to estimate aright. But there appears to be little doubt that the power of a piece in regard to any square that it is capable of commanding, is, other things being equal, in inverse ratio to the distance that intervenes; for, supposing the object be to check the king, if the chance be *one* that a square adjoining him shall be unoccupied, there will be only *half* the chance that this square and one farther away in the same direction will be vacant, only a *third* of the chance that still another will be vacant, and so on. And the same law holds good with regard to obstructing in all its phases. An illustration will best explain the matter.

The queen (X), in the position indicated in the diagram, is capable of obstructing the king as regards five directions. Now, we consider the chance to be *one* that she would be able to take up that position, and therefore *one* also that she should command the squares adjacent, since the one fact follows upon the other without fail. But the two adjacent squares indicated may, one or other or both, be occupied, and then, although the king's movements in respect of them are obstructed all the same, the queen's command of the more remote squares is curtailed. The chances in regard to the five squares are as indicated in the diagram, and we consider the queen's power in respect of such a position to be $1 + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{3} = 3\frac{1}{3}$, instead of 5, as it would have been but for the chance of interposition.



In the following diagram the queen gives check, and obstructs as regards five directions also; but, since she is in close proximity to the king, we regard her power (for reasons given above) as only half what it otherwise would be; that is, 3 instead of 6. This diagram will also serve to show, what must be borne in mind, that the adverse king is never an obstacle in the same sense that other pieces are. The power of the queen to obstruct the king as regards the square beyond him is the same as though he did not stand between.

The strongest position in which the queen, when unsupported, can be placed as regards the adverse king, is shown in the adjoining diagram. Here she insures his being obstructed in regard to three squares in any case, has a half-chance of checking him, and the same of obstructing him as to one square more. Her power, therefore, in respect of such a position, is $1 + 1 + 1 + \frac{1}{2} + \frac{1}{2} = 4$.



In estimating the relative value of the several pieces, so far as their power over the adverse king is concerned, we must suppose them to be equally liable to find the particular squares from which they could affect the king's movements already occupied, and equally liable to be captured by other pieces when placed thereon. Every square on the board must be regarded as neutral in these respects, the chances the same for all.

Let us then proceed to calculate the power of each piece as estimated by the average number of squares from which it can affect the adverse king's movements, and by the average amount of influence it exerts over him from each position. The labour is considerable, but the result is as follows:—

Queen = 74·9848, Knight = 37·2500,
Castle = 41·1729, Bishop = 16·0375,
Pawn = $\frac{1}{8}$ (7·3438) = ·917975.

Let it be observed, however, that this table exhibits, at best, the relative value of the several pieces when regarded from one point of view only; that is, as employed offensively against the adverse king. This is their chief office, but it is not the only one, since they have also to defend their own king, and in this aspect their relative value is very different, as we shall presently see.

The exaggerated power which the knight seems to have is due to the fact that he is the only piece that has the full power of checking; that is, of being placed in a position to check without being liable to the interposition of some other piece, and at the same time without incurring the risk of immediate seizure by the king. The small apparent power of the bishop is due to his being limited to squares of one colour; and the comparative insignificance of the pawn under this head, is due to his inability to move out of his column, except by capturing an enemy, and to his inability to move backwards.

Still, even regarding the pieces simply as weapons of offence, the table ought to be somewhat modified. The power of the several pieces has been deduced in great measure from the number of distinct positions whence each can affect the adverse king. But nothing has yet been said of the time taken by each piece in reaching this limited area from other parts of the board, nor of the varying conditions under which moves are made, so far as freedom is concerned, as the game advances.

The consideration of these two points will lead to a considerable modification of the table.

We will speak of the last mentioned first. There is a gradual diminution in the liability of the pieces to meet with obstructions in their path, which necessarily benefits the queen, castle, and bishop more than the knight and pawn. It is somewhat difficult to determine accurately the rate of diminution, or to deduce therefrom the average chances of obstruction during the whole game. But if we consider the chances to be equal at the beginning of the game, seeing that there are then the same number of squares occupied as unoccupied, they will be as 3 to 1 against a square being occupied when half the pieces are removed, and as 7 to 1 when only eight pieces remain on the board. The mean chance of finding any given square unoccupied, if reckoned from the beginning of the game until only the two kings are left (if such a thing ever occurs), is $5\frac{1}{3}$ to 1, although under the same circumstances the average number of squares unoccupied would be 47, to 17 occupied, or $2\frac{13}{17}$ to 1. But we must not estimate the value of pieces upon any such extreme condition of things as this. The game does not usually last until the board is cleared, nor are pieces removed from the board at regular intervals. Moreover, the chance of finding a square unoccupied increases with no uniform motion, but with rapidly accelerated strides, as piece after piece is removed. We must therefore adjust matters a little. A mean between the degree of freedom enjoyed on a clear board, and that which exists at the beginning of a game, when half the squares are occupied and half unoccupied, seems to be the fairest that can be selected. We have already given the average number of squares commanded by the several pieces in the various positions on the board when clear, and which we call their *range*. And we are now able to give the average number of squares which they command under circumstances of obstruction, when the chances are equal that a square will be occupied or unoccupied.

	Range on clear board.	Scope under equal chance of obstruction.	Mean between Range and Scope.
Queen	22·75	12·15	17·45
Castle	14·00	6·87	10·44
Bishop	8·75	5·12	6·94
Knight	5·25	5·25	5·25
Pawn	1·75	1·75	1·75

In which case, the pawn being 1, the knight is 3, the bishop 4, the castle 6, and the queen 10.

Applying the ratio between the *scope* and the *mean* in the above table to the figures in the table preceding, we find the mean effect produced upon the adverse king by the several pieces to be as follows :—

Queen = 107·7,	Bishop = 21·7,	Pawn = ·918.
Castle = 62·6,	Knight = 37·25,	

As to the extent of the moves of the several pieces, we may observe that that of the knight never varies, but may always be expressed by the length ($=\sqrt{5} = 2{\cdot}236$) of diagonal of a figure two squares in length and one square in breadth. The moves of the other pieces do vary in extent to a very considerable degree ; and it is somewhat difficult to apportion the proper value to the different modes of motion. Thus it is a question whether the bishops, who move diagonally, should be allowed the full value of such a movement, as expressed in the distance actually traversed, which, in respect of every square passed over, is $\sqrt{2}$ instead of 1. But we have given the knight the full benefit of a somewhat similar peculiarity, and there is a disadvantage at which the castle, and the queen, in so far as she moves like the castle, stand when compared with the bishop, which must not be disregarded. The chief obstacles to the passage of the superior pieces are necessarily (from their number) the pawns. The pawns move directly forward only, except when capturing. Consequently the castle, half of whose possible moves lie in a line with those of the pawns, is less likely to get rid of the obstruction than is the bishop, who has only the particular square on which the pawn stands obstructed, and who is freed the moment the pawn moves. The bishop can move through pawns when they are not in a row, though they stand on adjacent squares ; but the only hope of the castle, under similar circumstances, lies in one of the pawns being bribed to move out of the column in which it stands, by the offer of an adverse pawn, or, if that be not enough, of some superior piece ; and, after all, his worst obstacles are pawns of his own colour. It seems only right, therefore, to give the bishop the advantage which accrues from his diagonal motion, and to multiply the squares he moves over by $\sqrt{2}$. Moreover, we may regard the pawn as having, on the average, one opportunity of making the diagonal movement by capture, together with the power of making the two-square move at starting, and as having two single-square moves besides. It will then be fair to calculate the average moves of the bishop and castle by reference to their choice of moves when situated on the several squares of a clear board, and by giving to each possible move an equal probability. That of the queen will be the mean between the two.

The average moves of the various pieces, thus calculated, are as follows :—

Queen = 3·2434,	Knight = 2·2360,	Bishop = 3·4867 = $\sqrt{2}$ (2·4655).
Castle = 3·0000,	Pawn = 1·3540,	

Here the bishop has the pre-eminence, even over the queen, who can move precisely like him, but whose superiority consists in being able also to move like the castle, in which case the squares she passes over do not count for so much, and therefore give her a lower average.

If now we multiply the figures given in the Table of Mean Scope or Command by those now obtained for the average moves, we gain the following estimate—which may be considered a pretty fair one—for the comparative values of the several pieces for general purposes; that is, of an offensive character:—

Queen = 56·60,	Bishop = 24·20,	Pawn = 2·37.
Castle = 31·32,	Knight = 11·74,	

By raising these figures so as to make the same sum as those given in the Table of Comparative Influence over the adverse king—by multiplying each by 1·8235,—and then striking the difference between the two sets of figures, we obtain the following mean values of the pieces for all offensive purposes:—

Queen = 105·45,	Bishop = 32·92,	Pawn = 2·62;
Castle = 59·86,	Knight = 29·33,	

or, reducing the queen to 10, the values are as follows:—

Queen = 10·00,	Bishop = 3·12,	Pawn = 0·25.
Castle = 5·68,	Knight = 2·78,	

This is the mean comparative power of attack, referred to the queen as standard, rather than to the pawn, whose power is the least capable of being determined with any precision, as we shall see presently. Suffice it now to say, that when we consider the small chance that any particular pawn has of being in a position directly to affect the adverse king's movements, and the very limited number of squares that it can possibly come in contact with, we can readily understand why the average power assigned to the pawn, in its offensive capacity, should be so small.

As to the knight and bishop, it is of importance to remember that at the beginning of the game the knight is considerably superior to the bishop, and that it is not until about five pieces have been removed from the board that the bishop rises superior to him. In the matter of forcing the adverse king to move, the bishop never does rise superior to the knight.

We must now turn to consider the various pieces in their other aspect, viz., that of *defence*.

Defence is of two kinds—by interposition and by guarding. The former is the only method by which the king can be defended, but is rarely of avail unless the interposed piece take up a guarded position; that is, a square next the king, or one commanded by another friendly power. Interposed pieces are liable to be taken. Consequently, when interposition becomes necessary, and we have two or more pieces commanding a square in the enemy's line of attack, we naturally select the piece of least worth, unless there be some very good reason to the contrary. The same law obtains when exchanges occur.

For defensive purposes, therefore, the value of pieces bears a species of inverse ratio to that which they possess as agents of attack. Nevertheless, the ratio is not purely inverse, because the greater the range of a piece, upon which its value in attack mainly depends, the greater also its power of interposing in defence of the king and other pieces. Accordingly, we may regard the value for defensive purposes to be *inversely* as the entire value, and at the same time *directly* as the mean scope under average conditions. The entire value of a piece is the product of its separate values for offensive and defensive purposes, those separate values depending upon qualities that conduce to one end.

Let x be the entire value, a the value as agent of attack, and b the mean scope; then $\frac{ab}{x} = x$, whence $x = \sqrt{ab}$; consequently, to find the entire value of a piece, we multiply its value as an agent of attack by its mean scope, and take the square-root of the product. This gives the following results:—

Name of Piece.	Value in offensive capacity.	Mean scope under average conditions.	Entire Value.
Queen	10·00	10·0	10·00
Castle	5·68	6·0	5·84
Bishop.........	3·12	4·0	3·53
Knight	2·78	3·0	2·89
Pawn	0·25	1·0	C·50

But we have not taken into account the possibility of the pawn reaching the last row of squares and becoming a queen; and it is the difficulty of determining the value attaching to this possibility that renders the pawn, as we have before intimated, so unfit to be a standard by comparison with which to reckon the values of the other pieces. To *queen* a pawn is a matter of such infrequency, that it seems at first sight scarcely fair to introduce it as a modifying circumstance. On casually opening STAUNTON'S *Chess-Player's Handbook*, out of 24 games examined, only two contain an instance of the kind. In one of these, moreover, the pawn's aggrandisement is followed by instant defeat. And, even supposing that it occurs on the average once in every ten games, this is one pawn out of sixteen becoming queen once in ten games; and, supposing that the promoted pawn enters into the play during a fifth of the game, the mean value of the pawn will only be advanced from 0·5 to 0·512. If it occurred once in every game, and the promoted pawn played as queen during four-fifths of the game, the mean value of the pawn would not be doubled. At the same time, there can be no doubt that, although the probability of the pawn's aggrandisement is small, the possibility alone makes attention to the adverse pawns imperative, and lends a value to these humble agents which they would not otherwise possess. It may not seem unreasonable, therefore, to raise their mean value from ·5 to ·75. But this is quite arbitrary.

We have now accomplished the task we took in hand, and it remains only to express a wish that ere long some means may be devised of accurately registering the progress made during a game of chess. We can score in cricket and tennis and billiards, and positively tell what are good and bad hits; but the progress made in chess, except in so far as the capture of pieces is concerned, is in many instances a matter of the barest conjecture, until the final check puts an end to doubt.

Now, the mean scope of the several pieces, and the average effect they produce on the adverse king's movements, have largely entered as factors into the computation of their mean value. It is reasonable, therefore, to believe that in their actual scope at any time, and the actual command they have over the adverse king's squares, will be found the true solution of the difficulty, care being taken not to neglect the question of defence meantime.

Since writing the above, I have had the advantage of examining, at the Reading Room of the British Museum, several works on the subject of my paper, of which Mr. MILLER has been kind enough to make a brief catalogue for my use. The chief of these are PRATT'S or PHILIDOR'S *Chess Studies*, and TOMLINSON'S *Amusements in Chess*, the latter being little better in this particular than a reprint of the former. In fact, PHILIDOR appears to have been the pioneer in investigations of this sort, and TOMLINSON and STAUNTON have copied his results *verbatim*. The book I saw bore date 1817, and gave the values of the pieces as follows:—

Pawn 1·00, Knight 3·05, Bishop 3.50, Rook 5·48, Queen 9.94.

In the *Westminster Chess Club Papers* for July, 1876, Mr. PEIRCE, a correspondent, gives the following *résumé* of TOMLINSON'S treatment of the question (which I find to be a copy of PHILIDOR'S):—(1) Average power to move over open board. (2) Power of preventing pieces occupying any square in a particular line (about *middle* of game). Relative power of commanding squares. (3) Power in choosing what point to select as a position of attack. (4) Power to compel removal of assailed piece. (5) Power in giving mate.

Mr. PEIRCE points out that the various results are *added* to arrive at the final average. I am glad I had not seen the investigations of PHILIDOR or TOMLINSON before making my own independent calculations.

7456. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If $u = 0$ be the rational equation of a quadric referred to rectangular axes, prove that the locus of the point of concurrence of three tangent lines, at right angles to each other two and two, is $\frac{d^2 u^{\frac{1}{2}}}{dx^2} + \frac{d^2 u^{\frac{1}{2}}}{dy^2} + \frac{d^2 u^{\frac{1}{2}}}{dz^2} = 0.$

[The corresponding equation when the coordinate axes are inclined at angles α, β, γ is $\frac{d^2 u^{\frac{1}{2}}}{dx^2} + \frac{d^2 u^{\frac{1}{2}}}{dy^2} + \frac{d^2 u^{\frac{1}{2}}}{dz^2} = 2\cos\alpha \frac{d^2 u^{\frac{1}{2}}}{dy\, dz} + \ldots + \ldots$].

Solution by G. B. MATHEWS, B.A.; T. WOODCOCK, B.A.; *and others.*

$$\delta_x u^{\frac{1}{2}} = \tfrac{1}{2}u^{-\frac{1}{2}}\delta_x u, \quad \delta_x^2 u^{\frac{1}{2}} = \tfrac{1}{2}u^{-\frac{1}{2}}\delta_x^2 u - \tfrac{1}{4}u^{-\frac{3}{2}}(\delta_x u)^2 = \frac{1}{4u^{\frac{3}{2}}}\left[2u\,\delta_x^2 u - (\delta_x u)^2\right],$$

and similarly for $\delta_y^2 u^{\frac{1}{2}}, \delta_z^2 u^{\frac{1}{2}}$; thus, if $u \equiv (abcdfghlmn \between xyz1)^2$, so that

$$\delta_x^2 u = 2a, \quad \delta_x u = ax + hy + gz + l, \ \&c.,$$

the equation $\nabla^2 u^{\frac{1}{2}}$ becomes in its rational form

$$4(a+b+c)\,u - (\delta_x u)^2 - (\delta_y u)^2 - (\delta_z u)^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the tangent cone from (ξ, η, ζ) to the quadric may be written

$$4\,(abc \ldots \between \xi\eta\zeta 1)^2\, u - [x\delta_\xi u + y\delta_\eta u + z\delta_\zeta u + \ldots]^2 = 0,$$

and, if this have three generators at right angles, the sum of coefficients of

a^2, y^2, z^2 is zero, that is

$$4(a+b+c)(abc\ldots)(\xi\eta\zeta 1)^2-(\delta_\xi u)^2-(\delta_\eta u)^2-(\delta_\zeta u)^2=0,$$

and, considering (ξ, η, ζ) to be current coordinates, this agrees with (1); therefore, &c.

7473. (By R. Rawson.)—If v, u, X are given functions of x, show that $y = y_1 + y_2$ is the complete integral of

$$\frac{d^2y}{dx^2}+\left(v-\frac{du}{w\,dx}\right)\frac{dy}{dx}+\left(\frac{v^2-w^2}{4}+\frac{dv}{2dx}-\frac{v\,dw}{2w\,dx}\right)y=\mathrm{X}\ldots\ldots\ldots(1),$$

where y_1, y_2 satisfies the equations

$$\frac{dy_1}{dx}+\left(\frac{v+w}{2}\right)y_1=-\frac{\mathrm{X}}{w}+\frac{dy_2}{dx}+\left(\frac{v-w}{2}\right)y_2=w\ \ldots\ldots\ldots(2).$$

Solutions by (1) Professor Malet, F.R.S.; (2) *the* Proposer.

1. Consider the more general problem, to find the linear differential equation of which the solution shall be the sum of the solutions of the equations $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, and $\frac{dy}{dx}+\mathrm{R}y=\mathrm{S}$, where P, Q, R, S are functions of x. We have to eliminate y_1 and y_2 from

$$\frac{dy_1}{dx}+\mathrm{P}y_1=\mathrm{Q},\quad \frac{dy_2}{dx}+\mathrm{R}y_2=\mathrm{S},\quad y=y_1+y_2;$$

or y_1 from the first of these equations, and $\frac{dy}{dx}+\mathrm{R}y+(\mathrm{P}-\mathrm{R})\,y_1=\mathrm{Q}+\mathrm{S}$.

The result is $$\frac{d^2y}{dx^2}+\left\{\mathrm{R}+\mathrm{P}-\frac{\mathrm{P}'-\mathrm{R}'}{\mathrm{P}-\mathrm{R}}\right\}\frac{dy}{dx}+\left\{\mathrm{RP}+\frac{\mathrm{R}'\mathrm{P}-\mathrm{P}'\mathrm{R}}{\mathrm{P}-\mathrm{R}}\right\}y$$

$$=\mathrm{Q}'+\mathrm{S}'+\mathrm{QR}+\mathrm{PS}-(\mathrm{Q}+\mathrm{S})\frac{\mathrm{P}'-\mathrm{R}'}{\mathrm{P}-\mathrm{R}},\quad\text{where}\quad \mathrm{P}'=\frac{d\mathrm{P}}{dx},\ \&c.$$

For the values of P, Q, R, S given in queetion, the sinister of this equation is the same as Proposer's, and if we make the $\mathrm{S}=\mathrm{X}/w$, instead of the w in the question, the dexter reduces to X.

2. Differentiating (2), we have

$$\frac{d^2y_1}{dx^2}+\left(\frac{v+w}{2}\right)\frac{dy_1}{dx}+\left(\frac{dv}{2dx}+\frac{dw}{2dx}\right)y_1=-\frac{d\mathrm{X}}{w\,dx}+\frac{\mathrm{X}\,dw}{w^2\,dx},$$

$$\therefore\ \frac{d^2y_1}{dx^2}=\frac{(v+w)\,\mathrm{X}}{2w}-\frac{d\mathrm{X}}{w\,dx}+\frac{\mathrm{X}\,du}{w^2\,dx}+\left\{\frac{(v+w)^2}{4}-\frac{dw}{2dx}-\frac{dv}{2dx}\right\}y_1\ldots(3);$$

$$\frac{d^2y_2}{dx^2}=-\frac{(v-w)\,\mathrm{X}}{2w}+\frac{d\mathrm{X}}{w\,dx}-\frac{\mathrm{X}\,dw}{w^2\,dx}+\left\{\frac{(v-w)^2}{4}+\frac{du}{2dx}-\frac{dv}{2dx}\right\}y_2\ldots(4).$$

But $$\frac{dy}{dx}=\frac{dy_1}{dx}+\frac{dy_2}{dx}=-\frac{(v+w)}{2}y_1-\left(\frac{v-w}{2}\right)y_2\ldots\ldots\ldots\ldots\ldots(5),$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2} + \frac{d^2y_2}{dx^2}$$
$$= X + \left\{\frac{(v+w)^2}{4} - \frac{du}{2\,dx} - \frac{dv}{2\,dx}\right\} y_1 + \left\{\frac{(v-w)^2}{4} + \frac{dw}{2dx} - \frac{dv}{2dx}\right\} y_2 \ldots (6).$$

Then
$$\left(v - \frac{dw}{w\,dx}\right)\frac{dy}{dx}$$
$$= \left\{\frac{v\,dw}{2w\,dx} + \frac{du}{2dx} - \frac{v^2}{2} - \frac{wv}{2}\right\} y_1 + \left\{\frac{v\,dw}{2w\,dx} - \frac{dw}{2\,dx} - \frac{v^2}{2} + \frac{wv}{2}\right\} y_2 \ldots (7),$$
$$\left\{\frac{v^2-w^2}{4} + \frac{dv}{2dx} - \frac{v\,dw}{2w\,dx}\right\} y$$
$$= \left(\frac{v^2-w^2}{4} + \frac{dv}{2dx} - \frac{v\,dw}{2w\,dx}\right) y_1 + \left(\frac{v^2-w^2}{4} + \frac{dv}{2dx} - \frac{v\,dw}{2w\,dx}\right) y_2 \ldots\ldots (8).$$

Add (6), (7), (8), then the result in the question is obtained. Many interesting cases are included in (1) by giving special values to v, w.

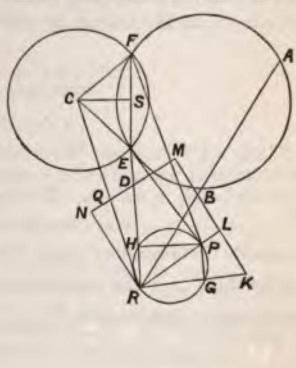
When $v = \dfrac{du}{w\,dx}$, $\quad \dfrac{d^2y}{dx^2} + \left\{\dfrac{d^2u}{2w\,dx^2} - \tfrac{3}{4}\left(\dfrac{dw}{w\,dx}\right)^2 - \dfrac{w^2}{4}\right\} y = X$ is soluble,

when $u = 2ax^{2n}$, $\quad \dfrac{d^2y}{dx^2} - \left(\dfrac{n(n+1)}{x^2} + a^2x^{4n}\right) y = X$ is soluble,

when $u = \dfrac{2a}{\beta}$, $\quad \dfrac{d^2y}{dx^2} + \left\{\tfrac{1}{4}\left(\dfrac{d\beta}{\beta\,dx}\right)^2 - \dfrac{d^2\beta}{2\beta\,dx^2} - \dfrac{a^2}{\beta^2}\right\} y = X$ is soluble.

7462. (By the Editor.)—Through two given points (A, B) draw a circle such that its points of intersection with a given circle (of centre C), and a third given point (P), shall form the vertices of a triangle of given area.

Solution by Rev. T. C. Simmons, M.A.; G. B. Mathews, B.A.; *and others.*



1. Through A, B draw any circle whose radical axis with the circle C meets AB in R; then R must evidently be a point on the chord of intersection of C with the required circle. On RP as diameter draw the circle RHP; take RP . RL = twice the given area; on RL construct a square RLMN; through R draw the line RDEF to meet MN and C so that RD = EF; let RD and RK perpendicular thereto meet the circle RP in H and G; then

FE . PH = RD . RG = RK . RG
= RP . RL = twice given area,

therefore ΔPEF = given area; hence the required circle is that drawn through the points A, B, E, F.

2. The following is an *algebraic* method for drawing the line RDEF so as to satisfy the condition RD = EF,—a problem which is proposed for a *geometric* solution in Question 7520:—Take R as origin, the line joining R to the centre of C as axis of x, and RL for axis of y; let the equation of C be $x^2+y^2+hxy+2gx+c^2=0$, and assume the equation of RD to be $y=mx$; then, at E and F, $x^2(1+m^2+mh)+2gx+c^2=0$; but the difference between the two values of x here equals a constant ($=2a$, say, = RQ), that is, putting $1+m^2+mh=p$, we have

$$(g^2-c^2p)^{\frac{1}{2}}/p=a \quad \text{or} \quad a^2p^2+c^2p-g^2=0,$$

giving two values of p, from each of which we obtain two values of m.

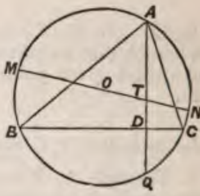
[If we put CE = CF = a = radius of given circle, RC = b, RP = c, $\angle$CRP = α, $\angle$CRE = θ, and k^2 = given area, we shall have

$$k^2=\tfrac{1}{2}\text{EF}\,.\,c\sin(\alpha-\theta)=c\sin(\alpha-\theta)\,(a^2-b^2\sin^2\theta)^{\frac{1}{2}},$$

whence θ and the line REF are determined.]

7457. (By Professor HUDSON, M.A.)—If I, O, T are the in-centre, circum-centre, and ortho-centre of a triangle, and r, R the in-radius and circum-radius; prove that $2\,\text{IT}^2-\text{OT}^2=4r^2-\text{R}^2$.

Solution by the Rev. T. C. SIMMONS, M.A.; R. NIXON, M.A.; *and others.*



Produce OT to meet the circumference in M, N; then we have

$$\text{R}^2-\text{OT}^2=\text{MT}\,.\,\text{TN}=\text{AT}\,.\,\text{TQ}$$
$$=2\text{AT}\,.\,\text{TD}=4r^2-2\,\text{IT}^2$$

(see Quest. 7179, *Reprint*, Vol. xxxix., p. 99); therefore $\quad 2\text{IT}^2-\text{OT}^2=4r^2-\text{R}^2$.

7429. (By Professor WOLSTENHOLME, M.A., D.Sc.)—The rectilinear asymptotes of the curve whose polar equation is $r(\sin\alpha-\sin\theta)=a\sin\alpha\cos\theta$ are $r\sin(\alpha\pm\theta)=a\sin\alpha$. The rectilinear asymptote of the curve

$$r=a\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta) \text{ is } r\cos\theta=2a.$$

Reconcile these results; since, if we put $\alpha=\frac{1}{2}\pi$ in the first equations, we get for the curve the equation $r=a\dfrac{\cos\theta}{1-\sin\theta}=a\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)$, and for the asymptote (the two then coinciding) $r\cos\theta=a$.

Solutions by (1) *the* PROPOSER, (2) G. HEPPEL, M.A., *and others.*

1. In the former curve $r = \infty$ when $\theta = \alpha$ or $\pi - \alpha$, and, taking $u = \frac{1}{r}$, the value of $\frac{du}{d\theta}$, when $\sin\theta = \sin\alpha$ is $-\frac{1}{a\sin\alpha}$; whence the asymptotes are as stated, and the curve is of the form in Fig. 1; α being taken between 0 and $\frac{1}{2}\pi$, O the origin and OA $= a$. The actual value of $\frac{du}{d\theta}$ is

$$\frac{1}{a\sin\alpha}\left(-1+\frac{\sin\theta\,(\sin\alpha-\sin\theta)}{\cos^2\theta}\right);$$

and, although the second term generally $= 0$ when $\sin\theta = \sin\alpha$, this is *not* the case when $\alpha = \frac{1}{2}\pi$, its limit then, when $\theta = \alpha$ (or $\pi - \alpha$), being $\frac{1}{2}$, so that $\frac{du}{d\theta}$ is then $-\frac{1}{2a}$, and the asymptote

$$r\cos\theta = 2a.$$

The discrepancy thus arises from neglecting to consider this term.

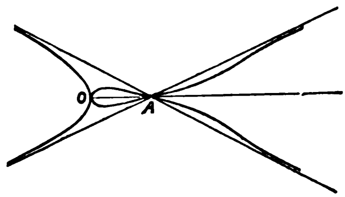


Fig. 1.

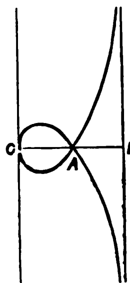


Fig. 2.

At the same time, it seems singular that, if we suppose α to change gradually from a value $< \frac{1}{2}\pi$ to a value the supplement of the former, the asymptotes should, as α passes through $\frac{1}{2}\pi$, take a sudden leap from A to B (Fig. 2), and back again; which they must do, since the curve is the same for $\pi - \alpha$ as for α.

It should be noticed that, when $\alpha = \frac{1}{2}\pi$, part of the locus of the equation $r(\sin\alpha - \sin\theta) = a\sin\alpha\cos\theta$ is the straight line $\theta = \frac{1}{2}\pi$, drawn through O at right angles to OAB.

2. Assuming the results to have been verified, let a line parallel to the axis of x, and at a distance y from it, have a portion c intercepted between the first curve and its asymptote.

Then $\quad c = x - a - y\cot\alpha = r\cos\theta - a - r\sin\theta\cot\alpha,$

therefore $\quad r(\cos\theta - \sin\theta\cot\alpha) = a + c;$

therefore $$\frac{\sin\alpha-\sin\theta}{\cos\theta-\sin\theta\cot\alpha} = \frac{a\sin\alpha\cos\theta}{a+c};$$

whence we obtain $$c = a\frac{\sin\frac{1}{2}(\alpha-\theta)\sin\theta}{\cos\frac{1}{2}(\alpha+\theta)}.$$

Now, if α have any value less than $\frac{1}{2}\pi$, $\theta = \alpha$ gives $c = 0$, as it should do. But, if $\alpha = \frac{1}{2}\pi$, then $c = a\sin\theta$, and then, if $\theta = \frac{1}{2}\pi$, $c = a$. Hence we are led to conclude that, if two intersecting asymptotes tend to coincide in consequence of the variation of some element, it does not follow that the single asymptote produced by this coincidence will pass through the point of intersection. Here the two, gradually closing like a pair of scissors, seem at the moment of closing to jump together to another position.

7414. (By R. Tucker, M.A.)—If from the "Brocard" points, O, O′, perpendiculars are drawn to the sides of the triangle, and their feet joined, two circumscribed triangles are obtained whose sides respectively make the same angles with the sides of the primitive triangle, and which have a common circumscribed circle; prove that the circumcentre, the centre of the "T. R." circle, and the point P, all lie on a straight line which bisects orthogonally the line OO′ in the centre of the above obtained circle. [The points O, O′ are got by making OBA = OCB = OAC = O′AB = O′BC = O′CA; the point P and the "T. R." circle are defined in the *Educational Times* for June, 1883, p. 178; and the minimum property is established in the *Ladies' and Gentlemen's Diary* for 1859, pp. 52—54.]

Solution by Charlotte A. Scott, B.Sc.

1. Let H be the circumcentre; then, to find the centre of the T. R. circle, we have $\angle$AFE′ = E′D′F′ (because E′D′F′F is circle) = C by known properties of the points D, E, F, &c.; therefore FE′ is perpendicular to AH. Also, AE′PF being parallelogram, AP bisects FE′ in α. Therefore, if T be the mid-point bisection of PH, Tα bisects FE′ at right angles, therefore the centre of the T. R. circle lies on αT. Similarly it lies on βT and γT, that is, T is centre of the T. R. circle.

2. If from A, B, C parallels be drawn to FD, DE, EF, we have

$\angle$DFB = DE′F′ = PE′F′,

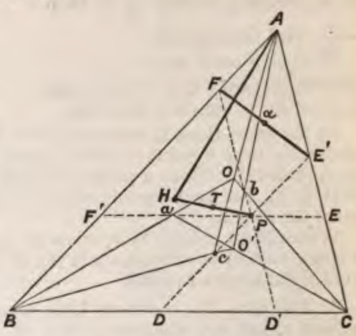
and $\angle$D′E′F′ = BAC;

therefore, since E′P, F′P, D′P meet in a point P, the three lines drawn as directed meet in a point O. Also

$\angle$PE′F′ = PF′D′ = PD′E′,

$\therefore$ $\angle$OAB = OBC = OCA,

and O is a Brocard-point for ABC; P being the corresponding B-point for E′F′D′. (Call O the *first* B-point.) Similarly lines drawn from ABC parallel to D′E′, &c. meet in a point O′, which is the *other* B-point for ABC, P being the corresponding B-point for FDE. (Call O′ the *second* B-point.)



Let BO and CO′ meet in a, &c. Then $\angle bO'a$ = E′D′F′ = C, and $\angle bOa$ = supplement of FED = supplement of C, therefore baOO′ is a circle, similarly c lies on it.

From H draw a perpendicular to BC, meeting F′E in a'. By (1), bisection PH to bisection DD′ is perpendicular to DD′, therefore PD′ = a'D, *i.e.*, a'D = EC; therefore a'C = ED, *i.e.*, a'B = ED, and is therefore parallel to ED; therefore a' is same as point called a above; and therefore HaP, HbP, HcP are all right angles, *i.e.*, circle abcOO′ is on HP as diameter, therefore the centre of Brocard's circle is at the mid-point of HP.

Now, P is the second B-point for FDE, and it is also the first B-point for E′F′D′. These triangles are equal in all respects, and have the same circumcentre; therefore the distance of the first B-point in each from circumcentre must be same. But the distance of the second B-point in one from circumcentre = distance of first B-point in other from circumcentre. Therefore in any triangle the two B-points will be at same distance from circumcentre, therefore HO = HO′, therefore OO′ is perpendicular to, and bisected by, HP.

7441. (By R. Russell, B.A.)—If from a point (x_1, y_1) four normals be drawn to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, prove that (1) the equation of the conic going through x_1, y_1, and the four centres of curvature on the normals, is

$$a^2x^2 + b^2y^2 + \frac{c^4xy}{x_1y_1} - \frac{b^2y_1^2}{x_1}x - \frac{a^2x_1^2}{y_1}y - c^4 = 0;$$

and (2) if $\omega^3 = 1$, the discriminant of this is

$$(a^2x_1^2 + b^2y_1^2 - c^4)(a^2x_1^2\omega + b^2y_1^2\omega^2 - c^4)(a^2x_1^2\omega^2 + b^2y_1^2\omega - c^4).$$

Solution by E. W. Symons, M.A.; R. W. Hogg, B.A.; *and others.*

The four centres of curvature are the four points of contact of the tangents from (x_1, y_1) to the evolute whose equation is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = c^{\frac{4}{3}}$, or rationally

$$(a^2x^2 + b^2y^2 - c^4)^3 + 27a^2b^2c^4x^2y^2 = 0 \quad\text{.....................(1).}$$

Also the equation of a tangent to the evolute at (x, y) is

$$\frac{a^{\frac{2}{3}}\xi}{x^{\frac{1}{3}}} + \frac{b^{\frac{2}{3}}\eta}{y^{\frac{1}{3}}} = c^{\frac{4}{3}} \quad (\xi, \eta \text{ being current coordinates}).$$

If this tangent pass through (x_1, y_1), we have

$$\frac{a^{\frac{2}{3}}x_1}{a^{\frac{1}{3}}} + \frac{b^{\frac{2}{3}}y_1}{y^{\frac{1}{3}}} = c^{\frac{4}{3}},$$

or, rationally, $(a^2x_1^3y + b^2y_1^3x - c^4xy)^3 + 27a^2b^2c^4x_1^3y_1^3x^2y^2 = 0$ ....(2).

(1) and (2) are then two curves passing through the four centres of curvature. Multiplying (1) by $x_1^3y_1^3$ and subtracting (2), we get, after extracting the cube root, $x_1y_1(a^2x^2 + b^2y^2 - c^4) = a^2x_1^3y + b^2y_1^3x - c^4xy$; a conic through the centres of curvature, and which is obviously the conic required for its equation is also satisfied by (x_1, y_1). Its discriminant is

$$-a^2b^2c^4x_1^3y_1^3 + \tfrac{1}{4}a^2b^2c^4x_1^3y_1^3 - \tfrac{1}{4}a^6x_1^7y_1 - \tfrac{1}{4}b^6x_1y_1^7 + \tfrac{1}{4}c^{12}x_1y_1$$
$$\equiv -\tfrac{1}{4}x_1y_1(a^6x_1^6 + b^6y_1^6 - c^{12} + 3a^2b^2c^4x_1^2y_1^2)$$
$$\equiv -\tfrac{1}{4}x_1y_1(a^2x_1^2 + b^2y_1^2 - c^4)(a^2x_1^2\omega + b^2y_1^2\omega^2 - c^4)(a^2x_1^2\omega^2 + b^2y_1^2\omega - c^4)$$
$$\text{if } \omega^3 = 1.$$

[If θ be the eccentric angle of one of the feet of normals from (x', y'), we have

$$a'x\sin\phi - b'y\cos\phi - c^2\sin\phi\cos\phi = 0,$$

$\therefore\ a^3x'^3\sin^3\phi - b^3y'^3\cos^3\phi - c^6\sin^3\phi\cos^3\phi - 3abc^2x'y'\sin^2\phi\cos^2\phi = 0;$

but $$\sin^6\phi + \cos^6\phi = 1 - 3\sin^2\phi\cos^2\phi;$$

hence, substituting $x = \frac{c^2}{a}\cos^3\phi$, $y = -\frac{c^2}{b}\sin^3\phi$,

we have the equation in (1), whose discriminant is that stated in (2).]

4675. (By Morgan Jenkins, M.A.)—Show that the number of pairs of numbers which have a given number G for their greatest common measure, and another number L (of course, a multiple of G) for their least common multiple, is 2^{n-1}, where n is the number of prime bases the product of whose powers is L/G.

Solution by W. J. Curran Sharp, M.A.

If A, B be any two quantities such as are required, we have $A = aG$, $B = bG$, $L = abG$, a and b being prime to each other; and the problem is, in how many ways L/G may be divided into two factors prime to each other. Let $L/G = \alpha^r . \beta^s . \gamma^t \ldots$, where α, β, γ, &c. are primes; then the sets of divisions 1 and $\alpha^r\beta^s\gamma^t\ldots$, α^r and $\beta^s\gamma^t\ldots$, β^s and $\alpha^r\gamma^t$, &c., $\alpha^r\beta^s$ and $\gamma^t\ldots$, &c., which give $1 + n + \frac{n(n-1)}{1 . 2} + \ldots = 2^n$ partitions, each of which occurs twice, and therefore the number of ways is 2^{n-1}.

7144. (By Professor Townsend, F.R.S.)—A concylic tetrad of foci of a system of bicircular quartic curves in a plane being supposed given; construct geometrically, for a given point in the plane,

(*a*) The directions of the two curves of the system that pass through it;

(*b*) Their remaining seven points of intersection at finite distances in the plane.

Solution by the Proposer.

If P, Q, R, S be the four foci of the given tetrad, O the centre of their containing circle C_0, X, Y, Z the three points of intersection of the three pairs of lines QR and PS, RP and QS, PQ and RS, and C_X, C_Y, C_Z the three circles orthogonal to C_0 having their centres at X, Y, Z; then, for an arbitrary point A in the plane, by known properties of confocal quartics of the bicircular class, if U and U′, V and V′, W and W′ be its six circles of connexion with QR and PS, RP and QS, PQ and RS, and if B, C, D, E be its four inverses with respect to the four circles C_0, C_X, C_Y, C_Z, and F, G, H the three inverses of any one of them B with respect to the remaining three circles C_X, C_Y, C_Z, the two curves of the system passing through A bisect internally and externally the three angles between the three pairs of connecting circles U and U′, V and V′, W and W′, and intersect again at the seven points B, C, D, E, F, G, H in the plane.

7216. (By F. Morley, B.A.)—If tangents to two similar epicycloids include a constant angle, prove that a straight line through their intersection, making a constant angle with either, will envelope a similar epicycloid.

Solution by the PROPOSER.

Let a, b, c be the sides of the evanescent triangle formed by the three straight lines; ρ_1, ρ_2, ρ_3 the radii of curvature of their envelopes, then

$$a\rho_1 + b\rho_2 + c\rho_3 = 0 \quad\ldots\ldots\ldots\ldots(1).$$

The equation to an epicycloid is of the form $\rho = \mathrm{A}\sin(a\omega+\beta)$. Let the two be $\rho_1 = \mathrm{A}_1 \sin(a\omega+\beta_1)$, $\rho_2 = \mathrm{A}_2\sin(a\omega+\beta_2)$.

Substituting in (1), we find for the envelope of the third side

$$\rho_3 = -\mathrm{B}_1\sin(a\omega+\beta_1) - \mathrm{B}_2\sin(a\omega+\beta_2) = \mathrm{A}_3\sin(a\omega+\beta_3),$$

if $\mathrm{B}_1\cos\beta_1 + \mathrm{B}_2\cos\beta_2 = -\mathrm{A}_3\cos\beta_3$, $\mathrm{B}_1\sin\beta_1 + \mathrm{B}_2\sin\beta_2 = -\mathrm{A}_3\sin\beta_3$.

This proves the property.

Consider a special case. Take two consecutive tangents to the *same* epicycloid; the envelope of the line bisecting the angle between them is the evolute of the curve. Hence the evolute of a cycloidal curve is a similar curve.

[Mr. WILLIAMSON has shown (*Reprint*, Vol. xxxiii., p. 67, and Vol. xxxvi., p. 63) that if two sides of a triangle moving in a plane envelop involutes of a circle, the third will also. This, of course, holds for any involute: the n^{th} involute is defined by

$$\frac{d^n\rho}{d\omega^n} = \text{constant} = k;$$

and from $a\rho_1 + b\rho_2 + c\rho_3 = 2\Delta$ we get

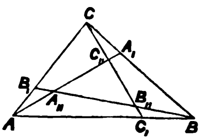
$$a\frac{d^n\rho_1}{d\omega^n} + b\frac{d^n\rho_2}{d\omega^n} + c\frac{d^n\rho_3}{d\omega^n} = 0;$$

whence, if two of the terms are constant, the third is also. Similarly, the n^{th} involute of an epicycloid is defined by $\dfrac{d^n}{d\omega^n}\rho = \mathrm{A}\sin(a\omega+\beta)$. And we see that if two sides envelope such curves, the third will also. The equiangular spiral $\rho = \mathrm{A}e^{\omega}$ has the same property.]

7452. (By G. B. MATHEWS, B.A.)—Prove that (1) if A′, B′, C′ divide the sides BC, CA, AB of the triangle ABC so that BA′ : A′C = CB′ : B′A = AC′ : C′B = $m : n$, the area of the triangle A″B″C″ inclosed by AA′, BB′, CC′ is $(m-n)^2 / (m^2+mn+n^2)\ \Delta$ ABC, and

(2) B″C″ : AA′ = C″A″ : BB′ = A″B″ : CC′ = $m^2 \sim n^2 : m^2+mn+n^2$.

Solution by A. MARTIN, B.A.; MARGARET T. MEYER; *and others.*



1. Taking AC, AB as axes, and putting

$$\angle\,\mathrm{CAB} = \omega,$$

the equations of AA′, BB′, CC′ are

$$y = \frac{mb}{nc}x,\quad \frac{x}{c} + \frac{y(m+n)}{bn} = 1,\quad \frac{x(m+n)}{mc} + \frac{y}{b} = 1;$$

Denoting by S any ellipsoid of the confocal system, by a, b, c its three semi-axes, by dS a differential element of its area at any point P of its surface, and by p the perpendicular from its centre O on its tangent plane at P; then since, by virtue of the property referred to, the product $\frac{d\phi}{dp} . dS$ is constant throughout the entire extent of the displacement filament of the mass determined by dS, and since, by the geometry of the ellipsoid, $\frac{d\phi}{dp} = \frac{d\phi}{d\lambda} . \frac{d\lambda}{dp} = \frac{d\phi}{d\lambda} . \frac{p}{\lambda}$, therefore the product $\frac{d\phi}{\lambda\, d\lambda} . p\, dS$ is constant throughout the entire extent of the filament; but, by the known properties of corresponding elements of confocal ellipsoids, the product $p\, dS$ throughout the entire extent of the filament is to the product abc in a constant ratio depending on the thickness of the filament, therefore the product $\frac{d\phi}{\lambda\, d\lambda} . abc$ is constant throughout the entire mass; hence, denoting its constant value by k, we have

$$d\phi = k\,[abc]^{-1}\,\lambda\, d\lambda, \; = k\,[(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)]^{-\frac{1}{2}}\,\lambda\, d\lambda,$$

where a_0, b_0, c_0 are the semi-axes of the particular ellipsoid of the system for which $\lambda = 0$, and therefore, observing that its value evidently $=0$ at infinity,

$$\phi = -k \int_\lambda^\infty [(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)]^{-\frac{1}{2}}\,\lambda\, d\lambda\,;$$

a form identical, as it ought, with that of the attraction potential, for the ordinary law of the inverse square of the distance, of a thin uniform ellipsoidal shell of semi-axes a_0, b_0, c_0, and mass k, the external equipotential surfaces of which are, as is well known, the external ellipsoids confocal with its surface, and therefore identical with those of the supposed irrotational strain of the question.

From the above we see that, if a thin uniform ellipsoidal shell of any expansible substance, holding in free equilibrium by its attraction for the ordinary law of the inverse square of the distance a surrounding mass of any incompressible liquid, receive a small expansion of volume deforming it into a confocal shell, and put in consequence the surrounding liquid into irrotational strain; the displacement lines of the resulting strain will be the intersections of the two systems of hyperboloids of opposite squares confocal with either shell.

5591. (By D. Edwardes.)—If r be the inscribed radius and s the semiperimeter of a triangle, prove that $s^2 < 27r^2$.

Solution by Professor Morel; J. O'Regan; *and others.*

Since $(a-b)^2+(b-c)^2+(c-a)^2$ is positive, $a^2+b^2+c^2 > ab+bc+ca$; hence, if a, b, c be the sides of a triangle, we have $s^2 > 3r^2+12Rr$. But the expression for the distance between the centres of inscribed and circumscribed circles shows that $R > 2r$; hence $s^2 > 3r^2+24r^2 > 27r^2$.

Similarly we can show that $27R^2 > 4s^2$; hence $27R^2 > 4s^2 > 108r^2$.

6983. (By Professor HADAMARD.) — Si m et n sont deux nombres entiers dont la somme, augmentée de 1, donne un nombre premier, on a

$$m!\,n! = \mathrm{M.\,de}\,(m+n+1) \pm 1.$$

Solution by the PROPOSER.

En effet, soit $m+n+1 = p$. Le théorème de Wilson donne:

$$(p-1)! = \mathrm{M.\,de}\,p-1\,;\ \text{mais}\ (p-1)! = (p-2)!\,(p-1),$$

et évidemment $\qquad p-1 = \mathrm{M.\,de}\,p-1.$

En retranchant membre à membre ces deux égalités, on obtient:

$$[(p-2)! - 1]\,(p-1) = \mathrm{M.\,de}\,p.$$

Mais p est premier avec $p-1$. Donc $(p-2)! = \mathrm{M.de}\,p+1$. Multiplions cette égalité par 2 et ajoutons-lui l'égalité $p-2 = \mathrm{M.\,de}\,p-2$, en remarquant que $(p-2)! = (p-3)!\,(p-2)$. Nous avons: $2\,(p-3)! = \mathrm{M.\,de}\,p-1$. De même remplaçons $(p-3)!$ par $(p-4)!\,(p-3)$; multiplions par 3 et retranchons $p-3 = \mathrm{M.de}\,p-3$; nous avons: $2\,.\,3\,(p-4)! = \mathrm{M.de}\,p+1$;

et en général: $\qquad (k-1)!\,(p-k)! = \mathrm{M.\,de}\,p \pm 1.$

Pour démontrer cette loi, comme elle a été vérifiée pour $k=2$, $k=3$, $k=4$, il suffit de montrer que, si elle existe pour une valeur de k, elle existe pour la valeur immédiatement supérieure. Soit donc

$$(k-1)!\,(p-k)! = \mathrm{M.\,de}\,p \pm 1,$$

ou $\qquad (k-1)!\,[p-(k+1)]!\,(p-k) = \mathrm{M.\,de}\,p \pm 1.$

Multiplions par k et ajoutons ou retranchons $p-k = \mathrm{M.de}\,p-k$. Nous avons $(k-1)!\,k\,...$, ou $\{k\,[(k+1)-1]!\,[p-(k+1)]! \pm 1\}\,(p-k) = \mathrm{M.\,de}\,p$,

ou, puisque p est premier avec $(p-k)$:

$$[(k+1)-1]!\,[p-(k+1)]! = \mathrm{M.de}\,p \mp 1,$$

ce qui démontre la loi.

Pour $k = p-m = n+1$, la formule devient $n!\,m! = \mathrm{M.\,de}\,p \pm 1$.

6737. (By Professor TOWNSEND, F.R.S.)—In the irrotational strain of an incompressible substance in a tridimensional space, if the equipotential surfaces of the strain be a system of confocal ellipsoids in the space, determine the form of the potential ϕ of the strain as a function of the parameter λ of the system.

Solution by the PROPOSER.

The circumstance of the incompressibility of the substance, from which it follows that for every displacement filament of the strain the product of the displacement into the transverse area is constant throughout the entire extent of the filament, enables us to determine the required form very readily as follows.

Denoting by S any ellipsoid of the confocal system, by a, b, c its three semi-axes, by dS a differential element of its area at any point P of its surface, and by p the perpendicular from its centre O on its tangent plane at P; then since, by virtue of the property referred to, the product $\frac{d\phi}{dp} . dS$ is constant throughout the entire extent of the displacement filament of the mass determined by dS, and since, by the geometry of the ellipsoid, $\frac{d\phi}{dp} = \frac{d\phi}{d\lambda} . \frac{d\lambda}{dp} = \frac{d\phi}{d\lambda} . \frac{p}{\lambda}$, therefore the product $\frac{d\phi}{\lambda\, d\lambda} . p\, dS$ is constant throughout the entire extent of the filament; but, by the known properties of corresponding elements of confocal ellipsoids, the product $p\, dS$ throughout the entire extent of the filament is to the product abc in a constant ratio depending on the thickness of the filament, therefore the product $\frac{d\phi}{\lambda\, d\lambda} . abc$ is constant throughout the entire mass; hence, denoting its constant value by k, we have

$$d\phi = k\,[abc]^{-1}\,\lambda\, d\lambda, \;= k\,[(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)]^{-\frac{1}{2}}\,\lambda\, d\lambda,$$

where a_0, b_0, c_0 are the semi-axes of the particular ellipsoid of the system for which $\lambda = 0$, and therefore, observing that its value evidently $= 0$ at infinity,

$$\phi = -\,k \int_\lambda^\infty [(a_0^2+\lambda^2)(b_0^2+\lambda^2)(c_0^2+\lambda^2)]^{-\frac{1}{2}}\,\lambda\, d\lambda\,;$$

a form identical, as it ought, with that of the attraction potential, for the ordinary law of the inverse square of the distance, of a thin uniform ellipsoidal shell of semi-axes a_0, b_0, c_0, and mass k, the external equipotential surfaces of which are, as is well known, the external ellipsoids confocal with its surface, and therefore identical with those of the supposed irrotational strain of the question.

From the above we see that, if a thin uniform ellipsoidal shell of any expansible substance, holding in free equilibrium by its attraction for the ordinary law of the inverse square of the distance a surrounding mass of any incompressible liquid, receive a small expansion of volume deforming it into a confocal shell, and put in consequence the surrounding liquid into irrotational strain; the displacement lines of the resulting strain will be the intersections of the two systems of hyperboloids of opposite squares confocal with either shell.

5591. (By D. Edwardes.)—If r be the inscribed radius and s the semiperimeter of a triangle, prove that $s^2 < 27r^2$.

Solution by Professor Morel; J. O'Regan; *and others.*

Since $(a-b)^2+(b-c)^2+(c-a)^2$ is positive, $a^2+b^2+c^2 > ab+bc+ca$; hence, if a, b, c be the sides of a triangle, we have $s^2 > 3r^2+12Rr$. But the expression for the distance between the centres of inscribed and circumscribed circles shows that $R > 2r$; hence $s^2 > 3r^2+24r^2 > 27r^2$.

Similarly we can show that $27R^2 > 4s^2$; hence $27R^2 > 4s^2 > 108r^2$.

7298. (By Captain MacMahon, R.A.)—Verify that the equation

$$(A+3Bx+3Cx^2+Dx^3)(A+3By+3Cy^2+Dy^3)(A+3Bz+3Cz^2+Dz^3)$$
$$= [A+B(x+y+z)+C(yz+zx+xy)+Dxyz]$$

leads to the differential relation

$$\frac{dx}{(A+3Bx+3Cx^2+Dx^3)^{\frac{2}{3}}}+\frac{dy}{(\dots)^{\frac{2}{3}}}+\frac{dz}{(A+3Bz+3Cz^2+Dz^3)^{\frac{2}{3}}}=0.$$

I. *Solution by the* Proposer.

Writing the equation $XYZ = P^3$, and differentiating logarithmically, we have

$$\frac{1}{X}\frac{dX}{dx}dx+\frac{1}{Y}\frac{dY}{dy}dy+\frac{1}{Z}\frac{dZ}{dz}dz=\frac{3}{P}\left\{\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz\right\},$$

or $\left(\frac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right)dx+\left(\frac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right)dy+\left(\frac{1}{3}P\frac{dZ}{dz}-Z\frac{dP}{dz}\right)dz=0.$

Now, if $X^{-\frac{2}{3}}dx+Y^{-\frac{2}{3}}dy+Z^{-\frac{2}{3}}dz=0$, we should have

$$Y\left\{\frac{1}{3}P\frac{dX}{dx}-X\frac{dP}{dx}\right\}^3-X\left\{\frac{1}{3}P\frac{dY}{dy}-Y\frac{dP}{dy}\right\}^3=0;$$

expanding the left-hand side, and multiplying out, it is found to be

$$(XYZ-P^3)(x-y)\Big[3(B^2-AC)^2-3(B^2-AC)(AD-BC)(x+y)$$
$$+\left\{(AD-BC^2)-(B^2-AC)(C^2-BD)\right\}(x^2+y^2)+\left\{(AD-BC)^2\right.$$
$$\left.+8(B^2-AC)(C^2-BD)\right\}xy-3(AD-BC)(C^2-BD)xy(x+y)$$
$$+3(C^2-BD)^2x^2y^2\Big],$$

which vanishes, and completes the verification.

II. *Solution by* G. B. Mathews, B.A.

Consider the curve

$$f\equiv y^3-(Ax^3+3Bx^2+3Cx+D)=0.$$

Corresponding to this, we have the Abelian function

$$\int\frac{dx}{f'(y)}=\int\frac{dx}{3y^2},\ \text{or}\ \int\frac{dx}{y^2}=\int\frac{dx}{(Ax^3+\dots)^{\frac{2}{3}}}.$$

The differential relation is

$$\frac{dx_1}{(Ax_1^3+\dots)^{\frac{2}{3}}}+\frac{dx_2}{(Ax_2^3+\dots)^{\frac{2}{3}}}+\frac{3}{(Ax_3^3+\dots)^{\frac{2}{3}}}=0,$$

and corresponding to this we have the integral relation expressing that the points (1, 2, 3) lie in a line, viz.,

$$(x_2-x_3)y_1+(x_3-x_1)y_2+(x_1-x_2)y_3=0,$$

or $\quad (x_2-x_3)(Ax_1^3+\dots)^{\frac{1}{3}}+(x_3-x_1)(Ax_2^3+\dots)^{\frac{1}{3}}+\dots=0.$

This is of the form $\quad P^{\frac{1}{3}}+Q^{\frac{1}{3}}+R^{\frac{1}{3}}=0,$

or, rationalized, $\quad (P+Q+R)^3=27PQR$ ....(α),

...writing x, y, z for x_1, x_2, x_3, we have

$$P = (y-z)^3 (Ax^3 + 3Bx^2 + 3Cx + D),$$

$$Q = (z-x)^3 (Ay^3 + \ldots), \quad R = (x-y)^3 (Az^3 + \ldots).$$

$$x(y-z) + \ldots + \ldots \equiv 0,$$

$$x^2(y-z)^3 + \ldots + \ldots = 3xyz(y-z)(z-x)(x-y).$$

$$x^3(y-z)^3 + \ldots + \ldots \equiv x^2[y^3 - z^3 - 3yz(y-z)] + \ldots$$

$$\equiv x^2(y^3-z^3) + \ldots + \ldots - 3xyz[x(y-z) + \ldots + \ldots]$$

$$\equiv (y-z)(z-x)(x-y)(yz+zx+xy),$$

$$x(y-z)^3 + \ldots \equiv (y-z)(z-x)(x-y)(x+y+z),$$

$$(y-z)^3 + \ldots \equiv 3(y-z)(z-x)(x-y),$$

$$P + Q + R \equiv 3(y-z)(z-x)(x-y)[Axyz + B(yz+zx+xy) + C(x+y+z) + D],$$

$$PQR \equiv (y-z)^3(z-x)^3(x-y)^3(Ax^3+3Bx^2+\ldots)(Ay^3+\ldots)(Az^3+\ldots),$$

...substituting in (a) and dividing out $27(y-z)^3(z-x)^3(x-y)^3$, we get

$$[Axyz + B(yz+zx+xy) + C(x+y+z) + D]^3$$

$$= (Ax^3+3Bx^2+3Cx+D)(Ay^3+3By^2+3Cy+D)(Az^3+3Bz^2+3Cz+D).$$

[Mr. Mathews states that for the method employed he is principally indebted to Professor Cayley's Lectures on "Abelian Functions."]

... (By Professor Cayley, F.R.S.)—Denoting by $x, y, z, \xi, \eta, \zeta$ ...ous linear functions of four coordinates, such that identically

$$x + y + z + \xi + \eta + \zeta = 0, \quad ax + by + cz + f\xi + g\eta + h\zeta = 0,$$

... $= bg = ch = 1$; show that $\sqrt{(x\xi)} + \sqrt{(y\eta)} + \sqrt{(z\zeta)} = 0$ is the ... of a quartic surface having the sixteen singular tangent planes ... touching it along a conic)

$$x = 0, \quad y = 0, \quad z = 0, \quad \xi = 0, \quad \eta = 0, \quad \zeta = 0,$$

$$\ldots = 0, \quad x + \eta + z = 0, \quad ax + by + cz = 0, \quad ax + g\eta + cz = 0,$$

$$\ldots = 0, \quad x + y + \zeta = 0, \quad f\xi + by + cz = 0, \quad ax + by + h\zeta = 0,$$

$$\ldots \frac{y}{\ldots ca} + \frac{z}{1-ab} = 0, \quad \frac{\xi}{1-gh} + \frac{\eta}{1-hf} + \frac{\zeta}{1-fg} = 0.$$

...on by W. J. Curran Sharp, M.A.

$$\sqrt{(x\xi)} + \sqrt{(y\eta)} + \sqrt{(z\zeta)} = 0,$$

$$\ldots \eta\zeta + 2zx\zeta\xi - x^2\xi^2 - y^2\eta^2 - z^2\zeta^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

... $(y\eta - z\zeta)^2 = 0$, or the plane $x = 0$ touches ... and similarly for the planes $y = 0$, $z = 0$,

... first equation of condition $\xi + \eta + \zeta = 0$, ... the equation (1) reduces to

$$4yz\eta\zeta \quad \text{or} \quad (z\eta - y\zeta)^2 = 0,$$

D... semi... surf... at P... $\frac{d\Phi}{dp}$... ment... ellip... stant... prop... $p\,dS$... a con... prod... its co... where... which... finity, a form... ordina... ellipsoi... potenti... confoca... irrotatio... From... expansib... for the o... mass of a... deformin... liquid in... strain wi... opposite s...

5591. (... semiperime...

Sol...

Since $(a$... hence, if a, b... expression for... scribed circles... $\ldots\ldots (a)$, Similarly we...

and the plane touches along the conic $x+y+z=0,\ z\eta-y\zeta=0$; and similarly for the next three planes. Again, if $ax+by+cz=0$, by the second identity $f\xi+g\eta+h\zeta=0$, and $x\xi=afx\xi=(by+cz)(g\eta+h\zeta)$, and (1) reduces to $[(by+cz)(g\eta+h\zeta)-(y\eta+z\zeta)]^2=4yz\eta\zeta$, or $(cgz\eta-bhy\zeta)^2=0$ by the given conditions; and therefore the plane $ax+by+cz=0$ touches (1) along the conic $ax+by+cz=0,\ cgz\eta-bhy\zeta=0$; and similarly the planes $ax+g\eta+cz=0,\ f\xi+by+cz=0,\ ax+by+h\zeta=0$ touch along conics, the equations to which may be derived from the above.

If $\dfrac{x}{1-bc}+\dfrac{y}{1-ca}+\dfrac{z}{1-ab}=0$, it is easy to show that

$$x:y:z=(b-c)(1-bc)\left\{-(a-b)\frac{\eta}{b}+(c-a)\frac{\zeta}{c}\right\}$$

$$:(c-a)(1-ca)\left\{-(b-c)\frac{\zeta}{c}+(a-b)\frac{\xi}{a}\right\}$$

$$:(a-b)(1-ab)\left\{-(c-a)\frac{\xi}{a}+(b-c)\frac{\eta}{b}\right\};$$

and therefore

$$x\xi:y\eta:z\zeta=a(b-c)(1-bc)\left\{-(a-b)\frac{\xi\eta}{ab}+(c-a)\frac{\zeta\xi}{ca}\right\}$$

$$:b(c-a)(1-ca)\left\{-(b-c)\frac{\eta\zeta}{bc}+(a-b)\frac{\xi\eta}{ab}\right\}$$

$$:c(a-b)(1-ab)\left\{-(c-a)\frac{\zeta\xi}{ca}+(b-c)\frac{\eta\zeta}{bc}\right\},$$

or $\quad A(-Z+Y):B(-X+Z):C(-Y+X)$, where $A+B+C=0$;
therefore the equation to the surface is

$$[A(Y-Z)]^{\frac12}+[B(Z-X)]^{\frac12}+[C(X-Y)]^{\frac12}=0,$$

therefore $A(Y-Z)+B(Z-X)+2[AB(Y-Z)(Z-X)]^{\frac12}=C(X-Y)$;

$\therefore\quad -(B+C)X+(A+C)Y-(A-B)Z+2[AB(Y-Z)(Z-X)]^{\frac12}=0,$

or $\quad A(X-Z)+B(Z-Y)+2[AB(Y-Z))Z-X)]^{\frac12}=0,$

and the plane touches along

$$[A(X-Z)]^{\frac12}+[B(Z-Y)]^{\frac12}=0 \quad\text{or}\quad AX+BY+CZ=0;$$

that is, $\quad a(b-c)(1-bc)\dfrac{\eta\zeta}{bc}+b(c-a)(1-ca)\dfrac{\zeta\xi}{ca}+c(a-b)(1-ab)\dfrac{\xi\eta}{ab}=0,$

a conic. And similarly in the case of the last plane.

[Mr. SHARP remarks that he does not see why Professor CAYLEY has not reckoned the six planes $\quad\dfrac{x}{1-cg}+\dfrac{\eta}{1-ca}+\dfrac{}{1-ag}=0,$

$$\frac{x}{1-gh}+\frac{\eta}{1-ah}+\frac{\zeta}{1-ag}=0,\qquad \frac{\xi}{1-bc}+\frac{y}{1-fc}+\frac{z}{1-bf}=0,$$

$$\frac{\xi}{1-gc}+\frac{\eta}{1-fc}+\frac{z}{1-fg}=0,\qquad \frac{x}{1-bh}+\frac{y}{1-ah}+\frac{\zeta}{1-ac}=0,$$

$\dfrac{\xi}{1-bh}+\dfrac{y}{1-fh}+\dfrac{\zeta}{1-bf}=0$, as tangent planes of the same kind.]

7454. (By Professor SYLVESTER, F.R.S.)—If I, an invariant of the i^{th} order of $(a_0, a_1, a_2 \dots)(x, y)^n$, becomes I′ when, for any suffix θ, a_θ becomes $a_{\theta+1}$, prove that $\quad \mathrm{I} = \phi \mathrm{I}'$, where $\phi = \Sigma \dfrac{\mathrm{E}_r^\lambda \,.\, \mathrm{E}_s^\mu \,.\, \mathrm{E}_t^\nu \dots}{\lambda \,.\, \pi\mu \,.\, \pi\nu \dots}$,
E in general signifying

$$a_0 \frac{d}{da_\epsilon} + \epsilon a_1 \frac{d}{da_{\epsilon+1}} + \frac{\epsilon(\epsilon+1)}{1\,.\,2} a_2 \frac{d}{da_{\epsilon+2}} + \frac{\epsilon(\epsilon+1)(\epsilon+2)}{1\,.\,2\,.\,3} a_3 \frac{d}{da_{\epsilon+3}} + \dots,$$

and $\lambda, \mu, \nu \dots r, s, t \dots$ being any positive integers satisfying the condition $\lambda r + \mu s + \nu t + \dots = i$.

Solution by ROBERT RUSSELL, B.A.

Let $$u = (a_0, a_1, a_2 \dots)(xy)^n,$$

$$\mathrm{U} = (a_0, a_1, a_2 \dots)(x + \theta y, y)^n \; (\mathrm{A}_0 \mathrm{A}_1 \dots)(x, y)^n, \quad \mathrm{V} = (a_1 a_2 \dots)(x, y)^n;$$

then, if we form the I-invariant of $\mathrm{V} + \theta \mathrm{U}$, the coefficient θ^i will be the original I-invariant of u,

$$\mathrm{I}(\mathrm{V} + \theta \mathrm{U}) = \left(1 + \theta\delta + \frac{\theta^2\delta^2}{2\,!} + \frac{\theta^3\delta^3}{3\,!} + \dots\right) \mathrm{I}' = e^{\theta\delta} \,.\, \mathrm{I}',$$

where $$\delta = \mathrm{A}_0 \frac{d}{da_1} + \mathrm{A}_1 \frac{d}{da_2} + \mathrm{A}_2 \frac{d}{da_3} + \dots = \mathrm{E}_1 + \theta \mathrm{E}_2 + \theta^2 \mathrm{E}_3 + \dots$$

Hence $\mathrm{I} =$ coefficient of θ^i in

$$e^{\theta \mathrm{E}_1 + \theta^2 \mathrm{E}_2 + \theta^3 \mathrm{E}_3 + \dots} \mathrm{I}' = \Sigma \frac{\mathrm{E}_r^\lambda \, \mathrm{E}_s^\mu \, \mathrm{E}_t^\nu \dots}{\lambda\,!\, \mu\,!\, \nu\,! \dots} \mathrm{I}', \text{ where } \lambda r + \mu s + \nu t \dots = i.$$

7484. (By Professor MALET, F.R.S.)—If two solutions of the linear differential equation (A) are the solutions of the equation (B),

$$\frac{d^3y}{dx^3} + \mathrm{Q}_1 \frac{d^2y}{dx^2} + \mathrm{Q}_2 \frac{dy}{dx} + \mathrm{Q}_3 y = 0, \quad \frac{d^2y}{dx^2} + \mathrm{P}_1 \frac{dy}{dx} + \mathrm{P}_2 y = 0 \dots (\mathrm{A}, \mathrm{B});$$

prove that (1)

$$\mathrm{P}_1 \mathrm{P}_2 (\mathrm{P}_1 - \mathrm{Q}_1) = \mathrm{P}_2 \left(\frac{d\mathrm{P}_1}{dx} + \mathrm{P}_2 - \mathrm{Q}_2 \right) = \mathrm{P}_1 \left(\frac{d\mathrm{P}_2}{dx} - \mathrm{Q}_3 \right),$$

and (2) the complete solution of (A) is the solution of

$$\frac{d^2y}{dx^2} + \mathrm{P}_1 \frac{dy}{dx} + \mathrm{P}_2 y = c\mathrm{P}_2 \, e^{-\int \frac{\mathrm{Q}_3}{\mathrm{P}_2} dx}.$$

Solution by R. RAWSON; G. B. MATHEWS, B.A.; *and others.*

In (A) substitute $y = y_1 \int v \, dx$, where (y_1) is a particular solution of (A),

then $$\frac{d^2v}{dx^2} + \left(3 \frac{dy_1}{y_1 \, dx} + \mathrm{Q}_1\right) \frac{dv}{dx} + \frac{3}{y_1} \left\{ \frac{d^2y_1}{dx^2} + \frac{2\mathrm{Q}_1}{3} \frac{dy_1}{dx} + \frac{\mathrm{Q}_2}{3} y_1 \right\} v = 0 \;\dots (3).$$

In (3) substitute $v = v_1 \int w\,dx$, where (v_1) is a particular solution of (3),

then $$\frac{dw}{w\,dx} + 2\frac{dv_1}{v_1\,dx} + 3\frac{dy_1}{y_1\,dx} + Q_1 = 0, \text{ or } w = \frac{e^{-\int Q_1 dx}}{v_1^2 y_1^3} \quad\text{.........(4).}$$

Since v_1 and $v_1 \int \frac{e^{-\int Q_1 dx}}{v_1^2 y_1^3}\,dx$ are each particular solutions of (3), it follows, therefore, that y_1, $y_1 \int v_1\,dx$, $y_1 \int \left\{ v_1 \int \frac{e^{-Q_1 dx}}{v_1^2 y_1^3}\,dx \right\} dx$ are each particular solutions of (A). In (1) substitute $Y_1 \int V dx$, where Y_1 is a particular solution of (1), then particular solutions of (1) are given by

$$Y_1,\ Y_1 \int \frac{e^{-\int P_1 dx}}{Y_1^2}\,dx.$$

Instead of (2) take $$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = ce^{\int R\,dx} \quad\text{........................(5),}$$

where R is a function of x, and in (5) substitute $y = Y_1 \int U dx$, where Y_1 is, of course, a solution of (1); therefore

$$cY_1 \int \left\{ \frac{e^{-\int P_1 dx}}{Y_1^2} \int Y_1 e^{\int (P_1 + R) dx}\,.\,dx \right\} dx \quad\text{.................(6)}$$

is a particular solution of (5). If, therefore, $y_1 = Y_1$, by the question,

$$\frac{d^3y_1}{dx^3} + Q_1\frac{d^2y_1}{dx^2} + Q_2\frac{dy_1}{dx} + Q_3 y_1 = 0,\quad \frac{d^2y_1}{dx^2} + P_1\frac{dy_1}{dx} + P_2 y_1 = 0 \quad\text{......(7).}$$

Differentiate the latter, and from the result take the former; then

$$\frac{d^2y_1}{dx^2} + \frac{\frac{dP_1}{dx} + P_2 - Q_2}{P_1 - Q_1}\frac{dy_1}{dx} + \frac{\frac{dP_2}{dx} - Q_3}{P_1 - Q_1} y_1 = 0 \quad\text{.................(8).}$$

Equation (8) cannot, therefore, be satisfied except

$$\frac{dP_1}{dx} + P_2 - Q_2 = P_1(P_1 - Q_1),\quad \frac{dP_2}{dx} - Q_3 = P_2(P_1 - Q_1) \quad\text{.........(9),}$$

which are the conditions given in the question.

Since $\frac{Q_3}{P_2} = \frac{dP_2}{P_2\,dx} - P_1 + Q_1$, equation (2) readily reduces to

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = ce^{\int (P_1 - Q_1) dx} \quad\text{.....................(10),}$$

which coincides with (5) when $R = P_1 - Q_1$.

Again, if the second particular solution of (A) is equal to the second solution of (1), then $$v_1 = \frac{e^{-\int P_1 dx}}{y_1^2}.$$

This value of (v_1) satisfies (3) by means of the conditions (9). Hence, the two first particular solutions of (A) are equal to the two particular solutions of (1).

Substitute the value of (v_1) above given in the third particular solution of (A), then it becomes $cy_1 \int \left\{ \frac{e^{-\int P\,dx}}{y_1^2} \int y_1 e^{\int (2P_1 - Q_1)\,dx} dx \right\} dx,$

which coincides with (6) when the proper substitutions, viz., $y_1 = Y_1$, $R = P_1 - Q_1$, are made. Hence, the truth of the proposition.

It follows, therefore, from the conditional equations, that

$$\frac{d^3y}{dx^2} + \left\{ P_2 e^{-\int \frac{Q_3}{P_2} dx} \int e^{\int \frac{Q_3}{P_2} dx} \left(\frac{Q_2}{P_2} - 1\right) dx + \frac{Q_3}{P_2} - \frac{dP_2}{P_2\,dx} \right\} \frac{d^2y}{dy^2} + Q_2 \frac{dy}{dx} + Q_3 y = 0$$

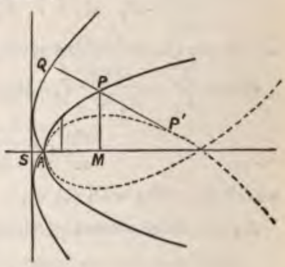
has, for its complete solution, the solution of

$$\frac{d^2y}{dx^2} + \left\{ P_2 e^{-\int \frac{Q_3}{P_2} dx} \int e^{\int \frac{Q_3}{P_2} dx} \left(\frac{Q_2}{P_2} - 1\right) dx \right\} \frac{dy}{dx} + P_2 y = ce^{-\int \frac{Q_3}{P_2} dx}.$$

7220. (By Professor Wolstenholme, M.A., D.Sc.)—If S be the given focus and A the given vertex of an ellipse, prove that (1) the straight line joining the second focus to the ends of the minor axis will envelope a curve of degree 4 and class 3, which is the involute starting from the vertex of the first negative pedal (with respect to the focus) of the parabola whose vertex is A, and whose directrix cuts SA at right angles in S; and (2) if Q, P be corresponding points on this curve and on the parabola, and PM be drawn perpendicular to the axis, PM = PQ, so that the circle with centre on the parabola which touches the axis will also envelope this same curve.

Solution by the Proposer.

Taking S as origin, axis of x along SA, and $SA = a$, then, if $2m$ be the major axis and e the excentricity of the ellipse, the equation of one of the lines is



$$\frac{x + 2me}{me} = \frac{y}{m(1-e^2)^{\frac{1}{2}}},$$

or, if $e = \frac{\lambda^2 - 1}{\lambda^2 + 1}$, the equation is

$$x\lambda + \tfrac{1}{2}y(1-\lambda^2) - a(1-\lambda^2)\lambda = 0,$$

or $\quad x - \tfrac{1}{2}y\left(\lambda - \frac{1}{\lambda}\right) = a(1-\lambda^2);$

whence, for the point of contact with the envelope,

$$\tfrac{1}{2}y\left(1 + \frac{1}{\lambda^2}\right) = 2a\lambda, \quad y = \frac{4a\lambda^3}{1+\lambda^2}, \quad \text{and} \quad x = \frac{a(1-\lambda^2)^2}{1+\lambda^2}.$$

Hence $\frac{dx}{d\lambda} = -\frac{2c\lambda(1-\lambda^2)(3+\lambda^2)}{(1+\lambda^2)^2}, \quad \frac{dy}{d\lambda} = \frac{4c\lambda^2(3+\lambda^2)}{(1+\lambda^2)^2};$

and there are cusps when $\lambda = 0$, or $\pm(-3)^{\frac{1}{2}}$; *i.e.*, at the points $(a, 0)$, $[-8a, \pm 6(-3)^{\frac{1}{2}}a]$; or, moving the origin to A, $(0, 0)$, $[-9a, \pm 6(-3)^{\frac{1}{2}}a]$; and the equations of the joining lines are $2x + y(-3)^{\frac{1}{2}} = 0$, $2x - y(-3)^{\frac{1}{2}} = 0$, $x + 9a = 0$. The values of these three quantities, at the point λ of the curve, are $\frac{2a\lambda^2}{1+\lambda^2}[\lambda + (-3)^{\frac{1}{2}}]^2$, $\frac{2a\lambda^2}{1+\lambda^2}[\lambda - (-3)^{\frac{1}{2}}]^2$, $\frac{a(\lambda^2+3)^2}{1+\lambda^2}$,

so that $[2x + y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} : [2x - y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} : (x+9a)^{-\frac{1}{2}}$

$$= [\lambda - (-3)^{\frac{1}{2}}] : \lambda + (-3)^{\frac{1}{2}} : \lambda 2^{\frac{1}{2}},$$

and the equation is therefore

$$[x + \tfrac{1}{2}y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} + [x - \tfrac{1}{2}y(-3)^{\frac{1}{2}}]^{-\frac{1}{2}} + 2(x+9a)^{-\frac{1}{2}} = 0.$$

The equation of the normal at the point λ is (A origin)

$$\left(x - \frac{a\lambda^2(\lambda^2-3)}{1+\lambda^2}\right)(\lambda^2-1) + \left(y - \frac{4a\lambda^3}{1+\lambda^2}\right)2\lambda = 0,$$

or

$$x(\lambda^2-1) + 2\lambda y = a\lambda^2(\lambda^2+3);$$

and the perpendicular on it from $(a, 0)$ is $(x-a)2\lambda = y(\lambda^2-1)$, hence at the foot of the perpendicular $x = a\lambda^2$, $y = 2a\lambda$, or the point lies on the parabola $y^2 = 4ax$. Thus the normal to the tricusp is a straight line drawn from a point of the parabola at right angles to the focal radius vector, and therefore envelopes the first negative pedal of the parabola with respect to the focus [whose equation is $27ay^2 = x(x-9a)^2$]. The tricusp is therefore an involute of this curve, starting from the vertex, since it has a cusp at that point.

If Q be the point λ on the tricusp, P, P′ the corresponding points on the parabola and its first negative pedal; we have already seen that

$$(1)\ x_1 = \frac{a\lambda^2(\lambda^2-3)}{1+\lambda^2},\quad y_1 = \frac{4a\lambda^3}{1+\lambda^2};\quad (2)\ x_2 = a\lambda^2,\quad y_2 = 2a\lambda;$$

and, by taking the envelope of the normal at Q, we get

$$x\left(1 + \frac{1}{\lambda^2}\right) = 3a(1+\lambda^2),\ \text{ or }\ x_3 = 3a\lambda^2,$$

and therefore

$$y_3 = a\lambda(3-\lambda^2).$$

Hence

$$x_1 - x_2 = -\frac{4a\lambda^2}{1+\lambda^2},\quad y_1 - y_2 = 2a\lambda\frac{\lambda^2-1}{\lambda^2+1},$$

so that $PQ^2 = 4a^2\lambda^2$; or $PQ = 2a\lambda = y_2$; and the circle with P as centre and QP as radius will touch the tricusp in Q, and will also touch the axis. Hence also the tangents at P, Q will meet on the axis of x. Hence the tricusp is also the envelope of the circle whose centre is on the parabola, and whose radius is the ordinate to the centre.

If s_1, s_2, s_3 be the arcs of the three curves measured from A,

$$\frac{ds_1}{d\lambda} = 2a\lambda\frac{(3+\lambda^2)}{1+\lambda^2} = 2a\lambda + \frac{4a\lambda}{1+\lambda^2},\quad s_1 = a[\lambda^2 + 2\log(1+\lambda^2)];$$

$$\frac{ds_2}{d\lambda} = 2a(1+\lambda^2)^{\frac{1}{2}},\ \text{ and }\ s_2 = a\{\lambda(1+\lambda^2)^{\frac{1}{2}} + \log[\lambda + (1+\lambda^2)^{\frac{1}{2}}\},$$

$$\frac{ds_3}{d\lambda} = 3a(1+\lambda^2),\quad s_3 = a(3\lambda + \lambda^3).$$

The equation of the tricusp is

$$\frac{2x}{x^2+\frac{3}{4}y^2}+\frac{2}{(x^2+\frac{3}{4}y^2)^{\frac{1}{2}}}=\frac{4}{x+9a},$$

or $$(3y^2+2x^2-18ax)^2=(x+9a)^2(4x^2+3y^2);$$

or $$y^4+x^2y^2-2ax(8x^2+9y^2)-27a^2y^2=0;$$

or, solving with respect to y^2,

$$2y^2+x^2-18ax-27a^2=[(x^2-18ax-27a^2)^2+64ax^3]^{\frac{1}{2}}=[(x+a)(x+9a)^3]^{\frac{1}{2}},$$

and expanding this in descending powers of x, $=\pm(x^2+14ax+37a^2)+\&c.$ Thus, at infinity, $y^2=16ax+32a^2$, a parabolic asymptote; and $x^2+y^2-4ax+10a^2=0$, a circular asymptote to the impossible branch which passes through the *cyclic* points, and itself altogether impossible. Coordinates of P, Q, P′ are

$$a(1+\lambda^2),\ 2a\lambda;\quad \frac{a(1-\lambda^2)^2}{1+\lambda^2},\ \frac{4a\lambda^3}{1+\lambda^2};\quad a(1+3\lambda^2),\ a\lambda(3-\lambda^2);$$

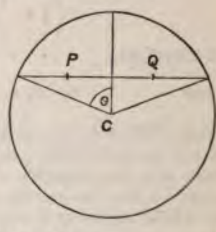
where the excentricity of the ellipse is $\frac{\lambda^2-1}{\lambda^2+1}$;

arc $AQ=a[\lambda^2+2\log(1+\lambda^2)]$, arc $AP=a\{\lambda(1+\lambda^2)^{\frac{1}{2}}+\log[\lambda+(1+\lambda^2)^{\frac{1}{2}}]\}$

and $$\text{arc } AP'=P'Q=a\lambda(3+\lambda^2).$$

7329. (By the late Professor Seitz, M.A.)—Show that the average area of a triangle drawn on the surface of a given circle of radius r, having its base parallel to a given line, and its vertex taken at random, is $\frac{256r^2}{525\pi}$.

Solution by D. Edwardes; Professor Roy, M.A.; *and others.*



Draw a chord parallel to the given direction, subtending an angle 2θ at the centre, and let P, Q be two points therein, and $PQ=z$. Let r be the radius, and p the perpendicular from a random point on the chord. Then, for the value of Σp, an investigation of the areas and positions of the centroids of the segments into which the chord divides the circle, gives easily

$$\Sigma p=r^3[\tfrac{4}{3}\sin^3\theta-2\theta\cos\theta+\pi\cos\theta+2\sin\theta\cos^2\theta].$$

Again, $$\Sigma PQ=\int_0^{2r\sin\theta}z\,dz\,(2r\sin\theta-z)=\tfrac{4}{3}r^3\sin^3\theta.$$

Then, Sum of areas $$=\int_0^{\frac{1}{2}\pi}\tfrac{1}{2}\Sigma p\,.\,\Sigma PQ\,.\,r\sin\theta\,d\theta$$

$$=\tfrac{2}{3}r^7\int_0^{\frac{1}{2}\pi}[-\tfrac{2}{3}\sin^7\theta-2\theta\cos\theta\sin^4\theta+\pi\cos\theta\sin^4\theta+2\sin^5\theta]\,d\theta.$$

Also if n be the number of ways the points are taken, so that the triangle may go through all possible variations, evidently

$$n = \int_0^{\frac{1}{2}\pi} \pi r^2 \,.\, r \sin\theta \, d\theta \,.\, 2r^2 \sin^2\theta = \tfrac{4}{5}\pi r^5 ;$$

hence the required average is

$$\frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi} \left[-\tfrac{1}{3}\sin^7\theta - \theta\cos\theta\sin^4\theta + \tfrac{1}{2}\pi\cos\theta\sin^4\theta + \sin^5\theta\right] d\theta = \frac{256r^2}{525\pi}.$$

7333. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—Approaching each other from rest at equal heights in the same normal section of two smooth planes, each making an angle θ with the horizon, slide by gravity two equal smooth spheres of homogeneous matter perfectly rigid and incompressible. About the lowest points p and q of their paths, the planes are scooped spherically in their inferior surface, so that the thickness at p and q vanishes. At the instant t of collision, two other spheres exactly like the former impinge by projection from below perpendicularly on the planes at the points p and q, with the same velocity $v \tan\theta$, v being the velocity acquired by the descending spheres. Required, for the peace of mind of Dr. Mustbeso, an orthodox account of the motion.

Solution by the Proposer.

Resolving along vertical and horizontal axes, the velocities of the four spheres, we obtain for each, at the time t, $v = 0$ in all directions; *i.e.*, the two descending spheres remain at rest at the points p and q, while the two others begin to descend at the time t from rest at those points by gravity. But $\Sigma mv^2 > 0$ being the sum of four positive quantities; how is this *vis viva* conserved? Not by thermal or other insensible vibrations of the spheres; for by definition they cannot be compressed and made to recoil or vibrate in any way, so as to communicate vibrations to any medium. Not by vibrations in the planes; for at the time t the planes receive no blow. The *vis viva* seems to be destroyed at the time t without compensation of any kind.

If it be denied that there are in the cosmos any perfectly rigid and incompressible spheres, what becomes of the atomic constitution of matter? If it be affirmed that particles of matter cannot actually collide by reason of a repulsion working at close quarters by *actio in distans* between those points without parts, their centres of gravity; I remark that, if the inscrutable Cause were to act in the same manner as at present in the lines connecting those ever-moving centres, in the absence of all matter, the facts of the universe would be to us exactly what they are now. Who can prove the presence of this wonderful matter? What clear question about the *data* of the cosmos given to us all is answered by the word *matter*? If any one informs me that he has a concept of matter, I reply that unfortunately I have none, but that, if he will kindly produce his concept, I will try to study it and to become wiser. As to the m for mass in Dynamics, what is it but a number determined by experiment, which would yield the same answer on the above supposition in the absence of matter? Is it enough to reply to these questions—that's all Metaphysics? Quite.

[To Mr. KIRKMAN's query, "How is this *vis viva* conserved?" Mr. CARR remarks that "the answer is contained in the formula $\infty \times 0 =$ finite magnitude. An increasing number of vibrations of diminishing amplitude fulfil the conditions, in the limit if an incompressible body (non-existent in fact), and give a thermal equivalent of Σmv^2. Why four spheres and two planes? Two spheres impinging in the same right line would illustrate the same thing."]

7057. (By J. GRIFFITHS, M.A.)—If

$$\cos\phi\cos\psi + \left(\frac{1-mnk^4}{1+mnk^2}\right)^{\frac{1}{2}}\sin\phi\sin\psi = \left(\frac{1-mn}{1+mnk^2}\right)^{\frac{1}{2}}.\,k\sin\phi;$$

and $\dfrac{1+k^2}{1+mnk^2} = \dfrac{2}{m+n}$, show (1) that

$$\frac{d\psi}{(1+mk\sin\psi\,.\,1-nk\sin\psi)^{\frac{1}{2}}} = \left(\frac{1+k^2}{1+mnk^2}\right)^{\frac{1}{2}}.\,\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}};$$

and (2) deduce LANDEN's transformation. [Take $k<1$ and $mn<1$.]

Solution by D. EDWARDES; *the* PROPOSER; *and others.*

1. The relation $\dfrac{1+k^2}{1+mnk^2} = \dfrac{2}{m+n}$ gives

$$2\,(1-mn)^{\frac{1}{2}}(1-mnk^4)^{\frac{1}{2}} = (m-n)(1+k^2);$$

hence the equation connecting the amplitudes gives

$$\cos^2\phi = \frac{[k(1-mn)^{\frac{1}{2}}-(1-mnk^4)^{\frac{1}{2}}\sin\psi]^2}{(1+k^2)[1-nk\sin\psi\,.\,1+mk\sin\psi]},$$

also
$$\tan\phi = \frac{(1+mnk^2)^{\frac{1}{2}}\cos\psi}{k(1-mn)^{\frac{1}{2}}-(1-mnk^4)^{\frac{1}{2}}\sin\psi};$$

whence, differentiating this last, we have

$$\frac{d\phi}{d\psi} = \left(\frac{1+mnk^2}{1+k^2}\right)^{\frac{1}{2}}\frac{(1-k^2\sin^2\phi)^{\frac{1}{2}}}{[1-nk\sin\psi\,.\,1+mk\sin\psi]^{\frac{1}{2}}},$$

the stated result.

2. Writing $-\phi$ for ϕ, and putting $m=0$, and making the transformation $\sin\psi = \dfrac{(1+k)^2-2(1+k^2)\sin^2\phi_1}{(1+k)^2-4k\sin^2\phi_1}$, we have, since $n=\dfrac{2}{1+k^2}$,

$$\frac{d\psi}{(1-nk\sin\psi)^{\frac{1}{2}}} = -\frac{2(1+k^2)^{\frac{1}{2}}\,d\phi_1}{[(1+k)^2-4k\sin^2\phi_1]^{\frac{1}{2}}}.$$

Also $\cos\psi = \dfrac{2(1-k)\sin\phi_1\cos\phi_1}{(1+k)(1-c^2\sin^2\phi_1)}$, where $c^2=\dfrac{4k}{1+k}$, and the equation connecting the amplitudes becomes $\sin(2\phi_1-\phi) = k\sin\phi$, and we have LANDEN's transformation, viz., $F(c,\phi_1) = \frac{1}{2}(1+k)F(k,\phi)$.

[If $m = 0$, $n = \frac{2}{1+k^2}$, the integral equation is $\cos(\psi - \phi) = k\sin\phi$, and we have, from the stated result,

$$\frac{d\psi}{(1+k^2-2k\sin\psi)^{\frac{1}{2}}} = \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}; \text{ or, if } \psi = 2\phi_1 - \tfrac{1}{2}\pi,$$

$$\frac{2d\phi_1}{[(1+k)^2-4k\sin^2\phi_1)]^{\frac{1}{2}}} = \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}, \text{ where } \sin(2\phi_1 - \phi) = k\sin\phi.$$

This is LANDEN's equation.]

7433 & 7443. (By the EDITOR.) — Show that the volume of the greatest parcel that can be sent by the Parcel Post is (1) $8/\pi = 2{\cdot}5468$ ft. when unlimited in form and therefore a right circular cylinder, and (2) 2 cubic feet when it is to be four-sided and plane.

Solution by W. M. MEE, B.A.; Professor ROY; *and others.*

1. The parcel must clearly be a right circular cylinder; thus, putting x for the length of its axis, and r for the radius of its base, we have length + girth $= 6$ ft. $= x + 2\pi y$, and volume $= \pi r^2 x =$ a maximum. These equations readily give $x =$ length $= 2$ ft., girth $= 2\pi y = 4$ ft., radius $= 2/\pi$ ft., and volume $= \pi x y^2 = 8/\pi$ cubic feet.

2. Here the parcel must evidently be a square parallelepiped, and in like manner to (1) it appears at once that its length must be 2 feet, each of its sides 1 foot, its girth 4 feet, and its volume 2 cubic feet.

7468. (By S. TEBAY, B.A.)—Find an integral value of a, such that $101^2 + a$ and $101^2 - a$ shall be rational squares.

Solution by G. HEPPEL, M.A.; W. G. LAX, B.A.; *and others.*

Let $$101^2 + a = (101+k)^2, \quad 101^2 - a = (101-l)^2,$$
$$a = k(202+k) = l(202-l);$$

hence l is greater than k; thus, putting $l = k + c$, we have

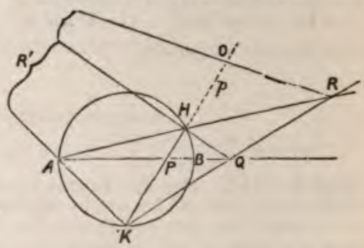
$$k(202+k) = (k+c)(202-k-c), \quad 2k^2 = 202c - 2kc - c^2.$$

Now, if c is a measure of k, the only values are 2 and 101; and neither of these lead to a solution. But if c does not measure k, it must measure $2k^2$, and hence the only other possible value of c is 4. This gives $k^2 + 4k - 396 = 0$, $(k+22)(k-18) = 0$; so that $k = 18$, $a = 3960$; and $101^2 + a = 119^2$, $101^2 - a = 79^2$.

7479. (By R. Tucker, M.A.)—P, Q are lines parallel to the directrix of a parabola; from any point p on P tangents are drawn to the curve cutting Q in r, s; through r, s lines are drawn parallel to the tangents, and meeting in t: prove that these lines envelope a parabola, and that pt passes through the pole of P.

Solution by G. B. Mathews, B.A.; R. Knowles, B.A., L.C.P.; *and others.*

The theorem can be easily proved by reciprocation with regard to the focus of the parabola, the reciprocal theorem being as follows:—

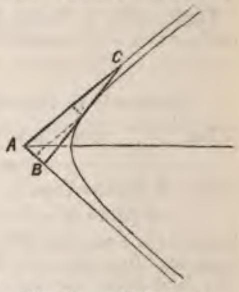


P, Q are any two points on the diameter AB of a given circle; HPK is a chord through P; AH, QK meet in R, and AK, QH in R′; then the locus of R, R′ is one and the same hyperbola which passes through A.

The chord HK corresponds to the point p and the line RR′ to the point t; so that the second part of the proposition amounts to showing that the point O where RR′, HK intersect lies on the polar of P, which is obvious, since KPHO is a harmonic range.

7505. (By G. Heppel, M.A.)—If three hyperbolas be described, to each of which one side of a given triangle is a tangent, and the other sides are asymptotes, show that the product of the three latera recta is equal to the cube of the diameter of the inscribed circle.

Solution by R. Knowles, B.A., L.C.P.; W. G. Lax, B.A.; *and others.*



Let α, β be the semiaxes, and e the eccentricity of the hyperbola opposite A; then, if a, b, c, A, B, C refer to the fixed triangle,

$$\tfrac{1}{4}bc = \tfrac{1}{4}(\alpha^2+\beta^2), \text{ or } bc = \alpha^2 e^2,$$

and
$$\cos\tfrac{1}{2}A = \frac{1}{e},$$

therefore
$$l_1 = \alpha(e^2-1) = \left(e - \frac{1}{e}\right)\alpha e$$
$$= (bc)^{\frac{1}{2}}\frac{\sin^2\frac{1}{2}A}{\cos\frac{1}{2}A} = \frac{s_2 s_3}{(ss_1)},$$

where $s_1 = s-a$, &c.; hence we have

$$l_1 l_2 l_3 = \frac{(ss_1s_2s_3)^{\frac{3}{2}}}{s^3} = \frac{\Delta^3}{s^3} = r^3, \text{ therefore } 2l_1 \cdot 2l_2 \cdot 2l_3 = (2r)^3.$$

7496. (By R. A. ROBERTS, M.A.)—A geodesic common tangent is drawn to two circular sections of an ellipsoid; show (1) that the perpendiculars from the centre on the tangent planes to the surface at the points of contact are equal; and hence (2) find the locus of the points of contact of the geodesic tangents drawn from an umbilic to the circular sections.

Solution by T. WOODCOCK, B.A.; Professor NASH, M.A.; *and others.*

1. Along any geodesic on the ellipsoid, pD = constant, p being the perpendicular upon the tangent plane at any point P, and D the semi-diameter parallel to the geodesic's tangent at P. Let p', D' be the lengths of the same lines for any other point P′, and let a, b, c be the semi-axes of the ellipsoid. If the same geodesic touch one circular section at P and another at P′, we have $D = b = D'$; therefore $p = p'$.

2. Along all geodesics through an umbilic, $pD = ac$. Therefore, at the points of contact of these geodesics with the circular sections $p = \frac{ac}{b}$. This equation represents a polhode on the surface of the ellipsoid.

7245. (By R. KNOWLES, B.A., L.C.P.)—Three normals are drawn from a point to a parabola, and tangents are then drawn at the points where the normals meet the curve; prove (1) that the area of the triangle formed by the tangents is *half* that formed by joining the points in the curve; (2) if the point moves on a given straight line, the locus of each of its angular points is the same hyperbola.

Solution by W. H. BLYTHE, M.A.; Professor MATZ, M.A.; *and others.*

1. This applies to any three points (m_1, m_2, m_3) on the parabola, $y^2 = 4ax$, putting the coordinates into the form $x = am^2$, $y = 2am$. Then the points of intersection of tangents become am_1m_2, $a(m_1+m_2)$, &c.; taking ordinary formulæ for the area of a triangle. First, for the three points $(am_1^2 . 2am_1)$, &c., we get $a^2(m_1-m_2)(m_2-m_3)(m_3-m_1)$; next, taking the points am_1m_2, $a(m_1+m_2)$, &c., we get

$$\tfrac{1}{2}a^2(m_1-m_2)(m_2-m_3)(m_3-m_1).$$

2. We take m_1, m_2, m_3 as the roots of the equation

$$am^3+m(h-2a)+k=0,$$

(hk) being the point at which the normals meet, and, since (hk) moves on a fixed line, we may take $k = bh+c$, b and c being constants; therefore (m_1, m_2, m_3) are the roots of $am^3+m(h-2a)+bh+c=0$ ....(α), where h varies. Take any one of the intersection of tangents $x = am_1m_2$, $y = a(m_1+m_2)$, since m_1, m_2 are two roots of (α), we obtain

$$(h-2a)a = y^2+ax, \quad abh+ac = xy;$$

whence, eliminating h, we obtain

$$by^2+abx+2a^2b = xy-ac, \text{ or } (y-ab)(by-x+ab^2)+a^2b^3+ac+2a^2b = 0,$$

an hyperbola with asymptotes $y = ab$ and $x = b(y+ab)$.

7275. (By D. EDWARDES.)—If $\tan\alpha\cot\frac{1}{2}(\beta+\gamma) = \tan\beta\cot\frac{1}{2}(\gamma+\alpha)$, prove that $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = 0$.

Solution by R. W. WHITE, B.A.; R. KNOWLES, B.A., L.C.P.; *and others.*

Reducing successively, we have

$$\frac{\sin\alpha\cos\frac{1}{2}(\beta+\gamma)}{\cos\alpha\sin\frac{1}{2}(\beta+\gamma)} = \frac{\sin\beta\cos\frac{1}{2}(\gamma+\alpha)}{\cos\beta\sin\frac{1}{2}(\gamma+\alpha)},$$

$$\frac{\sin\left[\alpha+\frac{1}{2}(\beta+\gamma)\right]}{\sin\left[\alpha-\frac{1}{2}(\beta+\gamma)\right]} = \frac{\sin\left[\beta+\frac{1}{2}(\alpha+\gamma)\right]}{\sin\left[\beta-\frac{1}{2}(\alpha+\gamma)\right]},$$

$$\cos\tfrac{1}{2}(3\alpha-\beta+2\gamma)-\cos\tfrac{1}{2}(\alpha-3\beta-2\gamma)-\cos\tfrac{1}{2}(\alpha+3\beta)+\cos\tfrac{1}{2}(3\alpha+\beta) = 0,$$

$$\sin(\beta-\alpha)\sin\left[\tfrac{1}{2}(\alpha+\beta)+\gamma\right]+\sin(\alpha+\beta)\sin\tfrac{1}{2}(\beta-\alpha) = 0,$$

$$2\cos\tfrac{1}{2}(\beta-\alpha)\sin\left[\tfrac{1}{2}(\alpha+\beta)+\gamma\right]+\sin(\alpha+\beta) = 0,$$

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta) = 0.$$

7425. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If ABCD be a tetrahedron in which $AB+AC = DB+DC$, prove that $\widehat{AB}+\widehat{AC} = \widehat{DB}+DC$, where $\widehat{AB}$ is the dihedral angle between the planes meeting in AB.

Solution by G. HEPPEL, M.A.; SARAH MARKS; *and others.*

Let the angles $\widehat{AB}, \widehat{AC}, \widehat{DB}, \widehat{DC} = \alpha, \beta, \gamma, \delta$; the edges AB, AC, DB, DC $= a, b, c, d$; $\widehat{AD} = \theta$, $AD = p$, $BC = q$; then,

$$\sin\alpha = \sin\theta\,\frac{\sin CAD}{\sin BAC},\quad \cos\beta = \frac{\cos BAD-\cos BAC\cos CAD}{\sin BAC\sin CAD},$$

therefore $\sin\alpha\cos\beta = \dfrac{\sin\theta}{\sin^2 BAC}\left[\cos BAD-\cos BAC\cos CAD\right]$.

Similarly, $\sin\beta\cos\alpha = \dfrac{\sin\theta}{\sin^2 BAC}\left[\cos CAD-\cos BAC\cos BAD\right]$,

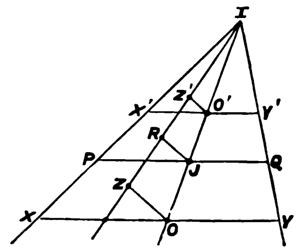
whence $$\sin(\alpha+\beta) = \frac{\sin\theta(\cos BAD+\cos CAD)}{1+\cos BAC},$$

therefore $$\frac{\sin(\alpha+\beta)}{\sin\theta} = \frac{(a+b)\,p^2+(a^2-c^2)\,b+(b^2-d^2)\,a}{(a+b)^2-q^2}.$$

By interchanging a and b with c and d, we obtain an expression for $\dfrac{\sin(\gamma+\delta)}{\sin\theta}$; and, if $a+b = c+d$, the denominators and first terms of the numerators are the same in each, and the difference of the remaining terms is $(a^2-c^2)(b+d)+(b^2-d^2)(a+c)$ or $(a+c)(b+d)(a+b-c-d)$; so that, if $a+b = c+d$, $\alpha+\beta = \gamma+\delta$.

7490. (By Professor WOLSTENHOLME, M.A., Sc.D.)—At each point of a central conic is described the rectangular hyberbola of closest contact; prove that the locus of its centre is the inverse of the conic with respect to the director-circle.

Solution by Professor TOWNSEND, F.R.S.



This pretty result is manifestly a particular case of the following more general property—that, if O and O′ (see figure) be the centres of any two conics U and U′ having double contact at any two points P and Q, I the intersection of their common tangents at those points, and J the middle point of their chord of contact PQ, the two points I and J being of course collinear with O and O′; then, when director-circle of U′ is evanescent, rectangle $OI-OO' =$ radius^2 of director-circle of U. Which may be readily proved as follows:—Since, manifestly, XY and X′Y′ being the parallels through O and O′ to PQ,

radius^2 of director-circle of $U = OI-OJ+OY-JQ$,

radius^2 of director-circle of $U' = O'I-O'J+O'Y'-JQ$,

therefore when the latter circle is evanescent, and when consequently by similar triangles $OI-O'J+OY-JQ = O$, then, by subtraction, radius^2 of director-circle of $U = OI\ (OJ-O'J) = OI-OO'$, and therefore, &c.

The analogous property in Geometry of three dimensions—viz., that, of all central quadrics having closest contact with a given central quadric at any point of its surface, the centre of that whose director-sphere is evanescent is the inverse of the point of contact with respect to the director-sphere of the given quadric—is manifestly also a particular case of the more general property that, if O and O′ (same figure) be the centres of any two quadrics U and U′ having triple contact at any three points P, Q, R, I the intersection of their common tangent-planes at those points, and J the centre of their conic of contact PQR, the two points I and J being of course again collinear with O and O′; then, when director-sphere of U′ is evanescent, rectangle $OI-OO' =$ radius^2 of director-sphere of U. Which again may be proved nearly similarly with its analogue as follows:—Since manifestly, for any pair of conjugate radii JQ and JR of the conic of contact, XYZ and X′Y′Z′ being the parallels through O and O′ to its plane PQR,

radius^2 of director-sphere of $U = OI-OJ+OY-JQ+OZ-JR$,

radius^2 of director-sphere of $U' = O'I-O'J+O'Y'-JQ+O'Z'-JR$.

Therefore, when the latter sphere is evanescent, and when consequently by similar triangles $OI-O'J+OY-JQ+OZ-JR = O$; then, by subtraction, radius^2 of director-sphere of $U = OI\ (OJ-O'J) = OI-OO''$, and therefore, &c.

The conic or quadric U′, in the above general property corresponding to the case, with the line or plane of contact PQ or PQR, being supposed to remain fixed, and U on the contrary to vary, and its centre O to assume in consequence every possible position on the fixed line IJ; it follows at once, from the above general equation corresponding to the case, viz., (radius)^2 of director circle or sphere of $U = OO'-OI$, that the system of director-circles or spheres determined by the variation of U has a common radical axis or plane, situated in each case midway between the two points O′ and I, which are in each case the two limiting points of the system,—a particular case, manifestly, of the still more general but probably better known property corresponding to the case, that, for every system of conics having four common tangent lines, or of quadrics having eight common tangent planes, the director circles or spheres determine a coaxal or coplanar system, the two limiting points of which are the centres of the two conics or quadrics of the system whose director circles or spheres are evanescent.

7465. (By G. Heppel, M.A.)—In a recent Cambridge Higher Local Examination, the following question was set:—"If n be a prime number, prove that $(x+y)^n - x^n - y^n$ is divisible by $n\,xy\,(x+y)$, (x^2+xy+y^2)." This being assumed, determine the general term of the quotient.

Solution by the Proposer.

Let P be the expression $(x+y)^n - x^n - y^n$; then

$$\frac{\mathrm{P}}{nxy} = x^{n-2} + \frac{n-1}{2\,!}\,x^{n-3}y + \frac{(n-1)(n-2)}{3\,!}\,x^{n-4}y^2 + \&\text{c.}$$

Now, let $A_1A_2A_3$, &c., $B_1B_2B_3$, &c., $C_1C_2C_3$, &c., be the respective coefficients of

$(x-y)\,\mathrm{P} \div n\,xy$, $\quad (x-y)\,\mathrm{P} \div n\,xy\,(x^3-y^3)$, $\quad (x-y)\,\mathrm{P} \div n\,xy\,(x^3-y^3)(x+y)$;

then $C_1C_2C_3$, &c. are the coefficients we require to know.

The following laws are evident from either multiplication or division

$$A_1 = 1,\ A_2 = \frac{n-3}{2\,!},\ A_3 = \frac{(n-1)(n-5)}{3\,!},\ A_4 = \frac{(n-1)(n-2)(n-7)}{4\,!},\ \&\text{c.};$$

$B_1 = A_1$, $B_2 = A_2$, $B = A_3$, $B_4 - B_1 = A_4$, $B_5 - B_2 = A_5$, $B_6 - B_3 = A_6$,

and thence we obtain the general relation

$$B_t = A_t + A_{t-3} + A_{t-6} + A_{t-9} + \&\text{c.}\ldots$$

Now C_t and the B coefficients are connected by $B_t = C_t + C_{t-1}$ which necessarily leads to $C_t = B_t - B_{t-1} + B_{t-2} - B_{t-3}$, &c. ..., and this gives as a final relation

$$C_t = (A_t - A_{t-1} + A_{t-2}) + (A_{t-6} - A_{t-7} + A_{t-8}) + (A_{t-12} - A_{t-13} + A_{t-14}) + \&\text{c.}$$

As a numerical illustration, let $n = 17$, and let it be required to find the seventh term of the quotient, then

$$C_7 = \frac{16.15.14.13.12.4}{7\,!} - \frac{16.15.14.13.6}{6\,!} + \frac{16.15.14.8}{5\,!} + 1$$

$= 416 - 364 + 224 + 1 = 277$, so that the seventh term is $277x^6y^6$.

With reference to the original question set in the Cambridge Examination, it may be noticed that the necessary and sufficient condition of divisibility is that n must be of the form $6n \pm 1$. This of course includes all primes.

C. F. Hodgson & Son, 1 Gough Square, Fleet Street, E.C.

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XLI.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1884.

*** Of this series forty-one volumes have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

LIST OF CONTRIBUTORS.

., M.A.; H.M. Inspector of Schools.
.A.J.C., M.A.; St. Peter's Coll., Camb.
rofessor GEO. J., LL.D.; Galway.
ALEX., B.A.; Queen's Coll., Galway.
EDWYN, M.A.; The Elms, Hereford.
E, Professor; Pesaro.
. STAWELL, LL.D., F.R.S.; Professor
nomy in the University of Dublin.
IH CHANDRA; Presid. Coll., Calcutta.
I, GIUSEPPE; Professore di Mate-
nell' Università di Roma.
:ORGE, B.A.; Clifton Ter., Kenilworth.
Professor; University of Pisa.
7AN DEN; Professor of Mathematics
olytechnic School, Delft.
H., M.A.; Cambridge.
BIGAH, M.A.; Delhi.
E, C.; Allerton Bywater.
; Gough H., Kingston-on-Thames.
. J. G., M.A.; London.
), ELIZABETH; Boulogne.
. H., B.A.; Egham.
', Dr. C. W.; Victoria Strasse, Berlin.
, R. H. M., M.A.; Fellow of St.
'ollege, Oxford.
W., M.A.; Bedford County School.
ofessor E.; Millersville, Pennsylvania.
CRUM, D.Sc.; Edinburgh.
'.COLIN; Andersonian Univ. Glasgow.
A., Ph.D.; Schol. New College, Oxford.
ARD, M.A.; Univ. Coll., Bristol.
W. S., M.A.; Professor of Mathe-
n the University of Dublin.
N., LL.B.; Bedford Square, London.
W. P., B.A.; Clonmel Gram. School.
B.A.; Endsleigh Gardens, London.
IN, LL.D., F.R.S.; Prof. of Higher
natics in the Catholic Univ. of Ireland.
Prof., M.A.; University of Upsala.
., B.A.; Magdalen College, Oxford.
, F.R.S.; Sadlerian Professor of Ma-
cs in the University of Cambridge,
r of the Institute of France, &c.
RTI, BYOMAKESA, M.A.; Professor
nematics, Calcutta.
INY EARLE, LL.D.; Professor of Phi-
in Haverford College.
olonel A. R., C.B., F.R.S.; Hastings.
. M., B.A.; Trinity College, Dublin.
rofessor; Paris.
r JAMES, M.A., F.R.S.; Ealing.
THUR, M.A., Q.C., M.P.; Holland Pk.
G., M.A.; University of St. Andrew's.
E, S.; Swinford Rectory, Mayo.
, J. H., M.A.; Royal School of Naval
cture, South Kensington.
LUIGI; Direttore della Scuola degli
eri, S. Pietro in Vincoli, Rome.
M. W., B.A., F.R.S.; Prof. of Math.
ch. in the R. M. Acad., Woolwich.
ILL, E.P., B.A.; Sch. of Trin. Coll., Dubl.
RTHUR HILL, LL.D., D.Sc.; Dublin.
Professor; Paris.
G., M.A.; Abingdon
F., B.A.; Wandsworth Common.
H. G., B.A.; Baymount, Dublin.
H. G., M.A.; Richmond Terr., Brighton.
NARENDRA LAL, M.A.; Calcutta.
., M.A.; Fellow of Caius Coll., Camb.
T., B.A.; Hexham Grammar School.
f. ARNOLD, M.A.; Porrentruy, Berne.
.C.; Professeur au Lycée d'Angoulême.
', W., B.A.; Grammar School, St. Asaph.
), G., M.A.; Saxonville, Massachusetts.
IELLE; Lockport, New York.
J., M.A.; Head Master of Aberdeen
iate School.
s, DAVID; Erith Villas, Erith, Kent.
E. B., M.A.; Fell. Queen's Coll., Oxon.
EXANDER J., F.R.S.; Kensington.
W. T. A.; Pembroke Coll., Oxford.
ALBERT, M.A.; Head Master of the
Hospital School, Greenwich.
L, EMMA; Coventry.

EVANS, Professor, M.A.; Lockport, New York.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in Queen's College, Belfast.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FINCH, T. H., B.A.; Trinity College, Dublin.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FRANKLIN, CHRISTINE LADD, M.A.; Professor of Natural Sciences and Mathematics, Union Springs, New York.
FUORTES, E.; University of Naples.
GALBRAITH, Rev. J., M.A.; Fell. Trin. Coll., Dublin.
GALE, KATE K.; Worcester Park, Surrey.
GALLATLY, W., B.A.; Earl's Court, London.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., B.A.; Stud. of Ch. Ch., Oxford.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GOLDENBERG, Professor, M.A.; Moscow.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Harold, B.A.; King's Coll., Cambridge.
HARLEY, Rev. R., F.R.S.; Huddersfield College.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HARRIS, J. R., M.A.; Clare College, Cambridge.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; The Grove, Hammersmith.
HERBERT, A., M.A.; King Alfred's Sch., Wantage.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
INGLEBY, C. M., M.A., LL.D.; London.
JENKINS, MORGAN, M.A.; London.
JOHNSON, J. M., B.A.; Radley College, Abingdon.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, SWIFT; Trin. Coll., Dublin.
JONES, L. W., B.A.; Merton College, Oxford.
KEALY, J. A., M.A.; Wilmington, Delaware.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rect.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
KNOWLES, R., B.A., L.C.P.; Tottenham.
LACHLAN, R., B.A.; Lewisham.
LARMOR, J., M.A.; Queen's College, Galway.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAX, W. G., B.A.; Trinity College, Cambridge.
LEIDHOLD, R., M.A.; Finsbury Park.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, A., D.Sc., F.R.S.E.; Examiner in Mathematics in the University of Edinburgh.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MACMAHON, Capt. P. A.; R. M. Academy.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.

McCay, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
McClelland, W. J., B.A.; Prin. of Santry School.
McColl, H., B.A.; 73, Rue Sibliquin, Boulogne.
McDowell, J., M.A.; Pembroke Coll., Camb.
McIntosh, Alex., B.A.; Bedford Row, London.
McLeod, J., M.A.; R.M. Academy, Woolwich.
McVicker, C. E., B.A.; Trinity Coll., Dublin.
Malet, Prof., M.A.; Queen's Coll., Cork.
Mannheim, M.; Prof. à l'Ecole Polytech., Paris.
Marks, Sarah; Cambridge Street, Hyde Park.
Martin, Artemas, M.A., Ph.D.; Editor & Printer of *Math. Visitor & Math. Mag.*, Erie, Pa.
Martin, Rev. H., D.D., M.A.; Edinburgh.
Mathews, G. B., B.A.; Colaba Lo., Leominster.
Matz, Prof., M.A.; King's Mountain, Carolina.
Mee, W. M., B.A.; Belturbet.
Merrifield, J., LL.D., F.R.A.S.; Plymouth.
Merriman, Mansfield, M.A.; Yale College.
Meyer, Mary S.; Girton College, Cambridge.
Miller, W. J. C., B.A., (Editor); The Paragon, Richmond-on-Thames.
Minchin, G. M., M.A.; Prof. in Cooper's Hill Coll.
Mitcheson, T., B.A., L.C.P.; City of London Sch.
Monck, Henry Stanley, M.A.; Prof. of Moral Philosophy in the University of Dublin.
Moncourt, Professor; Paris.
Moon, Robert, M.A.; Ex-Fell. Qu. Coll., Camb.
Moore, H. K., B.A.; Trin. Coll., Dublin.
Morel, Professor; Paris.
Morgan, C., B.A.; Salisbury School.
Morley, T., L.C.P.; Bromley, Kent.
Morley, F., B.A.; Woodbridge, Suffolk.
Morrice, G. G., B.A.; The Hall, Salisbury.
Moulton, J. F., M.A.; Fell. of Ch. Coll., Camb.
Muir, Thomas, M.A., F.R.S.E.; Bishopton.
Mukhopadhyay, Asutosh, M.A.; Bhowanipore.
Nash, A. M., M.A.; Prof. in Pres. Coll., Calcutta.
Nelson, R. J., M.A.; Naval School, London.
Newcomb, Prof. Simon, M.A.; Washington.
Nicolls, W., B.A.; St. Peter's Coll., Camb.
Openshaw, Rev. T. W., M.A.; Clifton.
O'Regan, John; New Street, Limerick.
Orchard, H. L., M.A., L.C.P.; Burnham.
Owen, J. A., B.Sc.; Tennyson St., Liverpool.
Panton, A. W., M.A.; Fell. of Trin. Coll., Dublin.
Pendlebury, Rev. C., M.A.; London.
Perryman, W.; Carbrook, Sheffield.
Phillips, F. B. W.; Balliol College, Oxford.
Pillai, C. K., M.A.; Trichy, Madras.
Pirie, A., M.A.; University of St. Andrew's.
Polignac, Prince Camille de; Paris.
Pollexfen, H., B.A.; Windermere College.
Potter, J., B.A.; Richmond-on-Thames.
Prudden, Frances E.; Lockport, New York.
Purser, Prof. F., M.A.; Queen's College, Belfast.
Putnam, K. S., M.A.; Rome, New York.
Rau, B. Hanumanta, B.A.; Head Master of the Normal School, Madras.
Rawson, Robert; Havant, Hants.
Raymond, E. Lancelot, B.A., Surbiton.
Read, H. T., M.A.; Brasenose Coll., Oxford.
Reeves, G. M., M.A.; Lee, Kent.
Renshaw, S. A.; Nottingham.
Reynolds, B., M.A.; Notting Hill, London.
Richardson, Rev. G., M.A.; Winchester.
Rippin, Charles R., M.A.; Woolwich Common.
Roberts, R. A., M.A.; Schol. of Trin. Coll., Dublin.
Roberts, S., M.A., F.R.S.; Tufnell Park, London.
Roberts, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
Roberts, W. R., M.A.; Ex-Sch. of Trin. Coll., Dub.
Robson, H. C., B.A.; Sidney Sussex Coll., Cam.
Rosenthal, L. H.; Scholar of Trin. Coll., Dublin.
Roy, Kaliprasanna, M.A.; Professor in St. John's College, Agra.
Rücker, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
Ruggero, Simonelli; Università di Roma.
Russell, J. W., M.A.; Merton Coll., Oxford.
Russell, R., B.A.; Trinity College, Dublin.
Rutter, Edward; Sunderland.
Salmon, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.

Sampson, C. H., M.A.; Balliol College, Oxford
Sanders, J. B.; Bloomington, Indiana.
Sanderson, Rev. T. J., M.A.; Royston, Camb
Sarkar, Nilkantha, M.A.; Calcutta.
Savage, Thomas, M.A.; Fell. Pemb. Coll., Cam
Scheffer, Professor; Mercersburg Coll., Pa.
Scott, A. W., M.A.; St. David's Coll., Lampet
Scott, Charlotte A.; Girton College, Camb.
Scott, R. F., M.A.; Fell. St. John's Coll., Cam
Serret, Professor; Paris.
Sharp, W. J. C., M.A.; Hill Street, London.
Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham
Shepherd, Rev. A. J. P., B.A.; Fellow Queen's College, Oxford.
Simmons, Rev. T.C., M.A.; Christ's Coll., Breco
Siverly, Walter; Oil City, Pennsylvania.
Smith, C., M.A.; Sidney Sussex Coll., Camb.
Stabenow, H., M.A.; New York.
Steggall, J. E. A., B.A.; Clifton.
Stein, A.; Venice.
Stephen, St. John, B.A.; Caius Coll., Cambridg
Stewart, H., M.A.; Framlingham, Suffolk.
Swift, C. A., B.A.; Grammar Sch., Weybridg
Sylvester, J. J., D.C.L., F.R.S.; Professor Mathematics in the University of Oxfor Member of the Institute of France, &c.
Symons, E. W., M.A.; Fell. St. John's Coll., Oxo
Tait, P. G., M.A.; Professor of Natural Phil sophy in the University of Edinburgh.
Tanner, Prof. H. W. L., M.A.; Bristol.
Tarleton, F. A., M.A.; Fell. Trin. Coll., Dub.
Taylor, Rev. C., D.D.; Master of St. John College, Cambridge.
Taylor, H. M., M.A.; Fell. Trin. Coll., Camb.
Taylor, W. W., M.A.; Ripon Grammar Schoo
Tebay, Septimus, B.A.; Farnworth, Bolton
Terry, Rev. T. R., M.A., Fell. Magd. Coll., Oxo
Thomas, Rev. D., M.A.; Garsington Rect., Oxfor
Thomson, Rev. F. D., M.A.; Ex-Fellow of St. John Coll., Camb.; Brinkley Rectory, Newmarke
Tirelli, Dr. Francesco; Univ. di Roma.
Todhunter, Isaac, F.R.S.; Cambridge.
Torelli, Gabriel; University of Naples.
Torry, Rev. A. F., M.A.; St. John's Coll., Cam
Townsend, Rev. R., M.A., F.R.S.; Professor Nat. Phil. in the University of Dublin, &c.
Traill, Anthony, M.A., M.D.; Fellow a Tutor of Trinity College, Dublin.
Trowbridge, David; Waterburgh, New Yor
Tucker, R., M.A.; Mathematical Master in Un versity College School, London.
Turrell, I. H.; Cumminsville, Ohio.
Turriff, George, M.A.; Aberdeen.
Vincenzo, Jacobini; Università di Roma.
Vose, G. B.; Professor of Mechanics and Ci Engineering, Washington, United States.
Walenn, W. H.; Mem. Phys. Society, Londo
Walker, J. J., M.A., F.R.S.; Hampstead.
Walmsley, J., B.A.; Eccles, Manchester.
Warburton-White, R., B.A., Salisbury.
Warren, R., M.A.; Trinity College, Dublin.
Watherston, Rev. A. L., M.A.; Bowdon.
Watson, Rev. H. W.; Ex-Fell. Trin. Coll., Caml
Wertsch, Fr.; Weimar.
White, J. R., B.A.; Worcester Coll., Oxford.
White, Rev. J., M.A.; Cowley College, Oxford.
Whiteside, G., M.A.; Eccleston, Lancashire.
Whitworth, Rev. W. A., M.A.; Fellow of S John's Coll., Camb.; Hammersmith.
Wickersham, D.; Clinton Co., Ohio.
Wilkins, W.; Scholar of Trin. Coll., Dublin.
Williamson, B., M.A.; Fellow and Tutor Trinity College, Dublin.
Wilson, J. M., M.A.; Head-master, Clifton Co
Wilson, Rev. J., M.A.; Rect. Bannockburn Aca
Wilson, Rev. J. R., M.A.; Royston, Cambs.
Woodcock, T., B.A.; Wellington, Salop.
Wolstenholme, Rev. J., M.A., Sc.D.; Profess of Mathematics in Cooper's Hill College.
Woolhouse, W. S. B., F.R.A.S., &c.; London
Wright, Dr. S. H., M.A.; Penn Yan, New Yor
Wright, W. E., B.A.; Herne Hill.
Young, John, B.A.; Academy, Londonderry.

CONTENTS.

Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7382. (By Professor SYLVESTER, F.R.S.) — If p and q are relative primes, prove that the number of integers inferior to pq which cannot be resolved into parts (zeros admissible), multiples respectively of p and q, is

$$\tfrac{1}{2}(p-1)(q-1).$$

[If $p = 4$, $q = 7$, we have $\tfrac{1}{2}(p-1)(q-1) = 9$; and 1, 2, 3, 5, 6, 9, 10, 13, 17 are the only integers inferior to 28, which are neither multiples of 4 or 7, nor can be made up by adding together multiples of 4 and 7.]

Solution by W. J. CURRAN SHARP, M.A.

If the product $(1+x^p+x^{2p}+\ldots+x^{pq})(1+x^q+x^{2q}+\ldots+x^{pq})$ be considered, each term between 1 and x^{pq} corresponds to a number less than pq, and of the form $mp+nq$; also $2x^{pq}$ is the middle term, and the coefficients from each end are the same. Hence twice the number of integers of the form $mp+nq$, and less than pq, is the value of the above product when $x=1$ with four deducted, since the terms involving x^1, x^{pq}, x^{2pq} are not included; and therefore the number of these integers is

$$\tfrac{1}{2}(p+1)(q+1)-2,$$

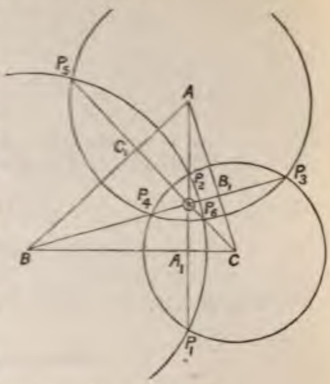
and the number of those which cannot be put into this form

$$= pq-1-[\tfrac{1}{2}(p+1)(q+1)-2] = \tfrac{1}{2}[pq-p-q+1] = \tfrac{1}{2}(p-1)(q-1).$$

1585. (By the late Professor CLIFFORD, F.R.S.)—If three circles are mutually orthotomic, prove that the circles on their common chords as diameters have a common radical axis.

Solution by Asútosh Mukhopádhyáy.

Let A, B, C be the centres of the three mutually orthotomic circles; P_1, P_2, P_3, P_4, P_5, P_6 their points of intersection. Then, from the ordinary theory of orthotomic circles, we know that the centre of each circle lies on the common chord of the other two; it can also be shown, from elementary geometry, that the line of centres of any two intersecting circles bisects their common chord at right angles; hence, we infer that, in the triangle ABC, AA_1, BB_1, CC_1 are the perpendiculars, and its orthocentre O is the radical centre of the system.



Moreover, calling the sides of the triangle ABC, a, b, c, as usual, we have

$$\left.\begin{array}{l} BA_1 = c\cos B \\ A_1C = b\cos C \end{array}\right\}, \quad \left.\begin{array}{l} CB_1 = a\cos C \\ B_1A = c\cos A \end{array}\right\}, \quad \left.\begin{array}{l} AC_1 = b\cos A \\ C_1B = a\cos B \end{array}\right\}.$$

Now, let us call S_1, S_2, S_3 the circles described on P_1P_2, P_3P_4, P_5P_6 as diameters; let their radii be ρ_1, ρ_2, ρ_3. Then we shall have

$$\rho_1^2 = A_1P_1^2 = BA_1 \, . \, A_1C \text{ (because, the circles being orthotomic, } \angle BP_1C \text{ is a right angle)}$$
$$= bc \cos B \cos C.$$

Similarly, $\rho_2^2 = AB_1 \, . \, B_1C = ca \, . \cos C \, . \cos A,$

and $\rho_3^2 = AC_1 \, . \, C_1B = ab \, . \cos A \, . \cos B.$

Taking A_1 as origin, A_1C, A_1A as the rectangular axes of x and y, the coordinates of the points A_1, B_1, C_1 will be (0, 0), ($c\cos A\cos C$, $a\cos C\sin C$), (*$b\cos A\cos B$, $a\cos B\sin B$), respectively. But, A_1, B_1, C_1 are the centres of the circles S_1, S_2, S_3 of radii ρ_1, ρ_2, ρ_3, therefore the equations of these circles are $x^2+y^2 = bc\cos B\cos C$.....................(1),

$$(x - c\cos A\cos C)^2 + (y - a\cos C\sin C)^2 = ca\cos C\cos A \text{ ......(2)},$$

$$(x - b\cos A\cos B)^2 + (y - a\cos B\sin B)^2 = ab\cos A\cos B \text{ ......(3)}.$$

Subtracting (2) from (1), attending to the relation $\sin A : a =$ &c., and cancelling like terms, we get, for the equation of the radical axis,

$$x\cos A + y\sin A = \tfrac{1}{2}(c\cos C + b\cos B - a\cos A) \text{ ....(4)}.$$

Subtracting (3) from (1), we get the same equation as (4). Therefore, the three circles have a common radical axis.

[This Question has been discovered by Mr. Walker to be erroneous. In fact, "solvitur delineando": if we draw (even mentally) the circles on P_1P_2, ... as diameters, we see they cannot have a common radical axis. The error of the above solution lies in the sign of the abscissa of C_1 (at * above) which should be *negative*, the abscissa being $-b\cos A\cos B$. The equations of the three radical axes are in fact

$$\pm x\cos A + y\sin A = \tfrac{1}{2}(-a\cos A + b\cos B + c\cos C)$$

and $(b\cos B + c\cos C)\cos A x + (-b\cos B + c\cos C)\sin A y$
$= (-b\cos B + c\cos C)(-a\cos A + b\cos B + c\cos C).$

The three have the same radical centre as the three orthogonal circles, viz. the point O. The common chords of the three derived circles are equal; in fact, the square of each is

$$2(a^2+b^2+c^2)\cos A\cos B\cos C = 2(b'c'+c'a'+a'b') - (a'^2+b'^2+c'^2),$$

where $a' = B_1 C_1 \ldots$, and the three chords are parallel respectively to the three radii drawn from the circum-centre of A, B, C to its corners. The subject is re-proposed for further discussion as Question 7605.]

7455 & 7481. (By Professor TOWNSEND, F.R.S.)—(7455). A system of plane waves, propagated by rectilinear vibrations perpendicular to the plane incidence, being supposed divided into two by refraction and reflexion at the surface of separation of two isotropic elastic solids in molecular contact with each other; determine, given the coefficients of resistance to compression and to distortion for both solids,—

(*a*) The relative amplitudes of vibration, for any angle of incidence, of the three systems of waves.

(*b*) The particular angle of incidence corresponding to the evanescence of the reflected vibrations.

(7481.) A system of plane waves, propagated either by normal or by transversal vibrations, being supposed divided into two by perpendicular refraction and reflexion at the surface of separation of two isotropic elastic solids in molecular contact with each other; show that, in either case, the *vis viva* of the original is divided without loss of total amount between the two derived systems of waves.

Solution by the PROPOSER.

(7455.) Denoting by k, k', k_1 the amplitudes of the incident reflected and refracted vibrations, by θ, θ', θ_1 the angles of incidence, reflexion and refraction of the wave systems, and by μ, μ', μ_1 and ν, ν', ν_1 the coefficients of resistance to compression and to distortion respectively in their propagation through the solids; then, the vibrations being manifestly perpendicular to the plane of incidence, and therefore to the direction of propagation in each derived as well as in the original system of waves, and all systems of waves propagated by transversal vibrations in isotropic media producing only distortion, but not compression of the molecules of the media, the six ordinary equations of condition, three geometrical and three dynamical, at the separating surface of the solids become reduced in this simple case to the two, one geometrical and one dynamical, viz., $k + k' = k_1$ and $\nu\cot\theta k + \nu'\cot\theta' k' = \nu_1\cot\theta_1 k_1$, or, since in the same case $\nu' = \nu$ and $\cot\theta' = -\cot\theta$, to the equivalent two $(k+k') = k_1$ and $\nu\cot\theta(k-k') = \nu_1\cot\theta_1 k_1$; from which, solving for k' and k_1, we get at once that $k' = k\dfrac{\nu\cot\theta - \nu_1\cot\theta_1}{\nu\cot\theta + \nu_1\cot\theta_1}$, and that $k_1 = k\dfrac{2\nu\cot\theta}{\nu\cot\theta + \nu_1\cot\theta_1}$, which giving in all cases the ratios of k' and k_1 to k in terms of θ and θ_1 and of ν and ν_1, and showing that $k' = 0$ when $\nu_1 \cot\theta_1 = \nu \cot\theta$, supply in consequence the solutions of both parts of the question.

The angle of incidence θ for which $k' = 0$, with the corresponding angle of refraction θ_1 for which $k_1 = 1$, may be readily determined, in terms of the coefficients ν and ν_1, and of the densities ρ and ρ_1 of the two solids, as follows. Since, in that case, as appears from the above, $\nu \cot\theta = \nu_1 \cot\theta_1$, and since in all cases, from the known laws of wave refraction,

$$\sin\theta : \sin\theta' = v : v' = \left(\frac{\nu}{\rho}\right)^{\frac{1}{2}} : \left(\frac{\nu_1}{\rho_1}\right)^{\frac{1}{2}},$$

where v and v' are the velocities of propagation for transversal vibrations in the solids, therefore here

$$(\rho\nu)^{\frac{1}{2}} . \cos\theta = (\rho_1\nu_1)^{\frac{1}{2}} . \cos\theta' \text{ and } (\rho\nu_1)^{\frac{1}{2}} . \sin\theta = (\rho_1\nu)^{\frac{1}{2}} . \sin\theta_1;$$

from which, by elimination successively of θ_1 and θ, it appears at once that

$$\tan^2\theta = \frac{\nu}{\nu_1} . \frac{\nu\rho - \nu_1\rho_1}{\nu\rho_1 - \nu_1\rho}, \text{ and that } \tan^2\theta_1 = \frac{\nu_1}{\nu} . \frac{\nu\rho - \nu_1\rho_1}{\nu\rho_1 - \nu_1\rho},$$

which give manifestly the values of the two angles in terms of the four quantities in question.

Multiplying together the two pairs of equivalents in the two general equations of condition at the separating surface of the media, viz., $(k+k') = k_1$, and $\nu \cot\theta\,(k-k') = \nu_1 \cot\theta_1 k_1$, we get at once the additional equation

$$\nu \cot\theta\,(k^2 - k'^2) = \nu_1 \cot\theta_1\, k_1^2;$$

or, remembering that $\nu : \nu_1 = \rho v^2 : \rho_1 v_1^2 = \rho \sin^2\theta : \rho_1 \sin^2\theta_1$, the equivalent equation $\rho \sin\theta\cos\theta\,(k^2 - k'^2) = \rho_1 \sin\theta_1 \cos\theta_1\, k_1^2$; from which it appears, by the usual mode of inference, that the *vis viva* of the entire motion is the same after as before the division of the original wave system into two at the separating surface of the media: a result technically termed *the preservation of the vis viva* in the division at the surface.

(7481.) For perpendicular incidence, either for transversal or for normal vibrations, the two equations of condition at the separating surface of the media assume alike the same simplified forms, viz.,

$$(k+k') = k_1 \text{ and } \rho v\,(k-k') = \rho_1 v_1 k_1;$$

where v and v_1 are the velocities of propagation corresponding to the species of vibration, whichever it be, in the media, and equal consequently to $\left(\frac{\nu}{\rho}\right)^{\frac{1}{2}}$ and $\left(\frac{\nu_1}{\rho_1}\right)^{\frac{1}{2}}$ for transversal, and to $\left(\frac{\mu + \frac{4}{3}\nu}{\rho}\right)^{\frac{1}{2}}$ and $\left(\frac{\mu_1 + \frac{4}{3}\nu_1}{\rho_1}\right)^{\frac{1}{2}}$ for normal vibrations. Therefore, for that incidence, in both cases alike, as appears at once by multiplication together of the two pairs of equivalents, $\rho v\,(k^2 - k'^2) = \rho_1 v_1 k_1^2$; and therefore, &c., as regards the property in question.

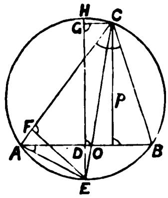
Solving for k' and k_1 from the two equations of condition, we see also that, in both cases alike,

$$k' = k\,\frac{\rho v - \rho_1 v_1}{\rho v + \rho_1 v_1} \text{ and } k_1 = k\,\frac{2\rho v}{\rho v + \rho_1 v_1},$$

from which it follows at once that, in both cases alike, $k' = 0$ and $k_1 = k$ when the products ρv and $\rho_1 v_1$ are equal in the media.

7357. (By Professor A. MOREL.)—Résoudre un triangle, connaissant une hauteur, le rayon du cercle inscrit, et le rayon du cercle circonscrit.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

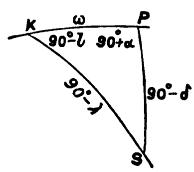


Let ABC be the triangle, then, denoting by a, b, c, s, R, r, p the three sides of the triangle, its semi-perimeter, the radii of its circumscribed and inscribed circles, and the altitude passing through the vertex C, we have pc = area $= rs$, therefore $s : c = r : p$ is known, and in the annexed well-known figure the ratio of CF $[= \frac{1}{2}(a+b)]$ to AD $(= \frac{1}{2}c)$ is known, or its equivalent CE : AE is known, and therefore $CE^2 : AE^2$ is known; but CE . EO $= AE^2$, therefore CE : AE = AE : EO, ∵ CE : EO = $CE^2 : AE^2$ is known, and therefore CE : CO or EG : p is known, therefore EG is known, and ED, which is equal to EG $- p$, is known. Draw, then, in the given circumscribed circle a diameter EH, take ED as found above, through D draw AB perpendicular to EH, take DG $= p$, draw GC parallel to AB, then ACB is the required triangle.

7510. (By Professor HAUGHTON, F.R.S.)—If α, δ denote right ascension and declination, and l, λ longitude and latitude; show that the inclination of the ecliptic is given by the equation

$$\cos\omega = \frac{\sin\lambda\tan l + \sin\delta\tan\alpha}{\sin\lambda\tan\alpha + \sin\delta\tan l}.$$

Solution by B. REYNOLDS, M.A.; Professor MATZ, M.A.; *and others.*



By twice applying the cot.-formula to the triangle here shown (avoiding the angle of position S), we get

$$\tan\lambda\sin\omega - \sin l\cos\omega = -\tan\alpha\cos l \;\;\ldots(1),$$

$$\tan\delta\sin\omega + \sin\alpha\cos\omega = \tan l\cos\alpha \;\;\ldots\ldots(2);$$

whence, eliminating $\sin\omega$, we find

$$\cos\omega = \frac{\tan\lambda\tan l\cos\alpha + \tan\delta\tan\alpha\cos l}{\tan\lambda\sin\alpha + \tan\delta\sin l}$$

$$= \frac{\sin\lambda\tan l\,\dfrac{\cos\alpha}{\cos\lambda} + \sin\delta\tan\alpha\,\dfrac{\cos l}{\cos\delta}}{\sin\lambda\tan\alpha\,\dfrac{\cos\alpha}{\cos\lambda} + \sin\delta\tan l\,\dfrac{\cos l}{\cos\delta}} = \frac{\sin\lambda\tan l + \sin\delta\tan\alpha}{\sin\lambda\tan\alpha + \sin\delta\tan l},$$

since $\dfrac{\cos\alpha}{\cos\lambda} = \dfrac{\cos l}{\cos\delta}$, by a well-known formula.

Solution by G. B. MATHEWS, B.A.; Prof. NASH, M.A.; *and others.*

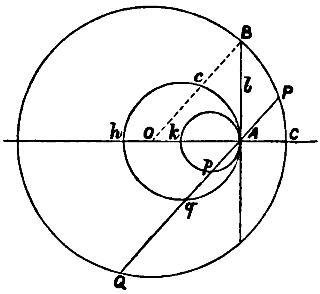


Fig. 1.

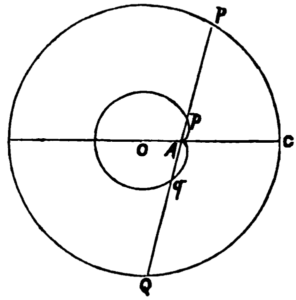


Fig. 2.

The construction throughout is to draw a chord PAQ of the fixed circle, and set off $Pp = Qq = b$. Then, if $OA = a$, $OC = c$, $AP = r$, $CAp = \theta$, we have
$$c^2 = OP^2 = OA^2 + AP^2 - 2OA \cdot AP \cos OA$$
$$= (b-r)^2 + a^2 - 2a(b-r)\cos\theta.$$

The first figure applies when $a^2+b^2 = c^2$, the second when $c = a+b$, and so for each of the other cases.

7421. (By R. KNOWLES, B.A.)—Two equal tangents OP, OQ are drawn to a parabola; prove that (1) the angle POQ is bisected by the axis, and (2) the distance of the centre of the circle OPQ from the vertex is constant and equal to one-half the latus rectum.

Solution by A. MARTIN, B.A.; KATE GALE; *and others.*

It is clear that the point O must be on the axis; hence POQ is bisected by the axis, and, as $SO = SP = SQ$, the centre of the circle must be the focus.

7534. (By the Rev. T. C. SIMMONS, M.A.)—A number is known to consist of four digits whose sum is 10; show that the odds are 154 : 65 in favour of the sum of the digits of twice the number being equal to 11.

Solutions by B. REYNOLDS, M.A.; A. MARTIN, B.A.; *and others.*

Of 4-digit numbers, those are favourable to the event specified which have *one* digit equal to or greater than 5. Unfavourable ones are those with two 5's (and two 0's therewith, of course), or with all the digits less than 5; hence, considering [illegible] each, and remembering that [illegible] and against are as follows:—

9100, 8200 [illegible]	[illegible]
8110, 6220 [illegible]	[illegible]
7111 gives	[illegible]
7210, 6310 [illegible]	[illegible]
6211, [illegible]	[illegible]
[illegible]	[illegible]
5500 gives	[illegible]
4222, [illegible]	[illegible]
4411 gives	[illegible]
4420 and [illegible]	[illegible]
4321 gives	[illegible]
3322 gives	[illegible]
[illegible]	[illegible]

7526. (By [illegible]) [illegible] vertical axis [illegible] from the tail [illegible] ballast, and the [illegible] bridge so that [illegible]

Solution by the Rev. [illegible]

Let w = weight [illegible] length of tail [illegible] require the minimum [illegible]

get [illegible]

5350. (By S. A. Renshaw.)—An ellipse and hyperbola have the same centre and directrices, and they have a common tangent which touches the ellipse in D and the hyperbola in E, and meets one of the directrices in H. A line from the common centre of the curves is drawn parallel to the common tangent and meeting the same directrix in R. Tangents RW, RV are drawn to the auxiliary circles of the ellipse and hyperbola. Show that, if FH, fH be joined, F and f being the foci of the curves belonging to the directrix RH,

$$DH \cdot HF : EH \cdot fH = WR \cdot VR$$

"SYMMEDIANS" AND THE "T. R." CIRCLE. By R. TUCKER, M.A.

M. MAURICE D'OCAGNE, in the *Nouvelles Annales* for October, 1883, (pp. 450—464,) has applied the name of "Symmedian" to the line AS, in a triangle, which makes the same angle with AC that the median line AM does with AB (take the points, to fix the figure, which is easily drawn, in the order B, M, S, C). On AC take $AB' = AB$, and on AB, $AC' = AC$, then the "Symmedian" through A (say S_a) bisects $B'C'$. From M, S, let fall perpendiculars MP, SU on AB and MQ, SV on AC,

then $\qquad SU/AS = MQ/AM$ and $SV/AS = MP/AM$,

therefore $\qquad SU/SV = MQ/MP = AB/AC.$

Now $\quad SU.AB/SV.AC = SB/SC, \quad \therefore SB/SC = AB^2/AC^2$.......(1).

From (1) it readily follows that S_a, S_b, S_c pass through a point (P say). It is evident from (1) that P is the point through which the lines DPE′, EPF′, FPD′ must be drawn parallel to AB, BC, CA to obtain the points of section by the "T. R." circle (see *Educational Times* for July). The point P might be called the "Symmedian" point; its determination is easy from the above definition. Many interesting results are obtained by M. d'Ocagne.

It is also worth remarking that the "Symmedian" point P is the radical centre of the circles about ABD′E, BCE′F, CAF′D.

7461. (By Professor GENESE, M.A.)—A plane triangle is constructed whose sides are arcs of equal circles. If these sides be measured by the angles which they subtend at the centres of the corresponding circles, prove geometrically that (as an extension of Euc. I. 32), with a certain convention of signs, $A + B + C = \pi + a + b + c$.

Solution by the PROPOSER.

The result in the question was obtained thus. Conceive the whole triangle, supposing its sides concave to the interior, rotated about the centre of BC through the angle a so that B comes upon C, then rotated about C, in the same sense, through the angle $\pi - C$, so that the old arc BC now lies along CA. It is clear that, if the process be continued as suggested, the triangle will ultimately come back to its old position, that is, will have turned through the angle 2π. Thus we have

$$a + \pi - C + b + \pi - A + c + \pi - B = 2\pi \text{ or } \pi + a + b + c = A + B + C.$$

The cases wherein sides are convex to the interior may be similarly treated, or may be deduced from the above by producing the arcs so as to get a triangle of the first case. It is found that for convex sides a negative value must be given to a, b, c.

[This Question is almost identical with Mr. HEPPEL's Quest. 6214, solved by Mr. WALKER on p. 58 of Vol. 34 of the *Reprints*.]

7487. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Given two conics U, U′, a tangent at P to U meets the polar of P with respect to U′ in P′; prove that the locus of P′ is the quartic $UV = U'^2$, where V is the polar reciprocal of U with respect to U′ so taken that the discriminants of U, U′, V are in geometrical progression.

Solution by T. WOODCOCK, B.A.; G. B. MATHEWS, B.A.; *and others.*

Using areal coordinates, we may write U and U′ in the forms

$$x^2+y^2+z^2=0 \quad \text{..............................(1)},$$

and $ax^2+by^2+cz^2=0$; the equation to V will be $a^2x^2+b^2y^2+c^2z^2=0$. We have to eliminate xyz between (1) and the pair $x\xi+y\eta+z\zeta=0$, $ax\xi+by\eta+cz\zeta=0$. From the last two, we have

$$\frac{x}{(b-c)\,\eta\zeta}=\frac{y}{(c-a)\,\zeta\xi}=\frac{z}{(a-b)\,\xi\eta};$$

therefore, by (1), $\quad \eta^2\zeta^2(b^2+c^2-2bc)+\ldots+\ldots=0.$

This may be put in the form $(\xi^2+\eta^2+\zeta^2)(a^2\xi^2+\ldots+\ldots)=(a\xi^2+\ldots+\ldots)^2$, which is the one required.

7525. (By T. MUIR, M.A., F.R.S.E.)—If in a determinant of the n^{th} order the elements in the main diagonal be all negative and all the others positive, prove that the number of positive terms in the development is $\quad \tfrac{1}{2}n!-(-2)^{n-2}(n-2).$

Solution by the PROPOSER; R. LACHLAN, B.A.; *and others.*

Let x, y be the numbers of positive and negative terms, respectively, and D the value of the determinant in question when each element is in magnitude equal to unity; then we have

$$x-y=D=(-2)^{n-1}(n-2) \text{ and } x+y=n!\text{; whence, \&c.}$$

7470. (By J. HAMMOND, M.A.)—Trace the curve

$$a^2+(b-r)^2-2a(b-r)\cos\theta=c^2,$$

with special reference to the cases (1) when $a^2+b^2=c^2$, (2) when $a\pm b\pm c=0$; and prove that it admits of an easy mechanical description.

[When $c=a$, the curve degenerates into a circle and a limaçon.]

[illegible]

than 5; hence, considering the numbers according to the highest digit in each, and remembering that 0 cannot stand as a first digit, the cases for and against are as follows:—

9100, 8200, 7300, 6400 give 6 each	= 24
8110, 6220 give 9 each	= 18
7111 gives	= 4
7210, 6310, 5410, 5320 give 18 each	= 72
6211, 5311, 5221 give 12 each..................	= 36
Total of favourable cases.........	=154
5500 gives	= 3
4222, 3331 give 4 each	= 8
4411 gives	= 6
4420 and 4330 give 9 each	= 18
4321 gives..	= 24
3322 gives..	= 6
Total of unfavourable cases......	= 65

7526. (By W. G. LAX, B.A.)—A swing-bridge is movable about a vertical axis on one bank of a river, and has a load of ballast suspended from the tail end of it; if the cost of bridge per ton be n times that of ballast, and the river a yards wide, find the length of the tail end of the bridge so that the cost of the whole may be a minimum.

Solution by the REV. T. C. SIMMONS, M.A.; BELLE EASTON; *and others.*

Let w = weight in tons of unit length, b = weight of ballast, x = required length of tail; then $\frac{1}{2}a \,.\, wa = \frac{1}{2}x \,.\, wx + x \,.\, b$, whence $b = \dfrac{wa^2}{2x} - \frac{1}{2}wx$, and we require the minimum of $y \equiv nw\,(a+x) + \dfrac{wa^2}{2x} - \frac{1}{2}wx$. Putting $\dfrac{dy}{dx} = 0$, we get $nw - \dfrac{wa^2}{2x^2} - \frac{1}{2}w = 0$, whence $x^2 = \dfrac{a^2}{2n-1}$.

5350. (By S. A. RENSHAW.)—An ellipse and hyperbola have the same centre and directrices, and they have a common tangent which touches the ellipse in D and the hyperbola in E, and meets one of the directrices in H. Also from the common centre of the curves S′R is drawn parallel to the common tangent and meeting the same directrix in R. Tangents RW, RV are drawn to the auxiliary circles of the ellipse and hyperbola. Show that, if FH, fH be joined, F and f being the foci of the curves belonging to the directrix RH,

$$\text{DH . HF : EH . }f\text{H} = \text{WR}' \text{ : VR}.$$

Now, if R_1 coincide with O, and $OR_1 = 0$, this becomes

$$OR\Sigma(OR_2 - OR)\dots(OR_p - OR)\,OR_{p+1}\dots OR_n = 0,$$

and one value of OR is zero, as it should be.

If R_2 also coincide with O, the equation reduces to

$$OR^2\Sigma(OR_3 - OR)\dots(OR_p - OR)\,OR_{p+1}\dots OR_n = 0, \text{ and so on.}$$

Hence the 2nd polar of an n-ic, if $R_1, R_2 \dots R_{n-3}$ all coincide with O, has for its equation

$$OR^{n-3}[(OR_{n-2} - OR)\,OR_{n-1} \,.\, OR_n + (OR_{n-1} - OR)\,OR_n \,.\, OR_{n-2} + (OR_n - OR)\,OR_{n-1}OR_{n-2}] = 0,$$

and $n-3$ values of OR are zero, and the other satisfies

$$\frac{3}{OR} = \frac{1}{OR_{n-2}} + \frac{1}{OR_{n-1}} + \frac{1}{OR_n};$$

and hence, if R coincide with R_{n-2}, R_{n-1}, or R_n, the line is divided harmonically; *i.e.*, if the line pass through an $(n-3)^{ic}$ point in the locus, and through an intersection of the locus and the second polar of the point, it is harmonically divided by the n-ic.

Now (1) a quartic curve is cut by the polar conic of a point on itself in five points, distinct from the point, the lines joining which to the original point are harmonically divided by the curve.

(2.) A quintic is cut by the polar cubic of a double point on it in $3 \times 5 - 4 \times 2 = 7$ points; and so many lines can be drawn through it so as to be harmonically divided by the curve, and generally a $(p+3)^{ic}$ is cut by the polar $(p+1)^{ic}$ of a p^{ic} point on it in $(p+1)(p+3) - (p+2)\,p = 2p+3$ points, distinct from the multiple point, and hence as many lines can be drawn through so as to be harmonically divided by the curve.

(3.) Since an n-ic surface meets any plane in an n-ic curve, the above argument shows that the curve of intersection of the second polar of an $(n-3)^{ic}$ point on it pierces the plane in $2n - 6 + 3 = 2n - 3$ points, and therefore this partial intersection is a $(2n-3)^{ic}$ curve; and therefore a quintic when the point is an ordinary point on a quartic surface.

5754. (By J. Hammond, M.A.)—Sum the series

$$\frac{1}{n}\cdot\frac{1}{2m+n} - 2m\,.\,\frac{1}{n+1}\cdot\frac{1}{2m+n-1} + \frac{2m(2m-1)}{1\,.\,2}\cdot\frac{1}{n+2}\cdot\frac{1}{2m+n-2} - \&c.,$$

where m is a positive integer, and the $(r+1)^{th}$ term is

$$(-)^r\frac{2m(2m-1)\dots(2m-r+1)}{1\,.\,2\,.\,3\dots r}\cdot\frac{1}{n+r}\cdot\frac{1}{2m+n-r}.$$

By W. J. C. Sharp, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

$$\frac{1}{n+r}\cdot\frac{1}{2m+n-r} = \frac{1}{2(m+n)}\cdot\left(\frac{1}{n+r} + \frac{1}{2m+n-r}\right),$$

[illegible]ed series may be written

$$[\text{illegible}]\left[\frac{1}{n} - 2m\,.\,\frac{1}{n+1} + \&c. + \frac{1}{2m+n} - 2m\,.\,\frac{1}{2m+n-1} + \&c.\right];$$

whence $a^2x^2\tan^4\theta - 2\,abxy\tan^3\theta$

$$+\left[a^2x^2+b^2y^2-(a^2-b^2)^2\right]\tan^2\theta - 2abxy\tan\theta + b^2y^2 = 0.$$

Whence, since coefficients of $\tan^3\theta$ and $\tan\theta$ are equal and of same sign,

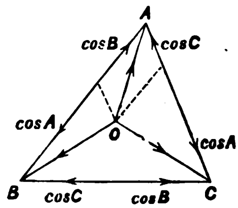
$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta = \tan\alpha\tan\beta\tan\gamma+\tan\alpha\tan\gamma\tan\delta+\ldots;$$

whence $$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}=0,$$

or $\tan(\alpha+\beta)+\tan(\gamma+\delta)=0$; whence generally $\alpha+\beta+\gamma+\delta=n\pi$; n an integer. Now, since the last term is positive, only two, or four, can be negative together; this consideration with the geometry of the figure will show that n can be only 3 or 5.

7488. (By Professor HUDSON, M.A.)—If O be the circumcentre of ABC, and forces act along OA, OB, OC proportional to BC, CA, AB; prove that their resultant passes through the in-centre.

Solution by C. MORGAN, B.A., R.N.; G. B. MATHEWS, B.A.; *and others.*



The force along OA is equivalent to two forces proportional to cos B, cos C along BA, CA respectively; hence, resolving the other forces similarly as in the figure, the trilinear equation of the resultant is

$$(\cos B-\cos C)\,\alpha+(\cos C-\cos A)\,\beta+(\cos A-\cos B)\,\gamma=0,$$

which goes through $\alpha=\beta=\gamma$; therefore &c.

5787 & 5945. (By W. J. C. SHARP, M.A.)—From an ordinary point on a quartic five straight lines can be drawn so as to be cut harmonically by two curves. How far is this modified when the point is a node?

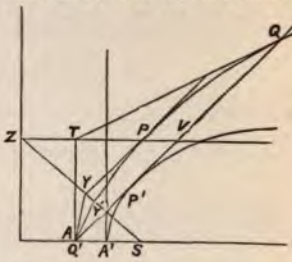
(5945.) From a double point on a quintic, a triple point on a sextic, or a p^{ic} point on a $(p+3)^{\text{ic}}$, prove that a limited number of lines can be drawn so as to be harmonically cut by the curve. (This is an extension of Question 5787, which may be extended to surfaces as follows):—Through an ordinary point on a quartic surface lines may be drawn so as to be cut harmonically by the surface; the points of section will trace out a quintic curve on the surface.

Solution by the PROPOSER.

The $(n-p)^{\text{th}}$ polar of O with respect to an n-ic curve or surface is the locus of a point R in OR_n, cutting the figure in $R_1, R_2, R_3 \ldots R_n$, such that $$\Sigma\,(OR_1-OR)\ldots(OR_p-OR)\,OR_{p+1}\ldots OR_n=0.$$

7519. (By R. Tucker, M.A.)—AP, A′P′ are two confocal and coaxial parabolas, the parameter of the former being twice that of the latter; prove that, if any chord QQ′ be tangential to the inner curve, then the distance of the mid-point of QQ′ from the directrix of AP is trisected where it meets AP, and the tangents at Q, Q′ pass through the other point of trisection.

Solution by R. Lachlan, B.A.; R. Knowles, B.A., L.C.P.; *and others.*



Let QQ′ be the chord touching A′P′ in P′, and V the mid-point of QQ′; and let the diameter through V meet the curve in P, the tangents at Q, Q′ in T, and the directrix of AP in Z. Then SZ must be perpendicular to QQ′, and the tangent at P; hence, if SZ meet QQ′ in Y′, and the tangent at P in Y, Y, Y′ must lie on the tangents at A, A′ to the parabolas; and evidently since

$$SA = 2SA', \; YY' = \tfrac{1}{4}SZ,$$

therefore $PV = \tfrac{1}{3}ZV$;

and since PT = PV it follows that ZV is trisected in P and T.

7366. (By C. Leudesdorf, M.A.)—Two particles A and C, each of mass m', are connected with each other by an elastic string whose modulus of elasticity is λ and whose unstretched length is l; and they are connected with another particle B of mass m by two massless rods, each of length a. The system lies on a smooth horizontal table, and is held so as to form a straight line ABC. The constraints at A and C are removed, and at the same instant each of the particles A and C is projected with velocity v in a direction at right angles to ABC. Find the stress along either rod at the moment when the particles form an equilateral triangle.

Solution by D. Edwardes; Professor Nash, M.A.; *and others.*

Since evidently m moves in a straight line, if x be its distance from a fixed point at time t, x', y' the coordinates of m', and θ the angle between the rods, then

$$(m+2m')\frac{dx}{dt} - 2am'\sin\theta\frac{d\theta}{dt} = 2m'v \text{ (no external forces).}$$

Again, since $x' = x + a\cos\theta$, $y' = a\sin\theta$, the kinetic energy is

$$\frac{2m'^2v^2 - 2a^2m'^2\sin^2\theta\dot{\theta}^2}{m+2m'} + m'a^2\dot{\theta}^2.$$

Subtracting $m'v^2$ from this, and equating the result to the work done by the tension, we get at once

$$\dot{\theta}^2 = \frac{mm'v^2 + 2\lambda(m+2m')\left(\frac{a^2}{l}\cos^2\theta - a + a\sin\theta\right)}{m'(m+2m')a^2 - 2a^2m'^2\sin^2\theta}.$$

Differentiating, and putting $\theta = 30°$,

$$\ddot{\theta} = \frac{\lambda(m+2m')\left\{a(m+\tfrac{1}{2}m') - \frac{a^2}{l}m\right\}\sqrt{3} + mm'^2v^2\sqrt{3}}{a^2m'(m+\tfrac{3}{2}m')^2}.$$

Now, when $\theta = 30°$, $\quad \dfrac{m+2m'}{am'}\ddot{x} = \sqrt{3}\,\dot{\theta}^2 + \ddot{\theta}$;

whence, substituting, we have, when $\theta = 30°$,

$$\frac{a}{\sqrt{3}}\ddot{x} = \frac{2mm'v^2 + \lambda\left\{\frac{a^2}{l}(m+\tfrac{3}{2}m') - 2am'\right\}}{(m+\tfrac{3}{2}m')^2}.$$

Let R be the stress along either rod, then $m\dfrac{d^2x}{dt^2} = \text{R}\sqrt{3}$,

therefore
$$\text{R} = m\frac{2mm'v^2 + \lambda\left\{\frac{a^2}{l}(m+\tfrac{3}{2}m') - 2am'\right\}}{a(m+\tfrac{3}{2}m')^2}$$
$$= 2m\frac{4mm'lv^2 + a\lambda\left\{2am + (9a-4l)m'\right\}}{al(2m+3m')^2}.$$

7494. (By W. J. C. SHARP, M.A.)—Show that
$$\int_{-1}^{1}\frac{d^{n-m}(x^2-1)^n}{dx^{n-m}}\cdot\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}\,dx = (-1)^m\frac{2^{2n+1}}{2n+1}(n!)^2.$$

Solution by D. EDWARDES; BELLE EASTON; *and others.*

Let $y = (1-x^2)^n$, and $\text{D} \equiv \dfrac{d}{dx}$. Then, integrating by parts,

$$u = \Big[\text{D}^{n+m-1}y\,\text{D}^{n-m}y - \text{D}^{n+m-2}y\,\text{D}^{n-m+1}y + \dots$$
$$\dots(-)^{m-1}\text{D}^n y\,\text{D}^{n-1}y\Big]_{-1}^{+1} + (-)^m\int_{-1}^{+1}\text{D}^n y\,\text{D}^n y\,dx.$$

Now, if P_n be the coefficient of z^n in the expansion of $(1+2zx+z^2)^{-\frac{1}{2}}$, by a well-known theorem $\displaystyle\int_{-1}^{+1}\text{P}_n\text{P}_n\,dx = \frac{2}{m+1}$, where $\text{P}_n = \dfrac{\text{D}^n(1-x^2)^n}{2^n\Gamma(n+1)}$. Also, from the equation $(1-x^2)\dfrac{dy}{dx} + 2nxy = 0$, we find, when $x = 1$ or -1, $\text{D}^r y = 0\;[r<n]$; hence $u = (-)^m\dfrac{2^{2n+1}}{2n+1}(n!)^2$.

7273. (By A. MCMURCHY, B.A.)—Prove that, if radii be drawn to a sphere parallel to the principal normals at every point of a closed curve of continuous curvature, the locus of their extremities divides the sphere into two equal parts.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

Let $d\theta$ be the angle between two consecutive osculating planes, then the angle in the spherical polygon formed by extremities of radii $= \pi - d\theta$, and sum of angles of polygon $= \Sigma(\pi - d\theta)$; hence

Area sought $= R^2[\Sigma(\pi - d\theta) - (n-2)\pi] = 2\pi R^2 - \Sigma d\theta$.

But, in going right round, the curve $d\theta$ is as often positive as negative; therefore $\Sigma d\theta = 0$, therefore area $= 2\pi R^2$, therefore &c.

7460. (By Professor WOLSTENHOLME, M.A., D.Sc.)—

If $$x^n = \cos^n\theta + \sin^n\theta,$$

prove that $$x^{2n-1}\left(\frac{d^2x}{d\theta^2} + x\right) = (n-1)(\sin\theta\cos\theta)^{n-2}.$$

Solution by J. S. JENKINS; R. KNOWLES, B.A.; *and others.*

$$x^{n-1}\frac{dx}{d\theta} = \sin^{n-1}\theta\cos\theta - \cos^{n-1}\theta\sin\theta;$$

and again, differentiating and multiplying each side by x^n, we have

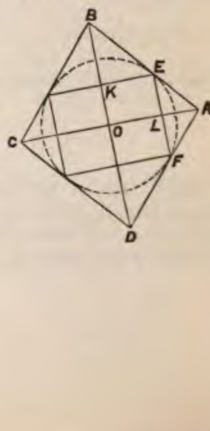
$$(n-1)\,x^{2n-2}\left(\frac{dx}{d\theta}\right)^2 + x^{2n-1}\frac{d^2x}{d\theta^2} = (n-1)(\sin^{n-2}\theta + \cos^{n-2}\theta)\,x^n - nx^{2n};$$

whence, by transposing and reducing, we have

$$x^{2n-1}\left(\frac{d^2x}{d\theta^2} + x\right) = (n-1)(\sin^{n-2}\theta\cos^n\theta + \cos^{n-2}\sin^n\theta) = \text{the result.}$$

7471. (By D. EDWARDES. Suggested by Quest. 7434.)—If an ellipse be inscribed in a rectangle, prove that the perimeter of the quadrilateral formed by joining the points of contact is constant.

Solution by the Rev. T. C. SIMMONS, M.A.; R. KNOWLES, B.A.; *and others.*



Let E be any point of contact on a side AB of the rectangle. Then an ellipse can be drawn through E, as in the figure, to touch the sides symmetrically, and with its centre at O. Also, since five conditions are given, viz., two consecutive points at E and three other tangents, this ellipse is the only one that can be drawn through E; that is, all the ellipses must be symmetrical with respect to the diagonals. Hence it follows at once that the quadrilateral joining the points of contact has its sides parallel to the diagonals; hence its perimeter is constant.

5421. (By Professor CAYLEY, F.R.S.) — Suppose $S_x = m_1(x-a_1)$, $m_2(x-a_2)$, $m_3(x-a_3)$, $m_4(x-a_4)$; where, for any given value of x, we write $+, -$, or 0, according as the linear function is positive, negative, or zero, and where the order of the terms is not attended to. If x is any one of the values a_1, a_2, a_3, a_4, the corresponding S is $0+++$, $0---$, $0++-$, or $0+--$; and if I denote indifferently the first or second form, and R denote indifferently the third or fourth form, then it is to be shown that the four S's are R, R, R, R, or else R, R, I, I.

Solution by W. J. C. SHARP, M.A.

If a_1, a_2, a_3, a_4 be in ascending order of magnitude, then, if the m's be all positive, the S's are I, R, R, I, being

0	+	+	+
−	0	+	+
−	−	0	+
−	−	−	0

and the signs in each column will change sign with the corresponding m. Now a change of sign in either outer column leaves the result R, R, I, I, and one in either or both the middle columns gives R, R, R, R; whilst these changes, in addition to the change of one or both the outer columns, give R, R, I, I.

7537. (By Professor TOWNSEND, F.R.S.)—An ellipsoidal shell being supposed, by a small movement of rotation round an arbitrary axis passing through the centre of its inner surface, to put into irrotational strain a contained mass of incompressible fluid completely filling its interior; investigate, in finite terms, the equations of the displacement line-system of the strain.

Solution by (1) C. GRAHAM, M.A.; (2) *the* PROPOSER.

1. Let $\omega_1, \omega_2, \omega_3$, be the component angular velocities about the axis of the ellipsoid, and V the velocity-potential referred to axes which at any instant coincide with the axes of the ellipsoid; then we have

$$\frac{dV}{dx}\frac{x}{a^2}+\frac{dV}{dy}\frac{y}{b^2}+\frac{dV}{dz}\frac{z}{c^2}$$
$$=(\omega_2 z-\omega_3 y)\frac{x}{a^2}+(\omega_3 x-\omega_1 z)\frac{y}{b^2}+(\omega_1 y-\omega_2 x)\frac{z}{c^2}$$

along the surface, and $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$ throughout the liquid; whence we easily deduce

$$V=\omega_1\frac{(b^2-c^2)}{b^2+c^2}yz+\omega_2\frac{(c^2-a^2)}{c^2+a^2}xz+\omega_3\frac{(a^2-b^2)}{a^2+b^2}xy;$$

which, by giving different values to V, represents a series of similar and concentric hyperboloids, having their common centre at the centre of the shell. Transform this system to its axes, and let its equation be $\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=K$, where K is the parameter; then the directions of the velocity of any particle of fluid are given by the equations

$$\frac{dx}{\frac{x}{\alpha}} = \frac{dy}{\frac{y}{\beta}} = \frac{dz}{\frac{z}{\gamma}}, \text{ or, integrating, } \frac{x^\alpha}{\lambda} = \frac{y^\beta}{\mu} = \frac{z^\gamma}{\nu};$$

where λ, μ, ν are arbitrary, and α, β, γ known constants.

2. Denoting by a, b, c the three semi-axes of the inner surface of the shell, by p, q, r the three components of the rotation with respect to their directions, by x, y, z the coordinates of any point of the fluid with respect to the same, and by ϕ the potential of the strain; then since, as is well-known,

$$\phi = p\frac{b^2-c^2}{b^2+c^2}yz + q\frac{c^2-a^2}{c^2+a^2}zx + r\frac{a^2-b^2}{a^2+b^2}xy = fyz + gzx + hxy,$$

we have, therefore, for the differential equations of the line-system in question, $\frac{dx}{gz+hy} = \frac{dy}{hx+fz} = \frac{dz}{fy+gx}$; the complete integrals of which in finite terms are, as is also well-known,

$$c_1(l_1x+m_1y+n_1z)^{\frac{1}{\lambda_1}} = c_2(l_2x+m_2y+n_2z)^{\frac{1}{\lambda_2}} = c_3(l_3x+m_3y+n_3z)^{\frac{1}{\lambda_3}};$$

where λ_1, λ_2, λ_3 are the roots of the equation $\lambda^3-(f^2+g^2+h^2)\lambda-2fgh=0$; $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ the corresponding values of l, m, n as given by the equations $hm+gn-l\lambda=0$, $fn+hl-m\lambda=0$, $gl+fm-n\lambda=0$; and c_1, c_2, c_3 any arbitrary constants the ratios of which are given for each particular line of the system with any single point $x'y'z'$ of its course.

7538. (By Professor HAUGHTON, F.R.S.)—Show that the law of propagation of heat in a solid sphere is $\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{2}{x}\frac{dv}{dx}\right)$.

Solution by C. GRAHAM, M.A.

The general law of propagation of heat in an isotropic solid is

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

where a depends on the conductivity and specific heat of the solid. Transforming this to polar coordinates by the well-known transformation, it becomes

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \frac{1}{r^2}\frac{d^2v}{d\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{d^2v}{d\theta^2}\right);$$

but v is independent of θ and ϕ since the solid is a sphere equally hot at points equidistant from the centre, therefore the equation reduces to the form given in the question.

7513. (By Professor MINCHIN, M.A.)—Give a simple geometrical proof of the existence and fundamental property of the Instantaneous-Acceleration Centre in the uniplanar motion of a rigid body.

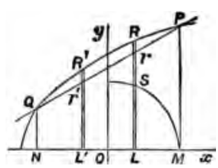
Solution by C. GRAHAM, M.A.

Suppose A and A′ to be two consecutive positions of the centre of instantaneous rotation, and P any point. If the acceleration of P is to be zero, we must have the velocity of P the same in direction and magnitude when the body is rotating about the two points A and A′. Therefore, if ω and ω' be the angular velocities in these two positions, we must have $AP \cdot \omega = A'P \cdot \omega'$ to make the velocities equal. Therefore P must lie on a known circle, since the ratio of AP to A′P is known; and to make the velocities parallel we must have AP parallel to A′P′ when P′ is the second position of P, and therefore $AA' \sin(PA'A) = AP \cdot \phi$ when ϕ is the small angle through which the body has turned in going from the first to its second position. This determines the angle PA′A, and therefore the position of P on the circle already found. So we see there is one position, and only one position, of P.

Again, since the acceleration of P is zero, the acceleration of any point relative to P is its absolute acceleration; but, if Q is any point, its acceleration relative to P along $PQ = PQ \cdot \omega^2$, and perpendicular to $PQ = PQ \cdot \frac{d\omega}{dt}$, and therefore the angle which the resultant acceleration makes with PQ is $= \tan^{-1}\left(\frac{d\omega}{dt} / \omega^2\right)$, which is independent of the position of Q.

7483. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In WALTON's *Mechanical Problems* (3rd ed., p. 19, "Centres of Gravity of Solids of Revolution," Ex. 10) it is stated that the centroid of the solid formed by scooping out a cone from a paraboloid of revolution, the bases and vertices of the two solids being coincident, bisects the axis; prove that (1) this is true for the volume formed by the revolution of any segment, cut off by a chord PQ, from any conic, about an axis of the conic, provided PQ does not cut the axis; also, more generally, (2) if PM, QN be drawn perpendicular to the axis, and a sphere be described on MN as diameter, the centroid of any part of the volume generated by the segment, intercepted between two planes perpendicular to the axis of revolution, is coincident with the centroid of the volume of the sphere intercepted between the same two planes.

Solution by T. WOODCOCK, B.A.; Professor MATZ, M.A.; *and others.*



1. Taking the axis MN for axis of x, and the middle point O of MN for origin, the equation to the conic may be written $y^2 = Ax^2 + Bx + C$. Let $PM = h$, $QN = h'$, $MN = 2a$. Consider two equally narrow strips RL, R′L′, drawn parallel to Oy, on opposite sides of it, and equidistant from it, meeting the curve in R, R′, the chord PQ in r, r', and the axis in L, L′. Let $OL = \lambda = OL'$. We have

$$h^2 = Aa^2 + Ba + C, \quad h'^2 = Aa^2 - Ba + C, \quad RL^2 = A\lambda^2 + B\lambda + C.$$

Also $rL = \frac{h-h'}{2a}\lambda + \frac{h+h'}{2}$; therefore $RL^2 - rL^2 = \frac{C - hh' - Aa^2}{2a^2}(a^2 - \lambda^2)$.

Similarly $RL'^2 - r'L'^2$ is equal to the same quantity. Therefore the volumes generated by the elements Rr and $R'r'$, when the figure revolves round the axis of x, are equal. Therefore the centre of gravity of the volume generated from the segment PQ is at O.

2. If the circle on MN as diameter meet RL in S, $SL^2 = a^2 - \lambda^2$, therefore $RL^2 - rL^2$ varies as SL^2. Therefore the volumes generated by the elements Rr and SL are always in the same proportion. Hence the second part of the theorem follows.

[Otherwise: $\bar{x} = \int x(y_2^2 - y_1^2)\,dx \div \int (y_2^2 - y_1^2)\,dx$, and $y_2^2 - y_1^2$ is a quadratic function of x vanishing at P, Q, when $x = x_1$ or x_2, therefore $\bar{x} = \dfrac{\int x(x-x_1)(x_2-x)\,dx}{\int (x-x_1)(x_2-x)\,dx}$, which is the expression for the centroid of the volume of the sphere on MN as diameter, either in part or whole.]

7413. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—Prove that no polyedron can have a seven-walled frame of pentagons.

Solution by the Proposer.

It should have been added, *if the ray-points of the frame are all triaces;* but it is best to consider only solids whose summits are all triaces, and whose least faces are pentagons. We have first to reproduce a theorem of Euler's. Let a solid having only triad summits have a_3 triangles, a_4 quadrilaterals, a_5 5-gons, ... a_m m-gons ... If from the four equations $e = s + f - 2$, $f = a_3 + a_4 + a_5 + \&c.$, $2e = 3s = 3a_3 + 4a_4 + 5a_5 + \&c.$, we eliminate e, s, and f, a_6 disappears, and we get Euler's result,

$$a_5 = 12 + a_7 + 2(a_8 - a_4) + 3(a_9 - a_3) + 4a_{10} + \dots + ma_{6+m} + \dots \ (m > 4),$$

which, when $a_3 = a_4 = 0$, becomes $a_5 = 12 + \Sigma m a_{6+m}$.

On all the edges of a $(6+r)$-gon A draw 6-gons, making $6+r$ summits 665, and $12+2r$ summits 655 with a circle of $12+2r$ pentagons, within which draw $6+r$ more 6-gons collateral with a central $(6+r)$-gon A′, and making with the same 5-gons the like summits. We thus get a $(26+4r)$-edron P, which has no frame, but a circle $C_{(12+2r)} = 1$.

If a $(6+r)$-walled frame F is possible, F can be imposed on the $(6+r)$-gon A′, and equally well on any $(6+r)$-gonal face of any polyedron. Let F be imposed on A′, a face of P. The frame F will have a contour $c_1, c_2, \dots c_{5+r}, c_{6+r}$, $(c_m \geqq 0)$. The 6-gonal wall which carries c_m ray-points becomes a $(6+c_m)$-gon, which will contribute by Euler's theorem c_m pentagons to the a_5 of the completed solid: that is, if $c_1 + c_2 + \dots = R$, the $(6+r)$ walls will contribute to that a_5 R pentagons. The $(6+r)$-gon A, the only non-pentagon > 6 besides these walls will contribute to that a_5 r pentagons, and the completed solid can have no more than $a_5 = 12 + r + R$ pentagons. In imposing F we have made no change in the $12+2r$ pentagons of P, and we have added to them not fewer than R more.

It follows that $12 + 2r + R \not> 12 + r + R$; Q.E.A., if $r > 0$.

7535. (By R. LACHLAN, B.A.)—Prove that, if $\alpha < \frac{1}{2}\pi$, and n be positive and < 1,
$$\int_0^\infty \frac{x^n dx}{1+2x\cos\alpha + x^2} = \frac{\pi}{\sin n\pi}\frac{\sin n\alpha}{\sin\alpha},$$
and
$$\int_0^\infty \frac{x^{n-1}dx}{1+2x\cos\alpha+x^2} = \frac{\pi}{\sin n\pi}\frac{\sin(1-n)\alpha}{\sin\alpha}.$$

Solution by Professor WOLSTENHOLME, Sc.D.; R. KNOWLES, B.A.; *and others.*

By a well-known formula, $\int_0^\infty \frac{x^{n-1}dx}{x+a} = \frac{\pi}{\sin n\pi} a^{n-1}$, if $n > 0 < 1$; for all values of a, real or impossible. Putting $a = \cos\alpha + i\sin\alpha$, and therefore $\frac{1}{x+a} = \frac{x+\cos\alpha - i\sin\alpha}{x^2+2x\cos\alpha+1}$, and denoting the integrals proposed by U, V, respectively, we have
$$U + (\cos\alpha - i\sin\alpha)V = \frac{\pi}{\sin n\pi}[\cos(n-1)\alpha + i\sin(n-1)\alpha];$$
whence $U + V\cos\alpha = \frac{\pi}{\sin n\pi}\cos(1-n)\alpha$, $V\sin\alpha = \frac{\pi}{\sin n\pi}\sin(1-n)\alpha$;
whence the results stated.

In these results, α is an angle determined as $\cos^{-1}(\cos\alpha)$, and the limits are accordingly 0 and π; but, since writing $-\alpha$ for α does not alter either member, the results will be true from $\alpha = -\pi$ to $\alpha = \pi$.

Both results are included in
$$\int_0^\infty \frac{x^{n-1}}{x^2+2x\cos\alpha+1}dx = \frac{\pi}{\sin n\pi}\frac{\sin(1-n)\alpha}{\sin\alpha},$$
if we take $n > 0 < 2$, $\alpha > -\pi < \pi$.

Writing x^p for x, $\frac{n}{p}$ for n, and $p\alpha$ for α, we get
$$\int_0^\infty \frac{x^{n-1}dx}{x^{2p}+2x^p\cos p\alpha+1} = \frac{\pi}{p\sin\frac{n\pi}{p}}\frac{\sin(p-n)\alpha}{\sin p\alpha},$$
where p is positive, $n > 0 < 2p$, and $p\alpha > -\pi < \pi$; and in this form, which is not really more general than the one proposed, the result will be found in WOLSTENHOLME's *Math. Problems* [1919 (50)].

If we put $\alpha = -\pi$ or π (in U or V), both members become ∞, but probably the limiting ratio of the two members as α tends to π is *not* one of equality. [See DE MORGAN's *Calculus*, p. 666.]

7439. (By R. RAWSON.)—Two inclined planes of the same height and inclination α, β, are placed back to back, with an interval between them ($2a$). Two weights P, Q are placed one on each inclined plane, and kept at rest by the connection of an inextensible string, indefinitely long, passing over two small tacks, one at the top of each inclined plane. A weight w, having a vertical velocity (c), is then placed on the string by a smooth ring at a point midway between the inclined planes. Show that the system thereby put in motion will come to rest at a point determined by a root of the quadratic

$$(4P^2 \sin^2 \alpha - w^2)\, s^2 - \frac{w}{g}(4ga P \sin \alpha + wc^2)\, s - \left(2Pa \sin \alpha + \frac{wc^2}{4g}\right)\frac{wc^2}{g} = 0.$$

Solution by D. EDWARDES; Professor MATZ, M.A.; *and others.*

Let a fixed horizontal plane below the system be taken as plane of reference, referred to which, let V be the original potential energy of the system P, Q, and h the common altitude of the wedges. Then the whole energy at first is $V + wh + \frac{1}{2}\frac{w}{g}c^2$. Now, since the *ring* is smooth, the tension is the same throughout, and w descends vertically. Let x be the distance it describes before reaching its position of instantaneous rest. Then the sum of the distances described by P and Q along the wedges is $2\left[(x^2+a^2)^{\frac{1}{2}} - a\right]$. There is evidently no impulse on the weights, and therefore no energy lost. Also, since the weights P and Q are in equilibrium at first, $P \sin \alpha = Q \sin \beta$. Hence the increase of potential energy of P and Q is $2P \sin \alpha \left[(x^2+a^2)^{\frac{1}{2}} - a\right]$. Therefore

$$V + wh + \tfrac{1}{2}\frac{w}{g}c^2 = \text{whole potential energy at last}$$
$$= w(h-x) + V + 2P \sin \alpha \left[(x^2+a^2)^{\frac{1}{2}} - a\right],$$

or
$$\tfrac{1}{2}\frac{w}{g}c^2 + wx = 2P \sin \alpha \left[(x^2+a^2)^{\frac{1}{2}} - a\right],$$

which, when rationalized, gives the required result.

7396. (By D. EDWARDES.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1 - \sin\theta\cos\phi)\sin\theta\, d\theta\, d\phi = \tfrac{1}{2}\pi\int_0^1 F(u)\, du.$$

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

The limits of integration show that the integral is extended to the surface of a quadrantal triangle traced on a sphere of radius unity, or, taking as axes of x, y, z the radii of the sphere drawn to the angular points of the triangle, and supposing the radius vector to any variable point on the surface to make with these axes angles α, β, γ, we have $\alpha = \theta$, $\cos\beta = \sin\theta\cos\phi$, $\cos\gamma = \sin\theta\sin\phi$; therefore, if $d\Omega$ denote the element of surface,

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\sin\theta\cos\phi)\sin\theta\, d\theta\, d\phi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\cos\beta)\, d\Omega$$

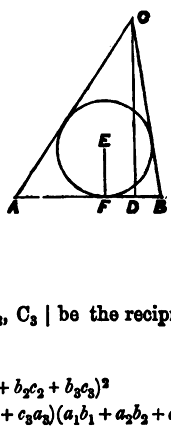
$$= \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\cos\beta)\sin\beta\, d\beta\, d\chi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(2\sin^2\tfrac{1}{2}\beta)\, 2\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\, d\beta\, d\chi,$$

or, putting $2\sin^2\frac{1}{2}\beta = u$, $\quad = \int_0^1\int_0^{\frac{1}{2}\pi} F(u)\, du\, d\chi = \frac{1}{2}\pi\int_0^1 F(u)\, du.$

7399. (By ASÛTOSH MUKHOPÂDHYÂY.)—A sphere is described round the vertex of a cone as centre; prove that the latus rectum of any section of the cone, made by any variable tangent plane to the sphere, is equal to the diameter of the sphere, multiplied by the tangent of the semi-vertical angle of the cone.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

It is geometrically evident that, if two given surfaces of the second degree touch along a conic, any plane cutting them both will cut the two surfaces in two conics which have double contact along the line in which this plane cuts the plane of contact of the two given surfaces; if, therefore, the plane *touch* one of the surfaces at one of its umbilici, it will cut the other surface in a conic with which this point of contact, an evanescent circle, will have (imaginary) double contact along the line in which this tangent plane cuts the plane of contact of the two given surfaces, or the point of contact is a focus of the conic and the line a directrix. It is therefore evident that the foci of any plane section of a right cone of revolution will be its points of contact with the two spheres which can be described touching it and touching the cone along a circle whose plane is perpendicular to the axis of the cone. Let then from C, the vertex of the cone, the perpendicular CD (p) be drawn to the plane of the conic, and let a plane be drawn through CD and EF (the line drawn from the centre of either of the spheres described as above indicated, to the point in which it touches the plane section), cutting the cone in the lines CA, CB; with the usual notation we have, in the triangle ACB,



$$\sin^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{ab} = \frac{(s-a)(s-b)\sin C}{p \cdot c};$$

hence we have

$$\frac{\text{Parameter}}{4} = \frac{\text{AF}\,.\,\text{FB}}{\text{AB}} = \frac{(s-a)(s-b)}{c} = \tfrac{1}{2}p\tan\tfrac{1}{2}C;$$

therefore Parameter $= 2p\tan\frac{1}{2}C$, but $2p$ is the diameter of the sphere referred to in the question.

7363. (By G. G. MORRICE, B.A.)—If $|\,A_1, B_2, C_3\,|$ be the reciprocal determinant of $|\,a_1, b_2, c_3\,|$, prove that

(1) $$\Sigma\,(a_1^2+a_2^2+a_3^2)(A_1^2+A_2^2+A_3^2)$$
$$= 3\,|\,a_1, b_2, c_3\,|^2 + 2\Sigma\,(a_1^2+a_2^2+a_3^2)(b_1c_1+b_2c_2+b_3c_3)^2$$
$$-6\,(b_1c_1+b_2c_2+b_3c_3)(c_1a_1+c_2a_2+c_3a_3)(a_1b_1+a_2b_2+a_3b_3).$$

(2) $$a_1(A_2-A_3)+a_2(A_3-A_1)+a_3(A_1-A_2)$$
$$=(b_1+b_2+b_3)(c_1a_1+c_2a_2+c_3a_3)-(c_1+c_2+c_3)(a_1b_1+a_2b_2+a_3b_3).$$

Solution by the PROPOSER.

$$(A_1^2+A_2^2+A_3^2)(a_1^2+a_2^2+a_3^2)-(A_1a_1+A_2a_2+A_3a_3)^2$$
$$=(a_2A_3-a_3A_2)^2+(a_3A_1-a_1A_3)^2+(a_1A_2-a_2A_1)^2.$$

But $\qquad a_2 A_3 - a_3 A_2 = a_2 (b_1 c_2 - b_2 c_1) - a_3 (b_3 c_1 - b_1 c_3)$
$= b_1 (a_2 c_2 + a_3 c_3) - c_1 (a_2 b_2 + a_3 b_3) = b_1 (a_1 c_1 + a_2 c_2 + a_3 c_3) - c_1 (a_1 b_1 + a_2 b_2 + a_3 b_3)$.

Adding the two similar expressions for $(a_3 A_1 - a_1 A_3)$ and $(a_1 A_2 - a_2 A_1)$, we get (2), and by squaring we get (1).

7475. (By J. O'Regan.)—The figures 142857 are arranged at random as the period of a circulating decimal, which is then reduced to a vulgar fraction in lowest terms; show that the odds are 119 : 1 against the denominator being 7.

Solution by A. Martin, B.A.; Rev. T. C. Simmons, M.A.; *and others.*

There are 6 ways in which the denominator can be 7, viz., when the decimal is $\cdot\dot{1}4285\dot{7}$, $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{2}8571\dot{4}$, $\cdot\dot{8}5714\dot{2}$, $\cdot\dot{5}7142\dot{8}$, $\cdot\dot{7}1428\dot{5}$. In all other cases the denominator is not 7. But there are 720 − 6, or 714, of these cases; hence the required odds are 714 : 6, or 119 : 1.

7236. (By the Rev. T. W. Openshaw, M.A.)—On AB, a chord of an ellipse, as diameter, a circle is drawn intersecting the ellipse again in C, D; if AB, CD are parallel to a pair of conjugate diameters: show that the locus of their intersection is $b^3 x + a^3 y = 0$.

Solution by the Proposer.

AB and CD must be parallel to equi-conjugate diameters, therefore their equations are of form $bx + ay + k = 0$, $bx - ay + k' = 0$; equation to conic through A, B, C, D is $(bx + ay + k)(bx - ay + k') = l\,(a^2y^2 + b^2x^2 - a^2b^2)$, the condition that this is a circle gives $l = \dfrac{b^2 + a^2}{b^2 - a^2}$; the coordinates of the centre are $\dfrac{b\,(k + k')\,(b^2 - a^2)}{4a^2b^2}$, $\dfrac{a\,(k' - k)(b^2 - a^2)}{4a^2b^2}$; if this is on $bx + ay + k = 0$, we get $(k' + k)\,b^2 = (k' - k)\,a^2$; whence, eliminating k, k', the locus is as stated.

7522. (By W. J. C. Sharp, M.A.)—Prove that (1) any two conics are polar reciprocals with respect to a third; (2) the same triangle is self-reciprocal with respect to all three, and the equation of the auxiliary conic, referred to this, may be derived from those to the other two by taking each coefficient proportional to the geometrical mean between the corresponding coefficients of the reciprocal conics; (3) the analogous proposition is true of quadrics.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

Referring the two conics to their common self-conjugate triangle, we may suppose their equations to be

$$x^2+y^2+z^2=0,\ (px)^2+(qy)^2+(rz)^2=0 \ \ldots\ldots\ldots\ldots(1,\ 2),$$

and, if we take another conic $lx^2+my^2+nz^2=0$ ....(3), with respect to which the triangle is self-conjugate, the reciprocal polars of (1), (2) with respect to (3) will be respectively $l^2x^2+m^2y^2+n^2z^2=0$, $\frac{l^2x^2}{p^2}+\frac{m^2y^2}{q^2}+\frac{n^2z^2}{r^2}=0$. Hence, if $\frac{l^2}{p^2}=\frac{m^2}{q^2}=\frac{n^2}{r^2}$, either of the conics (1), (2) is the reciprocal polar of the other with respect to (3). Thus there are *four* such auxiliary conics (a well-known result). Obviously, we shall have exactly similar equations for conicoids, and there will be eight auxiliary conicoids with respect to which either of two given conicoids is the reciprocal polar of the other. [The conics ϕ and F are polar reciprocals with respect to the same conics as U and V. And in space τ and T and τ' and T′ are polar reciprocals with respect to the same quadrics as U and V.]

7530. (By R. KNOWLES, B.A., L.C.P.)—From a point A a perpendicular AD is drawn to a straight line BC given in position, and the inscribed circle of the triangle ABC passes through the orthocentre; prove that the maximum value of its radius is one-half of AD.

Solution by the Rev. T. C. SIMMONS, M.A.; R. LACHLAN, B.A.; *and others.*

Let O be the orthocentre, and I the incentre; then $IO^2=2r^2-AO.OD$ (*Reprint*, Vol. 39, p. 99); therefore, if $IO=r$, $r^2=AO.OD$; and, since A and D are given, the maximum value of r is half of AD.

7458. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If n, r be positive integers, and

$$x^ry=\sin x,\quad x^{n+r}\frac{d^ny}{dx^n}=z,$$

prove that, according as $n+r$ is even or odd,

$$\frac{d^rz}{dx^r}=(-1)^{\frac{n+r}{2}}x^n\sin x,\quad\text{or}\quad(-1)^{\frac{n+r-1}{2}}x^n\cos x.$$

$$\left[\text{The results may be written } \frac{d^rz}{dx^r}=x^n\sin\left\{(n+r)\frac{\pi}{2}+x\right\}.\right]$$

Solution by J. HAMMOND, M.A.; G. B. MATHEWS, B.A.; *and others.*

If $D=\frac{d}{dx}$, we have, by the conditions of the question,

$$x^nD^{n+r}(x^ry)=x^n\sin\left[(n+r)\tfrac{1}{2}\pi+x\right],\quad D^r(x^{n+r}D^ny)=D^rz\ldots(1,\ 2),$$

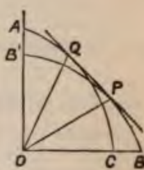
and it only remains to prove that $x^nD^{n+r}(x^ry)=D^r(x^{n+r}D^ny)$.

But, by LEIBNITZ's theorem, we have

$$D^r(x^{n+r}D^n y) = x^{n+r}D^{n+r}y + (n+r)\,r x^{n+r-1}D^{n+r-1}y$$
$$+ (n+r)(n+r-1)\frac{r(r-1)}{1\,.\,2}x^{n+r-2}D^{n+r-2}y + \ldots = x^n D^{n+r}(x^r y).$$

7155. (By T. WOODCOCK, B.A.)—If P, Q be the points in which the plane through the optic and ray axes intersects the circle of contact PQ of a tangent plane perpendicular to an optic axis of the wave surface of a biaxal crystal, and if a, c, the greatest and least axes of elasticity, be given; prove that, O being the centre of the wave surface, (1) the triangle POQ, (2) the circle of contact PQ, (3) the angle POQ will have their greatest values respectively, when the square of the mean axis b is (i.) the arithmetic, (ii.) the geometric, (iii.) the harmonic mean of a^2 and b^2; and the cone whose vertex is O and base the circle PQ will have its maximum volume when $b^2 = \frac{1}{8}\left[a^2+c^2+(a^4+14a^2c^2+c^4)^{\frac{1}{2}}\right]$.

Solution by the PROPOSER.



In the accompanying figure let BPB′ be a circle of radius b, and AQC an ellipse with axes $2a$, $2c$; and let D = semi-conjugate of OQ; then we have

$$PQ^2 = OQ^2 - OP^2 = a^2+c^2-D^2-b^2 = a^2+c^2-\frac{a^2c^2}{b^2}-b^2,$$

a maximum when $b^2 = ac$,

$$b^2\,.\,PQ^2 = (a^2+c^2)\,b^2 - a^2c^2 - b^4,$$

a maximum when $2b^2 = a^2+c^2$,

$$\tan^2 POQ = \frac{a^2+c^2}{b^2} - \frac{a^2c^2}{b^4} - 1,\ \text{a maximum when } b^2(a^2+c^2) = 2a^2c^2.$$

The quadratic in b^2, got by making the cone's volume a maximum, is

$$b^2(a^2+c^2)+a^2c^2-3b^4 = 0, \text{ therefore, \&c.}$$

7431. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If $2s \equiv \alpha+\beta+\gamma+\delta$, prove that

$$\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\delta)\left[\sin(s-\beta)+\sin(s-\gamma)-\sin(s-\alpha)-\sin(s-\delta)\right]^2$$
$$+\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta)\left[\sin(s-\gamma)+\sin(s-\alpha)-\sin(s-\beta)-\sin(s-\delta)\right]^2$$
$$+\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\left[\sin(s-\alpha)+\sin(s-\beta)-\sin(s-\gamma)-\sin(s-\delta)\right]^2$$
$$\equiv -16\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\delta)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta).$$

Solution by G. HEPPEL, M.A.; G. B. MATHEWS, B.A.; *and others.*

In a rough way this may be said to be shown by observing that the left-hand member vanishes when $\beta=\gamma$, and when $\alpha=\delta$. It is, however, a little troublesome to find the constant multiplier. A fuller and more satisfactory proof is obtained by putting

$$\sin\tfrac{1}{2}(\beta-\gamma)=u,\quad \cos\tfrac{1}{2}(\beta+\gamma)=u',\quad \sin\tfrac{1}{2}(\alpha-\delta)=p,\quad \cos\tfrac{1}{2}(\alpha+\delta)=p',$$
$$\sin\tfrac{1}{2}(\gamma-\alpha)=v,\quad \cos\tfrac{1}{2}(\gamma+\alpha)=v',\quad \sin\tfrac{1}{2}(\beta-\delta)=q,\quad \cos\tfrac{1}{2}(\beta+\delta)=q',$$
$$\sin\tfrac{1}{2}(\alpha-\beta)=w,\quad \cos\tfrac{1}{2}(\alpha+\beta)=w',\quad \sin\tfrac{1}{2}(\gamma-\delta)=r,\quad \cos\tfrac{1}{2}(\gamma+\delta)=r';$$

then, if U = left-hand member, we have

$$U=4pu\,(qv'+q'v)\,(rw'-r'w)+4qv\,(rw'+r'w)\,(pu'-p'u)$$
$$+4rw\,(pu'+p'u)\,(qv'-q'v),$$
$$\tfrac{1}{4}U=rw'\,[pq\,(uv'+u'v)+uv\,(pq'-p'q)]-r'w\,[qu\,(pv'+p'v)-pv\,(qu'-q'u)]$$
$$+rw\,(pu'+p'u)\,(qv'-q'v).$$

Now the following identities hold good :—

$$uv'+u'v=-\,w\cos\gamma,\quad pq'-p'q=w\cos\delta,$$
$$pv'+p'v=\ r\cos\alpha,\quad qu'-q'u=r\cos\beta;$$

therefore $$\frac{U}{4wr}=-\,pqw'\cos\gamma+uvw'\cos\delta-qur\cos\alpha+pvr'\cos\beta$$
$$+\,(pu'+p'u)\,(qv'-q'v)$$
$$=pq\,[u'v'-w'\cos\gamma]+uv\,[w'\cos\delta-p'q']$$
$$+qu\,[p'v'-r'\cos\alpha]-pv\,[q'u'-r'\cos\beta],$$

and every one of these four terms $=-pquv$, therefore $U=-\,16pqruvw$.

7343. (By Belle Easton.)—If a debating society has to choose one out of five subjects proposed, and 30 members vote each for one subject, show that (1) the votes can fall in 5^{30} ways, and (2) the chance that upwards of twenty votes fall to some one subject will be 5^{-20}.

Solution by W. W. Taylor, M.A.; Sarah Marks; *and others.*

1. Let v, w, x, y, z represent the subjects, then the possible combinations of votes and the relative probability of each combination will be represented by the coefficients in the expansion $(v+w+x+y+z)^{30}$; and the sum of these coefficients is $(1+1+1+1+1)^{30}=5^{30}$.

2. If upwards of 20 fall to one subject, 21 is the least number of votes that can be recorded for that subject; the remainder of the votes is 9. These can be given in 5^9 different ways, and the one subject can be chosen in five different ways; so, the conditions of this part of the problem can be satisfied in 5^{10} different ways, and therefore the chance required is 5^{-20}.

6907. (By S. Tebay, B.A.)—If A, B, C can do similar pieces of work in a, b, c hours respectively, $(a<b<c)$; and they begin simultaneously, and regulate their labour by mutual interchanges at certain intervals, so that the three pieces of work are finished at the same time: find the number of solutions.

Solution by the PROPOSER; SARAH MARKS; *and others.*

The parts done by each in x hours are $\frac{x}{a}, \frac{x}{b}, \frac{x}{c}$. Suppose now that B and C change works, while A continues. In y hours more they have done $\frac{y}{a}, \frac{y}{a}, \frac{y}{c}$. Therefore the work remaining to be done by each is

$$1-\frac{x}{a}-\frac{y}{a},\quad 1-\frac{x}{b}-\frac{y}{c},\quad 1-\frac{x}{c}-\frac{y}{b}.$$

There are two ways of completing the work according to the question, which may be set down as follows:—

A	finishes	B's	or A	finishes	C's,
B	,,	A's	,, C	,,	A's,
C	,,	C's	,, B	,,	B's.

Thus, if the times be equal, we have

$$b\left(1-\frac{x}{a}-\frac{y}{a}\right)=a\left(1-\frac{x}{b}-\frac{y}{c}\right)=c\left(1-\frac{x}{c}-\frac{y}{b}\right),$$

$$c\left(1-\frac{x}{a}-\frac{y}{a}\right)=b\left(1-\frac{x}{b}-\frac{y}{c}\right)=a\left(1-\frac{x}{c}-\frac{y}{b}\right).$$

Putting $s = bc+ca+ab$, $t = a+b+c$, these equations give

$$x=\frac{s^2-3tabc}{s(a-b)(c-b)}\,b,\quad y=\frac{s-3ab}{s(c-b)}\,bc \quad \ldots\ldots\ldots\ldots(1);$$

$$x=\frac{s^2-3tabc}{s(c-a)(c-b)}\,c,\quad y=\frac{3ac-s}{s(c-b)}\,bc \quad \ldots\ldots\ldots\ldots(2).$$

In the same way we have the following combinations. Reasoning as above, let C and A change works, while B continues; then let

B	finish	A's	or B	finish	C's,
A	,,	B's	,, C	,,	B's,
C	,,	C's	,, A	,,	A's.

These arrangements give

$$x=\frac{s^2-3tabc}{s(b-a)(c-a)}\,a,\quad y=\frac{s-3ab}{s(c-a)}\,ac \quad \ldots\ldots\ldots\ldots(3),$$

$$x=\frac{s^2-3tabc}{s(c-a)(c-b)}\,c,\quad y=\frac{3bc-s}{s(c-a)}\,ac \quad \ldots\ldots\ldots\ldots(4).$$

Again, let A and B change works, while C continues; then let

A	finish	C's	or B	finish	C's,
C	,,	A's	,, C	,,	B's,
B	,,	B's	,, A	,,	A's.

These arrangements give

$$x=\frac{s^2-3tabc}{s(b-a)(c-a)}\,a,\quad y=\frac{s-3ac}{s(b-a)}\,ab \quad \ldots\ldots\ldots\ldots(5),$$

$$x=\frac{s^2-3tabc}{s(a-b)}\,b,\quad y=\frac{s-3bc}{s(a-b)}\,ab \ldots\ldots\ldots\ldots(6).$$

Now $s^2-3tabc$ is always positive; and, since $a<b<c$, it will be seen that (1) and (6) are inadmissible. It is also evident that (2) and (5) cannot be

both relevant. The whole work is done in $3s^{-1}abc$ hours. If x and y be both positive, we must have $x+y<3s^{-1}abc$. Applying this test to (2), (3), (4), (5), we find

$$2bc>a(b+c),\ 2ab<c(a+b),\ 2ac<b(a+c),\ 2ac>b(a+b)\ldots(2', 5', 3', 4').$$

Here (2′) and (5′) are both possible, but only one is applicable. (3′) and (4′) are inconsistent, one only being applicable. There are therefore only two possible solutions.

If $b<\dfrac{2ac}{a+c}$, (2) and (4) are applicable; if $b>\dfrac{2ac}{a+c}$, (3) and (5) are applicable; if $b=\dfrac{2ac}{a+c}$, (2) and (5), and also (3) and (4), are identical, the work being completed in b hours.

Examples.—The harmonic mean between 4 and 12 is 6; so that, if $a=4$, $b=6$, $c=12$, the whole work is finished in 6 hours; A and C changing works at the end of 3 hours.

Take $a=4$, $b=5$, $c=12$; then

$x=2\frac{13}{112}$ hours, $y=1\frac{1}{14}$ hours; $x=2\frac{13}{112}$ hours, $y=2\frac{7}{16}$ hours...(2, 4),
the whole work being completed in $5\frac{5}{8}$ hours.

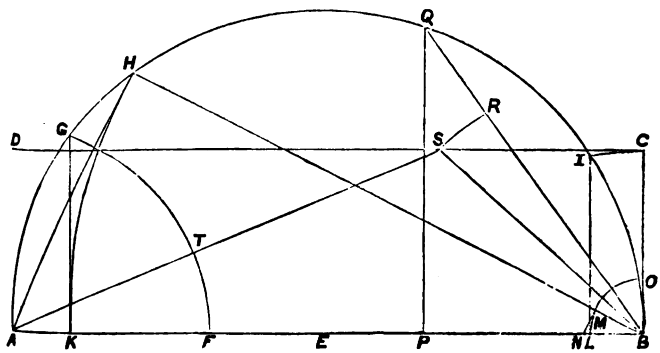
Take $a=4$, $b=7$, $c=12$; then

$x=2\frac{31}{80}$ hours, $y=2\frac{17}{20}$ hours; $x=2\frac{31}{80}$ hours, $y=\frac{14}{15}$ hour...(3, 5),
the whole work being completed in $6\frac{3}{10}$ hours.

