THE MATHEMATICAL THEORY

OF

ELECTRICITY AND MAGNETISM

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THE MATHEMATICAL THEORY OF ELECTRICITY AND MAGNETISM

BY

H. W. WATSON, D.Sc., F.R.S.
FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE

AND

S. H. BURBURY, M.A.
FORMERLY FELLOW OF ST. JOHN'S COLLEGE
CAMBRIDGE

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PREFACE.

The exhaustive character of the late Professor Maxwell's work on Electricity and Magnetism has necessarily reduced all subsequent treatises on these subjects to the rank of commentaries. Hardly any advances have been made in the theory of these branches of physics during the last thirteen years of which the first suggestions may not be found in Maxwell's book. But the very excellence of the work, regarded from the highest physical point of view, is in some respects a hindrance to its efficiency as a student's text-book. Written as it is under the conviction of the paramount importance of the physical as contrasted with the purely mathematical aspects of the subject, and therefore with the determination not to be diverted from the immediate contemplation of experimental facts to the development of any theory however fascinating, the style is suggestive rather than didactic, and the mathematical treatment is occasionally somewhat unfinished and obscure. It is possible, therefore, that the present work, of which the first volume is now offered to students of the mathematical theory of electricity, may be of service as an introduction to, or commentary upon, Maxwell's book. Its aim is to state the provisionally accepted two-fluid theory, and to develop it into its mathematical consequences, regarding that theory simply as an hypothesis, valuable so far as it gives formal expression and unity to experimental facts, but not as embodying an accepted physical truth.

The greater part of this volume is accordingly occupied with the treatment of this two-fluid theory as developed by Poisson, Green, and others, and as Maxwell himself has dealt with it. The success of this theory in formally explaining and co-ordinating experimental results is only equalled by the artificial and unreal character of the postulates upon which it is based. The electrical fluids are physical impossibilities, tolerable only as the basis of mathematical calculations, and as supplying a language in which the facts of experience have been expressed and results calculated and anticipated. These results being afterwards stated in more general terms may serve to suggest a sounder hypothesis, such for instance as we have offered to us in the displacement theory of Maxwell.

In the arrangement of the treatise the first three chapters are devoted to propositions of a purely mathematical character, but of special and constantly recurring application to electrical theory. By such an arrangement it is hoped that the reader may be able to proceed with the development of the theory in due course with as little interruption as possible from the intervention of purely mathematical processes. Few, if any, of the results arrived at in these three chapters contain anything new or original in them, and the methods of proof have been selected with a view to brevity and clearness, and with no attempt at any unnecessary modifications of demonstrations already generally accepted.

All investigations appear to point irresistibly to a state of polarisation of some kind or other, as the accompaniment of electrical action, and accordingly the physical properties of a field of polarised molecules have been considered at considerable length, especially in Chapter XI, in connection with the subject of specific induction and Faraday's hypothesis of a composite dielectric, and in Chapter XIV, with reference to Maxwell's displacement theory. The value of the last-mentioned hypothesis is now universally recognised, and it is generally regarded as of more promise than any other which has hitherto been suggested in the way of placing electrical theory upon a sound physical basis.

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CHAPTER I.

GREEN'S THEOREM.

ARTICLE 1.] LET S be any closed surface, u and u' any two functions of x, y, and z, which are continuous and single-valued everywhere within S. Then shall

$$\iiint \left\{ \frac{du}{dx}\frac{du'}{dx} + \frac{du}{dy}\frac{du'}{dy} + \frac{du}{dz}\frac{du'}{dz} \right\} dx\,dy\,dz$$
$$= \iint u' \frac{du}{d\nu} dS - \iiint u' \nabla^2 u \, dx\,dy\,dz$$
$$= \iint u \frac{du'}{d\nu} dS - \iiint u \nabla^2 u' \, dx\,dy\,dz;$$

in which the triple integrals are taken throughout the space enclosed by S, and the double integrals over the surface, $d\nu$ is an element of the normal to the surface inside of S, but measured outwards in direction, and ∇^2 stands for

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right).$$

For let a line parallel to x cut the surface in the points x_1, y, z and x_2, y, z. Then integrating by parts between $x = x_1$ and $x = x_2$, we have

$$\int_{x_1}^{x_2} u' \frac{d^2 u}{dx^2} dx = \left(u' \frac{du}{dx}\right)_{x_2} - \left(u' \frac{du}{dx}\right)_{x_1} - \int_{x_1}^{x_2} \frac{du}{dx}\frac{du'}{dx} dx.$$

Let $dy\,dz$ be the base of a prism of which the line between x_1 and x_2 is one edge. Then

$$dy\,dz \int_{x_1}^{x_2} u' \frac{d^2 u}{dx^2} dx$$
$$= dy\,dz \left(\left(u' \frac{du}{dx}\right)_{x_2} - \left(u' \frac{du}{dx}\right)_{x_1}\right) - dy\,dz \int_{x_1}^{x_2} \frac{du}{dx}\frac{du'}{dx} dx.$$

Now if l_1, m_1, n_1 be the direction-cosines of the normal to S drawn outwards at the point x_1, y, z, and if dS_1 be the element of area cut out at that point by the prism,

$$dy\,dz = -l_1 \, dS_1,$$

and using corresponding notation at the point x_2, y, z,

$$dy\,dz = l_2\,dS_2.$$

Therefore

$$dy\,dz\int_{x_1}^{x_2} u'\frac{d^2u}{dx^2}\,dx$$

$$= \left(u'\frac{du}{dx}\right)_{x_2} l_2\,dS_2 + \left(u'\frac{du}{dx}\right)_{x_1} l_1\,dS_1 - dy\,dz\int_{x_1}^{x_2}\frac{du}{dx}\frac{du'}{dx}\,dx.$$

Therefore, noting that x_1 and x_2 are functions of y and z, integrating and transposing, we obtain

$$\iiint\frac{du}{dx}\frac{du'}{dx}\,dx\,dy\,dz = \iint u'\,l\frac{du}{dx}\,dS - \iiint u'\frac{d^2u}{dx^2}\,dx\,dy\,dz,$$

in which the triple integrals comprise the whole space within S, and the surface integrals comprise the whole surface of S.

Similar equations, mutatis mutandis, hold for y and z.

Therefore

$$\iiint\left\{\frac{du}{dx}\frac{du'}{dx} + \frac{du}{dy}\frac{du'}{dy} + \frac{du}{dz}\frac{du'}{dz}\right\}dx\,dy\,dz$$

$$= \iint u'\left\{l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz}\right\}dS - \iiint u'\,\nabla^2 u\,dx\,dy\,dz$$

$$= \iint u'\frac{du}{d\nu}\,dS - \iiint u'\,\nabla^2 u\,dx\,dy\,dz;$$

$$= \iint u\,\frac{du'}{d\nu}\,dS - \iiint u\,\nabla^2 u'\,dx\,dy\,dz \text{ by symmetry.}$$

We have supposed the line through $y\,z$ to cut S in two points only, x_1, y, z and x_2, y, z. It may cut it in any even number of points, but all the reasoning would apply to each pair so long as x_1 relates to the point of ingress, and x_2 of egress. The equation will therefore hold equally where lines can be drawn cutting the surface in more than two points.

Further, the proof evidently holds for the space between two surfaces, S_1 and S_2, whereof S_2 completely encloses S_1.

On the Application of the Theorem to the Infinite External Space.

2.] Let us consider more closely the case of two surfaces, S_1 and S_2, of which S_2 completely encloses S_1.

Applying the theorem to the space between them we have

$$\iiint\left\{\frac{du}{dx}\frac{du'}{dx}+\frac{du}{dy}\frac{du'}{dy}+\frac{du}{dz}\frac{du'}{dz}\right\}dx\,dy\,dz$$

$$=\iint u'\frac{du}{d\nu}\,dS_1+\iint u'\frac{du}{d\nu}\,dS_2-\iiint u'\nabla^2 u\,dx\,dy\,dz\,;$$

in which the first of the double integrals relates to S_1, and the second to S_2, and the normals on S_1 are measured inwards as regards the space enclosed by S_1.

Now let S_2 be removed to an infinite distance. If in that case the surface integral $\iint u'\frac{du}{d\nu}\,dS_2$, extended over the infinitely distant surface S_2, vanishes, the theorem is true for the infinite space outside of S_1, as well as for the finite space within it, the normal being in this case measured inwards as regards S_1.

In order that $\iint u'\frac{du}{d\nu}\,dS_2$ should vanish, when extended over the infinitely distant surface, it is sufficient and necessary that uu' should be of less degree than -1. In the physical theorems with which we are concerned, this will generally be the case.

Generalisation of Green's Theorem.

3.] Let K be any continuous function of x, y, z. Then

$$\iiint K\left\{\frac{du}{dx}\frac{du'}{dx}+\frac{du}{dy}\frac{du'}{dy}+\frac{du}{dz}\frac{du'}{dz}\right\}dx\,dy\,dz$$

$$=\iint u'K\frac{du}{d\nu}\,dS$$

$$-\iiint u'\left\{\frac{d}{dx}\left\{K\frac{du}{dx}\right\}+\frac{d}{dy}\left\{K\frac{du}{dy}\right\}+\frac{d}{dz}\left(K\frac{du}{dz}\right)\right\}dx\,dy\,dz,$$

by the same process of partial integration as before. The condition for application of the theorem to the external space will in this case be that Kuu' must be of less degree than -1.

It will sometimes be convenient to denote the expression

$$\frac{d}{dx}\left(K\frac{du}{dx}\right)+\frac{d}{dy}\left(K\frac{du}{dy}\right)+\frac{d}{dz}\left(K\frac{du}{dz}\right)\text{ by }\nabla^2_K u.$$

4.] We have assumed u and u' to be *continuous* functions of x, y, and z. If at a certain surface within S, one of them, suppose u, is discontinuous but finite, and its differential coefficients $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, or one of them, are infinite, the theorem requires modification as follows:—

It still remains true, u being always finite, that

$$\int_{x_1}^{x_2} u\frac{d}{dx}(K\frac{du'}{dx})\,dx = (uK\frac{du'}{dx})_{x_2} - (uK\frac{du'}{dx})_{x_1} - \int_{x_1}^{x_2} K\frac{du}{dx}\frac{du'}{dx}\,dx;$$

and from this we may deduce Green's theorem in the form

$$\iiint K\left\{\frac{du}{dx}\frac{du'}{dx} + \&c.\right\}dx\,dy\,dz$$
$$= \iint uK\frac{du'}{d\nu}\,dS - \iiint u\nabla_K^2 u'\,dx\,dy\,dz.$$

But we cannot assert the truth of the theorem in the alternative form

$$\iiint K(\frac{du}{dx}\frac{du'}{dx} + \&c.)\,dx\,dy\,dz$$
$$= \iint u'K\frac{du}{d\nu}\,dS - \iiint u'\nabla_K^2 u\,dx\,dy\,dz.$$

If u become *infinite* at any point within S, we cannot include in the integration the point at which the infinite value occurs. But we may describe a surface S' completely enclosing, and very near to, that point, and apply the theorem to the space between S and S', regarding u' and its differential coefficients as constant throughout S'. For instance, let u become infinite at a point P within S. Let S' be a small sphere described about P, and let u'_p, $\frac{du'_p}{d\nu}$, and K_p be the values of u', $\frac{du'}{d\nu}$, and K in or on the surface of S'. Then we obtain

$$\iint uK\frac{du'}{d\nu}\,dS + K_p\frac{du'_p}{d\nu}\iint u\,dS' - \iiint u\nabla_K^2 u'\,dx\,dy\,dz$$
$$= \iint u'K\frac{du}{d\nu}\,dS + K_p u'_p\iint\frac{du}{d\nu}\,dS' - \iiint u'\nabla_K^2 u\,dx\,dy\,dz.$$

In this form the theorem can be made use of whenever the two surface integrals relating to S', namely $\iint u\,dS'$ and $\iint\frac{du}{d\nu}\,dS'$, are finite or zero. For instance, if $u = \frac{1}{r}$, where r is the distance of any point from P,

$$\iint u\,dS' = 0 \quad \text{and} \quad \iint \frac{du}{d\nu}\,dS' = -4\pi,$$

and the equation becomes

$$\iint \frac{K}{r}\frac{du'}{d\nu}\,dS - \iiint \frac{1}{r}\nabla^2{}_K u'\,dx\,dy\,dz$$
$$= \iint u' K \frac{d}{d\nu}\left(\frac{1}{r}\right) dS - \iiint u' \nabla^2_K \left(\frac{1}{r}\right) dx\,dy\,dz - 4\pi K_p u'_p.$$

The Correction for Cyclosis.

5.] We have assumed also that u and u' are *single-valued* functions of x, y, z; that is, that for any such function the line integral $\int \frac{du}{dl}\,dl$, taken round any closed curve that can be drawn within the space S to which Green's theorem is applied, is zero. The functions with which we shall have to deal in this treatise will generally satisfy this condition.

If however for any function u the condition $\int \frac{du}{dl}\,dl = 0$ be not satisfied for certain closed curves drawn within S, the statement of Green's theorem requires modification in the manner pointed out by Helmholz and Sir W. Thomson. The reader will find the subject fully treated in Maxwell's *Electricity and Magnetism*, Second Edition, Arts. 96 b–96 d.

It will be sufficient here to shew the modification required in a simple case. Suppose, for instance, S consist of an anchor-ring, Fig. 1, and that for any closed curve drawn within it, so as to embrace the axis, as $OPQO$, $\int \frac{du}{dl}\,dl = H$, but for closed curves not embracing the axis $\int \frac{du}{dl}\,dl = 0$. Let us suppose u to be measured from a section S_0 of the ring. Let O be a point in the

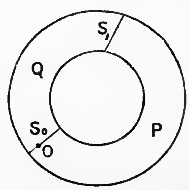

Fig. 1.

section S_0. Then, if we start from O, with u_0 for the value of u, and proceed round the curve $OPQO$, u will, on arriving again at O, have assumed by continuous variation the value u_0+H.

Let S_1 be any other section of the ring. Then S_0 and S_1 divide the space within the ring into two parts, S_0PS_1 and S_1QS_0. No curve embracing the axis can be drawn wholly within either S_0PS_1 or S_1QS_0. Therefore Green's theorem may be applied to either space. Applying it to S_0PS_1, we have

$$\iiint K\left\{\frac{du}{dx}\frac{du'}{dx}+\&c.\right\}dx\,dy\,dz$$
$$=\iint uK\frac{du'}{d\nu}dS+\iint u_0K\frac{du'}{d\nu}dS_0+\iint u_1K\frac{du'}{d\nu}dS_1$$
$$-\iiint u\nabla^2{}_K u'\,dx\,dy\,dz \;\ldots\;\ldots\;\ldots\;(1)$$

in which the first double integral relates to the surface of the ring, and the other two to the sections S_0 and S_1 respectively.

Again, applying the theorem to S_1QS_0, and regarding the normals to S_0 and S_1 as measured in the same direction as in the former case, that is inwards as regards the space now in question, we have

$$\iiint K\left(\frac{du}{dx}\frac{du'}{dx}+\&c.\right)dx\,dy\,dz$$
$$=\iint uK\frac{du'}{d\nu}dS-\iint u_1K\frac{du'}{d\nu}dS_1$$
$$-\iint (u_0+H)\,K\frac{du'}{d\nu}dS_0-\iiint u\,\nabla^2_K u'\,dx\,dy\,dz. \;\ldots\;\ldots\;\ldots\;(2)$$

If we now add the two equations (1) and (2) together, we obtain for the whole space within the ring

$$\iiint K\left(\frac{du}{dx}\frac{du'}{dx}+\&c.\right)dx\,dy\,dz$$
$$=\iint uK\frac{du'}{d\nu}dS-\iiint u\,\nabla^2_K u'\,dx\,dy\,dz-H\iint K\frac{du'}{d\nu}dS_0.$$

Hence $-H\iint K\frac{du'}{d\nu}dS_0$ is the correction for cyclosis in this case. Its value depends on the section of the ring arbitrarily chosen as the starting-point from which u is measured.

Deductions from Green's Theorem.

6.] Let u' be a constant. Then, since $\frac{du'}{dx}$, $\frac{du'}{dy}$, and $\frac{du'}{dz}$ are severally zero, we obtain the result that for any function u,

$$\iint K \frac{du}{d\nu} dS = \iiint \nabla^2{}_K u \, dx \, dy \, dz,$$

the integrals being taken over any closed surface S and the enclosed space.

7.] (*a*) *There exists one function u of x, y, and z which has arbitrarily assigned values at each point on a closed surface S, and satisfies the condition $\nabla^2_K u = 0$ at each point within S, K being everywhere positive.*

For evidently an infinite number of forms of the function u exist satisfying the condition that u has the assigned value at each point of S, irrespective of the value of $\nabla^2_K u$ within S.

For any function u let the integral

$$\iiint K \left\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dx \, dy \, dz$$

throughout the space enclosed by S be denoted by Q_u.

This integral is necessarily positive, and cannot be zero for any of the functions in question, unless the assigned values are the same at every point of S, in which case a function having that same constant value within S satisfies all the conditions of the problem.

If the assigned values of u be not the same at each point of S, then of all the functions which satisfy the surface conditions, there must be some one, or more, for which Q_u, being necessarily positive, is not greater than for any of the others. Let u be such function.

Let $u + u'$ be any other function which satisfies the surface conditions, so that $u' = 0$ at each point of S. Then also $u + \theta u'$ satisfies the surface conditions, if θ be any numerical quantity whatever, positive or negative.

Then

$$Q_{u+\theta u'}$$

$$= \iiint K \left\{ \left(\frac{d(u+\theta u')}{dx}\right)^2 + \left(\frac{d(u+\theta u')}{dy}\right)^2 + \left(\frac{d(u+\theta u')}{dz}\right)^2 \right\} dx\,dy\,dz$$

$$= Q_u + \theta^2 Q_{u'} + 2\theta \iiint K \left\{ \frac{du}{dx}\frac{du'}{dx} + \frac{du}{dy}\frac{du'}{dy} + \frac{du}{dz}\frac{du'}{dz} \right\} dx\,dy\,dz$$

$$= Q_u + \theta^2 Q_{u'} + 2\theta \left\{ \iint K u' \frac{du}{d\nu} dS - \iiint u' \nabla^2_K u\,dx\,dy\,dz \right\};$$

by Green's theorem,

$$= Q_u + \theta^2 Q_{u'} - 2\theta \iiint u' \nabla^2_K u\,dx\,dy\,dz,$$

because $u' = 0$ on S.

Now $Q_{u+\theta u'}$ is by hypothesis not less than Q_u, and therefore

$$\theta^2 Q_{u'} - 2\theta \iiint u' \nabla^2_K u\,dx\,dy\,dz$$

cannot be negative for any value of θ, or any value of u'.

But unless $\nabla^2{}_K u$ be zero at each point within S, it is possible to assign such values to u', consistently with its being zero on S, as to make

$$\iiint u' \nabla^2_K u\,dx\,dy\,dz$$

differ from zero. Therefore, it is possible to assign such a value to θ as to make

$$\theta^2 Q_{u'} - 2\theta \iiint u' \nabla^2_K u\,dx\,dy\,dz$$

negative.

It follows that $\nabla^2_K u = 0$ at each point within S, when u is a function satisfying the surface conditions for which Q_u is not greater than for any other function satisfying these conditions.

Corollary. If $u + u'$ be any other function satisfying the surface condition, but such that $\nabla^2_K u'$ is not zero at all points within S, evidently

$$Q_{u+u'} = Q_u + Q_{u'}.$$

(*b*) The theorem can also be applied to the infinite space outside of S with a certain modification, namely, *There exists a function u of x, y, and z which has arbitrarily assigned values at each point on S, and satisfies the condition $\nabla^2_K u = 0$ at each point outside of S, K being positive, and such that Ku^2 is of lower degree than -1.*

For of all the functions which satisfy the surface conditions on S and the condition as to degree, there must be some one or more for which the integral Q_u extended through the infinite external space is not greater than for any of the others.

Let u' be another function which is zero on S, and satisfies the condition as to degree. Then Green's theorem may be applied to the infinite external space with u and u' for functions. And it can be proved by the same process as used above that, unless $\nabla^2_K u = 0$ at every point in the external space, some value may be given to u' which will make $Q_{u+u'}$ less than Q_u. Therefore when Q_u has its least possible value for all functions satisfying the conditions, $\nabla^2_K u$ must be zero at all points outside of S.

8.] The theorems can be extended to the case where $\nabla^2{}_K u$, instead of being zero, has any given value ρ, a function of x, y, z, at each point within the limits of the triple integral, i.e. within or outside of S as the case may be.

For let V be a function of the required degree which satisfies $\nabla^2_K V = \rho$ at all points within the limits of the triple integral. Such a function always exists independently of the surface conditions *.

Then if σ be the assigned value of u on S, there exists, by Art. 7, a function W, having at each point on S the value $\sigma - V$, and such that $\nabla^2_K W = 0$, at all points within the limits of the triple integral. Let $u = W + V$.

Then u has at each point on S the value $\sigma - V + V$, that is, the required value σ, and

$$\begin{aligned} \nabla^2_K u &= \nabla^2_K V + \nabla^2_K W \\ &= \nabla^2_K V \\ &= \rho \end{aligned}$$

at each point within the limits of the triple integral.

9.] If the value of u be given at each point on S, and if the value of $\nabla^2{}_K u$, whether zero or any other assigned value, be given at each point within S, u has a *single and determinate* value at each point within S.

* $\iiint \frac{\rho \, dx' \, dy' \, dz'}{K\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$ is one such function.

For, let u and u' be two functions both satisfying the conditions.

Then $u=u'$ and $u-u'=0$ at each point on S; and

$$\nabla^2_K u - \nabla^2_K u' = 0, \text{ or } \nabla^2_K (u-u') = 0, \quad \ldots \ldots \quad (1)$$

at each point within S. Then

$$\iiint K\left\{\left(\frac{d(u-u')}{dx}\right)^2 + \left(\frac{d(u-u')}{dy}\right)^2 + \left(\frac{d(u-u')}{dz}\right)^2\right\} dx\,dy\,dz$$

$$= \iint K(u-u')\frac{d(u-u')}{d\nu}dS - \iiint (u-u')\nabla^2{}_K(u-u')\,dx\,dy\,dz \ldots (2)$$

$$= 0 \text{ by } (1).$$

It follows that

$$\frac{du}{dx} = \frac{du'}{dx}, \quad \frac{du}{dy} = \frac{du'}{dy}, \quad \frac{du}{dz} = \frac{du'}{dz},$$

at each point within S, and therefore u and u', being equal on S, have identical values at each point within S.

It follows as a corollary that, as stated above, if u be constant at each point on S, and if $\nabla^2_K u = 0$ everywhere within S, u has the same constant value everywhere within S. For the constant satisfies both the surface and internal conditions, and there can be no other function which does satisfy them.

The last theorem can be applied to the infinite space outside of S as well as to the space within it, if we add the condition that Ku^2 is of a less degree than -1, without which Green's theorem could not be applied to deduce (2).

10.] *There exists a function u of x, y, and z which satisfies the conditions following; namely—*

(1) *u has values constant but* arbitrary *over each of a series of closed surfaces S_1, S_2, … S_n, and* given *constant values over each of a second series of closed surfaces S_1', … S_m'.*

(2) *u is of lower degree than $-\frac{1}{2}$.*

(3) $\iint \frac{du}{d\nu}dS_1 = e_1$, $\iint \frac{du}{d\nu}dS_2 = e_2$, *&c.*,

and $\iint \frac{du}{d\nu}dS_n = e_n$,

where e_1, e_2, … e_n are given constants, and the double integrals are taken over S_1, S_2, … S_n.

(4) $\nabla^2 u = 0$ *at every point not within any of the surfaces* $S_1, S_2 \ldots, S_n$, *and* $S_1' \ldots S_m'$.

For, consider a function u which satisfies (1) and (2), and also satisfies

$$(a) \quad u_1 e_1 + u_2 e_2 + \ldots + u_n e_n = E, \text{ or } \Sigma ue = E,$$

where E is any arbitrary constant, and $u_1 \ldots u_n$ are the constant values assumed by u on $S_1 \ldots S_n$.

Evidently there exists an infinite variety of such functions.

For every such function u, the volume integral

$$Q_u = \iiint \left\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dx\, dy\, dz$$

extended throughout all space not within the surfaces, cannot be zero, if E be not zero, and is positive.

There must therefore be some one or more of such functions for which Q_u is not greater than for any other.

Let u be such function.

Let $u + u'$ be any other function satisfying (1), (2), and (a).

Then u' has constant values, u_1', u_2', &c. on the surfaces S_1, $S_2 \ldots S_n$ which satisfy $u_1'e_1 + u_2'e_2 + \&c. = 0$, is zero on each of the surfaces $S_1' \ldots S_m'$, and is of the required degree.

These are its only conditions. Also u' being of less degree than $-\frac{1}{2}$ may be used with u in Green's theorem for external space.

Let θ be any numerical quantity, positive or negative. Then $u + \theta u'$ also satisfies (1), (2), and (a). Then

$$Q_{u+\theta u'} = Q_u + \theta^2 Q_{u'} + 2\theta \iiint \left\{ \frac{du}{dx}\frac{du'}{dx} + \&c. \right\} dx\, dy\, dz$$

$$= Q_u + \theta^2 Q_{u'} + 2\theta \left\{ \iint u' \frac{du}{d\nu} dS_1 + \iint u' \frac{du}{d\nu} dS_2 + \&c. - \iiint u' \nabla^2 u \, dx\, dy\, dz \right\}$$

$$= Q_u + \theta^2 Q_{u'} + 2\theta \left\{ u_1' \iint \frac{du}{d\nu} dS_1 + u_2' \iint \frac{du}{d\nu} dS_2 + \&c. - \iiint u' \nabla^2 u \, dx\, dy\, dz \right\},$$

since u' is constant on each of the surfaces $S_1 \ldots S_n$, and is zero on each of the surfaces $S_1' \ldots S_m'$.

Now $Q_{u+\theta u'}$ cannot be less than Q_u, whatever u' may be, and whatever θ may be.

But unless the factor of 2θ in the last expression be zero, there must be some value of θ which makes $Q_{u+\theta u'}$ less than Q_u.

The quantity multiplied by 2θ must therefore be zero for all values of u' consistently with its conditions.

$\nabla^2 u$ must therefore be zero at all points within the triple integral, and

$$u_1'\iint \frac{du}{d\nu}dS_1 + u_2'\iint \frac{du}{d\nu}dS_2 + \&\text{c.} = 0$$

for all values of u_1', u_2', &c. consistent with

$$u_1' e_1 + u_2' e_2 + \&\text{c.} = 0.$$

Therefore we must have

$$\iint \frac{du}{d\nu} dS_1 = \mu e_1,\quad \iint \frac{du}{d\nu} dS_2 = \mu e_2,\ \&\text{c.},$$

where μ is some constant, the same for all the surfaces $S_1 \dots S_n$.

If the function u be found for any value of E, then μ is known from (a), and is proportional to E.

There must, therefore, be some value of E for which μ is unity, and the function u determined for that value of E satisfies (1), (2), (3), and (4).

11.] The theorem can be extended to the case where $\nabla^2 u$, instead of being zero at every point within the limits of the triple integral, has any assigned value ρ, a function of x, y, z.

For let V be a function of the required degree which has constant but arbitrary values on each of the surfaces $S_1 \dots S_n$, has the given constant values on $S_1' \dots S_m'$, and satisfies $\nabla^2 V = \rho$ at all points external to both series of surfaces. The existence of such a function is proved in Art. 8. V being so determined, let

$$\iint \frac{dV}{d\nu} dS_1 = e_1',\quad \iint \frac{dV}{d\nu} dS_2 = e_2',\ \&\text{c.}$$

Then, as we have proved, there exists a function W of the required degree having, some constant values on $S_1 \dots S_m$, the value zero on $S_1' \dots S_m'$, and satisfying

$$\iint \frac{dW}{d\nu} dS_1 = e_1 - e_1',\quad \iint \frac{dW}{d\nu} dS_2 = e_2 - e_2',\ \&\text{c.},$$

and $\nabla^2 W = 0$ at all points external to all the surfaces.

Let $u = W + V$.

Then u has some constant values on each of the surfaces $S_1 \dots S_n$, and the given constant values on each of the surfaces $S_1' \dots S_m'$.

Also
$$\iint \frac{du}{d\nu} dS_1 = e_1 - e_1' + e_1' = e_1.$$

Similarly,
$$\iint \frac{du}{d\nu} dS_2 = e_2,$$
$$\&c. = \&c.,$$

and
$$\nabla^2 u = \nabla^2 W + \nabla^2 V = \rho$$
at all external points.

12.] If u be a function which satisfies the conditions (1), (2), and (3) of Art. 10, and for which $\nabla^2 u$ has any assigned value, zero or otherwise, at every point not within any of the surfaces, then u has single and determinate value at each point in external space.

For let u and u' be two functions, each of degree less than $-\frac{1}{2}$, satisfying the surface conditions, so that u and u' are both constant on each surface, and

$$\iint \frac{du}{d\nu} dS_1 = \iint \frac{du'}{d\nu} dS_1, \text{ or } \iint \frac{d(u-u')}{d\nu} dS_1 = 0,$$

and so on for each of the surfaces.

Also $\nabla^2 u$ and $\nabla^2 u'$ both have the same given value at each point in the external space, and therefore $\nabla^2 (u - u') = 0$ at every point in that space.

Then

$$\iiint \left\{ \left(\frac{d(u-u')}{dx}\right)^2 + \left(\frac{d(u-u')}{dy}\right)^2 + \left(\frac{d(u-u')}{dz}\right)^2 \right\} dx\,dy\,dz$$
$$= \iint (u-u') \frac{d(u-u')}{d\nu} dS_1 + \iint (u-u') \frac{d(u-u')}{d\nu} dS_2 + \&c.$$
$$- \iiint (u-u') \nabla^2 (u-u')\, dx\,dy\,dz$$
$$= (u_1 - u_1') \iint \frac{d(u-u')}{d\nu} dS_1 + (u_2 - u_2') \iint \frac{d(u-u')}{d\nu} dS_2$$
$$- \iiint (u-u') (\nabla^2 u - \nabla^2 u')\, dx\,dy\,dz$$
$$= 0.$$

Therefore $\frac{du}{dx} = \frac{du'}{dx}$, $\frac{du}{dy} = \frac{du'}{dy}$, and $\frac{du}{dz} = \frac{du'}{dz}$,

and since $u = u'$ at an infinite distance, $u = u'$ at every point not within any of the surfaces.

13.] We proved in Art. 10 that if $\iint \frac{du}{d\nu} dS$ be given for each of the surfaces $S_1, S_2 \dots S_n$, and u be constant on each surface, and of degree lower than $-\frac{1}{2}$, then the triple integral Q_u has its least value when $\nabla^2 u = 0$ at each point in external space.

We can now prove that given $\iint \frac{du}{d\nu} dS$ as before for each surface and given $\nabla^2 u = \rho$ at each point not within any of the surfaces, and u of degree lower than $-\frac{1}{2}$, Q_u has its least value when u is constant over each surface. For let u be the function which satisfies the four conditions of Art. 10, u' any other function of degree less than $-\frac{1}{2}$ which satisfies conditions (3) and (4) of that Article, but is not constant on each of the surfaces $S_1 \dots S_n$. Then

$$u' = u + u' - u,$$

and if Q_u and $Q_{u'}$ denote the triple integral Q for u and u' respectively, we have, as in the preceding articles,

$$Q_{u'} = Q_u + \iiint \left\{ \left(\frac{d(u'-u)}{dx}\right)^2 + \left(\frac{d(u'-u)}{dy}\right)^2 + \left(\frac{d(u'-u)}{dz}\right)^2 \right\} dx\,dy\,dz$$
$$+ 2\left\{ u \iint \frac{d(u'-u)}{d\nu} dS - \iiint u\,\nabla^2(u'-u)\,dx\,dy\,dz \right\},$$

in which the double integral is understood to relate to each of the surfaces in succession. The second line is zero by the conditions, and therefore if u' differ from u,

$$Q_{u'} = Q_u + Q_{u'-u}.$$

The theorems of the last three articles can also be extended to the more general case in which the value of

$$\frac{d}{dx}\left(K\frac{du}{dx}\right) + \frac{d}{dy}\left(K\frac{du}{dy}\right) + \frac{d}{dz}\left(K\frac{du}{dz}\right), \quad \text{or} \quad \nabla_K^2 u$$

instead of $\nabla^2 u$ are given within the limits of the triple integrals, and

$$\iint K\frac{du}{d\nu} dS_1 = e_1, \quad \iint K\frac{du}{d\nu} dS_2 = e_2, \text{ \&c.},$$

where K is positive and constant for each surface, and Ku^2 of lower degree than -1.

For, we have only to replace $\nabla^2 u$ by the more general expression

$$\frac{d}{dx}\left(K\frac{du}{dx}\right)+\frac{d}{dy}\left(K\frac{du}{dy}\right)+\frac{d}{dz}\left(K\frac{du}{dz}\right), \quad \text{or} \quad \nabla^2{}_K u,$$

and Q_u by

$$\iiint K\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2\right\}dx\,dy\,dz,$$

and every step in the process applies as before.

14.] Again, if S be a closed surface, or series of closed surfaces external to each other, and if σ be a function having arbitrarily assigned values at each point on S, there always exists a function u satisfying the condition

(1) $\frac{du}{d\nu} = \sigma$ at each point on S,

(2) $\nabla^2 u = 0$ at each point in external space,

(3) u is of lower degree than $-\frac{1}{2}$.

For there must be an infinite variety of functions U which satisfy the conditions (4) $\iint U\sigma dS = E$, where E is any arbitrary quantity differing from zero, and (5) U is of lower degree than $-\frac{1}{2}$.

For any such function the integral Q_U must be greater than zero. There must therefore be some one or more of such functions for which this integral is not greater than for any other. Let u be any such function. Let $u+u'$ be any other function satisfying (4) and (5), and for which therefore $\iint u'\sigma dS = 0$.

Then it can be shewn by the same process as in Art. 10 that $\frac{du}{d\nu} \propto \sigma$, and $\nabla^2 u = 0$ at all points external to S, and that by properly choosing E we may make $\frac{du}{d\nu} = \sigma$ and $\nabla^2 u = 0$ as before, and that $Q_{u+u'} = Q_u + Q_{u'}$. This theorem also, as in the preceding, may be extended to the case in which $\nabla^2 u$, instead of being zero, has assigned values at all points in the external space.

Again, as there always exists a function u satisfying the conditions, so it can be shewn that it has single and determinate value at all external points.

For, if possible, let there be two functions u and u' of degree less than $-\frac{1}{2}$, both satisfying the conditions, so that $\frac{du}{d\nu} = \frac{du'}{d\nu}$ at each point on S, and $\nabla^2 u = \nabla^2 u'$, or $\nabla^2 (u-u') = 0$ at all external points. Then

$$\iiint \left\{\left(\frac{d(u-u')}{dx}\right)^2 + \&c.\right\} dx\,dy\,dz$$

$$= \iint (u-u')\left(\frac{du}{d\nu} - \frac{du'}{d\nu}\right) dS - \iiint (u-u')\nabla^2(u-u')\, dx\,dy\,dz$$

$$= 0,$$

and therefore $\frac{du}{dx} = \frac{du'}{dx}$, &c., and $u = u'$, since both vanish at an infinite distance.

15.] We can shew also by the same process that there exists a function u satisfying the condition that $\nabla^2 u = 0$ at all points in the *internal* space, and $\frac{du}{d\nu} = \sigma$ at all points on S, provided

$$\iint \sigma\, dS = 0.$$

For if that condition were not satisfied, the condition $\iint u\sigma\, dS = E$ might be satisfied by making u a constant, in which case Q_u would not have a minimum value greater than zero, and the proof would fail. In fact, if $\nabla^2 u = 0$ everywhere within S, $\iint \frac{du}{d\nu} dS = 0$; and therefore $\frac{du}{d\nu}$ cannot be equal to σ at all points on S unless $\iint \sigma\, dS = 0$.

If $\nabla^2 u$, instead of being zero, is to have the value ρ within S, the problem may be solved, provided $\iint \sigma\, dS = \iiint \rho\, dx\,dy\,dz$, as follows.

Let W be a function such that $\nabla^2 W = \rho$ at every point within S, and therefore that

$$\iint \sigma\, dS = \iint \frac{dW}{d\nu}\, dS.$$

Then there exists a function V such that

$$\frac{dV}{d\nu} = \sigma - \frac{dW}{d\nu}$$

at each point on S, and $\nabla^2 V = 0$ at each point within S. Let $u = V + W$. Then

$$\frac{du}{d\nu} = \frac{dV}{d\nu} + \frac{dW}{d\nu} = \sigma$$

at each point on S, and

$$\nabla^2 u = \nabla^2 V + \nabla^2 W = \rho$$

at each point within S.

It can easily be shown also that if u and u' be two functions both satisfying the conditions, $\frac{du}{dx} = \frac{du'}{dx}$, &c., at all points within S, and therefore u can only differ from u' by a constant.

16.] Let p, q, r be any functions of x, y and z, each of degree less than $-\frac{3}{2}$, satisfying the conditions

$$lp + mq + nr = \sigma \tag{1}$$

at every point on S, where σ is any arbitrary function, and

$$\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \tag{2}$$

at all points without S.

Then we know that there exists a function u of degree less than $-\frac{1}{2}$ satisfying the conditions

$$\frac{du}{d\nu} = l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} = \sigma$$

at each point on S, and

$$\nabla^2 u = \frac{d}{dx}\frac{du}{dx} + \frac{d}{dy}\frac{du}{dy} + \frac{d}{dz}\frac{du}{dz} = 0 \tag{3}$$

at all external points. Therefore the system

$$p = \frac{du}{dx}, \quad q = \frac{du}{dy}, \quad r = \frac{du}{dz},$$

satisfies (1) and (2).

It can now be shewn that the integral

$$\iiint (p^2 + q^2 + r^2)\, dx\, dy\, dz,$$

extended throughout the space external to S, has less value when $p = \frac{du}{dx}$, &c., than when p, q, and r are any other functions of degree less than $-\frac{3}{2}$ satisfying (1) and (2). For if

$$\frac{du}{dx} + \alpha, \quad \frac{du}{dy} + \beta, \quad \frac{du}{dz} + \gamma$$

be any other three functions of the required degree satisfying (1) and (2), α, β, and γ must satisfy

$$l\alpha + m\beta + n\gamma = 0 \tag{4}$$

at each point on S, and

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0 \tag{5}$$

at each point in external space.

Then

$$\iiint \left\{\left(\frac{du}{dx} + \alpha\right)^2 + \left(\frac{du}{dy} + \beta\right)^2 + \left(\frac{du}{dz} + \gamma\right)^2\right\} dx\,dy\,dz$$

$$= \iiint \left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\} dx\,dy\,dz + \iiint \{\alpha^2 + \beta^2 + \gamma^2\}\, dx\,dy\,dz$$

$$+ 2\iiint \left\{\alpha\frac{du}{dx} + \beta\frac{du}{dy} + \gamma\frac{du}{dz}\right\} dx\,dy\,dz.$$

By integrating the last term by parts, and attending to (4) and (5), we prove it to be zero. Because $u\alpha$, $u\beta$, and $u\gamma$ being of less degree than -2, the double integrals $\iint u\alpha\, dy\,dz$ &c. vanish for an infinitely distant surface. Hence the integral

$$\iiint \left\{\left(\frac{du}{dx}\right)^2 + \text{\&c.}\right\} dx\,dy\,dz$$

is less than

$$\iiint \left\{\left(\frac{du}{dx} + \alpha\right)^2 + \text{\&c.}\right\} dx\,dy\,dz.$$

A corresponding proposition can be proved for the space within S without restriction on the degree of p, q, r, and u.

17.] The propositions of Arts. 14 and 15 can be extended to the case in which $K\frac{du}{d\nu}$ is written for $\frac{du}{d\nu}$ at the surface, and $\nabla^2_K u$

for $\nabla^2 u$ at points in space; and Art. 16 may be similarly extended to prove that

$$\iiint \frac{1}{K}(p^2+q^2+r^2)\,dx\,dy\,dz$$

has a minimum value when

$$p = K\frac{du}{dx},\quad q = K\frac{du}{dy},\quad r = K\frac{du}{dz},$$

K being in each case a given positive function of x, y, and z, and such that Kp &c. are of lower degree than $-\frac{3}{2}$.

CHAPTER II.

SPHERICAL HARMONICS.

ARTICLE 18.] *Definition.*—If u be a homogeneous function of the n^{th} degree in x, y, and z, satisfying the condition $\nabla^2 u = 0$, where ∇^2 represents the operation

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2},$$

then u is said to be a *spherical harmonic function* of the n^{th} degree in x, y, and z.

If u be any function of x, y, z satisfying the condition $\nabla^2 u = 0$, then every partial differential coefficient of u, as $\frac{d^{\lambda+\mu+\nu}u}{dx^\lambda dy^\mu dz^\nu}$, will also satisfy the condition

$$\nabla^2 \frac{d^{\lambda+\mu+\nu}u}{dx^\lambda\, dy^\mu\, dz^\nu} = 0.$$

For since the order of partial differentiation is indifferent, it follows that

$$\nabla^2 \frac{d^{\lambda+\mu+\nu}u}{dx^\lambda\, dy^\mu\, dz^\nu} = \frac{d^{\lambda+\mu+\nu}}{dx^\lambda\, dy^\mu\, dz^\nu}\nabla^2 u$$
$$= 0.$$

19.] Let any point O be taken as origin of rectangular coordinates, and let the coordinates of P be x, y, z. Let $\phi(x, y, z)$ be any function of x, y, z. Let OH be any axis drawn from O and designated by h, and let Q be any point in this axis, and let $OQ = \rho$.

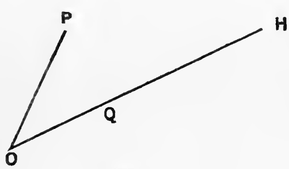

Fig. 2.

Let ξ, η, ζ be the coordinates of P referred to Q as origin, with axes parallel to the axes through O. Then the limiting value of the ratio

$$\frac{\phi(\xi, \eta, \zeta)-\phi(x, y, z)}{\rho},$$

as ρ is indefinitely diminished, is denoted by

$$\frac{d}{dh}\phi(x, y, z).$$

It is clear that

$$\frac{d}{dh}\phi(x, y, z), \text{ or } \frac{d\phi}{dh},$$

is itself generally a function of x, y, z; and therefore if another axis OH', denoted by h', be drawn from O, we may find by a similar process

$$\frac{d}{dh'}\left(\frac{d\phi}{dh}\right),$$

and so on for any number of axes.

If u be any function of x, y, z satisfying the condition $\nabla^2 u = 0$, and if $h_1, h_2, \ldots h_i$ denote any number of axes drawn from the origin, and the expression

$$\frac{d}{dh_1}\frac{d}{dh_2}\cdots\frac{d}{dh_i}u$$

be found according to the preceding definition, then

$$\nabla^2\left(\frac{d}{dh_1}\frac{d}{dh_2}\cdots\frac{d}{dh_i}\right)u = 0.$$

For let l_1, m_1, n_1 be the direction cosines of the axis h_1. Then by definition

$$\frac{du}{dh_1} = l_1\frac{du}{dx} + m_1\frac{du}{dy} + n_1\frac{du}{dz}.$$

But by hypothesis

$$\nabla^2 u = 0.$$

Therefore

$$\nabla^2\frac{du}{dx}, \quad \nabla^2\frac{du}{dy}, \quad \nabla^2\frac{du}{dz}$$

are severally equal to zero. Therefore

$$\nabla^2\frac{du}{dh_1} = 0;$$

and therefore by successive steps

$$\nabla^2\frac{d}{dh_1}\frac{d}{dh_2}\cdots\frac{d}{dh_i}u = 0.$$

20.] If

$$r^2 = (x^2+y^2+z^2),$$

$\frac{1}{r}$ is a spherical harmonic function of degree -1.

For

$$\frac{d}{dx}\left(\frac{1}{r}\right) = -\frac{x}{r^3},$$

$$\frac{d^2}{dx^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5}.$$

Similarly

$$\frac{d^2}{dy^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3y^2}{r^5},$$

and

$$\frac{d^2}{dz^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3z^2}{r^5};$$

whence

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\frac{1}{r} = -\frac{3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0.$$

21.] Whatever be the directions of the i-axes $h_1, h_2, \dots h_i$, the function

$$\frac{d}{dh_1}\frac{d}{dh_2}\cdots\frac{d}{dh_i}\left\{\frac{M}{r}\right\},$$

where M is any constant, is a spherical harmonic function of degree $-(i+1)$.

For it is evidently a homogeneous function of that degree, and since

$$\nabla^2\left(\frac{1}{r}\right) = 0,$$

it follows that

$$\nabla^2\frac{d}{dh_1}\frac{d}{dh_2}\cdots\frac{d}{dh_i}\left(\frac{M}{r}\right) = 0.$$

If we write this function in the form $\underline{|i}\,\frac{Y_i}{r^{i+1}}$, Y_i is a function of M, the direction cosines of the axes $h_1, h_2, \dots h_i$, and those of r. To fix the ideas we may conceive a sphere from the centre O of which are drawn in arbitrarily given directions the i-axes $OH_1, OH_2, \dots OH_i$ cutting the sphere in $H_1, H_2, \dots H_i$. Then if OQ be any radius, at every point P on OQ or OQ produced Y_i has a definite numerical value, being a function of the direction cosines of $OH_1, \dots OH_i$, and of OQ, and independent of r or OP. If $h_1, h_2, \dots h_i$ be the fixed axes of any harmonic, P any variable point, Y_i at P is spoken of as the harmonic at P with axes $h_1, h_2, \ldots h_i$.

Since each axis requires for the determination of its direction two independent quantities, Y_i will be a function of the two variable magnitudes determining the direction of r and the $2i$ arbitrary constant magnitudes determining the directions of the i-axes. Y_i may also be expressed in terms of the i-cosines $\mu_1, \mu_2, \ldots \mu_i$ of the angles made by r with the i-axes and the $\frac{i(i-1)}{2}$ cosines of the angles made by the axes with each other, and an expression for Y_i in this form may be found without much difficulty.

22.] If V_i be a spherical harmonic function of degree $-(i+1)$, and if $r = \sqrt{x^2+y^2+z^2}$, then $r^{2i+1}V_i$ will be a spherical harmonic function of degree i.

For by differentiation

$$\frac{d}{dx}(r^{2i+1}V_i) = (2i+1)r^{2i-1}xV_i + r^{2i+1}\frac{dV_i}{dx},$$

$$\begin{aligned}\frac{d^2}{dx^2}(r^{2i+1}V_i) = &(2i+1)(2i-1)r^{2i-3}x^2V_i \\ &+(2i+1)r^{2i-1}V_i \\ &+2.(2i+1)r^{2i-1}x\frac{dV_i}{dx} \\ &+r^{2i+1}\frac{d^2V_i}{dx^2}.\end{aligned}$$

Similar expressions hold for

$$\frac{d^2}{dy^2}(r^{2i+1}V_i) \quad \text{and} \quad \frac{d^2}{dz^2}(r^{2i+1}V_i).$$

Adding these expressions, and remembering that

$$x^2+y^2+z^2 = r^2,$$

we obtain

$$\begin{aligned}\nabla^2(r^{2i+1}V_i) = &(2i+1)(2i+2)r^{2i-1}V_i \\ &+2(2i+1)(r^{2i-1})\left(x\frac{dV_i}{dx}+y\frac{dV_i}{dy}+z\frac{dV_i}{dz}\right) \\ &+r^{2i+1}\nabla^2V_i,\end{aligned}$$

but $$x\frac{dV_i}{dx}+y\frac{dV_i}{dy}+z\frac{dV_i}{dz}=-(i+1)V_i,$$

and $$\nabla^2 V_i = 0.$$

Therefore

$$\nabla^2(r^{2i+1}V_i) = \{(2i+1)(2i+2)-2.(2i+1)(i+1\}\, r^{2i-1}V_i$$
$$= 0,$$

and $r^{2i+1}V_i$ is a homogeneous function of x, y, z of degree i: and is therefore a spherical harmonic function of degree i.

We have seen that $\frac{Y_i}{r^{i+1}}$, as above defined, is a spherical harmonic function of degree $-(i+1)$.

It follows then that

$$r^{2i+1}\frac{Y_i}{r^{i+1}} \text{ or } r^i Y_i$$

is a spherical harmonic function of degree i.

23.] Every possible spherical harmonic function of integral positive degree, i, can be expressed in the form $r^i Y_i$ if suitable directions be given to the axes $h_1, h_2, \ldots h_i$ determining Y_i.

For if H_i be a homogeneous function of the i^{th} degree it contains $\frac{(i+1)(i+2)}{2}$ arbitrary constants. Therefore $\nabla^2 H_i$ being of the degree $i-2$ contains $\frac{i(i-1)}{2}$ arbitrary constants.

In order that $\nabla^2 H_i$ may be zero for all values of x, y, and z, the coefficient of each term in $\nabla^2 H_i$ must be separately zero. This involves $\frac{i(i-1)}{2}$ relations between the constants in H_i. leaving $\frac{(i+1)(i+2)}{2}-\frac{i(i-1)}{2}$ or $2i+1$ of them independent. Therefore every possible harmonic function of degree i is to be found by attributing proper values to these $2i+1$ constants.

But the directions of the i-axes $h_1, h_2, \ldots h_i$ involve $2i$ arbitrary constants, making with the constant M, $2i+1$ in all. It is therefore always possible to choose the i-axes $h_1, h_2, \ldots h_i$ and the constant M, so as to make

$$r^{2i+1}\frac{d}{dh_1}\cdot\frac{d}{dh_2}\cdots\frac{d}{dh_i}\frac{M}{r}, \quad \text{or} \quad r^i Y_i,$$

equal to any given spherical harmonic function of degree i. Therefore $r^i Y_i$ is a perfectly general form of the spherical harmonic function of positive integral degree i.

Again, every possible spherical harmonic function of negative integral degree $-(i+1)$ can be expressed in the form $\frac{Y_i}{r^{i+1}}$.

For if V_i be any spherical harmonic function of degree $-(i+1)$, it follows from Art. 22 that $r^{2i+1}V_i$ is a spherical harmonic function of degree i. Hence, i being integral, it follows by the former part of this proposition that $r^{2i+1}V_i$ can always be expressed in the form $r^i Y_i$ by suitably choosing the axes of Y_i, and therefore that V_i may be expressed in the form $\frac{Y_i}{r^{i+1}}$.

Therefore $r^i Y_i$ and $\frac{Y_i}{r^{i+1}}$ are the most general forms of the spherical harmonic functions of the integral degrees i and $-(i+1)$ respectively.

Y_i is defined as the *surface spherical harmonic* of the *order* i, where i is always positive and integral; $r^i Y_i$ and $\frac{Y_i}{r^{i+1}}$ are called the *solid harmonics* of the order i.

24.] If Y_i and Y_j be any two surface spherical harmonics with the same origin O, and referred to the same or different axes, and of orders i and j respectively, and if $\iint Y_i Y_j \, dS$ be found over the surface of any sphere with centre O, then

$$\iint Y_i Y_j \, dS = 0, \text{ unless } i = j.$$

Let H_i and H_j be the solid spherical harmonics of degrees i and j respectively corresponding to the surface harmonics Y_i and Y_j, so that

$$H_i = r^i Y_i, \quad H_j = r^j Y_j.$$

Make U and U' equal to H_i and H_j respectively in the equation of Green's theorem taken for the space bounded by the aforesaid spherical surface, then

$$\iiint \left\{ \frac{dH_i}{dx}\frac{dH_j}{dx} + \frac{dH_i}{dy}\frac{dH_j}{dy} + \frac{dH_i}{dz}\frac{dH_j}{dz} \right\} dx\,dy\,dz$$

$$= \iint H_i \frac{dH_j}{d\nu}\, dS = \iint H_j \frac{dH_i}{d\nu}\, dS,$$

because $\nabla^2 H_i$ and $\nabla^2 H_j$ are each zero at every point within the sphere;

$$\therefore \quad \iint H_i \frac{dH_j}{d\nu}\, dS = \iint H_j \frac{dH_i}{d\nu}\, dS.$$

But $$\frac{dH_j}{d\nu} = \frac{dH_j}{dr} = \left(\frac{x}{r}\frac{d}{dx} + \frac{y}{r}\frac{d}{dy} + \frac{z}{r}\frac{d}{dz}\right) H_j = \frac{j}{r} H_j;$$

and similarly,

$$\frac{dH_i}{d\nu} = \frac{i}{r} H_i,$$

r being the radius of the sphere;

$$\therefore \quad \frac{j}{r}\iint H_i H_j\, dS = \frac{i}{r}\iint H_j H_i\, dS,$$

that is

$$j r^{i+j-1} \iint Y_i Y_j\, dS = i r^{i+j-1} \iint Y_j Y_i\, dS,$$

$$\text{or} \quad (i-j) \iint Y_i Y_j\, dS = 0;$$

therefore either

$$i = j, \quad \text{or} \quad \iint Y_i Y_j\, dS = 0.$$

25.] *Definition.*—The points in which the axes $h_1, h_2, \ldots h_i$ drawn from any origin O meet the spherical surface of radius unity round O as centre are called the *poles* of the axes $h_1, h_2, \ldots h_i$. When all these poles coincide, the corresponding spherical harmonics are called *zonal spherical harmonics* solid and superficial respectively, referred to the common axis, and the surface spherical harmonic of order i is in this case written Q_i.

If μ be the cosine of the angle between r and the common axis in the case of the surface zonal harmonic Q_i of order i, then Q_i is the coefficient of e^i in the expansion of

$$\frac{1}{\sqrt{1 - 2\mu e + e^2}}$$

in ascending powers of e.

Let OA be the common axis, and let OP be r and the angle POA be θ.

In OA take a point M at the distance ρ from O. Then if V_i be the solid zonal harmonic of degree $-(i+1)$ corresponding to the surface zonal harmonic Q_i, it follows from definition that

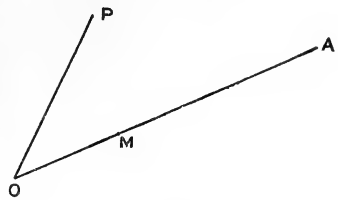

Fig. 3.

$$V_i = \left(\frac{d}{d\rho}\right)^i \frac{1}{PM},$$

when ρ is made equal to zero after differentiation.

Let $\rho = er$ and let $\cos\theta = \mu$.

Then $$V_i = \left(\frac{d}{d\rho}\right)^i \frac{1}{r\sqrt{1-2\mu e+e^2}} \text{ with } e = 0.$$

But $\dfrac{de}{d\rho} = \dfrac{1}{r}$ and is constant; therefore

$$\frac{d}{d\rho} = \frac{1}{r}\frac{d}{de},$$

$$V_i = \frac{1}{r^{i+1}}\left(\frac{d}{de}\right)^i \frac{1}{\sqrt{1-2\mu e+e^2}}, \quad \text{when } e = 0.$$

But if $\dfrac{1}{\sqrt{1-2\mu e+e^2}}$ be expanded in ascending powers of e, the coefficient of e^i in the expansion is, by Maclaurin's theorem,

$$\frac{1}{\underline{|i}}\left(\frac{d}{de}\right)^i \frac{1}{\sqrt{1-2\mu e+e^2}}, \quad \text{when } e = 0.$$

Let it be denoted by A_i.

Therefore $$V_i = \frac{\underline{|i}}{r^{i+1}} A_i.$$

But $$V_i = \frac{\underline{|i}}{r^{i+1}} Q_i;$$

Therefore $$Q_i = A_i.$$

Hence $Q_0 = A_0 = 1$ and $Q_1 = A_1 = \mu$. Also when $\mu = 1$

$$\frac{1}{\sqrt{1-2\mu e+e^2}} = \frac{1}{1-e} = 1 + e + e^2 + \&c.,$$

and therefore each coefficient Q is unity.

It is evident from definition that the zonal surface harmonic at P referred to OQ as axis is equal to the zonal surface harmonic at Q referred to OP as axis.

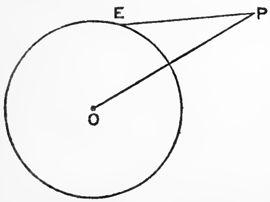

Fig. 4.

26.] Let a be the radius of a spherical surface S described round O as centre. Let P be any point within or without S. Let $OP = f$. And, E being any point on the surface, let $PE = D$, $\angle EOP = \theta$. