MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXXIV.

MATHEMATICAL QUESTIONS AND SOLUTIONS.

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES."

EDITED BY

D. BIDDLE, M.R.C.S.,

FELLOW OF THE ROYAL STATISTICAL SOCIETY;
FORMERLY STATISTICIAN TO THE GENERAL MEDICAL COUNCIL.

VOL. LXXIV.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1901.

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WARREN, A. T., M.A.; Usk Terrace, Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WASSON, J. C.; Kells, Meath.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. Trinity Coll., Camb.
WATTS, J. O.; Middlesbrough.
WEBB, G. H.; Halifax.
WEBB, H. A.; Trinity College, Cambridge.
WELLACOT, Rev. W. T., M.A.; Newton Abbot.
WERTSCH, FRANZ; Weimar.
WEST, J. WOODCOTE; New Cross.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHAPHAM, R. H. W., M.A.; Manchester.
WHITE, E. A., M.A.; Eckington, Pershore.
WHITESIDE, G. M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. W. A., M.A.; London.
WIGGINS, Rev. T., B.A.; Roehampton.
WILCOX, Rev. A. M., M.A.; Branhope, Leeds.
WILKINSON, J. F.; Bacup.
WILLIAMS, A. E., M.A.; Texas.
WILLIAMS, D. J., M.A.; Worcester Coll., Oxon.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub.
WILMOT, Rev. W. T., M.A.; Newton Abbot.
WILSON, Ven. Archdeacon J. M., M.A., F.G.S.
WILSON, G. D., B.A.; Cambridge.
WILSON, Rev. J., M.A.; Bannockburn.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOLTERS, J. B.; Groningen.
WOODALL, H. J., A.R.C.S.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WRIGHT, W. J.; New Jersey, U.S.A.
YOUNG, JOHN, B.A.; Portadown, Ireland.
ZERR, Professor, M.A.; Staunton, Virginia.

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Mathematical Papers, &c.

Solved Questions.

APPENDIX I.

APPENDIX II.

CORRIGENDA.

Vol. LXXIII.

P. 27, l. 12, *for* "A" *read* "Δ."
P. 49, l. 15, *for* "greater than" *read* "less than."
P. 66, l. 10 (from foot), *for* "OQ" *read* "OS."
P. 68, l. 4 (from foot), *for* "y_{m+1}" *read* "y^{m+1}."
P. 69, l. 8, *for* "$+x''y$" *read* "$+(x''y$."
P. 106, l. 17, *for* "$\frac{1}{3}a(m_1+m_2+m_3)$, $\frac{2}{3}a(m_1^2+m_2^2+m_3^2)$" *read* "$\frac{2}{3}a(m_1+m_2+m_3)$, $\frac{1}{3}a(m_1^2+m_2^2+m_3^2)$."
P. 112, l. 3 (from foot), *for* "$-y\sin\theta^2)$" *read* "$-y\sin\theta)^2$."

Vol. LXXIV.

P. 73, l. 13 (from foot), *for* "DGB, EGC" *read* "EGC, DGB."
P. 78, l. 15 (from foot), *for* "and in general" *read* "is in general"; and *for* "Zahrad*u*ick" *read* "Zahrad*n*ick."
P. 102, l. 9, *for* "$a\cos\frac{1}{3}(\theta-\pi)$" *read* "$a\cos\frac{1}{3}(\theta+\pi)$."
P. 73, l. 11, *for* "$\cos 2\theta\sin 3\phi$" *read* "$\cos 2\theta\sin 3\theta$."

MATHEMATICS

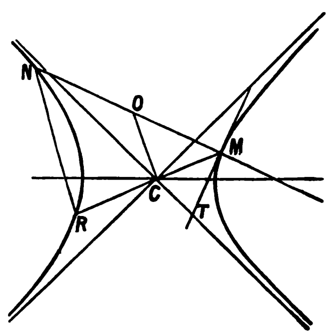
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

14424. (Professor Neuberg.)—Trouver le lieu des centres des hyperboles équilatères qui ont une corde normale commune MN.

Solution by Professor K. J. Sanjána, M.A.; Professor A. Droz-Farny; *and* H. W. Curjel, M.A.



The angle between the chord MN and the tangent at M is equal to the angle subtended by MN at the extremity R of the diameter through M.

Thus MRN is a right angle.

Draw CO parallel to RN to meet the chord. Then MCO is a right angle, and MO is half of MN.

Hence the locus of C is the circle on MO as diameter.

14501. (Rev. T. Wiggins, B.A.)—Given a triangle ABC, find a point D within it such that $DA^2+DB^2+DC^2$ is a minimum.

I. *Solution by* J. H. Taylor, M.A.

The general problem of which this is a particular case is given in Williamson's *Diff. Cal.* (1877), § 157.

Let A have coordinates h, k; B, 0, 0; C, a, 0; D, x, y.

$$\phi(u) = 3x^2 + 3y^2 - 2ax - 2hx - 2ky + a^2 + h^2 + k^2.$$

$$du/dx = 6x - 2a - 2h = 0, \text{ suppose.} \quad d^2u/dx^2 = 6 \; \ldots\ldots\ldots\ldots \; \text{(A)},$$

$$du/dy = 6y - 2k, \quad d^2u/dy^2 = 6 \; (c), \quad d^2u/(dx\,dy) = 0 \; \ldots\ldots\ldots \; \text{(B)}.$$

Therefore $x = \frac{1}{3}(a+h)$, $y = \frac{1}{3}k$ for the minimum, indicating clearly the centroid.

II. *Solution by* Professor JAN DE VRIES, Ph.D.

E being the middle point of BC, we get $BD^2 + CD^2 = 2DE^2 + \frac{1}{2}BC^2$.

Again, G being the centre of gravity, we have, by the theorem of STEWART, $\quad 2DE^2 + AD^2 = 3DG^2 + \frac{2}{3}AE^2.$

Combining with $\quad AB^2 + AC^2 = 2AE^2 + \frac{1}{2}BC^2,$

we obtain $\quad AD^2 + BD^2 + CD^2 = 3DG^2 + \frac{1}{3}(a^2 + b^2 + c^2).$

Hence $AD^2 + BD^2 + CD^2$ will be a minimum if D coincides with the centre of gravity.

III. *Solution by* F. H. PEACHELL, B.A.

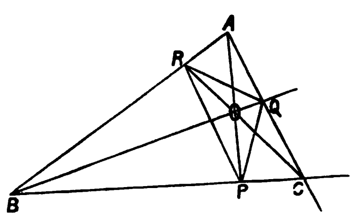
If A, B, C, &c., be any number of points, O their centroid, P any other point, then the sum of the squares of distances of P from A, B, C, &c., exceeds the sum of the squares of distances of O from these points by $n \cdot OP^2$, where n is the number of points. Then, for a minimum in the question given, D must evidently coincide with the centroid of the triangle.

[See CASEY's *Sequel*, 6th edition, p. 25.]

14410. (Rev. T. WIGGINS, B.A.)—Inscribe in a given triangle the triangle of least perimeter.

Solution by J. G. SMITH; W. J. GREENSTREET, M.A.; G. D. WILSON, B.A.; *and many others.*

The pedal triangle is that required.



For, because ROPB is cyclic, therefore RPB = ROB;

because AROQ is cyclic, therefore ROB = A;

because ABPQ is cyclic, therefore QPC = A.

Therefore RPB = QPC.

Therefore, if Q, R be fixed, QP + RP is minimum.

Similarly for the others. Therefore any change in the position of any vertex of PQR increases the perimeter. Therefore PQR has minimum perimeter.

14444. (P. MILNES.)—A conic cuts the sides of triangle ABC in D, D′, E, E′, F, F′ respectively; AD, AD′ intersect the conic again in d, d'; BE, BE′ in e, e'; CF, CF′ in f, f'. Show that the intersections of dd', ee', ff' with the polars of A, B, C respectively are collinear.

Solution by R. P. PARANJPYE, B.A.; *and* Professor SANJÁNA, M.A.

With the triangle ABC as the triangle of reference, let the conic be

$$0 = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation of the lines AD, AD′ is

$$by^2 + cz^2 + 2fyz = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The locus $ax^2 + 2gzx + 2hxy = 0$ passes through the intersection of (1) and (2). Therefore the equation of dd' is

$$ax + 2gz + 2hy = 0,$$

and we get similar equations for ee' and ff'.

Now we easily see that the line $x/f + y/g + z/k = 0$ passes through the intersection of dd' and BC, &c. Hence the three points in question are collinear.

[Mr. CURJEL solves the Question as follows:—

Let the polars of A, B, C cut dd', ee', ff' in P, Q, R. Now the polar of A cuts AD, AD′ in the harmonic conjugates of A with respect to Dd, D′d'. Therefore P lies on BC. Similarly Q and R lie on AC, AB.

Let O be the pole of DD′. Then OR, OQ, OA are the polars of C, B, P. Therefore OQ, AR and OR, AQ cut DD′ harmonically. Therefore OA, QR cuts DD′ harmonically; but OA, AP cut DD′ harmonically. Therefore QR cuts BC in P.]

14445. (Rev. J. CULLEN.)—Prove that

$$q^{2^{q-1}} - 1 \equiv 0 \quad [\text{mod } (q\,.\,2^q + 1)],$$

if $q\,.\,2^q + 1$ be a prime.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and* H. W. CURJEL, M.A.

Let $\qquad q\,.\,2^q + 1 = p$ (a prime).

Then, provided $q > 2$, $\qquad 2^{\frac{1}{2}(p-1)} \equiv +1.$

Also $\qquad (p-1)/2q = 2^{q-1}$ and $2^q\,.\,q \equiv -1 \pmod p$.

Therefore $\qquad (2^q)^{(p-1)/2q}\,.\,(q)^{(p-1)/2q} \equiv (-1)^{(p-1)/2q} \equiv +1.$

Therefore $\qquad 2^{\frac{1}{2}(p-1)}\,.\,q^{(p-1)/2q} \equiv +1,$ whence $q^{2^{q-1}} \equiv +1 \pmod p$.

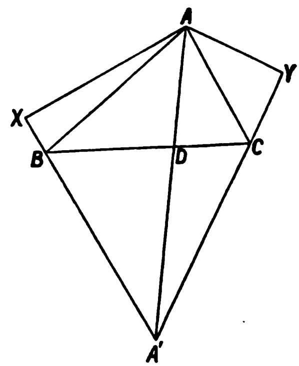
The proof fails when $q \not> 2$; but, when $q = 1$, $p = 3$, and it is obviously true; and when $q = 2$, $p = 9$, which is *not prime*. Unfortunately, the form $(q\,.\,2^q + 1)$ gives *no small primes*, none in fact when $q \not> 20$ (except $p = 3$).

14493. (J. H. Taylor, M.A.)—If A′, B′, C′ are vertices of similar isosceles triangles described all externally, or all internally, on the sides of any plane triangle BCA, the straight lines AA′, BB′, CC′ are concurrent.

Solution by W. L. Thomson *and the* Proposer.

Let A′BC be one of the isosceles triangles, the base angles being θ. Let AA′ cut BC in D. Let AX, AY be perpendiculars from A on BA′, CA′. Then

$$\frac{BD}{DC}=\frac{\triangle ABA'}{\triangle ACA'}=\frac{AX}{AY}=\frac{c\sin(B+\theta)}{b\sin(C+\theta)}.$$



Similarly, if BB′, CC′ cut AC, AB in E, F, respectively,

$$\frac{CE}{EA}=\frac{a\sin(C+\theta)}{c\sin(A+\theta)} \text{ and } \frac{AF}{FB}=\frac{b\sin(A+\theta)}{a\sin(B+\theta)}.$$

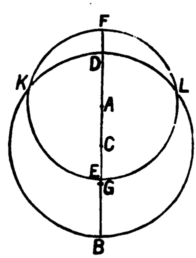
Therefore $\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}\cdot\dfrac{AF}{FB}=1.$

Therefore AA′, BB′, CC′ are concurrent.

14278. (I. Arnold.)—Two non-concentric spheres intersect, forming a shell. Find the centre of gravity of the larger shell and its distance from the centre of the larger sphere, the distance between the centres of the spheres being d, and the radii of the spheres being R and r.

Solution by the Proposer.

Let R^3 represent the weight of whole concentrated at centre of gravity C of larger sphere, and r^3 the weight of part at A, the centre of gravity of the smaller sphere; d being equal to AC, the distance of their centres.



The centre of gravity of shell will be in AC produced. Let it be G. Then, by a well known theorem,

$$CG : CA :: r^3 : R^3-r^3,$$

or $$CG : d :: r^3 : R^3-r^3;$$

therefore $$CG = dr^3/(R^3-r^3),$$

which determines the centre of gravity of the shell KBLE.

14512. (Professor Neuberg.) — Trouver dans le plan du triangle ABC un point M qui soit le centre de gravité de ses projections A′, B′, C′ sur BC, CA, AB, pour les poids donnés α, β, γ.

Solution by H. W. Curjel, M.A.; *and* Constance I. Marks, B.A.

Let A′M meet B′C′ in A″, and let $\angle$ BAM $= \theta$ and $\angle$ CAM $= \phi$. Then B′A″ : A″C′ $= \gamma : \beta$. Therefore $\sin\theta : \sin\phi = \beta \sin C : \gamma \sin B$.

Hence AM may be drawn as the line joining A to the intersections of two parallels to AB, AC at distances $\beta \sin C$, $\gamma \sin B$. Similarly, BM and CM may be drawn.

14201. (R. Tucker, M.A.)—P, Q, R are points on a parabola, such that PQ, QR are normals to the curve. If SP, SQ, SR are denoted by r_1, r_2, r_3, prove $(r_2-r_1)^3 = (r_1+r_2)^2(2r_2-r_1-r_3)$;

hence they cannot be in A.P. Show that the circle PQR is given by

$$m^2(m^2+2)^2(x^2+y^2)-2pqax-4pmay+m^2(m^2+2)(m^2+4)qa^2 = 0,$$

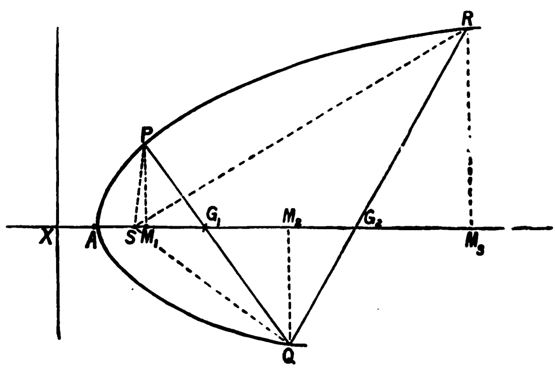
where P is the point $(am^2, 2am)$ and p, q stand respectively for m^4+4m^2+2, m^4+6m^2+4. Find also when the circle passes through the focus.

Solution by F. L. Ward, B.A.

With the usual notation, let PQ meet axis in G_1, and QR in G_2. Drop PM_1, QM_2, RM_3 perpendicular to axis. Denote P, Q, R by (x_1, y_1), &c.

Now $AM_1 AM_2 = AG_1^2$ or $x_1x_2 = (x_1+2a)^2$, since $M_1G_1 = 2a$.

Also $x_2x_3 = (x_2+2a)^2$, since $M_2G_2 = 2a$.



From these, and since $r_1 = x_1 + a$, &c.,

$$x_1(r_2-r_1) = 4ar_1 \quad\text{and}\quad x_2(r_3-r_2) = 4ar_2;$$

therefore $(r_3-r_2)/(r_2-r_1) = r_2x_1/r_1x_2 = r_2x_1^2/r_1(x_1+2a)^2$.

But $x_1(r_2-r_1) = 4ar_1$; therefore $(x_1+2a)/x_1 = (r_1+r_2)/2r_1$;

therefore $(r_3-r_2)/(r_2-r_1) = 4r_1r_2/(r_1+r_2)^2$,

which is the same as the result given.

This expresses the geometrical fact that $M_2M_3 : M_1M_2$ = ratio of harmonic mean of SG_2 and SG_1 to their arithmetic mean.

The equation to the line joining (x_1, y_1), (x_2, y_2) is

$$(y-y_1)(x_2-x_1)-(x-x_1)(y_2-y_1) = 0,$$

which reduces to $\quad y(y_1+y_2)-4ax-y_1y_2 = 0.$

The line through (x_3, y_3) making supplementary angles with the axis to the above is therefore

$$y(y_1+y_2)+4ax-y_3(y_1+y_2+y_3) = 0.$$

Therefore the required circle is

$$\{y(y_1+y_2)+4ax-y_3(y_1+y_2+y_3)\}\{y(y_1+y_2)-4ax-y_1y_2\}-\lambda(y^2-4ax)=0,$$

where the condition for a circle gives $\lambda = (y_1+y_2)^2+16a^2$; and the equation to the circle becomes

$$\begin{aligned}16a^2(x^2+y^2)-4ax\{y_1^2+y_2^2+y_3^2+y_1y_2+y_2y_3+y_3y_1+16a^2\}\\ +y\{(y_1+y_2)(y_3^2+y_1y_2+y_2y_3+y_3y_1)\}\\ -y_1y_2y_3(y_1+y_2+y_3) = 0\end{aligned}$$

or

$$\begin{aligned}(x^2+y^2)-ax(m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_3m_1+4)\\ +ay\tfrac{1}{2}(m_1+m_2)(m_3^2+m_1m_2+m_2m_3+m_3m_1)\\ -a^2m_1m_2m_3(m_1+m_2+m_3) = 0.\end{aligned}$$

But $\quad m_2 = -(2/m_1+m_1)$ and $m_3 = -(2/m_2+m_2)$;

therefore

$$m_1m_2 = -(2+m_1^2), \quad m_2m_3 = \frac{-(m_1^4+6m_1^2+4)}{m_1^2}, \quad m_3m_1 = \frac{m_1^4+6m_1^2+4}{m_1^2+2},$$

$$m_3 = \frac{m_1^4+6m_1^2+4}{m_1(m_1^2+2)}, \quad m_2 = -\left(\frac{m_1^2+2}{m_1}\right), \quad m_1m_2m_3 = \frac{-(m_1^4+6m_1^2+4)}{m_1}.$$

Substituting these values, we get the required result, *i.e.*,

$$m_1^2(m_1^2+2)^2(x^2+y^2)-2pqax-4pm_1ay+m_1^2(m_1^2+2)(m_1^2+4)qa^2 = 0.$$

If $(x = a,\ y = 0)$ lies on this circle, therefore

$$1-(m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_3m_1+4)-m_1m_2m_3(m_1+m_2+m_3) = 0$$

or $\quad m_1^2+m_2^2+m_3^2+m_1m_2+m_2m_3+m_3m_1+3+m_1m_2m_3(m_1+m_2+m_3) = 0,$

which reduces to $\quad \dfrac{2pq}{m^2(m^2+2)^2}-\dfrac{(m^2+4)q}{(m^2+2)} = 1$

or $\quad 2pq-m^2(m^2+2)(m^2+4)q = m^2(m^2+2)^2$

or $\quad \alpha^5+10\alpha^4+29\alpha^3+16\alpha^2-20\alpha-16 = 0$, where $\alpha = m^2$,

two solutions of which are $\alpha = -1$ or $m^2 = -1$, one solution of which is $\alpha = -4$ or $m^2 = -4$. The other two are $m^2 = -2\pm2\sqrt{2}$; the only real solutions being $m = \pm\sqrt{2(\sqrt{2}-1)}$.

14476. (Professor E. J. Nanson.)—If

$$\frac{a^2-bc}{a}+\frac{b^2-ca}{b}+\frac{c^2-ab}{c} = 0, \quad \text{then} \quad \frac{a}{a^2-bc}+\frac{b}{b^2-ca}+\frac{c}{c^2-ab} = 0.$$

Solution by J. BLAIKIE, M.A.; L. E. REAY, B.A.; *and many others.*

If $\Sigma \left\{(a^2-bc)/a\right\} = 0, \quad \Sigma (a^2bc-b^2c^2) = 0.$

Multiplying by $a+b+c$, we get

$$\Sigma \left\{a^3bc+ab^2c^2-a^3(b^2+c^2)\right\} = 0 \quad \text{or} \quad \Sigma\, a(b^2-ca)(c^2-ab) = 0.$$

Therefore $$\Sigma \left\{a/(a^2-bc)\right\} = 0.$$

14458. (J. A. THIRD, D.Sc.)—XYZ is a triangle inscribed in ABC and having its sides proportional to the medians of ABC. Show that the envelope of the circumcircle of XYZ is the LEMOINE ellipse of ABC.

Solution by Rev. J. CULLEN *and* G. N. BATES, B.A.

The sides of the pedal triangle XYZ of the symmedian point K are proportional to the medians. Hence, if the lines KX, KY, KZ revolve round K through an angle θ, the triangle formed by joining (KX_θ, BC), &c., is similar to XYZ, the modulus of similarity being sec θ. Therefore the envelope of the circle $X_\theta Y_\theta Z_\theta$ is the in-conic whose foci are K and G, the centroid, since in general, if the ranges forming the pedal triangles of Λ and its isogonal conjugate Λ' revolve through an angle θ in contrary directions, the six points are concyclic, the envelope of the circle being the in-conic whose foci are Λ and Λ'.

[Mr. BATES observes that this is a particular case of Mr. CULLEN's Quest. 14182; see Vol. LXXII., p. 65.]

14443. (R. KNOWLES.)—F, S are the foci of a rectangular hyperbola; from a point T on the circle whose diameter is FS, tangents TP, TQ are drawn to meet the curve in PQ; the circle TPQ cuts the curve again in CD; prove that (1) the diagonals of the quadrilateral PQCD intersect in the axis; (2) two of its sides are parallel.

Solution by Professor A. DROZ-FARNY.

Soient $x^2-y^2 = a^2$ et $x^2+y^2 = 2a^2$ les équations de l'hyperbole équilatère et du cercle FS. La polaire d'un point (x', y') par rapport à l'hyperbole étant $xx'-yy' = a^2$, l'équation d'un cercle TPQ sera de la forme $$(xx'-yy'-a^2)(xx'+yy'+\lambda)+\mu(x^2-y^2-a^2) = 0.$$
Cette conique sera un cercle passant par le point (x', y') du cercle $x^2+y^2 = 2a^2$ si $\lambda = \mu = -a^2$.

Les droites PQ et CD auront donc respectivement pour équations
$$xx'-yy' = a^2, \quad xx'+yy' = a^2.$$
Ces deux droites étant symétriquement disposées par rapport à l'axe transverse de l'hyperbole, le quadrilatère PQCD est un trapèze ayant cet axe comme médiane orthogonale. Il en résulte immédiatement les deux questions proposées.

14499. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Prove that the continued product $P = (a \mp b)(a^3 \mp b^3)(a^5 \mp b^5)\dots(a^{2n-1} \mp b^{2n-1})$ is divisible by $\{(2n)! \div (2^n . n!)\}^{n+1}$ if $(a \mp b)$ is divisible by $(2n)! \div (2^n . n!)$.

Solution by Rev. J. CULLEN; H. W. CURJEL, M.A.; *and the* PROPOSER.

Let $N = 1.3.5\dots 2n-1 = (2n)! \div (2^n . n!)$.

Now, if r is odd, we have

$$a^r \mp b^r = (a \mp b)(a^{r-1} \pm a^{r-2}b + \dots \pm b^{r-1}),$$

and, since $a \equiv \pm b \pmod N$,

$$a^{r-1} \pm a^{r-2}b + \dots \pm b^{r-1} \equiv rb^{r-1} \pmod N.$$

Therefore $a^r \mp b^r$ is divisible by rN, and hence P by $1.3.5\dots 2n-1.N^n$, *i.e.*, by N^{n+1}.

14516. (Professor JAN DE VRIES, Ph.D.)—For each conic of a given pencil the orthoptical circle (circle of MONGE) is constructed. How many of these circles will pass by a given point?

Solution by the PROPOSER.

Denoting by A_{kl} the minor of the determinant

$$\Delta = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix}$$

belonging to a_{kl}, the centre of the conic

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

is determined by $x_0 = A_{13} : A_{33}$, $y_0 = A_{23} : A_{33}$.

If the same conic is represented by the equation

$$b_{11}\xi^2 + b_{22}\eta^2 + b_{33} = 0,$$

we have $b_{33} = \Delta : A_{33}$; and the quantities

$$u_1 = -b_{33} : b_{11}, \quad u_2 = -b_{33} : b_{22}$$

will be determined by

$$A_{33}^3 u^2 + (a_{11} + a_{22}) A_{33} \Delta u + \Delta^2 = 0.$$

Since the radius ρ of the orthoptic circle is determined by

$$\rho^2 = u_1 + u_2 = -(a_{11} + a_{22}) \Delta : A_{33}^2,$$

the equation of that circle will be

$$(x - A_{13}/A_{33})^2 + (y - A_{23}/A_{33})^2 = -(a_{11} + a_{22}) \Delta / A_{33}^2.$$

It may be verified that

$$A_{13}^2 + A_{23}^2 + (a_{11} + a_{22}) \Delta = (A_{11} + A_{22}) A_{33}.$$

Hence the orthoptic circle is determined by

$$A_{33}(x^2 + y^2) - 2A_{13}x - 2A_{23}y + (A_{11} + A_{22}) = 0.$$

Now let the pencil of conics be determined by

$$(a_{11}+\lambda b_{11})\,x^2+2\,(a_{12}+\lambda b_{12})\,xy+\ldots=0,$$

then the corresponding orthoptic circles are represented by

$$\begin{aligned}&\left[(a_{11}+\lambda b_{11})(a_{22}+\lambda b_{22})-(a_{12}+\lambda b_{12})^2\right](x^2+y^2)\\&+2\left[(a_{13}+\lambda b_{13})(a_{22}+\lambda b_{22})-(a_{12}+\lambda b_{12})(a_{23}+\lambda b_{23})\right]x\\&+2\left[(a_{11}+\lambda b_{11})(a_{23}+\lambda b_{23})-(a_{12}+\lambda b_{12})(a_{13}+\lambda b_{13})\right]y\\&+(a_{22}+\lambda b_{22})(a_{33}+\lambda b_{33})-(a_{23}+\lambda b_{23})^2\\&+(a_{11}+\lambda b_{11})(a_{33}+\lambda b_{33})-(a_{13}+\lambda b_{13})^2=0.\end{aligned}$$

Hence by any point (x, y) will pass the two orthoptic circles belonging to the conics whose parameters λ are determined by the latter equation.

The system of orthoptic circles has therefore the index 2.

14027. (J. J. Barniville, M.A.)—Prove that

$$\frac{1}{2^4-1}+\frac{1}{3^4-1}+\frac{2}{5^4-1}+\frac{3}{8^4-1}+\ldots=\tfrac{1}{12}.$$

Solution by Professor Sanjána, M.A.

Let u_n denote the nth term of 1, 1, 2, 3, 5, 8, ... Then

$$\begin{aligned}u_{n+2}^4-u_{n+1}^4&=(u_{n+2}+u_{n+1})(u_{n+2}-u_{n+1})(u_{n+2}^2+u_{n+1}^2)\\&=u_{n+3}u_n\left\{u_{n+2}(u_{n+2}-2u_{n+1})+u_{n+1}(u_{n+1}+2u_{n+2})\right\}\\&=u_{n+3}u_n\left\{u_{n+2}(-u_{n-1})+u_{n+1}(u_{n+4})\right\}\\&=u_nu_{n+1}u_{n+3}u_{n+4}-u_{n-1}u_nu_{n+2}u_{n+3}.\end{aligned}$$

Also, $\quad u_{n+1}^4-u_n^4=u_{n-1}u_nu_{n+2}u_{n+3}-u_{n-2}u_{n-1}u_{n+1}u_{n+2}$;

and so on; $u_2^4-u_1^4=0$. Hence $u_{n+2}^4-u_1^4=u_nu_{n+1}u_{n+3}u_{n+4}$. Thus

$$\begin{aligned}\frac{u_n}{u_{n+2}^4-1}&=\frac{1}{u_{n+1}u_{n+3}u_{n+4}}=\frac{1}{2}\,\frac{2u_{n+2}}{u_{n+1}u_{n+2}u_{n+3}u_{n+4}}\\&=\frac{1}{2}\,\frac{u_{n+4}-u_{n+1}}{u_{n+1}u_{n+2}u_{n+3}u_{n+4}}=\frac{1}{2}\left\{\frac{1}{u_{n+1}u_{n+2}u_{n+3}}-\frac{1}{u_{n+2}u_{n+3}u_{n+4}}\right\}.\end{aligned}$$

As the given series is $\Sigma_1^\infty\dfrac{u_n}{u_{n+2}^4-1}$, its sum reduces to $\frac{1}{2}\dfrac{1}{u_2u_3u_4}$, *i.e.*, to $\frac{1}{12}$.

14474. (R. Knowles.)—Tangents from a point T meet a parabola in P, Q; the circle TPQ cuts the parabola again in C, D; the sides PC, QD of the quadrilateral PQCD meet in E; the diagonals in G; M is the mid-point of EG; MN_1, EN_2, GN_3 are drawn at right angles to the axis; MN_1 meets the parabola in K. Prove that $KN_1^2=EN_2.GN_3$.

Solution by R. Tucker, M.A.

Let P, Q, C, D be (m_1, m_2, m_3, m_4); then $\Sigma(m) = 0$. Now PC is $y(m_1+m_3)-2x = 2am_1m_3$, QD is $y(m_2+m_4)-2x = 2am_2m_4$...(i.);

therefore $2x_E = -a(m_1m_3+m_2m_4)$ and $2x_G = -a(m_1m_4+m_2m_3)$;

therefore $$4x_M = -a(m_1+m_2)(m_3+m_4).$$

Hence $$KN_1^2 = 4ax_M = a^2(m_1+m_2)^2.$$

From (i.), $y_E = a(m_1m_3-m_2m_4)/(m_1+m_3) = a(m_1+m_2)(m_2+m_3)/(m_1+m_3)$

and $y_G = a(m_2m_3-m_1m_4)/(m_2+m_3) = a(m_1+m_3)(m_1+m_2)/(m_2+m_3)$.

Therefore $$y_E\,y_G = KN_1^2.$$

5703. (Rev. W. G. Wright, Ph.D.)—A chord is drawn through the point (2, 0) in the ellipse whose semi-axes are 3 and 2. Find the locus of the intersection of normals from the ends of this chord.

Solution by F. H. Peachell, B.A.

The equation of the ellipse is $x^2/9+y^2/4 = 1$. The coordinates of the intersection of normals at the points whose eccentric angles are α, β will be found to be $3x = 5\cos\alpha\,.\cos\beta\,.\cos\tfrac{1}{2}(\alpha+\beta)/\cos\tfrac{1}{2}(\alpha-\beta)$ .... (1)

$$-2y = 5\sin\alpha\,.\sin\beta\,.\sin\tfrac{1}{2}(\alpha+\beta)/\sin\tfrac{1}{2}(\alpha-\beta). \quad \text{............ (2)}$$

If the chord joining α, β passes through (2, 0), we get

$$\cos\tfrac{1}{2}(\alpha+\beta)/\cos\tfrac{1}{2}(\alpha-\beta) = \tfrac{3}{2}.$$

Squaring, we get $1+\cos(\alpha+\beta) = \tfrac{9}{4}\{1+\cos(\alpha-\beta)\}$ .... (3)

From (1) $\tfrac{3}{5}x = \tfrac{3}{2}\cos\alpha\,.\cos\beta$, or $\tfrac{2}{5}x = \cos\alpha\,.\cos\beta$, or

$$\tfrac{4}{5}x = \cos(\alpha+\beta)+\cos(\alpha-\beta).$$

Substituting in (3), $1+\cos(\alpha+\beta) = \tfrac{9}{4}\{1+\tfrac{4}{5}x-\cos(\alpha+\beta)\}$, or

$$\tfrac{13}{4}\cos(\alpha+\beta) = \tfrac{9}{5}x+\tfrac{5}{4}, \text{ or } \cos(\alpha+\beta) = \tfrac{36}{65}x+\tfrac{5}{13}.$$

$$\cos(\alpha-\beta) = \tfrac{4}{5}x-\cos(\alpha+\beta) = \tfrac{16}{65}x-\tfrac{5}{13},$$

whence
$$\begin{aligned}\sin\alpha\,.\sin\beta &= \tfrac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\} \\ &= \tfrac{1}{2}(\tfrac{16}{65}x-\tfrac{5}{13}-\tfrac{36}{65}x-\tfrac{5}{13}) \\ &= -\tfrac{10}{65}x-\tfrac{5}{13}.\end{aligned}$$

Substituting in (2),

$$\begin{aligned}\tfrac{4}{25}y^2 &= (-\tfrac{10}{65}x-\tfrac{5}{13})^2\sin^2\tfrac{1}{2}(\alpha+\beta)/\sin^2\tfrac{1}{2}(\alpha-\beta), \\ &= \{\tfrac{1}{13}(2x+5)\}^2\{1-\cos(\alpha+\beta)\}/\{1-\cos(\alpha-\beta)\}, \\ &= \{\tfrac{1}{13}(2x+5)\}^2(20-18x)/(45-8x).\end{aligned}$$

Therefore locus is $338y^2(45-8x) = 25(10-9x)(2x+5)^2$.

14434. (Edward V. Huntington, A.M.)—An astroid, two nephroids, and four cardioids are drawn on the same fixed circle of radius a, their cusps lying at the quadrantal points of the circle. Prove: a line of length

2a sliding between either pair of opposite cardioids envelops that nephroid which has the same cusps; and a line of length 3a sliding between the two nephroids envelops the astroid. (Nephroid = two-cusped epicycloid; astroid = four-cusped hypocycloid.)

Solution by R. C. ARCHIBALD, M.A.

This theorem follows at once from the following modes of generation of the nephroid and astroid:—

I. The nephroid is the envelope of the diameter of a circle rolling on an equal circle.

II. The nephroid is an epicycloid generated by a point in the circumference of a circle rolling with internal contact on a circle of two-thirds its radius. A diameter of this generating circle envelops an astroid.

14449. (PAUL GIBSON.)—Given that, in reducing 1/N (N prime) in scale 17 to a pure circulator, five consecutive remainders formed are 1e, 4, 9, 21, 5 (in scale 17), to find N.

I. *Solution by* R. P. PARANJPYE, B.A.

Let p be the remainder just before 1e. Using, for convenience, the ordinary decimal notation, we have, obviously,

$$28+\lambda N = p\cdot 17, \qquad 4+\mu N = 28\cdot 17,$$
$$9+\nu N = 4\cdot 17, \qquad 35+\rho N = 9\cdot 17 \quad\ldots\ldots\ldots\ldots\ (2,\ 1),$$
$$5+\sigma N = 35\cdot 17,$$

where λ, μ, ν, ρ, σ are the successive figures in the decimal. They are obviously integers.

From (1) and (2) we obtain

$$9\cdot 9-35\cdot 4+(9\nu-4\rho)\,N = 0; \quad \text{therefore} \quad 59 = (4\rho-9\nu)\,N;$$

therefore, since 59 contains no factor, N = 59 (*i.e.*, 38 in the scale of 17).

It is easily seen that, with this value of N and suitable values of λ, μ, ..., all the above equations are satisfied.

II. *Arithmetical Solution by the* PROPOSER.

Lemma.—If r_0, r_1, r_2, ... are the remainders formed when 1/P (any scale), and P a prime in that scale, is turned into a pure circulator,

then, if $\qquad r_{n+1}-r_n = r_m, \quad r_{n+2}-r_{n+1} = r_{m+1},$

where the sign of equality means that the same remainder is obtained on division by P.

For, if $\qquad r_{n+1}-r_n = r_m,$ then $10r_{n+1}-10r_n = 10r_m.$

But $\qquad 10r_{n+1} = r_{n+2}$, &c.; therefore $r_{n+2}-r_{n+1} = r_{m+1}.$

By inspection we see $9-4 = 5$. Therefore by the lemma 4 − 1e must = 21, or 1e − 4 = the complement; therefore 17 + 21 = 38 = N.

[Mr. CURJEL remarks that any two consecutive remainders out of the five given would have been sufficient to determine N.]

14494. (Rev. T. ROACH, M.A.)—Along the hedge of a circular field of radius r are placed $2n$ heavy posts at equal distances. A man brings the posts together, one at a time, to one post. Show that the product of the $2n-1$ walks multiplied together $= 2^{2n} \cdot r^{2n-1} \cdot n$.

Solution by R. CHARTRES *and* LIONEL E. REAY, B.A.

Evidently the product

$$= (4r)^{2n-1} \sin \pi/2n \sin 2\pi/2n \ldots \sin \{(2n-1)\pi\}/2n$$
$$= 2^{4n-2} \cdot r^{2n-1} \cdot 2n/2^{2n-1} = 2^{2n} \cdot r^{2n-1} \cdot n.$$

14439. (H. MACCOLL, B.A.)—There are five possible hypotheses, H_1, H_2, &c., of which one must be, and only one can be, true; the chance of each being *one-fifth*. Each of the three H_1, H_2, H_3 implies that the chance that a statement A is true is ·52; whereas H_4 and H_5 lead each to the conclusion that this chance is ·06. From these data prove the paradoxical (but not absurd or impossible) conclusion that it is *probable but not true* that A is probable; and show that the chances that A is *probable and true*, *probable but not true*, *true but not probable*, *neither probable nor true*, are respectively ·312, ·288, ·024, ·376.

Solution by the PROPOSER.

Let the symbol A^p assert that A is *probable*; and let $A^{p\prime}$ be the denial of A^p, as $A^\prime$ is the denial of A. Let $A^{\alpha\beta}$ (as usual) mean $(A^\alpha)^\beta$ whatever the predicates α and β may be. Also, let A^a (when a is a *proper fraction*) be synonymous with $\left(\frac{A}{\epsilon} = a\right)$, and assert that the chance that A is true is a.

Since $H_1 + H_2 + H_3 + H_4 + H_5 = \epsilon$ (a *certainty*), we have

$$\frac{A^p}{\epsilon} = \frac{(H_1 + H_2 + \&c.)\,A^p}{\epsilon} = \frac{H_1}{\epsilon} \cdot \frac{A^p}{H_1} + \frac{H_2}{\epsilon} \cdot \frac{A^p}{H_2} + \&c.$$
$$= \tfrac{1}{5} \cdot \frac{A^p}{H_1} + \tfrac{1}{5} \cdot \frac{A^p}{H_2} + \&c. = \tfrac{1}{5}(1+1+1+0+0) = \tfrac{3}{5},$$
$$\frac{A}{\epsilon} = \frac{(H_1 + H_2 + \&c.)\,A}{\epsilon} = \tfrac{1}{5}\left(\frac{A}{H_1} + \frac{A}{H_2} + \&c.\right)$$
$$= \tfrac{1}{5}(\cdot 52 + \cdot 52 + \cdot 52 + \cdot 06 + \cdot 06) = \cdot 336.$$

Thus, the chance that A is *probable* is $\frac{3}{5}$; whereas the chance that A is *true* is only ·336. Hence, the paradoxical conclusion that *it is probable but not true that* A *is probable*. This conclusion implies no inconsistency; for do we not often find probable predictions falsified by the event? The explanation in this case is as follows:—The *a priori* chance that A is true is either ·52 or ·06; and the former value is more likely to be true than the latter. In this sense, we may say that *it is probable that* A *is probable*. But, on the other hand, if we repeatedly take one of the five hypotheses H_1, H_2, &c., at random (see note), and each time (assuming the H that turns up) try whether A turns out true or false; though the odds will be more frequently in favour of A than against, they will never be much in its favour; and when they are unfavourable they will be so heavily against it (94 to 6) that, *in the long run*, A will turn out false more frequently than true. In this sense we may say that *it is not true that* A *is probable*. Next, to find $\frac{A^pA}{\epsilon}$, $\frac{A^pA^\iota}{\epsilon}$, &c.,

$$\frac{A^pA}{\epsilon} = \frac{A^p}{\epsilon}\cdot\frac{A}{A^p} = \frac{A^{\cdot 52}}{\epsilon}\cdot\frac{A}{A^{\cdot 52}} = \tfrac{3}{5}(\cdot 52) = \cdot 312,$$

$$\frac{A^pA^\iota}{\epsilon} = \frac{A^p}{\epsilon}\cdot\frac{A^\iota}{A^p} = \tfrac{3}{5}\left(1-\frac{A}{A^p}\right) = \tfrac{3}{5}(1-\cdot 52) = \cdot 288,$$

$$\frac{AA^{p\iota}}{\epsilon} = \frac{A^{p\iota}}{\epsilon}\cdot\frac{A}{A^{p\iota}} = \tfrac{2}{5}\cdot\frac{A}{A^{\cdot 06}} = \tfrac{2}{5}(\cdot 06) = \cdot 024,$$

$$\frac{A^{p\iota}A^\iota}{\epsilon} = \frac{A^{p\iota}}{\epsilon}\cdot\frac{A^\iota}{A^{p\iota}} = \tfrac{2}{5}\left(1-\frac{A}{A^{p\iota}}\right) = \tfrac{2}{5}(1-\cdot 06) = \cdot 376.$$

Note.—By a simple experimental method, I found as the result of 100 trials that A turned out *probable* 54 times, and *true* only 28 times; which (so far as it goes) confirms the conclusions A^{pp} and $A^{p\iota}$. The statements A^pA, A^pA^ι, $AA^{p\iota}$, $A^{p\iota}A^\iota$ turned out true 28, 26, 0, 46 times respectively, instead of the theoretically most likely numbers, 31, 29, 2, 38.

14496. (G. H. Hardy, B.A.)—Prove that

$$\Sigma\,\sigma^2(u)\,\sigma_1^2(u)\,\sigma_2^2(v)\,\sigma_3^2(v) = \tfrac{3}{2}\left\{\wp(u+v)+\wp(u-v)\right\}\sigma^2(u+v)\,\sigma^2(u-v);$$

the notation being that of Weierstrass's theory of elliptic functions, and the summation applying to the six possible divisions into pairs of the functions σ, σ_1, σ_2, σ_3.

Solution by H. W. Curjel, M.A.

Consider the function

$$f(u) = \frac{\Sigma\,\sigma^2(u)\,\sigma_1^2(u)\,\sigma_2^2(v)\,\sigma_3^2(v)}{\sigma^2(u+v)\,\sigma^2(u-v)} = \frac{N(u)}{D(u)}\ \text{(say)}.$$

Since $\sigma(u)$, $\sigma_1(u)$, &c., are integral transcendental functions, all the poles of $f(u)$ are zeros of $D(u)$, *i.e.*, $\pm v + 2m\omega_1 + 2n\omega_2$ (each a double one). Also, if we change u into $u+2\omega_a$ ($a = 1, 2, 3$), we get

$$f(u+2\omega_a) = \frac{N(u+2\omega_a)}{D(u+2\omega_a)} = \frac{e^{8\eta a(u+\omega_a)}\,N(u)}{e^{8\eta a(u+\omega_a)}\,D(u)} = f(u);$$

therefore $f(u)$ is an elliptic function of periods $2\omega_1$, $2\omega_2$. Again,

$$\lim_{u=\mp v} \text{ of } (u\pm v)^2 f(u) = \frac{6\sigma^2(v)\,\sigma_1^2(v)\,\sigma_2^2(v)\,\sigma_3^2(v)}{\sigma^2(2v)} = \tfrac{3}{2};$$

therefore the infinite parts of the developments of $f(u)$ near the poles $\pm v$ are

$$\frac{3}{2(u-v)^2}+\frac{B}{u-v}, \quad \frac{3}{2(u+v)^2}+\frac{A}{u+v};$$

therefore $f(u) = A\zeta(u+v)+B\zeta(u-v)-\frac{3}{2}\{\zeta'(u+v)+\zeta'(u-v)\}$

$$= A\zeta(u+v)+B\zeta(u-v)+\tfrac{3}{2}\{\wp(u+v)+\wp(u-v)\},$$

where $A+B=0$. When $u=0$, we get

$$3\wp(v) = A\zeta(v)+B\zeta(-v)+3\wp(v); \quad \text{therefore } (A-B)\zeta(v)=0;$$

therefore, if $\zeta(v)\neq 0$, $A-B=0$; therefore $A = B = 0$; therefore

$$f(u) = \tfrac{3}{2}\{\wp(u+v)+\wp(u-v)\};$$

if $\zeta(v)=0$, substitute $v+2\omega_a$ for v; this does not change $f(u)$, and, as $\zeta(v+2\omega_a)$, $\zeta(v)$ cannot both be zero, the above proof can then be applied.

14452. (Professor UMES CHANDRA GHOSH.)—If Ω and Ω' are the BROCARD points of a triangle ABC, Δ_1, Δ_2, Δ_3 and Δ'_1, Δ'_2, Δ'_3 are the areas of the triangles ΩBC, ΩAC, ΩAB and Ω'BC, Ω'AC, Ω'AB, show that

(i.) $$\frac{\sin^2\omega}{4\Delta} = \frac{\Delta_1}{a^2c^2} = \frac{\Delta_2}{a^2b^2} = \frac{\Delta_3}{b^2c^2} = \frac{\Delta'_1}{a^2b^2} = \frac{\Delta'_2}{b^2c^2} = \frac{\Delta'_3}{a^2c^2};$$

(ii.) $$\frac{\sin\omega}{2\Delta} = \frac{\Omega A}{b^2c} = \frac{\Omega B}{ac^2} = \frac{\Omega C}{a^2b} = \frac{\Omega' A}{bc^2} = \frac{\Omega' B}{a^2c} = \frac{\Omega' C}{ab^2};$$

(iii.) $$\Omega\Omega' = \frac{abc}{a^2b^2+a^2c^2+b^2c^2}\sqrt{\{a^2(a^2-b^2)+b^2(b^2-c^2)+c^2(c^2-a^2)\}};$$

where ω is the BROCARD angle and Δ the area of the triangle ABC.

Solution by R. TUCKER, M.A.; A. F. VAN DER HEYDEN, B.A.; *and others.*

With the usual notation

(i.) $$\Delta_1 = \Omega BC = \tfrac{1}{2}\frac{2\Delta \cdot a^2c^2}{\lambda^2}; \quad \therefore \frac{\Delta_1}{a^2c^2} = \frac{\Delta}{\lambda^2} = \frac{4\Delta^2}{4\Delta\lambda^2} = \frac{\sin^2\omega}{4\Delta},$$

$$\Delta_1' = \Omega' BC = \tfrac{1}{2}\frac{2\Delta a^2b^2}{\lambda^2}; \quad \therefore \frac{\Delta'_1}{a^2b^2} = \frac{\sin^2\omega}{4\Delta}.$$

(ii.) $$\Omega A = cb^2/\lambda; \quad \therefore \frac{\Omega A}{cb^2} = \frac{1}{\lambda} = \frac{\sin\omega}{2\Delta},$$

and $$\frac{\Omega' A}{bc^2} = \frac{1}{\lambda} = \frac{\sin\omega}{2\Delta}, \text{ and so on.}$$

(iii.) The sinister side $= abc/\lambda^2\sqrt{(r^4-\lambda^2)} = abce/\lambda$;

where e is the eccentricity of the BROCARD circle.

Now $\quad \Omega\Omega' = 2a_1e$ (MILNE, p. 109)

$$= 2Re\sin\omega = 4R\Delta e/\lambda = abc/\lambda; \quad \therefore \text{\&c.}$$

The above results are easily got from my "triplicate-ratio" circle papers.

13978. (J. J. Barniville, M.A.)—Having formed the series

$$1,\ 1,\ 3,\ 7,\ 17,\ 41,\ 99,\ \dots\ (u_n = 2u_{n-1}+u_{n-2})$$

and

$$1,\ 1,\ 3,\ 11,\ 41,\ 153,\ \dots\ (u_n = 4u_{n-1}-u_{n-2}),$$

prove that

$$1-\frac{1}{1.3}+\frac{1}{3.7}-\frac{1}{7.17}+\frac{1}{17.41}-\dots=\frac{1}{\surd 2},$$

$$\frac{1}{1+\surd 2}+\frac{1}{3+\surd 2}+\frac{1}{17+\surd 2}+\frac{1}{99+\surd 2}+\dots=\frac{1}{\surd 2},$$

and

$$\frac{1}{1^2+2}+\frac{1}{3^2+2}+\frac{1}{11^2+2}+\frac{1}{41^2+2}+\dots=\frac{\surd 3}{4}.$$

Solution by Professor Sanjána, M.A.

Denoting the first sum by S_1 and transforming the series into a continued fraction, we get

$$S_1 = \frac{1}{1-}\ \frac{1^2}{1-3-}\ \frac{3^2}{-3+21-}\ \frac{21^2}{21-119-}\ \frac{119^2}{-119+697-}\ \dots$$

$$= \frac{1}{1-}\ \frac{1}{-2-}\ \frac{9}{18-}\ \frac{441}{-98-}\ \frac{17^2.7^2}{578-}\ \dots$$

$$= \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ \dots = \frac{1}{1+(\surd 2-1)} = \frac{1}{\surd 2}.$$

Again, for the first series,

$$u_n^2-u_{n-1}u_{n+1} = u_n(u_{n+2}-2u_{n+1})-u_{n+1}(u_{n+1}-2u_n)$$

$$= u_n u_{n+2}-u_{n+1}^2 = -(u_{n+1}^2-u_n u_{n+2});$$

but $u_2^2-u_1u_3 = -2$ and $u_3^2-u_2u_4 = 2$. Hence $u_n^2-u_{n-1}u_{n+1} = -2(-1)^n$, Thus, the second sum

$$S_2 = \frac{1-\surd 2}{1^2-2}+\frac{3-\surd 2}{3^2-2}+\frac{17-\surd 2}{17^2-2}+\frac{99-\surd 2}{99^2-2}+\dots$$

$$= -1+\frac{3}{1.7}+\frac{17}{7.41}+\dots+\surd 2-\surd 2\left\{\frac{1}{1.7}+\frac{1}{7.41}+\dots\right\}$$

$$= -1+\frac{1}{2}\left\{\frac{1}{1}-\frac{1}{7}+\frac{1}{7}-\frac{1}{41}+\dots\right\}$$

$$+\surd 2-\surd 2\left\{\frac{1}{6}\cdot\frac{6}{1.7}+\frac{1}{34}\cdot\frac{34}{7.41}+\dots\right\}$$

$$= -\frac{1}{2}+\surd 2-\frac{\surd 2}{2}\left\{\frac{1}{3}\left(\frac{1}{1}-\frac{1}{7}\right)+\frac{1}{17}\left(\frac{1}{7}-\frac{1}{41}\right)+\dots\right\}$$

$$= -\frac{1}{2}+\surd 2-\frac{\surd 2}{2}\left\{\frac{1}{1.3}-\frac{1}{3.7}+\frac{1}{7.17}-\frac{1}{17.41}+\dots\right\}$$

$$= -\frac{1}{2}+\surd 2-\frac{\surd 2}{2}\left(1-\frac{1}{\surd 2}\right)=\frac{1}{\surd 2}.$$

For the second series,

$$u_n^2 - u_{n+1}u_{n-1} = u_n(4u_{n+1} - u_{n+2}) - u_{n+1}(4u_n - u_{n+1}) = u_{n+1}^2 - u_{n+2}u_n;$$

but $u_2^2 - u_1u_3 = -2$. Hence $u_n^2 - u_{n-1}u_{n+1} = -2$, always. Hence

$$S_3 = \frac{1}{1.3} + \frac{1}{1.11} + \frac{1}{3.41} + \frac{1}{11.153} + \dots;$$

therefore $$4S_3 = \frac{4}{1.3} + \frac{1}{3}\cdot\frac{12}{1.11} + \frac{1}{11}\cdot\frac{44}{3.41} + \dots$$

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3.11} + \frac{1}{3.11} + \frac{1}{11.41} + \dots;$$

therefore $$\frac{4S_3+1}{2} = \frac{1}{1} + \frac{1}{3} + \frac{1}{33} + \frac{1}{451} + \dots$$

$$= \frac{1}{1-}\,\frac{1}{4-}\,\frac{9}{36-}\,\frac{1089}{484-}\dots = \frac{1}{1-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\dots$$

$$= \frac{1}{1-(2-\sqrt{3})} = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}.$$

Therefore $$S_3 = \tfrac{1}{4}\sqrt{3}.$$

14441. (Rev. T. Mitcheson, B.A.)—A regular polygon of an even number of sides is inscribed in a circle, and lines are drawn from one of the angular points to each of the others. Show that the sum of these lines $= (a\cot\pi/2n)/(\sin\pi/n)$ (a being a side of the polygon), and if the lines be h_1, h_2, h_3, &c., then

$$\tfrac{1}{2}(h_{n-1} + h_{n-2} + h_{n-3} + \dots) - \{h_{\frac{1}{2}(n-2)} + h_{\frac{1}{2}(n-4)} + h_{\frac{1}{2}(n-6)} + \dots) = R.$$

Solution by the Rev. T. Roach, M.A.; *and the* Proposer.

If n be the number of points, we have

$$S = 2R(\sin\pi/n + \sin 2\pi/n + \dots) \text{ to } n-1 \text{ terms}$$

$$= 2R\left[\{\sin\tfrac{1}{2}n\,.\,\pi/n\,\sin\tfrac{1}{2}(n-1)\pi/n\}/\sin\pi/2n\right]$$

$$= 2R\cot\pi/2n = (a\cot\pi/2n)/(\sin\pi/n).$$

Again $$\tfrac{1}{2}(h_{n-1} + h_{n-2} + \dots) - (h_{\frac{1}{2}(n-2)} + h_{\frac{1}{2}(n-4)} + h_{\frac{1}{2}(n-6)} + \dots)$$

$$= R(\sin\pi/n + \sin 2\pi/n + \dots)_1^{n-1} - 2R(\sin 2\pi/n + \sin 4\pi/n + \sin 6\pi/n)_1^{\frac{1}{2}(n-2)}$$

$$= R\left[\{\sin\tfrac{1}{2}n\,.\,\pi/n\,.\,\sin\tfrac{1}{2}(n-1)\pi/n\}/\sin\pi/2n\right]$$
$$- 2R\left[\{\sin\tfrac{1}{4}n\,.\,2\pi/n\,\sin\tfrac{1}{4}(n-2)\,2\pi/n\}/\sin\pi/n\right]$$

$$= R\cot\pi/2n - 2R\cot\pi/n = R\tan\pi/2n.$$

14481. (H. M. Taylor, M.A., F.R.S. Suggested by Quest. 14382.) —On the sides of a triangle A′B′C′, triangles B′C′A, C′A′B, A′B′C are constructed similar to three given triangles. Having given the triangle, ABC and the three triangles, reconstruct the triangle A′B′C′.

Solution by the Proposer.

ABC (Fig. 1) is the given triangle having been formed by constructing on the sides of the triangle A′B′C′ the triangles AB′C′, BC′A′, CA′B′ similar and homologous to the given triangles LXY, MXY, NXY

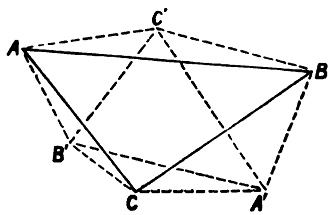


Fig. 1.

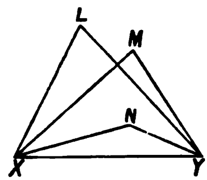


Fig. 2.

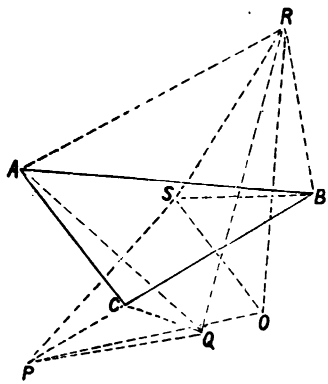


Fig. 3.

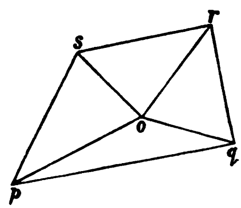


Fig. 4.

(Fig. 2) respectively; it is required, from these data, to reconstruct the triangle A′B′C′.

[The triangles ABC in Figs. 1 and 3 are identical, but they are drawn apart merely for the sake of clearness.]

Construction.—Take any point P (Fig. 3), and construct △PCQ similar and homologous to △XNY; and △QAR similar and homologous to △XLY; and △RBS similar and homologous to △XMY.

Next take a point *o* (Fig. 4), and describe △*poq* similar and homologous to △PCQ, △*qor* similar and homologous to △QAR, △*ros* similar and homologous to △RBS.

Now (in Fig. 3) describe $\triangle$POS similar and homologous to $\triangle pos$ of Fig. 4. We shall prove O to be coincident with A′.

Proof.—A rotation round C, through an angle equal to XNY, accompanied by a dilatation (or contraction) in the ratio NX : NY, would shift P to Q and A′ to B′. Next, a rotation round A through an angle XLY, accompanied by a dilatation in the ratio LX : LY, would shift Q to R and B′ to C′; and a rotation round B through an angle XMY, accompanied by a dilatation in the ratio MX : MY, would shift R to S and C′ to A′.

Hence by the three operations A′ is unmoved and P is shifted to S.

Hence, by a rotation round A′, through the sum of the angles XNY, XLY, XMY, *i.e.*, through the angle *pos* (Fig. 4), accompanied by a dilatation compounded of the three specified dilatations, *i.e.*, in the ratio *op* : *os* (Fig. 4), P would be shifted to S. But, from Fig. 3, a rotation round O, through the angle *pos*, accompanied by a dilatation in the ratio *op* : *os*, would shift P to S.

Therefore O is coincident with A′.

The points B′ and C′ can now be easily found.

14511. (John C. Malet, M.A., F.R.S.)—If, in the sextic algebraic equation $$x^6-p_1x^5+p_2x^4-p_3x^3+p_4x^2-p_5x+p_6 = 0,$$ the sum of three roots is equal to the sum of the other three, (1) prove $$4p_6Q_4-Q_4Q_3^2-p_6Q_2^2+p_5Q_2Q_3-p_5^2 = 0,$$ where $Q_2\equiv p_2-\frac{1}{4}p_1^2$, $Q_3\equiv p_3-\frac{1}{2}p_1p_2+\frac{1}{8}p_1^3$, $Q_4\equiv p_4-\frac{1}{2}p_1p_3+\frac{1}{4}p_1^2p_2-\frac{1}{16}p_1^4$; (2) solve the equation.

I. *Solution by the* Proposer; H. W. Curjel, M.A.; *and* Prof. Sanjána.

(1) Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the roots of the equation; then $$x_1+x_2+x_3 = x_4+x_5+x_6 = \tfrac{1}{2}p_1.$$

Let now $$x_1x_2+x_2x_3+x_3x_1 = u_1, \quad x_4x_5+x_5x_6+x_6x_4 = u_2,$$ $$x_1x_2x_3 = v_1, \quad x_4x_5x_6 = v_2;$$

and we find $$u_1+u_2 = Q_2 \quad \text{(i.)}.$$ $$v_1+v_2+\tfrac{1}{2}p_1(u_1+u_2) = p_3; \quad \text{therefore} \quad v_1+v_2 = Q_3 \quad \text{(ii.)}.$$ $$u_1u_2+\tfrac{1}{2}p_1(v_1+v_2) = p_4;$$

therefore $$u_1u_2 = Q_4, \quad v_1u_2+v_2u_1 = p_5, \quad v_1v_2 = p_6 \quad \ldots \text{(iii., iv., v.)}.$$

From (i.) and (iii.), we have $$2u_1 = Q_2+\sqrt{(Q_2^2-4Q_4)}, \quad 2u_2 = Q_2-\sqrt{(Q_2^2-4Q_4)};$$

and, from (ii.) and (v.), $$2v_1 = Q_3+\sqrt{(Q_3^2-4p_6)}, \quad 2v_2 = Q_3-\sqrt{(Q_3^2-4p_6)};$$

Substituting for u_1, u_2, v_1, v_2 in (v.) and rationalizing, we get the required condition.

(2) The roots of the sextic are the roots of the cubics $$2x^3-p_1x^2+\left\{Q_2\pm\sqrt{(Q_2^2-4Q_4)}\right\}x-\left\{Q_3\pm\sqrt{(Q_3^2-4p_6)}\right\} = 0.$$

II. *Solution by* G. H. Hardy, B.A.

It is interesting to consider this equation from the point of view of the Galois theory.

The function $\phi_1 \equiv x_1 + x_2 + x_3 \equiv [123]$ has, in general, 20 values, viz.:

[123], [124], [125], [126], [134], [135], [136], [145], [146], [156], [234], [235], [236], [245], [246], [256], [345], [346], [356], [456];

and satisfies an equation $g_{20}(\phi) = 0$ .... (i.).

The group of ϕ_1 is of order $(3\,!)^2 = 36$; and the solution of (i.) involves the complete solution of the sextic, for, if $x_1 + x_2 + x_3$, ... are known, so are $x_1, x_2, x_3, \ldots$.

If we know *one* root of (i.), *e.g.*, ϕ_1, and *adjoin* it, we can determine x_1, x_2, x_3 by the solution of a cubic, for $x_2x_3 + x_3x_1 + x_1x_2$ and $x_1x_2x_3$ are rational functions of ϕ_1. As we also know $x_4 + x_5 + x_6$ rationally, we can determine x_4, x_5, x_6 by the solution of a second cubic.

Thus, when the coefficients of the sextic are so conditioned that ϕ_1 is a rational function of them, the equation can be solved by means of square and cube roots. This remains true in the present case, although two values of ϕ, viz., [123], [456], are numerically equal.

Let $x^6 - p_1x^5 + p_2x^4 - p_3x^3 + p_4x^2 - p_5x + p_6$

$$\equiv (x^3 - \tfrac{1}{2}p_1x^2 + Ax - B)(x^3 - \tfrac{1}{2}p_1x^2 + A'x - B');$$

then
$$\left.\begin{array}{l} p_2 = \tfrac{1}{4}p_1^2 + A + A', \quad p_3 = \tfrac{1}{2}p_1(A + A') + B + B' \\ p_4 = \tfrac{1}{2}p_1(B + B') + AA', \quad p_5 = AB' + A'B, \quad p_6 = BB' \end{array}\right\} \ldots (a).$$

$$A + A' = p_2 - \tfrac{1}{4}p_1^2 = Q_2,$$

$$B + B' = p_3 - \tfrac{1}{2}p_1p_2 + \tfrac{1}{8}p_1^3 = Q_3, \quad AA' = p_4 - \tfrac{1}{2}p_1p_3 + \tfrac{1}{4}p_1^2p_2 - \tfrac{1}{16}p_1^4 = Q_4,$$

$$(A - A')(B - B') = Q_2Q_3 - 2p_5, \quad (Q_2Q_3 - 2p_5)^2 = (Q_2^2 - 4Q_4)(Q_3^2 - 4p_6);$$

i.e.,
$$4p_6Q_4 - Q_4Q_3^2 - p_6Q_2^2 + p_5Q_2Q_3 - p_5^2 = 0 \ldots\ldots\ldots\ldots\ldots \text{(ii.)}.$$

To solve the equation, we have to calculate

$$A, A' = \tfrac{1}{2}\left\{Q_2 \pm \sqrt{(Q_2^2 - 4Q_4)}\right\},$$

which must, in virtue of (ii.), be rational functions of the coefficients; and then determine B, B′ by equations (*a*). The roots are then found by solving the two cubics.

14222. (Professor Elliott, F.R.S.) – If $P + a_0Q$, in which P and Q are free from a_0, is annihilated by $a_0\dfrac{\partial}{\partial a_1} + 2a_1\dfrac{\partial}{\partial a_2} + 3a_2\dfrac{\partial}{\partial a_3} + \ldots$ to ∞, show that $\dfrac{\partial}{\partial a_1}Q = 0$, and that, when $m > 1$,

$$\frac{\partial}{\partial a_m}Q = -\tfrac{1}{2}\sum_{r=1}^{r=m-1}\frac{\partial^2}{\partial a_r\,\partial a_{m-r}}P.$$

Solution by G. D. Wilson, B.A.

By hypothesis, $\left(a_0\dfrac{\partial}{\partial a_1} + \sum_{n=1}^{\infty}(n+1)a_n\dfrac{\partial}{\partial a_{n+1}}\right)(a_0Q + P) \equiv 0,$

and P and Q are free from a_0. Hence

$$\frac{\partial Q}{\partial a_1} = 0, \quad \frac{\partial P}{\partial a_1} + \sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial Q}{\partial a_{n+1}} = 0, \quad \sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial P}{\partial a_{n+1}} = 0$$

...... (i., ii., iii.).

Therefore $$0 = \frac{\partial^2}{\partial a_r\, \partial a_{m-r}} \sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial P}{\partial a_{n+1}} \qquad \text{[from (iii.)]}$$

$$= \sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial^3 P}{\partial a_{n+1}\, \partial a_r\, \partial a_{m-r}} + (r+1) \frac{\partial^2 P}{\partial a_{r+1}\, \partial a_{m-r}} + (m-r+1) \frac{\partial^2 P}{\partial a_r\, \partial a_{m-r+1}}.$$

Therefore

$$\sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial}{\partial a_{n+1}} \left(\sum_{r=1}^{m-1} \frac{\partial^2 P}{\partial a_r\, \partial a_{m-r}} \right) = -2 \sum_{r=1}^{m-1} (r+1) \frac{\partial^2 P}{\partial a_{r+1}\, \partial a_{m-r}} = -2 \sum_{r=2}^{m} r \frac{\partial^2 P}{\partial a_r\, \partial a_{m+1-r}}.$$

But, from (ii.), $\displaystyle\sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial}{\partial a_{n+1}} \frac{\partial Q}{\partial a_m} = -\frac{\partial^2 P}{\partial a_1\, \partial a_m} - (m+1) \frac{\partial Q}{\partial a_{m+1}}$;

hence $$\sum_{n=1}^{\infty} (n+1)\, a_n \frac{\partial}{\partial a_{n+1}} \left(\sum_{r=1}^{m-1} \frac{\partial^2 P}{\partial a_r\, \partial a_{m-r}} + 2 \frac{\partial Q}{\partial a_m} \right)$$

$$= -2 \sum_{r=1}^{m} r \frac{\partial^2 P}{\partial a_r\, \partial a_{m+1-r}} - 2(m+1) \frac{\partial Q}{\partial a_{m+1}}$$

$$= -(m+1) \left(\sum_{r=1}^{m} \frac{\partial^2 P}{\partial a_r\, \partial a_{m+1-r}} + 2 \frac{\partial Q}{\partial a_{m+1}} \right).$$

Putting $m = 1, 2, \ldots$, &c., in this, we obtain the result stated.

14454. (Professor Sanjána, M.A.)—Solve, in rational numbers, the equation $M^2 - 2xN^2 = x^2 - 1$, where x stands for any one of the natural numbers 2, 3, 4, 5, 6, ... [The solution gives $N^4 + 1$ as the difference of two squares. I have reason to believe that 5 is the only small value of x admissible. For the method see Chrystal, xxxiii., §§ 15–19.]

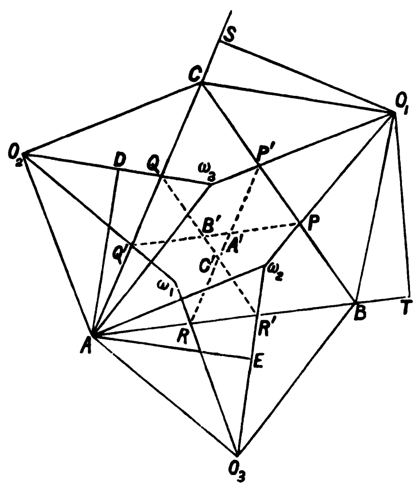
Solution by H. W. Curjel, M.A.; *and* Lt.-Col. Allan Cunningham, R.E.

x cannot be even, for $M^2 + 1$ cannot be divisible by 4; and, since $M^2 + 1 \equiv 0$, mod x, all the prime factors of x must be of the form $4n + 1$.

A large number of values of x will be excluded by the condition that $2x$ must be a quadratic residue of all factors which occur in $x^2 - 1$ to an odd power. This condition excludes the following values less than 100:—13, 25, 29, 41, 53, 61, 73, 85, 89, 97; $x = 5$, 17, 37, 65, 101 give solutions $x = 5$, $M = 8$, $N = 2$, the remaining solutions may be deduced with the help of $19^2 - 10 \times 6^2 = 1$; $x = 17$, $M = 72$, $N = 12$, the remaining solutions with the help of $35^2-34\cdot 6^2 = 1$; $x = 37$, all solutions are easily deduced from $43^2-5^2 . 74 = -1$ and $6^2-74 = -38$, since 74 is a non-residue of 9; $x = 65$, all solutions from $8^2-130 = -66$ and $57^2-5^2 . 130 = -1$; $x = 101$, all solutions from $10^2-202 = -102$, and $3141^2-202 \times 221^2 = -1$, since 202 is a non-residue of 25.

14534. (W. S. Cooney.)—Let O_1, O_2, O_3 be the centres of squares described externally, and ω_1, ω_2, ω_3 the centres of squares described internally, on the sides a, b, c, respectively, of triangle ABC. Join O_1 to ω_2 and ω_3, meeting side BC in P, P′; O_2 to ω_3 and ω_1, meeting CA in Q, Q′; O_3 to ω_1 and ω_2, meeting AB in R, R′. Prove that A′, B′, C′, the intersections of P′R, Q′P, R′Q are the centres of the insquares of ABC, and that, if AA′, BB′, CC′ meet sides of A′B′C′ in α, β, γ, then triangle $\alpha\beta\gamma$ is similar to ABC.

Solution by the Proposer.



From Quest. 14473 and from figure,

$$\triangle AO_2\omega_3 = \triangle A\omega_2O_3;$$

therefore the perpendiculars AD and AE are equal. Draw O_1S perpendicular to AC and O_1T to AB. $\triangle CQO_2$ is similar to $\triangle CP'O_1$, and ADQ is similar to CSO_1. Therefore

$$CQ/CP' = CO_2/CO_1 = AC/BC;$$

therefore QP′ is parallel to AB, and, similarly, PR′ and RQ′ are parallel respectively to CA and BC. ADQ and CSO_1 are similar. Therefore

$$AQ/AD = CO_1/SO_1;$$

also

$$AR'/AE = BO_1/TO_1;$$

but $AD = AE$ and $CO_1 = BO_1$; therefore

$$AQ/AR' = TO_1/SO_1,$$

therefore diagonal of completed parallelogram P′QAR′P passes through O_1 and also through A′ (since Q′R is parallel to PP′), and bisects QR′ in α. Similarly, BO_2 and CO_3 pass through B′ and C′, bisecting P′R and

PQ′ in β and γ. Therefore sides of $\alpha\beta\gamma$ are parallel to sides of ABC. Angles of $\triangle BR'O_2$ are 45°, 90° − B, and 45° + B. Therefore

$$BR' = (c\cos B)/(B), \quad \text{where} \quad (B) = \sin B + \cos B.$$

Also $$AR = (c\cos A)/(A) \quad \text{and} \quad RR' = (c\cos C)/(A)(B).$$

Similarly, $QQ' = (b\cos B)/(A)(C)$, $PR' = (b/c)BR' = (b\cos B)/(B)$.

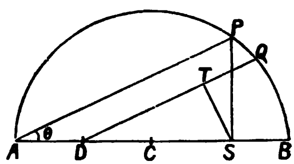
$$\frac{\sin BAB'}{\sin CAB'} = \frac{R'B'}{B'Q}\,\frac{AQ}{AR'} = \frac{PR'}{QQ'}\,\frac{TO_1}{SO_1} = \frac{(A)(C)}{(B)}\,\frac{\sin(B+45°)}{\sin(C+45°)} = \frac{(A)(C)}{(B)}\,\frac{(B)}{(C)}$$

$$= (A) = \frac{\sin(A+45°)}{\sin(45°)} = \frac{\sin BAO_2}{\sin CAO_2}.$$

Therefore $A.BB'CO_2$ is a harmonic pencil, as is also $C.BB'AO_2$. Therefore lines drawn through B′ parallel to AO_2 and CO_2 will make angles of 45° with AC, and have the parts intercepted by the sides AC and AB, and AC and CB bisected at B′. Therefore B′ is centre of insquare to AC. Therefore, &c.

14188. (Salutation.)—Bisect AB (= unity) in C, and AC in D; on AB describe a semicircle; from A, D draw parallel lines intersecting the semicircle in P, Q respectively; S, T being the projections of P on AB, and of S on DQ, prove that 4ST is the sine of an angle = 3PAB.

Solution by G. Birtwistle, B.A.; Professor T. Savage; *and many others.*



$$AP = \cos\theta;$$

therefore $AS = \cos^2\theta$;

therefore $DS = \cos^2\theta - \tfrac{1}{4}$;

therefore $ST = \sin\theta(\cos^2\theta - \tfrac{1}{4})$;

therefore $4ST = \sin\theta(3 - 4\sin^2\theta)$
$= \sin 3\theta$.

14532. (Rev. J. Cullen.)—Let Δ be any conic in the plane of a given triangle ABC. A point P is taken on Δ, and parallels through P to BC, CA, AB meet Δ again in A′, B′, C′. Prove that AP, BP, CP intersect B′C′, C′A′, A′B′ in three collinear points L, M, N. (A particular case is that the intersections of the symmedian lines with the corresponding sides of Brocard's triangle are collinear.)

Prove also that, if Δ be the circumcircle, then LMN is at right angles to the Simson-line of P.

Remarks by Professor Sanjána.

The first part is proved readily by trilinears. I append a geometrical proof of the last part.

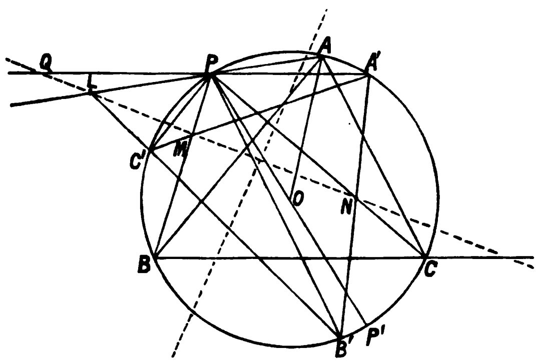
The triangle A′B′C′ is inversely similar to ABC. Hence

$$\angle MA'N = A = BAC = MPN;$$

therefore M, P, A′, N are concyclic; so also are N, P, L, B′ and L, P, M, C′.

Thus $\angle PMN = 180° - PA'N = 180° - PC'L = 180° - PML$,

so that LM, MN are in a straight line.



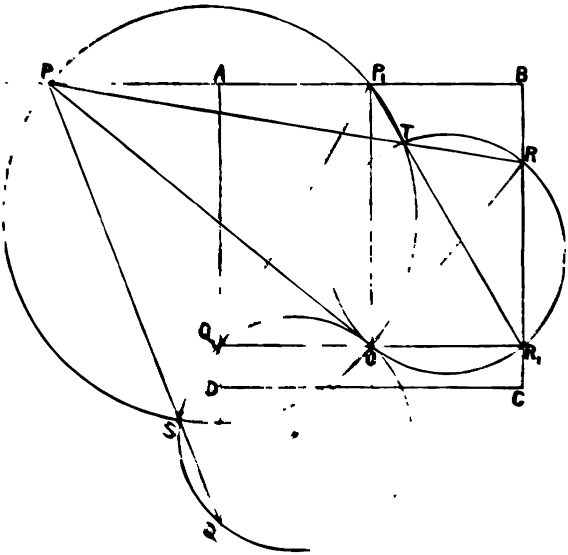
Let the arc PA subtend an angle α at the circumference; let LMN meet A′P in Q; and let POP′ be the diameter. The angle made by LMN with BC = $\angle PQL = PLM - APA' = PB'A' - APA' = \alpha = \frac{1}{2}POA$. But the angle which the Simson-line of P makes with BC = $\frac{1}{2}P'OA$; hence LMN is at right angles to the Simson-line. See the Lemma on p. 73, Vol. LXVII.

14299. (Rev. T. Mitcheson, B.A.)—Let $P_1Q_1R_1$ be an equilateral triangle such that P_1 is on one side of a square, Q_1 and R_1 on the adjacent sides, Q_1R_1 parallel to the other side, and O the mid-point of Q_1R_1; and let PQR be any other equilateral triangle, whose angular points are the same sides, QR passing through O, and let P_1Q_1 meet PQ in S, P_1R_1 meet PR in T. Then the circle passing through P, P_1, S, T touches QR in O, and circles passing through O, T, R, R_1 and O, S, Q, Q_1, respectively, are each one third of the first circle. (An echo of Quest. 14235.)

Solution by I. Arnold *and the* Proposer.

Let ABCD be the square, and $P_1Q_1R_1$ an equilateral triangle inscribed; O the mid-point of Q_1R_1, and QR passing through it meeting BC in R and AD in Q. From O draw OP perpendicular to QR meeting BA in P; then is RPQ the other equilateral triangle. A circle described on OP as diameter passes through $PSTP_1$, touching QR in O.

It is also evident that QO is the diameter of the circle passing through QSQ_1O, and that OR is the diameter of the circle passing through OR_1RT,



and these diameters are equal. For [illegible] is equal to [illegible], and circles being as the squares of their diameters, this makes either of the smaller circles one third of the larger circle.

[illegible] Théorème [illegible] : Lieu des points d'où l'on peut mener à l'ellipse quatre normales dont la somme des carrés soit constante.

[illegible]

[illegible]

[illegible] les coordonnées du point donné [illegible] pieds des normales [illegible]

[illegible]

[illegible]

[illegible] l'équation de l'ellipse

$b^2x^2+a^2y^2 = a^2b^2$ et celle de l'hyperbole d'Apollonius relative au point $\alpha\beta$, $c_2xy+b^2\beta x-a^2\alpha y = 0$, on trouve aisément

$$\Sigma x_1 = 2a^2\alpha/c^2 \; ; \; \Sigma y_1 = -2b^2\beta/c^2 \; ; \; \Sigma x_1^2 = 2a^2/c^4(a^2\alpha^2-b^2\beta^2+c^4) \; ;$$
$$\Sigma y_1^2 = 2b^2/c^4(b^2\beta^2-a^2\alpha^2+c^4).$$

En portant ces valeurs dans l'equation précédente elle devient après quelques transformations

$$\alpha^2(a^2-2b^2)+\beta^2(2a^2-b^2) = c^2(\mathrm{K}^2-a^2-b^2).$$

Pour $a>b\sqrt{2}$ le lieu est une ellipse réelle ou imaginaire.
Pour $a<b\sqrt{2}$ le lieu est une hyperbole.
Pour $a = b\sqrt{2}$ le lieu se compose de deux droites parallèles à l'axe des x.

14270. (H. MacColl, B.A.)—If k be a positive constant, and the variables x and y be each taken at random between 0 and 1, show that the chance that the fraction $k(1-x-y)/(1-y-ky)$ will also lie between 0 and 1 is $(k^2+1)/\{2k(k+1)\}$ or $(1+2k-k^2)/\{2(k+1)\}$ according as k is greater or less than 1.

Solution by the Proposer.

Let the symbol Q assert that the event whose chance is required will happen, while the symbol ϵ (expressing *certainty*) asserts the assumptions or data of the question. The symbol Q/ϵ (as in previous conventions) will then express the chance required. The symbols x_1, x_2, &c., y_1, y_2, &c., denote the limits of x and y, obtained as in my solution of Quest. 14210 (*Reprint*, Vol. lxxiii., p. 69), of which we may use the result, simply putting y for b and x for a (see the accompanying table).

$\epsilon = y_{1'.0}\,x_{1'.0}\,k^u$		
$y_1 = 1$	$x_1 = 1$	$k_1 = 1$
$y_2 = 1-x$	$x_2 = k/(k+1)$	
$y_3 = 1/(k+1)$	$x_3 = (k-1)/k$	
$y_4 = 1-k+kx$		

Using the symbol $y_{m'.n}\,x_{r'.s}$ as an abbreviation for the definite integral

$$\int_{y_n}^{y_m} dy \int_{x_s}^{x_r} dx,$$

we get for the cases k_1 and $k_{1'}$ respectively

$$Q/\epsilon = (y_{2'.0}+y_{1'.4})\,x_{1'.2}+y_{1'.2}\,x_{2'.0}+y_{4'.0}\,x_{2'.3} = (k^2+1)/(2k^2+2k) \; ;$$
$$Q/\epsilon = (y_{2'.0}+y_{1'.4})\,x_{1'.2}+(y_{1'.2}+y_{4'.0})\,x_{2'.0} = (1+2k-k^2)/(2k+2).$$

The symbol $y_{2'.0}$ indicates that the integral is to be taken between the limits y_2 and 0 ; and similarly for the symbols $y_{4'.0}$ and $x_{2'.0}$.

In this solution the order of variation is y, x, k. Another and fuller solution (with the order x, y, k) is as follows.

Taking the order of variation x, y, k (see second table of limits), and using the statement $N^uD^u(N-D)^r + N^rD^r(N-D)^u$, we get (see solution of Quest. 14210)

$Q = x_{2'.3}y_{2'} + x_{3'.2}y_2$.

$x_1 = 1$	$y_1 = 1$	$k_1 = 1$
$x_2 = 1-y$	$y_2 = 1/(k+1)$	
$x_3 = (y+k-1)/k$	$y_3 = 1-k$	

As the application of the formula

$$x_{m'.n} = x_{m'.n}(x_m - x_n)^u$$

introduces no fresh limit or factor into either term, we multiply by the certainty-factor $x_{1'.0}y_{1'.0}$, getting

$$Q = x_{2'.1'.3.0}y_{2'.1'.0} + x_{3'.1'.2\ 0}y_{1'.2.0}.$$

But $x_{2'.1'} = x_{2'}$; $x_{3.0} = x_3y_3 + x_0y_{3'}$; $x_{2\ 0} = x_2$; $x_{3'.1'} = x_{3'}$; $y_{2'.1'} = y_{2'}$; and $y_{2.0} = y_2$. Therefore

$$Q = x_{2'}(x_3y_3 + x_0y_{3'})\,y_{2'.0} + x_{3'.2}y_{1'.2} = x_{2'.3}y_{2'.3.0} + x_{2'.0}y_{3'.2'.0} + x_{3'.2}y_{1'.2}.$$

But $y_{3.0} = y_3k_{1'} + y_0k_1$, and $y_{3'.2'} = y_{3'}$. Therefore

$$Q = x_{2'.3}y_{2'}(y_3k_{1'} + y_0k_1) + x_{2'.0}y_{3'.0} + x_{3'.2}y_{1'.2}$$
$$= x_{2'.3}y_{2'.3}k_{1'} + x_{2'.3}y_{2'.0}k_1 + x_{2'.0}y_{3'.0} + x_{3'.2}y_{1'.2}.$$

But $y_{3'.0} = y_{3'.0}k_{1'}$; while $y_{2'.3}$, $y_{2'.0}$, and $y_{1'.2}$ introduce no fresh factors. Therefore

$$Q = (x_{2'.3}y_{2'.3} + x_{2'.0}y_{3'.0})\,k_{1'} + x_{2'.3}y_{2'.0}k_1 + x_{3'.2}y_{1'.2}(k_{1'} + k_1).$$

Hence, for the case k_1 we have

$$Q/\epsilon = x_{2'.3}y_{2'.0} + x_{3'.2}y_{1'.2} = x_{2'.3}y_{2'.0} + x_{2'.3}y_{2'.1}$$
$$= x_{2'.3}(y_{2'.0} + y_{2'.1}) = (k^2+1)/(2k^2+2k);$$

and for the case $k_{1'}$ we have

$$Q/\epsilon = x_{2'.3}y_{2'.3} + x_{2'.0}y_{3'.0} + x_{2'.3}y_{2'.1}$$
$$= x_{2'.3}(y_{2'.3} + y_{2'.1}) + x_{2'.0}y_{3'.0} = (1+2k-k^2)/(2k+2).$$

Note.—In working out integrations of multiple integrals with several limits, the following abbreviations (which I proposed some years ago in my paper on the "Limits of Multiple Integrals" in the *Proc. Lond. Math. Soc.*) will, I think, be found useful:—

The symbol $\phi(x)\,x_{m'.n}$ means $\int_{x_n}^{x_m} dx\,\phi(x)$; whereas the symbol $x_{m'.n}\,\phi(x)$ means $\phi(x_m) - \phi(x_n)$. Thus, for the case k_1 (that is, the case $k > 1$) we get

$$Q/\epsilon = x_{2'.3}(y_{2'.0} + y_{2'.1}) = (x_2 - x_3)(y_{2'.0} + y_{2'.1})$$
$$= 1/k\,\{1-(k+1)\,y\}\,(y_{2'.0} + y_{2'.}\ ) = (y_{2'.0} + y_{2'.1})\,1/k\,\{y - \tfrac{1}{2}(k+1)\,y^2\}$$
$$= 2/k\,\{y_2 - \tfrac{1}{2}(k+1)\,y_2^2\} - 1/k\,\{1 - \tfrac{1}{2}(k+1)\,1^2\}$$
$$= \frac{1}{k(k+1)} + \frac{k-1}{2k} = \frac{k^2+1}{2k(k+1)}.$$

The working for the case $k_{1'}$ (that is, $k <$) may be similarly simplified by abbreviated notation.

14436. (Rev. T. Roach, M.A. Suggested by 14376.)—If I, I_1, I_2, I_3 be in- and ex-centres of a triangle ABC, and o_1, o_2, o_3 circumcentres of II_2I_3, II_3I_1, II_1I_2 respectively, prove that $o_3I_1o_2I_3o_1I_2$ is an equilateral hexagon, and find the value of its angles.

Solution by A. F. van der Heyden, B.A.; *and* H. W. Curjel, M.A.

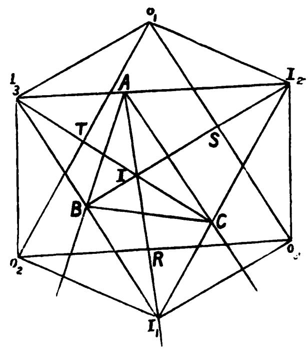
Let the sides of $\triangle o_1o_2o_3$ be cut in R, S, T by II_1, II_2, II_3. Then II_1, &c., are thereby bisected at right angles. Also, $o_2R = Ro_3$ (results required for Quest. 14376); therefore $o_2I_1 = I_1o_3$.

But $I_1o_3 = o_3I_2$ by definition.

Thus $o_2I_1o_3I_2o_1I_3$ is an equilateral hexagon. Also $o_2o_3I_2I_3$ is a parallelogram, &c. Thus opposite sides of the hexagon are parallel, and opposite angles equal. Now

$$\angle I_2o_1I_3 = 2\pi - 2\angle I_2II_3,$$
$$= 2\pi - 2\left(A + \tfrac{1}{2}B + \tfrac{1}{2}C\right),$$
$$= \pi - A = B + C.$$

Similarly, $\angle I_1o_3I_2 = \pi - C = A + B$, and $\angle I_1o_2I_3 = \pi - B = A + C$.



6630. (Professor Nash, M.A.)—If three tangents OP, OQ, OR be drawn to a semi-cubical parabola from any point O, prove that (1) the circle through P, Q, R meets the curve in three other points P′, Q′, R′, the tangents at which will meet in another point O′; (2) the middle point of OO′ always lies on a fixed straight line; and (3) the lines joining O, O′ to the cusp make equal angles with the axis.

I. *Solution by* Rev. J. Cullen *and* G. D. Wilson, B.A.

(1) For the semi-cubical parabola we have $x = a\mu^3$, $y = a\mu^2$, whence the tangent is $a\mu^3 - 3\mu y + 2x = 0 = \phi(\mu)$, say; therefore $\mu_1 + \mu_2 + \mu_3 = 0$, $\Sigma\mu_1\mu_2 = -(3y_1/a)$, $\mu_1\mu_2\mu_3 = -(2x_1/a)$, where (x_1, y_1) is the point O.

Now the equation of the circle PQR is of the form

$$\mu^6 + \mu^4 + g\mu^3 + f\mu^2 + c = 0;$$

therefore $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 = 0$; therefore $\mu_4 + \mu_5 + \mu_6 = 0$.

Also the tangents at P′, Q′, R′ are $\phi(\mu_4)$, $\phi(\mu_5)$, $\phi(\mu_6)$, but

$$\Sigma(\mu_5 - \mu_6)\,\phi(\mu_4) = 0;$$

therefore these three tangents meet in a point $O'(x_2, y_2)$.

(2) Since $\Sigma\mu_4 = 0$, we have $\Sigma\mu_1\mu_2 + \Sigma\mu_4\mu_5 = 1$; therefore

$$3(y_1 + y_2) + a = 0.$$

So that the mid-point of OO′ lies on the line $6y + a = 0$.

(3) $\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 = c = \dfrac{a + 3y_1}{3y_1}(\mu_1\mu_2\mu_3)^2$; $\therefore\ x_2 = \dfrac{a + 3y_1}{3y_1}x_1 = -\dfrac{y_2}{y_1}x_1$;

hence the lines $yx_1 - xy_1 = 0$ and $yx_2 - xy_2 = 0$ are equally inclined to the axis.

II. *Solution by* F. E. Cave.

Let O be α, β, and curve $ay^2 = x^3$. P, Q, R lie on

$$2a\beta y = 3\alpha x^2 - x^3 \quad \text{.............................. (1),}$$

and therefore their abscissæ are the roots of

$$x(x-3\alpha)^2 = 4a\beta^2 \quad \text{.............................. (2).}$$

The sextic which gives the abscissæ of the intersections of the curve (1) and a circle is satisfied by all the roots of (2) if the equation of the circle

is $$x^2 + y^2 + 2y\beta\frac{6\alpha + a}{3a} - 3x\alpha\frac{3\alpha + a}{2a^2} + 4\beta^2\frac{3\alpha + a}{3a\alpha} = 0.$$

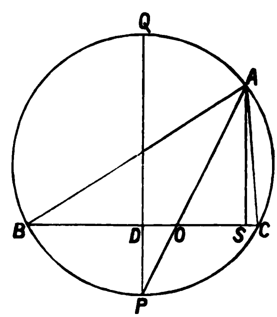
The same equation is obtained by starting with the point α', β', provided

$$3(\alpha + \alpha') + a = 0 \quad \text{and} \quad \beta'/\beta = -\alpha'/\alpha.$$

14460. (R. F. Davis, M.A.)—Given the base of a triangle in magnitude ($= 2a$) and position, and also the length ($= l$) of the line bisecting the vertical angle (vertex to base), prove that the locus of the vertex referred to the base as axis of x and a perpendicular to the base through its middle point as axis of y is

$$(x^2 + y^2 + a^2)^2 = 4x^2a^2 + l^4x^2/(l^2 - y^2).$$

Solution by F. H. Peachell, B.A.; R. Tucker, M.A.; *and others.*



Let BC be the given base, D the middle point; and let PDQ be perpendicular to BC, AS perpendicular to BC.

Then AO (the bisector) meets QD on the circle at P.

Now PO.OA = BO.OC,

and, from similar triangles,

PO : OA :: DO : OS.

Therefore

$$OA^2.DO/OS = BO.OC,$$

or $$l^2\frac{x - (l^2 - y^2)^{\frac{1}{2}}}{(l^2 - y^2)^{\frac{1}{2}}}$$

$$= \left\{a + x - (l^2 - y^2)^{\frac{1}{2}}\right\}\left\{a - x + (l^2 - y^2)^{\frac{1}{2}}\right\},$$

or $$l^2x/(l^2 - y^2)^{\frac{1}{2}} - l^2 = a^2 - x^2 + y^2 - l^2 + 2x(l^2 - y^2)^{\frac{1}{2}},$$

or $$\left\{l^2x/(l^2 - y^2)^{\frac{1}{2}} - 2x(l^2 - y^2)^{\frac{1}{2}}\right\}^2 = \left\{(x^2 + y^2 + a^2) - 2x^2\right\}^2,$$

or $l^4x^2/(l^2 - y^2) - 4l^2x^2 + 4x^2(l^2 - y^2) = (x^2 + y^2 + a^2)^2 + 4x^4 - 4x^4 - 4x^2y^2 - 4a^2x^2.$

So locus of vertex is $(x^2 + y^2 + a^2)^2 = 4a^2x^2 + l^4x^2/(l^2 - y^2).$

6587. (W. E. WRIGHT, B.A.)—From a point on a curve of the second degree tangents are drawn to another curve of the second degree. Find the envelope of their chord of contact.

Solution by A. HALL, A.R.C.S.

Let equation of curve on which the point lies be

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and of the other curve be $\quad Ax^2 + By^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$

Let coordinates of point be (x_1, y_1); therefore equation of chord of contact is $\quad Axx_1 + Byy_1 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$

Eliminate y_1 from (1) and $ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0$, and we get

$$x_1^2 (aB^2y^2 + bA^2x^2 - 2hAxBy) + 2x_1 (gB^2y^2 - fABxy - bAx + hBy)$$
$$+ (cB^2y^2 + 2fBy + b) = 0 \ldots (4).$$

Now the condition for envelope of (3) when x_1y_1 is on (1) is the same as condition for equal roots of (4) or

$$(aB^2y^2 + bA^2x^2 - 2hABxy)(cB^2y^2 + 2fBy + b) = (gB^2y^2 - fABxy - bAx)^2,$$

which, on simplification, reduces to

$$A^2x^2 (f^2 - bc) + B^2y^2 (g^2 - ac) + 2ABxy (ch - fg)$$
$$+ 2Ax (fh - bg) + 2By (gh - af) + h^2 - ab = 0,$$

which is the envelope required, and is a curve of second degree.

14402. (R. C. ARCHIBALD, M.A.)—Show that (1) the locus of the fourth harmonic point to P, S, P′, where PSP′ is any cuspidal chord of the cardioid $r = 2a(1 - \cos\theta)$, is the Cissoid of DIOCLES $r = 2a \sin\theta \tan\theta$; (2) if r and r' are the radii vectores respectively of the cardioid and cissoid for a given θ, $r : r' = \tan\frac{1}{2}\theta : \tan\theta$; (3) referred to $(-a, 0)$ as origin, the equation of the cissoid becomes $r/a = (1 + \tan^{\frac{2}{3}}\frac{1}{2}\theta)/(1 - \tan^{\frac{2}{3}}\frac{1}{2}\theta)$.

Solution by the PROPOSER.

(1) If (r', θ') be the coordinates of the fourth harmonic point, we have

$$2a(1 - \cos\theta') : 2a(1 + \cos\theta') = r' - 2a(1 - \cos\theta') : r' + 2a(1 + \cos\theta')$$

or $$r' = 2a \sin\theta' \tan\theta'.$$

(2) $r : r' = 2a(1 - \cos\theta) : 2a \sin\theta \tan\theta = \sin^2\frac{1}{2}\theta : \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta \tan\theta$;

therefore $$r : r' = \tan\tfrac{1}{2}\theta : \tan\theta.$$

(3) In Cartesian coordinates the polar equation $r = 2a \sin\theta \tan\theta$ becomes $y^2(2a - x) = x^3$; or, on changing the origin to $(-a, 0)$, $y^2(3a - x) = (x - a)^3$, which in polar coordinates may be written

$$r^3 \cos\theta - 3ar^2 + 3a^2 r \cos\theta - a^3 = 0$$

or $$(1 + \cos\theta)(r - a)^3 - (1 - \cos\theta)(r + a)^3 = 0,$$

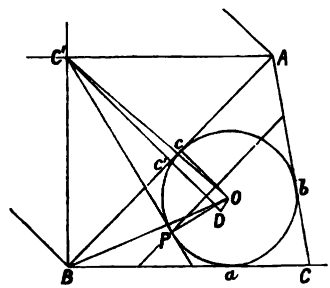
which becomes $$(r - a)^3 = \tan^2\tfrac{1}{2}\theta\,(r + a)^3$$

or $$r/a = (1 + \tan^{\frac{2}{3}}\tfrac{1}{2}\theta)/(1 - \tan^{\frac{2}{3}}\tfrac{1}{2}\theta).$$

14491. (R. Tucker, M.A.)—Squares are described externally on the sides of the triangle ABC, and tangents are drawn from their centres to the incircle of the triangle. Prove that

$$2\Sigma\,(\text{tangents})^2 = 2\Delta\,(2+3\cot\omega) - \Sigma\,(bc).$$

Solution by J. H. Taylor, M.A.; *the* Proposer; *and many others.*



A′, B′, C′ are the centres of squares described on the sides BC, CA, AB of the triangle ABC; a', b', c' the middle points; and a, b, c the points of contact of the incircle with those sides respectively. OD is parallel to AB. $Bc = s-b$, $Bc' = \frac{1}{2}c$; therefore $c'c = \frac{1}{2}(a-b)$.

Denote by C′P a tangent to the incircle t_c.

$$t_c^2 = \text{C}'\text{O}^2 - \text{OP}^2 = \text{C}'\text{D}^2 + c'c^2 - \text{OP}^2$$
$$= (\tfrac{1}{2}c + r)^2 + \left\{\tfrac{1}{2}(a-b)\right\}^2 - r^2$$
$$= \tfrac{1}{4}(a^2+b^2+c^2-2ab) + cr.$$

Therefore $2\,(t_a^2+t_b^2+t_c^2) = \frac{3}{2}(a^2+b^2+c^2) + 2\,(a+b+c)\,r - (ab+bc+ca)$
$$= 6\Delta\cot\omega + 4\Delta - (ab+bc+ca),$$

which is the result required, since

$$\cot\omega = \left\{\text{R}\,(a^2+b^2+c^2)\right\}/abc \quad\text{and}\quad \Delta = abc/4\text{R}.$$

14437. (R. P. Paranjpye, B.A.)—Show that there are six conics passing through three given points and having contact of the second order with a given conic; and, further, that these six conics all touch a quartic having the three points as nodes.

I. *Solution by* Professor K. J. Sanjána, M.A.

Taking the triangle formed by the three given points to be triangle of reference, and the given conic to be

$$\text{S} \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

we have one of the required conics of the form

$$\text{S}' \equiv 2f'\beta\gamma + 2g'\gamma\alpha + 2h'\alpha\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The invariants of (1) and the given conic are $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$,

$$\Delta' = 2f'g'h', \quad \Theta \equiv 2f'(gh-af) + 2g'(fh-bg) + 2h'(fg-ch),$$

and $\quad \Theta' \equiv -af'^2 - bg'^2 - ch'^2 + 2fg'h' + 2gh'f' + 2hf'g'.$

For contact of the second order we must have $\Theta+\Theta' = 3\Delta+\Theta = \Theta'+3\Delta'$; these give

$$4(Ff'+Gg'+Hh')^2 = 3\Delta(-af'^2-bg'^2-ch'^2+2fg'h'+2gh'f'+2hf'g')\ldots(2)$$

and $$2(Ff'+Gg'+Hh')(-af'^2-bg'^2-ch'^2+2fg'h'+2gh'f'+2hf'g') = 18\Delta f'g'h' \ldots (3).$$

From (2) and (3) the ratios $f' : g' : h'$ are found by a sextic; hence, in general there are six conics like (1). The quartic which has the triangle of reference as nodal triangle must be of the form

$$l\beta^2\gamma^2+m\gamma^2\alpha^2+n\alpha^2\beta^2+2p\alpha^2\beta\gamma+2q\beta^2\gamma\alpha+2r\gamma^2\alpha\beta = 0 \ldots\ldots\ldots (4).$$

The condition of tangency of (1) and (4) is found by the usual method to be $$f'^2L+g'^2M+h'^2N+2g'h'P+2h'f'Q+2f'g'R = 0 \ldots\ldots\ldots (5);$$

but (2) may be written in the form

$$f'^2(F^2+3BC)+g'^2(G^2+3CA)+h'^2(H^2+3AB)$$
$$+2g'h'(GH+3AF)+2h'f'(HF+3BG)+2f'g'(FG+3CH) = 0.$$

Hence, if we find l, m, n, p, q, r from the following six equations,

$$mn-p^2 = F^2+3BC,\quad ln-q^2 = G^2+3CA,\quad lm-r^2 = H^2+3AB,$$
$$qr-lp = GH+3AF,\quad rp-mq = HF+3BG,\quad pq-nr = FG+3CH,$$

we determine the quartic (4) uniquely, and the proposition in the second part follows from this.

II. *Solution by* Professor A. Droz-Farny.

Faisons une transformation homographique de manière à ce que deux des points donnés deviennent les ombilics du plan, il s'agira de prouver que par un point donné on peut mener six cercles osculateurs à une conique et que ces six cercles sont tangents à une quartique bicirculaire ayant un nœud au point donné.

Or dans son *Treatise on the Analytical Geometry*, second edition, p. 316, Casey a démontré que par un point quelconque du plan d'une conique on peut mener six cercles qui osculent cette conique et que leurs centres sont sur une autre conique.

Mais on sait que tous les cercles qui passent par un point fixe et ont leurs centres sur une conique enveloppent une anallagmatique du quatrième ordre, podaire de conique par rapport au point donné; d'où le théorème.

[The Proposer sends the following method of solution, which is interesting:—If we transform the general conic in homogeneous coordinates x, y, z by the substitution $\xi = 1/x$, $\eta = 1/y$, $\zeta = 1/z$, the transformed equation is that of a quartic with nodes at the angular points of the fundamental triangle. By the same substitution a line is transformed into a conic about the triangle of reference, and *vice versa*. Hence the given proposition can be reduced to the following: Show that a trinodal quartic has six points of inflexion and that the tangents at these points touch a conic. Now this is a well known property of these quartics. See Salmon (*Higher Plane Curves*). Hence the given proposition follows.]

14455. (Professor COCHEZ.)—Courbe $\rho^3 - 3\rho \tan\omega + 2 = 0$.

Solution by Rev. T. ROACH, M.A.

By CARDAN's solution,

$$\rho = \{-1 + \surd(1 - \tan^3\omega)\}^{\frac{1}{3}} + \{-1 - \surd(1 - \tan^3\omega)\}^{\frac{1}{3}}.$$

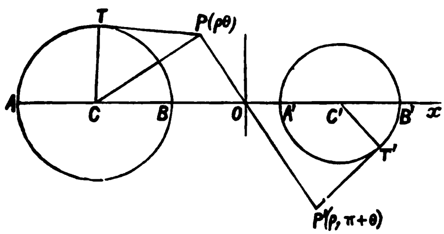
14265. (R. F. DAVIS, M.A.)—If O be the centre of inversion (constant $= \kappa^2$), investigate the formula of transformation

tangent from point P to the circle C
$= \lambda$ (tangent from inverse point P′ to inverse circle C′),

and show that $\lambda = \text{OP (or } \kappa^2/\text{OP}')/\text{tangent from O to C}'$.

Apply this to Quest. 13801. (See Vol. LXX., p. 73.)

Solution by E. W. REES, B.A.



$\text{OP} = \rho$, $\text{OP}' = \rho'$,
$\text{OC} = a$, $\text{OC}' = a'$,
$\text{CA} = r$, $\text{C}'\text{A}' = r'$.

Take O for origin and line CC′ for axis of x.

$$\text{PT}^2 = \rho^2 - 2a\rho\cos\theta + a^2 - r^2 = (\kappa^4/\rho'^2) - 2a^2(\kappa^2/\rho')\cos\theta + a^2 - r^2;$$

$$\text{P}'\text{T}'^2 = \rho'^2 - 2a'\rho'\cos\theta + a'^2 - r'^2.$$

But $\text{OA}.\text{OA}' = -(a+r)(a'-r') = \text{OB}.\text{OB}' = -(a-r)(a'+r') = -\kappa^2$;

therefore $\quad a + r = \kappa^2/(a'-r'), \quad a - r = \kappa^2/(a'+r')$;

therefore, by addition, $\quad a = \kappa^2 a'/(a'^2 - r'^2)$;

therefore
$$\begin{aligned}\text{PT}^2 &= \kappa^4/\rho'^2\,\{1 + (2a'\rho'\cos\theta)/(a'^2 - r'^2) + \rho'^2/(a'^2 - r'^2)\}\\ &= [\kappa^4/\{\rho'^2(a'^2 - r'^2)\}](\rho'^2 + 2a'\rho'\cos\theta + a'^2 - r'^2)\\ &= [\kappa^4/\{\rho'^2(a'^2 - r'^2)\}]\,\text{P}'\text{T}'^2\\ &= \{\kappa^4/(\rho'^2 t'^2)\}\,\text{P}'\text{T}'^2, \quad \text{where } t' = \text{tangent from O to C}',\end{aligned}$$

i.e., $\quad \text{PT}/\text{P}'\text{T}' = \kappa^2/\rho' t' = \lambda$,

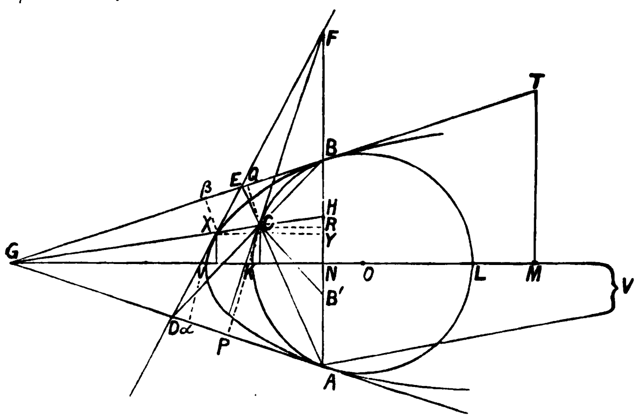
where $\quad \lambda = \text{OP (or } \kappa^2/\text{OP}')/\text{tangent from O to C}'$.

[The PROPOSER remarks as follows:—If U′ be the inverse of T (lying on C′), then the line PT inverts into a circle OU′P′ touching C′ at U′. Taking a point-circle at U′, the tangents from P′ to the circles C′, U′ are in the same ratio as the tangents from O. Hence $\text{P}'\text{T}' : \text{P}'\text{U}' = \lambda' : \text{OU}'$. But $\text{PT} : \text{P}'\text{U}' = \text{OP} : \text{OU}'$; therefore $\text{PT}/\text{P}'\text{T}' = \text{OP}/\lambda'$.]

12561. (C. E. Hillyer, M.A.)—(1) A, B are two fixed points on a circle whose centre is O, and C is any third point on the circumference; BC meets the tangent at A in D, CA the tangent at B in E, and AB the tangent at C in F. Prove that DEF is a straight line which envelops a conic of eccentricity ϵ, where $\epsilon^2 = 3OA^2/AB^2$. (2) Generalize the above and Quest. 12462 (solved in Vol. lxii., p. 89) by reciprocation.

Solution by the Proposer.

Draw CB′ equally inclined with CB to AB; then $\angle$FBC = CB′A, $\angle$FCB = CAB′; therefore the triangles FBC, CB′A are similar; therefore FC/FB = CA/B′C = CA/BC, and similarly DA/DC = AB/CA, EB/EA = BC/AB; therefore DA.EB.FC = DC.EA.FB.



But, since DB.DC = DA², we have DB/DC = DA²/DC², and similarly EC/EA = EB²/EA² and FA/FB = FC²/FB²; therefore

$$DB.EC.FA = DC.EA.FB.$$

Therefore DEF is a straight line.

Let the tangents at A and B meet in G, and draw GC meeting DE in X and AB in H. Draw from X $X\alpha$, $X\beta$, $X\gamma$, and from C CP, CQ, CR perpendicular respectively to GA, GB, BA; then G, X, C, H form a harmonic range; therefore $XG/CG = \frac{1}{2}(XH/CH)$; therefore $X\alpha/CP$ $X\beta/CQ = \frac{1}{2}(X\gamma/CR)$, and therefore $X\alpha . X\beta/CP.CQ = \frac{1}{4}(X\gamma^2/CR^2)$; but, since C is on the circumference of the circle, $CP.CQ = CR^2$; therefore $X\alpha . X\beta = \frac{1}{4}X\gamma^2$; therefore the locus of X is a conic touching GA, GB at A and B; also, since F, B, H, A form a harmonic range, GH is the polar of F with respect to the conic; therefore DEF touches the conic at X.

To find the eccentricity, let GO meet the circle in K, L, and AB in N; take V the harmonic conjugate of N with respect to G, K, and V′ with respect to G, L, M the mid-point of VV′, and draw MT perpendicular to

VV′ meeting GB in T; then V, V′ are the positions of X when C coincides with K, L respectively, and therefore VV′ is the major axis of the conic.

Thus, if a, b be the semi-axes of the conic, $a^2 = \text{MN}.\text{MG}$ and $b^2 = \text{NB}.\text{MT}$; therefore

$$\epsilon^2 = 1-(b^2/a^2) = 1-(\text{NB}.\text{MT}/\text{MN}.\text{MG})$$
$$= 1-(\text{NB}^2/\text{MN}.\text{NG}) = 1-(\text{NG}.\text{NO}/\text{MN}.\text{NG}) = \text{MO}.\text{MN}.$$

Now it can be shown* from the properties of a harmonic range that $\text{MO}/\text{MN} = \frac{3}{4}(\text{OG}/\text{GN})$; therefore

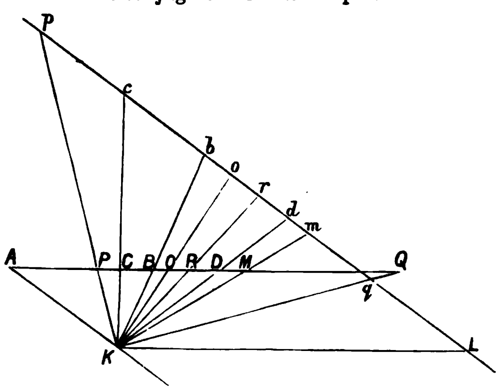
$$\epsilon^2 = \tfrac{3}{4}(\text{OG}/\text{GN}) = \tfrac{3}{4}(\text{OG}.\text{ON}/\text{GN}.\text{ON}) = \tfrac{3}{4}(\text{OA}^2/\text{AN}^2) = 3\text{OA}^2/\text{AB}^2.$$

If GN = NL, the point V′ is at an infinite distance, and the conic is a parabola; if GN < NL, V′ and V are on opposite sides of G, and the conic is a hyperbola; the least possible value of ϵ is $\frac{1}{2}\sqrt{3}$, and this will occur when AB is a diameter of the circle.

This proposition is the reciprocal of Quest. 12462 with respect to the given circle. By reciprocating either with respect to any other circle we can extend the other to the case of any conic. (See Vol. LXII., p. 89.)

**Lemma.*—If C, D be harmonic conjugates with respect to A, B, P the harmonic conjugate of B with respect to A, C, and Q the harmonic conjugate of B with respect to A, D, and O the mid-point of C, D, and M the mid-point of PQ, then $\text{OM}/\text{MB} = \frac{3}{4}(\text{OA}/\text{AB})$.

Take R the harmonic conjugate of B with respect to OA.



Take any point K in the plane, and let KP, KC, KB, KO, KR, KD, KM, KQ meet a parallel to KA in p, c, b, o, r, d, m, q. Draw KL parallel to AQ, meeting pq in L. Then $pc = cb = bd = dq$. Also $bo = or$. Now, since M is the mid-point of PQ, and KL parallel to PQ, p, m, q, L form a harmonic range; therefore $bm.b\text{L} = bq^2$, similarly $bo.b\text{L} = bd^2$; therefore $bm/bo = bq^2/bd^2 = \frac{4}{1}$, therefore $bm = 2br$, and therefore A, B, R, M form a harmonic range.

Therefore $\text{OM}/\text{MB} = \frac{1}{2}(\text{OR}/\text{RB}+\text{OA}/\text{AB})$, but, since A, B, O, R form a harmonic range, we have $\text{OR}/\text{RB} = \frac{1}{2}(\text{OA}/\text{AB})$; therefore $\text{OM}/\text{MB} = \frac{3}{4}(\text{OA}/\text{AB})$.

14394. (Professor THOMAS SAVAGE.)—Discuss, n being integral and positive, $(1+1/x)^n < 2$, but $(1+1/x)^{n+1} > 2$.

Solution by H. MACCOLL, B.A.

This question, as I understand it, will afford an instructive example of the application of symbolic logic to ordinary algebra. We are required to find the real limits of x. Let A denote the statement $(1+1/x)^n < 2$, and B the statement $(1+1/x)^{n+1} > 2$. My result is

$$n^e(\mathrm{AB} = x_{3'.1}) + n^o(\mathrm{AB} = x_{3'.1} + x_{0'.4}),$$

which asserts (see *Definitions* and *Table of Limits*) that either n is *even* and AB equivalent to the statement that x lies between the *superior* limit x_3 and the *inferior* limit x_1, or else n is *odd* and AB equivalent to the statement that x lies either between x_3 and x_1 or between *zero* and the *negative inferior* limit x_4.

Definitions. —The symbol n^e asserts that n is *even;* n^o that n is *odd;* a^u that a is *positive;* a^v that a is *negative;* $x_{m'}$ that x_m is a *superior limit* to x; x_m that x_m is an *inferior limit* to x; $x_{m'.n'.r.s}$ that x_m and x_n are *superior*, and x_r and x_s *inferior*, limits to x.

Table of Limits.

$x_0 = 0$	$y_0 = 0$
$x_1 = 1/y_1$	$y_1 = 2^{1/n} - 1$
$x_2 = 1/y_2$	$y_2 = -(2^{1/n} + 1)$
$x_3 = 1/y_3$	$y_3 = 2^{1/(n+1)} - 1$
$x_4 = 1/y_4$	$y_4 = -(2^{1/(n+1)} + 1)$

It will be convenient to put y for $1/x$, and first find the limits of y as follows:—

$$n^e\mathrm{A} = n^e\{(1+y)^n - 2\}^v = n^e\{(1+y) - 2^{1/n}\}^v\{(1+y) + 2^{1/n}\}^u$$
$$= n^e\{y - (2^{1/n} - 1)\}^v\{y + (2^{1/n} + 1)\}^u = n^e y_{1'.2};$$
$$n^e\mathrm{B} = n^e\{(1+y)^{n+1} - 2\}^u = n^e\{(1+y) - 2^{1/(n+1)}\}^u$$
$$= n^e\{y - (2^{1/(n+1)} - 1)\}^u = n^e y_3;$$

$$\therefore\quad n^e\mathrm{AB} = n^e y_{1'.2.3} = n^e y_{1'.3}, \text{ for } y_3 \text{ implies } y_2.$$

$$n^o\mathrm{A} = n^o\{(1+y)^n - 2\}^v = n^o\{(1+y) - 2^{1/n}\}^v$$
$$= n^o\{y - (2^{1/n} - 1)\}^v = n^o y_{1'};$$
$$n^o\mathrm{B} = n^o\{(1+y)^{n+1} - 2\}^u = n^o\{(1+y) - 2^{1/(n+1)}\}^u + n^o\{(1+y) + 2^{1/(n+1)}\}^v$$
$$= n^o\{y - (2^{1/(n+1)} - 1)\}^u + n^o\{y + (2^{1/(n+1)} + 1)\}^v = n^o(y_3 + y_{4'});$$

$$\therefore\quad n^o\mathrm{AB} = n^o y_{1'}(y_3 + y_{4'}) = n^o(y_{1'.3} + y_{1'.4'}) = n^o(y_{1'.3} + y_{4'}),$$

for $y_{4'}$ implies $y_{1'}$. Thus the statement for the limits of y is

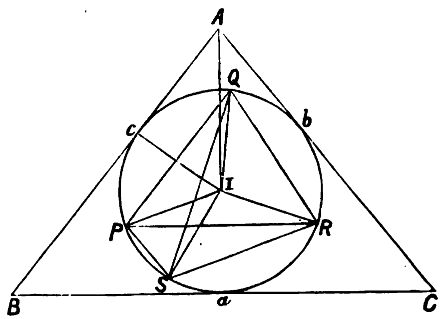
$$n^e(\mathrm{AB} = y_{1'.3}) + n^o(\mathrm{AB} = y_{1'.3} + y_{4'}).$$

From this statement, the limits of x are readily found; for, since $y = 1/x$, and y_1 and y_3 are positive and y_4 negative, it is clear that $y_{1'.3} = x_{3'.1}$, and that $y_{4'} = x_{0'.4}$.

The above contains every step—more than would be needed in actual practice—of the symbolic process for finding the limits. The whole reasoning presupposes but an elementary knowledge of common algebra.

14479. (Salutation.) — I is the incentre of the triangle ABC, of which A is the greatest angle. P is a point on the incircle, and through P lines are drawn parallel to the three sides of the triangle, and meeting the incircle again in Q, R, S, respectively. QR, RS being joined, prove that the quadrilateral PQRS is a maximum when AIP is a right angle, and find its mean area.

Solution by J. H. Taylor, M.A.; Rev. T. Roach, M.A.; *and* G. W. Preston, B.A.



The greatest quadrilateral must have the centre I within it. a, b, c are points of contact. Take a point P between cI and BI, and draw PQ, PR, PS parallel to AB, BC, CA, respectively. Let

$$c\text{IP} = \theta = c\text{IQ};$$

therefore

$$\text{QIP} = 2\theta;$$

$$\text{SIP} = 2(\text{A}-\theta).$$

$$\triangle\text{QIP} = \tfrac{1}{2}r^2 \sin 2\theta;$$

$$\triangle\text{SIP} = \tfrac{1}{2}r^2 \sin 2(\text{A}-\theta); \quad \triangle\text{SIR} = \tfrac{1}{2}r^2 \sin 2\text{C}; \quad \triangle\text{QIR} = \tfrac{1}{2}r^2 \sin 2\text{B}.$$

The convex quadrilateral PQRS is the sum of these four triangles and, since θ is the only variable, is a maximum when $\sin 2\theta + \sin 2(\text{A}-\theta)$ is a maximum, *i.e.*, when $\sin \text{A} \cos (2\theta - \text{A})$ is a maximum. These factors increase together until $2\theta - \text{A} = 0$, *i.e.*, till $\theta = \tfrac{1}{2}\text{A}$. But

$$\angle c\text{IA} = 90° - \tfrac{1}{2}\text{A}.$$

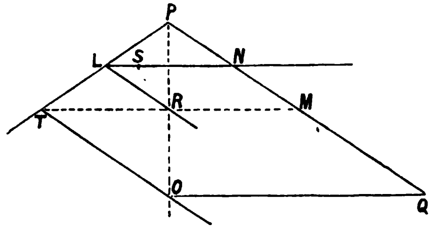
Therefore, when the quadrilateral PQRS is a maximum,

$$\angle\text{PIA} = 90°.$$

[The second part of the Question remains still unsolved.]

14504. (R. Knowles.)—The circle of curvature is drawn at a point P of a parabola; PQ is the common chord; an ordinate from P to the diameter through the focus meets the parabola in R, and a diameter through Q in O. If T be the pole of PQ with respect to the parabola, prove that TO, PQ, and the tangent at R are parallel.

Solution by Professor A. Droz-Farny; *and* J. H. Taylor, M.A.



Soit PL la tangente en P à la parabole. S, le sommet de cette dernière, est le point milieu de la soustangente. D'après le théorème suivant lequel les couples de côtés opposés d'un quadrilatère ayant pour sommet les points d'intersection d'un cercle avec une conique sont également inclinés sur les axes de la conique, on obtiendra la direction de la corde PQ en construisant la symétrique de PL par rapport à l'ordonnée. Le centre de gravité des quatre points d'intersection d'un cercle avec une parabole coincidant avec l'axe de cette dernière, comme trois des points d'intersection du cercle de courbure coincidant avec P et un avec Q, on obtiendra Q en prolongeant PN d'une longueur NQ = 3PN. Soit M le point milieu de PQ; comme PN = NM, le diamètre passant par M contiendra R et coupera PL au pôle T de PQ. Comme TR = RM, TL = LP et PR = RO, il est évident que les droites PQ, LR et TO sont parallèles.

Remarques.—LR est la tangente en R car elle est symétrique de PL par rapport à l'axe; TQ est la tangente en Q; on verrait que TQ divise RO dans le rapport de 1 : 2.

10358. (R. W. D. Christie.)—If ω_2 and ω_3 are irrational cube roots of unity, prove that, if $n+2$ is a prime number,

$$\omega_2^n + \omega^n + \frac{n+1}{2!}\left(\omega_2^{n-2} + \omega_3^{n-2}\right) + \frac{n\,.\,n+1}{3!}\left(\omega_2^{n-4} + \omega_3^{n-4}\right)$$

$$+\frac{n-1\,.\,n\,.\,n+1}{4!}\left(\omega_2^{n-6} + \omega_3^{n-6}\right) + \&c. = 0.$$

Solution by H. J. Woodall, A.R.C.S.

The series

$$= \left\{(\omega_2 + 1/\omega_2)^{n+2} + (\omega_3 + 1/\omega_3)^{n+2} - \omega_2^{n+2} - \omega_2^{-(n+2)} - \omega_3^{n+2} - \omega_3^{-(n+2)}\right\}/(n+2)$$

$$= \left[\left\{(\omega_2^2 + 1)^{n+2} - \omega_2^{2(n+2)} - 1\right\}/\omega_2^{n+2} + \left\{(\omega_3^2+1)^{n+2} - \omega_3^{2(n+2)} - 1\right\}/\omega_3^{n+2}\right]\Big/(n+2).$$

But, ω_2, ω_3 being irrational cube roots of unity, we have

$$(\omega_2^2+1)^{n+2} = (-\omega_2)^{n+2} = -\omega_2^{n+2} \quad\text{and}\quad (\omega_3^2+1)^{n+2} = -\omega_3^{n+2}$$

(n being odd). The proposed series becomes

$$= -\left[\left\{\omega_2^{2(n+2)} + \omega_2^{n+2} + 1\right\}/\omega^{n+2} + \left\{\omega_3^{2(n+2)} + \omega_3^{n+2} + 1\right\}/\omega_3^{n+2}\right]/(n+2)$$

$$= -\left\{\frac{\omega_2^{3(n+2)} - 1}{\omega_2^{n+2}(\omega_2^{n+2} - 1)} + \frac{\omega_3^{3(n+2)} - 1}{\omega_3^{n+2}(\omega_3^{n+2} - 1)}\right\}\Big/(n+2) = 0,$$

because the numerators are each zero, the denominators being always $\neq 0$.

[The PROPOSER observes :—The general term is

$$\frac{n+1\,!}{n-m+2\,!\;m\,!}\left(\omega_2^{n-2m+2} + \omega_3^{n-2m+2}\right).$$

Now $(\omega_2^n + \omega_3^n) = 2$ if $n = 3k$, and $= -1$ if $n = 3k \pm 1$ (*v.* TODHUNTER'S *Algebra*, p. 213).

And by a well known theorem (*v.* CARR'S *Synopsis*, 284, p. 94 Algebra), $S = 0$ if n be of the form $6m \pm 1$. The result necessarily follows.]

14425. (Professor U. C. GHOSH.)—Prove that

$$\int_0^\pi x\phi(\sin x)\,dx = \tfrac{1}{2}\pi\int_0^\pi \phi(\sin x)\,dx,$$

and hence evaluate $\displaystyle\int_0^\pi \frac{x\sin x\,(1-\sin^n x)}{1-\sin x}\,dx.$

Solution by H. W. CURJEL, M.A.; *and* CONSTANCE I. MARKS, B.A.

$$\int_0^\pi x\,\phi(\sin x) = \int_0^{\frac{1}{2}\pi}\left[x\,\phi(\sin x) + (\pi - x)\,\phi\left\{\sin(\pi - x)\right\}\right]dx$$

$$= \pi\int_0^{\frac{1}{2}\pi}\phi(\sin x) = \tfrac{1}{2}\pi\int_0^\pi \phi(\sin x)\,dx.$$

Hence $\displaystyle\int_0^\pi \frac{x\sin x\,(1-\sin^n x)}{1-\sin x}\,dx$

$$= \pi\int_0^{\frac{1}{2}\pi}\frac{\sin x\,(1-\sin^n x)}{1-\sin x}\,dx = \pi\int_0^{\frac{1}{2}\pi}(\sin x + \sin^2 x + \dots + \sin^n x)\,dx$$

$$= \pi\left\{1 + \tfrac{2}{3} + \tfrac{2}{3}\cdot\tfrac{4}{5} + \dots + \tfrac{1}{2}\pi\left(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{3}{4} + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6} + \dots\right)\right\}$$

$$= \pi\left(-1 + \tfrac{2}{1}\cdot\tfrac{4}{3}\cdot\tfrac{6}{5}\dots\frac{2m+1\pm1}{2m\pm1}\right) + \frac{\pi^2}{2}\left(-1 + \tfrac{3}{2}\cdot\tfrac{5}{4}\dots\frac{2m+1}{2m}\right),$$

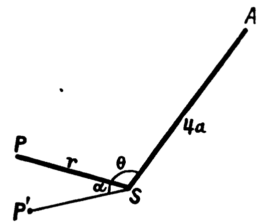
where $n = 2m + \tfrac{1}{2} \pm \tfrac{1}{2}.$

14372. (R. C. ARCHIBALD, M.A.)—Parabolas with a common focus pass through a fixed point. Show (1) that the locus of their vertices is a cardioid whose cusp is at the common focus and whose vertex is the fixed point; (2) that the locus of the points of intersection with the parabolas of the lines through the focus making a constant angle with their axes is a cardioid.

Solution by the Proposer.

(1) S is the common focus, A the fixed point, P the vertex of any parabola passing through A and with focus at S.

If $SA = 4a$, $\angle PSA = \theta$, $SP = r$, we have at once from the equation of the parabola, as the locus of P, $r = 2a(1+\cos\theta)$, which defines a cardioid. Geometrically, it is well known that the circle on the radius vector of a parabola as diameter is always tangent to the tangent at the vertex of the parabola. Hence, the locus of P is the pedal of the circle on SA as diameter with respect to the point S of its circumference: a well known cardioid definition.



(2) If P′ be a point of intersection and α the constant angle made by SP′ with SP, we easily find the locus of P′ to be the cardioid defined by the equation $r = \dfrac{4a\left\{1+\cos(\theta+\alpha)\right\}}{1+\cos\alpha}$.

14315. (B. N. Cama, M.A.)—If parabolas be described cutting an equiangular spiral orthogonally, and having their axes in the direction of the polar subtangent, the loci of the focus and the vertex are copolar spirals whose linear dimensions bear a constant ratio.

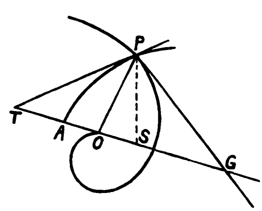
Solution by the Proposer.

Let S be the focus and vertex of one such parabola. Then, clearly,

$$OS = \tfrac{1}{2}(OG - OT)$$
$$= \tfrac{1}{2}(r\tan\alpha - r\cot\alpha) \propto r.$$

Also the vectorial angles of OS, OP differ by a right angle. Therefore S describes a spiral copolar with the given one.

Also $OA = \tfrac{1}{2}OT = \tfrac{1}{2}r\cot\alpha \propto r$.



Therefore locus of A is also a copolar spiral, the ratio of the linear dimensions of the two loci depending upon the angle of the original spiral, and therefore constant.

14461. (Rev. W. Allen Whitworth, M.A.)—If a straight line be divided at random into three parts x, y, z, show that the expectation of the volume $(y+z)(z+x)(x+y)$ is 14 times the expectation of the volume xyz.

I. *Solution by* H. W. Curjel, M.A.

If the straight line is taken of unit length $z = 1-x-y$, and if the integrals are taken over all positive values for which $x+y \not> 1$,

$$\frac{\text{expectation of } \pi(y+z)}{\text{expectation of } xyz}$$

$$= \frac{\iint (y+z)(z+x)(x+y)\,dy\,dx}{\iint xyz\,dy\,dx} = \frac{6\iint x^2z\,dy\,dx}{\iint xyz\,dy\,dx} + 2$$

$$= \frac{6\iint \{x^2(1-x)-yx^2\}\,dy\,dx}{\iint \{yx(1-x)-xy^2\}\,dy\,dx} + 2 = \frac{6\int_0^1 \frac{x^2-2x^3+x^4}{2}\,dx}{\int_0^1 \frac{x-3x^2+3x^3-x^4}{6}\,dx} + 2 = 14.$$

II. *Solution by* R. Chartres.

Let the length be one unit, and the parts x, y, z; then $1^3 = (x+y+z)^3$, which has ten terms each being of the same mean value, $\frac{1}{10}$; therefore mean value of $xyz = \frac{1}{60}$, and mean value of $(x+y)(y+z)(z+x)$, or

$$\Sigma(x^2y) + 2xyz = \frac{1}{5} + \frac{1}{30} = \frac{7}{30};$$

therefore mean value of the latter = 14 times that of the other.

[See the Proposer's *Expectation of Parts*.]

14464. (Edward V. Huntington, A.M.)—The angle between the principal axes of two given concentric ellipses is 90°, and $a+b = a'+b'$. Show that a line of length $a-b'$ (or $a'-b$) sliding between these curves envelops an astroid; and that any line rigidly connected with this sliding line envelops an involute of an astroid. (Astroid = hypocycloid of four cusps.)

Solution by R. C. Archibald, M.A.

Inside a circle of radius $a+b$ rolls a circle of half this radius. The ends of any given diameter of the rolling circle trace out two perpendicular diameters of the fixed circle, and in these diameters lie the axes of the ellipses traced by the points in the given diameter, distant from the circumference b and a', or b' and a. The diameter of the rolling circle envelops an astroid with the perpendicular diameters as axes of symmetry. Whence the theorem.

14432. (R. Tucker, M.A.)—PSQ is a focal chord of a parabola, and PQR is the maximum triangle in the segment cut off by PQ. Prove that the equation to the circle PQR is

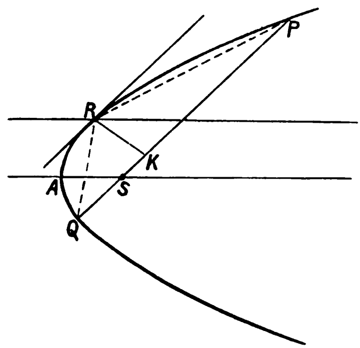
$$8(x^2+y^2) - 2(7p^2+20)\,ax + p(3p^2-4)\,ay + 6p^2a^2 = 0,$$

where $\quad p = m - 1/m \qquad$ (P is am^2, $2am$).

The locus of the centre is a cubic, and, if O is the fourth point of section, the locus of the mid-point of OR is a parabola, and the envelope of the chord OR is another parabola.

Solution by the PROPOSER; CONSTANCE I. MARKS, B.A.; *and* F. H. PEACHELL, B.A.

The triangle is a maximum when R is the vertex of the diameter corresponding to the chord PQ. The coordinates of P, Q, R are $(am^2, 2am)$, $(a/m^2, -2a/m)$, $(\frac{1}{4}ap^2, ap)$, where $p \equiv m - 1/m$.



By substitution it is readily verified that the circle through PQR is

$$8(x^2+y^2) - 2(7p^2+20)ax + p(3p^2-4)ay + 6p^2a^2 = 0.$$

The coordinates of the centre are h, k where $8h = (7p^2+20)a$,

$$16k = -p(3p^2-4)a;$$

therefore the locus is

$$343ak^2 = (2h-5a)(11a-3h)^2.$$

The fourth point (O) of section is found from $0 = 2am - 2a/m + ap + y$; therefore $y = -3ap$, and hence x.

The midpoint of OR is given by $y = -ap$, $x = \frac{5}{4}ap^2$; therefore its locus is the parabola $y^2 = \frac{4}{5}ax$.

The equation to OR is $2py + 4x = 3ap^2$; hence it envelops the parabola $y^2 + 12ax = 0$.

14419. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Find three sums of successive cubes which shall be in arithmetical progression.

Solution by the PROPOSER.

Let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$, $T_n = \frac{1}{2}n(n+1)$; then $S_r = T_r^2$ ... (1).

Then $S_x + S_z = 2S_y$, if S_x, S_y, S_z be in arithmetical progression. Hence

$$T_x^2 + T_z^2 = 2T_y^2, \quad 2T_y^2 - T_x^2 = T_z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

Assume $T_x = \xi . T_z$, $T_y = \eta . T_z$, where ξ, η are *both integers*.... .... (3)

(the possibility of this latter assumption is to be justified by the result).

Any solution of the Diophantine (2), which *also satisfies* both of (3)—for the same value of z—will be a solution. Here

$$(\xi, \eta) = (1, 1), (7, 5), (41, 29), (239, 169), \&c.,$$

are solutions of (2). The first ($\xi = 1$, $\eta = 1$) gives

$$T_x = T_y = T_z.$$

The next case ($\xi = 7$, $\eta = 5$) gives

$$T_x = 7 . T_z, \quad T_y = 5 . T_z.$$

The *lowest* solution is now easily found *by trial*. For, taking $z = 2$ gives $T_z = 3$, $T_x = 21$, $T_y = 15$, whence $x = 6$, $y = 5$, whence

$$(1^3+2^3+3^3+4^3+5^3+6^3)+(1^3+2^3) = 21^2+3^2 = 450 = 2 . 15^2$$
$$= 2 \times (1^3+2^3+3^3+4^3+5^3).$$

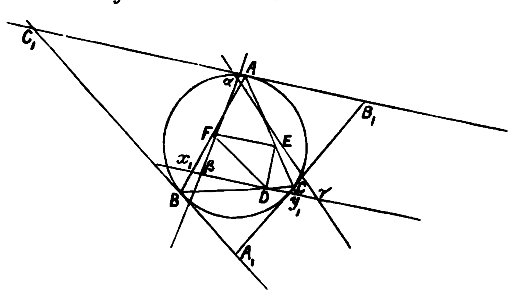
Other solutions probably exist, but seem difficult to find; thus it is easy to solve the equations $T_x = \xi . T_z$, $T_y = \eta . T_z$ *separately* (a general mode of doing this will be shown in the present writer's solution of Quest. 14413), but it is difficult to solve them for the same value of T_z.

It is worth noting that, if any second solution of (2) be found, say $T_x'^2 + T_z'^2 = 2T_y'^2$, the two solutions together give a solution of the more difficult problem where the sums of cubes do not start from 1^3, for they give

$$(T_x^2 - T_{x}'^2) + (T_z^2 - T_z'^2) = 2(T_y^2 - T_y'^2) \quad \text{or} \quad (S_x - S_{x'}) + (S_z - S_{z'}) = 2(S_y - S_{y'}).$$

14284. (Professor Neuberg.)—Soient O, I, I_a, I_b, I_c les centres des cercles circonscrit, inscrit et exinscrits au triangle ABC; soient D, E, F les pieds des hauteurs et A_1, B_1, C_1 les pôles de BC, CA, AB par rapport au cercle O. Les quatrièmes tangentes communes aux cercles (I, I_a), (I, I_b), (I, I_c) forment un triangle $\alpha\beta\gamma$ homothétique aux triangles $A_1B_1C_1$, DEF. Le centre d'homothétie des triangles $\alpha\beta\gamma$, $A_1B_1C_1$ partage la droite OI dans le rapport $R : r$, et est le conjugué isogonal du point de Gergonne de ABC; ses coordonnées normales par rapport au triangle $\alpha\beta\gamma$ sont $1/a$, $1/b$, $1/c$. Le centre d'homothétie des triangles $\alpha\beta\gamma$, DEF a pour coordonnées normales, dans ces triangles, $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$.

Solution by Professor Sanjána.



On AB take $Ax_1 = AC$, on AC take $Ay_1 = AB$; then x_1y_1 is the fourth tangent common to (I) and (I_a). Draw the two lines similar to x_1y_1, and let the three form the triangle $\alpha\beta\gamma$. As $\angle Ax_1y_1 = C$, $\angle Ay_1x_1 = B$, x_1y_1 is anti-parallel to BC, and therefore parallel to B_1C_1. Thus the triangles $A_1B_1C_1$, $\alpha\beta\gamma$ have corresponding sides parallel, and therefore are homothetic; so also are DEF, $\alpha\beta\gamma$. As x_1y_1, ..., are tangents to the incircle, the triangle $\alpha\beta\gamma$ has I for incentre;

and the triangle $A_1B_1C_1$ has O for incentre. Hence OI is the axis of perspective for these, and the centre of perspective divides OI in the ratio of the inradii of $A_1B_1C_1$ and $\alpha\beta\gamma$, *i.e.*, in the ratio $R:r$. The distance of this centre from BC

$$= (R\cos A \,.\, r + r\,.\,R)/(R+r) = a(s-a)/2(R+r);$$

thus this point is $a(s-a) : b(s-b) : c(s-c)$, and is therefore conjugate to $1/a(s-a) : 1/b(s-b) : 1/c(s-c)$, the GERGONNE point of ABC. The distance from A to x_1y_1 is the same as that from A to BC; thus, B_1C_1 and $\beta\gamma$ are at a distance p_1 apart. Hence the distance of the homothetic

centre from $\beta\gamma$ $= \{(p_1-R)\,r + r\,.\,R\}/(R+r) = p_1 r/(R+r)$,

and is therefore proportional to $1/a$; so also for its distances from $\gamma\alpha$, $\alpha\beta$. The distance of this centre from B_1C_1 is similarly seen to be $p_1R/(R+r)$; so that the point is $1/a : 1/b : 1/c$ in regard to $A_1B_1C_1$ also. Similarly, if r' be the inradius of DEF, q_1 the distance apart between EF and $\beta\gamma$, and H the incentre of DEF (the orthocentre of ABC), then the distance of the second centre of perspective from $\beta\gamma$ $= q_1 r/(r+r')$. Now

$$q_1 = 2R\sin B\sin C\,(1-\cos A) = 2R\sin B\sin C\sin A\,(1-\cos A)/\sin A$$
$$\propto \tan\tfrac{1}{2}A;$$

hence this point is $\tan\tfrac{1}{2}A : \tan\tfrac{1}{2}B : \tan\tfrac{1}{2}C$ with regard to $\alpha\beta\gamma$. So also with regard to DEF, the actual distances being now $q_1r'/(r+r')$, . . .

[The following analytical results are easily proved:—The equation to $\beta\gamma$ is $a\alpha + (b-c)(\beta-\gamma) = 0$; to B_1C_1, $b\gamma + c\beta = 0$; to $C_1\gamma$, $c(\alpha-\beta) + \gamma(a-b) = 3$; the centre of perspective of $A_1B_1C_1$, $\alpha\beta\gamma$ is $a(s-a)/2(R+r)$, $b(s-b)/2(R+r)$, $c(s-c)/2(R+r)$, or $\cos^2\tfrac{1}{2}A : \cos^2\tfrac{1}{2}B : \cos^2\tfrac{1}{2}C$. The equation to EF is

$$-\alpha\cos A + \beta\cos B + \gamma\cos C = 0;$$

to $F\gamma$, $\qquad \alpha\cos A - \beta\cos B + \gamma\cos C\,(a-b)/c = 0;$

$F\gamma$, $D\alpha$, $E\beta$ meet in the second centre of perspective

$$\cos^2\tfrac{1}{2}A/\cos A : \cos^2\tfrac{1}{2}B/\cos B : \cos^2\tfrac{1}{2}C/\cos C.]$$

6419. (The late J. J. WALKER, M.A., F.R.S.)—Three lines in space are determined each by a pair of planes

$$m_1 \equiv B_1y + C_1z + 1 = 0, \quad x + n_1 = 0, \quad (n_1 \equiv D_1y + E_1z)\ldots.$$

Prove that the equation to the pair of planes through the axis $y = 0$, $z = 0$, and one of the two lines meeting it and each of those three lines, is

$$\begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

Solution by Professor JAN DE VRIES, Ph.D.; *and* W. H. SALMON, B.A.

Let $qy = pz$, $x = ry + s$ be one of the lines meeting the axis OX and each of the lines $B_ky + C_kz + 1 = 0$, $x + D_ky + E_kz = 0$.

If it intersects the line $k = 1$ in the point (x_1, y_1, z_1), we have

$$(D_1+r)y_1+E_1z_1+s = 0, \quad B_1y_1+C_1z_1+1 = 0, \quad qy_1-pz_1 = 0.$$

Hence, by elimination of y_1, z_1,

$$pr-(B_1p+C_1q)s+(D_1p+E_1q) = 0.$$

Also we have

$$pr-(B_2p+C_2q)s+(D_2p+E_2q) = 0, \quad pr-(B_3p+C_3q)s+(D_3p+E_3q) = 0.$$

Eliminating pr and s, we get

$$\begin{vmatrix} 1 & B_1p+C_1q & D_1p+E_1q \\ 1 & B_2p+C_2q & D_2p+E_2q \\ 1 & B_3p+C_3q & D_3p+E_3q \end{vmatrix} = 0,$$

which agrees with the given result.

6514. (W. J. C. Miller, B.A.)—Find, to 4 decimals, the value of

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz\,\frac{3xyz\,(1-x-y-z)}{(1-x)(1-y)(1-z)(x+y+z)}.$$

Solution by J. O. Watts.

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz\,\frac{3xy\,(1-x-y-z)}{(1-x)(1-y)(1-z)(x+y+z)}$$

$$= \int_0^1\int_0^{1-x} dx\,dy\,\frac{3xy}{(1-x)(1-y)}\left(z+\frac{x+y}{x+y+1}\log\frac{1-z}{x+y+z}\right)_0^{1-x-y}$$

$$= \int_0^1\int_0^{1-x} dx\,dy\,\frac{3xy\,(1-x-y)}{(1-x)(1-y)} = \int_0^1 dx\,\frac{3x}{1-x}\left(x+y-\frac{x}{1-y}\right)dy$$

$$= \int_0^1 dx\,\frac{3x}{1-x}\left(\frac{1-x^2}{2}+x\log x\right) = \tfrac{5}{4}+\int_0^1\frac{3x^2\log x}{(1-x)}dx.$$

$$\int_0^1\frac{3x^2\log x\,dx}{1-x} = \int_0^1\frac{3\,(1-x)^2\log(1-x)\,dx}{x}$$

$$= 3\int_0^1\frac{\log(1-x)\,dx}{x}-6\int_0^1\log(1-x)\,dx+3\int_0^1 x\log(1-x)\,dx$$

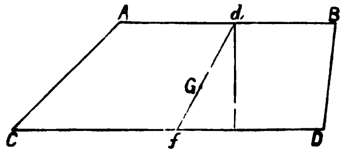
$$= \frac{-\pi^2}{2}+6\left(\frac{1}{1.2}+\frac{1}{2.3}+\dots\right)-3\left(\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots\right)$$

$$= \tfrac{1}{2}(-\pi^2)+6-\tfrac{3}{2}\cdot\tfrac{3}{2} = \tfrac{1}{2}(-\pi^2)+3\tfrac{3}{4}.$$

Therefore $\quad I = 5-\frac{1}{2}\pi^2 = 5-4\cdot9348 = \cdot0652.$

14219. (I. Arnold.)—If a and b be the two parallel sides of a trapezoid, and h the line which bisects those sides, the centre of gravity G of the trapezoid is in this line. It is required to find the distance of G from a in the line h in terms of a, b, and h.

Solution by the PROPOSER *and* Rev. T. MITCHESON, B.A.



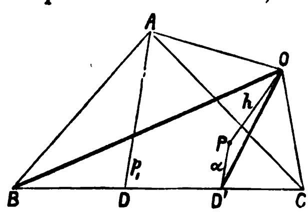
Let AB, CD be a, b, respectively, the parallel sides of the trapezoid: df be h, the line joining their mid-points; and G the centre of gravity.

From ARCHIMEDES' theorem, G lies in df and divides that line in the ratio of $2a+b : 2b+a$; therefore $fG : Gd :: 2a+b : 2b \pm a$, or $fd : Gd :: 3a+3b : 2b+a$;

therefore $$Gd = \tfrac{1}{3}h\,(a+2b)/(a+b).$$

13668. (D. BIDDLE.)—It is known that the opposite edges of a tetrahedron are equal, and the trilinear coordinates of the projection of the apex on the base are given. Find the cubic contents.

Solution by W. C. STANHAM, B.A.



Let OP be the perpendicular from the apex O on the base ABC, and let PD′ ($= \alpha$) be drawn perpendicular to BC. Then $\triangle$OCB is clearly similar to $\triangle$ABC and equal to it. Therefore OD′ (which is clearly perpendicular to BC) is equal to the perpendicular AD ($= p_1$) from A on BC. Therefore, if OP $= h$, and P is (α, β, γ),

$$h^2 + \alpha^2 = p_1^2 \;\ldots\ldots\ldots\ldots\; \text{(A)};$$

similarly $h^2 + \beta^2 = p_2^2$, $h^2 + \gamma^2 = p_3^2$, where p_2 and p_3 are the perpendiculars from B and C on AC and AB. And since $$a\alpha + b\beta + c\gamma = p_1 a = p_2 b = p_3 c = 2\Delta,$$
Δ being the area, and a, b, c the sides of ABC,

$$\alpha/p_1 + \beta/p_2 + \gamma/p_3 = 1.$$

Combining this with (A), we can find h, p_1, p_2, p_3 in terms of α, β, and γ. Also, if $2s = a+b+c$,

$$s = \Delta\,(1/p_1 + 1/p_2 + 1/p_3), \quad s - a = \Delta\,(1/p_2 + 1/p_3 - 1/p_1),$$

with similar expressions for $s-b$, $s-c$;

$$\Delta^2 = s\,(s-a)\,(s-b)\,(s-c);$$

therefore Δ can be found in terms of α, β, and γ, and therefore the cubic contents ($= h\Delta/3$) can be found in terms of α, β, γ, the only difficulty being the finding of h from the equation

$$\alpha/(h^2+\alpha^2)^{\frac{1}{2}} + \beta/(h^2+\beta^2)^{\frac{1}{2}} + \gamma/(h^2+\gamma^2)^{\frac{1}{2}} = 1.$$

14203. (V. R. Thyagaragaiyar, M.A.)—Show that the roots of the equation $$32x^5+16x^4-32x^3-12x^2+6x+1 = 0$$
are $\cos\frac{2}{11}\pi$, $\cos\frac{4}{11}\pi$, $\cos\frac{6}{11}\pi$, $\cos\frac{8}{11}\pi$, and $\cos\frac{10}{11}\pi$.

Solution by F. L. Ward, B.A.; Professor Sanjána, M.A.; *and others.*

Expanding,
$$\begin{aligned} \cos 5\theta &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta, \\ \cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1, \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta, \\ \cos 2\theta &= 2\cos^2\theta - 1, \\ \cos\theta &= \cos\theta. \end{aligned}$$

Therefore
$$2(\cos 5\theta + \cos 4\theta + \ldots + \cos\theta) + 1 = 32\cos^5\theta + 16\cos^4\theta - 32\cos^3\theta - 12\cos^2\theta + 6\cos\theta + 1.$$

But $$2(\cos 5\theta + \ldots + \cos\theta) + 1 = \frac{\sin\frac{11}{2}\theta - \sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} + 1 = \frac{\sin\frac{11}{2}\theta}{\sin\frac{1}{2}\theta}.$$

This is zero for the values $\theta = \frac{2}{11}\pi, \frac{4}{11}\pi, \frac{6}{11}\pi, \frac{8}{11}\pi, \frac{10}{11}\pi$. Therefore the cosines of these five angles are the solutions to the equation.

14456. (Professor N. Bhattacharyya.)—There are n smooth rings fixed to a horizontal plane, and a string, the ends of which are fastened to two of the rings, passes in order through them. In the loops formed by the successive portions of the string are placed a number of pulleys whose masses are m, $\frac{1}{2}m$, $\frac{1}{3}m$, $\frac{1}{4}m$, $\frac{1}{5}m$, &c. If, in the subsequent motion, all the portions of the string not in contact with the pulleys are vertical, show that the acceleration of the rth pulley is $\{(n-2r)/n\}g$. Discuss the case when n is even.

Solution by H. W. Curjel, M.A.

Let T be the tension of the string, and $\ddot{x}_r$ the downward acceleration of the rth pulley. Then
$$(m/r)\,\ddot{x}_r = (m/r)\,g - 2\mathrm{T},$$
and $2\Sigma_1^n x_r$ = length of string = constant;
$$\therefore\quad (n-1)\,g = (\mathrm{T}/m)\,n\,(n-1); \qquad \therefore\quad \ddot{x}_r = g\,(n-2r)/n.$$

When n is even ($= 2m$), then the mth pulley remains at rest until one of the pulleys cannot move any further up. Then the equations of motion are as if that pulley were left out of the system, and the new accelerations can be found the same way as before. If the pulley which first stops is the $(n-1)$th pulley, then the new equations are formed by substituting $n-1$ for n in the old equations, thus:
$$\ddot{x}'_r = g\,(n-1-2r)/(n-1).$$

14262. (Professor SANJÁNA.)—Prove that

$$\frac{1}{1}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4.5}+\frac{1}{3}\cdot\frac{1.2}{4.5.6}+\frac{1}{4}\cdot\frac{1.2.3}{4.5.6.7}+\ldots=\frac{6\pi^2-7^2}{36};$$

and show how to find the value of

$$\frac{1}{1}\cdot\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{n(n+1)}+\frac{1}{3}\cdot\frac{1.2}{n(n+1)(n+2)}+\ldots,$$

where n is any positive integer.

Solution by H. W. CURJEL, M.A.

The rth term of the former series

$$=\frac{6(r-1)!}{r(r+3)!}=\frac{6}{r^2(r+1)(r+2)(r+3)}$$

$$=\frac{1}{r^2}-\frac{1}{r}+\frac{1}{r+1}-\tfrac{1}{2}\left\{\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right\}$$

$$-\tfrac{2}{3}\left\{\frac{1}{r(r+1)(r+2)}-\frac{1}{(r+1)(r+2)(r+3)}\right\};$$

therefore sum to infinity

$$=\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}=\frac{6\pi^2-7^2}{36}.$$

Again, the rth term of the second series

$$=\frac{(n-1)!(r-1)!}{r(r+n-1)!}=\frac{(n-1)!}{r^2(r+1)(r+2)\ldots(r+n-1)}$$

$$=\frac{1}{r^2}-\frac{(r+1)(r+2)\ldots(r+n-1)-(n-1)!}{r^2(r+1)(r+2)\ldots(r+n-1)};$$

therefore sum to infinity $=\frac{1}{6}\pi^2-u_n$

where $\qquad u_n-u_{n-1}=\Sigma_1^\infty\frac{(n-2)!}{r(r+1)\ldots(r+n-1)}=\frac{1}{(n-1)^2}.$

But $\qquad u_4=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2};\qquad\therefore\quad u_n=\Sigma_1^{n-1}\frac{1}{r^2};$

$\therefore\qquad \text{sum}=\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\ldots-\frac{1}{(n-1)^2}$

which can be found; in fact, the series

$$=\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots\text{ to infinity.}$$

13391. (R. W. D. CHRISTIE.) — Prove that the solutions of $X^2-5Y^2=-4$ are $X=a^2-b^2+4ab$, $Y=a^2+b^2$, where a and b are any two successive terms of a *continuant*.

Solution by the PROPOSER.

Solving $5Y^2 - X^2 = 4$ by $X = a^2 - b^2 + 4ab$ and $Y = a^2 + b^2$, we get

$$a^2 - b^2 - ab = \pm 1.$$

Now, if we assume $a/b = p_n/q_n$ the convergents of the continued fraction,

$$F = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\dots = \frac{1}{2}\frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \dots,$$

it will be found that these convergents possess the property

$$a^2 - b^2 - ab = \pm 1$$

(*v.* Solutions to Quest. 13480 and 13533).

13339. (R. W. D. CHRISTIE.)—Prove that in general any integral square may be resolved into three integral squares in six ways at least.

Solution by the PROPOSER.

Assuming the truth of FERMAT's theorem that every integer may be resolved into four squares (proved by CAUCHY), we have

$$\begin{aligned} N^2 \equiv (a^2+b^2+c^2+d^2)^2 &\equiv (a^2+b^2-c^2-d^2)^2 + (2ac+2bd)^2 + (2ad-2bc)^2 \\ &\equiv (a^2+b^2-c^2-d^2)^2 + (2ac-2bd)^2 + (2ad+2bc)^2 \\ &\equiv (-a^2+b^2-c^2+d^2)^2 + (2ab+2cd)^2 + (2ad-2bc)^2 \\ &\equiv (-a^2+b^2-c^2+d^2)^2 + (2ab-2cd)^2 + (2ad+2bc)^2 \\ &\equiv (-a^2+b^2+c^2-d^2)^2 + (2ab+2cd)^2 + (2ac-2bd)^2 \\ &\equiv (-a^2+b^2+c^2-d^2)^2 + (2ab-2cd)^2 + (2ac+2bd)^2. \end{aligned}$$

E.g., $343^2 = 237^2 + 226^2 + 102^2 = 237^2 + 138^2 + 206^2 = 93^2 + 314^2 + 102^2$
$= 93^2 + 258^2 + 206^2 = 3^2 + 314^2 + 138^2 = 3^2 + 258^2 + 226^2$
(accidental) $= 294^2 + 147^2 + 98^2 = 279^2 + 186^2 + 62^2 =$ &c.

There are six other forms, but they give the same results as above.

12055. (Professor CLAYTON.)—ABCD is a quadrilateral figure formed by four lines of curvature on an ellipsoid. If p, q, r, s be the central perpendiculars on the tangent planes at A, B, C, D, respectively, then $pr = qs$.

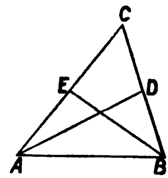
Solution by the PROPOSER.

Draw tangents to each line of curvature at the vertices A, &c. Let a_1, b_1 be the semi-axes of the central section parallel to tangent plane at A; a_2, b_2 corresponding lines for B, &c. Then, by a property of the surface $a_1 b_1 p = a_2 b_2 q = a_3 b_3 r = a_4 b_4 s$, and by a property of lines of curvature, $a_1 p = a_2 q$, $b_2 q = b_3 r$, $a_3 r = a_4 s$, $b_4 s = b_1 p$;

hence, at once, $pr = qs$.

14520 (Professor N. Bhattacharyya) and **14670** (E. W. Adair).—Required a *direct* proof of the old problem:—If the bisectors of the base angles of a triangle, being terminated at the opposite sides, be equal, show that the triangle is an isosceles one. (See Todhunter's *Euclid*.)

I. *Solution by* Rev. T. Roach, M.A.



Let $A = 2\theta + 2\phi, \quad B = 2\theta - 2\phi$;

$$\therefore \quad \frac{\sin(2\theta+2\phi)}{\sin(3\theta+\phi)} = \frac{BE}{BA} = \frac{AD}{AB} = \frac{\sin(2\theta-2\phi)}{\sin(3\theta-\phi)};$$

$$\therefore \quad \cos 2\theta \sin 2\phi \left\{\sin(3\theta-\phi) + \sin(3\theta+\phi)\right\}$$
$$= \sin 2\theta \cos 2\phi \left\{\sin(3\theta+\phi) - \sin(3\theta-\phi)\right\};$$

$\therefore \ \sin\phi = 0$ or

$$\cos 2\theta \sin 3\phi \,.\, 2\cos^2\phi = \sin 2\theta \cos 3\theta \cos 2\phi;$$

$$\therefore \quad 2\cos^2\phi \sin\theta = -2\sin\theta\cos\theta\cos 3\theta;$$

$\therefore \ \sin\theta = 0$, which is impossible, or $\cos 2\phi + 1 = -(\cos 4\theta + \cos 2\theta)$;

$\therefore \ \cos 2\phi + \cos 2\theta = -(1+\cos 4\theta)$; $\therefore \ 2\cos(\theta+\phi)\cos(\theta-\phi) = -2\cos^2 2\theta$,

which is impossible, as each factor is positive; therefore

$$\sin\phi = 0, \quad \text{and} \quad A = B.$$

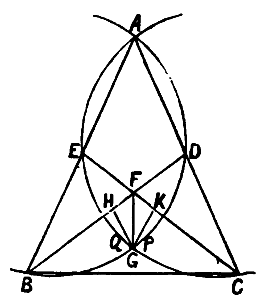
II. *Solution by* R. Chartres.

(i.) *Indirect.*—By *Euc.* vi. B, $ac - ab^2c/(a+c)^2 = ab - abc^2/(a+b)^2$,

or $a(c-b) = abc\left\{b/(a+c)^2 - c/(a+b)^2\right\}$,

of which $c-b$ is a factor; therefore $b = c$, or the triangle is isosceles. Indirect proofs are given in Todhunter's *Euclid*, p. 317, and in Nixon's *Euclid Revised*, p. 383.

The following is submitted as a direct proof:



(ii.) *Direct.* — Since BD = CE and they subtend the same angle A, therefore the circumcircles of ABD and AEC are equal; and F is clearly the incentre of the triangle ABC; therefore AF bisects A and passes through both P and Q, the middle points of the equal arcs DGB, EGC.

$\therefore$ rect. PF.FA = CF.FE (*Euc.* iii. 35)

and rect. QF.FA = BF.FD;

$\therefore$ rect. FA (PF − QF)

$= (CK^2 - FK^2) - (BH^2 - HF^2)$ (ii. 5)

$= HF^2 - FK^2$, and, since PK = QH,

$= FQ^2 - FP^2$,

which evidently requires PF = QF, or P and Q coincide at G, that is, ABC is isosceles.

[Dr. J. S. Mackay observes:—"A direct proof of this Question will be found in the *London, Edinburgh, and Dublin Philosophical Magazine* (Fourth Series), Vol. xlvii., pp. 354–7 (1874)."

Mr. R. Tucker further observes:—"This Question was proposed as

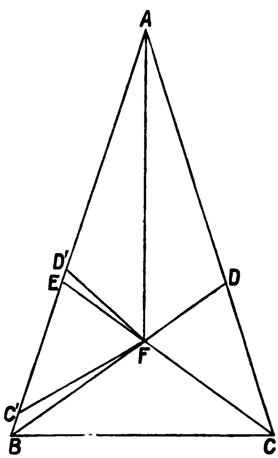
Quest. 1907 in *The Lady's and Gentleman's Diary* for 1856, and is solved on p. 58 (1857) by Messrs. T. T. Wilkinson, J. W. Elliott (the Proposer), and (analytically) by others. Mr. Wilkinson returns to the problem in his 'Notæ Geometricæ' in the *Diary* for 1859 (p. 87). A historical note is added on p. 88 which traces the Question back to the *Nouvelles Annales* for 1842. Professor Sylvester drew attention to the subject in the *Philosophical Magazine* for November, 1852. Dr. Adamson further discusses the matter in the *Philosophical Magazine* for April, May, and June, 1853. The best article I know on Quest. 1907 (*Diary*) appears in § 11 of Wilkinson's 'Horæ Geometricæ,' in the *Diary* for 1860, pp. 84–86, with a neat proof by the Rev. W. Mason. I find that the above references are given in Dr. Mackay's *Euclid*, p. 108. In the Key to this work Dr. Mackay prints a proof by M. Descube (*cf.* p. 92)."

And Mr. W. J. Greenstreet adds the following interesting information:—"For this and the similar theorem for two symmedians, *v.* *Intermédiaire des Mathématiciens*, Vol. II. (1895), pp. 151, 325. If the *external* bisectors of B and C are equal, it does not always follow that the triangle is isosceles. The data lead to $4Rr_1 = a^2 + bc$ in the triangle sides a, b, c (*v.* *Mathesis*, p. 261, 1895)."]

Editorial Note on Quests. 14520 and 14670.

As an alternative *geometrical solution* of the above, the following, notwithstanding its simplicity, may be deemed sufficient:—

Let the bisectors of B and C meet in F; then AF bisects A. About AF as axis, let AC be revolved until it coincides (at least in part) with AB; AC being considered shorter than AB, it is clear that FC must take up some such position as FC′; that is, FC < FB; also, since ∠C′FD′ = CFD = BFE, it is clear that FD must take up some such position as FD′; that is, FD > FE. Thus, we have FB + FD > FC + FE, contrary to the hypothesis. Therefore AC is not less than AB. If AB be supposed shorter than AC, revolve AB to lie on AC, when B′ and E′ will be above C and D respectively, and the same reasoning will apply.



When A > 60°, the longer line from F will lie below the shorter at E and D as well as at B and C, so that the same method of proof cannot be utilized. But we may adopt the same principle, by taking two axes of revolution, namely, BF and CF, bearing in mind that BC is now longer than either AB or AC Revolving CB about CF, it will lie on CA, and stretch some distance beyond A; and the same when BC is revolved about BF. But, if AC < AB, then BC − AB < BC − AC, and C′ lies nearer

A than B′ does. Consequently CF (= C′F) < BF (= B′F). Again, if AC < AB, it is clear (CE being equal to BD) that E is nearer to A than D is, that is, in this case, nearer the foot of the perpendicular from F; therefore DF > EF. Thus, as before, we have BF + DF > CF + EF, or one bisector of a base angle greater than the other, which is contrary to the hypothesis. Hence, under the conditions mentioned in the Question, the triangle is isosceles.

[It is the language alone, in the foregoing, that is not strictly Euclidean.]

14251. (R. Knowles, B.A.) — Prove that the sum of the first r coefficients in the expansion of $(1-x)^{-n}$ is $\{r(r+1)\dots(r+n-1)\}/n!$.

Solution by Rev. T. Mitcheson, B.A.; *and others.*

Todhunter shows that the sum of the first $r+1$ coefficients in this expansion is $(n+1)(n+2)-(n+r)/r!$. For r put $r-1$; then we have

$$(n+1)(n+2)-(n+r-1)/(r-1)! = (n+r-1)!/n!\,(r-1)!$$
$$= \{(n+r-1)(n+r-2)\dots r\}/n!$$

14250. (Robert W. D. Christie.)—Prove the following very general theorem:— $$x.10^{pn+k} = \frac{Pm+x}{\{\frac{1}{10}(XP+1)\}^k \bmod P} (\bmod P),$$ where x, n, k are any integers, P any odd prime, p the period of 1/P, m any integer required to make the remainder an integer (always possible).

Ex. gr.—(1) $x=3$, $k=5$, $P=7$, $X=1, 3, 7, 9$, when P ends in 9, 3, 7, 1, respectively. Therefore

$$3.10^{6n+5} = \frac{7m+3}{5^5 \bmod 7} (\bmod 7) = \frac{7m+3}{3} = 1 \bmod 7.$$

Thus $$3.10^{6n-1} = 1 \bmod 7.$$

(2) $n=7$, $k=1$, $P=19$.

$$7.10^{18n+k} = (19m+7)/2^k = 13 \bmod 19.$$

Thus $$7.10^{18n+1} = 13 \bmod 19.$$

Solution by Lt.-Col. Allan Cunningham, R.E.; *and the* Proposer.

Let a be any base prime to the prime P, and let p be the *least* exponent giving $a^p \equiv 1 \pmod P$. Let X be a number such that

$$(XP+1) \div a = \text{integer}.$$

Then $$x.a^{np+k}.\{(XP+1)/a\}^k = x.a_{\cdot du}(XP+1)^k \equiv x \pmod P$$

[because $a^{np} \equiv 1$, and $(XP+1)^k \equiv 1 \pmod P$]. Now substitute the residue of $(XP+1) \div a$ to mod P in the sinister, as is clearly admissible; therefore

$$x.a^{np+k}\left[\text{residue of } \{(XP+1)/a\}^k \bmod P\right] \equiv x \pmod P = (mP+x).$$

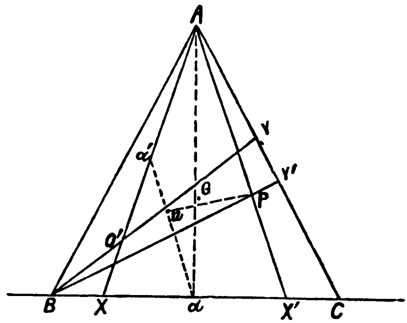
Now make $a = 10$, and divide by the expression in the brackets [...]; this gives the required result.

14430. (J. A. Third, D.Sc.)—A conic, whose centre is O, touches the sides BC, CA, AB of a triangle at X, Y, Z, and O′ is the point of concurrence of AX, BY, CZ. Show that O bears to ABC the same relation that the isotomic conjugate of O′ bears to the anticomplementary triangle of ABC (the triangle formed by parallels through A, B, C to the opposite sides).

Solution by Professor A. Droz-Farny; *and* Professor K. J. Sanjána, M.A.

Il suffit de démontrer que O est le point complémentaire du conjugué isotomique de O′, point de Gergonne de la conique inscrite.

Soit P un point quelconque du plan du triangle ABC et g une transversale quelconque passant par P. Cette droite coupe les côtés en A′, B′, C′. Soit A″ sur BC l'isotomique de A′; les trois points A″, B″, C″ sont sur une ligne droite g' la transversale réciproque de g (nomenclature de M. de Longchamps). Comme on le démontre aisément, lorsque g tourne autour de P, g' enveloppe une conique inscrite au triangle et touchant les côtés aux points X, Y, Z isotomiques des points d'intersection de PA, PB, PC respectivement avec BC, AC, AB.



Les points O′ et P sont donc conjugués isotomiques. Soient α le point milieu de BC et α' celui de AX. D'après une proposition connue, cas particulier du théorème de Newton sur le lieu des centres des coniques inscrites dans un quadrilatère, O est le point de croisement des droites $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$. Or, G étant le centre de gravité du triangle, soit u le point d'intersection de PG avec $\alpha\alpha'$. Les triangles AGP et $u\alpha$G étant semblables, $Gu : GP = G\alpha : GA = 1 : 2$; u est donc le complémentaire de P et par conséquent un point fixe sur PG par lequel passeront de même $\beta\beta'$ et $\gamma\gamma'$; u coincide donc avec O ; d'où la proposition.

14522. (J. H. Taylor, M.A.)—If A, B, C are vertices of equilateral triangles described all externally, or all internally, on the sides of a triangle A′B′C′, and Aa, Bb, Cc are diameters of circles circumscribing those equilateral triangles, then AA′, BB′, CC′ are equal and concurrent, and a, b, c form an equilateral triangle and are middle points, each of a pair of arcs, on sides of the triangles ABC, A′B′C′.

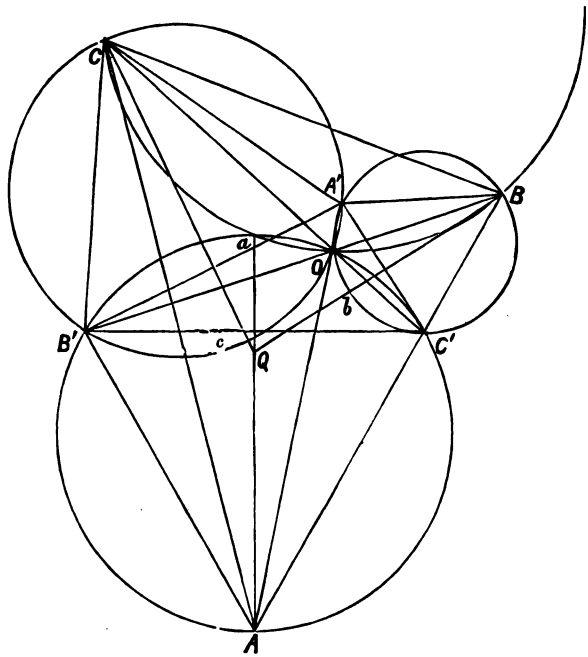
Solution by the Proposer *and* Professor Sanjána.

AA′, BB′, CC′ are concurrent, since the equilateral triangles are a particular case of similar isosceles triangles. (Quest. 14493.)

B'A', A'B = CA', A'C', each to each, and $\angle$ B'A'B = CA'C'; therefore B'B = CC' = AA' in like manner.

a, *b*, *c* are middle points of arcs containing angles of 120°, and therefore are centres of equilateral triangles described internally on the sides of the triangle B'C'A'; therefore *abc* is an equilateral triangle. (Quest. 14412.)

It has been shown (Quest. 14382) that AA', BB', CC' intersect at 60°.

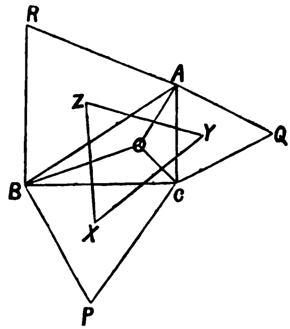


$\angle$ *a*OB' = *a*C'B' = 30° = *a*OC; therefore *a* is the mid-point of arc containing angle of 120° on CB, and it is also mid-point of a similar arc on C'B.

[Regarding this Question as well as Quest. 14412, Mr. Greenstreet observes:—For complete discussion of the numerous properties connected with these triangles, with copious bibliographical references, *v. Proc. Ed. Math. Soc.*, Vol. xv., p. 100 (Dr. J. S. Mackay on "Isogonic Centres").]

14412. (H. A. Webb.)—Three equilateral triangles are described outwards on the sides of any triangle as bases. Prove geometrically that the centres of these three equilateral triangles form the vertices of a fourth equilateral triangle.

Solution by J. G. SMITH; W. J. GREENSTREET, M.A.; *and many others.*



The three circles round the equilateral triangles meet in a point. For let two meet in O; then BOC = COA = 120°. Therefore BOA = 120°. Therefore O is on the circle ABR.

Join OA, OB, OC, and the centres of the circles X, Y, Z. Then OA is common chord of circles ARB and CAQ. Therefore YZ is perpendicular to OA. Similarly, XY is perpendicular to OC, ZX to OB; but OA, OB, OC are equally inclined to one another. Therefore YZ, ZX, XY are equally inclined. Therefore XYZ is equilateral.

Extensions.—(1) Same holds if the triangles are inscribed *inwards*.

(2) This theorem may be extended thus:—If *any similar triangles* be described inwards or outwards on the sides of any triangle so that each angle may be in turn vertical angle, then the centres of the circles round those triangles form the vertices of a new triangle similar to the described triangles.

The circles will meet in a point, and so on.

14463. (R. C. ARCHIBALD, M.A.)—Express the coordinates of any point on the cardioid as rational functions of a variable parameter, and show that the locus of a point which moves such that the triangle formed by joining the points of contact of the tangents drawn therefrom to the cardioid is of constant area and in general a curve of the eighth degree. [This theorem is due to Professor ZAHRADUIK.]

11427. (R. LACHLAN, M.A.)—If the points of contact of the three tangents which can be drawn from the point P to the cardioid $r = a(1+\cos\theta)$ be collinear, prove that (1) the locus of P is a circle $r + a\cos\theta = 0$; and (2), if the feet of the three normals which can be drawn from P be collinear, the locus of P is the circle $3r = a\cos\theta$.

Solution by Professor SANJÁNA, M.A.

We have $x = r\cos\theta = 2a\cos^2\tfrac{1}{2}\theta\cos\theta = 2a(1-t^2)/(1+t^2)^2$,

and $y = r\sin\theta = 2a\cos^2\tfrac{1}{2}\theta\sin\theta = 4at/(1+t^2)^2$,

where $t = \tan\tfrac{1}{2}\theta$.

The normal makes with the radius vector the angle $\tfrac{1}{2}\theta$; hence, its inclination to the axis of x is $\tfrac{3}{2}\theta$, that of the tangent $\tfrac{1}{2}\pi + \tfrac{3}{2}\theta$. Thus the equation of the normal at t is

$$y - \frac{4at}{(1+t^2)^2} = \frac{3t-t^3}{1-3t^2}\left(x - \frac{2a(1-t^2)}{(1+t^2)^2}\right),$$

or, on reduction, $y(1-3t^2)-x(3t-t^3)+2at = 0.$

Similarly, the equation of the tangent is

$$y(3t-t^3)+x(1-3t^2)-2a = 0.$$

From any point P (hk), let three tangents be drawn to the curve; then the three points of contact are given by the cubic

$$kt^3+3ht^2-3kt+2a-h = 0.$$

If these points are t_1, t_2, t_3, we get

$$\Sigma t_1 = -3h/k, \quad \Sigma t_2t_3 = -3, \quad \text{and} \quad t_1t_2t_3 = -(2a-h)/k.$$

The area of the triangle formed by any three points on the curve is

$$\Sigma\left(\frac{2a(1-t_1^2)}{(1+t_1^2)^2}\,\frac{4at_2}{(1+t_2^2)^2}-\frac{2a(1-t_2^2)}{(1+t_2^2)^2}\,\frac{4at_1}{(1+t_1^2)^2}\right)$$
$$= \frac{8a^2}{(1+t_1^2)^2(1+t_2^2)^2(1+t_3^2)^2}\Sigma(t_2-t_1)(1+t_1t_2)(1+t_3^2)^2$$
$$= \frac{8a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)}{(1+t_1^2)^2(1+t_2^2)^2(1+t_3)^2}\left\{3+t_1t_2t_3\Sigma t_1+(\Sigma t_1)^2-\Sigma t_2t_3\right\}.$$

If the three points are those of contact,

$$3+t_1t_2t_3\Sigma t_1+(\Sigma t_1)^2-\Sigma t_2t_3 = \frac{6(h^2+k^2+ah)}{k^2};$$

$$(1+t_1^2)^2(1+t_2^2)^2(1+t_3)^2$$
$$= \left\{1+(\Sigma t_1)^2-2\Sigma t_2t_3+(\Sigma t_2t_3)^2-2t_1t_2t_3\Sigma t_1+t_1^2t_2^2t_3^2\right\}^2$$
$$= \frac{16(4h^2+4k^2-4ah+a^2)^2}{k^4};$$

and

$$(t_1-t_2)(t_2-t_3)(t_3-t_1) = \sqrt{\left(-\frac{27}{k^6}(G^2+4H^3)\right)}$$

(see BURNSIDE and PANTON, § 42)

$$= \sqrt{\left(\frac{108}{k^4}(h^4+2h^2k^2+k^4-a^2k^2-2ahk^2-2ah^3)\right)}.$$

If this triangle has the constant area $8a^2c$, we get for the locus of P

$$\frac{\sqrt{\{108(h^4+2h^2k^2+k^4-a^2k^2-2ahk^2-2ah^3)\}}}{16(4h^2+4k^2-4ah+a^2)^2}\,6(h^2+k^2+ah) = c.$$

When the area is zero, the three points t are collinear, and the locus of P is the circle $h^2+k^2+ah = 0$, or $r+a\cos\theta = 0$.

If the three points are feet of normals,

$$3+t_1t_2t_3\Sigma t_1+(\Sigma t_1)^2-\Sigma t_2t_3 = \frac{2(3h^2+3k^2-ah)}{h^2};$$

$$(1+t_1^2)^2(1+t_2^2)^2(1+t_3^2)^2 = \frac{16(4h^2+4k^2-4ah+a^2)^2}{h^4};$$

and $(t_1-t_2)(t_2-t_3)(t_3-t_1)$

$$= \sqrt{\left(\frac{4}{h^4}(27h^4+54h^2k^2+27k^4-54ahk^2-8a\,h+12a^2h^2-54ah^3+9a^2k^2)\right)}.$$

If this triangle has the constant area $8a^2c$, we get for the locus of P

$$\frac{\surd\{4(27h^4+54h^2k^2+27k^4-\dots)\}}{16(4h^2+4k^2-4ah+a^2)^2}\,2(3h^2+3k^2-ah)=c.$$

When the area is zero, the three points t are collinear, and the locus of P is the circle $3h^2+3k^2-ah=0$ or $3r-a\cos\theta=0$.

In the general case the locus of P, for both tangents and normals, is a curve of the eighth degree, as is evident on squaring.

[For another solution of Quest. 11427, see Vol. LVIII., p. 42.]

11069. (J. J. BARNIVILLE.)—Prove that

$$1^5+(1^5+2^5)\,2^{-1}+(1^5+2^5+3^5)\,2^{-2}+\dots=2744,$$

$$1^3+(1^3+3^3)\,2^{-1}+(1^3+3^3+6^3)\,2^{-2}+(1^3+3^3+6^3+10^3)\,2^{-3}+\dots=6416,$$

$$1^2+(1^2+4^2)\,2^{-1}+(1^2+4^2+10^2)\,2^{-2}+(1^2+4^2+10^2+20^2)\,2^{-3}+\dots=2016;$$

$$n.1^3+(n-1)\,2^3+\dots+2(n-1)^3+1.n^3=\tfrac{1}{60}n(n+1)(n+2)(3n^2+6n=1);$$

in the figurate series 1, 7, 28, ..., $66u_n+26(u_{n+1}+u_{n-1})+u_{n+2}+u_{n-2})$

= a sum of consecutive fifth powers;

the ultimate term of the series 1, $\tfrac{1}{2}$, $\tfrac{2}{3}$, $\tfrac{3}{5}$, $\tfrac{5}{8}$, $\tfrac{8}{13}$, ... is $2\sin\tfrac{1}{10}\pi$.

Solution by H. W. CURJEL, M.A.

Let $\quad S(x)=1^5+(1^5+2^5)\,x+(1^5+2^5+3^5)\,x^2+\dots,$

and $\quad S_n=1^n+2^nx+3^nx^2+\dots,$

$$S_0=\frac{1}{1-x},\quad S_1=\frac{1}{(1-x)^2},\quad S_2=\frac{1}{1-x}(-S_0+2S_1)=\frac{1+x}{(1-x)^3},$$

$$S_3=\frac{1}{1-x}(3S_2-3S_1+S_0)=\frac{1+4x+x^2}{(1-x^2)^4},$$

$$S_4=\frac{1}{1-x}(4S_3-6S_2+4S_1-S_0)=\frac{1+11x+11x^2+x^3}{(1-x)^5},$$

$$S_5=\frac{1}{1-x}(5S_4-10S_3+10S_2-5S_1+S_0)=\frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6};$$

therefore $\quad S_x=\dfrac{1+26x+66x^2+26x^3+x^4}{(1-x)^7}$ ....(1);

therefore first series $=S(\tfrac{1}{2})=4328$.

Again, if $x=\tfrac{1}{2}$,

second series

$$=\frac{1}{1-x}\left\{1+3^3x+6^3x^2+\dots+\left(\frac{n(n+1)}{2}\right)^3x^{n-1}+\dots\right\}$$

$$=\frac{1}{(1-x)^2}\left(1+(3^3-1)\,x+(6^3-3^3)\,x^2+\dots+\frac{3n^5+n^3}{4}x^{n-1}+\dots\right)$$

$$=\frac{1}{(1-x)^2}\left(\frac{S_3}{4}+\frac{3S_5}{4}\right)=\frac{1+20x+48x^2+20x^3+x^4}{(1-x)^8}=6544.$$

Third series

$$= \frac{1}{1-x}\left\{1 + 4^2x + 10^2x^2 + \dots + \left(\frac{n(n+1)(n+2)}{6}\right)^2 x^{n-1} + \dots\right\}$$

$$= \frac{1}{(1-x)^2}\left(1 + (4^2-1)x + (10^2-4^2)x^2 + \dots + \frac{2n^5+5n^4+4n^3+n^2}{12}x^{n-1} + \dots\right)$$

$$= \frac{1}{12(1-x^3)^2}(2S_5 + 5S_4 + 4S_3 + S_2) = \frac{1+9x+9x^2+x^3}{(1-x)^8} = 2016.$$

Fourth series

$=$ coefficient of x^{n-1} in $(1+2x+3x^2+\dots)(1^3+2^3x+3^3x^2+\dots)$,

i.e., in S_1S_3 or

$$\frac{1}{(1-x)^4} + \frac{6x}{(1-x)^6} = \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)(n+2)(n+3)}{20}$$

$$= \frac{n(n+1)(n+2)}{60}(3n^2+6n+1).$$

Again, expanding $1/(1-x)^7$ in (1), and equating coefficients of x^{n+1}, we get $\quad 66u_n + 26(u_{n+1}+u_{n-1}) + u_{n+2} + u_{n-2} = \Sigma_1^{n+2}(n^5)$.

Also 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, &c., are the convergents of the continued fraction

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+\dots};$$

therefore ultimate term of the series is x, where

$$x = \frac{1}{1+x}; \quad \text{therefore } x = \frac{\sqrt{5}-1}{2} = 2\sin\frac{\pi}{10}.$$

14478. (Rev. T. MITCHESON, B.A.)—P, Q are the ends of conjugate semi-diameters of an ellipse, and a straight line drawn from the intersection of the normals at P and Q, through the centre C, meets PQ in S, whilst the tangents meet at the point (h, k); show that

$$CS = \frac{a^2b^2}{(a^4k^2+b^4h^2)^{\frac{1}{2}}}.$$

Solution by R. TUCKER, M.A.; F. H. PEACHELL, B.A.; *and others.*

Let the normals meet in O; then the equation to OC is

$$ax/(\cos\phi - \sin\phi) = by/(\cos\phi + \sin\phi) \quad \dots\dots\dots\dots \text{(i.)},$$

and to PQ is $\quad hx/a^2 + ky/b^2 = 1 \quad \dots\dots\dots\dots$ (ii.),

whence $\quad (h/a)(\cos\phi + \sin\phi) = (k/b)(\cos\phi - \sin\phi) \quad \dots\dots$ (iii.).

From (i.) and (iii.) $\quad a^2kx - b^2hy = 0$,

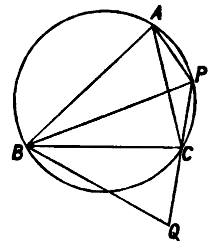
but (ii.) gives $\quad b^2hx + a^2ky = a^2b^2$;

hence OC is perpendicular to PQ; whence

$$CS = \sqrt{(x^2+y^2)} = \&c.$$

Alternative Proof of PTOLEMY'S *Theorem. By* R. F. DAVIS, M.A.

Let P be a point on the circumcircle of ABC, between A and C (say). Produce PC to Q so that the angle PBQ = ABC. Then, obviously, the triangles PBQ, ABC are similar; and so also the triangles CBQ, ABP.



Thus $\quad PQ = (b/c)\,PB;$

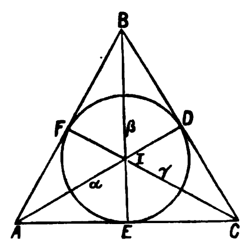
$\qquad QC = (a/c)\,PA.$

Since $\quad PQ = QC + PC,$

$\qquad b\,PB = a\,PA + c\,PC.$

14407. (Rev. T. MITCHESON, B.A.)—If α, β, γ be the distances of the incentre from the angular points of a triangle, the diameter of the incircle $= \alpha\beta\gamma \dfrac{(\alpha^{-1}\cos\frac{1}{2}A + \beta^{-1}\cos\frac{1}{2}B + \gamma^{-1}\cos\frac{1}{2}C)}{\alpha\cos\frac{1}{2}A + \beta\cos\frac{1}{2}B + \gamma\cos\frac{1}{2}C}.$

Solution by Rev. T. WIGGINS, B.A.; J. G. SMITH; W. J. GREENSTREET, M.A.; *and many others.*



Expression

$$= \frac{\beta\gamma\sin\frac{1}{2}(B+C) + \alpha\gamma\sin\frac{1}{2}(A+C) + \alpha\beta\sin\frac{1}{2}(A+B)}{AF + BD + CE}$$

$$= \frac{2\,(BIC + AIC + AIB)}{s} = \frac{2S}{s} = 2r.$$

14338. (Professor SANJÁNA, M.A.)—In Quest. 14110 denote $(2e-1)\,a^2 + b^2 + c^2$ by a_1; take b_1, c_1 similarly; call e the ratio of the TUCKER circle, and let $\lambda = (1-e)\tan\omega$. Then prove that (1) the equation of the TUCKER circle (whose ratio is e) XX′YY′ZZ′ is

$$\beta\gamma/bc + \gamma\alpha/ca + \alpha\beta/ab - (\alpha/a + \beta/b + \gamma/c)(1-e)\tan\omega + (1-e)^2\tan^2\omega = 0,$$

or $(\alpha - a\lambda)(\beta - b\lambda)(\gamma - c\lambda) = \alpha\beta\gamma$; (2) the envelope of this circle, as its ratio varies, is $\alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2 - 2\alpha\beta/ab - 2\beta\gamma/bc - 2\gamma\alpha/ca = 0$, or $\sqrt{(\alpha/a)} + \sqrt{(\beta/b)} + \sqrt{(\gamma/c)} = 0$, the BROCARD ellipse; (3) the radical axis of two TUCKER circles of radius f and g is $\alpha/a + \beta/b + \gamma/c = (2-f-g)\tan\omega$, so that the radical axis of the TUCKER circle e with itself is $\alpha/a + \beta/b + \gamma/c = 2\lambda$, which is also its chord of double contact with the envelope; (4) the radical axis of the TUCKER circle e and the circumcircle is $\alpha/a + \beta/b + \gamma/c = \lambda$, and the chord of contact of the circumcircle with the BROCARD ellipse is the LEMOINE line; (5) if $f + g$ = constant $= 1 + e$, the varying TUCKER circles f and g have a fixed radical axis, which is the radical axis of the circumcircle and the fixed TUCKER circle whose ratio is e; (6) if $f+g = 2\sin^2\omega$, the varying TUCKER circles f and g are of equal area; (7) the polar of the symmedian point with regard to the circle e is $\alpha/a+\beta/b+\gamma/c = (4e-1)/(2e)\,\lambda$, and, if $fg = \frac{1}{4}$, the varying TUCKER circles f and g have the same polar for this point; (8) the radical centre of the circles round AY′Z, ABC, and the TUCKER circle, lies on BC, the radical axis of the first two being $\beta c b_1 + \gamma b c_1 = 0$; so also for BZ′X, CX′Y; and these three radical centres, on BC, CA, AB respectively, are situated on the line $(\alpha a_1)/a + (\beta b_1)/b + (\gamma c_1)/c = 0$; and (9) the radical centre of the circles round AY′Z, BZ′X, C′Y is the point $\alpha/(b_1c_1\cos A) = \beta/(c_1a_1\cos\beta) = \gamma/(a_1b_1\cos C)$, which lies on the curve

$$\beta\gamma\sin 2A\sin(B-C)+\gamma\alpha\sin 2B\sin(C-A)+\alpha\beta\sin 2C\sin(A-B) = 0,$$

that circum-hyperbola of ABC which is the isogonal transformation of EULER's line. [The last result has been obtained by Rev. J. CULLEN in Quest. 13921.]

Solution by G. N. BATES, B.A.

For figure and notation see Quest. 14110.

We have $BX.BX' = c\rho\left\{a-(b^2/c)\,\rho\right\} = (2\Delta\lambda - b^2\lambda^2)/\sin^2 B$;

therefore equation of TUCKER circle is

$$abc\,\Sigma a\beta\gamma = \Sigma a\alpha.\Sigma\left\{(2\Delta\lambda - a^2\lambda^2)/\sin^2 A\right\}a\alpha \quad\text{or}\quad \Sigma\beta\gamma/bc - \lambda\Sigma\alpha/a + \lambda^2 = 0,$$

or $$(\alpha - a\lambda)(\beta - b\lambda)(\gamma - c\lambda) = \alpha\beta\gamma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From these equations (2), (3), (4); and (5) follow immediately.

Now $$R_f^2 = R^2\left\{f^2+(1-f)^2\tan^2\omega\right)$$

and $$R^2_{2\sin^2\omega-f} = R^2\left\{(2\sin^2\omega - f)^2 + (1-2\sin^2\omega+f^2)\tan^2\omega\right\}$$
$$= R_f^2 \text{ on reduction} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The polar of K ($\frac{1}{2}a\tan\omega$, &c.) is

$$\tfrac{1}{2}\tan\omega\Sigma\alpha/a - \tfrac{1}{2}\lambda\Sigma\alpha/a - \tfrac{1}{4}(3\lambda\tan\omega)+\lambda^2 = 0,$$

or $$\Sigma\alpha/a = (4e-1)\,\lambda/2e \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Now $$BA.BZ = c\left\{c-(b^2/a)\,\rho = c\left\{c - \frac{2(1-e)\,b^2c}{\Sigma a^2}\right\} = \frac{c^2b_1}{\Sigma a^2},\right.$$

and $$CA.CY' = b^2c_1/\Sigma a^2;$$

therefore AY′Z is $\quad a\Sigma a^2.\Sigma a\beta\gamma = \Sigma a\alpha\,(b_1c\beta + c_1b\gamma)$.

Similarly for BZ′X, CX′Y.

Therefore radical axis of ABC and AY′Z is $b_1c\beta + c_1b\gamma = 0$.

Radical axis of AY′Z and T_e is Y′Z, which intersects radical axis of T_e and circumcircle on BC (see Quest. 14159, Vol. LXXI.).

Therefore radical centre of AY′Z, ABC, and T_e is on BC, and is

$$\alpha/0 = \beta/bc_1 = \gamma/-cb_1.$$

This and the two corresponding points lie on the line

$$a_1\alpha/a + b_1\beta/b + c_1\gamma/c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The radical centre of AY′Z, BZ′X, and CX′Y is given by

$$(cb_1\beta + bc_1\gamma)/a = (ca_1\alpha + ac_1\gamma)/b = (ba_1\alpha + ab_1\beta)/c,$$

and is therefore the point $b_1c_1 \cos A$; $a_1c_1 \cos B$; $a_1b_1 \cos C$.

To find the locus of this point we have to eliminate between

$$\kappa \cos A/\alpha - \Sigma a^2 - 2(e-1)\,a^2 = 0$$

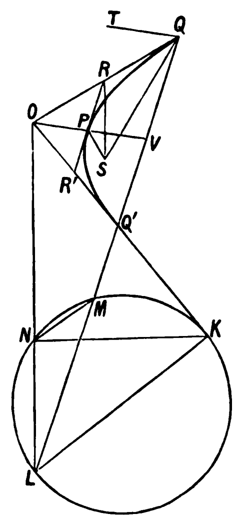
and two similar equations, giving

$$\Sigma\,(b^2-c^2) \cos A/\alpha = 0 \quad \text{or} \quad \Sigma \sin 2A \sin (B-C)/\alpha = 0,$$

the isogonal inverse of Euler's line.

14525. (J. Macleod, M.A.)—KL is a diameter of the circle KML. From L any two chords LM, LN on the same side of KL are drawn and produced to meet the tangent at K in Q′ and O. Through O a line is drawn parallel to MN, and LQ′ is produced to meet it in Q. QQ′ is bisected in V, and the straight line OV in P; through P a tangent is drawn to the parabola which is touched by OQ, OQ′ in the points Q, Q′, and meeting OQ, OQ′ in R, R′. Prove that the angle KOL is equal to the angle of the focal distances of P and R.

Solution by Constance I. Marks, B.A.; *and the* Proposer.



Draw QT parallel to VO.

At Q and P make angles OQS, R′PS, respectively equal to angles TQO, OPR′.

Therefore S is the focus of the parabola.

Join RS and NK.

Now ∠ KOL is complement of ∠ KLO.

Therefore

∠ KOL = ∠ NKL

= ∠ NML

= ∠ OQQ′ = ∠ ORR′

= difference between ∠ R′PS and ∠ RQS

= difference between ∠ R′PS and ∠ PRS

(△s PRS and QRS being similar)

= ∠ RSP.

13480. (R. W. D. Christie.)—Find a series which gives all the odd primes in order as factors.

Solution by the PROPOSER.

The series is 1, 4, 11, 29, 76, 199, 521, &c.

The law of formation $a_{n+1} \equiv 3a_n - a_{n-1}$,

$$(\omega_2+\omega_3)^{2n-1}+(\omega_4+\omega_5)^{2n-1} \equiv 1 \bmod 2n-1 \equiv a_n,$$

if $2n-1$ be prime and ω_n an unreal root of $x^5+1=0$,

i.e., $\{\frac{1}{2}(1+\surd 5)\}^{2n-1}+\{\frac{1}{2}(1-\surd 5)\}^{2n-1} = 1 \bmod 2n-1$

$$= \left\{\frac{1+pq+(\surd 5)^p}{2^p}\right\}+\left\{\frac{1-pq-(\surd 5)^p}{2^p}\right\} = 1 \bmod p,$$

where p is any *odd* prime,

$$= \frac{2+2p.q}{2^p} \text{ or } \frac{1+p.q}{2^{p-1}} = 1 \bmod p.$$

Now, by FERMAT's theorem, $2^{p-1}-1 = p.q$, when and when only p is prime.

If $p=2$, we have

$$\tfrac{1}{4}(1+2\surd 5+5)+\tfrac{1}{4}(1-2\surd 5+5) = 3 = 1 \bmod 2.$$

Thus the theorem is universally true for all primes.

13496. (Professor SANJÁNA, M.A.)—If p_r denote the cofficient of x^r in $(1-x)^{-\frac{1}{2}}$, prove that $\log 4 = p_1+\frac{1}{2}p_2+\frac{1}{3}p_3+\dots$,

$$\log(\surd 2+1) = p_1+\tfrac{1}{3}p_3+\tfrac{1}{5}p_5+\dots = 1-\tfrac{1}{3}p_1+\tfrac{1}{5}p_2-\tfrac{1}{7}p_3\dots.$$

Solution by Rev. J. CULLEN; H. W. CURJEL, M.A.; *and others.*

Since $$p_r = 2/\pi\int_0^{\frac{1}{2}\pi}\sin^{2r}\theta\,d\theta,$$

first series $= -2/\pi\int_0^{\frac{1}{2}\pi}\log(1-\sin^2\theta)\,d\theta = -4/\pi\int_0^{\frac{1}{2}\pi}\log\cos\theta\,d\theta = \log 4,$

second series $= 2/\pi\int_0^{\frac{1}{2}\pi}\log\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)d\theta = \log(\surd 2+1),$

third series $= 2/\pi\int_0^{\frac{1}{2}\pi}\tan^{-1}(\sin\theta)\,d\theta/\sin\theta.$

To integrate this, let $I = \int_0^{\frac{1}{2}\pi}\tan^{-1}(a\sin\theta)\,d\theta/\sin\theta$;

therefore $$dI/da = \int_0^{\frac{1}{2}\pi}d\theta/(1+a^2\sin^2\theta) = \tfrac{1}{2}\pi(1+a^2)^{\frac{1}{2}};$$

therefore $$I = \tfrac{1}{2}\pi\log(a+\sqrt{1+a^2}),$$

putting $a=1$, we have second series = third series = $\log(\surd 2+1)$.

11864. (Professor LUCAS.)—Mettre le nombre $x^{10}-5^5y^{10}$, où x et y sont des nombres entiers positifs, sous la forme d'un produit de trois facteurs rationnels.

Solution by H. W. CURJEL, M.A.; H. J. WOODALL, A.R.C.S.; *and others.*

$$x^{10}-5^5y^{10} = (x^2-5y^2)(x^8+5x^6y^2+25x^4y^4+125x^2y^6+625y^8)$$
$$= (x^2-5y^2)\left\{(x^4+15x^2y^2+25y^4)^2-(5x^3y+25xy^3)^2\right\}$$
$$= (x^2-5y^2)(x^4+5x^3y+15x^2y^2+25xy^3+25y^4)$$
$$\times(x^4-5x^3y+15x^2y^2-25xy^3+25y^4).$$

11075. (R. P. KATHERN, M.A.)—Solve the equation

$$\tan^2\theta+a\sec\theta+b\tan\theta+\sec^2\theta = 0.$$

Solution by H. J. WOODALL, A.R.C.S.

$$\tan^2\theta+a\sec\theta+b\tan\theta+\sec^2\theta = 0;$$

therefore $(2\tan^2\theta+b\tan\theta+1)^2 = a^2\sec^2\theta = a^2(\tan^2\theta+1)$;

therefore $4\tan^4\theta+4b\tan^3\theta+\tan^2\theta(4+b^2-a^2)+2b\tan\theta+(1-a^2) = 0$,

which requires solution.

11852. (Professor ZERR.)—If S represent the length of a quadrant of the curve $r^m = a$, $\cos m\theta$, ρ the radius of curvature of the path of the pole when the curve rolls along a straight line, S_1 the length of a quadrant of the first pedal of the curve, prove that

(1) $SS_1 = \rho\pi a^2/2r$; also (2) $\rho/r = (m+1)/m$ = a constant.

Solution by the PROPOSER.

The equation to the first pedal is $r^{m/(m+1)} = a^{m/(m+1)}\cos^{m/(m+1)}\theta$; hence (WILLIAMSON's *Int. Cal.*, Art. 156, Ex. 3)

$$S = \frac{a\sqrt{\pi}}{2m}\;\Gamma\left(\frac{1}{2m}\right)\Big/\Gamma\left(\frac{m+1}{2m}\right);$$

$$S_1 = \frac{(m+1)\,a\sqrt{\pi}}{2m}\;\Gamma\left(\frac{m+1}{2m}\right)\Big/\Gamma\left(1+\frac{1}{2m}\right);$$

therefore $SS_1 = \dfrac{(m+1)\,\pi a^2}{4m^2}\;\Gamma\left(\dfrac{1}{2m}\right)\Big/\Gamma\left(1+\dfrac{1}{2m}\right) = \dfrac{(m+1)\,\pi a^2}{2m}.$

The (p, r) equation to the curve is $r^{m+1} = a^mp$; by the theory of roulettes $(d/dp)(p/r) = 1/\rho$;

$$\frac{d}{dp}\left(\frac{p}{r}\right) = (a^mp)\frac{-1}{1+m}\left(\frac{m}{m+1}\right)\frac{1}{r}\left(\frac{m}{m+1}\right);$$

$$\therefore\; \rho = r(m+1)/m;\quad \therefore\; \frac{\rho}{r} = (m+1)/m = \text{a const.};\quad \therefore\; SS_1 = \frac{\rho\pi a^2}{2r}.$$

10364. (L. Bénézech.)—Si b est inférieur à $\sqrt{a}$, a et b étant des nombres entiers positifs, l'expression $a(1+a+2b)$ ne sera jamais un carré parfait.

Solution by H. J. Woodall, A.R.C.S.

Since $b < \sqrt{a}$, we may put $a = b^2 + k^2$. Therefore

$$a(1+a+2b) = (a+b)^2 + k^2 = (a+b+c)^2, \text{ say};$$

therefore $$k^2 = 2c(a+b) + c^2 = a - b^2;$$

therefore $b^2 + 2bc + c^2 = a(1-2c)$, *i.e.*, $(b+c)^2 = a(1-2c)$ (a negative quantity). This is impossible. Therefore $a(1+a+2b)$ cannot be a perfect square.

11762. (Professor Lucas.)—Démontrer l'identité

$$\frac{x-x^p}{(1-x)(1-x^p)} + \frac{x^p-x^{p^2}}{(1-x^p)(1-x^{p^2})} + \ldots + \frac{x^{p^n}-x^{p^{n+1}}}{(1-x^{p^n})(1-x^{p^{n+1}})} = \frac{x-x^{p^{n+1}}}{(1-x^p)(1-x^{p^{n+1}})}.$$

Solution by H. W. Curjel, M.A.

The given series

$$= \frac{1}{1-x} - \frac{1}{1-x^p} + \frac{1}{1-x^p} - \frac{1}{1-x^{p^2}} + \ldots + \frac{1}{1-x^{p^n}} - \frac{1}{1-x^{p^{n+1}}}$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{p^{n+1}}} = \frac{x-x^{p^{n+1}}}{(1-x)(1-x^{p^{n+1}})}.$$

14413. (Robert W. D. Christie.)—Find integral values of n to satisfy the equation $$T_x = nT_y,$$ and give general values for x and y. (T a triangular.)

Solution by Lt.-Col. Allan Cunningham, R.E.; *the* Proposer; *and others.*

$$T_x = \tfrac{1}{2}x(x+1) = \tfrac{1}{8}(X^2-1), \quad T_y = \tfrac{1}{2}y(y+1) = \tfrac{1}{8}(Y^2-1) \quad \ldots (1),$$

where $$X = 2x+1, \quad Y = 2y+1 \quad \ldots\ldots\ldots\ldots (2).$$

Then $T_x = n.T_y$ gives $nY^2 - X^2 = n-1$, (n supposed *integral*) ... (3).

Now suppose X_0, Y_0 to be *any* solution of this Diophantine ($X_0 = 1$, $Y_0 = 1$ is the obvious solution), and let ξ_r, η_r be any solution of $\xi^2 - n\eta^2 = 1$, all the solutions of which (infinite in number) are *known* by the ordinary rules. Then *all* the solutions (corresponding to X_0, Y_0) are given by $$X_r = \xi_r X_0 + n\eta_r Y_0, \quad Y_r = \xi_r Y_0 + \eta_r X_0.$$

From these x, y are given by (2), and T_x, T_y by (1).

The following are examples for the simple cases of $n = 2, 3, 5, 6$ given by $X_0 = 1$, $Y_0 = 1$:—

	$n = 2.$	$n = 3.$	$n = 5.$	$n = 6.$
$\xi =$	3, 17, 99, &c.	2, 7, 26, 97, &c.	9, 161	5, 49
$\eta =$	2, 12, 70, &c.	1, 4, 15, 56, &c.	4, 72	2, 20
$X =$	7, 41, 239, &c.	5, 19, 7, 265, &c.	29, 521	17, 169
$Y =$	5, 29, 169, &c.	3, 11, 41, 155, &c.	13, 233	7, 69
$x =$	3, 20, 119, &c.	2, 9, 35, 112, &c.	14, 260	8, 84
$y =$	2, 14, 84, &c.	1, 5, 20, 77, &c.	6, 116	3, 34
$T_x =$	6, 210, 7140, &c.	3, 45, 630, 8778, &c.	105, 33930	36, 3570
$T_y =$	3, 105, 3570, &c.	1, 15, 210, 2926, &c.	21, 6786	6, 595

A *similar* procedure is applicable when n is a *fraction*, say $n = \nu \div \mu$, so that the problem is to solve $\mu T_x = \nu T_y$,

which gives $\mu X^2 - \nu Y^2 = \mu - \nu$ .... (3A).

If now X_0, Y_0 be *any* solution of (3A), ($X_0 = 1$, $Y_0 = 1$ is the obvious one); and if ξ_r, η_r be *any* solution of $\xi^2 - \mu\nu . \eta^2 = 1$, all the solutions of which are known by the ordinary rules; then *all* the solutions of (3A) corresponding to X_0, Y_0 are given by $X_r = \xi_r X_0 + \nu\eta_r Y_0$, $Y_r = \mu\eta_r X_0 + \xi_r Y_0$. The values of x, y, T_x, T_y are thus given by (2) and (1). The two following examples are for the simple case of $n = \frac{2}{3}$, taking $X_0 = 1$, $Y_0 = 1$; here $\mu = 3$, $\nu = 2$, $\mu\nu = 6$.

ξ	η	X	Y	x	y	T_x	T_y
5,	2;	9,	11;	4,	5;	10,	15;
49,	20;	89,	109;	44,	54;	990,	1485.

Cor.—From the above may be solved the following *more general* problems (4) $T_x = T_a . T_y$;

(5) $T'_a . T_x = T^a . T_y$; (6) $T'_a . T'_b . T'_c \dots T_x = T_a . T_b . T_c = T_y$.

It suffices in (4) to take $T_a = n$; in (5) to take $T'_a = \mu$, $T_a = \nu$; in (6) to take $T'_a . T'_b . T'_c \dots = \mu$, $T_a . T_b . T_c \dots = \nu$; and the preceding solutions now apply.

11888. (Professor Zerr.)—A paraboloid floats in a liquid which fills a fixed paraboloidal shell; both the paraboloid and the shell have their axes vertical and their vertices downwards: the latus rectum of the paraboloid and shell are equal, and the axis of the shell is m times that of the paraboloid. If the paraboloid be pressed down until its vertex reaches the vertex of the shell, so that some of the liquid overflows, and then released, it is found that the paraboloid rises until it is just wholly out of the liquid, and then begins to fall. Prove that (1) the densities of the paraboloid and the liquid are in the ratio

$$2\left\{m^2 + m + 1 - (m+1)\sqrt{m^2 - 1}\right\} : 3\sqrt{(m+1)/(m-1)},$$

the free surface of the liquid being supposed to remain horizontal throughout the motion; and (2), if cone and conical be read for paraboloid and paraboloidal, the ratio is $3\left\{m^4-1-(m^3-1)\sqrt[3]{m^3-1}\right\} : 4\sqrt[3]{m^3-1}$, supposing the vertical angles of both equal.

Solution by the PROPOSER.

In the two positions in which the velocity of the paraboloid is zero, the heights of the centre of gravity of the paraboloid and liquid are equal. Let ρ be the density of the liquid, σ of the paraboloid, h the height of the paraboloid, x the height of the surface of the liquid in the second position. Then, if $y^2 = 4ax$ is the equation to the paraboloid, we get

$$2a\pi x^2 = 2a\pi m^2h^2 - 2a\pi h^2 = 2a\pi\,(m^2-1)\,h^2;$$

therefore $$x = \sqrt{m^2-1}\;h;$$

hence $$\tfrac{2}{3}mh\,.\,2a\pi m^2h^2\rho - \tfrac{2}{3}h\,.\,2a\pi h^2(\rho-\sigma)$$
$$= \tfrac{2}{3}\sqrt{m^2-1}\;h\,.\,2a\pi\,(m^2-1)\,h^2\rho + \left(\sqrt{m^2-1}+\tfrac{2}{3}\right)h\,.\,2a\pi h^2\sigma;$$
$$\tfrac{2}{3}m^3\rho - \tfrac{2}{3}(\rho-\sigma) = \tfrac{2}{3}\sqrt{(m^2-1)^3}\,\rho + \left(\sqrt{m^2-1}+\tfrac{2}{3}\right)\sigma;$$
$$2\left\{m^3-1-\sqrt{(m^2-1)^3}\right\}\rho = 3\sqrt{m^2-1}\;\sigma;$$

therefore $$2\left\{m^2+m+1-(m+1)\sqrt{m^2-1}\right\}\rho = 3\sqrt{(m+1)/(m-1)}\;\sigma;$$

$$\tfrac{1}{3}\pi x^3\tan^2\beta = \tfrac{1}{3}\pi m^3h^3\tan^2\beta - \tfrac{1}{3}\pi h^3\tan^2\beta; \quad 2\beta = \text{vertical angle};$$

therefore $x = \sqrt[3]{m^3-1}\;h$; hence

$$\tfrac{3}{4}mh\,.\,\tfrac{1}{3}\pi m^3h^3\tan^2\beta\,.\,\rho - \tfrac{3}{4}h\,.\,\tfrac{1}{3}\pi h^3\tan^2\beta\,(\rho-\sigma)$$
$$= \tfrac{3}{4}\sqrt[3]{m^3-1}\;h\,.\,\tfrac{1}{3}\left\{\pi\,(m^3-1)\,h^3\right\}\tan^2\beta\rho + \left(\sqrt[3]{m^3-1}+\tfrac{3}{4}\right)h\,.\,\tfrac{1}{3}\pi h^3\tan^2\beta\sigma;$$

therefore $$3\left\{m^4-1-(m^3-1)\sqrt[3]{m^3-1}\right\}\rho = 4\sqrt[3]{m^3-1}\;\sigma.$$

13663. (The late Professor WOLSTENHOLME, M.A., D.Sc.)—Normals to the parabola $y^2 = 4ax$ at the points P, Q, Q′ meet in a point, and QQ′ passes through the fixed point $(-c, 0)$: the envelope of the circle PQQ′ will be the pedal, with respect to the origin, of the parabola

$$4ay^2 = c^2(x-4a+c).$$

The origin (A) is a double focus of the envelope (which is a limaçon), and there are two single foci S_1, S_2 at the points $(-c, 0)$, $(c-c^2/4a, 0)$; the vector equation being, for the infinite branch,

$$(c-4a)\,.\,S_1P + 4a\,.\,S_2P = c\,.\,AP.$$

When $c = 4a$, the envelope is the pedal of the parabola itself with respect to the vertex (a cissoid), which is then a triple focus, and the vector equation becomes nugatory. When $c > 4a$, there is no oval branch. When $c = 8a$, the envelope becomes the straight line $x+4a = 0$, S_1, S_2 coincide, and the vector equation is $S_1P = AP$.

Solution by the PROPOSER.

The normal at (xy) passes through (XY) if

$$Y-y = -y/2a\,(X-y^2/4a);$$

so that, if y_1, y_2, y_3 be the roots of the equation,

$$y_1+y_2+y_3 = 0, \quad y_2y_3+y_3y_1+y_1y_2 = 4a\,(2a-X), \quad y_1y_2y_3 = 8a^2Y;$$

and the equation of QQ' is

$$4ax-y\,(y_2+y_3)+y_2y_3 = 0, \quad \text{whence} \quad y_2y_3 = 4ac.$$

Hence, if we put $y_2+y_3 = 2a\lambda$, $X = 2a-c+a\lambda^2$, $Y = -c\lambda$, and the equation of the circle PQQ',

$$x^2+y^2-x\,(2a+X)-\tfrac{1}{2}Yy = 0,$$

becomes $$x^2+y^2-x\,(4a-c+a\lambda^2)-\tfrac{1}{2}\lambda cy = 0;$$

so that the envelope is

$$c^2y^2+16ax\,(x^2+y^2-4ax+cx) = 0.$$

The equation of any tangent to the parabola $4ay^2 = c^2(x-4a+c)$ is

$$y = m\,(x-4a+c)+c^2/16am,$$

and the perpendicular from the origin is $x+my = 0$, whence the equation of the pedal is $$16ax\,(y^2+x^2-4ax+cx)+c^2y^2 = 0;$$

or coincides with the envelope. The straight line $x+iy = p$ will be a tangent if the equation

$$16ax\left\{(x-iy)\,p-(4a-c)\,x\right\}+c^2y^2 = 0$$

has equal roots giving the equation

$$16ac^2\,(p-4a+c)+64a^2p^2 = 0,$$

whose roots are $-c$, $c-c^2/4a$. The origin, being a node, is itself a double focus.

Also, at any point $P\,(x, y)$ on the curve,

$$x^2+y^2 = \frac{x^2\,(c-8a)^2}{c^2+16ax}, \quad (x+c)^2+y^2 = \frac{\left\{(c+8a)\,x+c^2\right\}^2}{c^2+16ax},$$

$$(x-c+c^2/4a)^2+y^2 = \frac{\left\{(3c-8a)\,x-c^2+c^3/4a\right\}^2}{c^2+16ax}.$$

Hence $$\frac{-AP}{x\,(8a-c)} = \frac{S_1P}{(c+8a)\,x+c^2} = \frac{-S_2P}{(3c-8a)\,x-c^2+c^3/4a};$$

the proper signs for the infinite branch being determined by putting $x = -c^2/16a$, which is the value at infinity on the curve. Thus we have the vector equation $$\lambda\,.\,S_1P+\mu\,.\,S_2P = \nu\,.\,AP,$$

where $\lambda\,(c+8a)-\mu\,(3c-8a)+\nu\,(8a-c) = 0$ and $\lambda = \mu\,(c/4a-1)$,

giving $$\lambda/(4a-c) = \mu/(-4a) = -\nu/c,$$

or $$(4a-c)\,S_1P-4a\,.\,S_2P+c\,.\,AP = 0.$$

This will be the vector equation for all values of c if we suppose that when $c<4a$, and P is on the loop, AP shall be counted negative. When $c = 8a$, the origin is the focus of the parabola

$$4ay^2 = c^2\,(x-4a+c) \quad \text{or} \quad y^2 = 16a\,(x+4a),$$

and the pedal is the tangent at the vertex $x = -4a$. The vector equation becomes $S_1P + S_2P = 2AP$, but, S_1, S_2 being coincident, this is equivalent to $S_1P = AP$, the straight line bisecting the distance between S_1, A at right angles.

14184. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{2^3} + \frac{1}{3^3-1} + \frac{2}{5^3+1} + \frac{3}{8^3-2} + \frac{5}{13^3+3} + \dots = \frac{17}{90},$$

$$\frac{1}{2\,.\,3} + \frac{7}{5\,.\,8} + \frac{57}{13\,.\,21} + \frac{285}{34\,.\,55} + \dots = 1,$$

$$\frac{7}{3\,.\,5} + \frac{31}{8\,.\,13} + \frac{183}{21\,.\,34} + \frac{835}{55\,.\,89} + \dots = \frac{3}{2}.$$

Solution by Professor Sanjána, M.A.

The first series

$$= \left(\frac{1}{8} + \frac{2}{126} + \frac{5}{2200} + \frac{13}{39312} + \dots\right) + \left(\frac{1}{26} + \frac{3}{510} + \frac{8}{9256} + \dots\right).$$

The first part

$$= \frac{1}{1\,.\,8} + \frac{1}{8\,.\,55} + \frac{1}{55\,.\,377} + \dots + \frac{1}{3\,.\,21} + \frac{1}{21\,.\,144} + \frac{1}{144\,.\,987} + \dots$$

$$= \frac{1}{8-}\,\frac{1}{7-}\,\frac{1}{7-}\dots + \frac{1}{63-}\,\frac{3^2}{7-}\,\frac{1}{7-}\ \text{(by the usual formulæ)}$$

$$= \frac{1}{8-\frac{1}{2}(7-3\sqrt{5})} + \frac{1}{63-\frac{3^2}{2}(7-3\sqrt{5})} = \frac{3-\sqrt{5}}{6} + \frac{7-3\sqrt{5}}{18}$$

$$= \frac{8}{9} - \frac{\sqrt{5}}{3};$$

and the second part

$$= \frac{1}{2\,.\,13} + \frac{1}{13\,.\,89} + \frac{1}{89\,.\,610} + \dots + \frac{1}{5\,.\,34} + \frac{1}{34\,.\,233} + \dots$$

$$= \frac{1}{26-2(7-3\sqrt{5})} + \frac{1}{170-\frac{25}{2}(7-3\sqrt{5})} = \frac{\sqrt{5}-2}{6} + \frac{5\sqrt{5}-11}{30}$$

$$= \frac{\sqrt{5}-2}{6} + \frac{5\sqrt{5}-11}{30} = \frac{\sqrt{5}}{3} - \frac{7}{10}.$$

Thus the whole series

$$= \frac{8}{9} - \frac{7}{10} = \frac{17}{90}.$$

The second series

$$= \frac{3-2}{2.3} + \frac{12-5}{5.8} + \frac{70-13}{13.21} + \frac{319-34}{34.55} + \dots$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{3}{2.5} - \frac{1}{8} + \frac{10}{3.13} - \frac{1}{21} + \frac{29}{5.34} - \frac{1}{55} + \dots$$

$$= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} - \frac{1}{8} + \frac{1}{3} - \frac{1}{13} - \frac{1}{21} + \frac{1}{5} - \frac{1}{34} - \frac{1}{55} + \dots = 1.$$

The third series

$$= \frac{10-3}{3.5} + \frac{39-8}{8.13} + \frac{204-21}{21.34} + \frac{890-55}{55.89} + \dots$$

$$= \frac{2}{3} - \frac{1}{5} + \frac{3}{8} - \frac{1}{13} + \frac{6}{21} - \frac{1}{34} + \frac{10}{55} - \frac{1}{89} + \dots$$

$$= 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{2} - \frac{1}{8} - \frac{1}{13} + \frac{1}{3} - \frac{1}{21} - \frac{1}{34} + \frac{1}{5} - \frac{1}{55} - \frac{1}{89} + \dots$$

$$= 1 + \frac{1}{2} = \frac{3}{2}.$$

[Mr. G. D. Wilson, B.A., sends the following interesting method of summing the first of the above series:—Denoting the series 1, 1, 2, 3, 5, ... by $v_1, v_2, \dots v_n, \dots$, the law of formation being $v_{n+1} = v_n + v_{n-1}$, we have the relations

$$v_{n-1} v_{n+1} - v_n^2 = (-1)^n \quad \dots\dots \quad \text{(i.)},$$

and

$$v_n v_{n+5} - v_{n+1} v_{n+4} = (-1)^{n+1} 3 \quad \dots\dots \quad \text{(ii.)}.$$

From (i.) we deduce that

$$v_{n+2}^3 - (-1)^n v_{n-1} = v_n v_{n+1} v_{n+5} \quad \dots\dots \quad \text{(iii.)}.$$

From (ii.) and (iii.),

$$\sum_{n=1}^{\infty} \frac{v^n}{v_{n+2}^3 - (-1)^n v_{n-1}} = \tfrac{1}{3} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1} v_n}{v_{n+1}} - \frac{(-1)^{n+5} v_{n+4}}{v_{n+5}} \right\}$$

$$= \tfrac{1}{3} \sum_{n=1}^{4} \frac{(-1)^{n+1} v_n}{v_{n+1}} = \tfrac{17}{90}].$$

14301. (J. J. Barniville, B.A.)—Sum the series

$$\frac{1}{1+3^2} + \frac{1}{2+4^2} + \frac{1}{3+5^2} + \dots,$$

$$\frac{1}{2+4^2} + \frac{1}{5+7^2} + \frac{1}{8+10^2} + \dots,$$

$$\frac{1}{2^2+3} + \frac{1}{3^2+7} + \frac{1}{4^2+11} + \dots,$$

$$\frac{1}{5^2+7} + \frac{1}{9^2+15} + \frac{1}{13^2+23} + \dots.$$

Solution by G. D. Wilson, B.A.; Professor Sanjána; *and others*.

$$\frac{1}{1+3^2}+\frac{1}{2+4^2}+\frac{1}{3+5^2}+\ldots=\frac{1}{3}\sum_1^\infty\left(\frac{1}{n+1}-\frac{1}{n+4}\right)=\frac{13}{36},$$

$$\frac{1}{2+4^2}+\frac{1}{5+7^2}+\frac{1}{8+10^2}+\ldots=\frac{1}{9}\sum_1^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{9},$$

$$\frac{1}{2^2+3}+\frac{1}{3^2+7}+\frac{1}{4^2+11}+\ldots=\frac{1}{6}\sum_1^\infty\left(\frac{1}{n}-\frac{1}{n+6}\right)=\frac{49}{120},$$

$$\frac{1}{5^2+7}+\frac{1}{9^2+15}+\frac{1}{13^2+23}+\ldots=\frac{1}{16}\sum_1^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{16}.$$

11830. (Professor Lebon.) — Deux plans parallèles à un cercle commun à plusieurs sphères et équidistants de ce cercle déterminent dans les sphères qu'ils coupent des segments équivalents.

Solution by H. W. Curjel, M.A.

Let h be the distance of each of the planes from the common section (radius a). Consider the sphere whose radius is R and whose centre is at distances x_1, x_2 from the two planes. Then

$$R^2=\left\{\tfrac{1}{2}(x_1+x_2)\right\}^2+a^2,\quad x_1-x_2=2h.$$

Therefore volume between the planes

$$=\tfrac{1}{6}\pi(x_1-x_2)\left\{3R^2-x_1^2-x_1x_2-x_2^2\right\}$$
$$=\tfrac{1}{3}\pi h\left\{3a^2+\tfrac{1}{4}(3x_1^2+6x_1x_2+3x_2^2-4x_1^2-4x_1x_2-4x_2^2)\right\}$$
$$=\tfrac{1}{3}\pi h(3a^2-h^2)=\text{constant}.$$

14158. (Lt.-Col. Allan Cunningham, R.E.) — Express $(3.2^{33}+1)$ in one or more of the forms $(c^2 \pm 2d^2)$, $(A^2 \pm 3B^2)$; or show that it does not admit of this expression.

Solution by the Proposer.

This number has been recently resolved by the Rev. J. Cullen into two *prime* factors: $N = 1893029.13613$.

These, being both of form $p = 24\varpi+5$, *cannot* be expressed in any of the proposed forms; so that their product (N) is also not so expressible.

14400. (R. F. Davis, M.A.)—Find positive integral values for N, x, y which will render N^2-3x^2, N^2-3y^2, $N^2-3(x+y)^2$, $N^2-3(x-y)^2$ perfect squares. [A special Christmas puzzle.]

Solution by the PROPOSER.

The following values satisfy the conditions:—

$$266^2-3.88^2 = 218^2, \qquad 266^2-3.153^2 = 23^2,$$
$$266^2-3.65^2 = 241^2, \qquad 266^2-3.\ 23^2 = 263^2.$$

14367. (Professor N. BHATTACHARYYA.)—Show that the product of three numbers representing the sides of a right-angled triangle is divisible by 60.

Solution by H. W. CURJEL, M.A.; *and many others.*

The numbers are of the form m^2-n^2, m^2+n^2, $2mn$. Their product is divisible by 4, since at least one of m, n, m^2-n^2 is divisible by 2. And $mn(m^2-n^2)$ is divisible by 3, since every square number $\equiv 1$ or 0 mod 3, and the produce of the three numbers is divisible by 5, since every square number $\equiv \pm 1$ or 0 mod 5. Therefore product is divisible by 60.

[Mr. PARANJPYE and Mr. MUGGERIDGE direct us to a solution of this Question in SMITH's *Algebra*, § 384, p. 489. The Question also appears in TODHUNTER's *Algebra* (1879) among the "Miscellaneous Examples," No. 268.]

14515. (J. A. THIRD, M.A., D.Sc.)—X, Y, Z are three points in the plane of a triangle ABC, such that the pairs AY and AZ, BZ and BX, CX and CY are equally inclined to the bisectors of the angles A, B, C respectively. Y moves on the straight line u_b, and Z on the straight line u_c. Prove the following statements:—(1) the locus of X is a straight line u_a; (2) if u_b pass through B, and u_c through C, u_a passes through A; (3) if u_b be perpendicular to CA, and u_c to AB, u_a is perpendicular to BC; (4) if L, M, N be the points where u_a, u_b, u_c meet BC, CA, AB respectively, AL, BM, CN meet in a point P; (5) AX, BY, CZ are concurrent in a point whose locus is, in general, a conic circumscribed to the triangle and passing through P; (6) if u_b, u_c meet on the cubic circumscribed to the triangle, and passing through every pair of isogonal points whose join passes through P, viz.,

$$l\,(y^2-z^2)/yz + m\,(z^2-x^2)/zx + n\,(x^2-y^2)/xy = 0,$$

where l, m, n are the coordinates of P, u_a, u_b, u_c are concurrent.

Solution by G. N. BATES, B.A.

Let the equations of AY, BZ, and CX be $y-\kappa z = 0$, $z-\lambda x = 0$, and $x-\mu y = 0$; $\quad u^b \equiv l_1x+m_1y+n_1z = 0$, $\quad u_c \equiv l_2x+my+n_2z$.

Then X is the point $(\lambda\mu, \lambda, \mu)$, Y is $(\kappa, \mu\kappa, \mu)$, and Z $(\kappa, \lambda, \kappa\lambda)$.

The locus of X is found by eliminating between

$$\left.\begin{aligned} l_1\kappa + m_1\kappa\mu + n_1\mu &= 0 \\ l_2\kappa + m_2\lambda + n_2\lambda\kappa &= 0 \end{aligned}\right\} \quad \text{(i.)}$$

and $\qquad x/y = \mu, \quad x/z = \lambda;$

i.e., $\qquad (m_1m_2 - n_1n_2)\,x + l_1m_2y - l_2n_1z = 0 \quad \text{(1).}$

If $m_1 = 0$ and $n_2 = 0$, this reduces to a straight line through A ... (2).

If $\quad m_1 = l_1 \cos C + n_1 \cos A \quad$ and $\quad n_2 = l_2 \cos B + m_2 \cos A$,

(1) reduces to $(l_1m_2 \cos C - n_1l_2 \cos B)\,x + l_1m_2y - l_2n_1z = 0$,

or $(l_1m_2 \cos C - n_1l_2 \cos B)(ax + by + cz) + (cl_1m_2 - bn_1l_2)(y \cos B - z \cos C) = 0$,

a straight line perpendicular to BC .... (3).

AL is $\quad l_1m_2y - l_2n_1z = 0, \quad$ *i.e.*, $\quad l_1(l_2x + m_2y) - l_2(l_1x + n_1z) = 0$,

a straight line through the intersections of BM and CN, viz., the point $(m_2n_1, -l_2n_1, -l_1m_2)$ .... (4).

AX, BY, CZ meet in the point (κ, λ, μ). The locus of this point is obtained by eliminating between the equations (i.) and $x/\kappa = y/\lambda = z/\mu$,

giving $\qquad n_2y\,(l_1x + n_1z) = m_1z\,(l_2x + m_2y),$

a circumconic passing through the intersection of

$$l_1x + n_1z = 0 \quad \text{and} \quad l_2x + m_2y = 0, \quad \textit{i.e.}, \text{ P} \quad \text{(5).}$$

u_a, u_b, u_c will be concurrent if

$$l_1x + m_1y + n_1z = 0, \quad l_2x + m_2y + n_2z = 0,$$
$$(m_1m_2 - n_1n_2)\,x + l_1m_2y - l_2n_1z = 0,$$

i.e., the point of intersection will be on the curve obtained by eliminating between these equations and $l/m_2n_1 = m/-l_2n_1 = n/-l_1m_2$,

i.e., $\qquad \left\{-m_2(l_1x + n_1z)/y + n_1(l_2x + m_2y)/z\right\} x + l_1m_2y - l_2n_1z = 0,$

or $\qquad l(y^2 - z^2)/yz + m(z^2 - x^2)/zx + n(x^2 - y^2)/xy = 0 \quad \text{(6).}$

14518. (Professor A. GOLDENBERG.)—Résoudre le système

$$(x + 2y)(x + 2z) = a^2, \quad (y + 2z)(y + 2x) = b^2, \quad (z + 2x)(z + 2y) = c^2.$$

Solution by JAMES BLAIKIE, M.A.; Rev. T. ROACH, M.A.; *and many others.*

$$(x + 2y)(x + 2z) = \left\{(x + y + z) + (y - z)\right\}\left\{(x + y + z) - (y - z)\right\}$$
$$= (x + y + z)^2 - (y - z)^2.$$

Thus, if we write s for $x + y + z$, the system becomes

$$y - z = \pm\sqrt{(s^2 - a^2)}, \quad z - x = \pm\sqrt{(s^2 - b^2)}, \quad x - y = \pm\sqrt{(s^2 - c^2)},$$

whence $\qquad \Sigma\left\{\pm\sqrt{(s^2 - a^2)}\right\} = 0;$

solving, we find $s^2 = \frac{1}{3}(a^2 + b^2 + c^2) \pm \frac{2}{3}\sqrt{(a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2)}$.

Also $3x = s-(z-x)+(x-y) = s \mp \sqrt{(s^2-b^2)} \pm \sqrt{(s^2-c^2)}$,
with similar values for $3y$ and $3z$.

E.g. $a = 1,\ b = 2,\ c = 3$;

$$x = \tfrac{2}{3}\sqrt{21} \pm \tfrac{1}{3}\sqrt{3},\quad y = \tfrac{2}{3}\sqrt{21} \pm \tfrac{2}{3}\sqrt{3},\quad z = \tfrac{2}{3}\sqrt{21} \mp \sqrt{3}.$$

[Professor Frank Morley, of Haverford College, U.S.A., refers to Vol. xxxii., p. 90, of the *Reprint*, where this Question (as 6028), having been set by himself (wrongly called T. Morley), is solved by the present Proposer.]

14482. (Professor Neuberg.)—Soient a, b, c, d les côtés AB, BC, CD, DA d'un quadrilatère sphérique ABCD circonscrit à un petit cercle. Démontrer la relation $\sin a \sin b \sin^2 \frac{1}{2}B = \sin c \sin d \sin^2 \frac{1}{2}D$.

Solution by Professor Sanjána *and* H. W. Curjel, M.A.

Since the quadrilateral is circumscribed to a small circle $a-b = d-c$,
therefore $\cos a \cos b + \sin a \sin b = \cos c \cos d + \sin c \sin d$;
but

$$\cos a \cos b + \sin a \sin b \cos B = \cos AC = \cos c \cos d + \sin c \sin d \cos D.$$

Hence $\sin a \sin b \sin^2 \frac{1}{2}B = \sin c \sin d \sin^2 \frac{1}{2}D$.

14528. (R. P. Paranjpye, B.A.)—Show that any triangle can be projected into an equilateral triangle whose centre of gravity is the projection of a given point.

Solution by G. D. Wilson, B.A.

Let ABC be the given triangle, P the given point. Let AP meet BC in A′. Take X so that (BA′CX) = −1, and similar constructions for Y and Z. Then X, Y, Z lie on a straight line. Project this line to infinity, and two of the angles of the triangle into angles of 60°; the projection of ABC is then an equilateral triangle, and has the projection of P for its centre of gravity.

14508. (W. H. Salmon, B.A.)—The frustum of a pyramid with quadrilateral base is such that the intersections of the opposite faces are coplanar (A); prove that (1) the diagonals of the frustum are concurrent (O); (2) each diagonal of the frustum is divided harmonically by O and its point of intersection with A; (3) the diagonals of each face are divided harmonically by their point of intersection and the plane A.

I. *Solution by the* Proposer.

The principle of continuity will allow us to project figures in three dimensions as in two dimensions. Hence in this problem the plane A may be projected to infinity, and the problem is reduced to: The diagonals of a parallelopiped are concurrent, and bisect each other; and the diagonals of the faces bisect each other.

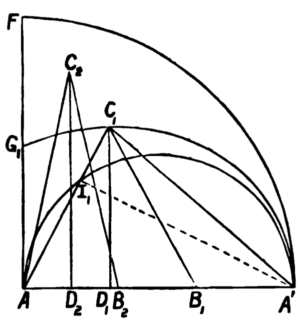
II. *Solution by* H. W. Curjel, M.A.

Let G be the vertex of the pyramid, and BCDE the base, and B′C′D′E′ the opposite face; GB′B, GC′C, GD′D, GE′E being straight lines. Since the opposite faces cut in a plane A, any two pairs of opposite faces cut in a point on A; *i.e.*, B′C′, ED, E′D′, BC meet in a point K on A, and B′E′, C′D′, BE, CD in a point H on A. Therefore the intersection of B′D, C′E is on the line joining G to the intersection of BD, CE, and B′D, EC′ cut on the same straight line; therefore C′E, B′D, EC′ meet in a point O.

Similarly, BD′ passes through O. B′E, C′D cut in M on the line of intersection of BB′E and CC′D; therefore M is in A, and therefore MK is in A. Therefore C′E, B′D are divided harmonically by O and A. Also B′E, B′E are divided harmonically by their point of intersection and GH.

6648. (H. Fortey, M.A.)—In a plane triangle ABC, bisect AB in D, and take DBA′ opposite to DA, or DAB′ opposite to DB, each equal to half the sum of AC and BC; and prove that the semi-perimeter AA′ or BB′ will subtend an acute angle at C if the base AB does not exceed half the sum of AC and BC.

Solution by D. Biddle.



It will be convenient to take AA′, the semi-perimeter of ABC, as unity, and on it describe a semicircle. Then, if we divide AA′ into three equal parts by B_1, D_1, and on AB_1 construct the equilateral triangle AB_1C_1, we have

$$\tfrac{1}{2}(AC_1 + B_1C_1) = D_1A' = AB,$$

which (as base) is thus at the limit stated. Now,

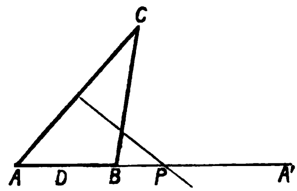
$$D_1A' = \tfrac{2}{3} \quad\text{and}\quad D_1C_1 = 1/\sqrt{3}.$$

Therefore $AA'.D_1A' = 2D_1C_1^2$,

and the circle on AA′ as diameter, namely, AI_1A', is the circle of curvature at A′ to the ellipse of which D_1A', D_1C_1 are the semiaxes. Consequently, of ellipses having one focus at A and the other in AA′, and whose major axes terminate at A′, $A'C_1G_1$ is the ellipse of greatest eccentricity, capable of touching the circle AI_1A' at A′ and wholly enclosing it. But this ellipse is the locus of C_1 when $\tfrac{1}{2}(AC_1 + B_1C_1) = D_1A'$, and, since in every position C_1 is outside the circle, therefore $\angle AC_1A'$ is acute. Taking $AB_2 < AB_1$, and making $AC_2 = B_2C_2 = D_2A'$, we can describe the ellipse of which

D_2A', D_2C_2 are the semiaxes, as the locus of C_2, and so on, until the foci coincide in A, and the ellipse becomes a circle. All these positions of C are clearly external to the circle AI_1A'; therefore, within the limits stated, $\angle ACA'$ is always acute.

[Mr. G. D. Wilson, B.A., sends the following trigonometrical solution of this Question:— Let the straight line bisecting AC at right angles meet AB in P. Then $\angle ACA'$ is acute if $AP > \frac{1}{2}AA'$, *i.e.*, if $b \sec A > s$.



This condition becomes, after some reduction, $\sin A > \tan \frac{1}{2}C$,

or $\quad 2s(s-c) > bc$

$$> sc - (s-b)\,c$$

or $\quad (s-b)\,c > s\,(3c-2s)$

or $\quad (s-b)\,c > 2s\left\{c - \frac{1}{2}(a+b)\right\}.$

The angle ACA' is therefore certainly acute if $c < \frac{1}{2}(a+b)$, *i.e.*, if $AB < DA'$.]

14447. (H. W. Curjel, M.A.)—If $f(x)$ is finite and continuous for all positive finite values of x except a finite number of values, then

$$\int_0^\infty \sin\{f(x)\}\,dx \quad \text{and} \quad \int_0^\infty \cos\{f(x)\}\,dx$$

are convergent or divergent according as $\lim_{x=\infty} \frac{df(x)}{dx}$ is infinite or finite; except in the case where $\lim_{x=\infty} f(x) = 0$ or $n\pi$, when $\int_0^\infty \sin\{f(x)\}\,dx$ may be convergent, and the case where $\lim_{x=\infty} f(x) = \frac{1}{2}\pi$ or $(2n+1)\frac{1}{2}\pi$, when $\int_0^\infty \cos\{f(x)\}\,dx$ may be convergent.

Solution by the Proposer.

First consider the case where $\lim_{x=\infty} \frac{df(x)}{dx}$ is infinite. Here it is obvious that to any assigned small quantity ϵ there corresponds a finite quantity m such that $f(x+\epsilon) - f(x) > \pi$ if $x > m$, and that the smaller ϵ is the greater the least value of m is. Hence, if $x > m$, as x increases $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\{f(x)\}$ changes signs at intervals which continually decrease and the greatest of which is less than ϵ. Hence $\left|\int_m^{m+n} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\{f(x)\}\,dx\right| <$ a series of terms whose signs are alternately positive and negative, of which the first and second are less than ϵ, and the third, fourth, fifth, &c., are each less than the preceding one; therefore

$$\left|\int_m^{m+n} \frac{\sin}{\cos}\left\{f(x)\right\} dx\right| < \epsilon;$$

therefore the integral is convergent.

Again, if $\lim_{x=\infty} \frac{df(x)}{dx} = a$, a finite quantity not zero, then the interval between the changes of sign of $\frac{\sin}{\cos}\left\{f(x)\right\}$ is always finite, and the integral can be reduced to a series of terms which are alternatively positive and negative, but never become indefinitely small; hence the integral is clearly divergent. This reasoning can be applied *a fortiori* to the case where $\lim_{x=\infty} \frac{df(x)}{dx} = 0$, except in the cases where $\lim_{x=\infty} \frac{\sin}{\cos}\left\{f(x)\right\} = 0$, *i.e.*, when $\lim_{x=\infty} f(x) = \begin{cases} -0 \text{ or } n\pi \\ \frac{1}{2}\pi \text{ or } n\pi + \frac{1}{2}\pi \end{cases}$, when the convergency of the integral clearly depends on the form of the function $f(x)$.

Note on Quest. 14447. *By* G. H. Hardy, B.A.

Mr. Curjel's solution does not seem to me to be entirely satisfactory. "The smaller ϵ is, the greater the least value of m is"; but it does not follow, so far as I can see, that, when m is fixed, the intervals between the points a_i at which $\sin f(x)$ changes sign in (m, ∞) will diminish continually as i increases, though they will certainly all be less than ϵ; and, even if they did, it would not follow that $\left|\int_{a_i}^{a_{i+1}} \sin\left\{f(x)\right\} dx\right|$ diminished continually as i increased. And, in fact, the theorem is untrue.

For, if $\frac{df(x)}{dx}$ be continuous and > 0 for all finite values of x,

$$\int^{H} \sin\left\{f(x)\right\} dx = \int^{f(H)} \sin y \frac{dy}{f'(x)} \qquad \left\{y = f(x)\right\}.$$

Now suppose $\qquad f'(x) = y^{\mu} - a \sin y \quad (0 < \mu < \frac{1}{2});$

i.e.,
$$x = \int_0^y \frac{dy}{y^{\mu} - a \sin y} \qquad \left\{y = f(x)\right\}.$$

Then $\int_0^\infty \sin\left\{f(x)\right\} dx$ converges or diverges with

$$\int_0^\infty \frac{\sin y\, dy}{y^{\mu} - a \sin y} \qquad (0 < \mu < \tfrac{1}{2}),$$

as x, y become infinite together. But it is not difficult to show that this integral *diverges*. It would take too long to enter into the proof at present; but it is easy to see the reason. The oscillations of the denominator *coincide in phase* with those of the numerator; so that, to speak roughly, all the positive elements are increased by the presence of the oscillating term in the denominator, and all the negative elements decreased.

The second part of Mr. CURJEL's theorem is true. But the integral may be determinate if $f'(x)$ does not tend to a limit at all, for $x = \infty$. But it *is* a sufficient condition that $f'(x)$ tend to ∞ *steadily; i.e.*, without oscillations.

[Mr. CURJEL having seen the above, and having been requested to send any further remarks he might deem requisite, says: "I have nothing to add, as Mr. HARDY's criticism is quite correct; my proof only applies when $f'(x)$ increases *steadily* to ∞, and consequently the theorem to a comparatively limited class of functions; *i.e.*, the condition given in the Question is not by itself sufficient for convergence, but that for divergence is sufficient."]

14509. (I. ARNOLD.)—Given two circles, one within the other, a point can be found such that the extreme portions of any right line cutting both circles shall subtend equal angles at the point.

Solution by J. H. TAYLOR.

Proof by Coaxial Circles.—PAP′, QBQ′, the given circles, are cut by chord PQQ′P′. Let O and C be the limiting points of the system of coaxial circles (Fig. 1) to which the given circles belong, and let another circle of the system touch PP′ in M. Then, since the last circle, the limiting point C, and the circle PAP′ are circles of the same coaxial system,

$$PC : PM = P'C : P'M.$$

(*Pitt Press Euclid*, p. 481.)

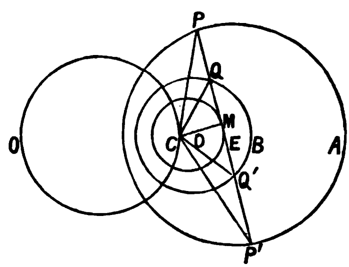


FIG. 1.

Therefore CM bisects $\angle PCP'$.

Similarly $QC : QM = Q'C : Q'M$. Therefore CM bisects $\angle QCQ'$. Therefore $\angle PCQ = P'CQ'$.

Similarly it may be proved that PQ, P′Q′ subtend equal angles at the other limiting point O.

If a point be taken within the circle QBQ′, and a chord PQQ′P′ cutting both circles be made to revolve round this point, the segments PQ, P′Q′ always subtend equal angles at both the limiting points.

Proof by Inversion.—Let PAP′, QBQ′ (Fig. 2) be the given circles cut by the chord PQQ′P′, and the circle OCF, with its centre on their common diameter, cut them both orthogonally.

[The points O, C, D, and the radii vectores in Figs. 2 and 3, are identical; but, for the sake of clearness, the inverses are drawn apart in Fig. 3.]

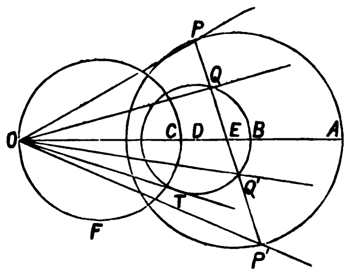


FIG. 2.

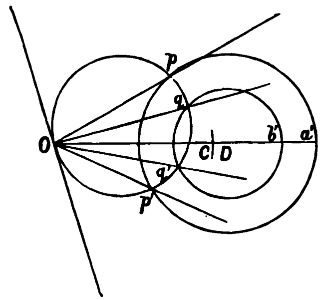


FIG. 3.

Invert with O as pole and OT as radius of inversion; then circle QBQ inverts into itself; circle PAP' inverts into a concentric circle.* The line PP' inverts into circle pqp', touching at the pole a line parallel to PP'. Since this circle cuts concentric circles, the arcs pq, $p'q'$ are equal. Hence the angles pOq, $p'Oq'$ are equal, and these are identical with the angles POQ, P'OQ'.

Similarly, it may be shown that the segments PQ, P'Q' subtend equal angles at C.

[Mr. H. W. CURJEL, M.A., solves the Question as follows:—

Let AB be a straight line meeting the two circles in AB, CD; the locus of points at which AC, BD subtend equal angles is the circle on E, F as diameter, where E, F are the common harmonic conjugates of AB, CD. This circle cuts the two given circles orthogonally, and therefore passes through the limiting points of the two given circles, either of which is the point P required.

The PROPOSER remarks as follows:—If PQQ'P' be produced to cut the radical axis of the two circles in R, and a tangent be drawn to either of the circles from R, then with R as centre and the tangent as radius a circle be described, it will cut the line joining the centres or its production in two points either of which fulfils the conditions of the Question. The points thus formed are the foci of involution. Of course they will correspond with C and O.]

14526. (R. C. ARCHIBALD, M.A.)—With reference to the centre of the fixed circle, the corresponding tangent and normal pedal curves (positive or negative) of the cardioid are similar.

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* Because the circle OCF, which cuts the given circles orthogonally, inverts into a straight line cutting OD at right angles; and a straight line cannot cut two circles, whose centres lie in a line at right angles to it, both orthogonally, unless those centres coincide. (*Pitt Press Euclid*, p. 461.)

Solution by the PROPOSER; J. H. TAYLOR, M.A.; *and others.*

This theorem is evident, from the fact that the centre of the fixed circle of a cardioid is also the centre of the fixed circles of all its evolutes.

Note.—The *first* positive tangent and normal pedal curves of a cardioid, with respect to the centre of its base, are similar to the cardioid's *radial* curve. (*Cf.* R. TUCKER, *Ed. Times*, Feb., 1863; *Ed. Times Reprint*, 1864, I., v., 16; 1865, II., 28.) If a be the radius of the base, the polar equation of the tangential pedal may be written $r = 3a\cos\frac{1}{3}\theta$; of the normal pedal $r = a\cos\frac{1}{3}(\theta-\pi)$; and of the radial curve, $r = \frac{1}{3}8a\cos\frac{1}{3}\theta$.

14344. (J. J. BARNIVILLE, B.A.)—Having $u_{n-1}+u_{n+1} = 4u_n$, prove that

$$\frac{1}{1+1}+\frac{1}{3+1}+\frac{1}{11+1}+\frac{1}{41+1}+\dots=\frac{\sqrt{3}}{2},$$

$$\frac{1}{1+3}+\frac{1}{3+3}+\frac{1}{11+3}+\frac{1}{41+3}+\dots=\frac{3\sqrt{3}}{10},$$

$$\frac{1}{1+11}+\frac{1}{3+11}+\frac{1}{11+11}+\frac{1}{41+11}+\dots=\frac{5\sqrt{3}}{38},$$

$$\frac{1}{2+\sqrt{6}}+\frac{1}{4+\sqrt{6}}+\frac{1}{14+\sqrt{6}}+\dots=\frac{\sqrt{2}-1}{\sqrt{3}-1},$$

$$\frac{1}{1+2}-\frac{1}{2+2}+\frac{1}{7+2}-\frac{1}{26+2}+\dots=\frac{1}{6},$$

$$\frac{1}{1+1}-\frac{1}{5-1}+\frac{1}{19+1}-\frac{1}{91-1}+\dots=\frac{1}{2\sqrt{3}}.$$

Solution by Professor SANJÁNA, M.A.

The first series

$$=\frac{1}{2}+\frac{1}{1.4}+\frac{1}{3.4}+\frac{1}{3.14}+\frac{1}{11.14}+\frac{1}{11.52}+\dots$$

$$=\frac{1}{2}+\frac{1}{4-}\,\frac{4^2}{4.4-}\,\frac{3^2.4^2}{3.3.6-}\,\frac{3^2.14^2}{14.14-}\,\frac{11^2.14^2}{11.11.6-}\dots$$

$$=\frac{1}{2}+\frac{1}{4-}\,\frac{1}{1-}\,\frac{1}{6-}\,\frac{1}{1-}\,\frac{1}{6-}\dots=\frac{1}{2}+\frac{1}{4-(3-\sqrt{3})}=\frac{1}{2}+\frac{1}{\sqrt{3}+1}=\frac{\sqrt{3}}{2}.$$

The second

$$=\frac{1}{4}+\frac{1}{44}+\frac{1}{2134}+\frac{1}{110774}\dots+\frac{1}{6}+\frac{1}{156}+\frac{1}{7956}\dots+\frac{1}{14}+\frac{1}{574}+\dots$$

$$=\frac{1}{1.4}+\frac{1}{4.11}+\frac{1}{11.194}+\frac{1}{194.571}+\dots+\frac{1}{2.3}+\frac{1}{3.52}+\frac{1}{52.153}+\dots$$

$$\dots+\frac{1}{1.14}+\frac{1}{14.41}+\frac{1}{41.724}+\dots$$

$$=\frac{1}{4-}\,\frac{4^2}{4^2.3-}\,\frac{4^2.11^2}{11^2.18-}\,\frac{11^2.194^2}{194^2.3-}\cdots+\frac{1}{6-}\,\frac{2^2.3^2}{3^2.18-}\,\frac{3^2.52^2}{52^2.3-}\cdots$$

$$\cdots+\frac{1}{14-}\,\frac{14^2}{14^2.3-}\,\frac{14^2.41^2}{41^2.18-}\cdots$$

$$=\frac{1}{4-}\,\frac{1}{3-}\,\frac{1}{18-}\,\frac{1}{3-}\cdots+\frac{1}{6-}\,\frac{4}{18-}\,\frac{1}{3-}\,\frac{1}{18-}\,\frac{1}{3-}\cdots+\frac{1}{14-}\,\frac{1}{3-}\,\frac{1}{18-}\,\frac{1}{3-}\cdots$$

$$=\frac{1}{4-(9-5\sqrt{3})}+\frac{1}{6-}\,\frac{4}{18-(9-5\sqrt{3})}+\frac{1}{14-(9-5\sqrt{3})}$$

$$=\frac{1}{5(\sqrt{3}-1)}+\frac{3}{10\sqrt{3}}+\frac{1}{5(\sqrt{3}+1)}=\frac{3\sqrt{3}}{10}.$$

The third series

$$\frac{1}{12}+\frac{1}{14}+\frac{1}{22}+\frac{1}{52}+\frac{1}{164}+\frac{1}{582}+\frac{1}{2142}+\frac{1}{7964}+\frac{1}{29692}+\frac{1}{110782}+\frac{1}{413414}+\cdots$$

may be written as the sum of five series

$$\frac{1}{12}+\frac{1}{582}+\frac{1}{413414}+\cdots+\frac{1}{14}+\frac{1}{2142}+\cdots+\frac{1}{22}+\frac{1}{7964}+\cdots$$

$$+\frac{1}{52}+\frac{1}{29692}+\cdots+\frac{1}{164}+\frac{1}{110782}+\cdots$$

$$=\frac{1}{3.4-}\,\frac{3^2.4^2}{3.3.66-}\,\frac{3^2.194^2}{194.194.11-}\cdots+\frac{1}{14-}\,\frac{14^2}{14.14.11-}\,\frac{14^2.153^2}{153.153.66-}\cdots$$

$$\cdots+\frac{1}{2.11-}\,\frac{2^2.11^2}{11.11.66-}\cdots+\frac{1}{1.52-}\,\frac{52^2}{52.52.11-}\cdots$$

$$\cdots+\frac{1}{4.41-}\,\frac{4^2.41^2}{41.41.66-}\cdots$$

$$=\frac{1}{12-}\,\frac{16}{66-}\,\frac{1}{11-}\,\frac{1}{66-}\cdots+\frac{1}{14-}\,\frac{1}{11-}\,\frac{1}{66-}\,\frac{1}{11-}\cdots+\frac{1}{22-}\,\frac{4}{66-}\,\frac{1}{11-}\cdots$$

$$\cdots+\frac{1}{52-}\,\frac{1}{11-}\,\frac{1}{66-}\cdots+\frac{1}{164-}\,\frac{16}{66-}\,\frac{1}{11-}\cdots$$

$$=\frac{1}{12-16(33-19\sqrt{3})/6}+\frac{1}{14-(33-19\sqrt{3})}+\frac{1}{22-4(33-19\sqrt{3})/6}$$

$$+\frac{1}{52-(33-19\sqrt{3})}+\frac{1}{164-16(33-19\sqrt{3})/6}$$

$$=\frac{3}{76(2\sqrt{3}-3)}+\frac{1}{19(\sqrt{3}-1)}+\frac{3}{38\sqrt{3}}+\frac{1}{19(\sqrt{3}+1)}+\frac{3}{76(2\sqrt{3}+3)}$$

$$=\frac{5\sqrt{3}}{38}.$$

The fourth series

$$= \frac{\sqrt{6}-2}{2} + \frac{4-\sqrt{6}}{10} + \frac{14-\sqrt{6}}{190} + \frac{52-\sqrt{6}}{2698} + \frac{194-\sqrt{6}}{37630} + \dots$$

$$= \frac{\sqrt{6}}{2} - 1 + \frac{1}{2}\left\{\frac{4}{5} + \frac{14}{95} + \frac{52}{1349} + \dots\right\}$$

$$- \sqrt{6}\left\{\frac{1}{1.10} + \frac{1}{10.19} + \frac{1}{19.142} + \frac{1}{142.265} + \dots\right\}$$

$$= \frac{\sqrt{6}}{2} - 1 + \frac{1}{2}\left\{1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{19} + \frac{1}{19} - \frac{1}{71} + \dots\right\}$$

$$- \sqrt{6}\left\{\frac{1}{10-}\ \frac{10^2}{10^2.2-}\ \frac{10^2.19^2}{19^2.8-}\ \frac{19^2.142^2}{142^2.2-} \dots\right\}$$

$$= \frac{\sqrt{6}}{2} - \frac{1}{2} - \sqrt{6}\left\{\frac{1}{10-(4-2\sqrt{3})}\right\} = \frac{\sqrt{6}+\sqrt{2}-2}{4}.$$

The result given in the Question seems inaccurate.

The fifth series

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{99} + \frac{1}{1353} + \dots - \left(\frac{1}{4} + \frac{1}{28} + \frac{1}{364} + \frac{1}{5044} + \dots\right)$$

$$= \frac{1}{3-}\ \frac{3}{4-}\ \frac{1}{4-}\ \frac{1}{4-} \dots - \left(\frac{1}{4-}\ \frac{1}{2-}\ \frac{1}{8-}\ \frac{1}{2-} \dots\right)$$

$$= \frac{1}{3-3(2-\sqrt{3})} - \frac{1}{4-(4-2\sqrt{3})} = \frac{\sqrt{3}+1}{6} - \frac{\sqrt{3}}{6} = \frac{1}{6}.$$

The sixth series may be similarly summed.

[Mr. H. J. Woodall, A.R.C.S., makes the following remarks regarding the last series :—

$u_{n-1} + u_{n+1} = 4u_n$ **gives** 1, 5, 19, **71**, 265, 989, 3691, 13775 ;

whence series is

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{20} - \frac{1}{70} + \frac{1}{266} - \frac{1}{988} + \frac{1}{3692} - \frac{1}{13774}.$$

Comparison with series $\frac{1}{a_1} \pm \frac{1}{a_2} + \frac{1}{a_3} \pm \frac{1}{a_4} + \dots$

(C. Smith, ***Algebra*****, p. 466) and its transformation**

$$= \frac{1}{a_1 \mp}\ \frac{a_1^2}{a_2 \pm a_1 \mp}\ \frac{a_2^2}{a_3 \pm a_2 \mp} \dots$$

gives the equivalent continued fraction

$$\frac{1}{2+}\ \frac{2^2}{2+}\ \frac{4^2}{16+}\ \frac{20^2}{50+}\ \frac{70^2}{196+}\ \frac{266^2}{722+}\ \frac{988^2}{2704+}\ \frac{3692^2}{10082+}$$

$$= \frac{1}{2+}\ \frac{2}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{2+}.$$

Now $\frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+} = y = \frac{1}{1+}\ \frac{1}{2+y}$ **gives** $y = \sqrt{3}-1$;

therefore series $= \frac{1}{2+}\ \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{2\sqrt{3}-2+2} = \frac{\sqrt{3}-1}{2\sqrt{3}}$.]

14541. (JOHN C. MALET, F.R.S.)—If the roots x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 of the equation $x^8-p_1x^7+p_2x^6-p_3x^5+p_4x^4-p_5x^3+p_6x^2-p_7x+p_8=0$ are connected by the relations

$$x_1+x_2+x_3+x_4 = x_5+x_6+x_7+x_8 \quad\text{and}\quad x_1x_2x_3x_4 = x_5x_6x_7x_8,$$

(*a*) prove $$p_7 = \sqrt{p_8}\,(p_3-\tfrac{1}{2}p_1p_2+\tfrac{1}{8}p_1^3),$$

$$(Q_2p_7-2p_5\sqrt{p_8}+2p_1p_8)^2 = (Q_2^2-4Q_4)\,(p_7^2-4p_8Q_6),$$

where $Q_2 \equiv p_2-\frac{1}{4}p_1^2$, $Q_4 \equiv p_4-p_1p_7/(2\sqrt{p_8})-2\sqrt{p_8}$, $Q_6 \equiv p_6-Q_2\sqrt{p_8}$; (*b*) solve the equation.

Solution by the PROPOSER *and* H. W. CURJEL, M.A.

We have $\Sigma x_1 = \Sigma x_5 = \frac{1}{2}p_1$; $x_1x_2x_3x_4 = x_5x_6x_7x_8 = \sqrt{p_8}$.

Let now $\Sigma x_1x_2 = u_1$, $\Sigma x_5x_6 = u_2$, $\Sigma x_1x_2x_3 = v_1$, $\Sigma x_5x_6x_7 = v_2$,

and we find $$u_1+u_2 = Q_2 \qquad (1),$$

$$\tfrac{1}{2}p_1(u_1+u_2)+v_1+v_2 = p_3;$$

therefore $$v_1+v_2 = p_3-\tfrac{1}{2}p_1p_2+\tfrac{1}{8}p_1^3 \qquad (2),$$

$$u_1u_2+\tfrac{1}{2}p_1(v_1+v_2)+2\sqrt{p_8} = p_4, \quad u_1v_2+u_2v_1 = p_5-p_1\sqrt{p_8} \ \ldots\ (3,\ 4),$$

$$(u_1+u_2)\sqrt{p_8}+v_1v_2 = p_6; \ \text{therefore}\ v_1v_2 = Q_6 \qquad (5);$$

$$(v_1+v_2)\sqrt{p_8} = p_7 \qquad (6).$$

From (6) and (2) we find

$$p_7 = \sqrt{p_8}\left\{p_3-\tfrac{1}{2}p_1p_2+\tfrac{1}{8}p_1^3\right\},$$

one of the required conditions. We have from (3) and (6)

$$u_1u_2 = Q_4 \qquad (7).$$

Hence, from (1) and (7),

$$2u_1 = Q_2+\sqrt{(Q_2^2-4Q_4)}, \qquad 2u_2 = Q_2-\sqrt{(Q_2^2-4Q_4)};$$

and from (5) and (6)

$$2v_1 = p_7/\sqrt{p_8}+\sqrt{(p_7^2/p_8-4Q_6)}, \qquad 2v_2 = p_7/\sqrt{p_8}-\sqrt{(p_7^2/p_8-4Q_6)}.$$

Substituting in (4) for u_1, u_2, v_1, v_2, and rationalizing, we find the second required condition and the roots of the given equation are the roots of the quartics

$$2x^4-p_1x^3+\left\{Q_2\pm\sqrt{(Q_2^2-4Q_4)}\right\}x^2-\left\{p_7/\sqrt{p_8}\pm\sqrt{(p_7^2/p_8-4Q_6)}\right\}x+2\sqrt{p_8} = 0.$$

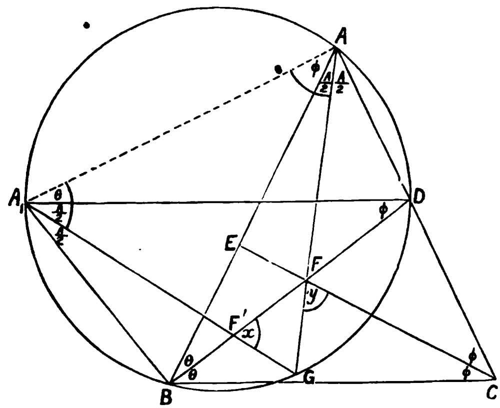
Further Note on Quests. 14520 *and* 14670 (*triangle isosceles if bisectors of base angles equal*).

Dr. J. S. MACKAY observes: "A direct proof of this Question, by F. G. HESSE, will be found in the *London, Edinburgh, and Dublin Philosophical Magazine* (Fourth Series), Vol. XLVII., pp. 354–7 (1874), and another by MOSSBRUGGER in GRUNERT'S *Archiv*, Vol. IV., pp. 330–1, 1844.

"The Question seems to have been proposed for the first time by Prof. LEHMUS, of Berlin, to JACOB STEINER in the year 1840. A proof, with extensions of the Question, was given by STEINER in CRELLE'S *Journal*, Vol. XXVIII., p. 375–379, and many other proofs will be found scattered through the volumes of GRUNERT'S *Archiv*. See, for example, Vols. XI.,

XIII., XV., XVI., XVIII., XX., XLI., XLII., &c. The proof given in TODHUNTER's *Euclid* is STEINER's."

Mr. R. CHARTRES sends the following further *direct proof*, which seems quite satisfactory:—Place the triangle AEC in the position A_1BD; then a circle will circumscribe $BDAA_1$, and the bisectors of BAD, BA_1D will

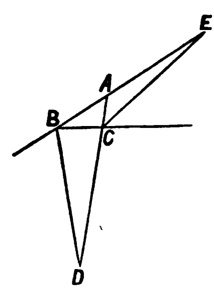


meet at G, the middle of the arc BGD, by a well known rider. Since $x \equiv y = \frac{1}{2}A + \phi$, and therefore $= GAA_1$, therefore a circle circumscribes $F'FAA_1$; and, since the chords of it FA, $F'A_1$ are identical, therefore (*Euc.* III. 14) $GA = GA_1$, or $\phi + \frac{1}{2}A = \theta + \frac{1}{2}A$; therefore $\theta = \phi$, or ABC is isosceles.

COR.—If the base angles be divided in a given ratio, then, if the dividing lines terminated by the opposite sides be equal and intersect on the bisector of the vertical angle, the triangle will be isosceles.

Professor K. J. SANJÁNA remarks: "In the algebraical solution by Mr. R. CHARTRES (p. 73), when the factor $c-b$ is removed, there is left

$$1 = -\frac{bc\,(a^2+b^2+c^2+bc+2ca+2ab)}{(a+b)^2(a+c)^2},$$



or $$a^4+2a^3b+2a^3c+a^2b^2+a^2c^2+5a^2bc+4ab^2c$$
$$+4abc^2+b^3c+2b^2c^2+bc^3 = 0.$$

Removing the factor $a+b+c$, we get

$$a^3+a^2b+a^2c+3abc+b^2c+bc^2 = 0.$$

This result is more readily obtained by *Euclid* VI.A. and trigonometry. We may write it thus

$$a^2(2s)+2abc+bc(2s) = 0,$$

or $a^2+bc+abc/s = 0$, *i.e.*, $a^2+bc = -4Rr$.

Thus there is no solution but $c = b$. If the external bisectors terminated by the sides produced be equal, we shall get by a similar process

$$a^3-a^2b-a^2c+3abc-b^2c-bc^2 = 0;$$

this may be written

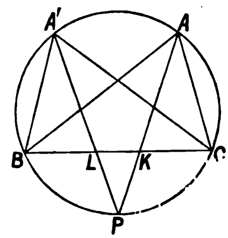
$$-a^2 \, . \, 2(s-a) + 2abc - bc \, . \, 2(s-a) = 0,$$

or $$a^2 + bc = abc/(s-a), \quad i.e., \quad a^2 + bc = 4Rr_1.$$

Thus in this case the triangle is not always isosceles.''

The accompanying diagram, supplied by Mr. Greenstreet, affords an instance in support of this view.

Mr. Tucker sends the following:—The above Question reduces to this: Construct a triangle on a given base, with a given vertical angle and given bisector of that angle. Let BC be the base; on it describe a segment containing the given angle, and let P be the mid-point of the arc remote from A. PKA is drawn so that AK = given bisector of angle. Now, by *Euclid* III., from P there can be drawn only one other line = PLA′ (= PKA); hence there can be only *two* congruent triangles, fulfilling the data of the problem; hence, &c.



Editorial Note on Quests. 2810, 3250 (the late Professor Sylvester).

Dr. Thomas Muir, of the Educational Department, Capetown, draws attention to the fact that a paper of his, "On a Class of Alternating Functions,'' in the *Trans. Roy. Soc. Edinb.*, Vol. XXXIII., pp. 309–312 (1887), contains a wide generalization of the late Professor Sylvester's theorem as enunciated in these Questions, which are identical. A solution by the late W. J. Curran Sharp, M.A., is to be found in the *Reprint*, Vol. XLVII., p. 21.

14384. (W. H. Salmon, B.A.)—If a chord of a circle S subtend a right angle at a fixed point O, show that its envelope is a conic S′; and that of the common tangents S and S′ two pairs intersect on the polar of O, one pair at the centre of S, and the other on a fixed line. Show also that O has the same polar for S and S′.

Solution by the Proposer.

Reciprocate with respect to O, and we have the well-known theorems:—

The chords of intersection of a conic and its director circle which are not diameters are directrices.

A conic and its director circle are concentric.

14682. (Professor E. N. Barisien.)—Soit ABC un triangle. Calculer le rayon d'un cercle tangent à la fois au cercle inscrit et aux côtés AB, AC.

Solution by Professor A. Droz-Farny; James Blaikie, M.A.; *and many others.*

Soient r le rayon du cercle inscrit, ρ_a, ρ'_a les rayons des deux cercles, le premier intérieur au triangle, tangents à la fois au cercle inscrit et aux côtés AB, AC; I, O, O′ les trois centres; T et T′ les deux points de contact. Menons les tangentes communes qui coupent le côté AB en D et D′; dans les triangles semblables ODI et ID′O′ on a

$$\text{OT} : \text{TI} = \text{IT}' : \text{T}'\text{O}', \quad \rho_a : r = r : \rho'_a;$$

d'où

$$\rho_a \rho'_a = r^2.$$

Dans les triangles rectangles ODT et TDI on a

$$\text{OT} = \text{DT} \tan \tfrac{1}{2}(180 - \text{A}), \quad \text{DT} = \text{TI} \tan \tfrac{1}{2}(180 - \text{A});$$

d'où

$$\rho_a = r\left\{\tan \tfrac{1}{2}(\pi - \text{A})\right\}^2, \quad \rho'_a = r/\left\{\tan^2 \tfrac{1}{2}(\pi - \text{A})\right\}.$$

Remarque. — De la relation $\Sigma\left\{\tan \tfrac{1}{2}(\pi - \text{A}) \tan \tfrac{1}{2}(\pi - \text{B})\right\} = 1$ on déduit aisément les deux relations pour les groupes de circonférences tangentes dans les trois angles du triangle

$$\sqrt{(\rho_a \rho_b)} + \sqrt{(\rho_a \rho_c)} + \sqrt{(\rho_b \rho_c)} = r, \quad 1/\sqrt{(\rho'_a \rho'_b)} + 1/\sqrt{(\rho'_a \rho'_c)} + 1/\sqrt{(\rho'_b \rho'_c)} = 1/r.$$

14683. (Professor P. Leverrier.)—Etant donnés un triangle ABC et un cercle O, on demande de couper le triangle par une transversale $\alpha\beta\gamma$, telle que les cercles $\alpha\beta$C et $\alpha\gamma$B soient égaux et que leur axe radical soit tangent au cercle O.

Solution by Professor A. Droz-Farny *and* R. F. Davis, M.A.

On sait que les quatre circonférences $\alpha\beta$C, $\alpha\gamma$B, $\beta\gamma$A et ABC se coupent en un même point F, foyer de la parabole inscrite au quadrilatère. Les deux cercles $\alpha\beta$C et $\alpha\gamma$B étant égaux, on a

$$\text{FB} = 2r \sin \text{F}\alpha\text{B}, \quad \text{FC} = 2r \sin \text{F}\alpha\text{C}; \quad \text{donc} \quad \text{FB} = \text{FC};$$

F est donc le point milieu de l'arc BC. On en déduit la construction suivante :—On mène de F une des tangentes à O; cette droite coupe BC en α. La circonférence FαB détermine γ sur AB.

14540. (Professor G. B. Mathews, F.R.S.)—Prove that, if

$$Q = \sum_{-\infty}^{+\infty} q^{\frac{1}{3}(3n+1)^2} + \sum_{-\infty}^{+\infty} q^{3n^2}, \quad \text{then} \quad Q^4 = \frac{\lambda^3 \lambda'^3}{16\kappa\kappa'},$$

where λ, λ' are the moduli into which κ, κ' are transformed by the change of q into q^3.

Solution by H. W. Curjel, M.A.

If $\quad q_0 = \prod_1^\infty (1 - q^{2n}), \quad q_1 = \prod_1^\infty (1 + q^{2n}), \quad q_2 = \prod_1^\infty (1 + q^{2n-1}),$

$$q_3 = \prod_1^\infty (1 - q^{2n-1}),$$

and Q_0, Q_1, Q_2, Q_3 are what q_0, q_1, q_2, q_3 become when q is changed into q^3. Then

$$Q = \frac{q^{\frac{1}{3}}\left\{1+\sum_1^\infty q^{3n^2}\left(q^{2n}+q^{-2n}\right)\right\}}{1+2\sum_1^\infty q^{3n^2}} = \frac{q^{\frac{1}{3}}Q_0\prod_1^\infty\left(1+q^{6n-1}\right)\left(1+q^{6n-5}\right)}{Q_0 Q_2^2} = \frac{q^{\frac{1}{3}}q_2}{Q_2^3};$$

therefore $$Q^{12} = q^4 q_2^{12}/Q_2^{36},$$
whence the result in the Question.

$$1+\lambda' = \frac{Q_0^2\left(Q_2^4+Q_3^4\right)}{Q_0^2 Q_2^4} = \frac{2Q_0^2 Q_1^2\prod_1^\infty\left(1+q^{6(2n-1)}\right)^4}{Q_0^2 Q_2^4};$$

therefore $$\lambda^8\lambda'\left(1+\lambda'\right)^4/256\kappa\kappa' Q^{12}$$

$$= \frac{2^6 q^{\frac{2}{3}}\dfrac{Q_1^{12}}{Q_2^{12}}\dfrac{Q_3^4}{Q_2^4}\dfrac{2^4 Q_1^8}{Q_2^{16}}\prod_1^\infty\left(1+q^{6(2n-1)}\right)^{16}}{2^8 . 2^2 q^{\frac{1}{3}}\dfrac{q_1^4}{q_2^4}\dfrac{q_3^4}{q_2^4}\dfrac{q^4 q_2^{12}}{Q_2^{36}}} = Q_1^{16}\prod_1^\infty\left(1+q^{12n-6}\right)^{16},$$

since $q_1 q_3 q_3 = 1$ and $Q_1 Q_2 Q_3 = 1$,

$$= \prod_1^\infty\frac{\left(1+q^{12n-6}\right)^{16}}{Q_2^{16}Q_3^{16}} = \prod_1^\infty\left(\frac{1+q^{12n-6}}{1-q^{12n-6}}\right)^{16},$$

which is clearly not equal to unity.

[The Proposer observes: "The proper result

$$Q^{12} = \frac{\lambda^3\lambda'^3}{16\kappa\kappa'}$$

easily comes from Mr. Curjel's result $Q^{12} = \dfrac{q^4 q_2^{12}}{Q_2^{36}}$,

because, with Weber's notation (*El. Funct.*, pp. 63, 149),

$$q_2^{12} = q^{\frac{1}{2}}f^{12}(\omega) = \frac{4q^{\frac{1}{2}}}{\kappa\kappa'}, \qquad Q_2^{36} = q^{\frac{3}{2}}f^{36}(3\omega) = \frac{64q^{\frac{3}{2}}}{\lambda^3\lambda'^3},$$

whence $$Q^{12} = \frac{\lambda^3\lambda'^3}{16\kappa\kappa'}."]$$

13746. (Lionel E. Reay, B.A.)—Find the area of the triangle with sides equal to the medians of a given triangle. Show whether such a triangle is always possible.

Solution by W. J. Greenstreet, M.A.

Since $m_1 = \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}$, the area may be found in the usual way. The condition $\sqrt{(2b^2+2c^2-a^2)}+\sqrt{(2c^2+2a^2-b^2)} > \sqrt{(2a^2+2b^2-c^2)}$ reduces to $b+c>a$. Therefore, &c.

14543. (Professor MORLEY.)—The greatest number of regions into which n spheres can divide space is $2n + \frac{1}{3}n(n-1)(n-2)$.

Solution by H. W. CURJEL, M.A.

The corresponding theorem for space of m dimensions is easily proved. If in space of m dimensions we call the hypersphere of highest possible dimensions a sphere, and a space of $(m-1)$ dimensions a plane, the theorem may be stated: The greatest number of regions into which n spheres can divide a space of m dimensions is

$$\{2(n-1)(n-2)\ldots(n-m)\}/m! \\ +2\left[1+(n-1)+\{(n-1)(n-2)\}/2!+\ldots\text{to } m \text{ terms}\right] = 2v_{m,n}+u_{m,n}$$

(using the notation of Quest. 13395, Vol. LXVIII., p. 39).

The greatest number of regions into which m planes can divide space of m dimensions is shown in Quest. 13395 to be $v_{m,n}+u_{m,n}$; if we invert with respect to a point in none of the planes, we see that the same is true of n spheres passing through a point. But the number of regions that are made to vanish by making the n spheres pass through a point is easily seen to be $v_{m,n}$; therefore the greatest number of regions

$$= 2v_{m,n}+u_{m,n}.$$

If we put $m = 3$, we get the result stated in the Question.

14546. (Professor NEUBERG.) — Si les angles des triangles ABC, A′B′C′ vérifient les égalités $A + A' = \pi$, $B = B'$, les côtés sont liés par la relation $aa' = bb' + cc'$.

Solution by R. P. PARANJPYE. B.A.; Professor IGNACIO BEYENS, Lt.-Col. du Génie à Cadix; *and many others.*

With centre C describe a circle, radius CA, cutting AB in D.

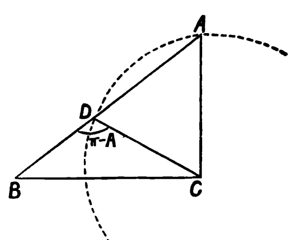
Then, obviously, DBC is similar to the triangle A′B′C′ of the enunciation.

Since the relation to be proved is homogeneous in a, b, c as well as a', b', c', we may prove

$$BC^2 = AC^2 + BA.BD,$$

which is a well-known proposition (see CASEY's *Sequel*).

Hence the required relation.



14547. Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Show that, if n be any positive integer greater than unity,

$$\frac{1^3+2^3+3^3+4^3+\ldots+n^3-(1^5+2^5+3^5+4^5+\ldots+n^5)}{(1+2+3+4+\ldots+n)^3-(1^5+2^5+3^5+4^5+\ldots+n^5)} = 4.$$

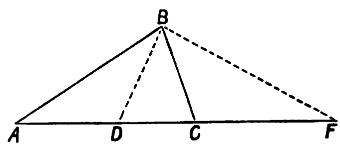
Solution by H. W. Curjel, M.A.; Lt.-Col. Allan Cunningham, R.E. *and many others.*

If $s_r = \sum_{n=1}^{n=n} n^r$, $s_3 = s_1^2$, and $s_5 = \frac{1}{3}(2n^2+2n-1)\,s_3$,

therefore $$\frac{s_5-s_3}{s_5-s_1^3} = \frac{\frac{1}{3}(2n^2+2n-1)-1}{\frac{1}{3}(2n^2+2n-1)-\frac{1}{2}n(n+1)} = 4.$$

14536. (I. Arnold.)—In any triangle the radius of the circumscribed circle is to the radius of the circle which is the locus of the vertex, when the base and the ratio of the sides are given, as the difference of the squares of those sides is to four times the area.

Solution by F. H. Peachell, B.A.; Ragunath Rau, B.A.; *and* W. J. Greenstreet, M.A.



The locus of vertex, when the ratio of the sides is given, is the semicircle on DF, where D, F divide AC internally and externally in ratio of the sides. Therefore

$$CD : DA = a : c,$$

or $$CD : AC = a : c+a;$$

therefore $CD = ab/(c+a)$, and similarly $CF = ab/(c-a)$; therefore radius of locus $= abc/(c^2-a^2)$.

Radius of locus : circum-radius $= abc/(c^2-a^2) : abc/4\Delta = 4\Delta : c^2-a^2$.

Note on Quest. 6144 (*Reprint*, Vol. lxxiii., p. 113). *By* Rev. Charles Taylor, D.D., Master of St. John's College, Cambridge.

For "circumscribed" read "circum-inscribed." This term is used in the *Ancient and Modern Geometry of Conics* (pp. 139, 140) in the sense *circumscribed* to one curve and *inscribed* to another.

14467. (G. H. Hardy, B.A.)—Prove that

$$\int_{-\infty}^{\infty} \{\phi(x-a)-\phi(x-b)\}\,dx = (b-a)\{\phi(\infty)-\phi(-\infty)\},$$

provided each side of the equation represents a determinate quantity. Deduce the values of

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh(x-a)\cosh(x-b)},\quad P\int_{-\infty}^{\infty} \frac{dx}{\sinh(x-a)\sinh(x-b)}.$$

Solution by the PROPOSER.

$$\int_{-\infty}^{\infty}\left\{\phi(x-a)-\phi(x-b)\right\}dx = \lim_{H=\infty}\int_{-H}^{H}$$

$$= \lim_{H=\infty}\left[\int_{-H-a}^{H-a}\phi(u)\,du-\int_{-H-b}^{H-b}\phi(u)\,du\right]$$

$$= \lim_{H=\infty}\left[\int_{H-b}^{H-a}\phi(u)\,du-\int_{-H-b}^{-H-a}\phi(u)\,du\right]$$

$$= (b-a)\left\{\phi(\infty)-\phi(-\infty)\right\},$$

if both sides of the equation be determinate.

If $$\phi(u) = \tanh u,$$

$$\phi(x-a)-\phi(x-b) = \frac{\sinh(b-a)}{\cosh(x-a)\cosh(x-b)},$$

and $$\int_{-\infty}^{\infty}\frac{dx}{\cosh(x-a)\cosh(x-b)} = \frac{b-a}{\sinh(b-a)}.$$

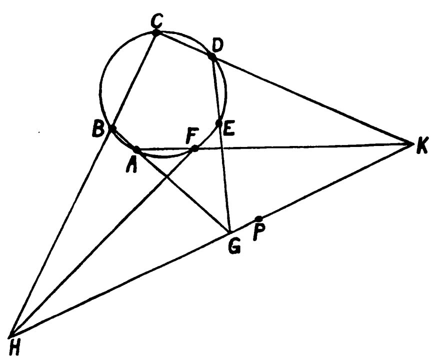
If $\phi(u) = \coth u$, we find

$$P\int_{-\infty}^{\infty}\frac{dx}{\sinh(x-a)\sinh(x-b)} = -\frac{b-a}{\sinh(b-a)}.$$

It is easy to see that the proof remains valid, although in the latter case only the principal value of the integral is determinate.

Euclidean Proof of PASCAL'S *Theorem.* *By* R. F. DAVIS, M.A.

Let ABCDEF be a cyclic hexagon. Produce AB, DE to meet in G, and AF, CD in K. Let BC and the circumcircle of DFK intersect GK in H, P respectively.



Then (1) P, D, B, G are concyclic, for

180°−DPG
= DPK = DFK
= 180°−DFA
= DBG.

Also (2) P, F, B, H are concyclic, for

FPH = 180°−FPK
= FDK = FBC
= 180°−FBH.

From (2)

BFH = BPH
= BPG
= BDE

from (1) = 180°−BFE;

therefore EF passes through H.

[In most geometrical conics PASCAL'S theorem for the conic is derived from the theorem for the circle by conical projection. The above proof is strictly *Euclidean:* it neither involves anharmonic ratios (CASEY's *Sequel*, NIXON's *Geometry Revised*, &c.) nor MENELAUS' *Transversal Theorem*, which is employed by CATALAN.]

14312. (Professor N. L. BHATTACHARYYA.)—A parabola slides between the two foci of an ellipse, such that the focus of the parabola always lies on the ellipse. Find the envelope of (1) the directrix, (2) the axis, of the parabola.

Solution by A. F. VAN DER HEYDEN, B.A.; H. W. CURJEL, M.A.; *and* Rev. J. CULLEN.

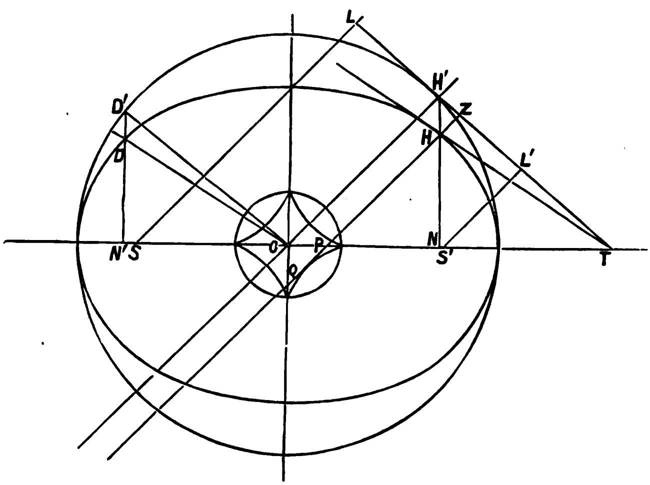
Let SL, S′L′, CH′ be perpendiculars to the directrix of the parabola, in any position, from the foci and centre of the ellipse. Then

$$SL + S'L' = 2CH'.$$

But $SL + S'L' = SH + S'H$ [if H is the focus of the parabola] $= 2CA$;

therefore $$CH' = CA;$$

therefore the envelope of the directrix is the auxiliary circle of the ellipse ....(1).



Let H, H′ be corresponding points; HT, H′T tangents. Then

$$SH : S'H = ST : S'T = SL : S'L'.$$

Hence, if H is the focus, TH′ is the directrix of the parabola. Let the axis ZH cut the axes of the ellipse in P and Q. Then, if CD be conjugate to CP, we have

$$CP = (a-b)/b\,N'D \quad \text{and} \quad CQ = (a-b)/b\,HN;$$

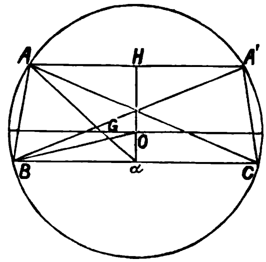
therefore $$PQ = a-b.$$

Therefore the envelope of the axis ZH of the parabola is the four-cusped hypocycloid generated by the rolling of a circle, radius $\frac{1}{4}(a-b)$, upon a circle, centre C, whose radius is $a-b$ ....(2).

[Mr. Curjel remarks that there are two parabolas with focus H passing through S, S′, the directrix of the second being the other tangent from T to the auxiliary circle; consequently the part of the axis intercepted between the axes of the ellipse $= a \pm b$; and its envelope is the two four-cusped hypocycloids $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a \pm b)^{\frac{2}{3}}$.]

14453. (Professor A. Droz-Farny.)—Construire un triangle, dont on connait la base, la hauteur correspondante et sachant que sa droite d'Euler est parallèle au côté donné.

Solution by Rev. J. Cullen; Professor Jan de Vries; *and many others.*



Let BC be the given base; through its mid-point α, draw αH perpendicular and equal to the given height; trisect αH in O, and through O and H draw parallels to BC; with O as centre and radius OB, describe a circle cutting the line through H in A and A′. Then ABC (or A′BC) is the required triangle, since αA is trisected by the line through O in G, Euler's line being OG.

14503. (Robert W. D. Christie.)—Show that the primitive roots of 331 are connected with the associated roots by the modular equations

$$r^m = \omega \bmod 331, \quad r_1^m = -\omega^2 \bmod 331,$$

where r is a primitive, and r_1 an associated, root; also ω signifies one of the roots of $x^3+1=0$, namely, $\frac{1}{2}\{1+\sqrt{(-3)}\}$ or $\frac{1}{2}\{1-\sqrt{(-3)}\}$; and generalize the result.

Solution by the Proposer *and* Lt.-Col. Allan Cunningham, R.E.

We have $$r^{3m}+1 = 0 \bmod 6m+1.$$

$$\therefore \quad r^{2m} - r^m + 1 = r_1^{2m} - r_1^m + 1 = 0 \bmod 6m+1.$$

$$\therefore \quad r^{2m} - r_1^{2m} - (r^m - r_1^m) = 0 \bmod 6m+1.$$

$\therefore\quad r^m + r_1^m - 1 = 0 \bmod 6m+1 \quad \text{and} \quad r^{2m} - r^m + 1 = 0 \bmod 6m+1.$

$\therefore\quad r^{2m} + r_1^m = 0 \bmod 6m+1 \quad \text{or} \quad (r^m)^2 + r_1^m = 0 \bmod 6m+1;$

i.e., $\omega^2 - \omega^2 = 0 \bmod 6m+1.$

Thus $r^m = \omega \bmod 6m+1, \qquad r_1^m = -\omega^2 \bmod 6m+1.$

14238. (Rev. W. Allen Whitworth, M.A.)—If a straight line be divided at random into any number of parts, the expectation of the square on any part taken at random is double of the expectation of the rectangle contained by any two of the parts taken at random. [This can be proved by algebra without the integral calculus.]

Solution by Rev. T. Roach, M.A.

The expectation of the square of any part $= 2s^2/\{n(n+1)\}$ (Whitworth, *Choice and Chance*, Supplement). The expectation of the product of any two parts $= s^2/\{n(n+1)\}$. Therefore

ex. squares = 2 ex. product.

14529. (Lt.-Col. Allan Cunningham, R.E.)—Show that $q^x \equiv 1 \pmod p$ where $x = \frac{1}{8} . 210^{4Q}$, $Q = q^4$, $p = Q . 210^{4Q} + 1 = prime.$

Solution by the Proposer.

Here $(p-1)/8Q = \frac{1}{8} . 210^{4Q} = x,$

and $\frac{1}{4}x$ is *even*. Also $p = 1 + (q . 210^Q)^4;$

therefore $210^{4Q} . q^4 \equiv -1 \pmod p;$

therefore $(210^{4Q} . q^4)^{\frac{1}{4}x} = 210^{Qx} . q^x \equiv (-1)^{\frac{1}{4}x} \equiv +1 \pmod p;$

therefore $(2 . 3 . 5 . 7)^{(p-1)/8} . q^x \equiv +1 \pmod p.$

Now $p = 1^2 + (210^{2Q} . q^2)^2 = (q^2 . 210^{2Q} - 1)^2 + 2(q . 210^Q)^2.$

These two partitions of p suffice to show that 2, 3, 5, 7 are *each* 8-ic residues of p when p is *prime*. [See two papers by Mr. C. E. Bickmore "On the Numerical Factors of $(a^n-1)^n$" in *Messenger of Mathematics*, Vol. xxv., p. 18, and Vol. xxvi., pp. 15, 18, 21.] Therefore

$$(2 . 3 . 5 . 7)^{(p-1)/8} \equiv +1,$$

leaving $q^x \equiv +1 \pmod p.$

14524. (R. F. Davis, M.A.)—If A, B, C, D be the angles of any convex quadrilateral,

$$\sin A \{\sin C + \sin B - \sin(A+D)\} : \sin C \{\sin A + \sin B - \sin(C+D)\}$$
$$= \sin A + \sin D - \sin(A+D) : \sin C + \sin D - \sin(C+D).$$

Solution by F. H. Peachell, B.A.

From the question $A+B+C+D = 2\pi$;

therefore $\sin(A+B) = -\sin(C+D)$, &c.

Also $\sin\frac{1}{2}(A+B) = \sin\frac{1}{2}(C+D)$, &c.

Now
$$\sin A\,\{\sin C + \sin B - \sin(A+D)$$
$$= \sin A\; 2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(B-C) + \sin(B+C)\}$$
$$= 2\sin A\sin\tfrac{1}{2}(B+C)\,\{\cos\tfrac{1}{2}(B-C) + \cos\tfrac{1}{2}(B+C)\}$$
$$= 4\sin A\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

Similarly,
$$\sin C\,\{\sin A + \sin B - \sin(C+D)\}$$
$$= 4\sin C\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B.$$

Therefore ratio of the two expressions is
$$\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B+C)/\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A+B).$$

Now
$$\sin A + \sin D - \sin(A+D)$$
$$= 2\sin\tfrac{1}{2}(A+D)\,\{\cos\tfrac{1}{2}(A-D) - \cos\tfrac{1}{2}(A+D)\}$$
$$= 4\sin\tfrac{1}{2}(A+D)\sin\tfrac{1}{2}A\sin\tfrac{1}{2}D = 4\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}A\sin\tfrac{1}{2}D.$$

Similarly,
$$\sin C + \sin D - \sin(C+D) = 4\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}C\sin\tfrac{1}{2}D.$$

Therefore the ratio of these two expressions is the same as that of the first two.

14132. (I. Arnold.)—Describe a square in a given sector, having two angular points on the arc and the other two on the radii.

Solution by the Proposer.

(1) Let OEG be the given sector; make $Oc = Ob$, and join bc. Take a line $O'A' = OE$, and on it describe the segment $O'B'A'$ containing an angle $O'B'A'$ equal to a right angle + Ocb. Also on it describe another segment $O'C'A'$ containing an angle $O'C'A'$ equal to $\frac{1}{2}$ right angle + Ocb,

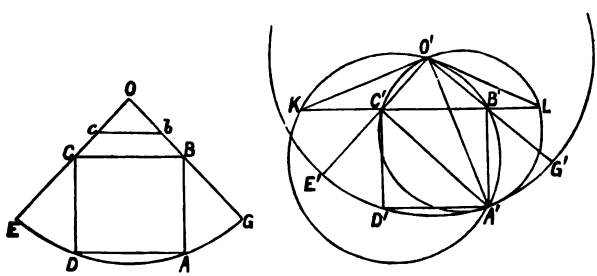


Fig. 2. Fig. 1.

and complete the circles. Draw $O'K$, cutting off a segment $KA'B'O'$ containing the angle $KB'O'$ equal to Ocb or Obc. Also draw the line

O′L, cutting off from the other circle a segment containing the angle O′C′L equal to O*bc*. Join KL, and produce O′C′, O′B′ to meet the circle described with O′A′ as radius in the points E′, G′.

Then is B′C′ the side of a square described in the sector E′O′G′, which is equal to the sector EOG. Next draw CB parallel to *cb* and equal to C′B′. Lastly, draw BA, CD each perpendicular to CB, and join AD. ABCD is a square, and it is inscribed in the sector EOG.

The analysis of the problem is derived from Fig. 1.

(2) When the sector AOB is greater than a semicircle.

Make O*f* = O*e*, and join *fe*. On *fe* describe the square *cdef*. Join O*c*, and produce to meet the circumference in C. Draw CF parallel to *cf*, and FE parallel to *fe*. Also draw CD parallel to FE, and join DE. CDEF is a square, and it is inscribed in the sector AOB, which is greater than a semicircle.

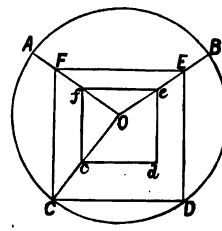


Fig. 3.

The quadrilateral figures CFEO and *cfe*O are symmetrical, and, since *fe* = *fc*, then FE = FC, and, the angle *cfe* being a right angle, EFC is also a right angle. Hence CDEF is a square, and it is inscribed in the sector AOB greater than a semicircle.

The foregoing are proposed as geometrical solutions of the problem.

[For another solution of this Question, see Vol. LXXI, p. 103.]

6498. (J. W. Russell, M.A.)—Show that

$$5(a+b+c)^2(a+b)^2(b+c)^2(c+a)^2-\Sigma a^2(a+b)^3(a+c)^3$$

is divisible by $(a+b+c)^3+abc$.

Solution by H. W. Curjel, M.A.; Professor Sanjána, M.A.; *and* J. O. Watts.

If, in the given expression, we put $a+b+c=-\omega(abc)^{\frac{1}{3}}=x$ (say), where $\omega^3=1$, it becomes

$$\Pi(a+b)^2\left\{5x^2-\Sigma\frac{a^2(ax+bc)}{(x-a)^2}\right\}$$

$$=\Pi(a+b)^2\left\{5x^2-\Sigma\frac{a(a^2x-x^3)}{(x-a)^2}\right\}=x\,\Pi(a+b)^2\left\{5x+\Sigma\frac{a(x+a)}{x-a}\right\}$$

$$=x\,\Pi(a+b)^2\left\{5x+\Sigma\left(-a+\frac{2ax}{x-a}\right)\right\}=2x^2\,\Pi(a+b)^2\left(2+\Sigma\frac{a}{x-a}\right)$$

$$=2x^2\,\Pi(a+b)\left\{2(x-a)(x-b)(x-c)+\Sigma a(x-b)(x-c)\right\}$$

$$=2x^2\,\Pi(a+b)\left\{2x^3-2x^3+2\Sigma bcx+2x^3+x^3-\Sigma a(b+c)x-3x^3\right\}$$

$$=0.$$

Therefore the expression is divisible by $(a+b+c)^3+abc$.

14538. (Salutation.)—Arrange in one plane two triangles of given dimensions in such manner that two specified vertices may coincide, and the other four be concyclic.

Solution by the Proposer *and* H. W. Curjel, M.A., *jointly.*

Analysis.—Let ABC, ADE (Fig. 1) be two triangles fulfilling the conditions, A being the common vertex, and B, C, D, E being on the

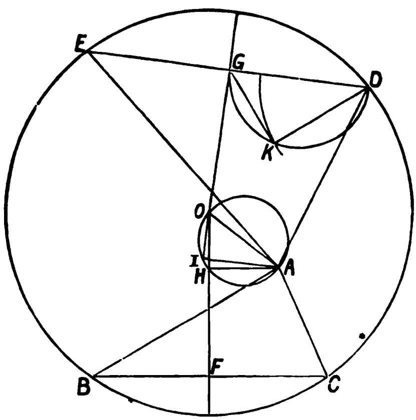


Fig. 1.

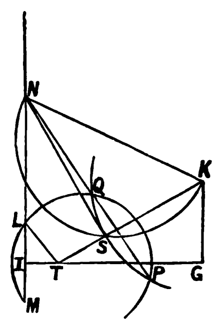


Fig. 2.

circumference of a single circle, of centre O. Let BC = $2a_1$, DE = $2a_2$; FH = h_1, GI = h_2; HA = d_1; IA = d_2; OF = x_1, OG = x_2. Then we clearly have

$$x_1^2 + a_1^2 = x_2^2 + a_2^2 \quad \text{.............................} \quad (\alpha),$$

and

$$(x_1 - h_1)^2 + d_1^2 = (x_2 - h_2)^2 + d_2^2 \quad \text{.....................} \quad (\beta).$$

The construction is therefore easy. Thus (Fig. 2), let IG = h_2, GK = $(a_2^2 - a_1^2)^{\frac{1}{2}}$, found as in Fig. 1, where DK = BF = a_1; also let IL = $(d_2^2 - d_1^2)^{\frac{1}{2}}$, found in the same way, GK and IL being perpendicular to IG. Moreover, let the arc QSP be drawn from centre K with radius = FH = h_1, and let IL be produced both ways, IM = IL. Then, with centre on IG, draw any circle passing through LM and cutting the arc QSP, say in Q, P; join PQ, and produce to meet IL in N; on KN describe a semicircle cutting QSP in S, and produce KS to meet IG in T. Then KT = x_1, the required height of O above BC (Fig. 1). For T is the centre of a circle passing through L, M and touching the arc QSP; therefore TL = TS = $x_1 - h_1$, and TI = $h_2 - x_2$; and we have the conditions stated in (α) and (β) fully carried out. (*Cf. Euc.* III. 36.)

Of course the example given is a particular case out of many, but the principle of construction is the same in all.

14109. (Professor Cochez.)—On donne les deux paraboles $y^2 = 2px$. $x^2 = 2qy$. Lieu des points M tels que les tangentes issues de ces points à chaque parabole soient rectangulaires.

Solution by W. J. Greenstreet.

$y - mx - p/2m = 0$ is a tangent to $y^2 = 2px$;

$my + x + k = 0$ is a tangent to $x^2 = 2qn$ if $k = q/2m$;

therefore for intersection we have

$$\frac{x}{pm+q} = \frac{y}{qm-p} = \frac{-1}{2m(1+m^2)} = \frac{x+n-y}{q(1+m^2)};$$

therefore $m = \dfrac{(px+qy)}{(qx-py)}$ and $\left(x + \dfrac{px+qy}{qx-py}y\right)\dfrac{2(px+qy)}{qx-py} + q = 0$;

therefore $2(px+qy)(x^2+y^2) + (qx-py)^2 = 0$ is the locus required.

6701. (Professor Crofton, F.R.S.)—Find a function X such that, whatever F be, $F(e^x D)X \equiv F(a)X$.

Solution by Rev. J. Cullen.

Let $u = -e^{-x}$; then $e^x Dy = dy/du$. Therefore $F(D')U = F(a)U$. Now consider a term $p_r x^r$ in $F(x)$; hence $D'^r U = a^r U$, which is satisfied by $U = e^{au}$ or $X = e^{-ae^{-x}}$.

[Mr. Curjel remarks:—"It is necessary and sufficient that

$$e^x DX = aX, \quad i.e., \quad X = Ce^{-ae^{-x}}."]$$

6632. (Professor Genese, M.A.)—Find the envelope of the asymptotes of conics inscribed in or circumscribed about a given quadrilateral.

Solution by W. J. Greenstreet, M.A.

Take the asymptotes of the equilateral hyperbola

$$xy + \kappa = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which passes through the four points, as axes of coordinates. Then

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

is any conic through the four points, with axes parallel to the coordinate axes; and $ax^2 + by^2 + 2\lambda xy + 2gx + 2fy + c + 2\lambda\kappa = 0$ .... (3)

is the general equation to conics through the four points.

If (1) were one of the hyperbolas through the four points, the axes of (3) would be oblique, but the form of (3) would be unchanged.

The locus of the centres of (3) is found to be

$$ax^2 - by^2 + dx - ey = 0 \quad \text{(the nine-point conic)} \quad (4),$$

and it is worth while noticing that, if (3) is a parabola, $\lambda = \pm\sqrt{ab}$, and the directions of the axes are given by $x\sqrt{a} = \pm y\sqrt{b}$. Hence the asymptotes of (4) are parallel to the axes of these two parabolas. The tangent at the origin is $\quad gx+fy = 0$ .... (5).

$y = mx+n$ meets (3) at infinity only if

$$2\lambda m + bm^2 + a = 0 = \lambda n + nbm + fm + g.$$

Hence, eliminating λ, we get $nbm^2 + 2fm^2 + 2gm - an = 0$; and, writing $m = -u/v$, $n = 1/v$, we get for envelope in tangential coordinates

$$2uv\,(fu - gv) + bu^2 - av^2 = 0 \qquad (6).$$

This is a curve of the third class and fourth order, and has three cusps. It touches the line at infinity where $bu^2 = av^2$, *i.e.*, the points of contact are on the axes of the parabolas mentioned above. It also touches the axes of coordinates. The third tangent from the origin has for angular coefficient $-u/v = -g/f$, *i.e.*, this tangent is the conjugate harmonic of (5) with reference to the axes of coordinates.

If (3) is a series of equilateral hyperbolas, (6) touches the line at infinity at the circules, and is a three-cusped hypocycloid.

When the quadrilateral is inscriptible, we have, if θ_1, θ_2, θ_3, θ_4 be the angles made by the normals with the axis of x, $\theta_1 + \theta_2 = \theta_3 + \theta_4$. Therefore,

in $\quad b^2\alpha^2m^4 - 2\alpha\beta b^2m^3 + m^2(a^2\alpha^2 + b^2\beta^2 - c^4) - 2a^2\alpha\beta m + a^2\beta^2 = 0$

we have $\quad (m_1 + m_2)/(1 - m_1m_2) = (m_3 + m_4)/(1 - m_3m_4)$ .... (a).

Writing this $\quad p/(1-q) = r/(1-s)$,

we have $\quad p + q = 2\beta/\alpha, \qquad pr + q + s = (a^2\alpha^2 + b^2\beta^2 - c^4)/b^2\alpha^2,$

$\qquad ps + qr = 2a^2\beta/b^2\alpha, \qquad qs = a^2\beta^2/b^2\alpha^2.$

Therefore $\quad p/(1-q) = r/(1-s) = (2\beta/\alpha)/\{2-(q+s)\}$.

Therefore

$$\left.\begin{aligned} \frac{(1-q)(1-s)\,2\beta}{\alpha\{2-(q+s)\}} + q + s &= \frac{a^2\alpha^2 + b^2\beta^2 - c^4}{b^2\alpha^2} \\ \frac{s(1-q)\,2\beta}{\alpha\{2-(q+s)\}} + \frac{q(1-s)\,2\beta}{\alpha\{2-(q+s)\}} &= \frac{2a^2\beta}{b^2\alpha} \\ qs &= a^2\beta^2/(b^2\alpha^2) \end{aligned}\right\} \qquad \text{(A)}.$$

Eliminating q and s from (A), we get

$$\frac{\beta^2}{\alpha^2}\left(\frac{a^2\beta^2 - b^2\alpha^2}{b^2\alpha^2}\right)\left(\frac{a^2\beta - \beta b^2}{b^2\alpha}\right)\left(\frac{\beta^2(a^2+b^2)^2}{b^4\alpha^2}\right)$$

$$= \frac{\beta^2}{\alpha^2}\left(\frac{a^2\beta^2 - b^2\alpha^2}{b^2\alpha^2}\right)^2\left[\left(\frac{a^2\alpha^2 + b^2\beta^2 - c^4}{b^2\alpha^2}\right)\left(\frac{\beta(a^2+b^2)}{b^2\alpha}\right) - 2\frac{a^2\beta^3}{b^2\alpha^3} - \frac{2a^2\beta}{b^2\alpha}\right].$$

This gives $\qquad \beta = 0, \quad a^2\beta^2 - b^2\alpha^2 = 0$;

viz., $\qquad y = 0, \quad a^2y^2 - b^2x^2 = 0$ .... (1), (2),

and $\qquad (x^2+y^2)^2a^2b^2 - c^2(a^2+b^2)(b^2x^2 - a^2y^2) = 0$.................. (3).

We see that, if $a^2\beta^2 = b^2\alpha^2$, then $m_1m_2m_3m_4 = 1$, which may be inconsistent with (a). Hence the locus is composed of the axis major of the ellipse and (3), which is the pedal of $a^2x^2 - b^2y^2 = (a^4 - b^4)$ with respect to the origin.

[For another solution of this Question, see Vol. III., p. 80 (1860).]

6642. (J. R. HARRIS, M.A.)—When two or more spherical soap-bubbles, blown from the same mixture, are allowed to coalesce into a single bubble, prove (1) that we obtain for the radius of the bubble an equation of the form

$$x^3-r_1^3-r_2^3-\ldots = a(r_1^2+r_2^2+\ldots-x^2),$$

$r_1, r_2, \ldots$ being the radii of the bubbles, and a some positive quantity; and (2) verify (what one would infer also from physical considerations) that this equation implies a reduction of the total surface.

Solution by H. W. CURJEL, M.A.

Let p_1, p_2, p_3, &c., be the pressures within the bubbles, and p the pressure within the single bubble of radius x. Then

$$px^3 = p_1r_1^3+p_2r_2^3+p_3r_3^3+\ldots, \quad \text{and} \quad p = 2t/x+\pi, \quad p_1 = 2t/r_1+\pi, \quad \text{\&c.},$$

where π is the atmospheric pressure and t is the surface tension;

therefore $\quad x^3-r_1^3-r_2^3-r_3^3-\ldots = 2t/\pi\,(r_1^2+r_2^2+r_3^2+\ldots-x^2)$,

where $2t/\pi = a$ is obviously positive. If the total surface is not reduced, Σr^2-x^2 is zero or negative. Therefore x is greater than any of the r's;

therefore $$x^3 \geqq x\Sigma r^2 > \Sigma r^3;$$

therefore $x^3-\Sigma r^3$ is positive, which is impossible if Σr^2-x^2 is zero or negative. Therefore the total surface is reduced.

14651. (Professor G. B. MATHEWS, F.R.S.)—Let α, β be any two given complex quantities, and let t be such that $(\alpha+t\beta)/(1+t)$ is real. Prove that, if $t = x+iy$, the locus of (x, y) is, in general, a circle. How is this to be reconciled with the fact that the line joining two imaginary points (α, β), (γ, δ) contains only one real point?

I. *Solution by* Professor E. B. ELLIOTT, F.R.S.

A "real" line contains ∞^2 points, of which ∞ are real; a "real" plane ∞^4 points, of which ∞^2 are real; and a "real" space ∞^6 points, of which ∞^3 are real.

The connector of two imaginary points on a "real" line is that line, and contains all its real points; that of two imaginary points in a "real" plane, but not on a "real" line in that plane, contains one real point; that of two imaginary points in "real" space, but not in a "real" plane, contains no real point.

First on a "real" line, if $x+iy : 1$ be the ratio in which a real point divides the intercept between two points whose distances from a real origin are $a+ia'$, $b+ib'$, we have, from the reality of

$$\frac{a+ia'+(x+iy)(b+ib')}{1+x+iy}, \quad \frac{a+bx-b'y}{1+x} = \frac{a'+b'x+by}{y},$$

i.e., the equation of circular form

$$b'(x^2+y^2+x)+a'(1+x)+(b-a)y = 0,$$

as the only relation limiting x and y. One of these may be taken at will, subject to the requirement that the resulting quadratic for the other have real roots, and the infiniteness of the number of ratios of division for real points is apparent. The solution $x+1=0$, $y=0$ is excluded.

Next in a "real" plane, if $(a_1+ia'_1,\ a_2+ia'_2)$ and $(b_1+ib'_1,\ b_2+ib'_2)$ be the coordinates referred to "real" axes of the points, the intercept between which is divided in the ratio $x+iy:1$, we have, for the reality of the dividing point,

$$b'_1(x^2+y^2+x)+a'_1(1+x)+(b_1-a_1)\,y=0$$

and
$$b'_2(x^2+y^2+x)+a'_2(1+x)+(b_2-a_2)\,y=0,$$

which give, besides the irrelevant $x=-1$, $y=0$, which refers to the point at infinity on the connector, a single pair of real finite values of x and y, and so a single ratio of division.

And generally in "real" space, referring to three "real" planes, we have three such equations of circles with a common point $x=-1$, $y=0$. They have as a rule no common second intersection; and so as a rule there is no real dividing point of our intercept.

II. *Solution by the* PROPOSER.

Let $\alpha=a+bi,\quad \beta=c+di,\quad t=x+yi.$

Then
$$\frac{\alpha+t\beta}{1+t}=\frac{(a+cx-dy)+(b+dx+cy)\,i}{(1+x)+yi},$$

and this is real when $(1+x)(b+dx+cy)-y(a+cx-dy)=0$,

or when $$d(x^2+y^2)+(b+d)\,x+(c-a)\,y+b=0 \quad\text{(i.)}.$$

Suppose now that $\gamma=a'+b'i,\quad \delta=c'+d'i.$

Then $(\gamma+t\delta)/(1+t)$ is real when

$$d'(x^2+y^2)+(b'+d')\,x+(c'-a')\,y+b'=0 \quad\text{(ii.)}.$$

The circles (i.) and (ii.) intersect at the *fixed* point $(-1,0)$, and at another point whose coordinates are rational functions of a, a', &c.

The first point gives $t=-1$, and this makes $(\alpha+t\beta)/(t+1)$ and $(\gamma+t\delta)/(t+1)$ both infinite; the other leads to the one real point on the line joining (α,γ) to (β,δ).

14028. (G. H. HARDY.)—Reduce the evaluation of $\int_0^\pi \frac{\cos(p/q)\,\phi\,d\phi}{1+2t\cos\phi+t^2}$, where p, q are integers, $p<q$ and $t<1$, to the integration of a rational fraction. Prove, in particular, that

$$\int_0^\pi \frac{\cos\frac{1}{2}\phi\,d\phi}{1+2t\cos\phi+t^2}=\frac{2}{1+t}\,\frac{\tanh^{-1}\sqrt{t}}{\sqrt{t}};$$

and deduce (and also prove independently) that

$$\int_0^\pi \tan^{-1}\left(\frac{2t\sin\phi}{1-t^2}\right)\frac{d\phi}{\sin\frac{1}{2}\phi}=8\tan^{-1}\sqrt{t}\,\tanh^{-1}\sqrt{t}.$$

Solution by the PROPOSER.

Since
$$\frac{1-t^2}{1+2t\cos\phi+t^2}=1+2\sum_1^\infty(-t)^n\cos n\phi,\quad t<1,$$

$$\int_0^\pi\frac{\cos a\phi\,d\phi}{1+2t\cos\phi+t^2}=\frac{1}{1-t^2}\int_0^\pi\left\{\cos a\phi+2\sum_1^\infty(-t)^n\cos a\phi\cos n\phi\right\}d\phi$$
$$=\frac{\sin a\pi}{1-t^2}\left\{\frac{1}{a}+2\sum_1^\infty\left(\frac{1}{n+a}-\frac{1}{n-a}\right)t^n\right\},$$

if $a<1$. That is to say,
$$=\frac{\sin a\pi}{1-t^2}\left\{t^{-a}\int_0^t\frac{t^{a-1}\,dt}{1-t}-t^a\int_0^t\frac{t^{-a}\,dt}{1-t}\right\}.$$

If $a=p/q$, this is
$$\frac{q\sin(p/q)\pi}{1-t^2}\left\{t^{-(p/q)}\int_0^{t^{1/q}}\frac{\tau^{p-1}\,d\tau}{1-\tau^q}-t^{p/q}\int_0^{t^{1/q}}\frac{\tau^{q-p-1}\,d\tau}{1-\tau^q}\right\}.$$

If $p=1,\ q=2$, we get
$$\frac{2}{1-t^2}\left\{t^{-\frac{1}{2}}\int_0^{t^{\frac{1}{2}}}\frac{d\tau}{1-\tau^2}-t^{\frac{1}{2}}\int_0^{t^{\frac{1}{2}}}\frac{d\tau}{1-\tau^2}\right\}=\frac{1}{1-t^2}\left(t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)\log\frac{1+t^{\frac{1}{2}}}{1-t^{\frac{1}{2}}}$$
$$=\frac{2}{1+t}\,\frac{\tanh^{-1}\sqrt{t}}{\sqrt{t}}.$$

(This integral and the corresponding integral with a sine in the numerator may also be easily obtained by contour integration applied to $\int\frac{z^{a-1}\,dz}{1+z}$.) That is to say,
$$\int_0^\pi\frac{\cos\frac{1}{2}\phi\,d\phi}{1+2t\cos\phi+t^2}=2\tanh^{-1}\sqrt{t}\,\frac{d}{dt}\left(2\tan^{-1}\sqrt{t}\right).$$

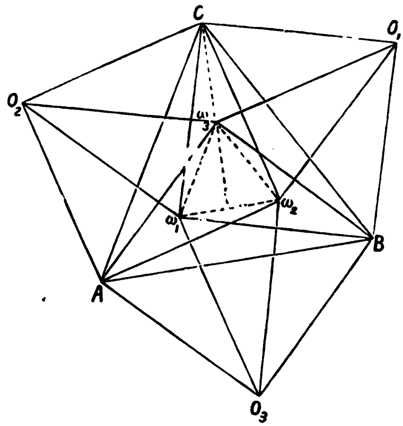
Similarly, $$\int_0^\pi\frac{\cos\frac{1}{2}\phi\,d\phi}{1-2t\cos\phi+t^2}=2\tan^{-1}\sqrt{t}\,\frac{d}{dt}\left(2\tanh^{-1}\sqrt{t}\right).$$

Adding and integrating from 0 to t, we find
$$\int_0^\pi\tan^{-1}\left(\frac{2t\sin\phi}{1-t^2}\right)\frac{d\phi}{\sin\frac{1}{2}\phi}=8\tan^{-1}\sqrt{t}\,\tanh^{-1}\sqrt{t}.$$

This may be verified independently, as follows:—
$$\int_0^\pi\tan^{-1}\left(\frac{2t\sin\phi}{1-t^2}\right)\frac{d\phi}{\sin\frac{1}{2}\phi}=2\sum_1^\infty\frac{t^{2n+1}}{2n+1}\int_0^\pi\frac{\sin(2n+1)\phi}{\sin\frac{1}{2}\phi}\,d\phi$$
$$=8\sum_1^\infty\left(1-\tfrac{1}{3}\ldots+\frac{1}{4n+1}\right)\frac{t^{2n+1}}{2n+1}$$
$$=8\left[t+\left(1-\tfrac{1}{3}+\tfrac{1}{5}\right)\tfrac{1}{3}t^3+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}\right)\tfrac{1}{5}t^5\ldots\right]$$
$$=8\left[\left(t^{\frac{1}{2}}-\tfrac{1}{3}t^{\frac{3}{2}}+\tfrac{1}{5}t^{\frac{5}{2}}\ldots\right)\left(t^{\frac{1}{2}}+\tfrac{1}{3}t^{\frac{3}{2}}+\tfrac{1}{5}t^{\frac{5}{2}}\ldots\right)\right]$$
$$=8\tan^{-1}\sqrt{t}\,\tanh^{-1}\sqrt{t}.$$

14473. (W. S. Cooney.)—Construct the triangle, being given *any three* of the following six points:—the centres of the squares described externally and internally on the sides.

Solution by the Proposer.



Let O_1, O_2, O_3, ω_1, ω_2, ω_3 be the centres of squares described externally and internally on sides of ABC.

By Quest. 13716, or easily from figure, O_1O_2 is perpendicular and equal to CO_3, for CO_2 and $C\omega_1$ are proportional to AC and BC, and $\angle O_2C\omega_1 = \angle C$; therefore $\triangle O_2C\omega_1$ is similar to ABC; therefore $O_2\omega_1 = AO_3$. Similarly $O_3\omega_1 = AO_2$; therefore $AO_2\omega_1O_3$ is a parallelogram; as are also $BO_1\omega_2O_3$, $CO_2\omega_3O_1$, $A\omega_2O_1\omega_3$, $B\omega_3O_2\omega_1$, and $C\omega_1O_3\omega_2$; therefore evidently $CO_3 = O_1O_2$, and $C\omega_3$ is also equal and perpendicular to $\omega_1\omega_2$; therefore, if O_1, O_2, O_3 or ω_1, ω_2, ω_3 be given, the perpendiculars of the triangles being drawn, the construction is obvious in each case. If ω_1, O_2, O_3 or O_1, ω_2, ω_3 be given, the completion of the parallelogram in each case gives A. If O_1, ω_1, ω_2 be given, B and C are known, which disposes of the twenty cases; therefore, &c.

This construction shows that the triangles $O_1O_2O_3$ and $\omega_1\omega_2\omega_3$ are so related that the perpendiculars of each bisect the sides of the other, and pass through A, B, C, for CO_3 bisects $\omega_1\omega_2$, and $C\omega_3$ bisects O_1O_2.

14329. (J. A. Third, M.A., D.Sc.)—L, L′, M, M′, N, N′ are points on a conic. LL′, MM′, NN′ form the triangle ABC; MN′, NL′, LM′ the triangle A′B′C′; and M′N, N′L, L′M the triangle A″B″C″. The straight line AA′A″ meets BC, B′C′, B″C″ in X, X′, X″ respectively; the straight line BB′B″ meets CA, C′A′, C″A″ in Y, Y′, Y″ respectively; and the straight line CC′C″ meets AB, A′B′, A″B″ in Z, Z′, Z″ respectively. Show that the following are triads of concurrent lines:—

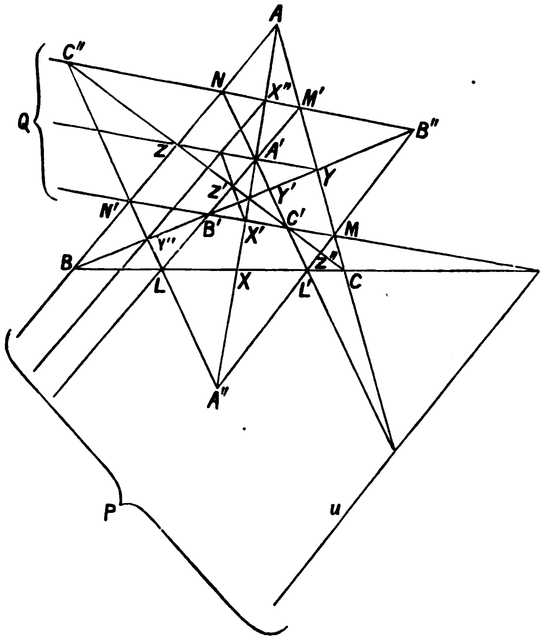
YZ, Z′X′, X″Y″; ZX, X′Y′, Y″Z″; XY, Y′Z′, Z″X″;
YZ, Z″X″, X′Y′; ZX, X″Y″, Y′Z′; XY, Y″Z″, Z′X′;

and that the points of concurrence lie on a conic.

Solution by the Proposer.

The pairs AB and A′B′, BC and B′C′, CA and C′A′ meet on the same

PASCAL line u. Let P be the point of intersection of AB and A′B′. Join



PX″, PY″. Then, since u is a diagonal of the quadrilaterals ANA′M′ and BLB′N′, PX″ and PY″ are harmonic conjugates of u with respect to PA and PA′, and therefore coincide. Thus, AB, A′B′, and X″Y″ are concurrent. Similarly, B′C′, B″C″, YZ are concurrent, say in Q. Hence the triangles X′Z′A′ and QZN are copolar with respect to C′. Therefore Z′X′ and YZ intersect on the same line as the pairs A′B′, AB and X′A′, QN. Thus the first triad consists of concurrent lines. A similar proof holds for each of the other triads.

Again, the triangles XYZ and X′Y′Z′ are obviously in perspective. Therefore the six points of intersection of the sides of the one with the non-corresponding sides of the other lie on a conic.

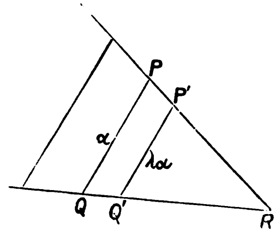
14173. (D. BIDDLE.)—The sides of a triangle being given, $a > b > c$, draw a line parallel to one of them, such that the quadrilateral formed shall have the maximum area possible in proportion to its perimeter, and find both area and perimeter.

Solution by W. C. STANHAM, B.A.

Let PQR be a triangle whose sides are α, β, γ. Parallel to PQ ($=\alpha$) draw P′Q′, so that $P'Q' = \lambda\alpha$. Then, if Δ and s denote area PQR and $\frac{1}{2}(\alpha+\beta+\gamma)$ respectively, and if $\alpha/s = \mu$,

area $\quad PP'Q'Q = \Delta(1-\lambda^2)$ ....(1),

perimeter $PP'Q'Q = 2s(1-\lambda)+2\lambda\alpha$...(2).



The ratio which is to be a maximum is therefore $(1-\lambda^2)/(1-\lambda+\mu\lambda)$, which for any value of λ is clearly a maximum when $a = c$, the least side. Differentiating, the value of λ which gives a maximum is found to be

$$\left[1/(1-\mu)-\left\{1/(1-\mu)^2-1\right\}^{\frac{1}{2}}\right].$$

Substituting this value of λ in (1) and (2), and putting $\alpha = c$, $\mu = c/s$, $s = \frac{1}{2}(a+b+c)$, the required values are obtained.

14519. (Professor U. C. GHOSH.)—Find the sum of the products of the terms of the geometric series a, a^2, a^3, a^4, ..., a^n, taken r at a time, r being less than n.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $\quad s_1 = a+a^2+a^3+\ldots+a^n, \quad s_r = a^r+a^{2r}+a^{3r}+\ldots+a^{nr}.$

Let S_r = required sum of products of the terms of s_1, taken r together.
Let X_r = sum of terms in S_r containing a *particular* term a^x.
Let S'_r = sum of terms in S_r free from a *particular* term a_x.
Let Σ denote summation with respect to x.

Here $\quad s_1 = \Sigma a^x = a\,.\,(a^n-1)/(a-1),$

$$s_r = \Sigma a^{rx} = a^r\,.\,(a^{nr}-1)/(a^r-1);$$

and $\quad X_r = a^x\,.\,S'_{r-1}, \quad S_r = \Sigma\,(a^x\,.\,S'_{r-1}).$

Hence $\quad S_1 = s_1, \quad S'_1 = S_1 - a^x,$

$$S_2 = \Sigma a^x(S_1-a^x) = S_1\,.\,\Sigma a^x - \Sigma a^{2x} = S_1 s_1 - s_2 = s_1^2 - s_2,$$

$$S'_2 = S_2 - a^x\,.\,S'_1, \quad S_3 = \Sigma a^x(S_2 - a^x\,.\,S'_1) = \Sigma(a^x\,.\,S_2 - a^{2x}\,.\,S_1 + a^{3x});$$

therefore $\quad S_3 = s_1S_2 - s_2S_1 + s_3 = s_1^3 - 2s_1s_2 + s_3,$

$$S'_3 = S_3 - a^x\,.\,S'_2, \quad S_4 = \Sigma a^x S'_3 = \Sigma(a^xS_3 - a^{2x}S_2 + a^{3x}S_1 - a^{4x});$$

therefore $\quad S_4 = s_1S_3 - s_2S_2 + s_3S_1 - s_4 = s_1^4 - 3s_1^2s_2 + s_2^2 + 2s_1s_3 - s_4.$

The law of formation of each sum (S_r) from the preceding (S_{r-1}) is now clear, all the terms being of equal weight (r)

$$S_r = s_1S_{r-1} - s_2S_{r-2} + s_3S_{r-3} - \&c.\ \ldots\ + (-1)^{r-1}s_r.$$

14549. (J. A. THIRD, M.A., D.Sc.)—K is a conic circumscribed to a triangle ABC; P is a point on it; Q is the isogonal conjugate of P with respect to the triangle; R is the point where PQ meets K again; L, M, N are the points where AR, BR, CR meet BC, CA, AB respectively; X, Y, Z are variable points, Y lying on QM and Z on QN, such that the pairs AY and AZ, BZ and BX, CX and CY are equally inclined to the bisectors of the angles A, B, C respectively. Prove that the locus of X is QL, and that the locus of the point of concurrence of AX, BY, CZ is K.

The construction usually given for KIEPERT's hyperbola (see CASEY's *Analytical Geometry*, p. 442) is a particular case of the foregoing.

Solution by Rev. J. CULLEN.

Taking ABC for the triangle of reference and P to be the point (x, y, z), we have $K \equiv \Sigma l/x = 0$. It is easy to see that R is the point

$$ax^2(y^2-z^2)/l = \ldots = \ldots \qquad (1).$$

Now, if X, Y, Z be the points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), then

$$AY \equiv \beta z_2 - \gamma y_2 = 0, \qquad AZ \equiv \beta z_3 - \gamma y_3 = 0, \quad \ldots.$$

These lines are equally inclined to the bisectors of A, ..., if

$$y_2 y_3 = z_2 z_3, \quad z_1 z_3 = x_1 x_3, \quad x_1 x_2 = y_1 y_2.$$

L, M, and N are given by putting, successively, $\alpha = 0$, $\beta = 0$, and $\gamma = 0$ in (1). Q is $\alpha x = \ldots = \ldots$. Hence

$$QM \equiv \begin{vmatrix} x_2, & y_2, & z_2 \\ \dfrac{1}{x}, & \dfrac{1}{y}, & \dfrac{1}{z} \\ \dfrac{l}{x^2(y^2-z^2)}, & 0, & \dfrac{n}{z^2(x^2-y^2)} \end{vmatrix} = 0 \equiv p_2 x_2 + q_2 y_2 + r_2 z_2,$$

and $QN \equiv p_3 x_3 + q_3 y_3 + r_3 z_3 = 0$, where $p_3, \ldots$, are the coefficients of $x_3, \ldots$, when a similar determinant is expanded.

Eliminating x_3 and y_3, we have $p_3 z_1 y_2 + q_3 x_1 z_2 + r_3 x_1 y_2 = 0$. Using $x_1 x_2 = y_1 y_2$ and $\Sigma p_2 x_2 = 0$, we get

$$x_1(r_2 r_3 - q_2 q_3) - y_1 q_3 p_2 + z_1 r_2 p_3 = 0 \qquad (2)$$

as the locus of X; or, expressed in terms of x, y, z and l, m, n, (2) may be written

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ \dfrac{1}{x}, & \dfrac{1}{y}, & \dfrac{1}{z} \\ 0, & \dfrac{m}{y^2(z^2-x^2)}, & \dfrac{n}{z^2(x^2-y^2)} \end{vmatrix} = 0,$$

which is the equation of QL.

From the equations

$$AX \equiv \beta z_1 - \gamma y_1 = 0, \quad BY \equiv \gamma x_2 - \alpha z_2 = 0, \quad CZ \equiv \alpha y_3 - \beta x_3 = 0,$$

we find the point of concurrence to be $\alpha y_1 z_2 = \beta z_1 x_2 = \gamma x_2 z_1$. Therefore

$$l/\alpha + m/\beta + n/\gamma = l y_1 z_2 + m z_1 x_2 + n x_2 y_1 \qquad (3).$$

Eliminating x_2 and z_2 by means of the preceding equations, we find that the dexter of (3) is the sinister of (2) multiplied by the factor $y(z^2-x^2)/x(y^2-z^2)$. Therefore the point of concurrence lies on $\Sigma l/\alpha = 0$, *i.e.*, K.

[The PROPOSER adds: "The theorem is true not merely when the pairs P and Q, AY and AZ, &c., are isogonal conjugates, but also when they are isotomic conjugates, and generally when the members of each pair are derived the one from the other by any reciprocal transformation (α, β, γ becoming $1/p\alpha$, $1/q\beta$, $1/r\gamma$). Quest. 14515 could be similarly generalized."]

14563. (R. KNOWLES.)—From a point T tangents TP, TQ are drawn to the parabola $y^2 = 4ax$. Prove that when the circle TPQ touches the parabola the locus of T is the parabola $y^2 = 4a(2a-x)$.

Solution by R. P. PARANJPYE, B.A.; G. D. WILSON, B.A.; *and many others.*

Let T be the point $\xi\eta$. The equation of PQ is $y\eta = 2a(x+\xi)$.

The tangent at the point where the circle touches the parabola is parallel to the line $y\eta = -2ax$ (since PQ and the tangent are equally inclined to the axis on opposite sides), and is therefore

$$y = -(2a/\eta)x - \tfrac{1}{2}\eta.$$

Therefore the equation of the circle is of the form

$$\{y\eta - 2a(x+\xi)\}\{y\eta + 2ax + \tfrac{1}{2}\eta^2\} + \lambda(y^2 - 4ax) = 0.$$

Since this is a circle, $\quad \eta^2 + \lambda = -4a^2$;

therefore $\quad \lambda = -(\eta^2 + 4a^2)$;

therefore the equation of the circle is

$$\{y\eta - 2a(x+\xi)\}\{y\eta + 2x + \tfrac{1}{2}\eta^2\} = (\eta^2 + 4a^2)(y^2 - 4ax).$$

But this passes through T (ξ, η); therefore, omitting the factor $\eta^2 - 4a\xi$,

we get $\quad \tfrac{3}{2}\eta^2 + 2a\xi = \eta^2 + 4a^2$; therefore $\eta^2 = 4a(2a-\xi)$,

whence the proposition.

4953. (W. J. C. MILLER, B.A.)—A king is placed at random on a clear chess board, and then, similarly, (1) a bishop, or (2) a rook. Find, in each case, the chance that the king is in check so as to be unable to take the attacking piece; and find also (3) the chance of check, with or without the power of taking, for any combination of two or three of the pieces. [If we estimate the powers of the pieces (α) by the chances of *simple* check, as investigated in the solution of Quest. 3314, *Reprint*, Vol. xv., pp. 50, 51, in January 1871; (β) by the chances of *safe* check, as shown in an interesting paper by H. M. TAYLOR in the *Philosophical Magazine* for March, 1876; (γ) by the *results* given in the *Berliner Schachzeitung*, we have the relative values of the knight, bishop, rook, queen as (α) 3 : 5 : 8 : 13; (β) 3 : $3\frac{1}{4}$: 6 : $9\frac{1}{4}$; (γ) 3 : $3\frac{1}{3}$: $4\frac{1}{2}$: $9\frac{1}{2}$.]

Solution by Professor SANJÁNA.

Simple check.—1. Knight.—When the king occupies one of the 16 squares marked *a*, this piece can check from 8 squares; on the 16 marked *b*, from 6 squares; on the 20 marked *c*, from 4 squares; on the 8 marked *d*, from 3 squares; and on each of the 4 corner squares, from 2 squares: altogether 336 squares. Thus the chance of checking the king is $336 \div 64 \times 63 = \frac{1}{12}$.

	d	*c*	*c*	*c*	*c*	*d*	
d	*c*	*b*	*b*	*b*	*b*	*c*	*d*
c	*b*	*a*	*a*	*a*	*a*	*b*	*c*
c	*b*	*a*	*a*	*a*	*a*	*b*	*c*
c	*b*	*a*	*a*	*a*	*a*	*b*	*c*
c	*b*	*a*	*a*	*a*	*a*	*b*	*c*
d	*c*	*b*	*b*	*b*	*b*	*c*	*d*
	d	*c*	*c*	*c*	*c*	*d*	

1. KNIGHT.

2. Bishop.—When the king occupies one of the four squares marked *a*, this piece can check from 13 squares; on the 12 marked *b*, from 11 squares; on the 20 marked *c*, from 9 squares; and on each of the 28 border squares, from 7 squares: altogether 560 squares. Thus the chance of checking the king is $560 \div 64 \times 63 = \frac{5}{36}$.

3. Rook.—This piece commands 14 squares when placed on any square. Hence the chance of checking the king is $14 \div 63 = \frac{2}{9}$.

4. Queen.—This piece commands, in any position, the squares commanded by both bishop and rook. These being mutually exclusive, we have $560 + 14 \times 64$ squares. Thus the chance of checking the king is

$$(560 + 14 \times 64) \div 64 \times 63 = \frac{13}{36}.$$

Multiplying each of these chances by 36, we find the relative values to be

3 : 5 : 8 : 13 (*a*).

	c	*c*	*c*	*c*	*c*	*c*	
	c	*b*	*b*	*b*	*b*	*c*	
	c	*b*	*a*	*a*	*b*	*c*	
	c	*b*	*a*	*a*	*b*	*c*	
	c	*b*	*b*	*b*	*b*	*c*	
	c	*c*	*c*	*c*	*c*	*c*	

2. BISHOP.

Safe check.—1. Knight.—As this piece is never close to the king when checking, the chance of a *safe* check is still $\frac{1}{12}$.

2. Bishop.—For each of the squares marked *a*, *b*, *c*, we have now to diminish the checking positions by 4; for the 24 border squares by 2; for the 4 corner squares by 1. Hence the chance of a *safe* check is $(4 \times 9 + 12 \times 7 + 20 \times 5 + 24 \times 5 + 4 \times 6) \div 64 \times 63 = \frac{13}{144}$.

3. Rook.—For each of the internal 36 squares, we have now to diminish the checking positions by 4; for the 24 border squares by 3;

for the 4 corner squares by 2. Hence the chance of a *safe* check is $(36 \times 10 + 24 \times 11 + 4 \times 12) \div 64 \times 63 = \frac{1}{6}$.

4. Queen.—The chance of a *safe* check is $\frac{13}{144} + \frac{1}{6} = \frac{37}{144}$.

Multiplying each of these chances by 36, we find the relative values to be $3 : 3\frac{1}{4} : 6 : 9\frac{1}{4}$ (β).

I have not seen any of the papers referred to, and nothing is given as to the *principle* on which the method of the *Berliner Schachzeitung* is based.

[For further remarks on the Relative Values of the Chessmen, see Vol. XL., pp. 85–97.]

14213. (Robert W. D. Christie.)—If

$$A_n = m^n - n.m^{n-2} + \frac{n\ n-3}{2\,!} m^{n-4} - \frac{n.n-4.n-5}{3\,!} m^{n-6} + \ldots$$

for all integral values of m and n, then

$$X^{2n} - A_n X^n + 1 = (X^2 - mX + 1)(a_1 X^{2n-2} + a_2 X^{2n-3} + a_3 X^{2n-4} + \ldots + a_2 X + 1),$$

where a_n = a series allied to A_n. *E.g.*—If $m = 5$, $n = 3$, then

$$x^6 - 110x^3 + 1 \equiv (x^2 - 5x + 1)(x^4 + 5x^3 + 24x^2 + 5x + 1).$$

There are two other allied theorems for positive values of A_n and m; it is required to establish them.

Solution by the Proposer.

Let $\alpha = m + \sqrt{(m^2-4)}/2$, $\beta = m - \sqrt{(m^2-4)}/2$; then we have

$$A_n = \alpha^n + \beta^n$$

$$= m^n - n.m^{n-2} + \frac{n.n-3}{2\,!} m^{n-4} - \frac{n.n-4.n-5}{3\,!} m^{n-6} + \ldots \frac{n.n-r-1\,!}{r\,!\ n-2r\,!} m^{n-2r}$$

(which is the generalized form of the "continuant" series 1, 1, 2, 3, 5, 8, 13, &c.).

$$\text{Also}\quad \frac{1}{n}\frac{du}{dm} A^n = m^{n-1} - n-2.m^{n-3} + \frac{n-3.n-4}{2\,!} m^{n-5} \ldots$$

$$\ldots \frac{n-r-1}{r\,!\ n-2r-1\,!} m^{n-2r-1}.$$

(i.) We have to show that $x^{2n} - A_n x^n + 1 = (x^2 - mx + 1)(M)$, where

$$(M) = a_0 x^{2n-2} + a_1 x^{2n-3} + a_2 x^{2n-4} + \ldots a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 x^0.$$

On multiplying this series (M) by $x^2 - mx + 1$, we find the law of the coefficients, excepting the middle one, is $a_n - ma_n + a_{n-2}$, and for the middle one $ma_n - 2a_{n-1}$. Now the values of these are respectively 0 and $\alpha^n + \beta^n$. Hence the theorem.

(ii., iii.) The theorems $x^{2n} + A_n x^n + 1 = (x^2 + mx + 1)$ (M) when n is odd, and $x^{2n} - A^n x_n + 1 = (x^2 + mx + 1)$ (M) when n is even, are solved in a similar way. The above is a generalization of several known theorems, *e.g.*,

$m = 1$ and $n = 6k \pm 1$, we have $x^{2n} - x^n + 1 = (x^2 - x + 1)$ (M) ... (1),

$n = 6k \pm 2$,, $x^{2n} + x^n + 1 = (x^2 - x + 1)$ (M) ... (2),

$m = -1$ and $n = 3k \pm 1$,, $x^{2n} + x^n + 1 = (x^2 + x + 1)$ (M) ... (3).

For odd powers we can establish cognate theorems, *e.g.*,

(iv.) $x^{2n+1} - A_n(x^{n+1}+x^n)+1 = (x^2-mx+1)$ (M), where A_n is the nth term of a cognate series.

(v.) In FERMAT's theorem we easily get q in terms of p (a prime), viz., $2^{p-1}-1 = pq = p\{2^{p-3} - \frac{p-3}{2!}2^{p-5} + \frac{p-4\,.\,p-5}{3!}2^{p-7} - \dots$

Other theorems are :—

(vi.) $x^{2n+1}+1 \equiv (x^2-x+1)$ (M) if $n = 3k+1$,

(vii.) $(x^{n+1}+1)(x^n+1) \equiv (x^2-x+1)$ (M) if $n = 6k+2$ or $6k+3$,

(viii.) $x^{2n+1}-(x^{n+1}+x^n)+1 \equiv (x^2-x+1)$ (M) if $n = 6k$ or $6k-1$,

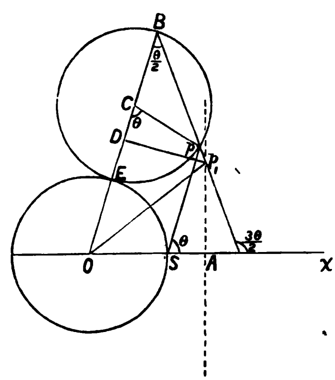
(ix.) $x^{2n+1}+2x^{n+1}+2x^n+1 \equiv (x^2+x+1)$ (M) if $n = 3k+1$,

(x.) $(x^{n+1}-1)(x^n-1) = (x^2+x+1)$ (M) if $n = 3k$ or $3k-1$,

&c., &c.

14495. (R. C. ARCHIBALD, M.A., Ph.D.)—The points p_1, p_2, p_3, where any three parallel tangents to a cardioid cut the double tangent, are joined to the centre O of the fixed circle. Prove that the angles p_1Op_2, p_2Op_3 are each equal to 60°.

Solution by the PROPOSER; H. M. TAYLOR, F.R.S.; J. H. TAYLOR, M.A.; *and* Professor SANJÁNA.



The cardioid is generated by a point P of the circumference of a circle, with centre C rolling on a fixed equal circle with centre O; their point of contact E formerly coincided with P at the point S on the fixed circle or base; S is the cusp of the cardioid. The line SP makes an angle θ with OS produced, and is parallel to the line OEC, which, when produced, meets the generating circle in B. Hence

$$\angle EOS = ECP = \theta,$$

and $\angle CBP = \frac{1}{2}\theta$.

But the line BP is tangent to the cardioid at P; it therefore makes with OS produced an angle $\frac{3}{2}\theta$. Hence *the line joining the points of contact of parallel tangents to a cardioid subtends an angle of* 120° *at the cusp.*

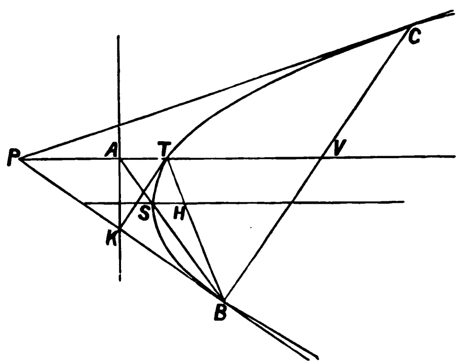
The line OS is produced to A, so that 2SA = OS; as is well known, the line through A perpendicular to OA is the double tangent of the cardioid traced by P. The perpendicular to OB at its middle point D meets the double tangent in p_1, the same point as that where the tangent to the cardioid at P meets it. For, joining Op_1, we have the triangles p_1OD, p_1OA, equal in all respects, since OD = OA. Hence

$$\angle p_1OA = p_1OD = DBp_1 = \tfrac{1}{2}\theta.$$

Since, then, $\angle SOp_1 = \frac{1}{2}\theta = \frac{1}{2}\angle PSA$, we deduce the required result from the theorem just given above. It may be noted, finally, that the points O, S, p_1, P lie on a circle.

14484. (Professor A. Droz-Farny.)—On joint un point A de la directrice d'une parabole au sommet S de cette dernière. AS coupe la courbe en un second point B. La tangente en B rencontre en P le diamètre de la parabole mené par A. Tirons la deuxième tangente PC. La droite CB est normale en B à la parabole.

I. *Solution by* Lionel E. Reay, B.A.; H. W. Curjel, M.A.; *and* Professor Sañjána.



It is a known theorem that, if THB be a focal chord and BSA be drawn, AT is parallel to SH. Hence, if ASB be drawn and AT be diameter through T, TB passes through H, the focus.

Therefore tangent at T meets PB at K, on the directrix, and TK is perpendicular to BK.

But TK is parallel to CB.

Therefore CB is perpendicular to PB.

II. *Solution by* J. H. Taylor, M.A.; *and* F. H. Peachell, B.A.

Equation of parabola $y^2 = 4ax$ .... (1).

Take point A $(-a, b)$; equation of AS: $y = -(b/a)x$. Coordinates of B are $4a^3/b^2$, $-4a^2/b$. Equation of tangent BP:

$$(-2a/b)\,y = x + 4a^3/b^2 \;\;\ldots\ldots\ldots\ldots\ldots\; (2).$$

Therefore $m = -b/2a$. Hence, coordinates of P are

$$x = -2a - 4a^3/b^2, \qquad y = b.$$

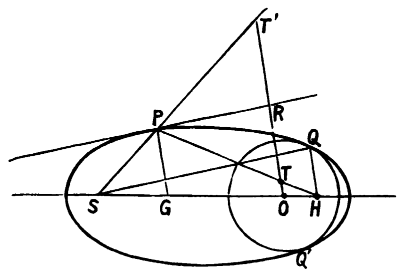
Therefore equation of chord of contact BC is

$$by = 2a\,(x - 2a - 4a^3/b^2) \;\;\ldots\ldots\ldots\ldots\ldots\; (3);$$

here $m' = 2a/b$, and $mm' = -1$; therefore $\angle CBP = 90°$, or CB is the normal at B.

14554. (R. F. Davis, M.A.)—Given a conic and a circle having double contact, prove that the envelope of a variable circle, whose centre lies on the first and which intersects orthogonally the second, consists of two fixed circles.

Solution by C. E. McVicker, M.A.



Let OQ be radius of fixed circle, P the centre of the variable circle, PR, PG the tangent and normal at P to the conic.

Draw OR perpendicular to tangent, meeting the focal radii SP, PH in T′, T respectively.

$OT : OH = GP : GH.$

$OT' : OS = GP : GS;$

therefore

$$OT.OT' : OS.OH = GP^2 : GS.GH$$
$$= \text{constant} = OQ^2 : OS.OH;$$

therefore $OT.OT' = OQ^2$, proving that T, T′ are inverse with respect to the fixed circle. It follows immediately that PT (or PT′) must be the radius of the variable circle.

Again, $ST'/SO = SP/SG = 1/e = SQ/SO;$

therefore $ST' = SQ$. Hence the variable circle, centre P and radius PT′, touches the fixed circle, centre S and radius SQ. Similarly, it touches another, centre H and radius HQ. The complete envelope is therefore a pair of circles concentric with the foci and co-intersecting at Q, Q′.

13754. (Professor Umes Chandra Ghosh.)—Aa, Bb, and Cc are the perpendiculars from the vertices of the triangle ABC on the opposite sides. O is the orthocentre of the triangle ABC. O_1, O_2, and O_3 are the points of intersection of the opposite connectors AO, bc; BO, ca; CO, ab of the tetrastigms AbOc, BaOc, and CbOa. AO_3, AO_2 cut BC at a_1 and a_2; BO_1, BO_3 cut AC at b_1 and b_2; and CO_1, CO_2 cut AB at c_2 and c_1. Prove that

(i.) $$Cc\,(1/cc_1 + 1/cc_2) + Bb\,(1/bb_1 + 1/bb_2) + Aa\,(1/aa_1 + 1/aa_2)$$
$$= 6\,Oc/Ac \;\; Bb/Cb \;\; Aa/Ba;$$

(ii.) $$Cc\,(1/cc_1 - 1/cc_2) + Bb\,(1/bb_1 - 1/bb_2) + Aa\,(1/aa_1 - 1/aa_2) = 0.$$

Solution by W. J. Greenstreet, M.A.

In the triangle AaC we have

$$aa_1 \,.\, Cb \,.\, AO = Oa \,.\, Ab \,.\, a_1C = Oa \,.\, Ab \,.\, (aC - aa_1).$$

Substituting obvious values for Cb, AO, Oa, Ab, aC, we get, after arranging, $\quad \mathrm{A}a/(aa_1) = b \sin \mathrm{C}/(aa_1) = \tan \mathrm{B} + 2 \tan \mathrm{C}.$

Similarly, $\quad \mathrm{A}a/(aa_2) = \tan \mathrm{C} + 2 \tan \mathrm{B}.$

Therefore

$$\Sigma \mathrm{A}a(1/aa_1 + 1/aa_2) = \Sigma 3 (\tan \mathrm{B} + \tan \mathrm{C}) = 6\Sigma \tan \mathrm{A} = 6\Pi \tan \mathrm{A} = 6\Pi \mathrm{A}a/\mathrm{B}a,$$

and $\quad \Sigma \mathrm{A}a(1/aa_1 - 1/aa_2) = \Sigma (\tan \mathrm{B} - \tan \mathrm{C}) = 0.$

5731. (R. A. Roberts, M.A.)—If from

$$(1)\ \ x^2/a^2 + y^2/b^2 - 4 = 0, \qquad (2)\ \ a^2x^2 + b^2y^2 - (a^2+b^2)^2 = 0,$$

tangents be drawn to $x^2/a^2 + y^2/b^2 - 1 \equiv \mathrm{S} = 0$, prove that they form, with their chord of contact, a triangle whose (1) centre of gravity, (2) intersection of perpendiculars, lies on $\mathrm{S} = 0$.

Solution by Professor Sanjána.

Let the tangents be drawn from T, and let α and β be the eccentric angles of the points of contact. Firstly, let $\bar{x}$, $\bar{y}$ be the coordinates of the centroid. Then T is given by

$$x/a = \cos \tfrac{1}{2}(\beta+\alpha)/\cos \tfrac{1}{2}(\beta-\alpha), \qquad y/b = \sin \tfrac{1}{2}(\beta+\alpha)/\cos \tfrac{1}{2}(\beta-\alpha);$$

hence, from equation (1), we get

$$1 = 4\cos^2 \tfrac{1}{2}(\beta-\alpha), \quad \text{or} \quad \cos \tfrac{1}{2}(\beta-\alpha) = \tfrac{1}{2}.$$

Now $3\bar{x} = a\cos\alpha + a\cos\beta + a\cos\tfrac{1}{2}(\beta+\alpha)/\cos\tfrac{1}{2}(\beta-\alpha) = 3a\cos\tfrac{1}{2}(\beta+\alpha)$, on reduction; so also

$$3\bar{y} = 3b\sin\tfrac{1}{2}(\beta+\alpha); \quad \text{therefore} \quad \bar{x}^2/a^2 + \bar{y}^2/b^2 = 1,$$

or the centroid lies on S. Secondly, let $\bar{x}$, $\bar{y}$ be the coordinates of the orthocentre. As T now lies on the locus (2), we get

$$a^4\cos^2\tfrac{1}{2}(\beta+\alpha) + b^4\sin^2\tfrac{1}{2}(\beta+\alpha) = (a^2+b^2)^2\cos^2\tfrac{1}{2}(\beta-\alpha) \ \ldots\ldots\ (3).$$

The perpendiculars from α and β on the tangents at β and α are respectively $\quad (y - b\sin\alpha)/(x - a\cos\alpha) = a\sin\beta/b\cos\beta,$

$$(y - b\sin\beta)/(x - a\cos\beta) = a\sin\alpha/b\cos\alpha;$$

so that $\quad \bar{x} = \dfrac{1}{a}\,\dfrac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}\left\{a^2 + a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta\right\},$

and $\quad \bar{y} = \dfrac{1}{b}\,\dfrac{\sin\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}\left\{b^2 + b^2\cos\alpha\cos\beta + a^2\sin\alpha\sin\beta\right\}.$

It will be found that

$$\bar{x} = \frac{1}{a}\,\frac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}\left\{(a^2+b^2)\cos^2\tfrac{1}{2}(\alpha-\beta) + (a^2-b^2)\sin^2\tfrac{1}{2}(\alpha+\beta)\right\}$$

$$= \frac{1}{a(a^2+b^2)}\,\frac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}\left\{(a^4-b^4)\sin^2\tfrac{1}{2}(\alpha+\beta) + a^4\cos^2\tfrac{1}{2}(\alpha+\beta) + b^4\sin^2\tfrac{1}{2}(\alpha+\beta)\right\},$$

$$= \frac{a^3}{a^2+b^2}\,\frac{\cos\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)} \quad \text{by means of (3);}$$

similarly, $\bar{y} = \dfrac{b^3}{a^2+b^2} \dfrac{\sin\frac{1}{2}(\beta+\alpha)}{\cos\frac{1}{2}(\beta-\alpha)}$.

Therefore $\bar{x}^2/a^2 + \bar{y}^2/b^2 = \dfrac{a^4\cos^2\frac{1}{2}(\beta+\alpha) + b^4\sin^2\frac{1}{2}(\beta+\alpha)}{(a^2+b^2)^2\cos^2\frac{1}{2}(\beta-\alpha)} = 1$,

or the orthocentre lies on S.

6679. (Rev. T. R. Terry, M.A., F.R.A.S.)—Show that the value of the continued fraction $\dfrac{N}{1+}\,\dfrac{N_1}{1+}\,\dfrac{N_2}{1+}\,\dfrac{N_3}{1+}\ldots$, where $N = nx$ and $N_r = \dfrac{n^2-r^2}{(2r-1)(2r+1)}x^2$, is $\dfrac{(1+x)^n-(1-x)^n}{(1+x)^n+(1-x)^n}$.

Solution by Professor Sanjána, M.A.

Take Euler's expression for $\tan nx$, viz.,

$$\frac{n\tan x}{1-}\,\frac{(n^2-1^2)\tan^2 x}{3-}\,\frac{(n^2-2^2)\tan^2 x}{5-}\ldots,$$

and put $\tan x = \iota x'$; thus we get

$$-\iota\tan n(\tan^{-1}\iota x') = \frac{nx'}{1+}\,\frac{(n^2-1)x'^2}{3+}\,\frac{(n^2-4)x'^2}{5+}\ldots$$

$$= \frac{nx'}{1+}\,\frac{\frac{1}{1.3}(n^2-1)x'^2}{1+}\,\frac{\frac{1}{3.5}(n^2-4)x'^2}{1+}\ldots.$$

Hence the given continued fraction $= -\iota\tan n(\tan^{-1}\iota x)$.

Let $x = \tanh u$; then $\tan^{-1}\iota x = \iota u$, and the fraction

$$= -\iota\tan n\iota u = \tanh nu$$

$$= \frac{C_1\tanh u + C_3\tanh^3 u + C_5\tanh^5 u + \ldots}{1 + C_2\tanh^2 u + C_4\tanh^4 u + \ldots}$$

$$= \frac{\frac{1}{2}\left\{(1+\tanh u)^n - (1-\tanh u)^n\right\}}{\frac{1}{2}\left\{(1+\tanh u)^n + (1-\tanh u)^n\right\}} = \frac{(1+x)^n-(1-x)^n}{(1+x)^n+(1-x)^n}.$$

14235. (R. Tucker, M.A.)—ABCD is a square. P, Q, R are points on AB, AD, BC, respectively, such that PQR is an equilateral triangle. Find maximum value of triangle. Show also that locus of intersection of AR, BQ, as P moves along AB, is a parabola.

Solution by Rev. T. Mitcheson, B.A.; *and the* Proposer.

(1) Let P_1 be mid-point of AB. With P_1 as centre, radius AB $(= 2a)$, describe circle meeting AD in Q_1, BC in R_1. Complete triangle $P_1Q_1R_1$, which is obviously equilateral.

Join P_1 and O_1 (the mid-point of Q_1R_1), and through O draw any intercept Q_2R_2 and OP_2 perpendicular to it, and complete triangle $P_2Q_2R_2$.
Since $\angle Q_1OQ_2 = \angle P_1OP_2$ (by Euc. VI. 4),

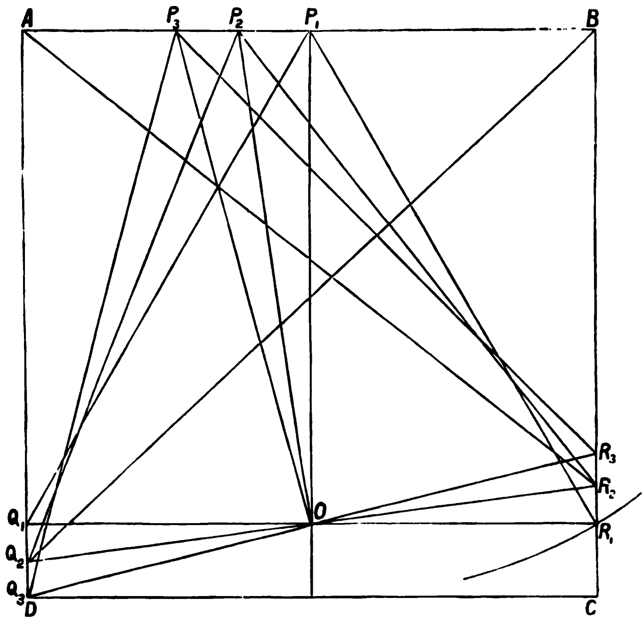
$$OQ_2 : OQ_1 = OP_2 : OP_1 \quad \text{or} \quad OQ_2 : OP_2 = OQ_1 : OP_1;$$

therefore (by Euc. VI. 6) OP_1Q_1 and OP_2Q_2 are similar, and OP_2Q_2 is equilateral.

Let Q_3 be at D, and complete (as before) the equilateral triangle $P_3Q_3R_3$. This latter is the *maximum*, while $P_1Q_1R_1$ is the *minimum*.

As $\angle P_3Q_3A = \angle R_3Q_3C$, each $= \frac{1}{12}\pi$ and $Q_3R_3 = 2a \sec \frac{1}{12}\pi$,

hence area of $P_3O_3R_3 = 2a^2 \sec^2 \frac{1}{12}\pi \sin \frac{1}{3}\pi$.



(2) Let $y_1 = Q_1Q_2 = R_1R_2$, and take O as origin. The equation to AR is

$$y - y_1 = \left\{(a\sqrt{3} - y_1)/-2a\right\}(x - a),$$

that of BQ is

$$y + y_1 = \left\{(a\sqrt{3} + y_1)/2a\right\}(x + a).$$

Eliminating y_1, we obtain

$$x^2\sqrt{3} = a(2y - a\sqrt{3}),$$

which is the locus of a parabola whose axis is collinear with P_1O, whose vertex cuts P_1O at its mid-point.

APPENDIX.

ON THE GEOMETRY OF CUBIC CURVES AND CUBIC SURFACES.

By W. H. BLYTHE, M.A.

WHEN investigating the properties of a cubic surface on which lie twenty-seven real straight lines I found that it was quite possible to define all such surfaces as the locus of the common vertex of six tetrahedra on six fixed bases among the volumes of which existed certain relations as to ratio, and that the plane sections of such surfaces could all be described by GRASSMAN's properties of a cubic curve.

It cannot be asserted that all plane cubic curves can actually be described by this method, nor is it fair to say that every cubic curve—for example, $y^2 = x^3$—has a real point of inflexion, nor yet that every cubic curve may be projected from one curve or one series of curves, but theoretically, and by use of imaginary points, certain projective properties true for one cubic are true for all. To put shortly, by means of analysis, what has been assumed, the equation of every cubic surface having twenty-seven real straight lines may be put into the form $\alpha\beta\gamma = \mu\delta\epsilon\eta$, where $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, $\epsilon = 0$, $\eta = 0$ represent the equations to planes and μ is a constant, and that any one of its plane sections has an equation of the same form taking the equations of straight lines in a plane, instead of planes in space.

The geometry of cubic curves and surfaces has been thoroughly investigated by REYE, in what is known as "Geometry of Position," but not, to the best of my knowledge, by any one by Euclidean geometry.

I suggest the following scheme:—

I. Plane cubic curves treated geometrically.
  (1) Circular cubics and their projections.
  (2) Central cubics.
II. Cubic surfaces having real straight lines.
  (1) Without nodes.
  (2) With nodes.
III. Cubic surfaces on which lie no real straight lines and their sections.

[NOTE.—In the following paper the first part of the first section only is given. I have also many interesting propositions on circular cubics, and a few on the second section, but little or nothing on the third, to which must be added a full investigation into the five different kinds of plane cubic curves from one of which any other can be projected by means of shadows, and the forms they take when projected.]

1. *Definition* (GRASSMAN).—A point P moves so that the straight lines PA, PE, PC, joining it to three fixed points A, E, C meet the sides of a triangle BDF in points lying in a straight line; then the locus of P is a cubic curve.

It will be found more convenient to assume that AB, CD, EF meet in a point K, and that A, B, C, D, E, F lie on a conic. This is only equivalent to assuming that the cubic curve has one real point of inflexion.

2. To show that P[ABEF] = P[CDEF].—Let PA meet BF in L, PC meet DF in M, and PE meet BD in N; then it is well known that the ratio compounded of the ratios of the alternate segments of the sides of the triangle BDF made by LMN is unity.

Observing that LB : LF :: area APB : area APF,
MF : MD :: area CPF : area CPD,
ND : NB :: area EPD : area EPB;
also area PLB : area PAB :: PL : PA,
area PLF : area PAF :: PL : PA,
so that LB : LF :: area PLB : area PLF;
and similar results for the points M and N, therefore the ratio compounded of the ratios area APB : area APF, area CPF : area CPD, area EPD : area EPB is unity; from which, by *Euclid* VI. 1, we arrive at the result P[ABEF] = P[CDEF].

3. It is possible from the result of the last article to determine when GRASSMAN's definition leads to the degeneration of a cubic curve into a conic and straight line.

For example: suppose the straight lines AB and EF coincide so that the four points A, B, E, F lie in one straight line; then the ratio P[ABEF] is constant, so that the ratio P[CDEF] is also constant. Therefore P moves in a fixed conic through C, D, E, F, or in the straight line ABEF.

4. A series of conics is described through four fixed points A, B, E, F, and the straight line EF is projected to infinity, one of the conics becoming a circle; then the series of conics becomes a series of circles described through two fixed points A, B. Then, if P[ABEF], P′[ABEF] be two anharmonic ratios determining any two conics of the series, and C_1, C_2 be the centres of the circles correspondiug to the conics,

$$P[ABEF] : P'[ABBF] :: OC_1 : OC_2$$

where O is a fixed point for the series.

Let M be the intersection of the diagonals of ABEF and let *mtt′* be one of the diagonals of the quadrilateral circumscribing one of the conics and touching it at A, B, E, F; then the direction of *mtt′* is the same for all conics of the series.

Let *mtt′* meet two conics of the series in P and P′ and EF in *p*, and let *t* and *t′* be the poles of EF with respect to the two conics; then it will be found that P[ABEF] : P′[ABEF] :: [*tt′pm*] : 1.

This problem being projective, it is more convenient to suppose ABEF a rectangle in the first case, when the above statement becomes evident.

Now project EF to infinity, and the conics into circles; p goes to infinity, t and t', the poles of the line at infinity, become the centres of the circles, and m a certain fixed point for the series. Therefore

$$P[ABEF] : P'_1[ABEF] :: OC_1 : OC_2,$$

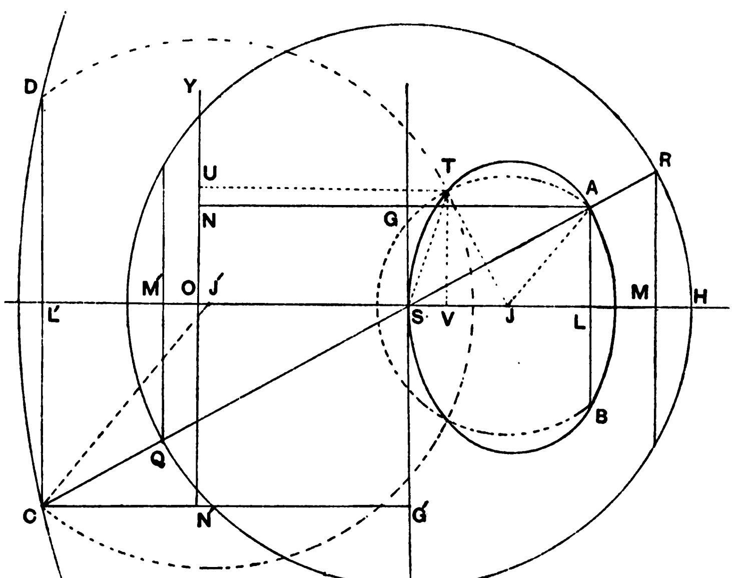
for the values of anharmonic ratios are unaltered by projection.

5. It is clear from the above that, if two series of conics be described through two sets of four points A, B, E, F, C, D, E, F, and each conic of one series intersects one of the other, so that P[ABEF] : P[CDEF] is a constant ratio, then P traces out a curve which projects into one described by the intersection of two series of circles each passing through two fixed points A, B and C, D, where, if L is the centre of any circle of one series and M the corresponding circle of the second series, then KL : HM in a fixed ratio, where K and H are fixed points in the straight lines along which L and M move.

6. One very simple well known case is that in which the ratio

$$P[ABEF] = P[CDEF];$$

for now K and H may be taken as the middle points of AB and CD after projection, and KL : HM :: AB : CD. Therefore, if EF is projected to infinity and the conics into circles, P becomes the locus of the intersection of similar segments of circles described upon two fixed straight lines AB and CD.



7. If we take GRASSMAN's definition of a cubic curve and the limitation that A, B, C, D, E, F lie on a conic and that AB, CD, EF meet at a point, and further project the conic into a circle and EF to infinity, we obtain the following construction for the projection of the cubic curve:—(Art. 6) A point P moves so that $\angle APB = CPD$, where AB, CD are two parallel chords of a circle, and, therefore, bisected at right angles by a fixed straight line OH. The curve is symmetrical with respect to OH, which may therefore be called its "axis."

AC and BD meet at S in OH; S is a point on the curve. A circle with centre S and radius SH, where $SH^2 = SA.SC$, may be called the "construction circle" of the curve. AB and CD meet OH in L and L′. OY is drawn at right angles to OH so that O and S are equidistant from the middle point of LL′. J and J′ are two points on LL′ such that CJ′ is parallel to AJ, so that, if circles are described with centres J and J′, and distances JA and J′C, then we have similar segments on AB and CD, and the point T at which they intersect is on the curve.

Let SA meet the "construction circle" in Q and R, and complete the figure as indicated, by perpendiculars and parallels to OH.

TS is supposed to meet the curve again in T′ and the construction circle in q and r, and perpendiculars are drawn from T, T′, q, r to meet OH in V, V′, m, and m'. V, V′, m, and m' are related to the chord TT′ as L, L′, M, and M′ are to AC.

8. To prove that $Tq.Tr : SH^2 :: OS : OV$.—We find by *Euc.*, Bk. II. that $SC^2 - ST^2 = 2J'S.VL'$; $SA^2 - ST^2 = 2JS.VL$.

$$\therefore \quad SC^2 - ST^2 : SA^2 - ST^2 :: J'S.VL' : JS.VL :: SL'.VL' : SL.VL.$$

$$\therefore \quad SC^2 - SA^2 : SA^2 - ST^2 :: SL'.VL' - SL.VL : SL.VL$$
$$:: LL'.OV : SL.VL.$$

Now, if we remember $SC.SA = SH^2$, and that $SA : SC :: SL : SL'$, we find that $SH^2 - SA^2 : SC^2 - SA^2 = LL' : SL$.

Compounding this ratio with that just found above,

$$SH^2 - SA^2 : SA^2 - ST^2 :: OV : VL;$$

$$\therefore \quad SH^2 - ST^2 : SH^2 - SA^2 :: OV + VL : OV :: OL : OV.$$

We know that

$$SH^2 - ST^2 = Tq.Tr \quad \text{and that} \quad SH^2 - SA^2 : SH^2 :: OS : OL;$$

$$\therefore \quad Tq.Tr : SH^2 :: OS : OV, \quad \text{and} \quad OV = TU.$$

COR.—This result may be written: The edges of a rectangular solid block being equal to TU, Tp, Tr, the solid is of constant volume. It is clear that OY is an asymptote to the curve, for as T becomes more distant from S the rectangle $Tp.Tr$ can be made to increase indefinitely, and, therefore, TU diminishes without limit, but can never actually vanish.

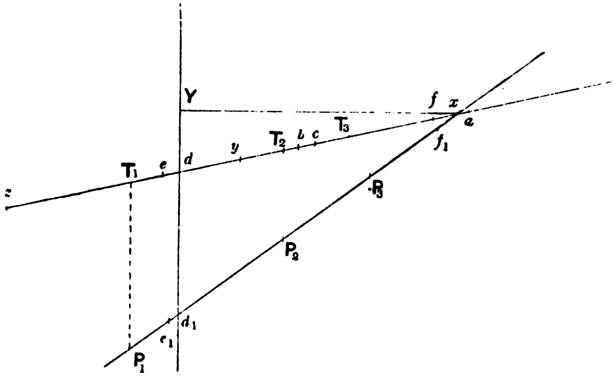
9. A straight line meets the curve in three points, two of which may be imaginary.

Let a straight line meet the construction circle in e and f and the asymptote in d. Then, if a point T be on this straight line and also on the curve, and if TU be perpendicular to OY, we must have

$$\text{rect. } Te.Tf : SH^2 :: OS : TU \qquad \text{(Art. 8)}$$

and TU : Td a fixed ratio for this straight line. If we now measure ez = SH, and dy of such a length that TU : Td :: OS : dy, then we have

$$\text{rect. } Te.Tf : ez^2 :: dy : Td.$$



Now, if we suppose T to be near e, Te.Tf is nearly zero, while Td is not infinitely greater than dy, but as T moves in the direction of z the ratio Te.Tf rapidly increases. We may assume, therefore, that there is at least one position of T for which this ratio is true, which we may call T_1. [A more obvious way of stating this is as follows:—The volume Te.Tf.Td increases as T moves from e in the direction of z from zero to any possible value, however great, and therefore, once at least must have the value $ez^2.dy$.]

If T be any other intersection of the line and curve, we find by compounding ratios $\quad$ rect. $Te.Tf : T_1e.T_1f :: T_1d : Td$;

$$\therefore \quad \text{rect. } T_1e.T_1f : Te.Tf - T_1e.T_1f :: Td : TT_1;$$

$$\therefore \quad \text{rect. } T_1e.T_1f = Td.Ta = ce^2 - Tc^2,$$

where ef is bisected in b, $ab = T_1b$, and ea is bisected in c.

If $T_1e.T_1f > ce^2$, there can be no real position of T.

If $T_1e.T_1f = ce^2$, there are two coincident positions of T at c.

If $T_1e.T_1f < ce^2$, there are two real positions of T measured at equal distances from c so that $Tc^2 = Te.Tf - ce^2$.

If x be any other fixed point in the line, it is clear that

$$xT_1 + xT_2 + xT_3 = xd + xe + xf \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$xT_1.xT_2 + xT_2.xT_3 + xT_3.xT_1 = xd.xe + xe.xf + xf.xd \ldots\ldots \quad (2),$$

$$xT_2.xT_3 : xe.xf - xl^2 :: xd : xT_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where $dy : xd :: xl^2 : ez^2$, so that xl is a length independent of T_1, T_2, T_3, and known when d, e, x, y, z are given.

10. To show that, if P and T be two points on the curve and TV, PV_2 be perpendiculars to OH, and Tq, Tr, Pq', Pr', segments of the chords of the construction circle through T and P, and x a point on OH the centre of a circle through T and P, then

$$Pr'.Pq' = 2Sx.OV_2$$

and

$$Tr.Tq = 2Sx.OV.$$

It has been shown by Art. 8 that

$$Tq.Tr : SH^2 :: OS : OV$$

and

$$Pq'.Pr' : SH^2 :: OS : OV_2;$$

$$\therefore \quad Tq.Tr : Pq'.Pr' :: 2Sx.OV_2 : 2Sx.OV \ldots\ldots\ldots\ldots\ldots (1);$$

and, since

$$Tq.Tr = SH^2 - ST^2,$$

$$Pq'.Pr' = SH^2 - SP^2,$$

$$\therefore \quad Tq.Tr - Pq'.Pr' = SP^2 - ST^2$$

$$= 2Sx.OV_2 - 2Sx.OV$$

(by *Euc.* II. 12 and 13).

Hence, from (1),

$$Tq.Tr : Tq.Tr - Pq'.Pr' :: 2Sx.OV_2 : 2Sx.OV_2 - 2Sx.OV;$$

$$\therefore \quad Tq.Tr = 2Sx.OV_2,$$

and

$$Pq'.Pr' = 2Sx.OV.$$

11. If TS meets the curve again in T′ and T′V′ is at right angles to OH, to show that $OV' = SV$, $OV = SV'$, and $OV' - OV = OS$, also $ST.ST' = SH^2$.

We know (Art. 8) that

$$Tq.Tr : SH^2 :: OS : OV$$

and

$$T'q.T'r : SH^2 :: OS : OV';$$

and, as Vm, Vm', Sm are proportional lengths to Tq, Tr, Sq made by perpendiculars to OH,

$$\therefore \quad OS : OV :: Vm.Vm' : Sm^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

for $SH = Sq$; $\therefore \quad OS : OV - OS :: Vm.Vm' : Vm.Vm' - Sm^2;$

$$\therefore \quad OS : SV :: Vm.Vm' : SV^2;$$

$$\therefore \quad Vm.Vm' = OS.SV \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly

$$V'm.V'm' = OS.SV';$$

$$\therefore \quad Vm.Vm' : V'm.V'm' :: SV : SV';$$

but, by Art. 8, $\quad Vm.Vm' : V'm.V'm' :: OV' : OV;$

$$\therefore \quad SV : SV' :: OV' : OV;$$

$$\therefore \quad SV : SV + SV' :: OV' : OV' + OV;$$

$$\therefore \quad SV = OV' \text{ and similarly } OV = SV'$$

and

$$OV - OV' = SV' - OV' = OS.$$

Since $SV = OV$,

$$\therefore \text{ by (2)} \quad Vm.Vm' = OS.OV';$$

$$\therefore \text{ by (1)} \quad OS : OV :: OS.OV' : Sm^2;$$

$$\therefore \quad OV.OV' = Sm^2; \quad \therefore \quad SV.SV' = Sm^2;$$

$$\therefore \quad ST.ST' = Sq^2 = SH^2 \text{ by proportionals.}$$

12. From the results of Art. 11 it follows that the curve consists of an oval and an infinite branch, each being the reciprocal of the other, with regard to the construction circle for $ST . ST' = SH^2$.

That one part of the curve is an oval is clear from the fact that the curve cannot cut the construction circle; therefore, if there be any points on the curve within the circle, they must form one or more closed curves within it, and there can be but one closed curve, or a straight line could be drawn to cut the circle in more than three points. The curve cannot cut the construction circle, because Tq or Tr cannot vanish for

$$Tq . Tr : SH^2 :: OS : OV.$$

That the part outside the construction circle consists of an infinite branch is not only evident from the fact that OY has been proved an asymptote, but also because it is the reciprocal of the oval which passes through S, the centre of the circle with regard to which reciprocation is performed.

13. Any circle is described on the chord T_2T_3 joining two points on the curve, and cuts the curve again in P_2 and P_3. Show that the chord P_2P_3 passes through a fixed point P_1.

Let the chord P_2P_3 meet T_2T_3 in x and construct as in Art. 9. x is not actually a fixed point, but may be considered so for the chords P_2P_3 and T_2T_3. Let xY be perpendicular to OY.

By supposition $\qquad TU : Td :: OS : dy;$

$\therefore \qquad xY : xd :: OS : dy:$

$\therefore \qquad xY : OS :: xd : dy;$

again $\qquad dy : xd :: xl^2 : ez^2;$

$\therefore \qquad OS : xY :: xl^2 : ez^2.$

But ez is a fixed length = SH; so is OS; and xY is the same for both chords; therefore the line known as xl is equal for both chords xP_1 and xT_1. Therefore

$$xT_2 . xT_3 : xe . xf - xl^2 :: xd : xT_1$$

and $$xP_2 . xP_3 : xe_1 . xf_1 - xl^2 :: xd_1 : xP_1 \qquad \text{(Art. 9).}$$

Now the construction circle passes through ef, e_1f_1; therefore

$$xe . xf = xe_1 . xf_1.$$

Again a circle passes through T_2, T_3, P_2, P_3; therefore

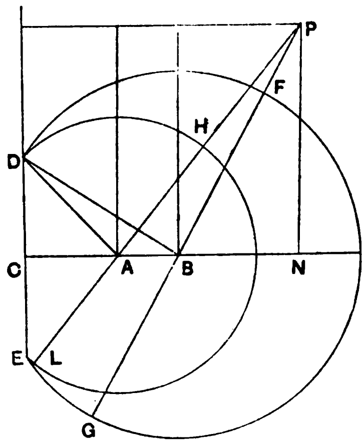
$$xT_2 . xT_3 = xP_2 . xP_3.;$$

therefore we obtain $\qquad xd : xd_1 :: xT_1 : xP_1;$

therefore $T_1 P_1$ is parallel to the asymptote, and, T_1 being a fixed point on the fixed chord $T_1T_2T_3$, the position of P_1 is also fixed.

14. Two series of circles are described on two fixed chords, one circle of one series intersecting one of the other so that their common chord passes through a fixed point. It is evident from the last proposition that their intersection traces out a cubic curve, for, if we draw any second fixed chord through T_1 and describe circles to meet the curve, these circles will also intersect the curve in chords passing through P_1; that is, in

the same chords at which the first series intersect it, and, therefore, they may be looked upon as tracing out the curve.



It is required to show that these circles may be described by the rule stated in the latter part of Art. 5. Let ED be a fixed chord upon which a series of circles is described and chords be drawn from a fixed point P to cut these circles. Let A and B be the centres of two such circles, and PHL, PFG chords through P.

Now $\quad PH \cdot PL = PH^2 - AH^2 = PA^2 - AD^2$

and $\quad PF \cdot PG = PB^2 - BF^2 = PB^2 - BD^2$;

therefore $\quad PH \cdot PL - PF \cdot PG = PA^2 - PB^2 + BD^2 - AD^2$

$$= AN^2 - BN^2 + BC^2 - AC^2;$$

where C is the middle point of ED, and PN perpendicular to CAB;

therefore $\quad PH \cdot PL - PF \cdot PG = 2NC \cdot AB.$

If there is a second series of circles described on another fixed chord having common chords through the fixed point P, each with one of the first series, it is evident that the rectangles under segments of chords drawn through P are equal for two intersecting circles; therefore, if a similar construction be made for the second fixed chord, writing A′, B′, C′, &c., for A, B, C, then $2NC \cdot AB = 2N'C' \cdot A'B'$;

therefore $\quad AB : A'B' :: N'C' : NC$;

therefore AB and A′B′ bear a constant ratio to one another. This can only be the case if the centres move along the lines CAB, C′A′B′ so that their distances measured along CAB, C′A′B′ from some fixed points in these lines bear a constant ratio to one another.

Hence we arrive at the conclusion that, if a curve be described by Grassman's definition, other points can be taken on the curve as described in Art. 5, such that $P[A_1B_1EF] : P[C_1D_1EF]$ is a constant ratio. Conversely, a curve so described is a cubic curve.

[NOTE.—This proposition is used for constructing cubic surfaces.]

COR.—The cubic may be described by a series of similar segments of circles described on the double ordinates through T and T′, the extremities of any chord through S.

15. *The Auxiliary Parabola.*—A parabola described with focus S and directrix OY is closely related to the circular cubic (Art. 7), and points on it may be used for drawing tangents and normals to the cubic.

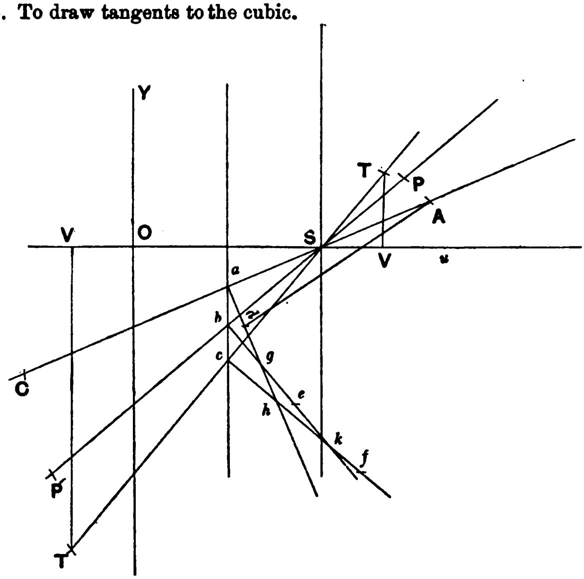
Since OV′ = SV (Art. 11), the middle points of all chords of the cubic through S lie on the tangent at the vertex of the auxiliary parabola.

Let *a*, *b*, *c* be three such middle points of the chords AC, PP′, TT′, and let *adgh*, *bgek*, *chkf* be tangents to the parabola touching it at *d*, *e*, *f*. Now it is evident, since *ag*, *bg* bisect AC, PP′ at right angles, that a circle centre *g* passes through A, C, P, and P′.

Hence the cubic curve may be very simply described. Take any fixed chord AC and the fixed tangent to the parabola, namely, *ag*; by drawing any other focal chord as PP′ to meet *abc* in *b*, and *bg* at right angles to it to meet *ag* in *g*, we find *g*. A circle centre *g*, distance *g*A or *g*C, cuts the chord PP′ in P and P′.

This construction assumes its simplest form if we take the tangent at the vertex as the "fixed tangent," for suppose the curve cuts the axis in *u*; then circles described with centres *a*, *b*, *c* and distances *au*, *bu*, *cu*, respectively, each cut the focal chords *a*S, *b*S, *c*S in points on the cubic.

16. To draw tangents to the cubic.



Construct as in Art. 15. Suppose b to move up to a and ultimately to coincide with it; then P and P′ move up to and coincide with A and C so that a circle described with centre d touches the cubic curve at A and C, for as b moves up to and coincides with a the point g, the centre of the circle, moves up to d.

Since a circle centre d touches the cubic at A and C, the normals at A and C meet at d, and the tangents may be drawn at right angles to them.

Cor.—It is known as a property of tangents to a parabola that

$$hg \,.\, hf = hd \,.\, kf.$$

If we then regard the points h, d, f as fixed, but gk a variable tangent to the parabola, we obtain the following:—g and k are the centres of two circles which move along two fixed straight lines, and each passes through two fixed points A, C, and T, T′. $hg : fk$ is a fixed ratio $= hd : hf$. Then the intersection of these circles is a cubic curve, namely, the variable points P, P′.

This is in effect the same result as Art. 14.

APPENDIX II.

NOTE ON THE REDUCTION OF FORMULÆ IN FACTORIZATION,

AFFORDING AN EASY MEANS OF FACTORIZING COMPOSITE NUMBERS, ESPECIALLY THOSE WHOSE FACTORS ARE OF KNOWN FORM.

By D. Biddle, M.R.C.S.

It is possible by careful inspection, without any elaborate process, to learn a good deal about any number.

Let $N = S^2 + A = (S+u)(S-v) = (2\Delta p+1)(2\Delta q+1)$,

where A, S, Δ are known. Then we easily arrive at the following facts:—

(i.) $A + uv = S(u-v)$.

(ii.) Since N is always odd, S, A are of opposite character, one odd and the other even.

(iii.) Since $S+u$, $S-v$ are both odd, their sum is even, and it follows that $u-v$ is invariably even.

(iv.) In consequence, u, v are both even or both odd.

(v.) From (iii.) it follows also that $A+uv$ is necessarily even.

(vi.) When A is odd, uv is odd also.

(vii.) When A is even, $uv \equiv 0 \pmod 4$.

(viii.) Since $S+u = 2\Delta p+1$, we have $u \equiv -(S-1) \pmod{2\Delta}$.

(ix.) Since $S-v = 2\Delta q+1$, we have $v \equiv (S-1) \pmod{2\Delta}$.

(x.) When A is odd, it follows from (ii.) and (iii.) that $A+uv \equiv 0 \pmod 4$.

(xi.) Since all odd numbers are of form $4n \pm 1$, we can at once find out (if A be odd) which of these A belongs to, and we then know that uv is the opposite.

(xii.) If uv be of form $4n+1$, then u, v are both of one form, either $4n+1$ or $4n-1$; and in these circumstances $u-v \equiv 0 \pmod 4$.

(xiii.) If uv be of form $4n-1$, then u is of one form and v of the other; and in this case $u-v \equiv 2 \pmod 4$.

(xiv.) Since $A+uv$ is a multiple of S (see i.), we have $uv \equiv -A \pmod S$, which we can combine with any congruence already discovered. Thus, having given $uv \equiv \beta \pmod B$, $\equiv \gamma \pmod C$, we have the equation $B\mu + \beta = C\kappa + \gamma$, and, the lowest value of κ' that will make μ integral being found and substituted in the equation, $B\mu' + \beta (= C\kappa' + \gamma)$ will be the residue of uv (mod L.C.M. of B, C).

Notation.—Let A_e, A_o denote that A is even, that A is odd, respectively. Then S_o, S_e will be corresponding characters of S. Let a, s equal $\frac{1}{2}A_e$, $\frac{1}{2}S_e$ respectively. Let u_1, v_1 represent $\frac{1}{2}u$, $\frac{1}{2}v$ respectively; u_2, v_2 represent $\frac{1}{2}u_1$, $\frac{1}{2}v_1$; and so on; and let the same suffixes supply to any quantities, even when bracketed.

Let H, h represent respectively the half-sum and the half-difference of the factors of N.

(xv.) $$H = S + \tfrac{1}{2}(u-v) = \Delta(p+q) + 1.$$

(xvi.) $$h = \tfrac{1}{2}(u+v) = \Delta(p-q).$$

(xvii.) First values of H can conveniently be found from the following formula:—$pq = \left\{\tfrac{1}{2}(N+1) - H\right\}/(2\Delta^2)$, where the right side is evidently integral, and in which H advances by steps of $2\Delta^2$.

(xviii.) $u-v = 2(H-S)$. Therefore the trial values of $u-v$ advance by steps of $4\Delta^2$.

(xix.) When $x+y$ is divisible by 2, we know that $x-y$ is even also, and *vice versa*. But this reciprocity does not extend to higher factors, unless x, y be both multiples thereof or of the half.

(xx.) During the process of reduction, uv becomes a function of a value which we may call μ, and $u-v$ becomes a function of a value which we may call κ. But all is done by known steps, so that the trial values of uv and $u-v$ corresponding with those of μ and κ can readily be found, and it is possible at once to compare any *composite* trial value of uv which is arrived at with the corresponding trial value of $u-v$, in order to see whether they suit each other.

(xxi.) It can be shown that μ and κ are functions of a common value ρ, so that uv and $u-v$ can be represented as functions of one unknown, namely, ρ. Consequently, u can be eliminated, and a quadratic in v be formed, having under the radical sign, in the root, the one unknown, ρ.

(xxii.) Since the quantity under the radical sign referred to is in reality h^2, we know, by (xvi.), that it is $\Delta^2(p-q)^2$. This fact enables us to find the residue of ρ in respect of the modulus Δ.

(xxiii.) Taking $\rho = \Delta\xi + b$, we can greatly simplify the quantity under the radical sign, and other means of shortening the process occur in particular cases.

The process of reduction may be set forth as follows:—

Take first the case of A_e, S_o; then we have

$$A_e + uv = S_o(u-v),\ \ A_e + 4u_1v_1 = S_o 2(u_1-v_1),\ \ a + 2u_1v_1 = S_o(u_1-v_1).$$

If a be even, u_1, v_1 are both even or both odd; but, if a be odd, they are one odd and the other even. Each case requires its own treatment, but we need describe only that in which a is odd. Take S_o from both sides;

then $2u_1v_1 - (S_o-a) = S_o(u_1-v_1-1)$, and $u_1v_1 - (S_o-a)_1 = S_o(u_1-v_1-1)_1$.

We know that u_1v_1 is even, and we are guided in our subsequent treatment by $(S_o-a)_1$. But an example will best show how the process is conducted.

Let $N = 559 = 23^2 + 30$; then $30 + uv = 23(u-v)$; $30 + 4u_1v_1 = 23.2(u_1-v_1)$; $15 + 2u_1v_1 = 23(u_1-v_1)$; u_1, v_1 are one even, one odd. Take 23 from each side; $2u_1v_1 - 8 = 23(u_1-v_1-1)$; $u_1v_1 - 4 = 23(u_1-v_1-1)_1$; $(u_1v_1)_1 - 2 = 23(u_1-v_1-1)_2$. We therefore know that $uv = 8\mu$, and that $u-v = 8\kappa+2$. Consequently $30 + 8\mu = 23(8\kappa+2)$, whence $\mu = 23\kappa+2$. Taking $\kappa = 1$, we have $\mu = 25$, $uv = 200$, $u-v = 10$, giving $u = 20$, $v = 10$, $N = 43.13$.

We might have had to take more trial values of κ. But the foregoing shows how to set about investigating the nature of N, when we have A_e, S_e.

Take next the case of A_o, S_e. Here we have $A_o + uv = S_e(u-v)$, and we know that the right side is a multiple of 4. Consequently, we know that $A_o + uv \equiv 0 \pmod 4$, whilst u, v are both odd. By reference to (xi.), (xii.), (xiii.), we can now render the formula as follows:— $A_o \pm 1 + 4\mu = 4s\,(2\kappa)$, or $4s\,(2\kappa \pm 1)$, and we can tell which is the correct quantity to put on the right, by examination of A and thence of uv. Let $a_1' = \frac{1}{4}(A_0 \pm 1)$. Then $a_1' + \mu = s\,(2\kappa)$, or $s\,(2\kappa \pm 1)$, and we know which. We also know whether s is odd or even, likewise a_1', so that we can often learn much regarding μ beyond what yet appears. To take another example, let $N = 1843 = 42^2 + 79$; then $79 + uv = 42\,(u-v)$. Clearly $79 = 4n-1$; therefore $uv = 4n+1$, and $u - v \equiv 0 \pmod 4$. Hence $uv - 1 \equiv 0 \pmod 8$, since S is even. $80 + uv - 1 = 21.8\kappa$; $10 + \mu = 21\kappa$; $\mu = 21\kappa - 10$; $uv = 8\mu + 1 = 168\kappa - 79$. The value on the right must be factorizable like that on the left. But, for most early values of κ, it is prime. However, for $\kappa = 3$, and next for $\kappa = 8$, we obtain composite values. For $\kappa = 3$, we obtain $425 = 17.25$; but $42 + 25$, $42 - 17$ yield only 1675. For $\kappa = 8$, we obtain $1265 = 23.55$; and $42 + 55$, $42 - 23$ are the proper factors of $N = 1843 = 97.19$.

But we might improve upon the process in this second example. Thus, $79 + uv = 42\,(u-v)$; therefore, taking $u = 4g \pm 1$, $v = 4k \pm 1$, we have $(79+1) + 16gk \pm 4\,(g+k) = 42.4\,(g-k)$. Since every term except $4\,(g+k)$ is divisible by 8 without question, that term itself must be, so that $(g+k)$ is clearly divisible by 2. Therefore $(g-k)$ is divisible by 2 likewise (see xix.). Consequently $10 + 2gk \pm (g+k)_1 = 21\,(g-k)$, $5 + gk \pm (g+k)_2 = 21\,(g-k)_1$, showing that $(g+k) \equiv 0 \pmod 4$, whence $k \equiv -g \pmod 4$. We now know that $uv - 1 \equiv 0 \pmod{16}$, and have $uv = 16\mu + 1 = 336\kappa - 79$, which gives us the correct value on the fourth trial of κ.

If into this example we introduce the consideration of Δ, which here is 3, we find that $u - v = 2\Delta r + 2 = 4\,(g-k)$, or $\Delta r + 1 = 2\,(g-k)$. Now we know that $g-k$ is even, and $2\,(g-k)$ is therefore a multiple of 4. In order to make $3r+1$ similarly a multiple of 4, we must take $r = 4n+1$. Taking $r = 1$, we have $g - k = 2$, and $u - v = 8$; but this makes $uv = 257$, a prime, which is impossible. Taking $r = 5$, $g - k = 8$, $u - v = 32$, and $uv = 1265$, which is the correct value. Thus we require two trials only. Of course, uv is found by (i.) when $u - v$ is given.

Let us next investigate a large number and see what can be known regarding it. Let $N = 329554457$. Here $S_o = 18153$, and $A_e = 23048$. $S_o = 3^2.2017$; $A_e = 2^3.43.67$. Moreover, the factors of N are of known form, $2.11m + 1$, Δ being 11. A can be written $2^3\,(2.11.2-1)(2.11.3+1)$; and, since $S - 1 \equiv 2 \pmod{2.11}$, we have, from (viii.) and (ix.), $u \equiv -2 \pmod{2.11}$, $v \equiv 2 \pmod{2.11}$, whence $uv \equiv -4 \pmod{4.11}$. Further, by (xiv.), we have $uv \equiv 13258 \pmod{18153}$.

Thus we have

$$2^3(2.11.2-1)\,(2.11.3+1) + 4.11m - 4 = 3^2.2017\left\{(2.11.g-2) - (2.11k+2)\right\},$$

$$2^2(2.11.2-1)\,(2.11.3+1) + 2.11m - 2 = 3^2.2017\left\{(11g-1) - (11k+1)\right\},$$

$$2\,(2.11.2-1)\,(2.11.3+1) + \quad 11m - 1 = 3^2.2017\left\{11\,(g-k)_1 - 1\right\} \,\ldots\ldots\ldots\, (a)$$

From (α) we find that $m \equiv 1 \pmod 3$; therefore

$$uv = 132\mu + 40 = 18153\kappa + 13258, \quad \text{and} \quad \mu = (18153\kappa + 13218)/132;$$

therefore $18153\kappa \equiv -18 \pmod{132}$, that is $69\kappa \equiv -18 \pmod{132}$, which is satisfied by $\kappa' = 38$, whence $\mu' = 5326$; therefore

$$uv \equiv 132.5326 + 40 \pmod{2^3.3^2.11.2017} \equiv 703072 \pmod{798732}$$
(see xiv.) ...... (β).

From the above equation (α) we obtain by reduction

$$m = 3^2.2017\,(g-k)_1 - 2174,$$

whence $\mu = 3.2017\,(g-k)_1 - 725$, in which $(g-k)_1 = (u-v+4)/44$. Therefore we have

$$uv = 3^2.2017\,(u-v+4) - 95660 = 2^2.3^2.11.2017\,(u-v-40)/44 + 703072,$$

which agrees with (β), and shows that $(u-v-40)/44$ is the multiplier of 798732 involved in (β). Let ρ be this multiplier; then $u-v = 44\rho+40$, and $u = 44\rho + 40 + v$. Consequently we have the following quadratic:— $uv = (44\rho+40)\,v + v^2 = 798732\rho + 703072$, whence

$$v + 22\rho + 20 = (4.11^2.\rho^2 + 799612\rho + 703472)^{\frac{1}{2}} = 2.11\,(\rho^2 + 1652\tfrac{1}{11}\rho + 1453\tfrac{5}{11})^{\frac{1}{2}} \quad \ldots\ldots (\gamma).$$

But $22\rho + 20 = \frac{1}{2}(u-v)$, and v added to this makes it $\frac{1}{2}(u+v) = h = \Delta(p-q)$. Therefore 11 is a proper factor of both sides of (γ), and the quantity under the radical sign must needs be integral; therefore by considering the two fractions under the radical sign we obtain

$$\rho \equiv -5 \pmod{11}.$$

Let $\rho = 11\xi + 6$. Then we have

$$v + 22\rho + 20 = 2.11\,(11^2\xi^2 + 18305\xi + 11402)^{\frac{1}{2}}$$
$$= 2.11^2\left(\xi^2 + 151\frac{34}{11^2}\xi + 94\frac{28}{11^2}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (\delta).$$

It can be proved that in the particular instance (see below) $h \equiv 0 \pmod{11^2}$. Therefore the quantity under the radical sign in (δ) is integral. Therefore $34\xi \equiv -28 \pmod{11^2}$, or $\xi \equiv 49 \pmod{11^2}$.

Let $\xi = 121\tau + 49$. Then we have

$$v + 22\rho + 20 = 2.11^2(11^4.\tau^2 + 30163\tau + 9908)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (\epsilon).$$

The quantity under the radical sign in (ϵ) is found to be a perfect square when $\tau = 4$. Therefore $v + 129138 = 2.11^2.604$, whence $v = 17030$, and $N = 1123.293459$.

In order to prove that in the present case $h \equiv 0 \pmod{11^2}$, let (α) be rendered thus,

$$8.6.11^2 - 4.11 - 2 + 11m - 1 = S.11\,(g-k)_1 - S$$
$$8.6.11^2 - 4.11 + 11m = S.11\,(g-k)_1 - (S-3)$$
$$= S.11\,(g-k)_1 - 2.11^2\,(M).$$

Since $S \equiv 3 \pmod{11^2}$, we have

$$8.6.11^2 - 4.11 + 11m - 3.11\,(g-k)_1 \equiv 0 \pmod{11^2},$$

that is, $$m - 4 - 3\,(g-k)_1 \equiv 0 \pmod{11}.$$

It is required to prove that $(g+k)_1$, a factor of h, is similarly divisible by 11. If so, $m-4+6k_1 \equiv 0 \pmod{11}$. Now $m = 11gk+g-k$, and $2k_1 = k$. Therefore, in order to fulfil the conditions,

$$11gk+g-k-4+3k \equiv 0 \pmod{11};$$

and finally $k \equiv 4 \pmod{11}$. This would render $v \equiv -31 \pmod{11^2}$.

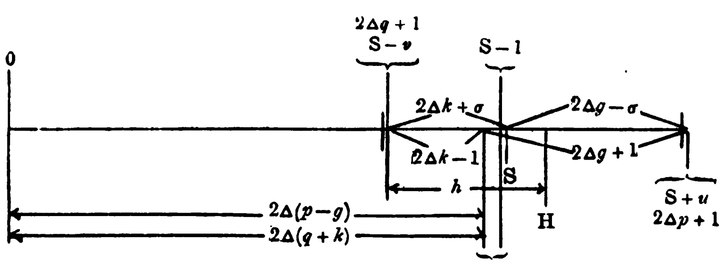
Let us then refer to (xvii.), bearing in mind that $\mathrm{H}-h = \mathrm{S}-v$. We find that $\mathrm{H} \equiv 34 \pmod{11^2}$; $\mathrm{S} \equiv 3 \pmod{11^2}$, as we have seen; and $3+31 = 34$. Therefore it is right to take $h \equiv 0 \pmod{11^2}$, in the particular instance.

Knowing that h is given by $v+2\Delta\rho+c$, as on the left side of (δ), we have $2\Delta\rho+c = \mathrm{H}-\mathrm{S}$, and since we are able, by aid of (xvii.), to find the residue (mod $2\Delta^2$) of $\mathrm{H}-\mathrm{S}$, we are able to find the residue (mod Δ) of ρ without difficulty, and thus to check the result obtained as in the last example.

It would be useful to be able to find the residue (mod Δ^2) of h when it is other than 0. Since $h = \Delta(p-q)$, we can take $h \equiv \Delta x \pmod{\Delta^2}$. Consequently, the quantity under the radical sign at the particular stage of the process, being presumably a perfect square, will, when diminished by $x^2 (= 0, 1, 4, 9, \&c.)$, be divisible by Δ. Owing to the combination of fractions (left by this division) amounting to an integral value, we are able to find the residue (mod Δ) of ξ, the unknown quantity specially concerned. We know that $x \equiv p-q \pmod{\Delta}$. Let $y \equiv p+q \pmod{\Delta}$, which can readily be found, because $\mathrm{N} = 2\Delta m+1$, and $m = 2\Delta pq+p+q$. In fact, $y \equiv m \pmod{\Delta}$. Therefore the sum of the residues of p and q, taken positively, cannot exceed $\Delta+y$; nor can their difference, namely, x, be outside the range $\pm\frac{1}{2}(\Delta-1)$. Consequently there will be only $\frac{1}{2}(\Delta-1)$ possible values of x^2 when x is other than 0.

But $p-q = g+k$, and it can be shown that $p+q \equiv g-k+\mathrm{D} \pmod{\Delta}$; for $u = 2\Delta g-\sigma$, and $v = 2\Delta k+\sigma$, where, taken positively, $\sigma \equiv \mathrm{S}-1 \pmod{2\Delta}$; moreover $\mathrm{H} = \Delta(p+q)+1 = \mathrm{S}+\Delta(g-k)-\sigma$. In order to obtain D, we first divide $\mathrm{S}-1$ by 2Δ—this gives remainder σ; we then double the quotient and divide again by Δ—this gives remainder D.

Let p' q', g', k' be the residues (mod Δ) of the separate quantities. Then $p'-q' = g'+k'$, and $p'+q' = g'-k'+\mathrm{D}$. It follows that $p'-g' = q'+k' = \frac{1}{2}\mathrm{D} = \pm d$, say. The accompanying diagram brings this out, and will explain more readily the relations between the various quantities.



Now we have the four following equations :—

$$p'+q' = m'\,; \quad g'-k' = m'-\mathrm{D}\,; \quad p'-g' = \tfrac{1}{2}\mathrm{D}\,; \quad q'+k' = \tfrac{1}{2}\mathrm{D}.$$

Let $\Omega = p'+q'+g'+k'$, the sum of the unknown quantities. Then, although they are (as shown by Kronecker, Molk, and G. B. Mathews) severally and collectively indeterminate by algebra, we obtain $\Omega = 2p'$ and $\Omega - m' = p'-q' = g'+k'$. Frequently, either g' or k' is identical with p', and the two remaining quantities sum to 0; but, whether this be so or not, we have $\Omega = 2p'$ in all cases. When $m' = 0$, we have $q' = -p'$, and $p'-k' = q'+g' = -\tfrac{1}{2}\mathrm{D}$. Again, when $m' = \mathrm{D}$, we have $g' = k'$, and $p'-q' = 2k'$. When $m' = \tfrac{1}{2}\mathrm{D}$, we have $q'+g' = 0$, and hence $p' = k'$. When $m' = \tfrac{3}{2}\mathrm{D}$, we have $q'+k' = g'-k' = \tfrac{1}{2}\mathrm{D}$, and $q'+g' = \mathrm{D}$; also $p'-g' = g'-k' = \tfrac{1}{2}\mathrm{D}$, and $p'-k' = \mathrm{D}$. When $\mathrm{D} = 0$, we have $p' = g'$, and $q' = -k'$. When $\mathrm{D} = \tfrac{1}{2}m'$, we have $p'-k' = q'+g' = \tfrac{3}{2}m'$, and $p'-q' = 2g'-\tfrac{1}{2}m'$, whilst $\Omega = 2g'+\tfrac{1}{2}m'$. When $p' = q'$, each $= \tfrac{1}{2}m'$. When $q' = k'$, each $= \tfrac{1}{4}\mathrm{D}$.

Let $p' = \tfrac{1}{2}m'+\eta$, $q' = \tfrac{1}{2}m'-\eta$; then $g' = \eta+\theta$, $k' = \eta-\theta$, where $\theta = \tfrac{1}{2}(m'-\mathrm{D})$. Thus we have $\Omega = m'+2\eta$; $p'-q' = g'+k' = \Omega - m' = 2\eta$.

The following particulars regarding certain specified numbers will afford a useful exercise :—

N	S	A	Δ	m	σ	D	m'	p'	q'	g'	k'	Ω	$p'-q'$
61811	248	307	7	4415	9	6	−2	−3	1	1	2	1	3
65869	256	333	11	2994	13	0	2	4	−2	4	2	−3	−5
71107	266	351	7	5079	13	1	4	3	1	−1	3	−1	2
138659	372	275	13	5333	7	2	3	4	−1	3	2	−5	5
217801	466	645	11	9900	3	9	0	−5	5	−4	5	1	1
447539	668	1315	13	17213	17	11	1	−6	−6	−5	5	1	0

Since $v \equiv \sigma \pmod{2\Delta}$, we may take $v = 2\Delta\lambda + \sigma$. Then by substitution in the equation just above (δ) on page 150, and division of both sides by 2Δ, we obtain an equation in λ and ξ, two unknowns. But by this means ξ^2 is eliminated, and we are able to define the value of ξ in terms of λ, arranged in fractional (though integral) form, such as $\xi = (\lambda^2+f\lambda-b)/(c-2\Delta\lambda)$, where b, c, f are known. This enables us to find limits, upper and lower, to λ, and greatly shortens the labour in cases which are not of the privileged order of the example referred to.

[Rev. J. Cullen, referring to equation (ϵ) in the foregoing, says: "It seems to me, if this form could be obtained in every case, we should have a very rapid method even if τ were large; for we could apply the prime moduli to the equation $a\tau^2+b\tau+c = \square$, giving on an average $\tfrac{1}{2}(p-1)$ cases to mod p. For instance, in (ϵ) we have $\tau^2+3\tau+3 \equiv 0, 1, 4 \pmod 5$, and a very large number ought to be manageable when τ is so small for N = 329554457.]

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