7552. (By the Editor.)—In a road parallel to a range, find, by elementary geometry, a point at which the sounds of the firing and of the hit of the bullet would be heard simultaneously.

Solution by (1) D. Biddle; (2) A. H. Curtis, LL.D., D.Sc.

1. Let A be the firing-point, B the target, CD the road, and AF the distance the sound will travel before the bullet reaches B. Draw BC at



right angles to AB; on AB the semi-circle AGHQB; CI with radius BC; IL perpendicular to AB; FG with radius AF; and GK perpendicular to AB. Draw LMO with radius BL, and KH with radius BK. Join HB cutting LMO in M, and draw MN parallel to HA, that is, at right angles to HB. Make NP = $\frac{1}{4}$AB, and draw PQ at right angles to AB, cutting the semicircle in Q. Join BQ, and make QR = $\frac{1}{2}$AF. Finally, with radius BR, draw RS, cutting CD in S. Then AS = BS + AF, and S is the required point.

For it will readily be seen that by construction, and the properties of circles and right-angled triangles, if AB = 1, then $AK = AF^2$, $BL = BC^2$, and $BP = BQ^2$. Now, BR = BS, and $BQ = BS + \frac{1}{2}AF$; therefore $BP = (BS + \frac{1}{2}AF)^2$. But $BP = BN + \frac{1}{4}AB^2$; hence we have

$$BN = (BS + \tfrac{1}{2}AF)^2 - \tfrac{1}{4}AB^2.$$

But $BM = BL = BC^2$ and $BN : BM = AB : BH = AB^2 : AB^2 - AF^2$,

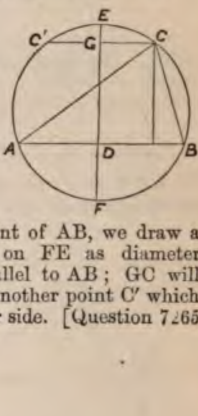
therefore $\qquad (BS + \frac{1}{2}AF)^2 - \frac{1}{4}AB^2 : BC^2 = AB^2 : AB^2 - AF^2$,

whence $\qquad BS = \left(\frac{1}{4}AB^2 + \frac{AB^2BC^2}{AB^2 - AF^2}\right)^{\frac{1}{2}} - \frac{1}{2}AF$,

which is the identical value arrived at from the equation

$$(BS^2 - BC^2)^{\frac{1}{2}} + [(BS + AF)^2 \quad BC^2]^{\frac{1}{2}} = AB.$$

Consequently, since it is plain that $(BS^2 - BC^2)^{\frac{1}{2}} + (AS^2 - BC^2)^{\frac{1}{2}} = AB$, it is equally evident that AS = BS + AF, and S is the required position.



2. The question is immediately reducible to the following:—Given, of a plane triangle, altitude, base, and difference of the other two sides, to construct it. Let ACB be the triangle, FE the diameter of its circumscribed circle, which is perpendicular to the base AB, and CG parallel to AB, then EG.DF + GD.DF, = ED.DF = AD^2 = $\frac{1}{4}AB^2$, is known, and EG.DF, = $\frac{1}{4}(AC - CB)^2$, is known; therefore GD.DF is known, and GD, = altitude, is known; therefore DF is known, and therefore GE. If therefore at D, the middle point of AB, we draw a perpendicular and measure off DF, DG, GE, on FE as diameter describe a circle, and through G draw GC parallel to AB; GC will intersect the circle in the required point C, and in another point C′ which is excluded, as, if B be the target, BC is the smaller side. [Question 7265 is related to this problem.]

7416. (By R. Rawson.)—In the Royal Society's *Transactions* (Part III., 1881, pp. 766, 767), Mr. J. W. L. Glaisher has shown, by the assumption of $\Sigma A_r x^{m+r}$ for all positive integral values of r, that (AU + BV) is the general integral of $\frac{d^2\omega}{dx^2} - a^2\omega = \frac{p(p+1)}{x^2}\omega$, where

$$U = x^{-p}\left\{1 - \frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.\,2\,!} - \&c.\right\},$$

$$V = x^{p+1}\left\{1 + \frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.\,2\,!} + \&c.\right\}.$$

Show that the restriction imposed upon r is unnecessary, and that, if $m = n-2p$, the general integral of the above differential equation is

$$\omega = A_0 x^{n-p}\left\{1 + \frac{a^2x^2}{(n+2)(m+1)} + \frac{a^4x^4}{(n+4)(n+2)(m+3)(m+1)} + \&c.\right\}$$

$$+\frac{n.m-1.A_0}{a^2}x^{n-p-2}\left\{1 + \frac{(n-2)(m-3)}{a^2x^2} + \frac{(n-4)(n-2)(m-5)(m-3)}{a^4x^4} + \&c.\right\}$$

Solution by the PROPOSER.

Let $\qquad \omega = \phi(x) = \Sigma A_r x^{2r+\beta}$

$= x^\beta[A_0 + A_1x^2 + A_2x^4 + \&c.] + x^{\beta-2}[A_{-1} + A_{-2}x^{-2} + A_{-3}x^{-4} + \&c.]\ldots(1).$

Differentiate (1), then

$$\frac{du}{dx} = \Sigma(2r+\beta)A_r x^{2r+\beta-1},\quad \frac{d^2\omega}{dx^2} = \Sigma(2r+\beta)(2r+\beta-1)A_r x^{2r+\beta-2},$$

$$-\frac{p(p+1)}{x^2}\omega = \Sigma - p(p+1)A_r x^{2r+\beta-2},\quad -a^2\omega = \Sigma - a^2A_r x^{2r+\beta};$$

hence $\qquad \frac{d^2\omega}{dx^2} - a^2\omega = \frac{p(p+1)}{x^2}\omega \ldots\ldots(2),$

if $\qquad (2r+\beta+p)(2r+\beta-p-1)A_r = a^2A_{r-1}\ldots\ldots(3),$

the summation extending to all positive and negative integral values of r.

The general integral of (2) is, therefore, $\phi(x)$, which must be of such a form that the coefficients of $x^{2r+\beta-2}$ and $x^{2r+\beta}$ shall satisfy (3).

Put $\qquad n = \beta + p,\quad$ and $\quad m = \beta - p = n - 2p\ldots\ldots(4).$

Equation (3) may now be written

$$(n+2r)(m+2r-1)A_r = a^2A_{r-1}\ \ldots\ldots(5).$$

From (1) the following values of A_1, A_2, &c., A_{-1}, A_{-2}, &c., may be readily obtained:—

$$A_1 = \frac{a^2A_0}{n+2\,.\,m+1},\quad A_2 = \frac{a^2A_1}{n+4\,.\,m+3} = \frac{a^4A_0}{n+4\,.\,n+2\,.\,m+3\,.\,m+1},$$

$$A_3 = \frac{a^2A_2}{n+6\,.\,m+5} = \frac{a^6A_0}{n+6\,.\,n+4\,.\,n+2\,.\,m+5\,.\,m+3\,.\,m+1},\ \&c.;$$

$$A_{-1} = \frac{n(m-1)A_0}{a^2},\quad A_{-2} = \frac{n-2\,.\,m-3\,.\,A_{-1}}{a^2} = \frac{n-2\,.\,n\,.\,m-3\,.\,m-1\,.\,A_0}{a^4},$$

$$A_{-3} = \frac{n-4\,.\,m-5\,.\,A_{-2}}{a^2} = \frac{n-4\,.\,n-2\,.\,n\,.\,m-5\,.\,m-3\,.\,m-1\,.\,A_0}{a^6},\ \&c.$$

Substitute these values in (1), then (ω) is the value assigned in the question. By making $n = 0$, and $m = 1$, successively, there results the two particular solutions obtained by Mr. GLAISHER.

7393. (By W. J. McClelland, B.A.)—If from any two points inverse to each other with respect to a given circle, perpendiculars are drawn on the sides of an inscribed polygon; show that the polygons formed by joining the feet of the perpendiculars are (1) similar, (2) to one another as the distances of their generating points from the circle's centre.

Note by T. A. Finch, M.A.

This is incorrect in every case, except for a triangle, in which case it becomes Mr. McCay's well-known theorem. The Proposer seems to have overlooked the fact that *all* corresponding lines in similar figures must be in the same ratio; that this condition is not fulfilled, can be easily seen by taking the ratios of the lines joining the feet of the perpendiculars from the inverse points on any two sides of the polygon which do not intersect on the circle.

7389. (By C. Leudesdorf, M.A.)—If O, I are the centres, R, r the radii, of the circumscribed and inscribed circles of a spherical triangle ABC, and P any point on the sphere; prove that

$$\cos IP = \frac{\cos OI}{\cos R} - \frac{\cos r}{\sin \frac{1}{2}(a+b+c)}\left[\sin a \sin^2 \tfrac{1}{2}(AP) + \sin b \sin^2 \tfrac{1}{2}(BP) + \sin c \sin^2 \tfrac{1}{2}(CP)\right].$$

Solution by D. Edwardes; Sarah Marks; *and others.*

If α, β, γ be the angles subtended by the sides at the pole of the circumscribed circle, and P any point on the sphere, it is easy to show that, since $\alpha+\beta+\gamma = 2\pi$,

$$\cos PA \sin\alpha + \cos PB \sin\beta + \cos PC \sin\gamma = 4 \cos OP \cos R \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}\gamma.$$

Applying this to the triangle DEF formed by joining the points of contact of the inscribed circle, and putting α, &c. for the angle FIE, &c., we have

$$\cos PD \sin\alpha + \cos PE \sin\beta + \cos PF \sin\gamma = 4 \cos IP \cos r \sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}\gamma.$$

Now $\alpha = 2AIE$; whence, by the right-angled triangle AIE, we get

$$\sin\alpha = 2\cos(s-a)\sin(s-a)\sin(s-b)\sin(s-c) \div \sin r \sin b \sin c,$$

and similarly for β and γ. Also

$$\sin\tfrac{1}{2}\alpha \sin\tfrac{1}{2}\beta \sin\tfrac{1}{2}\gamma = \frac{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}{\sin r \cos^2 r \sin a \sin b \sin c}, \text{ since } \tan r = \frac{n}{\sin s};$$

hence $\cos PD \sin a \cos(s-a) + \cos PE \sin b \cos(s-b) + \cos PF \sin c \cos(s-c)$

$$= \frac{2\cos IP \sin s}{\cos r}.$$

But, from the triangle PBC, we have

$$\cos PD \sin a = \cos PB \sin(s-c) + \cos PC \sin(s-b) \text{ \&c.}$$

Hence $\cos PA \sin a + \cos PB \sin b + \cos PC \sin c = \dfrac{2\cos IP \sin s}{\cos r}$.

Also (Todhunter's *Spherical Trigonometry*, Art. 144),

$$\sin a + \sin b + \sin c = \frac{2\cos OI \sin s}{\cos R \cos r},$$

$$\therefore \frac{\cos OI}{\cos R} - \frac{\cos r}{\sin s}\left[\sin a \sin^2\tfrac{1}{2}AP + \sin b \sin^2\tfrac{1}{2}BP + \sin c \sin^2\tfrac{1}{2}CP\right] = \cos IP.$$

7567. (By Professor Sylvester, F.R.S.)—Let nine quantities be supposed to be placed at the nine inflexions of a cubic curve, then they will group themselves in twelve sets of triads, which may be called collinear, and the product of each such triads may be called a collinear product. From the sum of the cubes of the nine quantities subtract three times the sum of their twelve collinear triadic products, and let the function so formed be called F. With another set of nine quantities form a similar function, say F′. Prove that FF′ will be also a similar function of nine quantities which will be lineo-linear functions of the other two sets, and find their values. [The inflexion-points are only introduced in order to make clear the scheme of the triadic combinations, so that the imaginariness of six of them will not matter to the truth of the theorem.]

Solution by R. Russell, B.A.

The expression denoted by F may symmetrically be written down thus:—

$$F \equiv a^3+b^3+c^3+l^3+m^3+n^3+x^3+y^3+z^3-3abc-3lmn-3xyz-3alx-3bmy$$
$$-3cnz-3a(mz+ny)-3b(nx+lz)-3c(ly+mx) \quad \ldots\ldots\ldots\ldots(1).$$

A little consideration suggests the transformation (where $w^3 = 1$)

$$\left.\begin{array}{lll} a+b+c=\alpha, & l+m+n=\lambda, & x+y+z=\xi \\ a+bw+cw^2=\beta, & l+mw+nw^2=\mu, & x+yw+zw^2=\eta \\ a+bw^2+cw=\gamma, & l+mw^2+nw=\nu, & x+yw^2+zw=\zeta \end{array}\right\} \ldots(2),$$

which reduces F to the very simple form,

$$\alpha\beta\gamma+\lambda\mu\nu+\xi\eta\zeta-\alpha\lambda\xi-\beta\mu\eta-\gamma\nu\zeta \quad \text{or} \quad F = \begin{vmatrix} \alpha, & \nu, & \eta \\ \zeta, & \beta, & \lambda \\ \mu, & \xi, & \gamma \end{vmatrix},$$

and proves the remarkable property that any determinant of the third order can be reduced linearly, as in (2), to the form of expression given in the question.

F′ can be expressed similarly; and FF′, being also a determinant of the third order whose constituents are lineo-linear functions of the two original sets of nine letters, can by the above be reduced at once to (1).

[The theorem is *the* analogue to Euler's theorem that the product of one sum of 4 squares by another is also a sum of 4 squares. In a precisely similar way, any determinant of the second order is reducible to a sum of 4 squares, with the aid of $\omega^2 = -1$.]

4925. (By the late Professor Clifford, F.R.S.)—Let U, V, W = 0 be the point equations, and $u, v, w = 0$ the plane-equations of three quadrics inscribed in the same developable, and let $u+v+w$ be identically zero. Then, if a tangent plane to U, a tangent plane to V, and a tangent plane to W, are mutually conjugate in respect of $au+bv+cw=0$, they will intersect on $\dfrac{U}{(b-c)^2}+\dfrac{V}{(c-a)^2}+\dfrac{W}{(a-b)^2}=0$,

which passes through the curves of contact of the developable with $au+bv+cw$ and one other quadric.

Solution by W. J. C. SHARP, M.A.

Let $\alpha_1 x+\beta_1 y+\gamma_1 z+\delta_1 w=0,\ \alpha_2 x+\beta_2 y+\gamma_2 z+\delta_2 w=0,$ and $\alpha_3 x+\beta_3 y+\gamma_3 z+\delta_3 w=0$, be the three tangent planes, and let (U), (V), (W) denote the values of U, V, and W, when the coordinates of the intersection of the planes are substituted for (x, y, z, w). Also let u_{11}, u_{22}, v_{33}, &c., denote the values of u, &c., when $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ are substituted for $\alpha, \beta, \gamma, \delta$; and $u_{12}, u_{23}, u_{31}, v_{12}$, &c., the conditions that these planes should be conjugate with respect to u, v, &c. Then, by the conditions, we have

$$\left.\begin{array}{c} u_{11}=0,\ v_{22}=0,\ w_{33}=0,\ au_{12}+bv_{12}+cw_{12}=0 \\ au_{23}+bv_{23}+cw_{23}=0,\quad au_{31}+bv_{31}+cw_{31}=0 \\ u_{11}+v_{11}+w_{11}=0,\ u_{22}+v_{22}+w_{22}=0,\ u_{33}+v_{33}+w_{33}=0 \\ u_{12}+v_{12}+w_{12}=0,\ u_{23}+v_{23}+w_{23}=0,\ u_{31}+v_{31}+w_{31}=0 \end{array}\right\}\ \ldots(\text{A}),$$

and $u_{11}u_{22}u_{33}+2u_{12}u_{23}u_{31}-u_{11}u_{23}^2-u_{22}u_{31}^2-u_{33}u_{12}^2=\Delta^2(\text{U})=(\text{U})$ if $\Delta=1$,

$$v_{11}v_{22}v_{33}+2v_{12}v_{23}v_{31}-v_{11}v_{23}^2-v_{22}v_{31}^2-v_{33}v_{12}^2=\Delta'^2(\text{V})=(\text{V})$$

if $\Delta'=1$; and if $\Delta''=1$, then

$$w_{11}w_{22}w_{33}+2w_{12}w_{23}w_{31}-w_{11}w_{23}^2-w_{22}w_{31}^2-w_{33}w_{12}^2=\Delta''^2(\text{W})=(\text{W}).$$

But, from the equations (A), we have

$$(c-a)^2 v_{33}u_{12}^2+(b-c)^2 v_{33}v_{12}^2=0,$$

$(a-b)^2 v_{11}v_{23}^2+(c-a)^2 w_{11}w_{23}^2=0$ and $(b-c)^2 w_{22}w_{31}^2+(a-b)^2 u_{22}u_{31}=0,$

and $$2u_{12}u_{23}u_{31}-u_{22}u_{31}^2-u_{33}u_{12}^2=(\text{U}),$$

$2v_{12}v_{23}v_{31}-v_{11}v_{23}^2-v_{33}v_{12}^2=(\text{V}),\quad 2w_{12}w_{23}w_{31}-w_{11}w_{23}^2-w_{22}w_{31}^2=(\text{W}),$

$$\therefore\ \frac{(\text{U})}{(b-c)^2}+\frac{(\text{V})}{(c-a)^2}+\frac{(\text{W})}{(a-b)^2}=2\left\{\frac{u_{12}u_{23}u_{31}}{(b-c)^2}+\frac{v_{12}v_{23}v_{31}}{(c-a)^2}+\frac{w_{12}w_{23}w_{31}}{(a-b)^2}\right\}=0,$$

since $u_{12}:v_{12}:w_{12}=u_{23}:v_{23}:w_{23}=\&\text{c.}=b-c:c-a:a-b.$

And if T and T′ be the ordinary covariants of U and V (Δ and Δ' still being $=1$), the point equation to $u+\lambda v=0$ is

$$\text{U}+\lambda\text{T}+\lambda^2\text{T}'+\lambda^3\text{V}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B}),$$

and, since $u+v+w=0$, $-\text{W}=\text{U}+\text{T}+\text{T}'+\text{V}$; also, at the points where $u+\lambda v=0$ touches the developable, the equation (B) gives equal values for λ, and therefore $\text{T}+2\lambda\text{T}'+3\lambda^2\text{V}=0$ and $3\text{U}+2\lambda\text{T}+\lambda^2\text{T}'=0$, and therefore along the curve of contact

$$\begin{vmatrix} \text{U}+\text{V}+\text{W}, & 3\lambda^2\text{V}, & 3\text{U} \\ 1, & 1, & 2\lambda \\ 1, & 2\lambda, & \lambda^2 \end{vmatrix}=0,\quad\text{or}\ -3\lambda^2(\text{U}+\text{V}+\text{W})+3\lambda^2\text{V}(2\lambda-\lambda^2)+3\text{U}(2\lambda-1)=0,$$

$$\text{or}\quad \text{U}(\lambda-1)^2+\text{V}\lambda^2(\lambda-1)^2+\text{W}\lambda^2=0,$$

and this is identical with $\dfrac{\text{U}}{(b-c)^2}+\dfrac{\text{V}}{(c-a)^2}+\dfrac{\text{W}}{(a-b)^2}=0,$

if $\lambda^2:1:(\lambda-1)^2=(b-c)^2:(c-a)^2:(a-b)^2$, or $\lambda=\dfrac{b-c}{a-c}$,

and $u+\lambda v=0$ the same as $au+bv+cw=0$.

7571. (By Professor HAUGHTON, F.R.S.)—A solid body is bounded by two infinite parallel planes kept constantly at the temperature of melting ice, and by a third plane, perpendicular to the first two planes, kept constantly at the temperature of boiling water. After the lapse of a very long time, show that the law of distribution of temperatures will be represented by the equations (between the limits $y = \pm \frac{1}{2}\pi$)

$$v = ae^{-x}\cos y + be^{-3x}\cos 3y + \&c., \quad 1 = a\cos y + b\cos 3y + \&c.$$

Solution by T. Woodcock, B.A.; Prof. Nash, M.A.; *and others.*

We have to find v in terms of x and y, knowing $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$, also $v = 0$ when $y = \pm \frac{1}{2}\pi$, and $v = 1$ when $x = 0$, as well as when $x = \infty$.

Try $v = \Sigma u \cos ry$, where u is independent of y. We must have $r =$ an odd integer $= 2m + 1$ say, and also $\frac{d^2u}{dx^2} - r^2u = 0$; $\therefore u = Ae^{r} + ae^{-rx}$. Now, when x is infinite, $v = 0$; $\therefore A = 0$; $\therefore v = \Sigma ae^{-(2m+1)x}\cos(2m+1)y$, the summation extending over all positive integral values of m. Putting $x = 0$, we have $1 = \Sigma a \cos(2m+1)y$, the limits of y being $\pm \frac{1}{2}\pi$.

7579. (By R. A. Roberts, M.A.)—Two uniform spherical shells attract according to the law of the inverse fifth power of the distance; show that, if they cut orthogonally, they will be in equilibrium under the influence of their mutual attraction.

Solution by Prof. Townsend, F.R.S.; J. A. Owen, B.Sc.; *and others.*

The attraction, for the law of the inverse fifth power of the distance, of a thin uniform spherical shell, upon a particle in its space, either external or internal to its mass, being directed towards the point of its surface nearest to the particle, and varying directly as the radial distance from its centre and inversely as the sixth power of the tangential distance from its surface; it follows at once that, for two such shells intersecting at right angles in a common space, if an elementary cone be supposed to diverge from the centre of either, it will intercept on the surface of the other two elements of mass, whose attractions by the former pass in opposite directions through its centre, and are to each other directly as the cubes of the radial distances from its centre, and inversely as the sixth powers of the tangential distances from its surface—that is, directly as the cubes of the radial distances from its centre, and inversely as the cubes of the perpendicular distances from its plane of intersection with the latter; and the two attractions being consequently equal in magnitude and opposite in direction, therefore, &c., as regards the property in question.

7569. (By Professor Townsend, F.R.S.)—In a tetranodal cubic surface in a space, show that—

(*a*) The four nodal tangent cones envelope a common quadric.

(*b*) Their four conics of intersection with the opposite faces of the nodal tetrahedron lie in a common quadric.

(*c*) The two aforesaid quadrics envelope each other along a plane having triple contact with the surface.

Solution by Professor MALET, F.R.S.; Prof. NASH, M.A.; *and others.*

The vertices of the tetrahedron of reference being the four nodes, the equation of the cubic is of the form $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}=0$; and the equations of the four nodal tangent cones are

$$\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}=0, \quad \frac{xy}{ab}+\frac{xw}{ad}+\frac{yw}{bd}=0,$$

$$\frac{xz}{ac}+\frac{xw}{ad}+\frac{zw}{cd}=0, \quad \frac{yz}{bc}+\frac{yw}{bd}+\frac{zw}{cd}=0;$$

which envelope the quadric

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d}\right)^2-4\left(\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}+\frac{xw}{ad}+\frac{yw}{bd}+\frac{zw}{cd}\right)=0,$$

and each of which meets the opposite face of the tetrahedron of reference in a conic situated on the quadric $\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}+\frac{xw}{ad}+\frac{yw}{bd}+\frac{zw}{dc}=0$. Now the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d}=0$ is a triple tangent plane to the cubic, the points of contact being $(a, b, -c, -d)$, $(a, -b, -c, d)$, $(a, -b, c, -d)$; therefore, &c.

NOTE ON INVERSE-COORDINATE CURVES, WITH SOLUTION OF QUEST. 6969.
By R. TUCKER, M.A.

The general discussion of the properties of these curves is given in SALMON'S *Higher Curves* (1st ed., p. 238, &c.; 2nd ed., p. 244). We have $xx'=yy'=c^2$, whence $\tan\theta=\cot\theta'$, *i.e.* $\theta+\theta'=\frac{1}{2}\pi$ and $rr'\sin 2\theta=2c^2$ (1, 2).

If $y=mx$, then i.-c. c. [*i.e.* inverse-coordinate curve] is $x=my$, *i.e.* a line through the origin has for i.-c. c. a line through the origin equally inclined to other axis (rectangular axes). Therefore, if any number of points on one curve are collinear with the origin, there will be the same number on the i.-c. c. collinear with the origin.

If $y=mx+b$, then i.-c. c. is $bxy=c^2(x-my)$. Hence, if parallel chords be drawn to the primitive curve, the corresponding points on "i.-c." lie on hyperbolas, the locus of whose centres is a straight line through the origin orthogonal to the above system of chords.

If ψ has the usual meaning, then $\cot\psi'\sim\cot\psi=\cot 2\theta\sim\cot 2\theta'=2\cot 2\theta$.

This enables us at once to solve the following Question (6969):—
"If, in a parabola vertical chords AP, AP', complementally inclined to the axis, make angles ψ, ψ' with the tangents at P, P',

then $\qquad \cot\psi'\sim\cot\psi=2\Delta\bar{A}PP'/L^2$."

The parabola $y^2 = 4ax$ will be its own "inverse" if $c = 4a = L$, then $2\Delta APP' = AP.AP' \sin PAP' = rr' \cos 2\theta = 2c^2 \cot 2\theta = 32a^2 \cot 2\theta$
$$= L^2 (2 \cot 2\theta) = L^2 (\cot \psi' \cot \psi).$$

Similar properties are readily obtained for other curves which are their own i.-c. c. as $2xy = a^2$ ($a^2 = 2c^2$), $x^3 = ay^2$ ($c = a$), &c.

[Another solution of Quest. 6969 is given on p. 114 of Vol. 37 of the *Reprint.*]

7194. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In the examination for the Mathematical Tripos, January 2, 1868, Question (6) is as follows:—"If there be n straight lines lying in one plane so that no three meet in one point, the number of groups of n of their points of intersection, in each of which no three points lie in one of the n straight lines, is $\frac{1}{2}(n-1)$." Prove that this is not true; but that, if "n-sided polygons" be written for "groups of n points, &c.," the result will be true: and calculate the correct answer to the question enunciated.

Solution by W. J. GREENSTREET, B.A.; A. MACMURCHY, B.A.; *and others.*

Denote the n straight lines by $1.2.3 \ldots n$. Make a group of n intersections in this way:—1 and 2, 2 and 3, ..., $n-1$ and n, n and 1. Then there are two, and only two, points on each straight line. Hence we must take two points on each straight line, for if not there would be more than two on some line or lines. So that we merely require now the number of ways the n intersections may be arranged in a ring, that is $\frac{1}{2}(n-1)$!

7247. (By Dr. CURTIS.)—Two magnets, whose intensities are I_1, I_2, and lengths a_1, a_2, are rigidly connected so as to be capable of moving only in a horizontal plane round a vertical line, which passes through the middle point of the line connecting the two poles of each magnet; if $2a$ denote the angle between the lines of poles of the two magnets in the direction of opposite poles, while θ denotes the inclination to the magnetic meridian of the line bisecting this angle, prove that (1) the positions of *stable* and *unstable* equilibrium (discriminating between them) are given by $\tan\theta = (I_1a_1 + I_2a_2) \tan a / (I_1a_1 - I_2a_2)$; and hence (2), if the intensities of the two magnets be inversely proportional to their lengths, the positions of equilibrium will be such that the lines of poles of the magnets will be equally inclined to the magnetic meridian.

Solution by W. M. COATES, B.A.; BELLE EASTON; *and others.*

The moment of the first magnet is I_1a_1, and the angle its axis makes with the magnetic meridian $= \theta - a$: therefore the moment of the couple tending to turn it is proportional to $I_1a_1 \sin(\theta - a)$.

Similarly the moment of the couple tending to turn the second magnet in the opposite direction is proportional to $I_2a_2 \sin(\theta + a)$. Hence, when the system is in equilibrium, $I_1a_1 \sin(\theta - a) = I_2a_2 \sin(\theta + a)$, whence $\tan\theta = (I_1a_1 + I_2a_2 \tan a) / (I_1a_1 - I_2a_2)$. [Whether the sign of the right-hand side of this equation be positive or negative, as the angle θ is sought from its tangent, there will be two solutions θ_1, θ_2, whose difference is 180°. The *stable* position is that in which the magnetic axis of each needle, in the direction of its *north-seeking* pole, makes an acute angle with the meridional line in its *northerly* direction, as in such position, if the system be turned through a small angle, the moment tending to return it to its original position is increased, and the opposite one diminished. The *unstable* position is derivable from the *stable* by turning the system through 180°.] If the condition $I_1 a_1 = I_2 a_2$ be fulfilled, $\theta = 90°$, from which it follows at once that the axes of the magnets are equally inclined to the meridian.

7508. (By Professor Sylvester, F.R.S.)—If m, n be any two square matrices of the same order $M = (mn-nm)^3$,

$$N = (m^2n-nm^2)(n^2m-mn^2)-(n^2m-mn^2)(m^2n-nm^2),$$

$$P = \begin{vmatrix} m^2, & mn+nm, & n^2 \\ m^2, & mn+nm, & n^2 \\ m^2, & mn+nm, & n^2 \end{vmatrix};$$ and D the determinant to the matrix

$$\alpha M+\beta N+\gamma P:$$

prove that D is an invariant to m, n; that is, remains unaltered when (supposing $pq'-p'q=1$) $pm+qn$ and $p'm+q'n$ are substituted for m and n.

Solution by the Proposer.

Let m become $m+\epsilon n$ where ϵ is infinitesimal; then the increment of P divided by ϵ is

$$\begin{vmatrix} mn+nm, & mn+nm, & n^2 \\ mn+nm, & mn+nm, & n^2 \\ mn+nm, & mn+nm, & n^2 \end{vmatrix} + \begin{vmatrix} m^2, & 2n^2, & n^2 \\ m^2, & 2n^2, & n^2 \\ m^2, & 2n^2, & n^2 \end{vmatrix}, \text{ i.e., } 0.$$

Again, for N, call $(m^2n-nm^2)=A$, $(n^2m-mn^2)=B$; then $N=AB-BA$,

and $$\frac{\delta A}{\epsilon}=(mn+nm)\,n-n\,(mn+nm)=B, \quad \frac{\delta B}{\epsilon}=n^2.n-n.n^2=0;$$

hence $$\delta N = \delta A\,.\,B - B\delta A = (B^2-B^2)\,\delta\epsilon = 0.$$

Finally, $\delta\,(mn-nm)=\epsilon\,(n-n)=0$; hence $\alpha M+\beta N+\gamma P$ is unaltered by the change of m into $m+\epsilon n$ and in like manner of n into $n+\epsilon m$; whence it follows, by the same reasoning as in the theory of ordinary invariance, that m and n may be changed into $pm+qn$ and $p'm+q'n$, provided $pq'-p'q=1$ without $\alpha M+\beta N+\gamma P$ undergoing a change, and consequently without its determinant changing, so that this latter is a binary invariant of the matrices m, n, as was to be proved.

[Professor Sylvester calls attention to the immense new horizon in the theory of Invariants opened out by this question, which forms part of a general theory of Matrices, including the algebraical theory of Quaternions as an insignificant single case; and in which he connects the subject with the ordinary concomitants of Ternary as well as of Binary forms, and in such a manner as to *react* upon the ordinary theory of Invariants. The theorem in the question is part of the solution of the prodigiously difficult subject of Involution of Matrices, now happily accomplished, or brought at least within a stone's throw of accomplishment.]

7558. (By W. J. C. SHARP, M.A.)—If A′, B′, C′, D′, be the feet of the perpendiculars from any point on the four faces of a tetrahedron ABCD, show that $AC'^2 - BC'^2 = AD'^2 - BD'^2$, &c., and conversely.

Solution by W. G. LAX, B.A.; MARGARET T. MEYER; *and others.*

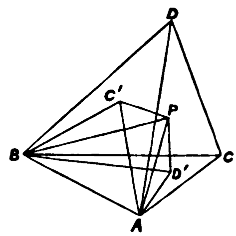
Join AP, BP, where P is the point from which the perpendiculars are drawn; then, since PC′B and PC′A are right angles,

$$AC'^2 - BC'^2 = AP^2 - PC'^2 - BP^2 + PC'^2$$
$$= AP^2 - BP^2.$$

Similarly, $AD'^2 - BD'^2 = AP^2 - BP^2$,

therefore $AC'^2 - BC'^2 = AD'^2 - BD'^2$,

and so on, and conversely.

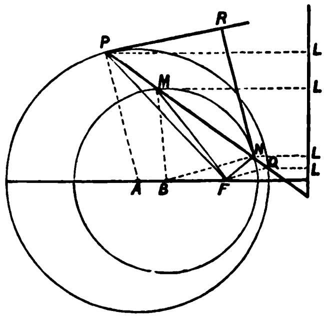


7364. (By W. S. M'CAY, M.A.)—If the line joining two points on two circles subtend a right angle at a limiting point, prove that the locus of the intersection of tangents at the points is a coaxal circle.

Solution by T. A. FINCH; SARAH MARKS; *and others.*

Let A and B be two circles, PN the chord subtending the right angle at F. Let PN meet the circles again in M and Q, and draw PL, ML, NL and QL perpendicular to L the radical axis of the circles and complete the Figure. Then

$$\frac{RP}{RN} = \frac{\sin RNP}{\sin RPN} = \frac{\cos BNM}{\cos APQ}$$
$$= \frac{MN}{2BM} \cdot \frac{2AP}{PQ}$$
$$= \frac{FM \cdot FN \sin MFN}{FP \cdot FQ \sin PFQ} \cdot \frac{AP}{BM},$$



and, from right angle, $\sin MFN = \cos PFM$; $\sin PFQ = \cos NFQ$, also $\angle PFM = \angle NFQ$ (since angles MFN and PFQ have same bisectors) therefore

$$\frac{RP^2}{RN^2} = \frac{FM^2 \cdot FN^2}{FQ^2 \cdot FP^2} \cdot \frac{AP^2}{BM^2} = \frac{BF \cdot ML \cdot BF \cdot NL}{AF \cdot QL \cdot AF \cdot PL} \cdot \frac{AP^2}{BM^2} = \frac{BF^2 \cdot AF^2}{AP^2 \cdot BM^2} = \text{const.}$$

Therefore, &c. [The PROPOSER remarks that this proof is much more elegant than his own. The theorem was originally derived by reciprocation from Question 5395, solved in *Reprint*, Vol. xxix., p. 23.]

1945. (By the late C. W. Merrifield, F.R.S.)—Find a rectangular parallelepiped such that its edges, the diagonals of its faces, and the diagonals of the solid, shall all be integral.

Solution by Asûtosh Mukhopâdhyây.

Let x, y, z be the edges of the solid; then the diagonals of its faces are $(x^2+y^2)^{\frac{1}{2}}$, $(y^2+z^2)^{\frac{1}{2}}$, $(z^2+x^2)^{\frac{1}{2}}$; also, the diagonals of the solid are equal to one another, and represented by $(x^2+y^2+z^2)^{\frac{1}{2}}$. We have, accordingly, to investigate whether it is possible to find positive integral values of x, y, z, which make x, y, z, $(x^2+y^2)^{\frac{1}{2}}$, $(y^2+z^2)^{\frac{1}{2}}$, $(z^2+x^2)^{\frac{1}{2}}$, $(x^2+y^2+z^2)^{\frac{1}{2}}$ all integral. Let $x^2+y^2 = (k^2+1)^2$, $y^2+z^2 = (l^2+1)^2$..................(1, 2),

$$z^2+x^2 = (m^2+1)^2,\quad x^2+y^2+z^2 = (n^2+1)^2 \;..............(3, 4).$$

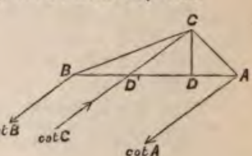
Then it is well-known that the solutions of (1, 2, 3) are

$$\left.\begin{matrix} x = 2k \\ y = k^2-1 \end{matrix}\right\} \left.\begin{matrix} y = 2l \\ z = l^2-1 \end{matrix}\right\} \left.\begin{matrix} z = 2m \\ x = m^2-1 \end{matrix}\right\}.$$

Now, substituting in (4) from (2), we get $x^2+(l^2+1)^2 = (n^2+1)^2$. Therefore $x = 2n$, $l^2+1 = n^2-1$, therefore $n^2-l^2 = 2$; hence the solution of the original problem depends on an equation of the form $(x+y)(x-y) = 2$. Now, a moment's consideration shows that this equation has no positive integral solution; for, assuming x, y to be positive integers, and since $(x+y)(x-y) = 2$, $(x+y)$, $(x-y)$ must be each a positive integral; and since the composition of 2 is unique (2×1), we must have $x+y = 2$, $x-y = 1$, which give $x = \frac{3}{2}$, $y = \frac{1}{2}$,—fractional values. Hence, it is demonstrated that the original system of equations has no positive integral solutions, and it is impossible to find the rectangular parallelepiped in question.

7573. (By Professor Hudson, M.A.)—Parallel forces act at the angular points of a triangle proportional to the cotangents of the angles. Can they be in equilibrium?

Solution by (1) B. Reynolds, M.A.; (2) W. J. Barton, B.A.



1. Yes: if $a^2+b^2 = 3c^2$, and if

$$BD' = DA = b\cos A,$$

CD′ being the direction of the forces.

For $\quad \cot C = \cot A + \cot B$,

$$\therefore\quad \frac{\cot C}{\sin C} = \frac{\sin C}{\sin A \sin B}, \text{ or } \cos C = \frac{c^2}{ab},$$

or $2ab\cos C = 2c^2$, whence $a^2+b^2 = 3c^2$.

Next, if $BD' = x$, we have $x\cot B = (c-x)\cot A$,

$$\text{or } x = c\,\frac{\cot A}{\cot A+\cot B} = c\,\frac{\cos A \sin B}{\sin(A+C)} = c\,\frac{\cos A\sin B}{\sin C} = b\cos A.$$

2. For equilibrium, one of the forces (at A say) must act in the opposite direction to the other two, whence $\cot A = \cot B + \cot C$; also, if the direction of force at A produced backwards cut BC in D, then, taking moments, $DC : BD = \cot B : \cot C$; and, if these conditions are

fulfilled, there is equilibrium and not otherwise. A particular case is when one of the angles (B say) is a right angle, so that the force of B vanishes; then we must have $A = C = \frac{1}{4}\pi$, and forces at A and C act in opposite directions along the hypotenuse.

[Mr. REYNOLDS states that he "inadvertently made the angle BCA obtuse, whereas it should be acute, since $c^2 < a^2 + b^2$."

Professor HUDSON remarks that "the figure is manifestly impossible; for, C being obtuse, cot C is $-$, and a force cot C in the opposite direction to cot A, cot B, is really in the same direction and cannot counterbalance them.

"In the case supposed by Mr. BARTON, the forces are proportional to $\cot A$, 0, $-\cot C$. Since for equilibrium one of the forces must act in the opposite direction to the other two, the proper inference is that the triangle is obtuse-angled.

"The condition of equilibrium is $\cot A + \cot B + \cot C = 0$, therefore $\frac{\sin(A+B)}{\sin A \sin B} + \frac{\cos C}{\sin C} = 0$, therefore $\frac{\sin^2 C}{\sin A \sin B} + \frac{\sin^2 A + \sin^2 B - \sin^2 C}{2 \sin AFB} = 0$, therefore $\sin^2 A + \sin^2 B + \sin^2 C = 0$, which is impossible]."

7495. (By S. TEBAY, B.A.)—Show that the mean length of the "Sailor's Knot," or geographical mile, in latitude λ, is approximately $1{\cdot}1566\,(1 - {\cdot}00667 \cos^2\lambda)$ mile.

Solution by the PROPOSER.

If the ellipsoid $(1-e^2)(x^2+y^2)+z^2 = b^2$ be cut by a diametral plane passing through a point in latitude λ, and inclined to the plane of the meridian at an angle θ, the equation to the section is

$$(1-e^2\cos^2\lambda)\,x^2 - 2e^2 \sin\lambda \cos\lambda \cos\theta\, xy + (1-e^2+e^2\cos^2\lambda\cos^2\theta)\,y^2 = b^2.$$

Let this be written $\quad Ax^2 - 2Cxy + By^2 = b^2.$

Differentiate twice with respect to x; thus

$$Ax - Cy - Cxp + Byp = 0, \quad A - 2Cp - Cyq + Bp^2 + Byq = 0.$$

Hence at a point on the meridian, putting $y = 0$ and $x = \frac{b}{A^{\frac{1}{2}}}$, we have

$$p = \frac{A}{C}, \quad q = \frac{A^{\frac{3}{2}}}{bC^3}(AB - C^2); \quad \therefore\ \rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{\left(A + \frac{C^2}{A}\right)^{\frac{3}{2}}}{AB - C^2}\,b$$

$$= \frac{a^2}{b}\left\{\frac{1-e^2(2-e^2)\cos^2\lambda}{1-e^2\cos^2\lambda}\right\}\ \frac{\left\{1 - \frac{e^4\sin^2\lambda\cos^2\lambda\sin^2\theta}{1-e^2(2-e^2)\cos^2\lambda}\right\}^{\frac{3}{2}}}{1-e^2\cos^2\lambda\sin^2\theta}$$

$$= H\frac{a^2}{b}\,\frac{(1-n\sin^2\theta)^{\frac{3}{2}}}{1-m\sin^2\theta}, \text{ suppose,}$$

$$= H\frac{a^2}{b}\left\{1 + (m - \tfrac{3}{2}n)\sin^2\theta + \left(m^2 - \tfrac{3}{2}mn + \frac{1}{2!}\tfrac{3}{2}n^2\right)\sin^4\theta\right.$$

$$\left. + \left(m^3 - \tfrac{3}{2}m^2n + \frac{1}{2!}\,\tfrac{3}{2}\,\tfrac{1}{2}\,mn^2 + \frac{1}{3!}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,n^3\right)\sin^6\theta + \&c.\right\}.$$

Let this be written $\rho = \mathrm{H}\frac{a^2}{b}(1 + \mathrm{N}_2\sin^2\theta + \mathrm{N}_4\sin^4\theta + \mathrm{N}_6\sin^6\theta + \ldots)$;

then, since $\int_0^{\frac{1}{2}\pi}\sin^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\frac{1}{2}\,\frac{1}{2}\pi$, ($n$ even),

the mean value of ρ is

$$\rho' = \left(\int^{\frac{1}{2}\pi}\rho\,d\theta\right) \div \tfrac{1}{2}\pi = 4\frac{a^2}{b}\left(1 + \tfrac{1}{2}\mathrm{N}_2 + \frac{3\,.\,1}{4\,.\,2}\mathrm{N}_4 + \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\mathrm{N}_6 + \&\text{c.}\right)$$

$$= \frac{a^2}{b}\left(1 - e^2\cos^2\lambda + \tfrac{3}{4}e^4\sin^2\lambda\cos^2\lambda + \tfrac{9}{16}e^6\sin^2\lambda\cos^4\lambda + \ldots\right), \text{ nearly.}$$

Thus the mean value of ρ at a point on the equator is b. Take

$$a = 3962{\cdot}824, \quad b = 3949{\cdot}585 \text{ miles;}$$

then the mean length of the knot at a point on the equator is

$$\frac{\pi b}{10800} = 1{\cdot}14889 \text{ mile} = 1 \text{ m. } 1 \text{ fur. } 42 \text{ yds.}$$

Approximately, the length of the knot is

$$\frac{\pi\rho'}{10800} = 1{\cdot}1566\,(1 - e^2\cos^2\lambda + \tfrac{3}{4}e^4\sin^2\lambda\cos^2\lambda + \tfrac{9}{16}e^6\sin^2\lambda\cos^4\lambda)$$

$$= 1{\cdot}1566\,(1 - {\cdot}00667\cos^2\lambda).$$

At 30°, 45°, 60° we have, respectively, $\rho' = 3956{\cdot}450,\ 3962{\cdot}881,\ 3969{\cdot}5$, and length of knot 1·1508, 1·1527, 1·1547 mile.

7581. (By C. Leudesdorf, M.A.)—If $\mathrm{A} + \mathrm{B} + \mathrm{C} = 180°$,

$$(y - z)\cot\tfrac{1}{2}\mathrm{A} + (z - x)\cot\tfrac{1}{2}\mathrm{B} + (x - y)\cot\tfrac{1}{2}\mathrm{C} = 0,$$

$$(y^2 - z^2)\cot\mathrm{A} + (z^2 - x^2)\cot\mathrm{B} + (x^2 - y^2)\cot\mathrm{C} = 0;$$

prove that $\dfrac{y^2 + z^2 - 2yz\cos\mathrm{A}}{\sin^2\mathrm{A}} = \dfrac{z^2 + x^2 - 2zx\cos\mathrm{B}}{\sin^2\mathrm{B}} = \dfrac{x^2 + y^2 - 2xy\cos\mathrm{C}}{\sin^2\mathrm{C}}$.

Solution by W. J. Barton, B.A.; Margaret T. Meyer; *and others.*

Let a, b, c be the sides of a triangle having opposite angles equal to A, B, C; then we have

$$\Sigma(b - c)\cot\tfrac{1}{2}\mathrm{A} \propto \Sigma(\sin\mathrm{B} - \sin\mathrm{C})\cot\tfrac{1}{2}\mathrm{A} \propto \Sigma\sin\tfrac{1}{2}(\mathrm{B} - \mathrm{C})\cos\tfrac{1}{2}\mathrm{A}$$

$$\propto \Sigma\sin\tfrac{1}{2}(\mathrm{B} - \mathrm{C})\sin\tfrac{1}{2}(\mathrm{B} + \mathrm{C}) \propto \Sigma\left[\sin\tfrac{1}{2}(2\mathrm{B}) - \sin\tfrac{1}{2}(2\mathrm{C})\right] = 0 \ldots (1),$$

$$\Sigma(b^2 - c^2)\cot\mathrm{A} \propto \Sigma\sin(\mathrm{B} + \mathrm{C})\sin(\mathrm{B} - \mathrm{C})\cot\mathrm{A} \propto \Sigma\sin(\mathrm{B} - \mathrm{C})\cos(\mathrm{B} + \mathrm{C})$$

$$\propto \Sigma(\sin 2\mathrm{B} - \sin 2\mathrm{C}) = 0 \ldots\ldots (2);$$

therefore, from (1) and (2) combined with the given equations,

$$\frac{b - c}{y - z} = \frac{c - a}{z - x} = \frac{a - b}{x - y}, \text{ and } \frac{b^2 - c^2}{y^2 - z^2} = \frac{c^2 - a^2}{z^2 - x^2} = \frac{a^2 - b^2}{x^2 - y^2} \ldots\ldots (3),$$

therefore $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$; therefore a triangle can be constructed with x, y, z for sides and A, B, C for angles; hence we obtain

$$\frac{x^2}{\sin^2\mathrm{A}} = \text{const.} = \frac{y^2 + z^2 - 2yz\cos\mathrm{A}}{\sin^2\mathrm{A}} = \frac{z^2 + x^2 - 2zx\cos\mathrm{B}}{\sin^2\mathrm{B}} = \&\text{c.}$$

3835. (By the EDITOR.)—The sides of a triangle ABC are BC = 6, CA = 5, AB = 4; and Q, R are points in AC, AB, such that CQ = 2; BR = 3. Show (1) by a general solution, that the distance from B to a point P in BC, such that $\angle$ CQP = BRP, is $BP = \frac{1}{3}(601^{\frac{1}{2}} - 13) = 3{\cdot}83843$; and (2) give a construction for finding the point P.

Solution by D. BIDDLE.; Prof. MATZ, M.A., Ph.D.; *and others.*

1. Draw (Fig. 1) PM, QS parallel to AB, and PN, RT parallel to AC; then the triangles QMP, RNP will be similar, and PN : PM = NR : QM;

but $PN = \dfrac{AC \,.\, BP}{BC}$, $PM = \dfrac{AB \,.\, PC}{BC}$,

$NR = \dfrac{AB \,.\, PT}{BC}$, $QM = \dfrac{AC \,.\, SP}{BC}$;

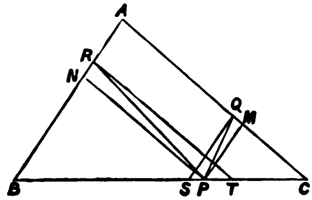


Fig. 1.

therefore $$AC \,.\, BP : AB \,.\, PC = AB \,.\, PT : AC \,.\, SP,$$
$$BP : PC = AB^2 . PT : AC^2 . SP;$$

but $PC = BC - BP$, $PT = \dfrac{BC \,.\, BR}{AB} - BP$, $SP = BP + \dfrac{CQ \,.\, BC}{AC} - BC$;

$\therefore BP : BC - BP = AB.BC.BR - AB^2 BP : AC^2 BP + AC.BC.CQ - AC^2 BC,$

and $$BP^2 + \frac{AC \,.\, BC \,.\, CQ - AC^2 BC + AB^2 BC + AB \,.\, BC \,.\, BR}{AC^2 - AB^2}\, BP = \frac{AB \,.\, BC^2 . BR}{AC^2 - AB^2};$$

whence
$$2(AC^2 - AB^2)BP = \left\{ \begin{matrix} (AC \,.\, BC \,.\, CQ - AC^2 BC + AB^2 BC + AB \,.\, BC \,.\, BR)^2 \\ + 4(AC^2 - AB^2)\, AB \,.\, BC^2 . BR \end{matrix} \right\}^{\frac{1}{2}} - (AC.BC \,.\, CQ - AC^2 BC + AB^2 BC + AB \,.\, BC \,.\, BR).$$

In the given case, $BP = \frac{1}{3}[(601^{\frac{1}{2}} - 13)] = 3{\cdot}83843$.

2. Draw (Fig. 2) QD perpendicular to BC; make DE = QD; join RE; make CF = BF (that is, from the mid-point of BC draw a perpendicular thereto). Draw EH parallel to BF, to meet AB in T; then TRE will be a triangle with an angle T = ABF = ABC − ACB, and the circle RPE, drawn round this triangle, will cut BC in the point P required, so that

$$\angle \text{ CQP} = \text{BRP}.$$

For $\angle$ SQD = SED = SRI,

$\angle$ CQD − BRI = ABC − ACB,

$\therefore$ SQC − BRS = ABC − ACB;

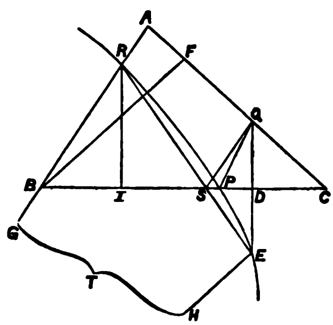


Fig. 2.

and, by construction, we have

$$\angle \text{PER} + \text{PRE} = \text{ABF} = \text{ABC} - \text{ACB}; \text{ but } \angle \text{PER} = \text{PQS},$$
$$\therefore \angle \text{PRE} + \text{PQS} = \text{ABC} - \text{ACB} = \text{SQC} - \text{BRS},$$

and $\qquad\qquad \text{PRE} + \text{BRS} = \text{SQC} - \text{PQS},$
hence $\qquad\qquad \angle\,\text{BRP} = \text{CQP}.$

[If BC $= a$, CA $= b$, AB $= c$, BR $= m$, CQ $= n$, and BP $= x$, we have

$$\cot\ \text{BRP} = \frac{m - x\cos\text{B}}{x\sin\text{B}} = \cot\text{CQP} = \frac{n-(a-x)\cos\text{C}}{(a-x)\sin\text{C}};$$

therefore $\quad \dfrac{m}{x}\ \text{cosec B} - \dfrac{n}{a-x}\ \text{cosec C} = \cot\text{B} - \cot\text{C}$, which gives

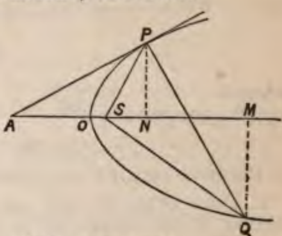
the quadratic equation $\quad \dfrac{mc}{bx} - \dfrac{n}{a-x} = \dfrac{c\cos\text{B} - b\cos\text{C}}{b} = \dfrac{c^2-b^2}{ab};$

whence, putting, for shortness' sake, $k = nb + mc - b^2 + c^2$, we obtain the general value $\quad x = \dfrac{a}{2(c^2-b^2)}\left\{k \pm [k^2 - 4mc\,(c^2-b^2)]^{\frac{1}{2}}\right\},$

which, with the numbers in the question, gives the specified result.]

7159. (By R. Knowles, B.A., L.C.P.)—In a parabola whose latus rectum is $4a$, if θ be the angle subtended at the focus S by a normal chord PQ, prove that the area of the triangle $\text{SPQ} = a^2\cot\frac{1}{2}\theta\,(\tan\frac{1}{2}\theta + 4\cot\frac{1}{2}\theta)^2$.

Solution by J. S. Jenkins; Sarah Marks; *and others.*



Let (x', y'), (x'', y'') be the coordinates of P and Q, and let $\theta = \angle\,\text{PSQ}$; then it can be readily shown that

$$\text{OM} = x'' = \frac{(x'+2a)^2}{x'},$$

$$\text{QM} = y'' = 2\left(\frac{a}{x'}\right)^{\frac{1}{2}}(x'+2a),$$

$$\Delta\,\text{PSQ} = 2\left(\frac{a}{x'}\right)^{\frac{1}{2}}(a+x')^2 \ldots (1);$$

$$\Delta\,\text{PSQ} = \tfrac{1}{2}(a+x')(a+x'')\sin\theta = \tfrac{1}{2}(a+x')\left(\frac{x'^2+4a^2+5ax'}{x'}\right)\sin\theta \ldots (2);$$

$$\therefore\ \sin\theta = \frac{4\,(ax')^{\frac{1}{2}}\,(a+x')}{x'^2+4a^2+5ax'} = \frac{8ay'}{y'^2+16a^2};\ \therefore\ y' = 4a\cot\tfrac{1}{2}\theta,\ \text{and}\ x' = 4a\cot^2\tfrac{1}{2}\theta.$$

This value of x' substituted in (1) gives the area

$$\Delta\,\text{PQS} = 2\left(\frac{a}{4a\cot^2\frac{1}{2}\theta}\right)^{\frac{1}{2}}(a+4a\cot^2\tfrac{1}{2}\theta)^2 = a^2\cot\tfrac{1}{2}\theta\left(\frac{1+4\cot^2\frac{1}{2}\theta}{\cot\frac{1}{2}\theta}\right)^2$$

$$= a^2\cot\tfrac{1}{2}\theta\,(\tan\tfrac{1}{2}\theta + 4\cot\tfrac{1}{2}\theta)^2.$$

7578. (By the Rev. T. C. Simmons, M.A.)—If a number have the sum of its digits equal to 10, find under what circumstances twice the number will have the sum of its digits equal to 11.

Solution by MARGARET T. MEYER; C. W. H. GREAVES; *and others.*

If all the digits are under 5, the double number will clearly have the sum of its digits = 20. If two of the digits are 5 (the rest of course all = 0), the double number will have sum of digits = 2. In all other cases, the sum will be 11 (or 20 − 9). [This is included in Quest. 7534, solved on p. 28 of this volume.]

7542. (By Professor MARTIN, M.A., Ph.D.)—Prove that for $n = \infty$,

$$\frac{\pi}{2n}\left\{\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{\pi}{2n}\right)}+\ldots+\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{n\pi}{2n}\right)}\right\} = \log_e 2.$$

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

$$\int_0^{\frac{1}{2}\pi}\frac{dx}{1+\sin x+\cos x} = \int_0^{\frac{1}{2}\pi}\frac{dx}{2\cos\frac{1}{2}x\,(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}$$
$$= \int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}\sec^2\frac{1}{2}x\,dx}{1+\tan\frac{1}{2}x} = \left[\log\left(1+\tan\tfrac{1}{2}x\right)\right]_0^{\frac{1}{2}\pi} = \log 2.$$

Writing this in the usual way, $dx = \pi/2n$, and the limit of the series in the question, when n is ∞, is $\log 2$.

Since $$\int_0^{\frac{1}{2}\pi}\frac{x}{1+\sin x+\cos x}\,dx = \int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}\pi-x}{1+\cos x+\sin x}\,dx,$$

we get $$\int_0^{\frac{1}{2}\pi}\frac{x}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+x\right)}\,dx = \tfrac{1}{4}\pi\log 2;$$

and this integral may be written

the limit of $$\frac{1}{n^2}\left\{\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{\pi}{2n}\right)}+\frac{2}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{2\pi}{2n}\right)}+\ldots\right.$$
$$\left.\ldots+\frac{n}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{n\pi}{2n}\right)}\right\},$$

when n tends to ∞ is $\frac{1}{\pi}\log 2$. This is not new, being only another way of getting at the well-known definite integral

$$\int_0^{\frac{1}{4}\pi}\log\left(1+\tan x\right)dx = \tfrac{1}{8}\pi\log 2.$$

7592. (By S. TEBAY, B.A.)—Find an integral value of a such that $(m^2+n^2)^2+a$ and $(m^2+n^2)^2-a$ shall be rational squares; m and n being positive integers.

Solution by MORGAN JENKINS, M.A.

If $(m^2+n^2)^2+a = (h+k)^2$ and $(m^2+n^2)^2-a = (h-k)^2$,

then $a = 2h\kappa$, and $(m^2+n^2)^2 = h^2+\kappa^2$. One solution of the last equation is $h = m^2-n^2$, $\kappa = 2mn$, whence $a = 4mn(m^2-n^2)$; and

$$(m^2+n^2)^2 \pm 4mn(m^2-n^2) = (m^2-n^2 \pm 2mn)^2.$$

The total number of primitive ways of expressing, as the sum of two different squares (zeros excluded), a number that contains no odd prime factors, and no power of 2 higher than 2^1, is 2^{t-1}, where t is the number of different odd prime factors which $\equiv 1$, mod. 4, and which are contained in the number. Hence the total number of solutions may be found when the mode of separating m^2+n^2 into factors is given; thus, if

$$m^2+n^2 = 2^\lambda . X^2 . \alpha^u . \beta^v . \gamma^w \dots,$$

where X is an odd number all of whose factors are of the form $4p+3$, and $\alpha, \beta, \gamma \dots$ are t prime factors, each of the form $4p+1$ (it being known that all the prime factors of X must be of even degree), then

$$(m^2+n^2)^2 = 2^{2\lambda} . X^4 . \alpha^{2u} . \beta^{2v} . \gamma^{2w} \dots,$$

and, if $\mu^2+\nu^2$ be one of the 2^{t-1} primitive representations of $\alpha^{2u} . \beta^{2v} . \gamma^{2w} \dots$ as the sum of two squares, $h = 2^\lambda . X^2\mu$ and $k = 2^\lambda . X^2\nu$. So, if $\mu^2+\nu^2$ be one of the 2^{t-1} primitive representations of $\alpha^{2u-2} . \beta^{2v} . \gamma^{2w} \dots$ as the sum of two squares, we have $h = 2^\lambda . X^2\alpha\mu$, $k = 2^\lambda . X^2\alpha\nu$, and so on; but the number of primitive representations of $\alpha^0\beta^v\gamma^w \dots$ will be only 2^{t-2}, and of $\alpha^0\beta^0\gamma^{2w} \dots$ will be only 2^{t-3}. Thus the total number of solutions will be the sum of the coefficients in the product

$$(\tfrac{1}{2}+\alpha^2+\alpha^4+\dots\alpha^{2u})(\tfrac{1}{2}+\beta^2+\beta^4+\dots\beta^{2v})(\tfrac{1}{2}+\gamma^2+\gamma^4\dots+\gamma^{2w})\dots\times 2^{t-1},$$

omitting the first term in the product, viz., $\dfrac{2^{t-1}}{2^t}$ or $\tfrac{1}{2}$, if we reject the solution $a = 0$; and therefore the total number of solutions will be

$$2^{t-1}(u+\tfrac{1}{2})(v+\tfrac{1}{2})(w+\tfrac{1}{2})\dots-\tfrac{1}{2} \text{ or } \tfrac{1}{2}\left[(2u+1)(2v+1)(2w+1)\dots-1\right].$$

[Since $101 = 10^2+1^2$, if we take $m = 10$, $n = 1$, we have $a = 3960$, and $101^2+3960 = 119^2$, $101^2-3960 = 79^2$.]

7351. (By Professor SYLVESTER, F.R.S.)—Let ν be the number of ways in which any number n can be composed with any i positive integers (all unequal), and let X_i represent the sum of the terms νx^n, which will be an *infinite* series. Also, let ν_j be the number of ways in which any number n can be composed with any i positive integers all unequal as before, but now *none greater* than j, and let $X_{i,j}$ represent the sum of the terms x^n which will be a *finite* series. Prove that

$$X_{i,j} = (1-x^j)(1-x^{j-1})\dots(1-x^{j-i+1})X_i.$$

Ex.—Let $i = 2$, $j = 3$; then

$$X_i = x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+\dots\dots$$

and $$X_{i,j} = x^3+x^4+x^5 = (1-x^2)(1-x^3)X_i.$$

Solution by W. J. C. SHARP, M.A.

1. Evidently X_i is the coefficient of z^i in the expansion of the infinite product $(1+xz)(1+x^2z)(1+x^3z)\ldots$, so that this is $1+X_1z+X_2z^2+X_3z^3+\&c.$; but, if $xz=y$, the same product

$= (1+y)(1+xy)(1+x^2y)(1+x^3y)\ldots = (1+y)(1+X_1y+X_2y^2+\&c.)$

$= (1+xz)(1+X_1xz+X_2x^2z^2+X_3x^3z^3+\&c.)$;

hence, equating coefficients, $X_1=(1+X_1)x$, $X_2=(X_1+X_2)x^2$, &c., &c.,

$X_i=(X_{i-1}+X_i)x^i$, so that $X_1=\dfrac{x}{1-x}$, &c.,

and
$$X_i=\frac{x^i}{1-x^i}X_{i-1}=\frac{x^{\frac{1}{2}i(i+1)}}{(1-x)(1-x^2)\ldots(1-x^i)}$$

[a value obtained in a different way in Art. 2]. Similarly $X_{i,j}$ is the coefficient of z^i in the expansion of $(1+xz)(1+x^2z)\ldots(1+x^jz)$, so that this product $=1+X_{1,j}z+X_{2,j}z^2+X_{3,j}z^3+\&c.$; and if, as before, $xz=y$, this

product $=\dfrac{1+y}{1+x^jy}(1+xy)(1+x^2y)\ldots(1+x^jy)$

$=\dfrac{1+xz}{1+x^{j+1}z}\left\{1+X_{1,j}xz+X_{2,j}x^2z^2+X_{3,j}x^3z^3+\&c.\right\}$;

and, multiplying by $1+x^{j+1}z$ and equating coefficients,

$$x^{j+1}+X_{1,j}=(1+X_{1,j})x,\quad x^{j+1}X_{1,j}+X_{2,j}=(X_{1,j}+X_{2,j})x^2,\ \&c.,$$

$$x^{j+1}X_{i-1,j}+X_{i,j}=(X_{i-1,j}+X_{i,j})x^i,\ \text{so that } X_{1,j}=\frac{x(1-x^j)}{1-x}\ \&c.,$$

$$X_{i,j}=\frac{x^i(1-x^{j-i+1})}{1-x^i}X_{i-1,j}=\frac{x^{\frac{i(i+1)}{2}}(1-x^j)(1-x^{j-1})\ldots(1-x^{j-i+1})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$=(1-x^j)(1-x^{j-1})\ldots(1-x^{j-i+1})X_i.$$

2. Evidently $\qquad X_1=x+x^2+x^3+x^4+\&c.=\dfrac{x}{1-x}$,

and $\qquad X_2=x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+\&c.$

$$=x^3(1+x)(1+2x^2+3x^4+4x^6+\&c.)=\frac{x^3(1+x)}{(1-x^2)^2}=\frac{x^3}{(1-x)(1-x^2)}.$$

Now $X_3=(x^3+x^6+x^9+\&c.)X_2$; for, if ν_n be the coefficient of x^n in X_3 and μ_n that in X_2, ν_n is made up of μ_{n-3}, the number of solutions of $p+q+r=n$ in which one value is 1 (p, q, and r being all unequal integers); of μ_{n-6}, the number of solutions in which one value is 2 and none 1, and so on. Therefore $X_3=\dfrac{x^3}{1-x^3}X_2=\dfrac{x^6}{(1-x)(1-x^2)(1-x^3)}$, and a repetition of the same argument leads to the result above stated.

7377. (By Professor SYLVESTER, F.R.S.)—Integrate the equation in differences $\qquad u_{n+1}=u_n+n(n-1)u_{n-1}+(2n-1)\omega_n$, where ω_n denotes the product of n terms of the fluctuating progression

$$1, 1, 3, 3, 5, 5, 7 \ldots\ldots$$

Solution by W. J. C. SHARP, M.A.

The equation $u_{n+1} = u_n + n(n-1)\,u_{n-1}$ is remarkable, as, though it is of the second order, when solved by successive substitution it only involves one constant. The solution is $u_n = \omega_n a$. This is easily verified as follows:—

Let $n = 2p-1$, then $\omega_n a + n(n-1)\,\omega_{n-1} a = 1\,.\,1\,.\,3 \ldots 2p-3\,.\,2p-1\,.\,a$

$$+\,2(p-1)(2p-1)\,.\,1\,.\,1 \ldots 2p-3\,.\,2p-3\,.\,a$$

$$= 1\,.\,1\,.\,3\,.\,3 \ldots 2p-3\,.\,2p-1\,.\,2p-1\,a = \omega_{n+1}\,a.$$

Let $n = 2p$, then $\omega_n a + n(n-1)\,\omega_{n-1} a = 1\,.\,1\,.\,3 \ldots 2p-1\,.\,2p-1\,.\,a$

$$+\,2p(2p-1)\,.\,1\,.\,1\,.\,3 \ldots 2p-1\,.\,a$$

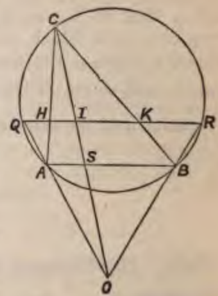
$$= 1\,.\,1\,.\,3\,.\,3 \ldots 2p-1\,.\,2p-1\,.\,2p+1\,.\,a = \omega_{n+1}\,a.$$

The solution of the given equation may therefore be put in the form $u_n = \omega_n a + v_n$, where v_n is the value of u_n obtained by putting $a = u_1 = 0$. Then $v_1 = 0$, $v_2 = 1$, $v_3 = 4$, $v_4 = 25$, $v_5 = 136$, $v_6 = 1041$, $v_7 = 7596$, &c. I have been unable to find a functional expression for v_n.

[The reduction in the number of constants only applies when n, &c. are integers, and seems to be due to the fact that, for such values, two successive equations, $(n = 1)\ u_2 = u_1$, $(n = 0)\ u_1 = u_2$, are of the first order.]

7544. (By the EDITOR.)—Construct a triangle, having given the base, the vertical angle, and the ratio of the segments of a given chord of the circumscribed circle drawn parallel to the base, cut off between the circle and the sides of the triangle.

Solution by MARGARET T. MEYER; D. BIDDLE; *and others.*



On the given base AB construct a segment of a circle containing an angle equal to the given vertical angle; and let QR, parallel to AB, be the given chord. Produce the chords QA, RB, to meet in O; divide AB in the given ratio at S, and through S draw OC, cutting the given chord in I, and the circumference in C. Then ABC will be the required triangle. For, let AC, CB meet QR in H and K; then, since QR is parallel to AB, we have

AS : SB = QI : IR = HI : IK = QH : KR.

7536. (By Professor SYLVESTER, F.R.S.)—If $3n-2$ points are given on a cubic curve, and through $3n-3\nu-2$ of these an $(n-\nu)$-ic be drawn, cutting the cubic in two additional points, and through these and the remaining 3ν given points a third curve of order $\nu+1$ be drawn, prove that its remaining intersection with the given cubic is a fixed point.

Solution by W. J. C. SHARP, M.A.

This theorem is a consequence of Professor SYLVESTER's theory of Residuation (*Reprint*, Vol. 34, p. 34). Taking any 3ν points, if A and B be the additional intersections of an $(n-\nu)$-ic through the other $3n-3\nu-2$ points, and C that of the $(\nu+1)$-ic, is through the 3ν points and A and B, the $3n-3\nu-2$ other points are coresidual to C and the original 3ν points. If A′, B′, C′ be corresponding points obtained by taking another $(n-\nu)$-ic through the same $3n-3\nu-2$ points, the 3ν points and C are coresidual to the same 3ν points and C′, and C and C′ denote the same point. This point is the single point coresidual to the original $3n-2$ points, for the $(n+1)$-ic system composed of the first $(n-\nu)$-ic and the corresponding $(\nu+1)$-ic is such that the $3n-2$ points are residual to A, A, B, B, and C. Also the n-ic through the $3n-2$ points and C will meet the cubic in a residual point P, which is, therefore, coresidual to A, A, B, B, and therefore residual to C, C (*i.e.* it is the tangential of C); and the $3n-2$ points are residual to P and C, and therefore coresidual to C, which is therefore the same, however the $3n-2$ points may be taken.

The theorem alluded to by Professor SYLVESTER is given in SALMON's *Higher Curves*, Art. 154, p. 131, and is shown to be a consequence of Professor SYLVESTER's theory of Residuation, Art. 160, p. 135; it is identical with Mr. J. J. WALKER's Quest. 7058.

If $n=3$ and $\nu=0$, the theorem becomes :—"If cubic curves be drawn through seven points on a given cubic, the lines joining the two remaining intersections of any of these with the original cubic will all pass through a fixed point upon it."

If $n=2$ and $\nu=1$, it becomes :—"If four points be given on a cubic and through any one of these a straight line be drawn meeting the cubic in two other points, the conic through these and the other three original points meets the cubic again in a fixed point; and, as a particular case of this, 'If a conic osculate a given cubic at a given point A and touch it at B, it will pass through the single point coresidual to the tangential of B, and three coincident points at A.'" In this way innumerable theorems may be deduced.

7512. (By Professor TOWNSEND, F.R.S.)—An ellipsoid and any inscribed polyhedron of maximum volume, or circumscribed polyhedron of minimum volume, being supposed to bound two solids of uniform density in their common space; show that both solids have the same principal axes at their common centre of inertia.

Solution by the PROPOSER; C. GRAHAM, M.A.; *and others.*

The polyhedron, whether inscribed or circumscribed, being always regular in the particular case of a sphere, therefore, for both solids, $\Sigma(yz\,dm)=0$, $\Sigma(zx\,dm)=0$, $\Sigma(xy\,dm)=0$, for every triad of rectangular planes passing through the common centre, in that particular case. And every triad of such planes for any mass or system of masses, transforming into a triad for which $\Sigma(y'z'\,dm) = 0$, $\Sigma(z'x'\,dm) = 0$, $\Sigma(x'y'\,dm) = 0$, in every transformation for which $x' = \lambda x$, $y' = \mu y$, $z' = \nu z$, where λ, μ, ν are constants; therefore, &c., for the general case.

7230. (By the Editor.)—On a square (A) of a chess-board, a knight is placed at random: find the probability that it can march (1) from that square (A) to a given square (B), as, for example, to one of the corner-squares, within a moves; and (2) over b squares in less than c moves, for instance, over the four corner-squares of the board.

Solution by D. Biddle.

In solving this problem, it is well to remember three things :—(a) That, according to his position, the knight's command of squares varies. When on one of the 16 central squares, he commands 8 squares; when on the 16 which flank the borders of the 4^2 central set, he commands 6 squares; when at the corners of the 6^2 set, he commands 4 squares, and when on any of the 4 middle squares of each side also, he commands 4 squares; when on a border square adjoining the corner on either side, he commands 3 squares; and when on a corner-square, he commands only 2. Consequently there are—

16	squares on which his range is		8,
16	,,	,,	6,
20	,,	,,	4,
8	,,	,,	3,
4	,,	,,	2,

and his average range is $5\frac{1}{4}$.

(b) That, as he moves from his original square 1, 2, 3 moves, his range undergoes a branching process; and his command of squares from one position overlaps that from another, being often partially similar, though never entirely the same. Thus, from either of the 4 central squares there are 8 once removed, but, instead of 8 times 8 twice removed, only 26.

(c) That the chess-board is so far symmetrical that an examination of the knight's progress from 10 out of the 64 squares is sufficient to give us data for the whole board. The 10 squares referred to are those which, roughly speaking, are included within the right-angled triangle whose hypotenuse is half a diagonal of the board starting from either corner. Each of those on the half-diagonal represents 4 similar ones, including itself; each of the other 6 represents 8 similar ones, including itself. Or, dividing the board into quarter blocks of 16 squares, E = B, I = C, N = D, K = G, O = H, and P = M. The only squares that have none corresponding in the same block are A, F, L, Q.

A	E	I	N
B	F	K	O
C	G	L	P
D	H	M	Q

The following table gives a precise account of the number of squares that can be reached by the knight from each of the 16 in 1, 2, 3, &c. moves :—

Original Square.	Reached in 1 move.	Reached in 2 moves.	Reached in 3 moves.	Reached in 4 moves.	Reached in 5 moves.	Reached in 6 moves.
A	8	26	24	5		
B	8	22	24	9		
C	6	20	26	11		
D	4	16	24	15	4	
E (=B)	8	22	24	9		
F	8	19	24	12		
G	6	17	25	14	1	
H	4	14	23	17	5	
I (=C)	6	20	26	11		
K (=G)	6	17	25	14	1	
L	4	13	26	18	2	
M	3	12	23	19	6	
N (=D)	4	16	24	15	4	
O (=H)	4	14	23	17	5	
P (=M)	3	12	23	19	6	
Q	2	9	20	21	10	1
Sum Totals	84	269	384	226	44	1
Average	$5\frac{1}{4}$ out of 63	$16\frac{13}{16}$ out of 63	24 out of 63	$14\frac{1}{8}$ out of 63	$2\frac{3}{4}$ out of 63	$\frac{1}{16}$ out of 63

1. From the foregoing table we find that the probability of (A) and (B), *both* taken at random, being within one move is $\frac{84}{1008} = \frac{1}{12}$; within 2 moves, $\frac{353}{1008}$; within 3 moves, $\frac{737}{1008}$; within 4 moves, $\frac{963}{1008}$; and within

5 moves, $\frac{1007}{1008}$. The opposite extremities of either diagonal are the only positions which take the knight *six* moves to march between. The chance of his being placed on a corner-square is $\frac{4}{64} = \frac{1}{16}$; and the chance of his having to march to the opposite corner, $\frac{1}{63}$. Consequently, $\frac{1}{16 \,.\, 63} = \frac{1}{1008}$ is the probability of his having to take 6 moves in marching from one position to another, and this is the remainder left by $1 - \frac{1007}{1008}$ already found.

But, in the question as stated, (B) is *given*, and (A) alone taken at random. Moreover, one of the corner-squares is specially selected for (B). Now in our table Q is the corner-square, and we are able to state that if $a = 1$ move, the probability is $\frac{2}{63}$; if $a = 2$ moves, $\frac{11}{63}$; if $a = 3$ moves, $\frac{31}{63}$; if $a = 4$ moves, $\frac{52}{63}$; if $a = 5$ moves, $\frac{62}{63}$; and if $a = 6$ moves, then $\frac{63}{63} = 1$, or certainty. The sum of the figures in the table, opposite the letter denoting the square selected for (B) (according to the question), up to and including those in the column devoted to the number of moves selected for a, will always be the numerator, and 63 always the denominator, of the probability required.

2. The second part of the question is more complicated, since it is impossible to tell which of the b squares may be nearest to (A); and the number of moves will vary not only according to the distance of (A) from the series b, but also according to the order in which the b squares are taken, unless, as in the instance given in the question, they are symmetrically placed. We can, however, discover the average moves taken by the knight in crossing from one position to another, both taken at random, by multiplying the sum totals given in the table by the numbers given at the head of the several columns; then taking the sum, and dividing by 1008 (their original grand total). Thus

$$\frac{1}{1008}\,(84 + 538 + 1152 + 904 + 220 + 6) = 2{\cdot}881 \text{ nearly},$$

or $\frac{2904}{1008}$. And this multiplied by b will give the average number of moves taken by the knight in marching from (A) to the series b and through it; because, though $b + 1$ = the number of squares marking the several positions of the knight from (A) onwards, yet the number of intervals $= b$ only. Of course, c must always equal or exceed b.

In the instance given, of the 4 corner-squares, we know that c must equal or exceed 15, to come within the range of probability, because 5 is the lowest number of moves taken by the knight in moving from one corner to another; and we will even allow that (A) may be itself a corner-square. In any case, the distance of (A) from a corner-square never exceeds 3 moves. Consequently, if c be 18 or more, the probability = 1 or certainty; and if c be 14 or under, the probability = 0. There are 16 squares from which the knight can reach one or other corner in 3 moves; 36 from which he takes 2 moves; 8 from which he takes 1 move; and 4 which are corner-squares, and in which the move must be reckoned 0. Therefore, if $c = 17$, the probability $= \frac{3}{4}$, that is $\frac{4 + 8 + 36}{64}$; if $c = 16$, the probability $= \frac{3}{16}$, that is $\frac{4 + 8}{64}$; and if $c = 15$, the probability $= \frac{1}{16}$, or $\frac{4}{64}$. Of course, if (A) cannot under the circumstances be a corner-square, the probabilities for 17, 16, and 15 will be $\frac{11}{15}$, $\frac{2}{15}$, and 0 respectively.

7622. (By SYAMA CHARAN BASU, B.A.)—PSQ is a focal chord of a parabola; tangents PR, QR intersect in R. Show that the third tangent parallel to PSQ bisects RS at right angles.

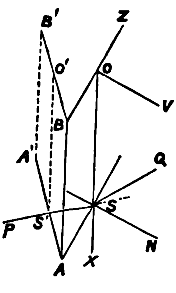
Solution by CHRISTINE LADD FRANKLIN, M.A.; KATE GALE; *and others.*

Any line parallel to PQ is at right angles to RS. That the parallel tangent bisects RS is a reciprocal of the proposition that a chord of a circle is bisected by the diameter perpendicular to it. Thus :—

A line through any point on a circle cuts the circle again in a point which is fourth harmonic to the first point, the point at infinity, and the intersection of the line with the diameter perpendicular to it.	The tangent parallel to a focal chord passes through the fourth harmonic to the point at infinity, the pole of that chord, and the focus.

6053. (By the Rev. A. J. C. ALLEN, B.A.)—A prism filled with fluid is placed with its edge vertical, and a beam of light is passed through an infinitely thin vertical slit, and is incident normally on the prism infinitely near its edge. The emergent beam is received on a vertical screen. If the refractive index of the fluid varies as the depth below a horizontal plane, find the nature and position of the bright curve formed in the screen.

Solution by J. J. WALKER, M.A.



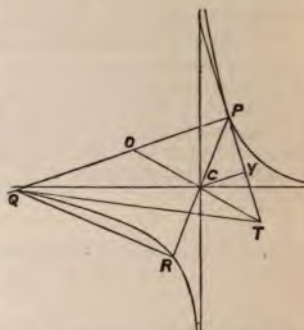
Let AB be the edge of the prism, ABB'A' the face on which a ray of the beam, as PS'S, is incident (at the point S') perpendicularly, emerging from the other face ABOS at the point S. The points of incidence lying on a line S'O' parallel to the edge AB, and consequently the points of emergence in another parallel SO, it is immaterial whether these points be supposed infinitely near the edge or not. Let SQ be the emergent ray, SN being the normal to the second face of the prism. Let YOZ be the horizontal plane to the distance of S from which the refractive index at S is proportional, OY being normal to and OZ lying on the face of the prism. Then, taking OSX, OY, OZ as axes, the equations to SQ are (OS = x') $\quad x = x',\ y \sin \text{QSN} = z \cos \text{QSN}$,

and ($\angle \text{SAS}' = \alpha$) $\quad \sin \text{QSN} = k'x' \sin \alpha = x' / k$, suppose;

so that, eliminating x' and $\angle$ QSN, there results $y^2x^2 = z^2(k^2 - x^2)$, as the equation of the surface generated by the emergent rays, and the bright curve formed on the vertical screen $y = mz + n$ will be a quartic curve.

7543. (By Professor Wolstenholme, M.A., Sc.D.)—In a rectangular hyperbola, PQ is a chord normal at P, and T is its pole: prove that CT will be at right angles to CP; that is, T is the extremity of the polar subtangent drawn from the centre C. [*Otherwise:* if O be the mid-point of PQ, the angle OCP will be a right angle.]

Solution by (1) J. A. Owen, B.Sc.; (2) Margaret T. Meyer.



1. By a known theorem, PQ is equal to the diameter of curvature at P, or

$$PO = \text{radius of curvature at P} = CP^2 / CY;$$

that is, $OP : PC = PC : CY$,

and the angles OPC, PCY are equal; hence the triangles OPC, PCY are similar, and the angle PCO is therefore a right angle; and also the angle PCT.

2. Let TC meet PQ in O, and PC meet the curve again in R; then QO = PO, RC = CP, hence TO is parallel to QR. But, since the hyperbola is rectangular, the angle QRP subtended by PQ at the other extremity of PCR, the diameter through P is equal to the angle between PQ and the tangent at P, that is, since PQ is a normal chord, QR is perpendicular to CP; hence CT is at right angles to CP, or OCP is a right angle.

7545. (By J. J. Walker, M.A., F.R.S.)—Prove that the points on a right line have a (1, 1) correspondence with the rays of a pencil in the same plane; show that the lines drawn from the points so as to make a given angle with their corresponding rays all touch a parabola, which is also touched by the given right line. [A generalisation of a theorem of Steiner's.]

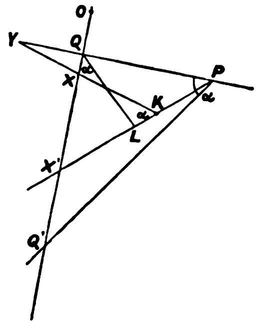
Solutions by (1) Prof. Wolstenholme, Sc.D.; (2) *the* Proposer.

1. If the ray corresponding to any point X on the right line meet the right line in X′, the points X, X′ will have a (1, 1) correspondence, and there will be a definite point O on the right line such that $OX(OX'+h)$ will be constant (c^2), h being a definite constant. Take O for origin and the right line for axis of x; and let (a, b) be the vertex of the pencil; then, if $\frac{x-a}{\cos\theta} = \frac{y-b}{\sin\theta}$ be any ray, and α the given angle, the straight line drawn through the point corresponding

to this ray will have for equation $y = \tan(\theta + \alpha)\left(x - \dfrac{c^2}{a + h - b\cot\theta}\right)$, which involves $\tan\theta$ in the second degree, so that the envelope is a conic; and, since the straight line is altogether at infinity when $a + h = b\cot\theta$, this conic must be a parabola.

The following further generalisation is obvious:—

If the points on a right line have a (1, 1) correspondence with the rays of a pencil, the straight line drawn through any point on the right line so as with its corresponding ray to divide a given segment in a given cross ratio, will envelope a conic, touching the given segment and the given right line.



2. The question may be solved more in the Steinerian manner as follows:—Let Q, Q′ be corresponding points on the right line which subtend at P, the vertex of the pencil, an angle equal to the given angle (α); X, X′ being any other corresponding points: viz., let PQ′, PX′ be the rays corresponding to Q, X. Let XK, making with PX′ the given angle (α), meet PQ in Y; and QL, meeting PX′ in L, make with the given right line the same angle (α). Then we have

$$\text{QY} : \text{QX} = \sin \text{X} : \sin \text{Y}$$

$$= (\text{PL} + \text{LX}')\sin \text{L} : \text{PX}'\sin \text{Q}'\text{PX}'$$

$$= \text{PQ}\sin \text{PQL} + \text{QX}'\sin\alpha : \text{X}'\text{Q}'\sin \text{Q}'$$

$$= \text{PQ}\sin \text{PQL} - \text{OQ}\sin\alpha + \text{OX}'\sin\alpha : \text{OQ}'\sin \text{Q}' - \text{OX}'\sin \text{Q}';$$

O being the point on the given range corresponding to the parallel-ray, and PO′ the ray corresponding to the point of the range at infinity; for which $\text{OX}' . \text{O}'\text{X} = c^2 = \text{OQ}' . \text{O}'\text{Q}$. Hence, multiplying both terms of the latter ratio by O′X, O′Q, and substituting, there results

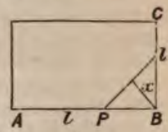
$$\text{QY} : \text{QX} = h . \text{O}'\text{X} + \text{O}'\text{Q}\sin\alpha . \text{QX}\sin \text{Q}',$$

where $c^2h = \text{O}'\text{Q}(\text{PQ}\sin \text{PQL} - \text{OQ}\sin\alpha)$, or $\text{QY}\sin \text{Q}' = h\text{O}'\text{X} + \text{O}'\text{Q}\sin\alpha$, or, if O″ is a point on PQ such that $\text{QO}''\sin \text{Q}' = \text{O}'\text{Q}\sin\alpha$, then $\text{O}''\text{Y} : \text{O}'\text{X}$ is a constant ratio; and consequently XY touches a given parabola, to which O′X and O″Y are also tangents.

[The parabolic envelope manifestly touches the given straight line, and its axis makes with the straight line that joins O to the vertex of the pencil an angle equal to the given angle. CHRISTINE LADD FRANKLIN remarks that, if we revolve the pencil through an angle equal to the given angle, the construction becomes the ordinary projective construction for a parabola,—which is, in fact, one of the two cases given by STEINER, viz., when the lines are drawn parallel to the rays of the pencil or perpendicular,—but he does not seem to have seen his way to the general case, though why it is difficult to conjecture.]

7611. (By B. REYNOLDS, M.A.)—A man, having to pass round the corner of a rectangular ploughed field, strikes across the field diagonally, at 45°, upon nearing the corner, to save time. If his velocity on the beaten path is u, and that on the field is $u-x$, where x is the perpendicular distance of the path chosen from the corner, find (1) where he should leave the beaten path, and (2) what value of x will make either route occupy the same time.

Solution by W. G. Lax, B.A.; C. Morgan, B.A.; *and others.*



1. Let l and l' be the lengths of the sides of the field adjacent to the angle the man cuts across; then

time from A to C $= \dfrac{l+l'-2\sqrt{2}\,.\,x}{u} + \dfrac{2x}{u-x}$,

and, for this to be a minimum, we have

$$\frac{-2\sqrt{2}}{u} + \frac{2(u-x)+2x}{(u-x)^2} = 0, \text{ whence } x = u\left(1-\frac{1}{\sqrt[4]{2}}\right).$$

2. Time round corner B $= \dfrac{l+l'}{u}$, and, for this to be same as across,

$$\frac{l+l'-2\sqrt{2}x}{u} + \frac{2x}{u-x} = \frac{l+l'}{u},\quad \frac{2\sqrt{2}x}{u} - \frac{2x}{u-x} = 0,\quad \text{and } x = u\left(1-\frac{1}{\sqrt{2}}\right).$$

[The Proposer remarks that he is "afraid that the assumed law of diminution of velocity is a very unscientific one, especially because u involves *time*, and x does not. If we make $u = 120$ *yards* per *minute*, the law seems reasonable, but with the same velocity, denoted as 2 yards per second, the law seems ridiculous. It seems also a pity for the diminished velocity $(u-x)$ to have a possibility of becoming zero or negative."]

7391. (By the Editor.)—Find the area of an inscriptible quadrilateral whose sides are roots of the equation $x^4+px^3+qx^2+rx+k=0$, and deduce therefrom a solution of Quest. 7330 (*Reprint*, Vol. 39, p. 111).

Solution by Dr. Curtis; S. Greenidge, M.A.; *and others.*

If $s = \frac{1}{2}(a+b+c+d) = -\frac{1}{2}p$, we have

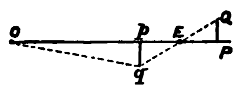
$$\begin{aligned}(\text{Area})^2 &= (s-a)(s-b)(s-c)(s-d),\\ &= s^4-(a+b+c+d)\,s^3+(ab+ac+\&c.)\,s^2-(abc+\&c.)\,s+abcd\\ &= -s^4+qs^2+rs+k = -\frac{p^4}{2^4}+\frac{p^2}{2^2}q-\frac{pr}{2}+k.\end{aligned}$$

When $k=0$, one side of the quadrilateral vanishes, the quadrilateral degenerates into a triangle, and we obtain the result in Quest. 7330.

7601. (By Professor Hudson, M.A.)—The lenses of a common astronomical telescope, whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ of an inch from the object-glass; find the magnifying power, the least distance of distinct vision being taken to be 8 inches.

Solution by B. H. RAU, M.A.; SARAH MARKS; *and others.*

Let F and f be the focal lengths of the object- and eye-glasses of the telescope. Then, by question, $F+f = 8\frac{1}{2}$ in., and $\frac{F}{f} = 16$; therefore $F = 8$ in., $f = \frac{1}{2}$ in. When arranged as a microscope, these glasses are interchanged in order.



Let O, E, PQ, pq, be the centres of the object-glass, eye-glass (of the telescope), the object viewed, and its image, respectively.

Then $Op = F = 8$ in.; $PE = \frac{5}{8}$ in.,

also $\frac{1}{EP}+\frac{1}{Ep} = \frac{1}{f}$; whence $\frac{Ep}{EP} = \frac{\frac{1}{2}}{\frac{5}{8}-\frac{1}{2}} = 4$. Now the magnifying

power $= \frac{pq}{Op} + \frac{PQ}{\text{distance of distinct vision}} = \frac{pq}{Op}\cdot\frac{8}{PQ} = \frac{8}{8}\cdot\frac{pE}{PE} = 4.$

7605. (By J. J. WALKER, M.A., F.R.S.)—Referring to Question 1585, show that (1) the circles drawn on the common chords of three mutually orthotomic circles as diameters have not a common radical axis (as erroneously stated in that Question), but have the same radical centre as those circles; and (2) their common chords are equal to one another, and (3) respectively parallel to the radii of the circle through the centres of the orthotomic triad, drawn to those centres.

Solution by ASÛTOSH MUKHOPÂDHYÂY.

My solution to Quest. 1585, is incorrect; the x-coordinate of C being in reality $-b\cos A\cos B$, so that the first term of equation (3) should be $(x+b\cos A\cos B)^2$. Equation (4) is correct. But, subtracting (3) *as corrected* from (1), we get, not the same equation as (4), but

$$-x\cos A + y\sin A = \tfrac{1}{2}(c\cos C + b\cos B - a\cos A)\ldots\ldots\ldots\ldots(5).$$

Hence, the circles have *not* the same radical axis. Solving for x and y from (4) and (5), we get for the coordinates of the radical centre of the circles (1), (2), (3), $x = 0$, $y = \frac{1}{2}\operatorname{cosec} A\,(c\cos C + b\cos B - a\cos A)$.

But these are well known to be the coordinates of O. Hence, the three circles on the common chords of the orthotomic triad as diameters, *instead of having a common radical axis*, have the same radical centre as the three original mutually orthotomic circles. [This can otherwise be proved by reasoning similar to what is followed in TOWNSEND's *Modern Geometry*, Vol. i., Art. 183, Cor. 3.]

Again, if P be the circumcentre of the triangle ABC, its coordinates are easily seen to be $-\frac{1}{2}a+b\cos C$, $\frac{1}{2}a\cot A$; but C is $(2\cos C, 0)$; hence the line PC is

$$\frac{x-b\cos C}{y} = \frac{\frac{1}{2}a}{-\frac{1}{2}a\cot A}, \text{ or } x\cos A+y\sin A = b\cos C\cos A \text{ ......(6).}$$

This is obviously parallel to (4), which is the radical axis of (1) and (2). Hence, we infer that the radical axes of the circles on the common chords of the mutually orthotomic circles as diameters are parallel to the radii of the circle passing through the centres of the orthogonal triad, drawn to these centres.

Again, we see that (4) and (5) differ only in this, that the intercepts they make on the axis of x are on opposite sides of the origin; hence, it follows, from elementary geometry, that the portions of these lines which form chords of (1) are equal,—as, indeed, can be shown by direct calculation, since $(\text{chord})^2 = 4bc\cos B\cos C-(c\cos C+b\cos B-a\cos A)^2$, in both cases; this interprets that the common chords of the three circles are equal.

7593. (By R. Knowles, B.A., L.C.P.)—A circle passes through the ends of a chord PQ of the parabola $y^2 = 4ax$ and its pole (hk); prove that (1) its equation is $x^2+y^2-\frac{k^2-2a^2}{a}x-\frac{k}{a}(a-h)y+h(2a-h) = 0$; (2) if PQ is perpendicular to the axis, the focus is the centre; (3) if the circle cuts the parabola again in CD, the middle point of the line joining the poles of PQ and CD, with respect to the parabola, is the focus.

Solution by Margaret T. Meyer; B. H. Rau, M.A.; *and others.*

The equation to PQ, the polar of (h, k) with respect to the parabola $y^2 = 4ax$ is $ky-2ax-2ah = 0$. The equation to CD must be of the form $ky+2ax+l = 0$ (since PQ and CD are equally inclined to the axis of the parabola), and that to any conic through P, Q, C, D is

$$y^2-4ax+\lambda(ky-2ax-2ah)(ky+2ax+l) = 0,$$

and if this represents a circle, we have

$$(1+\lambda k^2)+\lambda \,.\, 4a^2 = 0, \text{ whence } \lambda = -(k^2+4a^2)^{-1};$$

hence the equation to the circle is

$$(k^2+4a^2)(y^2-4ax)-k^2y^2+4a^2x^2+2ah(ky+2ax)-l(ky-2ax-2ah) = 0,$$

and $l=2a(2a-h)$, because the circle passes through (hk). Thus the circle becomes $x^2+y^2-\frac{k^2+2a^2}{a}x-\frac{k}{a}(a-h)y+h(2a-h) = 0$. If PQ is perpendicular to the axis, $k=0$, and the centre is the point $(a, 0)$, *i.e.* the focus. The pole of CD, *i.e.* $ky+2ax+2a(2a-h) = 0$, is the point $(2a-h, -k)$. The coordinates of the mid-point of the line joining the poles of PQ and CD with respect to the parabola are $\frac{1}{2}(h+2a-h)$ and $\frac{1}{2}(k-k)$, *i.e.* $(a, 0)$, *i.e.* those of the focus.

7294. (By A. McMurchy, B.A.)—Without knowing the angles of a triangular prism, show that its refractive index can be determined by observing the minimum deviations of rays passing in the neighbourhood of the three angles; and if these deviations be denoted by 2α, 2β, 2γ, then μ is given by

$$\mu^3 - \mu^2(\cos\alpha + \cos\beta + \cos\gamma) + \mu\left[\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta)\right] - \cos(\alpha+\beta+\gamma) = 0.$$

Solution by D. Edwardes; Professor Nash, M.A.; *and others.*

If θ, ϕ, ψ be the angles of the prism, and 2α the minimum deviation at one angle, then it is known that $\sin(\alpha + \frac{1}{2}\theta) = \mu \sin \frac{1}{2}\theta$, and similarly for 2β and 2γ. Also, since $\theta + \phi + \psi = 180°$,

$$\tan\tfrac{1}{2}\theta \tan\tfrac{1}{2}\phi + \tan\tfrac{1}{2}\phi \tan\tfrac{1}{2}\psi + \tan\tfrac{1}{2}\psi \tan\tfrac{1}{2}\theta = 1,$$

and $\tan\frac{1}{2}\theta = \dfrac{\sin\alpha}{\mu - \cos\alpha}$, &c. &c., whence, substituting and reducing, we have the result in the question.

7372. (By R. Russell, B.A.)—Determine $\theta(x)$ and $\phi(x)$ where they are of the form $\dfrac{Ax+B}{Cx+D}$, so that, by putting $y = \theta(x)$ or $\phi(x)$, the quartic $(abcde)(x, 1)^4 = 0$ and its Hessian may turn into the quartic $(abcde)(y, 1)^4 = 0$ and its Hessian.—(*a*) The determination of $\theta(x)$ and $\phi(x)$ depends on the solution of a cubic. (*b*) The roots of the quartic may be represented in the form α, $\theta(\alpha)$, $\phi(\alpha)$, $\theta[\phi(\alpha)]$.

Solution by the Proposer; Professor Matz, M.A.; *and others.*

If X, Y, Z be the quadratic factors of the G-covariant of the quartic, we know that $X^2 + Y^2 + Z^2 \equiv 0$, $\lambda X^2 + \mu Y^2 + \nu Z^2 = U$, and that X, Y, Z are connected harmonically in pairs. Hence (by the Question I have proposed), if we transform $(abcde)(x, y)^4$ by any of the substitutions,

$$\left.\begin{aligned}\xi &= \frac{dX}{dy}\\ \eta &= -\frac{dX}{cx}\end{aligned}\right\}(1),\qquad \left.\begin{aligned}\xi &= \frac{dY}{dy}\\ \eta &= -\frac{dY}{cx}\end{aligned}\right\}(2),\qquad \left.\begin{aligned}\xi &= \frac{dZ}{dy}\\ \eta &= -\frac{dZ}{cx}\end{aligned}\right\}(3),$$

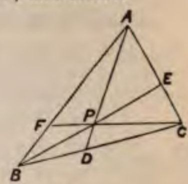
the quadratics X, Y, Z retain exactly their forms, and therefore the transformed quartic is $(abcde)(\xi, \eta)^4$.

7594. (By W. J. C. Sharp, M.A.)—If the circle inscribed in the triangle ABC touch the sides at the points D, E, F respectively, and P be the point of concurrence of the lines AD, BE, CF; and again, if D′, E′, F′, P′ be the corresponding points for the escribed circle opposite A, show that $\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=1, \quad -\frac{P'D'}{AD'}+\frac{P'E'}{BE'}+\frac{P'F'}{CF'}=1$ ...... (1, 2).

[In the second result, the lines are considered as signless magnitudes; if regard were had to the signs, the − should be omitted.]

Solution by Belle Easton; B. H. Rau, M.A.; *and others.*

1. This is true of any three straight lines AD, BE, CF passing through a common point P.



For $\frac{PD}{AD}=\frac{\Delta BPD}{\Delta BAD}=\frac{\Delta CPD}{\Delta CAD}=\frac{\Delta BPC}{\Delta ABC}$,

$$\frac{PE}{BE}=\text{\&c.},\quad \frac{PF}{CF}=\text{\&c.};$$

hence

$$\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=\frac{\Delta BPC+\Delta CPA+\Delta APB}{\Delta ABC}=\frac{\Delta ABC}{\Delta ABC}=1.$$

2. Since P′ is on the side of BC remote from A,

$$-\Delta PBC+\Delta PCA+\Delta PBA=\Delta ABC;$$

hence we have $$-\frac{P'D}{AD}+\frac{P'E}{BE}+\frac{P'F}{CF}=1.$$

7541. (By Professor Wolstenholme, M.A., Sc.D.)—The coordinates of a point being $x=a(m^2+m^{-2})$, $y=a(m-m^{-1})$, where m is the parameter, according to the usual rule the locus should be a quartic, since we get four values of m for determining the points in which the locus meets any proposed straight line. Nevertheless, the locus is the parabola $y^2=a(x-2a)$. Account for the discrepancy. Also, with the same values of (x, y), the equation of the tangent is $m^2x-2m(m^2-1)y+a(m^4-4m^2+1)=0$, which would make the class number 4.

Solution by W. J. C. Sharp, M.A.; R. Knowles, B.A.; *and others.*

The explanation is that $m^2+m^{-2}=(m-m^{-1})^2+2=t^2+2$, suppose, and so the equations may be written $x=a(t^2+2)$, $y=at$, which of course represents a unicursal curve of the second order; also the equation to the tangent may be put in the form $x-2(m-m^{-1})y+a(m^2-4+m^{-2})=0$ or $x-2ty+a(t^2-2)=0$, so that the curve is of the second class. If α be the inclination of the tangent at (x, y) to the axis, $m=-\tan\frac{1}{2}\alpha$ or $\cot\frac{1}{2}\alpha$.

