MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XLVII.

MATHEMATICAL QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND FOUR

APPENDICES.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XLVII.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1887.

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Questions Solved.

1856. (Professor Sylvester, F.R.S.)—Prove that the Jacobian of the time of a planet's describing any arc, the chord of the arc, and the sum of the two extreme distances from the sun, in respect to the eccentricity and two extreme eccentric anomalies, is zero; and hence deduce the time for a planet or comet in terms of the said chord and sum. 163

2391. (Professor Sylvester, F.R.S.) — Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order, and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call the *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it. 137

2810. (Professor Sylvester, F.R.S.)—Let S_i be used as the symbol of the sum of i-ary products; required to prove that, if i is $< n$,

$$\Sigma S_i(a_2, a_3, \ldots, a_n)\frac{(a_1-\alpha_1)(a_1-\alpha_2)\ldots(a_1-\alpha_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}$$
$$= S_{i+1}(a_1, a_2, \ldots, a_n) - S_{i+1}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

[For example, let $n = 3$, $i = 2$, then the theorem becomes

$$bc\frac{(a-\alpha)(a-\beta)(a-\gamma)}{(a-b)(a-c)} + ca\frac{(b-\alpha)(b-\beta)(b-\gamma)}{(b-a)(b-c)} + ab\frac{(c-\alpha)(c-\beta)(c-\gamma)}{(c-a)(c-b)}$$

$= abc - \alpha\beta\gamma$, which is obviously true.] 21

2832. (Professor Sylvester, F.R.S.)—Prove that the curve of intersection of two right cones with parallel axes is a spherical curve, and that a third right cone may be drawn through it. Prove also that a plane circular cubic may be found such that the distances of *any* two fixed points on it from *every* point in the curve of double curvature above-mentioned shall be in a fixed linear relation. 90

2866. (Professor Sylvester, F.R.S.)—Given the simultaneous equations $v_{x+1} = v_x + (x^2-x)v_{x-1}$, $u_{x+1} = u_x + (x^2-x)u_{x-1} + (2x+1)v_{x+1}$, prove that the general solution is of the form $v_x = \lambda\alpha_x + \mu\beta_x$,

$$u_x = \tfrac{1}{2}\lambda\left\{x^2+2x+\tfrac{1}{2}(-)^x\right\}\alpha_x + \tfrac{1}{2}\lambda\left\{x^2+2x-\tfrac{1}{2}(-)^x\right\}\beta_x + \nu\alpha_x + \pi\beta_x,$$

where λ, μ, ν, π are arbitrary constants, and determine the values of α_x, β_x. 37

2934. (Professor Sylvester, F.R.S.)—If $s_1, s_2, s_3, s_4, s_5, s_6$ represent the sum of *xyztuv*, and of their binary, ternary, quaternary, quinary, and sextic combinations respectively, and if E stands for the symbol of Emanation $a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz} + d\frac{d}{dt} + e\frac{d}{du} + f\frac{d}{dv}$, prove that the resultant of s_1, Es_2, s_3, Es_4, s_5, Es_6 is the product of

$$(a-b)^9(a+b-c-d)^3(a+b+c-d-e-f)^{12},$$

and of the similarly formed powers of products of the analogous linear functions of *abcdef*.. 101

2935. (Professor Sylvester, F.R.S.)—If $f(x)$ be a rational integral function of x of a higher order than the second, prove that it is impossible for $(fx)^2 + (f'x)^2$ to be a perfect square unless $f(x)$ contains at least two groups of equal factors.. 85

3427. (Professor Sylvester, F.R.S.)—If ϕ, ψ are quantics in x, y, each of degree μ; F a quantic in ϕ, ψ of degree m, and consequently in x, y of degree $m\mu$; and if J denote the Jacobian of ϕ, ψ, that is,

$$\frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx};$$

$D_{\phi,\psi}F$ the discriminant of F treated as a quantic in ϕ, ψ; $D_{x,y}F$ the discriminant of F, treated as a quantic in x, y; and if R be used as the symbol of "resultant in regard of x, y"; prove that

$$D_{x,y}F = 2[R(\phi,\psi)]^{m^2-2m}R(F, J)(D_{\phi,\psi}F)^{\mu}.$$

As a particular case of the foregoing theorem, show that the discriminant of F, any *symmetrical* quantic of an even degree in x, y, is of the form $F(1, 1)\,F(1, -1)\,\Omega^2$, where Ω is a rational integral function of the coefficients in F.

Find also what the general formula becomes when ϕ, ψ are taken linear functions of x, y. 140

3535. (Professor Sylvester, F.R.S.)—1. EULER has shown that the number of modes of composing n with i distinct numbers is equal to the denumerant (that is, the number of solutions in positive integers, zeros included) of the equation

$$x+2y+3z+\ \dots\ +i\omega = n-\tfrac{1}{2}(i^2+i).$$

(1) Show more generally that the number of modes of composing n with i numbers, all distinct except the largest, which is to be always taken j times, is the denumerant of the equation

$$jx+(j+1)y+\ \dots\ +i\omega = n-\tfrac{1}{2}(i-j+1)(i+j).$$

(2) Show also that, if all the partitions of n into i parts are distinct except the *least*, which is to be taken j times, then the number of such partitions is the denumerant of the equation

$$x+2y+\ \dots\ +(i-j)\phi+i\omega = n-\tfrac{1}{2}\left\{(i-j)^2+3i-j\right\}.$$ 138

3651. (Professor Sylvester, F.R.S.)—If through $3n+1$ given points on a cubic curve a curve of the order $N+n$ be drawn, and through the remaining $3N-1$ intersections of the two curves a third one be drawn of the order N; prove that this will intersect the cubic at a fixed point.

[This point may be called the opposite of the $3n+1$ given points; it becomes Dr. SALMON's opposite, as defined by him in the *Philosophical Transactions* for 1858, when $n=1$, $N=1$, and is independent of the value of N.] 137

5271. (Professor Cayley, F.R.S.)—If ω be an imaginary cube root of unity, show that, if

$$y=\frac{(\omega-\omega^2)x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)x^2},\ \text{then}\ \frac{dy}{(1-y^2)^{\frac{1}{2}}(1+\omega y^2)^{\frac{1}{2}}}=\frac{(\omega-\omega^2)\,dx}{(1-x^2)^{\frac{1}{2}}(1+\omega x^2)^{\frac{1}{2}}};$$

and explain the general theory. 141

5305. (Professor Sylvester, F.R.S.) — *Definition.* — Six right lines along which six forces can be made to equilibrate are said to be in Statical Involution.

Prove that any six right lines lying on a ruled cubic surface are in statical involution; and, *vice versâ*, if six right lines are in statical involution, a ruled cubic surface can be made to pass through them... 165

5420. (Professor Sylvester, F.R.S.) — From the expansion of $\{\log(1+x)\}^n$, in a series according to powers of x, prove that $S_{i,j}$ [the coefficient of t^j is the developed product of $(1+t)(1+2t)\ \dots\dots\ (1+it)$] is divisible by every prime number greater than $j+1$ contained in any term in the series $i+1$, i, $i-1$, ..., $i-j+1$. 144

5640. (W. J. Curran Sharp, M.A.) — SALMON says [*Higher Plane Curves*, p. 98] that the curve parallel to a given curve may be obtained (1) as the envelope of a circle of given radius whose centre moves on the given curve; (2) as the envelope of the parallel to the tangent to the given curve drawn at a constant distance. Prove that these processes are equivalent. 146

5643. (W. S. B. Woolhouse, F.R.A.S.) — Any two triangles being given, the first may always be orthogonally projected into a triangle similar to the second; determine the magnitude of the projected triangle geometrically by an easy construction with the ruler and compasses... 26

5755. (W. J. C. Sharp, M.A.) — If, through any point of inflection O on an n-ic, there be drawn three straight lines meeting the curve in $A_1, A_2 \dots A_{n-1}$; $B_1, B_2 \dots B_{n-1}$; $C_1, C_2 \dots C_{n-1}$, respectively; prove that every curve of the n^{th} degree through the $3n-2$ points O, $A_1, A_2 \dots A_{n-1}$; $B_1 \dots B_{n-1}$; $C_1 \dots C_{n-1}$ will have O for a point of inflexion. 146

5809. (D. Edwardes.)—If O be any point within a triangle ABC, prove that $OA^2 \sin 2A + OB^2 \sin 2B + OC^2 \sin 2C$ is least when O is the centre of the circumscribed circle. 146

5828. (Professor Darboux.) — On coupe une pyramide triangulaire SABC par un plan parallèle à la base; ce plan rencontre les arètes latérales SA, SB, SC en A′, B′, C′; on mène ensuite les plans CA′B′, AB′C′, BC′A′. Soit P leur point commun. Déterminer le lieu décrit par le point P lorsque le plan A′B′C′ se déplace en demeurant parallèle à la base. 148

5955. (Professor Sylvester, F.R.S.)—By a Cartesian Oval in space let us understand a curve the distances of whose points from three fixed points are linear functions of each other, or, which comes to the same thing, is the intersection of two surfaces of revolution, described by two plane Cartesians having a focus in common. Conversely, when two points can be found whose distances from any point in a space-Cartesian are linear functions of one another, let them be termed foci. Required to prove, that the locus of such foci is a plane *curve* of the 3rd degree. It will be observed that this curve for Cartesians of double curvature is the exact analogue of the three foci in a straight line for plane Cartesians... 69

6686. (W. J. C. Sharp, M.A.)—If a bar naturally curved be strained, the bending moment at a point, whose natural curvature is r^{-1} and strained curvature ρ^{-1}, is $E(\rho^{-1}-r^{-1})\{I+Ar^{-2}+Br^{-4}+x\}$.......... 148

6827. (Prof. Cayley, F.R.S.)—Consider a triangle ABC, and a point P; and let AP meet BC in M, and BP meet AC in N (if, to fix the ideas, P is within the triangle, then M, N are in the sides BC, AC, respectively, and the triangles APN, BPM are regarded as positive); find (1) the locus of the point P, such that the ratio $(\triangle APN + \triangle BPM) : \triangle ABC$ may have a given value; (2) drawing from each point P, at right angles to the plane of the triangle, an ordinate PQ of a length proportional to the foregoing ratio $(\triangle APN + \triangle BPM) : \triangle ABC$, trace the surface which is the locus of the point Q, a surface which has the loci in (1) for its contour lines; (3) find the volume of the portion standing on the triangle ABC as base; and (4) deduce the solution of the following case of the four-point problem, viz., taking the points P, P′ at random within the triangle ABC, what is the chance that A, B, P, P′ may form a convex quadrangle?... 149

7050. (D. Edwardes.)—If P be any point in the plane of a triangle ABC, and d its distance from the circumscribed centre, show that

$PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C = 4(R^2+d^2) \sin A \sin B \sin C$... 147

7069. (D. Edwardes.)—If x, y, z be the distances of a point P from the angular points of a triangle, prove that the mean value of $x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$, as P ranges over the circle about ABC, is three times the area of the triangle. 147

7169. (W. J. Greenstreet, B.A.)—The sum of the three sides of a right-angled spherical triangle is a quadrant: prove that (1) the minimum value of the hypotenuse is $\cos^{-1}\frac{4}{5}$, and (2) in this case the spherical excess is $\sin^{-1}\frac{1}{5}$.. 51

7270. (W. J. C. Sharp, M.A.)—Eliminate a, b, c, a', b', c' from

$$\begin{array}{r|r|r} a+b+c=d & a'+b'+c'=d' & aa'+bb'+cc'=\epsilon', \\ ab+bc+ca=s & a'b'+b'c'+c'a'=s' & a'bc+ab'c+abc'=\theta, \\ abc\quad=\delta & a'b'c'\quad=\delta' & ab'c'+a'bc'+a'b'c=\theta' \end{array}$$ 152

7301. (Professor Cayley, F.R.S.) — If the function $\dfrac{au+\beta}{\gamma u+\delta}=\phi u$ is periodic of the third order ($\phi^3 u = u$): given that the cubic equation $(a, b, c, d\between x, 1)^3 = 0$ has two roots u, v such that $v=\dfrac{au+\beta}{\gamma u+\delta}$, find u as a rational function of $a, b, c, d, \alpha, \beta, \gamma, \delta$; and examine the case in which u is not thus expressible. 153

7356. (Professor Wolstenholme, M.A., Sc.D. Suggested by Question 7285.)—At each point P of a given curve is drawn a straight line U, making a given angle with the tangent at P, and a straight line V, such that U, V are equally inclined to the ordinate at P; prove that the point of contact of U with its envelope is the projection upon U of the centre of curvature at P, and that the point of contact of V with its envelope is the projection upon V of the image with respect to the tangent at P of the centre of curvature at P. [That is, if O be the centre of curvature at P, and OPO′ be a straight line bisected in P, then, if OL, O′M be let fall perpendicular to U, V respectively, L, M will be the points of contact of U, V with their envelopes.] 64

7459. (Professor Wolstenholme, M.A., Sc.D.) — Prove that if, in a tetrahedron, any one or two of the equations $a \pm x = b \pm y = c \pm z$ be true, then will also the corresponding one or two equations of the set $A \pm X = B \pm Y = C \pm Z$ also be true.. 86

7661. (Rev. T. C. Simmons, M.A.)—If

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0,$$

show that $\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)$ cannot be greater than $\frac{1}{8}$. 124

7707. (Professor Malet, F.R.S.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\frac{\log\sin\theta\left\{\log\tan\theta-\log\Delta(\theta)\right\}}{\Delta(\theta)}\,d\theta=\tfrac{1}{2}K\left\{\log k\log k'+\tfrac{1}{4}\pi^2\right\}+\tfrac{1}{4}\pi K'\log k',$$

$$\int_0^{\frac{1}{2}\pi}\frac{\log\cos\theta\left\{\log\cot\theta-\log\Delta(\theta)\right\}}{\Delta(\theta)}\,d\theta=\tfrac{1}{8}\pi^2K,$$

$$\int_0^{\frac{1}{2}\pi}\frac{\log\Delta(\theta)\left\{\log\Delta(\theta)-\log\sin\theta\cos\theta\right\}}{\Delta(\theta)}\,d\theta=\tfrac{1}{2}K\log k\log k'+\tfrac{1}{4}\pi K'\log k',$$

where $\Delta(\theta)\equiv(1-k^2\sin^2\theta)^{\frac{1}{2}}$, $k^2+k'^2=1$, and K and K′ are complete elliptic integrals of the first kind with moduli respectively k and k'... 68

7987. (Rev. T. R. Terry, M.A.)—Four spheres, whose radii are a, b, c, d respectively, are such that each touches the other three externally. In the space between these four, another sphere of radius r is described touching all four externally. Show that

$$\frac{1}{r^2}-\frac{1}{r}\Sigma\left(\frac{1}{a}\right)+\Sigma\left(\frac{1}{a^2}\right)-\Sigma\left(\frac{1}{ab}\right)=0$$ ....(46).

8036. (R. Lachlan, B.A.)—If four spheres; radii a, b, c, d, touch one another externally; and r be the radius of the sphere which cuts them orthogonally, then $\frac{4}{r^2} = 2\Sigma\left(\frac{1}{ab}\right) - \Sigma\left(\frac{1}{a^2}\right)$ 46

8117. (Professor Wolstenholme, M.A., Sc.D.)—Two coincoids S, S′ have two common plane sections, and the poles of these planes with respect to S are the points P, P′; prove that (1) if S′ pass through P, it will also pass through P′; (2) also, in this case, the following relations must hold $(\Theta^2 - 3\Phi\Delta)(\Theta\Theta' - 6\Delta\Delta') + \Theta^2\Delta\Delta' = 0$, $\Theta^4 = 27\Delta^2(\Theta\Theta' - 6\Delta\Delta')$. The discriminant of $kS + S'$ is $\Delta k^4 + \Theta k^3 + \Phi k^2 + \Theta' k + \Delta'$.] 79

8135. (Rev. T. C. Simmons, M.A.)—If G be the centroid, I the in-entre, of a plane triangle, prove that

$$IG^2 = \tfrac{4}{9}R^2(1 + \cos A \cos B \cos C) - \tfrac{4}{3}Rr + \tfrac{4}{3}r^2.$$ 85

8204. (A. Gordon.)—If A, B, C, D, E are five points on a sphere, V_{12} the volume ACDE, V_{13} the volume ABDE, &c.: prove that, with a certain convention as to sign,

$$V_{12}(AB)^2 + V_{13}(AC)^2 + V_{14}(AD)^2 + V_{15}(AE)^2 = 0.$$ 155

8207. (W. J. C. Sharp, M.A.)—If A be the angle contained in the half of a small Circle of a sphere, angular radius a, by arcs of lengths b and c; *i.e.*, if ABC be a spherical triangle having the angle $A = B + C$; show that (1) $\cos b + \cos c = 1 + \cos a$; (2) $\cos A = -\tan\frac{1}{2}b \tan\frac{1}{2}c$; and (3) hence deduce Euclid I. 47 and III. 31. 62

8236. (Rev. T. R. Terry, M.A.)—A uniform lamina bounded by the arc of a parabola and its *latus rectum* is revolving with angular velocity ω about the *latus rectum*: suddenly the *latus rectum* becomes free and the vertex becomes fixed. Show that the angular velocity about the tangent to the vertex is $\frac{5}{7}\omega$.. 116

8237. (W. J. C. Sharp, M.A.)—If ${}_nC_r$ denote the number of combinations r together which can be formed out of n things; show, from *à priori* considerations, that

$${}_nC_r = {}_nC_{n-r}, \quad {}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1}, \quad {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1} \;.....\; (1, 2, 3),$$

or, more generally, ${}_nC_r = {}_{n-p}C_r + {}_pC_1\,{}_{n-p}C_{r-1} + {}_pC_2\,{}_{n-p}C_{r-2} + \&c.$ ... 156

8242. (Professor Sylvester, F.R.S.)—If by a Simplicissimum of the nth order be understood a figure in space of n dimensions formed by the indefinite protraction of the series of which a linear segment, a triangle, and a pyramid are the three first terms, prove that (1), when each edge is unity, the squared content is $\frac{(n+1)}{2^n(1.2.3...n)^2}$, and hence (2) deduce that the Cayleyan Persymmetrical Invertebrate Determinant of the squared edges by which such squared content is imaged must be diminished in the ratio of negative unity to $(-2)^n(1.2.3...n)^2$, in order that it may represent its absolute value. *Ex. gr.*, the determinant

$$\begin{vmatrix} . & (ab)^2 & (ac)^2 & 1 \\ (ba)^2 & . & (bc)^2 & 1 \\ (ca)^2 & (cb)^2 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

images the squared content of a triangle whose edges are (*ab*), (*ac*), (*bc*), for the triangle vanishes when this determinant vanishes, but the actual value of the squared content is this determinant diminished in the ratio of negative unity to $2^2(1 \cdot 2)^2$, *i.e.*, multiplied by $-\frac{1}{16} = (2)$. 53

8298. (R. Knowles, B.A.)—If p_r denote the coefficient of x^r, in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$1-\frac{p_1}{2^2}+\frac{p_2}{3^2}-\ldots+\frac{(-1)^n p_n}{(n+1)^n}=\frac{1}{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right).$$

.... 119

8302. (A. Gordon.)—Prove that the surface $x^2+y^2+z^2=c^2$ will represent a surface of revolution if $\cos^2\widehat{xy}=\cos^2\widehat{yz}=\cos^2\widehat{zx}$, where $\widehat{xy}$ denotes the angle between the axes of x and y. 117

8311. (Prof. R. Swaminatha Aiyar, B.A.)—Find how many numbers can be formed, having n for the sum of the digits (zero not being used as a digit). 64

8336. (Asparagus.)—From a point O are drawn two chords OPP′, OQQ′ of a given circle, and two tangents OA, OB; a conic is drawn touching the straight lines PQ, PQ′, P′Q, P′Q′ and passing through A or B; prove that this conic will touch the circle in A or B. 117

8380. (P. C. Ward, M.A.)—Prove that (1),

$$\begin{vmatrix} a+b-c, & 4a, & 6a, & 4a \\ 4b, & a+b-c, & 4a, & 6a \\ 6b, & 4b, & a+b-c, & 4a \\ 4b, & 6b, & 4b, & a+b-c \end{vmatrix}$$

$$\equiv (a+b+c)^4-8(a+b+c)^2(bc+ca+ab)-128abc(a+b+c)+16(bc+ca+ab)^2$$

= result of rationalizing $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}=0$; and hence (2) the above determinant is symmetrical with respect to a, b, c. 107

8385. (J. Brill, M.A.)—Three parabolas are drawn having a common focus; from a point T, external to all three, tangents TP and Tp are drawn to the first parabola, TQ and Tq to the second, and TR and Tr to the third; prove that Qr . Rp . Pq = qR . rP . pQ. 95

8405. (F. C. Wace, M.A.)—If $a_1+a_2+a_3+\ldots+a_n=s$, prove that

$$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a}-1\right)^{a_2}\ldots\ldots\left(\frac{s}{a_n}-1\right)^{a_n}<(n-1)^s. \quad \ldots\ldots \quad 66$$

8420. (Emily Perrin, B.Sc.)—Prove that the axis of perspective of a triangle and its pedal triangle is the common radical axis of the circum-circle, nine-point circle, and self-conjugate circle. 67

8423. (D. Edwardes.)—Prove that

$$(1)\ \int_0^{\frac{1}{2}\pi}\sin x(\log\sin x)^2\,dx=2+(\log 2)^2-2\log 2-\tfrac{1}{12}\pi^2,$$

$$(2)\ \int_0^{\frac{1}{2}\pi}\sin^2 x(\log\sin x)^2\,dx=\tfrac{1}{8}\pi\left\{2(\log 2)^2-2\log 2+\tfrac{1}{8}\pi^2-1\right\}. \ \ldots\ 119$$

8452. (Satis Chandra Rây, B.A.)—Two rods, of lengths a and b, are jointed by a smooth hinge and rest on the convex side of a parabolic arc whose axis is vertical; if, in the position of equilibrium, the rods include a right angle, show that the angle θ which the chord of contact makes with the axis is given by $\tan \frac{1}{2}\theta = a^2/b^2$. 52

8473. (Professor Booth, M.A.) — If a, b, c be the sides of a plane triangle, prove that the diameter of the circumscribing circle is a root of the equation

$$a\left[(x^2-b^2)(x^2-c^2)b\right]^{\frac{1}{2}}+\left[(x^2-c^2)(x^2-a^2)\right]^{\frac{1}{2}}+c\left[(x^2-a^2)(x^2-b^2)\right]^{\frac{1}{2}}=abc.$$
.... 89

8498. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—The equation

$$\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dt^2}-2\frac{dy}{dx}\cdot\frac{dy}{dt}\cdot\frac{d^2y}{dx\,dt}+\left(\frac{dy}{dt}\right)\frac{d^2y}{dx^2}=Q\cdot\left(\frac{dy}{dx}\right)^3$$

is integrable when (1) $Q=0$, (2) $Q = dx/dt$. Hence, obtain the complete primitive.. 61

8501. (R. Knowles, B.A.)—In any triangle, show that
$4\cos^2 A\cos^2 B-\cos(A-B)(3\cos A\cos B-\sin A\sin B)=\cos^2 C$... 75

8502. (B. Hanumanta Rau, B.A.)—Through any point K are drawn the straight lines B″KC′, C″KA′, and A″KB′ respectively parallel to the sides BC, CA, AB of a triangle; prove that (1) $\triangle A'B'C' = \triangle A''B''C''$; (2) parallels through A, B, C to A′B′, B′C′, C′A′ meet at a point O′; (3) parallels through A, B, C to A″C″, A″B″, C″B″ also meet at a point O′; (4) if the coordinates of O and O′ are (α, β, γ) and $(\alpha', \beta', \gamma')$ then $a\alpha : c\gamma' = b\beta : a\alpha' = c\gamma : b\beta'$; (5) if K is the Symmedian point, O and O′ become the Brocard points. 49

8521. (Professor Wolstenholme, M.A., Sc.D.)—The circle of curvature is drawn at a point P $(am^2, 2am)$ of the parabola $y^2 = 4ax$, RR′ is the common tangent of the parabola and circle, and meets PQ their common chord in T; prove that

$$(1)\ TR : TR' = 1+4m^2 : 1;\quad (2)\ RR' = 16am^3(1+m^2)^{\frac{3}{2}} = PQ^3/32a^2;$$
$$(3)\ TQ : TR' = TR' : TP = m^2 : 1+m^2;$$

and (4) the locus of T is a sextic having no real rectilinear asymptotes. 114

8537. (For Enunciation see Question 7169.) 51

8543. (R. Curtis, M.A.)—Prove (1) the following formula of transformation for the equation of a conic to a triangle of reference the sides of which are the polars of the vertices of the former triangle $(a, b, c, f, g, h)(x, y, z)^2 = \Delta^{-1}(A, B, C, F, G, H)(X, Y, Z)^2$; and hence (2) show that the equations of a conic referred to an inscribed triangle and to one circumscribed at the points of contact will be

$$fyz+gzx+hxy=0 \text{ and } (fX)^{\frac{1}{2}}\pm(gY)^{\frac{1}{2}}\pm(hZ)^{\frac{1}{2}}=0.$$ 111

8548. (Asparagus.)—Two straight lines turn about two fixed points O, O′ with angular velocities which are as 1 : 3, both straight lines coinciding with OO′ initially; if P be their point of intersection, prove that the envelope of a straight line drawn through P at right angles to OP will be a parabola whose directrix passes through O′ and is at right angles to OO′ and whose latus rectum is twice OO′............................... 65

8557. (Professor Mathews, M.A.)—If

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$$

are two confocal ellipses, such that polygons of r sides can be simultaneously inscribed in the second ellipse and circumscribed to the first, prove that $a = a' \operatorname{sn}\frac{(r-2)K}{r}$, $b = b' \operatorname{cn}\frac{2K}{r}$, where the modulus of the elliptic functions is equal to the eccentricity of the inner ellipse. Verify the above when $r = 3, 4, 5$ respectively.... 66

8559. (Professor Wolstenholme, M.A., Sc.D.)—From a fixed point O are drawn OP, OQ tangents to one of a system of confocal conics (foci S, S′, centre C), and from C are let fall perpendiculars on the normals at P, Q; prove that the envelope of the straight line joining the feet of these perpendiculars is the conic (parabola)

$$\{X(x\cos\alpha+y\sin\alpha)+Y(x\sin\alpha-y\cos\alpha)-c^2\cos\alpha\}^2 = 4XY(x\cos\alpha+y\sin\alpha)(x\sin\alpha-y\cos\alpha),$$

where C is origin, CS axis of x, (X, Y) the point O, SS′ = $2c$, and α is the sum of the angles which SO, S′O make with the axis of x.

[If $X^2-Y^2=c^2$, the straight line is fixed.] 128

8571. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.) — Show that the reciprocal polar of the evolute of the reciprocal polar of the evolute to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with respect to the circle described on the line joining the foci as diameter, is the curve

$$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}\left\{\left(\frac{x}{y}\right)^{\frac{1}{3}}\left(\frac{x}{b}\right)^{\frac{2}{3}}-\left(\frac{y}{x}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{2}{3}}\right\}^2=\left(\frac{a}{b}-\frac{b}{a}\right)^2 \text{.. 63}$$

8585. (R. F. Davis, M.A.)—If TP, TQ be tangents to a parabola, and PQ meet the directrix in Z, prove that ZT will be a mean proportional between ZP and ZQ.. 50

8600. (Professor Mahendra Nath Ray, M.A., LL.B.)—Through an indefinite point of a given hyperbola straight lines are drawn to meet the asymptotes; show that the hyperbola itself is the envelope of the locus of the middle points of the straight line. 50

8607. (R. Curtis, M.A.) — Between two curves, $f(x, y) = 0$ and $\phi(x, y) = 0$, of degrees m and n, show that there can be drawn mn straight lines of a given length l, parallel to a given straight line, from the curve $f(xy) = 0$ to the curve $\phi(xy) = 0$, and mn others from $\phi(xy) = 0$ to $f(xy) = 0$. [The lines must be drawn in a given direction, say from left to right.] 66

8623. (N'Importe.)—From the centre of similitude S, common radii vectores SPP′, SQQ′ are drawn to similar curves PQ, P′Q′. Having given the centre of gravity of the area SPQ, find that of PQQ′P′; and its limiting position when PQ, P′Q′ ultimately coincide. 50

8631. (Professor Sylvester, F.R.S.)—Find the discriminant of $x^3+y^3+z^3+3e\Sigma x^2y+6exyz$. 118, 156

8637. (Professor Hudson, M.A.) — The tangent and normal to $y/c=\frac{1}{2}(\epsilon^{x/c}+\epsilon^{-x/c})$ cut the axis of x in T, G, respectively; find the minimum value of GT. 161

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xu

8639. (Professor Wolstenholme, M.A., Sc.D.) — The distances of a point O (within a given triangle ABC) from the angular points are x, y, z respectively; prove that the volume of a tetrahedron in which a, b, c are the lengths of three conterminous edges, and x, y, z the lengths of the edges respectively opposite, is $\frac{1}{12}\left\{(a^2-x^2)(b^2-y^2)(c^2-z^2)\right\}^{\frac{1}{2}}$... 30

8643. (Professor Genese, M.A.)—Four complanar forces P_1, P_2, &c. are in equilibrium. Their lines of action (omitting one at a time) determine four triangles whose areas are Δ_1, Δ_2, &c., and circum-radii R_1, R_2, &c. Prove that $P_1 : P_2$, &c. $= \frac{\Delta_1}{R_1} : \frac{\Delta_2}{R_2}$, &c. 35

8646. (Emile Vigarié.)—On donne deux points A et B sur une circonférence de centre O; trouver sur cette circonférence un troisième point P, tel que les droites PA, PB coupent un diamètre fixe CC′ en des points M, N, tels que l'on ait OM = ON.. 36

8647. (R. W. D. Christie, M.A.)—Prove that $\Sigma + S = 2s^3$, if $s = 1^3+2^3+\ldots+n^3$, $S = 1^5+2^5+\ldots n^5$, $\Sigma = 1^7+2^7+\ldots n^7$. ... 33

8650 (E. M. Davys, M.A.)—Prove that, in any plane triangle,
$\cos^{12} A + 4\cos^{10} A \sin^2 A + 5\cos^8 A \sin^4 A - 5\cos^4 A \sin^8 A - 4\cos^2 A \sin^{10} A - \sin^{12} A = \cos 2A$.............................. 45

8656. (W. J. Greenstreet, B.A.)—In a spherical triangle, prove that $\sin^4 R = (\cos a - \cos^2 R)^2 + \sin^2 a \sin^2 R \sin^2 (S-A)$.......... 118

8658. (W. J. C. Sharp, M.A.)—BOOLE obtains the result

$$\iint \ldots X dx^n \equiv \frac{1}{1.2\ldots n-1}\left\{x^{n-1}\int X dx - (n-1)\,x^{n-2}\int X x\,dx + \frac{(n-1)(n-2)}{1.2}\,x^{n-3}\int X x^2 dx - \ldots \mp \int X x^{n-1} dx\right\}$$

by a symbolical method; show that it may also be obtained by ordinary methods. 71

8660. (B. Hanumanta Rau, M.A.) — Investigate the singular solution of the equation $4\left(x+y\frac{dy}{dx}\right)^2+25\left(y^2-xy\frac{dy}{dx}\right)=0$, and show that it is the envelope of a series of circles described on the subnormal of an ellipse as diameter.. 83

8661. (A. Gordon.)—Any curve of the 4th degree intersects a conic in 8 points. These are joined, forming an octagon $A_1A_2\ldots A_8$. Show that the 8 intersections of A_1 with A_4 and A_6, A_3 with A_6 and A_8, &c., &c., will lie on a conic.. 118

8671. (D. Edwardes.) — Prove that the *latus rectum* of a parabola is half the harmonic mean between any two focal chords at right angles to one another. 123

8676. (Professor Bordage.)—The three sides of a triangle forming an arithmetical progression, (a) being the shortest, (a') the longest; if the distance of the centres of the inscribed and circumscribed circles is designated by i, and the diameter of the nine-point circle by D, prove that $aa' = 3(D^2 - i^2)$. 107

8678. (Professor Genese, M.A.)—AB is a chord of a conic, BP, BQ are parallels to the asymptotes, APQ a variable transversal meeting the curve at R. Prove that the ratio PR : RQ is constant.................... 45

8681. (Professor Hudson, M.A.)—In the epicycloid of two cusps, if P be the describing point when Q is the point of contact, prove that the tangents at P, Q to the epicycloid and the fixed circle respectively meet on the line joining the cusps... 46

8683. (Professor Ignacio Beyens, M.A.)—Si dans un triangle la projection du côté BC sur BA qui est BH, et la projection du BA sur AC qui est AH′, sont egales; la bissectrice du A, la hauteur du C et la médiane de B se rencontrent en le même point. 96

8685. (The Editor.)—Given an angle at the base of a triangle, the sum of the two sides, and the distance between the given angular point and the point of contact of the escribed circle touching the base; construct the triangle.. 75

8692. (R. F. Davis, M.A. Suggested by Question 8559.)—If, from any point in a given fixed straight line passing through the focus of a parabola, tangents be drawn to the curve, prove that the envelope of the line joining the feet of the perpendiculars on these tangents from a given fixed point on the directrix is another parabola. 129

8696. (Emile Vigarié.)—Dans un cercle donné, par un point P pris sur la circonférence on mène trois cordes PA, PB, PC; sur chacune d'elles comme diamètre on décrit une circonférence; démontrer que les trois points de rencontre sont en ligne droit. 32

8697. (Captain H. Brocard.)—Les droites qui joignent le centre du cercle circonscrit aux milieux des segments interceptés par les perpendiculaires aux milieux des côtés sur les hauteurs du triangle sont perpendiculaires aux médianes correspondantes. 49

8700. (R. W. D. Christie, M.A.)—Given the sum of the expression $1^r + 2^r + 3^r + 4^r \ldots\ldots + n^r$, to find a method of writing down at once the sum of $1^{r+1} + 2^{r+1} + 3^{r+1} + 4^{r+1} \ldots\ldots + n^{r+1}$, where r is even or *vice versâ*.. 96

8708. (Maurice D'Ocagne.) — Si, dans le quadrilatère ABCD, les angles opposés B et D sont droits, la droite qui joint les pieds des perpendiculaires abaissées du sommet C sur les bissectrices intérieure et extérieure de l'angle A passe par le milieu de la diagonale BD. 34

8713. (Professor Steggall, M.A.) — Show that the solution in rational quantities of the equation $x^2+y^2+z^2=u^2$, is $x = k(a^2+b^2-c^2)$, $y = 2kac$, $z = 2kbc$, $u = k(a^2+b^2+c^2)$.. 31

8714. (Professor Genese, M.A.) — S is a focus of a conic, PN a fixed ordinate to the diameter through S, PQP′ a circle with centre S; a variable radius SQ meets PN at L and the conic at R. Prove that the cross ratio $\{\dot{S}L\dot{Q}R\}$ is constant... 107

8715. (Professor Hudson, M.A.)—Prove that
$\tan 37\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$. 88

8716. (Professor Mathews, M.A.) — Prove that the real common tangents of the circles $x^2+y^2-2ax=0$, $x^2+y^2-2by=0$ are represented by $2ab(x^2+y^2-2ax) = (by-ax+ab)^2$, or, which is the same thing, by
$2ab(x^2+y^2-2by) = (by-ax-ab)^2$. 51, 82

8718. (Professor Neuberg.)—L'angle BAC d'un triangle est fixe, et (1) le centre de gravité, (2) l'orthocentre, (3) le centre du cercle circonscrit parcourt une droite donnée. Trouver l'enveloppe du côté BC... 115

8719. (Professor Wolstenholme, M.A., Sc.D.) — The lengths of the edges OA, OB, OC of a tetrahedron OABC are denoted by *a*, *b*, *c*, those of the respectively opposite edges BC, CA, AB by *x*, *y*, *z*; and the dihedral angles *opposite* to these by A, B, C; X, Y, Z; prove that, if V be the volume expressed in terms of *a*, *b*, *c*, *x*, *y*, *z*, $\frac{\delta V}{\delta a} = \frac{1}{6} ax \cot A$, &c.; and thence that (1) when *a*, *x*; *b*, *y*, and $c+z$ are *given*, V will be a maximum when $C = Z$; (2) when *a*, *x*; *b*, *y*, and $c-z$ are given, V will be a maximum when $C+Z = 180^\circ$; (3) when *a*, *x*; $b+y$; $c+z$ are given, V will be a maximum when $B = X$ and $C = Z$ (*i.e.*, when $b = x$, $c = z$); (4) when *a*, *x*; $b+y$, $c-z$ are given, V will be a maximum when $B = X$, $C+Z = 180^\circ$; and (5) investigate if V can have a maximum when *a*, *x*; $b-y$, $c-z$ are given, or when $a-x$, $b-y$, $c-z$ are given; the given quantities being supposed always real and finite.

In case (4), prove also that 180° is the maximum value of $C+Z$ for variations in *b*, *y* (subject to the conditions stated), and that zero is the minimum value of $B-Y$ for variations in *c*, *z*.............................. 39

8724. (R. Tucker, M.A.) — If a constant line AB moves with ends on OX, OY, two rectangular axes, and on it a semicircle is described, then locus of mid-point of arc is one of two straight lines. Find locus of any other fixed point on the arc. [If C is the mid-point of AB, and P the mid-point of the arc, then the question may be enunciated for a bar rigidly connected with the bar AB.] 36

8725. (S. Tebay, B.A.)—If the sines of opposites dihedral angles of a tetrahedron be respectively proportional to the edges formed by these angles, the areas of the four faces are equal. 48

8726. (S. Roberts, M.A. Analogous to Quest. 3068.)—Show that, if some four of the roots of a quintic form an harmonic system, then

$$J^3 - 2^7 . 3^2 . JK + 2^{13} . 3^3 . L = 0,$$

where J, K, L are the three fundamental invariants of the orders 4, 8, 12 [see SALMON's *Higher Algebra*, 3rd ed., p. 211]. 33

8727. (B. Hanumanta Rau, B.A.)—The sides BC, CA, AB of a triangle ABC are produced to *a*, *b*, *c*, such that C*a*, A*b*, B*c* are respectively equal to BC, CA, AB. Prove that the centroid of the triangle *abc* coincides with that of ABC. 84

8728. (Captain H. Brocard.) — De chaque sommet d'un triangle on mène des perpendiculaires aux côtés adjacents, jusqu'à leurs rencontres avec le côté opposé. Les centres A′, B′, C′ des cercles circonscrits aux trois triangles ainsi formés sont les sommets d'un second triangle homologique avec le proposé. Le centre d'homologie est le centre O du cercle circonscrit au triangle donné. En d'autres termes, les nouveaux cercles sont tangents au cercle circonscrit, aux points A, B, C.................. 121

8729. (R. W. D. Christie, M.A.)—Of the series

$$1 . 1 + 3 . 5 + 5 . 13 + 7 . 25 + \&c.$$

to *n* terms, show that, (1) the sum is a square; (2) the square root is $= 1 + 3 + 5 + 7 +$ &c. to *n* terms; (3) each term is the product of two numbers which may be taken to represent the shorter side and hypothenuse of a right-angled triangle; (4) if unity be added to the nth term, it is divisible by $2n$, and if unity be subtracted it is divisible by $n-1$; and (5) all terms are odd numbers. 106

8737. (W. J. Greenstreet, B.A.)—ABC is a spherical triangle, the mid-points of its sides are the angular points of a triangle DEF; prove that (1) $\cos EF / \cos \frac{1}{2}a = \cos FD / \cos \frac{1}{2}b = \cos DE / \cos \frac{1}{2}c$;
and (2) if $\cos^2 \frac{1}{2}a = \cos \frac{1}{2}(b+c) \cos \frac{1}{2}(b-c)$, the angle D is a right angle. 108

8745. (W. W. Taylor, M.A. Suggested by Question 7938.)—Prove that (1) the areas of the Taylor-circles of the four triangles ABC, PBC, PCA, PAB are together equal to the area of the circumscribed circle, P being the orthocentre of the triangle ABC; also (2) the lines joining their centres are the Simson-lines of the middle points of the sides or of AP, BP, CP with respect to the pedal triangle. 100

8749. (F. Purser, M.A.)—Find (1) the cubic locus of the centre of a conic passing through the three vertices A, B, C of a triangle, and such that the three normals at these vertices are concurrent; and prove that (2) this cubic passes through the in-centre I, the three ex-centres I_a, I_b, I_c, the circum-centre O, the orthocentre H, the centroid G, and the symmedian G′, and cuts the three sides normally at their middle points L, M, N; also (3) tangents at (I, I_a, I_b, I_c), (A, B, C, G), (L, M, N, G′) are respectively concurrent in G, G′, O; and (4) the lines joining L, M, N meet the curve again in their intersections with the respective perpendiculars from the vertices.. 35

8754. (Professor Mahendra Nath Ray, M.A., LL.B.)—Prove that

(1) $\int_0^\pi \frac{x \sin^3 x\, dx}{1+\cos^2 x} = \pi(\frac{1}{2}\pi - 1)$, (2) $\int_0^\pi \frac{x \sin^5 x\, dx}{1+\cos^2 x} = \pi(\pi - \frac{8}{3})$. 123

8755. (Professor Neuberg.)—On prolonge les hauteurs du triangle ABC au delà des sommets des quantités AA′=BC, BB′=CA, CC′=AB; démontrer: (1) les triangles ABC, A′B′C′ ont même centre de gravité; (2) si α, α' sont les angles de Brocard de ces triangles, on a

$$A'B'C' = 2ABC(2+\cot\alpha),\ (A'B')^2+(B'C')^2+(C'A')^2 = 8ABC(3+2\cot\alpha),$$
$$\cot\alpha' = (2\cot\alpha+3)/(\cot\alpha+2);$$

(3) les points A, B, C sont les centres des carrés construits, intérieurement, sur les côtés du triangle A′B′C′; (4) les milieux des côtés de A′B′C′ sont les centres des carrés construits, extérieurement, sur les côtés de ABC... 55

8757. (Professor Edmund Bordage.)—If two fractions a/a', b/b' are such that $ab'-ba'=1$, prove that the simplest fraction included between the two given fractions is $(a+b)/(a'+b')$. 45

8763. (H. Stewart, M.C.P.)—Prove that

(1) if $u(1-x^2)^{\frac{1}{2}} + x(1-u^2)^{\frac{1}{2}} = a^2$, then $\frac{du}{dx} + \frac{(1-u^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} = 0$;

(2) if $(1-x^2)^{\frac{1}{2}} + (1-u^2)^{\frac{1}{2}} = a(x-u)$, then $\frac{du}{dx} = \frac{(1-u^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}$. ... 100

8765. (L. J. Rogers, M.A.) — If $x_1+x_2+x_3+a_1+a_2+a_3=0$, prove that (1)

$$\begin{vmatrix} \tan(x_1+a_1), & \tan(x_2+a_1), & \tan(x_3+a_1) \\ \tan(x_1+a_2), & \tan(x_2+a_2), & \tan(x_3+a_2) \\ \tan(x_1+a_3), & \tan(x_2+a_3), & \tan(x_3+a_3) \end{vmatrix} = 0;$$

and (2) the same is true if tangents are replaced by sines................ 88

8768. (R. F. Davis, M.A.)—Required an Analytical Proof of Feuerbach's Theorem.

[An analytical proof is given by the Rev. J. J. Milne, M.A., in the Introduction to his Weekly Problem Papers, but this proof may be modified and improved.] 131

8769. (J. Brill, M.A.)—An ellipse is inscribed in the triangle PQR so as to touch the sides QR, RP, PQ at P′, Q′, R′ respectively; prove that, if C be the centre of the ellipse,

$$\triangle QCR : \triangle Q'CR' = \triangle RCP : \triangle R'CP' = \triangle PCQ : \triangle P'CQ'.... \quad 34$$

8778. (W. J. C. Sharp, M.A.)—If $wy-vz$, $uz-wx$, and $vx-uy$ be cogredient to x, y, and z, the rectangular Cartesian coordinates of a point respectively, u, v, and w are so. 44

8783. (Captain H. Brocard.)—Les droites qui joignent l'orthocentre aux milieux des segments interceptés par les hauteurs du triangle sur les rayons du cercle circonscrit aboutissant aux sommets du triangle sont perpendiculaires aux symédianes correspondantes. 87

8784. (R. W. D. Christie, M.A.)—Prove that, if

$$s=1+2+3+\ldots+n,\quad S=1^2+2^2+3^2+\ldots+n^2,\quad S'=1^3+2^3+3^3+\ldots+n^3,$$
$$\Sigma=1^4+2^4+3^4+\ldots+n^4,\quad \sigma=1^5+2^5+3^5+\ldots+n^5,$$

then $$\frac{3\sigma+2s^3}{5\Sigma}=\frac{S'}{S} \ldots\ldots \quad 99$$

8786. (R. Tucker, M.A.) — Pp, Qq, Rr are focal chords of a parabola; if P, Q, R have a conormal point, then the centres of curvature for p, q, r are collinear. (This is another way of putting the first part of Question 8693.) Find the equation to this line. Prove that this *central* line envelopes (1) a hyperbola when tangents at p, q meet on the directrix, (2) an ellipse when the tangents meet on the latus rectum. 61

8789. (Professor Culley, M.A.) — Find by a geometrical construction the equation $SS'=P^2$ of the pair of tangents from T (x', y') to a circle $S \equiv x^2+y^2-z^2=0$, whose centre is O; and show that, if perpendiculars RL, RM, RK be drawn to the tangents and their chord of contact from a point R not on the circumference, RL . RM differs from RK^2 by $TO^2\cos^2\omega\cos^2\theta$, where 2ω and 2θ are the angles subtended by the circle at T and R respectively. 37

8793. (Professor Neuberg.) — Soient AB, CD deux droites, qui se coupent en O. Si l'on fait tourner AB autour de O, le point double de deux figures semblables construites sur AB et CD décrit une circonférence de cercle. 30

8797. (Professor Sáradáranjan Rây, M.A.) — OAP and OBQ are two fixed straight lines intersecting at O, and C is a circle touching them both at P and Q; prove that the perimeter of the triangle OAB, circumscribed by *any* circle passing through O and touching C externally, is constant. 89

8799. (Professor Hudson, M.A.)—Find the envelope of the straight line $au = f(\alpha)\cos(\theta-\alpha) + f'(\alpha)\sin(\theta-\alpha)$. 41

8801. (Rev. T. P. Kirkman, F.R.S. Suggested by Quest. 8325.)—Upon the most sacred of the undiscovered Egyptian pyramids, whose vertex P is at unequal distances from the angles of its scalene base ABC, is cut in the face PAC an ascending flight of steps from A to b, in PC; from b another flight mounts to c in PB; and a third leads up from c to a, at a given altitude h in PA. These three flights are equigradient. A second path from A mounts to b' in PC, and thence proceeding to c' in PB, ascends from c' to the above point a in PA. These three flights also are equigradient. By Abca Pharoah and his Grandees of Church and State were wont to mount to a platform at a for solemn religious rites; the admitted vulgar used only the path A$b'c'a$. Required, a plane projection of the pyramid and its six flights of stairs, and proof from that projection that the first three are equigradient, and the second three equigradient, up to the given altitude h. 24

8804. (S. Tebay, B.A.) — If (a, b, c) be conterminous edges of a tetrahedron, (x, y, z) the respective opposites, (A, X; B, Y; C, Z) corresponding dihedral angles, and V the volume; show that (1)

$$\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z};$$

also (2), if the areas of the four faces are equal, then

$$a = x,\quad b = y,\quad c = z,\quad A = X,\quad B = Y,\quad C = Z.$$

$$V = \tfrac{1}{12}\left\{2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)\right\}^{\frac{1}{2}},$$

$$(b-c)\sin A + (c-a)\sin B + (a-b)\sin C = 0.$$ 31

8807. (For Enunciation see Question 8745.) 100

8808. (F. R. J. Hervey.)—Find (1) in how many ways n lines of verse can be rhymed, so as to have r *different* rhymes, and no line unrhymed; and (2) show that, in the case of the *sonnet*, the numbers of ways with 2, 3, ... 7 rhymes are, respectively, 8177, 731731, 6914908, 12122110, 4099095, and 135135 75

8810. (Rev. T. R. Terry, M.A.) — Find the equation to the straight lines through the origin and the intersection of the conics

$$x^2-3xy-4x+y-1 = 0,\quad 2x^2+xy+4y^2+2x+13y+8 = 0.$$ ... 58

8813. (J. Griffiths, M.A.)—If we have

$$f(x) = A_0 + 2A_2x^2 + 2A_4x^4 + \dots \textit{ ad inf.},\quad \phi(x) = 1 - 2B_2x^2 - 2B_4x^4 - \dots \textit{ ad inf.}$$

where $A_0 = 1 + 2\Sigma(-)^s \operatorname{cn}^2\frac{sK}{n}$, $\quad B_2 = \Sigma(-)^s \operatorname{cn}^2\frac{sK}{n} + \operatorname{sn}^2\frac{sK}{n}$,

$$A_2 = k^2\,\Sigma(-)^s \operatorname{cn}^2\frac{sK}{n}\operatorname{sn}^2\frac{sK}{n},\qquad B_4 = \Sigma(-)^s \operatorname{cn}^3\frac{sK}{n} + \operatorname{sn}^4\frac{sK}{n},$$

$$A_4 = k^4\,\Sigma(-)^s \operatorname{cn}^2\frac{sK}{n}\operatorname{sn}^4\frac{sK}{n},\qquad B_6 = \Sigma(-)^s \operatorname{cn}^2\frac{sK}{n} + \operatorname{sn}^6\frac{sK}{n},$$

... ...

and s is a number from 1 to $n-1$, show that $f(x)\cdot\phi(x) = A_0$. 78

8814. (The Editor.) — Prove that the triangle formed by joining the mid-points of the altitudes of a triangle is one-fourth of the pedal triangle, and that the theorem holds good for any other three concurrent lines drawn through the vertices of the triangle.......................... 59

8816. (Captain H. Brocard.)—Ω, Ω' désignant les points de Brocard, S le milieu de $\Omega\Omega'$, S′ le point correspondant à S par droites symétriques, R le point de Steiner; démontrer que ce point R se trouve sur la droite SS′. (R est une des intersections du cercle ABC avec le diamètre perpendiculaire à l'axe d'homologie du triangle ABC et du premier triangle de Brocard. Ce point est aussi l'intersection commune des parallèles aux côtés du premier triangle de Brocard, menées par les sommets correspondants du triangle ABC.).. 22

8820. (Charles F. Lodge.) — A mirror, measuring 33 inches by 22 inches, is to have a frame of uniform width, whose area is to equal that of the glass; show that the width of the frame is $5\frac{1}{2}$ inches. 78

8823. (Professor Crofton, F.R.S.)—Two discs of any form are moveable in a plane round two fixed points A, B, respectively. Show that, when they are in such positions that the length of an endless band enveloping both is a maximum or a minimum, the portions of the band which form the common tangents will meet on AB if produced, and are equally inclined to AB.. 26

8830. (Professor Bordage.)—Given the equation

$$\sin 2\alpha \,.\, x^2 + 2\,(\sin\alpha + \cos\alpha)\,x + 2 = 0,$$

find, and solve, the equation of the *second degree* that has as roots the squares of the roots of the given equation. 110

8833. (Professor Neuberg.)—Un angle de grandeur constante tourne autour de son sommet A, ses côtés rencontrent une droite donnée XY aux points B et C. Trouver (1) les enveloppes de la médiane BB′ et de la symédiane BH; (2) les points où ces droites touchent leurs enveloppes. 38

8835. (Professor de Longchamps.)—On considère un triangle ABC et le cercle circonscrit Δ; soit μ une transversale rencontrant les côtés de ABC aux points A′, B′, C′. Par A′ on mène à Δ une tangente que l'on rabat sur BC de telle sorte que le point de contact vienne occuper sur BC une certaine position A″. Démontrer que la droite AA″ et les deux droites analogues BB″, CC″ vont passer par un même point M. 94

8837. (The Editor.)—On the sides of a regular n-gon, n points are taken at random, one on each, forming the apices of inscribed n-gon; again, inside the n triangles that lie outside this last polygon, n points are taken at random, one in each, forming, when joined, another n-gon; find the general average area of this last n-gon, and show therefrom that, in the cases of a hexagon and a square, the averages are $\frac{25}{36}$ and $\frac{16}{36}$ of the original figure respectively.. 41

8840. (W. S. M'Cay, M.A.)—Prove that (1) the locus of the mean centre of the four points, in which a line of given direction meets the faces of a tetrahedron, is a plane (diametral plane); and (2) if A, B, C, D be the areas of the faces of the tetrahedron, all the diametral planes envelope the quartic $(Ax)^{\frac12} + (By)^{\frac12} + (Cz)^{\frac12} + (Dw)^{\frac12} = 0$. 40

8843. (Rev. T. P. Kirkman, F.R.S.) — The angles *nabc* ... *lm* of any convex plane *n*-gon N are joined to a point P in its plane, within or without it. In P*a*, P*b*, ... P*n* are taken points $a_1 b_1 c_1 \ldots m_1 n_1$, such that $n a_1 b_1 \ldots m_1 n_1$ is a broken line L beginning and ending on P*n*. L is thus formed: first, $na_1 > a_1 a$; and next, if $d_1 e_1 f_1$ be any three consecutive points of it, the distances in the directions $f_1 e_1$ and $e_1 d_1$ of e_1 from the lines *fe* and *ed* in N are equal. The line $n_1 m_1$ meets the edge *nm* of N in *r*. Prove that, if from *s* in *an* produced we draw $sn_1 = rn_1$, sn_1 is parallel to the first portion na_1 of L. 58

8847. (W. Mills, B.A.)—Given the focus of a parabola, one tangent, and a point on it through which another tangent passes; prove that the locus of the point of intersection of the variable tangent with a diameter through the point of contact of the fixed tangent is a circle which touches the fixed tangent at the given point on it. 105

8851. (J. J. Walker, F.R.S.)—Show that either segment of a focal chord of a conic section is a mean proportional between its excess over the semi-latus rectum (or *vice versâ*) and the whole chord................ 41

8854. (Rev. T. R. Terry, M.A.)—Solve (1) the equation

$$w_{x+2} = (2x+5)\, w_{x+1} - (x^2+4x+3)\, w_x;$$

and show (2) that, if u_x and v_x both satisfy this equation, and if $u_1 = 2$, $u_2 = 10$, while $v_1 = 1$, $v_2 = 2$, then $2u_x = x(x+3)\, v_x$.................. 44

8856. (B. Hanumanta Rau, B.A.) — AB is the diameter of a semicircle; AC and BF tangents. CF cuts the semicircle in D. DG is drawn at right angles to CF, meeting AB in G. Prove that

$$AC \, . \, BF = AG \, . \, GB.$$ 112

8858. (Professor Cochez.)—Résoudre les équations

$$(xy+1)(x+y) = axy, \quad (x^2y^2+1)(x^2+y^2) = b^2x^2y^2.$$

Montrer que ce système est *quadratique*. Appliquer les formules au cas particulier suivant: $a = \frac{16}{3}$, $b = \frac{10}{3}$. 60

8862. (R. Curtis, S.J., M.A.) — Find (1) the locus of a point in a material lamina, such that if the lamina were moving without rotation in its own plane, and that point were suddenly fixed in space, the ensuing rotation of the lamina would be given; (2) where the point should be if the rotation produced is a maximum; and (3) where the point should be in order that the loss of kinetic energy would be given.................. 67

8863. (Ch. Hermite, Membre de l'Institut.) — Déterminer les intégrales définies

$$\int_0^{\pi} \frac{dx}{\sin(x+p)}, \quad \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{dx}{\sin(x+p)},$$

en supposant que *p* soit une quantité imaginaire quelconque. 53

8865. (Amiral de Jonquières.) — Soient trois nombres entiers, *a*, *b*, *c*, premiers entre eux, deux à deux, et vérifiant l'équation $a^n + b^n = c^n$. Démontrer que (l'exposant *n* premier et supérieur à 3): (1) *a* et *b* ne peuvent être, simultanément, premiers; (2) si *a*, supposé inférieur à *b*, est premier, $c = b+1$. 56

8866. (Professor Catalan.)—Démontrer les contributions au théorème de FERMAT:—*a* supposé premier, (1) $a - 1 = \mathfrak{M}(n)$; (2) $a^n - 1 = \mathfrak{M}(nb)$; (3) tout diviseur premier, de $c - a$, divise $a - 1$; (4) $a + b$ et $c - a$ sont premiers entre eux ; (5) $2a-1$ et $2b+1$ sont premiers entre eux ; (6) le nombre premier, a (s'il existe) est compris entre

$$(nb^{n-1})^{1/n} \text{ et } \{n(b+1)^{n-1}\}^{1/n};$$

(7) a et b surpassent n ; (8) le nombre b, qui satisfait à l'équation $(b+1)^n - b^n = a^n$, est compris entre

$$a(a/n)^{1/n-1} \text{ et } -1+a(a/n)^{1/n-1};$$

(9) soit b un nombre entier, supérieur au nombre entier n. Entre

$$(nb^{n-1})^{1/n} \text{ et } \{n(b+1)^{n-1}\}^{1/n},$$

il y a, tout au plus, un nombre entier ; (10) aucun des nombres $a+b$, $c-a$, $c-b$, n'est premier ; (11) chacun d'eux a la forme N, ou la forme $(1/n)$ N, N étant un nombre entier ; (12) soient, s'il est possible,

$$a+b = c'^n, \quad c-a = b'^n, \quad c-b = a'^n;$$

alors $c = \mathfrak{M}(n)$; (13) $(x+y)^n - x^n - y^n = nxy(x+y)\,\mathrm{P}$,

$$\mathrm{P} = \mathrm{H}_1 x^{n-3} + \mathrm{H}_2 x^{n-4} y + \ldots + \mathrm{H}_1 y^{n-3},$$

les coefficients sont donnés par la formule $\mathrm{H}_p = (1/n)[\mathrm{C}_{n-1,p} \pm 1]$, le signe + répondant au cas où p est pair, et le polynôme P est divisible par x^2+xy+y^2, et même par $(x^2+xy+y^2)^2$, si $n = \mathfrak{M}(6)+1$; (14) la différence des puissances $n^{\text{ièmes}}$ de deux nombres entiers consécutifs, a, $a+1$, étant diminuée de 1, est divisible par $na(a+1)(a^2+a+1)$; les facteurs a, $a+1$, a^2+a+1 sont premiers entre eux, deux à deux ; en outre, le troisième égale le produit des deux autres, augmenté de 1 ; (15) si, dans l'équation de Fermat, le nombre a est premier, on a, par le théorème 8865, de M. de Jonquières, $a^n - 1 = \mathfrak{M}[nb(b+1)(b^2+b+1)]$; et (16) c est compris entre $a+b$ et $\frac{1}{2}(a+b)$ 56

8869. (Professor Steggall, M.A.)—A shot of mass m is fired *in vacuo* from an air gun of length l, with a charge of air that at normal pressure p would occupy a volume v ; this air initially occupies a length b of the barrel of the gun. Show that the time of passage along the barrel and the velocity with which the shot leaves the gun are given by the equations $\mathrm{T} = (l + \frac{5}{8} l^{\frac{3}{5}} b^{\frac{2}{5}}) \div (5pv/m)^{\frac{1}{2}}$, $\mathrm{V} = (1 - b^{\frac{2}{5}}/2l^{\frac{2}{5}})(5pv/m)^{\frac{1}{2}}$, where b is small compared with l, and the ratio of the specific heats of air is taken as 1·4. 80

8873. (Professor Edmund Bordage.)—Given two relations,

$$(2\mathrm{S} = a+b+c),$$

$$\frac{x + y\cos c + z\cos b}{\cos(\mathrm{S}-a)} = \frac{y + z\cos a + x\cos c}{\cos(\mathrm{S}-b)} = \frac{z + x\cos b + y\cos a}{\cos(\mathrm{S}-c)} = 2m,$$

deduce therefrom $\dfrac{x}{\sin a} = \dfrac{y}{\sin b} = \dfrac{z}{\sin c} = \dfrac{m}{\sin \mathrm{S}}$. 91

8875. (Professor Nash, M.A.)—Professor Casey's Quest. 7839 may be enunciated as follows :—DD′, EE′, FF′ are the intersections of the sides of a triangle ABC with the cosine circle, the order of the letters being such that E′F, F′D, D′E are diameters. The circle round AE′F cuts the circles on AB, AC as diameters in the points a, a' ; and b, b', c, c' are similarly determined upon the circles BF′D, CD′E. To show that the circles round ACa, BAb, CBc pass through ω the positive Brocard-point of ABC, and the circles ABa', BCb', CAc' through the negative Brocard-point. Prove the following additional properties:—(1) The circles ACa, ABa' touch at A the sides AB, AC, and intersect again in a vertex of Brocard's second triangle; (2) the tangents to these circles at aa' bisect AB, AC; (3) the points B, C, a, a' are concyclic, and the circle through them touches OB, OC; (4) AaD and Aa'D$'$ are collinear; (5) Ca, Ba' intersect upon the symmedian AP, and Ca', Ba upon the perpendicular AL; (6) aE$'$, a'F meet BC at the foot of the symmedian AP; (7) aa', E$'$F, BC, and the radical axis of the circles ABC, AE$'$F, are concurrent; (8) the three points of concurrence of the three sets of lines in (7) lie upon a line parallel to $\omega\omega'$, and therefore the triangle formed by the lines aa', bb', cc' is in perspective with ABC; (9) the pole of aa' with respect to the circle AE$'$F lies upon the median of ADD$'$; (10) the pairs of points ac', ba', cb' are isogonal, and the inscribed conic whose foci are ac' touches CA at the foot of the perpendicular from B; (11) one directrix of this conic is the line joining C to the intersection of FD$'$ and DE$'$, and the other the line joining A to the intersection of EF$'$, FD$'$. 131

8876. (Professor Hudson, M.A.)—Prove that the normal chord which subtends a right angle at the focus of a parabola is divided by the axis in the ratio 2 : 3. 108

8886. (Rev. T. P. Kirkman, M.A., F.R.S.) — Two 28-edra have the following triangles, the summits being marked *abc* ... *npq*. Required, all the symmetry of both.

abg, abc, acd, ade, aef, afg, bjc, bih, bjh, big, cjl, clk, ckd, dkm, dfm, def, fmq, fgq, ghq, ghi, hpj, hpq, jlk, jkp, mnq, mnp, mpk, pqn.

abg, afg, afe, aed, abc, acd, bhg, bhi, bci, cji, cdj, djk, dkl, dfl, def, fkl, fkm, fgm, ghm, ijq, ihp, ipn, inq, jkn, jnq, kmn, hmn, hpn. 125

8887. (W. J. Greenstreet, B.A.)—In the "ambiguous case" of triangles, given a, b, and A; r, r' being the radii of the in-circles, ρ, ρ' of the escribed circles to a; prove that

$$rr'(b+a) = \rho\rho'(b-a), \quad r\rho(b-a) = r'\rho'c$$ 87

8888. (S. Tebay, B.A.) — D_1, D_2, D_3 are the shortest distances of opposite edges (a, a; b, b; c, c) of a tetrahedron, V the volume, and Δ the area of one of the equal faces; show that

$$V = \tfrac{1}{3}(D_1D_2D_3), \quad \Delta^2 = \tfrac{1}{8}(D_1^2a^2 + D_2^2b^2 + D_3^2c^2),$$

and $$D_1^2 + D_2^2 + D_3^2 = \tfrac{1}{2}(a^2+b^2+c^2) = D_1^2 + a^2 = \&c.$$ 161

8903. (Professor Genese, M.A.)—If $f(r, \theta)$ be the equation to a conic referred to any pole O, the value of $(df)/(dr)$ at any point P of the plane varies as (PQ + PR) / OQ . OR, Q, R being the intersections of OP with the conic; thus, for a point Q of the conic, $(df)/(dr) \propto$ QR/OQ . OR; and in particular, if O be a focus, $(df)/(dr)$ is constant over the curve. [In this case $f \equiv (l - er\cos\theta)^2 - r^2$.] 93

8905. (Professor Neuberg.)—Démontrer qu'à tout point f de la distance des foyers F, F$'$ d'une ellipse E correspond une droite D extérieure au plan de E et parallèle au petit axe, telle qu'il existe le rapport constant $c : a$ entre les distances d'un point quelconque de E à f et D. Lorsque f occupe toutes les positions sur FF$'$, D se déplace sur un cylindre qui a pour base une ellipse semblable à E. 113

8908. (Enunciation included in Question 8866.) 56

8912. (Professor Bordage.)—A square ABCD and a straight line Δ in its plane are given, and through the points A, B, C, D perpendiculars AA′, BB′, CC′, DD′ are drawn on Δ, A and C being two opposite summits; prove that (1) $(BB')^2+(DD')^2-2AA'.BB'$ is a constant quantity for every position of Δ; and (2) deduce therefrom an application for the envelope of the straight lines, such that the sum of the squares of the distances of one of them from two given points is constant. 109

8915. (Rev. T. C. Simmons, M.A.)—A point P being given in the plane of a triangle ABC, it is known that, with P as focus, five conics can be drawn, four of them circumscribing the triangle, and one inscribed in it. Show that for certain positions of P a sixth conic can be drawn, also having P for focus, and touching the other five; and find the locus of P when this is possible. [The Proposer suggests that conics with one focus common should be called *co-focal*, reserving the term confocal for conics having both foci common.] 95

8920. (R. Knowles, B.A.)—From a point O (x, y) tangents are drawn to meet the ellipse $a^2y^2+b^2x^2=a^2b^2$ in P and Q, a tangent parallel to the chord PQ meets OP, OQ in p and q respectively; prove that

$$pq : PQ = (a^2y^2+b^2x^2)^{\frac{1}{2}} : ab+(a^2y^2+b^2x^2)^{\frac{1}{2}}.$$ 115

8924. (Captain H. Brocard.)—Former l'équation des paraboles tangentes aux deux bissectrices de chaque angle du triangle et aux perpendiculaires aux côtés de cet angle en leurs milieux. Ces coniques admettent, comme on sait, pour directrices les médianes et pour foyers les sommets A″, B″, C″ du second triangle de Brocard. 70

8927. (S. Tebay, B.A.) — If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the faces of a tetrahedron, R the radius of the circumscribing sphere, and R_1, R_2, R_3, R_4 the radii of spheres passing through the centre of R and the angles of Δ_1, Δ_2, Δ_3, Δ_4; the volume

$$V=\tfrac{1}{3}R^2\left(\frac{\Delta_1}{R_1}+\frac{\Delta_2}{R_2}+\frac{\Delta_3}{R_3}+\frac{\Delta_4}{R_4}\right).$$ 110

8928. (Rev. T. R. Terry, M.A.) — Find the value of the definite integral $\int_0^\infty \frac{dx}{x}\sin px\,(a\cos^2 x+b\sin^2 x)$, when $p>2$. 157

8929. (R. Tucker, M.A.)—Find (1) the equation of the circle through the images of the centroid of a triangle with respect to the bisectors of angles; and show (2) that the sum of the squares of tangents to it from the angles (taken once) is $\frac{1}{3}(a^2+b^2+c^2)$. 88

8931. (Emile Vigarié.) — On donne deux points O et A et on considère toutes les paraboles ayant O pour sommet et qui passent par A. Trouver géométriquement le lieu (1) du point de concours des tangentes en A et O; (2) du point d'intersection de la normale en O et de la tangente en A (ces deux lieux sont tangents en O); (3) du point d'intersection de la tangente en O et de la normale en A; (4) du point de concours des normales en O et A (ces deux lieux sont tangents en A). ... 81

8933. (B. Hanumanta Rau, M.A.) — If a body describe an ellipse about a centre of force in the focus, the time of describing the arc between the extremities of the *latera recta* on the same side of the major-axis is to the periodic time as $\sin^{-1}(e) : \pi$. 119

8946. (W. J. C. Sharp, M.A.)—Show that

$$\begin{vmatrix} x, & -a, & -a, & \dots & -a, & -a \\ b, & x, & -a, & \dots & -a, & -a \\ b, & b, & x, & \dots & -a, & -a \\ \dots & \dots & \dots & & \dots & \\ b, & b, & b, & \dots & x, & -a \\ b, & b, & b, & \dots & b, & b \end{vmatrix} = b(x+a)^n,$$

where the determinant has $n+1$ rows. 122

8948. (W. J. C. Sharp, M.A.)—Show that

$$\sin n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\,\frac{d}{d\theta}\right)^{n-1}(\sin^2\theta),$$

and $$\cos n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\,\frac{d}{d\theta}\right)^{n-1}(\sin\theta\cos\theta).$$ 123

8977. (Professor Haughton, F.R.S.) — The form of the Terrestrial Radiation Function has been proved to be $A(\theta-\Theta_0)^n = a$, where A, Θ_0, n are unknown parameters, and θ, a are given by observation. The mean monthly observations at Greenwich, extending over thirty-six years, give January, $A(38{\cdot}9-\Theta_0)^n = 21{\cdot}4$; February, $A(40{\cdot}4-\Theta_0)^n = 35{\cdot}5$; March, $A(42{\cdot}8-\Theta_0)^n = 55{\cdot}9$. Find Θ_0, A, and n....... 69

8978. (Professor Sylvester, F.R.S.) — Show *algebraically* that, if a, b, c are the three sides of a triangle of reference, and A, B, C, the three perpendiculars on a variable line from the angles of that triangle, are regarded as its inverse coordinates, then the equation to the two circular points at infinity is

$a^2(A-B)(A-C)+b^2(B-A)(B-C)+c^2(C-A)(C-B) = 0$.... 157

8979. (Professor Brunel.)—Soient ABC un triangle, $A_1B_1C_1$ un autre triangle déduit du premier en menant par les sommets A, B, C des droites faisant avec les côtés du triangle et dans le même sens un angle ϕ. Du triangle $A_1B_1C_1$ l'on déduit, de même, un triangle $A_2B_2C_2$ et ainsi de suite, toujours avec le même angle ϕ. Démontrer que les points A, A_1, A_2 ... sont sur trois groupes de spirales logarithmiques ayant pour pôles les points de Brocard. Pour quelles valeurs de ϕ le triangle dérivé $A_1B_1C_1$ est-il égal au triangle proposé? 109

8980. (Professor Steggall, M.A.) — In a rectangular hyperbola OY is drawn at right angles to the tangent at P; prove that, if YX produced cuts SP in R, and Y be joined to X′, then XR = YX′, where X, X′ are the feet of the directrices.. 92

8981. (Professor Schoute.)—Given two conics; find (1) the locus of the vertex of a right angle circumscribed to these curves; and (2) consider the particular case of two homo-focal conics. [The problem in its general form may be otherwise stated thus:—Find the locus of points such that one of the tangents from it to a conic (2), together with one of the tangents to a second conic (3), form with the two tangents to a third conic (1) a harmonic pencil.] 102

8985. (Professor Byomakesa Chakravarti, M.A.)—A cylinder, weight W, radius r, is placed on a rough horizontal plane; a uniform plank, weight P, length 2Q, is inclined at an angle θ to the horizon, and rests with one end on the ground, the other *on* the cylinder (the plank being at right angles to the axis of the cylinder); if ψ be the angle made with the vertical by the re-action of the ground on the cylinder, prove that

$$\cot\psi = \cot\frac{\theta}{2} + \frac{r}{a} \cdot \frac{W}{P} \cdot \sec\theta. \text{ .... } 104$$

8994. (Charlotte A. Scott, B.Sc.)—A rectangular sheet of stiff paper, whose length is to its breadth as $\surd 2$ is to 1, lies on a horizontal table with its longer sides perpendicular to the edge and projecting over it. The corners on the table are then doubled over symmetrically so that the creases pass through the middle point of the side joining the corners, and make angles of 45° with it. The paper is then on the point of falling over; show that it had originally $\frac{25}{48}$ of its length on the table. 112

8996. (Mahendra Nath Ray, M.A., LL.B.) — If x, y, z denote the respective distances of any point in the plane of a given triangle ABC from the angular points; show that the following relation subsists among them, $(x^2+y^2+z^2+a^2+b^2+c^2)(a^2x^2+b^2y^2+c^2z^2) = 2a^2x^2(a^2+x^2)$
$+ 2b^2y^2(b^2+y^2) + 2c^2z^2(c^2+z^2) + a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 + a^2b^2c^2$. ... 158

8998. (Rev. T. R. Terry, M.A.)—Solve the equation

$$\frac{d^3y}{dx^3} - 6x\frac{d^2y}{dx^2} + (12x^2-6) - \frac{dy}{dx}4(2x^2-3)xy = 0. \text{ .... } 108$$

9002. (E. M. Langley, M.A.)—If a circle and a Simson-line of one of its points be both inverted with regard to that point, the two inverses will have the same relation to each and to the given point that the originals have. 109

9003. (R. F. Davis, M.A.)—If upon each side of a triangle a pair of points be taken so that the pairs on any two sides are concyclic, prove that all three pairs are concyclic. 93, 158

9008. (S. Roberts, M.A.)—Given three circles C_1, C_2, C_3, determine a circle cutting C_1 orthogonally, bisecting C_2, and bisected by C_3; and show that in general there are two such circles which may coincide or become imaginary. 159

9009. (Rev. T. P. Kirkman, M.A., F.R.S.)—P and Q are a regular 20-edron and 8-edron. KLMN are 4-edra, each on a base that covers a face of P or of Q. Required the number of polyedra, of which none is either the repetition or the reflected image of another, that can be made by laying one or more of KLMN on as many faces of P or of Q, with an account of the summits and symmetry of the constructed solids. 125

9010. (J. J. Walker, F.R.S.) — Prove that the area contained by two tangents to a central conic and the semi-diameters to points of contact is equal to $b^2x^2+a^2y^2-a^2b^2$. [See Quest. 3099, Vol. xiv., pp. 74, 75.]... 92

9011. (R. Lachlan, B.A.) — Show that the product of the three normals drawn from any point on a conic is equal to the product of the perpendiculars from the point on the asymptotes and the diameter of curvature at the point. 106

9013. (Emile Vigarié.)—Les projections orthogonales de deux *points inverses* M_1, M_2 (x', y', z') sur les trois côtés d'un triangle ABC, sont six points d'une même circonférence dont l'équation en coordonnées normales

est : $(yz \sin A + zx \sin B + xy \sin C)(x' \sin A + y' \sin B + z' \sin C)$
$\times (y'z' \sin A + z'x' \sin B + x'y' \sin C)$

$$= \sin A \sin B \sin C\,(x \sin A + y \sin B + z \sin C)\left\{\frac{xx'(y' + z' \cos A)(z' + y' \cos A)}{\sin A} + \frac{yy'(x' + z' \cos B)(z' + x' \cos B)}{\sin B} + \frac{zz'(x' + y' \cos C)(y' + x' \cos C)}{\sin C}\right\}.$$

[M. Vigarié remarks that "on appelle *points inverses* en France ce que M. Casey appelle *isogonal conjugate points* (*Sequel to Euclid*, 1886, p. 166)."] 93

9017. (A. Russell, B.A.)—If $x_1, x_2, \ldots, x_n = a_1, a_2, \ldots, a_n$, prove that

$$\frac{(a_1 - x_1)\ldots(a_1 - x_n)}{a_1(a_1 - a_2)\ldots(a_1 - a_n)} + \frac{(a_2 - x_1)\ldots(a_2 - x_n)}{a_2(a_2 - a_3)\ldots(a_2 - a_1)} + \ldots = 0. \;\ldots\ldots\; 124$$

9020. (F. Morley, B.A.)—ABDC is a parallelogram; O is any point on the line bisecting the angle A; CO, BO meet BD, CD in E, F; prove that BE = CF. 104

9024. (Professor Sylvester, F.R.S.)—For greater distinctness, the name of *Hyper-cartesian* (not to be confounded with a *hyper-cartesic*) being given to that particular form of the bicircular quartic in which four concyclic foci become collinear; prove that, if four points are given in a plane, the locus of the curve *in space* whose distances from any three of them are subject to a given homogeneous linear relation is a curve of the 4th order. (This space curve may be termed a Hyper-cartesic.) 160

9026. (Professor Catalan)—Soit

$$a(a+1)(a+2)\ldots(a+c) \pm b(b+1)(b+2)\ldots(b+c) = (a+b+c)\,\phi(a, b).$$

(Le signe +, si c est pair). (1) $\phi(a, b)$ est un polynôme entier, à coefficients entiers ; (2) si a, b sont remplacés par des nombres entiers, $\phi(a, b)$ devient un nombre entier ; (3) pour ces valeurs de a, b,

$$(a+b+c)\,\phi(a, b) = \mathfrak{M}\,[1 \,.\, 2 \,.\, 3 \ldots (c+1)];$$

(4) si, en outre, $a+b+c$ est premier, $\phi(a, b) = \mathfrak{M}\,[1 \,.\, 2 \,.\, 3 \ldots (c+1)]$. 101

9032. (For Enunciation see Question 8876.) 108

9060. (Maurice D'Ocagne.)—AB et MN étant deux diamètres d'un même cercle, si une parallèle quelconque à AB coupe la corde NA en B′, la corde NB en A′, et que les droites MA′, MB′ rencontrent le cercle O respectivement aux points A″ et B″, les droites AA″, BB″ se coupent au pied H de la perpendiculaire abaissée du point N sur la droite A′B′... 113

9063. (Mahendra Nath Ray, M.A., LL.B.)—If $a_1, a_2, a_3 \ldots a_{2n-1}$ be $2n-1$ positive numbers connected by the relation $a_1a_2a_3 \ldots a_{2n-1} = 1$; show, *by elementary algebra only*, that the minimum value of

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4)\ldots(1+a_{2n-1}) \text{ is } 2^{2n-1}. \;\ldots\ldots\; 161$$

APPENDIX I.