[The proper resultant of the two equations is $(y^2-ax+2a^2)^2=0$, so that the parabola should be considered as doubled, every straight line meeting it in two pairs of coincident points; and every tangent counting as 2. Every point on the curve and every tangent is given twice (for the values k, $-k^{-1}$ of m): this implies that every point of the locus is a node and every tangent a bitangent, which can only happen when a curve is double.

Of course the change of notation adopted above obviates all difficulty. Hence, writing down both resultants as determinants, it follows that

$$\begin{vmatrix} a, & 0, & -x, & 0, & a, & 0, & 0 \\ 0, & a, & 0, & -x, & 0, & a, & 0 \\ 0, & 0, & a, & 0, & -x, & 0, & a \\ a, & -y, & -a, & 0, & 0, & 0, & 0 \\ 0, & a, & -y, & -a, & 0, & 0, & 0 \\ 0, & 0, & a, & -y, & -a, & 0, & 0 \\ 0, & 0, & 0, & a, & -y, & -a, & 0 \\ 0, & 0, & 0, & 0, & a, & -y, & -a \end{vmatrix} = \begin{vmatrix} a, & 0, & 2a-x, & 0 \\ 0, & a, & 0, & 2a-x \\ 0, & a, & -y, & 0 \\ 0, & 0, & a, & -y \end{vmatrix}^2 = a^4(y^2-ax+2a^2)^2].$$

7492. (By W. J. C. Sharp, M.A.)—Show that at an inflexion on the curve $U = 0$, $\begin{vmatrix} u_{11}, & u_{12}, & u_1 \\ u_{12}, & u_{22}, & u_2 \\ u_1, & u_2, & 0 \end{vmatrix} = 0$. [This is an application of the form of the Hessian suggested at the end of the Solution of Question 5762.]

Solution by G. B. Mathews, B.A.; J. O'Regan; *and others.*

At an inflexion we have by Euler's theorem,

$$0 = \begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix} = \begin{vmatrix} u_{11}, & u_{12}, & xu_{11}+yu_{12}+zu_{13} \\ u_{21}, & u_{22}, & xu_{21}+yu_{22}+zu_{23} \\ u_{31}, & u_{32}, & xu_{31}+yu_{32}+zu_{33} \end{vmatrix} = \begin{vmatrix} u_{11}, & u_{12}, & (n-1)u_1 \\ u_{21}, & u_{22}, & (n-1)u_2 \\ u_{31}, & u_{32}, & (n-1)u_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_{11}, & u_{12}, & (n-1)u_1 \\ u_{21}, & u_{22}, & (n-1)u_2 \\ (n-1)u_1, & (n-1)u_2, & (n-1)[xu_1+yu_2+zu_3] \end{vmatrix}$$

similarly

$$= \begin{vmatrix} u_{11}, & u_{12}, & (n-1)u_1 \\ u_{21}, & u_{22}, & (n-1)u_2 \\ (n-1)u_1, & (n-1)u_2, & n(n-1)u \end{vmatrix};$$

or, since $u = 0$ at all points on the curve, the stated result follows.

7523. (By S. Tebay, B.A.)—Find the mean value of the radius of curvature for all points of an ellipse.

Solution by B. H. Rau, M.A.; A. Mukhopâdhyây; *and others.*

The radius of curvature at the point xy of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is $\rho = \frac{a^2}{b}\left(1-e^2\frac{x^2}{a^2}\right)^{\frac{3}{2}}$. Let ϕ be the complement of the eccentric angle of the point xy; then $x = a\sin\phi$, and $y = a\cos\phi$; therefore

$$\rho = \frac{a^2}{b}(1-e^2\sin^2\phi)^{\frac{3}{2}}.$$

The mean value of ρ is $\dfrac{\int \rho\, ds}{\int ds}$; and $\dfrac{ds}{d\phi} = a\,(1-e^2\sin^2\phi)^{\frac{1}{2}}$;

$$\therefore \quad s = 4a\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\phi)^{\frac{1}{2}}\,d\phi = 2\pi a\,(1-\tfrac{1}{4}e^2-\tfrac{3}{64}e^4-\tfrac{5}{256}e^6\ldots),$$

$$\int \rho\, ds = 4\,\frac{a^3}{b}\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\phi)^2\,d\phi = \frac{2\pi a^3}{b}\,(1-e^2+\tfrac{3}{8}e^4).$$

$$\therefore \quad \rho' = \frac{a^2}{b}\cdot\frac{1-e^2+\frac{3}{8}e^4}{1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\ldots} = \frac{a^2}{b}\,(1-\tfrac{3}{4}e^2+\tfrac{15}{64}e^4+\tfrac{11}{256}e^6+\ldots).$$

7587. (By Syama Charan Basu, B.A.)—If

$$\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\left(\frac{b}{c}+\frac{c}{b}\right)+4 = 0,$$

where α, β are the roots of $ax^2+bx^2+c = 0$, show that $\alpha = \beta = 2$.

Solution by Margaret T. Meyer; B. H. Rau, M.A.; *and others.*

Since $\dfrac{b}{c} = -\dfrac{\alpha+\beta}{\alpha\beta}$, therefore we have

$$\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\left(\frac{\alpha+\beta}{\alpha\beta}+\frac{\alpha\beta}{\alpha+\beta}\right) = 4,\quad \left(\frac{\alpha+\beta}{\beta^2}+\frac{\beta^2}{\alpha+\beta}-2\right)+\left(\frac{\alpha+\beta}{\alpha^2}+\frac{\alpha^2}{\alpha+\beta}-2\right) = 0,$$

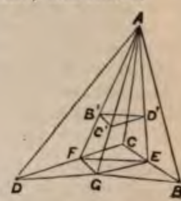
$$\left\{\frac{(\alpha+\beta)^{\frac{1}{2}}}{\beta}-\frac{\beta}{(\alpha+\beta)^{\frac{1}{2}}}\right\}^2+\left\{\frac{(\alpha+\beta)^{\frac{1}{2}}}{\alpha}-\frac{\alpha}{(\alpha+\beta)^{\frac{1}{2}}}\right\}^2 = 0,\ \therefore\ \alpha+\beta = \beta^2 = \alpha^2,$$

and, since $\alpha+\beta$ is not zero, we have $\alpha = \beta = 2$.

7576. (By the Editor.)—Two houses (A, B) stand 750 yards apart on the side of a hill of uniform slope, and at the respective distances of AC = 600 yards and BD = 150 yards from a brook that runs in a straight line CD along the foot of the hill. A man starts from the house A to go to the brook for water, which he is to carry to the house B. Supposing he can only walk 2 miles an hour in going up hill with the water, but 4 miles an hour in going down hill to the brook: show that (1), in order to perform his work in the shortest possible time, he must strike the brook at a point P such that CP = 546·124 yards, the distance he will travel is AP + PB = 811·494 + 159·298 = 970·79 yards, and the time the walking part of his journey will take is 6·916 + 2·715 = 9·631 minutes; also (2), if he start from B to return likewise to A, he will have to take the water at the mid-point (Q) of CD, the length of his return journey will be $450\sqrt{5}$ = 1006·23 yards, the time will be $\frac{1125}{176}\sqrt{5}$ = 14·293 minutes, and the two parts BQ, QA of his path will be perpendicular to each other.

7533. (By J. J. Walker, M.A., F.R.S.)—Prove that the common centre of the surface-mass of the four faces of a tetrahedron is the centre of the sphere inscribed in that determined by the four centres of the faces; and hence prove the obvious analogue in tri-dimensional space of Professor Hudson's Question 7488—which is true in any position of the point O, for forces proportional to OA sin A, OB sin B, OC sin C.

Solution by B. Hanumanta Rau, M.A.; J. O'Regan; *and others.*



Let A′, B′, C′, D′ be the centres of gravity of the faces opposite to the corners A, B, C, D of the tetrahedron ABCD, and let AB′, AC′, AD′ meet CD, DB, BC at their mid-points F, G, E; then, since

$$\frac{AC'}{AG} = \frac{AB'}{AF} = \frac{AD'}{AE} = \tfrac{2}{3},$$

therefore $\quad \Delta B'C'D' = \frac{4}{9}\Delta\, GEF = \frac{1}{9}\Delta\, BCD.$

Hence the mass at $A' = 9\Delta B'C'D'$, and the distance of H, the common centre of surface mass of the faces, from the plane BCD is equal to

$$\frac{\text{mass at } A' \times \text{perpendicular from } A' \text{ on } B'C'D'}{\text{sum of the masses at } A', B', C', D'}$$

$$= \frac{27 \times \text{vol. of tetrahedron } A'B'C'D'}{9 \times \text{sum of the faces}},$$

a symmetrical result. Hence H is the centre of the sphere inscribed in the tetrahedron A′B′C′D′.

[As to the generalization of Question 7488, the Proposer remarks as follows:—Through any point O let forces act along the lines OA′... OD′, proportional to OA′ × B′C′D′... OD′ × A′B′C′, their resultant will act along the line drawn to the common centre (H) of masses placed at A′ ... D′ and equal to the surface-masses B′C′D′... A′B′C′ respectively; and this resultant will be proportional to OH × surface of A′B′C′D′. But this mass-centre has been shown to coincide with the centre of the sphere inscribed in A′B′C′D′.]

7556. (By W. Nicholls, B.A.)—Two cubics U and V have the same points of inflexion. Show that the intersection of the tangent at any point on U and the polar of that point with respect to V lies on U.

Solution by G. B. Mathews, M.A.; Sarah Marks; *and others.*

Let the cubics be $\mathrm{U} \equiv x^3 + y^3 + z^3 + 6lxyz, \quad \mathrm{V} \equiv x^3 + y^3 + z^3 + 6mxyz,$

where (ξ, η, ζ) is any point on U; then the tangent is

$$(\xi^2 + 2l\eta\zeta)\,x + (\eta^2 + 2l\zeta\xi)\,y + (\zeta^2 + 2l\xi\eta)\,z = 0;$$

and the polar line of (ξ, η, ζ) with respect to V is

$$(\xi^2 + 2m\eta\zeta)\,x + \ldots + \ldots = 0;$$

hence the intersection is

$$x+y+z = \begin{vmatrix} \eta^2+2l\zeta\xi, & \zeta^2+2l\xi\eta \\ \eta^2+2m\zeta\xi, & \zeta^2+2m\xi\eta \end{vmatrix} : \ldots : = \xi(\eta^3-\zeta^3)+\eta(\zeta^3-\xi^3)+\zeta(\xi^3-\eta^3),$$

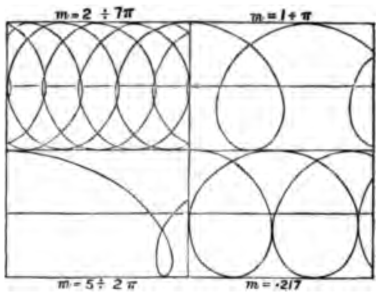
on reduction.

But this point lies on U; it is, in fact. the tangential of the point (ξ, η, ζ).

[See CAYLEY's *Memoir on Cubics*, or SALMON's *Higher Plane Curves.*]

7631. (By the late Professor CLIFFORD, F.R.S.)—A point moves uniformly round a circle while the centre of the circle moves uniformly with less velocity along a straight line in its plane; find the nodes of the curve which the point describes.

Solution by G. HEPPEL, M.A.; G. B. MATHEWS, B.A.; *and others.*



Let the path of the centre be the axis of x, and let the point be always supposed to start from the radius perpendicular to this, which is the axis of y. Then, if m be the ratio of the velocities, and the radius of the circle is taken as unity, the curve is given by $y = \cos\theta$, $x = m\theta + \sin\theta$. Hence the form of the curve depends solely on m. In the limit, when $m = 1$ the curve consists of a series of cycloids. The condition of a node is that $m\theta + \sin\theta = m(2k\pi - \theta) + \sin(2k\pi - \theta)$, or $\sin\theta = m(k\pi - \theta)$.

Now first suppose a line of nodes on the axis of x, then

$$\theta = \tfrac{1}{2}\pi,\ m(k\pi - \tfrac{1}{2}\pi) = 1, \text{ therefore } m = \frac{2}{(2k-1)\pi}.$$

Again, $\dfrac{dy}{dx} = \dfrac{-\sin\theta}{m+\cos\theta}$, and this becomes infinite if $\cos\theta = -m$. Hence, if loops touch at all, they touch below the axis of x, and the conditions of touching are that $\sin\theta = m(k\pi - \theta)$; $\cos\theta = -m$. Solving this approximately, if $k = 1$, $m = \cdot 217$; if $k = 2$, $m = \cdot 129$.

We thus arrive at the following results: $m = 1$, a series of cycloids; between 1 and $\frac{2}{\pi}$, one line of nodes below axis of x; $m = \frac{2}{\pi}$, one line on axis; between $\frac{2}{\pi}$ and ·217, one above; $m = \cdot 217$, one above and loops touching; between ·217 and $\frac{2}{3\pi}$, two below, one above; $m = \frac{2}{3\pi}$, one below, one on axis, and one above; between $\frac{2}{3\pi}$ and ·129, one below, two above; $m = \cdot 129$, one below, two above, and loops touching; between ·129 and $\frac{2}{5\pi}$, three below, two above, and so on.

The figure gives four examples for different values of m.

[Mr. HEPPEL thinks that some of the curves obtained in the way suggested in the Question might be utilized for Art-purposes.]

7597. (By Professor TOWNSEND, F.R.S.)—A system of plane waves, propagated by small parallel and equal rectilinear vibrations, being supposed to traverse in any direction an isotropic elastic solid, under the action of its internal elasticity only; show that the direction of vibration is necessarily either parallel or perpendicular to that of propagation, and determine the velocities of the latter corresponding to the two cases.

Solution by the PROPOSER.

Denoting by ξ, η, ζ the small displacements at any point x, y, z of the solid, by μ and ν its coefficients of resistance to compression and distortion respectively, and by ρ its density supposed approximately constant throughout the motion; the equations of motion of any small disturbance propagated through it by virtue of its internal elasticity only are, as is well known,

$$\rho\frac{d^2\xi}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dx} + \nu\nabla_2\xi,\quad \rho\frac{d^2\eta}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dy} + \nu\nabla_2\eta,$$
$$\rho\frac{d^2\zeta}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dz} + \nu\nabla_2\zeta,$$

where ω is the cubical dilatation $\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}$, and ∇_2 the familiar symbol of operation $\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2+\left(\frac{d}{dz}\right)^2$, at the point x, y, z of the solid.

Supposing now these equations to be satisfied for a system of plane waves propagated as in the question, and represented in consequence by the equations,

$$\xi = k\cos\alpha\,.f(lx+my+nz-vt) = k\cos\alpha\,.f(\phi),$$
$$\eta = k\cos\beta\,.f(\phi),\quad \zeta = k\cos\gamma\,.f(\phi),$$

where α, β, γ are the direction angles and k a small constant representing the absolute magnitude of vibration, l, m, n, and v the direction cosines and the velocity of propagation, and f any arbitrary periodic function oscillating between extreme limits of finite magnitude, and representing in consequence vibratory motion; we get, by substitution in the equations of propagation, the three following equations of connection between the several magnitudes involved, viz.,

$$\rho v^2\cos\alpha = (\mu+\tfrac{1}{3}\nu)\,l\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\alpha,$$
$$\rho v^2\cos\beta = (\mu+\tfrac{1}{3}\nu)\,m\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\beta,$$
$$\rho v^2\cos\gamma = (\mu+\tfrac{1}{3}\nu)\,n\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\gamma,$$

which give α, β, γ and v, when determinable, in terms of l, m, n, μ, ν, and ρ, which are supposed to be all given or known.

By elimination of v^2 between these latter equations in pairs, we get immediately the three following equations of connection between α, β, γ and l, m, n, viz.,

$$(m\cos\gamma-n\cos\beta)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$
$$(n\cos\alpha-l\cos\gamma)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$
$$(l\cos\beta-m\cos\alpha)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$

which show at once that, either

$$\cos\alpha : \cos\beta : \cos\gamma = l : m : n,\quad\text{or}\quad l\cos\alpha+m\cos\beta+n\cos\gamma = 0,$$

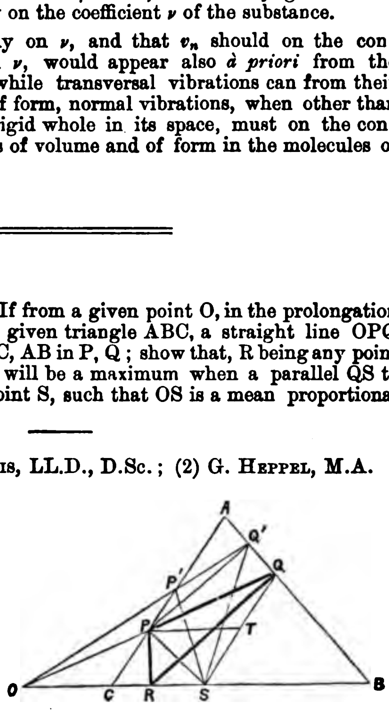
and establish in consequence the first part of the question.

Solving for v^2 in the two cases respectively, and denoting the corresponding velocities by v_n and v_t as corresponding to normal and to transversal vibrations respectively, we get, in answer to the second part of the question, that $(v_n)^2 = \frac{\mu + \frac{4}{3}\nu}{\rho}$ and that $(v_t)^2 = \frac{\nu}{\rho}$; which show that the former depends on the two coefficients μ and ν, and is always greater than the latter, which depends only on the coefficient ν of the substance.

That v_t should depend only on ν, and that v_n should on the contrary depend on both μ and ν, would appear also *à priori* from the obvious consideration that, while transversal vibrations can from their nature produce only change of form, normal vibrations, when other than those of the entire mass as a rigid whole in its space, must on the contrary produce at once changes of volume and of form in the molecules of the substance.

7638. (By the EDITOR.)—If from a given point O, in the prolongation through C of the base BC of a given triangle ABC, a straight line OPQ be drawn, cutting the sides AC, AB in P, Q; show that, R being any point in the base, the triangle PQR will be a maximum when a parallel QS to AC through Q cuts BC in a point S, such that OS is a mean proportional between OB and OC.

Solutions by (1) A. H. CURTIS, LL.D., D.Sc.; (2) G. HEPPEL, M.A.



1. Let S be the point such that $OS^2 = OB \cdot OC$; then, if SQ be drawn parallel to CA, SP will be parallel to BA, since

$$OQ : OP = OS : OC = OB : OS;$$

and, if OP′Q′ be any other cutting line, we have

$$\Delta SQP = SQ'P > SQ'P';$$

hence the triangle PSQ is the maximum of all triangles that have a common vertex at S, their base angles on AC, AB, and their bases passing through O; moreover, $\Delta P'RQ' : P'SQ' =$ distance of R from P′Q′ : distance of S from $P'Q' = OR : OS =$ a known ratio, hence these two triangles are together maxima.

2. Draw PT parallel to CB, and let it meet QS, the parallel through Q to AC, in T; then, putting $OB = b$, $OC = c$, $AC = h$, $OS = x$, the maximum value of ΔPQR depends upon that of $QS \cdot OR - PC \cdot OR$, or on that of $QS - PC$, or QT. Now we have

$$QS = \frac{h(b-x)}{b-c}, \text{ and } QT = \frac{h(b-x)(x-c)}{(b-c)x};$$

hence the maximum value depends upon that of $b+c-\frac{bc}{x}-x$, or of

$$(b^{\frac{1}{2}}-c^{\frac{1}{2}})^2-(b^{\frac{1}{2}}c^{\frac{1}{2}}x^{-\frac{1}{2}}-x^{\frac{1}{2}})^2;$$

and this evidently occurs when $x = b^{\frac{1}{2}}c^{\frac{1}{2}}$, or $x^2 = bc$, that is to say, when OS is a mean proportional between OB and OC.

7644. (By W. S. McCay, M.A.)—Prove that the three lines that join the mid-point of each side of a triangle to the mid-point of the corresponding perpendicular meet in a point.

Solutions by (1) A. H. Curtis, LL.D., D.Sc.; (2) Harold Harley, B.A.

1. The trilinear coordinates of the middle point of the side c, the axes being the sides of the triangle, are $\frac{1}{2}c\sin B$, $\frac{1}{2}c\sin A$, 0, while those of the middle point of the corresponding perpendicular p_3 are $\frac{1}{2}p_3\cos B$, $\frac{1}{2}p_3\cos A$, $\frac{1}{2}p_3$, and the equation of the line joining these points is

$$x\sin A - y\sin B - z\sin(A-B) = 0 \quad \text{.........................(1)},$$

while those of the two corresponding lines are

$$y\sin B - z\sin C - x\sin(B-C) = 0,\ z\sin C - x\sin A - y\sin(C-A) = 0 \ldots(2, 3).$$

If we multiply (1) by sin 2C, (2) by sin 2A, (3) by sin 2B, and add, the coefficients of x, y, z vanish identically; hence the lines meet in a point.

Again, if we add (2) and (3), we obtain $x\sin B - y\sin A = 0$..........(4) as the equation of a line passing through the intersection of (2) and (3), and obviously through the vertex C, while

$$y\sin C - z\sin B = 0 \ \text{ and } \ z\sin A - x\sin C = 0 \ \text{.........} \ (5, 6)$$

are the equations of the two corresponding lines through the common point of (1), (2), (3), and the vertices A and B. The form of the equation (4) shows that the line which it represents makes with the side a the same angle which the bisector of the side c makes with b; and hence it follows that the point of intersection of (1), (2), (3) is the second focus of the ellipse [Brocard's ellipse] inscribed in the triangle, and having for its first focus the point of intersection of the three lines joining the three vertices to the middle points of the opposite sides.

2. Let D, E, F be the mid-points of the sides of the triangle ABC, and G, H, K the mid-points of the perpendiculars from A, B, C on the sides of the triangle DEF; then, since ED is parallel to AB, therefore the right-angled triangles CEK, BFH are similar; hence we have

$$\frac{EK}{FH} = \frac{EC}{BF} = \frac{b}{c}, \quad \frac{FG}{DK} = \frac{c}{a}, \quad \frac{DH}{EG} = \frac{a}{b};$$

therefore DH . FG . EK = DK . EG . FH; whence, by Ceva's theorem, DG, EH, FK meet in a point.

[Mr. Tucker remarks that the theorem in the Question is given in §11, p. 7, of Neuberg's paper "Sur le Centre des Médianes antiparallèles," the point of intersection being what is known as the Point de Grèbe, or Symmedian point, of the triangle,—a fact unknown to the PROPOSER, who obtained the theorem from the three rectangles in Question 7612 having a common centre (the Symmedian point). The property in the question may be more generally enunciated as follows :—"If the mid-points of the portions intercepted on any three concurrent lines from the vertices of a triangle, between these and the opposite sides, be joined to the mid-points of the corresponding sides, the three connectors will pass through the same point," —a theorem which may be proved thus :—If D, E, F be the mid-points of the sides and AG, BH, CK any three concurrent lines meeting the sides in G, H, K, and g, h, k be the mid-points of AG, BH, CK; then g, h, k lie on EF, FD, DE respectively, and $Fg = \frac{1}{2}BG$, $gE = \frac{1}{2}GC$, $Ek = \frac{1}{2}Ak$, $kD = \frac{1}{2}KB$, &c. But BG.CH.AK = GC.HA.KB; hence $Fg.Dh.Ek = gE.hF.kD$, therefore &c.]

4516. (By the late T. COTTERILL, M.A.)—In a spherical triangle, of the five products

$$\cos a \cos A, \quad \cos b \cos B, \quad \cos c \cos C, \quad \cos a \cos b \cos c, \quad -\cos A \cos B \cos C,$$

one is negative, the other four being positive. In the solution of such triangles, what parts must be given that the affections of the remaining three can be determined by this theorem?

Solution by J. J. WALKER, M.A., F.R.S.

(1) If $\cos a$, $\cos b$, $\cos c$ are all positive, and $a > b > c$; then $\cos A$ alone may be negative, since both $\cos b$, $\cos c > \cos a$ and therefore *à fortiori* $> \cos c \cos a$, $\cos b \cos a$. But $-\cos A \cos B \cos C$ is opposite in sign to $\cos A$. Hence either the first or last of the five products alone will be negative.

(2) If $\cos a$ alone is negative, then $\cos B \cos C$ are both positive, but $\cos A$ is negative. Hence of the five products $\cos a \cos b \cos c$ alone will be negative.

(3) If $\cos a$, $\cos b$ are both negative, but $\cos c$ is positive, $a > b$; then $\cos A$ must be, and $\cos B$ may be, negative, $\cos C$ must be positive. Hence, of the five products, either $\cos b \cos B$ or $-\cos A \cos B \cos C$ alone will be negative.

(4) If $\cos a$, $\cos b$, $\cos c$ are all negative, then $\cos A$, $\cos B$, $\cos C$ are all necessarily negative. In this case, of the five products, $\cos a \cos b \cos c$ alone can be negative.

It follows from this theorem that, a, b, c being the given parts, if all are $< \frac{1}{2}\pi$, then one only of the three angles can be $> \frac{1}{2}\pi$; but if a, alone, $> \frac{1}{2}\pi$, then a must be $>$, B, C $< \frac{1}{2}\pi$; if two only of the given parts, as a, b, $> \frac{1}{2}\pi$, then one of the two angles A, B must, both may be, $> \frac{1}{2}\pi$; if all three of the given parts are $> \frac{1}{2}\pi$, then all three of the angles A, B, C must also be $> \frac{1}{2}\pi$.

The same things may be predicated *vice versâ* of angles and sides, save that, if two only of the given angles are obtuse, the opposite sides must also be obtuse, otherwise three of the five products would be negative.

In the other cases of solution, in which the theorem gives some clue to the affections of the parts to be found, the reservations are too numerous to make its application useful. . . . "His saltem accumulem donis. . . ."

7550. (By J. Griffiths, M.A.)—If $t = \frac{7}{2} + \frac{3}{4}\,\mathrm{sn}\,u\,.\,\mathrm{sn}\,(\mathrm{K}-u)$ and modulus $= \frac{1}{2}\sqrt{3}$, $\mathrm{K} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-\frac{3}{4}\sin^2\theta)^{\frac{1}{2}}}$; show that

$$\frac{dt}{[(t-2)\,(t-3)\,(t-4)\,(t-5)]^{\frac{1}{2}}} = du.$$

Solution by G. B. Mathews, B.A.; D. Edwardes; *and others.*

$$t-\tfrac{7}{2} = \tfrac{3}{4}\,\mathrm{sn}\,u\,\mathrm{sn}\,(\mathrm{K}-u) = \tfrac{3}{4}\,\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u},\quad t^2-7t+\tfrac{49}{4} = \tfrac{9}{16}\,\frac{\mathrm{sn}^2 u\,(1-\mathrm{sn}^2 u)}{1-\frac{3}{4}\,\mathrm{sn}^2 u},$$

$$(t-2)(t-5) = t^2-7t+10 = \tfrac{9}{16}\,\frac{s^2\,(1-s^2)}{1-\frac{3}{4}s^2}-\tfrac{9}{4} = -\,\tfrac{9}{16}\,\frac{(2-s^2)^2}{d^2},$$

$$(t-3)(t-4) = t^2-7t+12 = \tfrac{9}{16}\,\frac{s^2\,(1-s^2)}{d^2}-\tfrac{1}{4} = -\,\tfrac{1}{16}\,\frac{(2-3s^2)^2}{d^2},$$

therefore $\quad [(t-2)(t-3)(t-4)(t-5)]^{\frac{1}{2}} = \tfrac{3}{16}\,\dfrac{(2-s^2)(2-3s^2)}{d^2};$

but $\quad \dfrac{dt}{du} = \tfrac{3}{4}\,\dfrac{(\mathrm{cn}^2 u-\mathrm{sn}^2 u)\,\mathrm{dn}^2 u + k^2\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u}{\mathrm{dn}^2 u}$

$$= \tfrac{3}{4}\,\frac{(1-2s^2)(1-\frac{3}{4}s^2)+\frac{3}{4}s^2\,(1-s^2)}{1-\frac{3}{4}s^2}$$

$$= \tfrac{3}{16}\,\frac{4-8s^2+3s^4}{1-\frac{3}{4}s^2} = \tfrac{3}{16}\,\frac{(2-s^2)\,(2-3s^2)}{d^2} = [(t-2)(t-3)(t-4)(t-5)]^{\frac{1}{2}}.$$

Otherwise:—Referring to Mr. Griffiths's paper in the *Proc. Math. Soc.*, Feb. 8th, 1883, putting therein $\alpha=5$, $\beta=4$, $\gamma=2$, $\delta=3$, then $k=\frac{1}{2}\sqrt{3}$, $k'=\frac{1}{2}$, and $\mathrm{cn}\,u_0=0$, therefore $u_0=\mathrm{K}$ mod. $\frac{1}{2}\sqrt{3}$; also $\mathrm{M}=2$ and

$$t = \tfrac{7}{2}-\tfrac{3}{2}\,\mathrm{cn}\,u\,\mathrm{cn}\,(\mathrm{K}-u),$$

or, since $\mathrm{cn}\,u\,\mathrm{cn}\,(\mathrm{K}-u) = k'\,\mathrm{sn}\,u\,\mathrm{sn}\,(\mathrm{K}-u)$, $t = \frac{7}{2}-\frac{3}{4}\,\mathrm{sn}\,u\,\mathrm{sn}\,(\mathrm{K}-u)$,

and $\quad dt \div (t-2\,.\,t-3\,.\,t-4\,.\,t-5)^{\frac{1}{2}} = -du.$

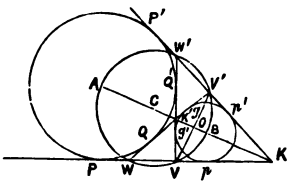
[Or, again, by putting $\alpha=5$, $\beta=4$, $\gamma=3$, $\delta=2$, the formula

$$\frac{dy}{[(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]^{\frac{1}{2}}} = \frac{2}{[(\alpha-\gamma)(\beta-\delta)]^{\frac{1}{2}}}\cdot\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}},$$

$$k^2 = \frac{\beta-\gamma\,.\,\alpha-\delta}{\alpha-\gamma\,.\,\beta-\delta}.\Big]$$

7511. (By Professor WOLSTENHOLME, M.A., Sc.D.)—A, B are the given centres of two circles; Pp, $P'p'$ the external common tangents, Qq, $Q'q'$ the internal common tangents, P, Q being on the same side of the axis; Pp, $Q'q'$ intersect at right angles in V, and $P'p'$, Qq at right angles in V′: prove that (1) P, Q, q', p' lie on one straight line, P′, Q′, q, p on another straight line, whose directions are fixed, and these two straight lines and VV′ meet in one point O; (2) the common tangents Pp, $P'p'$ are equal to the sum of the radii, and Qq, $Q'q'$ to the difference; (3) the points of contact lie on four fixed circles, and the common tangents pass through two fixed points; (4) PQ′, P′Q, pq', $p'q$ all intersect in one fixed point C bisecting AB; (5) PQ, P′Q′, pq, $p'q'$ are all of equal length, and the ratio $Pp' : Qq'$ is the duplicate ratio of $Pp : Qq$; (6) the ratios OP : p'O, OQ : Oq' are equal, and are equal to the ratio of the radii of the two circles; (7) the common tangents and the two straight lines through the eight points of contact all touch the same parabola, focus C, and directrix VV′.

Solution by W. J. C. SHARP, M.A.



Let AB meet Pp and Qq in K and K′ respectively, and let Qq meet Pp in W; then the points P′, p', Q′, q', V′, W′ are the reflexions of P, p, Q, q, V, W with respect to AB, — for the one-half of each circle is the reflexion of the other with respect to the same line; and, if C be the middle point of AB, the circle with centre C, passing through A and B, will also pass through V and W and their reflexions V′ and W′, for VB bisects the angle $q'Vp$, and VA bisects q'VA, and AVB is a right angle; similarly AWB is a right angle. Hence WW′ passes through C, and it and PP′, pp', QQ′, qq', and VV′ are all perpendicular to the line of centres.

Now let VK and VW′ be the positive directions of the rectangular Cartesian coordinates, and a, b, c be the radii of the circles with centres A, B, C, so that $(2c)^2 = (a-b)^2+(a+b)^2 = 2(a^2+b^2)$ or $2c^2 = a^2+b^2$, then the equations to the circles are

$$x^2+y^2+2ax-2ay+a^2 = 0 \equiv A, \quad x^2+y^2-2bx-2by+b^2 = 0 \equiv B,$$

and $$x^2+y^2+(a-b)\,x-(a+b)\,y = 0 \equiv C,$$

and therefore W is the point $-(a-b)$, 0, and W′ is 0, $a+b$, and the polar of W with respect to A is $bx-ay+ab = 0$, which is also that of W′ with respect to B, and $PQq'p'$ is a straight line, and therefore its reflexion $P'Q'qp$, the polars of W with respect to B and of W′ with respect to A, $ax+by-ab = 0$ and these meet on $(a-b)\,y = (a+b)\,x$, the perpendicular VOV′ from the origin on $y-b = -\dfrac{a-b}{a+b}(x-b)$, the line of centres and at the point O where these meet. This proves (1), and (2) follows at once from the values of the coordinates of the points of contact.

Evidently the circle through any two of the points P, Q, q', p', and with its centre in AB, passes through the reflexions of the two points, and there are six such circles, the two given circles being two of them (3). VB cuts the polar of V with respect to B (pq') at right angles, and, as they are the diagonals of the square $VpBq'$, bisects it; then pq' passes through C, and similarly so do PQ', $P'Q$, and $p'q$ (4). Again, $AW = BW = c\sqrt{2}$,

therefore $PQ = 2\dfrac{AP \cdot PW}{WA} = 2\dfrac{ab}{c\sqrt{2}}$ and $pq = 2\dfrac{Bp \cdot Wp}{WB} = 2\dfrac{ab}{c\sqrt{2}}$,

therefore $PQ = pq$, and these are equal to their reflexions $P'Q'$, $p'q'$.

And $\quad OP : p'O = PK : p'K = a : b,$

$\quad OQ : q'O = QK' : K'q = a : b$, which proves (6),

and $\quad Pp' : PO = a+b : a$ and $Qq' : QO = a-b : a,$

therefore $\quad Pp' : Qq' = (a+b)\, PO : (a-b)\, QO,$

and $PO : QO = Pp : Qq$ because the triangles PpO, QqO are similar, therefore $Pp' : Qq' :: Pp^2 : Qq^2$, which completes the proof of (5) and shows that the lines $PQq'p'$ and $pqQ'P'$ cut at right angles, and therefore touch the parabola in (7), as do the common tangents for the same reason.

The property, that the circle on the line of centres passes through the four points of intersection of the common tangents to two circles, which are not centres of similitude, is true at whatever angle the common tangents intersect.

The middle points of $W'V'$, $W'V$, WV', WV all lie on the radical axis of the two circles, which is the tangent at the vertex of the parabola.

C is the centre of the circles $PP'p'p$ and $QQ'qq'$, two of the four circles, since WV and Pp and $W'V'$ and $P'p'$ are bisected in the same points as are VW' and $Q'q'$ and $V'W$ and Qq.

Again, $\quad Cq : Cp' = \frac{1}{2}W'V' : \frac{1}{2}W'V' + V'p' = a-b : a+b :: Qq : Pp,$

and $\quad Cq : Cp' = Cq' : Cp = CQ : CP' = CQ' : CP,$

also VOV' is the polar of C with respect to B;

therefore $\quad BO \cdot BC = b^2$ and therefore $BO = \dfrac{b^2}{c}$,

and $\quad AO \cdot AC = (AB - AO)\, BC = 2c^2 - b^2 = a^2;$

and WW' is the polar of O with respect to A, and O and C are inverse points with respect to both circles; therefore, by Question 7209, the four circles described upon the common tangents all pass through these points, and the circles A and B will reciprocate, about either of these points, *into* confocal conics.

7648. (By D. Biddle.)—A series of isosceles triangles, beginning with the equilateral, is such that each in succession has two-thirds the vertical angle and two-thirds the base of its predecessor. Show that, when the base and vertical angle reach zero, the height of the last in the series is to the height of the first as $2\sqrt{3} : \pi$.

Solution by W. G. Lax, B.A.; Sarah Marks; *and others.*

Let $2a$ be the base of the first triangle, its vertical angle being $\frac{1}{3}\pi$ and height $= a\cot\frac{1}{6}\pi$; then the height of the $(n+1)^{\text{th}}$ triangle of the series $= (\frac{2}{3})^n\, a \cot\left[(\frac{2}{3})^n\, \frac{1}{6}\pi\right]$, and therefore the required ratio is

$$\mathrm{L}^{t}_{n=\infty} \frac{(\frac{2}{3})^n a \cot\left[(\frac{2}{3})^n \frac{1}{6}\pi\right]}{a \cot \frac{1}{6}\pi} = \mathrm{L}^{t}_{n=\infty} \frac{(\frac{2}{3})^n}{\sin\left[(\frac{2}{3})^n \frac{1}{6}\pi\right]} \, \frac{\cos\left[(\frac{2}{3})^n \frac{1}{6}\pi\right]}{\cot \frac{1}{6}\pi}$$

$$= \frac{1}{\sqrt{3}\,\frac{1}{6}\pi}\, \mathrm{L}^{t}_{n=\infty} \frac{(\frac{2}{3})^n \frac{1}{6}\pi}{\sin\left[(\frac{2}{3})^n \frac{1}{6}\pi\right]} = \frac{6}{\pi\sqrt{3}} = \frac{2\sqrt{3}}{\pi}.$$

[The PROPOSER remarks that, if a series of sectors of any circle be taken, with angles similarly diminishing to zero from 60°, the arcs will bear the same ratio to one another that the bases of the triangles in the question do; so that, if we suppose the height of the last of the series of triangles to correspond with the radius of the circle, $= 1$, the base of the first in the series will be $\frac{1}{3}\pi$ and its height $\frac{1}{6}\pi\sqrt{3}$; thus the ratio is

$$1 : \tfrac{1}{6}\pi\sqrt{3} = 2\sqrt{3} : \pi.]$$

7635. (By Professeur ANGELOT.)—Démontrer que

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{8} + \tan^{-1}\tfrac{1}{18} + \ldots + \tan^{-1}\frac{1}{2n^2} + \ldots \textit{ ad. inf.} = \tfrac{1}{4}\pi.$$

Solution by R. KNOWLES, B.A., L.C.P.; J. O'REGAN; *and others.*

It is easy to prove that the respective sums of 2, 3, 4 ... r terms are

$$\tan^{-1}\tfrac{2}{3},\ \tan^{-1}\tfrac{3}{4},\ \tan^{-1}\tfrac{20}{21}, \ldots \tan^{-1}\frac{r}{r+1};$$

hence the sum to infinity $= \tan^{-1} 1 = \frac{1}{4}\pi$.

7628. (By R. KNOWLES, B.A., L.C.P.)—If a, b, c represent the sides of a triangle, and $s_1 = s - a$, &c., prove that

$$bc - s_1^2 = ac - s_2^2 = ab - s_3^2 = r\,(r_1 + r_2 + r_3).$$

Solution by CHRISTINE L. FRANKLIN, B.A.; *and* W. J. GREENSTREET, B.A.

From $r = s_2 \tan \frac{1}{2}\mathrm{B}$, $r_1 = s_3 \cot \frac{1}{2}\mathrm{B}$, &c., we have

$$r\,(r_1 + r_2 + r_3) = s_2 s_3 + s_3 s_1 + s_1 s_2 = s_2 s_3 + a s_1$$
$$= \tfrac{1}{2}(ab + bc + ca) - \tfrac{1}{4}(a^2 + b^2 + c^2) = bc - s_1^2 = \&c.$$

7623. (By the EDITOR.)—If a knight is placed in a given square on a chess-board, show (1) how to move it 63 times, so that it may not occupy any square twice; and (2) how to solve the same problem when the number of squares is 49 or 81.

I. *Solution by* M. JENKINS, M.A.

In the problem of the knight's move I propose to show how to correct an imperfect arrangement of the moves by a method which I have never found to fail in any example where the side of the square has more than 5 places.

I have been told that the automaton at the Crystal Palace would, off-hand and very quickly, move the knight in the required manner, starting from any square which the spectator chose.

This would seem to indicate that the automaton had a simple rule for avoiding a false move.

The nearest approach I can make to such a rule in the case of the ordinary chess-board is "Start from a corner, and keep to the outside, not going into a corner unless the inlet and outlet are both unoccupied." This would give us 48 of the moves in the annexed square (Fig. 1), which shows an unbroken chain of moves. Since the 64 is a knight's move from 1, if we could commit the order of the numbers in Fig. 1 to memory, we could imitate the automaton. There is a defect in the tactical rule as stated, since it would lead us to go from 48 into 53 rather than into 49; but it would guide us fairly well from 49 to 64, if, for the purpose of ascertaining the outside, we suppose the rows and columns which have been completely filled to be cut off.

I will now explain the method I have referred to, which will help us where the imperfect rule fails.

Fig. 2, taken from the *Illustrated London News*, shows a broken chain of moves, which may be divided into 2 endless chains, viz., from 1 to 32, and from 33 to 64.

If we call two squares which are a knight's move from each other a link, there is a link in one of the two chains which may be connected with a link in the other chain, viz., 26, 27 with 59, 60. If therefore we pass from one chain to the other by means of the links, following the figures in the order indicated by the arrows appended to the circles accompanying Fig. 2, we shall obtain a single endless chain (Fig. 3).

For the square whose side has 6 places, if we start from a corner and keep to the outside, subject to the corner rule, we shall be able to move 34 moves without hindrance; the two squares which are left happen to be in connexion with the first corner. Joining them on and moving the figures two places backwards, we obtain a single broken chain, which can be divided into two endless chains, and then converted into a single endless chain (Fig. 4) just as in the previous example.

If the first trial is a bad one consisting of several chains broken or endless, with several detached single squares, I have found no difficulty in reducing these down to a single endless chain in the case of a square whose side has an even number of places >4, or to a single broken chain in the case of a square whose side has an odd number of places >5.

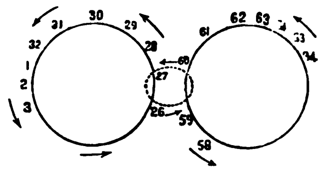
A knight could not move in a single endless chain in a square whose side has an odd number of places, because on a chequered board the colour of the square changes at each move. In a square of 49 places, the 49 would be of the same colour as 1, and could not therefore be a knight's move from it.

For the square of 49 places, starting from a corner as before, I get the square (Fig. 5) filled up with a broken chain 1 to 45, a link 46, 47, and another link 48, 49. The link 46, 47 can be connected with the link 32, 33, and the link 48, 49 with the link 36, 37, whence we obtain the single broken chain (Fig. 6).

For the square of 81 places, I first obtain Fig. 7, containing 3 broken chains, viz., from 1 to 73, from 74 to 77, and from 78 to 81. The extremities 81, 78 of the third chain may be connected with the first chain by means of the link 16, 17, and the second chain with the first chain by means of the link 69, 70, thus obtaining Fig. 8, which is a single unbroken chain.

26	23	38	9	36	21	48	7
39	10	25	22	53	8	35	20
24	27	12	37	60	49	6	47
11	40	63	52	57	54	19	34
28	13	56	59	50	61	46	5
41	64	51	62	55	58	33	18
14	29	2	43	16	31	4	45
1	42	15	30	3	44	17	32

FIG. 1.



With Figs. 2 and 3.

48	57	22	27	50	55	20	29
23	26	49	56	21	28	51	54
58	47	24	17	60	53	30	19
25	16	59	46	31	18	61	52
36	45	2	15	62	43	32	9
3	14	37	44	1	8	63	42
38	35	12	5	40	33	10	7
13	4	39	34	11	6	41	64

FIG. 2.

38	29	22	59	36	31	20	61
23	26	37	30	21	60	35	32
28	39	24	17	58	33	62	19
25	16	27	40	63	18	57	34
50	41	2	15	56	43	64	9
3	14	49	42	1	8	55	44
48	51	12	5	46	53	10	7
13	4	47	52	11	6	45	54

FIG. 3.

18	15	20	29	36	13
21	30	17	14	7	28
16	19	22	1	12	35
23	2	31	8	27	6
32	9	4	25	34	11
3	24	33	10	5	26

FIG. 4.

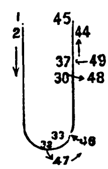
9	30	19	38	7	28	17
20	39	8	29	18	37	6
31	10	43	49	47	16	27
40	21	32	45	42	5	36
11	44	41	46	48	26	15
22	33	2	13	24	35	4
1	12	23	34	3	14	25

FIG. 5.

9	30	19	42	7	28	17
20	43	8	29	18	41	6
31	10	47	40	33	16	27
44	21	32	49	46	5	38
11	48	45	34	39	26	15
22	35	2	13	24	37	4
1	12	23	36	3	14	25

FIG. 6.

With Figs. 5 and 6.

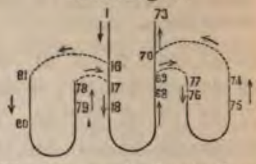


The loop marks the starting-point of a chain.

13	28	41	54	11	26	39	52	9
42	55	12	27	40	53	10	25	38
29	14	57	68	81	75	63	8	51
56	43	16	**74**	64	73	80	37	24
15	30	67	58	69	62	76	50	7
44	17	70	65	60	79	72	23	36
31	66	59	**78**	71	77	61	6	49
18	45	2	33	20	47	4	35	22
1	32	19	46	3	34	21	48	5

Fig. 7.

With Fig. 7.



The loop marks the starting-point of a chain.

13	32	45	58	11	30	43	56	9
46	59	12	31	44	57	10	29	42
33	14	61	72	17	76	67	8	55
60	47	16	77	68	81	18	41	28
15	34	71	62	73	66	75	54	7
48	21	78	69	64	19	80	27	40
35	70	63	20	79	74	65	6	53
22	49	2	37	24	51	4	39	26
1	36	23	50	3	38	25	52	5

Fig. 8.

II. *Solution by* D. Biddle.

(1) If A represent the *column* in which the knight stands, and B the *row*, then $A \pm 1$, $B \pm 2$, or $A \pm 2$, $B \pm 1$ will represent the position at the next move. The annexed diagram (Fig. 9) shows how on an ordinary chess-board the knight may proceed to every square once, and return to the square from which he started, which may be in any position, the cycle being complete. If we denote the columns by the first figures and the rows by the second figures in a series of numbers, we obtain the following :—

16	45	30	5	18	43	32	7
29	4	17	44	31	6	19	42
46	15	62	59	52	55	8	33
3	28	53	56	61	58	41	20
14	47	60	63	54	51	34	9
27	2	25	12	57	38	21	40
48	13	64	37	50	23	10	35
1	26	49	24	11	36	39	22

Fig. 9.

11, 23, 15, 27, 48, 67, 88, 76, 84, 72, 51, 43, 22, 14, 26, 18, 37, 58, 77, 85, 73, 81, 62, 41, 33, 21, 13, 25, 17, 38, 57, 78, 86, 74, 82, 61, 42, 63, 71, 83, 75, 87, 68, 47, 28, 16, 24, 12, 31, 52, 64, 56, 35, 54, 66, 45, 53, 65, 46, 34, 55, 36, 44, 32,

in which each figure up to 8 occurs 8 times as first and 8 times as second, and in which the successive numbers are formed by the addition of $\pm(10 \pm 2)$ or $\pm(20 \pm 1)$. Now $(10+2)$ is added 9 times and deducted 9 times ; $(10-2)$ is added 9 times also and deducted 9 times ; $(20+1)$ is added 8 times and deducted 8 times (if the knight completes the cycle by returning to its original square); and (20 − 1) is added 6 times and deducted 6 times. The figures denoting change of column amount to 46 − 46, those denoting change of row to 50 − 50. The balance is perfect. But, since the arrangement is unsymmetrical, it is evident that a difference in the course can be effected by simply starting from each of the 4 corners in rotation, by taking either of the two directions which lead out of the corners, and also in each of these 8 cases by reversing the course. Thus from one primary arrangement we obtain 16 distinct routes by which the knight can complete the round of the board and return to the square from which he started. We need not here consider the number of primary arrangements that could be made of this kind. But we may point out that to fulfil the requirements of the problem, as regards the ordinary chess-board, it is not necessary to be able to return to the original square. In Fig. 1 it is easy to see that by going forwards from 1 to 27 and then backwards from 64 to 28, we could finish on a remote square and yet traverse the whole board as required. Similarly, we could finish on 12, 48, or 58; and it is not improbable that, by modification of some one of the several primary arrangements (each with its 16 distinct routes), we could begin and end on any two specified squares of different colours.

(2) Where the number of squares on the board is odd, as in the given instances, 49 or 81, a complete cycle seems impracticable; that is, the knight cannot return to the square from which he started. The balance between the outgoing and return moves is necessarily imperfect, where there cannot be an equal number of each. But it is quite possible to comply with the requirements of the problem in regard to a 7^2 board, so far at least as 25 out of the 49 starting-points are concerned. The following diagrams (Figs. 10, 11) give two arrangements from which the 25 tours mentioned can easily be mapped out:—

27	16	5	46	25	14	3
6	47	26	15	4	45	24
17	28	35	40	37	2	13
48	7	38	**1**	34	23	44
29	18	41	36	39	12	33
8	**49**	20	31	10	43	22
19	30	9	42	21	32	11

FIG. 10.

19	4	29	6	21	8	11
28	37	20	39	10	31	22
3	18	5	30	7	12	9
36	**27**	38	17	40	23	32
45	2	47	26	**43**	16	13
48	35	44	41	14	33	24
1	46	**49**	34	25	42	15

FIG. 11.

Routes.	No. of similar starting-points.
1—49 (Fig. 10)	1
1—49 (Fig. 11)	4
49—1 (Fig. 10)	4
49—1 (Fig. 11)	8
27—49, 26—1 (Fig. 11)	4
43—1, 44—49 (Fig. 11)	4
	25

Treating the squares in the two arrangements as we treated Fig. 1, we find that, in Fig. 10, (10 + 2) is added 6 times and deducted 6 times, also (20 + 1) is added 6 times and deducted 6 times; but (10 − 2) is added 7 times and deducted only 5 times, and (20 − 1) is added 5 times and deducted 7 times: balance = −22 = 22 − 44 (the terminal squares in the chain). In Fig. 3 (10 + 2) is added 7 times and deducted 4 times; (10 − 2) is added 10 times and deducted 7 times; (20 + 1) is added 4 times and deducted 5 times; (20 − 1) is added 5 times and deducted 6 times: balance = + 20 = 31 − 11. About this latter there seems no regularity, which leads one to imagine that there is no real reason why the knight should be unable to complete his course from the remaining 24 squares of the 7^2 board. But these tours are certainly attended with greater difficulty.

On turning to the 9^2 board, we find that we can divide it into two portions, a central set of 5^2 squares, and an outer fringe two squares deep, and that these two portions can each be entirely traversed by the knight, without crossing the boundary line between the two, provided he start from a corner square. In Fig. 12, the simplest arrangement is laid down, and this would serve for a great number of distinct routes, with but slight modification. In Fig. 13 the two portions are used conjointly.

13	26	39	52	11	24	37	50	9
40	53	12	25	38	51	10	23	36
27	14	59	76	71	66	61	8	49
54	41	70	65	60	77	72	35	22
15	28	75	58	81	62	67	48	7
42	55	80	69	64	73	78	21	34
29	16	57	74	79	68	63	6	47
56	43	2	31	18	45	4	33	20
1	30	17	44	3	32	19	46	5

FIG. 12.

71	54	27	40	69	52	25	38	67
28	41	70	53	26	39	68	51	24
55	72	63	4	9	14	61	66	37
42	29	10	15	62	65	8	23	50
73	56	5	64	3	60	13	36	81
30	43	16	11	58	7	2	49	22
17	74	57	6	1	12	59	80	35
44	31	76	19	46	33	78	21	48
75	18	45	32	77	20	47	34	79

FIG. 13.

7040. (By Rev. T. R. TERRY, F.R.A.S.)—If p and q be two positive integers such that $p > q$, and if r be any positive integer, or any negative integer numerically greater than p, show that

$$1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{r(r-1)}{(p+r-1)(p+r-2)}-\&c.,$$
$$=\frac{p-q}{p}\cdot\frac{p+r}{p-q+r}.$$

[This identity has been suggested by Professor SYLVESTER's Quest. 6978, but a proof may be given independent of the theorem in that Question.]

Solution by W. J. C. SHARP, M.A.

It is easy to see that the equation holds for all values of q if $r=0$ or 1, and for all values of r if $q=0$ or 1. Suppose it to hold when $q-1$ and $r-1$ are written for q and r.

$$\therefore 1-\frac{q-1}{p-q+2}\cdot\frac{r-1}{p+r-2}+\frac{(q-1)(q-2)}{(p-q+2)(p-q+3)}\cdot\frac{(r-1)(r-2)}{(p+r-2)(p+r-3)}-\&c.$$
$$=\frac{p-q+1}{p}\cdot\frac{p+r-1}{p-q+r},$$

$$\therefore 1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{r(r-1)}{(p+r-1)(p+r-2)}-\&c.$$
$$=1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}\times\frac{p-q+1}{p}\cdot\frac{p+r-1}{p-q+r}=\frac{p-q}{p}\cdot\frac{p+r}{p-q+r};$$

and therefore by induction it holds for all positive values of q and r so long as $q<p$.

Again, if r be negative, $=-s$ say, the formula becomes

$$1-\frac{q}{p-q+1}\cdot\frac{s}{s-p+1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{s(s+1)}{(s-p+1)(s-p+2)}-\&c.$$
$$=\frac{p-q}{p}\cdot\frac{s-p}{s+q-p},$$

which holds for all values of $s>p$ if $q=0$ or $q=1$. Suppose it to hold for $q-1$ and $s+1$, put for q and s; therefore

$$1-\frac{q-1}{p-q+2}\quad\frac{s+1}{s-p+2}+\frac{(q-1)(q-2)}{(p-q+1)(p-q+2)}\cdot\frac{(s+1)(s+2)}{(s-p+2)(s-p+3)}-\&c.$$

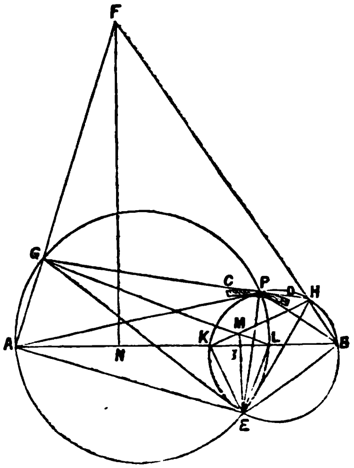
therefore
$$=\frac{p-q+1}{p}\cdot\frac{s-p+1}{s+q-p};$$

$$1-\frac{q}{(p-q+1)}\cdot\frac{s}{s-p+1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\quad\frac{s(s+1)}{(s-p+1)(s-p+2)}-\&c.$$
$$=1-\frac{q}{p-q+1}\cdot\frac{s}{s-p+1}\times\frac{p-q+1}{p}\cdot\frac{s-p+1}{s+q-p}=\frac{p-q}{p}\cdot\frac{s-p}{s+q-p},$$

and the formula holds for all values of $s>p$; therefore, &c.

6878, 7422, 7653. (By B. H. RAU, M.A.)—Given a concave spherical mirror, a luminous point, and the position of an eye perceiving one of the reflected rays; find the point of incidence and reflection on the mirror.

Solution by D. Biddle; Belle Easton; *and others.*



Let A be the luminous point, B the point through which the reflected ray is to pass, CD the concave spherical mirror, and E the centre of its curve. Then ∠APE = BPE, and AE, BE are similar chords of the circles EPA, EPB of which EP is a common chord. Draw AF, BF at right angles to AE, BE, and GH tangential to the reflecting surface of the mirror at P. Then, since EPG, EAG in the one circle, and EPH, EBH in the other circle, are right angles, the centres of the circles must be at the mid-points of EG, EH. Moreover, since AE, EB are similar chords, the diameters of the circles must bear the same ratio,

EG : EH = AE : BE,

and the complementary chords also must bear the same ratio, AG : BH = AE : BE.

We also have, given, ∠AEB; and AB, with the perpendiculars drawn to it, EI, FN; also AN and BN. Moreover, ∠ALE = APE, and ∠BKE = BPE. Consequently, EKL is an isosceles triangle and EI bisects KL. Again, ELG and EKH are right angles; and ∠EGL = EAB, and ∠EHK = EBA. Wherefore, the triangles ELG, EKH are similar to EIA and EIB, the sides being severally as EL : EI; and EM, the prolongation of EI, cuts off equal portions of each, and joins the apices of two triangles EGH, MGH, which have the same base GH, the height of one, EGH, being the radius EP of the curve of the reflecting surface. Now EI . EM = EL². Consequently, if EM be found, and a circle be drawn on EM as diameter, L and K will be its points of intersection with AB. Then G and H can be readily found, and GH will touch the reflecting surface in P, the required point.

Let EM = x, then, since ∠IEL = AEG, therefore we have

$\text{EA} : \text{EG} = \text{EI} : \text{EL} = \text{EL} : x$ and $\text{EA}^2 : \text{EA}^2 + \text{AG}^2 = \text{EI}.x : x^2 = \text{EI} : x$,

hence
$$= \frac{\text{EI}\,(\text{EA}^2 + \text{AG}^2)}{\text{EA}^2};$$

thus, putting AG = y, EA = 1, EB = a, then, by drawing perpendiculars to AB from G and H, we find

$$\text{GH}^2 = \left\{ \text{AB} - \left(\frac{\text{AN}}{\text{AF}} + \frac{\text{BN}a}{\text{BF}}\right) y \right\}^2 + \left(\frac{\text{FN}}{\text{AF}} - \frac{\text{FN}a}{\text{BF}}\right)^2 y^2.$$

Let
$$f = \frac{\text{AN}}{\text{AF}} + \frac{\text{BN}a}{\text{BF}}, \text{ and } g = \frac{\text{FN}}{\text{AF}} - \frac{\text{FN}a}{\text{BF}};$$

then $\text{GH}^2 = (\text{AB} - fy)^2 + g^2y^2$. Moreover, $\text{EG}^2 = 1 + y^2$ and $\text{EH}^2 = (1 + y)^2a^2$. Now, the area of the triangle EGH

$$= \tfrac{1}{4}\,(2\,\text{EG}^2\text{EH}^2 + 2\,\text{EH}^2\text{GH}^2 + 2\,\text{GH}^2\text{EG}^2 - \text{GH}^4 - \text{EG}^4 - \text{EH}^4)^{\frac{1}{2}}.$$

But the area of EGH $= \frac{1}{2}$ (GH . EP), also. Therefore

$$GH.EP = \tfrac{1}{2}\left(2EG^2EH^2 + 2EH^2GH^2 + 2GH^2EG^2 - GH^4 - EG^4 - EH^4\right)^{\frac{1}{2}},$$

and
$$4EP^2[(AB-fy)^2+g^2y^2] = 2a^2(1+y^2)^2 + 2a^2(1+y^2)[(AB-fy)^2+g^2y^2]$$
$$+2(1+y^2)[(AB-fy)^2+g^2y^2] - [(AB-fy)^2+g^2y^2]^2 - (1+y^2) - (1+y^2)^2a^4;$$

$$\therefore\ [2(1+a^2)(b^2+g^2)-(1-a^2)^2-(b^2+g^2)^2]\,y^4 - 4ABf[(1+a^2)-(b^2+g^2)]\,y^3$$
$$+2[(1+a^2)(f^2+g^2)-(1-a^2)^2+(1+a^2)AB^2-(3f^2+g^2)AB^2$$
$$-2(f^2+g^2)EP^2]\,y^2$$
$$-4ABf[(1+a^2)-(AB^2+2EP^2)]\,y + \{AB^2[2(1+a^2)-(AB^2+4EP^2)]$$
$$-(1-a^2)^2\} = 0.$$

This equation enables us to find G in any given instance, and then, if we draw a circle on EG as diameter, the reflecting surface is cut in the required point P.

Thus, to give an example, let AE $=1$, EB $=\cdot5429$, AB $=1\cdot4143$, and EP $=\cdot5357$; then AF $=1\cdot5143$, BF $=1\cdot7286$, FN $=1\cdot4429$, AN $=\cdot4429$, and BN $=\cdot9714$, whence $f=\cdot5976$, $g=\cdot4780$, and $a=$ EB $=\cdot5429$. Our equation then yields the following result, after reduction :—

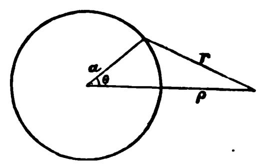
$$y^4 - 3\cdot5458y^3 - \cdot2529y^2 + 6\cdot3971y - 2\cdot3881 = 0;$$

whence $y=\cdot415=$ AG, and BH $=a$AG $=\cdot2253$. Find G and H by these, join GH, and the perpendicular EP will give P.

[A solution by Dr. CURTIS is given on p. 59 of Vol. 39 of *Reprints*.]

7669. (By Professor TOWNSEND, F.R.S.)—A thin uniform spherical shell being supposed to attract, according to the law of the inverse fifth power of the distance, a material particle moving freely in either region of its space external or internal to its mass; if, in either case, the current velocity of the particle be that from infinity under the action of the force, show that its trajectory will be an arc of a circle orthogonal to the surface of the shell.

Solutions by (1) A. H. CURTIS, LL.D., D.Sc.; (2) *the* PROPOSER.



1. If a denote the radius of the sphere, ρ the distance of the centre of the sphere from the attracted point, and r the distance of this point from any particle of the shell, and if V denote the function which, differentiated with regard to x, y, z, will give the corresponding components of attraction,

$$V = -\tfrac{1}{4}\iint \frac{\mu\, ds}{r^4} = -\tfrac{1}{4}\int_0^{\pi} \frac{\mu\, 2\pi a^2 \sin\theta\, d\theta}{[a^2+\rho^2-2a\rho\cos\theta]^2}$$
$$= \frac{\mu\pi a}{4\rho}\left\{\frac{1}{(a-\rho)^2} - \frac{1}{(a+\rho)^2}\right\} = \frac{\mu\pi a^2}{(a^2-\rho^2)^2}.$$

As this expression involves only ρ and constants, it shows, as also appears *à priori*, that the total attraction passes through the centre of the sphere, and for the orbit we must have

$$\frac{h^2}{p^2} = \int F\,d\rho = \int \frac{dV}{d\rho}\,d\rho = V = \frac{\mu\pi a^2}{(a^2-\rho^2)^2},$$

no constant being brought in by integration, as the velocity is that from infinity; or the equation of the orbit may be written $\rho^2 \sim a^2 = kp$. This curve will lie in a plane passing through the centre of the sphere and the line of initial velocity of the particle, and is plainly a circle as radius of curvature $\frac{\rho\,d\rho}{dp} = k$, and, when $p = 0$, $\rho = a$, therefore this circle cuts the section of the sphere made by its plane at right angles.

2. The potential of the attraction, for the law of the inverse fifth power of the distance, of a thin uniform spherical shell of mass m and radius a, at the distance r from its centre, being $= -\frac{1}{4}\frac{m}{(r^2-a^2)^2}$, and the square of the velocity from infinity at the same distance under the action of the force being consequently $= \frac{1}{2}\frac{m}{(r^2-a^2)^2}$; therefore, equating the latter to its equivalent $\frac{h^2}{p^2}$ in the trajectory of the particle, we have, for the relation between the p and r of the trajectory,

$$p^2 = 2\frac{h^2}{m}(r^2-a^2)^2 = k^2(r^2-a^2)^2;$$

which represents, as is well known, a circle, the tangential distance of whose circumference from the centre of the shell $= a$, and the reciprocal of whose diameter $= k$; and therefore, &c., as regards the property.

7404. (By Professor Wolstenholme, M.A.)—In a triangle whose sides are of lengths 57613·67, 50178·48, 34134·03, prove that the inscribed circle passes through the centre of the circumscribed circle and through the orthocentre.

Solution by George Heppel, M.A.

In a triangle of the kind suggested, we must have $IO = IP = r$. Now let $\Sigma(\cos A) = u$, $\Sigma(\cos A \cos B) = v$, $\cos A \cos B \cos C = w$. Then

$$r = R(u-1), \quad IO^2 = R^2(3-2u), \quad IP^2 = 4R^2(1-u+v-2w).$$

Also, since $A+B+C = 180°$, $u^2-2v+2w-1 = 0$. These equations give

$$u = \surd 2, \quad v = \tfrac{1}{4}(5-2\surd 2), \quad w = \tfrac{1}{4}(3-2\surd 2);$$

and $\cos A$, $\cos B$, $\cos C$ are the three roots of

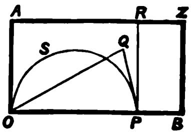
$$x^3 - \surd 2\,x^2 + \tfrac{1}{4}(5-2\surd 2)\,x - \tfrac{1}{4}(3-2\surd 2) = 0.$$

Here $x-\frac{1}{2}$ is a factor, and, solving the remaining quadratic, the three cosines are ·5, ·8080488, ·1061648. The angles are 60°, 36° 5′ 39″·4, 83° 54′ 20″·6. It will be found that the given triangle has its sides in the ratio of the sines of these angles.

[The angles of the triangle are 60°, $60° \pm \alpha$, where $\cos\alpha = \sqrt{2} - \frac{1}{2}$; the radii of the several circles of the triangle are
$R = 28970{\cdot}56$, $r = 12000$, $r_1 = 63789{\cdot}81$, $r_2 = 40970{\cdot}55$, $r_3 = 23121{\cdot}88$, and the distance between the orthocentre and the circumcentre is 23479·64.]

7603. (By the Editor.)—If on a rectangle AOBZ two random points (P, Q) be taken, P on the base OB, and Q on the surface OZ, show, by a general solution, that, OA remaining constant, (1) as OB increases indefinitely from zero to infinity, the probability that the triangle OPQ is acute-angled decreases from $\frac{1}{2}$ to 0; and (2) in the cases when OB = OA, $OB = \frac{3}{2}OA$, OB = 2OA, OB = 4OA, the probability will fall short of $\frac{1}{2}$ by the approximate values $\frac{22}{168}$, $\frac{33}{168}$, $\frac{44}{168}$, $\frac{66}{168}$, respectively.

Solutions by (I.) D. Biddle; (II.) *the* Proposer.



I. The point Q, in order to form with OP an acute-angled triangle, must be between the parallels AO, RP, and outside the semicircle OSP. The average length of $OP = \frac{1}{2}OB$, and the chance of Q being in the variable space AP (OB being fixed) is also $\frac{1}{2}$. Consequently the chance of Q being in position to form with OP an acute-angled triangle is $\frac{1}{2}$, when the space enclosed by the semicircle vanishes as it does when OB has diminished to zero. When OB is infinitely greater than OA, the semicircle absorbs on the average the whole of the space AP, and leaves no room for Q in the requisite position. Consequently the chance is then 0.

The actual chance at any limit of OB and any position of P is represented by the ratio subsisting between that portion of AP not included by the semicircle, and the entire rectangle AOBZ. And when the whole of the semicircle lies within the rectangle at all positions of P, that is, when OB does not exceed 2AO, the mean area of the semicircle can be deducted from the mean area of AP, and the remainder, in the ratio it bears to the entire rectangle, gives the required chance.

In these instances $OA \, . \, OP - \frac{1}{8}\pi \, . \, OP^2$ gives the area of the space in question for any single position of P, and the mean area for all positions of $P = OB \, . \, OA \left(\frac{1}{2} - \frac{1}{3} \cdot \frac{1}{8}\pi \frac{OB}{OA}\right)$, so that the probability of Q being in this space falls short of $\frac{1}{2}$ by $\cdot 1309 = \frac{21{\cdot}99}{168}$ when OB = OA, by $\cdot 19635 = \frac{32{\cdot}9868}{168}$ when $OB = \frac{3}{2}OA$, and by $\cdot 2618 = \frac{43{\cdot}98}{168}$ when OB = 2OA.

In the fourth instance given, viz., when OB = 4OA, the case is materially altered, since the semicircle (on OP when OP > 2OA) extends beyond the boundaries of the rectangle. However, we have already obtained the mean area of the said space whilst the semicircle is within the rectangle, that is, in the present instance, when $OP < \frac{1}{2}OB$. Therefore we have simply to find the area of the space left by the semicircle, when $OP > \frac{1}{2}OB$.

This area, for any single position of P, is $OP . OA - (\frac{1}{8}\pi . OP^2 - \text{segment})$. The segment of the semicircle when $OP = OB = 4OA$ is easily found, since its height $= OA$. The sector is accordingly just $\frac{2}{3}$ of the semicircle and the segment $= \frac{1}{12}\pi . OP^2 - OA\,(OP^2 - 4OA^2)^{\frac{1}{2}}$. From this it diminishes to zero when $OP = \frac{1}{2}OB$. To the same point, the mean area of the rectangle is $\frac{3}{4}OB . OA$, and of the semicircle $\frac{5}{7} . \frac{1}{8}\pi . OB^2$. Consequently, the mean required space for that portion of the series is

$$OB . OA \left(\tfrac{3}{4} - \tfrac{5}{7} . \tfrac{1}{8}\pi . \frac{OB}{OA}\right) + \text{mean segment.}$$

And the mean required space for the other portion of the series will be

$$OB . OA \left(\tfrac{1}{4} - \tfrac{1}{12} . \tfrac{1}{8}\pi . \frac{OB}{OA}\right).$$

The mean between these two will bear the same ratio to $OB . OA$ that the required chance bears to unity.

The mean segment can be found from a series ranging from $\frac{1}{4}$ diameter to zero in height, and each multiplied by $4 / (1-2h)^2$, since we have

$$h = (\tfrac{1}{2}OP - OA) / OP, \quad \text{and} \quad OP^2 : OA^2 = 4OA^2 : OA^2(1-2h)^2.$$

This gives a result of $\cdot 6596\, OA^2$ as the mean of the segments extending beyond the rectangle when $OP > \frac{1}{2}OB$. Hence we have

$$OB . OA \left(\tfrac{3}{4} - \tfrac{5}{7} . \tfrac{1}{8}\pi . \frac{OB}{OA}\right) + \cdot 6596 = \cdot 2876,$$

$$OB . OA \left(\tfrac{1}{4} - \tfrac{1}{12} . \tfrac{1}{8}\pi . \frac{OB}{OA}\right) = \cdot 4764,$$

$$\tfrac{1}{2}(\cdot 2876 + \cdot 4764) = \cdot 382 = \text{mean space required for Q.}$$

Now $4 . \cdot 382 = 168 : 16\cdot 044$; thus the chance is

$$\frac{16\cdot 044}{168} = \frac{1}{2} - \frac{67\cdot 956}{168}.$$

II. *Otherwise* :—The triangle OPQ will be acute-angled, if Q fall anywhere on an area (S, say) contained between the convex circumference of the semicircle OSP, the two tangents OA, PR, and the side AZ of the rectangle. Hence, for every position of P, the probability of an acute-angled triangle will be the ratio of the area (S) to the entire area of the rectangle on which Q must fall. Also the probability of P's falling on any portion of the side OB will be the ratio of that portion to the whole length of OB. Let the breadth OA of the rectangle = unity, its length $OB = 2\lambda$, and $OP = 2x$; then, putting S_1, S_2 for the respective values of S when $x < 1$ and > 1, we have

$$S_1 = 2x - \tfrac{1}{2}\pi x^2, \quad S_2 = 2x - x^2 \operatorname{cosec}^{-1} x - (x^2-1)^{\frac{1}{2}}.$$

Hence, putting p for the probability that the triangle APQ will be acute-angled, and, supposing the length of the rectangle not less than twice its breadth (λ not < 1), we shall have

$$p = \int_0^1 \frac{S_1}{2\lambda}\frac{2dx}{2\lambda} + \int_1^\lambda \frac{S_2}{2\lambda}\frac{2dx}{2\lambda}, \text{ or } 2\lambda^2 p = \int_0^1 S_1\,dx + \int_1^\lambda S_2\,dx \quad \ldots\ldots(\alpha).$$

But, if (2) the length does not exceed twice the breadth (λ not > 1), we have only to consider S_1; and then $2\lambda^2 p = \int_0^\lambda S_1\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$.

Now we readily find $\int S_1\,dx = x^2-\frac{1}{6}\pi x^3$, $\int_0^1 S_1\,dx = 1-\frac{1}{6}\pi$;

$$\int 3S_2\,dx = 3x^2-2x\,(x^2-1)^{\frac{1}{2}}-x^3\operatorname{cosec}^{-1}x+\log_e[x+(x^2-1)^{\frac{1}{2}}];$$

$$\int_1^\lambda S_2\,dx = \lambda^2-1-\tfrac{2}{3}\lambda\,(\lambda^2-1)^{\frac{1}{2}}+\tfrac{1}{6}\pi-\tfrac{1}{3}\lambda^3\operatorname{cosec}^{-1}\lambda+\tfrac{1}{3}\log_e[\lambda+(\lambda^2-1)^{\frac{1}{2}}].$$

Hence, when the length is not less than twice the breadth (λ not >1),

(a) gives $p = \frac{1}{2}-\dfrac{(\lambda^2-1)^{\frac{1}{2}}}{3\lambda}-\frac{1}{6}\lambda\operatorname{cosec}^{-1}\lambda+\dfrac{1}{6\lambda^2}\log_e[\lambda+(\lambda^2-1)^{\frac{1}{2}}]$ ......(a'),

an expression which, by putting $\lambda = \operatorname{cosec}\alpha$, may be written

$$p = \tfrac{1}{2}-\tfrac{1}{3}\cos\alpha-\tfrac{1}{6}\alpha\operatorname{cosec}\alpha+\tfrac{1}{6}\sin^2\alpha\,.\,\log_e\cot\tfrac{1}{2}\alpha \quad ....(\alpha'').$$

When the length is four times the breadth ($\lambda=2$, $\alpha=\frac{1}{6}\pi$), then, from equation (α') or (α''), we have

$$p=\tfrac{1}{6}(3-\sqrt{3})-\tfrac{1}{18}\pi+\tfrac{1}{24}\log_e(2+\sqrt{3}),\text{ or } p = \cdot 09166 = \tfrac{2}{21}\text{ nearly.}$$

When the length is not greater than twice the breadth (λ not >1), equation (β) gives $p = \frac{1}{2}-\frac{1}{12}\lambda\pi$ ....(β').

When the length is equal to twice the breadth ($\lambda=1$), then, from equation (β'), we have $p = \frac{1}{2}-\frac{1}{12}\pi$, or $p = \frac{5}{21}$ nearly.

When the rectangle is a square ($\lambda=\frac{1}{2}$), then we have

$$p = \tfrac{1}{2}-\tfrac{1}{24}\pi,\text{ or } p = \tfrac{31}{84},\text{ nearly.}$$

If we suppose the side OB (or 2λ) to increase without limit, the side OD remaining constant, then p decreases without limit, and becomes zero when λ is infinite; and, as OB decreases without limit, p increases up to $\frac{1}{2}$, which is its limit when λ is zero.

7676. (By J. J. Walker, M.A., F.R.S.)—If $F(xyz) = 0$ is the equation to any surface referred to rectangular axes, show that the equation to the curve in which it is cut by the plane $x\cos\alpha+y\cos\beta+z\cos\gamma=p$, referred to the foot of p as origin, and the line in which the plane is cut by that containing the line p and the axis of z, and a line at right angles thereto, as axes, is obtained by substituting for x, y, z, in $F(xyz) = 0$,

$$p\cos\alpha+(y\cos\beta-z\cos\gamma\cos\alpha)\operatorname{cosec}\gamma,$$

$$p\cos\beta-(y\cos\alpha+z\cos\beta\cos\gamma)\operatorname{cosec}\gamma,\quad p\cos\gamma+z\sin\gamma.$$

[It may readily be verified that these formulæ give, *e.g.*, as the equation to the section of $x^2+y^2+z^2-r^2 = 0$, $y^2+z^2-r^2+p^2 = 0$.]

Solution by W. J. Curran Sharp, M.A.

In a paper "On the Plane Sections of Surfaces, &c.," read before the London Mathematical Society, in December, 1883, I have shown that, if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be any three points, and if

$\frac{\lambda x_1+\mu x_2+\nu x_3}{\lambda+\mu+\nu}$, &c. be substituted for x, &c. in the equation to a surface, the resulting equation in $\lambda\mu\nu$ is the equation, in areal coordinates, to the section of the surface by the plane through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), these being the vertices of the triangle of reference. Mr. WALKER's origin of plane coordinates (A) is the point $(p\cos\alpha,\ p\cos\beta,\ p\cos\gamma)$, and his axes are the lines $\frac{x}{\cos\alpha}=\frac{y}{\cos\beta}=\frac{p-h\cos\gamma}{\sin^2\gamma}=\rho$,

$$\frac{p\cos\alpha-x}{\cos\beta}=\frac{y-p\cos\beta}{\cos\alpha}=\frac{z-p\cos\gamma}{0}=\sigma,$$

and let the two points B and C, which with A determine the plane, be chosen in these lines, in the negative direction, so that for B $\rho=r$, and for C $\sigma=s$. Then AB $=(r-p)\tan\gamma$ and AC $=s\sin\gamma$, also λ, μ, ν being the areal, and η and ζ the rectangular coordinates of any point in the plane, $\lambda:\mu:\nu=2\Delta ABC+AB.\eta+AC.\zeta:-AC.\zeta:-AB.\eta$

$=(r-p)s\tan\gamma\sin\gamma+(r-p)\tan\gamma.\eta+s\sin\gamma.\zeta:-s\sin\gamma.\zeta:-(r-p)\tan\gamma.\eta$

$=(r-p)s\sin\gamma+(r-p)\eta+s\cos\gamma.\zeta:-s\cos\gamma.\zeta:-(r-p)\eta.$

Therefore the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) being

$(p\cos\alpha,\ p\cos\beta,\ p\cos\gamma)$, $(r\cos\alpha,\ r\cos\beta,\ p\sin\gamma-r\tan\gamma\sin\gamma)$,

and $(p\cos\alpha-s\cos\beta,\ p\cos\beta+s\cos\alpha,\ p\cos\gamma)$,

$$x=(\lambda x_1+\mu x_2+\nu x_3)\div(\lambda+\mu+\nu)=\{[(r-p)s\sin\gamma+(r-p)\eta+s\cos\gamma.\zeta]p\cos\alpha$$
$$-s\zeta\cos\gamma.r\cos\alpha-(r-p)\eta(p\cos\alpha-s\cos\beta)\}\div(r-p)s\sin\gamma$$
$$=p\cos\alpha+(\eta\cos\beta-\zeta\cos\alpha\cos\gamma)\operatorname{cosec}\gamma,$$
$$y=(\lambda y_1+\mu y_2+\nu y_3)\div(\lambda+\mu+\nu)$$
$$=\{[(r-p)s\sin\gamma+(r-p)\eta+s\zeta\cos\gamma]p\cos\beta-s\zeta\cos\gamma.r\cos\beta$$
$$-(r-p)\eta(p\cos\beta+s\cos\alpha)\}\div(r-p)s\sin\gamma$$
$$=p\cos\beta-(\eta\cos\alpha+\zeta\cos\beta\cos\gamma)\operatorname{cosec}\gamma,$$
$$z=(\lambda z_1+\mu z_2+\nu z_3)\div(\lambda+\mu+\nu)$$
$$=\{[(r-p)s\sin\gamma+(r-p)\eta+s\zeta\cos\gamma]p\cos\gamma-s\zeta\cos\gamma(p\sin\gamma-\nu\tan\gamma\sin\gamma)$$
$$-(r-p)\eta.p\cos\gamma\}\div(r-p)s\sin\gamma$$
$$=p\cos\gamma+\zeta\sin\gamma;$$

and the metrical properties of the sections of surfaces may be investigated by means of this transformation, as I have attempted, in the paper above mentioned, to study the projective properties by the help of the transformation from which I have derived this.

[A solution may be effected by Quaternions as follows:—

If i, j, k be unit-vectors in the directions of the axes of x, y, z respectively, and i', j' k' others in those of p and the plane-axes of $-z'$ and $-y'$, as taken in the question, then

$$i'=\cos\alpha.i+\cos\beta.j+\cos\gamma.k,\quad k=\cos\gamma.i'-\sin\gamma.j',$$

therefore $j'=\operatorname{cosec}\gamma(\cos\alpha\cos\gamma.i+\cos\beta\cos\gamma.j-\sin^2\gamma.k)$,

therefore $k'=i'j'=\operatorname{cosec}\gamma[-\cos\gamma(\cos^2\alpha+\cos^2\beta-\sin^2\gamma)$

$$+(-\cos\beta\sin^2\gamma-\cos\beta\cos^2\gamma)i+(\cos\alpha\cos^2\gamma+\cos\alpha\sin^2\gamma)j$$
$$+(\cos\alpha\cos\beta\cos\gamma-\cos\alpha\cos\beta\cos\gamma)k]$$
$$=-\cos\beta\operatorname{cosec}\gamma.i+\cos\alpha\operatorname{cosec}\gamma.j.$$

Now, if ρ be the vector from the original origin to any point on the given plane, we have

$$\rho = xi + yj + zk = pi' - z'j' - y'k'$$
$$= [p \cos\alpha + (y' \cos\beta - z' \cos\alpha \cos\gamma) \operatorname{cosec}\gamma]\, i$$
$$+ [p \cos\beta - (y' \cos\alpha + z' \cos\beta \cos\gamma) \operatorname{cosec}\gamma]\, j$$
$$+ [p \cos\gamma + z' \sin\gamma]\, k;$$

whence the relations in the question follow at once.]

7619. (By M. Jenkins, M.A.)—Prove that the coefficient of x^n in $\dfrac{1}{(1-x)(1-x^2)(1-x^3)}$, is $\frac{1}{2}[n + R(\frac{1}{6}n)][1 + E(\frac{1}{6}n)] + E\,\frac{1}{6}[6 - R(\frac{1}{6}n)]$, where $E\left(\dfrac{n}{p}\right)$ is the integral quotient, and $R\left(\dfrac{n}{p}\right)$ the remainder, when n is divided by p.

Solution by the Proposer.

The required coefficient is the number of indefinite partitions of n into 3 parts, say ${}_nP_3$; and by dividing these into groups whose least element is 0, 1, 2, ... respectively, or by dividing the expansion of $\dfrac{1}{(1-x)(1-x^2)}$ by $1-x^3$ synthetically, it may be shown that

$${}_nP_3 = {}_nP_2 + {}_{n-3}P_2 + {}_{n-6}P_2 + \ldots \;\&c.$$

Now ${}_nP_2 = 1 + E(\frac{1}{2}n)$,

$$\therefore\; {}_nP_3 = [1 + E(\tfrac{1}{2}n)] + [1 + E\,\tfrac{1}{2}(n-3)] + \ldots \text{ repeated } 1 + E(\tfrac{1}{3}n) \text{ times}$$
$$= 1 + E(\tfrac{1}{3}n) + [E(\tfrac{1}{2}n) + E\,\tfrac{1}{2}(n-6) + E\,\tfrac{1}{2}(n-12) \text{ to } 1 + E(\tfrac{1}{6}n) \text{ terms}]$$
$$+ [E\,\tfrac{1}{2}(n-3) + E\,\tfrac{1}{2}(n-9) + \ldots \text{ to } 1 + E\,\tfrac{1}{6}(n-3) \text{ terms}];$$

whence, writing q for $E(\frac{1}{6}n)$, r for $R(\frac{1}{6}n)$,.

$${}_nP_3 = 1 + 2q + E(\tfrac{1}{3}r) + [3q + E(\tfrac{1}{2}r) + 3(q-1) + E(\tfrac{1}{2}r) + \ldots 3(0) + E(\tfrac{1}{2}r)]$$
$$+ [3(q-1) + E\,\tfrac{1}{2}(r+3) + 3(q-2) + E\,\tfrac{1}{2}(r+3) + \ldots 3(0) + E\,\tfrac{1}{2}(r+3)$$
$$+ E\,\tfrac{1}{2}(r-3), \text{ the last term being taken only if } r > 3, \text{ whence it may be denoted by } E\,\tfrac{1}{6}(r+1)],$$

$$\therefore\; {}_nP_3 = 1 + 2q + E(\tfrac{1}{3}r) + 3\,\tfrac{1}{2}q(q+1) + (1+q)E(\tfrac{1}{2}r) + 3\,\tfrac{1}{2}q(q-1)$$
$$+ qE\,\tfrac{1}{2}(r+3) + E\,\tfrac{1}{6}(r+1)$$
$$= 1 + 2q + 3q^2 + q[E(\tfrac{1}{2}r) + E\,\tfrac{1}{2}(r+3)] + E(\tfrac{1}{2}r) + E(\tfrac{1}{3}r) + E\,\tfrac{1}{6}(r+1).$$

Now $E(\frac{1}{2}r) + E\,\frac{1}{2}(r+3) = r+1$; and, by trial of all cases from $r = 0$ to $r = 5$, we may substitute $r + E\,\frac{1}{6}(6-r)$ for $1 + E(\frac{1}{2}r) + E(\frac{1}{3}r) + E\,\frac{1}{6}(r+1)$, therefore

$${}_nP_3 = 2q + 3q^2 + q(r+1) + r + E\,\tfrac{1}{6}(6-r)$$
$$= (3q+r)(1+q) + E\,\tfrac{1}{6}(6-r),$$

which proves the theorem, since $3q + r = \frac{1}{2}(6q + 2r) = \frac{1}{2}[n + R(\frac{1}{6}n)]$.

7506. (By S. Tebay, B.A.)—Find (1) the form of a when $x^2 + a$ and $x^2 - a$ are rational squares; also (2) deduce the simple values

$$x = (k-l)^2 + 4l^2, \quad a = 8l(k-3l)(k^2 - l^2);$$

and (3) give a neat method (with examples) of calculating corresponding numerical values of x and a when a only is integral.

Solution by G. B. Mathews, B.A.; *the* Proposer; *and others.*

Let $x^2+a=(x+m)^2$; then $x=\dfrac{a-m^2}{2m}$, and $x^2-a=\dfrac{a^2-6am^2+m^4}{4m^2}$.

Let $a^2-6am^2+m^4=\left(a-\dfrac{k}{l}m^2\right)$; then $m^2=\dfrac{2al(k-3l)}{k^2-l}$.

Take $a=\dfrac{2l(k-3l)}{k^2-l^2}b^2$, b being any arbitrary quantity; then

$$m=\frac{2l(k-3l)}{k^2-l^2}b,\ \text{and}\ \ x=\frac{(k-l)^2+4l^2}{2(k^2-l^2)}b.$$

Take $b=2(k^2-l^2)$, then $x=(k-l)^2+4l^2$, and $a=8(k-3l)(k^2-l^2)$.

Again, let $\dfrac{2l(k-3l)}{k^2-l^2}=\dfrac{ps^2}{qt^2}$ be a fraction in its lowest terms; thus

$$a=pq\left(\frac{s}{qt}b\right)^2.$$

Take $b=\dfrac{qt}{s}$; then $a=pq$, and $x=\dfrac{qt^2-ps^2}{2st}$.

Example I.—Let $k=7$, $l=2$; then $a=\frac{4}{45}b^2=\dfrac{1\,.\,2^2}{5\,.\,3^2}b^2$. Thus $p=1$, $q=5$, $s=2$, $t=3$; therefore $a=pq=5$, and $x=\frac{41}{12}$; thus the two squares are $(\frac{31}{12})^2$, $(\frac{49}{12})^2$.

II.—Let $k=5$, $l=1$; then $a=\frac{1}{6}b^2=\dfrac{1\,.\,1^2}{6\,.\,1^2}b^2$; and we find, as before, $a=6$, $x=\frac{5}{2}$, and the squares $(\frac{1}{2})^2$, $(\frac{7}{2})^2$.

III.—Let $k=9$, $l=1$; then $a=\dfrac{3\,.\,1^2}{5\,.\,2^2}b^2$; as before, $a=15$, $x=\frac{17}{4}$, and the squares $(\frac{7}{4})^2$, $(\frac{21}{4})^2$.

[If we assume $x^2+a=(m+n)^2$, $x^2-a=(m-n)^2$, we have $a=2mn$, $x^2=m^2+n^2=\square$; hence we may take $m=(k^2-l^2)$, $n=2kl$; then

$$x=k^2+l^2,\quad a=4kl(k^2-l^2).$$

If $l=2l'$, $k=(k'-l')$, we have $x=(k'-l')^2+4l'^2$,

$$a=8l'(k'-l')(k'+l')(k'-3l')=8l'(k'-3l')(k'^2-l'^2).$$

As an example, if $x=101=10^2+1^2$, we may take

$$k=10,\ \ l=1,\ \ a=40\,.\,99=3960.$$

See also solution of Quest. 7468, on p. 119 of Vol. 40 of *Reprints*.]

7598. (By Professor Wolstenholme, M.A., Sc.D.)—1. Circles are drawn with their centres on a given ellipse, and touching (α) the major axis, (β) the minor axis; prove that, if $2a$ be the major axis, and e the eccentricity, the whole length of the arc of the curve envelope of these
eccentricity, the whole length of the arc of the curve envelope of these

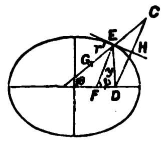
circles is $4a\left(1+\frac{1-e^2}{e}\log\frac{1+e}{1-e}\right)$, $4a\left((1-e^2)^{\frac{1}{2}}+2\frac{\sin^{-1}e}{e}\right)$ ......(α, β).

2. Circles are drawn with their centres on the arc of a given cycloid, and touching (α) the base, (β) the tangent at the vertex; prove that the curve envelope of these circles is (α) an involute of the cycloid which is the envelope of that diameter of the generating circle of the given cycloid which passes through the generating point; (β) a cycloid generated by a circle of radius $\frac{1}{2}a$ rolling on the straight line which is the locus of the centre of the generating circle (radius a) of the given cycloid.

3. Circles are drawn with centres on a given curve and touching the axis of x; prove that the arc of their curve envelope is $x-2\int y\,d\theta$, where x, y are the coordinates of the centre of the circle, and $\frac{dy}{dx}=\tan\theta$.

Solution by A. H. Curtis, LL.D., D.Sc.; Professor Rau, M.A.; *and others.*

The curve, envelope of circles described as in each of the cases included in this question, is by Quetelet's construction (see Salmon's *Higher Curves*) *an* involute of the caustic by reflexion of the curve due to rays perpendicular to the fixed line which the moving circle touches. In (1), (α) and (β), it is required to find the length of a certain involute of the curve which is the caustic of the ellipse due to rays parallel to an axis.



1. Let the figure represent the ellipse, let DE be any incident ray codirectional with the ordinate y, EF the normal and EG the reflected ray, making respectively with the axis major the angles ϕ and θ, and G be the corresponding point on the caustic; if, then, DC be perpendicular to the tangent at E, C is a point on the envelope sought; denoting EG by r', the radius of curvature of the ellipse by γ, and the chord of curvature along EG, or ED, by c, by the figure,

$$\phi+\phi-\theta=\tfrac{1}{2}\pi,\ \text{therefore}\ \tfrac{1}{2}\pi+\theta=2\phi,\ \text{and therefore}\ d\theta=2d\phi,$$

while, from the formula $\frac{1}{r}+\frac{1}{r'}=\frac{4}{c}$, as $r=\infty$, $r'=\frac{1}{4}c$, therefore, with the usual notation $r'=\frac{1}{4}c=\frac{1}{2}\gamma\sin\phi=\frac{b'^3}{2ab}\frac{py'}{b^2}=\frac{b'^2y'}{2b^2}$.

Now, if ds be the element of the locus of C, we have

$$ds=\text{CG}\,d\theta=(r'+y')\,d\theta,\ \text{and}\ s=8\int_0^{\frac{1}{2}\pi}(r'+y')\,d\theta=8\int_0^{\frac{1}{2}\pi}\left(\frac{b'^2}{2b^2}+1\right)y'\,d\phi,$$

or, as $\frac{\sin\phi}{p}=\frac{y'}{b^2}$,

$$s=8\int_0^{\frac{1}{2}\pi}\frac{b'^2+2b^2}{2p}\sin\phi\,d\phi=4\int_0^{\frac{1}{2}\pi}\left(\frac{a^2b^2}{p^3}+\frac{2b^2}{p}\right)\sin\phi\,d\phi$$

$$=4\int_0^{\frac{1}{2}\pi}\frac{a^2b^2\sin\phi\,d\phi}{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{3}{2}}}+8\int_0^{\frac{1}{2}\pi}\frac{b^2\sin\phi\,d\phi}{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}}$$

$$=4a^2b^2\int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}d(\tan^2\phi)}{(a^2+b^2\tan^2\phi)^{\frac{3}{2}}}-8b^2\int_0^{\frac{1}{2}\pi}\frac{d\cos\phi}{[b^2+(a^2-b^2)\cos^2\phi]^{\frac{1}{2}}}$$

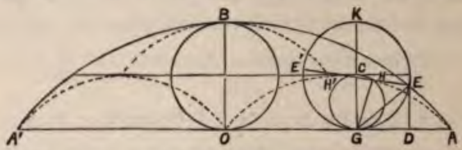
$$=4a^2\left(\frac{1}{a}\right)+4a\left(\frac{1-e^2}{e}\right)\log\left(\frac{1+e}{1-e}\right),\ \text{by the formula}$$

$$\int \frac{dz}{(1+z^2)^{\frac{1}{2}}} = \log\left[z + (1+z^2)^{\frac{1}{2}}\right], \quad \text{or} \quad s = 4a\left[1 + \frac{1-e^2}{e}\log\left(\frac{1+e}{1-e}\right)\right].$$

The discussion of (2) is similar, the difference in the form of the result from case (1) being due to the fact that the formula now employed is

$$\int \frac{dz}{(1-z^2)^{\frac{1}{2}}} = \sin^{-1} z.$$

2. This depends upon the fact that the caustic by reflexion of a cycloid due to rays perpendicular to its base consists of two cycloids as in figure,



each touching the given cycloid, and having for base one half of its base, and that this caustic is the envelope of the diameter referred to. This may be shown geometrically thus :—

Let ABA′ be the given cycloid, DE any incident ray, GEK the generating circle in the corresponding position, and C its centre, draw GH perpendicular to ECE′, and on GC as diameter describe a circle GHCH′, then ∠DEG = ∠CGE = ∠CEG, and therefore CE is the reflected ray and the radius through E of the generating circle in the corresponding position, and arc GH, subtending ∠GCH at circumference of circle GHC = arc GE, subtending same angle at the centre of the circle GEK, therefore = line GA; hence the locus of the point H is a cycloid to which HG is a normal, CE a tangent, and AO the base; thus (2, α) is proved. (2, β) appears from the fact that B is obviously a point on the locus, and it is therefore evident from the figure that the *particular* involute, in this case, is the cycloid stated in the question.

7699. (By R. Knowles, B.A., L.C.P.)—Prove that in any triangle

$$\frac{\cos A}{c \sin B} + \frac{\cos B}{a \sin C} + \frac{\cos C}{b \sin A} = \frac{1}{R} \quad \text{.....................(1).}$$

Solution by Maurice Pontonnier; J. Brill, B.A.; *and others.*

On a $\quad b \sin A = \dfrac{ab}{2R}, \quad c \sin B = \dfrac{bc}{2R}, \quad a \sin C = \dfrac{ac}{2R};$

donc (1) vient $\quad \dfrac{2R \cos A}{bc} + \dfrac{2R \cos B}{ac} + \dfrac{2R \cos C}{ab} = \dfrac{1}{R};$

d'où : $\quad a \cos A + b \cos B + c \cos C = \dfrac{abc}{2R^2};$

mais $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \text{......};$

on a alors : $$-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2 = \frac{a^2b^2c^2}{R^2},$$

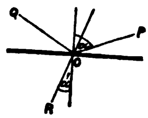
ou $$4b^2c^2\sin^2 A = \frac{a^2b^2c^2}{R^2}, \quad \text{ou} \quad \frac{a}{\sin A} = 2R,$$

formule connue, donc (1) est vérifiée.

$$\left[\frac{\cos A}{c\sin B}+\ldots+\ldots = \frac{a\cos A+\ldots}{2\Delta} = \frac{a\cos A+b\cos B+c\cos C}{a\,.\,R\cos A+b\,.\,R\cos B+c\,.\,R\cos C} = \frac{1}{R}.\right]$$

7602. (By Professor HUDSON, M.A.)—A ray proceeding from a point P, and incident on a plane surface at O, is partly reflected to Q and partly refracted to R: if the angles POQ, POR, QOR be in arithmetical progression, show that the angle of incidence is $\cot^{-1}\left(\frac{\mu-2}{\mu\sqrt{3}}\right)$.

Solution by C. MORGAN, B.A.; J. A. OWEN, B.Sc.; *and others.*



Let α, α' be the angles of incidence and refraction; then $POQ = 2\alpha$, $POR = \pi-\alpha+\alpha'$, $QOR = \pi-\alpha-\alpha'$, and, since these angles are in arithmetical progression, we have $2\alpha' = 3\alpha-\alpha'-\pi$, and $\alpha-\alpha' = \frac{1}{3}\pi$. But, if μ be the index of refraction, $\sin\alpha = \mu\sin\alpha' = \mu\sin(\alpha-60°)$, whence the stated result follows.

4904 & 6884. (By Dr. HART.)—Find the equation of the Cayleyan of the cubic $x^2y+y^2z+z^2x+2mxyz = 0$, and compute the invariants of this cubic.

Solution by W. J. CURRAN SHARP, M.A.

For this form of the equation to a cubic, the Hessian, the discriminant of the polar conic

$$yx'^2+zy'^2+xz'^2+2(mx+y)\,y'z'+2(my+z)\,z'x'+2(mz+x)\,x'y' = 0$$

is $$m^2\left[x^2y+y^2z+z^2x+2mxyz\right]-(x^3+y^3+z^3-3xyz) = 0,$$

and therefore, when this meets the curve,

$$(x+y+z)(x^2+y^2+z^2-yz-zx-xy) = 0,$$

an equation which represents one set of inflexional axes, of which one is real and the other two imaginary, and, since the discriminant $\frac{1}{3}(3^2-6m+4m^2)^2$ of the binary cubic determining the real inflexions is positive, the cubic cannot be cuspidal. In other cases the reduction to this form may be effected by identifying the inflexional axes with the above lines, and the curve is referred to such axes that (y, z), (z, x), (x, y) lie upon the curve, and each is the tangential of the one after it in cyclical order, and therefore its own third tangential. The Cayleyan is the condition that $\alpha x + \beta y + \gamma z = 0$ should cut the conics

$$U_1 \equiv z^2 + 2myz + 2xy, \quad U_2 \equiv x^2 + 2yz + 2mzx, \quad U_3 \equiv y^2 + 2zx + 2mxy,$$

in points in involution, *i.e.*,

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma - 3m(\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha) + 2m^2(\alpha\beta^2 + \beta\gamma^2 + \alpha^2\gamma) - 4m^3\alpha\beta\gamma = 0.$$

The invariants are

$$S \equiv -(m^4 + 3m) \quad \text{and} \quad T \equiv -(8m^6 + 36m^3 + 27)$$

(and therefore the cubic cannot be cuspidal); the discriminant is $27(8m^3 + 27)$. So that, if the cubic be nodal, $m^3 = -\frac{27}{8}$ and T is positive, *i.e.* (*Quarterly Journal*, Vol. xvi., p. 192) it is crunodal.

7427. (By Professor TOWNSEND, F.R.S.)—A lamina, setting out from any arbitrary position and moving in any arbitrary manner, being supposed to return to its original position after any number of complete revolutions in its plane; show that—

(*a*) All systems of points of the lamina which describe curves of equal area in the plane lie on circles fixed in the lamina;

(*b*) All systems of lines of the lamina which envelope curves of equal perimeter in the plane are tangents to circles fixed in the lamina;

(*c*) The two systems of circles, for different values of the area in the former case and of the perimeter in the latter case, are concentric, and have a common centre in the lamina.

Solution by G. B. MATHEWS, B.A.

The motion is determined by the rolling of a curve in the lamina upon a curve in the plane; thus the results at once follow from KEMPE's and McCAY's theorem [given on pp. 82 to 86 and p. 101 of MINCHIN's *Uniplanar Kinematics*].

7574. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If we denote by $F(x, n)$, the determinant of the nth order

$$\begin{vmatrix} x, & 1, & 0, & 0, & 0, & \dots\dots \\ 1, & x, & 1, & 0, & 0, & 0 \;\dots \\ 0, & 1, & x, & 1, & 0, & 0 \;\dots \\ \dots & \dots & \dots & \dots & & \\ 0, & 0, & 0 \dots\dots & 1, & x, & 1 \\ 0, & 0, & 0 \dots\dots & 0, & 1, & x \end{vmatrix}$$

prove that $F(x, 2r+1) \equiv xF(x^2-2, r)$,

$$F(x, 2r) \equiv F(x^2-2, r) + F(x^2-2, r-1),$$

$$F(x, n) \equiv \left(x - 2\cos\frac{\pi}{n+1}\right)\left(x - 2\cos\frac{2\pi}{n+1}\right) \times \left(x - 2\cos\frac{3\pi}{n+1}\right)\dots\left(x - 2\cos\frac{n\pi}{n+1}\right).$$

Solution by B. HANUMANTA RAU, M.A.; Prof. NASH, M.A.; *and others.*

By expanding the determinant, it is easily seen that

$$F(x, n) - xF(x, n-1) + F(x, n-2) = 0 \quad\dots\dots\dots\dots\dots\dots(1)$$

and, therefore, $F(x, n)$ is the coefficient of y^n in the expansion of $(1-xy+y^2)^{-1}$ in ascending powers of y. Let $x = 2\cos\theta$ and $\frac{\pi}{n+1} = a$,

then $$(1-2\cos\theta . y+y^2)^{-1} = (1-ye^{\theta\iota})^{-1}(1-ye^{-\theta\iota})^{-1}$$

$$= \frac{e^{\theta\iota}(1-ye^{-\theta\iota})-e^{-\theta\iota}(1-ye^{\theta\iota})}{(e^{\theta\iota}-e^{-\theta\iota})(1-ye^{\theta\iota})(1-ye^{-\theta\iota})} = \frac{1}{\sin\theta}[\sin\theta+x\sin 2\theta+x^2\sin 3\theta+\ldots],$$

$$\therefore\quad F(x, n) = \frac{\sin(n+1)\theta}{\sin\theta} = 2^n\sin(\theta+a)\sin(\theta+2a)\ldots\sin(\theta+na)\ldots(2).$$

But $$\sin(\theta+a)\sin(\theta+na) = \sin(\theta+a)\sin(a-\theta)$$

$$= \sin^2 a-\sin^2\theta = \cos^2\theta-\cos^2 a = (\cos\theta-\cos a)(\cos\theta-\cos na),$$

and so on. Therefore

$$F(x, n) = 2^n(\cos\theta-\cos a)(\cos\theta-\cos 2a)\ldots(\cos\theta-\cos na)$$

$$= (x-2\cos a)(x-2\cos 2a)\ldots(x-2\cos na).$$

If $n = 2r+1$, then $\cos(r+1)a = \cos\frac{1}{2}\pi = 0$,

and $(x-2\cos a)(x-2\cos na) = (x^2-4\cos^2 a) = (x^2-2-2\cos 2a)$,

$$\therefore F(x, 2r+1) = x(x^2-2-2\cos 2a)(x^2-2-2\cos 4a)\ldots$$

$$= x\left(x^2-2-2\cos\frac{\pi}{r+1}\right)\left(x^2-2-2\cos\frac{2\pi}{n+1}\right)\ldots = xF(x^2-2, r).$$

Again, from (1),

$$F(x, 2r+1)-xF(x, 2r)+F(x, 2r-1) = 0,$$

$$\therefore xF(x, 2r) = F(x, 2r+1)+F(x, 2r-1) = xF(x^2-2, r)+xF(x^2-2, r-1),$$

therefore $$F(x, 2r) = F(x^2-2, r)+F(x^2-2, r-1).$$

7410. (By W. J. C. Sharp, M.A.)—If $N : D$ be a fraction in its lowest terms, and $D \equiv 2^h . 5^k . a^l . b^m . c^n \ldots$, where a, b, c, &c. are prime numbers, the equivalent decimal will consist of h or k non-recurring figures (according as h or k is greatest), and of a recurring period, the number of figures in which is a measure of $a^{l-1}(a-1) . b^{m-1}(b-1) . c^{n-1}(c-1)\ldots$

Solution by George Heppel, M.A.

Let the equivalent decimal have p non-recurring and q recurring figures. Then $N : D = K : 2^p . 5^p . 10^q - 1$. Hence, obviously, p must be equal to the highest index of either 2 or 5 in the factors of D. Also, supposing D to have but one other prime factor a, then, from Fermat's theorem, the maximum value q can have is $a-1$. If q has any smaller value, then, since in actual division we have remainder 0 after every q nines used, and we have remainder 0 after using $(a-1)$ nines, therefore q measures $a-1$. Now, if D contains a factor a^l, suppose that the resulting period is one of c digits, and let $10^c = C$. Then l measures $C-1$, therefore it also measures the following:

$$(a-1)(c-1)+a \quad\text{or}\quad (a-1)C+1,$$

$$(a-2)(C^2-C)+(a-1)C+1 \quad\text{or}\quad (a-2)C^2+C+1,$$

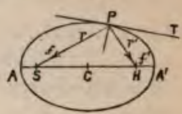
$$(a-3)(C^3-C^2)+(a-2)C^2+C+1 \quad\text{or}\quad (a-3)C^3+C^2+C+1.$$

Proceeding in this way, we see that l measures $C^{a-1}+C^{a-2}+\dots+C+1$; therefore it measures C^a-1 or $10^{ac}-1$, or q repeated ac times. Hence, the maximum period for a being $(a-1)$, for a^2 it is $a(a-1)$; for a^3 it is $a^2(a-1)$, and so on. Consequently the greatest possible period for D, as given in the question, must be $a^{l-1}(a-1)\,.\,b^{m-1}(b-1)\,.\,c^{n-1}(c-1)\,\dots$, and from the reasoning given above, the actual period always measures the maximum period.

7657. (By J. Crocker.)—If an ellipse be described under a force f to focus S and f_1 to focus H, and $SP = r$, $HP = r_1$; prove that

$$\frac{df_1}{dr_1}-\frac{df}{dr}=2\left(\frac{f}{r}-\frac{f_1}{r_1}\right).$$

Solution by D. Edwardes; W. G. Lax, B.A.; *and others.*



Resolving along the tangent, we have

$$v\frac{dv}{ds}=(f'-f)\frac{dr}{ds} \text{ or } v\frac{dv}{dr}=f'-f\text{ ......(1).}$$

Again, $\dfrac{v^2}{\rho}=(f+f')\sin SPT=(f+f')\dfrac{b}{(rr')^{\frac{1}{2}}}$,

that is, $$v^2=(f+f')\frac{rr'}{a}.$$

Differentiating this with respect to r, and remembering that $dr'+dr=0$,

$$2(f'-f)=\frac{rr'}{a}\left(\frac{df}{dr}-\frac{df'}{dr'}\right)+\frac{f+f'}{a}(r'-r),$$

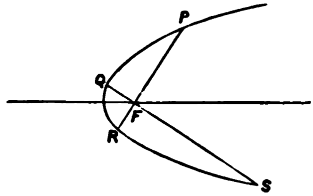
by (1). Putting $r+r'$ for $2a$ and reducing, we have the stated result.

[In a paper by Dr. Curtis "On Free Motion under the action of *several* Central Forces" (*Messenger of Mathematics*, New Series, No. 109, May 1880), this question is discussed, as a subordinate case, and the condition $\dfrac{1}{r^2}\dfrac{d}{dr}(Fr)-\dfrac{1}{r'^2}\dfrac{d}{dr'}(F'r')=0$, deduced, which is equivalent to the above.]

7547. (By R. Tucker, M.A.)—PFR, QFS, are two orthogonal focal chords of a parabola, and circles about PFQ, QFR, RFS, SFP cut the axis in points the ordinates to which meet the curve in P′, Q′, R′, S′: prove (1) locus of centres of mean position of P, Q, R, S is a parabola, (latus rectum $\frac{1}{4}$L); (2) $\Sigma\,(FP')+2L=2\Sigma\,(FP)$; and (3) if also normals at three of the points P, Q, R, S cointersect, then $y_4^{-1}\,\Sigma^3\,y^{-1})=-24L^{-2}$.

Solutions by the PROPOSER.

Let am_1^2, $2am_1$, &c., be the coordinates of P, Q, R, S; then equation to PQ is



$$(m_1+m_2)\,y-2x = 2am_1m_2 \ldots\ldots (1),$$

with condition

$$(1-m_1^2)(1-m_2^2)+4m_1m_2 = 0 \ldots (2).$$

The equation to PR is

$$(m_1+m_3)\,y-2x = 2am_1m_3 \ldots\ldots (3),$$

and it is a focal chord, $\therefore\ m_1m_3 = -1$,

similarly $\quad m_2m_4 = -1 \ldots\ldots\ldots (4)$;

(1) $2\bar{y} = a(m_1+m_2+m_3+m_4)$, $\quad 4\bar{x} = a\Sigma m^2$,

and $\quad \Sigma(m_rm_s) = -6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$

therefore $\quad 4\bar{y}^2 = 4a(\bar{x}-3a)$, that is, $\bar{y}^2 = a(\bar{x}-3a)$;

hence locus of mean centres is a parabola whose latus rectum $= \frac{1}{4}$L.

(2) The equation to circle round PFQ is

$$x^2+y^2-a(m_1^2+m_2^2)\,x-2a(m_1+m_2)\,y+a^2m_1m_2(4+m_1m_2) = 0,$$

therefore abscissa of $P' = am_1m_2(4+m_1m_2)$ and $FP' = a(m_1^2+m_2^2)$ by (2);

therefore $\quad \Sigma(FP')+2L = 2a\,[4+\Sigma m^2] = 2\Sigma(FP).$

(3) The equation to the normal through any point (x, y) is

$am^3+(2a-x)\,m-y = 0$, therefore $\Sigma(m) = 0$ for P, Q, R,

and $\dfrac{1}{y_4}\Sigma_1^3\left(\dfrac{1}{y}\right) = \Sigma_1^4\dfrac{1}{y_ry_s}-\Sigma_1^3\dfrac{1}{y_ry_s} = -\dfrac{6}{4a^2}$ by (5) $= -24L^{-2}$.

Or, (1) thus:—The equation to PR is $y = k(x-a)$,

therefore $\quad x_1+x_3 = 2a\left(1+\dfrac{2}{k}\right)$, $\quad y_1+y_3 = \dfrac{4a}{k}$.

Similarly $\quad x_2+x_4 = 2a(1-2k^2)$, $\quad y_2+y_4 = -4ak$,

therefore $\quad 4\bar{y} = \Sigma y = 4a\left(\dfrac{1}{k}-k\right)$, *i.e.*, $\bar{y} = a\left(\dfrac{1}{k}-k\right)$,

$$\bar{x} = a\left(1+k^2+\frac{1}{k^2}\right);$$

therefore $\quad \dfrac{\bar{y}^2}{a^2} = k^2+\dfrac{1}{k^2}-2 = \dfrac{\bar{x}}{a}-3$,

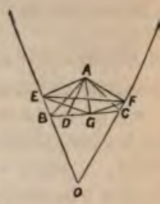
i.e., $\quad \bar{y}^2 = a(\bar{x}-3a)$.

[This is also the locus of the intersection of two orthogonal normals (see SMITH's *Conics*, p. 104).]

7658. (By S. CONSTABLE.)—The vertex of a triangle is fixed, the vertical angle given, and the base angles move on two parallel straight lines; construct the triangle when the base passes through a fixed point.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

For greater generality, let the lines not be parallel, let A be the fixed vertex and OBE, OCF the fixed straight line on which B and C, the extremities of base, move, D being the fixed point on base; draw AE, AF, AG perpendicular, respectively, to OB, OC, BC, then



$\angle EGF = AGE + AGF = ABE + ACF = BOC + BAC$,

and therefore known; hence one locus of G is a segment of a circle on EF containing this known angle, while another locus is a circle on AD as diameter; the intersection of the two loci gives point G, and determines the required base BC.

7476. (By D. EDWARDES.)—If $xyz = (2-x)(2-y)(2-z)$, show that

$$I \equiv \int_0^1 \int_0^1 xyz\, dx\, dy = \frac{\pi^2}{6} - \frac{5}{4}.$$

Solution by G. B. MATHEWS, B.A.; SARAH MARKS; *and others.*

From the equation, and then by putting $1-x$ for x, we have

$$I = \int_0^1 \int_0^1 \frac{(2-x)(2-y)\,xy}{2-x-y+xy}\, dx\, dy = \int_0^1 \int_0^1 \frac{(1-x^2)(1-y^2)}{1+xy}\, dx\, dy$$

$$= \int_0^1 (1-x^2)\, dx \int_0^1 \left(-\frac{y}{x} + \frac{1}{x^2} + \frac{1-\dfrac{1}{x^2}}{1+xy} \right) dy$$

$$= \int_0^1 dx\, (1-x^2) \left\{ -\frac{1}{2x} + \frac{1}{x^2} + \left(\frac{1}{x} - \frac{1}{x^3}\right) \log(1+x) \right\}$$

$$= \int_0^1 dx\, (1-x^2) \left\{ -\frac{1}{2x} + \frac{1}{x^2} + \left(\frac{1}{x} - \frac{1}{x^3}\right) \left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots\right) \right\}$$

$$= \int_0^1 dx\, (1-x^2) \left\{ 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \dots - \tfrac{1}{3} + \tfrac{1}{4}x - \tfrac{1}{5}x^2 + \dots \right\}$$

$$= 2\int_0^1 dx\, (1-x^2) \left\{ \frac{1}{1\,.\,3} - \frac{x}{2\,.\,4} + \frac{x^2}{3\,.\,5} - \frac{x^3}{4\,.\,6} + \dots \right\}$$

$$= 4\left(\frac{1}{1^2\,.\,3^2} - \frac{1}{2^2\,.\,4^2} + \frac{1}{3^2\,.\,5^2} - \frac{1}{4^2\,.\,6^2} + \dots \right).$$

$$= \left(\frac{1}{1^2} + \frac{1}{3^2} - \frac{2}{1\,.\,3} \right) - \left(\frac{1}{2^2} + \frac{1}{4^2} - \frac{2}{2\,.\,4} \right) + \left(\frac{1}{3^2} + \frac{1}{5^2} - \frac{2}{3\,.\,5} \right) - \&c.$$

$$= \frac{\pi^2}{6} - \frac{5}{4}.$$

7666. (By Professor HAUGHTON, F.R.S.)—Prove the following formula for finding the Moon's parallax in altitude in terms of her true zenith distance, viz., $\sin p = \sin P \sin z + \frac{1}{2}\sin^2 P \sin 2z + \frac{1}{3}\sin^3 P \sin 3z + \&c.$

Solution by D. EDWARDES; J. A. OWEN, B.Sc.; *and others.*

If Z be the observed zenith distance,

$$\sin p = \sin P \sin Z \quad \text{or} \quad \sin p = \sin P \sin (z+p),$$

therefore

$$e^{2ip} = \frac{1-\sin P\, e^{-iz}}{1-\sin P\, e^{iz}}.$$

Taking the logarithms, and expanding the right-hand member, we have

$$p = \sin P \sin z + \tfrac{1}{2}\sin^2 P \sin 2z + \tfrac{1}{3}\sin^3 P \sin 3z + \&c.$$

Now, if p be very small, $\sin p = p$ approximately.

7564. (By D. EDWARDES.)—If the sides, taken in order, of a quadrilateral inscribed in one circle, and circumscribed about another, are a, b, c, d; prove that the angle between its diagonals is $\cos^{-1}\dfrac{ac \sim bd}{ac+bd}$.

Solution by J. A. OWEN, B.Sc.; R. KNOWLES, B.A.; *and others.*

Let h, k be the diagonals, A the angle between b and c, and θ the required angle; then the area of the quadrilateral is $\frac{1}{2}hk\sin\theta =$ $\frac{1}{2}(ad+bc)\sin A$, $\therefore\ \sin\theta = \dfrac{ad+bc}{ac+bd}\sin A$; but $\cos A = \dfrac{b^2+c^2-a^2-d^2}{2(bc+ad)}$;

hence, remembering that $c+a = b+d$, we have

$$\sin^2 A = \frac{4abcd}{(ad+bc)^2}; \text{ hence } \cos\theta = \left\{1-\frac{4abcd}{(ac+bd)^2}\right\}^{\frac{1}{2}}, \ \cos\theta = \frac{ac\sim bd}{ac+bd}.$$

7575. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Two normals at right angles to each other are drawn respectively to the two (confocal) parabolas $y^2 = 4a(x+a)$, $y^2 = 4b(x+b)$; prove that the locus of their common point is the quartic

$$2y = (a^{\frac{1}{2}}+b^{\frac{1}{2}})\left[x-2(ab)^{\frac{1}{2}}\right]^{\frac{1}{2}} + (a^{\frac{1}{2}}-b^{\frac{1}{2}})\left[x+2(ab)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

which may be constructed as follows:—draw the two parabolas

$$y^2 = (a+b)x - 4ab \pm 2(ab)^{\frac{1}{2}}(x-a-b),$$

and let a common ordinate perpendicular to the axis meet these parabolas in P, p, Q, q, respectively, then the quartic bisects PQ, Pq, pQ, pq. Also the area included beween the quartic and its one real bitangent is $\frac{1}{2}a^2m^2(m+1)(m-1)^3$, where $a = bm^4$, and $a > b$. These results will only be real when ab is positive, or when the two confocals have their concavities in the same sense, but in all cases the rational equation of the quartic is $(y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0$.

[The quartic is unicursal, but has only one node at a finite distance ($x=a+b$, $y=0$); there is singularity at ∞, equivalent to two cusps. The class number is 4, and the deficiency 0, so that $2\delta+3\kappa=8$, $\delta+\kappa=3$, or $\delta=1$, $\kappa=2$.]

Solution by B. H. RAU, M.A.; Professor NASH, M.A.; *and others.*

The equations to the normals to the confocal parabolas $y^2=4a(x+a)$, $y^2=4b(x+b)$, are respectively

$$y=mx-a(m^3+m),\quad y=m_1x-b(m_1^3+m_1)\ \ldots\ldots\ldots\ldots\ldots(1,\ 2).$$

If these are at right angles to each other, $m_1=-\dfrac{1}{m}$, and therefore (2)

becomes $$y=-\frac{x}{m}+b\left(\frac{1}{m^3}+\frac{1}{m}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1)−(3), $$x\left(m+\frac{1}{m}\right)-am^2\left(m+\frac{1}{m}\right)-\frac{b}{m^2}\left(m+\frac{1}{m}\right)=0;$$

therefore $$am^2+\frac{b}{m^2}=x;$$

therefore $a^{\frac12}m+\dfrac{b^{\frac12}}{m}=[x+2(ab)^{\frac12}]^{\frac12}$, and $a^{\frac12}m-\dfrac{b^{\frac12}}{m}=[x-2(ab)^{\frac12}]^{\frac12}$.

Eliminating x from (1) and (3), we have $y=\dfrac{b}{m}-am$,

$$\therefore\ \ 2y=2\left(\frac{b}{m}-am\right)=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}+(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

since the radicals may be taken with either the positive or negative sign.

The quartic may be constructed with the help of the curves

$$y=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}\ \text{and}\ y=(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

which are the parabolas given by the equation

$$y^2=(a+b)x-4ab\pm2(ab)^{\frac12}(x-a-b).$$

Squaring the equation to the quartic, we have

$$4y^2=(a+b+2a^{\frac12}b^{\frac12})(x-2a^{\frac12}b^{\frac12})+(a+b-2a^{\frac12}b^{\frac12})(x+2a^{\frac12}b^{\frac12})+2(a-b)(x^2-4ab)^{\frac12};$$

or $$[2y^2-(a+b)x+4ab]^2=4(a-b)^2x^2-16ab(a-b)^2,$$

or $$(y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0,$$

which is the rational equation to the quartic.

7287 & 7353. (By Professor WOLSTENHOLME, M.A., D.Sc.)—(7278.) Two circles have radii a, b, the distance between their centres is c, and $a>b+c$; prove that, (1) if any straight line be drawn cutting both circles, the ratio of the squares of the segments made by the circles has the minimum value

$$a\left\{[(a+b)^2-c^2]^{\frac12}+[(a-b)^2-c^2]^{\frac12}\right\}:b\left\{[(a+b)^2-c^2]^{\frac12}-[(a-b)^2-c^2]^{\frac12}\right\};$$

and (2) the distances of the straight line corresponding to this minimum from the centres of the two circles will be in the same ratio.

(7353.) Prove that the maximum and minimum values of

$$u \equiv \frac{a^2-x^2}{b^2-(x-c\cos\theta)^2},$$

where x, θ are both variable, a, b, c are given positive constants, and $a > b+c$ are the roots of the quadratic $u^2b^2-u(a^2+b^2-c^2)+a^2=0$.

Solution by D. Edwardes.

(7353.) The conditions $\frac{du}{dx}=0$, $\frac{du}{d\theta}=0$ lead to $x=u(x-c\cos\theta)$, $\sin\theta(x-c\cos\theta)=0$. Also the sign of $\frac{d^2u}{dx^2}\frac{d^2u}{d\theta^2}-\left(\frac{d^2u}{dx\,d\theta}\right)^2$ is the same as that of $u\cos 2\theta$. If $x-c\cos\theta=0$, then $x=0$, $\theta=$ odd multiple of $\frac{1}{2}\pi$, and u is positive, so that this solution must be rejected. We have then $\sin\theta=0$ and $u=\frac{x}{x\pm c}$. Either sign gives the same quadratic for u, by the elimination of x, viz., $b^2u^2-u(a^2+b^2-c^2)+a^2=0$.

Moreover, both roots of this equation are positive, so that they are real maximum and minimum values, since $u\cos 2\theta$ is then positive. The minimum value is

$$\frac{a^2+b^2-c^2-\left[\left\{(a+b)^2-c^2\right\}\left\{(a-b)^2-c^2\right\}\right]^{\frac{1}{2}}}{2b^2}$$

or

$$\frac{a}{b}\,\frac{\left[(a+b)^2-c^2\right]^{\frac{1}{2}}-\left[(a-b)^2-c^2\right]^{\frac{1}{2}}}{\left[(a+b)^2-c^2\right]^{\frac{1}{2}}+\left[(a-b)^2-c^2\right]^{\frac{1}{2}}}.$$

(7278.) The foregoing value solves this Question, a, b being the radii of the circles, and x the perpendicular from the centre of the larger circle upon the cutting line. And $u=\frac{x}{x-c\cos\theta}$ or $\frac{x}{x-c}$ when u is a minimum by the above work, θ being the inclination of the cutting line to the line joining the centres. But $\frac{x}{x-c\cos\theta}$ is the ratio of the perpendiculars in the Question.

7446. (By R. Knowles, B.A., L.C.P.)—(Suggested by Question 7385.)—In an equilateral triangle ABC a circle is inscribed, and a tangent to the circle meets the sides CB, CA in the points A′, B′; the line joining the orthocentre of the triangle A′B′C with the centre of its circumscribing circle meets BC or AC in D; prove that, in either case, as A′B′ varies, the maximum and minimum values of DC are respectively two-ninths and two-thirds of a side of the equilateral triangle.

Solution by G. Heppel, M.A.; G. B. Mathews, B.A.; *and others.*

Let $a=$ side of triangle; $CA'=x$; $CB'=y$; $A'B'=u$.

Then $$u^2=(a-x-y)^2=x^2-xy+y^2;$$

be real when ab is positive, or when the two confocals have their concavities in the same sense, but in all cases the rational equation of the quartic is $(y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0$.

[The quartic is unicursal, but has only one node at a finite distance ($x=a+b$, $y=0$); there is singularity at ∞, equivalent to two cusps. The class number is 4, and the deficiency 0, so that $2\delta+3\kappa=8$, $\delta+\kappa=3$, or $\delta=1$, $\kappa=2$.]

Solution by B. H. Rau, M.A.; Professor Nash, M.A.; *and others.*

The equations to the normals to the confocal parabolas $y^2=4a(x+a)$, $y^2=4b(x+b)$, are respectively

$$y=mx-a(m^3+m),\quad y=m_1x-b(m_1^3+m_1)\quad\ldots\ldots\ldots\ldots(1,2).$$

If these are at right angles to each other, $m_1=-\dfrac{1}{m}$, and therefore (2) becomes

$$y=-\frac{x}{m}+b\left(\frac{1}{m^3}+\frac{1}{m}\right)\quad\ldots\ldots\ldots\ldots(3).$$

From (1)−(3), $$x\left(m+\frac{1}{m}\right)-am^2\left(m+\frac{1}{m}\right)-\frac{b}{m^2}\left(m+\frac{1}{m}\right)=0;$$

therefore $$am^2+\frac{b}{m^2}=x;$$

therefore $$a^{\frac12}m+\frac{b^{\frac12}}{m}=[x+2(ab)^{\frac12}]^{\frac12},\quad\text{and}\quad a^{\frac12}m-\frac{b^{\frac12}}{m}=[x-2(ab)^{\frac12}]^{\frac12}.$$

Eliminating x from (1) and (3), we have $y=\dfrac{b}{m}-am$,

$$\therefore\quad 2y=2\left(\frac{b}{m}-am\right)=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}+(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

since the radicals may be taken with either the positive or negative sign.

The quartic may be constructed with the help of the curves

$$y=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}\ \text{and}\ y=(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

which are the parabolas given by the equation

$$y^2=(a+b)x-4ab\pm2(ab)^{\frac12}(x-a-b).$$

Squaring the equation to the quartic, we have

$$4y^2=(a+b+2a^{\frac12}b^{\frac12})(x-2a^{\frac12}b^{\frac12})+(a+b-2a^{\frac12}b^{\frac12})(x+2a^{\frac12}b^{\frac12})+2(a-b)(x^2-4ab)^{\frac12};$$

or $$[2y^2-(a+b)x+4ab]^2=4(a-b)^2x^2-16ab(a-b)^2,$$

or $$(y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0,$$

which is the rational equation to the quartic.

7287 & 7353. (By Professor Wolstenholme, M.A., D.Sc.)—(7278.) Two circles have radii a, b, the distance between their centres is c, and $a>b+c$; prove that, (1) if any straight line be drawn cutting both circles, the ratio of the squares of the segments made by the circles has the minimum value

$$a\left\{[(a+b)^2-c^2]^{\frac12}+[(a-b)^2-c^2]^{\frac12}\right\}:b\left\{[(a+b)^2-c^2]^{\frac12}-[(a-b)^2-c^2]^{\frac12}\right\};$$

[illegible]

whence $y = \frac{a(a-2x)}{2a-3x}$. In triangle A'B'C,

$R = \frac{u}{\sqrt{3}}$; $\sin A' = \frac{y\sqrt{3}}{u}$; $\sin B' = \frac{x\sqrt{3}}{u}$; $\cos A' = \frac{2x-y}{2u}$; $\cos B' = \frac{2y-x}{2u}$.

Hence, with C as origin and CB as axis of x, the orthocentre is $\left\{\frac{y}{2}, \frac{2x-y}{2\sqrt{3}}\right\}$, and the circumcentre is $\left\{\frac{x}{2}, \frac{2y-x}{2\sqrt{3}}\right\}$. Forming the equation of the joining line and putting $y = 0$, we have

$$DC = \frac{x+y}{3} = \frac{a^2-3x^2}{3(2a-3x)}.$$

This is a maximum or minimum when $x = \frac{1}{3}a$, or a, and these give $\frac{2}{9}a$ and $\frac{2}{3}a$ as the values of DC. The first must be a maximum and the second a minimum, for, when $x = 0$ or $\frac{1}{2}a$, $DC = \frac{1}{6}a$; and when $x = \frac{2}{3}a$ or ∞, $DC = \infty$.

7499. (By R. Tucker, M.A.)—OA, OB are two fixed lines, A is a fixed peg, and B a peg movable along OB; an inextensible endless string, passing round AB, is kept stretched by a pencil C; find the envelope of the loci of the curves traced out by C, on the plane AB, by varying the position of B.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

Let the length of the loop be $2l$; then, as the distance of a focus of a conic from the remote extremity of the major axis is half the sum of the major axis and the distance between the foci, it follows that, if with A as centre and radius l a circle be drawn, this circle will touch, at the remote extremity of its axis major, each of the ellipses defined; consequently it is the envelope required.

7401. (By R. Russell, B.A.)—Find (1) $A_1, A_2, A_3 \dots A_{2n+1}$, such that $A_1(x-\alpha_1)^{2n+1} + A_2(x-\alpha_2)^{2n+1} + \dots + A_{2n+1}(x-\alpha_{2n+1})^{2n+1}$
$\equiv P(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{2n+1})$;
and (2) show that A_r is an invariant of the equation whose roots are the quantities $\alpha_1, \alpha_2, \dots \alpha_{2n+1}$ leaving out α_r.

Solution by the Proposer.

The answer is

$I_1\Delta_1(x-\alpha_1)^{2n+1} + I_2\Delta_2(x-\alpha_2)^{2n+1} + \dots = P(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{2n+1})$,

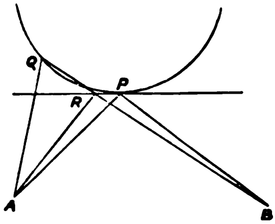
where Δ_r = product of differences of all the roots leaving out α_r, I_r = the invariant formed for the same roots which corresponds to

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\zeta)^2$$

for a sextic, and P = the product of differences of $\alpha_1\alpha_2 \dots \alpha_{2n+1}$.

7734. (By J. CROCKER.)—If A and B are fixed points; find, on a fixed circle, a point P such that AP + PB is a minimum.

Solutions by B. H. RAU, M.A.; DR. CURTIS; *and others.*



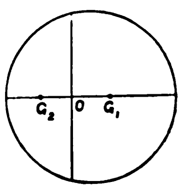
This is another form of Questions 6878, 7422, 7653, for, if P be such that PA and PB make equal angles with the tangent to the circle at the point P, and we take any point Q on the circle, and join QA, QB, the latter meeting the tangent at R, then we have

$$AQ + QB > AR + RB > AP + PB,$$

hence AP + PB is a minimum. [See *Reprint*, Vol. 39, p. 59, and Vol. 41.]

5522. (By Professor ASAPH HALL, M.A.)—If a planet be spherical and ϕ be the angle at the planet between the Earth and the Sun, and a the radius of the sphere; prove that the distance of the centroid of the planet's apparent disk from its true centre will be $\frac{8a}{3\pi}\sin^2\frac{1}{2}\phi$ when the planet is gibbous, and $\frac{8a}{3\pi}\cos^2\frac{1}{2}\phi$ when the planet is crescent.

Solution by the Rev. T. C. SIMMONS, M.A.



Take the case of the planet being gibbous; we have then to investigate the centroid of a figure consisting of a semicircle radius a joined to an ellipse whose semi-axes are a and $a\cos\phi$. Let O be true centre of disc, G_1 and G_2 the centroids of the semicircle and semi-ellipse; then $OG_1 = \frac{4a}{3\pi}$, $OG_2 = \frac{4a\cos\phi}{3\pi}$; therefore if $\bar{x}$ denote distance from O of centroid of apparent disc

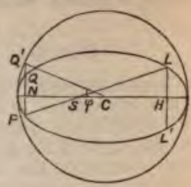
$$\bar{x} = \frac{\frac{4a}{3\pi}\cdot\frac{\pi a^2}{2} - \frac{4a\cos\phi}{3\pi}\cdot\frac{\pi a^2\cos\phi}{2}}{\frac{1}{2}\pi a^2 + \frac{1}{2}\pi a^2\cos\phi} = \frac{4a}{3\pi}\cdot\frac{1-\cos^2\phi}{1+\cos\phi} = \frac{8a}{3\pi}\sin^2\frac{\phi}{2}.$$

When the planet is crescent, we obtain

$$\bar{x} = \frac{4a}{3\pi}\cdot\frac{1-\cos^2\phi}{1-\cos\phi} = \frac{8a}{3\pi}\cos^2\frac{\phi}{2}.$$

7683. (By R. TUCKER, M.A.)—LSP and LHL′ are a focal chord and a latus rectum respectively of an ellipse, and the circle LL′P cuts the curve again in [illegible] ove that $\tan^2\frac{1}{2}(\phi) = (1+e)^3/(1-e)^3$.

Solution by W. G. LAX, B.A.; J. S. JENKINS; *and others.*



The circle through LPL′ cuts the curve again in Q, where QNP is perpendicular to the major axis. Produce NQ to cut the auxiliary circle in Q′, and join Q′ to C, then ∠ Q′CH = ϕ. Let a be (x, y); then, by combining the equations to the ellipse and to L′Q, we get

$$x = -\frac{ae\,(3+e^2)}{3e^2+1}, \quad \tan^2\tfrac{1}{2}\phi = \frac{1-\cos\phi}{1+\cos\phi} = \frac{(1+e)^3}{(1-e)^3}.$$

3873. (By J. B. SANDERS.)—The horizontal section of a cylindrical vessel is 100 square inches, its attitude is 36 inches, and it has an orifice whose section is $\frac{1}{10}$ of a square inch; find in what time, if filled with a fluid, it will empty itself, allowing for the contraction of the vein.

Solution by A. MARTIN, M.A.; Prof. EVANS, M.A.; *and others.*

Put $100 = \mathrm{K}$, $36 = h$, $\frac{1}{10} = k$, $v =$ velocity, and $x =$ altitude of the surface of the fluid at any time t.

Then (WALTON'S *Problems, Hydrodynamics*, p. 142),

$$v = \left(\frac{2gx}{1-\dfrac{k^2}{\mathrm{K}^2}}\right)^{\frac{1}{2}}, \quad \text{and} \quad kv\,dt = -\mathrm{K}\,dx \;\ldots\ldots\ldots\ldots\ldots(1, 2).$$

Therefore $$dt = -\frac{\mathrm{K}\,dx}{dv} = -\frac{\mathrm{K}\left(1-\dfrac{k^2}{\mathrm{K}^2}\right)^{\frac{1}{2}}dx}{k\,(2gx)^{\frac{1}{2}}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Integrating (3) between the limits $x=0$, $x=h$, we have

$$t = \frac{2\mathrm{K}\left(1-\dfrac{k^2}{\mathrm{K}^2}\right)^{\frac{1}{2}}h^{\frac{1}{2}}}{k\,(2g)^{\frac{1}{2}}} = \frac{2\left(\dfrac{\mathrm{K}^2}{k^2}-1\right)^{\frac{1}{2}}h^{\frac{1}{2}}}{(2g)^{\frac{1}{2}}} = 11 \text{ minutes } 36\cdot5 \text{ seconds},$$

taking the coefficient of efflux = 0·62.

7696. (By ALPHA.)—Two guns are fired at a railway station at an interval of 21 minutes, but a person in a train approaching the station observes that 20 minutes 14 seconds elapse between the reports; supposing that sound travels 1125 feet per second, show that the velocity of the train is 29·064 miles per hour.

Solution by B. Reynolds, M.A.; W. J. Greenstreet, B.A.; *and others.*

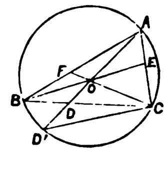
Space traversed by train during the interval between the hearings of the two reports = 46 × 1125 ft.; hence the velocity of train

$$= (46 \times 1125) \div 1214 \text{ ft. per second} = 29{\cdot}064 \text{ miles per hour.}$$

7660. (By R. Knowles, B.A., L.C.P.)—From the angular points of a triangle ABC, lines are drawn through the centre of the circum-circle to meet the opposite sides in D, E, F, respectively; prove that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}.$$

Solution by W. G. Lax, B.A.; J. S. Jenkins; *and others.*



Let O be the circum-centre; produce AOD to D′, and join CD′; then we have

$$\angle BAD = BCD' = \tfrac{1}{2}\pi - C;$$

$$AD = \frac{c \sin ABD}{\sin ADB} = \frac{c \sin B}{\sin(B + 90^\circ - C)}$$

$$= \frac{c \sin B}{\cos(C-B)} = \frac{2R \sin C \sin B}{\cos(C-B)};$$

$$\frac{1}{AD} = \frac{\cos(C-B)}{2R \sin C \sin B} = \frac{1}{2R}(1 + \cot B \cot C),$$

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{2R}[3 + \Sigma(\cot B \cot C)] = \frac{1}{2R}(3+1) = \frac{2}{R}.$$

[In the solution to Question 7594 (p. 80 of this volume), it is proved that

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1; \text{ hence } \frac{AD-R}{AD} + \frac{BE-R}{BE} + \frac{CF-R}{CF} = 1,$$

wherefrom the result immediately follows.]

7633. (By Professor Genese, M.A.)—A circle is inscribed in a segment of a circle containing an angle θ: the point of contact with the base divides it into segments h, k. Prove that (1) the radius of the inscribed circle is $\dfrac{hk}{h+k}\cot\tfrac{1}{2}\theta$; and hence (2) that the inscribed circle of a triangle touches the nine-point circle.

I. *Solution by* G. Heppel, M.A.; J. A. Owen, B.Sc.; *and others.*

(1) Let AB be the base of the segment, O and P the centres, and c and r the radii of the original and inscribed circles. Draw PC and OD perpendicular to AB, then AC = h, CB = k, AD = $c \sin\theta$, OD = $\pm c\cos\theta$, $h+k = 2c\sin\theta$. Also $PO^2 = (c\sin\theta - h)^2 + (r - c\cos\theta)^2 = (c-r)^2$, whence we obtain the stated result.

(2) In any triangle ABC, where b is $>c$, let A′ be the mid-point of BC D the foot of the perpendicular from A, and H the point of contact of the inscribed circle with BC. Then $BD = \frac{1}{2}a - \frac{b^2-c^2}{2a}$; $BH = \frac{1}{2}a - \frac{1}{2}(b-c)$; $BA' = \frac{1}{2}a$. And these are in order of magnitude, for $b+c > a$, and therefore H is always between D and A′. Applying the result just obtained to the segment cut off by DA′ from the nine-point circle, whose radius is $\frac{1}{2}R$, we have $\sin\theta = \frac{1}{2}DA' \div \frac{1}{2}R = \frac{DA'}{R} = \frac{b^2-c^2}{2aR} = \frac{\sin^2 B - \sin^2 C}{\sin A} = \sin(B-C)$;

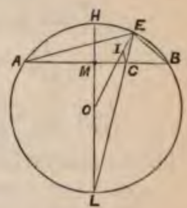
hence $\theta = B - C$; also $h = \frac{1}{2}(b-c)$; $k = \frac{b^2-c^2}{2a} - \frac{b-c}{2} = \frac{s_1(b-c)}{2a}$;

whence the radius of the circle touching the nine-point circle, and touching BC in H, is $s_1(b-c)\cot\frac{1}{2}(B-C) / (b+c) = s_1 \tan\frac{1}{2}A = r$.

II. *Solution by* Morgan Jenkins, M.A.

Professor Genese's result may be put in the following geometrical form:—

(1) If a circle be inscribed in a given segment, the diameter of this circle is equal to the product of the segments of the chord divided by the height of the supplementary segment.



If I be the centre of the in-circle touching the segment at E and the chord AB at C; and H, M, L, the mid-points of the arc of the given segment, the chord, and the supplementary arc, respectively; then EC passes through L, and EI through O, the centre of the circle AHBL;

and $$\frac{IC}{OL} = \frac{EC}{LE} = \frac{EC \cdot CL}{LE \cdot LC} = \frac{AC \cdot CB}{LM \cdot LH};$$

therefore $$IC = \frac{1}{2}\frac{AC \cdot CB}{ML}.$$

(2) Let I be the in-centre of the triangle ABC, KA′L the diameter perpendicular to BC, A′ being the mid-point of BC, IH, AD perpendiculars on BC, AN parallel to BC, vIu parallel to BC cutting KL in v, AD in u.

Then $Iu \cdot Iv : AN^2 = IA \cdot IL : LA^2$
$= KL \cdot r : LN \cdot LK$,

$Iu \cdot Iv : LN \cdot KN = KN \cdot r : LN \cdot KN$;

therefore $Iu \cdot Iv = KN \cdot r$.

Now, the centroid of the triangle ABC being the internal centre of similitude of the circumcircle and the nine-points circle, $NK = 2nk$, where nk is the height of the segment $A'kD$ of the nine-points circle, on the opposite side of A′D to I. Hence $\frac{A'H \cdot DH}{2nk} = r$, and therefore the in-circle touches the nine-points circle.

London: Printed by C. F. Hodgson & Son, 1 Gough Square, Fleet Street, E.C.

MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY ADDITIONAL

PAPERS AND SOLUTIONS

NOT PUBLISHED IN THE "EDUCATIONAL TIMES."

AND

AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

VOL. XLII.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1885.

*** Of this series there have now been published forty-two Volumes, each of which contains, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprises contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

LIST OF CONTRIBUTORS.

Aldis, J. S., M.A.; H.M. Inspector of Schools.
Allen, Rev. A. J. C., M.A.; St. Peter's Coll., Camb.
Allman, Professor Geo. J., LL.D.; Galway.
Anderson, Alex., B.A.; Queen's Coll., Galway.
Anthony, Edwyn, M.A.; The Elms, Hereford.
Armenante, Professor; Pesaro.
Ball, Robt. Stawell, LL.D., F.R.S.; Professor of Astronomy in the University of Dublin.
Basu, Satish Chandra; Presid. Coll., Calcutta.
Battaglini, Giuseppe; Professore di Matematiche nell' Università di Roma.
Bayliss, George, M.A.; Kenilworth.
Beltrami, Professor; University of Pisa.
Berg, F. J. van den; Professor of Mathematics in the Polytechnic School, Delft.
Besant, W. H., D.Sc., F.R.S.; Cambridge.
Bhut, Ath Bigah, M.A.; Delhi.
Bickerdike, C.; Allerton Bywater.
Biddle, D.; Gough H., Kingston-on-Thames.
Birch, Rev. J. G., M.A.; London.
Blackwood, Elizabeth; Boulogne.
Blythe, W. H., B.A.; Egham.
Borchardt, Dr. C. W.; Victoria Strasse, Berlin.
Bosanquet, R. H. M., M.A.; Fellow of St. John's College, Oxford.
Bourne, C. W., M.A.; Bedford County School.
Brooks, Professor E.; Millersville, Pennsylvania.
Brown, A. Crum, D.Sc.; Edinburgh.
Brown, Prof. Colin; Andersonian Univ. Glasgow.
Buchheim, A., M.A., Ph.D.; Schol. New Coll., Oxf.
Buck, Edward, M.A.; Univ. Coll., Bristol.
Burnside, W. S., M.A.; Professor of Mathematics in the University of Dublin.
Capel, H. N., LL.B.; Bedford Square, London.
Carmody, W. P., B.A.; Clonmel Gram. School.
Carr, G. S., B.A.; Endsleigh Gardens, London.
Casey, John, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
Cavallin, Prof., M.A.; University of Upsala.
Cave, A. W., B.A.; Magdalen College, Oxford.
Cayley, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge, Member of the Institute of France, &c.
Chakravarti, Byomakesa, M.A.; Professor of Mathematics, Calcutta.
Chase, Pliny Earle, LL.D.; Professor of Philosophy in Haverford College.
Clarke, Colonel A. R., C.B., F.R.S.; Hastings.
Coates, W. M., B.A.; Trinity College, Dublin.
Cochez, Professor; Paris.
Cockle, Sir James, M.A., F.R.S.; Bayswater.
Cohen, Arthur, M.A., Q.C., M.P.; Holland Pk.
Colson, C. G., M.A.; University of St. Andrew's.
Constable, S.; Swinford Rectory, Mayo.
Cotterill, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
Cremona, Luigi; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
Crofton, M. W., B.A., F.R.S.; Prof. of Math. and Mech. in the R. M. Acad., Woolwich.
Culverwell, E. P., B.A.; Sch. of Trin. Coll., Dubl.
Curtis, Arthur Hill, LL.D., D.Sc.; Dublin.
Darboux, Professor; Paris.
Davis, J. G., M.A.; Abingdon
Davis, R. F., B.A.; Wandsworth Common.
Dawson, H. G., B.A.; Baymount, Dublin.
Day, Rev. H. G., M.A.; Richmond Terr., Brighton.
Dey, Prof. Narendra Lal, M.A.; Calcutta.
Dick, G. R., M.A.; Fellow of Caius Coll., Camb.
Dobson, T., B.A.; Hexham Grammar School.
Droz, Prof. Arnold, M.A.; Porrentruy, Berne.
Dupain, J. C.; Professeur au Lycée d'Angoulême.
Easterby, W., B.A.; Grammar School, St. Asaph.
Eastwood, G., M.A.; Saxonville, Massachusetts.
Easton, Belle; Lockport, New York.
Edward, J., M.A.; Head Mas., Aberdeen Coll.
Edwardes, David; Erith Villas, Erith, Kent.
Elliott, E. B., M.A.; Fell. Queen's Coll., Oxon.
Ellis, Alexander J., F.R.S.; Kensington.
Emtage, W. T. A.; Pembroke Coll., Oxford.
Escott, Albert, M.A.; Head Master of the Royal Hospital School, Greenwich.
Essennell, Emma; *Coventry.*
Everett, *J. D., D.C.L.; Professor of Natural Philosophy in Queen's College, Belfast.*
Evans, Professor, M.A.; Lockport, New York.
Ficklin, Joseph; Prof. in Univ. of Missouri.
Finch, T. H., B.A.; Trinity College, Dublin.
Fortey, H., M.A.; Bellary, Madras Presidency.
Foster, F. W., B.A.; Chelsea.
Foster, Prof. G. Carey, F.R.S.; Univ. Coll., Lond.
Franklin, Christine Ladd, M.A.; Prof. of Nat. Sci. and Math., Union Springs, New York.
Fuortes, E.; University of Naples.
Galbraith, Rev. J., M.A., Fell. Trin. Coll., Dublin.
Gale, Kate K.; Worcester Park, Surrey.
Gallatly, W., B.A.; Earl's Court, London.
Galliers, Rev. I., M.A.; Kirkstead Rectory, Norwich.
Galton, Francis, M.A., F.R.G.S.; London.
Genese, Prof., M.A.; Univ. Coll., Aberystwith.
Gerrans, H. T., B.A.; Stud. of Ch. Ch., Oxford.
Glaisher, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
Goldenberg, Professor, M.A.; Moscow.
Graham, R. A., M.A.; Trinity College, Dublin.
Greenfield, Rev. W. J., M.A.; Dulwich College.
Greenstreet, W. J., B.A.; Framlingham.
Greenwood, James M.; Kirksville, Missouri.
Griffith, W.; Superintendent of Public Schools, New London, Ohio, United States.
Griffiths, G. J., M.A.; Fell. Ch. Coll., Camb.
Griffiths, J., M.A.; Fellow of Jesus Coll., Oxon.
Grove, W. B., B.A.; Perry Bar, Birmingham.
Hadamard, Professor, M.A.; Paris.
Haigh, E., B.A., B.Sc.; King's Sch., Warwick.
Hall, Professor Asaph, M.A.; Washington.
Hammond, J., M.A.; Buckhurst Hill, Essex.
Harkema, C.; University of St. Petersburg.
Harley, Harold, B.A.; King's Coll., Cambridge.
Harley, Rev. R., F.R.S.; Huddersfield College.
Harris, H. W., B.A.; Trinity College, Dublin.
Harris, J. R., M.A.; Clare College, Cambridge.
Hart, Dr. David S.; Stonington, Connecticut.
Hart, H.; R.M. Academy, Woolwich.
Haughton, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
Hendricks, J. E., M.A.; Des Moines, Iowa.
Heppel, G., M.A.; The Grove, Hammersmith.
Herbert, A., M.A.; King Alfred's Sch., Wantage.
Herman, R. A., M.A.; Trin. Coll., Cambridge.
Hermite, Ch.; Membre de l'Institut, Paris.
Hill, Rev. E., M.A.; St. John's College, Camb.
Hinton, C. H., M.A.; Cheltenham College.
Hirst, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
Holt, J. R., M.A.; Trinity College, Dublin.
Hopkins, Rev. G. H., M.A.; Stratton, Cornwall.
Hopkinson, J., D.Sc., B.A.; Kensington.
Hudson, C. T., LL.D.; Manilla Hall, Clifton.
Hudson, W. H. H., M.A.; Prof. in King's Coll., Lond.
Ingleby, C. M., M.A., LL.D.; London.
Jenkins, Morgan, M.A.; London.
Johnson, W. E., B.A.; King's Coll., Cambridge.
Johnson, Prof., M.A.; Annapolis, Maryland.
Johnston, Swift; Trin. Coll., Dublin.
Jones, H. S., M.A.; Llanelly.
Jones, L. W., B.A.; Merton College, Oxford.
Kealy, J. A., M.A.; Wilmington, Delaware.
Kennedy, D., M.A.; Catholic Univ., Dublin.
Kirkman, Rev. T. P., M.A., F.R.S.; Croft Rect.
Kitchin, Rev. J. L., M.A.; Heavitree, Exeter.
Kittudge, Lizzie A.; Boston, United States.
Knisely, Rev. U. J.; Newcomerstown, Ohio.
Knowles, R., B.A., L.C.P.; Tottenham.
Koehler, J.; Rue St. Jacques, Paris.
Lachlan, R., B.A.; Lewisham.
Larmor, J., M.A.; Queen's College, Galway.
Laverty, W. H., M.A.; Public Examiner in the University of Oxford.
Lawrence, E. J.; Ex-Fell. Trin. Coll., Camb.
Lax, W. G., B.A.; Trinity College, Cambrid-e.
Leidhold, R., M.A.; Finsbury Park.
Leudesdorf, C., M.A.; Fel. Pembroke Coll., Oxon.
Levett, R., M.A.; King Edw. Sch., Birmingham.
Lowry, W. H., M.A.; Blackrock, Dublin.
Macdonald, W. J., M.A.; Edinburgh.
Macfarlane, A., D.Sc., F.R.S.E.; Ontario.
Mackenzie, J. L., B.A.; Gymnasium, Aberdeen.
MacMahon, Capt. P. A.; R. M. Academy.
MacMurchy, A., B.A.; Univ. Coll., Toronto.

McAlister, Donald, M.A., D.Sc.; Cambridge.
McCay, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
McClelland, W. J., B.A.; Prin. of Santry School.
McColl, H., B.A.; 73, Rue Sibliquin, Boulogne.
McDowell, J., M.A.; Pembroke Coll., Camb.
McIntosh, Alex., B.A.; Bedford Row, London.
McLeod, J., M.A.; R.M. Academy, Woolwich.
McVicker, C. E., B.A.; Trinity Coll., Dublin.
Malet, Prof., M.A.; Queen's Coll., Cork.
Mannheim, M.; Prof. à l'Ecole Polytech., Paris.
Marks, Sarah; Cambridge Street, Hyde Park.
Martin, Artemas, M.A., Ph.D.; Editor & Printer of *Math. Visitor* & *Math. Mag.*, Erie, Pa.
Martin, Rev. H., D.D., M.A.; Edinburgh.
Mathews, G. B., B.A.; Univ. Coll., N. Wales.
Matz, Prof., M.A.; King's Mountain, Carolina.
Mee, W. M., B.A.; Belturbet.
Merrifield, J., LL.D., F.R.A.S.; Plymouth.
Merriman, Mansfield, M.A.; Yale College.
Meyer, Mary S.; Girton College, Cambridge.
Miller, W. J. C., B.A., (Editor); The Paragon, Richmond-on-Thames.
Minchin, G.M., M.A.; Prof. in Cooper's Hill Coll.
Mitcheson, T., B.A., L.C.P.; City of London Sch.
Monck, Henry Stanley, M.A.; Prof. of Moral Philosophy in the University of Dublin.
Moncourt, Professor; Paris.
Moon, Robert, M.A.; Ex-Fell. Qu. Coll., Camb.
Moore, H. K., B.A.; Trin. Coll., Dublin.
Morel, Professor; Paris.
Morgan, C., B.A.; Salisbury School.
Morley, Frank, B.A.; Bath Coll., Bath.
Morley, T., L.C.P.; Bromley, Kent.
Morrice, G. G., B.A.; The Hall, Salisbury.
Moulton, J. F., M.A.; Fell. of Ch. Coll., Camb.
Muir, Thomas, M.A., F.R.S.E.; Bishopton.
Mukhopadhyay, Asutosh, M.A.; Bhowanipore.
Nash, A. M., M.A.; Prof. in Pres. Coll., Calcutta.
Nelson, R. J., M.A.; Naval School, London.
Newcomb, Prof. Simon, M.A.; Washington.
Nicolls, W., B.A.; St. Peter's Coll., Camb.
Openshaw, Rev. T. W., M.A.; Clifton.
O'Regan, John; New Street, Limerick.
Orchard, H. L., M.A., L.C.P.; Burnham.
Owen, J. A., B.Sc.; Tennyson St., Liverpool.
Panton, A. W., M.A.; Fell. of Trin. Coll., Dublin.
Pendlebury, Rev. C., M.A.; London.
Perryman, W.; Carbrook, Sheffield.
Phillips, F. B. W.; Balliol College, Oxford.
Pillai, C. K., M.A.; Trichy, Madras.
Pirie, A., M.A.; University of St. Andrew's.
Polignac, Prince Camille de; Paris.
Pollexfen, H., B.A.; Windermere College.
Potter, J., B.A.; Richmond-on-Thames.
Prudden, Frances E.; Lockport, New York.
Purser, Prof. F., M.A.; Queen's College, Belfast.
Putnam, K. S., M.A.; Rome, New York.
Rau, B. Hanumanta, B.A.; Head Master of the Normal School, Madras.
Rawson, Robert; Havant, Hants.
Raymond, E. Lancelot, B.A., Surbiton.
Ray, Prof. Saradaranjan, M.A.; Dacca Coll., Bengal.
Read, H. T., M.A.; Brasenose Coll., Oxford.
Reeves, G. M., M.A.; Lee, Kent.
Renshaw, S. A.; Nottingham.
Reynolds, B., M.A.; Notting Hill, London.
Richardson, Rev. G., M.A.; Winchester.
Rippin, Charles R., M.A.; Woolwich Common.
Roberts, R. A., M.A.; Schol. of Trin. Coll., Dublin.
Roberts, S., M.A., F.R.S.; Tufnell Park, London.
Roberts, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
Roberts, W. R., M.A.; Ex-Sch. of Trin. Coll., Dub.
Robson, H. C., B.A.; Sidney Sussex Coll., Cam.
Rosenthal, L. H.; Scholar of Trin. Coll., Dublin.
Roy, Prof. Kaliprasanna, M.A.; St. John's Coll., Agra.
Rücker, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
Ruggero, Simonelli; Università di Roma.
Russell, J. W., M.A.; Merton Coll., Oxford.
Russell, R., B.A.; Trinity College, Dublin.
Rutter, Edward; Sunderland.
Salmon, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
Sampson, C. H., M.A.; Balliol College, Oxford.
Sanders, J. B.; Bloomington, Indiana.
Sanderson, Rev. T. J., M.A.; Royston, Cambs.
Sarkar, Nilkantha, M.A.; Calcutta.
Savage, Thomas, M.A.; Fell. Pemb. Coll., Camb.
Scheffer, Professor; Mercersbury Coll., Pa.
Scott, A. W., M.A.; St. David's Coll., Lampeter.
Scott, Charlotte A., B.Sc.; Manchester.
Scott, R. F., M.A.; Fell. St. John's Coll., Camb.
Serret, Professor; Paris.
Sharp, W. J. C., M.A.; Hill Street, London.
Sharpe, J. W., M.A.; The Charterhouse.
Sharpe, Rev. H. T., M.A.; Cherry Marham.
Shepherd, Rev. A. J. P., B.A.; Fellow of Queen's College, Oxford.
Simmons, Rev. T. C., M.A.; Christ's Coll., Brecon.
Siverly, Walter; Oil City, Pennsylvania.
Smith, C., M.A.; Sidney Sussex Coll., Camb.
Stabenow, H., M.A.; New York.
Steggall, Prof. J. E. A., M.A.; Clifton.
Stein, A.; Venice.
Stephen, St. John, B.A.; Caius Coll., Cambridge.
Stewart, H., M.A.; Framlingham, Suffolk.
Storr, G. G., B.A.; Blackburn.
Swift, C. A., B.A.; Grammar Sch., Weybridge.
Sylvester, J. J., D.C.L., F.R.S.; Professor of Mathematics in the University of Oxford, Member of the Institute of France, &c.
Symons, E. W., M.A.; Fell. St. John's Coll., Oxon.
Tait, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
Tanner, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
Tarleton, F. A., M.A.; Fell. Trin. Coll., Dub.
Taylor, Rev. C., D.D.; Master of St. John's College, Cambridge.
Taylor, H. M., M.A.; Fell. Trin. Coll., Camb.
Taylor, W. W., M.A.; Ripon Grammar School.
Tebay, Septimus, B.A.; Farnworth, Bolton.
Terry, Rev. T. R., M.A., Fell. Magd. Coll., Oxon.
Thomas, Rev. D., M.A.; Garsington Rect., Oxford.
Thomson, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Camb.; Brinkley Rectory, Newmarket.
Tirelli, Dr. Francesco; Univ. di Roma.
Todhunter, Isaac, F.R.S.; Cambridge.
Torelli, Gabriel; University of Naples.
Torry, Rev. A. F., M.A.; St. John's Coll., Camb.
Traill, Anthony, M.A., M.D.; Fellow and Tutor of Trinity College, Dublin.
Trowbridge, David; Waterburgh, New York.
Tucker, R., M.A.; Mathematical Master in University College School, London.
Turrell, I. H.; Cumminsville, Ohio.
Turriff, George, M.A.; Aberdeen.
Vincenzo, Jacobini; Università di Roma.
Vose, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
Walenn, W. H.; Mem. Phys. Society, London.
Walker, J. J., M.A., F.R.S.; Hampstead.
Walmsley, J., B.A.; Eccles, Manchester.
Warburton-White, R., B.A.; Salisbury.
Warren, R., M.A.; Trinity College, Dublin.
Watherston, Rev. A. L., M.A.; Bowdon.
Watson, Rev. H. W.; Ex-Fell. Trin. Coll., Camb.
Wertsch, Fr.; Weimar.
White, J. R., B.A.; Worcester Coll., Oxford.
White, Rev. J., M.A.; Cowley College, Oxford.
Whiteside, G., M.A.; Eccleston, Lancashire.
Whitworth, Rev. W. A., M.A.; Fellow of St. John's Coll., Camb.; Hammersmith.
Wickersham, D.; Clinton Co., Ohio.
Wilkins, W.; Scholar of Trin. Coll., Dublin.
Williamson, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
Wilson, J. M. [illegible] Clifton Coll.
Wilson, Rev [illegible] kburn Acad.
Wilson, P [illegible], Cambs.
Woodcoc [illegible] Salop.
Wolsten [illegible] D.; Professor of M [illegible] l College.
Woolho [illegible].; London
Wright [illegible], New York.
Wright [illegible]
Young, [illegible] londerry.

CONTENTS.

Mathematical Papers, &c.

Questions Solved.

Appendix.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7706. (By the late Professor CLIFFORD, F.R.S.)—What conditions must be fulfilled in order that the centre of pressure of a triangle wholly submerged in water may be at the intersection of perpendiculars?

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

When the vertices A, B, C of a triangle are sunk in a homogeneous liquid to depth h_1, h_2, h_3, respectively, then x, y, z, the coordinates of the centre of pressure of the triangle referred to the sides as axes, are given by the equations:—

$$x = \frac{p_1}{4}\left\{1+\frac{h_1}{h_1+h_2+h_3}\right\},\quad y = \frac{p_2}{4}\left\{1+\frac{h_2}{h_1+h_2+h_3}\right\},$$

$$z = \frac{p_3}{4}\left\{1+\frac{h_3}{h_1+h_2+h_3}\right\}$$

(see *Messenger of Mathematics*, No. 8, 1864); but the coordinates of the intersection of the perpendiculars p_1, p_2, p_3 are

$$x = p_1 - \frac{b\cos A}{\sin B} = p_1\left(1-\frac{\cos A}{\sin B\sin C}\right),$$

hence we have $p_1\left(1-\dfrac{\cos A}{\sin B\sin C}\right) = \dfrac{p_1}{4}\left(1+\dfrac{h_1}{h_1+h_2+h_3}\right)$,

$$\therefore\quad \frac{h_1}{h_1+h_2+h_3} = 3-\frac{4\cos A}{\sin B\sin C} = 3+\frac{4\cos(B+C)}{\sin B\sin C} = 4\cot B\cot C-1,$$

and two similar equations;

$\therefore\ h_1 : h_2 : h_3 :: 4\cot B\cot C-1 : 4\cot A\cot C-1 : 4\cot A\cot B-1.$

[For a discussion of this problem, and several other applications of the above formulæ for x, y, z, see *Messenger of Mathematics*, New Series, No. 12, 1872, where it is shown that, if the vertices of a triangle be sunk to depths h_1, h_2, h_3, and therefore the mid-points of the sides to depths

$H_1 = (h_2+h_3)$, $H_2 = \frac{1}{2}(h_1+h_3)$, $H = \frac{1}{2}(h_1+h_2)$, when the centre of pressure coincides with :

	Condition.
(1) Centre of gravity	$H_1 = H_2 = H_3$,
(2) Centre of circumscribing circle	$H_1 : H_2 : H_3 = \tan A : \tan B : \tan C$,
(3) Centre of inscribed circle	$H_1 : H_2 : H_3 = \cot \frac{1}{2}A : \cot \frac{1}{2}B : \cot \frac{1}{2}C$,
(4) Intersection of the three perpendiculars from vertices on opposite sides	$H_1 : H_2 : H_3 = 1 - 2\cot B \cot C : 1 - 2\cot A \cot C : 1 - 2\cot A \cot B$,
(5) Centre of inscribed square (alongside a)	$H_1 : H_2 : H_3 = 1 : \cot B : \cot C$,
(6) Centre of the nine-point circle	$H_1 : H_2 : H_3 = \sin 2A : \sin 2B : \sin 2C$;

hence it follows that, if a triangle be so immersed that its centre of pressure coincides with the intersection of the three perpendiculars from vertices on the opposite sides, then the centre of pressure of the triangle obtained from the given one by joining the middle points of its sides will coincide with the centre of the circle circumscribed to this derived triangle, and the centre of pressure of the triangle similarly derived from it will coincide with the centre of the nine-point circle of this last derived triangle.]

7708. (By Professor Townsend, F.R.S.)—A thin uniform spherical cap being supposed to attract according to the law of the inverse fifth power of the distance a material particle situated anywhere on the surface of the sphere; show that, for every position of the particle, the attraction (a) passes through the vertex of the cone which envelopes the sphere along the rim of the cap, (b) varies directly as the radial distance from the vertex of the cone, and inversely as the cube of the perpendicular distance from the base of the cap.

Solution by the Proposer.

Two pairs of planes inclined at elementary angles, one pair passing through the line of connection of the particle with the vertex of the cone, and the other pair passing through its polar with respect to the sphere, which latter lies of course in the rim-plane or base of the cap, will intercept in the mass of the cap a pair of quadrilateral elements whose attractions on the particle for the law of the inverse fifth power of the distance are easily seen to compound a resultant directed to the vertex of the cone; and, as the entire mass of the cap may manifestly be exhausted by pairs of such elements, therefore, &c., as regards the first part of the property.

As regards the second part. The integration based on the preceding method of division of the cap into elements leads to the result that the entire attraction of the cap on the particle $= \frac{m(a+b)b}{16a^5} \cdot \frac{r}{p^3}$; where a is the radius of the sphere, m the mass of the cap, b the distance of its base from the centre of the sphere, r the distance of the particle from the vertex of the cone, and p its distance from the base of the cap. And therefore, &c., as regards the second part also.

7709. (By Professor MINCHIN, M.A.)—A cylindrical bar of isotropic material is subject to uniform intensity of pressure over its *curved* surface; prove that, if M denote the "modulus of cylindric squeeze," while k and μ denote the resistances to cubical squeeze and to distortion, then $M = \dfrac{9k\mu}{3k+4\mu}$.

Solution by J. BRILL, B.A.; Prof. MATZ, M.A.; *and others.*

Taking the axis of z along the axis of the cylinder, measuring z from the middle point of the axis, and assuming $u = -ax$, $v = -ay$, $w = cz$, we have

$$N_1 = N_2 = (k - \tfrac{2}{3}\mu)(c-2a) - 2\mu a,$$
$$N_3 = (k - \tfrac{2}{3}\mu)(c-2a) + 2\mu c.$$

Since there is no force on the ends of the bar, we must have $N_3 = 0$,

therefore $0 = (k-\tfrac{2}{3}\mu)(c-2a) + 2\mu c$; therefore $c = 2a\dfrac{3k-2\mu}{3k+4\mu}$;

$$\therefore \quad M = \frac{[(k-\tfrac{2}{3}\mu)(c-2a)-2\mu a]}{-2a} = (k-\tfrac{2}{3}\mu)\left(1-\frac{3k-2\mu}{3k+4\mu}\right)+\mu$$
$$= \tfrac{1}{3}\left\{3k+\mu-\frac{(3k-2\mu)^2}{3k+4\mu}\right\} = \frac{9k\mu}{3k+4\mu}.$$

6251. (By the EDITOR.)—One of the diagonals of a regular quindecagon is drawn at random, and then the process is repeated; show that (1) the probability of the chosen diagonals being such as cross within the perimeter is $\frac{91}{267}$ if the two must be distinct, and $\frac{91}{270}$ if the second may be identical with the first; (2) the like probabilities for a regular $(2n+1)$-gon are $\frac{1}{3}(2n^2-n)$ divided, in the two cases respectively, by $[(2n+1)(n-1)-1]$ or $[(2n+1)(n-1)]$; and hence (3) the chance of two random chords meeting within a circle is $\frac{90}{270}$ or $\frac{1}{3}$.

Solution by D. BIDDLE, *and the* PROPOSER.

1. From each of the 15 angles there are 12 diagonals, or in all $180 \div 2 = 90$. The 6 which cross one-half of the figure are similar to the other 6. Taking the one 6 in order, we find that they are crossed by diagonals as follows:—I. by 12; II. by $(12-1)\times 2 = 22$; III. by $(12-2)\times 3 = 30$; IV. by $(12-3)\times 4 = 36$; V. by $(12-4)\times 5 = 40$; VI. by $(12-5)\times 6 = 42$. There are accordingly 6 differently circumstanced sets of diagonals, and the probability as to which of these is selected in the first diagonal drawn is $\frac{1}{6}$. For the second, there are 89 to choose from if the two must be distinct, 90 if the choice may be identical; and of these a particular number, as specified above, cross the first, but the rest do not. Hence the total probability that the chosen diagonals will cross within the perimeter is $(12+22+30+36+40+42)$, or 182, divided by $6 \cdot 89$ or $6 \cdot 90$ in the two cases respectively, which give the stated results.

2. For a regular $(2n+1)$-gon, there are $(n-1)$ different cases, all equally probable, since the first diagonal may cut off 1, 2, 3, ..., $(n-1)$ corners of

the polygon; and the probability that the second diagonal will cross the first is, in each of these several cases,

$$(2n-2),\ 2(2n-3),\ 3(2n-4),\ \ldots,\ (n-1)(2n-n),$$

divided (N being the total number of diagonals) by $N-1$ or N, according as we *ex*clude or *in*clude the first diagonal in the second drawing; and, as $N=(2n+1)(n-1)$, this leads at once to the results stated in the question.

3. If we suppose n to increase without limit, the polygon becomes, in the limit, a circle whereof the diagonals are chords, and the limit of the fraction that measures the probability is $\frac{1}{3}$.

Otherwise.—The crossing or not, within the circumference, of two random chords of a circle, will be governed by the same law. Let C be the circumference of a given circle, and A the arc of a given chord in it. Then $\frac{2A(C-A)}{C^2}$ will represent the proportion of chords crossing the given chord (the 2 in the numerator arising from C^2 in the denominator being divided by 2, to prevent our taking chords twice over, that is, from both ends). And we can form a series with $A=0$ at one extremity, and $A=\frac{1}{2}C$ at the other, a diameter being the longest chord possible. In the former $\frac{2A(C-A)}{C^2}=0$, in the latter $\frac{1}{2}$. Let $C=1$, then our formula becomes $2A(1-A)$, and, since $1-A=A$ in the first term of the descending series, we can construct a new formula, viz., $2(\frac{1}{2}-b)(\frac{1}{2}+b)$, where $b=$ the portion of C deducted from one factor and added to the other, to form any given term of the series; and this simplified is $2(\frac{1}{4}-b^2)$. Moreover, b^2 gradually increases from 0 to $\frac{1}{4}$, that is $(\frac{1}{2})^2$; and as $2(\frac{1}{4}-b^2)$ represents the probability, as to being crossed, of the chord of any given arc, we need only sum the series and divide by the number of terms, to obtain the average probability. Now the average of such terms, where each consists of two square numbers, equal at the beginning, but one constant and the other gradually diminishing to zero, is equivalent to the mean-section-area of the space intervening between a hollow cylinder, of given height and internal diameter, and a cone of the same height, whose base just fits the cylinder. Let the height $=1$, and the base $=\frac{1}{4}$, then the area of the cone $=\frac{1}{12}\cdot 1^3$, and its mean transverse section-area $=\frac{1}{12}\cdot 1^2=$ mean value of b^2 in above formula. Hence $2(\frac{1}{4}-b^2)$ is on the average $\frac{1}{3}$, the probability stated in the question.

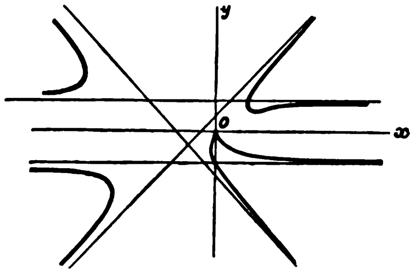
6607. (By Christine Ladd Franklin, B.A.)—Trace the curve whose equation is $(y^2-ax)^2=y^2(x^2-ay)$, and test the correctness of the form by applying Descartes' rule of signs to determine limits to the number of points of intersection of the curve by lines parallel to the axis of y.

Solution by the Rev. T. C. Simmons, M.A.

The given equation $\equiv y^4-x^2y^2-2ay^2x+ay^3+a^2x^2=0$, whence

$$x=\left[-a\pm(y^2+ay-a^3y^{-1})^{\frac{1}{2}}\right]/(1-a^2y^{-2})$$
$$=-y-\tfrac{3}{2}a-\tfrac{5}{8}a^2y^{-1}+\ldots \text{ or } y-\tfrac{1}{2}a+\tfrac{5}{8}a^2y^{-1}+\ldots,$$

whence $x = -y - \frac{3}{2}a$ and $x = y - \frac{1}{2}a$ are asymptotes, having the curves situated on the sides shown in the figure; also $y = \pm a$ are asymptotes.



Near the origin

$y^3 + ax^2 = 0$ approximately;

also

$$\frac{dy}{dx} = \frac{2y^2x + 2ay^2 - 2a^2x}{4y^3 - 2yx^2 + 3ay^2 - 4ayx}$$

$= 0$ at intersection of (1) with $y^2x + ay^3 - a^2x = 0$, which is the same as the intersection of $xy = a^2$ with

$$y^2 - ax + ay = 0,$$

giving $x = \frac{4}{3}a$, $y = \frac{3}{4}a$ nearly. The value of x shows that $y^4 + ay^3$ must be $> a^3y$, so that y cannot lie between 0 and $\frac{3}{4}a$.

At the points $(0, -a)$, (a, a), $dy : dx$ is -2 and 2 respectively. To find approximately the contour of the curve towards the left, putting $y = 2a$, x becomes $-\frac{13}{3}a$ nearly and $dy : dx = -\frac{8}{9}$; and putting $y = -2a$, x becomes $-\frac{10}{3}a$ nearly and $dy : dx = -\frac{8}{5}$; hence the figure is as annexed.

[As to the latter part of the question, writing the equation in the form $y^4 + ay^3 - (x^2 + 2ax)\,y^2 + a^2x^2 = 0$, we see that, by Descartes' rule of signs, when $x^2 + 2ax$ is positive, there cannot be more than two positive nor more than two negative values of y; and that, when $x^2 + 2ax$ is negative, that is, when x lies between 0 and $-2a$, there cannot be *any* such positive values, and not more than two negative ones.]

7710. (By Professor Cochez.)—Parmi les courbes planes uniformément pesantes de même longueur, passant par deux points fixes, quelle est celle dont le centre de gravité est le plus bas?

Solution by Arthur Hill Curtis, LL.D., D.Sc.

When any system of heavy particles are in equilibrium under the action of gravity alone, the position of stable equilibrium is such that the centre of gravity of the system is the lowest possible; as a particular case, a string of constant or variable density throughout will assume, as its position of equilibrium, the curve which satisfies this condition; the curve required in the question is consequently the catenary.

The Symmedian-Point Axis of a System of Triangles.

By R. Tucker, M.A.

Through the angular points of the triangle ABC straight lines are drawn parallel to the opposite sides forming the triangle A′B′C′: the process is repeated indefinitely. This is the system herein considered.

S = No. of summits $= a_3 + a_4 + a_5 +$ &c., $f =$ No. of faces $= f_3 + f_4 + f_5 +$ &c.,
$S + f = (a_3 + f_3) + a_4 + f_4 +$ &c. $= e + 2 = 2 + \frac{1}{4}[3(a_3 + f_3) + 4(a_4 + f_4) +$ &c.$]$,
therefore $a_3 + f_3 = 8 + a_5 + f_5 + 2(a_6 + f_6)$.

If $f_5 = f_6 = 0$, $f = f_3 = 8 - a_3 + a_5 + 2a_6 +$ &c. $= 8 + \Sigma(p-4)\,a_p$,
therefore $f - \Sigma(p-4)\,a_p = 8$, which proves the proposition.

7729. (By B. Reynolds, M.A.)—Show that the number of shortest routes from one corner of a chess-board to the opposite one, along the edges of the squares, is 12870.

Solution by D. Biddle.

The corners of squares being junctions, we may observe that from every junction, except on the two edges of the board which terminate in the goal, either of two directions may be taken; and that the number of shortest routes possible after reaching any junction, is the sum of the shortest routes possible from the two adjacent junctions over one or other of which we must next pass. Accordingly the total number of possible shortest routes from the extreme corner can be found by forming successive series of numbers, beginning with units and making the successive terms (except the first) of one series the successive differences of the next.

The following table gives the several series, and in so doing the number of shortest routes possible from every junction on the board, A being the goal:—

A	1,	1,	1,	1,	1,	1,	1,	1,
1,	2,	3,	4,	5,	6,	7,	8,	9,
1,	3,	6,	10,	15,	21,	28,	36,	45,
1,	4,	10,	20,	35,	56,	84,	120,	165,
1,	5,	15,	35,	70,	126,	210,	330,	495,
1,	6,	21,	56,	126,	252,	462,	792,	1287,
1,	7,	28,	84,	210,	462,	924,	1716,	3003,
1,	8,	36,	120,	330,	792,	1716,	3432,	6435,
1,	9,	45,	165,	495,	1287,	3003,	6435,	12870.

[Suppose we have to proceed, as quickly as possible, from O (the left-hand corner at bottom) to C (the right-hand corner at top of square). Let a denote the operation of moving horizontally over the edge of one square from left to right, and let b denote the operation of moving one step upwards. Then we must perform 8 a's and 8 b's in any order; hence the total number of ways is $(16!)/(8!\,8!)$, which reduces to 12870.]

7341. (By A. Martin, B.A.)—Solve the equations

$$yz(y+z-x) = a, \quad zx(z+x-y) = b, \quad xy(x+y-z) = c.$$

7667. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Prove that, if n be a positive integer,

$$\frac{1+\frac{1}{2}\sin^2\theta+\frac{1.3}{2.4}\sin^4\theta+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\sin^{2n}\theta}{1-n\frac{\cos^2\theta}{3}+\frac{n(n-1)}{2!}\frac{\cos^4\theta}{5}-\frac{n(n-1)(n-2)}{3!}\frac{\cos^6\theta}{7}+\ldots+(-1)^n\frac{\cos^{2n}\theta}{2n+1}}$$

$$=\frac{1+(\frac{1}{2})^2+\left(\frac{1.3}{2.4}\right)^2+\ldots+\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)^2}{1-n.\frac{1}{2.3}+\frac{n(n-1)}{2!}.\frac{1.3}{2.4.5}-\ldots+(-1)^n\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n(2n+1)}}$$

$$=\frac{3.5\ldots(2n+1)}{2.4.6\ldots 2n}.$$

Solution by G. HEPPEL, M.A.; E. BUCK, M.A.; *and others.*

Of the three members of the identity to be established, call the first $\frac{h}{k}$, the second $\frac{l}{m}$, and the third p; then we have

$$\frac{h}{k}=\frac{-h\cos\theta}{-k\cos\theta}=\frac{p\int\sin^{2n+1}\theta\,d\theta}{\int\sin\theta\,(1-\cos^2\theta)^n\,d\theta}=p,$$

also
$$\frac{l}{m}=\frac{\int_0^{\frac{1}{2}\pi}p\sin^{2n+1}d\theta}{\int_0^{\frac{1}{2}\pi}\sin\theta\,(1-\cos^2\theta)^n\,d\theta}=p.$$

7654. (By ASÛTOSH MUKHOPÂDHYÂY, M.A.)—A magnetic needle is free to revolve in a horizontal plane round a fixed point in the line joining its poles; if it is acted on by an indefinitely extending vertical galvanic current: find (1) the positions of equilibrium; (2) the cases wherein there is no position of equilibrium; and (3) when the positions of stable and unstable equilibrium are directly opposed.

Note by the EDITOR.

A full discussion of this Question, with the results sought—too long for our pages—is to be found in Dr. CURTIS's paper, published in The *Oxford, Cambridge, and Dublin Messenger of Mathematics* (No. 8, 1864).

7604. (By Rev. T. P. KIRKMAN, M.A., F.R.S.) — If a_p be the number of the p-aces on a triangular-faced n-edron, prove that $n-S_p(p-4)a_p$ is a cube.

Solution by EDWARD BUCK, M.A.

In any polyedron, if f_p be the number of p-gons (faces with p-sides),

$$e=\text{total number of edges}=\tfrac{1}{2}[3f_3+4f_4+5f_5+\&c.]=\tfrac{1}{2}[3a_3+4a_4+5a_5+\&c.]$$
$$=\tfrac{1}{4}[3(a_3+f_3)+4(a_4+f_4)+\&c.],$$

(1) express $(F_1)^n$ in terms F_1, F_2, F_3, &c.; and (2) I being an invariant of the i^{th} order of $(a_0, a_1, a_2 \dots a_n)(x, y)^n$, which becomes I' when every suffix is increased by unity, show that $I = \phi I'$, where

$$\phi = \Sigma \frac{E_r^\lambda . E_s^\mu . E_t^\nu \dots}{\lambda! \mu! \nu! \dots},$$

and $\lambda, \mu, \nu \dots r, s, t \dots$ are any integers satisfying the equation

$$\lambda r + \mu s + \nu t + \dots = i.$$

Solution by W. J. C. SHARP, M.A.

Let $F_1 = F_1' + F_1''$, where F_1 applies only to the function operated upon, and F_1'' only to the a's as involved in the operative symbols. Then

$$F_1''F_1' = 2!\,F_2', \quad (F_1'')^2 F_1' = 2F_1''F_2' = 3!\,F_3',$$

and generally $\quad (F_1'')^n F_1' = (n+1)!\,F_n'$;

therefore $\quad (F_1)^2 = (F_1' + F_1'')\,F_1 = 2!\left(\frac{F_1'^2}{2!} + F_2'\right),$

$$(F_1)^3 = (F_1' + F_1'')(F_1)^2 = 3!\left(\frac{F_1'^3}{3!} + F_1'F_2' + F_3'\right), \text{ \&c. \&c.}$$

and generally $\quad \frac{(F_1)^n}{n!} = \Sigma \frac{F_r'^\lambda . F_s'^\mu . F_t'^\nu \dots}{\lambda! \mu! \nu! \dots},$

where $\quad \lambda r + \mu s + \nu t + \dots = n.$

Again, I' is the same invariant of $v \equiv (a_1, a_2, \dots a_{n+1})(x, y)^n$, that I is of u, and the corresponding invariant of $v + \theta u \equiv (1 + \theta E_1)\,v$,

$$i.e., \left(1 + \frac{\theta E_1}{1!} + \frac{\theta^2 E_1^2}{2!} + \dots + \frac{\theta^i E_1^i}{i!}\right) I', \quad \therefore\ I = \frac{E_1^i}{i!} I'.$$

But, since I' is an invariant of v,

$$\left(a_1 \frac{d}{da_2} + 2a_2 \frac{d}{da_3} + \text{\&c.}\right) I' = 0,$$

and consequently $E_1 I' = F_1 I'$ and $E_1^i I' = F_1^i I$ and $E_r^\lambda I' = F_r^\lambda I'$, therefore

$$I = \frac{E_1^i}{i!} I' = \frac{F_1^i}{i!} I' = \Sigma \frac{F_r'^\lambda . E_s'^\mu . F_t'^\nu \dots}{\lambda! \mu! \nu! \dots} = \Sigma \frac{E_r^\lambda . E_s^\mu . E_t^\nu \dots}{\lambda! \mu! \nu! \dots} I',$$

where the E only operate on I'.

[A solution of Quest. 6218 is given in *Reprint*, Vol. 34, p. 99, but it is inaccurate, from the omission of the coefficients in the preliminary equations. A solution of Quest. 7454 is given on p. 113 of Vol. 40 of *Reprints*.]

7664. (By Professor CROFTON, F.R.S.)—Prove that the chance of heads turning up twice running during r tosses of a coin is equal to the chance of a run of three (either heads or tails) during $(r+1)$ tosses.

Solutions by (1) E. L. RAYMOND, M.A.; (2) D. BIDDLE.

1. Let α be the chance of two heads turning up in the first two tosses; β the chance of two heads turning up in the second and third tosses (and not before); ρ the chance that two heads turn up in the $(r-1)^{\text{th}}$ and r^{th} tosses (and not before); then the chance of turning up two heads running in r tosses is

$$\alpha+\beta+\ldots+\rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

Now the chance of turning up a head in the third throw is, of course $\frac{1}{2}$; hence $\frac{1}{2}\alpha$ is the chance of the first three throws being all heads; $\frac{1}{2}\beta$ is the chance of the second, third, and fourth throws being all heads; $\frac{1}{2}\rho$ is the chance of the $(r-1)^{\text{th}}$, r^{th}, and $(r+1)^{\text{th}}$ tosses being all heads; and thus the chance of turning up three heads running in $(r+1)$ tosses is

$$\tfrac{1}{2}(\alpha+\beta+\ldots+\rho) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

and, of course, the chance of three tails being turned up running in $(r+1)$ tosses is

$$\tfrac{1}{2}(\alpha+\beta\ldots+\rho) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

Hence the chance of a run of three (heads or tails) in $(r+1)$ tosses is

$$\alpha+\beta\ldots+\rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(D)},$$

which agrees with (A).

2. This proposition postulates two things which themselves require proof, viz., (1) that at every toss of a coin, there is an equal chance as to head or tail turning up, and (2) that all the possible results of r tosses are equally probable. But if *head* have already been turned up twice, it seems only reasonable to assign a greater probability to *tail* for the third, and as the average results must needs be equal for head and tail, in the long run, if a fair coin be fairly tossed, it seems only reasonable to consider the series of events, however short, which approximates to this average, as more probable than one which consists (for instance) of one sort only. The possible ways in which the events may occur double with every increase in the number (r), as may be seen in the adjoining diagram, in which × and 0 represent the only possible results of the first toss; × ×, 0 ×, × 0, and 0 0, of the second; and so on. Consequently, if we consider the probability to be equal as to head or tail at the third toss, no matter what the other two may have yielded, then × × × or 0 0 0 will one or other turn up as the sum of the results once in four times, as × × is supposed to have done when r was 2; and as $\frac{3}{8}$ has been the probability of × × occurring together when r is 3, $\frac{6}{16}=\frac{3}{8}$ is the probability of × × × or 0 0 0 occurring when r is 4. But the matter is open to grave doubt. How do we know that × 0 and 0 × are not twice as probable as × × and 0 0? and × × 0 or 0 0 × four times as probable as × × × or 0 0 0?

r				
1, 2, 3, 4	×	×	×	×
1, 2, 3, 4	0	×	×	×
2, 3, 4	×	0	×	×
2, 3, 4	0	0	×	×
3, 4	×	×	0	×
3, 4	0	×	0	×
3, 4	×	0	0	×
3, 4	0	0	0	×
4	×	×	×	0
4	0	×	×	0
4	×	0	×	0
4	0	0	×	0
4	×	×	0	0
4	0	×	0	0
4	×	0	0	0
4	0	0	0	0

Let us suppose such to be the case. The probability of head turning up in the first toss will then be, as before, $\frac{1}{2}$; but of × × in two, only $\frac{1}{6}$, and of × × × in three, only $\frac{1}{30}$, or of × × × or 0 0 0 (one or other), $\frac{1}{15}$.

Yet in each column there is an equal number of each, *e.g.* in set 3, 15 × + 15 0.

3	×oooooooo××××××oooooo××××××××o
	×××××oooooooooo××××××××××ooooo
	×××××××××××××××ooooooooooooooo
2	×oo××o
	×××ooo
1	×o

What makes us doubt the fairness of a coin which turns up heads (or tails) 100 times in succession, if this be no less probable, in a fair coin fairly tossed, than 50 of each alternately ?

7665. (By Professor Townsend, F.R.S.)—The motion of a system of waves, propagated by small rectilinear vibrations in an isotropic elastic solid under the action of its internal elasticity only, being supposed to produce irrotational strain of the substance throughout the entire space and time of vibration; determine, given the coefficients μ and ν of resistance to changes of volume and form of the solid, the differential equation for the potential of the strain at any instant of the motion.

Solution by the Proposer.

The strain of the substance being by hypothesis irrotational throughout the entire space and time of vibration, the differential equations of the propagation of the wave system assume in consequence the simplified forms
$$\frac{d^2\xi}{dt^2} = a^2\frac{d\omega}{dx}, \quad \frac{d^2\eta}{dt^2} = a^2\frac{d\omega}{dy}, \quad \frac{d^2\zeta}{dt^2} = a^2\frac{d\omega}{dz},$$
where $a^2 = \rho^{-1}(\mu + \frac{4}{3}\nu)$ = square of velocity of plane waves propagated by normal vibrations in the substance, and $\omega = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}$ = the cubical dilatation at the point xyz of its mass. But, for irrotational strain of the entire vibrating mass throughout the entire time of vibration, we have also, for all values of x, y, z, and t, within the space and time of vibration, $\xi = \frac{d\phi}{dx}$, $\eta = \frac{d\phi}{dy}$, $\zeta = \frac{d\phi}{dz}$, where ϕ = the required potential of the motion. Therefore the equations of propagation of the wave system, the original coordinates x, y, z of every vibrating molecule being of course independent of the time t of the motion, may be written in the forms
$$\frac{d}{dx}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dx}, \quad \frac{d}{dy}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dy}, \quad \frac{d}{dz}\left(\frac{d^2\phi}{dt^2}\right) = a^2\frac{d\omega}{dz},$$
from which it follows at once that
$$\frac{d^2\phi}{dt^2} = a^2\omega = a^2\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right);$$
which accordingly is the general equation, in partial differences of the second order, for the potential ϕ of the strain as a function of the coordinates x, y, z of the molecule and the time t of the motion : an equation identical

in form with that for the potential of the propagation of sound in a uniform medium of indefinite extent, and leading of course to the same consequences so familiar in that case.

For plane waves advancing in any common direction in the substance: since then $x\cos\alpha+y\cos\beta+z\cos\gamma=p$, where α, β, γ are the direction angles of propagation and p the distance of any wave plane from the origins, and since consequently $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=\frac{d^2\phi}{dp^2}$, therefore the equation for the determination of ϕ assumes the simplified form $\frac{d^2\phi}{dt^2}=a^2\frac{d^2\phi}{dp^2}$, the complete integral of which in finite terms, viz., $\phi=k.f(p\pm at)$, where k is a small constant representing the absolute amplitude of the vibrations, and f any arbitrary periodic function oscillating within finite limits as in vibratory motion generally, represents two systems of waves advancing in opposite directions with the common velocity a; the amplitudes of vibration undergoing no change for either system with its progress through the substance, and no wave of either system giving rise to a wave of the opposite system in its passage through any portion of the mass external to the region of the original disturbance giving rise to the motion.

For spherical waves diverging from any common centre in the substance: since then $x^2+y^2+z^2=r^2$, where r is the radius of any wave sphere of the system; and since consequently

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr},$$

therefore the equation for the determination of ϕ assumes again the simplified form $\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right)$, or, in consequence of the entire independence of r and t, the equivalent form $\frac{d^2(r\phi)}{dt^2}=a^2\frac{d^2(r\phi)}{dr^2}$; the complete integral of which in finite terms, viz., $r\phi=k.f(r\pm at)$, where k and f are as before, represents again two systems of waves diverging in opposite directions from the common centre with the common velocity a, the amplitudes of vibration varying at considerable distances from the centre inversely as the distance for each, and no wave of either system giving rise to a wave of the opposite system in its passage through any portion of the mass external to the region of the original disturbance giving rise to the motion.

7170. (By the EDITOR.)—If 10 cards are taken at random from a pack, show that the respective probabilities (p_1, p_2) that they will contain (1) exactly 4 cards with hearts, (2) not more than 4 cards with hearts, are

$$p_1=\frac{27417}{185932}=\cdot 14746=\frac{5}{34}\text{ nearly},\quad p_2=\cdot 94307=\frac{32}{34}\text{ nearly}.$$

Solution by D. BIDDLE.

1. The following considerations will enable us to build up the required fraction :—$52!$ = possible permutations of the whole pack; $10!\,42!$ = ways

of arranging the pack when a particular set of 10 cards is partitioned off and constantly taken first: *e.g.*, 4 hearts and 6 other cards; $\frac{13!}{9!}$ = ways in which 4 cards can be taken out of 13; $\frac{13!}{9!\,4!}$ = different sets of 4 that can be made out of 13, disallowing mere differences of arrangements; $\frac{39!}{33!}$ and $\frac{39!}{33!\,6!}$ = corresponding numbers in regard to 6 cards out of 39. $10!\,42!\,\frac{13!}{9!\,4!}\cdot\frac{39!}{33!\,6!}$ = number of ways in which 4 cards of one kind and 6 cards of other kinds can be arranged on one side of a partition, and the remainder of the pack on the other side, by allowing interchange of equivalent cards across the partition, and permutation on each side. Hence the probability that 10 cards drawn at random from a pack will contain exactly 4 hearts is $\frac{10!\,42!\,13!\,39!}{52!\,9!\,4!\,33!\,6!}$. This, reduced, is ·14746 nearly, or rather over $\frac{5}{34}$. Consequently, a person accepting odds of 6 to 1, laid against the occurrence, should win in the long run.

2. The probability that the 10 cards contain not more than 4 hearts is the sum of the respective probabilities concerning 4, 3, 2, 1, 0 which (as found by the formula in Art. 1) are as follows:—4 hearts = ·14746; 3 h. = ·27807; 2 h. = ·30334; 1 h. = ·17404; and 0 h. = ·04016. But the sum may be stated in the following manner:—

$$\frac{10!\,42!\,13!\,39!}{52!}\left(\frac{1}{9!\,4!\,33!\,6!}+\frac{1}{10!\,3!\,32!\,7!}+\frac{1}{11!\,2!\,31!\,8!}+\frac{1}{12!\,1!\,30!\,9!}+\frac{1}{13!\,29!\,10!}\right).$$

Hence *not more than* 4 *h.* = ·94307. This is slightly over $\frac{32}{34}$.

[Putting $C_n^{(r)}$ for the number of combinations of n things taken r together, the total number of ways and the number of favourable cases in (1) are respectively $C_{52}^{(10)}$, $C_{13}^{(4)}\times C_{39}^{(6)}$ (since each group of 4 hearts out of 13 may be combined with each group of 6 non-hearts out of 39); hence the probability in question is $C_{13}^{(4)}\times C_{39}^{(6)}\div C_{52}^{(10)}$, which gives the first result; and the sum of 5 such probabilities gives the second result.]

6456. (By G. Heppel, M.A.) — If the expansion of sec x be $1+\frac{u_2}{2!}x^2+\frac{u_4}{4!}x^4+$ &c., and that of 2 sec² x be $2+\frac{v_2}{2!}x^2+\frac{v_4}{4!}x^4+$ &c.; prove that the coefficients u_2, u_4, u_6, &c. may be found from the relations

$$u_{2n}=u_{2n-2}+\frac{2n-2!}{2!\,2n-4!}u_{2n-4}\cdot v_2+\frac{2n-2!}{4!\,2n-6!}u_{2n-6}\cdot v_4+\&c.,$$

and $$\tfrac{1}{4}v_{2n}=u_{2n}+\frac{2n!}{2!\,2n-2!}u_2u_{2n-2}+\frac{2n!}{4!\,2n-4!}u_4u_{2n-4}+\&c.,$$

taking care in the last series to stop before the first suffix exceeds the second, and to halve the coefficient when they become equal. Apply this method to verify the value of u_{10}, found (by another process) in De Morgan's *Differential Calculus*, to be 50521.

Solution by the Rev. T. C. Simmons, M.A.

Differentiating twice with respect to x the identity

$$\sec x \equiv 1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \ldots,$$

we have

$$\frac{u_{2n}}{2n-2!} = \text{coefficient of } x^{2n-2} \text{ in the expansion of } \sec x\,(2\sec^2 x - 1)$$

$$= \text{same coefficient in } \left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \ldots\right)\left(1 + \frac{v_2 x^2}{2!} + \frac{v^4 x^4}{4!} + \ldots\right)$$

$$= \frac{u_{2n-2}}{2n-2!} + \frac{u_{2n-4}}{2n-4!}\cdot\frac{v_2}{2!} + \frac{u_{2n-6}}{2n-6!}\cdot\frac{v_4}{4!} + \ldots,$$

whence follows the first required relation.

Again, $$\frac{1}{4}\,\frac{v_{2n}}{2n!} = \text{coefficient of } x^{2n} \text{ in } \tfrac{1}{2}\sec^2 x$$

$$= \text{same coefficient in } \frac{1}{2}\left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \ldots\right)\left(1 + \frac{u_2 x^2}{2!} + \frac{u_4 x^4}{4!} + \ldots\right)$$

$$= \frac{u_{2n}}{2n!} + \frac{u_{2n-2}}{2n-2!}\cdot\frac{u_2}{2!} + \frac{u_{2n-4}}{2n-4!}\cdot\frac{u_4}{4!} + \ldots,$$

with the reservation stated in the question.

Now put n successively equal to 2, 3, 4, 5; then, observing that u_0 and u_2 each $= 1$, we find $u_4 = 5$, $v_4 = 32$, $u_6 = 61$, $v_6 = 544$,

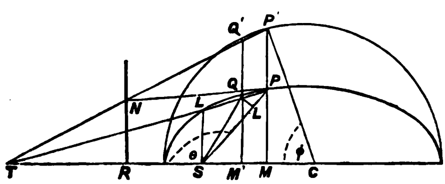
$$u_8 = 61 + 300 + 480 + 544 = 1385, \quad v_8 = 4\,(1385 + 1708 + 875) = 15872,$$

$$u_{10} = 1385 + 6832 + 11200 + 15232 + 15872 = 50521.$$

7733. (By H. J. Read, M.A.) — Transform $\int \frac{d\theta}{(1 + e\cos\theta)^{n+1}}$, by means of the geometry of the ellipse.

Solutions by (1) D. Edwardes; (2) H. L. Orchard, B.Sc., M.A.

1. Let P be any point on an ellipse, Q an adjacent point, P′, Q′ corres-



ponding points on the auxiliary circle; then, drawing QL perpendicular to SP, $\quad r = \text{SP} = a - ex = a\,(1 - e\cos\phi)$.

But $r = a\,(1-e^2)\,1\text{H}e\cos\theta$; $\therefore\ a\,(1-e^2) = a\,(1-e\cos\phi)(1-e\cos\theta)\ldots(\alpha)$.

Also $\quad \text{MM}' = \text{PQ}\cos\text{PTM} = \text{P}'\text{Q}'\cos\text{P}'\text{TM}$;

therefore
$$\frac{\text{PQ}}{\text{P}'\text{Q}'} = \frac{\text{TQ}}{\text{T}'\text{Q}'} = \frac{b'}{a},$$
where b' is the semi-diameter parallel to TQ. But
$$\text{QL} = r\,d\theta = \text{QP}\sin\text{QPL} = \text{PQ}\,\frac{b}{b'},$$
and $a\,d\phi = \text{P}'\text{Q}'$, therefore $\dfrac{\text{PQ}}{\text{P}'\text{Q}'} = \dfrac{r\,d\theta}{a\,d\phi}\,\dfrac{b'}{b} = \dfrac{b'}{a}$, or $r\,d\theta = b\,d\phi\ldots(\beta)$;

therefore $\displaystyle\int\frac{d\theta}{(1+e\cos\theta)^{n+1}} = \frac{1}{(1-e^2)^{n+\frac{1}{2}}}\int d\phi\,(1-e\cos\phi)^n$ from (α) and (β).

[The formula (β) seems to be useful; *e.g.*, area of ellipse
$$= 2\,.\,\tfrac{1}{2}\int_0^\pi r^2\,d\theta = \int_0^\pi r\,.\,r\,d\theta = ab\int_0^\pi (1-e\cos\phi)\,d\phi = \pi ab.]$$

2. *Otherwise:* by the geometry,
$$\text{SP} = e\text{PN} = e\text{MS} + e\text{SR},\quad i.e.,\ r = -er\cos\theta + \text{LS} = -er\cos\theta + l,$$
or,
$$1 + e\cos\theta = \frac{l}{r};$$
hence
$$\frac{l\,dr}{r^2} = e\sin\theta\,d\theta = \frac{\left[r^2(e^2-1) + 2lr - l^2\right]^{\frac{1}{2}}}{r}\,d\theta$$
$$= \left(\frac{2lar - lr^2 - l^2a}{a}\right)^{\frac{1}{2}}\frac{d\theta}{r}\ \left(\text{since } 1-e^2 = \frac{l}{a}\right);$$
whence
$$\int\frac{d\theta}{(1+e\cos\theta)^{n+1}} = \left(\frac{a}{l^{2n+1}}\right)^{\frac{1}{2}}\int\frac{r^n\,dr}{(2ar - r^2 - al)^{\frac{1}{2}}} = \frac{a^{n+1}}{b^{2n+1}}\int\frac{r^n\,dr}{(2ar - r^2 - b^2)^{\frac{1}{2}}},$$
a and b being the semi-axes.

5635. (By Elizabeth Blackwood.)—Two excursion trains, each m yards in length, may start with equal probability from their respective stations at any time between 2 o'clock and 10 minutes past 2, in directions at right angles to each other, each at a uniform rate v; find the chances of a collision, each being n yards distant from the point at which their lines cross, and both being ignorant of the risk they are running.

Solution by D. Biddle.

If v = velocity of each train in miles per hour, then $\frac{1}{6}v$ = distance in miles covered in 10 minutes; and, if m = length of train in yards, $\frac{1}{1760}m$ = length referred to a mile as unit. Also $\dfrac{6m}{1760v}$ = fraction of

10 minutes occupied in clearing the level crossing. $10-\frac{2.6m}{1760v}$ = time during which a collision can occur when train B starts as much as $\frac{6m}{1760v}$ before or after train A. $\frac{2.6m}{1760v}$ = time during which the chance is curtailed owing to limitation in the time of starting of the two trains, and when the average factor is reduced from 2 to $\frac{1}{2}(2+1)$. Consequently,

$$\frac{1}{10}\left\{\left(10-\frac{2.6m}{1760v}\right)\frac{2.6m}{1760v}+\frac{2.6m}{1760v}\cdot\frac{3.6m}{2.1760v}\right\} = \text{chance required}$$

$$= \frac{1}{10}\left(20-\frac{6m}{1760v}\right)\frac{6m}{1760v}.$$

Thus, let $m = 200$, and $v = 35$, then the chance of a collision

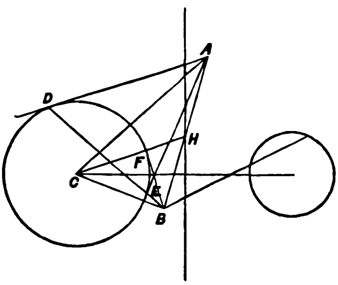
$= \frac{1}{10}\left(20-\frac{3}{154}\right)\frac{3}{154} = \cdot 03891$, or rather more than $\frac{10}{257}$.

The distance n of each train from the level crossing does not appear to affect the probability of a collision, under the other specified conditions. And the moral to be drawn is, that in such cases, the higher the speed and the shorter the train, the less the chance of disaster.

7730. (By W. J. Greenstreet, B.A.)—Prove that (1) the polar of a fixed point with regard to a series of circles having the same radical axis passes through another fixed point; and (2) these two points subtend a right angle at either limiting point.

Solution by A. H. Curtis, LL.D., D.Sc.; T. Brill, B.A.; *and others.*

Lemma. The line joining a point A to any point B, situated on the polar of A with respect to any circle, (1) is equal to the sum of the tangents to the circles from A and B, and (2) is equal to the double of the tangent drawn from its middle point H.



Let C be the centre of the circle and a its radius.

(1) $AB^2 = AD^2 + DB \cdot BE$
$= AD^2 + BF^2$.

(2) $4AH^2 = AB^2 = AD^2 + BF^2$
$= AC^2 + BC^2 - 2a^2 = 2CH^2 + 2AH^2 - 2a^2$,

therefore $AH^2 = CH^2 - a^2$ = square of tangent from H, therefore AB = the double of such tangent.

Let now the polars of A taken with respect to any two circles meet in B, bisect AB in H, then the tangents from H to the two circles are each = AH, therefore equal to each other; therefore H is on the radical axis of the circles, and, as AH = BH, A and B are equidistant from it; if we find then on DE a point B, whose distance from the radical axis = the distance of A from same, the point so formed is a point through which the polar of A taken with respect to any circle of the system must pass. Again, as AH = BH = tangent from H to any circle of the system = distance of H from either limiting point, the circle on AB as diameter will pass through the limiting points; therefore AB will subtend a right angle at either of these points.

6859. (By Professor SIMON NEWCOMB, M.A.)—Prove that

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\ldots-2\eta\cos x$$

$$-\tfrac{1}{2}\,.\,2\eta^2\cos 2x-\tfrac{1}{3}\,.\,2\eta^3\cos 3x-\ldots = \Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

Solution by ASÛTOSH MUKHOPÂDHYÂY.

By elementary trigonometry,

$$\frac{\eta\sin x}{1-2\eta\cos x+\eta^2} = \eta\sin x+\eta^2\sin 2x+\ldots,$$

therefore $\log(1-2\eta\cos x+\eta^2) = \int\frac{2\eta\sin x\,dx}{1-2\eta\cos x+\eta^2}$

$$= -2\left(\eta\cos x+\tfrac{1}{2}\eta^2\cos 2x+\tfrac{1}{3}\eta^3\cos 3x+\ldots\right).$$

Hence, since $\log(1+\eta^2) = \eta^2-\frac{1}{2}\eta^4+\frac{1}{3}\eta^6-\ldots$, we have

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = \left(-\eta^2+\tfrac{1}{2}\eta^4+\tfrac{1}{3}\eta^6+\ldots\right)-2\left(\eta\cos x+\tfrac{1}{2}\eta^2\cos 2x+\ldots\right)$$

$$= \Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

[For another solution, see *Reprint*, Vol. 36, p. 116.]

7673. (By Professor COCHEZ.)—La série

$$\frac{1}{2\log 2\,(\log\log 2)^a}+\frac{1}{3\log 3\,(\log\log 3)^a}+\ldots\frac{1}{n\log n\,(\log\log n)^a}$$

est convergente si $a>1$, divergente si $a<1$.

Solution by B. H. RAU, M.A.; BELLE EASTON; *and others.*

Let $\phi(x) = \dfrac{1}{x\log x\,(\log\log x)^a}$, then $\int\phi(x)\,dx = \dfrac{(\log\log x)^{1-a}}{1-a}$,

if a be not unity, and $=(\log\log\log x)$ if p be unity; hence

$$\int_2^\infty\phi(x)\,dx = -\frac{(\log\log 2)^{1-a}}{1-a},$$

if p be greater than unity, and is infinite if p be equal to unity or less than unity. But the definite integral and the given series are both finite or both infinite. Hence the series is convergent when $a > 1$ and divergent when $a < 1$.

7720. (By R. LACHLAN, B.A.)—Four circles, having their centres within the triangle ABC, are drawn to cut the side BC in angles α, 2σ, γ, β; the side CA in angles β, γ, 2σ, α; and the side AB in angles γ, α, β, 2σ respectively: where $2\sigma = \alpha+\beta+\gamma$. Let R_1, R_2, R_3, R_4 be their radii, and r the radius of the inscribed circle; also let these five circles cut any straight line in angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, θ; then prove that

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} = 4\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\cdot\frac{1}{r},$$

and $\cos\phi_1+\cos\phi_2+\cos\phi_3+\cos\phi_4 = 4\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}\cos\dfrac{\alpha+\beta}{2}\cdot\cos\theta.$

Solution by B. H. RAU, M.A.; Rev. T. C. SIMMONS, M.A.; *and others.*

Let x, y, z be the trilinear coordinates of the centre of the circle (R_1) which cuts the sides BC, CA, AB in angles α, β, γ; then $x = R_1\cos\alpha$, $y = R_1\cos\beta$, $z = R_1\cos\gamma$, $2\Delta = ax+by+cz$,

therefore $$R_1 = \frac{2\Delta}{a\cos\alpha+b\cos\beta+c\cos\gamma}.$$

Similarly $$R_2 = \frac{2\Delta}{a\cos 2\sigma+b\cos\gamma+c\cos\alpha}, \text{ and so on;}$$

therefore $$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} = \frac{a+b+c}{2\Delta}(\cos\alpha+\cos\beta+\cos\gamma+\cos 2\sigma).$$

$$= \frac{4}{r}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}.$$

If $lx+my+nz = 0$ be the equation to any straight line, then

$$\frac{R_1\cos\phi_1}{r\cos\theta} = \frac{\text{perp. from centre of } R_1}{\text{perp. from centre of } r} = \frac{lR_1\cos\alpha+mR_1\cos\beta+nR_1\cos\gamma}{lr+mr+nr};$$

therefore $$\frac{\cos\phi_1}{\cos\theta} = \frac{l\cos\alpha+m\cos\beta+n\cos\gamma}{l+m+n}.$$

Similarly $$\frac{\cos\phi_2}{\cos\theta} = \frac{l\cos 2\sigma+m\cos\gamma+n\cos\alpha}{l+m+n}, \text{ and so on,}$$

therefore $$\Sigma\frac{\cos\phi}{\cos\theta} = \cos\alpha+\cos\beta+\cos\gamma+\cos 2\sigma,$$

therefore $$\Sigma\cos\phi = 4\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)\cos\tfrac{1}{2}(\alpha+\beta)\cdot\cos\theta.$$

7646. (By W. G. LAX, B.A.)—If ABCD be a rectangle; E, G two points in AB, AD such that EOF, GOH meet each other and the diagonal

BD in O, and are parallel respectively to AD, AB; if also AB is taken to represent the external E . M . F of an electro-motor supplied with a current at constant E . M . F; EB that of the motor at a given speed, and the ratio BA : AD the resistance of the circuit: show from this figure what are the conditions for (1) the maximum efficiency, (2) the maximum rate of working; and (3) find expressions for the electrical energy wasted, and that used in work, each per unit of time.

Note by the EDITOR.

Mr. BUCK informs us that the figure is drawn and fully discussed as to all points in the question in Professor S. P. THOMPSON'S *Cantor Lectures*, delivered in December, 1882.

The PROPOSER remarks that the theorems were suggested to him by a question he happened to see in an examination paper, and that he had not the slightest idea that they had previously been investigated.

7314. (By G. HEPPEL, M.A.)—One side of a railway carriage is covered with a glass mirror. Show that, while the train is going round a level curve, the images of reflected objects appear to an observer sitting opposite to describe hyperbolas on the glass. Supposing the centre of the railway curve to be 1000 yards from the observer's eye and 900 yards from the object, the distance of the eye from the mirror to be 2 yards, and the height of the object above the eye 10 yards; find the position and magnitude of the axes of the hyperbola. Also discuss the path of the image of a star.

Solution by the PROPOSER.

Let horizontal and vertical axes be taken on the glass through the image of the observer's eye, which is a fixed point. Let the distance of this image from the eye be c, and from the centre of the railway curve be r. Let l be the distance measured horizontally between the object and the centre, and, taking the line joining these as an initial line, let θ be the angular distance travelled by the carriage. Let h be the height of the object above the horizontal plane through the eye. Then, by the first principles of reflection, we shall have

$$\frac{x}{c} = \frac{r - l\cos\theta}{l\sin\theta + c}; \qquad \frac{y}{h} = \frac{c}{l\sin\theta + c};$$

therefore $$l\sin\theta = \frac{c\,(h-y)}{y}; \quad l\cos\theta = r - \frac{hx}{y};$$

therefore $$\frac{c^2\,(h-y^2)}{y^2} + \left(r - \frac{hx}{y}\right)^2 = l;$$

$$(r^2 + c^2 - l^2)\,y^2 - 2rhxy + h^2x^2 - 2c^2hy + c^2h^2 = 0;$$

and this, except in the very exceptional case of l being less than c, is a hyperbola.

Taking the numbers given in the example, neglecting c^2 in comparison with l^2 and r^2, and expressing all lengths in inches, we find that the centre of the hyperbola is $x = -1\cdot\dot{7}$, $y = -\cdot001\dot{7}$; the inclination of the transverse axis is $3° 0' 22''$; and the semi-axes are 34·926 and 1·77. In the case of a star, if a be the altitude, l and h are both infinite, and $\frac{l}{h} = \tan a$; the hyperbola then becomes $y^2 = (x^2 + c^2) \tan^2 a$.

[Mr. BIDDLE remarks that "there are two arrangements by which the specified conditions can be met. The observer can be sitting with his back to the centre of the curve, the mirror being on the other side of the (saloon) carriage and tangential to the curve (A); or he may be facing a mirror whose plane contains the centre of the curve (B).

"The object, being nearer to the centre of the curve than the carriage, must lie within the circle of which the railway curve forms part; and it is immaterial whether the circular motion be effected by the *carriage* (with observer and mirror inside) or by the *object*, provided the centre of revolution be the same: the path of the image on the glass will be identical. Moreover, a circular ring representing the object in all possible positions at the same moment of time will, produce an image representing the path required, and this image will be identical with that which would be produced if the observer were looking through the glass at a similar circle on the other side; and, if he were to trace the outline in the manner recommended by Mr. RUSKIN (in his treatise on Perspective), he would find the curve to be *elliptical*, NOT *an hyperbola*.

"In fact, anyone who will take the trouble to draw a circle on the ceiling of his drawing-room, and look at it as reflected in the pier-glass (whether in the position A or B), will have an exemplification in miniature of the same thing.

"The path of the image of a star differs but slightly in principle from the foregoing. The star being outside the circle and at an infinite distance, so to speak, as compared with the diameter of the railway curve, would produce an image crossing the glass almost in a straight line. In reality, it would be a portion of an ellipse with comparatively small minor axis. In position (B) the mirror cannot reflect more than half any ring concentric with the railway curve at any one time."

[The PROPOSER "regrets that his question was not so clearly worded as to show without doubt that 'side' meant the partition against which the seats are placed, and not where the doors are. Adopting the method and illustration in the remarks, it may be stated that the curve on the glass is certainly an ellipse in case A, but this was not the case meant. In case B the effect would be the same as if an observer looked in the glass at the reflection of a circle on the ceiling, the centre of which was in the line of intersection of the wall and ceiling. If a cone be imagined, defined by such a circle and by the observer's eye as vertex, it is clear that, if the distance of the eye from the glass were less than the radius of the circle, the wall would cut the cone on both sides of the vertex, and the section would be a hyperbola; if equal, a parabola; and if greater, an ellipse. Returning from the illustration to the thing illustrated, it will be found that this is equivalent to considering whether the distance of the eye from the glass is less or greater than that of the object horizontally from the centre of the curve; and practically it is always less."]

7685. (By Rev. H. E. DAY, M.A.)—Find the probability of a piece at Chess being found on any particular square after having been moved at random an indefinitely long time. [By a well-known mathematical law, this limit is only strictly true after an *indefinitely* long time, but approximately true after a considerable number of moves. For the case of the Knight, see the solution of Quest. 4955, in *Reprint*, Vol. 25, p. 78.]

Solution by D. Biddle *and the* Proposer.

Let A be the chance of its being found on one particular square (with a outlets) communicating with squares having b, c, d, e, f outlets respectively, and whose several chances in regard to the piece are in like manner B, C, D, E, F.

Then $A = \frac{B}{b} + \frac{C}{c} + \frac{D}{d} + \&c.$ to a terms, since the square with chance A is one of the outlets of the squares with chance B, C, &c., and shares in due proportion their several chances, each outlet taking an equal share of the chance pertaining to the square of which it is an outlet. Equations for B, C, &c. can be obtained in like manner; and the several results arrived at, that is, the values of A, B, C, &c., though not identical, are constant; each is a fixed and invariable amount. But evidently any such equation is satisfied by $\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d} = \frac{E}{e}$ because all the squares on the board are outlets, proximate or remote, of one another, and the several equations carried out and reduced would give no other result. Such being the case,

$$\frac{A}{a} = \frac{A+B+C+\&c.}{a+b+c+\&c.} = \frac{\text{the sum of the chances for the whole board}}{\text{the sum of the outlets for the whole board}}$$

$$= \frac{1}{a+b+c+\&c.};$$

Consequently $A = \frac{a}{a+b+c+\&c.}$, and is proportional to the outlets (or which is the same thing, the *inlets*) of the particular square, so far as the particular piece is concerned.

7430. (By Professor Hudson, M.A.)—Given three points, determine in how many ways they may be the positions of an eye, a luminous point, and its image formed by reflexion at a plane mirror; and construct in each case the position of the mirror.

Solutions by (1) W. J. C. Sharp, M.A.; (2) D. Biddle.

1. Let A, B, C be the three points; then, if the eye be supposed to be at A, the mirror must be the perpendicular bisector of BC, and the object must be on the same side of this as A; hence there will be one arrangement with the eye at A if AB and AC are unequal, and none if these are equal. So for each of the other points; and the eye may be placed at each, at two, or at none of the points, according as the triangle is scalene, isosceles, or equilateral; and, whenever the eye occupies any angle, the mirror is the perpendicular bisector of the opposite side of the triangle formed by the points.

2. The optical requirements, in the above series of cases, are (1) that the eye and the object shall invariably be in front of the mirror, and the image behind it; and (2) that the reflecting surface, whilst placed directly between the eye and the image, shall occupy a plane which bisects at right angles the line joining the image and the object. This secures the needed equality of the angles of incidence and reflexion, and also respects the apparent equality which subsists between the perpendicular distance of the object in front and of the image behind from the plane of the mirror.

If the lines joining the three points form an equilateral triangle, it is impossible that the desired optical effect can be produced, unless the position of the eye be allowed to coincide with the plane of the mirror; and in no other case can the entire series of permutations, six in number, be accomplished. If the lines joining the three points form an isosceles triangle (not equilateral), there are two arrangements possible: for, if the angle at the apex be less than 60°, the points at the base may be those indifferently of the eye or the object; and, if the angle be greater than 60°, the points at the base may be those indifferently of the eye or the image. When the lines joining the three points form a scalene triangle, three arrangements are possible, the eye taking each of the three positions in turn, and the image the more distant of the two remaining. A diagram is scarcely needed to make this clear.

6553. (By the late G. F. WALKER, M.A.)—Solve the equations

$$x^2(y+z) = a^3,\quad y^2(z+x) = b^3,\quad z^2(x+y) = c^3.$$

Solution by the Rev. T. C. SIMMONS, M.A.

[As the final equations in the solution of this question given on p. 86 of Vol. xxxv. of our *Reprints*, are evidently *not* three ordinary equations for finding x, y, z, but only their *ratios*, the following method is suggested.]

Putting $xyz = -p$, the equations are

$$\frac{a^3}{px} + \frac{1}{y} + \frac{1}{z} = 0, \text{ \&c.} \quad\text{.....................(1), (2), (3),}$$

whence $\begin{vmatrix} \frac{a^3}{p} & 1 & 1 \\ 1 & \frac{b^3}{p} & 1 \\ 1 & 1 & \frac{c^3}{p} \end{vmatrix} = 0$, or $2p^3-(a^3+b^3+c^3)\,p^2+a^3b^3c^3 = 0$. Let any root of this equation be $-k$, then, solving (1), (2), (3), we obtain $x = mz$, $y = nz$, hence $mnz^3 = k$, therefore z, and consequently x and y, are determined.

[This method at once solves the more general equation $x^2(y+\lambda z) = a^3$, $y^2(z+\mu x) = b^3$, $z^2(x+\nu y) = c^3$, or, what is the same thing, the equations $x^2(ay+bz) = y^2(cz+dx) = z^2(ex+fy) = 1$, which can only be solved with great difficulty by the ordinary methods. A similar remark applies to Mr. SIMMONS's solution of Quest. 7341 (p. 29 of this volume), in the form

$$yz(ax+by+cz) = d,\quad zx(a'x+b'y+c'z) = d',\quad xy(a''x+b''y+c''z) = d'',$$

which possibly cannot be solved *at all* by the ordinary method. By this new determinant process a whole host of awkward-looking equations can be solved *at once*; whereof two specimens are hereunder :—

$$x^2(1+y^3)=ayz,\quad y^2(1+z^3)=bzx,\quad z^2(1+x^3)=cxy,$$
$$ax+by=x^2y^2,\quad cy+dz=y^2z^2,\quad ez+fx=z^2x^2.]$$

7700. (By W. J. McClelland, B.A.)—Prove that

$$\left(\frac{1+\cos a-\cos b-\cos c}{\sin\frac{1}{2}b\sin\frac{1}{2}c}\right)^2+(\dots)^2+\left(\frac{1+\cos c-\cos a-\cos b}{\sin\frac{1}{2}a\sin\frac{1}{2}b}\right)^2$$
$$+\frac{(1+\cos a-\cos b-\cos c)\dots(1+\cos c-\cos a-\cos b)}{2\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b\sin^2\frac{1}{2}c}=16.$$

Solution by B. H. Rau, M.A.; T. Trap, B.A.; *and others.*

Let $x, y, z \equiv 1-\cos a,\ 1-\cos b,\ 1-\cos c$, and $u\equiv x+y+z$; then

$$\text{Sinister}=\frac{4(u-2x)^2}{yz}+\frac{4(u-2y)^2}{zx}+\frac{4(u-2z)^2}{xy}+\frac{4(u-2x)(u-2y)(u-2z)}{xyz}$$
$$=\frac{4}{xyz}\{u^2(x+y+z)-4u(x^2+y^2+z^2)+4(x^3+y^3+z^3)$$
$$+u^3-2u^2(x+y+z)+4u(xy+yz+zx)-8xyz\}$$
$$=\frac{16}{xyz}\{x^3+y^3+z^3-2xyz-u(x^2+y^2+z^2-yz-zx-xy)\}=16.$$

[*Otherwise.*—Suppose a, b, c to represent the arcs of a spherical triangle, then $\dfrac{1+\cos a-\cos b-\cos c}{4\sin\frac{1}{2}b\sin\frac{1}{2}c}=\cos A'$, where A' is the angle of the corresponding chord at triangle, and since

$$A'+B'+C'=\pi,\quad \cos^2A'+\cos^2B'+\cos^2C'+2\cos A'\cos B'\cos C'=1.]$$

7722. (By Rev. H. G. Day, M.A.)—If there be m black squares, and n white ones, find the chance that $\alpha+\beta$ pieces placed at random will cover α black and β white.

Solution by the Proposer.

Let $(p)_q$ denote the number of combinations of p things q together; then the total number of ways in which the $\alpha+\beta$ pieces can be placed is $(m+n)_{\alpha+\beta}$. But the ways in which α black and β white squares can be covered is $(m)_\alpha(n)_\beta$; hence the required probability is $\dfrac{(m)_\alpha(n)_\beta}{(m+n)_{\alpha+\beta}}$.

For the chance of there being β obstacles between a bishop and king at opposite diagonal corners, we must take $m=56$, $n=6$, $\alpha+\beta=4$.

7583. (By Morgan Jenkins, M.A.)—Prove Gergonne's construction for describing a circle to touch three given circles without introducing, in the proof, two tangent circles.

Solution by the Proposer.

Let A, B, C be the centres of the three given circles, O of the orthogonal circle, T of one of the required circles; and let the circles, for brevity, be named by their centres. Let h, k be the points of contact of the circle T with the circles A and B respectively; $O\alpha$, $O\alpha'$ tangents from O to the circle A; $O\beta$, $O\beta'$ to the circle B. Let $\alpha\alpha'$ meet the common tangent at h to the circles A and T in H; and $\beta\beta'$ meet the common tangent at k in K, and Hh, Kk meet in t. Then, because Hh is the radical axis of the circles T and A, and $H\alpha\alpha'$ of the circles O and A, therefore H is on the radical axis of the circles O and T. Similarly K is on the same radical axis. Therefore HK is the radical axis of the circles O and T, and is perpendicular to the line OT. Again, since th and tk are tangents to the circles A and B, hk passes through a centre of similitude of these two circles. For the same reason, $\alpha\beta$ and $\alpha'\beta'$ pass through one centre of similitude of the circles A and B, $\alpha\beta'$ and $\alpha'\beta$ through the other centre of similitude. Of the two centres of similitude, one, say the former, must be that through which hk passes; let it be denoted by S: then, since rect. $S\alpha \,.\, S\beta =$ rect. $Sh \,.\, Sk$, S is on the radical axis of the circles O and T, that is, HK passes through S. Similarly, if γ, γ', l, L be points corresponding, for the circle C, to the points α, α', h, H, then HL is perpendicular to OT, that is, coincident with HK; and HKL passes through a centre of similitude of the circles A and C, and through a centre of similitude of the circles A and B. Therefore the radical axis of the circles O and T is an axis of similitude of the circles A, B, and C.

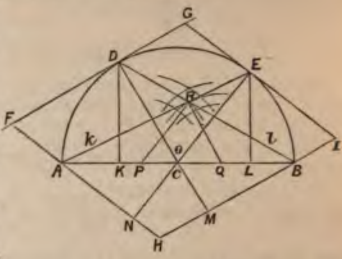
Again, since h is the pole of Hh, O of $\alpha\alpha'$ with regard to the circle A, therefore Oh is the polar of H, therefore the pole of HKL lies in Oh, therefore h may be determined by joining O to the pole, with regard to the circle A, of an axis of similitude of the three circles.

[If the three circles A, B, C cut one another two and two, the point O, the intersection of common chords, is inside each of the circles; the orthogonal circle and the six points α, α', β, β', γ, γ' are imaginary; the lines $\alpha\alpha'$, $\beta\beta'$, and $\gamma\gamma'$ are the polars of O with regard to the three circles respectively. If the circles A and B cut each other in q, q', and if SAB cut qq' in n, rect. $Sh.Sk = Sq^2 = Sn^2 + nq^2 = Sn^2 + On^2 + Oq\,.\,Oq'$

$= SO^2 +$ rect. $Oq.Oq' = SO^2 -$ square of radius of the imaginary circle, with centre O, cutting the three circles orthogonally. Therefore, as before, S is on the radical axis of the circles O and T, and the remaining part of the proof holds good.]

5672. (By Col. Clarke, C.B., F.R.S.)—P and Q are two points in a finite line AB. The parts PA, QB are rotated in opposite directions round P and Q respectively, until A and B meet in a point R. Supposing P and Q evenly distributed, determine the law of density of the points R.

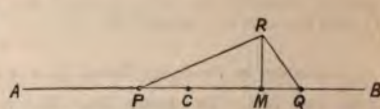
Solutions by (1) D. Biddle; (2) *the* Proposer.



1. In order to meet in a point R, PA and QB must bear a certain relation to each other; that is, Q must be within the same distance of C (the mid-point of AB) that P is of A; it must also be on the opposite side of C. Again, if we divide AC and CB into a number of equal parts, and draw a series of semicircles from each extremity of AB, with the diameters in arithmetical progression, it is easy to see that the spaces enclosed by arcs, though of different size and shape, will each contain the same number of possible positions for R, because the distance between any two adjoining centres on AB is the same, affording scope for equal numbers of P's and Q's.

Now, since the distance between any two semicircles drawn from one extremity of AB increases as they proceed in the same proportion as chords of the full-sized semicircle drawn on AB itself, we have a ready means of estimating for any position the proportionate intervals between adjacent semicircles in both series. They correspond with the chords which pass through the point in question, from A and B respectively, to the semicircle described on AB.

In order to estimate the relative extent of space occupied by R in its several positions, we must take a space bounded by arcs so minute as to be virtually straight lines, as in the diagram, where from P and Q a few concentric arcs are described and small portions of them magnified to form the quadrilateral FGIH. C being the centre of the semicircle, CD, CE are parallel to QR, PR; GI, FH are at right angles with the direction of PR, and FG, HI likewise with that of QR. Moreover GI, FH are separated by the chord AE, and FG, HI by the chord BD, lying in their normal positions. But the perpendicular distance between FG and HI = BK, and between GI and FH = AL; and these (AB being unity) are respectively BD^2 and AE^2. Also the angles F and I are each = DCE = PRQ. The area of FGIH = HI . DM = HI . BK = AL cosec I . BK = $AE^2 . BD^2$ cosec DCE. And the same rule applies to every position of R. If therefore θ = angle separating radii to upper extremities of the two chords k, l; then the extent of space occupied by R at the intersection of the chords may be represented by $k^2l^2 \operatorname{cosec}\theta$; and the density of points R in the same locality by $1 / k^2l^2 \operatorname{cosec}\theta$. We thus see that the density on the circumference and also on the base line is infinitesimal compared with the density just within the corners of the semicircle.



2. *Otherwise:* — Let half AB, that is AC, be taken as the unit of length. Let CM = x, MR = y. Also, let PC = u, CQ = v, then the double area of PQR is $y(u+v) = 2[uv(1-u-v)]^{\frac{1}{2}}$. Also expressing the difference of PM and QM gives us $x(u+v) = v - u$). Now, letting

u and v receive increments du and dv, R becomes one of the four corners of a small parallelogram whose coordinates are

$$\begin{array}{ll} x & y \\ x+\frac{dx}{du}du & y+\frac{dy}{du}du \\ x+\frac{dx}{dv}dv & y+\frac{dy}{dv}dv \\ x+\frac{dx}{du}du+\frac{dx}{dv}dv & y+\frac{dy}{du}du+\frac{dy}{dv}dv, \end{array}$$

the area of which is $\left(\frac{dx}{dv}\frac{dy}{du}-\frac{dx}{du}\frac{dy}{dv}\right)du\,dv,$

and the reciprocal of this is a measure of the density of distribution of the points. We find readily $\frac{dx}{dv}\frac{dy}{du}-\frac{dx}{du}\frac{dy}{dv}=\frac{y}{u+v}\cdot\frac{1}{1-u-v}.$

But, since $\qquad 2u=1-x-\frac{y^2}{1+x}$ and $2v=1+x-\frac{y^2}{1-x},$

the expression for the density becomes $\frac{y(1-x^2-y^2)}{(1-x^2)^2}.$

7572. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In the limaçon whose equation is $r=a\cos\theta+b$, where $b>a$, O is the origin, A, A′ the farther and nearer vertices, C a point of maximum curvature, P, P′ two points of the curve on the same side of the axis as C, such that OC is the harmonic mean between OP, OP′ (P coinciding with A when P′ coincides with A′); prove that (1) the difference of the angles AOP, A′OP′ is equal to the angle (ϕ) which the chord PP′ makes with the axis; the difference of the arcs AP, A′P′ is $4b\sin\frac{1}{2}\phi$; (2) the difference of the arcs AC, A′C is $4a$; (3) the locus of the intersection of the tangents at P, P′ is a cissoid; (4) taking the origin O at the single focus, and the equation $r^2-2r(a+b\cos\theta)+(b-a)^2=0$, the curve is its own inverse with respect to O, the radius of the circle of inversion being $b\sim a$; (5) if OPP′ be a chord through O so that P, P′ are inverse points, the locus of the point of intersection of the tangents at P, P′ is the cissoid $y^2(x+b-a)=(3a-b-x)^3$; (6) if we have a family of limaçons having a given single focus O and a given node S, $OS=b-a\equiv c$, then the locus of the centres of curvature at the points of maximum curvature is the cissoid $y^2(3c-x)=(x-c)^3$; and the envelope of the tangents at the points of inflexion is another cissoid $y^2(\frac{1}{3}c-x)=(x-\frac{1}{9}c)^3$, the origin in all these cases being at O, and the axis of x along OS.

Solution by Professor NASH, M.A.; SARAH MARKS; *and others.*

It can easily be shown that the radius of curvature is given by $\rho=\frac{(2bz-c^2)^{\frac{3}{2}}}{3bz-2c^2}$, where $c^2=b^2-a^2$, and thus that at the point of maximum curvature $bc = c^2$, $\cos\theta = -\frac{a}{b}$, $\rho = c$; hence, if (r_1, θ_1), (r_2, θ_2) be the coordinates of P, P′,

$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2b}{c^2}$, therefore $2ab(\cos\theta_1 \cos\theta_2 + 1) + (a^2 + b^2)(\cos\theta_1 + \cos\theta_2) = 0$,

therefore $$\frac{a^2+b^2}{\cos\theta_1\cos\theta_2+1} = \frac{-2ab}{\cos\theta_1+\cos\theta_2} = \frac{(b-a)^2}{(1+\cos\theta_1)(1+\cos\theta_2)}$$
$$= \frac{(b+a)^2}{(1-\cos\theta_1)(1-\cos\theta_2)},$$

$$\therefore \quad \frac{b-a}{\cos\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2} = \frac{b+a}{\sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2} = \frac{2b}{\cos\frac{1}{2}(\theta_2-\theta_1)} = \frac{-2a}{\cos\frac{1}{2}(\theta_2+\theta_1)} \ldots\text{(A)},$$

the positive sign being taken since $b > a$, and θ_1 and θ_2 each $< \pi$.

Again, if PP′ make an $\angle\phi$ with the axis,

$$\tan\phi = \frac{a(\sin 2\theta_2 - \sin 2\theta_1) + 2b(\sin\theta_2 - \sin\theta_1)}{a(\cos 2\theta_2 - \cos 2\theta_1) + 2b(\cos\theta_2 - \cos\theta_1)}$$
$$= -\frac{a\cos(\theta_2+\theta_1)\cos\frac{1}{2}(\theta_2-\theta_1) + b\cos\frac{1}{2}(\theta_2+\theta_1)}{a\sin(\theta_2+\theta_1)\cos\frac{1}{2}(\theta_2-\theta_1) + b\sin\frac{1}{2}(\theta_2+\theta_1)}$$
$$= -\frac{\cos\frac{1}{2}(\theta_2+\theta_1)\left[1-\cos(\theta_2+\theta_1)\right]}{\sin\frac{1}{2}(\theta_2+\theta_1)\left[1-2\cos^2\frac{1}{2}(\theta_2+\theta_1)\right]} = \tan(\theta_1+\theta_2) \text{ by (A)},$$

and $$\text{AOP} - \text{A'OP'} = \theta_1 - (\pi - \theta_2) = \theta_1 + \theta_2 - \pi = \phi.$$

Differentiating the last member of (A) and eliminating out, we get

$$\frac{d\theta_1}{\sin\theta_1} = \frac{-d\theta_2}{\sin\theta_2} = \frac{d(\theta_1+\theta_2)}{-2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)};$$

and again, from (A),

$$\frac{b}{\cos\frac{1}{2}(\theta_2-\theta_1)} = \frac{-a}{\cos\frac{1}{2}(\theta_2+\theta_1)} = \frac{(a^2+b^2+2ab\cos\theta_1)^{\frac{1}{2}}}{\sin\theta_1} = \frac{(a^2+b^2+2ab\cos\theta_2)^{\frac{1}{2}}}{\sin\theta_2}.$$

$$u = \text{arc AP} - \text{arc A'P'} = \int_0^{\theta_1} (a^2+b^2+2ab\cos\theta)^{\frac{1}{2}}\,d\theta - \int_{\theta_2}^{\pi} (a^2+b^2+2ab\cos\theta)^{\frac{1}{2}}\,d\theta;$$

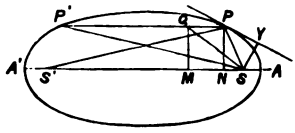
therefore $$\frac{du}{d\phi} = (a^2+b^2+2ab\cos\theta_1)^{\frac{1}{2}}\frac{d\theta_1}{d\phi} + (a^2+b^2+2ab\cos\theta_2)^{\frac{1}{2}}\frac{d\theta_2}{d\phi}$$
$$= \frac{-b\sin\theta_1}{\cos\frac{1}{2}(\theta_2-\theta_1)} \; \frac{\sin\theta_1}{2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)}$$
$$+ \frac{b\sin\theta_2}{\cos\frac{1}{2}(\theta_2-\theta_1)} \; \frac{\sin\theta_2}{2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)}$$
$$= \frac{b(\sin^2\theta_2 - \sin^2\theta_1)}{\sin(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2+\theta_1)} = 2b\sin\tfrac{1}{2}(\theta_1+\theta_2) = 2b\cos\tfrac{1}{2}\phi,$$

therefore $$u = 4b\sin\tfrac{1}{2}\phi,$$

Where P, P′ coincide with c, $\sin\frac{1}{2}\phi = \cos\theta$, therefore

$$\text{arc AC} - \text{arc A'C} = 4b\,\frac{a}{b} = 4a.$$

These results can be more easily obtained by inversion. If c be the radius of inversion, the inverse of the limaçon is an ellipse whose axes are $2b$, $2c$. The point of maximum curvature is the inverse of the extremity of the minor axis, and the line PP′ is parallel to the major axis.



(1) ASP − A′SP′ = ASP − AS′P = SPS′. Inverse of line PP′ is circle SPP′S′, and the angle at which this cuts SS′ is equal to the angle in the segment SPS′.

(2) Inverse of element of arc δs is $\dfrac{c^2\delta s}{r^2}$,

therefore u = arc AP − arc A′P′ is $c^2\displaystyle\int\left\{\frac{1}{r^2}-\frac{1}{(2b-r)^2}\right\}ds$ over arc AP.

Now, if $\quad SPS' = \phi,\ \cos\tfrac{1}{2}\phi = -\sin SPy = \dfrac{c}{[r(2b-r)]^{\frac{1}{2}}} =$ ....(B),

and $\quad \sin\tfrac{1}{2}\phi = \cos SPy = \dfrac{dr}{ds}$..............................(C).

Differentiating (B), and therefrom, by (C), we have

$$\tfrac{1}{2}\sin\tfrac{1}{2}\phi\frac{d\phi}{dr} = \frac{c(b-r)}{r^{\frac{3}{2}}(2b-r)^{\frac{3}{2}}},\qquad \frac{d\phi}{ds} = \frac{2c(b-r)}{r^{\frac{3}{2}}(2b-r)^{\frac{3}{2}}},$$

therefore $\quad u = c^2\displaystyle\int_0^{\phi}\frac{4b(b-r)}{r^2(2b-r)^2}ds = 2b\int_0^{\phi}\cos\tfrac{1}{2}\phi\,d\phi = 4b\sin\tfrac{1}{2}\phi.$

When P, P′ coincide, $\sin\tfrac{1}{2}\phi = \dfrac{a}{b}$, therefore $u = 4a$.

(3) The tangents at P, P′ invert into circles touching the ellipse at P, P′ and passing through the focus. These circles intersect in the point Q on PP′ where SQ bisects the angle PSP′. Drawing the ordinates PN, QM, P′N′, we have, successively,

$$\frac{MN}{SP} = \frac{MN'}{S'P} = \frac{2CM}{S'P-SP} = \frac{CN}{CA};\quad \text{but}\quad \frac{CS}{S'P-SP} = \frac{CA}{2CN},$$

$$\frac{CS}{CM} = \frac{CA^2}{CN^2},\quad \frac{CS}{CS-CM} = \frac{CA^2}{CA^2-CN^2} = \frac{CB^2}{PN^2},\quad \frac{SM}{CS} = \frac{QM^2}{CB^2},$$

therefore the locus of Q is a parabola whose vertex is S and latus rectum $(CB)^2/CS$, therefore the inverse is a cissoid.

(4) Inverting the equation SP + S′P = AA′ = $2b$, we get for the limaçon $2b$SP − $2a$OP = c^2, therefore the equation referred to the single focus is

$$4a^2r^2 - 4ar(a^2+b^2\cos^2\theta) + c^4 = 0 \text{ ....(D)},$$

and the curve is its own inverse with respect to O, the radius of inversion being $c^2/2a$.

(5) Inverting OPP′, we get a circle through S′PP′S, so that the locus is the same cissoid as in (3). Its equation referred to S is $y^2(a-x) = x^3$,

and referred to O it is $y^2\left(a-\dfrac{c^2}{2a}-x\right) = \left(x+\dfrac{c^2}{2a}\right)^3$ ....(E).

If we put $a = 2a'$, $\dfrac{b^2}{2a} = b'$, and therefore $\dfrac{c^2}{2a} = b'-a'$, so that (D) takes the form given in the question, (E) becomes $y^2(3a-b-x) = (x+b-a)^3$, and not $y^2(x+b-a) = (3a-b-x)^3$, as stated.

(6) Inverting, we have to find the locus of the inverse of the focus of a system of confocal conics with respect to the circle of curvature at the extremity of the minor axis, and this is easily shown to be the parabola $y^2 = cx$, the origin being the focus, and $2c$ the distance between the foci. Inverting again, c being the radius of inversion, the equation of the locus referred to S is $y^2(c-x) = x^3$. Transferring the origin to O, since $OS = \frac{1}{2}c$ in limaçon, the equation is

$$y^2(\tfrac{3}{2}c - x) = (x - \tfrac{1}{2}c)^3.$$

(7) Using the equation $r = a\cos\theta + b$ at the point of inflexion,

$$r = \frac{2c^2}{3b^2}, \quad \cos\theta = -\frac{2a^2+b^2}{3ab}, \sin\theta = \frac{c(4a^2-b^2)^{\frac{1}{2}}}{3ab}.$$

The tangent is $x(a\cos 2\theta + b\cos\theta) + y(a\sin 2\theta + b\sin\theta) = (a\cos\theta + b)^2$,

therefore the inflexional tangent is $cx(b^2+8a^2) + y(4a^2-b^2)^{\frac{3}{2}} + 4ac^3 = 0$,

or, if $$OS = k = \frac{c^2}{2a} = \frac{b^2-a^2}{2a},$$

$$(2k)^{\frac{3}{2}}x + (2k)^{\frac{1}{2}}a(9x+8k) + y(3a-2k)^{\frac{3}{2}} = 0,$$

a being variable.

Differentiating with respect to a, eliminating a, and then referring to O as origin, we have

$$y^2(x+\tfrac{2}{3}k) + (x+\tfrac{8}{9}k)^3 = 0, \quad \text{or} \quad y^2(x-\tfrac{1}{3}k) + (x-\tfrac{1}{9}k)^3 = 0.$$

6418. (By Professor MALET, M.A., F.R.S.)—Prove the following extension to surfaces of CHASLES' theorem for plane curves:—If to a surface of the class n any system of n parallel tangent planes be drawn, then the centre of mean position of their points of contact is fixed.

Solution by W. J. C. SHARP, M.A.

If $\frac{\lambda\alpha_1+\mu\alpha}{\lambda+\mu}$, $\frac{\lambda\beta_1+\mu\beta}{\lambda+\mu}$, $\frac{\lambda\gamma_1+\mu\gamma}{\lambda+\mu}$, $\frac{\lambda\delta_1+\mu\delta}{\lambda+\mu}$ be substituted for $\alpha, \beta, \gamma, \delta$ in $\Sigma = 0$, the tangential equation to a surface, and if

$$\Delta = \alpha\frac{d}{d\alpha_1} + \beta\frac{d}{d\beta_1} + \gamma\frac{d}{d\gamma_1} + \delta\frac{d}{d\delta_1},$$

the resulting equation in $\lambda : \mu$,

$$\lambda^n\Sigma_1 + \lambda^{n-1}\mu\Delta\Sigma_1 + \frac{1}{1.2}\lambda^{n-2}\mu^2\Delta^2\Sigma_1 + \&c. = 0,$$

determines the ratios of the perpendiculars upon the planes $(\alpha, \beta, \gamma, \delta)$ and $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ from points on the tangent planes drawn through the intersection of those planes, and therefore from the points of contact. Now,

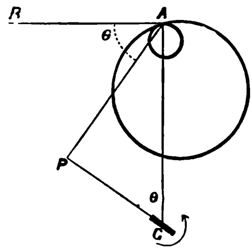
if $\Delta\Sigma_1 = 0$, or $$\alpha\frac{d\Sigma_1}{d\alpha_1} + \beta\frac{d\Sigma_1}{d\beta_1} + \gamma\frac{d\Sigma_1}{d\gamma_1} + \delta\frac{d\Sigma_1}{d\delta_1} = 0 \quad \text{.....................(1)}$$

(the equation to a point), the sum of these ratios is zero; let p, p', p'', &c. be the perpendiculars upon $(\alpha, \beta, \gamma, \delta)$, and p_1, p'_1, p''_1 those upon $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, then $\Sigma\left(\frac{p}{p_1}\right) = 0$, and, if $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be the plane infinity, p_1, p'_1, p''_1, &c. are equal and $\Sigma\,(p = 0)$; in this case also the tangent planes are all parallel. So that the sum of the perpendiculars from the points of contact on any plane through the point (1) is zero, and therefore that point is the mean centre of the points of contact.

5636. (By C. Leudesdorf, M.A.)—A polished uniform straight metal rod is held in a horizontal position with one end fixed at a point A, and is then allowed to swing under the action of gravity till it reaches a vertical position, when the end A is loosed, and the rod allowed to fall; find the locus traced out by the image of the fixed point A, as seen from any point by reflection at the rod during the motion of the latter.

Solution by the Rev. T. C. Simmons, M.A.

When the rod becomes vertical, it will have an angular velocity ω, which will afterwards remain constant; also its centre of gravity G will fall in a vertical line with acceleration g.



At any time t let θ be the angle the rod makes with the vertical, which will also be the angle which the perpendicular AP on its direction makes with the initial position AB, then $AG = \frac{1}{2}a + \frac{1}{2}gt^2$, $\theta = \omega t$, or, taking r, θ for the polar coordinates of the position of the image,

$$r = 2AP = 2AG\sin\theta = \left(a + \frac{g\theta^2}{\omega^2}\right)\sin\theta.$$

Now $\omega^2 = 3g/a$, so that gt^2 or $g\theta^2/\omega^2 = \frac{1}{3}a\theta^2$, and we have for the equation of the locus $r = (1 + \frac{1}{3}\theta^2)\,a\sin\theta$, consisting of a series of closed curves always returning through A and touching AB. They rapidly increase in size, and tend ultimately to assume the form of circles with centres lying vertically below A. The figure gives the outline of the first two branches from $\theta = 0$ to $\theta = 2\pi$, in accordance with the following table of numerical values:—

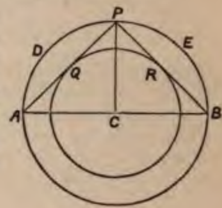
θ	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$
$\frac{r}{a}$	·8	1·8	2·0	4·4	8·4	7·2

It may be interesting to note that the result depends only on a, the length of the rod, and is wholly independent of g. Thus the same curve would be described on the sun, or on any of the planets. The *rate* of description, however, increases with the force of gravitation.

7723. (By D. BIDDLE.)—If a small marble be placed at random on a circular table with raised edge, and then propelled at random in any horizontal direction; show that the probability that it will rebound from the raised edge in a direction forming an obtuse angle with the line of incidence is $\frac{1}{2} - \frac{1}{\pi} = \cdot 1816901$ or $\frac{2}{11}$ nearly.

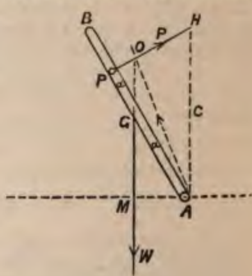
Solution by B. REYNOLDS, M.A.; D. EDWARDES; *and others.*

The point (P) struck by the marble, may be anywhere on the circumference, and the probability as to its position is the same for the whole raised edge. But to rebound from any given position P, at an obtuse angle with the line of incidence, the marble must have been placed originally outside the chords of quadrants AP or BP. The sum of the segments cut off by these is $\frac{1}{2}\pi - 1$, hence the probability is as stated.



7693. (By SYAMA CHARAN BASU, B.A.)—A heavy rod (weight W, length $2a$) capable of free motion, in a vertical plane, about a hinge at an extremity, has a small ring sliding on it. To the ring is attached a string, which passing over a smooth pin, vertically above the hinge at a distance c, supports a weight P, hanging freely. Show that in the position of equilibrium $\tan\theta = \frac{c}{a}\left\{\left[1+\left(\frac{Wa}{Pc}\right)^2\right]^{\frac{1}{2}} - \frac{P}{W}\right\}$, where θ is the inclination of the action on the hinge, to the horizon.

Solution by Rev. T. C. SIMMONS, M.A.; W. G. LAX, B.A.; *and others*



Let ϕ be the angle which the string makes with the horizon, or which the rod AB makes with the vertical; then, resolving vertically and horizontally,

$$R\sin\theta = W - P\sin\phi,$$
$$R\cos\theta = P\cos\phi,$$

whence $\tan\theta = \frac{W}{P}\sec\phi - \tan\phi$.

Again, taking moments about the hinge A,

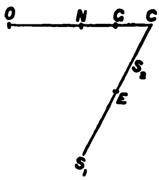
$$Wa\sin\phi = Pc\cos\phi, \quad \text{or} \quad \tan\phi = \frac{Pc}{Wa},$$

and $\tan\theta$ becomes $\frac{W}{P}\left[1+\left(\frac{Pc}{Wa}\right)^2\right]^{\frac{1}{2}} - \frac{Pc}{Wa} = \&c.$

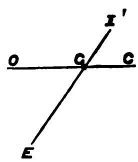
7684. (By D. EDWARDES.)—Given the in-circle and circumcircle of a triangle, prove that (1) the loci of the orthocentre and centroid are circles of the respective radii, $R-2r$, $\frac{1}{3}(R-2r)$, whose centres lie on the line joining the in-centre and circumcentre, and divide it harmonically; (2) the locus of the centroid of the perimeter is a circle whose centre is collinear with the two former centres, and radius $\frac{1}{2}(R-2r)$.

7728. (By Rev. T. C. SIMMONS, M.A.)—Given the circumcircle of a triangle, and any of the four circles touching the sides, show that the loci of the orthocentre and centroid are circles having the centres of the given circles as centres of similitude.

Solution by Rev. T. C. SIMMONS, M.A.; J. BRILL, B.A.; *and others.*



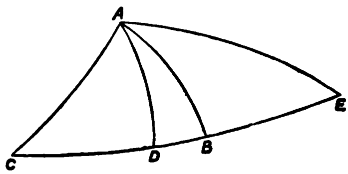
Let C be circumcentre, G centroid, N nine-point centre, O orthocentre, E the centre of the tangent circle, then, EN being constant, locus of N is a circle centre E. But $CG = \frac{2}{3}CN$, $CO = 2CN$, therefore loci of G and O are circles having, in common with the locus of N, C for one centre of similitude. Let S_1, S_2 be centres of these circles, then $[CS_2ES_1] = [CGNO]$, that is to say $[CS_1ES_2]$ is harmonic. But C is one centre of similitude, therefore E is the other.



Again, if A′, B′, C′ be the mid-points of the sides of the triangle, the centroid of its perimeter is the in-centre of A′B′C′. Also G is the common centroid, and likewise the centre of similitude of both triangles; whence, considering E originally the in-centre, and O′, I′ orthocentre and in-centre of A′B′C′, $GO' = \frac{1}{2}GO$, $GI' = \frac{1}{2}GE$, and O′ coincides with C, the original circumcentre: whence $EI' = \frac{3}{2}I'G$, so that the locus of I′ is similar and similarly situated to the locus of G, E being centre of similitude. Therefore the locus of I′ is a circle whose centre lies on EC and whose radius $= \frac{3}{2} \times \frac{1}{3}(R-2\rho) = \frac{1}{2}(R-2\rho)$.

7655. (By W. J. MCCLELLAND, B.A.)—Show that the sum of the cotangents of the intercepts made by the internal and external bisectors of the angles of a spherical triangle on the opposite sides is equal to zero.

Solution by B. H. RAU, B.A.; *the* PROPOSER; *and others.*



$$\cot CD \sin b = \cot\tfrac{1}{2}A \sin C + \cos b \cos C = (1+\cos A)\frac{\sin C}{\sin a} + \cos b \cos C;$$

$$\cot CD = \frac{\cos a \sin b + \sin C}{\sin a \sin b},$$

$$\cot CE = \frac{\cos a \sin b - \sin c}{\sin a \sin b};$$

therefore $\cot DE = \dfrac{\cot CD . \cot CE + 1}{\cot CD - \cot CE}$

$$= \frac{\sin a . \sin b}{2 \sin c} \left\{ 1 + \frac{\cos^2 a \sin^2 b - \sin^2 c}{\sin^2 a \sin^2 b} \right\} = \frac{\sin^2 b - \sin^2 c}{2 \sin a \sin b \sin c}.$$

Similarly for the cotangents of the other intercepts.

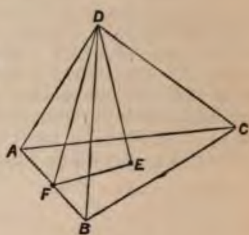
$$\text{Hence sum} = \frac{1}{2 \sin a \sin b \sin c} \left\{ \begin{array}{c} \sin^2 b - \sin^2 c + \sin^2 c - \sin^2 a \\ + \sin^2 a - \sin^2 b \end{array} \right\} = 0.$$

7656. (By W. J. C. Sharp, M.A.)—If ABCD be a tetrahedron, p_1, p_2, p_3, p_4 the perpendiculars from the vertices upon the opposite faces, then, denoting by (AB), &c., the dihedral angles between the faces which intersect in AB, &c., prove that (1)

$$\sin(AB) : \sin(BC) : \sin(CA) : \sin(AD) : \sin(BD) : \sin(CD)$$
$$= \frac{AB}{p_1 p_2} : \frac{BC}{p_2 p_3} : \frac{CA}{p_3 p_1} : \frac{AD}{p_1 p_4} : \frac{BD}{p_2 p_4} : \frac{CD}{p_3 p_4};$$

and (2) the equation to the sphere described about the tetrahedron may be written $ab \sin(ab)\, xy + bc \sin(bc)\, yz + \&c. = 0.$

Solution by W. G. Lax, B.A.; B. H. Rau, B.A.; *and others.*

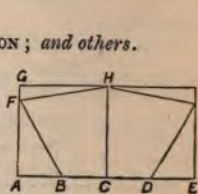


1. Draw DE, DF perpendicular to the face ABC and the edge AB; then

$$\sin(AB) = \frac{DE}{DF} = \frac{AB . DE}{2\Delta ABD}$$
$$= \frac{AB . p_4 . p_3}{6 \text{ vol. of tetrahedron}}$$
$$= \frac{AB}{p_1 p_2} . \frac{p_1 p_2 p_3 p_4}{6V} \propto \frac{AB}{p_1 p_2}.$$

7636. (By Professeur Cochez.)—Inscrire dans un rectangle un pentagone ayant les côtés égaux.

Solution by B. H. Rau, B.A.; Belle Easton; *and others.*



Let AE $= a$ and AG $= b$ be the sides of the rectangle; and put BD $= x$. If the pentagon is symmetrical, the mid-point c of a side BD will coincide with the mid-point of AE, and the opposite corner of the pentagon will be at the mid-point of the opposite side of the rectangle.

AB $= \frac{1}{2}(a-x)$; therefore AF $= [x^2 - \frac{1}{4}(a-x)^2]^{\frac{1}{2}}$.

Again, GH $= \frac{1}{2}a$, therefore GF $= (x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$;

therefore $\quad b = \mathrm{AF} + \mathrm{FG} = [x^2 - \frac{1}{4}(a-x)^2]^{\frac{1}{2}} + (x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$,

or $\quad b^2 + x^2 - \frac{1}{4}a^2 - 2b(x^2 - \frac{1}{4}a^2) = x^2 - \frac{1}{4}(a-x)^2$,

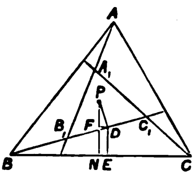
or $\quad \frac{1}{4}x^2 - \frac{1}{2}ax + b^2 = 2b(x^2 - \frac{1}{4}a^2)^{\frac{1}{2}}$,

or $\quad \frac{1}{16}x^4 - \frac{1}{4}ax^3 + (\frac{1}{4}a^2 + \frac{1}{2}b^2 - 4b^2)x^2 - ab^2x + b^4 + a^2b^2) = 0$,

a biquadratic for x.

7682. (By H. Fortey, M.A.)—Suppose three straight lines pass through the points A, B, C respectively, and turn about those points in the plane ABC with the same angular velocity and in the same direction. Find the locus of the centre of the circle described about the variable triangle thus formed, (1) when the lines through A, B, C are initially coincident with AB, BC, CA respectively; (2) when they initially coincide with AC, BA, CB; showing that the loci are two equal circles of radius $abc(\lambda - 16\Delta^2)^{\frac{1}{2}} / 16\Delta^2$ (where $\lambda = a^2b^2 + b^2c^2 + c^2a^2$ and Δ = area of ABC), that these circles touch each other at the centre of the circle about ABC, that (if $\mathrm{A} = a^4 - b^2c^2$, &c.) the equation to the line joining their centres is $(2\mathrm{A} + \mathrm{B} + \mathrm{C})\,bc\alpha + (2\mathrm{B} + \mathrm{C} + \mathrm{A})\,ca\beta + (2\mathrm{C} + \mathrm{A} + \mathrm{B})\,ab\gamma = 0$, and that this line touches the Brocard circle.

Solution by the Proposer.



Taking the first case; let

$$\angle \mathrm{BAB}_1 = \mathrm{CBC}_1 = \mathrm{ACA}_1 = \theta,$$

and let P be the centre of the circle about $\mathrm{A_1B_1C_1}$. Draw PN, PD perpendicular to BC, $\mathrm{BC_1}$ and DE, DF perpendicular to BC, PN. Let BN $= x$, PN $= y$, also let

$$a^2 + b^2 + c^2 = k, \quad a^4 + b^4 + c^4 = \nu.$$

Now, we have

$$\mathrm{BC}_1 = a\frac{\sin(\mathrm{C}-\theta)}{\sin \mathrm{C}} = a\left(\cos\theta - \frac{\cos \mathrm{C}}{\sin \mathrm{C}}\sin\theta\right) = a\left(\cos\theta - \frac{k-2c^2}{4\Delta}\sin\theta\right);$$

also $$\mathrm{BB}_1 = c\frac{\sin\theta}{\sin \mathrm{B}} = \frac{ac^2\sin\theta}{2\Delta};$$

$$\therefore \quad \mathrm{BD} = a\left(\frac{\cos\theta}{2} - \frac{k-4c^2}{8\Delta}\sin\theta\right), \quad \mathrm{B_1D} = a\left(\frac{\cos\theta}{2} - \frac{k}{8\Delta}\sin\theta\right),$$

$$\mathrm{PD} = \mathrm{B_1D}\cot \mathrm{A} = \mathrm{B_1D}\frac{k-2a^2}{4\Delta} = \frac{a(k-2a^2)}{4\Delta}\left(\frac{\cos\theta}{2} - \frac{k}{8\Delta}\sin\theta\right),$$

$$x = \mathrm{BN} = \mathrm{BE} - \mathrm{FD} = \mathrm{BD}\cos\theta - \mathrm{PD}\sin\theta,$$

$$y = \mathrm{PN} = \mathrm{DE} + \mathrm{PF} = \mathrm{BD}\sin\theta + \mathrm{PD}\cos\theta.$$

Substituting for BD and PD their values, and reducing, we get

$$\frac{\lambda - \mathrm{A}}{16\Delta^2} - \frac{2x}{a} = \frac{b^2-c^2}{4\Delta}\sin 2\theta - \frac{a^2k-\nu}{16\Delta^2}\cos 2\theta,$$

$$\frac{2y}{a} - \frac{2c^2-a^2}{4\Delta} = \frac{b^2-c^2}{4\Delta}\cos 2\theta + \frac{a^2k-\nu}{16\Delta^2}\sin 2\theta.$$

Squaring and adding,

$$\left(\frac{2x}{a}-\frac{\lambda-A}{16\Delta^2}\right)^2+\left(\frac{2y}{a}-\frac{2c^2-a^2}{4\Delta}\right)^2=\frac{(b^2-c^2)^2}{16\Delta^2}+\frac{(a^2k+\nu)^2}{256\Delta^4},$$

or

$$\left(x-\frac{a(\lambda-A)}{32\Delta^2}\right)^2+\left(y-\frac{a(2c^2-a^2)}{8\Delta}\right)^2=\frac{a^2b^2c^2(\lambda-16\Delta^2)}{256\Delta^4},$$

the equation to a circle whose radius is

$$\frac{abc(\lambda-16\Delta^2)^{\frac{1}{2}}}{16\Delta^2}.$$

Since the ordinate of the centre of the circle is $\frac{a(2c^2-a^2)}{8\Delta}$, if α, β, γ be the trilinear coordinates of this point,

$$\frac{\alpha_1}{a(2c^2-a^2)}=\frac{\beta_1}{b(2a^2-b^2)}=\frac{\gamma_1}{c(2b^2-c^2)}.$$

If (as in the second case) the revolving lines make equal angles with AC, BA, CB respectively, then, taking C as the origin and CB as the positive direction of the axis of x, the locus would be that given above, merely modified by the interchange of b and c. It is therefore a circle of the same radius; but the ordinate of the centre is now $\frac{a(2b^2-a^2)}{8\Delta}$, and if $\alpha_2, \beta_2, \gamma_2$ are the trilinear coordinates of this centre,

$$\frac{\alpha_2}{a(2b^2-a^2)}=\frac{\beta_2}{b(2c^2-b^2)}=\frac{\gamma_2}{c(2a^2-c^2)},$$

and the equation to the line through $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$ is

$$(2A+B+C)\,bc\alpha+(2B+C+A)\,ca\beta+(2C+A+B)\,ab\gamma=0 \quad \ldots\ldots(1).$$

It is easily shown that the centre of the circle about ABC lies in this line, and this centre is also clearly a point on both loci; therefore the two circles touch at that point.

Of course, the first circle passes through the first Brocard point of ABC and the second circle through the second Brocard point. Also, if ρ = radius of the Brocard circle and ω be Brocard's angle, then the radii of the above circles are each $= \rho\cot\omega$.

The equation to the line joining the Brocard points is

$$Abc\alpha+Bca\beta+Cab\gamma=0,$$

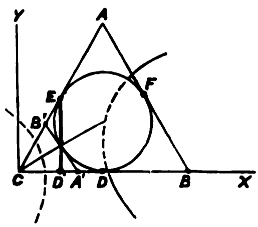
and this is parallel to (1). Therefore (1) touches the Brocard circle.

[For the Brocard circle, see *Reprint*, Vol. 40, p. 102, and *Quarterly Journal of Mathematics*, Vol. 19, p. 343.]

7385. (By Professor Wolstenholme, M.A., Sc.D.)—In an equilateral triangle ABC is inscribed a circle, any tangent to this circle meets the sides CB, CA in the points A′, B′; prove that (1) the centre of the circumscribed circle, and the centre of perpendiculars, of the triangle A′B′C have the same locus; (2) an hyperbola of which C is a focus, the centre of the circle is the farther vertex, and whose asymptotes are perpendicular to the sides CA, CB; (3) the centre of the circumscribed circle and the centre of perpendiculars are ends of a double ordinate to the transverse axis; (4) when they lie on the branch of which C is the exterior focus, the circle is the inscribed circle of the triangle; (5) when they lie on the branch of which C is the interior focus and between the radii drawn from C parallel to the asymptotes, the circle is the escribed circle opposite C; and (6) for the remainder of that branch the circle is one of the escribed circles opposite A′ or B′.

Solution by R. KNOWLES, B.A.; SARAH MARKS; *and others.*

1. Let AC $= a$, CB′ $= y'$, CA′ $= x'$, then the coordinates of A′, B′, the centre of perpendiculars, and of the circum-centre, of A′B′C, are respectively



$$(x', 0), \quad \left(\frac{y'}{2}, \frac{\sqrt{3}y'}{2}\right), \quad \left(\frac{y'}{2}, \frac{2x'-y'}{2\sqrt{3}}\right), \quad \left(\frac{x'}{2}, \frac{2y'-x'}{2\sqrt{3}}\right),$$

and $A'B'^2 = \frac{3}{4}y'^2 + (x' - \frac{1}{2}y')^2 = (a - x' - y')^2$; hence, substituting the values of the centres as above, the equation of the locus of each is $x^2 + y^2 = \frac{1}{4}(a - \sqrt{3}y - 3x)^2$ ......(α).

2. This locus is an hyperbola of which C is a focus, and as the axis is perpendicular to $3x + \sqrt{3}y = a$, and passes through C, its equation is

$$y = \frac{1}{\sqrt{3}} x \quad \text{.................................}(\beta).$$

From (α), (β) we find for the vertices $x = \frac{1}{6}a$ and $\frac{1}{2}a$; hence the centre of the circle is the farther vertex, and from (α)

$$y = -\frac{1}{\sqrt{3}} \frac{6x^2 - 6ax + a^2}{6x - 2a} = -\frac{1}{\sqrt{3}}(x - \tfrac{1}{3}a),$$

an asymptote at right angles to AC; and $6x = 2a$ gives another at right angles to BC.

3. The equation $y + \sqrt{3}\,x = \dfrac{x' + y'}{\sqrt{3}}$ passes through the centre of perpendiculars and of the circumscribed circle, and is perpendicular to $y = \dfrac{1}{\sqrt{3}}\,x$, the transverse axis.

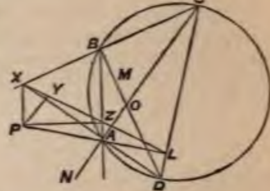
4. When they lie on the branch of which the centre is the vertex, A′, B′ lie on CB, CA produced, and the circle is the inscribed circle of the triangle.

5. The circle will be the escribed circle opposite C, when the line joining the two centres is within the triangle CED′, CD′ $= \frac{1}{2}a$; that is, when they lie on the branch between $x = 0$ and $y = -\dfrac{1}{\sqrt{3}}x$.

6. When A′, B′ lie on BC, AC produced, the circle is the escribed circle opposite A′ or B′.

7732. (By W. J. MCCLELLAND, M.A.)—On the sides of any quadrilateral inscribed in a circle, perpendiculars are drawn from the inverse of the point of intersection of the diagonals with respect to that circle; prove that the line of collinearity of the feet of the perpendiculars on the sides bisects at right angles the line joining the feet of the perpendiculars on the diagonals.

Solution by the PROPOSER; BELLE EASTON; *and others.*



Let ABCD be the quadrilateral, O and P inverse points, L, M, N and X, Y, Z the feet of the six perpendiculars from P on the sides and diagonals; then a circle circumscribes quadrilateral PXBM, hence

$$MX = PB \sin OBC;$$
$$NX = PC \sin OCB.$$

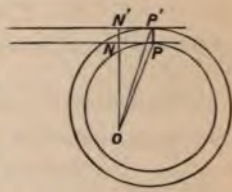
Therefore

$$\frac{MX}{NX} = \frac{PB}{PC} \cdot \frac{\sin OBC}{\sin OCB} = \frac{PB}{OB} \times \frac{OC}{PC} = 1;$$

hence $MX = NX$, also $MY = NY$, $MZ = NZ$, $ML = NL$, that is, X, Y, Z, L are four points on a line, which line bisects MN at right angles.

A NEW METHOD OF DERIVING LEGENDRE'S FORMULA $\int_0^{2\pi} p\, d\omega = L$.

By Professor CAVALLIN, M.A.



From the origin O, within a closed convex contour, let the perpendicular ON be drawn on the tangent PN, and the perpendicular ON′ on the tangent P′N′, parallel to PN, to a curve equidistant and infinitely near to the former, so that $PP' = \mu$ = constant.

If L, S denote respectively the length and area of the given curve, S′ the area of its pedal with respect to O as origin, and p, θ the length and direction of ON, we have immediately

$$S' = S + \tfrac{1}{2}\int_0^{2\pi} (PN)^2\, d\omega = S, \quad S' = \tfrac{1}{2}\int_0^{2\pi} (ON)^2 d\omega;$$

whence, by addition, we obtain the well-known formula

$$2S' = S + \tfrac{1}{2}\int_0^{2\pi} (OP)^2\, d\omega, \quad \text{or} \quad \int_0^{2\pi} (ON)^2\, d\omega = S + \tfrac{1}{2}\int_0^{2\pi} (OP)^2\, d\omega \;\ldots(1).$$

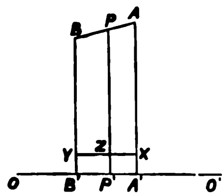
Varying (1), $$2\int_0^{2\pi} ON\, \delta ON\, d\omega = \delta S + \int_0^{2\pi} OP\, \delta\, OP\, d\omega \;\ldots\ldots\ldots\ldots(2),$$

and by substitution herein of the values $ON\, \delta\, ON = p\mu$, $\delta S = L\mu$, and $OP\, \delta\, OP = OP\,(OP' - OP) = OP \,.\, PP' \cos OP'P = \mu\, OP \sin OPN = p\mu$ in (2). LEGENDRE's formula follows at once.

7546. (By B. Reynolds, M.A.)—Prove that, when a and b are very large compared to their difference,

$$\left(\frac{ma+nb}{m+n}\right)^{p^{-1}} = \frac{ma^{p^{-1}}+nb^{p^{-1}}}{m+n}, \text{ nearly.}$$

Solution by F. C. Garner, M.A.; G. B. Mathews, B.A.; *and others.*



Draw BB′, AA′ perpendicular to any straight line OO′ and proportional to b, a respectively; and cut off B′Y, A′X so that

$$B'Y : B'B = b^{p^{-1}} : b,$$

and $$XA' : A'A = a^{p^{-1}} : a;$$

then, since the difference between a and b is very small, the difference between $a^{p^{-1}}$ and $b^{p^{-1}}$ is so small that it may be neglected, so that we may consider B′Y = A′X very nearly. Now take P in AB so that BP : PA = $m : n$; draw PP′ perpendicular to A′B′ and meeting XY in Z; then, since PP′ − BB′ is very small, we may consider

$$(PP')^{p^{-1}} = (BB')^{p^{-1}} \text{ very nearly};$$

hence we may consider $ZP' = (PP')^{p^{-1}}$ very nearly. But

$$PP' = \frac{m \,.\, AA' + n \,.\, BB'}{m+n} = \frac{ma+nb}{m+n},$$

and $$ZP' = \frac{m \,.\, A'X + n \,.\, B'Y}{m+n} = \frac{ma^{p^{-1}}+nb^{p^{-1}}}{m+n}; \text{ therefore, \&c.}$$

[*Otherwise* :—

$$\left(\frac{ma+nb}{m+n}\right)^{p^{-1}} = \left(a+\frac{n(b-a)}{m+n}\right)^{p^{-1}} = a^{p^{-1}}\left(1+\frac{n(b-a)}{(m+n)a}\right)^{p^{-1}}$$

$$= a^{p^{-1}}\left(1+\frac{n}{p}\cdot\frac{b-a}{(m+n)a}\right) \text{ to first order}$$

$$= \frac{a^{p^{-1}}}{m+n}\left\{m+n\left(1+\frac{1}{p}\cdot\frac{b-a}{a}\right)\right\}$$

$$= \frac{a^{p^{-1}}}{m+n}\left\{m+n\left(1+\frac{b-a}{a}\right)^{p^{-1}}\right\} \text{ to first order} = \frac{ma^{p^{-1}}+nb^{p^{-1}}}{m+n}.$$

As a special case, we have, when a is very nearly $= b$,

$$\left\{\tfrac{1}{3}(2a+b)\right\}^{\frac{1}{3}} = \tfrac{2}{3}a^{\frac{1}{3}} + \tfrac{1}{3}b^{\frac{1}{3}}.]$$

7739. (By W. G. Lax, B.A.)—If x, y; r, θ be the rectangular and polar coordinates of a point respectively, and if $\left(\frac{dx}{dr}\right)$ and $\left(\frac{dr}{dx}\right)$ be the as the resultant of multiples of the other systems, and, if the forces be so chosen that

$$\begin{vmatrix} P_1, & Q_1, \dots & U_1 \\ P_2, & Q_2, \dots & U_2 \\ \&c. & \&c. & \\ P_{i-1}, & Q_{i-1}, \dots & U_{i-1} \end{vmatrix} = 1, \quad m_i = 1, \&c.,$$

all the other m's are integral.

Similarly, if the points lie on a plane reticulation, and the distances of the points of application from the fixed point, measured parallel to the threads, be $a_1, a_2 \dots a_i$; and $b_1, b_2 \dots b_i$ respectively, then from equations (1) which still hold, and from the corresponding system in the b's, the same system (2) follows, and similarly for a solid reticulation. So far it has been assumed that all the separate systems of forces are parallel; if this be not the case, $3(i-1)$ sets of forces may be taken, $i-1$ sets being parallel to each of the coordinate axes, and these, with the components of any other set, will form i sets of forces parallel to each of the axes, and so the last set may, by what precedes, be expressed in terms of the $3(i-1)$ other sets.

7741. (By the late Professor CLIFFORD, F.R.S.)—The motion of a point is compounded of two simple harmonic motions at right angles to one another, which are very nearly equal in period, but whose amplitudes are slowly diminishing at a uniform rate; find the general shape of the curve which the point will describe.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

Let the motion of the particle be defined by

$$x = A(1-\alpha t)\cos(kt+\alpha),$$

$$y = B(1-\alpha t)\cos[k(1+m)t+\beta] = B(1-\alpha t)\cos(kt+kmt+\beta);$$

then $$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(kmt+\beta-\alpha) = (1-\alpha t)^2\sin^2(kmt+\beta-\alpha),$$

an ellipse whose axes are constantly changing in magnitude and position, but the ratio of which is periodic, for, if ϕ be the angle which an axis makes with X, $\tan 2\phi = [2AB\cos(kmt+\beta-\alpha)]/(A^2-B^2)$ representing an oscillatory motion, whose period $= 2\pi/km$, at the expiration of which period the equation of the curve of vibration, for the instant, becomes

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(\beta-\alpha) = \left(1-\frac{2\pi\alpha}{km}\right)^2\sin^2(\beta-\alpha),$$

an ellipse similar and similarly placed to the initial one,

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos(\beta-\alpha) = \sin^2(\beta-\alpha),$$

and thus, at the expiration of each period, the linear magnitude of the ellipse is diminished in geometrical progression, while the direction of its axis major returns to its initial position, which may be taken to be that corresponding to any instant during its motion; results which can easily be verified experimentally by the kaleidophone.

5481. (By Professor BURNSIDE, M.A.)—Trace the relation between the characteristics of a curve of the mth degree having the maximum number of double points, and the curve enveloped by the line

$$(a_0, a_1, a_2, \dots a_m)(\theta, 1)^m = 0,$$

where $a_0, a_1, a_2, \dots a_m$ are linear functions of the coordinates, and θ a variable parameter.

Solution by W. J. C. SHARP, M.A.

If the given equation $(a_0, a_1, \dots a_m)(\theta, 1)^m = 0$ be identical with

$$\alpha x + \beta y + \gamma z = 0,$$

then $$\lambda\alpha = (p_0, p_1, \dots p_m)(\theta, 1)^m,\ \lambda\beta = (q_0, q_1, \dots q_m)(\theta, 1)^m,$$
$$\lambda\gamma = (r_0, r_1, \dots r_m)(\theta, 1)^m,$$

where p_0, p_1, &c. ... r_m are constant, and the envelope, the tangential equation of which is the eliminant of these three equations, will be a unicursal curve of the m^{th} order if α, β, γ be looked upon as ordinary coordinates. And therefore, when they are tangential coordinates, $i+\tau = \frac{1}{2}(m-1)(m-2)$, as in the case of a unicursal curve $k+\delta = \frac{1}{2}(m-1)(m-2)$, and, using accented letters for the unicursal curve, and unaccented for the envelope,

$$i = k',\ k = i',\ \tau = \delta',\ \delta = \tau',\ m = n',\ n = m'.$$

In fact, from the considerations above, it appears that every envelope of the class here given is the polar reciprocal, with respect to $x^2+y^2+z^2=0$ of the unicursal curve

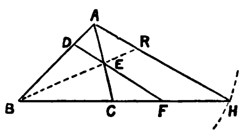
$$\lambda x = (p_0, p_1, \dots p_m)(\theta, 1)^m,\quad \lambda y = (q_0, q_1, \dots q_m)(\theta, 1)^m,$$
$$\lambda z = (r_0, r_1, \dots r_m)(\theta, 1)^m,$$

where $$a_n \equiv p_n x + q_n y + r_n z.$$

7675. (By the EDITOR.)—Draw a transversal DEF to cut the sides AB, AC, BC of a triangle ABC in D, E, F respectively, in suchwise that, if M, N, P, Q be given lines, we shall have (1) BD : DE = M : N and CE : EF = P : Q; or (2) that BD : DE : EC = M : N : P.

Solution by W. G. LAX, B.A.; SARAH MARKS; *and others.*

1. Let (Fig. 1) a circle with centre A, and radius AH, such that CA : AH = P : Q, be drawn cutting BC produced in H; join AH; take AR so that BA : AR = M : N; join BR cutting AC in E; and draw DEF parallel to AH; then BD : DE = BA : AR = M : N, and CE : EF = CA : AH = P : Q.

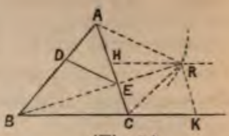


(Fig. 1.)

2. Take (Fig. 2) CH so that

$$BA : CH = M : P;$$

draw HR parallel to BC; with centre A and radius AR, such that $BA : AR = M : N$, draw a circle cutting HR in R; join AR; draw RK parallel to AC; join BR cutting AC in E; and draw DE parallel to AR; then



(Fig. 2.)

$$DE : AR = BE : BR = BD : BA \text{ and } CE : KR = BE : BR,$$

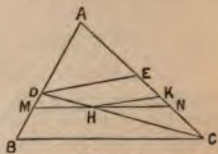
therefore $\quad BD : DE : EC = BA : AR : RK = M : N : P.$

[*Otherwise* :—Make (Fig. 3) $BM = M$ and $CK = P$; draw MN parallel to BC; inflect $KH = N$ to MN; draw CHD to AB; and through D draw DE parallel to HK; then

$$CD : CH = BD : BM = DE : HK = CE : CK,$$

therefore $BD : DE : EC$

$$= BM : HK : CK = M : N : P.]$$



(Fig. 3.)

7515. (By Professor Wolstenholme, M.A., Sc.D.)—If normals OP, OQ, OR be drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ from the point O (whose coordinates are X, Y), and the tangents at P, Q, R form a triangle P′Q′R′; prove that the ratio $k : 1$ of the triangles PQR, P′Q′R′ is given by

$$\left\{ k^2 + (k - \tfrac{1}{2}) \frac{a^2X^2 + b^2Y^2}{c^4} \right\}^2 = (\tfrac{1}{4} - k) \left(\frac{a^2X^2 - b^2Y^2}{c^4} \right)^2.$$

Solution by R. Lachlan, B.A.; Sarah Marks; *and others.*

Let α, β, γ be eccentric angles of P, Q, R: then we find

$$-2k = 1 + \cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) \text{ ....(1)};$$

also α, β, γ are three of the roots of

$$aX \sin\phi - bY \cos\phi = c^2 \sin\phi \cos\phi \text{ ....(2)},$$

i.e., of

$$c^4 \cos^4\phi - 2ac^2 X \cos^3\phi + (a^2X^2 + b^2Y^2 - c^4) \cos^2\phi + 2ac^2 X \cos\phi - a^2X^2 = 0,$$

or of $\quad c^4 \sin^4\phi + 2bc^2 Y \sin^3\phi + \&c. = 0;$

and, if ϕ be the fourth root of (2), we find

$$\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta = \frac{a^2X^2 + b^2Y^2 - c^4}{c^4} - \frac{2aX}{c^2}\cos\phi + \cos^2\phi,$$

$$\sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta = \frac{a^2X^2 + b^2Y^2 - c^4}{c^4} + \frac{2bY}{c^2}\sin\phi + \sin^2\phi.$$

Thus (1) may be written

$$aX\cos\phi - bY\sin\phi = c^2\left(k + \frac{a^2X^2+b^2Y^2}{c^4}\right) \quad\text{.................(3).}$$

From (2) and (3), by multiplication, $-ab\,XY = c^4\,k\sin\phi\cos\phi$ ....(4), also, squaring (2) and (3), we get, by addition,

$$a^2X^2 + b^2Y^2 - 4ab\,XY\sin\phi\cos\phi = c^4\sin^2\phi\cos^2\phi + c^4\left(k + \frac{a^2X^2+b^2Y^2}{c^4}\right)^2.$$

Substituting from (4), we have, after a little reduction, the stated result.

[Since four normals can be drawn from (X, Y), there are four values of k; and, if these be k_1, k_2, k_3, k_4, the equation proves that

$$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\frac{1}{k_4} = 4,$$

$$k_1^2+k_2^2+k_3^3+k_4^2-2(k_2k_3+\ldots)+2(k_1+k_2+k_3+k_4) = 0.$$

When $a^2X^2 = b^2Y^2$, the four points P, Q, R, S may be taken so that PQ, RS are parallel to each other and to one of the equal conjugate diameters; in which case the triangles PQR, PQS will be equal to each other, and so also P′Q′R′, P′Q′S′; PRS, QRS; P′R′S′, Q′R′S′.

The equation will also have equal roots when (X, Y) lies on the evolute. When k is positive, it is of course $< \frac{1}{4}$; and, since

$$k = -2\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\alpha-\beta),$$

its real negative values must be > -2, or, more exactly, its value for real triangles > -2; which is the value at the centres of maximum and minimum curvature.

It may easily be proved, from the quartic in k, that any positive value of $k < u^{\frac{1}{2}}(1-u^{\frac{1}{2}})$; and any positive value of $-k < u^{\frac{1}{2}}(1-u^{\frac{1}{2}})$, u being always < 1 for real triangles, limits which are closer than $\frac{1}{4}$ and 2.]

7484. (By Professor MALET, F.R.S.)—If two solutions of the linear differential equation

$$\frac{d^3y}{dx^3} + Q_1\frac{d^2y}{dx^2} + Q_2\frac{dy}{dx} + Q_3y = 0 \quad\text{..................(A)}$$

are the solutions of the equation $\dfrac{d^2y}{dx^2} + P_1\dfrac{dy}{dx} + P_2y = 0$; prove that (1)

$$P_1P_2(P_1-Q_1) = P_2\left(\frac{dP_1}{dx} + P_2 - Q_2\right) = P_1\left(\frac{dP_2}{dx} - Q_3\right);$$

and (2) the complete solution of (A) is the solution of

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = CP_2e^{-\int\frac{Q_3}{P_2}dx.}$$

Solution by R. LACHLAN, B.A.; NILKANTA SARKAR, B.A.; *and others.*

If U, V, W be any particular solutions of (A), the general solution of A is of the form $y = C_1U + C_2V + C_3W$; hence, differentiating and eliminating C_1, we get $\left(\frac{d}{dx}-X_1\right)y = C_2V' + C_3W'$, where $X_1 = \frac{1}{U}\frac{dU}{dx}$. Similarly, eliminating C_2, C_3, we shall get

$$\left(\frac{d}{dx}-X_3\right)\left(\frac{d}{dx}-X_2\right)\left(\frac{d}{dx}-X_1\right)y = 0.$$

Also, if U, V be particular solutions of (B), then (B) must be of the form

$$\left(\frac{d}{dx}-X_2\right)\left(\frac{d}{dx}-X_1\right)y = 0;$$

hence (A) is of the form $\left(\frac{d}{dx}-X\right)\left(\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y\right) = 0$; thus, comparing this with (A), we find $X = P_1 - Q_1 = \dfrac{P_2 - Q_2 + \frac{dP_1}{dx}}{P_1} = \dfrac{\frac{dP_2}{dx} - Q_3}{P_2}$, and the form of this equation shows that the solution of (A) must be

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y = ce^{\int X\,dx} = cP_2e^{-\int\frac{Q_3}{P_2}dx}.$$

[Mr. LACHLAN believes this solution to be preferable to that given by Mr. RAWSON on pp. 113 to 115 of Vol. xl. of *Reprints*, from the fact that the method here employed is capable of more general application.]

7639. (By CHRISTINE LADD FRANKLIN, B.A.)—If, in a certain lot of objects, the a's are identical with the non-x's which are b's together with the y's which are non-b's, and the c's are not identical with the x's which are non-d's together with the non-y's which are d's, what relation exists between a, b, c, d?

Solution by DR. MACFARLANE ; *the* PROPOSER ; *and others.*

The first and second conditions are

$$a = (1-x)\,b + y\,(1-b), \quad c+v = x\,(1-d)+(1-y)\,d+w \ldots\ldots (1, 2),$$

where v and w are determinate, but such that both cannot be null.

These equations may be written in the form

$$xb - y\,(1-b)+a-b = 0,\; x\,(1-d)-yd+d+w-c-v = 0 \ldots\ldots (3, 4).$$

Multiply (3) by b and (4) by $(1-d)$,

$$xb+b\,(a-1) = 0,\; x\,(1-d)+1-d\,(w-c-v) = 0 \ldots\ldots\ldots (5, 6);$$

$$\therefore\; b\,(1-d)(w-c-v) = b\,(1-d)(a-1),\; i.e.,\; b\,(1-d)[a+v = 1-c+w] \ldots (7).$$

Again, multiply (3) and (4) by $(1-b)$ and d respectively, then

$$-y\,(1-b)+a\,(1-b) = 0,\; -yd+d\,(1-c+w-v) = 0 \ldots\ldots\ldots (8, 9);$$

$$\therefore\quad (1-b)\,d\,[1-c+w-v = a],\; i.e.,\; (1-b)\,d\,[a+v = 1-c+w] \ldots\ldots (10).$$

As $b(1-d)$ and $(1-b)d$ are mutually exclusive, (7) and (10) can by addition be combined into the equivalent equation

$$[b(1-d)+(1-b)d](a+v = 1-c+w) \quad\text{..................(11).}$$

Of the b's which are not d or the non-b's which are d, those which are a are not identical with those which are not c.

7554. (By T. Muir, M.A., F.R.S.E.)—Prove that

$$\begin{aligned}&(C-B)(C^6-B^6)(D-A)(D^2-A^2)A^2D^2\\&\qquad+(C-B)(C^2-B^2)(D-A)(D^6-A^6)B^2C^2\\&+(C^4-B^4)(C^3-B^3)(D^4-A^4)(D-A)AD\\&\qquad+(C^4-B^4)(C-B)(D^4-A^4)(D^3-A^3)BC\\&=(C^2-B^2)(C^5-B^5)(D^2-A^2)(D^3-A^3)AD\\&\qquad+(C^2-B^2)(C^3-B^3)(D^2-A^2)(D^5-A^5)BC.\end{aligned}$$

Solution by G. B. Mathews, B.A.; Prof. Matz, M.A.; *and others.*

Dividing both sides by $(C-B)(C^2-B^2)(D-A)(D^2-A^2)$, the equation becomes

$$\begin{aligned}&(C^4+C^2B^2+B^4)A^2D^2+(D^4+D^2A^2+A^4)B^2C^2\\&+(C^2+B^2)(C^2+CB+B^2)(D^2+A^2)AD\\&\qquad+(C^2+B^2)(D^2+DA+A^2)(D^2+A^2)BC\\&=(C^4+C^3B+C^2B^2+CB^3+B^4)(D^2+DA+A^2)AD\\&\qquad+(D^4+D^3A+D^2A^2+DA^3+A^4)(C^2+CB+B^2)BC\text{...(A).}\end{aligned}$$

Observing that $C^4+C^2B^2+B^4=(C^2-CB+B^2)(C^2+CB+B^2)$, the left-hand side may be written

$$\begin{aligned}&AD(D^2+DA+A^2)(C^4+C^2B^2+B^4)+AD.CB(C^2+CB+B^2)(D^2+A^2)\\&+BC(C^2+CB+B^2)(D^4+D^2A^2+D^4)+BC.AD(D^2+DA+A^2)(C^2+B^2)\end{aligned}$$

and this, by adding diagonally, becomes

$$\begin{aligned}&AD(D^2+DA+A^2)[C^4+C^2B^2+B^4+BC(C^2+B^2)]\\&+BC(C^2+CB+B^2)[D^4+D^2A^2+A^4+AD(D^2+A^2)]\\&=\text{right-hand side of (A).}\end{aligned}$$

5501. (By Professor Ball, LL.D., F.R.S.)—If in an equation x be changed into $k+x'^{-1}$, show that any semi-invariant of the transformed will be a covariant in k of the original equation.

Solution by W. J. C. Sharp, M.A.

If $f(x)\equiv(a_0, a_1...a_n)(x,1)^n = 0$ be the given equation, the transformed will be

$$x'^n f\left(k+\frac{1}{x'}\right)\equiv f(k)x'^n+f'(k)x'^{n-1}+\frac{1}{1.2}f''(k)x^{n-2}+\&c. = 0,$$

or, say, $$(A_0, A_1 \dots A_n)(x, 1)^n = 0,$$
where $$A_0 = (a_0, a_1 \dots a_n)(k, 1)^n,$$

$$nA_1 = n(a_0, a_1, a_2 \dots a_{n-1})(k, 1)^{n-1} = \left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) A_0,$$

$$n(n-1)A_2 = n(n-1)(a_0, a_1 \dots a_{n-2})(k, 1)^{n-2}$$
$$= n\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) A_1, \ \&c. \ \&c.$$

But $$nA_1 = f'(k) = \frac{d}{dk} A_0, \quad n(n-1)A_2 = f''(k) = n\frac{d}{dk} A_1, \&c. \&c.;$$

therefore, for any function of A_0, A_1, &c., the operations

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) \text{ and } \frac{d}{dk}$$

give identical results. Now, if F $(A_0, A_1 \dots)$ be a function of the differences of the roots α', β', &c. of the transformed equation F_0, the coefficient of the highest power of k will be a function of the differences of the roots $\alpha, \beta \dots$ of the given equation.

Since $$\alpha' - \beta' = -(\alpha - \beta) \Big/ \left\{ \left(\frac{\alpha}{k} - 1\right)\left(\frac{\beta}{k} - 1\right) \right\},$$

and therefore $$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right) F_0 = 0,$$

and therefore F is a covariant in k of the original equation.

[See SALMON's *Higher Algebra*, 2nd Edition, p. 119.]

7260. (By ELIZABETH BLACKWOOD.)—A pack of n different cards is laid face downwards on a table. A person names a certain card. That and all the cards above it are shown to him, and removed. He names another; and the process is repeated until there are no cards left. Find the chance that, in the course of the operation, a card was named which was (at the time) at the top of the pack.

Solution by D. BIDDLE; BELLE EASTON; *and others.*

The card named may with equal probability occupy any position in that portion of the pack not yet removed.

There are three positions which instantly decide the issue, namely, the first (or top), the last, and the last but one; for the first and the last but one equally command success, since in the latter case only one card will remain, and this it is easy to name next time. The last or lowest position is the only one that insures failure. If the card named be in the lowest position but two, an equal chance of success or failure will be left for the next trial.

If it be in the last but three, there will in the next trial be two chances of success to one of failure; and, if in the last but four, there will be five (or $2\frac{1}{2}$) chances of success to three (or $1\frac{1}{2}$) of failure. In other words,

$P_1 = 1$, $P_2 = \frac{1}{2}$, $P_3 = \frac{2}{3}$, $P_4 = \frac{5}{8}$. But, counting from the top of the pack, the probabilities attaching to the several positions are as follows:—

$$1,\ P_{n-2},\ P_{n-3},\ P_{n-4} \dots P_1,\ 0;$$

and
$$P_n = \frac{1}{n}[1 + P_{n-2} + P_{n-3} + P_{n-4} + \&c.],$$

but similarly
$$P_{n-1} = \frac{1}{n-1}[1 + P_{n-3} + P_{n-4} + \&c.],$$

$\therefore P_n = \frac{1}{n}[(n-1)P_{n-1} + P_{n-2}]$. But this resolves itself into

$$P_n = \frac{1}{n}[(n-1)P_{n-2} + P_{n-3}];\ \therefore\ P_{n-2} - P_{n-3} = (n-1)(P_{n-2} - P_{n-1}),$$

which shows that the probabilities are alternately greater and less, but that the differences between them rapidly become infinitesimal as n increases. Above P_{10} the probabilities are alike to six places of decimals, viz., ·633388, or rather more than $\frac{19}{30}$, each probability being resolvable, according to the foregoing statements, into n terms of the series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \&c.$$

6740. (By Professor Asaph Hall, M.A.)—Given

$$z = a \sin(x + \alpha) + b \sin(y + \beta),$$

reduce z to the form $z = D \sin(x + \alpha + y + \beta + \delta)$.

Solution by Asûtosh Mukhopâdhyây.

We have $z = a\sin(x+\alpha) + b\sin(y+\beta)$; put $a = p+q$, $b = p-q$, then

$$z = p[\sin(x+\alpha) + \sin(y+\beta)] + q[\sin(x+\alpha) - \sin(y+\beta)]$$
$$= 2p \sin\tfrac{1}{2}(x+\alpha+y+\beta)\cos\tfrac{1}{2}(x+\alpha-y-\beta)$$
$$+ 2q\cos\tfrac{1}{2}(x+\alpha+y+\beta)\sin\tfrac{1}{2}(x+\alpha-y-\beta).$$

If we determine θ, D, so that

$$2p\cos\tfrac{1}{2}(x+\alpha-y-\beta) = D\cos\tfrac{1}{2}\theta,\quad 2q\sin\tfrac{1}{2}(x+\alpha-y-\beta) = D\sin\tfrac{1}{2}\theta,$$

we have $Z = D\sin\frac{1}{2}(x+\alpha+y+\beta+\theta)$. Finally, if we determine δ from

$$\delta = \tfrac{1}{2}\theta - \tfrac{1}{2}(x+\alpha+y+\beta),\ \text{we get}\ Z = D\sin(x+\alpha+y+\beta+\delta),$$

which is the form required.

7643. (By Rev. H. G. Day, M.A.)—A and B sit down to play for a shilling per game, the odds being $k:1$ on B; they have m and n shillings respectively, and agree to play till one is ruined: find A's chance of success.

Solution by the Proposer.

Let u_r be A's chance when he is r games ahead; then his probability of scoring the next game is $\frac{1}{1+k}$, and of losing it $\frac{k}{1+k}$; but in these cases his chances become u_{r+1} and u_{r-1} respectively; hence

$$(1+k)\,u_r = u_{r+1}+ku_{r-1}.$$

Solving this equation, $u_r = Ck^r + C_1$; but $u_n = 1$ and $u_{-m} = 0$; therefore

$$u_r = \frac{k^r - k^{-m}}{k^n - k^{-m}} = \frac{k^{r+m}-1}{k^{n+m}-1}; \text{ and A's chance } u_0 = \frac{k^m-1}{k^{n+m}-1}.$$

If k is unity, the solution becomes $u_r = Cr + C_1$; but $1 = Cn + C_1$, and $0 = -Cm + C_1$, therefore $u_r = \dfrac{r+m}{n+m}$ and $u_0 = \dfrac{m}{n+m}$;
that is, the players' chances are directly as the sums they risk.

7677. (By W. E. Johnson, B.A.)—If p and n be any integers, and $\omega_1, \omega_2 \ldots \omega_{n-1}$ are all the n^{th} roots of unity except unity itself, show that the remainder, when p is divided by n, is

$$F(p) \equiv \frac{n-1}{2} + \omega_1^p \frac{\omega_1}{1-\omega_1} + \omega_2^p \frac{\omega_2}{1-\omega_2} + \ldots + \omega_{n-1}^p \frac{\omega_{n-1}}{1-\omega_{n-1}}.$$

Solution by B. Hanumanta Rau, M.A.; E. Rutter; *and others.*

1. If $p = mn$, the expression becomes

$$\frac{n-1}{2} + \frac{\omega_1}{1-\omega_1} + \frac{\omega_2}{1-\omega_2} + \ldots + \frac{\omega_{n-1}}{1-\omega_{n-1}}.$$

If now $y = \dfrac{x}{1-x}$, where $x^n = 1$, we have $x = \dfrac{y}{1+y}$, $\therefore\ y^n = (1+y)^n$.
Corresponding to $x = 1$, we have $y = \infty$; the remaining values of y are given by $ny^{n-1} + \frac{1}{2}n(n-1)\,y^{n-2} + \ldots = 0$; thus the sum of roots $= -\frac{1}{2}(n-1)$; and hence, when $p = mn$, the given expression vanishes.

2. $F(p) - F(p-1) = -(\omega_1^p + \omega_2^p + \ldots + \omega_{n-1}^p)$. This equals 1, unless $p = mn$, in which case it equals $-(n-1)$. Hence the proposition is completely proved.

7292. (By Dr. Curtis.)—Two heavy particles P and Q are connected by a flexible and inextensible cord, which rests on a pulley of infinitesimal radius; P is restricted to the circumference of a smooth circle, whose centre is vertically under the pulley, or, *more generally*, of a smooth Cartesian oval, one of whose foci coincides with the pulley, and whose axis is vertical; it is required to prove that the curve to which Q should be restricted, in order that equilibrium should exist for all possible positions of P and Q, is a Cartesian oval.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

finity, the orbit is $u = (2a)^{-1} \sec(\theta-\varpi)$, where $h^2 = \frac{\mu}{8a^2}$ ....(1).

Expressing that the part of $\frac{du}{d\theta}$ which arises from the variation of the constants vanishes, we have $\quad \operatorname{cosec}(\theta-\varpi)\frac{da}{d\theta} + a\sec(\theta-\varpi)\frac{d\varpi}{d\theta} = 0$......(2).

Also $\frac{du}{d\theta} = (2a)^{-1}\frac{\sin(\theta-\varpi)}{\cos^2(\theta-\varpi)}$, whence

$$\frac{d^2u}{d\theta^2} = \frac{1+\sin^2(\theta-\varpi)}{2a\cos^3(\theta-\varpi)}\left(1-\frac{d\varpi}{d\theta}\right) - \frac{\sin(\theta-\varpi)}{2a^2\cos^2(\theta-\varpi)}\frac{da}{d\theta},$$

and, since for the disturbed orbit, $\frac{d^2u}{d\theta^2} + u - \frac{\mu u^3}{h^2} = \frac{f}{h^2u}$, this gives by (2)

$$-\frac{4a^2}{h^2} f\cos^2(\theta-\varpi) = \frac{1}{2a\cos^3(\theta-\varpi)}\frac{d\varpi}{d\theta},$$

so that $\quad \operatorname{cosec}(\theta-\varpi)\frac{da}{d\theta} = -a\sec(\theta-\varpi)\frac{d\varpi}{d\theta} = \frac{8a^4f}{h^2}\cos^4(\theta-\varpi).$

But $$\frac{d\theta}{dt} = hu^2 = \frac{1}{8a^3}\left(\frac{\mu}{2}\right)^{\frac{1}{2}}\sec^2(\theta-\varpi);$$

$$\therefore \quad \operatorname{cosec}(\theta-\varpi)\frac{da}{dt} = -a\sec(\theta-\varpi)\frac{d\varpi}{dt} = 4fa^3\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\cos^2(\theta-\varpi).$$

7498. (By A. MARTIN, B.A.)—If a straight line be drawn from the focus of an ellipse to make a given angle α with the tangent, show that the locus of its intersection with the tangent will be a circle which touches or falls entirely without the ellipse according as $\cos\alpha$ is less or greater than the excentricity of the ellipse.

Solution by Rev. J. L. KITCHIN, M.A.; J. O'REGAN; *and others.*

The equations of the tangent and of a line through the positive focus at angle α with tangent are

$$y = mx + (m^2a^2+b^2)^{\frac{1}{2}},\quad y = \frac{m+\tan\alpha}{1-m\tan\alpha}(x-ae) \quad ....(1,\ 2),$$

therefore $\quad m = \frac{y-kx+aek}{ky+x-ae}$, putting k for $\tan\alpha$.

Hence, by (1), $\quad y - \frac{xy-kx^2+aekx}{ky+x-ae} = \left\{a^2\left(\frac{y-kx+aek}{ky+x-ae}\right)^2 + b^2\right\}^{\frac{1}{2}},$

or $\;[kx(x-ae)+y(ky-ae)]^2 = a^2(y-kx+aek)^2 + (a^2-a^2e^2)(ky+x-ae)^2,$

which becomes $\quad k^2(x^2+y^2) - 2ae\,ky + a^2e^2 = a^2(1+k^2)$ ....(3).

For intersection of this circle with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$\frac{a^2e^2y^2}{b^2} + \frac{2aey}{k} + \frac{b^2}{k^2} = 0 = \left(\frac{aey}{b} + \frac{b}{k}\right)^2.$$

Solution by the PROPOSER.

Let O, the centre of the pulley, be taken as origin, the axis of y being vertical; then, by the principle of virtual velocities, if y, y' be the coordinates of P and Q. $Py + Qy' = \text{const.}$, or $y + \lambda y' = A$; also, r and r' being the radii vectores of P and Q, $r + r' = \text{const.} = B$; but, as P is restricted as defined in the question,

$$r^2 - 2Lr + My = N, \text{ or } (B-r')^2 - 2L(B-r') + M(A-\lambda y') = N,$$

which establishes a linear relation between y' and r', and therefore represents a Cartesian oval as defined.

7615. (By W. NICOLLS, B.A.)—If $u_1 + u_2 + u_3 = c$ represent a surface of revolution, the origin being the centre of revolution, and u_1, u_2, u_3 containing respectively all the terms of the first, second, and third degrees in x, y, z; prove that u_1 is perpendicular to the axis of revolution and a factor of u_3.

Solution by J. P. JOHNSTON, B.A.; ELIZABETH BLACKWOOD; *and others.*

Considering the equation of the cubic surface in cylindrical coordinates (z, r, θ) as an equation for r, and taking the axis of z as the axis of revolution, it is evident that it can only contain even powers of r since all sections perpendicular to the axis are circles. Therefore the equation is of the form $z\,[z^2 + r^2 f_1(\theta)] + az^2 + r^2 f_2(\theta) + bz + c = 0$, where $f_1(\theta)$ and $f(\theta)$ are quadratic functions of $\cos\theta$ and $\sin\theta$. Hence it appears that if his be put in the form $u_3 + u_2 + u_1 = c$, u_1 is perpendicular to the axis of revolution and a factor of u_3.

[Mr. NICOLLS' theorem may be slightly generalised thus:—All cubics of revolution can be written in the form $LC + C' + L = 0$, where C and C′ are cones and L a plane perpendicular to the axis of revolution.]

7445. (By C. LEUDESDORF, M.A.)—A particle, describing a circular orbit about a centre of attractive force μ (distance)$^{-3}$ tending to a point on the circumference, is disturbed by a small force f tending to the same point; prove that the variations of the diameter $(2a)$ and of the inclination to a fixed straight line in the plane (ϖ) of that diameter which passes through the centre of force are given by the equations

$$-\operatorname{cosec}(\theta-\varpi)\frac{da}{dt} = a\sec(\theta-\varpi)\frac{d\varpi}{dt} = 4fa^3\left(\frac{2}{\mu}\right)^{\frac{1}{2}}.$$

Solution by D. EDWARDES; MARGARET T. MEYER; *and others.*

If the attraction be μD^{-5} and the velocity of projection that from in-

Solution by MARGARET T. MEYER; Professor NASH, M.A.; *and others.*

The points of contact are $a\cos\alpha$, $b\sin\alpha$, $a\cos\beta$, $b\sin\beta$, $a\cos\gamma$, $b\sin\gamma$, where (α, β, γ) are the eccentric angles; and the equations of the tangents are $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha = 1$, &c.

$$\text{Area of triangle in the ellipse} = \tfrac{1}{2}\begin{vmatrix} a\cos\alpha, & b\sin\alpha, & 1 \\ a\cos\beta, & b\sin\beta, & 1 \\ a\cos\gamma, & b\sin\gamma, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}ab\left[\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)\right].$$

Area of triangle contained by the three tangents is

$$\tfrac{1}{2}\begin{vmatrix} \frac{1}{a}\cos\alpha, & \frac{1}{b}\sin\alpha, & 1 \\ \frac{1}{a}\cos\beta, & \frac{1}{b}\sin\beta, & 1 \\ \frac{1}{a}\cos\gamma, & \frac{1}{b}\sin\gamma, & 1 \end{vmatrix}^2 \div \begin{vmatrix} \frac{1}{a}\cos\alpha, & \frac{1}{b}\sin\alpha \\ \frac{1}{a}\cos\beta, & \frac{1}{b}\sin\beta \end{vmatrix} \times \begin{vmatrix} \&c. \\ \&c. \end{vmatrix} \times \begin{vmatrix} \&c. \\ \&c. \end{vmatrix}$$

$$= \tfrac{1}{2}ab\,\frac{\left[\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)\right]^2}{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)};$$

therefore the ratio is
$$\frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{\sin(\alpha-\beta)+2\sin\frac{1}{2}(\beta-\alpha)\cos\left[\frac{1}{2}(\alpha+\beta)-\gamma\right]}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{2\sin\frac{1}{2}(\alpha-\beta)\left\{\cos\frac{1}{2}(\alpha-\beta)-\cos\left[\frac{1}{2}(\alpha+\beta)-\gamma\right]\right\}}$$

$$= \frac{\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$= 2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha) = 2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi\cos\tfrac{1}{2}\psi.$$

7698. (By R. LACHLAN, B.A.)—Show that (1) four circles can be drawn cutting the sides of a triangle in angles α, β, γ respectively; (2) if their radii be ρ, ρ_1, ρ_2, ρ_3, and they cut any other straight line in angles ϕ, ϕ_1, ϕ_2, ϕ_3, then

$$\frac{1}{\rho} = \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}, \quad \cos\phi = \cos\phi_1+\cos\phi_2+\cos\phi_3.$$

Solution by Rev. T. C. SIMMONS, M.A.; B. HANUMANTA RAU, M.A.; *and others.*

Consider first the circle whose centre lies *within* the triangle; let d_1, d_2, d_3 denote the distances of its centre from the sides, and ρ its radius; then

$$d_1 = \rho\cos\alpha, \quad d_2 = \rho\cos\beta, \quad d_3 = \rho\cos\gamma; \text{ but } ad_1+bd_2+cd_3 = 2\Delta,$$

whence
$$\frac{1}{\rho} \cdot \frac{a\cos\alpha+b\cos\beta+c\cos\gamma}{2\Delta}.$$

Hence the locus is a circle which intersects the ellipse in two coincident points, *i.e.*, touches the ellipse; but, since $\frac{aey}{b}+\frac{b}{k}=0$ can only be true, as regards the ellipse, as long as $y =$ or $< -b$; hence the circle touches or falls without the ellipse according as $\frac{b^2}{ae}k <$ or $> b$, or as $\frac{1}{k} <$ or $> \frac{ae}{b}$, or as $\frac{\cos\alpha}{\sin\alpha} <$ or $> \frac{ae}{b}$, or as

$$\frac{\cos\alpha}{(\sin^2\alpha+\cos^2\alpha)^{\frac{1}{2}}} < \text{ or } > \frac{ae}{\left[b^2+\frac{a^2(a^2-b^2)}{a^2}\right]^{\frac{1}{2}}} \left\{=\frac{ae}{a}=e\right\},$$

i.e., according as $\cos\alpha <$ or $> \epsilon$.

7689. (By N'Importe.)—If the two roots of the equation $x^2-a_1x+a_2=0$ are whole and positive numbers, prove that (1) $\frac{1}{36}a_2(1+a_1+a_2)(1+2a_1+4a_2)$ is a whole number decomposable into the sum of a_2 squares; (2) $\frac{1}{16}a_2^2(1+a_1+a_2)^2$ is a whole number decomposable into the sum of a_2 cubes; (3) $a_2^2(1+2a_1+4a_2)$ is decomposable into the algebraic sum of $4a_2$ squares.

Solution by B. Hanumanta Rau, M.A.; R. Knowles, B.A.; *and others.*

Let m, n be the roots of the equation $x^2-a_1x+a_2=0$, then $a_1=m+n$ and $a_2=mn$.

1. $\frac{1}{36}a_2(1+a_1+a_2)(1+2a_1+4a_2) = \frac{1}{36}mn(1+m)(1+n)(1+2m)(1+2n)$
$= \frac{1}{6}m(m+1)(2m+1)\,.\,\frac{1}{6}n(n+1)(2n+1)$
$= [1^2+2^2+\ldots+m^2][1^2+2^2+\ldots+n^2] =$ sum of mn squares.

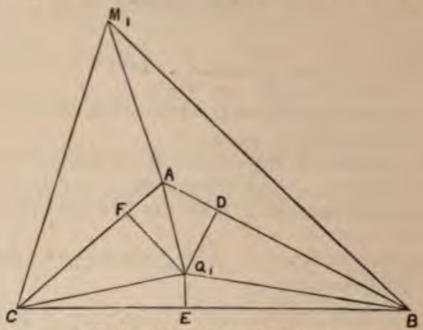
2. $\frac{1}{16}a_2^2(1+a_1+a_2)^2=\frac{1}{16}m^2n^2(1+m)^2(1+n)^2=[\frac{1}{2}m(m+1)]^2\,.\,[\frac{1}{2}n(n+1)]^2$
$= [1^3+2^3+\ldots+m^3][1^3+2^3+\ldots+n^3] =$ sum of mn cubes.

3. The expression $m^2n^2[1+2(m+n)+4mn] = m^2(2m+1)\,.\,n^2(2n+1)$;
now $m^2(2m+1) = 2\times\frac{1}{6}[2m(2m+1)(4m+1)]-4[\frac{1}{6}m(m+1)(2m+1)]$
$= 2\,.\,1^2+2^2+2\,.\,3^2+\ldots 2(2m-1)^2+(2m)^2 =$ algebraic sum of $2m$ squares.

Thus $a_2^2(1+2a_1+4a_2) =$ (sum of $2m$ squares) $\times$ (sum of $2n$ squares)
$=$ algebraic sum of $4mn$ or $4a_2$ squares.

7563. (By Rev. T. C. Simmons, M.A.)—Show that the ratio of the area of a triangle inscribed in an ellipse to the area of its polar triangle depends only on θ, ϕ, ψ, the differences between the eccentric angles of the points of contact, and is equal to $2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi\cos\frac{1}{2}\psi$.

Solution by D. Biddle, *Member of the Aëronautical Society.*



(Scale, 400 yards to the inch.)

Let Q_1, M_1 be where plumb-lines let down from the balloons would at the moment of observation strike the earth; then

$$\left.\begin{array}{l} \cot 84^\circ\,10'\,10'' \;= \cdot 1021151 = \text{rel. val. of } Q_1A \\ \cot 76^\circ\,13'\,46{\cdot}5'' = \cdot 2450762 = \quad ,, \quad ,, \; Q_1B \\ \cot 79^\circ\,35'\,5{\cdot}5'' \;= \cdot 1838074 = \quad ,, \quad ,, \; Q_1C \end{array}\right\},$$

$$\left.\begin{array}{l} \cot 84^\circ\;\,2'\,50'' \;= \cdot 1042710 = \text{rel. val. of } M_1A \\ \cot 75^\circ\,57'\;\,1'' \;= \cdot 2502499 = \quad ,, \quad ,, \; M_1B \\ \cot 79^\circ\,22'\,12'' \;= \cdot 1876869 = \quad ,, \quad ,, \; M_1C \end{array}\right\}.$$

Moreover, since AB = 553, BC = 791, and CA = 399, therefore

$$\left(\cos A = \frac{AB^2 + CA^2 - BC^2}{2AB\,.\,CA}\right),$$

$$\cos A = -\cdot 3640771 = \cos 111^\circ\,21'\;\,3'',$$

$$\cos B = \;\;\cdot 8827700 = \cos\;\,28^\circ\;\,1'\,18'',$$

$$\cos C = \;\;\cdot 7589600 = \cos\;\,40^\circ\,37'\,39''.$$

Moreover, $\sin A = \cdot 9313687$; $\sin B = \cdot 4698150$; $\sin C = 6514951$.

With these data, proceeding to find the position of Q_1, draw perpendiculars therefrom to the sides of the triangle cutting AB in D, BC in E, and CA in F; then

$$AQ_1^2 - AD^2 = BQ_1^2 - (AB - AD)^2,$$

$$BQ_1^2 - BE^2 = CQ_1^2 - (BC - BE)^2, \quad CQ_1^2 - CF^2 = AQ_1^2 - (CA - CF)^2;$$

$$AD = \frac{AB}{2} - \frac{BQ_1^2 - AQ_1^2}{2AB}, \quad BD = \frac{AB}{2} + \frac{BQ_1^2 - AQ_1^2}{2AB},$$

$$BE = \frac{BC}{2} + \frac{BQ_1^2 - CQ_1^2}{2BC}, \quad CE = \frac{BC}{2} - \frac{BQ_1^2 - CQ_1^2}{2BC},$$

$$CF = \frac{AC}{2} + \frac{CQ_1^2 - AQ_1^2}{2AC}, \quad AF = \frac{AC}{2} - \frac{CQ_1^2 - AQ_1^2}{2AC}.$$

Now $$\frac{AD}{AQ_1} = \cos BAQ_1 \quad \frac{AF}{AQ_1} = \cos CAQ_1,$$

So, if ρ_1 be the radius of that circle whose centre lies beyond the side a,

$$\frac{1}{\rho_1} = -\frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{2\Delta},$$

with similar expressions for $\frac{1}{\rho_2}$ and $\frac{1}{\rho_3}$, hence $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}$; and, since the expressions always give real values for the four radii, four circles can always be drawn.

Again, let the equation to the new line referred to the given triangle as triangle of reference be $\lambda x + \mu y + \nu z = 0$,
and let the perpendiculars on it from the four centres be respectively p, p_1, p_2, p_3; then

$$p = \frac{\lambda d_1 + \mu d_2 + \nu d_3}{\text{some denominator D}}, \quad \text{or} \quad \frac{p}{\rho} = \frac{\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma}{\text{D}};$$

Similarly $\frac{p_1}{\rho_1} = \frac{-\lambda\cos\alpha + \mu\cos\beta + \nu\cos\gamma}{\text{D}}$, &c.; $\therefore \frac{p}{\rho} = \frac{p_1}{\rho_1} + \frac{p_2}{\rho_2} + \frac{p_3}{\rho_3}$;

that is, $\cos\phi = \cos\phi_1 + \cos\phi_2 + \cos\phi_3$.

7695. (By J. O'Regan.)—Two persons play for a stake, each throwing two dice. They throw in turn, A commencing. A wins if he throws 6, B if he throws 7: the game ceasing as soon as either event happens. Show that A's chance is to B's as 30 to 31.

Solution by D. Biddle; W. J. Greenstreet, B.A.; *and others.*

Out of 36 ways of throwing two dice, 6 may be turned up in 5 ways, viz., $1+5$, $2+4$, $3+3$, $4+2$, $5+1$; and 7 may be turned up in 6 ways, viz., $1+6$, $2+5$, $3+4$, $4+3$, $5+2$, $6+1$. There are therefore 31 chances against throwing 6, but only 30 against throwing 7. The probability that B will have a throw after A is accordingly $\frac{31}{36}$; but that A will throw again after B, only $\frac{30}{36}$.

1192. (By the Editor.)—In order to ascertain the heights of two balloons (Q, M), their angles of elevation as set forth hereunder are observed, at the same instant, from three stations (A, B, C) on the horizontal plane, whose distances apart are AB = 553, BC = 791, CA = 399 yards, (Q, A) denoting the elevation of Q at A, &c.:—

(Q, A) = 84° 10′ 10″	(M, A) = 84° 2′ 50″
(Q, B) = 76° 13′ 46·5″	(M, B) = 75° 57′ 1″
(Q, C) = 79° 35′ 5·5″	(M, C) = 79° 22′ 12″

It is also observed that only *one* of the balloons (Q) is vertically over the triangle ABC. Show that the heights of the balloons Q, M are 1874·8, 3339·4, and that their distance apart is 1560·4.

1966. (By the late SAMUEL BILLS.)—Find values of x, y that will make $S \equiv (p^2+q^2)^4 + 64p^2q^2(p^2-q^2)^2$ a perfect square.

Solution by ASÛTOSH MUKHOPÂDHYÂY.

The expression S can be at once seen to be a perfect square, when (1) $p = q$; (2) $p = 0$, $S = (q^4)^2$; (3) $q = 0$, $S = (p^4)^2$. Developing S, and arranging it in ascending powers of q, we have

$$S = p^8 + 68p^6q^2 - 122p^4q^4 + 68p^2q^6 + q^8,$$

which can easily be put into the form

$$(p^4 + 34p^2q^2 - 639q^4)^2 + 68 \times 640\, p^2q^6 - [(639)^2 - 1]q^8;$$

hence, when S is a perfect square, $68 \times 640\, p^2q^6 = (639^2 - 1)q^8$; whence (4) we have $p^2 = \dfrac{319}{34}q^2$.

Verification.—Substituting for p^2 this value, we get

$$S = q^8\left\{\left(\frac{353}{34}\right)^4 + \frac{34 \times 64 \times 319}{34^2}\left(\frac{285}{34}\right)^2\right\}$$

$$= \frac{71909249281}{34^4}q^8 = \frac{(268159)^2}{34^4}q^8;$$

therefore $$p = \sqrt{\frac{319}{34}}\,q \text{ gives } S = \left\{\frac{268159}{1156}q^4\right\}^2.$$

Similarly, if we arrange S in descending powers of q, from the symmetry of the expression we at once see the condition in this case to be $q^2 = \frac{319}{34}p^2$, and the value of S is easily inferred from (4) to be

$$\frac{(268159)^2}{34^4}p^8 = \left\{\frac{268159}{34^2}\,\frac{34^2}{319^2}q^4\right\}^2 = \left\{\frac{268159}{101761}q^4\right\}^2.$$

NOTE ON BIOT'S FORMULA. *By* ASÛTOSH MUKHOPÂDHYÂY.

1. A magnetic needle suspended on its centre of gravity is constrained to move in a vertical plane, making an angle ψ with the magnetic meridian; θ is the dip in the magnetic meridian, and ϕ the dip in the vertical plane in which we suppose the motion to be.

Refer the system to rectangular axes, x being at right angles to the plane of the magnetic meridian, y horizontal and in that plane, and z vertical. Then the vertical and horizontal components are $F = M\sin\theta$, $H = M\cos\theta$, and the statical equation of virtual velocities is $F\delta z + H\delta y = 0$. Now, $y = a\cos\phi\cos\psi$, $z = a\sin\phi$, where a is the distance of any particle of the needle from its centre; therefore $\delta y = -a\sin\phi\cos\psi\,\delta\phi$, $\delta z = a\cos\phi\,\delta\phi$, therefore $F\cos\phi = H\sin\phi\cos\psi$, or $\sin\theta\cos\phi = \cos\theta\sin\phi\cos\psi$, therefore

$$\cos\phi = \cot\theta\cos\psi \quad\text{.................................(1)},$$

which determine ϕ and thence the position of equilibrium.

The equation $F\cos\phi = H\sin\phi\cos\psi$ gives $F = M\cos\theta\tan\phi\cos\psi$, which gives F, when M, &c. are known.

By logarithmic differentiation, $\frac{\Delta F}{F} = \frac{\Delta M}{M} - \frac{\sin\theta}{\cos\theta}\Delta\theta + \frac{\sec^2\phi}{\tan\phi}\Delta\phi$, a formula different from, but equivalent to, that of Biot.

Biot's Formula is easily obtained from the same source. For,

$$F = M\cos\theta\tan\phi\cos\psi \quad \text{and} \quad \sin\theta\cos\phi = \cos\theta\sin\phi\cos\psi,$$

therefore $\cos^2\phi - \cos^2\phi\cos^2\theta = \cos^2\theta\cos^2\psi\,(1-\cos^2\phi)$,

therefore $\cos^2\theta\cos^2\psi = \cos^2\phi\,(1-\cos^2\theta\sin^2\psi)$,

therefore $\frac{\cos\theta\cos\psi}{\cos\phi} = (1-\cos^2\theta\sin^2\psi)^{\frac{1}{2}}$,

therefore $F = M\sin\phi\,(1-\cos^2\theta\sin^2\psi)^{\frac{1}{2}}$. Taking the logarithmic differential, we have $\frac{\Delta F}{F} = \frac{\Delta M}{M} + \cot\phi\,\Delta\phi + \frac{1}{2}\cdot\frac{\sin 2\theta\sin^2\psi}{1-\cos^2\theta\sin^2\psi}\cdot\Delta\theta$, which is Biot's Formula.

2. When the needle is counterpoised by means of a brass weight so as to remain in equilibrium in a given position, we have only to substitute, in the statical equation, $F-\mu$ for F, where μ is a constant moment, therefore $F = \mu + M\cos\theta\tan\phi\cos\psi$. When the position is accurately horizontal, $\phi = 0$, therefore $F = \mu$ or $M\sin\theta = \mu$; and therefore, when the needle deviates from horizontality by a very small angle $\Delta\phi$, we have nearly

$$F = \mu + \mu\cot\theta\cos\psi\,\Delta\phi \quad \text{and} \quad \frac{F-\mu}{\mu} = \cot\theta\cos\psi\,\Delta\phi.$$

To make this formula available, it appears that the difficulty is to estimate ψ, and this may be accomplished as follows:—Let the needle be slightly moved in the vertical plane; then the vertical component called into action will be $F-\mu$, or $\mu\cot\theta\cos\psi\,\Delta\phi$; or, since $\Delta\phi$ is the arc described, the *pendulum force* will be $\mu\cot\theta\cos\psi$, and let T be the time of vibration. Afterwards, let the needle be so adjusted as to vibrate in a horizontal plane under the action only of the horizontal component $M\cos\theta$, and let T_1 be the time of vibration. Then we have, by the Theory of Pendulums, $\frac{T_1^2}{T^2} = \frac{\mu\cot\theta\cos\psi}{\mu\cot\theta} = \cos\psi$; hence $\frac{\Delta F}{F} = \frac{T_1^2}{T^2}\cot\theta\,\Delta\phi$.

The utility of this last formula is thus explained. If we could place the needle so as to vibrate exactly in the magnetic meridian, so much the better; in this case, $T_1 = T$, and we have simply $\Delta F / F = \cot\theta\,\Delta\phi$. But, since we can never be sure of this, the deviation may be estimated from the ratio $(T' : T)^2$.

7711. (By Professor Wolstenholme, M.A., Sc.D.)—Prove that (1) the locus of the points of contact of tangents drawn from a given point O to a series of confocal parabolas is a circular cubic, whose equation with O as origin is $r = 2a\sin\theta\cos\theta / \sin(\theta+\alpha)$, where $a = OS$, and 2α is the acute angle which SO makes with the common axis; S is the common focus, and the initial line is parallel to the straight line bisecting the acute angle between SO and the axis; (2) if, instead of a series of parabolas, we have a system of central conics with given foci S, S′, and centre C, the locus of the point of contact of tangents from a given point O is exactly the same, where $a = \text{SO} \,.\, \text{S}'\text{O} \,/\, 2\text{CO}$, and α is the angle which OC makes with the straight line bisecting the angle SOS′, which is the initial line; (3) the shape of this cubic will be the same for all points O lying on the lemniscate whose equation (with C origin and CS initial line) is $r^2 \sin 2\alpha = c^2 \sin 2(\theta+\alpha)$, where $c = \text{CS}$, and α has the same meaning as before; and the foci of these lemniscates lie on the rectangular hyperbola whose foci are S, S′; (4) if any circle be described with centre O, the points of intersection of common tangents to this circle and any one of the conics whose foci are S, S′ is also this cubic, a remarkable instance of a definite locus of points, whose position (appearing to depend on two variable parameters) would be expected to be arbitrary.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

(1) Let OT be any tangent to the parabola, OC and TR parallel to the axis SV, and OV the *initial* line, then

$$\phi = \angle \text{STO} = \angle \text{RTL} = \angle \text{COT},$$

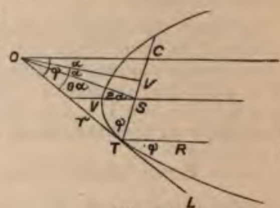
$$\therefore \quad \angle \text{OSC} = \phi + \angle \text{SOT}$$

$$= \phi + \phi - 2\alpha = 2(\phi - \alpha) = \theta,$$

and $\quad r \sin\phi = \text{OS} \,.\, \sin \text{OSC} = a \sin 2\theta,$

or $\quad r \sin(\theta+\alpha) = 2a \sin\theta \cos\theta,$

therefore $\quad r = 2a \sin\theta \cos\theta \,/\, \sin(\theta+\alpha).$

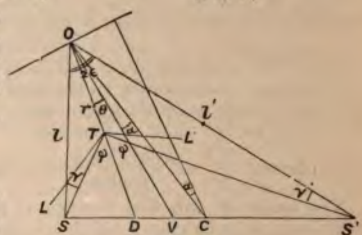


(Fig. 1.)

(2) and (5) If OT be a tangent to any conic of the confocal system, T being the point of contact, OT will be the bisector of the angle STS′, and if TL, TL′ be two tangents to any other conic of the system,

$$\angle \text{STL} = \angle \text{S}'\text{TL}',$$

and, therefore, OT also bisects $\angle$ LTL′, therefore TL, TL′ will each be equidistant from O, and therefore touch a circle with O as centre, so that the loci defined in (1) and in (5) are one and the same, and will be the curve locus of point T such that the line bisecting the $\angle$ STS′ shall pass through the fixed point O. This may be determined as follows:—



(Fig. 2.)

Let $\angle \text{SOS}' = 2\epsilon$, $\angle \text{OST} = \gamma$, $\angle \text{OS}'\text{T} = \gamma'$, $\text{OS} = l$, $\text{OS}' = l'$, $\angle \text{STD} = \phi = \angle \text{S}'\text{TD}$, then $l \sin\gamma = r \sin\phi = l' \sin\gamma'$, $\gamma - \gamma' = 2\theta$, $\gamma + \gamma' + 2\epsilon = 2\phi$, hence $\epsilon - \theta = \phi - \gamma$, $\epsilon + \theta = \phi - \gamma'$, therefore

$$l \sin(\epsilon - \theta) = l \sin(\phi - \gamma) = l(\sin\phi \cos\gamma - \cos\phi \sin\gamma),$$

$$l' \sin(\epsilon + \theta) = l' \sin(\phi - \gamma') = l'(\sin\phi \cos\gamma' + \cos\phi \sin\gamma'),$$

$$\therefore \quad l' \sin(\epsilon+\theta) - l \sin(\epsilon - \theta) = (l' \cos\gamma' - l \cos\gamma) \sin\phi$$

$$= \frac{ll'}{r}(\sin\gamma \cos\gamma' - \cos\gamma \sin\gamma') = \frac{ll'}{r} \sin(\gamma - \gamma') = \frac{ll'}{r} \sin 2\theta,$$

or $$\frac{2ll'}{r} \sin\theta \cos\theta = (l' + l) \cos\epsilon \sin\theta + (l' - l) \sin\epsilon \cos\theta;$$

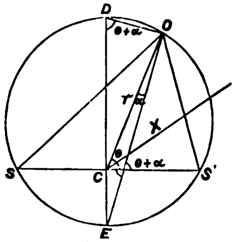
but it is geometrically evident that

$$(l'+l)\cos\epsilon = OC\cos\alpha,\quad (l'-l)\sin\epsilon = OC\sin\alpha,$$

$$\therefore\quad \frac{2ll'}{r}\sin\theta\cos\theta = 2OC(\cos\alpha\sin\theta+\sin\alpha\cos\theta) = 2OC\sin(\theta+\alpha),$$

therefore $$r = \frac{ll'}{OC}\,\frac{\sin\theta\cos\theta}{\sin(\theta+\alpha)} = \frac{2a\sin\theta\cos\theta}{\sin(\theta+\alpha)}.$$

3. The *shape* is invariable, though the *magnitude* of the curve varies with a, so long as α is constant, and we must find the locus of the vertex of a triangle the base of which is given and the bisector of whose vertical angle makes a constant angle α with the line drawn from the vertex to the mid-point of the base: by figure,



(Fig. 3.)

$$\frac{r^2}{c^2} = \frac{CO^2}{CS.CS'} = \frac{CO^2}{CD.CE} = \frac{CO}{CD}\cdot\frac{CO}{CE}$$

$$= \frac{\sin(\theta+\alpha)}{\cos\alpha}\,\frac{\cos(\theta+\alpha)}{\sin\alpha} = \frac{\sin 2(\theta+\alpha)}{\sin 2\alpha},$$

or $r^2\sin 2\alpha = c^2\sin 2(\theta+\alpha)$, or, taking for line of reference X a line through C inclined to SS' at $\frac{1}{4}\pi$, we have $\theta = \frac{1}{4}\pi+\theta'$, and the equation of the locus becomes $r^2 = \dfrac{c^2}{\sin 2\alpha}\cos 2(\theta'+\alpha)$, the lemniscata required, the polar coordinates of a focus of which being f and α, we have $f^2 = \dfrac{c^2}{2\sin 2\alpha} = \dfrac{c^2}{4\sin\alpha\cos\alpha}$, therefore the locus of the foci, when α varies, referred to X and a line through C perpendicular to it is, $xy = \frac{1}{4}c^2$, an equilateral hyperbola whose foci are S and S'.

Again, it may be remarked that, as a tangent from any point to a conic is, at the point of contact, normal to a confocal conic, the same circular cubic will be the locus of the extremities of all *normals* drawn from a given point O to a system of confocals.

The equation of this cubic may also be obtained by combining the equation of the pair of tangents to a conic from a point, referred to *elliptic coordinates*, with the equation of the polar of the point similarly referred.

7350. (By Dr. Curtis.)—If there be circumscribed to a given conic a polygon of m sides, such that the arcs between the consecutive points of contact subtend equal angles at a focus, and 2α denote the angle which the axis of the conic makes with the radius vector drawn to any one of the points of contact, prove that (1) the product of the squares of the perpendiculars from the focus on the sides of the polygon varies inversely as $C-\cos 2m\alpha$, where C is a constant which becomes unity when the conic is a parabola; and (2) any symmetrical function of the positive powers of the squares of the reciprocals of the perpendiculars of a degree inferior to the m^{th} remains constant, however the polygon may change consistently with the conditions of its construction. [An extension of Quest. 7317.]

Solution by the PROPOSER.

The following theorem is due to the late Professor MACCULLAGH :— If any point O, taken on the circumference of a circle of radius b, be joined to all the angular points of a regular polygon of m sides inscribed in a concentric circle of radius a, and the angle subtended at the centre by the point O, and any angular point of the polygon, *e.g.*, the adjacent one A, be denoted by 2α, any symmetric function, inferior to the m^{th}, of the squares of the lines drawn from O to the vertices of the polygon is constant, however α may vary.

Let r_k denote the line joining the point θ to the k^{th} angular point in succession to the point A, then

$$z = r_k^2 = a^2 + b^2 - 2ab\cos\left(2\alpha + \frac{2k\pi}{m}\right). \quad \therefore \; 2\cos\left(2\alpha + \frac{2k\pi}{m}\right) = \frac{a^2+b^2-z}{ab};$$

but, if $2\cos\phi = u$, $2\cos m\phi = u^m - mu^{m-2} + \frac{m\,.\,m-3}{1\,.\,2}u^{m-4} - \&c.$, therefore

$$2\cos(2m\alpha) = 2\cos(2m\alpha + 2k\pi) = 2\cos m\left(2\alpha + \frac{2k\pi}{m}\right)$$

$$= \left(\frac{a^2+b^2-z}{ab}\right)^m - m\left(\frac{a^2+b^2-z}{ab}\right)^{m-2} + \frac{m\,.\,m-3}{1\,.\,2}(a^2+b^2-z)^{m-4} - \&c.$$

$$= Lz^m + Mz^{m-1} + Nz^{m-2} + \&c. + 2C,$$

or
$$Lz^m + Mz^{m-1} + Nz^{m-2} + \&c. + 2(C - \cos 2m\alpha) = 0.$$

Now the only coefficient in this equation which contains α is the last, or absolute term, and, as all symmetric functions of the roots of this equation, of a degree inferior to the m^{th}, can be expressed in terms of the other coefficients, the theorem is proved.

Again, $L = (-1)^m \times \frac{1}{a^m b^m}$, and therefore the product of the roots of the equation for $z = 2a^m b^m (C - \cos 2m\alpha)$.

If, taking O as origin, we reciprocate these theorems, we obtain the theorems proposed, the reciprocal to the circle of radius a being an ellipse, parabola, or hyperbola, according as $b \lesseqgtr a$, and

$$2C = \left(\frac{a^2+b^2}{ab}\right)^m - m\left(\frac{a^2+b^2}{ab}\right)^{m-2} + \&c.,$$

or $2a^m b^m\, C = (a^2+b^2)^m - m(a^2+b^2)^{m-2}a^2b^2 - \&c. = a^{2m} + b^{2m}$

(see MURPHY's *Theory of Equations*, p. 32), and $C = \frac{a^{2m}+b^{2m}}{2a^m b^m}$, which becomes unity when, and only when, $b = a$.

7804. (By Professor CAYLEY, F.R.S.)—1. If (a, b, c, f, g, h) are the six coordinates of a generating line of the quadric surface $x^2+y^2+z^2+w^2=0$, then $a=f,\ b=g,\ c=h$, or else $a=-f,\ b=-g,\ c=-h$, according as the line belongs to the one or the other system of generating lines.

2. If a plane meet the quadriquadric curve, $Ax^2+By^2+Cz^2+Dw^2=0$, $A'x^2+B'y^2+C'z^2+D'w^2=0$ in four points, and if (a, b, c, f, g, h) are the coordinates of the line through two of them, (a', b', c', f', g', h') of the line through the other two of them, then

$$af'+a'f=0,\quad bg'+b'g=0,\quad ch'+c'h=0.$$

Solution by W. J. C. SHARP, M.A.

Let $\dfrac{x-\alpha w}{a}=\dfrac{y-\beta w}{b}=\dfrac{z-\gamma w}{c}$ be the equation to the line whose coordinates are a, b, c, f, g, h, so that $f=c\beta-b\gamma$, $g=a\gamma-c\alpha$, $h=b\alpha-a\beta$, and $af+bg+ch=0$, and let $\alpha:\beta:\gamma:1$ be a point, the surface

$$\alpha^2+\beta^2+\gamma^2+1=0,$$

and, if the line be a generator, $x=\alpha w'+a\rho$, $y=\beta w'+b\rho$, $z=\gamma w'+c\rho$, and $w=w'$ will satisfy the equation to the quadric for all values of w' and ρ, and therefore, since $\alpha^2+\beta^2+\gamma^2+1=0$, $a\alpha+b\beta+c\gamma=0$; and $a^2+b^2+c^2=0$, $af+bg+ch=0$; also

$$(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)-(f^2+g^2+h^2)\equiv(a\alpha+b\beta+c\gamma)^2=0,$$

therefore $f^2+g^2+h^2=0$, therefore $(a\pm f)^2+(b\pm g)^2+(c\pm h)^2=0$ (all the double signs to be read alike), or $a=f,\ b=g,\ c=h$, or $a=-f$, $b=-g,\ c=-h$.

Also two lines of the same system cannot meet, for if

$$cy-bz+aw=0,\quad az-cx+bw=0 \text{ meet } c'y-b'z+a'w=0,$$

and
$$a'z-c'x+b'w=0,$$

$$\begin{vmatrix} -c, & 0, & a, & b \\ 0, & c, & -b, & a \\ -c', & 0, & a', & b' \\ 0, & c', & -b', & a' \end{vmatrix}=0,\quad\text{or}\quad (ac'-a'c)^2+(bc'-b'c)^2=0,\quad\text{or}\quad \frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'},$$

and the lines are identical. If

$$\frac{x-\alpha w}{a}=\frac{y-\beta w}{b}=\frac{z-\gamma w}{c}\quad\text{and}\quad\frac{x-\alpha' w}{a'}=\frac{y-\beta' w}{b'}=\frac{z-\gamma' w}{c'}$$

be the lines a, b, c, f, g, h and a', b', c', f', g', h', then, if these are coplanar,

$$\begin{vmatrix} 1, & \alpha, & \beta, & \gamma \\ 0, & a, & b, & c \\ 1, & \alpha', & \beta', & \gamma' \\ 0, & a', & b', & c' \end{vmatrix}=0,\quad\text{or}\quad af'+a'f+bg'+b'g+ch'+c'h=0.$$

Now, if these lines be those in part (2) of the question, and $\alpha, \beta, \gamma, 1$ $\alpha', \beta', \gamma', 1$ be two of the points on the quadriquadric curve,

$$A\alpha^2+B\beta^2+C\gamma^2+D=0,\qquad A'\alpha^2+B'\beta^2+C'\gamma^2+D'=0$$
$$A\alpha'^2+B\beta'^2+C\gamma'^2+D=0,\qquad A'\alpha'^2+B'\beta'^2+C'\gamma'^2+D'=0,$$

and
$$2(Aa\alpha+Bb\beta+Cc\gamma)\rho w+(Aa^2+Bb^2+Cc^2)\rho^2=0,$$
$$2(A'a\alpha+B'b\beta+C'c\gamma)\rho w+(A'a^2+B'b^2+C'c^2)\rho^2=0,$$

are simultaneously true, or

$$bcf(BC'-B'C)+cag(CA'-C'A)+abh(AB'-A'B)=0.$$

Similarly $b'c'f'(BC'-B'C)+c'a'g'(CA'-C'A)+a'b'h'(AB'-A'B)=0$,

therefore $\quad BC'-B'C : CA'-C'A : AB'-A'B$

$$=aa'(b'gch'-bg'c'h) : bb'(c'haf'-ch'a'f) : cc'(a'fbg'-af'b'g).$$

But the ratios of $BC'-B'C : CA'-C'A : AB'-A'B$,

which are constant throughout the curve, must be independent of the coordinates of two arbitrary lines each joining two points upon it, therefore

$$\frac{af'}{a'f}=\frac{bg'}{b'g}=\frac{ch'}{c'h}=\mu \text{ suppose,}$$

and if the lines be coplanar

$$(a'f+b'g+c'h)(1+\mu)=0 \text{ and } \mu=-1,$$

and therefore $\quad af'+a'f=0,\ bg'+b'g=0,\ ch'+c'h=0.$

7705. (By Professor SYLVESTER, F.R.S.)—Prove that to any given set of i quantities there corresponds a set of i other quantities, such that every symmetrical function of the differences of the first set is a function of all the successive power-sums from the second to the i^{th} inclusive of the second set. [Professor SYLVESTER calls Σa^{ω} an ω^{th} power-sum of a's.]

Solution by W. J. C. SHARP, M.A.

Let the first i quantities $\alpha, \delta \dots \lambda$ be the roots of the equation

$$z^i+ix_1z^{i-1}+\frac{i(i-1)}{1.2}x_2z^{i-2}+\dots=0.$$

Then, if U be a symmetrical function of the differences of α, β, &c., $\left(\frac{d}{dx_1}+2x_1\frac{d}{dx_2}+\dots\right)U=0$, and U is a function of any $(i-1)$ solutions of

the system $$dx_1=\frac{dx_2}{2x_1}=\frac{dx_3}{3x_2}=\&c. \quad\dots\dots\dots\dots(A).$$

Now, if there be i quantities $y_1, y_2, \dots y_i$ such that the system of equations

$$\left.\begin{array}{l} y_1dy_1+y_2dy_2+\dots+y_idy_i=0,\quad y_1^2dy_1+y_2^2dy_2+\dots+y_i^2dy_i=0 \\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ y_1^{i-1}dy_1+y_2^{i-1}dy_2+\dots+y_i^{i-1}dy_i=0 \end{array}\right\}\dots(B)$$

is a transformation of the system (A), U can be expressed as stated.

But, if $y_1=x_1+\alpha$, $y_2=x_1+\beta$, &c., $y_i=x_1+\lambda\dots$, any symmetrical function of the differences of $\alpha, \beta, \dots \lambda$ is a symmetrical function of the differences of $y_1, y_2 \dots y_i$, and therefore the system of equations

$$d\Sigma y_1=\frac{d\Sigma y_1y_2}{2\Sigma y_1}=\frac{d\Sigma y_1y_2y_3}{3\Sigma y_1y_2}=\&c. \quad\dots\dots\dots\dots(C)$$

will be equivalent to the system (A), and, since for these values of y_1, y_2, &c., $\Sigma y_1=0$, therefore $d.\Sigma y_1=0$, $d.\Sigma y_1y_2=0$, $d.\Sigma y_1y_2y_3=0$, &c...(D),

and it appears by differentiating Newton's formulæ that the system (D) is equivalent to the system (B), and U is a function of the power-sums of $y_1, y_2 \dots y_i$, from the second to the i^{th} inclusive.

4390. (By the Editor.)—Two gamesters, A and B, play together, A having the power to fix the stakes. Whenever A loses a game, he increases the last stake by a shilling for the next game, and diminishes it by a shilling after every gain. When they leave off playing, A has gained £13; and, had each won the same number of games, A would still, by following the above principle in regulating his stakes, have gained 10*s*. If the first stake be 30*s*., show that A won 15 and lost 5 games.

Solutions by (1) D. Biddle; (2) *the* Proposer.

1. A formula for solving questions of this sort is the following:—

$$(x-y)\left[a-\tfrac{1}{2}b\,(x-y-1)\right]+by=c,$$

where x = games won by A, y = games won by B, a = original stake, b = difference between successive stakes, c = total amount won by A. In the present instance, $a=30$, $b=1$, and, when x and y are equal, $c=10$. Such being the case, the first term of the equation $=0$, and $by=c$; hence $y=10$, and $x+y=20$.

Now, substituting $20-x$ for y, and 260 for c, the equation becomes

$$(2x-20)\left[30-\tfrac{1}{2}(2x-21)\right]+20-x=260,$$

whence $x^2-50x=-525$, and $x=15$; thus A has won 15 games, and B 5. The construction of the equation may readily be elucidated by taking the stakes won by A in one order, and those lost in the reverse order, turning about at each change of sign. We can arrange those won and lost, in any way: the result will be the same.

(2) −31	(3) +32		(12) +25		
(1) −30	(4) +31		(13) +24		
			(14) +23		
	(5) +30				
	(6) +29		(15) +22	(18) −21	(19) +22
	(7) +28		(16) +21	(17) −20	(20) +21
	(8) +27				
	(9) +26				
	(10) +25	(11) −24			
−61	+228	−24	+115	−41	+43

$$=386-126=260.$$

For here $30+29+28+\dots+21=30+(30-1)+(30-2)+\dots+(30-9)$

$$=10\left[30-\tfrac{1}{2}(10-1)\right]=(x-y)\left[a-\tfrac{1}{2}b\,(x-y-1)\right];$$

and $(32-31)+(31-30)+(25-24)+(22-21)+(21-20)=5\times 1=by.$

No matter in what order the 20 games have been won or lost, the stakes taken by A in the 10 surplus games will make a descending series from 30 to 21 inclusive; and he will have won besides a balance of 1 from each of the five pairs of games remaining.

2. *Otherwise*:—Let a be the first stake, c the sum added or subtracted, s_1 the gain of A on winning x and losing y games, and s_2 his gain if each player had won $\frac{1}{2}(x+y)$ games; then i standing, as usual, for $\surd(-1)$, the successive stakes and gains (or losses) may be represented as follows:

Stakes.	*Gains.*
a	$i^{\alpha}a$
$a-i^{\alpha}c$	$i^{\beta}a-i^{\alpha}i^{\beta}c$
$a-i^{\alpha}c-i^{\beta}c$	$i^{\gamma}a-i^{\alpha}i^{\gamma}c-i^{\beta}i^{\gamma}c$
$a-i^{\alpha}c-i^{\beta}c-i^{\gamma}c$	... ...
... ...	

where the quantities $\alpha, \beta, \gamma, \ldots$ are subject to the condition that x of them must be even and y odd, to correspond respectively to the x games that A wins, and the y games that he loses. If, therefore, we put m for the sum of the $(x+y)$ quantities $\alpha, \beta, \gamma, \ldots$, and n for the sum of the products of every two of them, we shall have $s_1 = am-cn$. Now m, n will be the coefficients of z^{x+y-1}, z^{x+y-2} in the equation $(z-1)^x(z+1)^y = 0$, which has x roots each $= +1$ and y roots each $= -1$; thus we have

$$m = x-y, \quad n = \tfrac{1}{2}[(x-y)^2-(x+y)].$$

The values of x, y are therefore given by the equations

$$2a(x-y)+c(x+y)-c(x-y)^2 = 2s_1, \quad c(x+y) = 2s_2.$$

Adding, we get $\qquad c(x-y) = a \pm [a^2-2c(s_1+s_2)]^{\frac{1}{2}}$,

therefore $\qquad 2cx = 2s_2 + a \pm [a^2-2c(s_1+s_2)]^{\frac{1}{2}}$.

In the particular case proposed, we have $a = 30$, $c = 1$, $s_1 = 260$, $s_2 = 10$; hence $x = 15$, and $y = 5$, as stated in the question.

7807. (By the late Professor Townsend, F.R.S.)—A triangle in the plane of a conic being supposed self-reciprocal with respect to the curve; show that an infinite number of triangles could be at once inscribed to the conic and circumscribed to the triangle, or conversely.

Solution by W. J. C. Sharp, M.A.

If the given triangle be taken as triangle of reference, and

$$ax^2+by^2+cz^2 = 0$$

be the conic, then, $x-\lambda y = 0$, $y-\mu z = 0$, and $z-\nu x = 0$ being the sides of a triangle circumscribed to the triangle of reference, its vertices will be

$$1 : \mu\nu : \nu, \quad \lambda : 1 : \lambda\nu, \quad \lambda\mu : \mu : 1,$$

and therefore if it be also inscribed in the conic

$$a+b\mu^2\nu^2+c\nu^2 = 0, \quad a\lambda^2+b+c\lambda^2\nu^2 = 0, \quad a\lambda^2\mu^2+b\mu^2+c = 0 \ldots (1, 2, 3),$$

any one of which is easily deduced from the other two, *e.g.*, from (1) and (2)

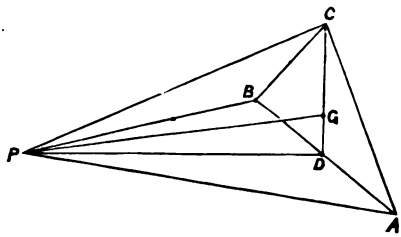
$$[(a\lambda^2+b)(b\mu^2+c)-ac\lambda^2]\,\nu^2=0. \quad \text{Therefore } a\lambda^2\mu^2+b\mu^2+c=0,$$

and the system of equations (1), (2), (3) admits of an infinite number of solutions.

The converse follows at once by reciprocation.

7771. (By the late Professor CLIFFORD, F.R.S.)—Find the locus of a point P which moves so that the length of the resultant of the translations PA, PB, PC is constant—the points A, B, C being fixed.

Solution by B. HANUMANTA RAU, M.A.; N. SARKAR, B.A.; *and others.*



Bisect BA at D and take G, so that CG = 2GD, then the resultant of PA, PB, PC = result. of 2PD and PC = 3PG = constant; and, since G is fixed, being the centroid of ABC, the locus of P is a sphere whose centre is the centroid of A, B, C, and radius one-third of the given resultant.

[If ρ, α, β, γ be the vectors drawn to P, A, B, C from any origin, $\alpha-\rho+\beta-\rho+\gamma-\rho$ will be the vector representing the resultant, and by the question $T(3\rho-\alpha-\beta-\gamma)=3c$ or $T[\rho-\frac{1}{3}(\alpha+\beta+\gamma)]=c$, whence the stated result follows.

7769. (By Professor SYLVESTER, F.R.S.)—Prove algebraically that, if ABC..., A′B′C′... are two superposed projective point-series which do not possess self-conjugate points, then the segment between any two corresponding points, as AA′, BB′..., will subtend the same angle at a point properly chosen outside the line in which the point-series lie.

Solution by the PROPOSER.

Let the given line and any line perpendicular to it be assumed as axes. Call a, b, c, a', b', c', λ, λ' the distances from the origin of A, B, C, A′, B′, C′, and of the *umbra* in each series (respectively) of the infinite point in the other.

Then $(\lambda-a)(\lambda'-a')=(\lambda-b)(\lambda'-b')=(\lambda-c)(\lambda'-c')$, say $=-u$, and, if there are self-conjugate points, $(\lambda-e)(\lambda'-e)=-u$ must give real values for e. Hence, if there are not self-conjugate points, $u>[\frac{1}{2}(\lambda-\lambda')]^2$, *i.e.*, $\lambda\lambda' + u > [\frac{1}{2}(\lambda+\lambda')]^2$. Also, for determining λ, λ', we have the equations $(\lambda\lambda'+u)-a\lambda'-a'\lambda+aa' = 0$, $(\lambda\lambda'+u)-b\lambda'-b'\lambda+bb' = 0$,

$$(\lambda\lambda'+u)-c\lambda'-c'\lambda+cc' = 0.$$

But, if ω is the constant angle in question, we have

$$\frac{\frac{a-x}{y}-\frac{a'-x}{y}}{1-\frac{a-x}{y}\cdot\frac{a'-x}{y}} = \tan\omega, \quad \text{or, calling } y\cot\omega = v,$$

$$y^2+x^2-(a+a')\,x-(a-a')\,v+aa' = 0,$$

and similarly $\quad y^2+x^2-(b+b')\,x-(b-b')\,v+bb' = 0,$

$$y^2+x^2-(c+c')\,x-(c-c')\,v+cc' = 0.$$

Hence $y^2+x^2 = \lambda\lambda'+u$, $x+v = \lambda'$, $x-v = \lambda$, and $x = \frac{1}{2}(\lambda+\lambda')$; and x is always real, for

$$\lambda'(\lambda-a)-a'\lambda+aa' = \lambda'(\lambda-b)-b'\lambda+bb' = \lambda'(\lambda-c)-c'\lambda+cc'$$

gives $\quad \dfrac{(a'-b')\,\lambda+(bb'-aa')}{b-a} = \dfrac{(a'-c')\,\lambda+(cc'-aa')}{c-a}.$

In fact, $\quad \lambda = \dfrac{bc\,(c'-b')+ca\,(a'-c')+ab\,(b'-a')}{(ab'-a'b)+(bc'-b'c)+(ca'-c'a)},$

and $\quad \lambda' = \dfrac{b'c'\,(c-b)+c'a'\,(a-c)+a'b'\,(b-a)}{(a'b-ab')+(b'c-bc')+(c'a-ca')},$

and if $\lambda\lambda+u > [\frac{1}{2}(\lambda+\lambda')]^2$, we have $y^2+x^2 > x^2$, and y is real,
but if $\lambda\lambda+u < [\frac{1}{2}(\lambda+\lambda')]^2$, $y^2+x^2 < x^2$, and y is imaginary.
Hence the points x, y are real or imaginary according as the self-conjugate points are imaginary or real, subject to the trivial limitation that, if the two self-conjugate points are real but coincident, the point (x, y) is real, but lies on the line containing the two series, at which point the angle subtended by each segment such as AA′ will be zero.

7514. (By Professor Wolstenholme, M.A., Sc.D.)—Prove that the centroid of the arc of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, included between the positive coordinate axes, if $e = \left(1-\frac{b^2}{a^2}\right)^{\frac{1}{2}}$, and $a > b$, is given by

$$\bar{x} = \frac{a}{16e^2}\,\frac{1}{1-(1-e^2)^{\frac{3}{2}}}\left\{(3+e^2)(3e^2-1)+\frac{3\,(1-e^2)^3}{2e}\log\left(\frac{1+e}{1-e}\right)\right\},$$

$$\bar{y} = \frac{b}{16e^2}\,\frac{(1-e^2)^{\frac{1}{2}}}{1-(1-e^2)^{\frac{3}{2}}}\left\{(4e^2-3)(2e^2+1)+\frac{3\sin^{-1}e}{e\,(1-e^2)^{\frac{1}{2}}}\right\}.$$

Solution by D. Edwardes; B. Hanumanta Rau, M.A.; *and others.*

Putting $x = a\cos^3\phi$, $y = b\sin^3\phi$, we get

$$s = \int_0^{\frac{1}{2}\pi} \tfrac{3}{2}a\,(1-e^2\sin^2\phi)^{\frac{1}{2}}\,d\sin^2\phi = ae^{-2}\,[1-(1-e^2)^{\frac{3}{2}}];$$

hence $e^{-2}[1-(1-e^2)^{\frac{3}{2}}]\,\bar{x} = 3\int_0^{\frac{1}{2}\pi}\cos^4\phi\,\sin\phi\,(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}\,d\phi$

$$= 3\int_0^1 x^4\,[b^2+(a^2-b^2)\,x^2]^{\frac{1}{2}}\,dx = -\tfrac{3}{64}a^{-5}e^{-5}\int_b^{a(1-e)}\frac{(b^4-z^4)^2\,(b^2-z^2)^2}{z^7}\,dz$$

$$\left(\text{where } x = \frac{b^2-z^2}{2aez}\right)$$

$$= -\tfrac{3}{64}a^{-5}e^{-5}\left[\frac{(z^6-b^6)(z^2+b^2)(z^4-4b^2z^2+b^4)}{6z^6}+4b^6\log z\right]_b^{a(1-e)},$$

which gives the result stated on the Question.
Also

$$ae^{-2}[1-(1-e^2)^{\frac{3}{2}}]\,\bar{y} = 3b\int_0^{\frac{1}{2}\pi}\sin^4\phi\,\cos\phi\,(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}\,d\phi$$

$$= 3ab\int_0^1 x^4\,(1-e^2x^2)^{\frac{1}{2}}\,dx = 3abe^{-5}\int_0^{\sin^{-1}e}\sin^4\theta\cos^2\theta\,d\theta$$

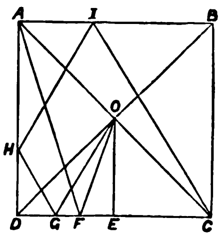
$$= 3abe^{-5}\left[\tfrac{3}{8}(\theta-\tfrac{1}{2}\sin^2\theta)-\tfrac{1}{4}\sin^2\theta\cos\theta+\tfrac{1}{6}\sin^5\theta\cos\theta-\tfrac{5}{6}\int\sin^4\theta\,d\theta\right]_0^{\sin^{-1}e}$$

$$= 3abe^{-5}\left[\tfrac{1}{16}\sin^{-1}e+\frac{e(1-e^2)^{\frac{1}{2}}(4e^2-3)(2e^2+1)}{48}\right],$$ whence the result.

5200. (By S. Tebay, B.A.)—A small marble is thrown at random on a square table having an elevated rim. If it be struck at random in any direction, determine the probability that it impinges (1) on two opposite sides; (2) on two adjacent sides, and one opposite; (3) on three consecutive sides; (4) on the four sides in succession.

Solution by D. Biddle.

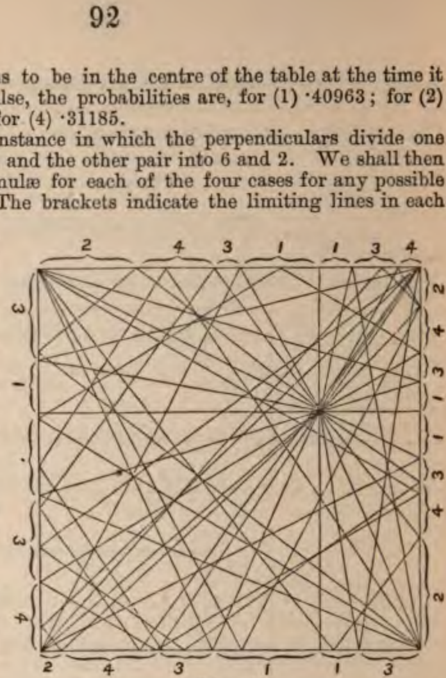
The probability in each of the four cases is the ratio borne by the average sum of eight variable angles to 360°. The eight angles (two for each side of the square, since the marble can be sent from any given position against any one side in either of two directions, that is, on either side of the perpendicular) represent the limits within which the particular event specified can occur, according to the law of the equality of the angles of incidence and reflection. Let us see what the eight angles are, in each case, when the marble is originally placed in the centre of the square.



(Fig. 1.)

Here the eight angles for each case are equal and symmetrically placed. Each of them for case (1) = EOF; for case (2) = *zero*, since this case requires that the marble shall strike the first side at a less angle than 45°, which is at present impossible; for case (3) = FOG; and for case (4) = DOG. Now (taking AD = 1) EF : DF = OE : AD = ½ : 1. ∴ EF = ⅓ED, and DF = ⅓ = tan ∠EOF = tan 18° 26′. Similarly, BI = 2EG = ⅗ = tan EOG = tan (FOG + EOF) = tan 30° 58′, ∴ ∠FOG = 12° 32′. And ∠DOG = 45° − (FOG + EOF) = 14° 2′. Consequently, when the marble happens to be in the centre of the table at the time it receives the random impulse, the probabilities are, for (1) ·40963; for (2) ·00000; for (3) ·27852; for (4) ·31185.

Let us next take an instance in which the perpendiculars divide one pair of sides into 5 and 3, and the other pair into 6 and 2. We shall then be able to construct formulæ for each of the four cases for any possible position of the marble. The brackets indicate the limiting lines in each case.



(FIG. 2.)

To find the tangent of the angle formed by the particular limiting line with the perpendicular, in any of the eight directions appertaining to each of the four cases, we proceed as follows:—Let a = distance of marble from particular side, b = distance of perpendicular from particular corner of table between which and perpendicular the limiting line lies. Then it is easy to see that in case (1), $x = \frac{b}{1+a}$; in case (2), $x = \frac{b}{a}$, since the limiting line extends in this instance to the corner, and the compound angle, whose tangent is now under consideration, includes the individual angles of (1), (3), and (4), as well as (2); in case (3), $x = \frac{1+b}{2+a}$, or, when $\frac{1+b}{2+a} > \frac{b}{a}$, then $x = \frac{b}{a}$; and in case (4), $x = \frac{1+b}{1+a}$ when $a < b$, but when $a > b$, then $x = \frac{b}{a}$. Thus, it will be seen that (1) takes precedence of (3), (3) of (4), and (4) of (2), no tangent greater than $\frac{b}{a}$ being possible; and if we make equations to suit every possible instance, for the tangents of these compound angles, then for (1), $x = \frac{b}{a}\left(1 - \frac{1}{1+a}\right)$; for (3), $x = \frac{b}{a}\left(1 - \frac{2b-a}{2b+ab}\right)$; for (4), $x = \frac{b}{a}\left(1 - \frac{b-a}{b(1+a)}\right)$; and for (2), $x = \frac{b}{a}$.

In (3) and (4), if the second term has no real value [that is, if in (4) $a > b$, or in (3) $a > 2b$], then this term is discarded and $x = \frac{b}{a}$. Now the tangent of the compound angle formed by the limiting line with the perpendicular is, in (2), the tangent for (1) + (3) + (4) + (2); is, in (4), the tangent for (1) + (3) + (4); and is, in (3), the tangent for (1) + (3). The tangent in case (1) is the only proper tangent of the angle specially referred to by number. But it is easy to see that, if we have these several tangents, we are virtually in possession of the angles belonging to them, and can readily arrive at the component parts by successive subtractions. The same must be true of the mean tangents, and from these we can deduce the mean angles; and, if we take the mean sum of the eight angles appertaining to each of the four cases, we have the several numerators of the probabilities required. Now a and b in the equations last given can be transposed, and, not only so, $(1-a)$ can take the place of a and $(1-b)$ of b, or $1-a$ of b and $(1-b)$ of a. In the following table are given the eight tangents for the four cases:—

	I.	II.
Perpendicular ...	a	a
Portion of Side...	b	$1-b$
(1)	$\frac{b}{a}\left(1-\frac{1}{1+a}\right)$	$\frac{1-b}{a}\left(1-\frac{1}{1+a}\right)$
(2)	$\frac{b}{a}$	$\frac{1-b}{a}$
(3)	$\frac{b}{a}\left(1-\frac{2b-a}{2b+ab}\right)$	$\frac{1-b}{a}\left(1-\frac{2-2b-a}{(1-b)(2+a)}\right)$
(4)	$\frac{b}{a}\left(1-\frac{b-a}{b+ab}\right)$	$\frac{1-b}{a}\left(1-\frac{1-b-a}{(1-b)(1+a)}\right)$

	III.	IV.
Perpendicular ...	b	b
Portion of Side...	a	$1-a$
(1)	$\frac{a}{b}\left(1-\frac{1}{1+b}\right)$	$\frac{1-a}{b}\left(1-\frac{1}{1+b}\right)$
(2)	$\frac{a}{b}$	$\frac{1-a}{b}$
(3)	$\frac{a}{b}\left(1-\frac{2a-b}{2a+ab}\right)$	$\frac{1-a}{b}\left(1-\frac{2-2a-b}{(1-a)(2+b)}\right)$
(4)	$\frac{a}{b}\left(1-\frac{a-b}{a+ab}\right)$	$\frac{1-a}{b}\left(1-\frac{1-a-b}{(1-a)(1+b)}\right)$

	V.	VI.
Perpendicular ...	$1-a$	$1-a$
Portion of Side...	b	$1-b$
(1)	$\frac{b}{1-a}\left(1-\frac{1}{2-a}\right)$	$\frac{1-b}{1-a}\left(1-\frac{1}{2-a}\right)$
(2)	$\frac{b}{1-a}$	$\frac{1-b}{1-a}$
(3)	$\frac{b}{1-a}\left(1-\frac{2b-1+a}{b\,(3-a)}\right)$	$\frac{1-b}{1-a}\left(1-\frac{a+1-2b}{(1-b)(3-a)}\right)$
(4)	$\frac{b}{1-a}\left(1-\frac{b-1+a}{b\,(2-a)}\right)$	$\frac{1-b}{1-a}\left(1-\frac{a-b}{(1-b)(2-a)}\right)$

	VII.	VIII.
Perpendicular ...	$1-b$	$1-b$
Portion of Side...	$1-a$	a
(1)	$\frac{1-a}{1-b}\left(1-\frac{1}{2-b}\right)$	$\frac{a}{1-b}\left(1-\frac{1}{2-b}\right)$
(2)	$\frac{1-a}{1-b}$	$\frac{a}{1-b}$
(3)	$\frac{1-a}{1-b}\left(1-\frac{b+1-2a}{(1-a)(3-b)}\right)$	$\frac{a}{1-b}\left(1-\frac{2a-1+b}{a\,(3-b)}\right)$
(4)	$\frac{1-a}{1-b}\left(1-\frac{b-a}{(1-a)(2-b)}\right)$	$\frac{a}{1-b}\left(1-\frac{a-1+b}{a\,(2-b)}\right)$

This table, with the aid of a table of natural tangents, will enable us to find the respective probabilities in the four cases, for any given position of the marble. To find the average probability for all positions, we must find the mean tangents. In the annexed diagram is shown in what proportion of positions cases (2) and (4) are possible. If DC be the side of the table against which the marble is sent, and D the corner between which and the perpendicular it impinges; then, to fulfil the requirements of case (2), it must have been placed originally below the diagonal DB, that is, somewhere in the triangle DBC; and, to

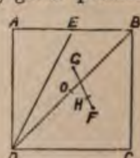


Fig. 3.

fulfil the requirements of case (4), it must have been placed to the right of the line DE (joining D and the mid-point of AB). Thus, in only $\frac{3}{4}$ of the possible positions is case (4) practicable, and in only $\frac{1}{2}$ is case (2). Now the average positions for the fulfilment of the several requirements are the mathematical centres of the spaces in which they are possible. For case (1) and case (3), the centre of the table is the average position, and the mean value of a and b is $\frac{1}{2}$ each. For case (2), the centre of the triangle is the average position, and within the said limits the mean value of a is $\frac{1}{3}$, of b $\frac{2}{3}$. For case (4), the centre of the trapezoid is the average position, and the mean value of a is $\frac{4}{9}$, of b, $\frac{11}{18}$. Under these circumstances, and since the second terms within brackets in the equations for (4) and (3) are governed by (2) and (4) respectively, the average value of $\frac{b-a}{b(1+a)}$, in the equation for case (4), in those instances in which the term counts at all, will be $\frac{\frac{2}{3}-\frac{1}{3}}{\frac{2}{3}(1+\frac{1}{3})} = \frac{3}{8}$; and of $\frac{2b-a}{2b+ab}$, in the equation for case (3), will be $\frac{\frac{22}{18}-\frac{8}{18}}{\frac{22}{18}+\frac{44}{162}} = \frac{63}{121}$. But we must multiply these by the factors denoting their proportionate occurrence, to find their mean values, namely, for case (4), $\frac{3}{8}\times\frac{1}{2}=\frac{3}{16}$; and, for case (3), $\frac{63}{121}\times\frac{3}{4}=\frac{189}{484}$.

Then the mean tangents of the compound angles will be as follows:— 1·000000, ·912500, ·609504, and of the angle for case (1), ·333333. The angles corresponding to these tangents are 45°, 42° 22′ 49$\frac{3}{5}$″, 31° 21′ 45″, and 18° 26′ 0$\frac{2}{5}$″. Consequently, the mean angles for each separate case are, (1) 18° 26′ 0$\frac{2}{5}$″, (2) 2° 37′ 10$\frac{2}{5}$″, (3) 12° 55′ 44$\frac{3}{5}$″, (4) 11° 1′ 4$\frac{3}{5}$″. And these represent the following probabilities:—(1) ·4096, (2) ·0582, (3) ·2873, (4) ·2449.

7607. (By T. Muir, M.A., F.R.S.E.)—Prove that [see Quest. 7574]

$$\begin{vmatrix} a, b & \dots & \dots & \cdot & \cdot \\ c, a, b & \dots & & \cdot & \cdot \\ \cdot\ c, a & \dots & & \cdot & \cdot \\ \dots & \dots & \dots & & \\ \cdot & \cdot & \cdot & \dots & a, b \\ \cdot & \cdot & \cdot & \dots & c, a \end{vmatrix}_n = \left(a-2(bc)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a-2(bc)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\dots \dots\left(a-2(bc)^{\frac{1}{2}}\cos\frac{n\pi}{n+1}\right),$$

or what is equally general,

$$K(a^b a^b \dots {}^b a)_n = \left(a-2(-b)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\dots\left(a-2(-b)^{\frac{1}{2}}\cos\frac{n\pi}{n+1}\right).$$

Solution by B. Hanumanta Rau, M.A.; Sarah Marks; *and others.*

Let $a = (bc)^{\frac{1}{2}}x$, and let D_n represent the given determinant; then

$$D_n = aD_{n-1} - bcD_{n-2} \text{ or } D_n - (bc)^{\frac{1}{2}}xD_{n-1} + bcD_{n-2} = 0,$$

therefore D_n is the coefficient of y^n in the expansion of

$$[1-(bc)^{\frac{1}{2}}xy+bcy^2]^{-1} \text{ or } (bc)^{\frac{1}{2}n}\times\text{coefficient of } z^n \text{ in } (1-xz+z^2)^{-1},$$

$$\therefore \quad D_n = (bc)^{\frac{1}{2}n} \left(x - 2\cos\frac{\pi}{n+1}\right)\left(x - 2\cos\frac{2\pi}{n+1}\right)\dots\left(x - 2\cos\frac{n\pi}{n+1}\right)$$

[see Solution of Question 7574]

$$= \left(a - 2(bc)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a - 2(bc)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\dots .$$

Again, $K(a^b a^b \dots {}^b a)_n = \begin{vmatrix} a, & b, & 0, & 0 & \dots \\ -1, & a, & b, & 0 & \dots \\ 0, & -1, & a, & b & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = D_n$ where $c = -1$

$$= \left(a - 2(-b)^{\frac{1}{2}}\cos\frac{\pi}{n+1}\right)\left(a - 2(-b)^{\frac{1}{2}}\cos\frac{2\pi}{n+1}\right)\dots .$$

7766. (By R. Tucker, M.A.)—If ρ, ρ', ω are the "T. R." and Brocard radii, and Brocard angle respectively of a triangle, prove that (1) $\frac{\cos 3\omega}{\cos\omega} = \left(\frac{\rho'}{\rho}\right)^2$; and (2), if ρ_1, ρ_2 are the "T. R." radii in the ambiguous case of triangles, then $\rho_1 \cos\omega_1 = \rho_2 \cos\omega_2$.

Solution by the Proposer; B. Hanumanta Rau, M.A.; *and others.*

Referring to equations (xii.) and (xviii.) of the article on the "Triplicate-Ratio Circle" (*Quarterly Journal of Mathematics*, Vol. xix., No. 76), we see that

$$\cos\omega = \frac{k}{2\lambda}, \text{ therefore } \frac{\cos 3\omega}{\cos\omega} = 4\cos^2\omega - 3 = \frac{k^2}{\lambda^2} - 3 = \left(\frac{\rho'}{\rho}\right)^2;$$

we also get at once $\rho'^2 + 3\rho^2 = R^2$, a result which is obtained geometrically by Dr. Casey in his *Sequel to Euclid*, 3rd ed., p. 167. From the same article, we get $2\rho_1\cos\omega_1 = R$, $2\rho_2\cos\omega_2 = R'$. Hence in *all* cases when the circum-radii are equal, as they are in the ambiguous case,

$$\rho_1\cos\omega_1 = \rho_2\cos\omega_2.$$

7415. (By the Rev. T. C. Simmons, M.A.)—Two conics have a common focus, about which one of them is turned. Prove that the enveloping conic of the common chord depends only on the positions of the directrices, and the ratio of the eccentricities, of the original conics; and hence, when these are known, give an easy method of constructing it. [See Question 4417, *Reprint*, Vol. 39, p. 117.]

Solutions by (1) Dr. Curtis; (2) *the* Proposer.

1. The conics are represented by the equations

$$r = e(x\cos\alpha + y\sin\alpha - p), \quad r = e'(x\cos\beta + y\sin\beta - p') \dots\dots\dots (1, 2),$$

where α, p, p' are given, and β is variable; hence their common chord and its envelope are

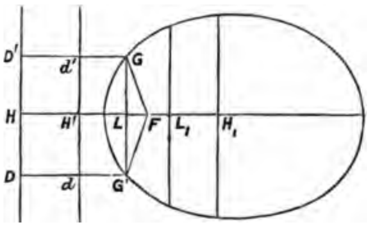
$$e'x\cos\beta + e'y\sin\beta - [e'p' + e(x\cos\alpha + y\sin\alpha - p)] = 0,$$

$$e'^2(x^2+y^2) = [e'p' + e(x\cos\alpha + y\sin\alpha - p)]^2,$$

or
$$r = \frac{e}{e'}\left\{x\cos\alpha + y\sin\alpha - \left(p - \frac{e'}{e}p'\right)\right\},$$

a conic confocal with the other two, whose eccentricity is e/e' and whose directrix is parallel to that of the fixed conic, at a distance from it $= e'p'/e$, and nearer to the focus.

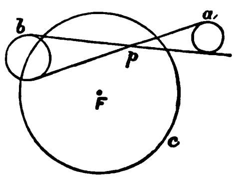
To construct this conic, let us suppose $\beta = \alpha$, then G, G′, the points of intersection of the conics, will lie on a line parallel to the common direction of DD′, dd', the directrices of the fixed and of the movable conic, respectively, cutting FH′H, the perpendicular from the common focus F on the directrices, in L, so that HL : H′L :: e' : e.



Again, suppose $\beta = \pi + \alpha$, then the directrix of the movable conic will be parallel to DD′ and at a distance FH_1 from F, where $FH_1 = FH'$; find L_1 so that $HL_1 : H_1L_1 :: e' : e$, then it is plain that L and L_1 will be the extremities of the axis major of the required conic, and, as one of its foci F is known, it is thus fully determined.

2. Dr. Curtis's solution, admirable in other respects, leaves out of sight the other chord of intersection, with respect to which, as is seen below, the theorem also holds. In fact, as r is really a surd quantity in x and y, equations (1) and (2) ought perhaps in strictness to be squared before subtracting.

Let A be the fixed, B the movable conic, and let them be reciprocated with respect to the common focus F into two circles a and b; then a is fixed and the centre of b revolves in a circle centre F; and, since the centre of b is the pole of the directrix of B, if we take this as the reciprocating circle, the directrix of B will always touch it.



Now, since p, the intersection of the common tangents of a and b, always divides the line joining the centres of a and b in the same ratio, the locus of p is a circle which in conjunction with C has the centre of a for one centre of similitude. That is to say, the common tangents of a and b intersect in a circle whose common tangents with the circle C intersect in the centre of a.

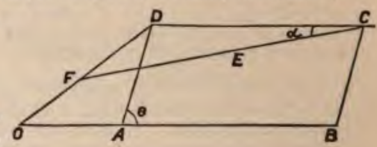
Hence in the original figure the chord of intersection of A and B touches a conic with the same focus, and whose intersection with the circular locus of the foot of the directrix of B lies on the directrix of A. We have, then, two points on the enveloping conic depending only on the positions of the directrices of A and B; and since, as shown above, its vertex depends only on the ratio of the eccentricities of A and B, and its focus is known, it is completely determined, and the required result follows. It will also be seen that the above reasoning applies equally to either of those two chords of intersection which in the initial position are perpendicular to the common axis.

If the directrix $dH'd'$ of the movable conic is further from the focus than DHD′ the fixed directrix, describe a circle centre F and radius FH′. The two points where it meets DHD′ will lie on both of the enveloping conics, a striking result, since the points depend only on the distances of the nearer directrices from the common focus, and are entirely independent of the nature of the original conics. It can be also seen analytically (but for one of the enveloping conics only) from Dr. Curtis's equation

$$r = \frac{e}{e'}\left\{x\cos\alpha + y\sin\alpha - \left(p - \frac{e'}{e}p'\right)\right\} \text{ by putting } x\cos\alpha + y\sin\alpha - p = 0.$$

7617. (By D. Biddle.)—Let a parallelogram ABCD have one side AB fixed and the other three capable of movement in one plane by hinge-action; and within the parallelogram let CE form a given angle with CD; then, if O be a fixed point in BA produced, and F, F′, &c. the points of intersection of CE, C′E′ with OD, O′D′, &c.; find the locus of F′.

Solution by B. Hanumanta Rau, M.A.; J. O'Regan; *and others.*



Let $OA = a$, $OB = b$, $AD = c$; then the equations to OD, CF are $\dfrac{y}{x} = \dfrac{c\sin\theta}{a + c\cos\theta}$,

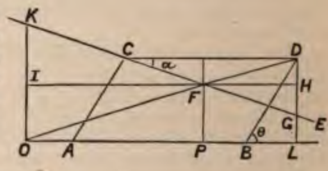
$y - c\sin\theta = (x - b - c\cos\theta)\tan\alpha$,

and, as the coordinates of O satisfy both these equations, we get

$$c\sin\theta = y + \frac{(a-b)\tan\alpha\,.\,y}{x\tan\alpha - y} \text{ and } c\cos\theta = x - a + \frac{(a-b)\tan\alpha\,.\,x}{x\tan\alpha - y},$$

therefore $c^2 = \left\{y + \dfrac{(a-b)\tan\alpha\,.\,y}{x\tan\alpha - y}\right\}^2 + \left\{(x-a) + \dfrac{(a-b)\tan\alpha\,.\,x}{x\tan\alpha - y}\right\}^2,$

the equation to a circular quartic.



[The Proposer remarks that, if C and D change places, the above reasoning will still apply, although one or two of the signs will need altering; thus, let $OB = a$, $OA = b$, $AC = c$, $OP = x$, $FP = y$; then,

$DL = c\sin\theta$; $BL = c\cos\theta$;

$IK = x\tan\alpha$; $DG = (a-b)\tan\alpha$.

But $\qquad DH : DG = FP : IK + FP,$

therefore $\qquad \dfrac{DG\,.\,FP}{IK + FP} = DH, \quad DL = FP + \dfrac{DG\,.\,FP}{IK + FP};$

i.e., $c\sin\theta = y + \dfrac{(a-b)\tan\alpha\,.\,y}{x\tan\alpha + y}$; similarly $PL : OP = DG : IK + FP$;

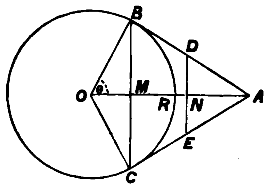
therefore $\dfrac{DG.OP}{IK+FP} = PL$, and $BL = \dfrac{DG.OP}{IK+FP} - PB$;

i.e., $$c\cos\theta = (x-a) + \frac{(a-b)\tan\alpha . x}{x\tan\alpha + y},$$

and $$c^2 = \left\{y + \frac{(a-b)\tan\alpha . y}{x\tan\alpha + y}\right\}^2 + \left\{(x-a) + \frac{(a-b)\tan\alpha . x}{x\tan\alpha + y}\right\}^2\Big].$$

7146. (By Professor GENESE, M.A.)—AB, AC are two tangents to a circle, and DE bisects them; show (1) that DE cannot meet the circle; and hence prove (2) that for acute angles $\theta - \sin\theta < \tan\theta - \theta$.

Solution by B. REYNOLDS, M.A.; Professor NASH, M.A.; *and others.*



$$ON = OM + \tfrac{1}{2}MA = r\cos\theta + \tfrac{1}{2}BA\sin\theta$$
$$= r(\cos\theta + \tfrac{1}{2}\tan\theta\sin\theta)$$
$$= r\frac{2\cos^2\theta + \sin^2\theta}{2\cos\theta},$$
$$= r\frac{1+\cos^2\theta}{2\cos\theta} = r\frac{(1-\cos\theta)^2 + 2\cos\theta}{2\cos\theta},$$
$$= r\left\{1 + \frac{(1-\cos\theta)^2}{2\cos\theta}\right\},$$

and is thus greater than r, θ being always acute. Hence DNE cannot cut the circle.

Again, BD + DE + EC > arc BRC, or BA + BM > arc BRC,

or $r\tan\theta + r\sin\theta > 2r\theta$, whence $\theta - \sin\theta < \tan\theta - \theta$.

[DE is the radical axis of the circle and a point-circle at A, and cannot therefore meet the circle.]

7752. (By ASPARAGUS.)—From a point on one of the common chords perpendicular to the transverse axis of two confocal conics are drawn tangents OP, OQ, OP′, OQ′ to the two conics: prove that the straight lines PP′, PQ′, P′Q, P′Q′ each pass through one of the common foci.

Solution by DR. CURTIS; Professor NASH, M.A.; *and others.*

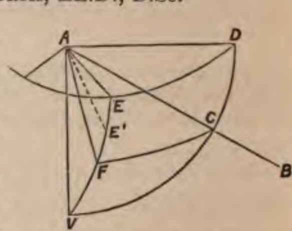
It is plain that, if through the intersection of a centre of similitude of two non-intersecting circles a straight line be drawn intersecting them, the tangents at each pair of corresponding points are parallel; and, if we reciprocate this theorem with regard to any circle whose centre is at one of the limiting points of the two circles, we obtain the proposed theorem.

[A *direct* proof of this theorem is not difficult, and may readily be found.]

7688 & 7761. (By W. J. C. Sharp, M.A.)—(7688.) Find the curve in which a kite-string will hang, when acted on by a uniform wind blowing in a direction inclined to the horizon.

(7761.) A flexible string is suspended slackly from two fixed points, and acted upon by a uniform horizontal wind, blowing in a direction making any angle with the horizontal projection of the line joining the points. Find the curve in which the string hangs and the tension at any point.

Solution by Arthur Hill Curtis, LL.D., D.Sc.



Let A, B be the fixed points; with A as centre, let a sphere of radius unity be drawn, ADCV being the vertical great circle whose plane contains AB, and AVE the great circle whose plane is parallel to AE, the direction of the wind, supposed horizontal, let f denote the accelerating force of the wind on the element and g that of gravity; the combined action of the two will be in the direction AF, where $\tan VF = f/g$, the great circle CF determines the plane in which the string will rest, and, as each equal element is acted on by a constant force in the invariable direction, AF, the curve of equilibrium will be a catenary, the same as would result if the plane AFB were turned round AB into a vertical position, and then turned along with AB round its axis until AF were vertical; all the known relations for the catenary as to tension, &c., hold, g in the usual formulæ being replaced by $h \equiv (f^2+g^2)^{\frac{1}{2}}$.

If the known direction of the wind be not supposed to be horizontal, let AE′ denote its direction, and AF will be determined by the condition $\frac{\sin VF}{\sin FE'} = \frac{f}{g}$, and $h \equiv (f^2+g^2+2fg\cos VE')^{\frac{1}{2}}$, all else being as before.

7744. (By Professor Cochez.)—Parmi les courbes isoperimètres planes passant par deux points fixes, quelle est celle qui par sa révolution autour de l'axe des x, engendre la surface maximum ou minimum?

Solution by Dr. Curtis; Professor Matz, M.A.; *and others.*

As, by Guldin's theorem, the area of the generated surface is equal to the length of the curve multiplied by the circumference of the circle, or known proportional part of the circumference of the circle, described by the centre of gravity of the curve, this area will be maximum, or minimum, when the centre of gravity of the curve is at the greatest, or least, distance from the axis of X, that is, in the forms which would be the positions of equilibrium assumed by the string if each element were acted on by a constant force perpendicular to X, *from* it for maximum, and *towards* it for minimum, viz., a catenary concave towards X in the first case, and convex in the second.

7836. (By Professor SYLVESTER, F.R.S.)—If p, q be two matrices (to fix the ideas, suppose of the third order), which have one latent root in common, and let λ', λ''; μ', μ'' be the other latent roots of p, q; prove that the product $(p-\lambda')(p-\lambda'')X(q-\mu')(q-\mu'')$ (where X is an arbitrary matrix) is of invariable *form*, the only effect of the intermediate arbitrary matrix being to alter the value of each term of the product in a constant ratio; *i.e.*, in the nomenclature of the New Algebra,

$$(p-\lambda')(p-\lambda'')X(q-\mu')(q-\mu'')$$

is constant to a *scalar* multiple *près*.

For the benefit of the learner, I recall that if $p = \begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$ the roots of the algebraical equation

$$\begin{vmatrix} a-\lambda & b & c \\ a' & b'-\lambda & c' \\ a'' & b'' & c''-\lambda \end{vmatrix} = 0$$

are called the *latent roots* of p, the equation itself being called the latent equation, and the function equated to 2-en the latent function.

Solution by the PROPOSER.

To show that, if p, q (two ternary matrices) have the latent roots $\lambda = \mu$ in common, and λ', λ''; μ', μ'' are the other two latent roots of p, q respectively, $(p-\lambda')(p-\lambda'')X(q-\mu')(q-\mu'')$ is constant to a scalar factor *près*. Let u be the matrix above written, then $(p-\lambda)u = 0$, $u(q-\mu) = 0$, hence $pu-uq = 0$. If, then, we write out the constituents of p, q, and of u, we shall obtain from the above equation 9 homogeneous linear equations for determining the last-named 9, and the resultant of these equations will be identical with the resultant of the latent functions of p and q; the ratios of the 9 constituents of u will be determined from selecting any 8 out of the 9 equations which are perfectly independent of X; hence u is known to a scalar factor *près* in terms of the elements of p and q.

Of course, the same reasoning applies to matrices of any order ω; the corresponding u, it should be noticed, will always have $\omega-1$ degrees of nullity, *i.e.*, all the minor determinants to u of the second order will be equal to zero. We may write $u = c(p-\lambda')(p-\lambda'')(q-\mu')(q-\mu'')$, and obtain an identical equation to u, of which the coefficients will be rational integer functions of the coefficients to the identical equations to p and q respectively.

7838. (By the late Professor CLIFFORD, F.R.S.)—Prove that a string will rest in the form of a circle if it be repelled from a point in the circumference with a force inversely as the cube of the distance.

Solution by ÂSÛTOSH MUKHOPÂDHYÂY.

Suppose that the string has assumed the form of a circle of radius a, under the action of a repulsive central force at any point O on the circumference; and, since the osculating plane at every point contains the centre of force, and since two consecutive osculating planes have a tangent line to the string common, it easily follows that the string lies wholly in one plane. Let r be the radius vector from the origin O to any point P, θ being its inclination to the diameter OA, which is the initial line; and, if p be the perpendicular from O on the tangent at P, we have $p = r\cos\theta$, and $r = 2a\cos\theta$, which gives $r^2 = 2ap$; or, taking u as the reciprocal of the radius vector, we have $u^2 = (2ap)^{-1}$, and

$$du = d(p^{-1}) / 4au \quad \text{..............................(1).}$$

Again, if T and T + dT be the tensions at the extremities of an element of the string at P, we have, as usual, $\mathrm{T}p = h$ ....(2), and, if F be the force, and m the element of mass of the string, we have

$$d\mathrm{T} = -m\mathrm{F}dr \quad \text{..............................(3).}$$

(See Professor TOWNSEND's Classical Paper "On the Analogy between the Curve of Free Equilibrium of a String under a Central *Repulsive* Force, and the Free Orbit of a Particle under a Central *Attractive* Force," *Quarterly Journal of Mathematics*, Vol. xiii., p. 217.)

Eliminating T between (2) and (3), we have

$$\mathrm{F}\,du = \frac{h}{m}u^2 d\left(\frac{1}{p}\right) \quad \text{..............................(4).}$$

Eliminating du between (1) and (4), we see that $\mathrm{F} \propto u^3$.

According to the analogy pointed out in Professor TOWNSEND's paper, it follows that the law of force under which a particle would freely describe the same circle is that of the inverse *fifth* power of the distance. (See NEWTON's *Principia*, Lib. 1, Prop. 7.)

The inverse problem—viz., given the law of force, to determine the orbit—is also easily solved. In fact, from what has been said above, joined with the fact that $r^4 = p^2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}$, it follows at once that the orbit is $d\theta = \dfrac{\mathrm{C}\,dr}{r\,(\mathrm{T}^2r^2 - \mathrm{C}^2)^{\frac{1}{2}}}$, where C is a constant. Hence it follows that, if the force varies inversely as the μth power of the distance, the tension is equal to $\dfrac{mdf}{\mu - 1}\cdot\dfrac{1}{r^{\mu-1}}$, where d is a constant, and f the magnitude of the force exerted at unit distance on the unit mass of matter. Substituting for T, we get the general equation to the curve in the form

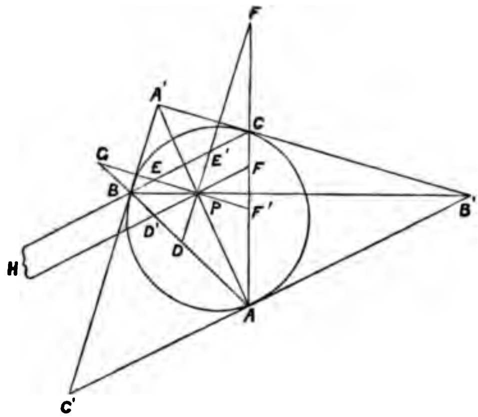
$$\left(\frac{d}{r}\right)^{\mu-2} = \cos(\mu - 2)\,\theta.$$

If $\mu = 3$, we get $r = d\cos\theta$, which denotes a circle, of diameter d, the origin being on the circumference, which is Professor CLIFFORD's theorem.

7612. (By W. S. McCAY, M.A.)—If O be the centre of perspective of the triangle ABC, and the triangle formed by the tangents at the vertices to the circumcircle; and if through O parallels be drawn to the tangents cutting the sides internally in six points, and externally in three points; prove that (1) the six internal points lie on a circle whose centre is P and radius $abc / (a^2 + b^2 + c^2)$; (2) these points are vertices of two equal triangles, similar to ABC; (3) these points are vertices of three rectangles inscribed in ABC having a common circum-circle; (4) the three external intersections of the sides with the lines through P are collinear.

I. *Solution by* HANUMANTA RAU, M.A.; G. G. STORR, B.A.; *and others.*

Let A'B'C' be the triangle formed by the tangents. Then P is the point of intersection of AA', BB', CC'. Taking ABC as the triangle of



reference, the equations to B'C', C'A', A'B'; AA', BB', CC' are

$$\frac{\beta}{b}+\frac{\gamma}{c}=0,\quad \frac{\gamma}{c}+\frac{\alpha}{a}=0,\quad \text{and}\quad \frac{\alpha}{a}+\frac{\beta}{b}=0.$$

$$\frac{\gamma}{c}-\frac{\beta}{b}=0,\quad \frac{\gamma}{c}-\frac{\alpha}{a}=0,\quad \frac{\alpha}{a}-\frac{\beta}{b}=0,$$

hence the coordinates of P are given by

$$\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}=\frac{2\Delta}{a^2+b^2+c^2},$$

and P is thus seen to be the *point de Grebe* of the triangle.

If DEF be drawn through P parallel to A'C', then

$$\text{PD}=\frac{\gamma}{\sin \text{PDB}}=\frac{\gamma}{\sin \text{C}}=\frac{2\Delta\,.\,\text{C}}{\sin \text{C}\,(a^2+b^2+c^2)}=\frac{abc}{a^2+b^2+c^2}=\text{PE}.$$

Similarly, the other internal points of intersection are at the same distance from P; also $\angle \text{EDF} = \frac{1}{2}\text{EPF} = \frac{1}{2}(\pi-\text{B}') = \text{CBA}$,

$$\angle \text{D'E'F'} = \tfrac{1}{2}\text{D'PF'} = \tfrac{1}{2}(\pi-\text{B}') = \text{CBA},$$

and so on; hence the triangles EDF, D'E'F' are similar to ABC, and, since they are inscribed in the same circle, they are equal.

This is obvious from the fact that any two diameters of a circle determine an inscribed rectangle. [See Question 7747 and its solution.]

The equation to DE'F, which passes through P (a, b, c) and is parallel to C'A' $\left(\frac{\alpha}{a}+\frac{\gamma}{c}=0\right)$, is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ a, & b, & c \\ \frac{b}{c}, & \frac{c}{a}-\frac{a}{c}, & -\frac{b}{a} \end{vmatrix}=0.$$

Putting $\beta = 0$, the equation to BF becomes

$$\left(a-\frac{b^2}{a}-\frac{c^2}{a}\right)a+\left(c-\frac{a^2}{c}-\frac{b^2}{c}\right)\gamma = 0,$$

or $$\frac{2bc\cos A}{a}a+\frac{2ab\cos C}{c}\gamma = 0, \quad \text{or} \quad \frac{a\cos A}{a^2}+\frac{\gamma\cos C}{c^2} = 0.$$

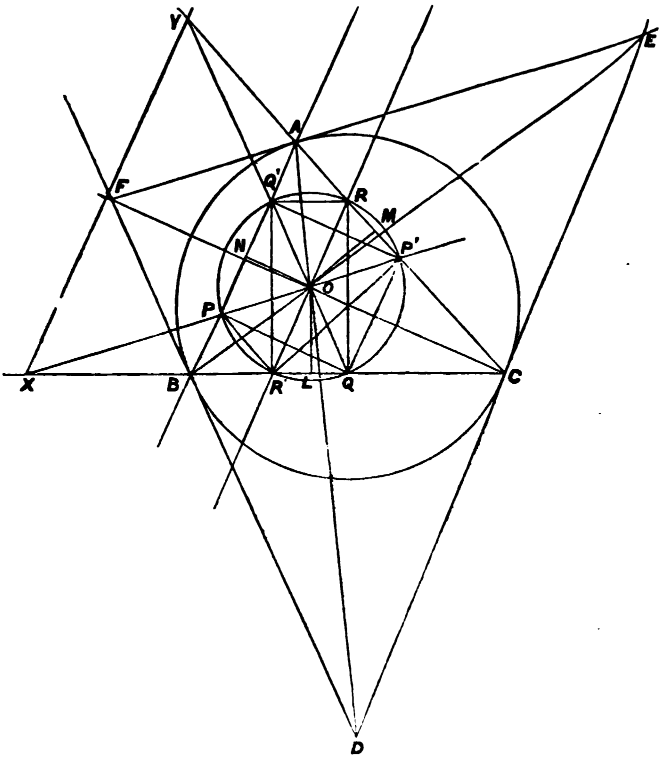
Similarly the lines CG and AH are represented by

$$\frac{a\cos A}{a^2}+\frac{\beta\cos B}{b^2} = 0, \quad \text{and} \quad \frac{\beta\cos B}{b^2}+\frac{\gamma\cos C}{c^2} = 0.$$

The points F, G, H therefore lie on the straight line

$$\frac{a\cos A}{a^2}+\frac{\beta\cos B}{b^2}+\frac{\gamma\cos C}{c^2} = 0.$$

II. *Solution by* J. Brill, B.A.; Nilkanta Sarkar, M.A.; *and others.*



1. Draw OL, OM, ON perpendicular to the sides; then
$\angle ABD = 180° - ABF = 180° - C$, similarly $\angle ACD = 180° - B$;

$$\therefore \quad \frac{AD}{BD} = \frac{\sin ABD}{\sin BAD} = \frac{\sin C}{\sin BAD}; \text{ also } \frac{AD}{CD} = \frac{\sin ACD}{\sin CAD} = \frac{\sin B}{\sin CAD}.$$

Now BD = CD, therefore $\dfrac{\sin BAD}{\sin CAD} = \dfrac{\sin C}{\sin B}$,

But OM = OA sin CAD, and ON = OA sin BAD,

therefore $\dfrac{OM}{ON} = \dfrac{\sin CAD}{\sin BAD} = \dfrac{\sin B}{\sin C} = \dfrac{b}{c}$; similarly $\dfrac{OL}{OM} = \dfrac{a}{b}$,

therefore $\dfrac{OL}{a} = \dfrac{OM}{b} = \dfrac{ON}{c} = \dfrac{a.OL + b.OM + c.ON}{a^2+b^2+c^2} = \dfrac{2\Delta}{a^2+b^2+c^2}$.

Hence we have $\qquad OL = \dfrac{2a\Delta}{a^2+b^2+c^2} = \dfrac{abc \sin A}{a^2+b^2+c^2}$.

Now $\angle OQL = \angle CBD = A$, therefore $OQ \sin A = OL = \dfrac{abc \sin A}{a^2+b^2+c^2}$,

$\therefore \quad OQ = \dfrac{abc}{a^2+b^2+c^2} = OQ' = OP = OP' = OR = OR'$ (from symmetry).

2. We have proved that P, Q, R, P′, Q′, R′ all lie on a circle having its centre at O. Now PP′ is a diameter of this circle, therefore ∠PRP′ is a right angle. Similarly it may be proved that the angles QPQ′ and RQR′ are right angles, therefore the sides of the triangle PQR are respectively perpendicular to those of the triangle ABC, therefore the two triangles are similar.

In like manner it can be proved that the triangle P′Q′R′ is also similar to the triangle ABC.

Moreover OP = OP′ and OQ = OQ′, also ∠POQ = ∠P′OQ′, therefore PQ = P′Q′; similarly it may be proved that QR = Q′R′, and RP = R′P′. Thus we have two equal triangles PQR and P′Q′R′, each of them similar to the triangle ABC.

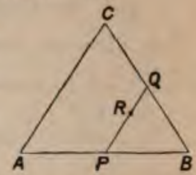
3. Since POP′ and QOQ′ bisect each other, they are the diagonals of a parallelogram, therefore PQP′Q′ is a parallelogram. But the angles PQP′, QP′Q′, P′Q′P, Q′PQ are right angles, being angles in semicircles, therefore PQP′Q′ is a rectangle. Similarly, QRQ′R′ and RPR′P′ are rectangles, therefore the points P, Q, R, P′, Q′, R′ are the vertices of three rectangles inscribed in ABC, and having a common circumcircle.

4. We have $\qquad \angle BPX = \angle APP' = \angle PAF = C$;
therefore $\angle BPP' + \angle BCP' = \angle BPP' + \angle BPX = 2$ right angles;
therefore circle passes round BPP′C; therefore BX.CX = PX.P′X, Therefore X is a point on the radical axis of the two circles. Similarly it may be shown that Y and Z lie on the radical axis; *i.e.*, X, Y, Z are collinear.

[The circle PQRP′Q′R′ makes intercepts on the sides proportional to the cosines of the angles, whence Professor Casey (in the 3rd edition of his *Sequel to Euclid*) proposes to call this circle the "Cosine-Circle."]

7849. (By the Rev. T. C. Simmons, M.A.)—If from a random point within an equilateral triangle perpendiculars are drawn on the sides, show that the respective chances that they can form (1) *any* triangle, (2) an acute-angled triangle, are $p_1 = \frac{1}{4}$, $p_2 = 3 \log_e 2 - 2 = \cdot 07944 = \frac{5}{63}$ nearly.

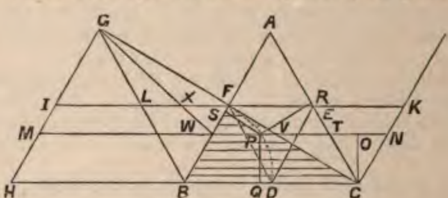
Solutions by (1) H. McColl, B.A.; (2) D. Biddle.



1. Let ABC be the equilateral triangle. In AB take any random point P. Draw PQ parallel to AC, and meeting BC at Q. In PQ take a random point R. Then R may be taken as the random point spoken of. Taking AB as unity, let $x = AP$, and let $y = PR$. The perpendiculars from R upon the sides are proportional to the algebraic quantities x, y, $1-x-y$. The chance that *no* triangle can be formed by the perpendiculars is therefore 3 times the chance that $1-x-y$ will be greater than $x+y$, that is to say, 3 times the chance that $y-\frac{1}{2}+x$ will be negative, all values of x between 0 and 1, and of y between 0 and $1-x$, being equally probable. The chance that $y-\frac{1}{2}+x$ will be negative is a fraction the numerator of which is the value of the integral $\int dx \int dy$ when the limits for x are 0 and $\frac{1}{2}$, and those for y are 0 and $\frac{1}{2}-x$, and the denominator of which is the value of the same integral when the limits of x are 0 and 1, and those of y are 0 and $1-x$. The chance of *no* triangle is therefore $\frac{3}{4}$, and of *some* triangle $\frac{1}{4}$.

The chance that *no acute-angled triangle* can be formed by the perpendiculars is 3 times the chance that $(1-x-y)^2$ will be greater than x^2+y^2, that is to say, 3 times the chance that $y-\dfrac{1-2x}{2-2x}$ will be negative. The chance that this will be negative is a fraction, the denominator of which will be the same as in the former case, but the numerator of which will be the integral $\int dx \int dy$, between the limits 0 and $\frac{1}{2}$ for x, and 0 and $\dfrac{1-2x}{2-2x}$ for y. The chance of *no acute-angled triangle* is therefore $3(1-\log 2)$; and the chance of an acute-angled triangle is $1-3(1-\log 2)$, which $= 3\log 2 - 2$. [This value, in decimals, is ·07944154. See the solution of Question 3342, *Reprint*, Vol. xv., p. 58, and Vol. xvi., p. 68.]

2. *Otherwise*:—To meet the first-named conditions, it is evident the point must lie within the space bounded by the lines joining the mid-



points of the sides of the triangles, since these lines form the locus of equality between one perpendicular and the sum of the other two. The area of the said space is one-fourth of the whole triangle. Hence $p_1 = \frac{1}{4}$.

To find p_2 we must discover the locus of points from which perpendiculars may be drawn so that the sum of the squares of two shall equal the square of the third; in other words, the locus for the formation of right-angled triangles, which are the limit of acute. The mid-points of the sides are, as before, the starting points, but the course consists of three curves instead of straight lines. Let ABC be the triangle, D, E, F the mid-points of its sides, and P a point placed so that $PQ^2+PS^2 = PR^2$. It is easily seen that PS : PW = PR : PT = PQ (= CO) : CT (= NT). But PW + PT + NT = WN = BC = 1. Consequently the problem resolves itself into dividing a given line into three portions, so as to form a right-angled triangle, or rather a series of right-angled triangles with one side increasing arithmetically from 0 to ½ (represented by NT, between 0, at C, and ½, at RK). Now $WP^2 = PT^2 - NT^2$, and $PT = 1 - WP - NT$, therefore $WP = \frac{1-2NT}{2-2NT}$. If, then, we produce CB to H, and on BH describe BGH = ABC; and if we join GC, cutting AB in its mid-point F, and draw KI through F parallel to CH, we are able to find P for any given height (less than half that of the triangle) above BC. Thus, by drawing NM at the given height parallel to CH, and drawing FP parallel to GW, we cut NM in the point required. For, NV = 2NT, MV = 2 − 2NT, WV = 1 − 2NT, and IF = 1. Therefore

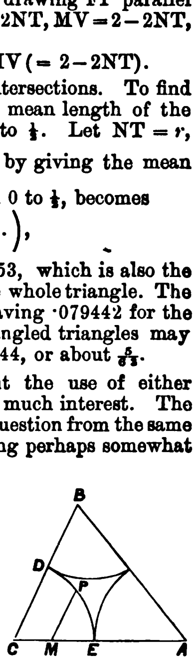
$$XF\,(=WP) : WV\,(=1-2NT) = IF\,(=1) : MV\,(=2-2NT).$$

The curve DPF is indicated by a series of similar intersections. To find the area of the space cut off by it, we must find the mean length of the lines represented by WP, when NT varies from 0 to ½. Let NT = r, then $\frac{1-2r}{2-2r} = \tfrac{1}{2}(1-r-r^2-r^3-r^4-\ldots-r^{\infty})$, which, by giving the mean values of the several powers of r, when r varies from 0 to ½, becomes

$$\tfrac{1}{2}\left(1-\frac{1}{2\,.\,2}-\frac{1}{4\,.\,3}-\frac{1}{8\,.\,4}-\frac{1}{16\,.\,5}-\&c.\right),$$

or, to six places of decimals, ½ (1 − ·386294) = ·306853, which is also the proportionate area of the space as compared with the whole triangle. The proportionate area of the three spaces = ·920558, leaving ·079442 for the space within which P must lie in order that acute-angled triangles may be formed according to clause (2). Hence $p_2 = \cdot07944$, or about $\frac{5}{63}$.

[Mr. Biddle's solution, effected as it is without the use of either Analytical Geometry or the Integral Calculus, is of much interest. The following solution, by their aid, while regarding the question from the same point of view, has been sent by Mr. Simmons, as being perhaps somewhat simpler :—



After proving as above, that $p_1 = \frac{1}{4}$, take any point P on the curve $\alpha^2+\beta^2=\gamma^2$, and let PM drawn parallel to BC = y, CM = x. Then $\alpha = x \sin 60°$, $y = \beta \sin 60°$. Whence, since

$$\alpha^2+\beta^2 = (a \sin 60° - \alpha - \beta)^2,$$

we have $x^2+y^2 = (a-x-y)$ or $y = \frac{a^2-2ax}{2(a-x)}$, so that the area of CDPE is

$$\frac{a}{2}\sin 60° \int_0^{\frac{1}{2}a} \frac{a-2x}{a-x}\,dx = \frac{\sqrt{3}a}{4}(a - a\log 2) = \Delta\,(1-\log 2);$$

and the whole space within ABC for which an acute-angled triangle is impossible is $3\Delta\,(1-\log 2)$; hence the chance of an acute-angled triangle is $[\Delta - 3\Delta\,(1-\log 2)]\,/\,\Delta$ or $3\log 2 - 2$.

Mr. Simmons states that he was not aware, before seeing Mr. McColl's solution, that the question had been previously proposed by him as Question 3342.]

7326. (By Professor HUDSON, M.A.)—Prove that, in the curve $r = a + b\cos\theta$, the polar subtangent cannot have a maximum or minimum value for a finite value of r, unless $a > b$.

Solution by B. HANUMANTA RAU, M.A.; J. O'REGAN; *and others.*

$$r = a + b\cos\theta, \text{ polar subtangent } = u = r^2\frac{d\theta}{dr} = -\frac{(a+b\cos\theta)^2}{b\sin\theta},$$

and, for a maximum or minimum value of u,

$$\frac{du}{d\theta} = 0, \text{ therefore } 2(a+b\cos\theta) + \frac{(a+b\cos\theta)^2\cos\theta}{b\sin^2\theta} = 0,$$

or $2b(1-\cos^2\theta) + (a+b\cos\theta)\cos\theta = 0$, therefore $\cos\theta = \dfrac{a \pm (a^2+8b^2)^{\frac{1}{2}}}{4b}$.

In order that θ may be possible, we must have

$$4b \nless a \pm (a^2+8b^2)^{\frac{1}{2}}, \text{ or } 16b^2 - 8ab + a^2 \nless a^2 + 8b^2, \text{ therefore } b \nless a.$$

4038. (By the Rev. T. P. KIRKMAN, M.A., F.R.S.)—Prove that (1) a triangle can be partitioned into 13 triangles in 457 ways, two ways being reckoned the same if one is in any position the reflected image of the other, the size of the partitions being of no consequence; and find (2) in how many ways an equilateral triangle can be partitioned into 13 triangles of equal area.

Solution by the PROPOSER.

1. The number of partitions is the number of different triangles symmetric and asymmetric, which are found upon the 34 symmetrical and the 16 asymmetrical 9-acral 14-edra. These triangles are given by their edges, which are rapidly enumerated in the processes of my Theory of the Polyedra, nearly 25 years old.

There are 8 zoned polar edges (33), 7 zoneless polars, 30 epizonals, and 64 zonals; from which, and their zonal signatures, we find that there are 4 zoned polar, and 64 monozone, in all 68 symmetric, triangles on 34 symmetrical solids zoned and zoneless, namely, on 9 zoned polar, 7 zoneless polar, and 18 monozone solids. And on these 34 symmetric solids are 244 asymmetric edges, and 165 asymmetric triangles. This is evident from the signatures of symmetry, before the construction of a single figure.

The entire number of asymmetric edges on all the 9-acral 14-edra is entered, after a glance of inspection at the table preceding, thus, in two words,
$$(33)\,5\mathbf{12} = {}_0344^{4}{}_{48} + 236_1 = 580,$$
where 344 is the number of asymmetric diagonals that can be drawn in the 4-gons of the 9-acral 13-edra, and the 236 are asymmetric edges (33), across all of which lies a triangular section along an effaceable. Subtracting from these 580 the asymmetric edges found in the 34 symmetric solids, we get $580-244 = 336 = 21.16$; that is, there are 16 asymmetric solids. Every zonal and every zoneless polar edge is in one asymmetric

triangle, and every asymmetric edge is taken to be in two; and $2.580+64+7 = 1231$ is our number of these edges, which is to be diminished by one asymmetric edge for every monozone triangle. This gives $1231-64 = 1167 = 3.389$, proving that there are 389 asymmetric triangles. The sum of these and the 68 symmetric ones gives us 457 as the complete number of different triangles upon the 50 9-acral 14-edra. All this, and much more, is deduced and registered by the method of the Theory, before any attempt at construction by lines.

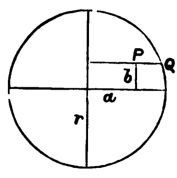
On any one p of the triangles upon any given one P of the 50 solids, the remaining 13 faces of P can be projected. The 13 triangles may by their shapes and unequal areas be varied in ways innumerable, while all the different figures will alike be the projection of the same P on this one face of it, p, taken for base. And can there be a reason why the 13 partitions should not all be of equal area, whether p be equilateral or not; so that 457 is the correct answer to both (1) and (2) in the question?

2. I cannot find nor recall my solution of (2), but I believe the answer to be $457-N$, where N is the number of different triangles on those 9-acral 14-edra which have fewer than two triaces. See a simpler question on this matter, which, I hope, will soon appear, and deserve discussion.

[That the preceding will be quite intelligible to our readers, Mr. KIRKMAN cannot flatter himself, limited as he is here in space. A clearer view of the subject may be obtained from a Memoir in the Proceedings of the Literary and Philosophical Society of Liverpool, Vol. XXXII., 1877–78, of which Memoir the title is "*The Enumeration and Construction of the 9-acral 9-edra.*"]

5218. (By the EDITOR.)—A circular target revolves uniformly around a vertical axis, lying in its plane and passing through its centre; and a shot is fired at the target in (1) a given or (2) a random direction: find, in the first case, the chance that the shot will hit the target, and show therefrom that, in the second case, the chance is $2/\pi$.

Solution by D. BIDDLE.



1. Let P be the point which the bullet would hit if the target were stationary and fully fronting the marksman, and a, b its coordinates. Then it is evident that, as the target revolves, the bullet, if it strikes the face at all, must hit it in the line PQ, and therefore during that portion of the revolution which, if r be the radius of the target, is represented by $2\cos^{-1}\left(\frac{a}{(r^2-b^2)^{\frac{1}{2}}}\right)$; and, if hitting the target back or front be allowed, then the angle is doubled, and this, divided by 2π, gives $\frac{2}{\pi}\cos^{-1}\left(\frac{a}{(r^2-b^2)^{\frac{1}{2}}}\right) = p_1$.

2. If an infinite number of horizontal lines be drawn on the target, a random shot may hit any one of those lines, and any point on that particular

line, supposing the target to be stationary as before. C
revolution takes place, the probability that the bullet
proportion borne by the mean apparent area of the t
area, and this is the ratio between mean cosine and rad

6960. (By Dr. MacAlister.)—Show from first princ motion of a particle the tangential force be measure second at which momentum is *increased*, the normal forc units be measured by the rate per second at which mom

---

Solution by Âsûtosh Mukhopâdhyây.

Consider the motion of a particle of given mass un given forces; then, the motion is completely defined as two essentially distinct circumstances about it—viz., t path described, and the curvature of the path at any poi as soon as we know the equation connecting the intrin (s) and the angle of deflection (ϕ) measured from a defined origins.

Regarding a curve as the limit of a polygon, it f definition of a tangent as the line forming two co that the *length* of the arc depends on the tangential force that is, its curvature, depends on the normal force; an force is independent of the tangential force and *vice ver* once that, if we *define* the tangential force as the time *magnitude* ("increase") of the momentum, the other forc independent of this one, viz., the normal force, must be effect, that is, the normal force is the time variation of th flection") of the momentum. This is identically the sa that the *length* and the *form* (curvature) are the two dis pendent elements which are necessary and sufficient to It follows that this is the statement of the analytical theo equation $M\frac{d^2s}{dt^2} = \frac{d}{dt}(Mv)$, is identically true, we have

$$M\frac{v^2}{\rho} = M\left(\frac{ds}{dt}\right)^2\frac{d\phi}{ds} = (Mv)\frac{d\phi}{dt},$$

where the equation to the curve $s = f(\phi)$ does not involv elements, but only the intrinsic elements s, ϕ.

[The Proposer remarks that he does not understand wh the "time variation of the *position* of the momentum." If the momentum at any moment increases the *position* of t demurs to the proposition; if it does not mean that, then a lit is wanted. Again, if the reasoning is so clear and direct as the above, ought it not to be possible to derive $\frac{Mv^2}{\rho}$ from fir little more inevitably?]

7803. (By the EDITOR.)—Trace the curve $y^2(x-a) = x^3-b^3$.

Solution by the REV. T. C. SIMMONS, M.A.; N. SARKAR, M.A.; *and others.*

The asymptotes are $\pm y = x+\frac{1}{3}a$ and $x = a$. Developing further, we see that, when x is positive, the curve lies above the first asymptote, and below when x is negative, also that it meets it when $x = \dfrac{4b^3-a^3}{3a^2}$. Now

$$\frac{dy}{dx} = \frac{2x^3-3ax^2+b^3}{2y(x-a)^2} = \infty \text{ at } (b,\ 0) \text{ and at } (a,\ 0),$$

also $$= \pm\left(\frac{b^3}{4a^3}\right)^{\frac{1}{2}} \text{ at } \left\{0,\ \pm\left(\frac{b^3}{a}\right)^{\frac{1}{2}}\right\}.$$

At $(b, 0)$ the curve is of the form $\eta^2 = \dfrac{3b^2}{b-a}\xi$. Moreover, it is symmetrical with respect to Ox, and when a and b are unequal there is no real value of x intermediate to a and b.

The two figures are for (i.) $b > a$, (ii.) b positive and $< a$. When $b = a$, the curve becomes the line $x = a$ and a rectangular hyperbola. When $b = 0$, there is a cusp at the origin.

Fig. (i.)

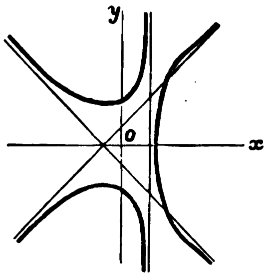
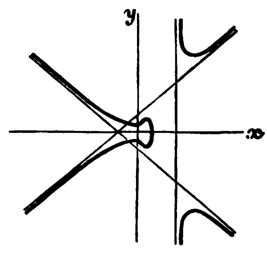


Fig. (ii.)



7797. (By D. EDWARDES.)—If

$$V_n = \int_0^1 [\log(1+x)]^n\,dx, \quad \text{prove that } V_n + nV_{n-1} = 2(\log_e 2)^n.$$

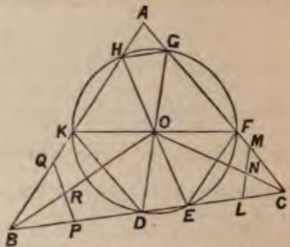
Solution by G. G. STORR, B.A.; W. T. MITCHELL, M.A.; *and others.*

Put $1+x = y$, then $V_n = \int_1^2 (\log y)^n\,dy$, hence we have

$$V_n = \left\{(\log y)^n y\right\}_1^2 - \int_1^2 n(\log y)^{n-1}\,dy = 2(\log 2)^n - nV_{n-1}.$$

7747. (By the Editor.)—Show that (1) in a triangle there can be inscribed three rectangles having each a side on one of the sides of the triangle, and their diagonals equal and crossing at their mid-points; and (2), if a, b, c be the sides of the triangle, the length of these equal diagonals is $2abc / (a^2+b^2+c^2)$.

Solutions by (1) J. McDowell, M.A.; (2) A. H. Curtis, LL.D., D.Sc.

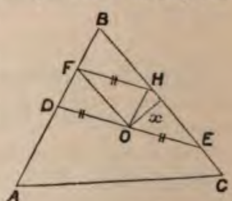


1. Let ABC be any triangle. Draw any straight line LM cutting BC, CA in L, M, and making $\angle$CLM = A, and therefore CML = B, and any straight line PQ making $\angle$BPQ = A, and therefore BQP = C. Let N be mid-point of LM, and R the mid-point of PQ, and let CN, BR meet in O. Through O draw DG parallel to LM and EH parallel to PQ, therefore the angles ODE and OED each = A; therefore DOE and GOH are identically equal isosceles triangles, and therefore HG is parallel to BC. Let the circle with centre O and radius OD, which, of course, passes through D, E, G, H, meet AB and AC again in K and F; then $\angle$OKH = OHK = C, and OFG = OGF = B, hence therefore KOF is a straight line. Hence the three rectangles with their centres at O and equal diagonals are DEGH, FGKD, HKEF.

2. Since HG is parallel to BC, &c., let HG = la, DK = mb, EF = nc; $\therefore$ AH = lc, HK = nc, KB = mc; $\therefore$ $lc+nc+mc$ = AH + HK + KB = c; therefore $l+m+n=1$; hence, putting x for a diagonal, we have $\angle$OFE = OEF = C, therefore $\frac{1}{2}$EF or $\frac{1}{2}nc = \frac{1}{2}x\cos C$; $\therefore n = x\cos C / c$; similarly $\qquad l = x\cos A / a,\ \ m = x \cos B / b$;

$$\therefore \quad x\left(\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}\right) = l+m+n = 1; \quad \therefore x = \frac{2abc}{a^2+b^2+c^2}.$$

Otherwise.—The locus of the mid-point of the diagonal of all rectangles inscribed in a triangle, and having a side along a, is the line joining the mid-point of a to the mid-point of the perpendicular from A on a. This locus and the two corresponding ones for the other two sides meet in a point (Quest. 7644) the trilinear coordinates of which are proportional to sin A, sin B, sin C, or are equal to Ka, Kb, Kc, and, as $2\Delta = ax+by+cz$ $= K(a^2+b^2+c^2)$, we have $K = \dfrac{2\Delta}{(a^2+b^2+c^2)}$; in the figure let O be this point, and let DE be drawn so as to be bisected at O, then DE will be one of the diagonals in question; draw OF, OH parallel to a and c, then BF = OH



$= \dfrac{x}{\sin B} = \dfrac{Ka}{\sin B}$, similarly $BH = \dfrac{Kc}{\sin B}$,

$\therefore DE^2 = 4FH^2 = 4(BF^2+BH^2-2BF.BH\cos B)$

$$= \frac{4K^2}{\sin^2 B}(a^2+c^2-2ac\cos B) = \frac{4K^2b^2}{\sin^2 B}, \text{ therefore } DE = \frac{2Kb}{\sin B}$$

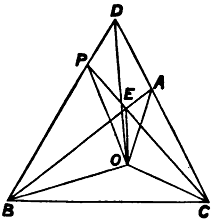
$$= \frac{4\Delta b}{(a^2+b^2+c^2)\sin B} = \frac{2abc}{(a^2+b^2+c^2)}.$$

The symmetry of the result shows that the three lines thus drawn are equal, each pair being the diagonals of one of the three rectangles referred to in the question.

7758. (By J Brill, B.A.)—If ABC be any triangle, and O a point within it; prove that

$$\frac{OA\,.\,BC}{\sin(BOC-BAC)} = \frac{OB\,.\,AC}{\sin(COA-CBA)} = \frac{OC\,.\,AB}{\sin(AOB-ACB)}.$$

Solution by R. Lachlan, B.A.; R. Knowles, B.A.; *and others.*



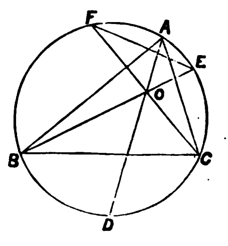
Let the circles AOC, AOB cut AB, AC in E and D; and let CE, BD meet in P; then
$\angle PBC = AOB-ACB$, $\angle PCB = AOC-ABC$,
therefore $\angle BPC = BOC-BAC$.
Again, $\angle PBO = OAC$,
also $\angle ODB = OAB = OCP$;
therefore P, O, C, D are concyclic, therefore
$\angle BPO = OCA$;
hence the triangles PBO, CAO are similar,
therefore $\frac{OB}{BP} = \frac{OA}{AC}$, therefore $\frac{OA\,.\,BC}{BC} = \frac{OB\,.\,AC}{PB} = \frac{OC\,.\,AB}{PC}$,
by symmetry, whence follows the result in the question.

[If $OA = x$, $OB = y$, $OC = z$, $\angle BOC = \alpha$, $\angle COA = \beta$, $\angle AOB = \gamma$,
we have $a^2 = y^2+z^2-2yz\cos\alpha$, $b^2 = z^2+x^2-2zx\cos\beta$............(1, 2),
$c^2 = x^2+y^2-2xy\cos\gamma$ ....(3),
$bc\sin A = ca\sin B = ab\sin C = yz\sin\alpha + zx\sin\beta + xy\sin\gamma$......(4),
(2) + (3) − (1) gives $bc\cos A = x^2-xy\cos\gamma-zx\cos\beta+yz\cos\alpha$ ....(5).
Eliminating yz from (4), (5), we have, since $\alpha+\beta+\gamma = 2\pi$,

$$bc\sin(\alpha-A) = x^2\sin\alpha - xy\sin(\alpha+\gamma) - xz\sin(\alpha+\beta)$$
$$= x(x\sin\alpha + y\sin\beta + z\sin\gamma),$$

$$\therefore\ \frac{ax}{\sin(\alpha-A)} = \frac{abc}{x\sin\alpha+y\sin\beta+z\sin\gamma} = \frac{by}{\sin(\beta-B)} = \frac{cz}{\sin(\gamma-C)},$$

by symmetry.



Geometrically, if AO, BO, CO cut the circumcircle in D, E, F; then, from the triangles OEF, OBC, we have

$$\frac{EF}{BC} = \frac{EO}{OC} = \frac{EO\,.\,OB}{B\,.\,OC};$$

hence $$\frac{EF}{AO\,.\,BC} = \frac{FD}{BO\,.\,CA} = \frac{DE}{CO\,.\,AB};$$

also $\angle ECF = BOC-CEB$, when follows the result in the question.

This theorem enables us to find the distances from the angular points of the triangle of the in-centre, es-centres, ortho-centre, and in fact of any point when the angles are known, and the angles when the distances are known.]

7802. (By W. J. Greenstreet, B.A.)—Prove that the sum to infinity of the series $\log\frac{2\,.\,4}{3^2}+\log\frac{4\,.\,6}{5^2}+\log\frac{6\,.\,8}{7^2}+\dots$ is $\log\frac{\pi}{4}$.

Solution by W. T. Mitchell, M.A.; G. G. Storr, B.A.; *and others.*

$$\text{The sum} = \log\frac{2\,.\,4^2\,.\,6^2\,.\,8^2\dots}{3^2\,.\,5^2\,.\,7^2\dots} = \log\,.\,\frac{2\,.\,4\,.\,4\,.\,6\dots}{3\,.\,3\,.\,5\,.\,5\dots} = \log\frac{\pi}{4},$$

by Wallis's theorem.

7795. (By C. E. McVicker, B.A.)—Prove that the distance between the instantaneous centre of rotation of a movable line, and the centre of curvature of its envelope is, in any position, $dx/d\omega$, where x is the distance of any carried point on the line from the point of contact, and ω the angle of rotation.

Solution by W. J. C. Sharp, M.A.; *the* Proposer; *and others.*

The instantaneous centre for any position of the line is its point of contact with the envelope; if P and P′ be successive points upon this, $dx = PP' = ds$ in the envelope, and, the successive positions of the line being the tangents at P and P′, $\rho = ds/d\omega = dx/d\omega$.

7794. (By J. Brill, B.A.)—Prove that in any triangle

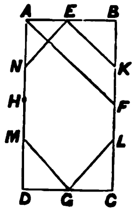
$$a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B) = 3abc.$$

Solution by Rev. D. Thomas, M.A.; G. G. Storr, B.A.; *and others.*

$$\begin{aligned} a^3\cos(B-C)+\dots &= a\,[a\cos B\,a\cos C + a\sin B\,a\sin C]+\dots \\ &= a\,[(b-c\cos A)(c-b\cos A)+bc\sin^2 A]+\dots \\ &= a\,[bc-(b^2+c^2)\cos A+bc]+\dots = 6abc-bc\,(b\cos C+c\cos B)-\dots \\ &= 6abc-3abc = 3abc. \end{aligned}$$

7801. (By B. Hanumanta Rau, M.A.)—Inscribe a regular hexagon in a rectangle whose sides are a and b; and find the ratio of a to b in order that the polygon may be also equiangular. [Suggested by Question 7636.]

Solution by Arthur Hill Curtis, LL.D., D.Sc.



Let ABCD be the given rectangle, in which AB = a, BC = b, and, suppose $b>a$; let E, F, G, H be the middle points of the sides; on EF as base construct a triangle, EKF, whose vertex shall be on BC, and such that EK = 2KF; take NH = HM = LF = FK, and EKLGMN will be an *equilateral* hexagon; it will moreover be a *regular* hexagon if $\angle$ EKB = $\frac{1}{3}\pi$, or if

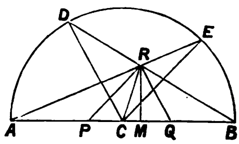
$$KB = \tfrac{1}{2}KE = KF,$$

and therefore AF is parallel to EK, therefore

$$\frac{b}{2a} = \frac{FB}{BA} = \tan\tfrac{1}{6}\pi = \frac{1}{3^{\frac{1}{2}}}, \text{ or } 2a = 3^{\frac{1}{2}}b.$$

Note on the Solutions of Question 5672.

(See pp. 45—47 of this Volume.)



Mr. Biddle's result, $1/k^2l^2 \operatorname{cosec}\theta$, may be thus translated into terms of coordinates of R in respect of centre of circle:—In the figure, we have AE = k, BD = l, $\angle$DCE = PRQ = θ, CM = x, RM = y, AC = 1; but

$$AE = \cos A \,.\, AB, \quad BD = \cos B \,.\, AB,$$

therefore $k = 2\cos A$, $l = 2\cos B$.

Also $\sin A = y / [(1+x)^2 + y^2]^{\frac{1}{2}}$,

$$\sin B = y / [(1-x)^2+y^2]^{\frac{1}{2}}, \quad \cos A = (1+x) / [(1+x)^2+y^2]^{\frac{1}{2}},$$
$$\cos B = (1-x) / [(1-x)^2+y^2]^{\frac{1}{2}}.$$

Moreover, $\operatorname{cosec}\theta = 1/\sin\theta = 1/\sin(180^\circ - \theta) = 1/\sin(\text{RPQ}+\text{RQP})$

$$= 1/\sin 2(A+B) = 1/2\sin(A+B)\cos(A+B)$$
$$= 1/[2\sin A\cos A(\cos^2 B - \sin^2 B) + 2\sin B\cos B(\cos^2 A - \sin^2 A)]$$
$$= \frac{(1-x^2)^2 + y^4 + 2(1+x^2)y^2}{4y(1-x^2-y^2)};$$

moreover, $k^2l^2\operatorname{cosec}\theta = 16\cos^2 A\cos^2 B\operatorname{cosec}\theta = \dfrac{16(1-x^2)^2}{4(1-x^2-y^2)}$,

and $$\frac{1}{k^2l^2\operatorname{cosec}\theta} = \frac{y(1-x^2-y^2)}{4(1-x^2)^2},$$

which, to the constant 4 *près*, agrees with Colonel Clarke's result.

7800. (By E. BUCK, B.A.)—Without involving the Integral Calculus, prove the formula $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \&c.$

Solution by G. G. STORR, B.A.; B. HANUMANTA RAU, M.A.; *and others.*

Since $\sin^{-1} 0 = 0$ and $\sin^{-1}(-x) = -\sin^{-1} x$, therefore $\sin^{-1} x$ is an odd function of x; hence, putting $\sin^{-1} x = Ax + B\frac{x^3}{3} + C\frac{x^5}{5} + \&c.$, and, differentiating, $\frac{1}{(1-x^2)^{\frac{1}{2}}} = A + Bx^2 + Cx^4 + \&c. = 1 + \frac{1}{2} \cdot x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \&c.$ (by Algebra). Hence, equating coefficients and substituting, we have the stated formula.

3826. (By J. B. SANDERS.)—The heights of the ridge and eaves of a house are 40 feet and 32 feet respectively, and the roof is inclined at 30° to the horizon. Find where a sphere rolling down the roof from the ridge will strike the ground, and also the time of descent from the eaves.

Solution by Professor MARTIN, M.A.; Professor WICKERSHAM; *and others.*

Let u be the velocity of the sphere at the eaves equal to the velocity of the sphere acquired by falling freely through 8 feet; and equal

$$(16 \times 32{\cdot}15945)^{\frac{1}{2}} = 22{\cdot}6837 \text{ feet.}$$

The velocity in a vertical direction at the eaves is $= 11{\cdot}3418$ feet. And the horizontal velocity at the eaves is $= u\cos 30° = 22{\cdot}6837 \times \frac{1}{2}\sqrt{3}$. Let t be the time of the sphere falling from the eaves. Then in the time t the sphere falls through $\frac{1}{2}ut$ by the constant velocity $\frac{1}{2}u$; and in the time t it falls through $\frac{1}{2}gt^2$ by the accelerated velocity commencing at the eaves, where $g = 32{\cdot}15945$ feet; then we have the equation

$$\tfrac{1}{2}gt^2 + \tfrac{1}{2}ut = 32 \text{ or } t^2 + \frac{ut}{g} = \frac{64}{g}, \text{ and } t = -\frac{u}{2g} + \left(\frac{u^2}{4g^2} + \frac{64}{g}\right)^{\frac{1}{2}};$$

and $t = -0{\cdot}352675 + 1{\cdot}454119 = 1{\cdot}101444$ seconds, the time of descent from eaves. And $tu\cos 30° = 1{\cdot}101441 \times 22{\cdot}6837 \times 0{\cdot}866025 = 21{\cdot}6375$ feet. And the sphere will strike the ground 21·6375 feet plus half the radius of the sphere from a vertical line from the eaves.

After the sphere leaves the eaves, it describes a parabola whose radius of curvature is 18·475 feet at the eaves; therefore a sphere of 2 or 3 feet radius, or less, will not touch the eaves after a radius of the sphere becomes perpendicular to the roof at the eaves. Friction and the resistance of the air are not considered.

7774. (By Professor WOLSTENHOLME, M.A., Sc.D.)—The lengths of the edges OA, OB, OC of a tetrahedron OABC are respectively

9·257824, 8·586, and 8·166; those of the respectively opposite edges BC, CA, AB are 8·996, 9·587, and 9·997. Prove that the dihedral angles opposite to OA and BC are equal to each other (each $= 7^\circ 19' 18''$). Denoting the lengths by a, b, c, x, y, z, and the dihedral angles respectively opposite by A, B, C, X, Y, Z, find what relation must subsist between a, b, c, x, y, z in order that A may be equal to X.

Solution by W. H. BLYTHE, M.A.

1. We have $\angle BAC = 54^\circ 38' 32''$, $CAO = 51^\circ 19' 17''\cdot 6$, $BAO = 52^\circ 47' 37''\cdot 4$; and if we denote the dihedral angle opposite BC by M, and the above angles by p, q, r, and half their sum by s, then

$$4 \tan \tfrac{1}{2} M = 10 + \tfrac{1}{2} [L \sin (s-q) + L \sin (s-r)] - \tfrac{1}{2} [L \sin s + L \sin (s-p)];$$

hence we obtain the stated result for M.

By a similar process, we find the dihedral opposite OA, from the values $OBA = 59^\circ 10' 49''\cdot 8$, $OBC = 55^\circ 17' 24''\cdot 3$, $CBA = 60^\circ 21' 29''\cdot 4$.

2. $$\sin A = \frac{(3 \text{ volume})\, x}{2 (\text{area ABC}) (\text{area OBC})}, \quad \sin X = \frac{(3 \text{ volume})\, a}{2 (\text{area OBA}) (\text{area CAO})},$$

and, if these angles are equal,

$$a^2 (b+c+x)(b+c-x)(b+x-c)(x+c-b) \times (x+y+z)(x+y-z)(z+x-y)(y+z-x)$$
$$= x^2 (a+b+z)(a+b-z)(a+z-b)(b+z-a) \times (a+c+y)(a+c-y)(a+y-c)(y+c-a).$$

7652. (By G. HEPPEL, M.A.)—Show that the square root of

$$2E \equiv 2(1 + \cos\alpha\cos\beta - \cos\alpha\cos\gamma - \cos\alpha\cos\delta - \cos\beta\cos\gamma - \cos\beta\cos\delta + \cos\gamma\cos\delta - \sin\alpha\sin\beta\sin\gamma\sin\delta + \cos\alpha\cos\beta\cos\gamma\cos\delta)$$

is $$2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma-\delta) \sim 2\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma+\delta).$$

Solution by B. HANUMANTA RAU, M.A.; SARAH MARKS; *and others.*

$$2E = (1+\cos\alpha)(1+\cos\beta)(1-\cos\gamma)(1-\cos\delta) + (1-\cos\alpha)(1-\cos\beta)(1+\cos\gamma)(1+\cos\delta)$$
$$= (4\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\delta)^2 + (4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\delta)^2,$$
$$2\sin\alpha\sin\beta\sin\gamma\sin\delta$$
$$= 2 \cdot 4\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\delta \times 4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\delta,$$
$$\therefore (2E)^{\frac{1}{2}} = 4(\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\delta \sim \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\delta) = \&c.$$

4165. (By J. CONWILL.)—A fish is floating in a cubical glass tank filled with water, with its head in one corner, and its tail towards the one diagonally opposite; describe the appearance which will be presented to an eye looking towards the corner in the direction of the length of the fish, and in the same horizontal plane with it.

Solution by the Rev. J. L. Kitchin, M.A.; A. Martin, M.A.; *and others*.

The fish, looked at along the diagonal in which it is placed, will be seen by light refracted out of a denser medium into a rarer. Hence each point in the two sides will appear shifted from its position *from* the diagonal, and the nearer the point the greater will the displacement appear, because it is seen under a greater angle. The two sides will then appear to recede from the diagonal, and the appearance presented is that of two fish united by their heads at the further corner of the cube.

4737. (By Professor Artemas Martin, M.A., Ph.D.)—Three equal circles, each 4 inches in diameter, are drawn at random on a circular slate whose diameter is 12 inches; find the probability that each circle intersects the other two.

Solution by D. Biddle.

In order that three complete circles, each 4 inches in diameter, may lie on a circular slate 12 inches in diameter, the centre of each must be within the circumference of a circle concentric with the slate and 8 inches in diameter. Moreover, in order that two of the smaller circles may intersect each other, their centres must lie within a radius of 4 inches of each other; and the probability of this occurring is the proportion borne to the entire area of the 8-inch circle by the mean space cut off by another 8-inch circle whose centre may be anywhere between the former's centre and circumference. This mean space is the average for all positions within the 8-inch circle, and is found by taking a series of segments from half-radius to radius in height, doubling each, and multiplying also by the difference between the height and radius, which difference is half the distance of the centre of the invading circle from the centre of the slate, and is proportioned to the ring (concentric with the slate) for all parts of which the particular segment will serve; and we finally divide the sum of the several products by the sum of the multipliers above referred to. This gives $\frac{2278}{3937} = \cdot 57856$, as the probability that a 4-inch circle drawn at random on the slate will cut one of similar size also drawn at random on the slate. Now, if A, B, C be the three 4-inch circles, the above is the probability that A will cut B, that B will cut C, that C will cut A. But, as these events must concur to produce the mutual intersection of all three circles, therefore $(\cdot 57856)^3$ is the probability required $= \cdot 19366$, or rather more than $\frac{6}{31}$.

6746. (By Âsútosh Mukhopâdhyây.)—A certain number of candidates apply for a situation, to whom the voters attribute every degree of merit between the limits 0 and ψ; find the mean value of all the candidates' merits.

Solution by H. Fortey, M.A.; G. Eastwood, M.A.; *and others*.

Let n be the number of candidates, and $t_1, t_2, t_3 \dots t_n$ their respective

Now, if n be an odd number, $t_{\frac{1}{2}(n+1)}$ will be the middle term, and its mean value $\frac{1}{2}\psi$. If n be an even number, $t_{\frac{1}{2}n}$, $t_{\frac{1}{2}(n+2)}$ are the two middle terms; their values are $\frac{\psi}{2}+\frac{\frac{1}{2}\psi}{n+1}$ and $\frac{\psi}{2}-\frac{\frac{1}{2}\psi}{n+1}$, and the mean of these is $\frac{1}{2}\psi$. The sum of a mean value is $\frac{1}{2}\psi n$, and, therefore, the mean of all the values is $\frac{1}{2}\psi$.

7780. (By the Rev. T. R. TERRY, M.A.)—Prove that the mean value of the fourth powers of the distances from the centre of all points inside an ellipsoid whose axes are $2a$, $2b$, $2c$, is

$$\mathrm{A} = \tfrac{1}{35}\left[(a^2+b^2+c^2)^2 + 2(a^4+b^4+c^4)\right].$$

Solution by NILKANTA SARKAR, M.A.; D. EDWARDES; *and others.*

We have $\quad \mathrm{A} = \frac{1}{6}\pi abc\,\mathrm{A} = \iiint (x^2+y^2+z^2)^2\,dx\,dy\,dz,$

over the positive compartment. Integrating with respect to z, and then putting, successively, $x = ax' = ar\cos\theta$, $y = by' = br\sin\theta$, the dexter becomes

$$abc\int_0^1\int_0^{(1-x'^2)^{\frac{1}{2}}}\left[(1-x'^2-y'^2)^{\frac{1}{2}}(a^2x'^2+b^2y'^2)^2+\tfrac{2}{3}(a^2x'^2+b^2y'^2)c^2(1-x'^2-y'^2)^{\frac{3}{2}}\right.$$
$$\left.+\tfrac{1}{5}c^4(1-x'^2-y'^2)^{\frac{5}{2}}\right]dx'\,dy' =$$

$$abc\int_0^{\frac{1}{2}\pi}\int_0^1\left[r^4(1-r^2)^{\frac{1}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)^2+\tfrac{2}{3}c^2r^2(1-r^2)^{\frac{3}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)\right.$$
$$\left.+\tfrac{1}{5}c^4(1-r^2)^{\frac{5}{2}}\right]r\,dr\,d\theta.$$

Putting $r = \sin\phi$, and integrating with respect to ϕ, this becomes

$$abc\int_0^{\frac{1}{2}\pi}\left[\frac{8}{3\,.\,35}(a^2\cos^2\theta+b^2\sin^2\theta)^2+\frac{4c^2}{3\,.\,35}(a^2\cos^2\theta+b^2\sin^2\theta)+\frac{1}{35}c^4\right]d\theta,$$

and, integrating with respect to θ, we have

$$\tfrac{1}{6}\pi abc = \tfrac{1}{6}\pi\,.\,\tfrac{1}{35}abc\left[(a^2+b^2+c^2)^2+2(a^4+b^4+c^4)\right].$$

[By LEJEUNE DIRICHLET's Theorem, as given in TODHUNTER's chapter on Definite Integrals, we have, by taking integrations over one-eighth of the ellipsoid,

$$\iiint x^4\,dx\,dy\,dz = \frac{a^5bc}{8}\,\frac{\Gamma(\frac{5}{2})\,.\,\Gamma(\frac{1}{2})\,.\,\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})} = \frac{\pi a^5bc}{70},$$

$$\iiint y^2z^2\,dx\,dy\,dz = \frac{ab^3c^3}{8}\,\frac{\Gamma(\frac{1}{2})\,.\,\Gamma(\frac{3}{2})\,.\,\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{2})} = \frac{\pi ab^3c^3}{210};$$

therefore $\frac{1}{6}\pi abc\,\mathrm{A} = \frac{1}{210}\pi abc\left[3(a^4+b^4+c^4)+2(b^2c^2+c^2a^2+a^2b^2)\right]$; whence the result follows.]

7792. (By ASPARAGUS.)—The tangent at any point of a parabola meets the axis in T and the latus rectum in t; prove that Tt is equal to one-fourth of the parallel normal chord.

merits, t_1 being the merit attributed to the candidate whom the voters deem most worthy, t_2 the merit ascribed to the next candidate, and t_r the merit attributed to the candidate in the rank r.

First, suppose that t_1 may have any positive value comprised between the limits 0 and ψ, and that the mean of all such positive values of t_1 is required. Let w be the whole number of the values, and Σt_1 their sum; then, if M be the required mean, $M = \frac{\Sigma t_1}{w}$. Suppose that the values of t_1 receive equal increments Δ, and that they become 0, Δ, 2Δ, $3\Delta \ldots \psi$; then $w = \frac{\Sigma\Delta}{\Delta}$; in this case $M = \frac{\Delta\Sigma t_1}{\Delta\Sigma}$; but, because t_1 is supposed to receive all values between 0 and ψ, the quantity Δ may be considered indefinitely small, and may consequently be represented by dt_1. Hence

$$M = \frac{\int_0^\psi t_1 dt_1}{\int_0^\psi dt_1} = \frac{\frac{1}{2}\psi^2}{\psi} = \frac{\psi}{2};$$

hence the mean of all these values is the mean of the two extreme values.

Next suppose t_1, t_2 to be the merits of two candidates according to the opinion of any voter, t_1 being the merit attributed to the more worthy candidate, t_1 therefore being greater than t_2. Here t_2 may have any values between 0 and t_1; and t_1 any values between $t_2 = 0$ and $t_2 = t_1$. The aggregate of all the values of t_1 contained between $t_2 = 0$ and $t_2 = t_1$ is equal to $\frac{t_1 \int_0^{t_1} dt_2}{dt_2}$, and the sum of all the values of t_1 is equal to $\frac{\int_0^\psi \left(t_1 dt_1 \int_0^{t_1} dt_2\right)}{dt_1 dt_2}$; the number of these values is $\frac{\int_0^\psi dt_1 \int_0^{t_1} dt_2}{dt_1 dt_2}$; hence the mean value of t_1 is $\frac{\int_0^\psi \int_0^{t_1} t_1 dt_1 dt_2}{\int_0^\psi \int_0^{t_1} dt_1 dt_2}$.

If t_r be the merit attributed to the candidate in rank r,

$$\text{the mean value of } t_1 = \frac{\int_0^\psi \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{n-1}} t_r dt_1 dt_2 \ldots dt_n}{\int_0^\psi \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{n-1}} dt_1 dt_2 \ldots dt_n}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n-r} \cdot \frac{1}{1-r+2} \cdots \frac{1}{n+1} \psi^{n+1}}{\frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n} \psi^n} = \frac{n-r+1}{n+1} \psi.$$

When $n = e^2$ and $\phi = \frac{1}{2}\pi$, we have

$$\Pi(e^2, e) + \Pi(1, e) - F(e) = \frac{\pi}{2 \ldots [2(1+e^2)]},$$

which by mistake I wrote $2\Pi(e^2, e) - F(e) = \dfrac{\pi}{2(1+e^2)}$.

This relation, however, is of no use here, as it will not enable us to get rid of the third-order functions. We could, by it, eliminate $F(e)$, and thus obtain a result containing only one kind of elliptic functions.

[Professor MARTIN states that he suspects this question was suggested to the PROPOSER by his own solution of Problem 200 in the *Mathematical Visitor*, Vol. ii., No. 1, p. 20, the result of which is erroneous, owing to a mistake made in attempting to eliminate $\Pi(e^2, e)$ by the formula referred to in CAYLEY's *Elliptic Functions*.]

1208 & 6695. (By the EDITOR.)—(1208.) Show that the values of x, y, z, from the equations

$$x^2 + 4xy + 6y^2 = 28,\ x^2 + 4xz + 14z^2 = 60,\ 3y^2 + 2yz + 7z^2 = 40 \ldots (1, 2, 3),$$

are given by

$$x^2 = (\pm\sqrt{5}-1)(\pm 5\sqrt{2}-6)(\pm\sqrt{10}-2),$$
$$y^2 = \tfrac{1}{6}(\pm\sqrt{5}+1)(\pm 5\sqrt{2}-6)(\pm\sqrt{10}+2),$$
$$z^2 = \tfrac{1}{14}(\pm\sqrt{5}+1)(\pm 5\sqrt{2}+6)(\pm\sqrt{10}-2).$$

(6695.) The sides of a triangle are 40, 30, 14, and x^2, y^2, z^2 are the radii of three circles respectively inscribed in the angles opposite to the sides 40, 30, 14, such that each touches the other two and two sides of the triangle; show that the values of these radii are given by

$x^2 = 1{\cdot}539,\ y^2 = 2{\cdot}982,\ z^2 = 3{\cdot}583,\ x^2 = 83{\cdot}416,\ y^2 = 8{\cdot}194,\ z^2 = 1{\cdot}278,$
$x^2 = 49{\cdot}163,\ y^2 = 13{\cdot}901,\ z^2 = {\cdot}110,\ x^2 = 17{\cdot}893,\ y^2 = {\cdot}257,\ z^2 = 5{\cdot}958,$

of which triads of values the first gives the radii of the *inscribed* circles, and the other three those of the three triads of *escribed* circles.

Solutions by (1) *the* PROPOSER; (2) *the late* S. BILLS.

1. The equations in Question 1208 may be written thus:—

$$x^2\cot\alpha + y^2\cot\beta + 2xy = 4(\cot\alpha + \cot\beta),$$
$$x^2\cot\alpha + z^2\cot\gamma + 2xz = 4(\cot\alpha + \cot\gamma),$$
$$y^2\cot\beta + z^2\cot\gamma + 2yz = 4(\cot\beta + \cot\gamma),$$

α, β, γ being three angles such that $\frac{1}{2}\tan\alpha = 3\tan\beta = 7\tan\gamma = 1 \ldots (A)$.

The solution of the above equations is (see HYMERS's *Trigonometry*, p. 153, 3rd edition) $\lambda^2x^2 = \mu^2y^2 = \nu^2z^2 = 2\lambda\mu\nu$ ....(B),

where $\lambda - \tan\frac{1}{2}\alpha = \mu - \tan\frac{1}{2}\beta = \nu - \tan\frac{1}{2}\gamma = 1$ ....(C).

From the groups of equations (A), (C), we have

$$\lambda = \tfrac{1}{2}(\pm\sqrt{5}+1),\quad \mu = \pm\sqrt{10}-2,\quad \nu = \pm 5\sqrt{2}-6;$$

and then (B) gives the results stated in the question.

Multiplying out the radicals, the values of one triad are

$$x^2 = 38-20\sqrt{2}+2\sqrt{5}-4\sqrt{10}, \quad y^2 = \tfrac{1}{6}(38-20\sqrt{2}-2\sqrt{5}+4\sqrt{10}),$$
$$z^2 = \tfrac{1}{14}(38+20\sqrt{2}-2\sqrt{5}-4\sqrt{10}).$$

A verification may be obtained as follows :—

$$xy = 2\sqrt{50}-12, \quad xz = 2\sqrt{10}-4, \quad yz = \sqrt{5}+1;$$
$$x^2+4xy+6y^2 = (\sqrt{50}-6)(2\sqrt{50}+12) = 28,$$
$$x^2+4xz+14z^2 = (\sqrt{10}-2)(10\sqrt{10}+20) = 60,$$
$$3y^2+2yz+7z^2 = (\sqrt{5}+1)(10\sqrt{5}-10) = 40.$$

2. *Otherwise* :—From (1) + (2) − 2 (3) we have

$$x^2+2xy+2xz-2yz = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let $y = mx$, $z = nx$; then equations (1), (2), (4) become

$$x^2(1+4m+6m^2) = 28, \quad x^2(1+4n+14n^2) = 60 \ldots\ldots\ldots\ldots (5, 6),$$
$$x^2(1+2m+2n-2mn) = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (5) ÷ (6) and (5) ÷ (7) we have

$$\frac{1+4m+6m^2}{1+4n+14n^2} = \tfrac{7}{15}, \quad \frac{1+4m+6m^2}{1+2m+2n-2mn} = 7 \quad \ldots\ldots\ldots\ldots (8, 9).$$

From (9), $n = \dfrac{3+5m-3m^2}{7(m-1)}$; and, this being substituted in (8), the result is

$$36m^4-24m^3-34m^2-4m+1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Assume $\quad (10) = (6m^2+Am+1)(6m^2+Bm+1)$

$$= 36m^4+6(A+B)m^3+(AB+12)m^2+(A+B)m+1 \quad \ldots\ldots\ldots (11);$$

then, comparing the coefficients of (10) and (11), we have $A+B = -4$, $AB = -46$; whence $A = -2+5\sqrt{2}$ and $B = -2-5\sqrt{2}$. Hence the four roots of (10) are those of the two quadratics $6m^2-(2\pm5\sqrt{2})m+1 = 0$; these roots give, therefore, for $12m$, the values

$$2+5\sqrt{2}+2\sqrt{5}+\sqrt{10}, \quad 2-5\sqrt{2}+2\sqrt{5}-\sqrt{10}, \quad 2-5\sqrt{2}-2\sqrt{5}+\sqrt{10},$$
$$2+5\sqrt{2}-2\sqrt{5}-\sqrt{10}.$$

Hence putting, for shortness,

$$a = 19+10\sqrt{2}+\sqrt{5}+2\sqrt{10}, \quad b = 19+10\sqrt{2}-\sqrt{5}-2\sqrt{10},$$
$$c = 19-10\sqrt{2}+\sqrt{5}-2\sqrt{10}, \quad d = 19-10\sqrt{2}-\sqrt{5}+2\sqrt{10},$$

the values of x^2, y^2, z^2 are

$$x^2 = 2c, \quad y^2 = \tfrac{1}{3}d, \quad z^2 = \tfrac{1}{7}b \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$
$$x^2 = 2a, \quad y^2 = \tfrac{1}{3}b, \quad z^2 = \tfrac{1}{7}d \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$
$$x^2 = 2b, \quad y^2 = \tfrac{1}{3}a, \quad z^2 = \tfrac{1}{7}c \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$
$$x^2 = 2d, \quad y^2 = \tfrac{1}{3}c, \quad z^2 = \tfrac{1}{7}a \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The given equations (in 1208) may be otherwise solved as follows :—

Put $84-x^2 = n^2$; then, from (1), (2), we have $6y = -2x\pm n\sqrt{2}$, $14z = -2x\pm n\ldots10$; and, these values of y, z being substituted in (3), the resulting reduced equation in x is

$$x^4-4(19\pm\sqrt{5})x^2+168(3\mp\sqrt{5}) = 0.$$

In a similar manner, we find for y and z the equations

$$y^4-\tfrac{2}{3}(19\pm2\ldots10)y^2+\tfrac{28}{9}(7\pm3\ldots10) = 0,$$
$$z^4-\tfrac{2}{7}(19\pm10\ldots2)z^2+\tfrac{12}{49}(43\pm30\ldots2) = 0.$$

The values of x^2, y^2, z^2 found from these are the same as those given above.

7585 & 7616. (By W. J. C. Sharp, M.A.)—(7585.) If two straight lines cut the sides of a triangle ABC in the points D and D′, E and E′, F and F′, respectively, and points d and d' be taken in BC harmonically conjugate to D and D′, e and e' in CA conjugate to E and E′, and f and f' in AB conjugate to F and F′, prove that each of the sets of six points (d, d', E, E′, F, F′), (D, D′, e, e', F, F′), (D, D′, E, E′, f, f'), and (d, d', e, e', f, f') will lie on a conic, and be such that the lines drawn to them from the opposite vertices form two pencils, each composed of three concurrent lines.

(7616.) If the vertices of the triangle of reference be joined to a point $(\alpha_1, \beta_1, \gamma_1)$, and a circle be described through the three points in which these lines intersect the opposite sides, prove that (1) the point of concurrence of the other three lines drawn from the angles to the intersections of the circle with the opposite sides is determined by the equations

$$\alpha_1\alpha(\beta_1\sin B+\gamma_1\sin C)(\beta\sin B+\gamma\sin C)=\beta_1\beta(\gamma_1\sin C+\alpha_1\sin A)$$
$$\times(\gamma\sin C+\alpha\sin A)=\gamma_1\gamma(\alpha_1\sin A+\beta_1\sin B)(\alpha\sin A+\beta\sin B);$$

and (2) if $\alpha_1=\alpha$, $\beta_1=\beta$, and $\gamma_1=\gamma$, these equations determine the points of concurrence of lines drawn from the vertices to the opposite points of contact of the inscribed and escribed circles.

Solution by the Proposer; R. Knowles, B.A.; *and others.*

Let $ax^2+by^2+cz^2-2fyz-2gzx-2hxy=0$ be the equation to the two straight lines; then the points (d, d'), (e, e'), (f, f') are determined by

$$\left.\begin{array}{l}(x=0,\ by^2+2fyz+cz^2=0)\\(y=0,\ cz^2+2gzx+ax^2=0)\\(z=0,\ ax^2+2hxy+by^2=0)\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

So that these all lie on the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0;$$

and, since $abc-2fgh-af^2-bg^2-ch^2=0$, this meets the sides of the triangle in points which are the feet of two sets of concurrent lines through the vertices. (See Question 6813, *Reprint*, Vol. 37, p. 21.)

If (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ be the points of concurrence, the equations (A)

are equivalent to $\quad\left(\dfrac{y}{\beta}-\dfrac{z}{\gamma}\right)\left(\dfrac{y}{\beta_1}-\dfrac{z}{\gamma_1}\right)=0$, &c.,

and therefore

$$a:b:c:-2f:-2g:-2h=\frac{1}{\alpha\alpha_1}:\frac{1}{\beta\beta_1}:\frac{1}{\gamma\gamma_1}:\left(\frac{1}{\beta\gamma_1}+\frac{1}{\gamma_1\beta}\right):\&c.;$$

and, if the conic be a circle,

$$b\sin^2 C+c\sin^2 B-2f\sin B\sin C=c\sin^2 A+a\sin^2 C-2g\sin C\sin A$$
$$=a\sin^2 B+b\sin^2 A-2h\sin A\sin B,$$

$$\therefore\left(\frac{\sin C}{\beta}+\frac{\sin B}{\gamma}\right)\left(\frac{\sin C}{\beta_1}+\frac{\sin B}{\gamma_1}\right)=\left(\frac{\sin A}{\gamma}+\frac{\sin C}{\alpha}\right)\left(\frac{\sin A}{\gamma_1}+\frac{\sin C}{\alpha_1}\right)=\&c.,$$

which proves Question 7616 (1), if $\alpha_1=\alpha$, &c.; also

$$\frac{\sin C}{\beta}+\frac{\sin B}{\gamma}=\pm\left(\frac{\sin A}{\gamma}+\frac{\sin C}{\alpha}\right)=\pm\left(\frac{\sin B}{\alpha}+\frac{\sin A}{\beta}\right),$$

which give the four points required.

London: Printed by C. F. Hodgson & Son, Gough Square, Fleet Street, E.C.

APPENDIX.

RATIO RATIONIS:

Or that primary faculty of human nature which finds exercise alike in Logic, in Induction, and in the various processes of Mathematics.

AN ESSAY BY D. BIDDLE.

ALL things admit of both *comparison* and *contrast*. No two things are *equal* in all respects, for in that case they would be indistinguishable, and could not be known to be two; nor are any two things in all respects *unequal*, for in that case they could not exist together even in thought. The mind cannot conceive of any two things which have not some qualities in common, and also some qualities peculiar to each. 0 and ∞, the infinite and the infinitesimal, are very unlike each other, but they agree in concerning quantity, and in being at the extremes of it.

By the aid of perception and memory, the mind makes a natural classification (more or less correct) of all things coming within its range; and (taking the term in its widest signification) *Logic*, as a science, deals with the laws which govern this classification, and, as an art, teaches us to compare and contrast methodically, showing how, by successive steps, intricate comparisons and contrasts may be rendered more and more simple and intelligible, and how, on the other hand, upon a few simple facts we may build a superstructure, apparently far exceeding them in magnitude and importance, but consisting in reality only of deductions from them.

That which characterises a conscious being is the possession of a power to *feel*. This *sentient power* receives, from force transmitted to it, various *impressions*, which are agreeable or disagreeable; and it exerts an influence in promoting or preventing their recurrence accordingly. It would appear that a disagreeable impression is produced by anything which interferes with the harmonious exercise of the being's powers, and that all other impressions are agreeable.

The powers which are in immediate relation with the sentient power are the only ones of whose actions it is ever properly cognizant. Through them it receives all the force by which it is impressed, and upon them also it reacts. They communicate with (or form part of) a special organism, by which force is transformed into what are called *ideas*, and which, accordingly, we may term the *idea-framing organ*. Every impression which the sentient power receives is communicated to it by the idea-framing organ. Undisturbed by this organ, it is not conscious, we have reason to believe, even of its own existence. For self-consciousness is the result of a complex operation. When the sentient power has an idea presented to it, it instantaneously reacts upon the idea-framing organ, and gives rise to another idea. This contains a notion of the primary idea, and also of its having produced an impression upon the sentient power. And therefore, when it is itself presented, it renders the sentient power indirectly self-conscious; the sentient power then *feels that it feels* (or that it *has felt*), and this is its self-consciousness. The sentient power perceives nothing respecting itself, but in this indirect manner. Moreover, the tendency of the evidence within our reach is to prove that the sentient power is dependent upon the idea-framing organ, not only for the *reception* of impressions, but for their *continuance* also; in other words, that it does not itself retain impressions, but, like a mirror, loses them instantaneously when that by which they are communicated ceases to operate. It is owing to this qualification of the sentient power (a qualification perfect of its kind, though apparently negative) that ideas can be received by it in such quick succession, without interfering with one another, or producing the effect of dissolving views.*

The Reason depends for its data upon Perception (internal and external) and Memory (including habit), and these are, from various causes, so different in different people that it is scarcely to be wondered at, that the testimony of two persons so seldom coincides regarding any occurrence, even when they have had, to all outward seeming, the same opportunity of forming an opinion. Wilful liars are by no means those who are least capable of telling the truth, although they are certainly those who when discovered are in future least credited.

One of the most potent factors in the production of false impressions is that *habit* (referred to above as included under Memory) which leads us, often prematurely and falsely, to fill in outlines of sensation (visual, aural, or otherwise) with what to us have been their usual details. It is this habit that the conjuror takes advantage of, and may be instanced by crossing the middle and forefinger and placing a marble between, when instantly a sensation is felt as of two marbles being there, the two fingers being touched on the sides usually farthest apart. It is very common to mistake a person who is seen in the distance. I have known a fellow-student dart from my side, and give a vigorous pat on the back to a perfect stranger, in the belief that he was thus warmly greeting an old friend. He was, of course, undeceived when the stranger turned round.

Then there are the more subtle illusions which defy the more ordinary means to dispel, and which have been known to hold even the civilised world enthralled. I refer to such as that which refers the rising and setting of the sun to his own movement, and such as that which makes plants and animals to grow of themselves. These things should teach us to exercise extreme caution in regard to even the most commonly received notions. The last half of most men's lives is engaged in unlearning what has been acquired during the first.

Now, in his excellent treatise on Logic, Whately says:—"Complaints have been made that Logic leaves untouched the greatest difficulties, and those which are the sources of the chief errors in reasoning, viz., the ambiguity or indistinctness of terms, and the doubts respecting the degrees of evidence in various propositions: an objection which is not to be removed by any such attempt as that of Watts to lay down 'rules for forming clear ideas, and for guiding the judgment'; but by replying that no art is to be censured for not teaching more than falls within its

* A considerable portion of the above is taken from "The Spirit Controversy" (Williams & Norgate), and was written by me as far back as 1867.

province, and indeed more than can be taught by any conceivable art. Such a system of universal knowledge as should instruct us in the full meaning of every term, and the truth or falsity, certainty or uncertainty, of every proposition, thus superseding all other studies, it is most unphilosophical to expect or even to imagine. And to find fault with Logic for not performing this, is as if one should object to the science of optics for not giving sight to the blind; or as if (like the man of whom Warburton tells a story in his 'Divine Legation') one should complain of a reading-glass for being of no service to a person who had never learned to read."

This is perfectly correct, but at the same time, when we leave symbols and attempt to reduce Logic to practice, we find that a syllogism is only a link in a chain of argument; and it is a fact never to be forgotten that a chain is no stronger than its weakest link. Every proposition, every premise, nay every well-defined perception, is the result of a whole series of syllogisms. For instance, I cannot truthfully say that "This which I see before me is a red ball," until my mind (by unconscious cerebration, it may be) has come to at least as many conclusions as there are words in the proposition.

Moreover, Logic makes use of universal affirmatives and negatives, whereas in practice we have to reduce these to strong probabilities at most; or, in case we make use of the syllogistic form, we have to insert the qualifying "if." Thus, Enoch and Elijah are men; therefore, *if* all men are mortal, Enoch and Elijah are mortal.

But let us consider the fundamental principle upon which the reason proceeds. The most intimate faculty of the human mind (next to that of bare *feeling*, above referred to) consists in the detection (however imperfect and undefined) of similarities and dissimilarities in the various objects of which it takes cognizance. This in its simplest form is a matter of impression or perception which defies further analysis. For, though we can perceive the difference between the impressions produced, for instance, by two colours such as red and blue, we cannot adequately describe the difference, much less the impressions themselves; and, for aught we know, our impression of red may be totally unlike that produced on another person, and this without any colour-blindness either on our part or his. It matters little, provided we can distinguish red from other colours, as well as our fellows. But it is more than probable that the impression produced by red is compound. If therefore it is difficult to describe the compound impression, how impossible must it be to describe the simpler impressions which compose it!

It is the same with all elementary impressions: we cannot describe them to other persons. But we can distinguish between them, and we can select and classify objects which produce various combinations of them: we can also on the same principle classify events.

Now the primary and fundamental axiom, as laid down by Euclid, is, that "things which are equal to the same thing are equal to one another." But this is apt to mislead if taken as indicating the simplest act of the reason. For the human mind can, in numberless instances, perceive the equality of two things without reference to any third thing, and could apparently recognize that equality, if no third thing of the kind existed. The human mind starts with a recognized (or at least assumed) equality between things, and the converse of Euclid's axiom would more accurately describe its mode of procedure, viz., that "things which are equal to one another are equal to the same thing." This can be variously rendered: "The equality of equals to other things is coextensive;" "That which can be predicated of one of two things which are equal, can with equal truth be predicated of the other;" "Of two things which in any given respect are equal to each other, one cannot be equal and the other unequal, *in that respect*, to any third thing."

This, in reality, is the axiom underlying the construction and proof in the first proposition of Euclid. For when he says, "From the centre A, at the distance AB, describe the circle BCD,"—unless we regard as a species of *hocus pocus* the postulate demanding that such a thing may be considered possible,—we must imagine it as done by something comparable to a pair of compasses of infinite perfection. But, if so, the extremities of the imaginary instrument must be fixed at a distance apart equal to AB, after one of them has been accurately placed on the point A; and all this requires us to believe in the perception of an equality of two things without reference to any third thing. Again, when the two circles are described, and all their radii are determined by the interval between the compasses, this interval being equal to AB, the radii will be equal to AB; and as, of AB and AC, which are equal (in length), AB is equal (in length) to BC, therefore AC also is equal (in length) to BC, and the triangle which the three form is equilateral.

The same may be said of the superposition or application of one figure to another, had recourse to in the fourth proposition; it proceeds upon the assumption that the mind can intuitively perceive the equality of two things: otherwise there ought to be proof that AB and DE, AC and DF, respectively coincide throughout their whole length, just as BC and EF are proved to do. The definition of a straight line, as "that which lies evenly between its extreme points," does not provide for the coincidence of two straight lines in their intermediate points until their extreme points are shown to coincide. The practicability of it, however, is assumed, and we find little difficulty in making the concession. But the strict method of application of the triangle ABC to DEF would be first to place A accurately on D, allowing ABC to lie anywhere in the same plane as DEF, then to wheel ABC round so that B and C should describe circles with centre D (A) and radii AB (= DE), and AC (= DF) respectively. There would then be no doubt that E and F would be points on these circles, so that the application of B to E and C to F would be clearly practicable, and the remaining proof would present no difficulty. This may appear to some persons to be a case of hair-splitting; and it may be regarded as utterly useless to contend for the pre-eminence of one axiom over another, especially when those axioms are so closely related as the two we are considering, viz., "things which are equal to the same thing are equal to one another," and "things which are equal to one another are equal to the same thing." But it cannot be too strongly asserted that the mind starts, and does not merely conclude, with the equality of things, and that the intuitive perception by the mind of equality, and also of inequality, is the grand basis of reason. Now, no doubt, the idea may be regarded as implied in Euclid's axiom, when he says, "Things which are equal to the same thing;" but not only is there nothing to show how this equality is arrived at; on the contrary, the final clause of the axiom might indicate that the equality of things to one another, including of course that of each to the "same thing," was to be relegated back and back in an endless vista, its origin being lost in the dim distance, or in other words, that the mind always needed a third thing for the purpose of comparing any two. It is of importance to establish the fact that reason has its origin in intuitive perception, and this, I trust, has now been done.

The well-known symbols $A=B=C=D=$, &c., fairly represent the axiom that "the equality of equals to other things is coextensive."

There is another faculty of the mind, taken account of in the Differential and the Integral Calculus, and also to some extent in the Doctrine of Probabilities, viz., that faculty which gauges similarity where there is not absolute equality, and is content to ignore infinitesimal inequalities as making no practical difference in the work of life. But, when once these infinitesimal inequalities are discarded, the process is much the same as in the former case. Moreover, as a rule, such inequalities (relatively infinitesimal) can be diminished to any extent, though never entirely eliminated, except by arbitrary deletion.

To sum up, the mind takes cognizance of equality and of inequality, and so far gauges similarity as to accept as practically equal, things between which it perceives or concludes there is in the given respect only an infinitesimal inequality. Moreover, the mind can assume these things or take them for granted, where it cannot directly perceive them. And it proceeds from the equality or inequality of two things which it perceives, to the equality or inequality of these with other things, although it cannot directly compare and contrast them all. And this brings us to the greatest of all the laws of reason, viz., the Law of Substitution :—

Things which are equal in a given respect are in that respect equivalent, and may be substituted the one for the other, when the given respect only is concerned.

Thus, when it is said, "A pound sterling will cover the cost," we may with equal truth substitute for "A pound sterling," twenty shillings, or eight half-crowns.

This law governs most of the operations of Algebra and Geometry, where the given respect in which things are considered is simple and unambiguous, being generally that of number, extent, figure, and angular relation.

It also explains nearly all the "axioms" of Euclid. Thus, to take Axiom I. :—"Things which are equal to the same thing are equal to one another." Let $A = B$, then, if $B = C$, C can be substituted for B in the former equation and $A = C$.

Axiom II. :—"If equals be added to equals, the wholes are equal." $A+B = A+B$ being identical, but let $A = C$, and $B = D$, then by substitution $A+B = C+D$.

Axiom III. :—"If equals be taken from equals, the remainders are equal." $A-B = A-B$ being identical, but let $A = C$, and $B = D$, then $A-B = C-D$.

Axiom IV. :—"If equals be added to unequals, the wholes are unequal." Let $A\pm X = B$, and $C = D$; it is required to find the result of adding A to C and B to D. $B+C = B+C$ being identical, and by substitution $B+C = B+D = A\pm X+C$, so that $B+D$ is just so much greater or less than $A+C$, as B is greater or less than A, and the difference is represented by $\pm X$.

Axiom V. :—"If equals be taken from unequals, the remainders are unequal." Let $A\pm X = B$, and $C = D$; it is required to find the result of taking C from A, and D from B. $B-C = B-C$ being identical, and by substitution $B-C = B-D = A\pm X-C$, so that $B-D$ is just so much greater or less than $A-C$, as B is greater or less than A, and the difference is represented by $\pm X$.

Axiom VI. :—"Things which are double of the same are equal." Let $A = B$, then $A+A = A+A$ being identical, but by substitution

$A+A = B+B$, or $2A = 2B$. The same would hold good for any multiple.

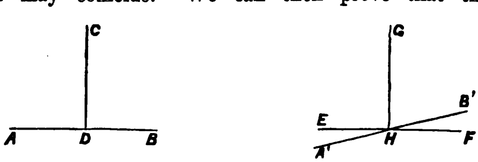
Axiom VII. :—" Things which are halves of the same are equal to one another." This is more difficult to explain than any of the preceding, and is more like a self-evident truth incapable of logical proof. But definition will clear up much. The two halves of a thing are equal to each other, and are together equal to the whole. The doubles of each half are also equal (by Axiom VI.), and severally equal to the whole (by definition). And, of course, the law of substitution would allow us to substitute B for A on one side of the equation $\frac{1}{2}A = \frac{1}{2}A$, provided $A = B$. But the axiom before us involves the equality of the four quarters, and of the eight half-quarters, &c., of any single thing. One quarter might not be equal to another quarter, unless it were half of the same half. But $\frac{1}{2}A + \frac{1}{2}B = \frac{1}{2}A + \frac{1}{2}B$ being identical, and $\frac{1}{2}A + \frac{1}{2}B - B = \frac{1}{2}A + \frac{1}{2}B - A$ (Axiom III.); $\therefore \frac{1}{2}A - \frac{1}{2}B = \frac{1}{2}B - \frac{1}{2}A$, the two sides of which are identical with what would result if we assumed that $\frac{1}{2}A$ and $\frac{1}{2}B$ were equal and might be substituted for each other on one side of the undisputed equation $\frac{1}{2}A - \frac{1}{2}B = \frac{1}{2}A - \frac{1}{2}B$. Moreover, if $m-n$ can equal $n-m$, unless $m = n$, let us suppose that $n = m+x$; then $m-(m+x) = m+x-m$, that is $(m-m)-x = (m-m)+x$. Now no appreciable quantity can be added and subtracted in this way without making an appreciable difference; $\therefore x = 0$, and there is no difference between m and n, or between $\frac{1}{2}A$ and $\frac{1}{2}B$ in the cases instanced.

Axiom VIII. :—" Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another." This cannot be regarded as a distinct axiom, but as an application of Axiom I.; for things which fill the same space must be equal to the same thing. But the recognition of coincidence is of different kinds and degrees. Thus, in Book I., Prop. iv., when the one triangle is applied to the other, the coincidence of the point A with D is a matter, if not of perception, of pure assumption; but the coincidence of B with E is recognized only after an act of the reason; that of the lines AC and DF, and of the points C and F, only after further efforts of the reason; and, finally, that of the lines BC and EF, only after elaborate argument.

Axiom IX. :—" The whole is greater than its part." This again contains more of definition than of distinct axiomatic truth. For what is a *part* of anything? It is that which, if it be deducted from the whole, leaves another portion behind. Again, when is one thing *greater* than another? When it contains more. As Dr. Thomson, in his edition of Euclid, says, " the whole is equal to all its parts taken together "; for, if it be greater, it must contain something not forming a part of itself, and, if it be less, the parts must have increased spontaneously in size or number. Dr. Thomson simply defines a term, the *whole*.

Axiom X. :—" Two straight lines cannot enclose a space." Euclid's definition of a straight line is that of a *finite* straight line, since he says that it " lies evenly between its extreme points "; but in Postulate ii. he says, " Let it be granted, that a terminated straight line may be produced to any length in a straight line," and this gives us the idea of an *infinite* straight line. But do we require more than two infinite straight lines in one plane, each passing through certain given points in that plane, to determine a space? Or again, are we to consider the space between the lines to be unenclosed, when we assume the existence of two infinite straight lines parallel to each other, and do not further limit the figure? If an infernal spirit were a point moving in one plane, and his prison a portion of that plane, bounded by two impassable straight lines, parallel to each other, and extended infinitely in both directions, it would take him a long time to gain his liberty; in fact, he would be practically shut in, not to say, "enclosed": he could never get to the extra-linear portions of the plane. It is evident, however, that Euclid, in the axiom before us, alludes to finite space and finite straight lines. But it is singular that he nowhere defines a "space," although he defines a *figure* as "that which is enclosed by one or more boundaries." We can scarcely regard the two as synonymous, but a limited space, defined by boundaries, is no doubt a figure. The simplest figures, bounded by lines alone, are on a *plane superficies*, and this is defined as "that in which, any two points being taken, the straight line between them lies wholly in that superficies." But it is of importance to observe that a *line* is "length without breadth," that a *superficies* is "that which hath only length and breadth," and that the latter is meant by Euclid when he says that two straight lines cannot enclose a "space." This is evident from the use he makes of the axiom in Prop. iv., where he supposes two straight lines to be drawn between the same two points, and adduces this axiom to prove that the two straight lines must needs coincide. But (though it is difficult to see how *space* can be enclosed on a plane surface) when once we know that the hypothetical "space" referred to has finite length and breadth, we need no distinct axiom to register the fact that two straight lines could not suffice for its boundaries; we simply require a proper definition of *breadth* as distinct from *length*. Let us suppose the length of a plane figure to be represented by a given straight line AB, then its breadth, if represented at all, must be represented by a straight line running in a direction transverse to AB. In this line (CD) an infinite number of points can be taken, to any one of which (E) straight lines can be drawn from A and B respectively; but AE and BE will not be in the same straight line (or AE produced will not reach B) unless E be in AB, and the "space" be thus eliminated. Much more is it clear that, if AC, AD, BC, BD be joined, (which will form the simplest rectilinear boundaries that can be conceived), more than two straight lines will result. Hence two straight lines cannot alone bound a figure. But this is proof, not axiom.

Axiom XI. :—"All right angles are equal to one another." This again is capable of proof. The definition of a right angle is as follows: "When a straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it." Let us then take two sets of such lines and apply one to the other, so that the feet of the perpendiculars and the perpendiculars themselves may coincide. We can then prove that the remain-



ing lines must also coincide, or at least that the angle separating them is *nil*. Thus, let CD, AB and GH, EF be straight lines at right angles with each other. Apply D to H and let DC lie along HG; it is required to prove that AB will lie along EF, and that the angle EHA′ or FHB′ will be *nil*. In the first place, if two straight lines, EF and A′B′, pass through the same point, H, they must either coincide or cut one another. This is as clear as many of the so-called axioms, but admits of proof. Let us on the one hand imagine that A′H can approach EF at an angle, and then coalesce with it along HF, A′HF and EHF, both remaining straight. This supposition is considered in a corollary to Prop. xi., Book I., but its absurdity can be shown at once; for, if we were required to produce the line FH in a backward direction Postulate ii. would be of no avail, unless it were further pointed out along which of the two, HE or HA′, it was to be continued; in fact, the line might proceed in any random direction. Let us next imagine that A′B, instead of cutting EF, can approach it at an angle during the first portion of its course, so as to touch it at H, and then recede from it *whilst still on the same side*. The absurdity of this conception is shown by taking into account, that the second portion of the line would, in receding from EF, reach (if produced) any given distance from it, and that a straight line parallel to EF and cutting A′H would thus meet that second portion, and two straight lines would enclose a space, cutting each other twice. Consequently, if the first portion of A′B′ be inclined at an angle to EF in somewhat the position of A′H, we are reduced to the necessity of regarding the second portion as lying in somewhat the position of HB′, that is, somewhere between the perpendicular GH and HF. But, by hypothesis, GHB′ = GHA′, and GHE = GHF, and, by the law of substitution, we have the following propositions:

$$GHA' > GHE\ (= GHF) > GHB'\ (= GHA').$$

Therefore, if A′B′ be inclined to EF so as to form appreciable angles, EHA′ and FHB′, we are driven to the absurd conclusion that a thing can be greater than itself. Hence we conclude that, applied as stated, AB would coincide with EF; and thus is it proved that all right angles are equal.*

Axiom XII.:—"If a straight line meets two straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles." Dr. Simson, admitting that this is not self-evident, demonstrates the truth of it by the aid of *five* different propositions! But it can be made clear on principles which are acknowledged to be valid at the very commencement of Book I. Superposition will aid us here as elsewhere. Let us suppose two parallel straight lines AB, CD to be placed at any distance apart, and also two sets of straight lines EF, GH,

---

* Lest it should be thought that the corollary to Prop. xi. alone justifies Postulate ii., I beg to state that Prop. xi. itself (from which the corollary is supposed to derive its stability) depends upon previous propositions in which Post. ii. forms an important feature. For Prop. viii., one of the direct assistants of Prop. xi., is dependent on Prop. vii., in which Post. ii. is used, and Prop. vii. is further dependent on Prop. v., in which Post. ii. is used. Moreover, Prop. iii., another of the direct assistants of Prop. xi., is dependent on Prop. ii., in which Post. ii. is used. Now, if Post. ii. depend for its justification on the corollary to Prop. xi., we are reasoning in a circle; and if, on the other hand, Post. ii. be granted on independent grounds, there is no need to postpone the corollary, to Prop. xi.

and IK, LM, in which (Fig. 2) equal angles are formed. Then, if

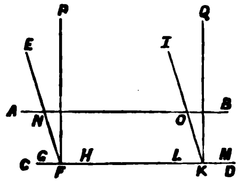


FIG. 2.

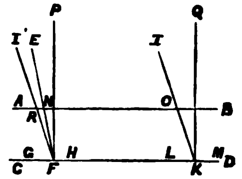


FIG. 3.

GH and LM be made to coincide with CD, as in the figure, ∠ IKC = EFC; and, if K be now made to approach F and coincide with it, then IK will coincide with EF, and both will cut AB in N. Let now IK, LM be withdrawn from this position, along CD, until they regain the position given in the figure. Is it not evident that every part of IK, LM is withdrawn from the corresponding part of EF, GH to an equal distance, and that accordingly ON = KF? And is it not also evident that this would be true, if FE and KI were produced indefinitely, and AB placed at any distance from CD? Such being the case, EF and KI, if produced, would never meet, but always maintain the same distance from each other; they are therefore parallel. Again, let PF, QK be imaginary perpendiculars to CD. Then IKC = EFC = PFC − PFE, and EFD = PFD + PFE; therefore, by addition, EFD + IKC = PFD + PFC, *i.e.*, the "two interior angles" = two right angles.

Let us next consider the case specially referred to in the "axiom," and in which the "two interior angles" EFD, IKC (Fig. 3) are less than two right angles. If we apply IK, LM to EF, GH now, IK will not, as before, coincide with EF, but take up the position indicated by I′F; that is ∠ I′FC will be less than EFC, for I′FC = IKC, and, by the hypothesis, EFD + IKC are less than two right angles, therefore I′FC + EFD are less than PFD + PFC, not greater, as they would have to be to place I′F on the other side of EF. ON, therefore, on AB, is less than OR, which is the distance (measured in a direction parallel to CD) between the parallels I′F, IK; and, as RN must needs widen, the more remote AB is from CD, it will gradually encroach upon the constant OR, until ON is reduced to a point by the meeting of FE and KI produced.

We have thus considered, in detail, the twelve Axioms of Euclid, Book I. The axioms prefixed to some of the succeeding books are all seen to be amenable to the Law of Substitution, without the slightest difficulty.

And, step by step, this Law of Substitution can bring us to the highest pinnacles of mathematical knowledge, all that is requisite being a clear definition and perception of terms, and a careful inspection, lest a step should be taken for which the law gives no warrant.

But when we leave the domain of mathematics, where the data, being of an abstract character, can be accurately defined, and where, assuming that A = B, it is also true that B = A, and when we come down to the Logic of Common Life, where it is well-nigh impossible to declare the absolute equality of any two things in any one respect, where an element of doubt clings to all our data, where in consequence of proceeding from particulars to generals, instead of from generals to particulars, we can rarely arrive at a universal proposition, and where, even granting that "all A is

B," we can only deduce therefrom that "some B is A,"—although the Law of Substitution still holds good, we need additional safeguards against error.

Still, as Whately says, "the rules of Logic have nothing to do with the truth or falsity of the premises, but merely teach us to decide (not whether the premises are *fairly laid down*, but) whether the conclusion *follows fairly from* the premises or not. The *degree of evidence* for any proposition we originally assume as a premiss, is not to be learned from Logic, but is the province of whatever science furnishes the subject-matter of the argument. *E.g.*, from examination of many horned animals, as sheep, cows, &c., a Naturalist finds that they have cloven feet; now *his skill as a Naturalist* is to be shown in judging whether these animals are likely to resemble in the form of their feet all other horned animals; and it is the exercise of this judgment, together with the examination of individuals, that constitutes what is usually meant by the *Inductive Process*, which is that by which we gain *new truths*, and which is not connected with Logic, being not what is strictly called *Reasoning*, but *Investigation*."

It is essential, however, as Whately further says, "that we should abstract that portion of any object presented to the mind which is important to the argument in hand. There are expressions in common use which have a reference to this caution; such as, 'this is a question, not as to the nature of the object, but the magnitude of it,' 'this is a question of time, or of place, &c.'; that is, 'the subject must be referred to this or to that category.' The categories enumerated by Aristotle, are *οὐσία* (being, essence), *πόσον* (how great?), *ποῖον* (of what kind?), *πρός τι* (toward what?), *ποῦ* (where?), *πότε* (when?), *κεῖσθαι* (is lying?), *ἔχειν* (is holding?), *ποιεῖν* (is doing?), *πάσχειν* (is suffering?)—which are usually rendered, as adequately as perhaps they can be in our language, Substance, Quantity, Quality, Relation, Place, Time, Situation, Possession, Action, Suffering, and may all be ultimately referred to the two heads of *Substance* and *Attribute* (or Accident)."

But when care is taken to consider things in one definite respect at a time, so as to compare and contrast properly, much help may be derived from the following method of classification:—

A { B, or A non Z { C, or AB non Y, / Y, or AB non C, } / Z, or A non B }
Z, or A non B } X.
W, or X non Z }

Let A be the most comprehensive class or order of things referred to in the argument, whether as subject or predicate, under one given aspect; and let A comprise under it two genera only, B and Z, which are mutually exclusive; also let B comprise under it two species (or individual and remainder) C and Y, mutually exclusive. Moreover, let Z, which may contain any number of individuals from one upwards, provided they be A non B's, be wholly comprehended under a different order of things, X, so that the thing or things indicated can be referred to as AXZ or XAZ, as most convenient; and let W be the remainder of X. We can now illustrate the nineteen legitimate forms of Syllogism (including twelve different *moods*), under the four divisions (called *figures*) defined by the position of the "middle term."

I. 1. *Barbara.*—All B's are A's; all C's are B's; ∴ all C's are A's.

I. 2. *Celarent.*—No B's are Z's; all Y's are B's; ∴ no Y's are Z's.

I. 3. *Darii.*—All Z's are X's; some A's are Z's; ∴ some A's are X's.

I. 4. *Ferio.*—No Z's are B's; some X's are Z's; ∴ some X's are not B's.

N.B.—It is important here to note that B and W are not mutually exclusive, nor yet co-extensive.

II. 1. *Cesare.*—No Z's are B's; all C's are B's; ∴ no C's are Z's.

II. 2. *Camestres.*—All Y's are B's; no Z's are B's; ∴ no Z's are Y's.

II. 3. *Festino.*—No Y is C; some B's are C; ∴ some B's are not Y's.

II. 4. *Baroko.*—All C's are B's; some A's are not B's; ∴ some A's are not C's.

III. 1. *Darapti.*—All Z's are A's; all Z's are X's; ∴ some X's are A's.

III. 2. *Disamis.*—Some B's are Y; all B's are A's; ∴ some A's are Y.

III. 3. *Datisi.*—All B's are A's; some B's are C's; ∴ some C's are A's.

N.B.—This conclusion is particular where it might have been universal.

III. 4. *Felapton.*—No Z's are W's; all Z's are A's; ∴ some A's are not W's.

N.B.—This does not assert that no W's are A's.

III. 5. *Bokardo.*—Some B's are not Y's; all B's are A's; ∴ some A's are not Y's.

III. 6. *Feriso.*—No B's are Z's; some B's are Y's; ∴ some Y's are not Z's.

N.B.—This again is particular where it might be universal; but the minor proposition might have been (within the truth) "some B's are A's"; and then "some A's are not Z's" could not have been made universal.

IV. 1. *Bramantip.*—All C's are B's; all B's are A's; ∴ some A's are C's.

IV. 2. *Camenes.*—All C's are B's; no B's are Z's; ∴ no Z's are C's.

IV. 3. *Dimaris.*—Some A's are Z's; all Z's are X's; ∴ some X's are A's.

IV. 4. *Felapo.*—No W's are Z's; all Z's are A's; ∴ some A's are not W's.

IV. 5. *Fresison.*—No Z's are B's; some B's are C's; ∴ some C's are not Z's.

N.B.—The same remarks apply here as after III. 6.

Before proceeding further, I may observe that the relation of Z to X is a most important one. Z may be selected from the A's, not because it belongs to the X's, but on other grounds, such as for the purpose of experiment, or even by accident; but having been selected, and possibly subjected to a particular treatment, it proves to belong to a distinct order of things, and the class X is founded. Other A's being similarly examined or tested, may or may not produce the same result, but, in any case, fresh knowledge is acquired, a discovery is made leading to experiment in various directions, and possibly an invention is originated. Thus are Logic and Induction seen to be very closely related, if not identical processes.

But, of course, X will equally serve to represent in other instances the quality, or class of qualities, to which Z's owe their distinction from B's, or any other quality or attribute which Z's possess in addition to that indicated by A.

A few instances of reasoning conducted by the method now outlined, will not be out of place. There is the famous discussion, referred to by Whately, regarding *Prudence*, dating from the time of Aristotle, and continued in later days by Hutcheson and Adam Smith. Hutcheson placed all virtue in benevolence; but Adam Smith replied in terms of the following syllogism:—Prudence has for its object the benefit of the individual; prudence is a virtue; therefore some virtue has for its object the benefit of the individual.

Let X = virtues, and A = things having for their object the benefit of the individual. Then, if it be granted that Prudence is XAZ, or one of the virtues having for its object the benefit of the individual, Adam Smith's conclusion follows, by *Darapti* (III., 1).

But Aristotle wished to prove the virtues inseparable (*Ethics*, Book VI.), and made use of the following singular syllogism, which is employed by Whately to illustrate the first and easiest mood (I., 1):—"He who possesses prudence, possesses all virtue; he who possesses one virtue, must possess prudence; therefore, he who possesses one, possesses all."

Let A = the possessors of at least one virtue, B = the possessors of prudence, and C = the possessors of all the virtues. This is the natural classification of the three terms, and that which, *primâ facie*, we are bound to make; for, until we know that the virtues are inseparable, and not variously distributed, we must perforce assume that persons of whom it may simply be said that they are not without any virtue, will form the more comprehensive class, and those having all the virtues, the more select. But the Stagirite reverses the order of things, and those having all the virtues are made to include those having one. The syllogism is thus made to stand on its head instead of its feet; and the conclusion certainly cannot be maintained, unless, with the natural classification given above, proof can be given that AB non C's and A non B's have no existence. But is it true that there are no prudent persons except those who have all the virtues? Is it true that those who have not prudence, have no virtue? According to Aristotle, the class A is (so far as virtue is concerned) indivisible, so that, if we be convinced that a man has one virtue, we may safely take the remainder of his nature on trust! A strange *prudence* this! It is true, no doubt, that the word has had many meanings. Thus, according to Paley, some philosophers have held that *Benevolence* proposes good ends, *Prudence* suggests the best means of attaining them, while *Fortitude* and *Temperance* complete the list of virtues. But this makes prudence to be mere practical wisdom, a matter of the intellect. "Virtue is distinguished by others," says Paley again, "into two branches only, *prudence* and *benevolence*—prudence, attentive to our own interest; benevolence, to that of our fellow creatures." But if this, agreeing with Adam Smith's definition (given above), be correct, there have been many noble and apparently virtuous deeds, from which the suspicion of prudential motives would have seriously detracted. If it be a virtue to risk life, and to encounter death itself, for the sake of another, in what consists the prudence, unless it be in what Butler calls "the principle of reasonable and *cool* self-love"? Are there no beings who entirely forget themselves in their anxiety for others' good? Or, if such exist, must we take from them their patent of nobility as the supremely virtuous? Human nature, as at present constituted, is far from perfect, and, even in the best men, the virtues are by no means evenly balanced. But the man who errs on the side of Benevolence and Self-sacrifice, is surely to be preferred to one who errs on the side of selfinterest, or even of practical wisdom; for there is such a thing as cultivating prudence to the destruction of the other virtues. Such being the case, it seems monstrous to consider prudence as the essential virtue without which no other can exist. A man who has one virtue may have prudence, but the man (of the A non B's) who is imprudently brave in a good cause is of a much nobler sort. Now, a well-grounded suspicion that even one A is not B, or one AB not C, that is, that a man not devoid of virtue may be imprudent, or a prudent man not wholly virtuous, is sufficient to upset an argument of such extremely unstable equilibrium as that we have considered.

Our next instance shall be taken from Paley's celebrated disquisition on a *moral sense*, its existence in man or otherwise, given in the work on Moral Philosophy (Book I., Chap. 5). He begins by giving the case of Caius Toranius, who betrayed his own father to arrest and death; and, after depicting the deed in all its malignity, he says, "The question is, whether, if this story were related to the wild boy caught some years ago in the woods of Hanover, or to a savage without experience, and without instruction, cut off in his infancy from all intercourse with his species, and, consequently, under no possible influence of example, authority, education, sympathy, or habit; whether such a one would feel, upon the relation, any degree of that sentiment of disapprobation of Toranius' conduct which we feel or not?" And that we may be in no doubt as to what he considers to be the matter in dispute, he further says, "They who maintain the existence of a moral sense; of innate maxims; of a natural conscience; that the love of virtue and the hatred of vice are instinctive, or the perception of right and wrong intuitive (all which are only different ways of expressing the same opinion), affirm that he would. They who deny the existence of a moral sense, &c., affirm that he would not." After saying that "what would be the event can only be judged of from probable reasons," he proceeds in the most lucid language to give the various reasons adduced on either side. Thus, the one party assert that a certain approbation of noble deeds and a corresponding condemnation of vice, are instantaneous and without deliberation; and also uniform and universal. But the other side show that nearly every form of vice has at some time or in some country been countenanced by public opinion, even by philosophers and others in high position; that we ourselves do not perfectly agree as to what is right and what is wrong; and that the general though not universal approval of certain lines of conduct may be accounted for in various ways. For instance, "having experienced at some time, a particular conduct to be beneficial to ourselves, or observed that it would be so, a sentiment of approbation rises up in our minds, which sentiment afterwards accompanies the idea or mention of the same conduct, although the private advantage which first excited it no longer exist." By these means the custom of approving certain actions *commenced:* it is kept up by authority, by imitation, by inculcation, by habit. Besides, say they, none of the so-called *innate maxims* are absolutely and universally true, but all *bend* to circumstances. Thus, veracity, which seems, if any be, a natural duty, is excused in many cases towards an enemy, a thief, or a madman; and so with the obligation to keep a promise. Nothing is so soon made as a maxim: Aristotle laid down, as a fundamental and self-evident maxim, that nature intended barbarians to be slaves. "Upon the whole," says Paley, "it seems to me, either that there exist no such instincts as compose what is called the moral sense, or that they are not now to be distinguished from prejudices and habits; on which account they cannot be depended upon in moral reasoning; that is, it is not a safe way of arguing, to assume certain principles as so many dictates, impulses, and instincts of nature, and then to draw conclusions from these principles, as to the rectitude or wrongness of actions, independent of the tendency of such actions, or of any other considerations whatever"; and he finishes by dismissing the question as of no concern except to the curious.

But a very different complexion is put upon the matter by a careful classification of the chief terms. *Morals* may be divided into our own and other people's, and under both these heads we may place on one side *overt acts*, *habits*, &c., and on the other side, what are summed up under the designation of *motives*—those secret springs of thought and action which may be inferred, but cannot be perceived, by outsiders. These motives act in the higher regions of the being's nature, in those parts which are in immediate relation with the sentient power, and they produce an impression, agreeable or otherwise, according to their harmony or discord with what the being himself accepts as *right*. As the rain-drops descend upon the sides of a mountain, and, percolating through the several strata, reach the central reservoir whence the streams receive their supply, and as the set of the strata determines in great measure the particular side of the mountain on which the spring will appear, so a man's deeds are the resultants of the various influences brought to bear upon him, and, in his reaction upon the outer world, he is able, by his Will, to determine more or less the character of his acts. It is at this juncture that the conscience comes in, its province being to perceive the equality or inequality of a nascent act to the being's accepted standard of right, that is, to the degree of *light* he possesses. If, at the critical moment, temptation prevail, a painful impression is produced, but, if the temptation be withstood and overcome, the result is pleasing. In these respects the moral sense is like the other senses, which perceive equality or inequality in things which concern them, and produce corresponding impressions. But the conscience or moral sense of one man is not concerned with the overt acts, much less the motives, of another man. The overt acts of others may be judged of by the Reason, and, if good, followed, if bad, shunned; but it must not be forgotten that what is good, or at least harmless, for one man, may be extremely blameworthy in another. The rules that suit everybody are broad indeed. Caius Toranius may have been, and probably was, the greatest blackguard imaginable; but to reprobate his conduct will not mend matters for me. The question for *my* conscience is, how far *my* present conduct tallies with *my* present light. Moreover, the moral sense can be blunted and destroyed, or educated and refined, much as any other. This and various circumstances concur to produce at different times, and in different localities, habits and customs which differ greatly on the score of morality. But to deny the existence of a moral sense on this account, is like denying the sense of hearing, because the accepted music of one nation is discord and confusion to another; or like denying the sense of sight, because one man beholds beauty where another sees only so much canvas and paint.

There are cases in which we can so divide any class of things under consideration, as to give the exact or approximate proportion borne by the particular genus or species selected, to the class of which it forms part. Scientific observations and the results of experiments often admit of careful division of this sort; and thus some idea is gained of the relative importance of events. The Method of Classification now advocated is also useful in the Calculation of Chances, especially in cases of a mixed character. The following instance, by way of illustration, is from Boole's "Laws of Thought":—"The chances of two causes A_1 and A_2 are c_1 and c_2 respectively. The chance that, if the cause A_1 present itself, an event E will accompany it, whether as a consequence of the cause A_1 or not, is p_1; and the chance that, if the cause A_2 present itself, the event E will accompany it, whether as a consequence of it or not, is p_2. Moreover, the event E cannot appear in the absence of both the causes A_1 and A_2. Required the chance of the event E." Here we may leave out the things signified, and merely put the proportions; and then, taking unity as the class-total for each cause, under which to put c_1 or c_2, the probability of its occurrence, and $(1-c_1)$ or $(1-c_2)$, the probability of its non-occurrence; and, dividing these probabilities again, according to the question, we obtain the following scheme:—

$$1\left\{\begin{array}{l} (1-c_1) \\ \\ c_1 \ldots\ldots \left\{\begin{array}{l} c_1 p_1 \ldots\ldots \left\{\begin{array}{l} y \\ z \end{array}\right. \\ c_1(1-p_1) \end{array}\right. \end{array}\right. \qquad \left.\begin{array}{r} (1-c_2) \\ \left.\begin{array}{r} \left.\begin{array}{r} x \\ y \end{array}\right\} \ldots\ldots p_2 c_2 \\ (1-p_2)\, c_2 \end{array}\right\} \ldots\ldots c_2 \end{array}\right\} 1$$

in which z = chance that E will occur when A_1 alone is present, x = chance when A_2 alone is present, y = chance when both A_1 and A_2 are present. Let u = total chance $= x + y + z$. Now, in the question there is nothing to show how E is affected by the combination of the two causes. Consequently, y must be regarded as a variable quantity, and the other two as varying more or less with it. If E occurs only when A_1 and A_2 are present separately, then $y = 0$; if one of the causes be inoperative unless the other be present, then either $x = 0$, or $z = 0$, but this cannot be the case with both, unless $c_1 p_1 = c_2 p_2$. In any case, $u + y = c_1 p_1 + c_2 p_2$. If we assume that E is inevitable when both causes are present, then $y = c_1 c_2$, and $u = c_1 p_1 + c_2 p_2 - c_1 c_2$. But if we assume that the occurrence of E, when both causes are present, simply bears to its total occurrence the same proportion that their combined occurrence bears to their total occurrence, then $y : u = c_1 c_2 : c_1 + c_2 - c_1 c_2$ and $u = (c_1 p_1 + c_2 p_2)\left(1 - \frac{c_1 c_2}{c_1 + c_2}\right)$. If we take $c_1 = \cdot 1$, $c_2 = \cdot 2$, $p_1 = \cdot 6$, and $p_2 = \cdot 7$, we shall find that, under the former assumption, $u = \cdot 18$, and, under the latter assumption, $u = \cdot 18'6'$. Professor Boole's equation for finding the *exact* value of the required chance (based apparently upon the assumption that A_1 and A_2 are independent, but that E is very little more probable when both are present) is as follows:—

$$\frac{(u - c_1 p_1)(u - c_2 p_2)}{c_1 p_1 + c_2 p_2 - u} = \frac{\{1 - c_1(1 - p_1) - u\}\{1 - c_2(1 - p_2) - u\}}{1 - u},$$

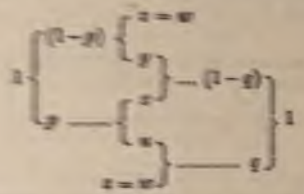
which, translated to suit our present scheme, is as follows:—

$$\frac{xz}{y} = \frac{(1 - c_1 - x)(1 - c_2 - z)}{1 - u},$$

showing that the non-occurrences play a prominent part in its formation, a fact rather difficult to account for. Giving the above numerical values to the terms of Professor Boole's equation, we find that $u = \cdot 19068$. This nearly approaches the result obtained by the following abstract ratio:—

$\therefore u = c_1p_1c_2p_2 - c_1p_1 + c_2p_2 - c_1p_1c_2p_2$, whence $y = c_1p_1c_2p_2$, and $u = \cdot 1916$, in which case the two causes might be regarded as totally unhelpful of each other; for, under the conditions, the chance of E occurring rises the nearer y approaches to zero, and $c_1p_1c_2p_2$ would represent the lowest assignable value of y, unless special circumstances were known to exist rendering the combination of A_1 and A_2 impossible.

Another instance may be taken from the same source (Boole's "Laws of Thought"). "The chance that a witness A speaks the truth is p, the chance that another witness B speaks the truth is q, and the chance that they disagree in a statement is r. What is the chance that, if they agree, their statement is true?" Here taking A and B to restrict their answers to *Yes* or *No*, or to give what is equivalent to an affirmation or denial of a statement submitted to both alike, we can easily produce the following scheme:—



in which u = chance that A and B will agree truthfully, and y = chance that they will agree falsely. Therefore $u+y = 1-r$, and $\frac{u}{u+y} = \frac{u}{1-r}$ = chance required. Now $x+y = 1-q$, $\therefore u-x = q-r$, but $u+x = p$, $\therefore u = \frac{p+q-r}{2}$, and chance required $= \frac{p+q-r}{2(1-r)}$. Of course, if A and B are wholly independent (which is not assumed in the question), $u = pq$, and $y = (1-p)(1-q)$, and the chance required $= \frac{pq}{pq+(1-p)(1-q)}$. If, therefore, we have had an opportunity of gauging p and q, before the present series of questions is put to A and B, we are able to estimate the degree of suspicion which should attach to their present answers on the score of collusion, by noting during the series how much $(1-r)$ exceeds $\{pq+(1-p)(1-q)\}$.

A further instance may be taken from the *Educational Times* for Dec. 1884, Question 7957 (by Rev. T. C. Simmons, M.A.):—"Solve the equations $x^2-yz = a^2$, $y^2-zx = b^2$, $z^2-xy = c^2$." This is simplified as follows:—

$$(x+y+z)^2 \begin{cases} x(x+y+z) \begin{cases} x^2 = x^2 \\ xy = z^2-c^2 \\ xz = y^2-b^2 \end{cases} \\ \\ y(x+y+z) \begin{cases} xy = z^2-c^2 \\ y^2 = y^2 \\ yz = x^2-a^2 \end{cases} \\ \\ z(x+y+z) \begin{cases} xz = y^2-b^2 \\ yz = x^2-a^2 \\ z^2 = z^2 \end{cases} \end{cases}$$

Then $$x^2+y^2+z^2-b^2-c^2 = x(x+y+z),$$
$$x^2+y^2+z^2-a^2-c^2 = y(x+y+z),$$
$$x^2+y^2+z^2-a^2-b^2 = z(x+y+z),$$
and $$x(x+y+z)-a^2 = y(x+y+z)-b^2 = z(x+y+z)-c^2.$$
Let $x+y+z = \mathrm{S}$, and $u = \mathrm{S}x-a^2 = \mathrm{S}y-b^2 = \mathrm{S}z-c^2$; then, by addition, $\mathrm{S}^2-3u = a^2+b^2+c^2$, and $u = \dfrac{\mathrm{S}^2-(a^2+b^2+c^2)}{3}$.

Therefore $$x = \frac{u+a^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - a^2\right),$$
$$y = \frac{u+b^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - b^2\right),$$
$$z = \frac{u+c^2}{\mathrm{S}} = \frac{\mathrm{S}}{3} - \frac{1}{\mathrm{S}}\left(\frac{a^2+b^2+c^2}{3} - c^2\right),$$
and, a^2, b^2, c^2 being constants, S is a variable, and x, y, z vary with it, forming the coordinates of a curve which is not in one plane. But, as S increases, x, y, z approximate to equality, for $y = x - \dfrac{a^2-b^2}{\mathrm{S}}$, and $z = x - \dfrac{a^2-c^2}{\mathrm{S}}$.

In conclusion, I cannot do better than advise those who wish to be carried safely and expeditiously through a long argument, or through certain calculations such as those preceding Multiple Integration, to use Mr. Hugh McColl's System of Notation, which he calls, "The Calculus of Equivalent Statements." It is highly ingenious, yet very simple, and needs only to be gathered under one cover, to be very widely used. At present, its head, trunk, and limbs lie scattered in different publications, of which the chief are as follows:—An article on "Symbolical Reasoning" in *Mind* (No. 17, Jan. 1880); "On the growth and use of a Symbolical Language" in the *Memoirs of the Manchester Literary and Philosophical Society* (1880—81); and four papers which have appeared at various times in the *Proceedings of the London Mathematical Society* (Vol. ix., Nos. 122, 123, read Nov., 1877; No. 135, read June, 1878; Vol. x., Nos. 141, 142; and Vol. xi., No. 163). But one of the earliest papers was published in the *Educational Times* (Vol. xxviii., p. 20). *The London Mathematical Society* has other papers under consideration at the present time from the same hand, and which it is to be hoped will soon be forthcoming.

London: C. F. Hodgson & Son, Printers, Gough Square, Fleet Street, E.C.

$y : u = c_1p_1c_2p_2 : c_1p_1 + c_2p_2 - c_1p_1c_2p_2$, whence $y = c_1p_1c_2p_2$, and $u = \cdot 1916$, in which case the two causes might be regarded as totally unhelpful of each other; for, under the conditions, the chance of E occurring rises the nearer y approaches to zero, and $c_1p_1\,c_2p_2$ would represent the lowest assignable value of y, unless special circumstances were known to exist rendering the combination of A_1 and A_2 impossible.

Another instance may be taken from the same source (Boole's "Laws of Thought"). "The chance that a witness A speaks the truth is p, the chance that another witness B speaks the truth is q, and the chance that they disagree in a statement is r. What is the chance that, if they agree, their statement is true?" Here taking A and B to restrict their answers to *Yes* or *No*, or to give what is equivalent to an affirmation or denial of a statement submitted to both alike, we can easily produce the following scheme:—

$$1\left\{\begin{array}{l}(1-p)\left\{\begin{array}{l} z=w \\ y \end{array}\right. \\[2em] p \;\ldots\ldots\left\{\begin{array}{l} x \\ u \end{array}\right. \\ \qquad\qquad z=w \end{array}\right. \qquad \left.\begin{array}{l}\left.\begin{array}{l} y \\ x \end{array}\right\}\ldots(1-q) \\[2em] \left.\begin{array}{l} u \\ z=w \end{array}\right\}\ldots\ldots\ldots\; q \end{array}\right\}1$$

in which u = chance that A and B will agree truthfully, and y = chance that they will agree falsely. Therefore $u+y = 1-r$, and $\frac{u}{u+y} = \frac{u}{1-r}$ = chance required. Now $x+y = 1-q$, $\therefore u-x = q-r$, but $u+x = p$, $\therefore u = \frac{p+q-r}{2}$, and chance required $= \frac{p+q-r}{2\,(1-r)}$. Of course, if A and B are wholly independent (which is not assumed in the question), $u = pq$, and $y = (1-p)\,(1-q)$, and the chance required $= \frac{pq}{pq+(1-p)\,(1-q)}$. If, therefore, we have had an opportunity of gauging p and q, before the present series of questions is put to A and B, we are able to estimate the degree of suspicion which should attach to their present answers on the score of collusion, by noting during the series how much $(1-r)$ exceeds $\{pq+(1-p)\,(1-q)\}$.

A further instance may be taken from the *Educational Times* for Dec. 1884, Question 7957 (by Rev. T. C. Simmons, M.A.):—"Solve the equations $x^2-yz = a^2$, $y^2-zx = b^2$, $z^2-xy = c^2$." This is simplified as follows:—

$$(x+y+z)^2\left\{\begin{array}{l} x\,(x+y+z)\left\{\begin{array}{l} x^2 = x^2 \\ xy = z^2-c^2 \\ xz = y^2-b^2 \end{array}\right. \\[1em] y\,(x+y+z)\left\{\begin{array}{l} xy = z^2-c^2 \\ y^2 = y^2 \\ yz = x^2-a^2 \end{array}\right. \\[1em] z\,(x+y+z)\left\{\begin{array}{l} xz = y^2-b^2 \\ yz = x^2-a^2 \\ z^2 = z^2 \end{array}\right. \end{array}\right.$$