Solutions of Questions 8886 and 9009, by Rev. T. P. Kirkman, M.A., F.R.S. 125

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Miscellaneous Solutions and Questions, by R. F. Davis, M.A. 128

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APPENDIX IV.

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CORRIGENDA.

p. 99, line 5 from bottom, *for* $\frac{S}{S}$ *read* $\frac{S'}{S}$.

p. 161, line 7 from bottom, *for* $c/(\ldots)$ *read* $4c(\ldots)$.

p. 162, last line should be thus:—

$$\tfrac{1}{36} D_1^2 D_2^2 D_3^2 \times 4 = (\tfrac{1}{3} D_1 D_2 D_3)^2.$$

p. 163, line 10, *read* $r = \ldots = a(1 - e\cos u)$.

" " 23, *for* T *read* J.

" " 24, *read* $(\ldots \sin u_2)^2$.

" " 31, *read* $2e(\sin u_1 - \sin u_2)$ times.

p. 164, line 3, *read* $(\ldots \cos u_2)$, $(\ldots + \sin u_2)$.

" " 4, *for* T *read* J.

" " 11, *read* $\ldots(\ldots)\frac{1-\cos(u_2-u_1)}{1+\cos(u_2-u_1)}$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2810. (By Professor SYLVESTER, F.R.S.)—Let S_i be used as the symbol of the sum of i-ary products; required to prove that, if i is $< n$,

$$\Sigma S_i(a_2, a_3, \ldots, a_n)\frac{(a_1-\alpha_1)(a_1-\alpha_2)\ldots(a_1-\alpha_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}$$
$$= S_{i+1}(a_1, a_2, \ldots, a_n) - S_{i+1}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

[For example, let $n = 3$, $i = 2$, then the theorem becomes

$$bc\frac{(a-\alpha)(a-\beta)(a-\gamma)}{(a-b)(a-c)} + ca\frac{(b-\alpha)(b-\beta)(b-\gamma)}{(b-a)(b-c)} + ab\frac{(c-\alpha)(c-\beta)(c-\gamma)}{(c-a)(c-b)}$$
$$= abc - \alpha\beta\gamma,$$ which is obviously true.]

Solution by W. J. CURRAN SHARP, M.A.

Assume $(x-a_1)(x-a_2)\ldots(x-a_n) - (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$

$$\equiv P_1(x-a_2)(x-a_3)\ldots(x-a_n) + P_2(x-a_3)(x-a_4)\ldots(x-a_n)(x-a_1)$$
$$+ \ldots + P_n(x-a_1)(x-a_2)\ldots(x-a_{n-1});$$

then, by putting $x = a_i$, it appears that

$$-(a_i-\alpha_1)(a_i-\alpha_2)\ldots(a_i-\alpha_n) = P_i(a_i-a_{i+1})\ldots(a_{i-}-a_1).$$

Hence we have $$P_i = -\frac{(a_i-\alpha_1)(a_i-\alpha_2)\ldots(a_i-\alpha_n)}{(a_i-a_{i+1})\ldots(a_i-a_1)\ldots(a_i-a_{i-1})},$$

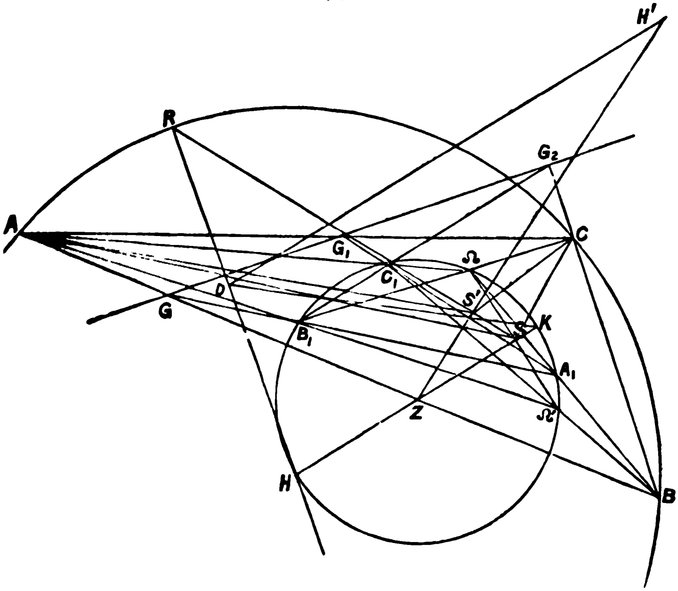
and equating the coefficients of $n-i-1$,

$$(-1)^{i+1}\left\{S_{i+1}(a_1, a_2\ldots a_n) - S_{i+1}(\alpha_1, \alpha_2\ldots\alpha_n)\right\}$$
$$= (-1)^i\Sigma_1^n P_i S_i(a_1, a_2\ldots a_{i-1}, a_{i+1}, a_n);$$

therefore $$\Sigma S_i(a_2, a_3\ldots a_n)\frac{(a_1-\alpha_1)(a_1-\alpha_2)\ldots(a_1-\alpha_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_2-a_n)}$$
$$= S_{i+1}(a_1, a_2\ldots a_n) - S_{i+1}(\alpha_1, \alpha_2\ldots\alpha_n).$$

8816. (By Captain H. Brocard.)—Ω, Ω' désignant les points de Brocard, S le milieu de $\Omega\Omega'$, S′ le point correspondant à S par droites symétriques, R le point de Steiner ; démontrer que ce point R se trouve sur la droite SS′. (R est une des intersections du cercle ABC avec le diamètre perpendiculaire à l'axe d'homologie du triangle ABC et du premier triangle de Brocard. Ce point est aussi l'intersection commune des parallèles aux côtés du premier triangle de Brocard, menées par les sommets correspondants du triangle ABC.)

Solutions by (1) D. Biddle ; (2) Professor De Longchamps.



1. In the triangle ABC, let Ω, Ω' be the Brocard-points ; ARCB, the circumscribed circle, with centre H ; $H\Omega\Omega'$, the Brocard-circle ; $A_1B_1C_1$, the first triangle of Brocard ; GG_2, the axis of homology of the triangles ABC and $A_1B_1C_1$; RN, the diameter of the circumscribed circle perpendicular to GG_2 ; S, the mid-point of $\Omega\Omega'$, and S′ its corresponding point. It is required to prove that S, S′, R are in one straight line. This will be the case if $(R_a - S'_a) : (S'_a - S_a) = (R_b - S'_b) : (S'_b - S_b)$, in which R_a, R_b... are the distances of the points from BC, AC respectively. Let Δ = area of ABC ; $a^2+b^2+c^2 = m^2$; $a^2b^2+a^2c^2+b^2c^2 = n^4$; $a^4+b^4+c^4 = p^4$. Now S_a, S_b, S'_a, S'_b are well known, and are as follows :—

$$S_a = \frac{\Delta}{n^4}\, a\,(b^2+c^2), \qquad S_b = -\frac{\Delta}{n^4}\, b\,(a^2+c^2),$$

$$S' = \frac{2\Delta\,(a^2+b^2)(a^2+c^2)}{(3n^4+p^4)\,a}, \quad S'_b = -\frac{2\Delta\,(a^2+b^2)(b^2+c^2)}{(3n^4+p^4)\,b}.$$

In order to find R_a and R_b, we have H, the circum-centre of ABC, with H_a and H_b known, HR the radius of the circum-circle, and D, the centre of homology of ABC and $A_1B_1C_1$, with D_a and D_b, known.

$$HR = \frac{abc}{4\Delta};\ H_a = \frac{2\Delta}{2n^4-p^4}a(m^2-a^2);\ H_b = -\frac{2\Delta}{2n^4-p^4}b(m^2-2b^2);$$

$$D_a = \frac{2\Delta a^2b^2c^2}{n^4a^3};\ D_b = -\frac{2\Delta a^2b^2c^2}{n^4b^3}.\ \text{Also } HD = \frac{abc}{4\Delta}\cdot\frac{p^4-n^4}{n^4}.$$

Now $R_a = H_a + (D_a - H_a)\,HR\,/\,HD = \dfrac{2\Delta\left[a^2b^2c^2 - a^4(m^2-2a^2)\right]}{a^3(p^4-n^4)}$

$$R_b = H_b + (D_b - H_b)\,HR\,/\,HD = -\frac{2\Delta\left[a^2b^2c^2 - b^4(m^2-2b^2)\right]}{b^3(p^4-n^4)};$$

therefore $(R_a - S'_a)\,/\,(S'_a - S_a)$

$$= \frac{2n^4\left\{\left[a^2b^2c^2-a^4(m^2-2a^2)\right](3n^4+p^4)a - a^3(p^4-n^4)(a^2+b^2)(a^2+c^2)\right\}}{a^3(p^4-n^4)\left\{2n^4(a^2+b^2)(a^2+c^2)-(3n^4+p^4)a^2(b^2+c^2)\right\}},$$

and $(R_b - S'_b)\,/\,(S'_b - S_b)$

$$= \frac{2n^4\left\{b^3(p^4-n^4)(a^2+b^2)(b^2+c^2) - \left[a^2b^2c^2-b^4(m^2-2b^2)\right](3n^4+p^4)b\right\}}{b^3(p^4-n^4)\left\{(3n^4+p^4)b^2(a^2+c^2)-2n^4(a^2+b^2)(b^2+c^2)\right\}}.$$

On the first reduction, $(R_a - S'_a)\,/\,(S'_a - S_a) : (R_b - S'_b)\,/\,(S'_b - S_b)$

$$= \frac{\left[b^2c^2-a^2(m^2-2a^2)\right](3n^4+p^4)-(p^4-n^4)(a^2+b^2)(a^2+c^2)}{2n^4(a^2+b^2)(a^2+c^2)-(3n^4+p^4)a^2(b^2+c^2)} :$$

$$\frac{\left[a^2c^2-b^2(m^2-2b^2)\right](3n^4+p^4)-(p^4-n^4)(a^2+b^2)(b^2+c^2)}{2n^4(a^2+b^2)(b^2+c^2)-(3n^4+p^4)b^2(a^2+c^2)}$$

$$= \frac{2n^4(2a^4-a^2b^2-a^2c^2+2b^2c^2)-2p^4(a^2c^2+a^2b^2)}{n^4(2a^4-a^2b^2-a^2c^2+2b^2c^2)-p^4(a^2c^2+a^2b^2)} :$$

$$\frac{2n^4(2b^4-a^2b^2-b^2c^2+2a^2c^2)-2p^4(b^2c^2+a^2b^2)}{n^4(2b^4-a^2b^2-b^2c^2+2a^2c^2)-p^4(b^2c^2+a^2b^2)} = 2 : 2.$$

Wherefore, R, S′, and S are in one straight line. Moreover, since $2n^4\,/\,(p^4-n^4)$ was the common factor left out on the first reduction,

$$RS' : S'S = 4n^4 : p^4 - n^4.$$

2. *Otherwise :*—Le point S, milieu de la droite qui joint les points de Brocard, est représenté en coordonnées barycentriques par

$$a^2(b^2+c^2),\quad b^2(c^2+a^2),\quad c^2(a^2+b^2).$$

Le point S′, le *point inverse*, suivant la dénomination du Colonel Mathieu, a donc pour coordonnées $(b^2+c^2)^{-1},\ (c^2+a^2)^{-1},\ (a^2+b^2)^{-1}$.

Enfin, le point de Steiner R a pour coordonnées, comme l'on sait,

$$(b^2-c^2)^{-1},\quad (c^2-a^2)^{-1},\quad (a^2-b^2)^{-1}.$$

On conclut de là que l'équation de RS′ est

$$\begin{vmatrix} \alpha & \beta & \gamma \\ (b^2-c^2)^{-1}, & (c^2-a^2)^{-1}, & (a^2-b^2)^{-1} \\ (b^2+c^2)^{-1}, & (c^2+a^2)^{-1}, & (a^2+b^2)^{-1} \end{vmatrix} = 0,$$

ou, après développement, $\Sigma\alpha\,(a^4-b^2c^2)(b^4-c^4) = 0$ .... (1).

D'après cela, on voit que la droite Δ qui correspond à cette équation passe (1) par le point S, parce que l'on a

$$\Sigma a^2 (b^2+c^2)(a^4-b^2c^2)(b^4-c^4) \equiv 0;$$

(2) par le point J dont les coordonnées sont :

$$b^2c^2+a^4, \quad c^2a^2+b^4, \quad a^2b^2+c^4.$$

Cette dernière propriété a été signalée déjà par M. Brocard au Congrès de Rouen. [Voyez *l'Annuaire de l'Association Française pour l'avancement des Sciences*, 1883, p. 104.]

Un cinquième point, situé sur Δ, apparaît immédiatement d'après l'équation (1) ; nous voulons parler d'un point U dont les coordonnées sont $\quad (a^4-b^2c^2)^{-1}, \quad (b^4-c^2a^2)^{-1}, \quad (c^4-a^2b^2)^{-1}.$

Ce point U se rencontre fréquemment dans la géométrie triangulaire et il est certainement remarqué de tous ceux qui la connaissent un peu.

Lorsqu'on cherche l'équation de la droite G, axe d'homologie du triangle de référence ABC et du triangle de Brocard $A_1B_1C_1$, on trouve

$$\alpha / (a^4-b^2c^2) + \beta / (b^4-c^2a^2) + \gamma / (c^4-a^2b^2) = 0.$$

Cette droite est *harmoniquement associée* au point U_0 dont les coordonnées sont $\quad (a^4-b^2c^2), \quad (b^4-c^2a^2), \quad (c^4-a^2b^2);$

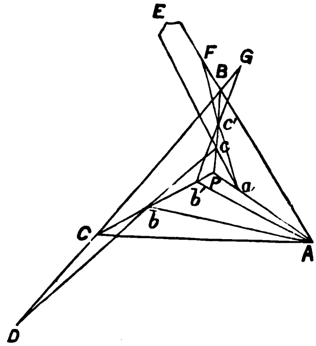
U_0 est donc, dans le sens que nous avons donné autrefois à ce mot, sens qui est aujourd'hui généralement adopté, croyons-nous, le *réciproque* de U.

[Le terme *harmoniquement associé* a été proposé par nous pour représenter une droite μ, associée à un point donné M, dans les conditions suivantes :—La droite AM rencontre BC en un certain point A' ; soit A'' le point qui forme avec BA'C une ponctuelle harmonique ; les trois points A'', B'', C'' sont situés sur une droite μ que nous appelons l'*harmonique associée de M*.]

L'équation (1) donne ainsi 5 points en ligne droite : le milieu de la droite qui joint les points de Brocard, le point inverse de celui-ci, le point de Steiner, le point de concours de la droite qui joint le centre du cercle circonscrit à l'orthocentre avec la droite qui joint le réciproque du point de Lemoine à son inverse (le point J), et enfin le point U_0, ci-dessus défini.

8801. (By Rev. T. P. Kirkman, F.R.S. Suggested by Quest. 8325.)—Upon the most sacred of the undiscovered Egyptian pyramids, whose vertex P is at unequal distances from the angles of its scalene base ABC, is cut in the face PAC an ascending flight of steps from A to *b*, in PC ; form *b* another flight mounts to *c* in PB ; and a third leads up from *c* to *a*, at a given altitude *h* in PA. These three flights are equigradient. A second path from A mounts to *b'* in PC, and thence proceeding to *c'* in PB, ascends from *c'* to the above point *a* in PA. These three flights also are equigradient. By A*bca* Pharoah and his Grandees of Church and State were wont to mount to a platform at *a* for solemn religious rites ; the admitted vulgar used only the path A*b'c'a*. Required, a plane projection of the pyramid and its six flights of stairs, and proof from that projection that the first three are equigradient, and the second three equigradient, up to the given altitude *h*.

Solution by the PROPOSER.



Let the distance bA of every point b of PC in the projection from A be greater than bB, and also than bC. We can with the ruler find through any point b in PC the equigradient path $Abca$, so that if $Pa = x$ and $Pb = y$, x is given with y, one value of x for every value of y, and that x and y vanish together; for, if we take P for our b, the lines bD and bC in the projection coincide, and P is also our point c, *i.e.*, $x = 0$ follows from $y = 0$. And evidently there cannot be two equigradient descending paths from a to A, about the pyramid, for no two lines ac and ac' across the triangle APB can be equigradient with the same Ab. That is, for every x there is one y, and one only. To find x when y is given is very easy. To find y when x is given is Mr. BIDDLE's problem in Quest. 8325, and to work out his solution of it is not easy. We must content ourselves with the method of interpolation.

We can write $$y = \phi x = Ax + Bx^2 + \ldots + Nx^n.$$

Next, taking any n points $b_1, b_2 \ldots b_n$ on PC in our projection, we get with the ruler the corresponding $a_1, a_2 \ldots a_n$ on PA, and write ($Pb_1 = y_1$, $Pa_1 = x_1$), $y_1 = \phi x_1$, $y_2 = \phi x_2 \ldots$, $y_n = \phi x_n$; from which n equations we obtain AB ... N in ϕx. Then, if $Pa = X$ be our given distance from P, $Y = \phi X$ is our required distance Pb, and when the point b is given, the path $Abca$ is easily found by the ruler, and this solves our problem with accuracy increasing without limit with n, the order of the parabola so found in the face CPA, if PA and PC are our axes of x and y. This parabola passes through (0, 0) and ($x = PA$, $y = PC$).

The second path $ac'b'A$ cannot be all through a descending path; but nothing prevents its being equigradient, if $c'b'$ is an ascent, while ac' and $b'A$ are downwards. Let b' in our projection be any point of PC. In CB produced, at G make $b'G = b'A$, cutting PB in c'. In AB produced at F make $c'F = c'G$, and produce it to a' in AP. This a' may or not be the a before found; although in the figure, by a few trials, a and a' are made to coincide. It is evident that for every b' in PC we can find an a' in PA.

The lines on the pyramid whose projections are $b'c'$ and $b'A$ are the sides of an isosceles triangle drawn to the base in the planes CPB and CPA from b' in both, and are therefore equigradient. For a like reason $c'a'$ and $c'b'$ in the planes BPA and CPB are equigradient, and the up and down path $Ab'c'a'$ is equigradient to the altitude of a'. Putting $Pa' = u$ and $Pb' = v$, we have the right, as above shown, to write

$$v = \psi u = A'u + B'u^2 + \ldots + N'u^n.$$

Taking any n points $d_1, d_2 \ldots d_n$ in PC we get easily the corresponding points $a'_1, a'_2 \ldots a'_n$ in PA; and by n equations ($Pd_1 = v_1$, $Pa_1 = u_1$), $v_1 = \psi u_1$, $v_2 = \psi u_2$, &c., we obtain the coefficients of ψu_1, so that, if our given distance $Pa = X$, we get $V = \psi X$, our required distance $Pd = Pb' = V$ on PC, which gives by the ruler the requisite up and down path $Ab'c'a$ to the given altitude of a in PA. The two flights bc and $b'c'$ must have a common step at some point p; on which, before or after the descent of the vulgar, a fitting carpet would be laid for the foot of ascending majesty.

If the *descent* of the "vulgar" took place on the third face instead of on the second, the two paths would not cross each other, and the gradient of the "common" steps might be lessened.

From the solution of Quest. 8843, it is evident that, with the concession first above asked, the solution of this problem of two equigradient paths about the pyramid from A to a in PA, is equally in our power, whatever be the polygonal base of it, and whether the vertex P is vertically over a point inside or outside the polygon.

8823. (By Professor CROFTON, F.R.S.)—Two discs of any form are moveable in a plane round two fixed points A, B, respectively. Show that, when they are in such positions that the length of an endless band enveloping both is a maximum or a minimum, the portions of the band which form the common tangents will meet on AB if produced, and are equally inclined to AB.

Solutions by (1) Professor STEGGALL, M.A.; (2) Professor SIRCOM, M.A.

1. Let the discs be rotated through infinitesimal angles $d\theta$, $d\phi$ about A, B; let p_1, p_2; q_1, q_2 be the perpendiculars from A, B on the straight parts of the string. Then the amount of slack is easily seen by a diagram to be $(p_1-p_2)\,d\theta+(q_1-q_2)\,d\phi$. This must vanish for a minimum length, and therefore $p_1=p_2$, $q_1=q_2$, since $d\theta$, $d\phi$ are independent, whence the result immediately follows.

2. *Otherwise*:—Let the discs be smooth, and the band elastic. In the position of equilibrium the length of the band will be a maximum or a minimum. But then each disc is acted on by equal forces at its rim; the common direction of the resultants will therefore pass through A, B, and be equally inclined to the directions of the forces. Hence the theorem.

5643. (By W. S. B. WOOLHOUSE, F.R.A.S.)—Any two triangles being given, the first may always be orthogonally projected into a triangle similar to the second; determine the magnitude of the projected triangle geometrically by an easy construction with the ruler and compasses.

Solutions by the PROPOSER.

1. Let a triangle ABC, having the sides a, b, c, be orthogonally projected vertically into the triangle A″B″C″, having the sides a'', b'', c'', in a hori-

zontal plane. Then, considering the neutralizing result of changes of altitude in passing first from B to C, then from C to A, and thence from A to the starting point B, there arises the following relation,

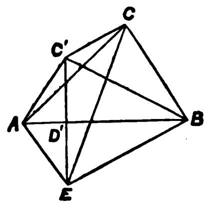
$$\sqrt{(a^2-a''^2)}+\sqrt{(b^2-b''^2)}+\sqrt{(c^2-c''^2)}=0 \text{..................(1).}$$

This relation can subsist only when the three terms of the expression are either all real or all imaginary. In the latter case the imaginary character is removed by writing it thus,

$$\sqrt{(a''^2-a^2)}+\sqrt{(b''^2-b^2)}+\sqrt{(c''^2-c^2)}=0 \text{ ....(2).}$$

But the form (1) implies that ABC is projected *into* A″B″C″; and (2) implies that ABC is projected *from* A″B″C″.

Hence, when (1) and (2) are cleared of radicals, the involved algebraical relation amongst the sides a, b, c, and a'', b'', c'', will be symmetrical and identically the same whether the triangle ABC be projected into or projected from the triangle A″B″C″.



According to the problem, A″B″C″ is required to be similar to another given triangle. Upon AB construct a triangle ABC′ similar to this given triangle; and let the sides of the triangle so constructed be a', b', c. The sought triangle A″B″C″ being similar to this, we can assume that $a''=ka'$, $b''=kb'$, $c''=kc$,

in which the common factor k remains to be determined. The relation (1) now becomes

$$\sqrt{(a^2-k^2a'^2)}+\sqrt{(b^2-k^2b'^2)}+\sqrt{(c^2-k^2c^2)}=0 \text{ ....(3).}$$

If Δ, Δ' denote the respective areas of the two triangles ABC, ABC′;

then
$$\left.\begin{aligned}16\Delta^2&=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4\\16\Delta'^2&=2a'^2b'^2+2a'^2c^2+2b'^2c^2-a'^4-b'^4-c^4\end{aligned}\right\}\text{ ....(4).}$$

To free (3) from surds, conceive the three terms to be the sides of a supposed triangle; then, according to the stated condition, the area of such triangle is obviously zero. Hence, substituting, in (4),

$$0=2(a^2-k^2a'^2)(b^2-k^2b'^2)+2(a^2-k^2a'^2)(c^2-k^2c^2)+2(b^2-k^2b'^2)(c^2-k^2c^2)$$
$$-(a^2-k^2a'^2)^2-(b^2-k^2b'^2)^2-(c^2-k^2c^2)^2,$$

which, arranged according to powers of k, is, by (4),

$$0=16\Delta'^2k^4-2\left\{(b^2+c^2-a^2)\,a'^2+(a^2+c^2-b^2)\,b'^2+(a^2+b^2-c^2)\,c^2\right\}k^2+16\Delta^2,$$

or
$$k^4-\frac{(b^2+c^2-a^2)\,a'^2+(a^2+c^2-b^2)\,b'^2+(a^2+b^2-c^2)\,c^2}{8\Delta'^2}k^2+\frac{\Delta^2}{\Delta'^2}=0\text{...(5).}$$

But, joining CC′ and denoting this line by γ, then

$$2c^2\gamma^2=2c^2\left\{b^2+b'^2-2bb'\cos(\mathrm{A}\mp\mathrm{A}')\right\},$$

according as C, C′ are on the same or on opposite sides of the base AB.

That is,
$$\begin{aligned}2c^2\gamma^2&=2c^2(b^2+b'^2)-4c^2bb'(\cos\mathrm{A}\cos\mathrm{A}'\pm\sin\mathrm{A}\sin\mathrm{A}')\\&=2c^2(b^2+b'^2)-(b^2+c^2-a^2)(b'^2+c^2-a'^2)\mp16\Delta\Delta'\\&=(b^2+c^2-a^2)\,a'^2+(a^2+c^2-b^2)\,b'^2+(a^2+b^2-c^2)\,c^2\mp16\Delta\Delta'.\end{aligned}$$

Denoting CE, CC′, the two values of γ, by γ_1, γ_2 respectively, we have therefore

$$c^2(\gamma_1^2+\gamma_2^2)=(b^2+c^2-a^2)\,a'^2+(a^2+c^2-b^2)\,b'^2+(a^2+b^2-c^2)\,c^2,$$
$$c^2(\gamma_1^2-\gamma_2^2)=16\Delta\Delta'.$$

Hence (5) may be stated thus,

$$k^4 - \frac{c^2(\gamma_1^2+\gamma_2^2)}{8\Delta'^2}k^2 + \frac{c^4(\gamma_1^2-\gamma_2^2)^2}{256\Delta'^4} = 0,$$

the solution of which is

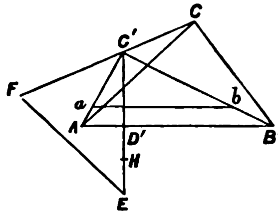
$$k^2 = c^2\frac{\gamma_1^2+\gamma_2^2}{16\Delta'^2} \mp c^2\frac{2\gamma_1\gamma_2}{16\Delta'^2} = \frac{c^2}{16\Delta'^2}(\gamma_1\mp\gamma_2)^2;$$

therefore $$k = \frac{c}{4\Delta'}(\gamma_1 \mp \gamma_2) = \frac{\gamma_1\mp\gamma_2}{2p'},$$

and the perpendicular of the sought triangle is $p'' = kp' = \frac{1}{2}(\gamma_1\mp\gamma_2)$.

We have therefore the following

Construction. — Let ABC be the first of the given triangles; and on the base AB describe ABC′ similar to the other given triangle. Then the projection of ABC is required to be similar to ABC′, and is constructed as follows:

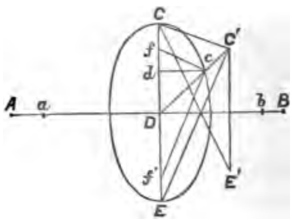


From C′ demit the perpendicular C′D′, and in continuation of it make D′E equal to C′D′. Draw a straight line through the vertices C and C′, and on it take CF equal to CE; then on C′E take C′H equal to C′F; and lastly bisect C′H by the perpendicular line *ab*, thereby cutting off from ABC′ the required projected triangle.

It may be observed that C and E need not be joined, and also that the problem admits of three essentially different projections, for BCA, CAB may each in like manner be projected into triangles respectively similar to ABC′.

2. *Otherwise, Geometrically*:—Take the plane of the paper for the plane of projection. Let *ab* be the projection of AB, the triangle being turned round the perpendicular CD as an axis so as to bring the point A vertically above *a* and B vertically below *b*. Then, denoting by ϵ the angle of inclination of the side AB with the plane of projection, we shall have $Da = DA\cos\epsilon$, $Db = DB\cos\epsilon$,

$$ab = AB\cos\epsilon.$$



To completely generalize the arrangement, now conceive the triangle ABC in its new position to turn about the side AB as an axis. The vertex C will describe a circle having its plane inclined to the plane of projection at an angle equal to the complement of ϵ. This circle will therefore project into an ellipse the semi-diameters of which are $\alpha = CD$, $\beta = CD\sin\epsilon$; and the excentricity is hence $e = \cos\epsilon$.

The periphery of the ellipse is the locus of the projection of C. Let *c* be the actual projection of C, and *abc* will then be the resulting projection of the triangle ABC from the position it has finally acquired. As this projection is required to be similar to the triangle ABC′, the projected vertex *c* must fall on the straight line DC′, making $Dc = DC'\cos\epsilon = e \,.\, DC'$. Let f, f' be the foci of the ellipse; draw C′H perpendicular to AB, making

D′H = CD′; *cd* perpendicular to CE; and other lines joining various points as represented in the diagram. Then $Df = e\,.\,DC$, $Df' = e\,.\,DE$; therefore cf, cf' are respectively parallel to C′C, C′E.

$$\text{By Conics,}\quad Dd\,(=x) = \frac{cf'-cf}{2e} = \tfrac{1}{2}(C'E - C'C) = \tfrac{1}{2}(CE' - CC'),$$

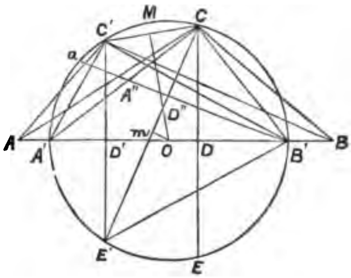
which is the perpendicular of the sought triangle *abc*, and determines its magnitude by a simple and easy construction.

3. From the preceding article we have

$$\cos\epsilon = \frac{Dd}{C'D'},\quad \cos E = \frac{abc}{ABC} = \frac{Dd\,.\,ab}{CD\,.\,AB} = \frac{Dd}{CD}\cos\epsilon = \frac{Dd^2}{CD\,.\,C'D'},$$

which determine the respective inclinations of the side AB and of the plane of the triangle ABC with the plane of projection.

4. *Another Geometrical Solution* may be obtained thus:—Let ABC be the triangle to be projected from, and on the base AB let ABC′ be constructed similar to the triangle into which ABC is to be projected. The plane of the paper and of the triangle ABC′ being taken as the plane of projection, the position of the triangle in space may be assumed to be suitably located so as to cause the projection of AB to fall on the line AB, and such that the perpendicular of the projected triangle shall fall on C′D′.



In effect, as regards the projected triangle, the point D is removed to D′, the segments DA, DB are proportionally shortened by projection, and the perpendicular is the same as C′D′ when diminished in a like proportion. Now, if we retain the various positions indicated and in either case conceive AB to be an indefinite line, and assume in it any two points A′ and B′, the distances DA′, DB′ will obviously be reduced by projection in exactly the same ratio. It hence follows that, wherever the points A′, B′ are taken in AB, the triangle A′B′C will be projected into a triangle similar to A′B′C′. Taking advantage of this circumstance, such points may be conveniently chosen so as to make the angles at C and C′ both right angles; thus, join CC′, bisect it with the perpendicular MO meeting AB in O, and with O as a centre describe the circle CC′A′B′.

A consideration of what has been advanced establishes the general proposition that any projection of the figure which makes A′B′C similar to A′B′C′ will, at the same time, make the triangle ABC similar to ABC′ as required. As the angle C of the triangle A′B′C after projection must remain a right angle, the most simple and direct mode of projection is arrived at by drawing B′A″ meeting A′C, and making the angle A″B′C equal to A′B′C′. For, if A′B′C merely turn round B′C as a horizontal axis until A″ becomes the projection of A′, then the projection of A′B′C will be A″B′C, which is obviously similar to A′B′C′.

Produce the perpendiculars CD, C′D′ to meet the circle again in E, E′, and join B′E′; also join CE′ intersecting A″B′ in D″ and bisect it in *m*. The angle E′CB′ is equal to E′C′B′ or the complement of A′B′C′ or the complement A″B′C′. Therefore CE′ is perpendicular to A″B′ and

CD″ is the perpendicular of the projected triangle. By the geometry of th diagram,

$$Cm = \tfrac{1}{2}CE', \quad mD'' = \tfrac{1}{2}CC';$$

therefore $$CD'' = \tfrac{1}{2}(CE' - CC'), \quad ED'' = \tfrac{1}{2}(CE' + CC').$$

These are wholly independent of the position of the points A, B in the line AB. The former is the perpendicular of the projected triangle; the latter is the perpendicular of the triangle similar to ABC′ from which ABC may be projected.

Notes.—If E denote the inclination of the plane of the triangle with the horizontal plane, then

$$\cos E = \frac{A''C}{A'C} = \frac{CD''}{ED''} = \frac{CE' - CC'}{CE' + CC'}.$$

The line CA′ bisects the angle C′CE′; the line CB′ is perpendicular to it, and therefore bisects the exterior angle C of the triangle C′CE′. It has been shown that the latter of these lines determines the direction of the intersection of the projecting plane with the horizontal plane of projection.

[An attempt at a solution, the only one heretofore received, is given on p. 24 of Vol. xxx.]

8793. (By Professor Neuberg.)—Soient AB, CD deux droites, qui se coupent en O. Si l'on fait tourner AB autour de O, le point double de deux figures semblables construites sur AB et CD décrit une circonférence de cercle.

Solution by Professors Schoute, Beyens, *and others.*

For any position of AB the centre of similitude of the two similar figures on AB and CD is evidently the second point of intersection P of the two circles AOB, COD. Therefore transformation of the problem by reciprocal radies with the centre O brings it in the following form:—Two lines A′B′, C′D′ intersect at O; to find the locus of the point P′ common to A′C′ and B′D′, when A′B′ rotates about O. Considering A′C′P′ as a transversal of the triangle OB′D′, we find

$$(A'B' / A'O) \cdot (C'O / C'D') \cdot (P'D' / P'B') = 1,$$

and as A′B′ / A′O and C′O / C′D′ are constant, the same may be said of P′D′ / P′B′. Thus the locus of P′ is a circle, for which D′ is a centre of similitude with respect to the circle described by B′, and the locus of P is also a circle.

8639. (By Professor Wolstenholme, M.A., Sc.D.)—The distances of a point O (within a given triangle ABC) from the angular points are x, y, z respectively; prove that the volume of a tetrahedron in which a, b, c are the lengths of three conterminous edges, and x, y, z the lengths of the edges respectively opposite, is $\frac{1}{12}\left\{(a^2-x^2)(b^2-y^2)(c^2-z^2)\right\}^{\frac{1}{2}}$.

Solution by Professor NEUBERG; Professor GENESE, M.A.; *and others*.

Soit V le volume du tétraèdre en question; on sait que

$$144V^2 = \Sigma a^2x^2(b^2+y^2+c^2+z^2-a^2-x^2)-x^2y^2z^2-a^2b^2z^2-b^2c^2x^2-c^2a^2y^2.$$

Soit V′ le volume d'un tétraèdre dont x, y, z sont les arêtes d'un angle solide, et a, b, c les arêtes respectivement opposés; on aura

$$144V'^2 = \Sigma a^2x^2(b^2+y^2+c^2+z^2-a^2-x^2)-a^2b^2c^2-a^2y^2z^2-b^2z^2x^2-c^2x^2y^2.$$

D'où $$144(V^2-V'^2) = (a^2-x^2)(b^2-y^2)(c^2-z^2).$$

Dans la question proposée, on a $V' = 0$.

8713. (By Professor STEGGALL, M.A.)—Show that the solution in rational quantities of the equation $x^2+y^2+z^2 = u^2$, is $x = k(a^2+b^2-c^2)$, $y = 2kac$, $z = 2kbc$, $u = k(a^2+b^2+c^2)$.

Solution by W. J. BARTON, M.A.; SARAH MARKS; *and others*.

Let $y/z = a/b$; substituting for y, we get $(a^2+b^2)z^2 = b^2(u^2-x^2)$. This equation is satisfied by $u-x = kc^2$, $u+x = k(a^2+b^2)$; which give $u = \frac{1}{2}k(a^2+b^2+c^2)$, $x = \frac{1}{2}k(a^2+b^2-c^2)$, $z = kbc$, $y = kac$.

8804. (By S. TEBAY, B.A.)—If (a, b, c) be conterminous edges of a tetrahedron, (x, y, z) the respective opposites, (A, X; B, Y; C, Z) corresponding dihedral angles, and V the volume; show that (1)

$$\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z};$$

also (2), if the areas of the four faces are equal, then

$$a = x, \quad b = y, \quad c = z, \quad A = X, \quad B = Y, \quad C = Z.$$

$$V = \tfrac{1}{12}\left\{2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)\right\}^{\frac{1}{2}},$$

$$(b-c)\sin A + (c-a)\sin B + (a-b)\sin C = 0.$$

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

(1) If $\alpha_1, \beta_1, \gamma_1$ be the plane angles at the point where a, b, c meet, X the dihedral angle opposite x, $\cos X = \dfrac{\cos\alpha_1 - \cos\beta_1\cos\gamma_1}{\sin\beta_1\sin\gamma_1}$,

whence $\sin X = \dfrac{(1-\cos^2\alpha_1-\cos^2\beta_1-\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}}{\sin\beta_1\sin\gamma_1}$,

but volume (V) $= \frac{1}{6}abc(1-\cos^2\alpha_1-\cos^2\beta_1-\cos^2\gamma_1+2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}$;

also, if R_1, R_2, R_3, R_4 be the radii of the circles circumscribing the triangular faces, $\sin\beta_1 = \frac{y}{2R_2}$, $\sin\gamma_1 = \frac{z}{2R_3}$, &c.,

whence $\sin X = \frac{24 V R_2 R_3}{abc\,yz}$; and $\frac{x}{\sin X} = \frac{abc\,xyz}{24 V R_2 R_3}$;

and, similarly, $\frac{a}{\sin A} = \frac{abc\,xyz}{24 V R_1 R_4}$;

or $$\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z} = \frac{(abc\,xyz)^2}{576 V^2 R_1 R_2 R_3 R_4}.$$

If A_1, A_2, A_3, A_4 denote the areas of the faces,

$$R_1 = \frac{xyz}{4A_1},\quad R_2 = \frac{xbc}{4A_2},\quad R_3 = \frac{ayc}{4A_3},\quad R_4 = \frac{abz}{4A_4};$$

or $$\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z} = \frac{4}{9}\,\frac{A_1 A_2 A_3 A_4}{V^2}.$$

(2) In an equifacial tetrahedron, $a = x$, $b = y$, $c = z$; $A = X$, $B = Y$, $C = Z$; and $\alpha_1 + \beta_1 + \gamma_1 = \pi$, so that the volume

$$= \tfrac{1}{6} abc\,(1 - \cos^2\alpha_1 - \cos^2\beta_1 - \cos^2\gamma_1 + 2\cos\alpha_1\cos\beta_1\cos\gamma_1)^{\frac{1}{2}}$$
$$= \tfrac{1}{3} abc\,(\cos\alpha_1 \cos\beta_1 \cos\gamma_1)^{\frac{1}{2}}$$
$$= \tfrac{1}{12}\left[2\,(b^2 + c^2 - a^2)\,(c^2 + a^2 - b^2)\,(a^2 + b^2 - c^2)\right]^{\frac{1}{2}}.$$

The equations of (1) become $\frac{a^2}{\sin^2 A} = \frac{b^2}{\sin^2 B} = \frac{c^2}{\sin^2 C} = \frac{4}{9}\,\frac{A^4}{V^2}$,

where A is the area of any face; whence

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{2}{3}\,\frac{A^2}{V},$$

and $$(b - c)\sin A + (c - a)\sin B + (a - b)\sin C = 0.$$

In such a tetrahedron, the radius of the circumscribed sphere is $\tfrac{1}{4}\left[2\,(a^2 + b^2 + c^2)\right]^{\frac{1}{2}}$, a result I sent to the *Educational Times* long ago. I cannot discover any single length connected with the tetrahedron whose square is equal to $ax / \sin A \sin X$, and I do not know that we have any right to expect it. I have much admiration of this question.

The equations

$$\left(\frac{ax}{\sin A \sin X} = \frac{by}{\sin B \sin Y} = \frac{cz}{\sin C \sin Z} = \frac{4}{9}\,\frac{A_1 A_2 A_3 A_4}{V}\right)$$

are strictly analogous to the equations $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{4S}$,

and, writing the numerical coefficients $(\frac{2}{3})^2$, $(\frac{1}{2})^2$, it would suggest that the general form in geometry of n dimensions is $\{(n-1)/n\}^2$. This I leave to those who are experienced in such high matters.

8696. (By E. Vigarié.)—Dans un cercle donné, par un point P pris sur la circonférence on mène trois cordes PA, PB, PC; sur chacune d'elles comme diamètre on décrit une circonférence; démontrer que les trois points de rencontre sont en ligne droit.

Solution by R. F. DAVIS, M.A.; H. BRAKSPEAR; *and others.*

For, if a, b, c be the feet of the perpendiculars from P upon the sides of the triangle ABC, the circle described on PA as diameter passes through b, c; while a, b, c are collinear.

8647. (By R. W. D. CHRISTIE, M.A.)—Prove that $\Sigma + S = 2s^2$, if

$$s = 1^3 + 2^3 + \ldots + n^3, \quad S = 1^5 + 2^5 + \ldots n^5, \quad \Sigma = 1^7 + 2^7 + \ldots n^7.$$

Solution by Rev. D. THOMAS, M.A.; Professor BEYENS; *and others.*

Assuming the relation to be true for n, it will be true for $n+1$ if

$$\Sigma + S + (n+1)^5 + (n+1)^7 = 2\left[s^2 + 2s(n+1)^3 + (n+1)^6\right],$$

or if

$$4s = (n+1)^4 + (n+1)^2 - 2(n+1)^3 = [n(n+1)]^2,$$

and this is the case, therefore, &c.

8726. (By S. ROBERTS, M.A. Analogous to Quest. 3068.)—Show that, if some four of the roots of a quintic form an harmonic system, then

$$J^3 - 2^7 \cdot 3^2 \cdot JK + 2^{13} \cdot 3^3 \cdot L = 0,$$

where J, K, L are the three fundamental invariants of the orders 4, 8, 12 [see SALMON's *Higher Algebra*, 3rd ed., p. 211].

Solution by the PROPOSER.

In the *Proceedings of the London Mathematical Society*, Vol. XIV., Prof. M. J. M. HILL gave the equation determining the anharmonic ratios of the roots of a quintic. This is of the order 24 in the coefficients, and, substituting therein any value for the variable, we get the condition that some four of the roots of the quintic may form an anharmonic ratio having that value.

But in our particular case the condition is of the reduced order 12, and is therefore of the form $J^3 + kJK + lL = 0$. Taking the form

$$x(ax^4 + 10cx^4 + 5e)$$

(which we can do, since it only implies the evanescence of HERMITE's Skew Invariant of the order 18), we have for the T of the quartic factor

$c\,(ae-\frac{5}{3}c^2)$, and, substituting $\frac{5}{3}c^2$ for ae in

$$J = 16e\,(ace+3c^3),$$
$$K = -2a^3e^5+22a^2c^2e^4-38ac^4e^3+18c^6e^2,$$
$$L = 4ce^3\,(ae-c^2)^4,$$

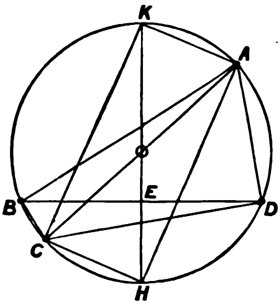
we get $$J = \frac{512}{9}c^3e,\quad K = \frac{2432}{9^3}e^2c^6,\quad L = 2\,\frac{512}{9^4}e^3c^9,$$

giving $(9\cdot 512)^2+k\,2432+2l = 0$. But also, if $a = 0$, the roots of

$$10cx^3y+5exy^3 = 0$$

form an harmonic system, so that $48^3+k\,18\,.\,48+4l = 0$. The required invariant condition is therefore $J^3-\frac{1}{2}\,(48)^2JK+2\,.\,48^3\,L = 0$.

8708. (By MAURICE D'OCAGNE.)—Si, dans le quadrilatère ABCD, les angles opposés B et D sont droits, la droite qui joint les pieds des perpendiculaires abaissées du sommet C sur les bissectrices intérieure et extérieure de l'angle A passe par le milieu de la diagonale BD.



Solution by R. F. DAVIS, M.A.; G. G. MORRICE, M.A.; *and others.*

Bisect AC, BD in O and E respectively; then O is the circumcentre of the quadrilateral. If H, K be the extremities of the diameter OE of this circle, AH and AK are the bisectors of the angle A, and the angles CHA, CKA are right angles.

8769. (By J. BRILL, M.A.)—An ellipse is inscribed in the triangle PQR so as to touch the sides QR, RP, PQ at P′, Q′, R′ respectively; prove that, if C be the centre of the ellipse,

$$\Delta QCR : \Delta Q'CR' = \Delta RCP : \Delta R'CP' = \Delta PCQ : \Delta P'CQ'.$$

Solution by ISABEL MADDISON; BELLE EASTON; *and others.*

Let $x^2/a^2+y^2/b^2 = 1$ be the equation of the ellipse; (x_1y_1), (x_2y_2), and $(x_3\,y_3)$ the coordinates of P′, Q′, R′. Then the coordinates of P are $a^2\,(y_3-y_2)\,/\,X_1$, $b^2\,(x_2-x_3)\,/\,X_1$, where $X_1 = x_2y_3-x_3y_2$, $X_2 = x_3y_1-x_1y_3$, $X_3 = x_1y_2-x_2y_1$, and the coordinates of Q and R are found by substituting in cyclic order. After reducing, we find

$\Delta QCR : \Delta Q'CR' = (X_1+X_2+X_3)\,/\,X_1X_2X_3$, which is symmetrical;

hence $\Delta QCR : \Delta Q'CR' = \Delta RCP : \Delta R'CP' = \Delta PCQ : \Delta P'CQ'$.

8749. (By F. PURSER, M.A.)—Find (1) the cubic locus of the centre of a conic passing through the three vertices A, B, C of a triangle, and such that the three normals at these vertices are concurrent; and prove that (2) this cubic passes through the in-centre I, the three excentres I_a, I_b, I_c, the circum-centre O, the orthocentre H, the centroid G, and the symmedian G′, and cuts the three sides normally at their middle points L, M, N; also (3) tangents at (I, I_a, I_b, I_c), (A, B, C, G), (L, M, N, G′) are respectively concurrent in G, G′, O; and (4) the lines joining L, M, N meet the curve again in their intersections with the respective perpendiculars from the vertices.

Solution by W. J. GREENSTREET, B.A.; Professor BEYENS; *and others.*

(1) Let the chord joining the points α, β make an angle θ with diameter which bisects it, then we find $\cot\theta = \dfrac{c^2}{2ab}(\sin\alpha+\beta)$, therefore normals are concurrent if $\cot\theta_1+\cot\theta_2+\cot\theta_3=0$, where sides of Δ are at angles θ_1, θ_2, θ_3 to diameters bisecting them. Now let α, β, γ be perpendiculars from a point on the sides of the triangle, then

$$\cot\theta_1 = \frac{\beta\sin B-\gamma\sin C+\alpha\sin(B-C)}{2\alpha\sin B\sin C},\quad \cot\theta_2 = \&c.,\quad \cot\theta_3 = \&c.;$$

$$\therefore\qquad \frac{\beta\sin B-\gamma\sin C+\alpha\sin(B-C)}{2\alpha\sin B\sin C}+\ldots\ldots=0,$$

$$\text{or}\qquad \frac{\alpha}{\sin A}(\beta^2-\gamma^2)+\frac{\beta}{\sin B}(\gamma^2-\alpha^2)+\ldots\ldots=0,$$

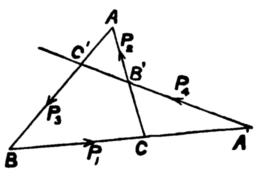
the cubic locus required.

(2) This obviously passes through the centroid, the ex-centres, the in- and circum-centres, and symmedian point. Put $\alpha=\beta=\gamma-0$, we find it cuts the sides at the same points as the lines $\alpha\sin A=\beta\sin B=\gamma\sin C$, viz., the mid-points. [The other parts of Quest. not solved.]

8643. (By Professor GENESE, M.A.) — Four complanar forces P_1, P_2, &c. are in equilibrium. Their lines of action (omitting one at a time) determine four triangles whose areas are Δ_1, Δ_2, &c., and circumradii R_1, R_2, &c. Prove that $P_1 : P_2$, &c. $= \dfrac{\Delta_1}{R_1} : \dfrac{\Delta_2}{R_2}$, &c.

Solution by ASPARAGUS; D. EDWARDES; *and others.*

Let the forces P_1, P_2, P_3 act along the sides BC, CA, AB of the triangle ABC, and P_4 along the straight line A′B′C′. The conditions of equilibrium will be satisfied if the moments of all the forces about A, B, C vanish. Taking moments about A, we get

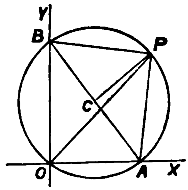


$$P_1\frac{\Delta A{\cdot}BC}{BC} = P_4\frac{\Delta AB'C'}{B'C'}.$$

But $\Delta ABC = \Delta_4$, $BC = 2R_4 \sin A$, $\Delta AB'C' = \Delta_1$, and $B'C' = 2R_1 \sin A$, hence $P_1 : P_4 = \frac{\Delta_1}{R_1} : \frac{\Delta_4}{R_4}$. Similarly, by taking moments about B and C, we get $P_2 : P_3 : P_4 = \frac{\Delta_2}{R_2} : \frac{\Delta_3}{R_3} : \frac{\Delta_4}{R_4}$.

8724. (By R. Tucker, M.A.)—If a constant line AB moves with ends on OX, OY, two rectangular axes, and on it a semicircle is described, then locus of mid-point of arc is one of two straight lines. Find locus of any other fixed point on the arc. [If C is the mid-point of AB, and P the mid-point of the arc, then the question may be enunciated for a bar rigidly connected with the bar AB.]

Solution by C. E. Williams, M.A.; Professor Mathews, M.A.; *and others.*



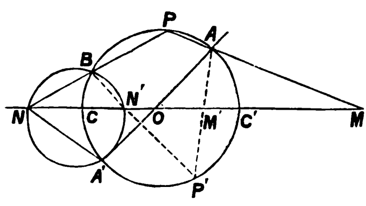
A circle passes round AOBP, therefore angle POA = PBA = $\frac{1}{4}\pi$, therefore P must lie on one of the bisectors of OX, OY. If P be any fixed point on the arc, we still have

$$POA = PBA = \text{constant},$$

and therefore P lies on one of two straight lines.

8646. (By E. Vigarié.)—On donne deux points A et B sur une circonférence de centre O; trouver sur cette circonférence un troisième point P, tel que les droites PA, PB coupent un diamètre fixe CC′ en des points M, N, tels que l'on ait OM = ON.

Solution by Professor Mathews, M.A.; Professor Schoute; *and others.*



Produce AO to meet given circle in A′; through B, A′ draw a circle BNA′N′ so that the segment BNA′ may be similar to BA′A; produce NB to P, etc., as in figure. Then ∠A′NB = supplement of BPA, hence NA′ is parallel to PAM; and, since OA′ = OA, ON = OM. Similarly, if BN′ be produced to P′ and P′A meet CC′ in M′, M′ = ON′.

2866. (By Professor SYLVESTER, F.R.S.)—Given the simultaneous equations $v_{x+1} = v_x + (x^2 - x)\, v_{x-1}$, $u_{x+1} = u_x + (x^2 - x)\, u_{x-1} + (2x+1)\, v_{x+1}$, prove that the general solution is of the form $v_x = \lambda\alpha_x + \mu\beta_x$,

$$u_x = \tfrac{1}{2}\lambda \left\{x^2 + 2x + \tfrac{1}{2}(-)^x\right\} \alpha_x + \tfrac{1}{2}\lambda \left\{x^2 + 2x - \tfrac{1}{2}(-)^x\right\} \beta_x + \nu\alpha_x + \pi\beta_x,$$

where λ, μ, ν, π are arbitrary constants, and determine the values of α_x, β_x.

Solution by Professor SEBASTIAN SIRCOM, M.A.

The values of α_x, β_x are given in Quest. 2845 [Vol. XIII., p. 50], and may be written $\alpha_x = \Pi\left\{x - \tfrac{1}{2} - (-)^x \tfrac{1}{2}\right\}$, $\beta_x = \Pi\left\{x - \tfrac{1}{2} + (-)^x \tfrac{1}{2}\right\}$.

The solution of the equation in u_x is easily verified.

Writing $x(x+1)$ in the form $\left\{x - \tfrac{1}{2} - (-1)^x \tfrac{1}{2}\right\} \left\{x - 1 + \tfrac{1}{2} - (-)^{x-1} \tfrac{1}{2}\right\}$, and assuming $u_x = \lambda\gamma_x\, \alpha_x$, the equation for γ_x will be

$$\left\{x + 1 - \tfrac{1}{2} + (-)^{x+1} \tfrac{1}{2}\right\} \gamma_{x+1} - \gamma_x - \left(x - 1 + \tfrac{1}{2} - (-)^{x-1} \tfrac{1}{2}\right\} \gamma_{x-1}$$
$$= (2x+1) \left\{x + 1 - \tfrac{1}{2} - (-)^{x+1} \tfrac{1}{2}\right\},$$

whence γ_x must be of the form $a_x + (-1)^x c$. Substituting, and equating the terms that do, and those that do not, contain $(-)^x$ separately,

$$(x + \tfrac{1}{2})\, a_{x+1} - a_x - (x - \tfrac{1}{2})\, a_{x-1} = (2x+1)(x + \tfrac{1}{2}) \ldots\ldots\ldots\ldots\ldots (1),$$

$$a_{x+1} - a_{x-1} - 4c = 2x + 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating a_{x+1}, we have $a_x - a_{x-1} = 4c\,(x + \tfrac{1}{2})$ and $a_{x+1} - a_x = 4c\,(n + \tfrac{3}{2})$, whence $a_{n+1} - a_n = 8cn + 8c$, which agrees with (2) if $c = \tfrac{1}{4}$; then $a_x - a_{x-1} = x + \tfrac{1}{2}$, and $a_x = \tfrac{1}{2}(x^2 + 2x) + \nu$, $\gamma_x = \tfrac{1}{2}\left\{x^2 + 2x + (-)^x \tfrac{1}{2}\right\} + \nu$. Similarly for $u_x = \mu\delta_x\,\beta_x$.

8789. (By Professor CULLEY, M.A.)—Find by a geometrical construction the equation $SS' = P^2$ of the pair of tangents from $T(x', y')$ to a circle $S \equiv x^2 + y^2 - z^2 = 0$, whose centre is O; and show that, if perpendiculars RL, RM, RK be drawn to the tangents and their chord of contact from a point R not on the circumference, RL . RM differs from RK^2 by $TO^2 \cos^2\omega \cos^2\theta$, where 2ω and 2θ are the angles subtended by the circle at T and R respectively.

Solution by Professors WOLSTENHOLME, BEYENS, *and others.*

If TP, TQ be tangents from the point (XY) to the circle $x^2 + y^2 = r^2$, and (x, y) be any point P on either tangent (say TQ),

$PQ^2 = x^2 + y^2 - r^2$, $TQ^2 = X^2 + Y^2 - r^2$, and $PQ^2 : TQ^2 = PM^2 : TN^2$, where PM, TN are perpendiculars on QQ'. But the equation of QQ' being $xX + yY = r^2$, $PM : QN = xX + yY - r^2 : X^2 + Y^2 - z^2$,

or $\quad x^2 + y^2 - r^2 : X^2 + Y^2 - r^2 = (xX + yY - r^2)^2 : (X^2 + Y^2 - r^2)$,

giving the required equation

$$(x^2 + y^2 - r^2)(X^2 + Y^2 - r^2) = (xX + yY - r^2)^2.$$

[Almost exactly the same proof applies to the central conics.]

8833. (By Professor Neuberg.)—Un angle de grandeur constante tourne autour de son sommet A, ses côtés rencontrent une droite donnée XY aux points B et C. Trouver (1) les enveloppes de la médiane BB′ et de la symédiane BH ; (2) les points où ces droites touchent leurs enveloppes.

Solutions by Professors (1) Schoute, (2) de Longchamps.

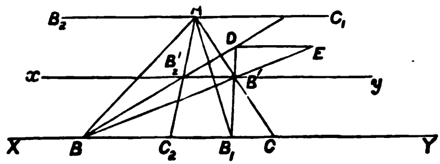


Fig. 1.

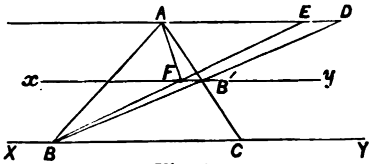


Fig. 2.

1. As the points B, B′ (Fig. 1) determine on the parallels XY, xy two homographic divisions, the line BB′ envelopes a conic that touches the parallels and has A for a focus. When B_1AC_1 and B_2AC_2 are the positions of the angle BAC, in which AC_1 and B_2A are parallel to XY, B_1 and B_2' are the points where XY and xy are touched by this conic. And by applying the theory of Brianchon's hexagon to the triangle YBB′y circumscribed to the conic, the point E, where BB′ touches it, is easily found.

When BE (Fig. 2) is a symmedian of triangle BAC, the angles ABE and B′BC are equal, and the triangles ABE and ADB are similar. Then the corresponding medians AF and AB′ of these triangles are isogonal with respect to AB and AE. Thus $\angle$ FAE = $\angle$ BAC = constant. This proves that all the lines BE pass through the fixed point F. And as in the same manner the symmedians with respect to C pass through another fixed point G of xy, and there is evidently a correspondence (1, 1) between these two pencils F and G, the locus of the point of Lemoine K of the triangle ABC will be a conic through F, G, touching AF, AG. Only when the rotating angle BAC is right the points F and G coincide, and all the triangles BAC have the same point of Lemoine. (N.B.—*Symédiane* is introduced by M. D'Ocagne for *antimédiane.*)

2. *Otherwise* :—Généralisons la question en supposant que les côtés de l'angle A rencontrent : l'un, une droite fixe Δ, au point B ; l'autre, une seconde droite fixe Δ', au point C. Soit B′ le milieu de AC ; le lieu décrit par B′ est une droite δ parallèle à Δ', et les points B, B′ décrivent sur Δ et sur δ deux divisions homographiques. On sait que, dans les conditions, BB′ enveloppe une conique tangente à Δ et à δ en de certains points M, m, que l'on sait déterminer.

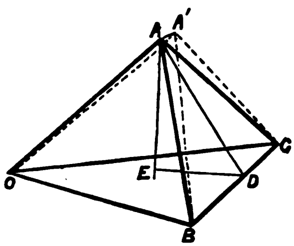
Si l'on imagine le point R commun à BB′ et à Mm, la polaire de R coupe BB′ au point cherché ρ. Il est visible que ρ est le conjugué harmonique de R sur le segment BB′.

Remarque.—Si, dans l'énoncé donné, il faut supposer que BH désigne la *hauteur* du triangle ABC ; alors, en observant que BH fait avec BA un angle constant, on reconnaît que BH enveloppe une parabole. Le point de contact de BH avec son enveloppe se trouve, comme l'on sait, sur la circonférence qui passe par A et par B tangentiellement à Δ.

8719. (By Professor Wolstenholme, M.A., Sc.D.)—The lengths of the edges OA, OB, OC of a tetrahedron OABC are denoted by a, b, c, those of the respectively opposite edges BC, CA, AB by x, y, z; and the dihedral angles *opposite* to these by A, B, C; X, Y, Z; prove that, if V be the volume expressed in terms of a, b, c, x, y, z, $\frac{\delta V}{\delta a} = \frac{1}{6}ax \cot A$, &c.; and thence that (1) when a, x; b, y, and $c+z$ are *given*, V will be a maximum when C = Z; (2) when a, x; b, y, and $c-z$ are given, V will be a maximum when C + Z = 180°; (3) when a, x; $b+y$; $c+z$ are given, V will be a maximum when B = X and C = Z (*i.e.*, when $b = x$, $c = z$); (4) when a, x; $b+y$, $c-z$ are given, V will be a maximum when B = X, C + Z = 180°; and (5) investigate if V can have a maximum when a, x; $b-y$, $c-z$ are given, or when $a-x$, $b-y$, $c-z$ are given; the given quantities being snpposed always real and finite.

In case (4), prove also that 180° is the maximum value of C + Z for variations in b, y (subject to the conditions stated), and that zero is the minimum value of B − Y for variations in c, z.

Solution by Professors Mathews, Beyens, *and others.*



Draw AD perpendicular to BC, and AE perpendicular to OBC.

Let the vectors OA, OB, OC be represented by α, β, γ.

Then, if Δ be the volume OABC,

$$6\Delta = S\alpha\beta\gamma.$$

In order to find $\frac{\delta\Delta}{\delta a}$, turn the triangle ABC through a small angle about BC, so that A comes to A′; this changes a, and leaves b, c, x, y, z unaltered. AA′ is ultimately perpendicular to ABC, so that we may put

$$d\alpha = \text{vector AA}' = h\,(V\beta\gamma + V\gamma\alpha + V\alpha\beta) \quad\text{.................(1)},$$

where h is a small scalar quantity.

$$\text{Hence } 6\,\frac{\delta\Delta}{\delta a} = \frac{dS\alpha\beta\gamma}{dT\alpha} = \frac{S\beta\gamma d\alpha}{\frac{S\alpha d\alpha}{T\alpha}} = T\alpha\,\frac{S\beta\gamma\,(V\beta\gamma + V\gamma\alpha + V\alpha\beta)}{S\alpha\beta\gamma} \quad\text{......(2)};$$

by (1), observing that $S\alpha\,(V\beta\gamma + V\gamma\alpha + V\alpha\beta) = S\alpha\beta\gamma$.

$$\text{Now}\quad S\beta\gamma\,(V\beta\gamma + V\gamma\alpha + V\alpha\beta) = S\,.\,V\beta\gamma\,(V\beta\gamma + V\gamma\alpha + V\alpha\beta)$$
$$= 4\text{OBC}\,.\,\text{ABC}\,.\,\cos A = 2\text{OBC}\,.\,\text{BC}\,.\,\text{AD}\cos A$$
$$= 2\text{OBC}\,.\,\text{BC}\,.\,\text{AE}\cot A = 6\Delta x \cot A,$$

hence (2) becomes

$$6\,\frac{\delta\Delta}{\delta a} = a\,\frac{6\Delta x \cot A}{6\Delta} = ax\cot A, \text{ therefore } \frac{\delta\Delta}{\delta a} = \tfrac{1}{6}ax\cot A.$$

If Δ is to be made a maximum or a minimum by the variation of a, b, c, x, y, z,

$$\frac{\delta\Delta}{\delta a}da + \ldots + \ldots + \frac{\delta\Delta}{\delta x}dx + \ldots + \ldots = 0,$$

or $$ax\,(\cot A\,.\,da + \cot X\,.\,dx) + \ldots + \ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Case (1) gives $da = dx = db = dy = 0$, $dc + dz = 0$, so that (3) reduces to $cz\,(\cot C - \cot Z) = 0$, $c = 0$, $z = 0$ make $\Delta = 0$ a minimum; hence $\cot C = \cot Z$, or $C = Z$ makes Δ a maximum.

(2), (3), and (4) may be similarly treated.

8840. (By W. S. M'Cay, M.A.)—Prove that (1) the locus of the mean centre of the four points, in which a line of given direction meets the faces of a tetrahedron, is a plane (diametral plane); and (2) if A, B, C, D be the areas of the faces of the tetrahedron, all the diametral planes envelope the quartic $(Ax)^{\frac{1}{2}} + (By)^{\frac{1}{2}} + (Cz)^{\frac{1}{2}} + (Dw)^{\frac{1}{2}} = 0$.

Solution by Professor Schoute.

1. When α, β, γ, δ represent the four given planes, L the point at infinity indicating the direction of the lines, and π, ϕ, ψ the planes that are separated harmonically from L by the couples of planes α and β, γ and δ, π and ϕ, respectively, then it is evident that in the determination of the locus in question the planes α and β may be replaced by π counted twice, the planes γ and δ by ϕ counted twice, and the planes π and ϕ each of them counted twice by ψ counted four times. This proves that ψ is the locus.

2. If P represents a vertex of the given tetrahedron, Q the point of intersection of the line LP with the opposite face of the tetrahedron, and R the point on PQ for which $PR = \frac{1}{4}PQ$, the locus of the mean centre passes through R. When a, b, c, d are the coordinates of P, which, as P is at infinity, satisfy the condition $Aa + Bb + Cc + Dd = 0$, the locus contains the four points

$$(-3a,\ b,\ c,\ d),\quad (a,\ -3b,\ c,\ d),\quad (a,\ b,\ -3c,\ d),\quad (a,\ b,\ c,\ -3d).$$

Therefore its equation is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} + \dfrac{w}{d} = 0.$

By differentiation the coordinates of the point, where this plane touches its envelope, are found to be proportional to

$$Aa^2,\quad Bb^2,\quad Cc^2,\quad Dd^2;$$

which, in due combination with the condition $Aa + Bb + Cc + Dd = 0$, proves that the equation of the envelope is

$$(Ax)^{\frac{1}{2}} + (By)^{\frac{1}{2}} + (Cz)^2 + (Dw)^2 = 0.$$

Thus this envelope is the *Roman surface of Steiner*, the quadric surface of which the three lines, that join the centres of the opposite edges of the given tetrahedron, are double edges.

[If a system of parallel lines pierce the faces of the tetradedron at angles α, β, γ, δ, the equation of the corresponding diametral plane is obviously $\dfrac{x}{\sin\alpha} + \dfrac{y}{\sin\beta} + \dfrac{z}{\sin\gamma} + \dfrac{w}{\sin\delta} = 0$; the projection of the faces gives $A\sin\alpha + B\sin\beta + C\sin\gamma + D\sin\delta = 0$, whence the result.]

8799. (By Professor HUDSON, M.A.) — Find the envelope of the straight line $au = f(\alpha)\cos(\theta-\alpha) + f'(\alpha)\sin(\theta-\alpha)$.

Solution by ASPARAGUS; Professor MATZ; *and others.*

The equation $au = f(\alpha)\cos(\theta-\alpha) + f'(\alpha)\sin(\theta-\alpha)$ represents the tangent line to the curve $au = f(\theta)$ at the point where $\theta = \alpha$, and of course the envelope of the straight line is the curve $au = f(\theta)$. By the usual rule, we get $0 = [f''(\alpha) + f(\alpha)]\sin(\theta-\alpha)$. Hence, unless $f''(\alpha) + f(\alpha) = 0$, we shall have for the point of contact with the envelope $\theta = \alpha$, giving for the equation of the envelope $au = f(\theta)$. In the particular case when

$$f''(\alpha) + f(\alpha) = 0, \quad f(\alpha) = A\cos\alpha + B\sin\alpha, \quad f'(\alpha) = -A\sin\alpha + B\cos\alpha,$$

and the equation becomes $au = A\cos\theta + B\sin\theta$, or represents a definite straight line, and no envelope exists.

8851. (By J. J. WALKER, F.R.S.)—Show that either segment of a focal chord of a conic section is a mean proportional between its excess over the semi-latus rectum (or *vice versâ*) and the whole chord.

Solution by W. J. GREENSTREET, B.A.; Professor BEYENS; *and others.*

The semi-latus rectum is an harmonic mean between the segments of a focal chord; hence

$$SP \cdot SP' = RR' \cdot PP', \quad \text{or} \quad SP\,(PP'-SP) = RR' \cdot PP',$$

or

$$SP^2 = PP'\,(SP - RR');$$

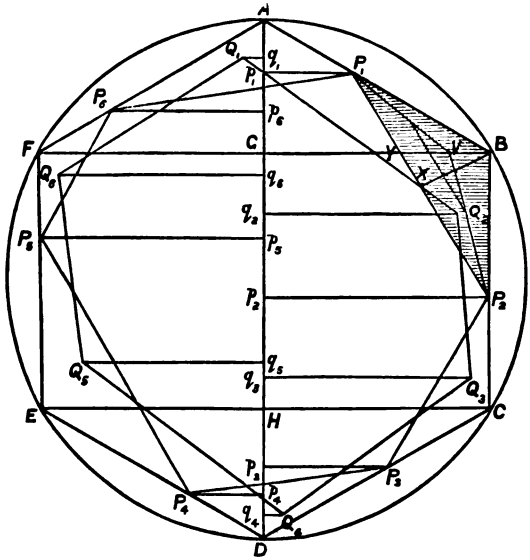
similarly for the other segment SP'.

8837. (By the EDITOR.)—On the sides of a regular n-gon, n points are taken at random, one on each, forming the apices of inscribed n-gon; again, inside the n triangles that lie outside this last polygon, n points are taken at random, one in each, forming, when joined, another n-gon; find the general average area of this last n-gon, and show therefrom that, in the cases of a hexagon and a square, the averages are $\frac{25}{36}$ and $\frac{16}{36}$ of the original figure respectively.

Solution by D. BIDDLE.

If the mid-points of the sides of the original n-gon be first joined, and then the centroids of the resulting triangles, a polygon is formed whose area represents the area required. Accordingly, its comparative value may be thus defined $(\frac{2}{3} + \frac{1}{3}\cos\alpha)^2$, where α is the angle subtending a side of the original n-gon. But the following solution gives an independent view of the case, and the concluding equation has an interesting bearing on the value of π, when the n-gon is regarded as carried to its limit in the circle.

Through the angle A draw AD, a diameter of the circumscribing circle, as a line of reference, and reckon perpendiculars to it as positive when to the right, negative when to the left of it; also distances measured on it as positive when upwards, negative when downwards, from the several feet of the perpendiculars drawn from the angles of the original n-gon.



Taking the case of the hexagon, let P_1, P_2, ... P_6 be the points taken on the perimeter, and Q_1, Q_2, ... Q_6 be the points taken in the triangles. Then the n-gon formed by joining Q_1Q_2, Q_2Q_3, &c.

$$= \tfrac{1}{2}\big[Q_2q_2(Gq_1-Gq_2) - Q_6q_6(Gq_1-Gq_6) + Q_1q_1(Gq_6-Gq_2) + Q_3q_3(Hq_3-Hq_4) - Q_5q_5(Hq_5-Hq_4) + Q_4q_4(Hq_3-Hq_5) + (Q_2q_2+Q_3q_3)(Hq_2-Hq_3) - (Q_5q_5+Q_6q_6)(Hq_6-Hq_5)\big]$$

$$= \tfrac{1}{2}\big[Q_1q_1(Gq_6-Gq_2) + Q_2q_2(Gq_1-Gq_3) + Q_3q_3(Hq_2-Hq_4) + Q_4q_4(Hq_3-Hq_5) + Q_5q_5(Hq_4-Hq_6) + Q_6q_6(Gq_5-Gq_1)\big].$$

From this it is evident that the area of any n-gon so formed is

$$\tfrac{1}{2}\big[Q_1q_1(Dq_n-Dq_2) + Q_2q_2(Dq_1-Dq_3) + \ldots + Q_nq_n(Dq_{n-1}-Dq_1)\big] \ldots \text{(A)},$$

in which the q-terms within brackets are respectively the one before and the one after that corresponding to the factor outside.

Now it is easy to see, from an examination of the shaded triangle in the diagram, that the average position of Q is the centroid, X, and this can be given in terms of P. For, if we suppose Q_2 to be at X, then

$$Q_2q_2 = \tfrac{1}{2}(P_1p_1+P_2p_2) + \tfrac{1}{3}\big\{BG - \tfrac{1}{2}(P_1p_1+P_2p_2)\big\} = \tfrac{1}{3}(BG+P_1p_1+P_2p_2),$$

and $Dq_2 = \frac{1}{2}(Dp_1 + Dp_2) + \frac{1}{3}\left\{DG - \frac{1}{2}(Dp_1 + Dp_2)\right\} = \frac{1}{3}(DG + Dp_1 + Dp_2)$.

Let l_1, l_2, l_3, &c. = DA, DG, DH, &c., and k_1, k_2, k_3, &c. = (0), BG, CH, &c. Also, let $x_1 = AP_1/AB$, $x_2 = BP_2/BC$, &c. Then

$$P_1 p_1 = k_1 + (k_2 - k_1)\, x_1, \quad P_2 p_2 = k_2 + (k_3 - k_2)\, x_2, \text{ \&c.,}$$

and

$$Dp_1 = l_1 + (l_2 - l_1)\, x_1, \quad Dp_2 = l_2 + (l_3 - l_2)\, x_2, \text{ \&c.}$$

Consequently,

$$Q_1 q_1 = \tfrac{1}{3}\left\{k_1 + \left[k_n + (k_1 - k_n)\, x_n\right] + \left[k_1 + (k_2 - k_1)\, x_1\right]\right\},$$

$$Q_2 q_2 = \tfrac{1}{3}\left\{k_2 + \left[k_1 + (k_2 - k_1)\, x_1\right] + \left[k_2 + (k_3 - k_2)\, x_2\right]\right\}, \text{ \&c.,}$$

and

$$Dq_1 = \tfrac{1}{3}\left\{l_1 + \left[l_n + (l_1 - l_n)\, x_n\right] + \left[l_1 + (l_2 - l_1)\, x_1\right]\right\},$$

$$Dq_2 = \tfrac{1}{3}\left\{l_2 + \left[l_1 + (l_2 - l_1)\, x_1\right] + \left[l_2 + (l_3 - l_2)\, x_2\right]\right\}, \text{ \&c.}$$

Now, bearing in mind that $k_n = -k_2$, $k_{n-1} = -k_3$, &c., and $l_n = l_2$, $l_{n-1} = l_3$, &c., by symmetry, let $k_2 - k_1 = m_1$, $k_3 - k_2 = m_2$, &c., and let $l_1 - l_2 = f_1$, $l_2 - l_3 = f_2$, &c. Then,

$$Q_1 q_1 = k_1 + \tfrac{1}{3}\left\{m_1 x_1 - m_n (1 - x_n)\right\}, \quad Q_2 q_2 = k_2 + \tfrac{1}{3}\left\{m_2 x_2 - m_1 (1 - x_1)\right\}, \text{ \&c.,}$$

and

$$Dq_1 = l_1 + \tfrac{1}{3}\left\{f_n (1 - x_n) - f_1 x_1\right\}, \quad Dq_2 = l_2 + \tfrac{1}{3}\left\{f_1 (1 - x_1) - f_2 x_2\right\}, \text{ \&c.,}$$

where

$$f_n = -f_1, \quad f_{n-1} = -f_2, \text{ \&c.}$$

Consequently, $\frac{1}{2}\left[Q_1 q_1 (Dq_n - Dq_2) + Q_2 q_2 (Dq_1 - Dq_3) + \ldots\right]$

$$= \tfrac{1}{2}\iint\Big[\left\{k_1 + \tfrac{1}{3}\left[m_1 x_1 - m_n (1 - x_n)\right]\right\}\left\{l_n + \tfrac{1}{3}\left[f_{n-1}(1 - x_{n-1}) - f_n x_n\right] - l_2 - \tfrac{1}{3}\left[f_1 (1 - x_1) - f_2 x_2\right]\right\} + \text{\&c.}\Big]$$

$$\begin{aligned} = \tfrac{1}{2}\iint\Big[\Big\{ & k_1 l_n - k_1 l_2 + \tfrac{1}{3}k_1 f_{n-1}(1 - x_{n-1}) - \tfrac{1}{3}k_1 f_n x_n \\ & - \tfrac{1}{3}k_1 f_1 (1 - x_1) + \tfrac{1}{3}k_1 f_2 x_2 + \tfrac{1}{3}l_n m_1 x_1 - \tfrac{1}{3}l_n m_n (1 - x_n) \\ & + \tfrac{1}{9}m_1 f_{n-1} x_1 (1 - x_{n-1}) - \tfrac{1}{9}m_1 f_n x_1 x_n - \tfrac{1}{9}m_n f_{n-1}(1 - x_n)(1 - x_{n-1}) \\ & + \tfrac{1}{9}m_n f_n x_n - \tfrac{1}{9}m_n f_n x_n^2 - \tfrac{1}{3}l_2 m_1 x_1 + \tfrac{1}{3}l_2 m_n (1 - x_n) \\ & - \tfrac{1}{9}m_1 f_1 x_1 + \tfrac{1}{9}m_1 f_1 x_1^2 + \tfrac{1}{9}n_1 f_2 x_1 x_2 + \tfrac{1}{9}m_n f_1 (1 - x_n)(1 - x_1) \\ & - \tfrac{1}{9}m_n f_2 x_2 (1 - x_n)\Big\} + \text{\&c.}\Big] \end{aligned}$$

$$\begin{aligned} = \tfrac{1}{2}\Big[\Big\{ & k_1 (l_n - l_2) + \tfrac{1}{6}k_1 (f_{n-1} - f_n - f_1 + f_2) + \tfrac{1}{6}(l_n - l_2)(m_1 - m_n) \\ & + \tfrac{1}{36}(m_1 - m_n)(f_{n-1} + f_2) + \tfrac{1}{36}(m_n f_1 - m_1 f_n) + \tfrac{1}{54}(m_n f_n - m_1 f_1)\Big\} + \text{\&c.}\Big] \end{aligned}$$

$$\begin{aligned} = \tfrac{1}{216}\Big[\Big\{ & (3k_2 l_{n-1}) + (12k_1 l_{n-1} + 12k_2 l_n) + (47k_1 l_n + k_2 l_1 + 3k_n l_{n-1}) \\ & + (13k_n l_n - 13k_2 l_2) - (47k_1 l_2 + 3k_2 l_3 + k\ l_1) \\ & - (12k_1 l_3 + 12k_n l_2) - (3k_n l_3)\Big\} + \text{\&c.}\Big]. \end{aligned}$$

From this formula we are able [by joining a series, the several terms of which are made to correspond with the respective terms in (A), by being all raised one degree at a time, k_n to k_1, k_1 to k_2, l_n to l_1, l_1 to l_2, &c., until the n terms of (A) are satisfied] to calculate the numerator of a fraction of which the area of the original n-gon is the denominator; and this fraction is the mean area required.

The above formula is easily reducible to the following, in which the

Greek ν at the foot of the several factors represents each figure from 1 to n in succession,

$$\tfrac{1}{72} k_\nu (l_{\nu-3}+8l_{\nu-2}+17l_{\nu-1}-17l_{\nu+1}-8l_{\nu+2}-l_{\nu+3}).$$

In the case of the hexagon, taking the radius of the circumscribing circle as unity, and the area of the original figure consequently as $3\sqrt{3}/2$, $k_1 = 0$, $k_2 = \frac{1}{2}\sqrt{3}$, $k_3 = \frac{1}{2}\sqrt{3}$, $k_4 = 0$, $k_5 = -\frac{1}{2}\sqrt{3}$, $k_6 = -\frac{1}{2}\sqrt{3}$; $l_1 = 2$, $l_2 = 1\frac{1}{2}$, $l_3 = \frac{1}{2}$, $l_4 = 0$, $l_5 = \frac{1}{2}$, $l_6 = 1\frac{1}{2}$; and the required area is $\frac{75}{72}\sqrt{3}/\frac{3}{2}\sqrt{3} = \frac{25}{36}$.

In the case of the square, taking the diagonal $= 2$, and the area also as 2, $k_1 = 0$, $k_2 = 1$, $k_3 = 0$, $k_4 = -1$, $l_1 = 2$, $l_2 = 1$, $l_3 = 0$, $l_4 = 1$; and the required area $= \frac{64}{72}/2 = \frac{4}{9} = \frac{16}{36}$.

8854. (By Rev. T. R. TERRY, M.A.)—Solve (1) the equation

$$w_{x+2} = (2x+5)\, w_{x+1} - (x^2+4x+3)\, w_x;$$

and show (2) that, if u_x and v_x both satisfy this equation, and if $u_1 = 2$, $u_2 = 10$, while $v_1 = 1$, $v_2 = 2$, then $2u_x = x(x+3)\, v_x$.

Solution by Professors SIRCOM *and* BEYENS.

Dividing by $(x+3)!$ the equation may be written

$$\frac{1}{x+3}\left\{\frac{w_{x+2}}{(x+2)!} - \frac{w_{x+1}}{(x+1)!}\right\} = \frac{1}{x+2}\left\{\frac{w_{x+1}}{(x+1)!} - \frac{w_x}{x!}\right\} = \text{constant};$$

whence $\dfrac{w_x}{x!} - \dfrac{w_{x-1}}{(x-1)!} = C(x+1)$, and $w_x = x!\, Cx\, \tfrac{1}{2}(x+3) + x!\, C_1$;

then, if $u_x = x!\, Cx\, \frac{1}{2}(x+3)$, $v_x = x!\, C_1$, the given values of u and v make $C = 1 = C_1$, then $2u_x = x(x+3)\, v_x$.

8778. (By W. J. C. SHARP, M.A.) — If $wy-vz$, $uz-wx$, and $vx-uy$ be cogredient to x, y, and z, the rectangular Cartesian coordinates of a point, respectively, u, v, and w are so.

Solution by D. EDWARDES; SARAH MARKS; *and others.*

We have
$$\begin{aligned} WY - VZ &= l_1(wy-vz) + l_2(uz-wx) + l_3(vx-uy) \\ &= W(m_1x+m_2y+m_3z) - V(n_1x+n_2y+n_3z), \end{aligned}$$

whence
$$m_1W - n_1V = l_3v - l_2w, \quad m_2W - n_2V = l_1w - l_3u,$$
$$m_3W - n_3V = l_2u - l_1v;$$

multiplying these by m_1, m_2, m_3, and adding, we have

$W = (m_3l_2 - m_2l_3)\, u + (m_1l_3 - m_3l_1)\, v + (l_1m_2 - l_2m_1)\, w$, or $W = n_1u + n_2v + n_3w$.

Similarly for U and V.

8650. (By E. M. DAVYS, M.A.)—Prove that, in any plane triangle,

$$\cos^{12} A + 4\cos^{10} A \sin^2 A + 5\cos^8 A \sin^4 A - 5\cos^4 A \sin^8 A - 4\cos^2 A \sin^{10} A - \sin^{12} A = \cos 2A.$$

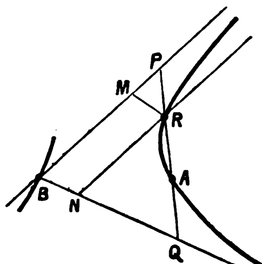
Solution by Rev. T. R. TERRY, M.A.; HANNAH MOTLEY; *and others.*

Quotient of left-hand side by right-hand side

$$= (\cos^8 A + \cos^4 A \sin^4 A + \sin^8 A) + 4\sin^2 A \cos^2 A (\cos^4 A + \sin^4 A) + 5\cos^4 A \sin^4 A = (\cos^2 A + \sin^2 A)^4 = 1.$$

8678. (By Professor GENESE, M.A.)—AB is a chord of a conic, BP, BQ are parallels to the asymptotes, APQ a variable transversal meeting the curve at R. Prove that the ratio PR : RQ is constant.

Solution by R. F. DAVIS, M.A.; Professor CHAKRAVARTI; *and others.*



Let parallels through R to the asymptotes meet BP, BQ in M and N respectively; and let ϖ, ϖ' be the points at ∞ on the curve through which BP, BQ respectively pass. Then

$$R\{AB\varpi\varpi'\} \text{ is constant;}$$

or, estimating upon $BP\varpi$,

$$\{PB\varpi M\} \text{ is constant;}$$

whence also so is PM : BM or PR : RQ.

8757. (By Professor EDMUND BORDAGE.) — If two fractions a/a', b/b' are such that $ab' - ba' = 1$, prove that the simplest fraction included between the two given fractions is $(a+b)/(a'+b')$.

Solution by H. L. ORCHARD, B.Sc., M.A.; Professor BEYENS; *and others.*

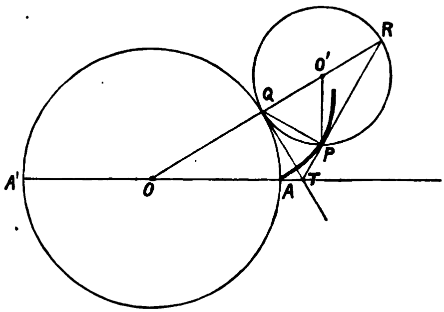
If $ab' - ba' = 0$, then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{a+b}{a'+b'}$.

Now $\dfrac{a+b}{a'+b'}$ is $< \dfrac{a + \dfrac{b'}{a'}a}{a'+b'} < \dfrac{a}{a'}$, and is $> \dfrac{b + \dfrac{b}{b'}a'}{a'+b'} > \dfrac{b}{b'}$.

8681. (By Professor HUDSON, M.A.)—In the epicycloid of two cusps, if P be the describing point when Q is the point of contact, prove that the tangents at P, Q to the epicycloid and the fixed circle respectively meet on the line joining the cusps.

Solution by R. F. DAVIS, M.A.; Rev. J. WHITE, M.A.; *and others.*

Let O, O′ be the centres of the fixed and moving circles respectively; R the extremity of the diameter of the latter through the point of contact



Q, so that PR, PQ are tangential and normal to the curve at P. Then, if A be the cusp, or point on fixed circle through which P was originally coincident, arc AQ = arc PQ, and OQ = 2O′Q; hence

$$\angle AOQ = \tfrac{1}{2}\angle PO'Q = \angle PRQ.$$

But OQ = QR, hence QT (perpendicular to OQ) meets PR on OA produced.

7987 & 8036. (By the Rev. T. R. TERRY, M.A.)—(7987.)—Four spheres, whose radii are a, b, c, d respectively, are such that each touches the other three externally. In the space between these four, another sphere of radius r is described touching all four externally. Show that

$$\frac{1}{r^2} - \frac{1}{r}\Sigma\left(\frac{1}{a}\right) + \Sigma\left(\frac{1}{a^2}\right) - \Sigma\left(\frac{1}{ab}\right) = 0.$$

(8036.)—(By R. LACHLAN, B.A.)—If four spheres, radii a, b, c, d, touch one another externally; and r be the radius of the sphere which cuts them orthogonally, then $\frac{4}{r^2} = 2\Sigma\left(\frac{1}{ab}\right) - \Sigma\left(\frac{1}{a^2}\right)$.

Solution by W. W. TAYLOR, M.A.

Inverting with respect to the point of contact of one pair of the four spheres, this pair of spheres becomes two parallel planes (1, 2); the other pair of spheres and the two spheres that touch the four original spheres become four equal spheres (3, 4, 5, 6), whose centres form two equilateral triangles; and the sphere of Quest. 8036 becomes the plane (7) passing through the four points of contact. This inverse system can be represented by the equations

$$y \pm A = 0, \quad x^2+y^2+z^2 \pm 2Ax = 0 \ldots\ldots\ldots\ldots(1, 2, 3, 4),$$
$$x^2+y^2+z^2 \pm 2\sqrt{3}\,Az + 2A^2 = 0, \quad z = 0 \ldots\ldots\ldots\ldots(5, 6, 7).$$

Now, the radius of the inverse of the sphere $f(x, y, z)$

$$\equiv x^2+y^2+z^2+2Ax+2By+2Cz+D = 0,$$

with respect to the point h, k, l (constant of inversion = 4), is

$$\frac{4}{f(h, k, l)}\sqrt{(A^2+B^2+C^2-D)}.$$

Invert with reference to the point h, k, l, and call the radii corresponding to these seven equations $a, b, c, d, r_1, r_2, r_3$; then

$$\frac{1}{a} = \tfrac{1}{2}(A+k), \quad \frac{1}{b} = \tfrac{1}{2}(A-k) \ldots\ldots\ldots\ldots(8, 9),$$

$$\frac{1}{c} = \frac{1}{4A}(h^2+k^2+l^2+2Ah), \quad \frac{1}{d} = \frac{1}{4A}(h^2+k^2+l^2-2Ah) \ldots(10, 11),$$

$$\frac{1}{r_1} = \frac{1}{4A}(h^2+k^2+l^2+2\sqrt{3}\,Al+2A^2) \ldots\ldots\ldots\ldots(12),$$

$$\frac{1}{r_2} = \frac{1}{4A}(h^2+k^2+l^2-2\sqrt{3}\,Al+2A^2), \quad \frac{1}{r_3} = \frac{l}{2} \ldots\ldots\ldots(13, 14).$$

From (8), (9), $\quad A = \frac{1}{a} + \frac{1}{b}, \quad k = \frac{1}{a} - \frac{1}{b} \ldots\ldots\ldots(15, 16)$;

from (10), (11), we have

$$h^2+k^2+l^2 = 2A\left(\frac{1}{c}+\frac{1}{d}\right) = 2\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right),$$
$$h = \left(\frac{1}{c}-\frac{1}{d}\right) \ldots\ldots\ldots\ldots(17, 18);$$

from (10), (12), and (15), (18),

$$l\sqrt{3}-h = 2\left(\frac{1}{r_1}-\frac{1}{c}\right)-A, \quad l\sqrt{3} = \frac{2}{r_1}-\Sigma\frac{1}{a} \ldots\ldots\ldots(19);$$

also, by (16), (17), (18),

$$l^2 = 2\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right)-\left(\frac{1}{a}-\frac{1}{b}\right)^2-\left(\frac{1}{c}-\frac{1}{d}\right)^2 = 2\Sigma\frac{1}{ab}-\Sigma\frac{1}{a^2} \ldots\ldots\ldots(20);$$

therefore, by (19), (20), $\quad \frac{4}{r_1^2}-\frac{4}{r_1}\Sigma\frac{1}{a}+\Sigma\frac{1}{a^2}+2\Sigma\frac{1}{ab} = 6\Sigma\frac{1}{ab}-3\Sigma\frac{1}{a^2}$,

therefore $\quad \frac{1}{r_1^2}-\frac{1}{r_1}\Sigma\frac{1}{a}+\Sigma\frac{1}{a^2}-\Sigma\frac{1}{ab} = 0.$

For r_3 substitute for l in (20), from (14), $\frac{4}{r_3^2} = 2\Sigma\frac{1}{ab}-\Sigma\frac{1}{a^2}$.

[These results are most readily obtained from the general formula given by the late Prof. CLIFFORD (on p. 335 of his *Mathematical Papers*) for finding the radius of a sphere which cuts five given spheres at given angles:—viz., if the spheres whose radii are r_x, r_y cut at the angle $\omega_{x,y}$, then

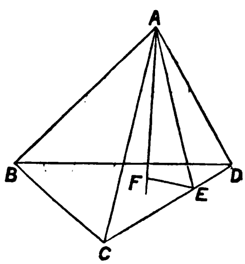
$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \\ \frac{1}{r_1}, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4}, & \cos\omega_{1,5}, \\ \frac{1}{r_2}, & \cos\omega_{2,1}, & -1, & \&c., & \dots & \dots \\ \frac{1}{r_3}, & \&c., & \dots & -1, & \dots & \dots \\ \frac{1}{r_4}, & \dots & \dots & \dots & -1, & \dots \\ \frac{1}{r_5}, & \dots & \dots & \dots & \dots & -1, \end{vmatrix} = 0$$

By taking $\omega_{1,2} = \omega_{1,3} = \&c. = 0$, a result equivalent to (7987) is at once obtained; and by taking $\omega_{1,2} = \omega_{1,3} = \omega_{1,4} = \omega_{2,3} = \omega_{2,4} = \omega_{3,4} = 0$, and $\omega_{1,5} = \omega_{2,5} = \omega_{3,5} = \omega_{4,5} = \frac{1}{2}\pi$, a result equivalent to (8036) is obtained.

The above theorem is a particular case of a more general theorem given in the Royal Society's *Proceedings* for March 11th, 1886.]

8725. (By S. TEBAY, B.A.)—If the sines of opposites dihedral angles of a tetrahedron be respectively proportional to the edges formed by these angles, the areas of the four faces are equal.

Solution by Professors MATHEWS, BEYENS, *and others.*



Denoting by (CD) the dihedral angle of which the containing planes intersect in CD, we have

$$\sin(CD) = AF / AE$$

$$= \frac{3 \text{ vol. } ABCD}{\text{area } BCD} \Big/ \frac{2 \text{ area } ACD}{CD}$$

$$\frac{\sin(CD)}{CD} = \frac{3 \text{ vol. } ABCD}{2 \,.\, ACD \,.\, BCD},$$

so also

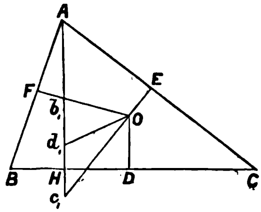
$$\frac{\sin(AB)}{AB} = \frac{3 \text{ vol. } ABCD}{2\, CAB \,.\, DAB}, \ \&c.$$

Hence, if $\frac{\sin(CD)}{CD} = \frac{\sin(AB)}{AB}$, &c., $ACD \,.\, BCD = CAB \,.\, DAB$;

with two similar equations: whence $BCD = CDA = DAB = ABC$.

8697. (By Captain H. Brocard.)—Les droites qui joignent le centre du cercle circonscrit aux milieux des segments interceptés par les perpendiculaires aux milieux des côtés sur les hauteurs du triangle sont perpendiculaires aux médianes correspondantes.

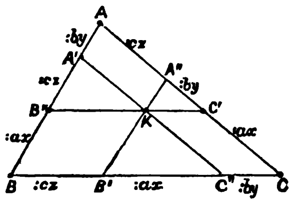
Solution by Professor Schoute; R. F. Davis, M.A.; *and others.*



Let O be the circum-centre and D, E, F the middle points of the sides of the triangle. Let OE, OF meet the perpendicular AH on BC in the points c_1, b_1, and let d_1 be the middle point of b_1c_1. Then the triangles Ob_1c_1, ABC are obviously directly similar, corresponding lines being at right angles. Hence the median line Od_1 is perpendicular to the corresponding line AD.

8502. (By B. Hanumanta Rau, B.A.)—Through any point K are drawn the straight lines B″KC′, C″KA′, and A″KB′ respectively parallel to the sides BC, CA, AB of a triangle; prove that (1) $\triangle A'B'C' = \triangle A''B''C''$; (2) parallels through A, B, C to A′B′, B′C′, C′A′ meet at a point O′; (3) parallels through A, B, C to A″C″, A″B″, C″B″ also meet at a point O′; (4) if the coordinates of O and O′ are (α, β, γ) and $(\alpha', \beta', \gamma')$ then $a\alpha : c\gamma' = b\beta : a\alpha' = c\gamma : b\beta'$; (5) if K is the Symmedian point, O and O′ become the Brocard points.

Solution by R. F. Davis, M.A.; Professor Sarkar, M.A.; *and others.*



Let x, y, z be the trilinear coordinates of K. Then it will be seen that each side of the triangle ABC is divided similarly into three parts $= ax : by : cz$.

(1) $\triangle A'B'C' =$ sum of $\triangle$s B′KC′ C′KA′, A′KB′ = semi-sum of parallelograms A′A″, B′B″, C′C″ = $\triangle A''B''C''$.

(2) Since $BB' : B'C'' = cz : ax$, a line through A parallel to A′B′ divides BC in the same ratio, and its equation is $\beta / \gamma = ax / bz$. Similarly the equations to lines through B, C parallel to B′C′, C′A′ respectively are $\gamma / \alpha = by / cx$ and $\alpha / \beta = cz / ay$. These meet in a point O,

$$\alpha : \beta : \gamma = czx : axy : byz.$$

(3) Similarly for O′, $\alpha' : \beta' : \gamma' = bxy : cyz : azx$.
(4) Whence $a\alpha : c\gamma' =$ &c.
(5) If $x : y : z = a : b : c$ then $\alpha : \beta : \gamma = c/b : a/c : b/a$, &c.

8585. (By R. F. DAVIS, M.A.)—If TP, TQ be tangents to a parabola, and PQ meet the directrix in Z, prove that ZT will be a mean proportional between ZP and ZQ.

Solution by Rev. T. R. TERRY, M.A.; Prof. BEYENS, M.A.; *and others.*

If the coordinates of Z are $(-a, h)$, and equation to ZQP be

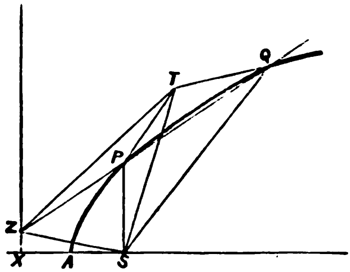
$$\frac{y-h}{\sin\theta} = \frac{x+a}{\cos\theta} = c \text{.........(1)},$$

then

$$ZP \, . \, ZQ = (h^2+4a^2)\operatorname{cosec}^2\theta \text{...(2)}.$$

Also, from (1), coordinates of T are $a + h\cot\theta,\ 2a\cot\theta$, therefore

$$ZT^2 = (h^2+4a^2)\operatorname{cosec}^2\theta \text{...(3)}.$$



[Since SZ bisects externally the angle PSQ,

$$ZP \, . \, ZQ = SZ^2 + SP \, . \, SQ = SZ^2 + ST^2 = ZT^2.]$$

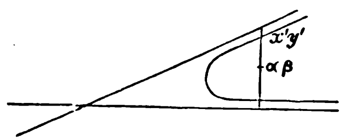
8600. (By Professor MAHENDRA NATH RAY, M.A., LL.B.)—Through an indefinite point of a given hyperbola straight lines are drawn to meet the asymptotes; show that the hyperbola itself is the envelope of the locus of the middle points of the straight line.

Solution by Professors SCOTT *and* IGNACIO BEYENS.

Let $xy = k^2$ be the hyperbola. The line is $x\alpha + y\beta = 2\alpha\beta$, where (α, β) is the middle point; it passes through x', y', therefore

$$x'\alpha + y'\beta = 2\alpha\beta,$$

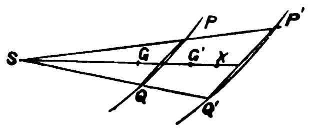
also
$$y' = \frac{k^2}{x},$$



hence $\alpha x'^2 - 2\alpha\beta x' + \beta k^2 = 0$ and $\alpha^2\beta^2 = \alpha\beta k^2$ or $\alpha\beta = k^2$; therefore $xy = k^2$ is the envelope.

8623. (By N'IMPORTE.)—From the centre of similitude S, common radii vectores SPP′, SQQ′ are drawn to similar curves PQ, P′Q′. Having given the centre of gravity of the area SPQ, find that of PQQ′P′; and its limiting position when PQ, P′Q′ ultimately coincide.

Solution by Rev. H. London, M.A.; Professor Ignacio Beyens; *and others.*



Let G be the centre of gravity of SPQ, and let

$$SP : SP' = SQ : SQ' = 1 : k,$$

$\therefore$ area SPQ : area SP′Q′ = $1 : k^2$.

Let G′ be centre of gravity of SP′Q′ and x that of PQQ′P′, and m = mass of SPQ. Therefore, taking moments about S,

$$k^2m \,.\, SG' = m \,.\, SG + (k^2-1)\, m \,.\, SX;$$

but $$SG' : SG = k : 1, \quad SG\,(k^3-1) = SX\,(k^2-1),$$

$$SX = \frac{k^3-1}{k^2-1} \,.\, SG = \frac{k^2+k+1}{k+1} \,.\, SG = \tfrac{3}{2}SG \text{ in the limit.}$$

8716. (By Professor Mathews, M.A.)—Prove that the real common tangents of the circles $x^2+y^2-2ax = 0$, $x^2+y^2-2by = 0$ are represented by $2ab\,(x^2+y^2-2ax) = (by-ax+ab)^2$, or, which is the same thing, by

$$2ab\,(x^2+y^2-2by) = (by-ax-ab)^2.$$

Solution by C. E. Williams, M.A.; W. J. Greenstreet, B.A.; *and others.*

Two tangents from (x', y') to the circle $x^2+y^2-2ax = 0$ are given by

$$[xx'+yy'-a\,(x+x')]^2 = (x^2+y^2-2ax)(x'^2+y'^2-2ax').$$

But (x', y') is in this case the external centre of similitude whose coordinates are easily found to be $ab/(b-a)$, $ab/(a-b)$. Substituting these values for (x', y'), we get $(by-ax-ab)^2 = 2ab\,(x^2+y^2-2ax)$, and the other expression is obtained similarly from the other circle.

7169 & 8537. (By W. J. Greenstreet, B.A.)—The sum of the three sides of a right-angled spherical triangle is a quadrant: prove minimum value of hypotenuse is $\cos^{-1}\frac{4}{5}$, and that in this case the spherical excess is $\sin^{-1}\frac{1}{9}$.

Solution by Rev. T. R. Terry, M.A.; J. Young, M.A.; *and others.*

Since $$\cos c = \cos a \cos b, \text{ and } a+b+c = \tfrac{1}{2}\pi \quad \text{..................(1, 2),}$$

therefore, for minimum value of c,

$$\tan a\, da + \tan b\, db = 0, \quad \text{and} \quad da+db = 0,$$

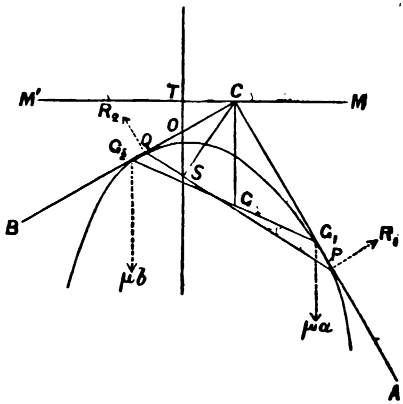
therefore $a = b$, whence, by (1) and (2), $\cos a = \cos b = 2/\sqrt{5}$, $\cos c = \frac{4}{5}$. This is obviously a minimum. Also

$$\sin E = \sin\,(2A - \tfrac{1}{2}\pi) = \tfrac{1}{9}, \quad \text{since} \quad \cos A = \tfrac{2}{3}.$$

[For another solution, see Vol. xxxviii., p. 89.]

8452. (By Satis Chandra Rây, B.A.)—Two rods, of lengths a and b are jointed by a smooth hinge and rest on the convex side of a parabolic arc whose axis is vertical; if, in the position of equilibrium, the rods include a right angle, show that the angle θ which the chord of contact makes with the axis is given by $\tan\frac{1}{2}\theta = a^2/b^2$.

Solution by W. J. Barton, M.A.; Sarah Marks, B.Sc.; *and others.*



Let CA, CB be the rods, of lengths a and b; G_1, G_2 their middle points; μa, μb their weights; S the focus, MOM′ the directrix of the parabola; P, Q the points of contact of the rods; ST the axis; R_1, R_2 the reactions at P and Q. Then TSP $= \theta$; and therefore MCA $= \frac{1}{2}\theta$, M′CB $= \frac{1}{2}\pi - \frac{1}{2}\theta$.

(1) Taking moments about C for the equilibrium of each rod separately, we get

$$\mu a \,.\, \tfrac{1}{2}a\cos\tfrac{1}{2}\theta = R_1 t_1,$$

$$\mu b \,.\, \tfrac{1}{2}b\sin\tfrac{1}{2}\theta = R_2 t_2,$$

where $CP = t_1$, $CQ = t_2$,

whence $$\frac{R_1}{R_2} = \frac{t_2 a^2}{t_1 b^2}\cot\tfrac{1}{2}\theta \quad\text{.........(1).}$$

Resolving horizontally for the equilibrium of the system,

$$R_1\sin\tfrac{1}{2}\theta = R_2\cos\tfrac{1}{2}\theta,\quad\text{whence}\quad \frac{R_1}{R_2} = \cot\tfrac{1}{2}\theta \quad\text{.........(2).}$$

From (1) and (2) we get $t_1/t_2 = a^2/b^2$.

If CB meet the axis in O, since SO = SQ and OSP $= \theta$, the angle CQP $= \frac{1}{2}\theta$; therefore $t_1/t_2 = \tan\frac{1}{2}\theta$; whence $\tan\frac{1}{2}\theta = a^2/b^2$.

[The equations to EF, FD, DE are

$$-\alpha\cos A + \beta\cos B + \gamma\cos C = 0,\quad \alpha\cos A - \beta\cos B + \gamma\cos C = 0\text{...(1, 2)},$$

$$\alpha\cos A + \beta\cos B - \gamma\cos C = 0 \quad\text{.........(3).}$$

Hence the points of intersection of (1), (2), (3) with $\alpha = 0$, $\beta = 0$, $\gamma = 0$, respectively, lie on the line

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0 \quad\text{.........(4).}$$

If $S \equiv a\beta\gamma + b\gamma\alpha + c\alpha\beta$, then the equations of the circumscribed, nine-point, and self-conjugate circles are, respectively,

$$S = 0,\quad 2S - (a\alpha + b\beta + c\gamma)(\alpha\cos A + \beta\cos B + \gamma\cos C) = 0,$$

$$S - (a\alpha + b\beta + c\gamma)(\alpha\cos A + \beta\cos B + \gamma\cos C) = 0;$$

hence (4) is the common radical axis of these circles.]

8863. (By CH. HERMITE, Membre de l'Institut.)—Déterminer les intégrales définies $\int_0^{\pi} \frac{dx}{\sin(x+p)}$, $\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{dx}{\sin(x+p)}$, en supposant que p soit une quantité imaginaire quelconque.

Solution by the PROPOSER.

La détermination de ces intégrales définies s'obtient comme conséquence des résultats suivants auxquels conduisent facilement les méthodes élémentaires.

Soit a une quantité réelle, positive, et différente de zéro, on a :

$$\text{I.} \quad \int_0^{\pi} \frac{dx}{\sin(x-ia)} = +\int_0^{\pi} \frac{dx}{\sin(x+ia)} = 2\log\frac{1+e^{-a}}{1-e^{-a}};$$

$$\text{II.} \quad \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{dx}{\sin(x-ia)} = -\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \frac{dx}{\sin(x+ia)} = 4i\,\text{arc tg}\,e^{-a}.$$

Remarquons ensuite qu'en remplaçant a par une variable imaginaire $\alpha+i\beta$, dont la partie réelle soit positive et différente de zéro, et par conséquent dans toute la région au-dessus de l'axe des abscisses, les diverses intégrales définies sont des fonctions holomorphes de cette variable. Observons maintenant que z étant à l'intérieur d'une circonférence de rayon égal à l'unité et dont le centre est à l'origine, la fonction $\log(1+z/1-z)$ et arc tg z sont aussi holomorphes. Or on obtient en posant $z=e^{-a}$, des valeurs qui remplissent cette condition, et l'extension des relations (I.) et (II.) à toutes les quantités $a=\alpha+i\beta$, où α est positif et différent de zéro, est la conséquence immédiate de la proposition bien connue de RIEMANN : Deux fonctions, uniformes, holomorphes ou n'ayant dans une aire donnée que des discontinuités en nombre fini, sont égales en tous les points de cette aire, si elles coincident le long d'une ligne de grandeur finie.

8242. (Professor SYLVESTER, F.R.S.)—If by a Simplicissimum of the nth order be understood a figure in space of n dimensions formed by the indefinite protraction of the series of which a linear segment, a triangle, and a pyramid are the three first terms, prove that (1), when each edge is unity, the squared content is $\frac{(n+1)}{2^n(1.2.3\ldots n)^2}$, and hence (2) deduce that the Cayleyan Persymmetrical Invertebrate Determinant of the squared edges by which such squared content is imaged must be diminished in the ratio of negative unity to $(-2)^n(1.2.3\ldots n)^2$, in order that it may represent its absolute value. *Ex. gr.*, the determinant

$$\begin{vmatrix} . & (ab)^2 & (ac)^2 & 1 \\ (ba)^2 & . & (bc)^2 & 1 \\ (ca)^2 & (cb)^2 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

images the squared content of a triangle whose edges are (ab), (ac), (bc), for the triangle vanishes when this determinant vanishes, but the actual value of the squared content is this determinant diminished in the ratio of negative unity to $2^2(1.2)^2$, *i.e.*, multiplied by $-\frac{1}{16}=(2)$.

Solution by W. J. C. SHARP, M.A.

If orthogonal coordinates be employed, the contents of the successive Simplicissima are

$$\frac{1}{1!}\begin{vmatrix} x_1, & 1 \\ x_2, & 1 \end{vmatrix}, \quad \frac{1}{2!}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2 & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \quad \frac{1}{3!}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}, \text{ \&c.,}$$

and in n-dimension space $\frac{1}{n!}\begin{vmatrix} x_1, & y_1, & \ldots\ldots & 1 \\ x_2, & y_2, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_{n+1}, & y_{n+1}, & \ldots\ldots & 1 \end{vmatrix} = \text{V}$ suppose.

Now, if the origin be taken at the centre of the hyper-sphere radius R which passes through the $n+1$ vertices, and if the edges be denoted by (1.2), (2.3), &c.,

$$\begin{array}{ll} x_1^2 + y_1^2 + \ldots\ldots = \text{R}^2, & 2(x_1x_2 + y_1y_2 + \ldots\ldots) = 2\text{R}^2 - (1.2)^2, \\ x_2^2 + y_2^2 + \ldots\ldots = \text{R}^2, & 2(x_2x_3 + y_2y_3 + \ldots\ldots) = 2\text{R}^2 - (2.3)^2, \\ \ldots \ldots \ldots \ldots \ldots & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ x_{n+1}^2 + y_{n+1}^2 \ldots\ldots = \text{R}^2, & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{array}$$

and

$$\text{V} = \frac{1}{n!}\begin{vmatrix} 1, & 0, & 0, & \ldots\ldots & 0, & 0 \\ 0, & x_1, & y_1, & \ldots\ldots & t_1, & 1 \\ 0, & x_2, & y_2, & \ldots\ldots & t_2, & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & x_{n+1}, & y_{n+1} & \ldots\ldots & t_{n+1}, & 1 \end{vmatrix} = \frac{(-1)^{2n+1}}{2^n n!}\begin{vmatrix} 0, & 0, & \ldots\ldots & 0, & 1 \\ 1, & 2x_1, & \ldots\ldots & 2t_1, & 0 \\ 1, & 2x_2, & \ldots\ldots & 2t_2, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & 2x_{n+1}, & \ldots\ldots & 2t_{n+1}, & 0 \end{vmatrix}$$

therefore $\text{V}^2 = \frac{(-1)^{2n+1}}{2^n (n!)^2}$

$$\times \begin{vmatrix} 0, & 1, & 1, & \ldots\ldots \\ 1, & 2(x_1^2 + y_1^2 \ldots t_1^2), & 2(x_1x_2 + y_1y_2 \ldots t_1t_2) & \ldots\ldots \\ 1, & 2(x_1x_2 + y_1y_2 \ldots t_1t_2), & 2(x_2^2 + y_2^2 \ldots t_2^2) & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1, & 2(x_1x_{n+1} + y_1y_{n+1} \ldots t_1t_{n+1}), & 2(x_2x_{n+1} + y_2y_{n+1} \ldots t_2t_{n+1}) & \ldots\ldots \end{vmatrix}$$

$$= \frac{(-1)^{2n+1}}{2^n (n!)^2}\begin{vmatrix} 0, & 1, & 1, & \ldots\ldots \\ 1, & 2\text{R}^2, & 2\text{R}^2 - (1.2)^2, & \ldots\ldots \\ 1, & 2\text{R}^2 - (1.2)^2, & 2\text{R}^2, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1, & 2\text{R}^2 - (1.n+1)^2, & 2\text{R}^2 - (2.n+1)^2 & \ldots\ldots \end{vmatrix}$$

$$= \frac{(-1)^{2n+1}}{2^n (n!)^2}\begin{vmatrix} 0, & 1, & 1, & \ldots\ldots \\ 1, & 0, & -(1.2)^2, & \ldots\ldots \\ 1, & -(1.2)^2, & 0, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1, & -(1.n+1)^2, & -(2.n+1)^2, & \ldots\ldots \end{vmatrix}$$

$$= \frac{(-1)^{3n+5}}{2^n (n!)^2}\begin{vmatrix} 0, & 1, & 1, & \ldots\ldots \\ 1, & 0, & (1.2)^2, & \ldots\ldots \\ 1, & (1.2)^2, & 0, & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1, & (1.n+1)^2, & (2.n+1)^2, & \ldots\ldots \end{vmatrix},$$

which proves (2); also, if $(1\,.\,2)^2 = (2\,.\,3)^2 = \&c. = 1$, the determinant is $= (-1)^{n+1}(n+1)$, and $V^2 = \dfrac{(n+1)}{2^n (n\,!)^2}$, which proves (1).

[If the $n+1$ points, which determine the Simplicissimum in space of n dimensions, lie on a linear locus in space of $(n-1)$ dimensions, the determinant V vanishes, and therefore the corresponding Cayleyan Persymmetrical Invertebrate Determinant; and this equated to zero gives a relation between the mutual distances of $n+1$ points on a linear locus in space of $n-1$ dimensions, as of three points in a straight line, four in a plane, five in a space, &c.

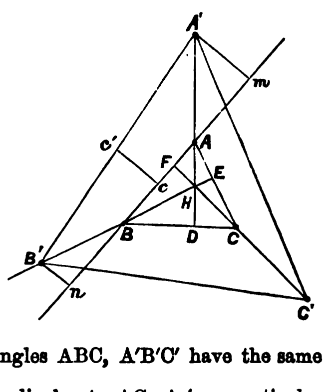
Mr. Sharp considers this to be an *epoch-making question*, as he believes that it will lead to the application to space of higher dimensions of methods analogous to those of trilinear and quadriplanar coordinates.

8755. (Professor Neuberg.)—On prolonge les hauteurs du triangle ABC au delà des sommets des quantités AA′ = BC, BB′ = CA, CC′ = AB; démontrer: (1) les triangles ABC, A′B′C′ ont même centre de gravité; (2) si α, α' sont les angles de Brocard de ces triangles, on a

$$A'B'C' = 2ABC\,(2+\cot\alpha),\ \ (A'B')^2+(B'C')^2+(C'A')^2 = 8ABC\,(3+2\cot\alpha),$$
$$\cot\alpha' = (2\cot\alpha+3)\,/\,(\cot\alpha+2);$$

(3) les points A, B, C sont les centres des carrés construits, intérieurement, sur les côtés du triangle A′B′C′; (4) les milieux des côtés de A′B′C′ sont les centres des carrés construits, extérieurement, sur les côtés de ABC.

Solution by R. F. Davis, M.A.; Prof. Aiyar, M.A.; *and others.*



Let fall the perpendiculars $A'm$, $B'n$ upon AB. Then, since the triangles $AA'm$, CBF are equal in all respects, $A'm = BF$ and $Am = CF$. Similarly, $B'n = AF$, and $Bn = CF$. Hence c, the middle point of AB, is also the middle point of mn; and if cc' be drawn perpendicular to AB meeting A′B′ in c', c' will be the middle point of A′B′, and

$$cc' = \tfrac{1}{2}(A'm + B'n) = Ac \text{ or } Bc.$$

Thus (4) c' is the centre of the square described externally on AB.

Since cc' is parallel to and one-half of CC′; Cc, $C'c'$ intersect in a point G which divides each of them in the ratio 2 : 1, and therefore (1) the triangles ABC, A′B′C′ have the same centre of gravity G.

Since BB′, Bc' are equal and perpendicular to AC, Ac', respectively, $B'c'$ is also equal and perpendicular to Cc', and (3) C is the centre of the square described internally on A′B′.

(2) Since $HA' = a + 2R\cos A = a(1+\cot A)$, $HB' = b(1+\cot B)$,

$\angle A'HB' = \pi - C$, area $A'HB' = \Delta(1+\cot A)(1+\cot B)$.

Hence area $A'B'C' = \Delta\{(1+\cot A)(1+\cot B) + \ldots + \ldots\}$

$= \Delta(4 + 2\cot a) = 2\Delta(2+\cot a)$,

since $\cot B \cot C + \ldots + \ldots = 1$, $\cot a = \cot A + \ldots + \ldots$.

Again, $(A'B')^2 = (A'm - B'n)^2 + mn^2 = (BF - AF)^2 + 4cm^2$

$= 4\{cF^2 + (Ac + CF)^2\} = 4\{Cc^2 + Ac^2 + 2\Delta\} = 2(a^2+b^2) + 8\Delta$,

hence $(A'B')^2 + (B'C')^2 + (C'A')^2 = 24\Delta + 4(a^2+b^2+c^2) = 8\Delta(3 + 2\cot a)$,
whence $\cot a'$ as stated.

It is to be observed, when ABC is equilateral, then so also is A'B'C', and $\cot a = \cot a' = \sqrt{3}$.

8865. (Amiral DE JONQUIÈRES.) — Soient trois nombres entiers, a, b, c, premiers entre eux, deux à deux, et vérifiant l'équation $a^n + b^n = c^n$. Démontrer que (l'exposant n premier et supérieur à 3): (1) a et b ne peuvent être, simultanément, premiers; (2) si a, supposé inférieur à b, est premier, $c = b + 1$.

8866 & 8908. (Professor CATALAN.)—Démontrer les contributions au théorème de FERMAT:—a supposé premier, (1) $a - 1 = \mathfrak{M}(n)$; (2) $a^n - 1 = \mathfrak{M}(nb)$; (3) tout diviseur premier, de $c - a$, divise $a - 1$; (4) $a + b$ et $c - a$ sont premiers entre eux; (5) $2a - 1$ et $2b + 1$ sont premiers entre eux; (6) le nombre premier, a (s'il existe) est compris entre

$$(nb^{n-1})^{1/n} \text{ et } \{n(b+1)^{n-1}\}^{1/n};$$

(7) a et b surpassent n; (8) le nombre b, qui satisfait à l'équation $(b+1)^n - b^n = a^n$, est compris entre

$$a(a/n)^{1/n-1} \text{ et } -1 + a(a/n)^{1/n-1};$$

(9) soit b un nombre entier, supérieur au nombre entier n. Entre

$$(nb^{n-1})^{1/n} \text{ et } \{n(b+1)^{n-1}\}^{1/n},$$

il y a, tout au plus, un nombre entier; (10) aucun des nombres $a + b$, $c - a$, $c - b$, n'est premier; (11) chacun d'eux a la forme N, ou la forme $(1/n)$ N, N étant un nombre entier; (12) soient, s'il est possible,

$$a + b = c'^n, \quad c - a = b'^n, \quad c - b = a'^n;$$

alors $c = \mathfrak{M}(n)$; (13) $(x+y)^n - x^n - y^n = nxy(x+y)P$,

$$P = H_1 x^{n-3} + H_2 x^{n-4} y + \ldots + H_1 y^{n-3},$$

les coefficients sont donnés par la formule $H_p = (1/n)[C_{n-1,p} \pm 1]$, le signe + répondant au cas où p est pair, et le polynôme P est divisible par $x^2 + xy + y^2$, et même par $(x^2+xy+y^2)^2$, si $n = \mathfrak{M}(6) + 1$; (14) la différence des puissances $n^{\text{ièmes}}$ de deux nombres entiers consécutifs, a, $a+1$, étant diminuée de 1, est divisible par $na(a+1)(a^2+a+1)$; les facteurs a, $a+1$, a^2+a+1 sont premiers entre eux, deux à deux; en outre, le troisième égale le produit des deux autres, augmenté de 1; (15) si, dans l'équation de FERMAT, le nombre a est premier, on a, par le théorème 8865, de M. de Jonquières, $a^n - 1 = \mathfrak{M}[nb(b+1)(b^2+b+1)]$; et (16) c est compris entre $a + b$ et $\frac{1}{2}(a+b)$.

Solution by Samuel Roberts, F.R.S.

It seems convenient to take these three questions together. The particular results are so numerous, that brevity must be specially consulted in each instance. They are, however, in general, simple deductions from the considerations indicated by Barlow (*Theory of Numbers*, Cap. vi.) and by Abel (*Works*, Holmboë's Edition, p. 264, Vol. ii.).

We cannot have $a = b$; but if a, b are prime the conditions $c-a = c-b = 1$ are necessary, and c cannot be prime, since $c^n = (a+b)\,\phi$, $\phi > 1$. If a (the least number) is prime, $c-b = 1$.

(1) By Fermat's theorem applied to $c^n - a^n - b^n = 0$, we have $a+b-c = \mathfrak{M}n = a-1$.

(2) $a^n - 1 = c^n - b^n - (c-b)^n = \mathfrak{M}b\,(b+1)\,n$.

(3) $(c-a)+(a-1) = b$; $(c-a)\,\phi = b^n$; $(c-a)$ is of the form r^n or $r^n n^{n-1}$, r or rn being a factor of b, therefore $a-1$ contains the prime factors of $(c-a)$.

(4) $(a+b) = k\,(c-a)$ makes a not prime to b, contrary to hypothesis.

(5) $2a-1 = (a+b)-(c-a)$, $2b+1 = (a+b)+(c-a)$.

(6) By (1) since a is > 1, $b > a$.

(7) $(b+1)^n - b^n > nb^{n-1}$, $(b+1)^n - \{(b+1)-1\}^n < n\,(b+1)^{n-1}$.

(8) By (7), $\dfrac{b^{n-1}}{a^{n-1}} < \dfrac{a}{n}$, $\dfrac{(b+1)^{n-1}}{a^{n-1}} > \dfrac{a}{n}$.

(9) If a lies between $(nb^{n-1})^{1/n}$ and $\{n\,(b+1)^{n-1}\}^{1/n}$,

we have $$a+1 > (nb^{n-1})^{1/n} + 1 > \{n\,(b+1)^{n-1}\}^{1/n},$$

or $$(nb^{n-1})^{1/n} + \left(1-\frac{1}{n}\right), \text{ since } b > n.$$

Similarly, $$(a-1) < \{n\,(b+1)^{n-1}\}^{1/n} - 1 < \left[n\{(b+1)-1\}^{n-1}\right]^{1/n}$$
$$< \{n\,(b+1)^{n-1}\}^{1/n} - \left(1-\frac{1}{n}\right).$$

Therefore $a+1$, $a-1$ lie outside the limits.

(10) Each of these quantities is of the form r^n or $r^n n^{n-1}$.

(11) As above, see Barlow, Cap. vi. (Barlow's mistake occurs further on).

(12) I do not understand this. If c contains n as a factor, $a+b$ contains n as a factor and is of the form $r^n n^{n-1}$. For $\dfrac{a^n+b^n}{a+b}$ is prime to n if $a+b$ is prime to n, and, if $a+b$ contains n, $\dfrac{a^n+b^n}{a+b}$ contains n. It has been assumed all along that a, b, c are prime to one another.

(13) $$(x+y)^n - x^n - y^n = (x+y)\left\{(x+y)^{n-1} - (x^{n-1} + x^{n+2}y + \ldots + y^{n+1})\right\}$$
$$= xy\,(x+y)\left\{\left[(n-1)+1\right]x^{n-3} + \left(\frac{(n-1)(n-2)}{1\,.\,2} - 1\right)x^{n-4}y + \ldots + \left[(n-1)+1\right]y^{n-3}\right\}.$$

Also $(x+y)^n - x^n - y^n$, when n is of the forms $6k \pm 1$, is made zero by putting ωy for x, ω being an imaginary cube root of unity. If n is of the form $6k+1$, the derived function $n(x+y)^{n-1} - nx^{n-1}$ is made zero by the same substitution. The function $(x+y)^n - x^n - y^n$ contains therefore $(x-\omega y)(x-\omega^2 y)$ or x^2+xy+y^2, and in the last case twice over.

(14), (15) These are direct consequences of the foregoing.

(16) $(c-a)+(c-b) > 0$, $c^n = a^n + b^n < (a+b)^n$.

8810. (Rev. T. R. Terry, M.A.) — Find the equation to the straight lines through the origin and the intersection of the conics

$$x^2 - 3xy - 4x + y - 1 = 0, \quad 2x^2 + xy + 4y^2 + 2x + 13y + 8 = 0.$$

Solution by H. Fortey, M.A.; *the* Proposer; *and others.*

In the most general case, the equations to the conics are of the form

$$\left.\begin{aligned} u_2 + u_1 + u_0 &= 0 \\ v_2 + v_1 + v_0 &= 0 \end{aligned}\right\},$$

whence $\qquad u_2 v_0 - v_2 u_0 = -(u_1 v_0 - v_1 u_0)$, or $W_2 = W_1$;

therefore the equation to the four lines required is

$$u_2 W_1^2 + u_1 W_1 W_2 + u_0 W_2^2 = 0.$$

In the particular case this reduces to $(x-y)^2(2x+y)^2 = 0$, *i.e.*, the four lines coincide two and two with the lines $x-y=0$ and $2x+y=0$.

8843. (Rev. T. P. Kirkman, F.R.S.) — The angles $nabc \ldots lm$ of any convex plane n-gon N are joined to a point P in its plane, within or without it. In Pa, Pb, ... Pn are taken points $a_1 b_1 c_1 \ldots m_1 n_1$, such that $na_1 b_1 \ldots m_1 n_1$ is a broken line L beginning and ending on Pn. L is thus formed: first, $na_1 > a_1 a$; and next, if $d_1 e_1 f_1$ be any three consecutive points of it, the distances in the directions $f_1 e_1$ and $e_1 d_1$ of e_1 from the lines fe and ed in N are equal. The line $n_1 m_1$ meets the edge nm of N in r. Prove that, if from s in an produced we draw $sn_1 = rn_1$, sn_1 is parallel to the first portion na_1 of L.

Solution by the Proposer.

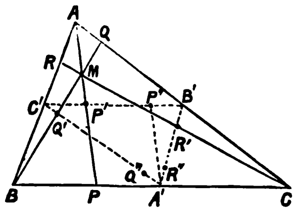
The described figure is the orthogonal projection of a pyramid whose vertex P is anywhere over the base N, and on which the line over $na_1 \ldots m_1 n_1$ crosses every face about P, in a path equigradient from n to n_1. For, as the lines drawn from e_1 are equal, they are projected sides of an isosceles triangle drawn from e_1 in Pe to edges of N, in the faces about P whose intersection is Pe; and the sides of this triangle make equal angles with the base, or are equigradient. Hence sn_1 and na_1 are equigradients directed upwards from s and n in the same plane aPn, and are therefore parallels.

Thus we can find (Quest. 8325) with the ruler the fly's upward path from n through a_1 to a point n_1 in Pn. A few trials with the ruler *in plano*, using a_2, a_3, &c. instead of a_1, will bring us to n_2, n_3, &c., one of which is sure to be near enough for engineering purposes to the starting point E.

As the solution of Mr. Biddle's interesting Quest. 8325 is not likely to be completely worked out, this easy approximation to it by plane geometry may be worth having.

8814. (The Editor.)—Prove that the triangle formed by joining the mid-points of the altitudes of a triangle is one-fourth of the pedal triangle, and that the theorem holds good for any other three concurrent lines drawn through the vertices of the triangle.

Solution by Professor De Longchamps; D. Biddle; *and others.*



1. *Geometrically* :—Soient A′, B′, C′ les milieux des côtés du triangle considéré, PQR le triangle pédal correspondant à M. Les droites AP, BQ, CR rencontrent les côtés de A′B′C′ respectivement en des points P′, Q′, R′. Considérons aussi les points P″, Q″, R″ *isotomiques* de P′, Q′, R′; c'est à dire des points tels que B′P″ = P′C′, etc.... La droite A′P″ est parallèle à AP, et d'une façon plus générale, on peut dire que la figure A′B′C′P″Q″R″ est homothétique à la figure ABCPQR; le centre de gravité de ABC étant le centre de l'homothétie et le rapport d'homothétie étant égal à $\frac{1}{2}$. De là, nous concluons d'abord que l'aire du triangle P″Q″R″ est le $\frac{1}{4}$ de celle du triangle PQR. D'autre part, on sait (*Journal de Mathématiques élémentaires*, 1877, p. 224), par une propriété que nous avons indiquée autrefois, que *deux triangles isotomiques* P′Q′R′, P″Q″R″ *sont équivalents.*

Ainsi, en résumé, le triangle P′Q′R′ est le $\frac{1}{4}$ de PQR.

2. *Analytically* :—Si nous considérons trois points quelconques M_0, M_1, M_2, dont les coordonnées barycentriques sont, respectivement,

$$(\alpha_0, \beta_0, \gamma_0), \quad (\alpha_1, \beta_1, \gamma_1), \quad (\alpha_2, \beta_2, \gamma_2),$$

la surface Σ du triangle $M_0M_1M_2$ est donnée, comme l'on sait, par la formule $\pm S^2\Sigma = \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}$, S désignant l'aire du triangle de référence ABC; $\alpha_0, \beta_0, \gamma_0$ ... représentant *les coordonnées absolues* des points considérés. Prenons un point μ dont les coordonnées soient $(\alpha', \beta', \gamma')$; les droites A$\mu$, B$\mu$, C$\mu$ rencontrent les côtés du triangle ABC en des points

M_0, M_1, M_2 ayant pour coordonnées

$$0,\ \frac{S\beta'}{\beta'+\gamma'},\ \frac{S\gamma'}{\beta'+},\ \text{et l'on a: } \pm S^2\Sigma = \begin{vmatrix} 0, & \frac{S\beta'}{\beta'+\gamma'}, & \frac{S\gamma'}{\beta'+\gamma'} \\ \frac{S\alpha'}{\alpha'+\gamma'}, & 0, & \frac{S\gamma'}{\alpha'+\gamma'} \\ \frac{S\alpha'}{\alpha'+\beta'}, & \frac{S\beta'}{\alpha'+\beta'}, & 0 \end{vmatrix},$$

Σ désignant maintenant l'aire du triangle pédal $M_0\,M_1\,M_2$. Cette formule donne, finalement, $\pm\Sigma = 2S\dfrac{\alpha'\beta'\gamma'}{(\alpha'+\beta')(\beta'+\gamma')(\gamma'+\alpha')}$ ....(1).

Considérons maintenant les points μ_0, μ_1, μ_2 milieux des segments AM_0, AM_1, AM_2. Les coordonnées barycentriques absolues de ces points sont

$$\frac{S}{2},\quad \frac{S}{2}\frac{\beta'}{\beta'+\gamma'},\quad \frac{S}{2}\frac{\gamma'}{\beta'+\gamma'}.$$

La surface Σ' du triangle $\mu_0\,\mu_1\,\mu_2$ est donnée par la formule

$$\pm S^2\Sigma' = \begin{vmatrix} \frac{S}{2}, & \frac{S}{2}\frac{\beta'}{\beta'+\gamma'}, & \frac{S}{2}\frac{\gamma'}{\beta'+\gamma'} \\ \frac{S}{2}\frac{\alpha'}{\alpha'+\gamma'}, & \frac{S}{2}, & \frac{S}{2}\frac{\gamma'}{\beta'+\gamma'} \\ \frac{S}{2}\frac{\alpha'}{\alpha'+\beta'}, & \frac{S}{2}\frac{\beta'}{\alpha'+\beta'}, & \frac{S}{2} \end{vmatrix},$$

donc $$\pm\Sigma' = \frac{S}{8}\,\frac{1}{(\alpha'+\beta')(\beta'+\gamma')(\gamma'+\alpha')}\begin{vmatrix} \beta'+\gamma', & \beta', & \gamma' \\ \alpha', & \alpha'+\gamma', & \gamma' \\ \alpha', & \beta', & \alpha'+\beta' \end{vmatrix}.$$

Le déterminant qui entre dans cette formule étant développé, ou trouve facilement (mais ce résultat peut être vu aussi par des voies plus directes et plus élégantes) qu'il est égal à $4\alpha'\beta'\gamma'$.

Ecrivons donc $\pm\Sigma' = \dfrac{S}{2}\,\dfrac{\alpha'\beta'\gamma'}{(\alpha'+\beta')(\beta'+\gamma')(\gamma'+\alpha')}$ ....(2).

La comparaison des formules (1) et (2) prouve que $\Sigma = 4\Sigma'$.

8858. (Professor COCHEZ.)—Résoudre les équations

$$(xy+1)(x+y) = axy, \quad (x^2y^2+1)(x^2+y^2) = b^2x^2y^2.$$

Montrer que ce système est *quadratique*. Appliquer les formules au cas particulier suivant: $a = \frac{16}{3}$, $b = \frac{10}{3}$.

Solution by Professor STEGGALL; R. F. DAVIS, M.A.; *and others.*

Let $x+1/x = \xi$, $y+1/y = \eta$; then the equations become $\xi+\eta = a$, $\xi^2+\eta^2 = b^2+4$, whence ξ, η are easily found. In the particular case $\xi = \frac{10}{3}$ or 2; $\eta = 2$ or $\frac{10}{3}$; and we have $x = 1$, $y = 3$ or $\frac{1}{3}$, $y = 1$, $x = 3$ or $\frac{1}{3}$.

8498. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—The equation

$$\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dt^2}-2\frac{dy}{dx}\cdot\frac{dy}{dt}\cdot\frac{d^2y}{dx\,dt}+\left(\frac{dy}{dt}\right)\frac{d^2y}{dx^2}=\mathrm{Q}\cdot\left(\frac{dy}{dx}\right)^3$$

is integrable when (1) $\mathrm{Q}=0$, (2) $\mathrm{Q}=dx/dt$. Hence, obtain the complete primitive.

Solution by Professor Lloyd Tanner, M.A.

The proposed equation is a transformation of $\frac{d^2x}{dt^2}+\mathrm{Q}=0$, where x, y are independent variables. The solutions required are, therefore,

$x=\phi y+t\psi y$ when $\mathrm{Q}=0$; and $x=\phi y+e^{-t}\psi y$ when $\mathrm{Q}=dx/dt$.

[See *Messenger of Mathematics*, New Series, Vol. v., p. 71.]

Note on Analllagmatic Curves. *By* Professor Wolstenholme, Sc.D.

In a curve which is its own inverse with respect to a point O, if any straight line through O meet the curve in two points P, Q so that the rectangle OP . OQ is constant, a circle can be drawn touching the curve in P, Q. Hence, when we suppose Q to coincide with P, the bitangent circle will in the limit have four-point contact with the curve at P, and will therefore be a circle of curvature of maximum or minimum radius. The points of greatest or least curvature (other than the vertices) in any such curve will therefore be the points of contact of tangents drawn from a centre of inversion of the curve.

I do not remember ever to have seen this simple deduction drawn. I ought to have myself noticed it years ago, as I have two particular cases of it in my book of problems; but in both I obtained the result in a much more laborious way. No doubt many interesting particular cases might be considered. I have not much hope of doing anything at it myself, so should be glad to have the above note published, *pro bono publico.*

8786. (R. Tucker, M.A.) — Pp, Qq, Rr are focal chords of a parabola; if P, Q, R have a conormal point, then the centres of curvature for p, q, r are collinear. (This is another way of putting the first part of Question 8693.) Find the equation to this line. Prove that this *central* line envelopes (1) a hyperbola when tangents at p, q meet on the directrix, (2) an ellipse when the tangents meet on the latus rectum.

Solution by R. Knowles, B.A.; Prof. Beyens; *and others.*

In the solution of Quest. 8693 it is shown that the centres of curvature of three points on a parabola are collinear when $\Sigma y^{-1}=0$; if x_1, y_1, &c.,

be the coordinates of PQR, those of p, q, r are

$$\frac{a^2}{x_1},\quad \frac{-4a^2}{y_1}, \text{ \&c., and } \Sigma y^{-1} = -4a^2(y_1+y_2+y_3) = 0,$$

since P, Q, R have a conormal point.

The coordinates of the centres of curvature at p, q, r are

$$\frac{3a^2+2ax_1}{x_1},\quad \frac{16a^4}{y_1^3}, \text{ \&c.,}$$

and if the tangents at p, q meet in the directrix $y_1y_2 = -4a^2$. The equation to the central line

$$y - \frac{16a^4}{y_3^3} = \frac{4a(y_1^2+y_1y_2+y_2^2)}{3y_1y_2(y_1+y_2)}\left(x - \frac{3a^2+2ax_3}{x_3}\right)$$

becomes $$y_1+y_2 = -y_3 \qquad\qquad (1),$$

$$-2ay_3^2 + 3ay_3^3 - 48a^5 = (y_3^2+4a^2)(y_3^2x - 12a^3 - 12ay_3^2) \qquad (2),$$

if the tangents at pq meet in the latus rectum $y_1y_2 = 4a^2$, and (1) becomes

$$3ay_3y - 48a^5 = -(y_3^2-4a^2)(y_3^2x - 12a^3 - 2ay_3^2) \qquad (3),$$

and the envelopes of (2) and (3) are respectively the hyperbola

$$9y^2 - 16x^2 + 112ax - 160a^2 = 0,$$

and the ellipse $$9y^2 + 16x^2 - 16ax + 32a^2 = 0.$$

8207. (W. J. C. Sharp, M.A.)—If A be the angle contained in the half of a small Circle of a sphere, angular radius a, by arcs of lengths b and c; *i.e.*, if ABC be a spherical triangle having the angle A = B + C; show that (1) $\cos b + \cos c = 1 + \cos a$; (2) $\cos A = -\tan\frac{1}{2}b \tan\frac{1}{2}c$; and (3) hence deduce Euclid I. 47 and III. 31.

Solution by Rev. T. Galliers, M.A.; Prof. Matz, M.A.; *and others.*

Let AO = OB = OC = R; then, from triangles AOC, AOB,

$$\cos b = \cos AO \,.\, \cos OC + \sin AO \,.\, \sin OC \cos AOC$$
$$= \cos^2 R + \sin^2 R \cos \widehat{AOC} \qquad (1),$$
$$\cos c = \cos AO \,.\, \cos OB + \sin AO \sin OB \cos AOB$$
$$= \cos^2 R + \sin^2 R \cos \widehat{AOB} \qquad (2).$$

From (1) and (2) by addition, and since $\widehat{AOC} + \widehat{AOB} = 180°$,

$$\cos b + \cos c = 2\cos^2 R = 1 + \cos 2R = 1 + \cos a \qquad (3);$$

also, from triangle ABC,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos b + \cos c - 1 - \cos b \cos c}{\sin b \sin c}$$
$$= -\frac{2\sin^2\frac{1}{2}b \,.\, 2\sin^2\frac{1}{2}c}{4\sin\frac{1}{2}b\cos\frac{1}{2}b\sin\frac{1}{2}c\cos\frac{1}{2}c} = -\tan\frac{1}{2}b\tan\frac{1}{2}c \qquad (4).$$

Now, the formulæ in Plane Trigonometry corresponding to (3) and (4) are found by writing $a = \alpha/r$, $b = \beta/r$, $c = \gamma/r$ and making r infinite, when, α, β, γ become the sides of a plane triangle.

Thus (3) becomes $\cos\frac{\beta}{r} + \cos\frac{\gamma}{r} = 1 + \cos\frac{a}{r}$ .... (5).

But

$$\cos\frac{\beta}{r} = 1 - \frac{1}{2}\cdot\frac{\beta^2}{r^2} + \frac{1}{4!}\cdot\frac{\beta^4}{r^4} - \dots,\quad \cos\frac{\gamma}{r} = 1 - \frac{1}{2}\cdot\frac{\gamma^2}{r^2} + \frac{1}{4!}\cdot\frac{\gamma^4}{r^4} - \dots,$$

$$\cos\frac{a}{r} = 1 - \frac{1}{2}\cdot\frac{a^2}{r^2} + \frac{1}{4!}\cdot\frac{a^4}{r^4} - \dots\dots$$

Substituting in (5), we have, after reduction, $\beta^2 + \gamma^2 = a^2 + \lambda\frac{1}{r^2}$, where λ is a quantity which remains finite when r is made infinite. Thus, when r is infinite, $\beta^2 + \gamma^2 = a^2$, a result which agrees with Euc. I. 47.

Also, when $b = \frac{\beta}{r}$, $c = \frac{\gamma}{r}$, in (4), we have $\cos A = -\frac{\sin(\beta/2r)\sin(\gamma/2r)}{\cos(\beta/2r)\cos(\gamma/2r)}$.

But $\sin\frac{\beta}{2r} = \frac{\beta}{2r} - \frac{1}{3!}\left(\frac{\beta}{2r}\right)^3 + \dots,\quad \sin\frac{\gamma}{2r} = \frac{\gamma}{2r} - \frac{1}{3!}\left(\frac{\gamma}{2r}\right)^3 + \dots\dots;$

also

$$\cos\frac{\beta}{2r} = 1 - \frac{1}{2}\left(\frac{\beta}{2r}\right)^2 + \frac{1}{4!}\left(\frac{\beta}{2r}\right)^4 - \dots,\quad \cos\frac{\gamma}{2r} = 1 - \frac{1}{2}\left(\frac{\gamma}{2r}\right)^2 + \frac{1}{4!}\left(\frac{\gamma}{2r}\right)^4 - \dots\dots\dots$$

Therefore $\cos A = -\frac{\lambda'.(1/r^2)}{1+\mu.(1/r^2)}$, where λ' and μ remain finite when r is made infinite; therefore, when r is infinite, $\cos A = 0$, or $A = 90°$, a result which agrees with first part of Euc. III. 31.

8571. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.) — **Show that the reciprocal polar of the evolute of the reciprocal polar of the evolute of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with respect to the circle described on the line joining the foci as diameter, is the curve**

$$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}\left\{\left(\frac{x}{y}\right)^{\frac{1}{3}}\left(\frac{x}{b}\right)^{\frac{2}{3}} - \left(\frac{y}{x}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{2}{3}}\right\}^2 = \left(\frac{a}{b} - \frac{b}{a}\right)^2.$$

Solution by Profs. Sircom, Nilkantha Sarkar; *and others.*

The reciprocal polar of the evolute of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with respect to $x^2 + y^2 = a^2 - b^2$ is the locus of xy, where $xx_1 + yy_1 = a^2 - b^2$, $\frac{x}{y} = -\frac{a^2 y_1}{b^2 x_1}$ and $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, whence $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$. The reciprocal polar of the evolute of the last locus is the locus of xy, where

$$xx_1 + yy_1 = a^2 - b^2,\quad \frac{x}{y} = -\frac{b^2 x_1^3}{a^2 y_1^3},\quad \frac{a^2}{x_1^2} + \frac{b^2}{y_1^2} = 1 \;\dots\dots\; (1, 2, 3),$$

whence, from (1), $$\frac{y_1}{x_1} = -\left(\frac{a^2x}{b^2y}\right)^{\frac{1}{3}},$$

and, from (2), $$\frac{a^2-b^2}{x_1} = x - \left(\frac{a^2x}{b^2y}\right)^{-\frac{1}{3}} y, \quad \frac{a^2-b^2}{y_1} = y - \left(\frac{b^2x}{a^2y}\right) x.$$

Substituting in (3) and reducing, we obtain the required result.

8311. (R. Swaminatha Aiyar, B.A.)—Find how many numbers can be formed, having n for the sum of the digits (zero not being used as a digit).

Solution by G. G. Storr, M.A.; *the* Proposer.

Let P_n represent the number of numbers that have n for the sum of the digits. Of these P_n, those that have 1 for their first digit are P_{n-1}, those that have 2 for their first digit are P_{n-2}, those that have 3 for their first digit are P_{n-3}, and so on. Therefore

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} \ldots\ldots + P_{n-9}.$$

Thus P_n is the coefficient of x^n in the expansion in powers of x of

$$\frac{1}{1-x-x^2-\ldots-x^9}, \text{ that is, of } \frac{1-x}{1-2x+x^{10}}.$$

$$\frac{1-x}{1-2x+x^{10}} = (1-x)\left\{1-x(2-x^9)\right\}^{-1}$$

$$= (1-x)\left\{1+x(2-x^9)+x^2(2-x^9)^2+\ldots\ldots x^n(2-x^9)^n+\ldots\ldots\right\}.$$

The coefficient of x^n is $2^n-(n-3)\,2^{n-10}+\dfrac{(n-18)(n-19)}{2!}\,2^{n-20}-$ &c.

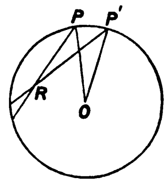
$$-\left\{2^{n-1}-(n-10)\,2^{n-11}+\frac{(n-19)(n-20)}{2!}\,2^{n-21}-\&c.\right\}.$$

It will be noticed that, if n be not greater than 9, or if the radix of the scale of notation be not less than $n+1$, $P_n = 2^{n-1}$.

7356. (Professor Wolstenholme, M.A., Sc.D. Suggested by Quest. 7285.)—At each point P of a given curve is drawn a straight line U, making a given angle with the tangent at P, and a straight line V, such that U, V are equally inclined to the ordinate at P; prove that the point of contact of U with its envelope is the projection upon U of the centre of curvature at P, and that the point of contact of V with its envelope is the projection upon V of the image with respect to the tangent at P of the centre of curvature at P. [That is, if O be the centre of curvature at P, and OPO′ be a straight line bisected in P, then, if OL, O′M be let fall perpendicular to U, V respectively, L, M will be the points of contact of U, V with their envelopes.]

Solution by Profs. Nash, M.A.; Nilkanta Sarkar, M.A.; *and others.*

1. Let P, P′ be two adjacent points on the curve, PR, P′R the positions of the line U, and O the centre of curvature at P, then ultimately P, P′ may be considered as points lying on the circle of curvature. Now, since $\angle OPR = \angle OP'R$,

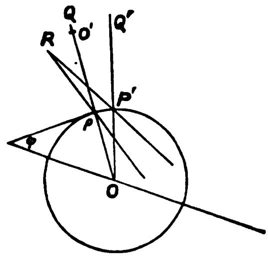


therefore PR′OR are concyclic, therefore

$$\angle PRO = \pi - \angle PP'O.$$

But in the limit PP′ is a tangent, and PP′O a right angle, therefore OR is at right angles to PR.

2. The line V makes with the normal an angle $\frac{1}{2}\pi - \alpha - 2\phi$, where ϕ is the inclination of the tangent to the axis of x.



In the figure,

$$POP' = \delta\phi, \quad QPR = \tfrac{1}{2}\pi - \alpha - 2\phi,$$

$$Q'P'R = \tfrac{1}{2}\pi - \alpha - 2\phi + 2\delta\phi,$$

$$PRP' = RP'Q' - RPQ - POP' = \delta\phi,$$

therefore $\dfrac{RP}{PP'} = \dfrac{\sin RP'P}{\sin \delta\phi}$,

and $PP' = OP\delta\phi = OP \sin \delta\phi$,

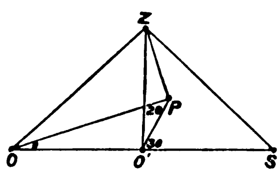
therefore $RP = OP \cos RPQ$; therefore, if $PO' = OP$, R is projection of O upon V.

8548. (Asparagus.) — Two straight lines turn about two fixed points O, O′ with angular velocities which are as 1 : 3, both straight lines coinciding with OO′ initially; if P be their point of intersection, prove that the envelope of a straight line drawn through P at right angles to OP will be a parabola whose directrix passes through O′ and is at right angles to OO′ and whose latus rectum is twice OO′.

Solution by R. F. Davis, M.A.; G. G. Storr, B.A.; *and others.*

Produce OO′ to S, so that $OO' = O'S$; and let $\angle POS = \theta$,

$$\angle PO'S = 3\theta, \quad \angle OPO' = 2\theta.$$



Draw PZ, O′Z perpendicular to OP, OO′ respectively. Then, since O, O′, P, Z are concyclic,

$$\angle PZO' = POO' = \theta,$$

and $\angle SZO' = OZO' = OPO' = 2\theta$.

Hence PZ bisects the $\angle SZO'$, and therefore envelopes a parabola with focus S and directrix O′Z.

8607. (R. CURTIS, M.A.)—Between two curves, $f(x, y) = 0$ and $\phi(x, y) = 0$, of degrees m and n, show that there can be drawn mn straight lines of a given length l, parallel to a given straight line, from the curve $f(xy) = 0$ to the curve $\phi(xy) = 0$, and mn others from $\phi(xy) = 0$ to $f(xy) = 0$. [The lines must be drawn in a given direction, say from left to right.]

Solution by Rev. T. R. TERRY, M.A.; Professor BEYENS; *and others.*

As the degree of the curves is not altered by changing the axes, we may take the axis of x parallel to the given straight line.

Since f is of the m^{th} degree, and ϕ of the n^{th}, there are mn points of intersection of the curves $f(x, y) = 0$ and $\phi(x-l, y) = 0$, therefore mn straight lines can be drawn from f to ϕ parallel to axis of x and from left to right. Similarly, the other result.

8405. (F. C. WACE, M.A.)—If $a_1 + a_2 + a_3 + \ldots + a_n = s$, prove that

$$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a}-1\right)^{a_2}\ldots\ldots\left(\frac{s}{a_n}-1\right)^{a_n} < (n-1)^s.$$

Solution by R. F. DAVIS, M.A.; Rev. T. R. TERRY, M.A.; *and others.*

If $a_1, a_2, \ldots\ldots a_n$ be positive integers, take $a_1, a_2, \ldots\ldots a_n$ quantities each equal respectively to $\quad \frac{s}{a_1}-1, \quad \frac{s}{a_2}-1, \quad \frac{s}{a_n}-1 \ldots\ldots,$

in all s quantities whose sum $= ns - s = (n-1)s$. The inequality of the question is the expression of the fact that the Arithmetical mean of these quantities is greater than the Geometrical mean.

8557. (Professor MATHEWS, M.A.)—If

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$$

are two confocal ellipses, such that polygons of r sides can be simultaneously inscribed in the second ellipse and circumscribed to the first, prove that $\quad a = a' \operatorname{sn}\frac{(r-2)K}{r}, \quad b = b' \operatorname{cn}\frac{2K}{r},$

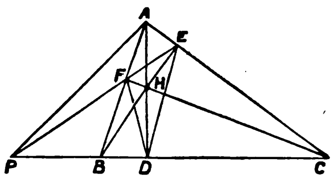
where the modulus of the elliptic functions is equal to the eccentricity of the inner ellipse. Verify the above when $r = 3, 4, 5$ respectively.

Solution by JOHN GRIFFITHS, M.A.

The solution of this question is contained in my paper "On the Derivation of Elliptic Function Formulæ from Confocal Conics," given in the Proceedings of the London Mathematical Society, Vol. xiv., p. 47.

8420. (EMILY PERRIN.)—Prove that the axis of perspective of a triangle and its pedal triangle is the common radical axis of the circumcircle, nine-point circle, and self-conjugate circle.

Solution by H. O. S. DAVIES, M.A.; W. J. BARTON, M.A.; *and others.*



Let ABC be the triangle, H its orthocentre, DEF the pedal triangle; and let EF meet BC produced in P. Then, since BFEC is a cyclic quadrilateral,

$$PB \,.\, PC = PE \,.\, PF$$

and $PA^2 = PB \,.\, PC + AE \,.\, AC.$
Hence

$$PB.PC = PA^2 - AH.AD = PH^2 - HD^2 + AD^2 - AH.AD = PH^2 + AH.HD.$$

Thus P is a radical centre of the three circles of the question. Similarly, if FD, CA meet in Q, and DE, AB in R; then Q, R are also radical centres and the axis of perspective PQR is the common radical axis of the three circles.

8862. (R. CURTIS, S.J., M.A.)—Find (1) the locus of a point in a material lamina, such that if the lamina were moving without rotation in its own plane, and that point were suddenly fixed in space, the ensuing rotation of the lamina would be given; (2) where the point should be if the rotation produced is a maximum; and (3) where the point should be in order that the loss of kinetic energy would be given.

Solution by Professors STEGGALL *and* BEYENS.

Taking the axes of x, y through the centre of gravity, and that of x parallel to the direction of motion, u being the initial velocity, w the final velocity about the point x, y, arrested; then $uy = w(k^2+x^2+y^2)$, whence locus (1) is a circle; point (2) is given by $x = 0$, $y = k$, as may be seen geometrically; locus (3) is given by $u^2 - (k^2+x^2+y^2)\,w^2 =$ constant, or $\dfrac{y^2}{b^2} - \dfrac{x^2}{k^2} = 1$, where b is arbitrary.

7707. (Professor S. MALET, F.R.S.)—Prove that

$$\int_0^{\frac12\pi} \frac{\log\sin\theta\left\{\log\tan\theta-\log\Delta(\theta)\right\}}{\Delta(\theta)}\,d\theta=\tfrac12\mathrm{K}\left\{\log k\log k'+\tfrac14\pi^2\right\}+\tfrac14\pi\mathrm{K}'\log k',$$

$$\int_0^{\frac12\pi} \frac{\log\cos\theta\left\{\log\cot\theta-\log\Delta(\theta)\right\}}{\Delta(\theta)}\,d\theta=\tfrac18\pi^2\mathrm{K},$$

$$\int_0^{\frac12\pi} \frac{\log\Delta(\theta)\left\{\log\Delta(\theta)-\log\sin\theta\cos\theta\right\}}{\Delta(\theta)}\,d\theta=\tfrac12\mathrm{K}\log k\log k'+\tfrac14\pi\mathrm{K}'\log k',$$

where $\Delta(\theta)\equiv(1-k^2\sin^2\theta)^{\frac12}$, $k^2+k'^2=1$, and K and K′ are complete elliptic integrals of the first kind with moduli respectively k and k'.

Solution by D. EDWARDES; SARAH MARKS, B.Sc.; *and others.*

Let $\theta=am\,u$, $\mathrm{A}=\frac{\pi}{\mathrm{K}}\frac{q^{\frac14}}{k^{\frac12}}$, $\mathrm{B}=2k'^{\frac12}\frac{q^{\frac14}}{k^{\frac12}}$, $\mathrm{C}=k'^{\frac12}$; we have

$$\log\frac{\pi}{\mathrm{AK}}\,\mathrm{sn}\,u=-\frac{1-q}{1+q}\cos\frac{\pi u}{\mathrm{K}}-\tfrac12\frac{1-q^2}{1+q^2}\cos\frac{2\pi u}{\mathrm{K}}-\tfrac13\frac{1-q^3}{1+q^3}\cos\frac{3\pi u}{\mathrm{K}}-\&c.,$$

whence
$$\int_0^{\mathrm{K}}\log\mathrm{sn}\,u\,du=\tfrac12\mathrm{K}\log\frac1k-\tfrac14\pi\mathrm{K}',$$

$$\int_0^{\mathrm{K}}\left\{\log\frac{\pi}{\mathrm{AK}}\,\mathrm{sn}\,u\right\}^2du=\tfrac12\mathrm{K}\left[\left(\frac{1-q}{1+q}\right)^2+\frac1{2^2}\left(\frac{1-q^2}{1+q^2}\right)^2+\frac1{3^2}\left(\frac{1-q^3}{1+q^3}\right)^2+\&c.\right]$$

$$\log\mathrm{cn}\,u=\log\tfrac12\mathrm{B}+\frac{1+q}{1-q}\cos\frac{\pi u}{\mathrm{K}}-\tfrac12\frac{1-q^2}{1+q^2}\cos\frac{2\pi u}{\mathrm{K}}+\tfrac13\frac{1+q^3}{1-q^3}\cos\frac{3\pi u}{\mathrm{K}}-\&c.,$$

$$\log\mathrm{dn}\,u=\log\mathrm{C}+4\left(\frac{q}{1-q^2}\cos\frac{\pi u}{\mathrm{K}}+\tfrac13\frac{q^3}{1-q^6}\cos\frac{3\pi u}{\mathrm{K}}+\&c.\right).$$

Hence
$$\int_0^{\mathrm{K}}\left[(\log\mathrm{sn}\,u)^2-\log\mathrm{sn}\,u\log\mathrm{cn}\,u\,\mathrm{dn}\,u\right]du=-\mathrm{K}\left(\log\frac{\pi}{\mathrm{AK}}\right)^2$$
$$-2\log\frac{\pi}{\mathrm{AK}}\int_0^{\mathrm{K}}\log\mathrm{sn}\,u\,du+\mathrm{K}\log\tfrac12\mathrm{BC}\log\frac{\pi}{\mathrm{AK}}+\mathrm{K}\left(1+\frac1{3^2}+\frac1{5^2}+\&c.\right),$$

or, substituting for A, B, C, and $\int_0^{\mathrm{K}}\log\mathrm{sn}\,u\,du$ their values, this reduces to the stated result. In the same way we find

$$\int_0^{\mathrm{K}}\left[(\log\mathrm{cn}\,u)^2-\log\mathrm{cn}\,u\log\mathrm{sn}\,u\,\mathrm{dn}\,u\right]du=\tfrac18\pi^2\mathrm{K}.$$

And putting $\mathrm{K}-u$ for u in this, and subtracting from the first integral, we get

$$-\log k'^2\int_0^{\mathrm{K}}\log\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\,du+\int_0^{\mathrm{K}}\left[(\log\mathrm{dn}\,u)^2-\log\mathrm{dn}\,u\log\mathrm{sn}\,u\,\mathrm{cn}\,u\right]du.$$

But
$$\int_0^{\mathrm{K}}\log\frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u}\,du=\int_0^{\mathrm{K}}\log\frac{\mathrm{cn}\,u}{\mathrm{sn}\,u\,\mathrm{dn}\,u}\,du\ (\mathrm{K}-u\text{ for }u)=0,$$

and therefore, &c.

8977. (Professor HAUGHTON, F.R.S.)—The form of the Terrestrial Radiation Function has been proved to be $A(\theta-\Theta_0)^n = a$, where A, Θ_0, n are unknown parameters, and θ, a are given by observation. The mean monthly observations at Greenwich, extending over thirty-six years, give

$$\text{January, } A(38{\cdot}9-\Theta_0)^n = 21{\cdot}4\,; \quad \text{February, } A(40{\cdot}4-\Theta_0)^n = 35{\cdot}5\,;$$
$$\text{March, } A(42{\cdot}8-\Theta_0)^n = 55{\cdot}9. \quad \text{Find } \Theta_0, \text{ A, and } n.$$

Solution by A. R. JOHNSON, M.A.

Θ_0 is found from $\log\dfrac{40{\cdot}4-\Theta_0}{38{\cdot}9-\Theta_0}\Big/\log\dfrac{42{\cdot}8-\Theta_0}{40{\cdot}4-\Theta_0} = \log\dfrac{35{\cdot}5}{21{\cdot}4}\Big/\log\dfrac{55{\cdot}9}{35{\cdot}5}$,

or, as a tentative approximation, $\dfrac{1{\cdot}5}{79{\cdot}3-2\Theta_0}\Big/\dfrac{2{\cdot}4}{83{\cdot}2-2\Theta_0} = \dfrac{21981}{19718}$,

or $162{\cdot}5-4\Theta_0 = \dfrac{82331}{23177}\times 3{\cdot}9$; therefore $\Theta_0 = 37{\cdot}15$.

Substituting this value in the equation

$$n\log\frac{42{\cdot}8-\Theta_0}{38{\cdot}9-\Theta_0} = \log\frac{55{\cdot}9}{21{\cdot}4}, \text{ we have } n = 0{\cdot}819 = [\bar{1}{\cdot}91341].$$

Lastly, find A from the equation

$$\log A + n\log 5{\cdot}65 = \log 55{\cdot}9, \text{ whence } A = 13{\cdot}53 = [1{\cdot}13129].$$

On using these values for A, Θ_0, n, there result the values 21·40, 35·53, 55·90 in the left-hand sides of the observation equations, so that, to judge by the standard of accuracy taken, the results are sufficiently near the true ones to dispense with further approximation; summarily,

$$A = 13{\cdot}53, \quad n = 0{\cdot}819, \quad \Theta_0 = 37{\cdot}15.$$

[The PROPOSER remarks that this question involves, *for the first time*, Θ_0 as an unknown quantity equal to the control temperature that guides radiation.]

5955. (Professor SYLVESTER, F.R.S.)—By a Cartesian Oval in space let us understand a curve the distances of whose points from three fixed points are linear functions of each other, or, which comes to the same thing, is the intersection of two surfaces of revolution, described by two plane Cartesians having a focus in common. Conversely, when two points can be found whose distances from any point in a space-Cartesian are linear functions of one another, let them be termed foci. Required to prove, that the locus of such foci is a plane *curve* of the 3rd degree. It will be observed that this curve for Cartesians of double curvature is the exact analogue of the three foci in a straight line for plane Cartesians.

Solution by W. J. C. SHARP, M.A.

Let the Cartesian in space be the intersection of the surfaces generated by the revolution of the plane Cartesians $mr + r' = n$ and $m'r + r'' = n'$ about the lines joining their foci; let the axes be taken so that the quantities r, r', r'' are measured from the points $(0, 0, 0)$, $(h, k, 0)$, and $(h', 0, 0)$ respectively. Then, if (x, y, z) be any point on the curve,

$$r^2 = x^2+y^2+z^2, \quad r'^2 = x^2+y^2+z^2-2hx-2ky+h^2+k^2,$$
$$r''^2 = x^2+y^2+z^2-2h'x+h'^2,$$

and therefore $$2x = \frac{r^2-r''^2+h'^2}{h'} = \frac{r^2(1-m'^2)+2m'n'r+h'^2-n'^2}{h'}$$
$$= Ar^2+2Br+C, \text{ say,}$$
where A, B, and C are constants; and
$$2y = \frac{r^2-r'^2+h^2+k^2-2hx}{k} = \frac{[(1-m^2)r^2+2mnr+h^2+k^2-n^2-2hx]}{k}$$
$$= A'r^2+2B'r+C', \text{ say, where } A', B', \text{ and } C' \text{ are constants.}$$
If $$\rho^2 = x^2+y^2+z^2-2\xi x-2\eta y+\xi^2+\eta^2,$$
$$\rho^2 = r^2-\xi(Ar^2+2Br+C)-\eta(A'r^2+2B'r+C')+\xi^2+\eta^2$$
$$= r^2(1-A\xi-A'\eta)-2r(B\xi+B'\eta)+\xi^2+\eta^2-C\xi-C'\eta$$
$$= (Hr+K)^2 \text{ (where H and K are functions } \xi, \eta \text{ and constants),}$$
if $$(1-A\xi-A'\eta)(\xi^2+\eta^2-C\xi-C'\eta) = (B\xi+B'\eta)^2,$$
and therefore if (ξ, η) lie upon this plane circular cubic, $\rho = Hr+K$, and if (ξ', η') be any other point upon the plane curve, $\rho' = H'r+K'$, where H' and K' are functions of (ξ', η'), and constants, and therefore
$$H'\rho-H\rho' = H'K-HK',$$
and therefore, &c. The three given foci lie upon the circular cubic.

8924. (Captain BROCARD.) — Former l'équation des paraboles tangentes aux deux bissectrices de chaque angle du triangle et aux perpendiculaires aux côtés de cet angle en leurs milieux. Ces coniques admettent, comme on sait, pour directrices les médianes et pour foyers les sommets A″, B″, C″ du second triangle de Brocard.

Solution by Professor BOUBALS; R. F. DAVIS, M.A.; *and others.*

Prenons pour axes les côtés ACx, ABy du triangle. L'équation générale des coniques tangentes aux deux bissectrices AE′, AE″ est
$$\lambda(y^2-x^2)+(y+\mu x+\nu)^2 = 0.$$
Elle représentera une parabolé si $\lambda = \mu^2-1$, ce qui donne
$$(\mu y+x)^2+2\nu(\mu x+y)+\nu^2 = 0.$$
Les perpendiculaires aux milieux des côtés AC, AB (ou médiatrices) ont pour équations $x+y\cos A-\frac{1}{2}b = 0$, $x\cos A+y-\frac{1}{2}c = 0$. La parabole devant être tangente à ces deux droites, on a les conditions
$$\nu\sin^2 A+b(\mu-\cos A) = 0, \quad \nu\sin^2 A+c(1-\mu\cos A) = 0,$$
d'où l'on tire $$\mu = \frac{c+b\cos A}{b+c\cos A}, \quad \nu = \frac{-bc}{b+c\cos a}.$$

L'équation de la parabole est donc

$$\left[(c+b\cos A)\,y+(b+c\cos A)\,x\right]^2$$
$$-2bc\left[(c+b\cos A)\,x+(b+c\cos A)\,y\right]+b^2c^2=0.$$

Si maintenant l'on prend le triangle ABC pour triangle de référence, les formules de transformation

$$x=\frac{bc\gamma}{a\alpha+b\beta+c\gamma},\quad y=\frac{bc\beta}{a\alpha+b\beta+c\gamma},$$

donneront pour l'équation en coordonnées trilinéaires de la parabole

$$\left[(c\beta+b\gamma)\cos A-a\alpha\right]^2=(b^2-c^2)(\beta^2-\gamma^2)\sin^2 A.$$

L'équation $a\alpha-(c\beta+b\gamma)\cos A=0$ représente la corde de contact des deux tangentes rectangulaires AE′, AE″. D'où l'identité

$$b^2+c^2-a^2=(bc+bc)\cos A=2bc\cos A,$$

qui exprime que cette corde passe par le point A″.

Autre vérification :—La symédiane AA″ doit être perpendiculaire à la corde de contact. *Or*, la symédiane a pour équation $c\beta-b\gamma=0$. On doit donc avoir identiquement (CASEY, *A Treat. on Conic Sect.*, p. 53),

$$(b^2-c^2)\cos A-\cos A\,(bc-bc)-\cos B\,(-ab)-\cos C\,(ac)=0,$$

ou $$(b^2-c^2)\cos A+a\,(b\cos B-c\cos C)=0,$$

ou $$(b^2-c^2)(b^2+c^2-a^2)+b^2(a^2+c^2-b^2)-c^2\,(a^2+b^2-c^2)=0,$$

ou $$(b^2-c^2)(b^2+c^2-a^2+a^2)+c^4-b^4=0,$$

ce qui est bien une identité.

8658. (W. J. C. SHARP, M.A.)—BOOLE obtains the result

$$\iint\ldots X\,dx^n\equiv\frac{1}{1\,.\,2\ldots n-1}\left\{x^{n-1}\int X\,dx-(n-1)\,x^{n-2}\int Xx\,dx\right.$$
$$\left.+\frac{(n-1)(n-2)}{1\,.\,2}\,x^{n-3}\int Xx^2dx-\ldots\mp\int Xx^{n-1}\,dx\right\}$$

by a symbolical method; show that it may also be obtained by ordinary methods.

Solution by Professors BEYENS, STONE, *and others*.

Cette formule est certain pour $n=2$, $n=3$, etc., et nous allons à démontrer que si elle est vérifiée pour le cas (n) elle sera pour le suivant. Supposons-nous que pour $(n-1)$ intégrales on aie

$$\iiint\ldots X\,dx^{n-1}$$
$$=\frac{1}{1\,.\,2\,.\,3\ldots(n-2)}\left\{x^{n-2}\int X\,dx-(n-2)\,x^{n-3}\int Xx\,dx\ldots\pm\int Xx^{n-2}dx\right\},$$

nous aurons $\iiiint\ldots X\,dx=\int dx\iiint^{(n-1)}\ldots X\,dx^{n-1}=$

$$\int dx\,\frac{1}{1\,.\,2\,.\,3\ldots(n-2)}\left\{x^{n-2}\int X\,dx-(n-2)\,x^{n-3}\int Xx\,dx\ldots\pm\int Xx^{n-2}\,dx\right\}$$

Mais, appliquant la méthode d'intégration par parties, nous aurons

$$\int dx\,.\,x^{n-2}\int X\,dx = \frac{x^{n-1}}{n-1}\int X dx - \int\frac{x^{n-1}}{n-1}X dx$$
$$= \frac{1}{n-1}\left\{x^{n-1}\int X dx - \int X x^{n-1} dx\right\},$$

$$-(n-2)\int dx\,.\,x^{n-3}\int X x dx = -x^{n-2}\int X x dx + \int X x^{n-1} dx,$$

$$\frac{(n-2)(n-3)}{1\,.\,2}\int dx\,.\,x^{n-4}\int X x^2 dx = \frac{n-2}{1\,.\,2}x^{n-3}\int X x^2 dx - \frac{n-2}{1\,.\,2}\int X x^{n-1} dx,$$

et ainsi de suite nous aurons finalement,

$$\pm\int dx\int X x^{n-2} dx = \pm x\int X x^{n-2} dx \mp \int X x^{n-1} dx.$$

Remplaçant ces valeurs en (1), le coefficient de $\int X x^{n-1} dx$ sera $\Big($après avoir séparé le facteur commun $\dfrac{1}{1\,.\,2\,.\,3\ldots(n-1)}\Big)$,

$$-1+(n-1)-\frac{(n-1)(n-2)}{1\,.\,2}\ldots\mp(n-1);$$

mais, d'après une propriété bien connue,

$$-1+(n-1)-\frac{(n-1)(n-2)}{1\,.\,2}\ldots\mp(n-1)\pm 1 = 0,$$

suivant $(n-1)$ pair ou impair, et par suite la formule (1), deviendra le théorème proposé.

[Let $\dfrac{d^n y}{dx^n} = X$, therefore $\displaystyle\iint\ldots X dx^n = y$,

and, by integration by parts,

$$\int X x^p dx = \int x^p\frac{d^n y}{dx^n}dx = x^p\frac{d^{n-1}y}{dx^{n-1}} - px^{p-1}\frac{d^{n-2}y}{dx^{n-2}} - p(p-1)x^{p-2}\frac{d^{n-3}y}{dx^{n-3}} - \&c.$$

Now, if all the values 0, 1, ... $n-1$ be given to p, the result of eliminating the successive differential coefficients of y will be the given formula.]

Note on a Probability-Question. *By the* Editor.

The problem in Question 7624 having excited much interest, we give space to the subjoined additional remarks and developments in regard to the solutions and notes that have been published in our previous volumes.

Mr. Simmons' solution seems to Mr. Putnam to be "only applicable when A is the first man to enter the train, and B the second. In that case there are $q-1$ vacancies in the car A has entered, and $pq-1$ vacancies in the train. The chance is then $(q-1)/(pq-1)$. It is impossible, however, to find any other condition in which this will prove true. Instead of throwing out m, Mr. SIMMONS' solution assumes $m = pq-1$. By the conditions of the problem, the chance under this assumption is the least possible. Although Mr. BIDDLE's solution is intricate, I do not see how it can be avoided, if the solution is to follow mathematical principles.

"If we may eliminate any of the data, let q be thrown out. Then, if there are vacant seats anywhere, the carriage in which A is seated will have an equal chance with the rest, and the number of vacancies belonging to it will be m/p, and the chance of B's taking that carriage will be $1/m \times m/p = 1/p$, or the same chance there will be that B will enter the same carriage A is seated in without reference to vacancies."

In further reply to Mr. SIMMONS, who, in Vol. XLVI., p. 37, has given what he considers to be test examples to prove his case,—viz., that m, in the question, is an immaterial quantity,—Mr. BIDDLE wishes to say that he is equally convinced that the subject cannot be so summarily dismissed." A very similar (not to say *identical*) question occurs in Mr. WHITWORTH's work on *Choice and Chance* (No. 229, p. 234); but, although answers are given to most of the questions, none is given to this, which would scarcely have been the case if the author had considered the matter so simple as Mr. SIMMONS does.

"One thing is clear, the particular passenger, A, must not be regarded as being selected at the time the new-comer, B, enters the train; nor, on the other hand, must he be regarded as invariably the first passenger who takes a seat. So far as our information goes, he is simply one of $pq-m$ passengers, who, before the train began to be filled, had an equal chance of occupying any carriage and any seat therein. We must therefore consider the number of ways in which $pq-m$ passengers can be distributed in p carriages, each containing q seats, and the probability of each kind of arrangement. We must next take the total number of *occupied* carriages given in the full tale of these arrangements, and find what proportion of them have one vacant seat, what two, what three, and so on. These will be the chances, as regards vacant seats, for any particular occupied carriage, such as that which A will occupy, because the *a priori* chances are the same for all. And here it may be observed, that it is only *a posteriori* that we can say that A has twice as much chance of being where there are two passengers as where there is only one.

The rest is easy, and the examples given by Mr. SIMMONS will equally serve the purpose now in view. "Let us take the particular instance of 3 compartments, each of 3 seats, and test the numerical results."

Here, taking 0 to represent a vacant seat, and X a passenger, the possible contents of any one carriage will be

(*a*)	(*b*)	(*c*)	(*d*)
X X X	X X 0	X 0 0	0 0 0

(i.) "Suppose that, besides A, *two* other passengers, P and Q, are already seated."

Then the essentially different ways in which the train may be made up can be represented as follows, the probability of each being given at the end:—

$(a+2d)$	X X X	0 0 0	0 0 0	$\frac{1}{28}$
$(b+c+d)$	X X 0	X 0 0	0 0 0	$\frac{18}{28}$
$(3c)$	X 0 0	X 0 0	X 0 0	$\frac{9}{28}$.

"In the first arrangement, the first passenger entering decides the carriage, then $\frac{2.1}{8.7} = \frac{1}{28}$ = the probability that the other two join him. In the second arrangement, the two first passengers may enter as they list; if they go together, of which the probability is $\frac{2}{8}$, the chance that the third enters another carriage is $\frac{6}{7}$; if, on the contrary, they enter distinct carriages, of which the probability is $\frac{6}{8}$, the chance that the third joins one of them is $\frac{4}{7}$. Therefore $\frac{2.6}{8.7} + \frac{6.4}{8.7} = \frac{18}{28}$ = the probability of the second arrangement. In the third arrangement, the chance that the second passenger does not join the first is $\frac{6}{8}$, and that the third does not join either is $\frac{3}{7}$. Therefore $\frac{6}{8} . \frac{3}{7} = \frac{9}{28}$ = the probability of the third arrangement. We now see that, where the first arrangement occurs once, the second occurs 18 times, and the third 9 times. Consequently, the state (*a*) of a single carriage occurs only once, whilst (*b*) occurs 18 times, (*c*) 45 times, and (*d*) 20 times. Disregarding (*d*), because A cannot be in a carriage where the seats are all vacant, and also (*a*), except for addition to the denominator, because B cannot enter where no seats are vacant, we find that, of occupied carriages, the probability that the state shall be (*b*) $= \frac{18}{64}$; that it shall be (*c*) $= \frac{45}{64}$. Therefore, since there are, in all, 6 vacant seats for B to choose from, the probability that he joins A $= \frac{18}{64} . \frac{1}{6} + \frac{45}{64} . \frac{2}{6} = \frac{9}{32}$, instead of $\frac{8}{32}$ or $\frac{1}{4}$, the probability given by Mr. Simmons."

(ii.) "Suppose that, besides A, there are already *three* other passengers, P, Q, and R."

Here the specific arrangements are also three in number, and their probabilities as follows:—

$(a+c+d)$,	X X X	X 0 0	0 0 0,	$\frac{4}{28}$
$(b+2c)$,	X X 0	X 0 0	X 0 0,	$\frac{18}{28}$
$(2b+d)$,	X X 0	X X 0	0 0 0,	$\frac{6}{28}$.

From which we gather that the chances of any carriage being as (*a*), (*b*), (*c*), (*d*) are as 4, 30, 40, 10. Consequently, as the vacant seats are 5 in number, the probability that B joins A $= \frac{30}{74} . \frac{1}{5} + \frac{40}{74} . \frac{2}{5} = \frac{11}{37}$, instead of $\frac{11}{44}$ or $\frac{1}{4}$, the probability given by Mr. Simmons.

(iii.) "Moreover, if there had been only one passenger P besides A," the specific arrangements would be two in number, and their probabilities as follows:—

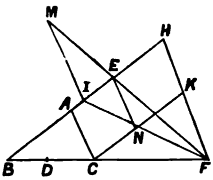
$(b+2d)$	X X 0	0 0 0	0 0 0	$\frac{2}{8}$
$(2c+d)$	X 0 0	X 0 0	0 0 0	$\frac{6}{8}$.

"Hence we see that the chances of any carriage being as (*a*), (*b*), (*c*), (*d*) are as 0, 2, 12, 8. Consequently, as the vacant seats are 7 in number, the present probability that B joins A $= \frac{2}{14} . \frac{1}{7} + \frac{12}{14} . \frac{2}{7} = \frac{13}{49}$, instead of $\frac{13}{52}$ or $\frac{1}{4}$, as given by Mr. Simmons.

"In fact, the only case in which the same probability is certainly arrived at by both methods, is where A is the sole passenger in the train when B enters. But if it was a mistake, as Mr. Putnam intimates, to say that the probability would be the same according to either method, in the case of only one vacant place remaining, the mistake arose from judging of the probabilities of A's position by an *a posteriori* process."

8685. (The Editor.)—Given an angle at the base of a triangle, the sum of the two sides, and the distance between the given angular point and the point of contact of the escribed circle touching the base; construct the triangle.

Solution by the Proposer.



Let L and l represent the two given lines. Draw the $\angle$ EBD = the given $\angle$, making EB = L and BD = l. Produce BD to F, so that DF = L + l: then BD + DF = BF = L + $2l$.

Also produce BE to H, so that BH = BF. Bisect BH in I: draw IF; and from I apply IM = IH to meet FE produced; also draw EN ∥ MI, and through N draw KC ∥ BH, and CA ∥ EN. The Δ required is ABC. For, by construction, $\angle$ ABC = given $\angle$, and BD = l.

Moreover NK : IH = FN : FI = NE : IM = NE : IH, $\therefore$ NK = NE = NC; but AC = AE; therefore AB + AC = EB = L.

Again, CF = CK = 2CN = 2AC, $\therefore$ BC + CF = BC + 2AC = L + $2l$; but AB + AC = L; therefore, adding, AB + 3 AC + BD + DC = 2 (L + l); but AB + AC + BD = L + l; therefore, subtracting,

$$2AC + DC = L + l = AB + AC + BD;$$

therefore

$$AB + BD = AC + DC.$$

Now, it is well known that the angular point opposite the base of a triangle and the point of contact of the escribed circle with the base bisect its perimeter. Therefore D is the point of contact of the escribed circle with the base BC.

8501. (R. Knowles, B.A.)—In any triangle, show that

$$4\cos^2 A\cos^2 B - \cos(A-B)(3\cos A\cos B - \sin A\sin B) = \cos^2 C.$$

Solution by Isabel Madison, B.Sc.

Identically, $4x^2 - (x+y)(3x-y) = (x-y)^2$, and putting $x = \cos A\cos B$, $y = \sin A\sin B$, the required theorem is proved.

8808. (F. R. J. Hervey.) — Find in how many ways n lines of verse can be rhymed, so as to have r *different* rhymes, and no line unrhymed; and show that, in the case of the *sonnet*, the numbers of ways with 2, 3, ... 7 rhymes are, respectively, 8177, 731731, 6914908, 12122110, 4099095, and 135135.

Solution by the Proposer.

Let $\phi(n, r)$ denote the required number; we shall obtain all the case as follows. Take all the cases of $n-1$ lines with r rhymes, and append to each a line rhyming with some that precede; this gives $r\phi(n-1, r)$ ways. Again, take all the cases of $n-1$ lines with r distinct endings *one* of which is unrhymed, and append to each a line rhyming with the odd line. The number of these is $= (n-1)\,\phi(n-2, r-1)$.

Hence $\qquad \phi(n, r) = r\phi(n-1, r) + (n-1)\,\phi(n-2, r-1).$

Form a table, as below. Write the first row (of units), and first *cypher* of each subsequent row (forming an oblique column); and continue these rows both ways, satisfying the relation $\phi_r(n+1) = r\phi_r n + n\phi_{r-1}(n-1)$; $\phi_r n$ denoting throughout the number of r^{th} row which falls under the number n: thus $\phi_r n = \phi(n, r)$ so long as $n > r$. The values of $\phi_r n$ from $n = r$ to $n = 0$ follow an obvious law, which, it is easy to show, must hold for each subsequent row; we have, then, $\phi_r 0 = (-1)^{r-1} / r!$.

0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
−/2	−1	−1	0	3	10	25	56	119	246	501
/3!	/2	1	1	0	0	15	105	490	1918	6825
−/4!	−/3!	−/2	−1	−1	0	0	0	105	1260	9450

The complete solution of the equation $u_{n+1} - ru_n = \psi n$, ψ being a rational integral function of the k^{th} degree, is
$u_n = Cr^n - \left\{ / (r-1) + \Delta / (r-1)^2 + \ldots + \Delta^k / (r-1)^{k+1} \right\} \psi n = Cr^n - \chi n$, say; whence, if an integral function be required, the only solution is $u_n = -\chi n$. If the given equation be $u_{n+1} - ru_n = a^{n+1}\psi n$, we have u_n/a^n as before, giving
$u_n = Cr^n - a^n \chi n$, where $\chi n = s\,(1 + s\Delta + \ldots + s^k\Delta^k)\,\psi n$ and $s = a / (r-a)$.
Again, if $u_{n+1} - ru_n =$ the sum of several terms of the form $a^{n+1}\psi n$ (which denote by $\Sigma a^{n+1}\psi n$), we may find u_n by solving as many equations of the preceding form and adding the results; giving a solution of the form $u_n = Cr^n - \Sigma a^n \chi n$.

To find $\phi_{r+1} n$ we must solve the equation $u_{n+1} - (r+1)\,u_n = n\phi_r(n-1)$.

Thus $\phi_1 n = 1$, $\phi_2 n = 2^{n-1} - (n+1)$, $2\phi_3 n = 3^{n-1} - 2^{n-1}(n+2) + n^2 + n + 1$,

$$6\phi_4 n = 4^{n-1} - 3 \,.\, 3^{n-2}(n+3) + 3 \,.\, 2^{n-3}(n^2 + 3n + 4) - (n^3 + 2n + 1);$$

generally, it will be shown that

$$r!\,\phi_r n = r^n - C_{r,1}(r-1)^{n-1}P + C_{r,2}(r-2)^{n-2}Q - \ldots \pm C_{r,r-1}T,$$

[otherwise, $(r-1)!\,\phi_r n = r^{n-1} - C_{r-1,1}(r-1)^{n-2}P + \ldots$], where $C_{r,m}$ is the coefficient of x^m in $(1+x)^r$, and P, Q, ... T are functions of n with unity for coefficient of highest power; these must first be determined.

Unity, in the first term of the above expression, may be regarded as the starting-point of a new series of these functions, each formed from the preceding, which are to appear in the 2nd term of ϕ_{r+1}, the 3rd of ϕ_{r+2}, &c.; the law of formation being

$$f_k n = \left\{1 + r\Delta / k + \ldots\right\} \left\{n f_{k-1}(n-1)\right\},$$

which, inverted, gives $(1-r\Delta/k)f_k n = nf_{k-1}(n-1)$ ....(1).

Thus $f_0 n = 1$, $f_1 n = n+r$, $nf_1(n-1) = n^2+(r-1)n$, giving

$$f_2 n = n^2+(2r-1)n+r^2,\quad nf_2(n-1) = n^3+(2r-3)n^2+(r^2-2r+2)n,$$

giving $\quad f_3 n = n^3+(3r-3)n^2+(3r^2-3r+2)n+r^3$, and so on.

Thus written, in powers of n, the law is not evident; but we may *observe* (and, for the moment, *assume*) the relation

$$f_k n = nf_{k-1}(n-1)+rf_{k-1}n \text{ ....(2).}$$

From (1) and (2), $\Delta f_k n = kf_{k-1}n$, whence $\Delta^2 f_k n = k(k-1)f_{k-2}n$, and so on; also, from (2), $f_k 0 = rf_{k-1}0 = \ldots = r^k$; hence

$$f_k n = f_k 0 + n\Delta f_k 0 + \ldots = r^k + C_{k,1}r^{k-1}n + C_{k,2}r^{k-2}n(n-1)+\ldots$$
$$\ldots + n(n-1)\ldots(n-k+1).$$

This is the function sought; for it will be found to satisfy (1), which, f_{k-1} being supposed known, is an equation of differences for finding f_k, having only one such solution. Denoting it in future by $f(r,k)$, assuming the form previously suggested for ϕ_r, and applying the general process to form ϕ_{r+1}, we have

$$r!\,\phi_r n = r^n - C_{r,1}(r-1)^{n-1}f(r-1,1) + C_{r,2}(r-2)^{n-2}f(r-2,2) - \ldots$$
$$\ldots \pm C_{r,r-1}f(1,r-1),$$

$$r!\,\phi_{r+1}n = A(r+1)^n - r\,.\,r^{n-2}f(r,1) + \frac{r-1}{2}C_{r,1}(r-1)^{n-3}f(r-1,2)$$
$$-\frac{r-2}{3}C_{r,2}(r-2)^{n-4}f(r-2,3)+\ldots \mp \frac{1}{r}C_{r,r-1}f(1,r),$$

$$(r+1)!\,\phi_{r+1}n = B(r+1)^n - C_{r+1,1}r^{n-1}f(r,1)$$
$$+C_{r+1,2}(r-1)^{n-2}f(r-1,2) - \ldots \mp C_{r+1,r}f(1,r),$$

$$(r+1)!\,\phi_{r+1}0 = (-1)^r = B - C_{r+1,1} + C_{r+1,2} - \ldots \mp C_{r+1,r};$$

whence $B = 1$, and the above formula is completely established; n being any integer, positive or negative. It may also be written

$$(r-1)!\,\phi_r n = r^{n-1} - C_{r-1,1}(r-1)^{n-2}f(r-1,1)$$
$$+C_{r-1,2}(r-2)^{n-3}f(r-2,2) - \ldots \pm f(1,r-1).$$

If we collect from the last the complete coefficient of a given factorial, $n(n-1)\ldots(n-m+1)$, we shall find it (by help of the relation $C_{r,m+t}C_{m+t,t} = C_{r,m}C_{r-m,t}$) to be $\pm C_{r-1,m}\Delta^{r-m-1}1^{n-m-1}$ (derived from $1^{n-m-1}, 2^{n-m-1}, \ldots$). Hence, restricting n to be not $< r$, we may easily throw $\phi_r n$ into the form

$$\phi_r n = \Delta^{(r)}0^n - C_{n,1}\Delta^{(r-1)}0^{n-1} + C_{n,2}\Delta^{(r-2)}0^{n-2} - \ldots \pm C_{n,r-1}\Delta^{(1)}0^{n-r+1},$$

where $\Delta^{(r)}$ means $\Delta^r/r!$. From this, and from the fact that $\phi_r n = 0$ if $n > r < 2r$, it may be inferred that $\Delta^{(r)}0^{r+p}$, p being constant and > 0, is an integral function Fr, of degree $2p$, containing the factors $r(r+1)\ldots(r+p)$, (which is otherwise known); and that $\phi_r(r+p)$ or $\phi(r+p,r)$ is the $(r+p)^{\text{th}}$ difference of the first zero value $F(-p)$. [This result is easily connected with the meaning of $\Delta^{(r)}0^n$ previously found.]

The case $n = 2r$ is noticeable. By the construction of the table, $\phi_r(2r)$ = the product of odd numbers $1.3\ldots(2r-1)$. This may also be inferred as follows. Suppose it specified that p are lines to rhyme together, q others to do so, and so on; where $p+q+\ldots=n$. The number of ways will be $n!/\{p!\,q!\ldots\}$ if $p, q\ldots$ are all different; but, if k of them are equal, we must divide by $k!$; and so on. Now, in the above case, the lines necessarily rhyme in pairs; hence $\phi(2r, r) = (2r)!/(2^r.r!)$. It follows that the coefficient of r^{2p} in $\mathrm{F}r$ is $/(2^p.p!)$.

	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			3	10	25	56	119	246	501	1012	2035	4082	8177
3					15	105	490	1918	6825	22935	74316	235092	731731
4							105	1260	9450	56980	302995	1487200	6914908
5									945	17325	190575	1636635	12122110
6											10395	270270	4099095
7													135135

8820. (Charles F. Lodge.)—A mirror, measuring 33 inches by 22 inches, is to have a frame of uniform width, whose area is to equal that of the glass; show that the width of the frame is $5\frac{1}{2}$ inches.

Solution by D. Biddle; H. L. Orchard, B.Sc.; *and others.*

Lines drawn perpendicular to the sides of the mirror, from its centre, divide not only the mirror itself (m), but its frame (f), into quarters. Moreover, $\frac{1}{4}m = \frac{1}{4}f$; $\frac{1}{4}m = 2\times3\times(5\frac{1}{2})^2$; therefore $\frac{1}{4}m+\frac{1}{4}f = 4\times3\times(5\frac{1}{2})^2$. But $4 = 3+1$, and $3 = 2+1$. Consequently $1\times5\frac{1}{2}$ = the width of frame required, in inches.

8813. (J. Griffiths, M.A.)—If we have

$$f(x) = \mathrm{A}_0 + 2\mathrm{A}_2x^2 + 2\mathrm{A}_4x^4 + \ldots \textit{ ad inf.},\quad \phi(x) = 1 - 2\mathrm{B}_2x^2 - 2\mathrm{B}_4x^4 - \ldots \textit{ ad inf.}$$

where
$$\mathrm{A}_0 = 1 + 2\Sigma(-)^s\,\mathrm{cn}^2\frac{s\mathrm{K}}{n},\qquad \mathrm{B}_2 = \Sigma(-)^s\,\mathrm{cn}\frac{s\mathrm{K}}{n} \div \mathrm{sn}^2\frac{s\mathrm{K}}{n},$$

$$\mathrm{A}_2 = k^2\,\Sigma(-)^s\,\mathrm{cn}^2\frac{s\mathrm{K}}{n}\mathrm{sn}^2\frac{s\mathrm{K}}{n},\qquad \mathrm{B}_4 = \Sigma(-)^s\,\mathrm{cn}^2\frac{s\mathrm{K}}{n} \div \mathrm{sn}^4\frac{s\mathrm{K}}{n},$$

$$\mathrm{A}_4 = k^4\,\Sigma(-)^s\,\mathrm{cn}^2\frac{s\mathrm{K}}{n}\mathrm{sn}^4\frac{s\mathrm{K}}{n},\qquad \mathrm{B}_6 = \Sigma(-)^s\,\mathrm{cn}^2\frac{s\mathrm{K}}{n} \div \mathrm{sn}^6\frac{s\mathrm{K}}{n},$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\qquad\qquad\ldots\quad\ldots\quad\ldots\quad\ldots$$

and s is a number from 1 to $n-1$, show that $f(x)\,.\,\phi(x) = \mathrm{A}_0$.

Solution by D. EDWARDES ; SARAH MARKS, B.A. ; *and others.*

Referring to Mr. GRIFFITHS' Paper (*Proceedings of London Math. Soc.*, Nov. 12, 1885), we have

$$\left\{ y = \text{sn}\,(\text{M}u, \lambda),\ x = \text{sn}\,(u, k),\ u_0 = \frac{s\text{K}}{n},\ v_0 = \frac{2t-1}{2n}\,\text{K}, \right.$$

$$\text{mod. } k,\ s = 1 \text{ to } n-1,\ t = 1 \text{ to } n,$$

$$\left. \text{M} = \Pi\,\text{sn}^2\frac{s\text{K}}{n} + \Pi\,\text{sn}^2\frac{2t-1}{2n}\,\text{K},\ \ \lambda = k^{2n}\,\Pi\,\text{sn}^4\,\frac{2t-1}{2n}\,\text{K} \right\}$$

$$\frac{\text{M}\lambda}{k^2}\,y = \frac{x\,(1-x^2)^{\frac{1}{2}}}{(1-k^2x^2)^{\frac{1}{2}}}\left\{1 + 2\Sigma\,\frac{(-)^s\,\text{cn}^2\,\dfrac{s\text{K}}{n}}{1-k^2\text{sn}^2\dfrac{s\text{K}}{n}\,x^2}\right\}$$

$$= \frac{x\,(1-x^2)^{\frac{1}{2}}}{(1-k^2x^2)^{\frac{1}{2}}}\left\{\text{A}_0 + 2\text{A}_2x^2 + 2\text{A}_4x^4 + \ldots\ldots\right\}.$$

Changing y, x into $1/\lambda y$, $1/kx$, we have

$$\frac{\text{M}}{y} = \frac{(1-k^2x^2)^{\frac{1}{2}}}{x\,(1-x^2)^{\frac{1}{2}}}\left\{1 - 2\Sigma\,\frac{(-)^s\,\text{cn}^2\,\dfrac{s\text{K}}{n}\,x^2}{\text{sn}^2\,\dfrac{s\text{K}}{n}\left[1 - x^2\Big/\left(\text{sn}^2\,\dfrac{s\text{K}}{n}\right)\right]}\right\}$$

$$= \frac{(1-k^2x^2)^{\frac{1}{2}}}{x\,(1-x^2)^{\frac{1}{2}}}\left\{1 - 2\text{B}_2x^2 - 2\text{B}_4x^4 - \ldots\ldots\right\}.$$

Hence, $\text{M}^2\lambda/k^2 = fx\,.\,\phi x$, and, when x and y are small, $\text{M}^2\lambda/k^2 = \text{A}_0$, and therefore, &c.

8117. (Professor WOLSTENHOLME, M.A., Sc.D.)—Two conicoids S, S′ have two common plane sections, and the poles of these planes with respect to S are the points P, P′; prove that (1) if S′ pass through P, it will also pass through P′; (2) also, in this case, the following relations must hold $(\Theta^2 - 3\Phi\Delta)(\Theta\Theta' - 6\Delta\Delta') + \Theta^2\Delta\Delta' = 0$, $\Theta' = 27\Delta^2(\Theta\Theta' - 6\Delta\Delta')$. [The discriminant of $k\text{S} + \text{S}'$ is $\Delta k^4 + \Theta k^3 + \Phi k^2 + \Theta' k + \Delta'$.]

Solution by Professor NASH, M.A.

Let $\text{S} = ax^2 + by^2 + cz^2 + dw^2 = 0$, and let P, P′ be the points (0, 0, 0, 1), (X, Y, Z, W), then the equation of S′ is

$$ax^2 + by^2 + cz^2 + dw^2 + 2w\,(ax\text{X} + by\text{Y} + cz\text{Z} + dw\text{W}) = 0.$$

If this passes through (0, 0, 0, 1), $1 + 2\text{W} = 0$, and S′ becomes

$$ax^2 + by^2 + cz^2 + 2w\,(ax\text{X} + by\text{Y} + cz\text{Z}) = 0,$$

which is satisfied by the coordinates (X, Y, Z, W).

The discriminant of $kS + S'$ is

$$\begin{vmatrix} (k+1)\,a, & 0, & 0, & aX \\ 0, & (k+1)\,b, & 0, & bY \\ 0, & 0, & (k+1)\,c, & cZ \\ aX, & bY, & cZ, & kd \end{vmatrix} = 0,$$

or $\quad k\,(k+1)^3\,abcd - (k+1)^2\,abc\,(aX^2 + bY^2 + cZ^2) = 0;$

putting $\quad aX^2 + bY^2 + cZ^2 = dP,$

therefore $\quad abcd\,\{k^4 + 3k^3 + (3-P)\,k^2 + (1-2P)\,k - P\} = 0;$

therefore $\quad \dfrac{\Delta}{1} = \dfrac{\Theta}{3} = \dfrac{\Phi}{3-P} = \dfrac{\Theta'}{1-2P} = \dfrac{\Delta'}{-P},$

therefore $\quad \Theta^2 - 3\Delta\Phi = 3P\Delta^2, \quad \Theta\Theta' - 6\Delta\Delta' = 3\Delta^2.$

From these we obtain the given relations

$$(\Theta^2 - 3\Delta\Phi)(\Theta\Theta' - 6\Delta\Delta') + \Theta^2\Delta\Delta' = 0, \quad \Theta^4 = 27\Delta^2(\Theta\Theta' - 6\Delta\Delta').$$

[It would be better if the coordinates of the second point P′ were written X : Y : Z : W; so that X, Y, Z, W may involve an arbitrary multiplier; otherwise the equation $1 + 2W = 0$ will seem to limit the position of P′. The general form of S′, when S is given, is

$$2\left(X_2\frac{dS_1}{dX_1} + Y_2\frac{dS_1}{dY_1} + Z_2\frac{dS_1}{dZ_1} + W_2\frac{dS_1}{dW_1}\right)S \mp \left(X_1\frac{dS}{dx} + \dots\right)\left(X_2\frac{dS}{dx} + \dots\right),$$

where $(X_1Y_1Z_1W_1)$, $(X_2Y_2Z_2W_2)$ are the two points P, P′.]

8869. (Professor Steggall, M.A.)—A shot of mass m is fired *in vacuo* from an air gun of length l, with a charge of air that at normal pressure p would occupy a volume v; this air initially occupies a length b of the barrel of the gun. Show that the time of passage along the barrel and the velocity with which the shot leaves the gun are given by the equations $\quad T = (l + \tfrac{5}{8}l^{\frac{3}{5}}b^{\frac{2}{5}}) \div (5pv/m)^{\frac{1}{2}}, \quad V = (1 - b^{\frac{2}{5}}/2l^{\frac{2}{5}})(5pv/m)^{\frac{1}{2}},$
where b is small compared with l, and the ratio of the specific heats of air is taken as 1·4.

Solution by D. Edwardes; Charlotte A. Scott, B.Sc.; *and others.*

If P be the pressure at time t on the shot, the equation of motion is $m\ddot{x} = P = Cx^{-\gamma}$ (γ ratio of specific heats), on the supposition that there is no communication of heat. By the Question $C = pvb^{\gamma-1}$.

Hence, integrating between l and b, we have

$$mV^2 = 5pv\left(1 - \frac{b^{\frac{2}{5}}}{l^{\frac{2}{5}}}\right), \quad \text{or} \quad V = \left(\frac{5pv}{m}\right)^{\frac{1}{2}}\left(1 - \frac{b^{\frac{2}{5}}}{2l^{\frac{2}{5}}}\right) \text{ approximately;}$$

$$\left(\frac{5pv}{m}\right)^{\frac{1}{2}} dt = \frac{dx}{(1 - b^{\frac{2}{5}}/x^{\frac{2}{5}})^{\frac{1}{2}}},$$

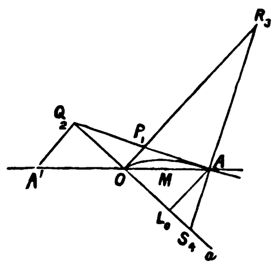
therefore $$\left(\frac{5pv}{m}\right)^{\frac{1}{5}} \mathrm{T} = 5\int_{b^{\frac{1}{5}}}^{l^{\frac{1}{5}}} \frac{z^5 dz}{(z^3 - b^{\frac{2}{5}})^{\frac{1}{2}}}$$

$$= 5l^{\frac{4}{5}}(l^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{1}{2}} - \tfrac{20}{3} l^{\frac{2}{5}}(l^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{3}{2}} + \tfrac{8}{3}(l^{\frac{2}{5}} - b^{\frac{2}{5}})^{\frac{5}{2}}$$

$$= l\left(1 - \frac{b^{\frac{2}{5}}}{2l^{\frac{2}{5}}}\right)\left(1 + \tfrac{4}{3}\frac{b^{\frac{2}{5}}}{l^{\frac{2}{5}}}\right) = l\left(1 + \tfrac{5}{6}\frac{b^{\frac{2}{5}}}{l^{\frac{2}{5}}}\right) \text{ approx.} = l + \tfrac{5}{6}l^{\frac{3}{5}}b^{\frac{2}{5}}).$$

8931. (Emile Vigarié.)—On donne deux points O et A et on considère toutes les paraboles ayant O pour sommet et qui passent par A. Trouver géométriquement le lieu (1) du point de concours des tangentes en A et O; (2) du point d'intersection de la normale en O et de la tangente en A (ces deux lieux sont tangents en O); (3) du point d'intersection de la tangente en O et de la normale en A; (4) du point de concours des normales en O et A (ces deux lieux sont tangents en A).

Solutions by (I.) Professor Schoute; (II.) R. F. Davis, M.A.



I. In the diagram P_1, Q_2, R_3, S_4 are the points that describe the four loci, when the axis a of the parabola through O and A rotates about O; M is the centre of the segment OA, O of the segment A′A; AL_0 is perpendicular to the rotating axis.

(1). As the locus of L_0 is the circle described on OA as diameter and

$$OP_1 = -\tfrac{1}{2}AL_0,$$

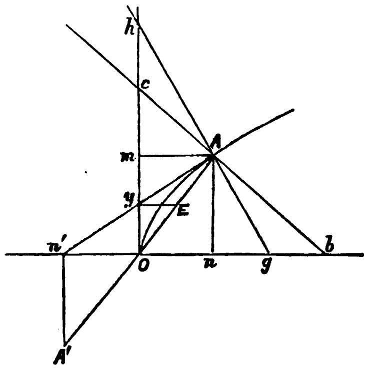
the locus of P_1 is the circle described on OM as diameter.

(2) The locus of Q_2 is the circle described on A′O as diameter.

(3) and (4). The couples of points P_1 and R_3, Q_2 and S_4 are so situated on radii vectores through O, that the segments P_1R_3 and Q_2S_4 appear from A under right angles. Therefore the loci of R_3 and S_4 are the curves that correspond to the loci of P_1 and Q_2 in the involutive quadratic transformation of the points on radii vectores through O and conjugated to one another with respect to the point circle A. So we find for these loci unicursal circular cubics with the isolated point O, that touch in A the perpendicular to OA. The nine points common to these two cubics are O counted four times, A counted twice, the two circular points at infinity, and the point at infinity common to the perpendiculars to OA. The real asymptotes of these curves are different; the first intersects OA in a point B, so that AB = OA, the second in a point B′, so that AB′ = A′A.

The particular kind of quadratic transformation made use of in this solution is treated analytically by Prof. G. de Longchamps ("Etude d'une Transformation Réciproque," *Journal de Mathématiques Spéciales*, 1882, p. 19), and geometrically by me ("Sur la Construction des

Cubiques Unicursales," *Annuaire de l'Association Française pour l'Avancement des Sciences*, Congrès de Grenoble, 1885, p. 169).



II. *Otherwise:* Let An be the ordinate of A to the axis of any parabola having O for its vertex. Then if, on this axis, On' be taken = On, An' is the tangent at A, and y, its middle point, lies on the tangent at the vertex. If yE be drawn parallel to the axis, it bisects OA in the fixed point E. Now (1) the locus of y is the fixed circle described upon OE as diameter. (2) If AO be produced to A' so that OA' = OA, the locus of n' is the fixed circle described upon OA' as diameter. [These circles touch each other in O.] (3) If a fixed straight line through A perpendicular to OA meet the axis in b, then $An^2 = On \,.\, bn$, and bn = latus rectum. The normal at A will therefore bisect bn in g. Now, the locus of n is the fixed circle upon OA as diameter, while Ab is the tangent A to the same circle. Therefore, since O, n, g, b are collinear and $bg = ng$, the locus of g is a certain curve symmetrical with respect to OA touching the circle in A. [This curve is a circular cubic; its polar equation is $r = a\,(\sec\theta + \cos\theta)$ and its Cartesian equation $y^2(x-a) = x^2(2a-x)$ referred to O as origin and OA as axis of x. It has for an asymptote the line through E perpendicular to OA. OA = $2a$.] (4) Since g bisects bn, it may be easily shown that c bisects hm. But O, m, c, h are collinear, while the locus of m is the fixed circle upon OA as diameter, and that of c the tangent bAc to the same circle at A. Therefore, as before, h describes a certain curve $[r = 2a\,(2\sec\theta - \cos\theta)]$ symmetrical with respect to OA. These two curves touch each other at A, and lie upon opposite sides of their common tangent.

8716. (Professor Mathews, M.A.) — Prove that the real common tangents of the circles $x^2+y^2-2ax = 0$, $x^2+y^2-2by = 0$ are represented by $2ab\,(x^2+y^2-2ax) = (by-ax+ab)^2$, or, which is the same thing, by

$$2ab\,(x^2+y^2-2by) = (by-ax-ab)^2.$$

Solution by W. J. Barton, M.A.; Kate Gale, B.Sc.; *and others.*

By elementary geometry we have, if (x_0, y_0) be coordinates of the point of intersection of the common tangents,

$$\frac{x_0-a}{x_0} = \frac{a}{b} = \frac{y_0}{y_0-b}, \quad \text{whence} \quad x_0 = -y_0 = -\frac{ab}{a-b}.$$

Substituting these values in the equation to the pair of tangents from (x_0, y_0) to $x^2+y^2-2ax = 0$, viz.,

$$(x^2+y^2-2ax)(x_0^2+y_0^2-2ax_0) = \{xx_0+yy_0-a(x+x_0)\}^2,$$

and dividing by $\dfrac{a^2}{(a-b)^2}$, we get, for the equation to the real common tangents,

$$2ab(x^2+y^2-2ax) = (by-ax+ab)^2.$$

8660. (By B. Hanumanta Rau, M.A.)—Investigate the singular solution of the equation $4\left(x+y\dfrac{dy}{dx}\right)^2+25\left(y^2-xy\dfrac{dy}{dx}\right) = 0$, and show that it is the envelope of a series of circles described on the subnormal of an ellipse as diameter.

Solution by Robert Rawson, F.R.A.S.

From the given equation, we have

$$p^2-\frac{17}{4}\frac{x}{y}p+\frac{x^2}{y^2}+\frac{25}{4} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then, the (p) and (c) equations are readily found to be

$$8yp-17x\mp 5(9x^2-16y^2)^{\frac{1}{2}} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$C-\log\left\{(9x^2-16y^2)^{\frac{1}{2}}\pm 5x\right\}\mp\frac{5x}{(9x^2-16y^2)^{\frac{1}{2}}\pm 5x} = 0\ldots\ldots\ldots(3).$$

From (2), $$\frac{dp}{dy} = \frac{-17x\mp(9x^2-16y^2)^{\frac{1}{2}}}{8y^2}\mp\frac{10}{(9x^2-16y^2)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots(4).$$

Now, (4) becomes (∞) when $y = 0$ and $4y = \pm 3x$, neither of which is a solution of (1).

Again, $$\frac{d}{dx}\left(\frac{1}{p}\right) = \frac{8\left\{\mp 45x-17(9x^2-16y^2)^{\frac{1}{2}}\right\}y}{(9x^2-16y^2)^{\frac{1}{2}}\left\{17x\pm 5(9x^2-16y^2)^{\frac{1}{2}}\right\}} \quad\ldots\ldots\ldots\ldots\ldots(5).$$

Now, (5) becomes (∞) when $4y = \pm 3x$ and $4x^2+25y^2 = 0$, neither of which satisfies (1).

The condition of equal roots of (2) *and* (3) *fails to give a solution.*

It appears, therefore, that neither of the above recognised methods avails in this case in finding the singular solution. The present theory of singular solutions of differential equations is not, I fear, free from the reproach of not being in such a satisfactory state as it might be, considering the attention which has been given to it by every mathematician of eminence since the days of Dr. Brooke Taylor. Indeed it appears, from the researches of Sir James Cockle and others, that

$$\frac{dy}{dC} = 0, \qquad \frac{dx}{dC} = 0, \quad \text{derived from the C-equation,}$$

$$\frac{dp}{dy} = \infty, \quad \frac{d}{d}\left(\frac{1}{p}\right) = \infty, \quad \text{derived from the } p\text{-equation.}$$

The condition of equal roots of the C-equation and p-equation can have but a limited application. Another solution besides (3) can be obtained by the following consideration:—

Let $$C + R \pm f(S^{\frac{1}{n}}) = 0 \qquad \text{(6)}$$
be the C-equation, where R, S are functions of x, y.

The p-equation, from (6), becomes

$$p + \frac{n\frac{dR}{dx}S^{\frac{n-1}{n}} \pm f'(S)^{\frac{1}{n}}\frac{dS}{dx}}{n\frac{dR}{dy}S^{\frac{n-1}{n}} \pm f'(S)^{\frac{1}{n}}\,dS} = 0 \qquad \text{(7)}.$$

Now (6) is, therefore, the complete primitive of (7), which is rendered exact by the multiplier $1/nS^{\frac{n-1}{n}}$. But (7) is also satisfied by
$$S = 0 \qquad \text{(8)}.$$

Hence (6) and (8) are both solutions of (7), and if (6) does not include (8) by giving a constant value to (C), then (8) is a singular solution by the usual definition. The solution (8) may, or may not, be of the envelope species; this depends upon the functions R, S.

Compare (6) with (3), then
$$R = \mp \frac{5x}{(9x^2 - 16y^2)^{\frac{1}{2}} \pm 5x}, \qquad S = \{(9x^2 - 16y^2)^{\frac{1}{2}} \pm 5x\}^n,$$
when $S = 0$, then $$(9x^2 - 16y^2)^{\frac{1}{2}} \pm 5x = 0 \qquad \text{(9)}.$$
From (9), $$x^2 + y^2 = 0 \qquad \text{(10)}.$$

Now (10) satisfies (1), and is not included in (3) as a particular integral. It must, therefore, be a singular solution by the definition.

Professor Cayley, a very great authority in this matter, would I apprehend reject this solution as being singular. At all events, whether it is singular or otherwise, it is a solution of (1), and is not included as a particular case of (3) by giving a constant value to (C).

The locus of (10) is two straight lines, viz., $y = \pm ix$, where $i = \sqrt{-1}$, but the relation of (10) to the complete primitive is not readily observed. It is not difficult to show that (10) is the envelope of a series of circles described upon the subnormal, as diameter, of an ellipse or hyperbola whose excentricities satisfy $e^2 + 1 = 0$.

8727. (B. Hanumanta Rau, B.A.)—The sides BC, CA, AB of a triangle ABC are produced to a, b, c, such that Ca, Ab, Bc are respectively equal to BC, CA, AB. Prove that the centroid of the triangle abc coincides with that of ABC.

Solution by Rev. D. Thomas, M.A.; C. E. Williams, M.A.; *and others.*

Let α, β, γ be coinitial vectors of A, B, C respectively and α', β', γ' of a, b, c respectively; then $\alpha' = \beta + 2(\gamma - \beta)$ and $\Sigma\alpha' = \Sigma\alpha$, therefore, &c.

8135. (Rev. T. C. SIMMONS, M.A.)—If G be the centroid, I the incentre, of a plane triangle, prove that

$$IG^2 = \tfrac{4}{9}R^2(1+\cos A\cos B\cos C) - \tfrac{4}{3}Rr + \tfrac{5}{3}r^2.$$

Solution by Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

If r = distance between two points whose trilinear coordinates are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, we know that

$$r^2 = \frac{abc}{4\Delta^2}\left\{a\cos A(\alpha_1-\alpha_2)^2 + b\cos B(\beta_1-\beta_2)^2 + c\cos C(\gamma_1-\gamma_2)^2\right\},$$

or $$= \frac{\sin 2A(\alpha_1-\alpha_2)^2 + \sin 2B(\beta_1-\beta_2)^2 + \sin 2C(\gamma_1-\gamma_2)^2}{2\sin A\sin B\sin C}.$$

For I we have $\alpha_1 = 2\beta_1 = \gamma_1 = r$, and for G, $\alpha_2 = \tfrac{1}{3}p_1$, $\beta_2 = \tfrac{1}{3}p_2$, $\gamma_2 = \tfrac{1}{3}p_3$, where p_1, p_2, p_3 are the lengths of the perpendiculars from A, B, C on opposite sides.

Also $p_1 = c\sin B = 2R\sin B\sin C$, and so for p_2 and p_3.

Therefore $$2\sin A\sin B\sin C\,.\,IG^2 = (\sin 2A + \sin 2B + \sin 2C)\,r^2$$
$$-\tfrac{8}{3}Rr\sin A\sin B\sin C(\cos A+\cos B+\cos C)$$
$$+\tfrac{4}{9}R^2(\sin 2A\sin^2 B\sin^2 C + \sin 2B\sin^2 C\sin^2 A + \sin 2C\sin^2 A\sin^2 B);$$

but $$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C,$$

therefore $$IG^2 = 2r^2 - \tfrac{4}{3}Rr(\cos A+\cos B+\cos C)$$
$$+\tfrac{4}{9}R^2(\cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C).$$

Now, $\cos A+\cos B+\cos C = 1 + 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = 1 + r/R$,

and the trigonometrical multiplier of

$$\tfrac{4}{9}R^2 = \sin C\sin(A+B) + \sin A\sin B\cos C = 1+\cos A\cos B\cos C;$$

therefore, finally,

$$IG^2 = 2r^2 - \tfrac{4}{3}r^2 - \tfrac{4}{3}Rr + \tfrac{4}{9}R^2(1+\cos A\cos B\cos C),$$

or $$= \tfrac{4}{9}R^2(1+\cos A\cos B\cos C) - \tfrac{4}{3}Rr + \tfrac{2}{3}r^2,$$

the required result.

2935. (Professor SYLVESTER, F.R.S.)—If $f(x)$ be a rational integral function of x of a higher order than the second, prove that it is impossible for $(fx)^2 + (f'x)^2$ to be a perfect square unless $f(x)$ contains at least two groups of equal factors.

Solution by W. J. C. SHARP, M.A.

If $f(x)$ be of order n, and $F(x) \equiv \{f(x)\}^2 + \{f'(x)\}^2$, $F(x)$ is of order $2n$. Now, if $F(x)$ be a perfect square, it measures its own Hessian (Quest. 8296), which only differs from $2nF\,.\,F'' - (2n-1)F'^2$ by a positive factor (Quest. 5762); therefore F must measure F'^2 and $F' = 2f'(f+f'')$, therefore $f^2+f'^2$ measures $f'^2(f+f'')^2$ and $f'^2(f+f'')^2 \equiv (f^2+f'^2)(f'^2+u)$, where u is a function of x of an order not higher than $2n-4$ (and hence all that follows implies $n>2$); therefore

$$f'^2\left\{2ff''+f''^2-f'^2-u\right\}\equiv f^2u \quad\text{.......................(1)},$$

and either f and f' have a common measure, and f has one group, at least, of equal factors, or u is a multiple of f'^2; but f'^2 is of order $2n-2$, and u at most of $2n-4$, and therefore there is at least one group of equal factors.

Now, let $f(x)\equiv(x-a)^m\phi(x)$ and $f'(x)=(x-a)^{m-1}\phi_1(x)$,
and $f''(x)=(x-a)^{m-2}\phi_2(x)$,

then ϕ, ϕ_1, and ϕ_2 are each of order $n-m$, and, if $\phi(x)$ does not contain $x-a$ as a factor, neither do $\phi_1(x)$ or $\phi_2(x)$. From (1),

$$(x-a)^{2m-2}\phi_1^2\left\{2(x-a)^{2m-2}\phi\phi_2+(x-a)^{2m-4}\phi_2^2-(x-a)^{2m-2}\phi_1^2-u\right\}$$
$$\equiv(x-a)^{2m}\phi^2u,$$

or $\phi_1^2\left\{2(x-a)^{2m-2}\phi\phi_2+(x-a)^{2m-4}\phi_2^2-(x-a)^{2m-2}\phi_1^2-u\right\}\equiv(x-a)^2\phi^2u$,

and, therefore, u must be divisible by $(x-a)^{2m-4}$. Let $u\equiv(x-a)^{2m-4}v$, then v is at most of order $2(n-m)$, and

$$\phi_1^2\left\{2(x-a)^2\phi\phi_2+\phi_2^2-(x-a)^2\phi_1^2-v\right\}\equiv(x-a)^2\phi^2v,$$

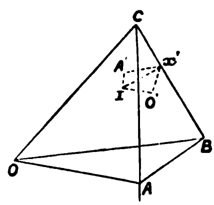
and either ϕ_1 and ϕ have a common measure, or $v\equiv\phi_1^2w$, where w is independent of x, and

$$2(x-a)^2\phi\phi_2+\phi_2^2-(x-a)^2\phi_1^2-w\phi_1^2\equiv(x-a)^2\phi^2w,$$

an identity which requires that the two leading coefficients of ϕ should vanish. Therefore ϕ_1 and ϕ have a common measure, and therefore ϕ and ϕ' for $\phi_1=m\phi+(x-a)\phi'$, and there must be a second group of equal factors.

7459. (Professor WOLSTENHOLME, M.A., Sc.D.)—Prove that if, in a tetrahedron, any one or two of the equations $a\pm x=b\pm y=c\pm z$ be true, then will also the corresponding one or two equations of the set $A\pm X=B\pm Y=C\pm Z$ also be true.

Solution by the PROPOSER.



In the tetrahedron OABC, OA $=a$, OB $=b$, OC $=c$; BC $=x$, CA $=y$, AB $=z$; suppose a sphere can be drawn to touch the four edges a, b, x, y (none produced), then, if t_1, t_2, t_3, t_4 be the lengths of tangents to this sphere from O, A, B, C, $a=t_1+t_2$, $x=t_3+t_4$; $b=t_1+t_3$, $y=t_2+t_4$, whence $a+x=b+y=t_1+t_2+t_3+t_4$. Thus such a sphere can be drawn if $a+x=b+y$. Let I be the centre of the sphere; p_1, p_2, p_3, p_4 the perpendiculars IO′, IA′, IB′, IC′ on the faces; a', b', x', y' the points of contact with the corresponding edges; then the dihedral angle A (*opposite* to the edge a) is the supplement of the angle O′IA′; and if 2α, 2β, 2γ, 2δ be the angles subtended at I by diameters of the circles in which the sphere is met by the faces, the angle

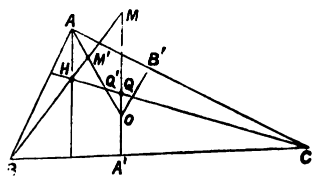
$$O'IA' = \alpha+\beta, \text{ or } A = \pi-\alpha-\beta; \text{ so } X = \pi-\gamma-\delta;$$

or $$A+X = 2\pi-\alpha-\beta-\gamma-\delta = B+Y.$$

The proof that, when $a-x = b-y$, then will also $A-X = B-Y$, is exactly similar, the sphere then touching the produced edges.

8783. (Captain H. Brocard.)—Les droites qui joignent l'orthocentre aux milieux des segments interceptés par les hauteurs du triangle sur les rayons du cercle circonscrit aboutissant aux sommets du triangle sont perpendiculaires aux symédianes correspondantes.

Solution by Professor Ignacio Beyens, M.A.



Soit O le centre du cercle circonscrit, H l'orthocentre. OA' perpendiculaire sur BC coupe les hauteurs BH, CH en M et Q; les triangles ABC, HMQ sont semblables, leurs côtés étant perpendiculaires; et l'angle MHQ = A.

Soient M', Q' les points de rencontre de BH, CH avec le rayon OA, nous avons HM'Q' = M'OB' = B, et par suite M'Q' est antiparallèle par rapport au côté QM du triangle HQM, c'est-à-dire, que la droite qui joint H au milieu de M'Q' sera la symédiane qui part du sommet H du triangle HMQ, et par suite elle sera perpendiculaire à la symédiane de l'angle A du triangle proposé, car ces lignes sont homologues en figures semblables qui ont leurs côtés perpendiculaires.

8837. (W. J. Greenstreet, B.A.)—In the "ambiguous case" of triangles, given a, b, and A; r, r' being the radii of the in-circles, ρ, ρ' of the escribed circles to a; prove that

$$rr'(b+a) = \rho\rho'(b-a), \quad r\rho(b-a) = r'\rho'c.$$

Solution by H. L. Orchard, B.Sc., M.A.; Sarah Marks, B.Sc.; *and others.*

Let c and c' be the two values of c. Then $cc' = b^2-a^2$, and

$$\frac{r}{\rho'} = \frac{c(s'-a)}{c's} = \frac{c}{a+b}, \quad \frac{r'}{\rho} = \frac{c'}{a+b} = \frac{b-a}{c};$$

whence $\quad rr'(b+a) = \rho\rho'(b-a)$, and $r\rho(b^2-a^2) = r'\rho'c^2$, &c.

8715. (Professor HUDSON, M.A.)—Prove that
$$\tan 37\tfrac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-\sqrt{2}-2.$$

Solution by G. G. STORR, M.A.; Rev. T. R. TERRY, M.A.; *and others.*

If $x = \tan 37\frac{1}{2}^\circ$, then $2x/(1-x^2) = \tan 75^\circ = 2+\sqrt{3}$; whence, solving $x = \pm(\sqrt{6}-\sqrt{2})-(2-\sqrt{3})$, the upper sign gives $\tan 37\frac{1}{2}^\circ$, the lower sign gives $\tan 127\frac{1}{2}^\circ$.

8929. (R. TUCKER, M.A.)—Find (1) the equation of the circle through the images of the centroid of a triangle with respect to the bisectors of angles; and show (2) that the sum of the squares of tangents to it from the angles (taken once) is $\frac{1}{3}(a^2+b^2+c^2)$.

Solution by the PROPOSER.

1. Since the median and symmedian lines of a triangle make equal angles with the bisectors of angles, therefore the images in question lie on the symmedians and the circle is concentric with the in-circle. Assume its equation to be

$$abc\,(\alpha\beta\gamma+\ldots+\ldots) = (a\alpha+\ldots+\ldots)\left[a\,(s-a)^2\,\alpha+\ldots+\ldots\right]+\lambda\,(a\alpha+\ldots+\ldots)^2 \quad \text{............(i.)};$$

then, since the centroid is on (i.), we have

$$a^2+b^2+c^2 = 3\left[(s-a)^2+\ldots+\ldots\right]+9\lambda,$$

therefore $9\lambda = -2\,(a^2+b^2+c^2)+3s^2$, and $36\lambda = 6\Sigma ab-5\Sigma a^2$.

The circle then can be put under the form

$$abc\,(\alpha\beta\gamma+\ldots+\ldots)$$
$$= (a\alpha+\ldots+\ldots)\left[a\alpha\,(s-a)^2+\ldots+\ldots+\tfrac{1}{36}(6\Sigma ab-5\Sigma a^2)(a\alpha+\ldots+\ldots)\right].$$

2. If t_a, t_b, t_c are the tangents from the vertices, we have (see *Mathematics from Educational Times*, Vol. 46, *Appendix* II., p. 138)

$$t^{a2} = (s-a)^2+\tfrac{1}{36}(6\Sigma ab-5\Sigma a^2) = \tfrac{1}{36}\left[9\,(b+c-a)^2+6\Sigma ab-5\Sigma a^2\right]$$
$$= \tfrac{1}{36}\left[4\Sigma a^2+24bc-12ab-12ac\right] = \tfrac{1}{9}\left[\Sigma a^2+6bc-3ab-3ac\right],$$

therefore $$t_a^2+t_b^2+t_c^2 = \tfrac{1}{3}\Sigma a^2 = \tfrac{1}{3}(a^2+b^2+c^2).$$

8765. (L. J. ROGERS, M.A.)—If $x_1+x_2+x_3+\alpha_1+\alpha_2+\alpha_3 = 0$, prove that (1)

$$\begin{vmatrix} \tan(x_1+\alpha_1), & \tan(x_2+\alpha_1), & \tan(x_3+\alpha_1) \\ \tan(x_1+\alpha_2), & \tan(x_2+\alpha_2), & \tan(x_3+\alpha_2) \\ \tan(x_1+\alpha_3), & \tan(x_2+\alpha_3), & \tan(x_3+\alpha_3) \end{vmatrix} = 0;$$

and (2) the same is true if tangents are replaced by sines.

Solution by F. R. J. Hervey; Belle Easton, B.A.; *and others.*

1. Each element is a product of three tangents which may be replaced by their sum; moreover, each tangent appears in two elements with contrary signs: hence the whole vanishes.

2. Any element represented by $\sin a \sin b \sin c$ may be replaced by $\sin a \cos b \cos c + \ldots + \ldots$; which being done throughout, the complete coefficient of a given *cosine* = 0; for instance, that of $\cos(x_1 + a_1)$ is

$$\sin(x_2+a_2)\cos(x_3+a_3)+\cos(x_2+a_2)\sin(x_3+a_3) \\ -\left\{\sin(x_3+a_2)\cos(x_2+a_3)+\cos(x_3+a_2)\sin(x_2+a_3)\right\} = 0;$$

and the same for any other, by interchange of suffixes. Hence, if each term $\sin a \cos b \cos c$ be taken twice, the whole can be arranged in nine groups of four mutually destructive terms, and vanishes identically.

8797. (Professor Sáradáranjan Rây, M.A.)—OAP and OBQ are two fixed straight lines intersecting at O, and C is a circle touching them both at P and Q; prove that the perimeter of the triangle OAB, circumscribed by *any* circle passing through O and touching C externally, is constant.

Solution by Professor Wolstenholme, M.A., Sc.D.

One of the trigonometrical questions in my book of problems is: "The radius of a circle touching two sides AB, AC of a triangle ABC and the circumscribed circle is $r\sec^2\frac{1}{2}A$ or $r_1\sec^2\frac{1}{2}A$, according as the contact of the two circles is internal or external." Hence, with the notation of the question, the perimeter of the triangle OAB will be

$$2OP\cos^3\tfrac{1}{2}POQ / \sin\tfrac{1}{2}POQ;$$

and similarly, if the contact were internal, the excess of the two sides OA, OB over AB would be $2OP\cos^3\frac{1}{2}POQ / \sin\frac{1}{2}POQ$.

8473. (Professor Booth, M.A.)—If a, b, c be the sides of a plane triangle, prove that the diameter of the circumscribing circle is a root of the equation

$$a\left[(x^2-b^2)(x^2-c^2)b\right]^{\frac{1}{2}}+\left[(x^2-c^2)(x^2-a^2)\right]^{\frac{1}{2}}+c\left[(x^2-a^2)(x^2-b^2)\right] = abc.$$

Solution by Isabel Maddison; Rosa H. W. Whapham; *and others.*

If x be the diameter of the circum-circle, we have $x^2-a^2 = a^2\cot^2 A$, &c., and the equation becomes $\cot B\cot C+\cot C\cot A+\cot A\cot B = 1$, which is correct.

2832. (Professor SYLVESTER, F.R.S.)—Prove that the curve of intersection of two right cones with parallel axes is a spherical curve, and that a third right cone may be drawn through it. Prove also that a plane circular cubic may be found such that the distances of *any* two fixed points on it from *every* point in the curve of double curvature above-mentioned shall be in a fixed linear relation.

Solutions by (1) Professor SIRCOM, M.A.; (2) W. J. C. SHARP, M.A.

1. The right cones $x^2+y^2 = mz^2$, $x^2+(y-b)^2 = n(z-c)^2$ have their axes parallel, and the sphere

$$(n+1)(x^2+y^2-mz^2) = (m+1)\left\{x^2+(y-b)^2-n(z-c)^2\right\}$$

passes through their intersection. Also the surface

$$x^2+y^2-mz^2 = \lambda\left\{x^2+(y-b)^2-n(z-c)^2\right\}$$

is a right cone if its discriminant vanishes, that is, if $\lambda = \dfrac{m}{n}\cdot\dfrac{b^2-nc^2}{b^2-mc^2}$.

The circular cubic in the plane $x=0$ passing through the vertices of the three cones, and having for foci the four points where the spherical curve meets $x=0$, satisfies the given condition.

For, taking the origin and x, y for the two fixed points, and $x_1 y_1 z_1$ a point on the spherical curve, we must have, by the question,

$$\left\{x_1^2+(y_1-y)^2+(z_1-z)^2\right\}^{\frac{1}{2}}+\lambda\,(x_1^2+y_1^2+z_1^2)^{\frac{1}{2}} = \mu$$

for every point $x_1y_1z_1$. Eliminating x_1y_1 by means of the equations of the curve, and rationalizing, we obtain

$$\left\{(1-\lambda^2)(m+1)\,b+(n-m)\,y\right\}z_1^2-2\left\{bz+ncy+\lambda\mu\,(m+1)^{\frac{1}{2}}\,b\right\}z_1$$
$$+\,b\,(y^2+z^2)-(b^2-nc^2)\,y-b\mu^2 = 0;$$

identically; hence, equating the coefficients to 0, and eliminating λ, μ, we obtain

$$\left\{(n-m)\,y+(m+1)\,b\right\}\left\{b\,(y^2+z^2)-(b^2-nc^2)\,y\right\} = (bz+ncy)^2,$$

the required circular cubic.

2. *Otherwise*:—If the vertex of one of the cones be taken as origin of rectangular coordinates, its axis as axis of z, and the plane of the axes as plane of yz; then, the other vertex being $(0, k, l)$, the equations to the cones are $x^2+y^2+z^2 = m^2z^2$, $x^2+(y-k)^2+(z-l)^2 = n^2(z-l)^2$; therefore, at intersection,

$$(n^2-m^2)(x^2+y^2+z^2)+2m^2ky+2m^2lz-m^2k^2-m^2l^2 = m^2n^2\,(2lz-l^2),$$

and the curve of intersection lies on the sphere

$$x^2+y^2+z^2+2ay+2bz = c,$$

if $\qquad \dfrac{m^2k}{n^2-m^2} = a, \quad \dfrac{m^2l\,(1-n^2)}{n^2-m^2} = b, \quad \dfrac{m^2\left\{k^2+(1-n^2)\,l^2\right\}}{n^2-m^2} = c,$

and k and l are determined by the equations

$$ak+bl = c \quad\text{and}\quad m^2kl = a\,(1-m^2)\,l-bk,$$

so that a third cone passes through the curve. If $(0, k', l')$ be the vertex of this cone, and n' the corresponding value of n, at every point of the curve of intersection, $r = mz$, $r' = n\,(z-l)$, and $r'' = n'\,(z-l')$, where r, r', r'' are the distances of (x, y, z) from the vertices $(0, 0, 0)$, $(0, k, l)$, and $(0, k', l')$, and $nr-mr' = nl$ and $n'r-mr'' = n'l'$, so that the curve is a Cartesian oval in space (see Quest. 5955).

Now, if ρ be the distance of (x, y, z) from the point $(0, \eta, \zeta)$,

$$\rho^2 = x^2+y^2+z^2-2\eta y-2\zeta z+\eta^2+\zeta^2 = m^2z^2-\frac{\eta}{a}(c-2bz-m^2z^2)-2\zeta z+\eta^2+\zeta^2$$

(if the point is on the curve)

$$= m^2\left(1+\frac{\eta}{a}\right)z^2+2\left(\frac{b\eta}{a}-\zeta\right)z+\eta^2+\zeta^2-\frac{\eta c}{a},$$

and $$\rho = \mathrm{A}z+\mathrm{B},$$

if $$m^2\left(1+\frac{\eta}{a}\right)\left(\eta^2+\zeta^2-\frac{c\eta}{a}\right) = \left(\frac{b\eta}{a}-\zeta\right)^2;$$

hence, if any two points be taken on this curve, and ρ and ρ' be the radii vectores from them, $\rho = \mathrm{A}z+\mathrm{B}$, $\rho' = \mathrm{A}'z+\mathrm{B}'$, therefore

$$\mathrm{A}'\rho-\mathrm{A}\rho' = \mathrm{A}'\mathrm{B}-\mathrm{A}\mathrm{B}'.$$

The circular cubic passes through each of the vertices.

8873. (Professor Edmund Bordage.)—Given two relations,

$$(2\mathrm{S} = a+b+c),$$

$$\frac{x+y\cos c+z\cos b}{\cos(\mathrm{S}-a)} = \frac{y+z\cos a+x\cos c}{\cos(\mathrm{S}-b)} = \frac{z+x\cos b+y\cos a}{\cos(\mathrm{S}-c)} = 2m,$$

deduce therefrom $$\frac{x}{\sin a} = \frac{y}{\sin b} = \frac{z}{\sin c} = \frac{m}{\sin \mathrm{S}}.$$

Solution by W. J. C. Sharp, M.A.; Professor Matz, M.A.; *and others.*

$$2m = \frac{(x+y\cos c+z\cos b)\,\mathrm{A}+(y+z\cos a+x\cos c)\,\mathrm{B}+(z+x\cos b+y\cos a)\,\mathrm{C}}{\mathrm{A}\cos(\mathrm{S}-a)+\mathrm{B}\cos(\mathrm{S}-b)+\mathrm{C}\cos(\mathrm{S}-c)}.$$

If $\mathrm{A}\cos c+\mathrm{B}+\mathrm{C}\cos a = 0$, and $\mathrm{A}\cos b+\mathrm{B}\cos a+\mathrm{C} = 0$,

$$\mathrm{A}:\mathrm{B}:\mathrm{C} = -\sin^2 a : \cos c-\cos a\cos b : \cos b-\cos a\cos c,$$

and the numerator $= -x\left\{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\right\}$.

The denominator

$$= \left\{-2\sin^2 a\cos(\mathrm{S}-a)\sin\mathrm{S}+2(\cos c-\cos a\cos b)\cos(\mathrm{S}-b)\sin\mathrm{S}\right.$$
$$\left.+2(\cos b-\cos a\cos c)\cos(\mathrm{S}-c)\sin\mathrm{S}\right\} \div 2\sin\mathrm{S}$$

$$= \left\{-\sin^2 a\left[\sin(b+c)+\sin a\right]+\left[\cos c-\cos a\cos b\right]\left[\sin(a+c)+\sin b\right]\right.$$
$$\left.+\left[\cos b-\cos a\cos c\right]\left[\sin(a+b)+\sin c\right]\right\} \div 2\sin\mathrm{S}$$

$$= \left\{-\sin a\,(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)\right.$$
$$+\sin b\,(-\sin^2 a\cos c+\cos c-\cos a\cos b+\cos a\cos b-\cos^2 a\cos c)$$
$$\left.+\sin c\,(-\sin^2 a\cos b+\cos b-\cos a\cos c+\cos a\cos c-\cos^2 a\cos b)\right\}$$
$$\div 2\sin\mathrm{S}$$

$$= \left\{-\sin a\,(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\right\} \div 2\sin\mathrm{S},$$

therefore $$2m = \frac{2x}{\sin a}\times\sin\mathrm{S},$$

or $$\frac{x}{\sin a} = \frac{m}{\sin S} = \frac{y}{\sin b} = \frac{z}{\sin c}, \text{ by symmetry.}$$

[Observing that $2 \sin S \cos (S-a) = \sin a + \sin (b+c)$ and for brevity's sake writing k for $m/\sin S$, we see that the given relation
$$(x + y \cos c + z \cos b) \sec (S-a) = 2m$$
is equivalent to
$$(x - k \sin a) + (y - k \sin b) \cos c + (z - k \sin c) \cos b = 0;$$
and similarly, from the other given relations, we have
$$(y - k \sin b) + (z - k \sin c) \cos a + (x - k \sin a) \cos c = 0,$$
$$(z - k \sin c) + (x - k \sin a) \cos b + (y - k \sin b) \cos a = 0,$$
whence $$x - k \sin a = 0, \quad y - k \sin b = 0, \quad z - k \sin c = 0.$$

Hence, if a, b, c be the sides of a spherical triangle, any quantities x, y, z which satisfy the given relations are proportional to the sines of the angles, and conversely.]

8980. (Professor STEGGALL, M.A.)—In a rectangular hyperbola OY is drawn at right angles to the tangent at P; prove that, if YX produced cuts SP in R, and Y be joined to X′, then XR = YX′, where X, X′ are the feet of the directrices.

Solution by R. KNOWLES, B.A.; SARAH MARKS, B.Sc.; *and others.*

If $x^2 - y^2 = a^2$ be the equation to the hyperbola, that of the tangent at P $(x'y')$ is $x'x - y'y = a^2$, that of OY $x'y + y'x = 0$; therefore the coordinates of Y are $a^2x'/(x'^2+y'^2)$, $-a^2y'/(x'^2+y'^2)$; those of X, X′, S are $(a/2^{\frac{1}{2}}, 0)$, $(-a/2^{\frac{1}{2}}, 0)$, $(2^{\frac{1}{2}}a, 0)$, respectively; whence the equations to SP and XY are
$$y'x + (2^{\frac{1}{2}}a - x')\, y = 2^{\frac{1}{2}}ay' \quad \text{.........................(1)},$$
$$2^{\frac{1}{2}}ay'x - (x'^2 + y'^2 - 2^{\frac{1}{2}}ax')\, y = a^2y' \quad \text{.....................(2)}.$$
From (1), (2), we have
$$x = \frac{ax'\,(2^{\frac{3}{2}}x - 3a)}{(2^{\frac{1}{2}}x - a)^2}, \qquad y = \frac{a^2y'}{(2^{\frac{1}{2}}x' - a)^2},$$
the coordinates of R, whence we find XR and YX′ each
$$= a\,(2^{\frac{1}{2}}x' + a)^{\frac{1}{2}} / 2^{\frac{1}{2}}.$$

9010. (J. J. WALKER, F.R.S.)—Prove that the area contained by two tangents to a central conic and the semi-diameters to points of contact is equal to $b^2x^2 + a^2y^2 - a^2b^2$. [See Question 3099, Vol. XIV., pp. 74, 75.]

Solution by C. E. WILLIAMS, M.A.; R. KNOWLES, B.A.; *and others.*

Projecting the ellipse into the auxiliary circle, the $(\text{area})^2$ would have to be multiplied by $\frac{a^2}{b^2}$, and as $X = x$, $Y = \frac{a}{b}y$, the expression becomes

$$(\text{area})^2 = a^2X^2 + a^2Y^2 - a^4 = a^2(X^2 + Y^2 - a^2),$$

or area = (rad. × tang.), which is evidently true for the circle.

8903. (Professor GENESE, M.A.)—If $f(r, \theta)$ be the equation to a conic referred to any pole O, the value of $(df)/(dr)$ at any point P of the plane varies as $(PQ + PR)/OQ.OR$, Q, R being the intersections of OP with the conic; thus, for a point Q of the conic, $(df)/(dr) \propto QR/OQ.OR$; and in particular, if O be a focus, $(df)/(dr)$ is constant over the curve. [In this case $f \equiv (l - er\cos\theta)^2 - r^2$.]

Solution by R. F. DAVIS, M.A.; Professor CHAKRAVARTI, M.A.; *and others.*

The polar equation of the conic referred to O is of the form

$$f(r, \theta) = Ar^2 + Br + C = 0,$$

where A, B are functions of θ alone, and C is constant. For any definite value of θ, we have $r_1 + r_2 = -2B/A$, $r_1r_2 = C/A$.

Now $$(df)/(dr) = 2(Ar + B) = 2A(r + B/A)$$
$$\propto \{2r - (r_1 + r_2)\}/r_1r_2 \propto (PQ + PR)/OQ.OR.$$

If P coincide with Q, PQ = 0, &c. If O be a focus, $(df)/(dr) =$ constant, which is *a priori* evident, since the semi-latus-rectum is a harmonic mean between OQ, OR.

9003 & 9013. (9003.)—(R. F. DAVIS, M.A.)—If upon each side of a triangle a pair of points be taken so that the pairs on any two sides are concyclic, prove that all three pairs are concyclic.

9013. (EMILE VIGARIÉ.)—Les projections orthogonales de deux *points inverses* M_1, $M_2(x', y', z')$ sur les trois côtés d'un triangle ABC, sont six points d'une même circonférence dont l'équation en coordonnées normales est:

$$(yz\sin A + zx\sin B + xy\sin C)(x'\sin A + y'\sin B + z'\sin C)$$
$$\times (y'z'\sin A + z'x'\sin B + x'y'\sin C)$$
$$= \sin A\sin B\sin C\,(x\sin A + y\sin B + z\sin C)\left\{\frac{xx'(y' + z'\cos A)(z' + y'\cos A)}{\sin A}\right.$$
$$\left. + \frac{yy'(x' + z'\cos B)(z' + x'\cos B)}{\sin B} + \frac{zz'(x' + y'\cos C)(y' + x'\cos C)}{\sin C}\right\}.$$

[M. VIGARIÉ remarks that "on appelle *points inverses* en France ce que M. CASEY appelle *isogonal conjugate points* (*Sequel to Euclid*, 1886, p. 166)."]

Solution by A. R. Johnson, M.A.; C. E. Williams, M.A.; *and others.*

It seems best to take these together, as they are both easy deductions from the form of the equation in trilinears to any circle, viz.:

$$4R^2\left\{yz\sin A+zx\sin B+xy\sin C\right\}\sin A\sin B\sin C$$
$$=\left\{x\sin A+y\sin B+z\sin C\right\}\left\{p^2x\sin A+q^2y\sin B+r^2z\sin C\right\}\ldots\ldots(1),$$

where p, q, r are the tangents from the vertices to the circle.

Let D, D′; E, E′; F, F′ be three pairs of points on the sides, and concyclic in pairs. Then, if we take $p^2 = AE \, . \, AE' = AF \, . \, AF'$;

$$q^2 = BF \, . \, BF' = BD \, . \, BD'; \quad r^2 = CD \, . \, CD' = CE \, . \, CE',$$

the circle represented by (1) must cut the sides in D, D′, E, E′, F, F′; so that all six points are concyclic.

Now let D, E, F; D′, E′, F′ be the projections on the sides of the points x', y', z'; k/x', k/y', k/z', where

$$k\left(\frac{\sin A}{x'}+\frac{\sin B}{y'}+\frac{\sin C}{z'}\right)=x'\sin A+y'\sin B+z'\sin C\ldots\ldots\ldots(2).$$

Then $AE \, . \, AE' = AF \, . \, AF' = k\left\{y'+z'\cos A\right\}\left\{z'+y'\cos A\right\}/y'z'\sin^2 A$,

so that the projections of the points are concyclic in pairs, and therefore all lie on the circle for which $p^2 = k\,(y'+z'\cos A)(z'+y'\cos A)/y'z'\sin^2 A$, and q^2 and r^2 have analogous values. Substituting in (1) for p^2, q^2, r^2, k, there results the equation sought.

8835. (Professor de Longchamps.) — On considère un triangle ABC et le cercle circonscrit Δ; soit μ une transversale rencontrant les côtés de ABC aux points A′, B′, C′. Par A′ on mène à Δ une tangente que l'on rabat sur BC de telle sorte que le point de contact vienne occuper sur BC une certaine position A″. Démontrer que la droite AA″ et les deux droites analogues BB″, CC″ vont passer par un même point M.

Solution by Professor Ignacio Beyens, M.A.

Soit A′M′ la tangente menée de A′; nous aurons

$$A'B \, . \, A'C = (A'M')^2 = (A'A'')^2, \quad C'A \, . \, C'B = C'C''^2, \quad B'A \, . \, B'C = B'B'^2,$$

et aussi
$$\frac{BA''}{A''C}=\frac{A'A''-BA'}{A'C-A'A''}=\frac{(A'B \, . \, A'C)^{\frac12}-BA'}{A'C-(A'B \, . \, A'C)^{\frac12}},$$

ou
$$\frac{BA''}{A''C}=\frac{(A'B \, . \, A'C)^{\frac12} \, . \, A'C-A'B \, . \, A'C+A'B \, . \, A'C-BA'(A'B \, . \, A'C)^{\frac12}}{(A'C)^2-A'B \, . \, A'C},$$

$$\frac{BA''}{A''C}=\frac{(A'B \, . \, A'C)^{\frac12}(A'C-A'B)}{A'C\,(A'C-A'B)}=\frac{(A'B \, . \, A'C)^{\frac12}}{A'C}=\frac{(A'B)^{\frac12}}{(A'C)^{\frac12}}.$$

De la même manière, on a $\dfrac{CB''}{AB''}=\dfrac{(B'C)^{\frac12}}{(B'A)^{\frac12}}$ et $\dfrac{AC''}{BC''}=\dfrac{(AC')^{\frac12}}{(BC')^{\frac12}}$.

Multipliant ces proportions,

$$\frac{BA''}{A''C} \cdot \frac{CB''}{AB''} \cdot \frac{AC''}{BC''} = \left(\frac{A'B \,.\, B'C \,.\, AC'}{A'C \,.\, B'A \,.\, BC'}\right)^2 = 1,$$

$$\left(\text{parce que } \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{AC'}{BC'} = 1\right);$$

donc les trois droites AA″, BB″, CC″ se rencontrent au même point.

8385. (J. Brill, M.A.)—Three parabolas are drawn having a common focus; from a point T, external to all three, tangents TP and Tp are drawn to the first parabola, TQ and Tq to the second, and TR and Tr to the third; prove that $Qr \,.\, Rp \,.\, Pq = qR \,.\, rP \,.\, pQ$.

Solution by the Proposer; Belle Easton, B.A.; *and others.*

Let S be the common focus of the three parabolas, then we have

$$(SP) \,.\, (Sp) = (SQ) \,.\, (Sq) = (SR) \,.\, (Sr) = (ST)^2;$$

therefore

$$(Qr) \,.\, (Rp) \,.\, (Pq) + (qR) \,.\, (rP) \,.\, (pQ)$$
$$= \{(Sr)-(SQ)\}\{(Sp)-(SR)\}\{(Sq)-(SP)\}$$
$$+ \{(SR)-(Sq)\}\{(SP)-(Sr)\}\{(SQ)-(Sp)\}$$
$$= \{(SP)+(Sp)\}\{(SR) \,.\, (Sr)-(SQ) \,.\, (Sq)\}$$
$$+ \{(SQ)+(Sq)\}\{(SP) \,.\, (Sp)-(SR) \,.\, (Sr)\}$$
$$+ \{(SR)+(Sr)\}\{(SQ) \,.\, (Sq)-(SP) \,.\, (Sp)\} = 0.$$

Hence it follows that $Qr \,.\, Rp \,.\, Pq = qR \,.\, rP \,.\, pQ$.

8915. (Rev. T. C. Simmons, M.A.)—A point P being given in the plane of a triangle ABC, it is known that, with P as focus, five conics can be drawn, four of them circumscribing the triangle, and one inscribed in it. Show that for certain positions of P a sixth conic can be drawn, also having P for focus, and touching the other five; and find the locus of P when this is possible. [The Proposer suggests that conics with one focus common should be called *co-focal*, reserving the term confocal for conics having both foci common.]

Solution by R. F. Davis, M.A.

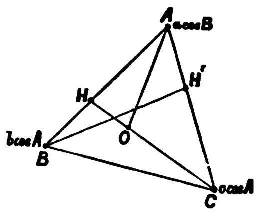
If, the system consisting of a triangle abc, its one circumscribed and four inscribed circles be reciprocated with respect to any point P, we obtain a corresponding system consisting of a triangle ABC and five conics each having P as focus, four of them circumscribing the triangle ABC and one inscribed in it.

The nine-point circle of the triangle *abc* touches each of the four inscribed circles; and in the particular case when one of the angles is a right angle it also touches the circumscribed circle.

Hence, if one of the sides of ABC subtend a right angle at P, the reciprocal of this nine-point circle is a conic (focus P) touching each of the above five conics. P must therefore lie on one of the three circles described upon the sides of ABC as diameters.

8683. (Professor Ignacio Beyens, M.A.)—Si dans un triangle la projection du côté BC sur BA qui est BH, et la projection du BA sur AC qui est AH′, sont egales; la bissectrice du A, la hauteur du C et la médiane de B se rencontrent en le même point.

Solution by R. F. Davis, M.A.; R. Knowles, B.A.; *and others.*



Since BH = AH′, $a\cos B = c\cos A$. Let the bisector of the angle A meet the perpendicular CH in O, so that

$$OH : OC = AH : AC = \cos A : 1.$$

Then O may be regarded as the centroid of the masses $a\cos B$, $b\cos A$, $c\cos A$, placed at the angular points A, B, C respectively. For, since

$$c = a\cos B + b\cos A = BH + AH,$$

the masses at A and B may be replaced by a mass c at H; which with $c\cos A$ at C is finally equivalent to a mass $c(1+\cos A)$ at O. Hence BO divides AC in the ratio $c\cos A : a\cos B = 1 : 1$, *i.e.*, it bisects it.

8700. (R. W. D. Christie, M.A.) — Given the sum of the expression $1^r + 2^r + 3^r + 4^r \ldots\ldots + n^r$, to find a method of writing down at once the sum of $1^{r+1} + 2^{r+1} + 3^{r+1} + 4^{r+1} \ldots\ldots + n^{r+1}$, where r is even or *vice versâ.*

Solution by Prof. Genese, M.A.; Prof. Nash, M.A.; *and others.*

$$1^r + 2^r + 3^r + \ldots\ldots + n^r = \frac{n^{r+1}}{r+1} + An^r + \&c. = \phi(n) \text{ say},$$

therefore $$(1+x)^r + (2+x)^r + \ldots (n+x)^r = \phi(n+x) - \phi(x).$$

This equality holding for all integral values of x (and therefore for more than r values), *is an identity.* Equating coefficients of x,

$$r(1^{r-1} + 2^{r-1} + \ldots\ldots + n^{r-1}) = \phi'(n) - \text{coefficient of } x \text{ in } \phi(x)$$
$$= \phi'(n) - \phi'(0).$$

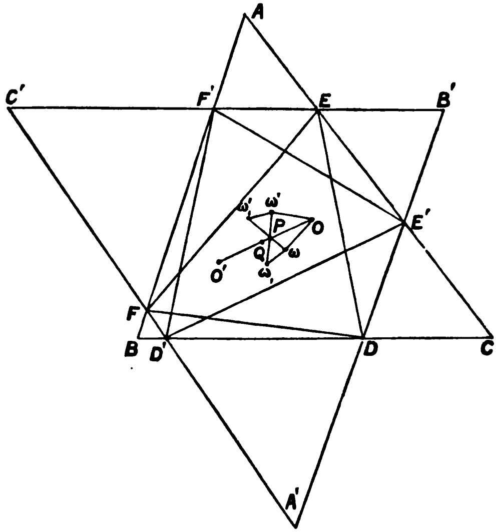
Hence, to find Σx^{r+1} from Σx^r, multiply by $(r+1)$, and integrate with respect to n. The constant of integration can always be determined by the condition that the expression equals unity when n is unity, and is in fact nB_n, when B_n represents Bernouilli's numbers.

To find Σx^r from Σx^{r+1} reverse the process and omit the term without n.

[Prof. GENESE adds that the above occurred to him last year as the proper method of dealing with the series $(a+d)^r+(a+2d)^r+(a+3d)^r+$ &c., obtained by putting x above equal to d/a].

NOTE ON QUEST. 8375 (VOL. XLV., p. 98). *By* Professor NASH, M.A.

Most of the theorems enumerated in this Question are true, not only in the particular case considered, but also when the lines ωD, ω'D, &c. make any constant angle with the sides. Let the angles CDω, AEω,



BFω, BD$'\omega'$, CE$'\omega'$, AF$'\omega'$, measured in the direction of the letters, be all $=\theta+\omega$, where ω is the Brocard angle of the triangle ABC. Therefore $\omega EC = \pi-(\theta+\omega) = \pi - \omega DC$; therefore ωDCE are concyclic, and similarly ωEAF, ωFBD, ω'D$'$CE$'$, ω'E$'$AF$'$, ω'F$'$BD$'$ are concyclic; therefore $\omega DE = \omega CA = \omega$, $\omega EF = \omega AB = \omega$, &c.; therefore ω is positive Brocard-point of DEF, and ω' is negative Brocard-point of D$'$E$'$F$'$.

Also $\omega DF = \omega BA = B-\omega$, therefore FDE = B.

Similarly DEF = C, EFD = A, F$'$D$'$E$'$ = C, D$'$E$'$F$'$ = A, E$'$F$'$D$'$ = B, therefore DEF, D′E′F′ are similar to one another and to ABC, and ω is centre of similitude of ABC, DEF, ω' that of ABC, D′E′F′.

Because B, D are corresponding points, the ratio of similitude is

$$\omega D / \omega B = \sin\omega / \sin(\omega+\theta),$$

similarly for D′E′F′, ABC the ratio is $\omega' D' / \omega' C = \sin\omega / \sin(\omega+\theta)$, therefore the two triangles are equal to one another. Again, the angle

$$CD'E' = \pi - C - CE'D' = \pi - C - \theta = \pi - C - CDE = CED.$$

Therefore D′ lies in the circumcircle of DEF, and similarly for E′F′, so that the two triangles have a common circumcircle (a Tucker's circle).

Let Q be the centre of this circle, then Q corresponds to O, whether considered as belonging to DEF or D′E′F′, therefore

$$\omega Q / \omega O = \omega' Q / \omega' O = \sin\omega / \sin(\omega+\theta),$$

therefore Q lies on line OP which bisects the angle $\omega O \omega'$. If rR be the radii of the circles DEF, ABC, therefore

$$r / R = \sin\omega / \sin(\omega+\theta) = \omega Q / \omega O,$$

therefore the radius of a Tucker's circle is proportional to the distance of its centre from the Brocard-points. Also, since $O\omega Q$ is similar to $B\omega D$, therefore $O\omega Q = \theta$. If ω_1 be negative Brocard-point of DEF, ω_1 corresponds to ω', therefore $\omega'\omega\omega_1 = \theta$, $\omega\omega'\omega_1 = \omega = \omega\omega' P$; therefore ω_1 lies on the fixed line $\omega' P$, and similarly the positive Brocard point of D′E′F′ lies upon ωP. If p, p' be the symmedian points of DEF, D′E′F′, then $P\omega p$ is similar to $O\omega Q$, therefore $\omega P p = \omega$.

Therefore Pp is parallel to $\omega\omega'$, and Pp' of course coincides with Pp.

Since the Brocard-points are equidistant from the circumcentre, therefore Q is the centre of the circle $\omega\omega'\omega_1\omega'_1$, therefore

$$\omega Q \omega_1 = 2\omega\omega'\omega_1 = 2\omega = \omega P \omega_1,$$

therefore P lies on the Brocard circles of DEF and D′E′F′.

Again, from the figure, $E'EF' = \pi - E'D'F' = \pi - C$, therefore EF′ is parallel to BC; and because AE . AE′ = AF . AF′, therefore E′F is antiparallel to BC. Also E′F subtends at the circumference an angle $E'F'F = \pi - \theta$, therefore $E'F = 2r\sin\theta$; therefore the three antiparallels E′F, F′D, D′E are all equal, and this common length is proportional to $OQ = \dfrac{2R}{O\omega}\sin\omega \,.\, OQ$. DD′ subtends at the circumference an angle $DE'D' = CE'D' - CE'D = \theta - A$, therefore $DD' = 2r\sin(\theta - A)$, therefore the intercepts DD′, EE′, FF′ are proportional to $\sin(\theta - A)$, $\sin(\theta - B)$, $\sin(\theta - C)$.

If $\theta = \frac{1}{2}\pi$, the intercepts are proportional to cos A, cos B, cos C, and the circle DEF is the cosine circle.

In this case the sides of the triangle DEF, D′E′F′ are perpendicular to the homologous sides of ABC, and, since $O\omega Q = \frac{1}{2}\pi = O\omega P$, therefore the centre of the cosine circle is the symmedian point.

The radius of the circle is $R\tan\omega$.

In the case considered by Mr. Simmons, $\theta = \frac{1}{2}\pi - \omega$, $\therefore O\omega Q = O\omega\omega'$, therefore Q is the mid-point of $\omega\omega'$, and the radius of the circle is $R\sin\omega$.

The lines EF′, FD′, DE′ form a new triangle A′B′C′ similar to ABC, and DEF, D′E′F′ are Tucker's triangles with respect to A′B′C′. Therefore the symmedian point of A′B′C′ is the bisection of pp' as in the case of ABC, *i.e.*, the symmedian points of ABC, A′B′C′ coincide, therefore P is the centre of similitude of ABC, A′B′C′ and also the centre of perspective.

If R′ be the radius of the circle A′B′C′, it is evident that

$$\frac{r}{R'} = \frac{\sin\omega}{\sin(\pi-\theta+\omega)} = \frac{\sin\omega}{\sin(\theta-\omega)}, \text{ therefore } \frac{R'}{R} = \frac{\sin(\theta-\omega)}{\sin(\theta+\omega)}$$

$$= \text{ratio of similitude of A'B'C' to ABC.}$$

But $P\omega_1 / P\omega = \sin P\omega\omega_1 / \sin P\omega_1\omega = \sin(\theta-\omega)/\sin(\theta+\omega)$,

therefore $P\omega_1 / P\omega' = P\omega_1' / P\omega = R/R'$;

therefore ω_1' and ω_1 are positive and negative Brocard-points of A′B′C′. The circumcentre of A′B′C′ is a point O′ on PO such that $Q\omega_1O' = \pi-\theta$.

Therefore $QO'/\sin\theta = Q\omega_1/\sin\omega = Q\omega/\sin\omega = QO/\sin\theta$,

therefore O′, O are equidistant from Q.

If the circle DEF is the T. R. circle, the lines EF′, FD′, DE′ pass through P, so that the triangle A′B′C′ reduces to a point, therefore $\sin(\theta-\omega) = 0$ and $\theta = \omega$, and the centre of the T. R. circle is the mid-point of OP. Its radius is $\dfrac{R\sin\omega}{\sin 2\omega} = \dfrac{R}{2\cos\omega}$.

If EP, F′P meet BC in d', d, FP, D′P meet CA in e', e, DP, E′P meet AB in f', f, then, since E, d' are corresponding points of the triangles A′B′C′, ABC, therefore $PE / Pd' = R'/R$, and similarly for the others.

Therefore def, $d'e'f'$ form another pair of Tucker's triangles, the sides of which make, with the homologous sides of ABC, an angle $\pi-\theta$. If q be the centre and r' the radius of the corresponding Tucker's circle,

$$O\omega q + O\omega Q = \pi \text{ or } q\omega P = Q\omega P,$$

therefore q, Q are conjugate points with respect to the Brocard-circle of ABC.

Also $r'/r = q\omega / Q\omega = qP/QP = qO/QO = \sin(\theta-\omega)/\sin(\theta+\omega)$.

If two Tucker-circles be taken whose centres are equidistant from the Brocard-points, their radii will be equal, and therefore also the corresponding Tucker's triangles. As a particular case of this, we see that two triangles can be inscribed in ABC which are equal to it in all respects, and that the sides of these triangles make an angle $\pi-2\omega$ with those of ABC.

8784. (By R. W. D. Christie, M.A.)—Prove that, if

$$s = 1+2+3+\ldots+n,\quad S = 1^2+2^2+3^2+\ldots+n^2,\quad S' = 1^3+2^3+3^3+\ldots+n^3,$$
$$\Sigma = 1^4+2^4+3^4+\ldots+n^4,\quad \sigma = 1^5+2^5+3^5+\ldots+n^5,$$

then
$$\frac{3\sigma+2s^3}{5\Sigma} = \frac{S'}{S}.$$

Solution by H. L. Orchard, M.A., B.Sc.; Professor Beyens; *and others.*

By summing the series by the method of Undetermined Coefficients, or otherwise, we find $5\Sigma S'S^{-1} - 2s^3 = \frac{1}{4}n^2(n+1)^2[2n(n+1)-1]$; hence, we have to show that $n^2(n+1)^2[2n(n+1)-1] = 12(1^5+2^5+\ldots+n^5)$.

In the left-hand member put $n = m$ and $m+1$ successively, and subtract. We thus obtain $2(m+1)^3[(m+2)^3-m^3]-(m+1)^2[(m+2)^2-m^2]$, which reduces to $12(m+1)^5$. Hence, if the theorem holds for $n = m$, it holds for $n = m+1$, and so on, universally. But it evidently holds for $n = 1$, and for $n = 2$; therefore, &c.

8763. (By H. Stewart, M.C.P.)—Prove that

(1) if $$u(1-x^2)^{\frac12}+x(1-u^2)^{\frac12}=a^2, \quad \text{then} \quad \frac{du}{dx}+\frac{(1-u^2)^{\frac12}}{(1-x^2)^{\frac12}}=0;$$

(2) if $$(1-x^2)^{\frac12}+(1-u^2)^{\frac12}=a(x-u), \quad \text{then} \quad \frac{du}{dx}=\frac{(1-u^2)^{\frac12}}{(1-x^2)^{\frac12}}.$$

Solution by F. R. J. Hervey; W. J. Greenstreet, B.A.; *and others.*

Putting, for brevity, $(1-x^2)^{\frac12}=s$, $(1-u^2)^{\frac12}=t$, $du/dx=u'$, we have

$$(1)\quad u's-u\frac{x}{s}+t-x\frac{u}{t}u'=0,\quad u'\left(s-\frac{xu}{t}\right)=\frac{xu}{s}-t,\quad u'=-\frac{t}{s};$$

$$(2)\quad -\frac{x}{s}-\frac{u}{t}u'=a(1-u')=\frac{s+t}{x-u}(1-u'),\quad u'\left(\frac{s+t}{x-u}-\frac{u}{t}\right)=\frac{s+t}{x-u}+\frac{x}{s};$$

but $t(s+t)-u(x-u)=st+1-xu=s(s+t)+x(x-u)$, $\therefore\ u'=t/s$.

8745 & 8807. (W. W. Taylor, M.A. Suggested by Quest. 7938.)—Prove that (1) the areas of the Taylor-circles of the four triangles ABC, PBC, PCA, PAB are together equal to the area of the circumscribed circle, P being the orthocentre of the triangle ABC; also (2) the lines joining their centres are the Simson-lines of the middle points of the sides or of AP, BP, CP with respect to the pedal triangle.

Solution by R. F. Davis, M.A.; Professor Beyens; *and others.*

The four triangles of the question have all the same circum-radius; while their angles are (A, B, C), $(\pi-A,\ \tfrac12\pi-B,\ \tfrac12\pi-C)$, &c. Hence, employing the known form for the radius of the Taylor-circle

$$= R\left\{\sin^2 A\sin^2 B\sin^2 C+\cos^2 A\cos^2 B\cos^2 C\right\}^{\frac12},$$

we get sum of squares of the Taylor-radii

$$= R^2(\sin^2 A+\cos^2 A)(\sin^2 B+\cos^2 B)(\sin^2 C+\cos^2 C)=R^2.$$

The second part follows at once from the fact the Simson-line of the middle point of any side with respect to the pedal triangle passes through the Taylor-centre. In this case the four triangles have all the same peda triangle.

2934. (Professor Sylvester, F.R.S.) — If $s_1, s_2, s_3, s_4, s_5, s_6$ represent the sum of $xyztuv$, and of their binary, ternary, quaternary, quinary, and sextic combinations respectively, and if E stands for the symbol of Emanation $a\frac{d}{dx}+b\frac{d}{dy}+c\frac{d}{dz}+d\frac{d}{dt}+e\frac{d}{du}+f\frac{d}{dv}$, prove that the resultant of s_1, Es_2, s_3, Es_4, s_5, Es_6 is the product of

$$(a-b)^9(a+b-c-d)^2(a+b+c-d-e-f)^{12},$$

and of the similarly formed powers of products of the analogous linear functions of $abcdef$.

Solution by Professor Sircom, M.A.

The equations $s_1=0$, $s_3=0$, $s_5=0$ are equivalent to $\Sigma x=0$, $\Sigma x^3=0$, $\Sigma x^5=0$, whence we shall have 15 systems of solutions, such as $x+t=0$, $y+u=0$, $z+v=0$. Now, $Es_2=0$, $Es_4=0$, $Es_6=0$ are equivalent to $\Sigma ax=0$, $\Sigma ax^3=0$, $\Sigma ax^5=0$, whence, substituting for tuv and writing l, m, n for $a-d$, $b-e$, $c-f$, we have

$$lx+my+nz=0,\ lx^3+my^3+nz^3=0,\ lx^5+my^5+nz^5=0\ldots(1,2,3),$$

the resultant of which is found by multiplying together the three factors $lx_1^5+my_1^5+nz_1^5$, &c., where $y_1:x_1$, $z_1:x_1$, &c., are roots of the cubics, obtained from (1) and (2),

$$m(m^2-n^2)y^3+3lm^2y^2x+3l^2myx^2+l(l^2-n^2)x^3=0\ \ldots\ldots\ldots(4),$$
$$n(n^2-m^2)z^3+3ln^2z^2x+3l^2nzx^2+l(l^2-m^2)x^3=0\ \ldots\ldots\ldots\ldots(5).$$

Substituting from (1) and (2), we shall find

$$lx_1^5+my_1^5+nz_1^5=lx_1^5\left(\frac{y_1^2}{x_1^2}-1\right)\left(\frac{z_1^2}{x_1^2}-1\right);$$

from (4), $\quad(y_1^2/x_1^2-1)(y_2^2/x_2^2-1)(y_3^2/x_3^2-1)$

$$=(m^2-l^2)(l+m+n)(m+n-l)(l-m+n)(l+m-n)\,/\,m^2(m^2-n^2);$$

so for $z_1:x_1$, &c. Whence, observing that the resultant is of the degree $3.5+1.3+1.5=23$ in the coefficients, its expression in terms of a, b, c, &c., is

$$(a-d)^3(b-e)^3(c-f)^3(a+b-d-e)(a+d-b-e)\ldots(a+b+c-d-e-f)^2\ldots.$$

There are 15 expressions of this kind corresponding to the 15 systems of solutions, and as there are 15 factors $a-b$, the degree of each is 9 in the final resultant. Also the resultant is of the degree 6.15 in the factors $a+b-c-d$, and 6.15 terms of this kind can be formed, but half of these differ only in sign from the rest, hence each is of the degree 2. Similarly, the resultant is of the degree $8.15=120$ in the factors $a+b+c-d-e-f$, of which 10 only are distinct, hence each is of the degree 12 in the resultant. Hence the theorem.

9026. (Professor Catalan.)—Soit

$$a(a+1)(a+2)\ldots(a+c)\pm b(b+1)(b+2)\ldots(b+c)=(a+b+c)\,\phi(a,b).$$

(Le signe +, si c est pair). (1) $\phi(a,b)$ est un polynôme entier, à coefficients entiers; (2) si a, b sont remplacés par des nombres entiers, $\phi(a,b)$

devient un nombre entier ; (3) pour ces valeurs de a, b,

$$(a+b+c)\cdot\phi(a, b) = \mathfrak{M}\left[1\,.\,2\,.\,3 \ldots (c+1)\right];$$

(4) si, en outre, $a+b+c$ est premier, $\phi(a, b) = \mathfrak{M}\left[1\,.\,2\,.\,3 \ldots (c+1)\right]$.

Solution by W. J. C. Sharp, M.A.; Rev. J. G. Birch, M.A.; *and others.*

If $a+b+c=0$, we have $b = -a-c$, and

$$a(a+1)\ldots(a+c) \pm b(b+1)\ldots(b+c) = a(a+1)\ldots(a+c)\left\{1 \pm (-1)^{c+1}\right\},$$

which vanishes for the + or − sign as c is even or odd; therefore

$$a(a+1)\ldots(a+c) + (-1)^c b(b+1)\ldots(b+c) \equiv (a+b+c)\,\phi(a, b),$$

where $\phi(a, b)$ is an integral function of a and b. Also, since $(a+b+c)\,\phi(a, b)$ has integral coefficients for all values of a and b, the coefficients of $\phi(a, b)$ are integral, which proves (1) and (2).

Again, where a and b are whole numbers,

$$a(a+1)\ldots(a+c) + (-1)^c b(b+1)\ldots(b+c)$$
$$\equiv (c+1)!\left\{{}^{a+c}C_{c+1} + (-1)^c\, {}^{b+c}C_{c+1}\right\}$$

(where ${}^nC_r$ denotes the number of combinations which can be formed out of n things taken r together), which proves (3); and (4) follows at once.

8981. (Professor Schoute.)—Given two conics; find (1) the locus of the vertex of a right angle circumscribed to these curves; and (2) consider the particular case of two homo-focal conics. [The problem in its general form may be otherwise stated thus:—Find the locus of points such that one of the tangents from it to a conic (2), together with one of the tangents to a second conic (3), form with the two tangents to a third conic (1) a harmonic pencil.]

Solution by A. R. Johnson, M.A.

The condition that the two lines (1′), together with one of the lines (2′), and one of the lines (3′),

$$a_1x^2 + 2b_1xy + c_1y^2 = 0, \quad a_2x^2 + 2b_2xy + c_2y^2 = 0, \quad a_3x^2 + 2b_3xy + c_3y^2 = 0, \quad \ldots\ldots\ (1', 2', 3'),$$

may form a harmonic pencil, is

$$0 = h_{11}^2\left\{h_{23}^2 - 4h_{22}h_{33}\right\} - 2h_{11}h_{12}h_{13}h_{23} + h_{12}^2h_{13},$$

where $\quad h_{11} = a_1c_1 - b_1^2, \quad h_{11} = a_1c_2 + a_2c_1 - 2b_1b_2$; etc.

Now the equations to the tangents from a point to (1), (2), (3) may be thrown into the forms (1′), (2′), (3′) respectively, and from the significations of h_{11}, h_{12}, etc. it is seen at once that, in terms of the coordinates of the point, $h_{11} = s_{11}$, $h_{12} = s_{12}$, etc.; where s_{11} denotes $(B_1C_1 - F_1^2)\,x^2 + \&c.$; $s_{12} \equiv (B_1C_2 + B_2C_1 - 2F_1F_2)\,x^2 + \&c.$ in Salmon's notation; the equation to the locus is therefore $\quad 0 = s_{11}^2\left\{s_{23}^2 - 4s_{22}s_{33}\right\} - 2s_{11}s_{12}s_{13}s_{23} + s_{12}^2s_{13}^2 \ \ldots\ldots$ (A), representing an octavic curve. It is seen, from (A), that the conics $s_{12} = 0$, $s_{13} = 0$, as is geometrically evident, touch the curve where they meet (1), and intersect it also where they meet the common tangents to (2) and (3), the equation to these last being $s_{23}^2 - 4s_{22}s_{33} = 0$. The same is true of the quartic $$s_{12}s_{13} - 2s_{11}s_{23} = 0.$$

Also, from the form of (A), it is seen that (1) touches the curve where it meets $s_{12} = 0$, $s_{13} = 0$. Hence it is seen that the octavic has eight double points all lying on (1), which also touches a tangent at each double point. The other tangents at the double points form two groups which touch, respectively, $s_{12} = 0$, $s_{13} = 0$ at the corresponding double points.

Again, we may throw the equation (A) into the form

$$(s_{11}s_{23} - s_{12}s_{13})^2 - 4s^2\, s_{22}s_{33} = 0,$$

showing that the conics (2) and (3) touch the octavic at their eight points of intersection with the quartic $s_{11}s_{23} - s_{12}s_{13} = 0$.

If (1), (2), and (3) have common tangents, these must form part of the locus; for at each point of them three out of the six tangents to the conics coincide. Dividing out by the equation to these common tangents, the degree of the curve locus becomes correspondingly depressed.

Suppose, now, that (1) is inscribed to the circumscribing quadrilateral of (2) and (3), so that ($\Sigma = 0$ denoting a tangential equation)

$$\Sigma_1 = k_2\Sigma_2 + k_3\Sigma_3, \text{ therefore } s_{11} = k_2^2 s_{22} + k_2k_3s_{23} + k_3^2 s_{33},$$

$$s_{12} = \frac{\delta s_{11}}{\delta k_2}, \quad s_{13} = \frac{\delta s_{11}}{\delta k_3},*$$

and the equation (A) becomes

$$0 = \left\{s_{11}^2 - 2k_2^3k_3s_{23}s_{22} - 2k_2k_3^3s_{23}s_{33} - k_2^2k_3^2(s_{23}^2 + 4s_{22}s_{33})\right\}\left\{s_{23}^2 - 4s_{22}s_{33}\right\},$$

i.e., $$0 = \left\{k_2^2s_{22} - k_3^2s_{33}\right\}^2\left\{s_{23}^2 - 4s_{22}s_{33}\right\},$$

representing the four common tangents and the twice repeated conic

$$k_2^2s_{22} - k_3^2s_{33} = 0 \quad \text{.............................} \quad \text{(B)}$$

passing through the intersections of (2) and (3), as is otherwise geometrically evident, since the tangents to two conics at a point of intersection are conjugate with respect to any conic inscribed to the four common tangents.

Since $\Sigma_3 = \frac{1}{k_3}\Sigma_1 - \frac{k_2}{k_3}\Sigma_2$, therefore $k_3^2s_{33} = s_{11} - k_2s_{12} + k_2^2s_{22}$,

and (B) may be written $k_2s_{12} - s_{11} = 0$, or $k_3s_{13} - s_{11} = 0$, so that (B) cuts (1) in the four points of contact of (1) with the four common tangents.

Now, let Σ_1 consist of two points, then $s_{11} = 0$ is the twice repeated joining line, and $s_{12} = 0$, $s_{13} = 0$ will have double contact with one another at the points. If the points be the circular points at infinity, then $s_{12} = 0$, $s_{13} = 0$ are the director circles of (2) and (3), and s_{11} is the square of the line at infinity; thus leading to the well-known result for the orthoptic locus of confocal conics.

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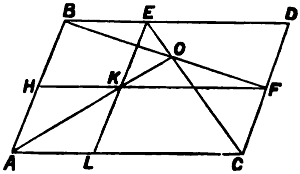
* Hence $2s_{11} = k_2s_{12} + k_3s_{13}$; *i.e.*, if (1), (2), (3) be any conics inscribed to the same quadrilateral, then the director conics of (1), (2); (1), (3) intersect in four points on (1). It seems natural to call $s_{12} = 0$ the director conic of (1) and (2). The four points are the points of contact of (1) with the four common tangents.

9020. (F. MORLEY, B.A.)—ABDC is a parallelogram; O is any point on the line bisecting the angle A; CO, BO meet BD, CD in E, F; prove that BE = CF.

Solution by Rev. D. THOMAS, M.A.; Prof. BEYENS; *and others.*



Let α, β be unit-vectors along AC, AB respectively, and AC $= a$, AB $= b$.

So that AO $= m(\alpha+\beta)$,

AE $= b\beta + x\alpha = (1-y)\,a\alpha + my(\alpha+\beta)$;

therefore $b = my$,

$$x = a(1-y)+my$$

$$= b + a\left(1-\frac{b}{m}\right) = a+b-\frac{ab}{m} = \text{BE}.$$

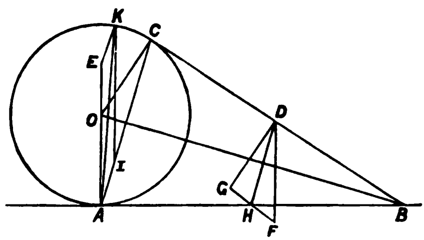
Similarly, $$\text{CF} = a+b-\frac{ab}{m}.$$

[If we draw EKL parallel to AB, and join KF, we have, from the rhombus HL, BE = AL = AH; but, clearly, LF is a parallelogram, hence CF = AH = BE.]

8985. (Professor BYOMAKESA CHAKRAVARTI, M.A.)—A cylinder, weight W, radius r, is placed on a rough horizontal plane; a uniform plank, weight P, length 2Q, is inclined at an angle θ to the horizon, and rests with one end on the ground, the other *on* the cylinder (the plank being at right angles to the axis of the cylinder); if ψ be the angle made with the vertical by the re-action of the ground on the cylinder, prove

that $$\cot\psi = \cot\frac{\theta}{2} + \frac{r}{a}\cdot\frac{\text{W}}{\text{P}}\cdot\sec\theta.$$

Solution by D. BIDDLE.



Let A be the point of contact with the ground of a section-area of the cyclinder, in the same plane as D the mid-point of the plank, as B the point of contact of the plank with the ground, and as C the point of contact of the plank with the cylinder. Also, let AE = W, and DF = P. O being the centre of the section-area of the cylinder, join CO and CA, and draw DG, GF, DH parallel to CO, CB, CA respectively, and cut off AI = $\frac{1}{2}$DH; draw IK parallel and equal to AE, and join EK (parallel to AC and equal to AI), also join AK. Then $\angle$OAK = ψ.

Now, in the diagram, the plank is tangential to the cylinder. In this case, $\cot\psi = (\text{EK}\cos\frac{1}{2}\theta + \text{W})/(\text{EK}\sin\frac{1}{2}\theta) = \cot\frac{1}{2}\theta + \text{W}/(\text{EK}\sin\frac{1}{2}\theta)$

$$= \cot\tfrac{1}{2}\theta + \text{W}/(\tfrac{1}{2}\text{DH}\sin\tfrac{1}{2}\theta) = \cot\tfrac{1}{2}\theta + \text{W}/(\tfrac{1}{2}\text{P}\cos\theta\sec\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta)$$

$$= \cot\tfrac{1}{2}\theta + \text{W}\sec\theta/(\tfrac{1}{2}\text{P}\tan\tfrac{1}{2}\theta),$$

which, by taking $\frac{1}{2}r\tan\frac{1}{2}\theta = a$, reduces to $\cot\frac{1}{2}\theta + \dfrac{r}{a}\cdot\dfrac{\text{W}}{\text{P}}\cdot\sec\theta$.

But the plank can rest (*on* the cylinder) lower than C, say C′. Let $\angle\text{OAC}' = \phi$. Then

$$\cot\psi = (\text{EK}'\cos\phi + \text{W})/(\text{EK}'\sin\phi) = \cot\phi + \text{W}/(\text{EK}'\sin\phi)$$

$$= \cot\phi + \text{W}/\{\tfrac{1}{2}\text{P}\cos\theta\sec(\theta-\phi)\sin\phi\}$$

$$= \cot\phi + \frac{2\text{W}}{\text{P}}(\cot\phi + \tan\theta).$$

But, in this case,

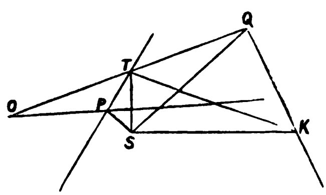
$$a = r\sec\theta \Big/ \left\{2(\cot\phi + \tan\theta) + \frac{\text{P}}{\text{W}}(\cot\phi - \cot\tfrac{1}{2}\theta)\right\}$$

$$= r \Big/ \left\{2\sin\theta + 2\cos\theta\left[\left(\frac{\text{Q}\sin\theta}{r-\text{Q}\sin\theta}\right)^{\frac{1}{2}} + \frac{\text{P}}{2\text{W}}\left\{\left(\frac{\text{Q}\sin\theta}{r-\text{Q}\sin\theta}\right)^{\frac{1}{2}} - \frac{1+\cos\theta}{\sin\theta}\right\}\right]\right\},$$

which fits neither $\frac{1}{2}r\tan\text{OAC}$, nor $\frac{1}{2}r\tan\text{OAC}'$, nor $\frac{1}{2}r\tan\frac{1}{2}\theta$.

8847. (W. Mills, B.A.)—Given the focus of a parabola, one tangent, and a point on it through which another tangent passes; prove that the locus of the point of intersection of the variable tangent with a diameter through the point of contact of the fixed tangent is a circle which touches the fixed tangent at the given point on it.

Solution by C. E. Williams, M.A.; Prof. Sarkar, M.A.; *and others.*



Let S be the focus, PT the fixed tangent touching at P and T the fixed point in it, through which passes the variable tangent OTQ touching the parabola at Q and meeting the diameter through P at O. Let TK, QK at right angles to PT, TQ meet at K; then STP = SQT, therefore STK = SQK; therefore S, T, Q, K are concyclic and TSK a right angle; but, ST being constant in position and magnitude, therefore TK is, and the locus of Q is a circle touching PT at T. And since TO = TQ, therefore the locus of O is another equal circle touching PT at T on the other side.

9011. (R. Lachlan, B.A.) — Show that the product o the three normals drawn from any point on a conic is equal to the product of the perpendiculars from the point on the asymptotes and the diameter of curvature at the point.

Solution by A. R. Johnson, M.A.; Prof. Matz, M.A.; *and others.*

This is one of a numerous class of deductions that may be drawn from the theorem that the product of the $m+n$ normals drawn to a curve of order m and class n is equal to the product of the n tangents and m perpendiculars on the asymptotes from the point.

Let the point approach the curve; two tangents and one normal become evanescent, and the product of the tangents is equal in the limit to the product of the normal and the diameter of curvature; thus, then, the product of the $m+n-1$ normals to a curve from any point on it is equal to the continued product of the $n-2$ tangents, the m perpendiculars on the asymptotes, and the diameter of curvature at the point.

Since the product of the perpendiculars on the asymptotes of any conic from a point on it is constant, the result might be stated,—The product of the three normals to a central conic from any point on it varies inversely as the curvature at the point. Analogous is the theorem,—The product of the four normals to a central conic from any point on a similar and similarly situated conic varies as the product of the tangents from the same point.

In the cubic the product of the perpendiculars on the asymptotes from any point on the curve varies as the perpendicular on the satellite of the line at infinity; and generally the product of the perpendiculars varies as the quotient of the normals to the $(m-2)$-ic through the $m(m-2)$ points of intersection of the asymptotes with the curve, divided by the product of the distances of the point from the real foci of the curve.

[By the aid of these considerations many new theorems may be obtained.]

8729. (R. W. D. Christie, M.A.)—Of the series

$$1.1+3.5+5.13+7.25+\&c.$$

to n terms, show that, (1) the sum is a square; (2) the square root is $=1+3+5+7+\&c.$ to n terms; (3) each term is the product of two numbers which may be taken to represent the shorter side and hypothenuse of a right-angled triangle; (4) if unity be added to the nth term, it is divisible by $2n$, and if unity be subtracted it is divisible by $n-1$; and (5) all terms are odd numbers.

Solution by G. G. Storr, M.A.; Rev. G. H. Hopkins, M.A.; *and others.*

The nth term is $(2n-1)(2n^2-2n+1)$, whence obviously (4) and (5). This term $=n^4-(n-1)^4$, whence (1) and (2). Also (3) is the interpretation of the fact that $(2n^2-2n+1)^2-(2n-1)^2 \equiv \{2n(n-1)\}^2$ = perfect square.

8380. (P. C. WARD, M.A.)—Prove that (1),

$$\begin{vmatrix} a+b-c, & 4a, & 6a, & 4a \\ 4b, & a+b-c, & 4a, & 6a \\ 6b, & 4b, & a+b-c, & 4a \\ 4b, & 6b, & 4b, & -a+b-c \end{vmatrix}$$

$$\equiv (a+b+c)^4 - 8(a+b+c)^2(bc+ca+ab) - 128abc(a+b+c) + 16(bc+ca+ab)^2$$

= result of rationalizing $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}} = 0$; and hence (2) the above determinant is symmetrical with respect to a, b, c.

Solution by R. F. DAVIS, M.A.; SATIS CHANDRA BASU; *and others.*

Let $b = \lambda^4 . a$; then $c = a(1+\lambda)^4$, or $a+b-c+4a\lambda+6a\lambda^2+4a\lambda^3 = 0$. Multiplying successively by λ, λ^2, λ^3, and putting for λ its value b/a, four linear equations in the variables λ, λ^2, λ^3 are obtained. Eliminating these variables, determinant required = 0.

Or thus; if $\Sigma_1 = a+b+c$, $\Sigma_2 = bc+ca+ab$, $\Sigma_3 = abc$,

$$\Sigma_1 = a+b+c = 2b^{\frac{1}{2}}c^{\frac{1}{2}} + \ldots + \ldots, \quad \Sigma_1^2 = 4\left\{bc+ca+ab+2a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}\right)\right\},$$

$$(\Sigma_1^2 - 4\Sigma_2)^2 = 64\Sigma_3(a+b+c+2b^{\frac{1}{2}}c^{\frac{1}{2}}+\ldots+\ldots) = 64\Sigma_3(\Sigma_1+\Sigma_1) = 128\Sigma_1\Sigma_3.$$

8714. (Professor GENESE, M.A.) — S is a focus of a conic, PN a fixed ordinate to the diameter through S, PQP′ a circle with centre S; a variable radius SQ meets PN at L and the conic at R. Prove that the cross ratio $\{\dot{S}L\dot{Q}R\}$ is constant.

Solution by C. E. WILLIAMS, M.A.; SARAH MARKS, B.Sc.; *and others.*

Let $\qquad SP = K = SQ, \; QSC = \phi,$

then $\qquad SL = (K-l)/(e\cos\phi), \; SR = l/(1-e\cos\phi),$

and the ratio SL . QR : SQ . LR reduces by substitution to $(K-l) : K$.

8676. (Professor BORDAGE.)—The three sides of a triangle forming an arithmetical progression, (a) being the shortest, (a') the longest; if the distance of the centres of the inscribed and circumscribed circles is designated by i, and the diameter of the nine-point circle by D, prove that $aa' = 3(D^2 - i^2)$.

Solution by R. F. DAVIS, M.A.; R. KNOWLES, B.A.; *and others.*

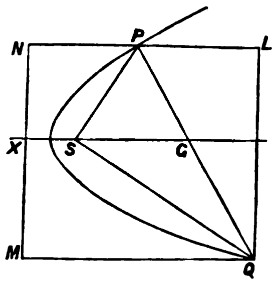
Let b be the mean side, so that $2b = a+a'$, $2s = 3b$, $2aa's = 3aba'$.

Then $\qquad 2Rr = 2(aba'/4S)(S/s) = aba'/2s = aa'/3;$

but $\qquad i^2 = R^2 - 2Rr = D^2 - aa'/3$; therefore, &c.

8876 & 9032. (Professor Hudson, M.A.)—Prove that the normal chord which subtends a right angle at the focus of a parabola is divided by the axis in the ratio 2 : 3.

Solution by G. G. Storr, M.A.; D. E. Shorto, M.A.; *and others.*



Draw PQ a normal to a parabola at P, and QL perpendicular to the diameter through P; then

$$SP = PL = \tfrac{1}{2}NL = \tfrac{1}{2}SQ;$$

therefore, since $SG = SP$, we have

$$QG \cdot QP = SQ^2 - SP^2 = 3SP^2,$$
$$QP^2 = SQ^2 + SP^2 = 5SP^2;$$

therefore $QG : QP = 3 : 5$,

and $PG : GQ = 2 : 3$.

8737. (W. J. Greenstreet, B.A.)—ABC is a spherical triangle, the mid-points of its sides are the angular points of a triangle DEF; prove that (1) $\cos EF / \cos \tfrac{1}{2}a = \cos FD / \cos \tfrac{1}{2}b = \cos DE / \cos \tfrac{1}{2}c$; and (2) if $\cos^2 \tfrac{1}{2}a = \cos \tfrac{1}{2}(b+c) \cos \tfrac{1}{2}(b-c)$, the angle D is a right angle.

Solution by Rev. T. R. Terry, M.A.; A. Gordon; *and others.*

1. $$\cos EF = \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \,[\cos c - \cos a \cos b] \operatorname{cosec} b \operatorname{cosec} c$$
$$= (\cos^2 \tfrac{1}{2}a + \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c - 1) / (2 \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c)$$

whence the first result.

2. Also D is a right angle if $\cos EF = \cos ED \cos DF$; that is, if
$$\cos^2 \tfrac{1}{2}a = \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c - 1 = \cos^2 \tfrac{1}{2}b \cos^2 \tfrac{1}{2}c - \sin^2 \tfrac{1}{2}b \sin^2 \tfrac{1}{2}c$$
$$= \cos \tfrac{1}{2}(b+c) \cos \tfrac{1}{2}(b-c).$$

8998. (Rev. T. R. Terry, M.A.)—Solve the equation

$$\frac{d^3y}{dx^3} - 6x\frac{d^2y}{dx^2} + (12x^2 - 6) - \frac{dy}{dx} 4(2x^2 - 3)\,xy = 0.$$

Solution by D. Edwardes; G. G. Storr, M.A.; *and others.*

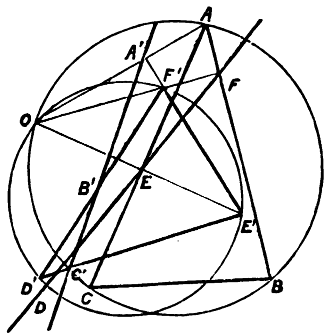
If $\pi = \dfrac{d}{dx} - 2x$, the equation becomes $\pi^3 y = 0$.

And $\pi^{-1} 0 = Ce^{x^2}$, $\pi^{-1} Ce^{x^2} = e^{x^2}(Cx + C_2)$.

Therefore, finally, $y = e^{x^2}(C_1 x^2 + C_2 x + C_3)$.

9002. (E. M. Langley, M.A.)—If a circle and a Simson-line of one of its points be both inverted with regard to that point, the two inverses will have the same relation to each and to the given point that the originals have.

Solution by the Proposer.



Let O be the point, OD, OE, OF the perpendiculars from O to the sides BC, CA, AB of a triangle inscribed in the circle; and let A′, B′, C′, D′, E′, F′ be inverses of A, B, C, D, E, F. Then, since E, F, A lie on a circle through O, therefore E′, F′, A′ lie in a straight line; and since A, A′, F, F′ are concyclic, therefore $\angle$ OA′F′ = OFA, therefore OA′ is perpendicular to E′F′. Similarly OB′ is perpendicular to F′D′, and OC′ to D′E′. But, since D, E, F are in a straight line, therefore D′, E′, F′ lie on a circle through O, therefore A′, B′, C′ lie on a Simson-line of circle D′E′F′.

8912. (Professor Bordage.)—A square ABCD and a straight line Δ in its plane are given, and through the points A, B, C, D perpendiculars AA′, BB′, CC′, DD′ are drawn on Δ, A and C being two opposite summits; prove that (1) $(BB')^2+(DD')^2-2AA'.BB'$ is a constant quantity for every position of Δ; and (2) deduce therefrom an application for the envelope of the straight lines, such that the sum of the squares of the distances of one of them from two given points is constant.

Solution by C. E. Williams, M.A.; Prof. Byomakesa Chakravarti, M.A.; *and others.*

1. This is proved in Casey's *Sequel to Euclid*, Book II., Prop. 8, the constant being the area of the square.

2. If $(BB')^2+(DD')^2$ is constant, then AA′. CC′ is constant, and the line envelopes a conic whose foci are A and C.

8979. (Professor Brunel.)—Soient ABC un triangle, $A_1B_1C_1$ un autre triangle déduit du premier en menant par les sommets A, B, C des droites faisant avec les côtés du triangle et dans le même sens un angle ϕ. Du triangle $A_1B_1C_1$ l'on déduit, de même, un triangle $A_2B_2C_2$ et ainsi de suite, toujours avec le même angle ϕ. Démontrer que les points A, A_1, A_2 ... sont sur trois groupes de spirales logarithmiques ayant pour pôles les points de Brocard. Pour quelles valeurs de ϕ le triangle dérivé $A_1B_1C_1$ est-il égal au triangle proposé ?

Solution by Profs. NASH, SCHOUTE, *and others.*

The construction shows that ABC is a Tucker triangle of $A_1B_1C_1$, $A_1B_1C_1$ a Tucker triangle of the same species of $A_2B_2C_2$, and so on. Hence the triangles have one or other of the Brocard points as a common centre of similitude, and $\omega A/\omega A_1 = \omega A_1 / \omega A_2 = \&c. = \sin(\omega+\phi)/\sin\omega$, where ω is the Brocard angle. Also the angles $A\omega A_1$, $A_1\omega A_2$, &c. are all equal to ϕ. Therefore A, A_1, A_2, &c. all lie upon an equiangular spiral whose pole is ω. ω will be the positive or negative Brocard point according as the line through A makes an angle ϕ with AB or AC. The triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$, &c. will all be equal if $\sin(\omega+\phi) = \sin\omega$, *i.e.*, if $\phi = \pi - 2\omega$.

8830. (Professor BORDAGE.)—Given the equation

$$\sin 2a \,.\, x^2 + 2(\sin a + \cos a)\, x + 2 = 0,$$

find, and solve, the equation of the *second degree* that has as roots the squares of the roots of the given equation.

Solution by W. J. BARTON, M.A.; W. J. GREENSTREET, B.A.; *and others.*

If α, β are the roots $\alpha+\beta = -\dfrac{2(\sin a+\cos a)}{\sin 2a}$, $\alpha\beta = \dfrac{2}{\sin 2a}$,

whence $\alpha^2+\beta^2 = \dfrac{4}{\sin^2 2a}$, $\alpha^2\beta^2 = \dfrac{4}{\sin^2 2a}$,

and equation required is $\sin^2 2a\, x^2 - 4x + 4 = 0$;

of which the roots are $\dfrac{2(1\pm\cos 2a)}{\sin^2 2a}$, or $\dfrac{1}{\sin^2 a}$, $\dfrac{1}{\cos^2 a}$;

as might have been expected, since the roots of original equation are $-1/\sin a$, $-1/\cos a$.

8927. (S. TEBAY, B.A.)—If Δ_1, Δ_2, Δ_3, Δ_4 be the areas of the faces of a tetrahedron, R the radius of the circumscribing sphere, and R_1, R_2, R_3, R_4 the radii of spheres passing through the centre of R and the angles of Δ_1, Δ_2, Δ_3, Δ_4; the volume

$$V = \tfrac{1}{3}R^3\left(\frac{\Delta_1}{R_1}+\frac{\Delta_2}{R_2}+\frac{\Delta_3}{R_3}+\frac{\Delta_4}{R_4}\right).$$

Solution by W. J. C. Sharp, M.A.; Professor Beyens, M.A.; *and others.*

If O, O_1, O_2, O_3, O_4 be the centres of the spheres whose radii are R, R_1, R_2, R_3, R_4, OO_1 cuts the plane of the triangle BCD, area Δ_1, in the centre of the circum-circle K_1; hence

$$OK_1 + O_1K_1 = R_1,\quad OK_1^2 + CK_1^2 = R^2,\quad O_1K_1^2 + CK_1^2 = R_1^2,$$

therefore $$(OK_1 + O_1K_1)(OK_1 - O_1K_1) = R^2 - R_1^2,$$

therefore $$OK_1 - O_1K_1 = \frac{R^2}{R_1} - R_1,\ \text{therefore } OK_1 = \frac{R^2}{2R_1},$$

$\therefore$ the tetrahedron $OBCD = \frac{1}{3}OK_1 . \Delta_1 = \frac{1}{6}R^2\frac{\Delta_1}{R_1}$, $OCDA = \frac{1}{6}R^2\frac{\Delta_2}{R_2}$, &c.,

and the volume $$= \tfrac{1}{6}R^2\left\{\frac{\Delta_1}{R_1} + \frac{\Delta_2}{R_2} + \frac{\Delta_3}{R_3} + \frac{\Delta_4}{R_4}\right\}.$$

[A corresponding formula holds for all simplicissima. (See Quest. 8242.) Thus, if R_1, R_2, R_3 be the radii of the circles through two vertices of a triangle, and the centre of the circumcircle (of radius R), we have

$$\text{Area} = \tfrac{1}{4}R^2\,(aR_1^{-1} + bR_2^{-1} + cR_3^{-1});$$

and generally, if V be the content of a simplicissima in space of n dimensions, R_1, R_2, &c. the radii of the hyper-spheres through all the vertices but one and through the centre of the circumscribed hyper-sphere (of radius R), and V_1, V_2, &c., the contents of the simplicissima (in space of $n-1$ dimensions) whose vertices coincide with all but one of those of V,

we shall have $$V = \frac{R^2}{n\,!}\,\Sigma\,\frac{V_1}{R_1}.]$$

8543. (R. Curtis, M.A.)—Prove (1) the following formula of transformation for the equation of a conic to a triangle of reference the sides of which are the polars of the vertices of the former triangle $(a, b, c, f, g, h)(x, y, z)^2 = \Delta^{-1}(A, B, C, F, G, H)(X, Y, Z)^2$; and hence (2) show that the equations of a conic referred to an inscribed triangle and to one circumscribed at the points of contact will be

$$fyz + gzx + hxy = 0 \text{ and } (fX)^{\frac{1}{2}} \pm (gY)^{\frac{1}{2}} \pm (hZ)^{\frac{1}{2}} = 0.$$

Solution by the Proposer; Professor Beyens; *and others.*

$$ax + hy + gz = X,\quad hx + by + fz = Y,\quad gx + fy + cz = Z,$$

therefore $$\Delta x = AX + HY + GZ,\quad \Delta y = HX + BY + FZ,$$
$$\Delta z = GX + FY + CZ;$$

therefore $$xX + yY + zZ \equiv (a, b, c, f, g, h)(x, y, z)^2,$$

and $$\Delta\,(xX + yY + zZ) \equiv (A, B, C, F, G, H)(X, Y, Z)^2$$

is $$(a, b, c, f, g, h)(x, y, z)^2 \equiv \frac{1}{\Delta}(A, B, C, F, G, H)(X, Y, Z)^2.$$

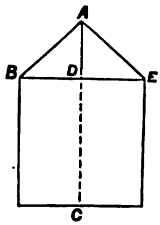
Again, if $a=0$, $b=0$, $c=0$, then $A=-f^2$, $B=-g^2$, $C=-h^2$, $F=gh$, &c.; therefore, by above, $2fyz+2gzx+2hxy=0$ becomes

$$\frac{1}{2fgh}\left(-f^2X^2-g^2Y^2-h^2Z^2+2ghYZ+2hfZX+2fgXY\right)=0,$$

or $$(fX)^{\frac{1}{2}}+(gY)^{\frac{1}{2}}+(hZ)\ =0.$$

8994. (Charlotte A. Scott, B.Sc.)—A rectangular sheet of stiff paper, whose length is to its breadth as $\sqrt{2}$ is to 1, lies on a horizontal table with its longer sides perpendicular to the edge and projecting over it. The corners on the table are then doubled over symmetrically so that the creases pass through the middle point of the side joining the corners, and make angles of 45° with it. The paper is then on the point of falling over; show that it had originally $\frac{25}{48}$ of its length on the table.

Solution by C. E. Williams, M.A.; W. J. Greenstreet, B.A.; *and others.*



Let $\quad BD=AD=DE=a,\ \ AC=2\sqrt{2}a.$

Taking moments about A, we have

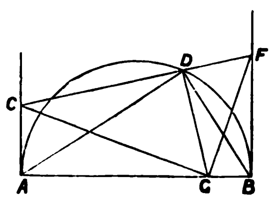
$$4\sqrt{2}\,a^2 \,.\, x = 2a^2 \,.\, \tfrac{2}{3}a + 2a\,(2\sqrt{2}-1)\,a \,.\, [a+(\sqrt{2}-\tfrac{1}{2})\,a]$$

$$=\frac{25}{3}\,a^3,\quad x=\frac{25}{12\sqrt{2}}\,a=\frac{25}{48}\text{ of AC.}$$

8856. (B. Hanumanta Rau, B.A.)—AB is the diameter of a semicircle; AC and BF tangents. CF cuts the semicircle in D. DG is drawn at right angles to CF, meeting AB in G. Prove that

$$AC\,.\,BF = AG\,.\,GB.$$

Solution by A. W. Cave, M.A.; W. J. Greenstreet, B A.; *and others.*



Circles can be drawn round the figures ACDG, DGBF,

therefore $\angle ACG = \angle ADG = 90° - \angle GDB$
$= 90° - \angle GFB = \angle FGB,$

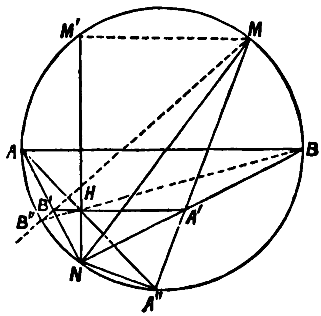
therefore triangles CAG and BFG are equiangular; therefore

$$AC : AG = GB : BF;$$

therefore $\ AC\,.\,BF = AG\,.\,GB.$

9060. (MAURICE D'OCAGNE.)—AB et MN étant deux diamètres d'un même cercle, si une parallèle quelconque à AB coupe la corde NA en B′, la corde NB en A′, et que les droites MA′, MB′ rencontrent le cercle O respectivement aux points A″ et B″, les droites AA″, BB″ se coupent au pied H de la perpendiculaire abaissée du point N sur la droite A′B′.

Solution by Rev. D. THOMAS, M.A.; J. H. STOOPS, M.A.; *and others.*



Let the figure be drawn, and join AH, HA″,

$$NHA'' = NAA''$$
$$= \tfrac{1}{2}(NA'' + MB)$$
$$= \tfrac{1}{2}(NA'' + AM'),$$
$$ANH = \tfrac{1}{2}(AM'),$$

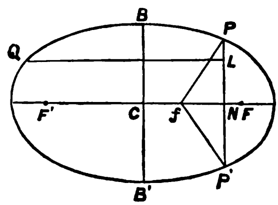
therefore

$$NHA'' - ANH = \tfrac{1}{2}(NA''),$$

which shows that AHA″ are collinear, similarly, BHB″ are collinear.

8905. (Professor NEUBERG.)—Démontrer qu'à tout point f de la distance des foyers F, F′ d'une ellipse E correspond une droite D extérieure au plan de E et parallèle au petit axe, telle qu'il existe le rapport constant $c : a$ entre les distances d'un point quelconque de E à f et D. Lorsque f occupe toutes les positions sur FF′, D se déplace sur un cylindre qui a pour base une ellipse semblable à E.

Solution by R. F. DAVIS, M.A.; C. E. WILLIAMS, M.A.; *and others.*



Let PNP′ be a double ordinate of the ellipse, the normals at whose extremities pass through f; x, y the coordinates of P. With centre f and radius fP, describe a circle having double contact with E. Then, if Q be any other point on E, the tangent from Q to the circle = $e \times$ perpendicular QL on PP′, or

$$Qf^2 - Pf^2 = e^2 . QL^2, \quad Qf^2 = e^2 (QL^2 + Pf^2/e^2).$$

If, therefore, the line D be parallel to PP′ (or BB′) at a height z above the plane of E = Pf/e, which is of course independent of Q, we shall have $Qf = e \times$ perpendicular from Q on D.

Since $z^2 = Pf^2/e^2 = (b^2/a^2e^2)(a^2 - e^2x^2)$, we have $x^2/a^2 + z^2/b^2 = 1/e^2$ as the equation of the trace of the cylinder (swept out by D) upon the plane through the major axis of E perpendicular to E. This is obviously a similar ellipse passing through the feet of the directrices of E.

The above semi-geometrical method is only applicable when f lies between the limits $x = \pm ae^2$ and the circle is real; analysis would, however, not require any distinction to be made between the cases when the circle was real and when imaginary.

When f lies between $x = ae^2$ and $x = ae$, although the circle is imaginary, the chord of contact is nevertheless real, lying outside the ellipse.

The focus F itself may be regarded as a point-circle having imaginary double contact with the ellipse, the chord of contact being the directrix. In this limiting case, when f reaches F the line D lies in the plane of E, and in fact coincides with the F directrix.

[Let (x, y) be the point P, $(k, 0)$ the point f, and $(x = \text{X}, z = \text{Z})$ the line D: then $\text{P}f^2 = (x-k)^2 + y^2 = a^{-2}c^2x^2 - 2xk + k^2 + b^2$ and (distance from P to D$)^2 = (x-\text{X})^2 + \text{Z}^2$; hence, comparing coefficients, Pf : distance from P to D $= c : a$, if $\text{X} = a^2c^{-2}k$; $\text{X}^2 + \text{Z}^2 = a^2c^{-2}(k^2+b^2)$; and, eliminating k, we get, for the base of the cylinder described by D, $a^{-2}\text{X}^2 + b^{-2}\text{Z}^2 = a^2c^{-2}$.]

8521. (Professor Wolstenholme, M.A., Sc.D.) — The circle of curvature is drawn at a point P $(am^2, 2am)$ of the parabola $y^2 = 4ax$, RR′ is the common tangent of the parabola and circle, and meets PQ their common chord in T; prove that

$$(1)\ \text{TR} : \text{TR}' = 1 + 4m^2 : 1; \quad (2)\ \text{RR}' = 16am^3(1+m^2)^{\frac{3}{2}} = \text{PQ}^3/32a^2;$$
$$(3)\ \text{TQ} : \text{TR}' = \text{TR}' : \text{TP} = m^2 : 1 + m^2;$$

and (4) the locus of T is a sextic having no real rectilinear asymptotes.

Solution by R. Knowles, B.A.; Rev. T. Galliers, M.A.; *and others.*

The equation to the circle of curvature, centre O, is

$$x^2 + y^2 - 2a(2+3m^2)x + 4am^3y - 3a^2m^4 = 0;$$

putting $x = y^2/4a$, this reduces to $(y-2am)^3(y+6am) = 0$; hence the coordinates of Q are $9am^2, -6am$; therefore $\text{PQ} = 8am(1+m^2)^{\frac{1}{2}}$ ...... (1),

and its equation is $\qquad x + my - 3am^2 = 0$ .... (2).

The equation to the tangent to the parabola at R′$(x'y')$ is

$$y'y - 2a(x+x') = 0,$$

and, since this is also a tangent to the circle, we have

$$a(2+3m^2) + m^3y' - (1+m^2)(4ax' + 4a^2)^{\frac{1}{2}} = 0;$$

putting $x' = y'^2/4a$, this reduces to $(y'-2am)^3\left[y' + 2am(4m^2+3)\right] = 0$;

therefore the coordinates of R′ are $am^2(4m^2+3)^2, -2am(4m^2+3)$, hence

$$\text{RR}' = (\text{OR}'^2 - \text{OR}^2)^{\frac{1}{2}} = 16am^3(1+m^2)^{\frac{3}{2}} \text{ .... (3)},$$

and its equation is $x + m(4m^2+3)y + am^2(4m^2+3)^2 = 0$ .... (4).

From (2), (4) we obtain $x = am^2(2m^2+3)(4m^2+3)/(2m^2+1)$,

$$y = -2am(4m^4+6m^2+3)/(2m^2+1),$$

coordinates of T. Whence we have

$$TP = 8am\,(1+m^2)^{\frac{3}{2}}/(2m^2+1), \quad TQ = 8am^5\,(1+m^2)^{\frac{1}{2}}/(2m^2+1) \ldots (5, 6).$$

$$TR' = 8am^3\,(1+m^2)^{\frac{3}{2}}/(2m^2+1), \ldots\ldots (7).$$

$$TR = RR' - TR' = 8am^3\,(1+m^2)^{\frac{3}{2}}(1+4m^2)/(2m^2+1) \ldots\ldots (8),$$

From (1), (3), (5), (6), (7), (8) we obtain the results in parts (1), (2), (3) of the question.

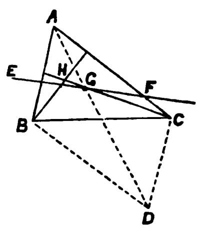
(4) The locus of T is obtained by eliminating m between equations (2) and (4), and we get, after some labour, the sextic

$$8y^6 + 162a^2y^4 + 729a^4y^2 + 72axy^4 + 16x^2y^4 + 108a^2x^2y^2 + 324a^3xy^2 + 48ax^3y^2 - 2916a^5x - 3888a^4x^2 - 432a^3x^3 - 576a^2x^4 - 64ax^5 = 0,$$

whose rectilinear asymptotes are given by the equations $y = \infty$, $y = \pm(-2)^{\frac{1}{2}}x + B$, and are therefore not real.

8718. (Professor NEUBERG.)—L'angle BAC d'un triangle est fixe, et (1) le centre de gravité, (2) l'orthocentre, (3) le centre du cercle circonscrit parcourt une droite donnée. Trouver l'enveloppe du côté BC.

Solution by the PROPOSER.



1° Lorsque le centre de gravité G parcourt la droite EF, le sommet du parallélogramme CABD décrit une droite E′F′ parallèle à EF. Par conséquent, BC enveloppe une parabole tangente à AB, AC aux points de rencontre de ces droites avec E′F′.

2° Lorsque l'orthocentre H parcourt EF les hauteurs BH, CH marquent sur AB et AC des ponctuelles semblables. Donc l'enveloppe de BC est encore une parabole.

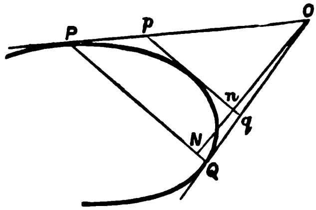
3° Les projections du centre O du cercle ABC, lorsque O se meut sur EF, marquent sur AB et AC des ponctuelles semblables. Donc BC enveloppe une parabole.

8920. (R. KNOWLES, B.A.)—From a point O (x, y) tangents are drawn to meet the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ in P and Q, a tangent parallel to the chord PQ meets OP, OQ in p and q respectively; prove that

$$pq : PQ = (a^2y^2 + b^2x^2)^{\frac{1}{2}} : ab + (a^2y^2 + b^2x^2)^{\frac{1}{2}}.$$

Solution by GERTRUDE POOLE, B.A.

Let ON, On be the perpendiculars from O (x, y) on PQ, pq; then PQ is
$$\frac{Xx}{a^2}+\frac{Yy}{b^2}-1=0,$$



$$\therefore\quad ON=\frac{a^2y^2+b^2x^2-a^2b^2}{(a^4x^2+b^4y^2)^{\frac{1}{2}}};$$

pq is $y+\dfrac{b^2x}{a^2y}\,x=\dfrac{ab}{a^2y}\,(b^2x^2+a^2y^2)^{\frac{1}{2}}$,

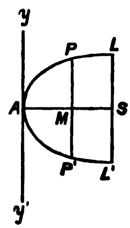
$$\therefore\ On=\frac{a^2y^2+b^2x^2+ab\,(b^2x^2+a^2y^2)^{\frac{1}{2}}}{(a^4x^2+b^4y^2)^{\frac{1}{2}}},$$

and PQ : pq = ON : On, therefore

$$PQ : pq=(a^2y^2+b^2x^2-a^2b^2) : \left\{a^2y^2+b^2x^2+ab\,(b^2x^2+a^2y^2)^{\frac{1}{2}}\right\}$$
= stated result.

8236. (Rev. T. R. TERRY, M.A.)—A uniform lamina bounded by the arc of a parabola and its *latus rectum* is revolving with angular velocity ω about the *latus rectum*: suddenly the *latus rectum* becomes free and the vertex becomes fixed. Show that the angular velocity about the tangent to the vertex is $\frac{2}{5}\omega$.

Solution by Rev. T. GALLIERS, M.A.; C. MORGAN, M.A.; *and others.*



Let the equation of the parabola which forms the curved boundary of the lamina be $y^2=4ax$, and let PMP′ be a double ordinate of the curve at a distance AM = x from the vertex, yAy' the tangent at the vertex.

The only dynamical condition is that the moment of momentum of the lamina about yAy' *before* the vertex becomes fixed is equal to the moment of momentum of the lamina about yAy' *after* the vertex becomes fixed. Let ω' = angular velocity about tangent at vertex after the vertex becomes fixed.

Now, in the first case, the velocity of every point in PMP′ = $(a-x)\,\omega$ perpendicular to plane of lamina, therefore moment of momentum of lamina about yAy' *before* A becomes fixed
$$=k\int_0^a x\,(a-x)\,\omega\,.\,y\,dx,$$
where k is a constant depending on the nature of the lamina. Also moment of momentum of the lamina about yAy' *after* A becomes fixed
$$=k\int_0^a x\,.\,x\,.\,\omega'\,.\,y\,dx,\quad\text{hence}\quad \omega\int_0^a (ax-x^2)\,y\,dx=\omega'\int_0^a x^2y\,dx,$$
or, by (1), $\omega\displaystyle\int_0^a (ax^{\frac{3}{2}}-x^{\frac{5}{2}})\,dx=\omega'\int_0^a x^{\frac{5}{2}}dx$, $\therefore\ (\frac{2}{5}-\frac{2}{7})\,\omega=\frac{2}{7}\omega'$, or $\omega'=\frac{2}{5}\omega$.

8302. (A. GORDON.) — Prove that the surface $x^2+y^2+z^2 = c^2$ will represent a surface of revolution if $\cos^2\widehat{xy} = \cos^2\widehat{yz} = \cos^2\widehat{zx}$, where $\widehat{xy}$ denotes the angle between the axes of x and y.

Solution by Professor IGNACIO BEYENS; *the* PROPOSER; *and others.*

Let $x^2+y^2+z^2$ become $A(X^2+Y^2)+CZ^2$ by change to rectangular axes, therefore $x^2+y^2+z^2-\lambda\,(x^2+y^2+z^2+2xy\cos\widehat{xy}+2yz\cos\widehat{yz}+2zx\cos\widehat{zx})$

becomes $$A(X^2+Y^2)+CZ^2-\lambda\,(X^2+Y^2+Z^2);$$

but the latter equated to 0, and hence the former, represents two coincident planes when $\lambda = A$, viz.,

$$(C-A)\,Z^2 = 0 \quad\text{or}\quad (lx+my+nz)^2 = 0,$$

therefore $$1-\lambda = l^2 = m^2 = n^2, \quad -\lambda\cos\widehat{xy} = lm,$$

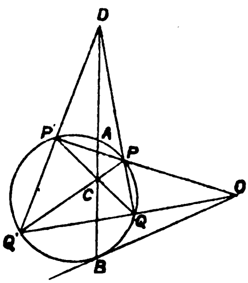
$$-\lambda\cos\widehat{yz} = mn, \quad -\lambda\cos\widehat{zx} = nl,$$

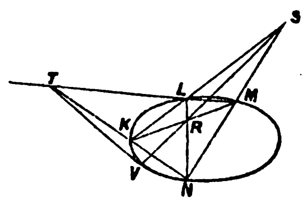
from which $$\cos^2\widehat{xy} = \cos^2\widehat{yz} = \cos^2\widehat{zx}.$$

8336. (ASPARAGUS.)—From a point O are drawn two chords OPP′, OQQ′ of a given circle, and two tangents OA, OB; a conic is drawn touching the straight lines PQ, PQ′, P′Q, P′Q′ and passing through A or B; prove that this conic will touch the circle in A or B.

Solution by H. LONDON, M.A.; R. KNOWLES, B.A.; *and others.*

Given a conic touching PQ, PQ′, P′Q, P′Q′, and passing through B.





Let K, L, M, N be points on the reciprocal conic whose polars are P′Q′, PQ, P′Q, and PQ. Then T, R, S, O are the poles of DC, PP′, QQ′, RS. But, if SR meet the conic at V, TV is tangent by the properties of a quadrilateral; therefore T is the pole of SV, therefore DC is the polar of O. Hence OB is the tangent to the given conic at B.

8656. (W. J. GREENSTREET, B.A.)—In a spherical triangle, prove that $\sin^4 R = (\cos a - \cos^2 R)^2 + \sin^2 a \sin^2 R \sin^2 (S-A)$.

Solution by Rev. T. GALLIERS, M.A.; SARAH MARKS, B.Sc.; *and others.*

We have $\tan R \cos (S-A) = \tan \frac{1}{2}a$,

therefore $\sin^2 (S-A) = 1 - \tan^2 \frac{1}{2}a \cot^2 R$,

therefore $(\cos a - \cos^2 R)^2 + \sin^2 a \sin^2 R \sin^2 (S-A)$

$$= \cos^2 a - 2\cos a \cos^2 R + \cos^4 R + \sin^2 a \sin^2 R - \sin^2 a \sin^2 R \tan^2 \tfrac{1}{2}a \cot^2 R$$

$$= \cos^2 a + 1 - 2\sin^2 R + \sin^4 R + \sin^2 R - \cos^2 a \sin^2 R - 1 + \sin^2 R$$

$$- \cos^2 a + \cos^2 a \sin^2 R = \sin^4 R.$$

8661. (A. GORDON.) — Any curve of the 4th degree intersects a conic in 8 points. These are joined, forming an octagon $A_1A_2 \dots A_8$. Show that the 8 intersections of A_1 with A_4 and A_6, A_3 with A_6 and A_8, &c., &c., will lie on a conic.

Solution by H. G. DAWSON, M.A.; Professor NASH, M.A.; *and others.*

Dr. SALMON (*Higher Plane Curves*, 3rd ed., p. 19) proves that, if a polygon of $2n$ sides be inscribed in a conic, the $n(n-2)$ points where each odd side meets the non-adjacent even sides will lie on a curve of the $n-2^{\text{th}}$ degree; therefore put $n = 4$, and we have the required theorem. There does not appear, then, to be any necessity for the 8 points to lie on a quartic.

8631. (Professor SYLVESTER, F.R.S.)—Find the discriminant of

$$x^3 + y^3 + z^3 + 3e\Sigma x^2y + 6exyz.$$

Solution by Professor MATHEWS, M.A.

Let $u \equiv x^3 + y^3 + z^3 + 3e\Sigma x^2y + 6exyz \equiv e(x+y+z)^3 - (e-1)(x^3+y^3+z^3)$.

Then, if the curve $u = 0$ have a double point,

$$e(x+y+z)^2 - (e-1)x^2 = 0, \qquad e(x+y+z)^2 - (e-1)y^2 = 0,$$

$$e(x+y+z)^2 - (e-1)z^2 = 0,$$

therefore $$\pm\left(\frac{e}{e-1}\right) \pm \left(\frac{e}{e-1}\right) \pm \left(\frac{e}{e-1}\right)^{\frac{1}{2}} = 1,$$

and the discriminant is found by rationalising this equation.

Now, the rationalised form of $w^{\frac{1}{2}} = \pm x^{\frac{1}{2}} \pm y^{\frac{1}{2}} \pm z^{\frac{1}{2}}$ is

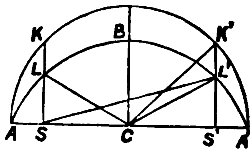
$$(w^2 + x^2 + y^2 + z^2 - 2wx - \dots - 2yz - \dots)^2 - 64xyzw = 0,$$

and, putting $w = e-1$, $x = y = z = e$, the left-hand side becomes

$$\{(e-1)^2 - 6e(e-1) - 3e^2\}^2 - 64e^3(e-1) = (1 + 4e - 8e^2)^2 - 64e^3(e-1) = 1 + 8e.$$

8933. (B. Hanumanta Rau, M.A.)—If a body describe an ellipse about a centre of force in the focus, the time of describing the arc between the extremities of the *latera recta* on the same side of the major-axis is to the periodic time as $\sin^{-1}(e) : \pi$.

Solution by Professor Stone, M.A.; C. E. Williams, M.A.; *and others.*



The times are as the areas described round the focus; but area LSL′ = LCL′ and LCL′ : area of ellipse = KCK′ : area of aux. circle

$$= \text{angle } K'CB : \pi = \sin^{-1}(e) : \pi.$$

8298. (R. Knowles, B.A.)—If p_r denote the coefficient of x^r, in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$1 - \frac{p_1}{2^2} + \frac{p_2}{3^2} - \dots + \frac{(-1)^n p_n}{(n+1)^n} = \frac{1}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right).$$

Solution by Professor Sircom, M.A.; Professor Matz, M.A.; *and others.*

We have $$1 - \frac{p_1}{2^2} + \frac{p_2}{3^2} - \dots + \frac{(-1)^n p_n}{(n+1)^n} = \int_0^1 \frac{1}{x}\int_0^x (1-y)\, dy\, dx$$

$$= \frac{1}{n+1}\int_0^1 \frac{1}{x}\left\{1-(1-x)^{n+1}\right\} dx;$$

or, writing $1-x$ for x, we have the integral

$$= \frac{1}{n+1}\int_0^1 \frac{1-x^{n+1}}{1-x}\, dx = \frac{1}{n+1}\left(1 + \frac{1}{2} + \dots + \frac{1}{n+1}\right).$$

8423. (D. Edwardes.)—Prove that

(1) $$\int_0^{\frac{1}{2}\pi} \sin x\,(\log\sin x)^2\, dx = 2 + (\log 2)^2 - 2\log 2 - \tfrac{1}{12}\pi^2,$$

(2) $$\int_0^{\frac{1}{2}\pi} \sin^2 x\,(\log\sin x)^2\, dx = \tfrac{1}{8}\pi\left\{2(\log 2)^2 - 2\log 2 + \tfrac{1}{6}\pi^2 - 1\right\}.$$

Solution by Professor Sebastian Sircom, M.A.

1. $$\int_0^{\frac{1}{2}\pi} \sin x\,(\log\sin x)^2\, dx = \tfrac{1}{4}\int_0^1 \left\{\log(1-x^2)\right\}^2 dx$$

$$= \tfrac{1}{4}\int_0^1 \left[\left\{\log(1-x)\right\}^2 + 2\log(1-x)\log(1+x) + \left\{\log(1+x)\right\}^2\right] dx,$$

$$\int_0^1 \{\log(1+x)\}^2 dx = 2(\log 2)^2 - 4\log 2 + 2 \text{ (by Quest. 7797)} \ldots (a),$$

$$\int_0^1 \{\log(1-x)\}^2 dx = \int_0^1 \left(\log\frac{1}{y}\right)^2 dy = 2 \quad \ldots\ldots\ldots\ldots (b).$$

Also $\log(1+x)\log(1-x) = -(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\ldots)\log(1+x)$,

and $$\int_0^1 x^n \log(1+x)\,dx$$

$$= \frac{1}{n+1}\log 2 - \frac{1}{n+1}\left\{\frac{1}{n+1}-\frac{1}{n}+\ldots(-1)^{n-1}+(-1)^n\log 2\right\},$$

hence $$\int_0^1 \log(1+x)\log(1-x)\,dx = -2\log 2\left(\frac{1}{2\,.\,3}+\frac{1}{4\,.\,5}+\ldots\right)$$

$$+\frac{1}{1\,.\,2}(\tfrac{1}{2}-1)+(\tfrac{1}{3}-\tfrac{1}{2})(\tfrac{1}{3}-\tfrac{1}{2}+1)+(\tfrac{1}{4}-\tfrac{1}{3})(\tfrac{1}{4}-\tfrac{1}{3}+\tfrac{1}{2}-1)+\&c.$$

$$= 2\log 2(\log 2 - 1) - 2\left(\frac{1}{2^2}+\frac{1}{3}+\frac{1}{4^2}+\ldots\right)-1$$

$$+\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\ldots - 2\Sigma\frac{(-1)^{m+n}}{mn}\,(n>m),$$

therefore $$2\int_0^1 \log(1+x)\log(1-x)\,dx = 2(\log 2)^2 - 4\log 2 + 4 - \tfrac{1}{3}\pi^2 \quad \ldots (c),$$

since $$(\log 2)^2 = \tfrac{1}{6}\pi^2 + 2\Sigma\frac{(-1)^{m+n}}{mn}\,(n>m).$$

Adding (a), (b), (c), we obtain the required result.

2. We have $(\log\cos x)^2 = \{\log(2\cos x) - \log 2\}^2$

$$= \{\cos 2x - \tfrac{1}{2}(\cos 4x) + \tfrac{1}{3}(\cos 6x) - \ldots\}^2$$
$$-2\log 2\{\cos 2x - \tfrac{1}{2}(\cos 4x) + \tfrac{1}{3}(\cos 6x) - \ldots\} + (\log 2)^2.$$

Expanding, and putting

$$2\cos 2mx \cos 2(m+1)x = \cos 2x + \cos 2(2m+1)x,$$

we have $$(\log\cos x)^2 = \tfrac{1}{12}\pi^2 + (\log 2)^2 - \cos 2x\left(2\log 2 + \frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\ldots\right)$$

+ terms in cosines of multiples of $2x$

$$= \tfrac{1}{12}\pi^2 + (\log 2)^2 - \cos 2x(2\log 2 + 1) + \ldots\ldots\ldots\ldots (a).$$

Integrating between $\frac{1}{2}\pi$ and 0, we obtain the known result

$$\int_0^{\frac{1}{2}\pi} (\log\cos x)^2 dx = \tfrac{1}{24}\pi^3 + \tfrac{1}{2}\pi(\log 2)^2.$$

Multiplying (a) by $\cos 2x$ and integrating,

$$\int_0^{\frac{1}{2}\pi} \cos 2x(\log\cos x)^2 dx = -\tfrac{1}{2}\pi\log 2 - \tfrac{1}{4}\pi,$$

then $$\int_0^{\frac{1}{2}\pi} \sin^2 x(\log\sin x)^2 dx = \int_0^{\frac{1}{2}\pi} \cos^2 x(\log\cos x)^2 dx$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (1+\cos 2x)(\log\cos x)^2 dx$$

$$= \tfrac{1}{8}\pi\{2(\log 2)^2 - 2\log 2 + \tfrac{1}{6}\pi^2 - 1\}.$$

[Referring to the solution of Question 8465 (Vol. XLVI., p. 75), we have

$$\iint \sin\theta\,(\log\sin\theta\cos\phi)^2\,d\theta\,d\phi = \iint \sin\omega\,(\log\omega)^2\,d\omega\,d\mu = \tfrac{1}{2}\pi\int_0^1 (\log z)^2\,dz = \pi.$$

The limits are 0 and $\frac{1}{2}\pi$ throughout. Also, as is proved in the solution referred to,

$$\int \sin\theta\,\log\sin\theta\,d\theta = \log 2 - 1;$$

and we have the known results

$$\int \log\cos\phi\,d\phi = -\tfrac{1}{2}\pi\log 2,\quad \int (\log\cos\phi)^2\,d\phi = \tfrac{1}{2}\pi\left\{(\log 2)^2 + \tfrac{1}{12}\pi^2\right\},$$

and hence the first result immediately follows.

Again, $\iint \sin\theta\cos\theta\,(\log\sin\theta\cos\phi)^2\,d\theta\,d\phi = \iint \sin^2\omega\,\sin\mu\,(\log\cos\omega)^2\,d\omega\,d\mu.$

And
$$\int \sin\theta\cos\theta\,(\log\sin\theta)^2\,d\theta = \int_0^1 z\,(\log z)^2\,dz = \tfrac{1}{4},$$

$$\int \sin\theta\cos\theta\,(\log\sin\theta)\,d\theta = -\tfrac{1}{4},\quad \int \sin\theta\cos\theta\,d\theta = \tfrac{1}{2},$$

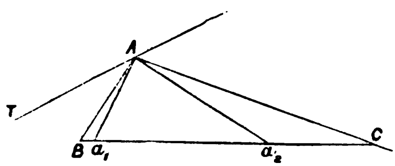
so that

$$\tfrac{1}{8}\pi - \tfrac{1}{2}\int \log\cos\phi\,d\phi + \tfrac{1}{2}\int (\log\cos\phi)^2\,d\phi = \int (\log\cos\omega)^2\,d\omega - \int \cos^2\omega\,(\log\cos\omega)^2\,d\omega,$$

giving $\displaystyle\int \cos^2\omega\,(\log\cos\omega)^2\,d\omega = \tfrac{1}{8}\pi\left\{2\,(\log 2)^2 - 2\log 2 - \tfrac{1}{6}\pi^2\right)\}$].

8728. (Captain H. Brocard.) — De chaque sommet d'un triangle on mène des perpendiculaires aux côtés adjacents, jusqu'à leurs rencontres avec le côté opposé. Les centres A′, B′, C′ des cercles circonscrits aux trois triangles ainsi formés sont les sommets d'un second triangle homologique avec le proposé. Le centre d'homologie est le centre O du cercle circonscrit au triangle donné. En d'autres termes, les nouveaux cercles sont tangents au cercle circonscrit, aux points A, B, C.

Solutions by (1) C. E. Williams, M.A.; (2) Professor de Longchamps.



1. As BAa_2, CAa_1 are right angles, we have the angles BAa_1, CAa_2 equal.

If TA be the tangent at A to the circumscribed circle

$$TAB = ACB,$$

therefore

$$TAa_1 = ACB + CAa_2 = Aa_2a_1,$$

therefore TA is also the tangent to the circle circumscribed to Aa_1a_2, therefore the circles touch and their centres are in a line with A.

2. *Otherwise*:—Soit ABC le triangle proposé. Les perpendiculaires élevées au point A aux côtés AC, AB rencontrent BC en des points a_1, a_2, et il faut démontrer que les cercles Aa_1a_2, ABC sont tangents.

En coordonnées barycentriques, et d'après la notation que nous avons proposée (Mémoire "Sur un nouveau Cercle remarquable du plan d'un triangle," *Journal de Mathématiques Spéciales*, 1886), le cercle Aa_1a_2 peut-être représenté par l'équation

$$(\alpha+\beta+\gamma)(u\alpha+v\beta+w\gamma)-\zeta=0 \quad \text{.....................(A).}$$

Dans cette égalité u, v, w représentent, en grandeur et en signe, les puissances des sommets A, B, C par rapport au cercle Aa_1a_2, et ζ le premier membre de l'équation du cercle ABC.

Nous avons, dans le cas présent,

$$u=0, \quad v=-Ba_1 \,.\, Ba_2=\frac{c}{\cos B}\left(a-\frac{b}{\cos C}\right), \quad \text{ou} \quad v=-c^2\frac{\cos A}{\cos B\cos C}.$$

On a, de même, $$w=-b^2\frac{\cos A}{\cos B\cos C},$$

et, finalement, l'équation de Aa_1a_2 est

$$\frac{\cos A}{\cos B\cos C}(\alpha+\beta+\gamma)(c^2\beta+b^2\gamma)+\zeta=0.$$

L'axe radical Δ des cercles ABC, Aa_1a_2 est donc représenté par l'équation $c^2\beta+b^2\gamma=0$; c'est-à-dire que Δ est, précisément, la tangente en A au cercle ABC.

Remarque.—Cette proposition très simple peut, naturellement, faire l'objet de démonstrations élémentaires ; mais, en proposant la solution précédente, nous avons voulu montrer comment l'équation (A) s'applique immédiatement, et presque sans effort, à ces nombreuses questions de la géométrie moderne dans lesquelles, considérant différents cercles dérivés du triangle, on demande d'observer quelques unes de leurs propriétés.

8946. (W. J. C. Sharp, M.A.)—Show that

$$\begin{vmatrix} x, & -a, & -a, & \dots & -a, & -a \\ b, & x, & -a, & \dots & -a, & -a \\ b, & b, & x, & \dots & -a, & -a \\ \dots & \dots & \dots & & \dots & \\ b, & b, & b, & \dots & x, & -a \\ b, & b, & b, & \dots & b, & b \end{vmatrix} = b(x+a)^n,$$

where the determinant has $n+1$ rows.

Solution by Isabel Maddison.

Multiplying the last row by ab^{-1}, and adding to each of the other rows, the determinant becomes

$$\begin{vmatrix} x+a, & 0, & 0 & \dots & 0, & 0 \\ b+a, & x+a, & 0 & \dots & 0, & 0 \\ b+a, & b+a, & x+a & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b+a, & b+a, & b+a & \dots & x+a, & 0 \\ b, & b, & b, & & b, & b \end{vmatrix} = b(x+a)^n.$$

8948. (W. J. C. SHARP, M.A.)—Show that

$$\sin n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\,\frac{d}{d\theta}\right)^{n-1}(\sin^2\theta),$$

and
$$\cos n\theta = \frac{1}{n-1!\,\sin^n\theta}\left(\sin^2\theta\,\frac{d}{d\theta}\right)^{n-1}(\sin\theta\cos\theta).$$

Solution by D. EDWARDES.

$\left(\frac{d}{dt}\right)^{n-1}\frac{1}{t+j} = (-)^{n-1}\frac{(n-1)!}{(t+j)^n}$. Put $t = -\cot\theta$, $j^2 = -1$, then the left side becomes $-\left(\sin^2\theta\,\frac{d}{d\theta}\right)^{n-1}\sin\theta\,(\cos\theta + j\sin\theta)$, and the right side becomes $-(n-1)!\,\sin^n\theta\,(\cos n\theta + j\sin n\theta)$. Equating real and imaginary parts, we have the formulæ in question.

8754. (Professor MAHENDRA NATH RAY, M.A., LL.B.)—Prove that

$$\int_0^\pi \frac{x\sin^3 x\,dx}{1+\cos^2 x} = \pi\,(\tfrac{1}{2}\pi - 1), \qquad \int_0^\pi \frac{x\sin^5 x\,dx}{1+\cos^2 x} = \pi\,(\pi - \tfrac{8}{3}).$$

Solution by A. GORDON; W. J. GREENSTREET, B.A.; *and others.*

Calling Z the first integral, Y the second, we have

$$2Z = \pi\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx, \qquad 2Y = \pi\int_0^\pi \frac{\sin^5 x}{1+\cos^2 x}\,dx,$$

$$2Z = \pi\int_0^{\frac{1}{2}\pi}\left(\frac{\sin^3 x}{1+\cos^2 x} + \frac{\cos^3 x}{1+\sin^2 x}\right)dx$$

$$= 2\pi\int_0^1 dz\,\frac{1-z^2}{1+z^2} = 2\pi\int_0^1\left(-1+\frac{2}{1+z^2}\right)dz;$$

therefore $Z = \pi\,(-1+\frac{1}{2}\pi)$. Similarly, $Y = \pi\,(-3+\frac{1}{3}+\pi) = \pi\,(\pi - \frac{8}{3})$.

8671. (D. EDWARDES.)—Prove that the *latus rectum* of a parabola is half the harmonic mean between any two focal chords at right angles to one another.

Solution by the Rev. T. GALLIERS, M.A.; W. P. GOUDIE, B.A.; *and others.*

From the equation

$$r = \frac{2a}{1+\cos\theta} \text{ we get } f = \frac{4a}{1-\cos^2\theta} \text{ and } f' = \frac{4a}{1-\sin^2\theta},$$

therefore
$$\frac{1}{f} + \frac{1}{f'} = \frac{1}{4a}.$$

9017 & 2810. (A. RUSSELL, B.A.)—(9017.) If $x_1, x_2, \ldots, x_n = a_1, a_2, \ldots, a_n$, prove that

$$\frac{(a_1 - x_1)\ldots(a_1 - x_n)}{a_1(a_1 - a_2)\ldots(a_1 - a_n)} + \frac{(a_2 - x_1)\ldots(a_2 - x_n)}{a_2(a_2 - a_3)\ldots(a_2 - a_1)} + \ldots = 0.$$

Solution by THOMAS MUIR, LL.D.

9017. For convenience in writing, and not for simplification of the proof, let $n = 4$. Then the common denominator of the fractions is

$$a_1 a_2 a_3 a_4 \, \zeta^{\frac{1}{2}}(a_1 a_2 a_3 a_4),$$

and the total numerator

$$= -a_2 a_3 a_4 \, \zeta^{\frac{1}{2}}(a_2 a_3 a_4)\left[a_1^4 - a_1^3 \Sigma x_1 + a_1^2 \Sigma x_1 x_2 - a_1^1 \Sigma x_1 x_2 x_3 + x_1 x_2 x_3 x_4\right]$$
$$+ a_1 a_3 a_4 \, \zeta^{\frac{1}{2}}(a_1 a_3 a_4)\left[a_2^4 - a_2^3 \Sigma x_1 + a_2^2 \Sigma x_1 x_2 - a_2 \Sigma x_1 x_2 x_3 + x_1 x_2 x_3 x_4\right]$$
$$- \ldots\ldots + \ldots\ldots.$$

This, when the coefficients of Σx_1, $\Sigma x_1 x_2$, &c. are collected, becomes

$$|a_1^1 a_2^2 a_3^3 a_4^4| - \Sigma x_1 |a_1^1 a_2^2 a_3^3 a_4^3| + \Sigma x_1 x_2 |a_1^1 a_2^2 a_3^3 a_4^2|$$
$$- \Sigma x_1 x_2 x_3 |a_1^1 a_2^2 a_3^3 a_4^1| + x_1 x_2 x_3 x_4 |a_1^1 a_2^2 a_3^3 a_4^0|$$

The first determinant here $= a_1 a_2 a_3 a_4 \, \zeta^{\frac{1}{2}}(a_1 a_2 a_3 a_4)$, the last $= -\zeta^{\frac{1}{2}}(a_1 a_2 a_3 a_4)$, and the others vanish; hence Mr. RUSSELL's expression is equal to

$$1 - \frac{x_1 x_2 x_3 x_4}{a_1 a_2 a_3 a_4}.$$

2810. A generalisation including Professor SYLVESTER's theorem in Quest. 2810 (see Vol. XLV., p. 129) may, in like manner, be readily obtained (see *Proc. Roy. Soc.*, Edinburgh, 1886–87).

7661. (Rev. T. C. SIMMONS, M.A.)—If

$$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0,$$

show that $\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha)$ cannot be greater than $\frac{1}{8}$.

Solution by the PROPOSER.

Let PQR be a triangle touching an ellipse in points A, B, C whose eccentric angles are α, β, γ; then, by Quest. 7563 (Vol. XLII., p. 73.)

$$\Delta ABC : \Delta PQR = 2\cos\tfrac{1}{2}(\alpha - \beta)\cos\tfrac{1}{2}(\beta - \gamma)\cos\tfrac{1}{2}(\gamma - \alpha) : 1.$$

Now, let α, β, γ be such that $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$, then the normals at A, B, C will be concurrent, and in this case $\Delta ABC : \Delta PQR$ cannot be $> \frac{1}{4}$; hence, subject to the above condition,

$$2\cos\tfrac{1}{2}(\alpha - \beta)\cos\tfrac{1}{2}(\beta - \gamma)\cos\tfrac{1}{2}(\gamma - \alpha) \text{ cannot be } > \tfrac{1}{4}.$$

[A solution is desired by a direct method.]

APPENDIX I.

SOLUTION OF QUESTIONS 8886 AND 9009.

By Rev. T. P. KIRKMAN, M.A., F.R.S.

8886. (By Rev. T. P. Kirkman, M.A., F.R.S.)—Two 28-edra have the following triangles, the summits being marked *abc* ... *npq*. Required, all the symmetry of both.

abg, abc, acd, ade, aef, afg, bjc, bih, bjh, big; cjl, clk, ckd, dkm, dfm, def, fmq, fgq, ghq, ghi, hpj, hpq, jlk, jkp, mnq, mnp, mpk, pqn.

abg, afg, afe, aed, abc, acd, bhg, bhi, bci, cji, cdj, djk, dkl, dfl, def, fkl, fkm, fgm, ghm, ijq, ihp, ipn, inq, jkn, jnq, kmn, hmn, hpn.

9009. (Rev. T. P. Kirkman, M.A., F.R.S.)—P and Q are a regular 20-edron and 8-edron. KLMN are 4-edra, each on a base that covers a face of P or of Q. Required the number of polyedra, of which none is either the repetition or the reflected image of another, that can be made by laying one or more of KLMN on as many faces of P or of Q, with an account of the summits and symmetry of the constructed solids.

Solution.

(9009.) The required solids are, one 7-acron, three 8-acra, three 9-acra; six 10-acra; one 13-acron, six 14-acra, thirteen 15-acra, and forty-two 16-acra: in all 75 solids. Their summits and symmetry are as follows, where the terms preceding the colons are the summit-signatures; *e.g.*, the 7-acron has the signature $5^3 4^3 3$, showing three pentaces, three tessaraces, and one triace; the first 15-acron has $8 7^2 6^2 5^7 3^3$, showing one octace, two heptaces, two hexaces, seven pentaces, and three triaces.

One 7-acron:

1; $5^3 4^3 3$: 3-zoned monaxine heteroid.

Three 8-acra:

1, 2; $6^2 5^2 4^2 3^2$, $6 5^4 4 3^2$: both 2-zoned monaxine heteroids.

3; $5^6 3^2$: 3-zoned monarchaxine homozone, which has one janal 3-zoned axis, with repeating zones, and three like contrajanal 2-ple zoneless axes.

Three 9-acra:

1, 2; $7 6^2 5^2 4 3^2$, $6^3 5^3 3^3$: two monozones.

3; $6^3 5^3 3^3$: 3-zoned monaxine heteroid.

Six 10-acra:

1; $8 6^4 4 3^4$: 4-zoned monaxine heteroid.

2; $7^3 5^3 3^4$: 3-zoned monaxine heteroid.

3; $7^2 6^2 5^2 3^4$: 2-ple zoneless monaxine heteroid.

4; $7 6^4 5 3^4$: monozone.

5; $6^6 3^4$: zoned triaxine, having three unlike janal 2-zoned axes.

6; $6^6 3^4$: homozone triaxine, having one janal 2-zoned axis, with like zones, and two like janal 2-ple zoneless axes.

One 13-acron:

1; $6^3 5^9 3$: 3-zoned monaxine heteroid.

Six 14-acra:

1, 2; $7^2 6^4 5^6 3^2$, $6^6 5^6 3^2$: both 2-zoned monaxine heteroids.

3, 4; $7 6^6 5^7 3^2$, $6^6 5^6 3^2$: both monozones.

5; $6^6 5^6 3^2$: 2-ple zoneless monaxine heteroid.

6; $6^6 5^6 3^2$: 3-zoned monarchaxine homozone.

Thirteen 15-acra:

1, 2, 3, 4, 5, 6, 7; $8 7^2 6^4 5^7 3^3$, $8 7 6^4 5^6 3^3$, $7^2 6^5 5^5 3^3$, $7^2 6^5 5^5 3^3$, $7^2 6^7 5^4 3^3$, $7^2 6^7 5^4 3^3$, $6^9 5^3 3^3$: seven monozones.

8; $7^3 6^3 5^6 3^3$: 3-zoned monaxine heteroid.

9; $6^9 5^3 3^3$: 3-ple zoneless monaxine heteroid.

10, 11, 12, 13; $7^3 6^3 5^6 3^3$, $7^2 6^5 5^5 3^3$, $7 6^7 5^4 3^3$, $7 6^7 5^4 3^3$: four asymmetricals.

Forty-two 16-acra:

1, 2, 3, 4, 5, 6, 7, 8, 9; $9 7^3 6^2 5^6 3^4$, $8 7^3 6^3 5^5 3^4$, $8 7^2 6^5 5^4 3^4$, $8 7^2 6^5 5^4 3^4$, $8 7 6^7 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$: nine monozones.

10, 11; $8^3 3^3 5^6 3^4$, $7^3 6^6 5^3 3^4$: two 3-zoned monaxine heteroids.

12, 13, 14, 15; $8^2 7^2 6^2 5^6 3^4$, $7^4 6^4 5^4 3^4$, $7^2 6^8 5^2 3^4$, $7^2 6^8 5^2 3^4$: four 2-ple zoneless monaxine heteroids.

16, 17; $8^2 6^6 5^4 3^4$, $7^2 6^8 5^2 3^4$: two 2-zoned monaxine heteroids.

18; $7^4 6^4 5^4 3^4$: zoned triaxine.

19; $7^3 6^6 5^3 3^4$: 3-ple zoneless monaxine heteroid.

20; $7^2 6^8 5^2 3^4$: 2-ple monaxine monozone, having a contrajanal 2-ple zoneless axis.

21, 22, 23, 24, 25, 26, 27; $8^2 7 6^4 5^5 3^4$, $8 7^3 6^3 5^5 3^4$, $8 7^3 6^3 5^5 3^4$, $8 7^2 6^5 5^4 3^4$, $8 7^2 6^5 5^4 3^4$, $8 7^2 6^5 5^4 3^4$, $8 7^2 6^5 5^4 3^4$;

28, 29, 30, 31, 32, 33, 34; $8 7 6^7 5^3 3^4$, $8 7 6^7 5^3 3^4$, $7^4 6^4 5^4 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$;

35, 36, 37, 38, 39, 40, 41; $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^3 6^6 5^3 3^4$, $7^2 6^8 5^2 3^4$, $7^2 6^8 5^2 3^4$, $7 6^{10} 5 3^4$: twenty-one asymmetricals.

42; $6^{12} 3^4$: zoneless tetrarchaxine, which has four principal 3-ple heteroid, and three secondary 2-ple janal, axes, all seven zoneless.

There is nothing tentative in the enumeration or construction of these solids on the bases P and Q, to one familiar with the resulting changes of symmetry.

(8886.)—The answers are the 15-acra, 16-acra 42 and 13 above given. In first 24-edron, *i*, *e*, *l*, *n* occur thrice only, and in the second *elqp* thrice only. These summits are the vertices of the imposed 4-edra. Either figure is easily drawn on a triangle whose summits all occur more than three times.

If we construct the first on *dfm*, we have a zoneless tetrarchaxine whose four like 3-ple principal axes have for polar summits *e*, *i*, *l*, *n*; and for opposite 3-ple polar faces, the triangles *hpj*, *dkm*, *fgq*, and *abc*. The four janal secondary 2-ple faces bisect the pairs of edges *ag*, *kp*; *bj*, *fm*; *cd*, *hq*. If we efface the twelve bases of the triangles whose vertices are *e*, *i*, *l*, *n*, we obtain another zoneless tetrarchaxine 16-edral 16-acron, with signature $4^{12}\,3^4$, autopolar; and if we remove next the four triaces *e*, *i*, *l*, *n*, we get a zoned tetrarchaxine 8-edron, which is obtained by cutting away the summits of a 4-edron. The reciprocals of all these are readily made from the hexarchaxine 12-edron. By cutting away four of its summits so as to make twelve 6-gons, we construct the reciprocal of No. 42, the first in Quest. 8886.

If we construct the second in Quest. 8886 on its face *abg*, we have a 2-ple zoneless monaxine heteroid, whose axis bisects the edges *bg* and *jk*.

I have only recently learned, in turning over the leaves of Vol. LXVIII. of CRELLE's *Journal*, that in this and Vol. LXVI. there is a treatise on the symmetry of Polyedra. Above twenty years ago, its French author did me the honour to converse with me in my house, in fluent and good English, on Groups and Polyedra. I was more amused than surprised to find that he seems to write, in CRELLE's *Journal*, quite unaware that his subject had been previously and thoroughly discussed in *Phil. Tran.* 1862. My complete familiarity with all the aspects of polyedral symmetry lacks the appetite required for a real study of M. JORDAN's Memoirs upon it; but they ought to be useful to the student who wishes to verify the above solution of Quest. 9009. I may here, in mentioning CRELLE's *Journal*, be permitted to request the student's help in the solution of a puzzle that beats me—how did the *Cambridge and Dublin Mathematical Journal*, Vol. II., N. S. (VI. O. S.), p. 191, contrive to steal so much from a later paper in CRELLE's *Journal*, Vol. LVI., p. 326, on exactly the same problem in combinations?

The following letter to me is before me, gummed into my copy of the learned and noble writer's life:—

"41, Chalcot Villas, Adelaide Road, N.W.;
"*April* 18*th*, 1862.

"My dear Sir,—I have to thank you for several papers, which I have looked at from time to time. Your excessively complicated subject will take shape in your hands I see. You have ascertained that the French Institute is mainly founded for the purpose of advancing the claims of Frenchmen—either by this means or that. But why go and buy your experience, when you might have had it for nothing from history? By printing in England you may save yourself from having your theorems given to a Frenchman. Yours truly,

A. DE MORGAN."

The "printing" is that of my Theory of the Polyedra, which is also the "subject" spoken of. The subject I have long, long ago exhausted, at least so far that I know how to answer any question that I know how to ask in it. The printing has been a distinguished failure. Yet I am satisfied that enough has seen the light in London and Liverpool to prevent the giving either of my theorems or of my methods to another.

The Theory of Knots, which Professor Tait and I have pretty nearly wound up in the *Phil. Trans. and Proc. Edin.*, is a brief supplementary chapter of the Polyedra. Every knot is a polyedron, or a polyedral reticulation, which has only tessarace summits, and which has or has not 2-gonal faces.

APPENDIX II.

MISCELLANEOUS SOLUTIONS AND NOTES.

By R. F. DAVIS, M.A.

8559. (Professor WOLSTENHOLME, M.A., Sc.D.)—From a fixed point O are drawn OP, OQ tangents to one of a system of confocal conics (foci S, S′, centre C), and from C are let fall perpendiculars on the normals at P, Q; prove that the envelope of the straight line joining the feet of these perpendiculars is the conic (parabola)

$$\left\{X(x\cos\alpha+y\sin\alpha)+Y(x\sin\alpha-y\cos\alpha)-c^2\cos\alpha\right\}^2$$
$$=4XY(x\cos\alpha+y\sin\alpha)(x\sin\alpha-y\cos\alpha),$$

where C is origin, CS axis of x, (X, Y) the point O, $SS'=2c$, and α is the sum of the angles which SO, S′O make with the axis of x.

[If $X^2-Y^2=c^2$, the straight line is fixed.]

*Solution.**

Let θ, ϕ be the eccentric angles of the points P, Q for any conic of the system whose semi-axes are a, b, where $a^2-b^2=c^2$. To find the coordinates of L, the foot of the perpendicular from the centre on the normal at P, we have

$$bx\cos\theta+ay\sin\theta=0,$$
$$ax\sin\theta-by\cos\theta=c^2\sin\theta\cos\theta,$$

therefore
$$(a^2\sin^2\theta+b^2\cos^2\theta)\,x=c^2a\sin^2\theta\cos\theta,$$
$$(\ldots\ldots\ldots\ldots\ldots\ldots)\,y=-c^2b\sin\theta\cos^2\theta.$$

Similarly for M, the foot of the perpendicular from the centre on the normal at Q.

The equation, therefore, to LM (whose envelope is required) is

$$\begin{vmatrix} x & y & 1 \\ c^2a\sin^2\theta\cos\theta, & -c^2b\sin\theta\cos^2\theta, & a^2\sin^2\theta+b^2\cos^2\theta \\ c^2a\sin^2\phi\cos\phi, & -c^2b\sin\phi\cos^2\phi, & a^2\sin^2\phi+b^2\cos^2\phi \end{vmatrix}=0 \quad\ldots(1),$$

or
$$Ax+By+C=0 \quad\ldots\ldots\ldots\ldots(2),$$

where, after reduction, it is found

$$A=-c^2b(\sin\theta-\sin\phi)\left\{b^2\cos^2\theta\cos^2\phi-a^2\sin\theta\sin\phi(1+\sin\theta\sin\phi)\right\},$$
$$B=c^2a(\cos\phi-\cos\theta)\left\{a^2\sin^2\theta\sin^2\phi-b^2\cos\theta\cos\phi(1+\cos\theta\cos\phi)\right\},$$
$$C=c^4ab\sin\theta\sin\phi\cos\theta\cos\phi\sin(\phi-\theta).$$

* For this Solution I am indebted to the EDITOR.

The relations (useful generally) connecting θ, ϕ with X, Y are found to be

$$(\sin\phi-\sin\theta) : (\cos\theta-\cos\phi) : \sin(\phi-\theta) = b\mathrm{X} : a\mathrm{Y} : ab \quad\text{.........(3)},$$

and $\cos\theta\cos\phi : \sin\theta\sin\phi : 1 = a^2(b^2-\mathrm{Y}^2) : b^2(a^2-\mathrm{X}^2) : (b^2\mathrm{X}^2+a^2\mathrm{Y}^2)$...(4), where (3) follows from the consideration that the tangents at θ, ϕ pass through X, Y; and (4) from the consideration that θ, ϕ are the points of intersection of the polar of X, Y with the ellipse.

By means of (3) and (4) we are enabled to substitute completely for θ, ϕ in (2); thus,

$$\left\{(b^2-\mathrm{Y}^2)^2-(a^2-\mathrm{X}^2)(b^2+\mathrm{Y}^2)\right\}\mathrm{X}x-\left\{(a^2-\mathrm{X}^2)^2-(b^2-\mathrm{Y}^2)(a^2+\mathrm{X}^2)\right\}\mathrm{Y}y$$
$$+c^2(a^2-\mathrm{X}^2)(b^2-\mathrm{Y}^2) = 0 \quad\text{..........................(5)}.$$

If we now put $a^2-\mathrm{X}^2 = \lambda-\kappa$, $b^2-\mathrm{Y}^2 = \lambda$, where λ is indeterminate and $\mathrm{X}^2-\mathrm{Y}^2-c^2 = \kappa$, then $a^2+\mathrm{X}^2 = \lambda-\kappa+2\mathrm{X}^2$, $b^2+\mathrm{Y}^2 = \lambda^2+2\mathrm{Y}^2$; and (5) reduces to

$$c^2\lambda^2+\left\{(\kappa-2\mathrm{Y}^2)\mathrm{X}x+(2\mathrm{X}^2+\kappa)\mathrm{Y}y-c^2\kappa\right\}\lambda+2\kappa\mathrm{Y}^2\mathrm{X}x-\kappa^2\mathrm{Y}y = 0 \quad\text{...(6)}.$$

Now, with the notation of the question,

$$\alpha = \cot^{-1}\frac{\mathrm{X}-c}{\mathrm{Y}}+\cot^{-1}\frac{\mathrm{X}+c}{\mathrm{Y}} = \cot^{-1}\frac{\kappa}{2\mathrm{XY}},$$

therefore $\kappa = 2\mathrm{XY}\cot\alpha.$

Then equation (6) becomes

$$c^2\lambda^2+2\mathrm{XY}\left\{(\mathrm{X}\cot\alpha-\mathrm{Y})x+(\mathrm{X}+\mathrm{Y}\cot\alpha)y-c^2\cot\alpha\right\}\lambda$$
$$+4\mathrm{X}^2\mathrm{Y}^3\cot\alpha(x-y\cot\alpha) = 0 \quad\text{......(7)},$$

and the envelope required is

$$\left\{(\mathrm{X}\cot\alpha-\mathrm{Y})x+(\mathrm{Y}\cot\alpha+\mathrm{X})y-c^2\cot\alpha\right\}^2 = 4c^2\mathrm{Y}\cot\alpha(x-y\cot\alpha) \quad\text{...(8)},$$

or
$$\left\{\mathrm{X}(x\cos\alpha+y\sin\alpha)-\mathrm{Y}(x\sin\alpha-y\cos\alpha)-c^2\cos\alpha\right\}^2$$
$$= 4c^2\mathrm{Y}\cos\alpha(x\sin\alpha-y\cos\alpha) \quad\text{...(9)},$$

which can be easily shown to agree with Professor WOLSTENHOLME's result.

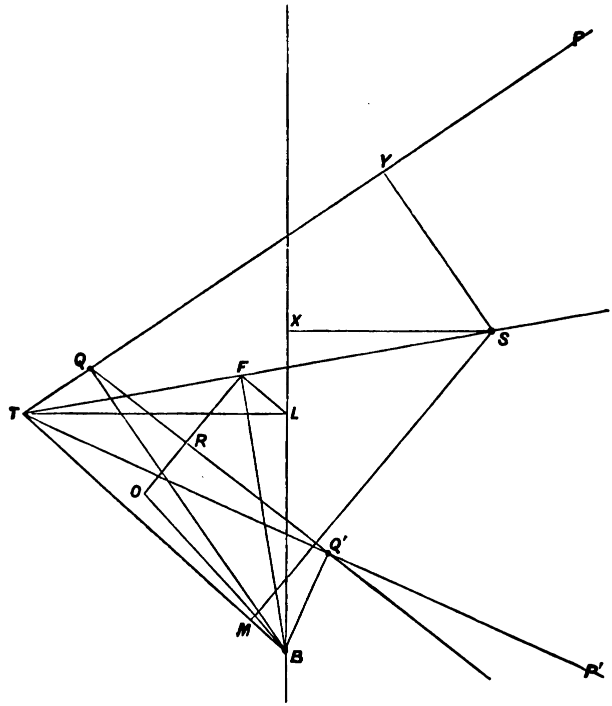
In a former Note (see Vol. XLV., Appendix III., pp. 169—172) I have shown that, if O be a fixed point from which tangents OP, OQ are drawn to any one of a system of confocal conics, then (1) the normals at P, Q touch the same parabola, which is the envelope of PQ, whose focus F is found by making the $\angle$FCS = $\angle$OCS and taking CF . CO = CS^2, and whose directrix is OC; (2) these same normals intersect on a fixed straight line FH perpendicular to OF. I therefore re-proposed Professor WOLSTENHOLME's question in the form given below, the geometrical solution of which can be reconciled with the equation to the envelope given above (8). (See Appendix, Section 17.)

8692. (R. F. DAVIS, M.A. Suggested by Question 8559.)—If, from any point in a given fixed straight line passing through the focus of a parabola, tangents be drawn to the curve, prove that the envelope of the line joining the feet of the perpendiculars on these tangents from a given fixed point on the directrix is another parabola.

Solution by the PROPOSER.

From any point T in the given fixed straight line ST, let there be drawn two tangents TP, TP′ to the parabola focus S. Let B be a fixed point on the directrix; draw BQ, BF, BQ′ perpendicular to TP, TS, TP′ respectively, and let the diameter through T meet the directrix in L. Then F is fixed point; and Q, F, L, Q′ all lie on the circle described upon BT as diameter.

Let fall the perpendicular FR on QQ′; then we shall show that the locus of R is a fixed straight line, and consequently the envelope of QQ′ is another parabola having its focus at F.



Since from the point T two tangents are drawn, ∠PTS = ∠P′TL or ∠QTF = ∠LTQ′, therefore FQ = LQ′ and FL is parallel to QQ′ and perpendicular to FR. Now FR = FQ sin STP′ = BT sin STP sin STP′ = BT (SY / ST) (SY / SP) = BT (SA / ST). Produce FR to O so that FO = 2FR, and draw SM perpendicular to BT. Then FO . ST = BT . SX and FO/SX = BT/ST = BF/SM, hence FO/BF = SX/SM.

Again, ∠BFO = complement of ∠BFL, or of ∠BTL = ∠TBL = ∠MSX, since S, X, M, B are concyclic. Thus the triangles BFO, MSX are similar, and ∠OBF = ∠XMS = ∠XBS. Hence the locus of O is a straight line through B, making with BF (the perpendicular from B on the given fixed straight line ST) an angle OBF equal to the angle XBS between BX and BS. Also, since F is a fixed point and FR = $\frac{1}{2}$FO, the locus of R is a fixed straight line through the middle point of BF. Hence the envelope of QQ′ is a parabola focus F.

8768. (R. F. Davis, M.A.)—Required an Analytical Proof of Feuerbach's Theorem.

[An analytical proof is given by the Rev. J. J. Milne, M.A., in the Introduction to his Weekly Problem Papers, but this proof may be modified and improved.]

Solution.

Let ABC be a triangle inscribed in a circle S, and t_1, t_2, t_3 be the lengths of the tangents from its angular points to another circle S′. Then, if
$$at_1 \pm bt_2 \pm ct_3 = 0,$$
these circles will touch one another.

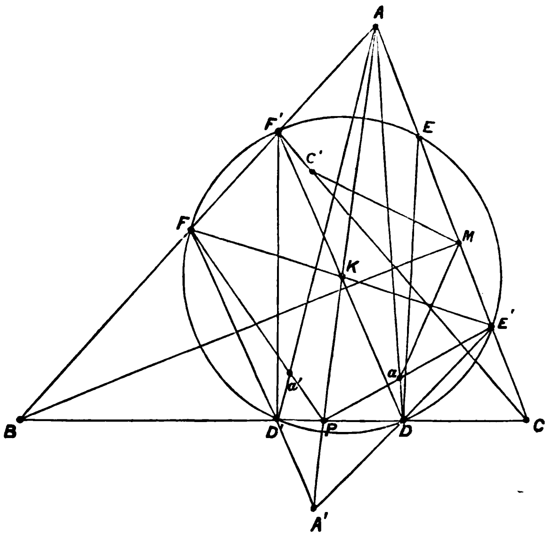
For if p_1, p_2, p_3 be the perpendiculars from A, B, C on the radical axis (L) of the two circles, t_1^2, t_2^2, t_3^2 are proportional to p_1, p_2, p_3 respectively. Hence $a\sqrt{p_1}+b\sqrt{p_2}+c\sqrt{p_3} = 0$, and this is the known condition that L should touch S, or that the points of intersection of S and S′ should coincide. Now the tangents from the middle points of the sides of any triangle to the inscribed circle may easily be shown to satisfy the above relation; hence the circle through these middle points touches the inscribed circle.

8875. (Professor Nash, M.A.)—Professor Casey's Quest. 7839 may be enunciated as follows:—DD′, EE′, FF′ are the intersections of the sides of a triangle ABC with the cosine circle, the order of the letters being such that E′F, F′D, D′E are diameters. The circle round AE′F cuts the circles on AB, AC as diameters in the points a, a'; and b, b', c, c' are similarly determined upon the circles BF′D, CD′E. To show that the circles round ACa, BAb, CBc pass through ω the positive Brocard-point of ABC, and the circles ABa', BCb', CAc' through the negative Brocard-point. Prove the following additional properties:—(1) The circles ACa, ABa' touch at A the sides AB, AC, and intersect again in a vertex of Brocard's second triangle; (2) the tangents to these circles at aa' bisect AB, AC; (3) the points B, C, a, a' are concyclic, and the circle through them touches OB, OC; (4) AaD and Aa'D′ are collinear; (5) Ca, Ba' intersect upon the symmedian AP, and Ca', Ba upon the perpendicular AL; (6) aE′, a'F meet BC at the foot of the symmedian AP; (7) aa', E′F, BC, and the radical axis of the circles ABC, AE′F, are concurrent;

(8) the three points of concurrence of the three sets of lines in (7) lie upon a line parallel to $\omega\omega'$, and therefore the triangle formed by the lines aa', bb', cc' is in perspective with ABC; (9) the pole of aa' with respect to the circle AE′F lies upon the median of ADD′; (10) the pairs of points ac', ba', cb' are isogonal, and the inscribed conic whose foci are ac' touches CA at the foot of the perpendicular from B; (11) one directrix of this conic is the line joining C to the intersection of FD′ and DE′, and the other the line joining A to the intersection of EF′, FD′.

Solution.

Let ABC be a triangle, K its symmedian point, DD′, EE′, FF′ the intersections of the cosine circle (centre K) with the sides of the triangle so that E′KF, F′KD, D′KE are diameters.



Upon AD, AD′ take points a, a' such that

$$Aa \cdot AD = Aa' \cdot AD' = AF \cdot AB = AE' \cdot AC.$$

Then, since F, a', a, E′ are the inverses (with respect to A) of the four collinear points B, D′, D, C, the circumcircle of AE′F passes through a, a'. Also, since B, F, a, D are concyclic, the $\angle BaD = \angle BFD = \frac{1}{2}\pi$, by a known property of the cosine circle. Hence a lies on the circle described on AB as diameter; similarly a' lies on the circle described on AC.

Since D, a, E′, C are concyclic, and DE′ is parallel to AB, $\angle AaC = \pi - \angle CaD = \pi - \angle CE'D = \pi - A$. Similarly $\angle Aa'B = \pi - A$. Hence the circumcircles of ACa, ABa' pass one through the positive Brocard-point of ABC and one through the negative Brocard-point; while

(1) They intersect again in a vertex of BROCARD's second triangle. Moreover, since $\angle BAa = \angle aDE' = \angle aCA$, AB is a tangent to the first circle.

(2) The circles ACa, ABa may be considered as the inverses (with respect to A) of the perpendicular lines DE′, DF. They therefore intersect orthogonally, and the tangent at a to the first circle passes through the centre of the second circle which is the middle point of AB.

(3) Since F, D′, D, E′ are concyclic, their inverses B, a', a, C are also concyclic. Now $\angle BaC = \angle BaD + \angle CaD = \frac{1}{2}\pi + A$; and if O be the circum-centre $\angle OBC = \frac{1}{2}\pi - A$, hence OB touches the circle $Ba'aC$.

(4) The points a, a' were assumed to lie on AD, AD′ respectively.

(5) Since DE′, FD′ are parallel to AB, AC respectively, they intersect in a point A′ on the symmedian KA produced such that KA′ = KA. Then, since $AE'.AC = AF.AB$ and $A'D.A'E' = A'D'.A'F$, hence AKA is the radical axis of the circles $CE'aD$, $BFa'D'$.

In the remainder of this solution frequent use will be made of the well-known theorem that the radical axes of any three circles (taken two at a time) are concurrent.

From the circles $CE'aD$, $BFa'D'$, $Ba'aC$ we infer that Ca, Ba' meet on the symmedian AKA′; and from the circles ACa', ABa, $Ba'aC$, that Ca', Ba meet on the perpendicular AL.

(6) From the circles $CE'aD$, $BFa'D'$, AE′F we infer that aE', $a'F$ meet on the symmedian AKA′.

Moreover, since $\angle aBD = \frac{1}{2}\pi - \angle aDB = \frac{1}{2}\pi - \angle aE'C = \angle D'E'a$, the four points B, D′, a, E′ are concyclic. Then, from the circles $BD'aE'$, $BFa'D'$, AE′F, we infer that aE', $a'F$ meet on BC. But they have already been shown to meet on AKA′; they therefore meet at the foot P of the symmedian; and Ca, CA′ divide the angle at C harmonically.

(7) From the circles BFE′C, $Ba'aC$, AE′F we infer that aa', E′F, BC meet in a point α; which, since $\alpha B . \alpha C = \alpha E' . \alpha F$, is also a point on the radical axis of the circles ABC, AE′F. If β, γ be similarly determined, α, β, γ lie on a straight line which is the Pascal line of the hexagon DD′EE′FF′.

(8) This line $\alpha\beta\gamma$ is obviously the radical axis of the cosine- and circum-circles, whose direction must be perpendicular to KO or parallel to $\omega\omega'$.

(9) The pole of aa' with respect to the circle AE′F is obviously the symmedian line through A of the triangle Aaa'; or the median line of the triangle ADD′, since aa', DD′ are antiparallel.

(10) Draw Bc' perpendicular to CF′, and BM to AC. Then, since $\angle BaC + \angle Bc'C = \frac{1}{2}\pi + A + \frac{1}{2}\pi = \pi + A$; and similarly $\angle AaB + \angle Ac'B = \pi + C$; hence a, c' are isogonal conjugates. Moreover, since $\angle aMB = \angle aAB = \angle c'CB = \angle c'MB$, M is the point of contact of the side CA.

(11) Since $\angle BMC$ is a right angle, and $\angle BMa = \angle BAa = \angle aDE' = \angle aCM$; therefore $\angle MaC$ is a right angle and C a point on the directrix. Since CA, CB are tangents and a the focus, CA′ (the fourth harmonic to CA, Ca, CB) is the directrix.

[For information respecting the "cosine-circle" see the forthcoming volume, entitled "Companion to the Weekly Problem Papers," by the Rev. J. J. MILNE, M.A. (Macmillans), which has an admirable section devoted to the new geometry of the triangle, contributed by our correspondent the Rev. T. C. SIMMONS, M.A.]

Note on the Brocardal Ellipse.

1. **If a point Q be taken on the tangent at any point P of an ellipse (Fig. 1), such that the angle SQP is constant ($=\beta$), the locus of Q will be a circle whose centre O lies on the minor axis of the ellipse, and which has double contact with the ellipse.**

For, drawing SY perpendicular on the tangent, since $\angle QSY = \frac{1}{2}\pi - \beta$, and $SQ = SY \operatorname{cosec} \beta$, the locus of Q is a circle of radius $a \operatorname{cosec} \beta$. Let O be a point on the minor axis, such that $\angle SOC = \beta$, then the triangles CSY, OSQ are similar and in the linear proportion of $1 : \operatorname{cosec} \beta$. Hence the locus of Q is a circle whose centre is O, and whose radius $= SO/e =$ the normal to the ellipse passing through O, and consequently has double contact (real or imaginary) with the ellipse.

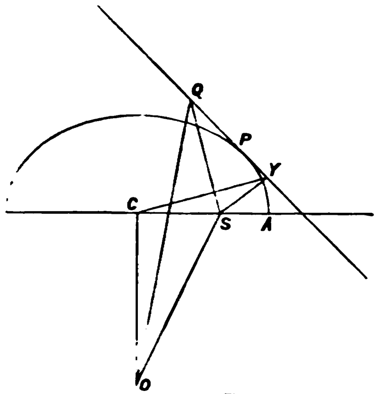


Fig. 1.

It is easily seen from the above proof that, when SY revolves in the *positive* direction through an angle $\frac{1}{2}\pi - \beta$ into the position SQ, the locus of Q has its centre O on the minor axis at a distance $ae \cot \beta$ *below* the centre; for the *negative* direction of rotation, O is at the same distance *above* the centre. There is no restriction as to the value of β, and by varying it we get a series of circles having double contact with the ellipse.

2. If the tangent to the ellipse at P (Fig. 2) meet a particular circle in Q_1, and the other tangent from Q_1 to the ellipse meet the circle again in Q_2, then $\angle SQ_2Q_1 = \beta$; and so on. Starting from any point P on the ellipse in this way, we obtain an UNCLOSED polygon $PQ_1Q_2Q_3 \ldots Q_n$ of any number of sides we please, whose vertices lie on the circle, and whose sides touch the ellipse, such that

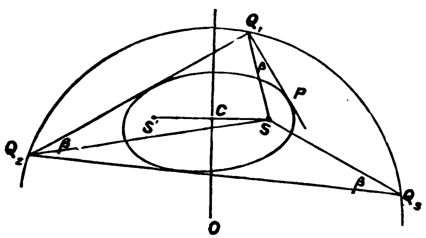


Fig. 2.

$$\angle SQ_1P = \angle SQ_2Q_1 = \angle SQ_3Q_2 = \ldots = \angle SQ_nQ_{n-1} = \beta;$$

and consequently, if S′ be the other focus,

$$\angle S'Q_1Q_2 = \angle S'Q_2Q_3 = \ldots = \angle S'Q_{n-1}Q_n = \beta.$$

3. If now we determine β, by means of the invariant condition, that it should be possible to inscribe figures of 3, 4, 5, 6...sides in the circle which at the same time circumscribe the ellipse, $Q_1Q_2 \ldots Q_n$ (of the pre-

vious article) is no longer an unclosed figure. The two foci become in this case the two Brocard points, and β becomes the Brocard angle ω.

Taking the equation to the two curves in the forms

$$x^2 / a^2 + y^2 / b^2 = 1 \quad \text{.................................} (\Sigma'),$$

and
$$x^2 + (y + ae \cot\omega)^2 = a^2 \operatorname{cosec}^2\omega,$$

$$x^2 + y^2 + 2aey \cot\omega - (a^2 + b^2 \cot^2\omega) = 0 \quad \text{..................} (\Sigma).$$

The equation determining κ, which causes $\Sigma + \kappa\Sigma'$ to break up into straight lines, is (after reduction)

$$(\kappa + a^2)^2 (\kappa + b^2 \operatorname{cosec}^2\omega) = 0.$$

Hence Δ', Θ', Θ, Δ are proportional to 1, $2a^2 + b^2 \operatorname{cosec}^2\omega$, $a^4 + 2a^2b^2 \operatorname{cosec}^2\omega$, $a^4b^2 \operatorname{cosec}^2\omega$, or as 1, $2+z$, $1+2z$, z, where $z = b^2 \operatorname{cosec}^2\omega / a^2$.

Now, the condition that it should be possible to inscribe triangles to Σ that are circumscribed to Σ' is $\Theta'^2 - 4\Delta'\Theta = 0$.

Hence $\qquad (2+z)^2 - 4(1+2z) = 0$, or $z^2 - 4z = 0$, $z = 4$.

Thus $\qquad \dfrac{b^2 \operatorname{cosec}^2\omega}{a^2} = 4$; $\quad \sin\omega = \dfrac{b}{2a}$; $\quad e^2 = 1 - \dfrac{b^2}{a^2} = 1 - 4\sin^2\omega$.

Quadrilateral: $\qquad \Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta'^2\Delta = 0.$

$$(2+z)^3 - 4(2+z)(1+2z) + 8z = 0, \quad z^3 - 2z^2 = 0, \quad z = 2.$$

Thus $\qquad \dfrac{b^2 \operatorname{cosec}^2\omega}{a^2} = 2$ or $\sin\omega = \dfrac{b}{\sqrt{2}a}$.

Hexagon:

$$\{\Theta'^2 - 4\Delta'\Theta\}^3 = 4\{\Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta'^2\Delta\}\{\Theta'^3 - 4\Theta\Theta'\Delta' + 16\Delta'^2\Delta\},$$

whence $z = \frac{4}{3}$.

4. Summarizing results:—If R be the circum-radius and ω the Brocard angle of a

(i.) Triangle: the semi-major axis of the Brocard ellipse is $R\sin\omega$, the semi-minor axis $2R\sin^2\omega$, and the eccentricity $= \sqrt{(1 - 4\sin^2\omega)}$.

(ii.) Quadrilateral: $R\sin\omega$, $\sqrt{2}R\sin^2\omega$, $\sqrt{(\cos 2\omega)}$.

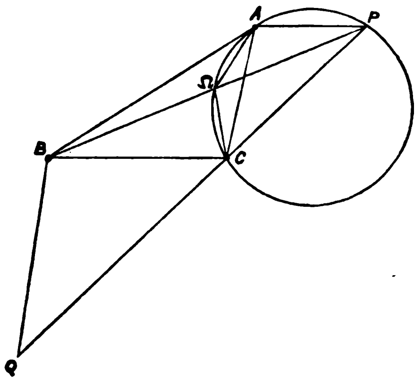
(iii.) Hexagon: $R\sin\omega$, $\dfrac{2}{\sqrt{3}}R\sin^2\omega$, $\sqrt{(1 - \frac{4}{3}\sin^2\omega)}$.

The above note was suggested by, and is an amplification of, No. 1056 in Professor WOLSTENHOLME's larger collection of Problems. Published so far back as 1878, it contains (as was first pointed out to me by the Rev. T. C. SIMMONS) a precise and complete statement of the existence of the Brocard ellipse.

Geometrical Construction for the Brocard-Angle, &c.

I. Let ABC be a triangle. Describe a circle touching AB in A and passing through C. Let AP be the chord of this circle parallel to BC. Join BP, meeting the circle in Ω, which will be the positive Brocard-point.

For, since AB touches the circle in A, $\angle \Omega AB = \angle \Omega CA$, and also $= \angle \Omega PA = \angle \Omega BC$. An analogous construction gives the negative Brocard-point Ω'.



II. Since $\angle ACB = \angle CAP$ and $\angle BAC = \angle APC$, the triangle PCA is inversely similar to ABC. Hence a simple ruler-construction can be stated for the graphical determination of the Brocard-angle of a triangle.

This construction (on the authority of the Rev. T. C. Simmons, M.A.) is already known to continental mathematicians.

III. Similarly, if on BC a triangle CQB be described inversely similar to ABC, $\angle AQB = \angle APB = \omega$, the Brocard-angle; and, since $\angle ACP = B$ and $\angle BCQ = A$, the points P, C, Q are collinear; also $PC : AC = AB : BC$ and $QC : BC = AB : AC$, therefore

$$PC \cdot QC = AB^2.$$

Now P, Q both lie on the circle Σ, which is the locus of points at which AB subtends a constant angle ω, supposing AB and ω only given. If $AB = a$, the centre of Σ lies on the line bisecting AB at right angles at a distance therefrom $= \frac{1}{2}a \cot \omega$; while radius $= \frac{1}{2}a \operatorname{cosec} \omega$. Moreover, since $PC \cdot QC = a^2 =$ constant, the locus of C is a circle concentric with Σ, whose radius ρ is given by the equation

$$\tfrac{1}{4}a^2 \operatorname{cosec}^2 \omega - \rho^2 = a^2 \quad \text{or} \quad 4\rho^2 = a^2(\cot^2 \omega - 3).$$

The locus of C under these circumstances is termed Neuberg's circle; and its equation, referred to AB and the perpendicular bisector of AB as axes, is $x^2 + (y - \frac{1}{2}a \cot \omega)^2 = \frac{1}{4}a^2(\cot^2 \omega - 3)$ or $x^2 + y^2 - ay \cot \omega + \frac{3}{4}a^2 = 0$.

APPENDIX III.

SOLUTIONS OF SOME OLD QUESTIONS.

By W. J. CURRAN SHARP, M.A.

2391. (Professor SYLVESTER, F.R.S.)—Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order, and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call the *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

3651. (Professor SYLVESTER, F.R.S.)—If through $3n+1$ given points on a cubic curve a curve of the order $N+n$ be drawn, and through the remaining $3N-1$ intersections of the two curves a third one be drawn of the order N; prove that this will intersect the cubic at a fixed point.

[This point may be called the opposite of the $3n+1$ given points; it becomes Dr. SALMON's opposite, as defined by him in the *Philosophical Transactions* for 1858, when $n=1$, $N=1$, and is independent of the value of N.]

Solution.

(1) If $\mu=3i-1$. Since every n-ic curve through $3n-1$ points in a cubic cuts it in the same additional point (SALMON's H. P. C., p. 133), any i-ic through the μ points will intersect the cubic again in the same point A (the residual). Let this be $V=0$.

Now if $U=0$	be a	ν-ic	through the	μ	points and	μ_1 others,
$U_1=0$	,,	ν_1-ic	,,	μ_1	,,	μ_2 ,,
$U_2=0$	,,	ν_2-ic	,,	μ_2	,,	μ_3 ,,
...	...	...	...	...	...	...
$U_{2r}=0$	,,	ν_{2r}-ic	,,	μ_{2r}	,,	1 ,,

this one point will be A. For evidently $3\nu_r=\mu_r+\mu_{r+1}$, and if $i+\nu_1+\nu_3 \ldots +\nu_{2r-1} = m,$

$$\begin{aligned} 3m &= 3i+3\nu_1+3\nu_3+ \ldots\ldots\ldots\ldots +3\nu_{2r-1} \\ &= 1+\mu+\mu_1+\mu_2+\mu_3+ \ \ldots\ +\mu_{2r-1}+\mu_{2r} \\ &= 1+3\nu+3\nu_2+ \ldots\ldots\ldots\ldots +3\nu_{2r-2}+3\nu_{2r}-1 \\ &= 3(\nu+\nu_2+ \ \ldots\ldots\ldots\ldots\ldots +\nu_{2r}) \end{aligned}$$

and $\mu+\mu_1+\mu_2+ \ \ldots\ +\mu_{2r} = 3m-1;$

therefore $\mathrm{V} . \mathrm{U}_1 . \mathrm{U}_3 \ldots \mathrm{U}_{2r-1} = 0$ and $\mathrm{U} . \mathrm{U}_2 \ldots \mathrm{U}_{2r} = 0$ are m-ic curves through the same $3m-1$ points on the cubic, and therefore through the same additional point, which is A, the residual on the first, and the remaining intersection of $\mathrm{U}_{2r} = 0$ on the second; which proves the principle for this case.

(2) In the other case it is advisable, first, to show that every n-ic curve through $3n-2$ points on a cubic cuts it again in two, such that the line joining them passes through a fixed point.

For, if $\mathrm{V} = 0$ be one n-ic through the $3n-2$ points and A and B
$\mathrm{W} = 0$ be another ,, ,, ,, and C and D
and $\alpha = 0$ be the line AB and $\beta = 0$ the line CD.

$\beta\mathrm{V} = 0$ and $\alpha\mathrm{W} = 0$ are two $(n+1)$-ics through the same $3n+2$ points (viz., the given ones and A, B, C, and D), and therefore their remaining intersections with the curve must coincide, *i.e.*, $\alpha = 0$ and $\beta = 0$ meet the cubic in a fixed point.

If $\mu = 3i+1$, and V be $(i+1)$-ic through the μ points and A and B, and $\alpha = 0$ be the line AB, the third intersection of $\alpha = 0$ is fixed.

Now let $\mathrm{U} = 0$ be a ν-ic through the μ points and μ_1 others
$\mathrm{U}_1 = 0$,, ν_1-ic ,, μ_1 ,, μ_2 ,,
... ...
$\mathrm{U}_{2r-1} = 0$,, ν_{2r-1}-ic ,, μ_{2r-1} ,, 1 ,,

then, if $m = 1+\nu+\nu_2+ \ \ldots\ +\nu_{2r-2},$

$$\begin{aligned} 3m &= 3+3\nu+3\nu_2+\ldots+3\nu_{2r-2} = 3+\mu+\mu_1+\mu_2+\mu_3 \ldots +\mu_{2r-1} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \because\ 3\nu_r = \mu_r+\mu_{r+1} \\ &= 3i+4+3\nu_1+3\nu_3 \ldots +3\nu_{2r-1}-1 = 3\left(1+i+\nu_1+\nu_3 \ldots +\nu_{2r-1}\right), \end{aligned}$$

therefore $\alpha\ \mathrm{U} . \mathrm{U}_2 \ldots \mathrm{U}_{2r-2} = 0$ and $\mathrm{V} . \mathrm{U}_1 . \mathrm{U}_3 \ldots \mathrm{U}_{2r-1} = 0$ are m-ic curves through A, B, and the $\mu+\mu_1 \ldots +\mu_{2r-1}$ points on the cubic, *i.e.*, through the same $3m-1$ points, and therefore the same additional point, which is the third intersection of $\alpha = 0$ (AB).

This important theorem is thus fully proved. In Question 3651, the single point is the one co-residual to the $3n+1$, and is therefore fixed by (2) above.

3535. (Professor Sylvester, F.R.S.)—1. Euler has shown that the number of modes of composing n with i distinct numbers is equal to the denumerant (that is, the number of solutions in positive integers, zeros included) of the equation

$$x+2y+3z+\ldots+i\omega=n-\tfrac{1}{2}(i^2+i).$$

(1) Show more generally that the number of modes of composing n with i numbers, all distinct except the largest, which is to be always taken j times, is the denumerant of the equation

$$jx+(j+1)y+\ldots+i\omega=n-\tfrac{1}{2}(i-j+1)(i+j).$$

(2) Show also that, if all the partitions of n into i parts are distinct except the *least*, which is to be taken j times, then the number of such partitions is the denumerant of the equation

$$x+2y+\ldots+(i-j)\phi+i\omega=n-\tfrac{1}{2}\{(i-j)^2+3i-j\}.$$

Solution.

1. The required number of ways of composing n with i numbers all distinct except the largest, which is to be taken j times, is the coefficient of $x^n z^i$ in

$$x^j z^j+(1+xz)\,x^{2j}z^j+(1+xz)(1+x^2z)\,x^{3j}z^j+\&c.$$
$$=x^j z^j\{A_0+A_1z+A_2z^2+\&c.\}\text{ suppose.}$$

That is, it is the coefficient of $x^{n-j}z^{i-j}$ in

$$1+(1+xz)\,x^j+(1+xz)(1+x^2z)x^j+\&c.=A_0+A_1z+A_2z^2+\&c.,$$

where $A_0=\dfrac{1}{1-x^j}$.

Let $xz=y$, therefore

$$1+(1+y)\,x^j+(1+y)(1+xy)\,x^{2j}+\&c.=A_0+\frac{A_1}{x}\,y+\frac{A_2}{x^2}\,y^2+\&c.;$$

but $\quad 1+(1+y)\,x^j+(1+y)(1+xy)\,x^{2j}+\&c.$

$$=1+(1+y)\,x^j\{1+(1+xy)\,x^j+(1+xy)(1+x^2y)\,x^{2j}+\&c.\}$$
$$=1+(1+y)\,x^j\{A_0+A_1y+A_2y^2+\&c.\},$$

and, equating coefficients,

$$\frac{A_1}{x}=x^j(A_0+A_1),\ \ \frac{A_2}{x^2}=x^j(A_1+A_2),\text{ and generally }\frac{A_n}{x^n}=x^j(A_{n-1}+A_n);$$

therefore $\qquad A_1=\dfrac{x^{j+1}}{1-x^{j+1}}A_0,\quad A_2=\dfrac{x^{j+2}}{1-x^{j+2}}A_1,$ &c.,

$$A_n=\frac{x^{nj+\frac{1}{2}n(n+1)}}{(1-x^j)(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+n})},$$

and therefore the number required, the coefficient x^{n-j} in A_{i-j}, is the coefficient of $\quad x^{n-\frac{1}{2}(i-j+1)(i+j)}$ in $\dfrac{1}{(1-x^j)(1-x^{j+1})(1-x^{j+2})\ldots(1-x^i)}$,

which is the denumerant of

$$jx+(j+1)\,y+(j+2)\,z+\ldots+i\omega=n-\tfrac{1}{2}(i-j+1)(i+j).$$

2. If all the partitions of n into i parts are distinct, except the least, which is always to be taken j times, the number of these is the coefficient of $x^n z^i$ in $x^j(z^j)(1+x^2z)(1+x^3z)\ldots+x^{2j}z^j(1+x^3z)(1+x^4z)\ldots+\&c.,$

or that of $x^{n-j}z^{i-j}$ in

$$(1+x^2z)(1+x^3z)\ldots+x^j(1+x^3z)(1+x^4z)\ldots+\&c.=A_0+A_1z+A_2z^2+\&c.,$$

suppose.

If $xz=y$ this becomes

$$(1+xy)(1+x^2y)(1+x^3y)\ldots+x^j(1+x^2y)(1+x^3y)\ldots+\&c.$$
$$=A_0+A_1\frac{y}{x}+A_2\frac{y^2}{x^2}+\&c.,$$

but the first member $=(1+xy)(1+x^2y)(1+x^3y)\ldots$

$$+x^j\left\{(1+x^2y)(1+x^3y)\ldots+x^j(1+x^3y)(1+x^4y)\ldots+\&c.\right\}$$
$$=1+\frac{x}{1-x}y+\frac{x^3}{(1-x)(1-x^2)}y^2+\&c.+x^j\left\{A_0+A_1y+A_2y^2+\&c.\right\},$$

and therefore $\dfrac{A_n}{x^n}=\dfrac{x^{\frac{1}{2}n(n+1)}}{(1-x)(1-x^2)\ldots(1-x^n)}+A_nx^j,$

$$A_n=\frac{x^{\frac{1}{2}n(n+3)}}{(1-x)(1-x^2)\ldots(1-x^n)(1-x^{j+n})},$$

and therefore the number required, the coefficient of x^{n-j} in A_{i-j}, is the coefficient of $x^{n-\frac{1}{2}[(i-j)^2+3i-j]}$ in $\dfrac{1}{(1-x)(1-x^2)\ldots(1-x^{i-j})(1-x^i)}$,

which is the denumerant of

$$x+2y+\ldots+(i-j)\,\phi+i\omega=n-\tfrac{1}{2}\left\{(i-j)^2+3i-j\right\}.$$

3427. (Professor SYLVESTER, F.R.S.)—If ϕ, ψ are quantics in x, y, each of degree μ; F a quantic in ϕ, ψ of degree m, and consequently in x, y of degree $m\mu$; and if J denote the Jacobian of ϕ, ψ, that is,

$$\frac{d\phi}{dx}\frac{d\psi}{dy}-\frac{d\phi}{dy}\frac{d\psi}{dx};$$

$D_{\phi,\psi}$ F the discriminant of F treated as a quantic in ϕ, ψ; $D_{x,y}$ F the discriminant of F, treated as a quantic in x, y; and if R be used as the symbol of "resultant in regard of x, y"; prove that

$$D_{x,y}\,F=2\left[R(\phi,\psi)\right]^{m^2-2m}R(F,J)\,(D_{\phi,\psi}F)^{\mu}.$$

As a particular case of the foregoing theorem, show that the discriminant of F, any *symmetrical* quantic of an even degree in x, y, is of the form F (1, 1) F (1, -1) Ω^2, where Ω is a rational integral function of the coefficients in F.

Find also what the general formula becomes when ϕ, ψ are taken linear functions of x, y.

Solution.

By the Question, $F\equiv(\phi-\alpha\psi)(\phi-\beta\psi)(\phi-\gamma\psi)$ to m factors, therefore $D_{x,y}F=$ the product of the discriminants of $\phi-\alpha\psi$, $\phi-\beta\psi$, &c. $\times$ the squares of the resultants of $\phi-\alpha\psi$, $\phi-\beta\psi$, &c. taken two and two.

Now the resultant of $\phi-\alpha\psi$ and $\phi-\beta\psi$ is $(\alpha-\beta)^{\mu}\,\mathrm{R}(\phi,\psi)$, (SALMON'S *Higher Algebra*, Arts. 180, &c., whence this solution, due to Prof. SYLVESTER, *Comptes Rendus*, LVIII., 1078, is mainly derived.)

Therefore $\mathrm{D}_{x,y}\,\mathrm{F}=$ the product of the discriminants of $\phi-\alpha\psi$, $\phi-\beta\psi$, &c.

$$\times(\alpha-\beta)\quad(\alpha-\gamma)^{2\mu}\,\ldots\ldots\ldots\,[\mathrm{R}(\phi,\psi)]^{m(m-1)},$$

$$=\text{the above product}\times(\mathrm{D}_{\phi,\psi}\mathrm{F})^{\mu}\,[\mathrm{R}(\phi,\psi)]^{m(m-1)}.$$

Again, the discriminant of $\phi-\alpha\psi$ is the resultant of

$$\frac{d\phi}{dx}-\alpha\frac{d\psi}{dx}=0\quad\text{and}\quad\frac{d\phi}{dy}-\alpha\frac{d\psi}{dy}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\;(\mathrm{I.})$$

or, by Euler's theorem, of $\phi-\alpha\psi=0$, $\dfrac{d\phi}{dx}\cdot\dfrac{d\psi}{dy}-\dfrac{d\phi}{dy}\dfrac{d\psi}{dx}\equiv\mathrm{J}=0$,

unless $\phi=0$, $\psi=0$, or the x or y introduced vanish, which last suppositions only give numerical factors, and the discriminant can only differ by such a factor from $\mathrm{R}(\phi-\alpha\psi,\,\mathrm{J})\div\mathrm{R}(\phi,\psi)$, and therefore the product of the discriminants $=\mathrm{R}(\mathrm{F},\mathrm{J})\div[\mathrm{R}(\phi,\psi)]^{m}$;

and therefore $\quad\mathrm{D}_{x,y}\,\mathrm{F}=[\mathrm{R}(\phi,\psi)]^{m^2-2m}\,\mathrm{R}(\mathrm{F},\mathrm{J})\,(\mathrm{D}_{\phi,\psi}\mathrm{F})^{\mu}$.

If F be a symmetrical quantic of even degree, let $\phi=x^2+y^2$, $\psi=xy$;

therefore $\quad\mathrm{J}=2(x^2-y^2)$ and $\mathrm{R}(\mathrm{F},\mathrm{J})=\mathrm{F}(1,1)\,\mathrm{F}(1,-1)$,

$$\mathrm{R}(\phi,\psi)=1\text{ and }(\mathrm{D}_{\phi,\psi}\mathrm{F})^{\mu}=(\alpha-\beta)^{2\mu}(\alpha-\gamma)^{2\mu}\ldots\ldots=\Omega^2,$$

where Ω is a rational integral function of the coefficients of F.

If $\quad\phi=\lambda x+\mu y,\quad\psi=\lambda' x+\mu' y,\quad\mathrm{R}(\phi,\psi)=\lambda'\mu-\lambda\mu'$,

$$\mathrm{J}=\lambda'\mu-\lambda\mu',\quad\text{and}\quad\mathrm{R}(\mathrm{F},\mathrm{J})=(\lambda'\mu-\lambda\mu')^{m};$$

therefore, by substitution, $\quad\mathrm{D}_{x,y}(\mathrm{F})=(\lambda'\mu-\lambda\mu')^{m(m-1)}\,(\mathrm{D}_{\phi,\psi}\mathrm{F})$,

which is the evidently correct result, since the discriminant is an invariant, and $\lambda'\mu-\lambda\mu'$ the modulus of the linear transformation.

5271. (Professor CAYLEY, F.R.S.)—If ω be an imaginary cube root of unity, show that, if

$$y=\frac{(\omega-\omega^2)\,x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)\,x^2},\ \text{then}\ \frac{dy}{(1-y^2)^{\frac12}(1+\omega y^2)^{\frac12}}=\frac{(\omega-\omega^2)\,dx}{(1-x^2)^{\frac12}(1+\omega x^2)^{\frac12}};$$

and explain the general theory.

Solution.

If
$$\frac{dy}{\left\{(1-y^2)(1-\lambda^2y^2)\right\}^{\frac12}}=\frac{m\,dx}{\left\{(1-x^2)(1-\kappa^2x^2)\right\}^{\frac12}}\;\ldots\ldots\ldots\ldots\ldots\;(1),$$

y must be a function of x. Let $y=\phi(x)$.

Now if, in (1), $-y$ and $-x$ be written for y and x, the equation is unaltered; therefore $-y\equiv\phi(-x)$, and therefore xy and y/x are even functions of x. Let $y/x\equiv\psi(x^2)$ .... (2).

(This form is chosen in order that y and x may vanish together.)

Again, if, in (1), $\frac{1}{\lambda y}$ and $\frac{1}{\kappa x}$ be written for y and x, the equation is unaltered; therefore $$\frac{\kappa x}{\lambda y} \equiv \psi\left(\frac{1}{\kappa^2 x^2}\right),$$

and therefore $$\frac{\kappa}{\lambda} \equiv \psi(x^2)\,\psi\left(\frac{1}{\kappa^2 x^2}\right) \quad \text{.......................} \quad (3).$$

It therefore appears that $\psi(x^2)$ is fractional. Now, assuming that the transformation is rational, let

$$\psi(x^2) = \frac{\alpha(x^2)}{\beta(x^2)} \equiv \frac{A\,(a_1-x^2)\,(a_2-x^2)\,\ldots\,(a_m-x^2)}{B\,(b_1-x^2)\,(b_2-x^2)\,\ldots\,(b_n-x^2)};$$

and, from (3), $$\kappa\beta(x^2)\,\beta\left(\frac{1}{\kappa^2x^2}\right) \equiv \lambda\,\alpha(x^2)\,\alpha\left(\frac{1}{\kappa^2x^2}\right),$$
or

$$\kappa B^2(b_1-x^2)(b_2-x^2)\ldots(b_n-x^2) \times \left(b_1-\frac{1}{\kappa^2x^2}\right)\left(b_2-\frac{1}{\kappa^2x^2}\right)\ldots\left(b_n-\frac{1}{\kappa^2x^2}\right)$$

$$\equiv \lambda A^2(a_1-x^2)(a_2-x^2)\ldots(a_m-x^2) \times \left(a_1-\frac{1}{\kappa^2x^2}\right)\left(a_2-\frac{1}{\kappa^2x^2}\right)\ldots\left(a_m-\frac{1}{\kappa^2x^2}\right),$$

an identity which involves $m = n$ (unless some of the quantities $a_1 a_2 \ldots a_m$, $b_1 b_2 \ldots b_n$ be zero), and a series of conditions of the form $b_r - x^2 = 0$, equivalent to $a_r - x^2 = 0$, or $a_r - [1/(\kappa^2x^2)] = 0$. Each equivalence of the first class serves to reduce the order of the transformation, while each of the second class gives $b_r = 1/(a_r\kappa^2)$,

and therefore $$\frac{y}{x} = \frac{A}{B}\,\frac{(a_1-x^2)(a_2-x^2)\ldots(a_n-x^2)}{\left(\frac{1}{a_1\kappa^2}-x^2\right)\left(\frac{1}{a_2\kappa^2}-x^2\right)\ldots\left(\frac{1}{a_n\kappa^2}-x^2\right)}$$

$$= \frac{A}{B}\,(ab\ldots l)^4\,\kappa^{2n}\,\frac{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x^2}{b^2}\right)\ldots\left(1-\frac{x^2}{l^2}\right)}{(1-a^2\kappa^2x^2)(1-b^2\kappa^2x^2)\ldots(1-l^2\kappa^2x^2)}$$

if $a_1 = a^2$, $a_2 = b^2$, &c. &c.;

and, from (3), $\frac{\kappa}{\lambda} = \frac{A^2}{B^2}(ab\ldots l)^2\kappa^{2n}$; therefore $\frac{A}{B} = \frac{1}{(\kappa^{2n-1}\lambda)^{\frac{1}{2}}(ab\ldots l)^2}$,

$$\frac{y}{x} = \left(\frac{\kappa^{2n+1}}{\lambda}\right)^{\frac{1}{2}}(ab\ldots l)^2\,\frac{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x^2}{b^2}\right)\ldots\left(1-\frac{x^2}{l^2}\right)}{(1-a^2\kappa^2x^2)(1-b^2\kappa^2x^2)\ldots(1-l^2\kappa^2x^2)},$$

while, by substitution in (1) and putting $x = 0$, it appears that

$$\left(\frac{\kappa^{2n+1}}{\lambda}\right)(ab\ldots l)^4 = m^2, \text{ \&c.},$$

therefore $$y = mx\,.\,\frac{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x^2}{b^2}\right)\ldots\left(1-\frac{x^2}{l^2}\right)}{(1-a^2\kappa^2x^2)(1-b^2\kappa^2x^2)\ldots(1-l^2\kappa^2x^2)},$$

which agrees with the form given by Prof. CAYLEY (*Elliptic Functions*, p. 169), and the quantities $a, b, \dots l$ might be found from the identity

$$(1-x^2)(1-\kappa^2x^2)(dy)^2 = (1-y^2)(1-\lambda^2y^2)\,m^2(dx)^2.$$

This would, however, be a very long process, and an equivalent result may be obtained as follows.

Since $(1-y^2)(1-\lambda^2y^2)$ must be equal to $(1-x^2)(1-\kappa^2x^2)$ multiplied by a square factor, assume either

$$1-y = \frac{\{f(x)\}^2(1-x)}{\beta(x^2)} \quad\text{or}\quad \frac{\{f(x)\}^2(1+x)}{\beta(x^2)},$$

then

$$1+y = \frac{\{f(-x)\}^2(1+x)}{\beta(x^2)} \quad\text{or}\quad \frac{\{f(-x)\}^2(1-x)}{\beta(x^2)},$$

$$1-\lambda y = \frac{\lambda m x a(x^2)}{\beta(x^2)}\cdot\frac{\left\{f\left(\frac{1}{\kappa x}\right)\right\}^2(1-\kappa x)}{\kappa x\beta\left(\frac{1}{\kappa^2x^2}\right)} \quad\text{or}\quad \frac{\lambda m x a(x^2)}{\beta(x^2)}\cdot\frac{\left\{f\left(\frac{1}{\kappa x}\right)\right\}^2(1+kx)}{\kappa x\beta\left(\frac{1}{\kappa^2x^2}\right)}$$

$$1+\lambda y = \frac{\lambda m x a(x^2)}{\beta(x^2)}\cdot\frac{\left\{f\left(-\frac{1}{\kappa x}\right)\right\}^2(1+\kappa x)}{\kappa x\beta\left(\frac{1}{\kappa^2x^2}\right)}$$

or

$$\frac{\lambda m x a(x^2)}{\beta(x^2)}\cdot\frac{\left\{f\left(-\frac{1}{\kappa x}\right)\right\}^2(1-\kappa x)}{\kappa x\beta\left(\frac{1}{\kappa^2x^2}\right)},$$

of which either set fulfils the condition, while the equation assumed

$$1-y = \frac{\{f(x)\}^2(1-x)}{\beta(x^2)} \text{ or } \frac{\{f(x)\}^2(1+x)}{\beta(x^2)}$$

gives just enough conditions to determine $a, b, \dots l$, and give a relation, the modular equation, between κ and λ.

In the particular case where $n=1$ — the cubic transformation (assuming the second form that the quantities may be real),

let
$$1-y = \frac{(1-hx)^2(1+x)}{1-a^2\kappa^2x^2} \equiv \frac{1-a^2\kappa^2x^2-\left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}}(a^2-x^2)\,x}{1-a^2\kappa^2x^2}$$

and let $\kappa = u^4$, $\lambda = v^4$.

Therefore
$$2h-1 = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}}a^2 = \frac{u^6}{v^2}a^2,\quad 2h-h^2 = a^2\kappa^2 = u^8a^2,$$

$$h^2 = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}} = \frac{u^6}{v^2};$$

therefore
$$\frac{2h-h^2}{v^2} = (2h-1)\,u^2, \text{ or } 2h(1-u^2v^2) = h^2-u^2v^2;$$

therefore
$$\pm 2\frac{u^3}{v}(1-u^2v^2) = \frac{u^6}{v^2}-u^2v^2, \text{ or } \pm 2uv(1-u^2v^2) = u^4-v^4,$$

the known form of the modular equation (CAYLEY's *Functions*, p. 188).

Returning to the form $y = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}} a^2 x \, \frac{1 - \frac{x^2}{a^2}}{1 - a^2\kappa^2 x^2}$,

and comparing it with the given equation $y = \frac{(\omega - \omega^2)\, x + \omega^2 x^2}{1 - \omega^2 (\omega - \omega^2)\, x^2}$.

It appears that $m = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}} a^2 = \omega - \omega^2$, $\frac{1}{a^2} = -\frac{\omega^2}{\omega - \omega^2}$,

and $a^2\kappa^2 = \omega^2 (\omega - \omega^2)$; therefore $\kappa^2 = -\omega^4 = -\omega$, and $\left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}} = -\omega^2$;

therefore $\frac{\kappa^3}{\lambda} = \omega^4 = -\kappa^2$, and $\kappa = -\lambda$, and $\lambda^2 = \kappa^2 = -\omega$;

and therefore $$\frac{dy}{(1-y^2)^{\frac{1}{2}}(1+\omega y^2)^{\frac{1}{2}}} = \frac{(\omega-\omega^2)\, dx}{(1-x^2)^{\frac{1}{2}}(1+\omega x^2)^{\frac{1}{2}}},$$

a result which may be obtained from the equations which lead to the modular equation, viz.,

$$2h - 1 = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}} a^2, \quad 2h - h^2 = a^2\kappa^2, \quad h^2 = \left(\frac{\kappa^3}{\lambda}\right)^{\frac{1}{2}}.$$

In these let $\kappa = -\lambda$;

therefore $2h - 1 = \kappa(-1)^{\frac{1}{2}} a^2, \quad 2h - h^2 = \kappa^2 a^2, \quad h^2 = \kappa(-1)^{\frac{1}{2}}$,

therefore $(2h-1)\, h^2 + 2h - h^2 = 0$,

and, since h is not zero, $h^2 - h + 1 = 0$, therefore $h = -\omega$,

therefore $\kappa^2 = -h^4 = -\omega^4 = -\omega = \lambda^2$,

and $\kappa^2 a^2 = 1 - (1-h)^2 = -2\omega - \omega^2 = 1 - \omega = \omega^2(\omega - \omega^2)$,

and $a^2 = -\omega(\omega - \omega^2) = -\frac{\omega - \omega^2}{\omega^2}$,

which give the same transformation.

5420. (Professor Sylvester, F.R.S.) — From the expansion of $\{\log(1+x)\}^n$, in a series according to powers of x, prove that $S_{i,j}$ [the coefficient of t^j is the developed product of $(1+t)(1+2t)\ldots\ldots(1+it)$] is divisible by every prime number greater than $j+1$ contained in any term in the series $i+1$, i, $i-1$, ..., $i-j+1$.

Solution.

Let $z = \log_e(1+x)$, therefore $1+x = e^z$, and

$$e^y = (1+x)^{y/z}$$

$$= 1 + \frac{y}{1}\,\frac{x}{z} + \frac{y(y-z)}{1.2}\left(\frac{x}{z}\right)^2 + \frac{y(y-z)(y-2z)}{1.2.3}\left(\frac{x}{z}\right)^3 + \&c.$$

$$= 1 + \frac{y}{z}\left\{\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.\right\}$$

$$+ \frac{y^2}{z^2}\left\{\frac{x^2}{1.2} - S_{2,1}\frac{x^3}{1.2.3} + S_{3,2}\frac{x^4}{4!} - \&c.\right\} + \&c.$$

$$+ \frac{y^n}{z^n}\left\{\frac{x^n}{n!} - S_{n,1}\frac{x^{n+1}}{(n+1)!} + S_{n+1,2}\frac{x^{n+2}}{(n+2)!} - \&c.\right\} + \&c.$$

$$= 1 + \frac{y}{1} + \frac{y^2}{1.2} + \ldots + \frac{y^n}{n!} + \&c.;$$

therefore $\{\log(1+x)\}^n = z^n = x^n\left\{1 - \frac{S_{n,1}}{n+1}x + \frac{S_{n+1,2}}{(n+1)(n+2)}x^2 + \&c.\right\}$,

and hence $\frac{S_{i,j}}{(i-j+2)\ldots(i+1)}(-1)^j$ is the coefficient of x^{i+1} in

$$\{\log(1+x)\}^{i-j+1} = x^{i-j+1}\left\{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \&c.\right\}^{i-j+1},$$

which proves the proposition, as the coefficient of x^{i+1} in this last expansion cannot involve any prime factor (in the denominator) greater than $j+1$, and will involve $i-j+1$ (in the numerator), if that number be a prime, unless $n(i-j+1) = i+1$ when $i+1 = \frac{nj}{n-1}$ and $i-j+1 = \frac{j}{n-1}$ and $i-j+1$ is not greater than $j+1$.

[Prof. SYLVESTER observes that his theorem ought to be stated as follows:—$S_{i,j}$ contains every divisor of the product $(i+1)\,i\,(i-1)\ldots(i-j+1)$ which has no prime factor less than $j+2$. He also observes that the well-known theorem ordinarily associated with WILSON's, viz., that $S_{p-1,j}$ when p is a prime number and $j < p-1$ contains p, is an immediate consequence of his theorem, which teaches this, but much more besides; that is to say, his theorem is at once a generalization and an extension of the theorem associated with WILSON's theorem.

In general the coefficient of x^j in $\left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^j}{j+1} + \&c\right)^{i-j+1}$ consists of two groups of fractions. In one group each fraction $(i-j+1)$ appears as a factor in its numerator, in the other group the factor $(i-j+1)$ will not so appear.

But this latter group has no existence unless j contains $i-j+1$ as a factor, which cannot be the case when $i-j+1$ is greater than j nor *à fortiori* when $i-j+1$ is greater than $j+1$. In every fraction of either group the denominator contains only powers of 2, 3, ... $(j+1)$.

Hence, striking out from the product of

$$(i+1)\,i\,.\,(i-1)\ldots(i-j+2)\,(i-j+1)$$

all powers which it contains of the numbers 2, 3, ... $(j+1)$, or, which is the same thing, of all the prime numbers not inferior to $j+2$, the quantity which remains must be a divisor of $S_{i,j}$.

5640. (W. J. Curran Sharp, M.A.)—Salmon says [*Higher Plane Curves*, p. 98] that the curve parallel to a given curve may be obtained (1) as the envelope of a circle of given radius whose centre moves on the given curve; (2) as the envelope of the parallel to the tangent to the given curve drawn at a constant distance. Prove that these processes are equivalent.

Solution.

On the first supposition, we have to find the envelope of

$$(\xi-x)^2+(\eta-y)^2=k^2 \text{ where } \phi(xy)=0 \text{ or } L+M\frac{dy}{dx}=0 \quad ......(1,\ 2).$$

Now, from (1), $(\xi-x)+(\eta-y)\dfrac{dy}{dx}=0$, therefore $\dfrac{\xi-x}{L}=\dfrac{\eta-y}{M}$,

and $\quad n\phi(xy)\equiv Lx+My+N=0$ and $(\xi-x)^2+(\eta-y)^2=k^2$;

$$\therefore\ \frac{\xi-x}{L}=\frac{\eta-y}{M}=\frac{k}{\sqrt{L^2+M^2}},\quad \therefore\ \frac{L(\xi-x)+M(\eta-y)}{L^2+M^2}=\frac{k}{\sqrt{L^2+M^2}},$$

therefore $\quad L\xi+M\eta+N=k\sqrt{L^2+M^2}$,

the equation to the line enveloped upon the second supposition.

5755. (W. J. C. Sharp, M.A.)—If, through any point of inflection O on an n-ic, there be drawn three straight lines meeting the curve in $A_1, A_2 \dots A_{n-1}$; $B_1, B_2 \dots B_{n-1}$; $C_1, C_2 \dots C_{n-1}$, respectively; prove that every curve of the n^{th} degree through the $3n-2$ points O, $A_1, A_2 \dots A_{n-1}$; $B_1 \dots B_{n-1}$; $C_1 \dots C_{n-1}$ will have O for a point of inflexion.

Solution.

If we call the second line, which, with the inflexional tangent at O, makes up the polar conic, a supra-harmonic polar, and if the three lines meet this in A, B, and C, these points are also common loci of supra-harmonic means of the point O with regard to all curves through the $3n-2$ points. This locus, then, which in general is a conic, must, since these three points of it are in a right line, be for all those curves the same right line; and, therefore, the point O must be a point of inflexion on each of them.

In the above I have called OR the supra-harmonic mean between $OR_1, OR_2 \dots OR_{n-1}$ f $\quad \dfrac{n-1}{OR}=\dfrac{1}{OR_1}+\dfrac{1}{OR_2}+\dots+\dfrac{1}{OR_{n-1}}$.

5809. (D. Edwardes.)—If O be any point within a triangle ABC, prove that $OA^2\sin 2A+OB^2\sin 2B+OC^2\sin 2C$ is least when O is the centre of the circumscribed circle.

7050. (D. EDWARDES.)—If P be any point in the plane of a triangle ABC, and d its distance from the circumscribed centre, show that

$$PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C = 4 (R^2 + d^2) \sin A \sin B \sin C.$$

7069. (D. EDWARDES.)—If x, y, z be the distances of a point P from the angular points of a triangle, prove that the mean value of $x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$, as P ranges over the circle about ABC, is three times the area of the triangle.

Solutions.

(7050.) If the centre of the circum-circle be taken as origin of rectangular Cartesian coordinates, and (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) denote P, A, B, C.

$$PA^2 = (x-x_1)^2 + (y-y_1)^2 = R^2 + d^2 - 2(xx_1 + yy_1),$$
$$PB^2 = (x-x_2)^2 + (y-y_2)^2 = R^2 + d^2 - 2(xx_2 + yy_2),$$
$$PC^2 = (x-x_3)^2 + (y-y_3)^2 = R^2 + d^2 - 2(xx_3 + yy_3),$$

therefore
$$\begin{vmatrix} PA^2, & x_1, & y_1 \\ PB^2, & x_2, & y_2 \\ PC^2, & x_3, & y_3 \end{vmatrix} = (R^2 + d^2) \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix},$$

or $2PA^2 . \triangle BOC + 2PB^2 . \triangle COA + 2PC^2 . \triangle AOB = 2(R^2 + d^2) . \triangle ABC$...(A), and $2\triangle BOC = R^2 \sin 2A$, $2\triangle COA = R^2 \sin 2B$, $2\triangle AOB = R^2 \sin 2C$; and therefore

$$2\triangle ABC = R^2 (\sin 2A + \sin 2B + \sin 2C) = 4R^2 \sin A \sin B \sin C,$$

therefore $PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C = 4(R^2 + d^2) \sin A \sin B \sin C$.

(5809.) And $PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C$ is a minimum when $d = 0$, *i.e.*, when P coincides with O the circum-centre.

In space of n dimensions, if V be the content of a simplicissimum ABC, &c. (Quest. 8242), and V_1 be the content of the simplicissimum OBC..., V_2 of OAC..., &c., the formula

$$PA^2 . V_1 + PB^2 . V_2 + PC^2 . V_3 + \ldots = (R^2 + d^2) V$$

may be established in the same way as the formula (A), and it follows that the sinister is a minimum when P coincides with O, the centre of the circumscribed spheric (hyper-sphere).

(7069.) From the above, we have

$$x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C = 4 (R^2 + d^2) \sin A \sin B \sin C,$$

and the mean value

$$= 4 \sin A \sin B \sin C \times 4 \int_0 \int_0^{(R^2-x^2)^{\frac{1}{2}}} (x^2 + y^2 + R)^2 dx\,dy \div 4 \int_0^R \int_0^{(R^2-x^2)^{\frac{1}{2}}} dx\,dy$$

$$= 4R^2 \sin A \sin B \sin C \int_0^1 \int_0^{(1-x^2)^{\frac{1}{2}}} (x^2 + y^2 + 1)\, dx\,dy \div \int_0^1 \int_0^{(1-x^2)^{\frac{1}{2}}} dx\,dy$$

$$= 4R^2 \sin A \sin B \sin C \times \frac{3\pi}{8} \div \frac{\pi}{4} = 3 \times 2R^2 \sin A \sin B \sin C$$

$= 3\triangle ABC$. [This integration may be effected thus :—

$$\int_0^1 \int_0^{(1-x^2)^{\frac{1}{2}}} (x^2 + y^2 + 1)\, dx\,dy = \int_0^1 \left(\frac{2x^2}{3} + \frac{4}{3}\right) (1 - x^2)^{\frac{1}{2}}\, dx$$

$$= \frac{2}{3}\int_0^1 x^2(1-x^2)^{\frac{1}{2}}\,dx + \frac{4}{3}\int_0^1 (1-x^2)^{\frac{1}{2}}dx$$

$$= \frac{1}{6}\int_0^1 \frac{x^2}{(1-x^2)^{\frac{1}{2}}}\,dx + \frac{4}{3}\times\frac{\pi}{4} = \frac{1}{6}\int_0^1 (1-x^2)^{\frac{1}{2}}dx + \frac{\pi}{3}$$

$$= \frac{\pi}{24} + \frac{\pi}{3} = \frac{3\pi}{8}.$$

We may, however, more readily obtain the mean value as

$$= 4 \sin A \sin B \sin C \int_0^R \int_0^{2\pi} (r^2+R^2)\,r\,dr\,d\theta \div \int_0^R \int_0^{2\pi} r\,dr\,d\theta = \&c.$$

From Question 7050 it appears that, if

$$PA^2.\ \triangle BOC + PB^2.\ \triangle COA + PC^2.\ \triangle AOB$$

is constant, P lies on a circle whose centre is O and &c. in higher space.]

6686. (W. J. C. SHARP, M.A.)—If a bar naturally curved be strained, the bending moment at a point, whose natural curvature is r^{-1} and strained curvature ρ^{-1}, is $E(\rho^{-1}-r^{-1})\{I + Ar^{-2} + Br^{-4} + x\}$.

Solution.

Using the notation of MINCHIN's Statics, p. 510, if PP′ be the portion of any fibre intercepted between two near normal sections, PP″ the same fibre in its unstrained state, nn' the portion of the mean fibre intercepted between these sections, and r and ρ the initial and strained radii of curvature, we have

$$\frac{PP'}{nn'} = \frac{\rho+y}{\rho},\quad \frac{PP''}{nn'} = \frac{r+y}{r},\ \text{therefore}\ \frac{PP'-PP''}{PP''} = y\left(\frac{1}{\rho}-\frac{1}{r}\right)\frac{r}{r+y},$$

and consequently the longitudinal stress on the prism standing on a small area $\delta\sigma$

$$= E\left(\frac{1}{\rho}-\frac{1}{r}\right) r\,\frac{y}{r+y}\,\delta\sigma,$$

and therefore the bending moment $= E\left(\frac{1}{\rho}-\frac{1}{r}\right)\int \frac{y^2}{1+\frac{y}{r}}\,d\sigma$

$$\left[= E\left(\frac{1}{\rho}-\frac{1}{r}\right) I,\ \text{if the bar be of small section}\right]$$

$$= E\left(\frac{1}{\rho}-\frac{1}{r}\right)\left\{I + \int\frac{y^4}{r^2}d\sigma + \int\frac{y^6}{r^4}d\sigma + \&c.\right\}\ \text{in any case}$$

$$= E\left(\frac{1}{\rho}-\frac{1}{r}\right)\left\{I + \frac{A}{r^2} + \frac{B}{r^4} + \&c.\right\}.$$

5828. (Professor DARBOUX.)—On coupe une pyramide triangulaire SABC par un plan parallèle à la base ; ce plan rencontre les arêtes latérales SA, SB, SC en A′, B′, C′ ; on mène ensuite les plans CA′B′, AB′C′, BC′A′. Soit P leur point commun. Déterminer le lieu décrit par le point P lorsque le plan A′B′C′ se déplace en demeurant parallèle à la base.

Solution.

Taking the given tetrahedron as tetrahedron of reference, and the faces opposite to S, A, B, C, respectively, as $\lambda = 0$, $\mu = 0$, $\nu = 0$, $\pi = 0$, the coordinates being tetrahedral, so that $\lambda + \mu + \nu + \pi = 0$ is the equation to infinity; let $\lambda - a(\lambda + \mu + \nu + \pi) = 0$ be the equation to A′B′C′,

then at A′, $\nu = 0$, $\pi = 0$, $\lambda - a(\lambda + \mu) = 0$,

B′, $\pi = 0$, $\mu = 0$, $\lambda - a(\lambda + \nu) = 0$,

C′, $\mu = 0$, $\nu = 0$, $\lambda - a(\lambda + \pi) = 0$;

then the equations to CA′B′, AB′C′, BC′A′ are

$$a(\lambda + \mu + \nu) = \lambda,\quad a(\nu + \pi + \lambda) = \lambda,\quad \text{and}\quad a(\pi + \lambda + \mu) = \lambda,$$

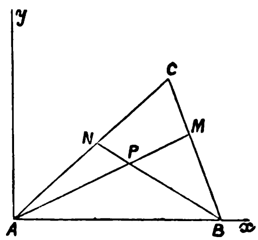
and therefore at P the common point of these planes, $\mu = \nu = \pi$, and the locus of P is the line drawn from S to the centroid of the face ABC.

A similar result may be obtained in exactly the same way in space of n dimensions. It may be enunciated as follows:—If SABC... be a simplicissimum (Quest. 8242) in space of n dimensions, and a linear locus A′B′C′... parallel to the face ABC... meet the edges SA, SB, SC... in A′, B′, C′... &c., respectively; the locus of the point P in which the linear loci AB′C′..., BA′C′..., CA′B′..., &c., meet, will be the line joining S to the centroid of the opposite face ABC...

6827. (Prof. Cayley, F.R.S.)—Consider a triangle ABC, and a point P; and let AP meet BC in M, and BP meet AC in N (if, to fix the ideas, P is within the triangle, then M, N are in the sides BC, AC, respectively, and the triangles APN, BPM are regarded as positive); find (1) the locus of the point P, such that the ratio ($\triangle$APN + $\triangle$BPM) : $\triangle$ABC may have a given value; (2) drawing from each point P, at right angles to the plane of the triangle, an ordinate PQ of a length proportional to the foregoing ratio ($\triangle$APN + $\triangle$BPM) : $\triangle$ABC, trace the surface which is the locus of the point Q, a surface which has the loci in (1) for its contour lines; (3) find the volume of the portion standing on the triangle ABC as base; and (4) deduce the solution of the following case of the four-point problem, viz., taking the points P, P′ at random within the triangle ABC, what is the chance that A, B, P, P′ may form a convex quadrangle?

Solution.

Take A as origin of rectangular Cartesian coordinates, and AB as axis of x, and let ξ, η be the coordinates of P.



Then N is the intersection of $y = \tan\alpha \, . \, x$ (AC) and $y = \dfrac{\eta}{\xi - c}(x - c)$, (BP),

therefore $\triangle$APN = $\triangle$ABN − $\triangle$APB

$$= \frac{c}{2}\left\{\frac{c\eta\tan\alpha}{\eta + (c-\xi)\tan\alpha} - \eta\right\}$$

$$= \frac{c\eta}{2}\left\{\frac{-\eta + \xi\tan\alpha}{\eta + (c-\xi)\tan\alpha}\right\}.$$

Similarly, M is the intersection of

$$y = -(x-c)\tan\beta \ldots (\mathrm{BC}) \text{ and } y = (\eta/\xi)\,x \ldots (\mathrm{AP}),$$

therefore $\triangle\mathrm{BPM} = \triangle\mathrm{ABM} - \triangle\mathrm{APB}$

$$= \frac{c}{2}\left\{\frac{c\eta\tan\beta}{\eta+\xi\tan\beta} - \eta\right\} = \frac{c\eta}{2}\left\{\frac{(c-\xi)\tan\beta-\eta}{\eta+\xi\tan\beta}\right\};$$

hence, if $\triangle\mathrm{APN} + \triangle\mathrm{BPM} : \triangle\mathrm{ABC} :: r : 1$ and $p = b\sin\alpha$ (ξ and η being the current coordinates), the equation to the locus of P is

$$\eta\left\{\frac{-\eta+\xi\tan\alpha}{\eta-\xi\tan\alpha+c\tan\alpha} + \frac{(c-\xi)\tan\beta-\eta}{\eta+\xi\tan\beta}\right\} = rp,$$

or $(2\eta+rp)(\eta-\xi\tan\alpha+\tfrac{1}{2}c\tan\alpha)(\eta+\xi\tan\beta-\tfrac{1}{2}c\tan\beta)$

$$+\tfrac{1}{2}c\left\{\eta\left[rp(\tan\alpha+\tan\beta)+c\tan\alpha\tan\beta\right]+\tfrac{1}{2}rpc\tan\alpha\tan\beta\right\} = 0 \equiv \mathrm{M},$$

a cubic through A and B, the asymptotes of which are the parallels to AC and BC through the middle point of AB and the line $2\eta+rp = 0$, a parallel to AB at a distance depending on r; and therefore, as the satellite of infinity is also a parallel to AB at a distance depending upon r, the asymptote $2\eta+rp = 0$ touches at an inflexion at infinity.

Any node there may be will lie upon $d\mathrm{M}/d\xi = 0$,

or $$(2\eta+rp)\left\{(\tan\beta-\tan\alpha)\,\eta-(\xi-\tfrac{1}{2}c)\tan\alpha\tan\beta\right\} = 0,$$

and there is no node on $2\eta+rp$; therefore, if there is one, it lies upon

$$(\tan\beta-\tan\alpha)\,\eta-(\xi-\tfrac{1}{2}c)\tan\alpha\tan\beta = 0;$$

but, where this cuts the curve,

$$(2\eta+rp)\,\eta^2+\tfrac{1}{2}c\eta\left[rp(\tan\alpha+\tan\beta)+c\tan\alpha\tan\beta\right]+\tfrac{1}{4}rpc^2\tan\alpha\tan\beta = 0,$$

so that the cubic is non-singular. With the same origin and axes of x and y, and the line at right angles to the plane of the triangle at A as axis of z, the equation to the locus of Q is

$$(2y+\mu z)(y-x\tan\alpha+\tfrac{1}{2}c\tan\alpha)(y+x\tan\beta-\tfrac{1}{2}c\tan\beta)$$
$$+\tfrac{1}{2}c\left\{y\left[\mu z(\tan\alpha+\tan\beta)+c\tan\alpha\tan\beta\right]+\tfrac{1}{2}c\mu z\tan\alpha\tan\beta\right\} = 0;$$

or, denoting $2y+\mu z$ by L, $y-x\tan\alpha$ by M, and $y+x\tan\beta$ by N, this may be written

$$\mathrm{LN}(\mathrm{M}+c\tan\alpha)-c\,\frac{(\mathrm{M}\tan\beta+\mathrm{N}\tan\alpha)^2}{\tan\alpha+\tan\beta} = 0,$$

so that the planes $\mathrm{L}\equiv 2y+\mu z = 0$, $\mathrm{M}+c\tan\alpha\equiv y-(x-c)\tan\alpha = 0$, $\mathrm{N}\equiv y+x\tan\beta = 0$ all meet the surface where it meets the plane $\mathrm{M}\tan\beta+\mathrm{N}\tan\alpha\equiv py = 0$, and the lines thus determined lie wholly on the surface, and any planes through these lines meet the surface on the three systems of quadrics,

$$\mathrm{LN} = \lambda c(\mathrm{M}\tan\beta+\mathrm{N}\tan\alpha),\quad \mathrm{L}(\mathrm{M}+c\tan\alpha) = \lambda c(\mathrm{M}\tan\beta+\mathrm{N}\tan\alpha),$$
$$\mathrm{N}(\mathrm{M}+c\tan\alpha) = \lambda c(\mathrm{M}\tan\beta+\mathrm{N}\tan\alpha),$$

which are hyperbolic paraboloids. The curves (1) are the sections by planes parallel to $z = 0$, the plane of the triangle and the plane $2y+\mu z = 0$ will meet the surface at inflexions in these sections, and the planes $y-x\tan\alpha+\tfrac{1}{2}c\tan\alpha = 0$ and $y+x\tan\beta-\tfrac{1}{2}c\tan\beta = 0$ will always contain two of the asymptotes to these sections; as all these sections are non-singular, the surface will not have any multiple point or multiple line. If the equation be written in the form

$$\mu z = y\left\{\frac{-y+x\tan\alpha}{y-x\tan\alpha+c\tan\alpha} + \frac{c\tan\beta-x\tan\beta-y}{y+x\tan\beta}\right\},$$

it appears that there is only one value of z for any values of x and y, and that z cannot become infinite for any finite values of x and y unless they make $y + x\tan\beta = 0$, or $y - x\tan\alpha + c\tan\alpha = 0$; and the traces of these planes are the parallels to BC through A and to AC through B, and these are always external to the triangle, and therefore z does not become infinite within the limits of the integration required for the solution of (3).

The volume required in (3) is the integral of

$$u \equiv \frac{y}{\mu}\left\{\frac{-y+x\tan\alpha}{y-x\tan\alpha+c\tan\alpha} + \frac{c\tan\beta - x\tan\beta - y}{y+x\tan\beta}\right\}$$

$$\equiv \frac{1}{\mu}\left\{-2y + c(\tan\alpha+\tan\beta) + c\tan^2\alpha\,\frac{x-c}{y-(x-c)\tan\alpha} - c\tan^2\beta\,\frac{x}{y+x\tan\beta}\right\}$$

with respect to x and y over the whole area of the triangle ABC,

and therefore $= \left\{\int_0^{b\cos\alpha}\int_0^{x\tan\alpha} + \int_{b\cos\alpha}^{c}\int_0^{(c-x)\tan\beta}\right\} u\,dx\,dy$;

and $\int u\,dy = \dfrac{1}{\mu}\left\{-y^2 + cy(\tan\alpha+\tan\beta) + c(x-c)\tan^2\alpha\log\left[y-(x-c)\tan\alpha\right]\right.$
$\left. - cx\tan^2\beta\log(y+x\tan\beta)\right\}$,

therefore the volume $= \dfrac{1}{\mu}\displaystyle\int_0^{b\cos\alpha}\left\{-x^2\tan^2\alpha + cx(\tan\alpha+\tan\beta)\tan\alpha\right.$

$$\left. + c(x-c)\tan^2\alpha\log\frac{c}{c-x} - cx\tan^2\beta\log\frac{\tan\alpha+\tan\beta}{\tan\beta}\right\}dx$$

$$+ \frac{1}{\mu}\int_{b\cos\alpha}^{c}\left\{-(c-x)^2\tan^2\beta + c(c-x)(\tan\alpha+\tan\beta)\tan\beta\right.$$

$$\left. + c(x-c)\tan^2\alpha\log\frac{\tan\alpha+\tan\beta}{\tan\alpha} - cx\tan^2\beta\log\frac{c}{x}\right\}dx$$

(or writing x for $c-x$ in the second integral)

$$= \frac{1}{\mu}\int_0^{b\cos\alpha}\left\{-x^2\tan^2\alpha + cx(\tan\alpha+\tan\beta)\tan\alpha + c(x-c)\tan^2\alpha\log\frac{c}{c-x}\right.$$
$$\left. - cx\tan^2\beta\log\frac{\tan\alpha+\tan\beta}{\tan\beta}\right\}dx$$

$$+ \frac{1}{\mu}\int_0^{a\cos\beta}\left\{-x^2\tan^2\beta + cx(\tan\alpha+\tan\beta)\tan\beta + c(x-c)\tan^2\beta\log\frac{c}{c-x}\right.$$
$$\left. - cx\tan^2\alpha\log\frac{\tan\alpha+\tan\beta}{\tan\alpha}\right\}dx;$$

and the first integral

$$= -\frac{b^3\cos\alpha\sin^2\alpha}{3} + \frac{b^2c}{2}\cdot\frac{\sin\gamma\sin\alpha}{\cos\beta} + \frac{a^2c}{2}\cdot\frac{\cos^2\beta\sin^2\alpha}{\cos^2\alpha}\log\frac{\sin\gamma}{\sin\alpha\cos\beta}$$

$$+ \frac{a^2c\cos^2\beta\sin^2\alpha}{4\cos^2\alpha} - \frac{b^2c}{2}\cdot\frac{\cos^2\alpha\sin^2\beta}{\cos^2\beta}\log\frac{\sin\gamma}{\sin\beta\cos\alpha} - \frac{c^3}{4}\cdot\frac{\sin^2\alpha}{\cos^2\alpha};$$

and the second is derived from it by interchanging a and b and α and β.

Therefore the volume $= \dfrac{1}{\mu}\left\{-\dfrac{b^2\sin^2\alpha}{3}b\cos\alpha\right.$

$$+ \frac{bc\sin\alpha}{2}\sin\gamma\frac{b}{\cos\beta} + \frac{a^2c}{2}\cdot\frac{\cos^2\beta\sin^2\alpha}{\cos^2\alpha}\log\frac{\sin\gamma}{\sin\alpha\cos\beta}$$

$$-\frac{c\sin^2\alpha}{4\cos^2\alpha}(c^2-a^2\cos^2\beta)-\frac{b^2c}{2}\cdot\frac{\cos^2\alpha\sin^2\beta}{\cos^2\beta}\log\frac{\sin\gamma}{\sin\beta\cos\alpha}$$

$$-\frac{a^2\sin^2\beta}{3}a\cos\beta+\frac{ac\sin\beta}{2}\sin\gamma\frac{a}{\cos\alpha}+\frac{b^2c}{2}\cdot\frac{\cos^2\alpha\sin^2\beta}{\cos^2\beta}\log\frac{\sin\gamma}{\sin\beta\cos\alpha}$$

$$\left.-\frac{c\sin^2\beta}{4\cos^2\beta}(c^2-b^2\cos^2\alpha)-\frac{a^2c}{2}\cdot\frac{\cos^2\beta\sin^2\alpha}{\cos^2\alpha}\log\frac{\sin\gamma}{\sin\alpha\cos\beta}\right\}$$

$$=\frac{1}{\mu}\left\{-\frac{p^2c}{3}+\frac{pc^2\sin\gamma}{2\cos\alpha\cos\beta}-\frac{pc^2}{4}(\tan\alpha+\tan\beta)\right.$$

$$\left.-\frac{pc}{4\cos\alpha\cos\beta}(a\sin\alpha\cos^2\beta+b\sin\beta\cos^2\alpha)\right\}$$

$$=\frac{1}{\mu}\left\{-\frac{p^2c}{3}+\frac{pc^2\sin\gamma}{4\cos\alpha\cos\beta}-\frac{pc^2}{4\cos\alpha\cos\beta\sin\gamma}\right.$$

$$\left.\left[\sin^2(\alpha+\beta)-2\sin\alpha\sin\beta\cos\alpha\cos\beta\right]\right\}$$

$$=\frac{p^2c}{\mu}\left\{-\tfrac{1}{3}+\tfrac{1}{2}\right\}=\frac{p^2c}{6\mu}\ (\text{where } p=a\sin\beta \text{ or } b\sin\alpha).$$

If, in the figure, APP′B is to be a convex quadrilateral, P′ must lie in one of the triangles APN or BPM; and, therefore, for any position of P, the chance for a convex quadrilateral is $(\triangle APN+\triangle BPM)/\triangle ABC = r$ and $\mu z = rp$, and the chance is $\mu z/p$; and consequently the chance for any position of P in the triangle $=\iint\frac{\mu z}{p}\,dx\,dy+\iint dx\,dy$, both integrals being taken over the triangle, and by what precedes this

$$=\frac{\mu\,.\,p^2c}{p\times 6\mu}+\frac{pc}{2}=\frac{1}{3};$$

and the chance for a re-entrant quadrilateral may be at once deduced as $1-\frac{1}{3}=\frac{2}{3}$, or it might be obtained, in a similar way, by finding the locus of P where the ratio $\triangle APB$ + fig. CNPM : $\triangle ABC$ is fixed, and describing a surface of which the ordinates should be proportional to the ratio for each point, and then integrating as above. This surface, if described on the opposite face of ABC, would coincide with the surface in the question if the latter were removed a distance unity towards ABC.

[The above result may be confirmed as follows:—The mean values of $\triangle APB$, $\triangle ANB$, $\triangle AMB$ are $\frac{1}{3}\triangle ABC$, $\frac{1}{2}\triangle ABC$, $\frac{1}{2}\triangle ABC$; and the mean value of $(\triangle ANP+\triangle BMP)$ = mean value of $(\triangle ANB+\triangle AMB-2\triangle APB)$ $=\frac{1}{3}\triangle ABC$, which gives the same result as above (see § 66 of Prof. CROFTON's article on *Probability*, in the *Encyclopædia Britannica*.)]

7270. (W. J. C. SHARP, M.A.)—Eliminate a, b, c, a', b', c' from

$$\begin{array}{r|r|r} a+b+c=d & a'+b'+c'=d' & aa'+bb'+cc'=e', \\ ab+bc+ca=s & a'b'+b'c'+c'a'=s' & a'bc+ab'c+abc'=\theta, \\ abc=\delta & a'b'c'=\delta' & ab'c'+a'bc'+a'b'c=\theta'. \end{array}$$

Solution.

(a, b, c) are the respective roots of $x^3-dx^2+sx-\delta=0$ ....(1),

(a_1, b_1, c_1) ,, ,, ,, $x^3-d_1x^2+s_1x-\delta_1=0$ ....(2),

(ab_1, bc_1, ca_1) ,, ,, ,, $x^3-sx^2+d\delta x-\delta^2=0$ ....(3),

(a_1b_1, b_1c_1, c_1a_1) ,, ,, $x^3-s_1x^2+d_1\delta_1x-\delta_1^2=0$ ....(4).

Let G, Δ, G_1, Δ_1 have their usual meanings for (1) and (2), and g, $\delta^2\Delta$, g_1, $\delta_1^2\Delta_1$ for (3) and (4); then (BURNSIDE's *Theory of Equations*, p. 113)

$$aa_1+bb_1+cc_1=e=\tfrac{1}{3}dd_1+3\left[\tfrac{1}{2}(-G+\sqrt{\Delta})\cdot\tfrac{1}{2}(-G_1-\sqrt{\Delta_1})\right]^{\frac{1}{3}}$$
$$+3\left[\tfrac{1}{2}(-G-\sqrt{\Delta})\cdot\tfrac{1}{2}(-G_1+\sqrt{\Delta_1})\right]^{\frac{1}{3}},$$
$$a_1bc+ab_1c+abc_1=\theta=\tfrac{1}{3}d_1s+3\left[\tfrac{1}{2}(-g+\delta\sqrt{\Delta})\cdot\tfrac{1}{2}(-G_1-\sqrt{\Delta_1})\right]^{\frac{1}{3}}$$
$$+3\left[\tfrac{1}{2}(-g-\delta\sqrt{\Delta})\cdot\tfrac{1}{2}(-G_1+\sqrt{\Delta_1})\right]^{\frac{1}{3}},$$

with a similar equation for θ_1. These three, being rationalised, give the results of the elimination.

[If
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=m,$$
$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=m'$$
be two concentric quadrics referred to rectangular axes,

$$a+b+c\equiv d,\qquad a'+b'+c'\equiv d',$$
$$ab+bc+ca-f^2-g^2-h^2\equiv s,\qquad a'b'+b'c'+c'a'-2f'^2-2g'^2-2h'^2\equiv s',$$
$$abc+2fgh-af^2-bg^2-ch^2\equiv\delta,\qquad a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2\equiv\delta',$$
$$aa'+bb'+cc'-2ff'-2gg'-2hh'=e,$$
$$a'bc+ab'c+abc'+2f'gh+2fg'h+2fgh'-a'f^2-b'g^2-c'h^2$$
$$-2aff'-2bgg'-2chh'=\theta,$$
$$ab'c'+a'bc'+a'b'c+2fg'h'+2f'gh'+2f'g'h-af'^2-bg'^2-ch'^2$$
$$-2a'f'f-2b'g'g-2c'h'h=\theta',$$

are their invariants and co-variants for rectangular transformations; and the given values of d, s, δ, d', s', δ', e, θ, and θ' are those which they assume when the quadrics are coaxal (*Proceedings of the London Mathematical Society*, Vol. XIII., Nos. 193—4), and are referred to their principal axes, and hence the resultants so elegantly obtained by Prof. SIRCOM are the invariant conditions that the quadrics should be coaxal.]

7301. (Professor CAYLEY, F.R.S.) — If the function $\dfrac{\alpha u+\beta}{\gamma u+\delta}=\phi u$ is periodic of the third order ($\phi^3u=u$): given that the cubic equation $(a, b, c, d \between x, 1)^3=0$ has two roots u, v such that $v=\dfrac{\alpha u+\beta}{\gamma u+\delta}$, find u as a rational function of $a, b, c, d, \alpha, \beta, \gamma, \delta$; and examine the case in which u is not thus expressible.

Solution.

If
$$v=\frac{\alpha u+\beta}{\gamma u+\delta}\equiv\phi(u),\quad \phi^2(u)\equiv\frac{(\alpha^2+\beta\gamma)u+(\alpha+\delta)\beta}{(\alpha+\delta)\gamma u+(\delta^2+\beta\gamma)},$$

and $$\phi^3(u) \equiv \frac{(\alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta)\, u + \beta\,(\alpha^2 + \delta^2 + \alpha\delta + \beta\gamma)}{\gamma\,(\alpha^2 + \delta^2 + \alpha\delta + \beta\gamma)\, u + (\delta^3 + 2\beta\gamma\delta + \alpha\beta\gamma)},$$

and therefore, in order that $\phi^3(u) \equiv u$,

$$\alpha^3 + 2\alpha\beta\gamma + \beta\gamma\delta = \delta^3 + 2\beta\gamma\delta + \alpha\beta\gamma, \text{ and } \alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0,$$

which are both satisfied if $\quad \alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0$

(for $\beta = 0$, $\gamma = 0$, $\alpha = \delta$ the other conditions make $\phi(u) = u$), and this is the necessary and sufficient condition that ϕ should be a periodic function of the third order.

Again, if $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ have two roots u and v such that $$v = \frac{\alpha u + \beta}{\gamma u + \delta},$$

the equation

$$a(\alpha x + \beta)^3 + 3b(\alpha x + \beta)^2(\gamma x + \delta) + 3c(\alpha x + \beta)(\gamma x + \delta)^2 + d(\gamma x + \delta)^3 = 0,$$

or
$$\begin{aligned}&(a\alpha^3 + 3b\alpha^2\gamma + 3c\alpha\gamma^2 + d\gamma^3)\, x^3\\ &+ 3\left\{a\alpha^2\beta + b(\alpha^2\delta + 2\alpha\beta\gamma) + c(2\alpha\gamma\delta + \beta\gamma^2) + d\gamma^2\delta\right\} x^2\\ &+ 3\left\{a\alpha\beta^2 + b(\beta^2\gamma + 2\alpha\beta\delta) + c(2\beta\gamma\delta + \alpha\delta^2) + d\gamma\delta^2\right\} x\\ &+ (a\beta^3 + 3b\beta^2\delta + 3c\beta\delta^2 + d\delta^3) = 0.\end{aligned}$$

Say $a'x^3 + 3b'x^2 + 3c'x + d' = 0$ has the root v in common with the original equation $(a, b, c, d \between x, 1)^3 = 0$, and therefore the resultant

$$\begin{aligned}&+ (ad' - a'd)^3 - 9(ad' - a'd)^2(bc' - b'c) + 27(ac' - a'c)^2(cd' - c'd)\\ &27(bd' - b'd)^2(ab' - a'b) - 81(ab' - a'b)(bc' - b'c)(cd' - c'd)\\ &- 27(ad' - a'd)(ab' - a'b)(cd' - c'd) = 0\end{aligned}$$

must be satisfied by the above values of a', b', c', and d'. The root may be found by finding the G. C. M. of

$$(a, b, c, d \between x, 1)^3 \text{ and } (a', b', c', d' \between x, 1)^3,$$

which will be found to be

$$\begin{aligned}&\left\{9c(ab' - a'b)^2 - 9b(ab' - a'b)(ac' - a'c) + 3a(ac' - a'c)^2 \right.\\ &\left. \qquad - a(ab' - a'b)(ad' - a'd)\right\} x,\\ &+ \left\{3d(ab' - a'b)^2 - 3b(ab' - a'b)(ad' - a'd) - a(ac' - a'c)(ad' - a'd)\right\},\end{aligned}$$

and therefore $v =$

$$\frac{-\left\{3d(ab' - a'b)^2 - 3b(ab' - a'b)(ad' - a'd) - a(ac - a'c)(ad' - a'd)\right\}}{9c(ab' - a'b)^2 - 9b(ab' - a'b)(ac' - a'c) + 3a(ac' - a'c)^2 - a(ab' - a'b)(ud' - a'd)},$$

and $u = -\dfrac{\delta v - \beta}{\gamma v - \alpha}$, so that both u and v are expressed rationally in terms of a, b, c, d and α, β, γ, δ.

If, however, the two quantics $(a, b, c, d \between x, 1)^3$ and $(a', b', c', d' \between x, 1)^3$ have a common quadratic factor, v as found above $= 0/0$.

The G. C. M. is $\quad 3(ab' - a'b)\, x^2 + 3(ac - a'c)\, x + (ad' - a'd)$,

and the common roots obtained from a quadratic equation are not expressed rationally in terms of the quantities a, b, c, d; a', b', c', d'.

In this case, if u, v, w are the roots of $(a, b, c, d \between x, 1)^3$,

$$\frac{\alpha u + \beta}{\gamma u + \delta},\ \frac{\alpha v + \beta}{\gamma v + \delta},\ \frac{\alpha w + \beta}{\gamma w + \delta} \text{ are the roots of } (a', b', c', d' \between x, 1)^3,$$

and we have not only $v = \dfrac{\alpha u + \beta}{\gamma u + \delta}$, but also either $w = \dfrac{\alpha v + \beta}{\gamma v + \delta}$ or else $u = \dfrac{\alpha w + \beta}{\gamma w + \delta}$; and, if $\dfrac{\alpha u + \beta}{\gamma u + \delta} \equiv \phi(u)$, the roots are either u, $\phi(u)$, and $\phi^2(u)$, or w, $\phi(w)$, and $\phi^2(w)$.

And hence, if the function be periodic of the third order, the transformed equation will be identical with the original one; so that, if

$$\alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0,$$

and $a\alpha^3 + 3b\alpha^2\gamma + 3c\alpha\gamma^2 + d\gamma^3 : a\alpha^2\beta + b(\alpha^2\delta + 2\alpha\beta\gamma) + c(2\alpha\gamma\delta + \beta\gamma^2) + d\gamma^2\delta$
$: a\alpha\beta^2 + b(\beta^2\gamma + 2\alpha\beta\delta) + c(2\beta\gamma\delta + \alpha\delta^2) + d\gamma\delta^2 : a\beta^3 + 3b\beta^2\delta + 3c\beta\delta^2 + d\delta^3$
$:: a : b : c : d,$

the equation $(a, b, c, d \between x, 1)^3 = 0$ has its roots u, $\phi(u)$, and $\phi^2(u)$, where ϕ is a periodic function of the third order, and is unchanged by the transformation $\qquad y = \dfrac{\alpha x + \beta}{\gamma x + \delta}$

(where $\alpha : \beta : \gamma : \delta$ may be found from the above relations, and also the condition which must be fulfilled.)

8204. (Alice Gordon.)—If A, B, C, D, E are five points on a sphere, V_{12} the volume ACDE, V_{13} the volume ABDE, &c.: prove that, with a certain convention as to sign,

$$V_{12}(AB)^2 + V_{13}(AC)^2 + V_{14}(AD)^2 + V_{15}(AE)^2 = 0.$$

Solution.

If (x_1, y_1, z_1), (x_2, y_2, z_2)......, (x_5, y_5, z_5) be the rectangular Cartesian coordinates of A, B, C, D, and E,

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2, & x_4, & y_4, & z_4, & 1 \\ x_5^2+y_5^2+z_5^2, & x_5, & y_5, & z_5, & 1 \end{vmatrix} = 0;$$

therefore, multiplying the second, third, and fourth columns by $2x_1$, $2y_1$, $2z_1$, and subtracting the sum from the first,

$$\begin{vmatrix} -x_1^2-y_1^2-z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2-2x_1x_2-2y_1y_2-2z_1z_2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2-2x_1x_3-2y_1y_3-2z_1z_3, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2-2x_1x_4-2y_1y_4-2z_1z_4, & x_4, & y_4, & z_4, & 1 \\ x_5^2+y_5^2+z_5^2-2x_1x_5-2y_1y_5-2z_1z_5, & x_5, & y_5, & z_5, & 1 \end{vmatrix} = 0;$$

then, subtracting the top line from each of the others,

$$\begin{vmatrix} (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2, & x_2-x_1, & y_2-y_1, & z_2-z_1 \\ (x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2, & x_3-x_1, & y_3-y_1, & z_3-z_1 \\ (x_1-x_4)^2+(y_1-y_4)^2+(z_1-z_4)^2, & x_4-x_1, & y_4-y_1, & z_4-z_1 \\ (x_1-x_5)^2+(y_1-y_5)^2+(z_1-z_5), & x_5-x_1, & y_5-y_1, & z_5-z_1 \end{vmatrix} = 0,$$

which proves the property.

This method of proof applies equally in space of any dimensions, and the general property may also be proved by means of simplicissimum content coordinates, as I have done in a paper on the "Properties of Simplicissima," &c., read before the London Mathematical Society in April, 1887.

8237. (W. J. C. Sharp, M.A.)—If ${}_nC_r$ denote the number of combinations r together which can be formed out of n things; show, from *à priori* considerations, that

$${}_nC_r = {}_nC_{n-r}, \quad {}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1}, \quad {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1} \ldots\ldots (1, 2, 3),$$

or, more generally, ${}_nC_r = {}_{n-p}C_r + {}_pC_1\,{}_{n-p}C_{r-1} + {}_pC_2\,{}_{n-p}C_{r-2} + \&c.$

Solution.

The given equations, which easily follow from the known formula for ${}^nC_r$, may also be proved from *à priori* considerations.

Since every set of r formed out of n things implicitly forms a set of $n-r$ (those not taken), and that every change in the one set involves a corresponding change in the other, ${}^nC_r = {}^nC_{n-r}$. If one of the things be removed, ${}^{n-1}C_{r-1}$ sets of $r-1$ things, and ${}^{n-1}C_r$ sets of r things, may be formed out of the remaining $n-1$; and, since the missing thing may be prefixed to each of the ${}^{n-1}C_{r-1}$ sets of $r-1$ things, there are ${}^{n-1}C_{r-1}$ sets of r things in which a particular one appears, and ${}^{n-1}C_r$ in which it does not; hence ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$; also, considering each of the n things in turn, $n\,{}^{n-1}C_{r-1}$ includes all the sets of r things, so that each occurs r times, once in virtue of each of its components; therefore $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$. If p of the things be removed, the number of sets of r will be made up of those which contain none of the p, one of the p, two of the p, &c., or of ${}^{n-p}C_r$, ${}^{n-p}C_{r-1}$, and ${}^pC_1$, since each of p may be taken with each set of $r-1$ of the $n-p$, ${}^{n-p}C_{r-2}$, and ${}^pC_2$, &c.; therefore

$${}^nC_r = {}^{n-p}C^r + {}^pC_1\,{}^{n-p}C_{r-1} + {}^pC_2\,{}^{n-p}C_{r-2} + \&c.$$

8631. (Professor Sylvester, F.R.S.)—Find the discriminant of

$$x^3 + y^3 + z^3 + 3e\Sigma x^2y + 6exyz.$$

Solution.

The discriminant is the resultant of the equations,

$$x^2 + 2ex(y+z) + e(y+z)^2 = 0, \quad y^2 + 2ey(z+x) + e(z+x)^2 = 0,$$
$$z^2 + 2ez(x+y) + e(x+y)^2 = 0,$$

which are equivalent to the system

$$e(x+y+z)^2 = (e-1)x^2 = (e-1)y^2 = (e-1)z^2 \ldots\ldots\ldots\ldots\ldots (A),$$

and from the first equation

$$x^2+ey^2+ez^2 = -2e\,(xy+yz+zx),$$

$$\therefore \quad \left\{(x^2+ey^2+ez^2)^2-4e^2\,(x^2y^2+y^2z^2+z^2x^2)\right\}^2 = 64e^4x^2y^2z^2\,(x+y+z)^2$$

from the equations (A),

therefore $\quad \left\{e^2\,(1+2e)^2-4e^2\times 3e^2\right\}^2 = 64e^7\,(e-1),$

therefore $\quad \left\{(1-e)^4\,(1+4e-8e^2)\right\}^2+64e^3\,(1-e)^9 = 0,$

and this expression is the discriminant; and the invariants T and S are

$$(1-e)^4\,(1+4e-8e^2) \text{ and } e\,(1-e)^3,$$

results easily confirmed by substitution in the general values given in SALMON's *Higher Plane Curves*, pp. 184—5.

8928. (Rev. T. R. TERRY, M.A.)—Find the value of the definite integral $\int_0^\infty \frac{dx}{x} \sin px\,(a\cos^2 x + b\sin^2 x)$, when $p > 2$.

Solution.

Denoting the integral by I,

$$4\frac{dI}{dp} = 4\int_0^\infty \cos px\,(a\cos^2 x+b\sin^2 x)\,dx$$

$$= (a-b)\int_0^\infty \left\{\cos(p+2)\,x+\cos(p-2)\,x\right\}+2\,(a+b)\int_0^\infty \cos px\,dx$$

$= 0$, unless $p = \pm 2$ or 0, in which case it is infinite.

Therefore, if p is not ± 2 or 0, I is independent of p. Let $p = 1$, therefore

$$2I = \int_0^\infty \frac{dx}{x}\sin x\left\{a+b+(a-b)\cos 2x\right\}$$

$$= (a+b)\int_0^\infty \sin x\,\frac{dx}{x}+\frac{(a-b)}{2}\int_0^\infty (\sin 3x-\sin x)\,\frac{dx}{x}$$

$= (a+b)\,\frac{1}{2}\pi+\frac{1}{2}\,(a-b)(\frac{1}{2}\pi-\frac{1}{2}\pi)$ (GREGORY's *Examples*, p. 480),

therefore $\quad I = (a+b)\,\frac{1}{2}\pi.$

8978. (Professor SYLVESTER, F.R.S.)—Show *algebraically* that, if a, b, c are the three sides of a triangle of reference, and A, B, C, the three perpendiculars on a variable line from the angles of that triangle, are regarded as its inverse coordinates, then the equation to the two circular points at infinity is

$$a^2\,(A-B)\,(A-C)+b^2\,(B-A)\,(B-C)+c^2\,(C-A)\,(C-B) = 0.$$

Solution.

In areal coordinates the circular points at infinity are represented by

$$a^2\mu\nu+b^2\nu\lambda+c^2\lambda\mu=0, \quad \text{and} \quad \lambda+\mu+\nu=0 \quad \text{..................(1, 2)},$$

and if $l\lambda+m\mu+n\nu=0$ represent any straight line, the square of the perpendicular upon it from $[\lambda'\mu'\nu'=Q\,(l\lambda'+m\mu'+n\nu')]^2$; as I have shown in a paper read before the *London Mathematical Society*, in April last; therefore $A:B:C::l:m:n$, and the equation to the line may be written

$$A\lambda+B\mu+C\nu=0 \quad \text{.................................(3)},$$

therefore from (2) and (3) $\lambda:\mu:\nu::B-C:C-A:A-B$, and therefore from (1), $a^2(A-B)(A-C)+b^2(B-C)(B-A)+c^2(C-B)(C-A)=0$.

8996. (Mahendra Nath Ray, M.A., LL.B.)—If x, y, z denote the respective distances of any point in the plane of a given triangle ABC from the angular points; show that the following relation subsists among them,

$$(x^2+y^2+z^2+a^2+b^2+c^2)(a^2x^2+b^2y^2+c^2z^2) = 2a^2x^2(a^2+x^2)$$
$$+2b^2y^2(b^2+y^2)+2c^2z^2(c^2+z^2)+a^2y^2z^2+b^2z^2x^2+c^2x^2y^2+a^2b^2c^2.$$

Solution.

If the volume of the tetrahedron ABCD, whose edges are $DA=x$, $DB=y$, $DC=z$, $BC=a$, $CA=b$, $AB=c$, is (Quest. 8242) V,

$$V^2=\frac{1}{8\times 36}\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & x^2, & y^2, & z^2 \\ 1, & x^2, & 0, & c^2, & b^2 \\ 1, & y^2, & c^2, & 0, & a^2 \\ 1, & z^2, & b^2, & a^2, & 0 \end{vmatrix};$$

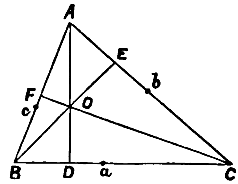
and therefore, if D lie in the plane of ABC,

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & x^2, & y^2, & z^2 \\ 1, & x^2, & 0, & c^2, & b^2 \\ 1, & y^2, & c^2, & 0, & a^2 \\ 1, & z^2, & b^2, & a^2, & 0 \end{vmatrix}=0,$$

which, when expanded, gives the result required. The same result is obtained in a different way in Salmon's *Geometry of Three Dimensions*, p. 32.

9003. (R. F. Davis, M.A.)—If upon each side of a triangle a pair of points be taken so that the pairs on any two sides are concyclic, prove that all three pairs are concyclic.

Solution.



If A_1, A_2; B_1, B_2; C_1, C_2 be the points in which BC, CA, and AB are divided,

$$AB_1 . AB_2 = AC_1 . AC_2,$$
$$CA_1 . CA_2 = CB_1 . CB_2,$$
and $$BA_1 . BA_2 = BC_1 . BC_2.$$

Now, let the circle through C_1, C_2, B_1, B_2 cut BC in A_1', A_2'; then

$$BA_1' . BA_2' = BC_1 . BC_2 = BA_1 . BA_2 \text{ .... (1)},$$
$$CA_1' . CA_2' = CB_1 . CB_2 = CA_1 . CA_2,$$
or $$(BC - BA_1')(BC - BA_2') = (BC - BA_1)(BC - BA_2),$$
and therefore $$BA_1' + BA_2' = BA_1 + BA_2 \text{ .... (2)},$$

therefore, from (1) and (2), BA_1' and BA_2' are the roots of the same quadratic as BA_1 and BA_2, and A_1' and A_2' coincide with A_1 and A_2.

If a, b, c be the middle points of the sides of the triangle, and D, E, F the feet of the perpendiculars from the vertices upon the opposite sides,

$$\left.\begin{array}{l} Ab . AE = \tfrac{1}{2}bc \cos A = Ac . AF \\ Cb . CE = \tfrac{1}{2}ab \cos C = Ca . CD \\ Ba . BD = \tfrac{1}{2}ac \cos B = Bc . BF \end{array}\right\} \text{ .... (A)};$$

and therefore the points a, b, c, D, E, F are concyclic, or the circle about a, b, c passes through D, E, and F. And if O be the intersection of AD, BE, and CF, the circle through D, E, and F, the feet of the perpendiculars from the vertices upon the opposite sides of the triangle AOB (viz., D, E, and F) will bisect the sides, and so the circle about DEF passes through a, b, c and bisects AO, BO, CO. (I think this is, perhaps, the simplest proof of the existence of the nine-point circle.)

The equations (A) may be proved by elementary Geometry as follows: The circle on BC as diameter will pass through F and E (Euclid iii. 31), therefore AE . AC = AF . AB (iii. 36), or 2AE . Ab = 2AF . Ac, and similarly for the others.

9008. (S. Roberts, M.A.)—Given three circles C_1, C_2, C_3, determine a circle cutting C_1 orthogonally, bisecting C_2, and bisected by C_3; and show that in general there are two such circles which may coincide or become imaginary.

Solution.

Let the axes of rectangular Cartesian coordinates be taken so that $x^2 + y^2 - a^2 = 0 \equiv C_1$, $(x-h)^2 + y^2 - b^2 = 0 \equiv C_2$, $(x-h')^2 + (y-k')^2 - c^2 = 0 \equiv C_3$, and $(x-\xi)^2 + (y-\eta)^2 - R^2 = 0 \equiv S$, the required circle. Then, since C_1 and S cut orthogonally, $\xi^2 + \eta^2 = R^2 + a^2$ .... (1);
since S bisects C_2, $(\xi - h)^2 + \eta^2 = R^2 - b^2$ .... (2);
and, since C_3 bisects S, $(\xi - h')^2 + (\eta - k')^2 = c^2 - R^2$ .... (3);
from (1) and (2), $$\xi = \frac{a^2 + b^2 + h^2}{2h},$$

and therefore, from (3), $2k'\eta = h'^2 + k'^2 - c^2 + a^2 + 2R^2 - \frac{h'}{h}(a^2+b^2+h^2)$

$$\equiv A + 2R^2 - \frac{h'}{h}B, \text{ say,}$$

and $$2k'\xi = \frac{k'}{h}B;$$

$$\therefore \quad 4k'^2(\xi^2+\eta^2) = 4k'^2(R^2+a^2) = \left(A+2R^2-\frac{h'}{h}B\right)^2 + \frac{k'^2}{h^2}B^2,$$

$$\therefore \quad 4R^4 + 4R^2\left\{A - \frac{h'}{h}B - k'^2\right\} + \left(A - \frac{h'}{h}B\right)^2 + \frac{k'^2}{h^2}B^2 - 4a^2k'^2 = 0,$$

and the values of R^2 are real, equal, or imaginary,

according as $\left\{A - \frac{h'}{h}B - k'^2\right\}^2 > = < \left(A - \frac{h'}{h}B\right)^2 + \frac{k'^2}{h^2}B^2 - 4a^2k'^2,$

„ „ $-2\left(A - \frac{h'}{h}B\right) + k'^2 > = < \frac{B^2}{h^2} - 4a^2,$

„ „ $-2h'^2 - k'^2 + 2c^2 + \frac{2h'}{h}(a^2+b^2+h^2) > = < \frac{(a^2+b^2)^2}{h^2} + 2b^2 + h^2,$

„ „ $2(c^2+hh')h^2 > = < (a^2+b^2-hh')^2 + h^2(2b^2+h'^2+k'^2+h^2).$

When the values of R^2 are real, those of η are so, and those of ξ are always so, and therefore, &c.

9024. (Professor SYLVESTER, F.R.S.)—For greater distinctness, the name of *Hyper-cartesian* (not to be confounded with a *hyper-cartesic*) being given to that particular form of the bicircular quartic in which four concyclic foci become collinear; prove that, if four points are given in a plane, the locus of the curve *in space* whose distances from any three of them are subject to a given homogeneous linear relation is a curve of the 4th order. (This space curve may be termed a Hyper-cartesic.)

Solution.

If the plane of the foci be taken for that of xy, and the axes be so chosen that they are $(0, 0, 0)$, $(h, 0, 0)$, $(h', k', 0)$, and $(h'', k'', 0)$, and if p, q, r, s be the four focal distances of the point (x, y, z) on the curve,

$$p^2 = x^2+y^2+z^2, \quad q^2 = p^2 - 2hx + h^2, \quad r^2 = p^2 - 2h'x - 2k'y + h'^2 + k'^2,$$
$$s^2 = p^2 - 2h''x - 2k''y + h''^2 + k''^2,$$

and $$lp + mq + nr = 0 \text{ and } l'p + m'q + n's = 0,$$

or $$2l^2m^2p^2q^2 + 2m^2n^2q^2r^2 + 2n^2l^2r^2p^2 - l^4p^4 - m^4q^4 - n^4r^4 = 0 \quad\ldots\ldots\ldots\ldots(1),$$
$$2l'^2m'^2p^2q^2 + 2m'^2n'^2q^2s^2 + 2n'^2l'^2s^2p^2 - l'^4p^4 - m'^4q^4 - n'^4r^4 = 0\ldots(2),$$

or $$ap^4 + 2Lp^2 + S = 0 \text{ and } a'p^4 + 2L'p^2 + S' = 0,$$

where L and L′ are linear and S and S′ quadratic functions of x and y; and their resultant, which is the trace of the curve on the plane of xy, is

$$(aS'-a'S)^2+4(a'L-aL')(LS'-L'S)=0,$$

a quartic, to every point of which two points, one above and another below the plane, correspond, so that these lie on two quartics, each of which is the reflexion of the other.

9063. (MAHENDRA NATH RAY, M.A., LL.B.)—If $a_1, a_2, a_3 \dots a_{2n-1}$ be $2n-1$ positive numbers connected by the relation $a_1a_2a_3 \dots a_{2n-1}=1$; show, *by elementary algebra only*, that the minimum value of

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4)\dots(1+a_{2n-1}) \text{ is } 2^{2n-1}.$$

Solution.

If x be any positive quantity $x+1/x > = < 2$,

according as $(x-1)^2 > = < 0$,

but $(x-1)^2$ is $\not< 0$, therefore $x+1/x \not< 2$,

i.e., 2 is the minimum value of $x+1/x$.

Now, if $a_1a_2\dots a_n=1$,

$$(1+a_1)(1+a_2)\dots(1+a_n)=(a_1a_2\dots a_n)^{\frac{1}{2}}(a_1^{\frac{1}{2}}+a_1^{-\frac{1}{2}})(a_2^{\frac{1}{2}}+a_2^{-\frac{1}{2}})\dots(a_n^{\frac{1}{2}}+a_n^{-\frac{1}{2}})$$
$$=(a_1^{\frac{1}{2}}+a_1^{-\frac{1}{2}})(a_2^{\frac{1}{2}}+a_2^{-\frac{1}{2}})\dots(a_n^{\frac{1}{2}}+a_n^{-\frac{1}{2}}),$$

and the minimum value of each factor is 2; therefore the minimum value of product is 2^n. Of course the quantities $a_1^{\frac{1}{2}}$, $a_2^{\frac{1}{2}}$, &c. are the positive or signless square roots.

8637. (Professor HUDSON, M.A.) — The tangent and normal to $y/c=\frac{1}{2}(\epsilon^{x/c}+\epsilon^{-x/c})$ cut the axis of x in T, G, respectively; find the minimum value of GT.

Solution.

Let the tangent be inclined to the axis of x at an angle ϕ; then

$$GT=c\sec\phi(\tan\phi+\cot\phi)=c/(\sin\phi-\sin^3\phi);$$

this is least when $\sin\phi=1/\sqrt{3}$, and therefore $GT=\frac{3}{2}c\sqrt{3}$.

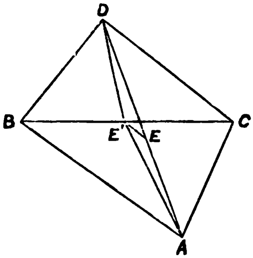
8888. (S. TEBAY, B.A.)—D_1, D_2, D_3 are the shortest distances of opposite edges (a, a; b, b; c, c) of a tetrahedron, V the volume, and Δ the area of one of the equal faces; show that

$$V=\tfrac{1}{3}(D_1D_2D_3), \quad \Delta^2=\tfrac{1}{8}(D_1^2a^2+D_2^2b^2+D_3^2c^2),$$

and $$D_1^2+D_2^2+D_3^2=\tfrac{1}{2}(a^2+b^2+c^2)=D_1^2+a^2=\&c.$$

Solution.

Let ABCD be the tetrahedron, and let BC = DA = a, CA = DB = b, AB = DC = c bisect BC in E′ and AD in E; then, since DB = AC and BE′ = CE′, and DBE′ = DBC = BCA = ACE′, DE′ = E′A, and therefore EE′ is at right angles to AD; similarly, it is at right angles to BC, and it is therefore the shortest distance between them. Therefore



$$DE^2 + EE'^2 = DE'^2,$$

and $$DE'^2 = \tfrac{1}{2}\left(b^2 + c^2 - \tfrac{1}{2}a^2\right),$$

therefore

$$D_1^2 = EE'^2 = DE'^2 - DE^2 = \tfrac{1}{2}\left(b^2 + c^2 - a^2\right),$$

and, similarly,

$$D_2^2 = \tfrac{1}{2}\left(c^2 + a^2 - b^2\right), \quad D_3^2 = \tfrac{1}{2}\left(a^2 + b^2 - c^2\right),$$

therefore

$$D_1^2 + D_2^2 + D_3^2 = \tfrac{1}{2}\left(a^2 + b^2 + c^2\right) = D_1^2 + a^2 = \&c.$$

$$\Delta^2 = \tfrac{1}{16}\left\{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right\}$$

$$= \tfrac{1}{16}\left\{a^2\left(b^2 + c^2 - a^2\right) + b^2\left(c^2 + a^2 - b^2\right) + c^2\left(a^2 + b^2 - c^2\right)\right\}$$

$$= \tfrac{1}{8}\left\{D_1^2a^2 + D_2^2b^2 + D_3^2c^2\right\};$$

also (Quest. 8242), $$V^2 = \frac{1}{8 \times 36}\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & c^2, & b^2, & a^2 \\ 1, & c^2, & 0, & a^2, & b^2 \\ 1, & b^2, & a^2, & 0, & c^2 \\ 1, & a^2, & b^2, & c^2, & 0 \end{vmatrix},$$

or (replacing the 2nd column by the sum of the 2nd and 5th less that of the 3rd and 4th, replacing the 3rd column by the sum of the 3rd and 2nd less that of the 4th and 5th, replacing the 4th column by the sum of the 4th and 2nd less that of the 3rd and 5th, which gives 4 times the determinant),

$$= \frac{1}{4 \times 8 \times 36}\begin{vmatrix} 0, & 0, & 0, & 0, & 1 \\ 1, & a^2 - b^2 - c^2, & c^2 - b^2 - a^2, & b^2 - c^2 - a^2, & a^2 \\ 1, & b^2 + c^2 - a^2, & c^2 - a^2 - b, & a^2 + c^2 - b^2, & b^2 \\ 1, & b^2 + c^2 - a^2, & a^2 + b^2 - c^2, & b^2 - a^2 - c^2, & c^2 \\ 1, & a^2 - b^2 - c^2, & a^2 + b^2 - c^2, & a^2 + c^2 - b^2, & 0 \end{vmatrix}$$

$$= \frac{D_1^2D_2^2D_3^2}{144}\begin{vmatrix} 1, & -1, & -1, & -1 \\ 1, & 1, & -1, & 1 \\ 1, & 1, & 1, & -1 \\ 1, & -1, & 1, & 1 \end{vmatrix} = \frac{D_1^2D_2^2D_3^2}{36}\begin{vmatrix} 1, & -1, & 1 \\ 1, & 1, & -1 \\ -1, & 1, & 1 \end{vmatrix}$$

$$= \frac{D_1^2D_2^2D_3^2}{9} \times T = \frac{D_1D_2D_3}{9}.$$

1856. (Professor SYLVESTER, F.R.S.)—Prove that the Jacobian of the time of a planet's describing any arc, the chord of the arc, and the sum of the two extreme distances from the sun, in respect to the eccentricity and two extreme eccentric anomalies, is zero; and hence deduce the time for a planet or comet in terms of the said chord and sum.

Solution.

With the usual notation (see Tait and Steele's *Dynamics*), we have

$$\cos\theta = \frac{\cos u - e}{1 - e\cos u}, \text{ therefore } 1 + e\cos\theta = \frac{1-e^2}{1-e\cos u},$$

$$\sin\theta = \sqrt{\left\{1 - \left(\frac{\cos u - e}{1-e\cos u}\right)^2\right\}} = \frac{(1-e^2)^{\frac{1}{2}}\sin u}{1-e\cos u},$$

$$= \frac{a(1-e^2)}{1+e\cos\theta} = a(1-e\cos u).$$

Now, if t_1, t_2 be the times from perihelion to the ends of the arc, T the time of describing it, u_1, u_2 the eccentric anomalies of the extremities, r_1, r_2, θ_1, θ_2 the corresponding radii vectores and true anomalies, S the sum of r_1 and r_2, and C the chord, P being the periodic time,

$$T = t_2 - t_1 = \frac{P}{2\pi}\left\{u_2 - e\sin u_2 - (u_1 - e\sin u_1)\right\} \;\ldots\ldots\ldots\ldots(1),$$

$$S = r_1 + r_2 = a(1 - e\cos u_1 + 1 - e\cos u_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$C^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2) = r_1^2 + r_2^2 - 2r_1r_2\left\{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right\}$$

$$= a^2\left[(1-e\cos u_1)^2 + (1-e\cos u_2)^2 - 2(1-e\cos u_1)(1-e\cos u_2) \times\left\{\frac{\cos u_1 - e}{1-e\cos u_1}\,\frac{\cos u_2 - e}{1-e\cos u_2} + \frac{(1-e^2)\sin u_1\sin u_2}{(1-e\cos u_1)(1-e\cos u_2)}\right\}\right]$$

$$= a^2\left[2 - 2\cos u_1\cos u_2 - 2\sin u_1\sin u_2 - e^2(\sin u_1 - \sin u_2)^2\right] \;\ldots\ldots\ldots(3).$$

Again, the Jacobian of C^2, T, S is merely that of C, T, S multiplied by 2C, and therefore they will vanish together unless $C = 0$, a case which may be disregarded. Now, T (C^2, T, S)

$$= \frac{Pa^3}{2\pi}\begin{vmatrix} -2e(\sin u_1 - \sin u_2), & 2\sin u_1\cos u_2 - 2\cos u_1\sin u_2 - 2e^2\cos u_1(\sin u_1 - \sin u_2), & -2\sin u_1\cos u_2 + 2\cos u_1\sin u_2 + 2e^2\cos u_2(\sin u_1 - \sin u_2) \\ \sin u_1 - \sin u_2, & -1 + e\cos u_1, & 1 - e\cos u_2, \\ -(\cos u_1 + \cos u_2), & e\sin u_1, & e\sin u_2 \end{vmatrix}$$

= (adding $2e$ times the second row to the first)

$$\frac{Pa^3}{\pi}\begin{vmatrix} 0, & \sin u_1\cos u_2 - \cos u_1\sin u_2 - e(\sin u_1 - \sin u_2), & -\sin u_1\cos u_2 + \cos u_1\sin u_2 + e(\sin u_1 - \sin u_2) \\ \sin u_1 - \sin u_2, & -1 + e\cos u_1, & 1 - e\cos u_2 \\ -(\cos u_1 + \cos u_2), & e\sin u_1, & e\sin u_2 \end{vmatrix}$$

= (adding the third column to the second)

$$\frac{Pa^2}{\pi}\left\{-\sin u_1\cos u_2+\cos u_1\sin u_2 + e(\sin u_1-\sin u_2)\right\}\begin{vmatrix} \sin u_1-\sin u_2, & e(\cos u_1-\cos u_2) \\ -(\cos u_1+\cos u_2, & e(\sin u_1-\sin u_2)\end{vmatrix}\equiv 0;$$

and therefore $\qquad T(C, T, S) = 0,$

and T is a function of e, u_1, and u_2 only as it is a function of S and C (Boole's *Diff. Equa.*, Sup. Vol., p. 56), and therefore it is possible to eliminate e, u_1, and u_2 from the equations (1), (2), (3).

From (2), $\qquad e = \dfrac{2a-S}{a(\cos u_1+\cos u_2)},$

therefore, from (3),

$$C^2 = a^2\left\{2-2\cos(u_2-u_1)-\left(\frac{2a-S}{a}\right)^2\left(\frac{\sin u_1-\sin u_2}{\cos u_1+\cos u_2}\right)^2\right\}$$

$$= a^2\left\{2-2\cos(u_2-u_1)-\left(\frac{2a-S}{a}\right)^2\frac{1-\cos(u_1-u_2)}{1+\cos(u_1-u_2)}\right\},$$

a quadratic in $\cos(u_2-u_1)$, whence

$$\cos(u_2-u_1) = \left(1-\frac{S-C}{2a}\right)\left(1-\frac{S+C}{2a}\right)+\sqrt{\left\{\left(1-\frac{S-C}{2a}\right)^2\left(1-\frac{S+C}{2a}\right)^2 - \frac{C^2-2a^2+(2a-S)^2}{2a^2}\right\}}$$

$$= \left(1-\frac{S-C}{2a}\right)\left(1-\frac{S+C}{2a}\right)+\sqrt{\left\{\left[1-\left(1-\frac{S-C}{2a}\right)^2\right]\times\left[1-\left(1-\frac{S+C}{2a}\right)^2\right]\right\}}$$

$$= \cos z\cos z'+\sin z\sin z' = \cos(z-z'),$$

if $\qquad 1-\dfrac{S-C}{2a} = \cos z \quad\text{and}\quad 1-\dfrac{S+C}{2a} = \cos z'.$

So that z and z' are functions of S and C. Therefore

$$\tan\left(\frac{u_2-u_1}{2}\right) = \tan\frac{z-z'}{2} = \frac{\sin z-\sin z'}{\cos z+\cos z'}.$$

Now, from (1), (2),

$$T = \frac{P}{2\pi}\left\{u_2-u_1-\frac{2a-S}{a}\,\frac{\sin u_2-\sin u_1}{\cos u_1+\cos u_2}\right\}$$

$$= \frac{P}{2\pi}\left\{z-z'-\frac{2a-S}{a}\tan\frac{u_2-u_1}{2}\right\},$$

and $\qquad \dfrac{2a-S}{a} = \cos z+\cos z' \quad\text{and}\quad \tan\dfrac{u_2-u_1}{2} = \tan\dfrac{z-z'}{2},$

therefore $\qquad T = \dfrac{P}{2\pi}\{z-z'-\sin z+\sin z'\}$

a formula given by Lambert, as an extension of one of Euler's for parabolic motion, viz., $\quad T = \dfrac{1}{6\sqrt{\mu}}\{(S+C)^{\frac{3}{2}}-(S-C)^{\frac{3}{2}}\},$

which is proved in Tait and Steele. See also Pontecoulant, *Système du Monde*, Vol. I., pp. 277 and 287—291.

5305. (Professor SYLVESTER, F.R.S.)—*Definition.*—Six right lines along which six forces can be made to equilibrate are said to be in Statical Involution.

Prove that any six right lines lying on a ruled cubic surface are in statical involution; and, *vice versâ,* if six right lines are in statical involution, a ruled cubic surface can be made to pass through them.

Solution.

If $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$; $a_2, b_2, c_2, \alpha_2, \beta_2, \gamma_2$, &c. be the direction cosines, and the coordinates of fixed points upon the six lines, and P_1, P_2, &c. the forces acting along them; the conditions of equilibrium are

$$\Sigma Pa = 0, \quad \Sigma Pb = 0, \quad \Sigma Pc = 0,$$

$$\Sigma P(c\beta - b\gamma) = 0, \quad \Sigma P(a\gamma - c\alpha) = 0, \quad \text{and} \quad \Sigma P(b\alpha - a\beta) = 0,$$

and the necessary and sufficient condition for statical involution is

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \\ c_1, & c_2, & c_3, & c_4, & c_5, & c_6 \\ c_1\beta_1 - b_1\gamma_1, & c_2\beta_2 - b_2\gamma_2, & c_3\beta_3 - b_3\gamma_3, & c_4\beta_4 - b_4\gamma_4, & c_5\beta_5 - b_5\gamma_5, & c_6\beta_6 - b_6\gamma_6 \\ a_1\gamma_1 - c_1\alpha_1, & a_2\gamma_2 - c_2\alpha_2, & a_3\gamma_3 - c_3\alpha_3, & a_4\gamma_4 - c_4\alpha_4, & a_5\gamma_5 - c_5\alpha_5, & a_6\gamma_6 - c_6\alpha_6 \\ b_1\alpha_1 - a_1\beta_1, & b_2\alpha_2 - a_2\beta_2, & b_3\alpha_3 - a_3\beta_3, & b_4\alpha_4 - a_4\beta_4, & b_5\alpha_5 - a_5\beta_5, & b_6\alpha_6 - a_6\beta_6 \end{vmatrix} = 0.$$

Now, if $a\lambda + b\mu + c\nu + (c\beta - b\gamma)\rho + (a\gamma - c\alpha)\sigma + (b\alpha - a\beta)\tau = 0$ be satisfied by any point in the line

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c},$$

it is satisfied by every point in the line, *i.e.*,

$$a\lambda + b\mu + c\nu + (cy - bz)\rho + (az - cx)\sigma + (bx - ay)\tau = 0;$$

and, if there be six such lines, the determinant above will be transformed into the equation to a ruled cubic surface, upon which each of the lines lies, by writing x for each of the α's, y for each of the β's, and z for each of the γ's. Also any ruled cubic surface may be reduced to this form, the components of each column being the coordinates of any six lines upon it, and so the proposition is proved.

APPENDIX IV.

NOTE ON THE USE OF COMMON LOGARITHMS IN THE NUMERICAL SOLUTION OF EQUATIONS OF THE HIGHER ORDERS. By Major-General P. O'CONNELL.

The problem to be solved may be stated as follows:

Given $$f(x) = ax^m + bx^n + cx^p + \dots + l = 0,$$

where $a, b, c, \dots l$; m, n, p, &c. are constants; m, n, p, &c. being large numbers consisting either of whole numbers, whole numbers and fractions, or whole numbers and a few places of decimals arranged in descending order of magnitude, *i.e.*, m is the largest and every other is smaller than any of those preceding it.

To find a value of x which, when substituted in $f(x)$, shall reduce that function to as small a value as the extent of the tables of logarithms used and the true value of x permit it.

When m, n, p, &c. are small whole numbers, I know no method of solution better than Horner's, which has been very fully explained by the late Professor De Morgan, by Todhunter, and others.

When m, n, p, &c., are large whole numbers, the solution may be obtained in the manner shown by that able mathematician and computer, the late Mr. Thomas Weddle, in a pamphlet published in 1842, by Hamilton, Adams, & Co., Paternoster Row. Mr. Weddle does not use any table of logarithms, and his computations are therefore very long.

The method here used is nearly that of Mr. Oliver Byrne, as explained in his "Dual Arithmetic," published by Bell & Dalby, 186, Fleet Street, 1864. It differs therefrom, however, in the use of common logarithms, and in presenting the calculations in a more condensed form, a single page being sufficient for the solution of an equation of a very high order, as may be seen by reference to my Solution of Question 9185 [which will appear hereafter in Vol. 48], where an equation of the 622nd degree is solved, and a value of x is found containing six places of decimals or seven figures in all.

In Mr. Weddle's, Mr. Byrne's, and in the present method, x is conceived to be resolved into factors. In Mr. Weddle's method,

$$x = \text{R}(1+r_1)(1+r_2)(1+r_3)\dots,$$

where R is the first significant figure of x multiplied by some power of 10, and r_1, r_2, r_3, &c. form a series of decimals each consisting of one significant figure preceded by a decimal point, and 0, 1, or more zeros, the number of zeros increasing as we pass along successive values of r from left to right. In Mr. BYRNE's method

$$x = M\left(1+\frac{1}{10}\right)^{m_1}\left(1+\frac{1}{10^2}\right)^{m_2}\left(1+\frac{1}{10^3}\right)^{m_3}, \text{ \&c.},$$

where M contains one or more significant figures of x multiplied by some power of 10, and m_1, m_2, m_3, &c. may have as values any suitable whole numbers or zero. In the present method;

$$x = a_1(1+a_2)(1+a_3)\ldots,$$

where a_1 is the first approximation to the value of x, and may contain only a whole number of one, two, or three figures, or a whole number and a decimal, $a_1(1+a_2)$ is the second approximate value of x, $a_1(1+a_2)(1+a_3)$ is the third approximate value of x, and so on, a_2 being less than unity, a_3 less than $a_2\ldots$, and a_n less than $a_{n-1}\ldots$, &c.; one advantage of using a table of logarithms being, that each approximate value will generally be found to contain nearly twice as many correct figures as the one preceding it.

In all these methods, the values of the factors into which x is conceived to be resolved are discovered successively, commencing with the left-hand one, then the one next to it on the right, and so on.

The first factor is usually obtained, in using WEDDLE's or BYRNE's method, by a process of guessing and trial, or by some simple transformation or other algebraic artifice.

In using the method about to be described, it will often be found that a value of the first factor containing three figures may be obtained by the use of logarithmic slide rules. This facilitates its application and shortens the whole process, and is therefore recommended to all who are accustomed to use these instruments. It may, however, be conveniently applied by those who do not use such aids.

The method of passing from the first to the second approximation in the value of x, due to Sir ISAAC NEWTON, and, as far as I know, generally followed until it was slightly altered by Mr. OLIVER BYRNE, is thus described in TODHUNTER's *Theory of Equations* (4th Edition, 1880), page 142:—

"Let $f(x) = 0$ be an equation which has a root between certain limits α and β, the difference of which is a small fraction; let c be a quantity between α and β which is assumed as a first approximation to the required root; and let $c+h$ denote the exact value of the root, so that h is a small fraction which is to be determined.

"Thus, $f(c+h) = 0$, that is, by article 10,

$$f(c) + hf'(c) + \frac{h^2}{1.2}f''(c) + \ldots \frac{h^n}{\underline{|n}}f^n(c) = 0.$$

Now, since h is supposed to be a small fraction, h^2, h^3, &c. will be small compared to h; if we neglect the squares and higher powers of h in the above equation, we obtain $f(c) + h.f'(c) = 0$, thus

$$h = -\frac{f(c)}{f'(c)}.$$

Supposing, then, that we thus obtain an approximation to the value of h, we have $c - \frac{f(c)}{f'(c)}$ as a new approximation to the root of the proposed equation."

The simple alteration introduced into the above process by Mr. BYRNE is, to express the second approximation to the root, not as $c - \frac{f(c)}{f'(c)}$, but as $c\left(1 - \frac{f(c)}{cf'(c)}\right)$. This slight variation suffices to change the method to one suited to logarithmic calculation.

The first step in the process is to find the logarithmic values of the several terms of the given equation when a_1 (the first approximation to its value) is substituted for x. When this has been done, we have the set of terms,

$$\log\{a(a_1)^m\}, \qquad \log\{b(a_1)^n\}, \qquad \log\{c(a_1)^p\} \ldots \log l,$$

to which, adding

$$\log\left\{1 - \frac{f(a_1)}{a_1 f'(a_1)}\right\}^m, \quad \log\left\{1 - \frac{f(a_1)}{a_1 f'(a_1)}\right\}^n, \quad \log\left\{1 - \frac{f(a_1)}{a_1 f'(a_1)}\right\}^p,$$

we obtain

$$\log\left[a\{a_1(1+a_2)\}^m\right], \ \log\left[b\{a_1(1+a_2)^n\right], \ \log\left[c\{a_1(1+a_2)\}^p\right] \ldots \log l.$$

Before we can add the second line of logarithms, shown above, we must first obtain the value of $a_2 = -\frac{f(a_1)}{a_1 f'(a_1)}$. This can be conveniently effected by arranging the calculation in the form shown in the following example, where the three operations of finding (1) the logarithmic values of the several terms of the given equation, (2) the natural numbers corresponding thereto, and (3) the value of the next factor in the value of the unknown quantity, are recorded in different parts of the same central columns, a separate column on the left hand being reserved for the logarithms of the required powers of the successive factors of the value of the root and for the final value of the root, the several portions being numbered so as to facilitate reference, as well as correction if necessary.

In this way the computation circulates from the left-hand column down the central one, and back again to the left-hand column, and so on again until it is complete.

The example worked out is (WEDDLE, No.7, page 27) an equation having four real roots, of which two are positive. One of the positive roots is given by Mr. WEDDLE, and the other is here found.

This equation in x has first been transformed to one in $y\left\{= \left(\frac{x}{10}\right)^{60}\right\}$, in which a first approximation to the value of y is easily guessed.

It will be seen that only three factors are obtained, and that

$$y = 10^{-5} \times 1{\cdot}5\,(1{\cdot}0372)(1{\cdot}00064574),$$

an ordinary seven-figure logarithm table having been used. If more factors be required, VEGAS' ten-figure table, or PETER GRAY's twenty-four-figure table, may be used. When the latter table is used, a good deal of room and calculation are needed for passing from a large natural number to its logarithm and back from a large logarithm to its natural number, so that it is better to keep this portion of the computation separate and transfer the results alone to the form here used.

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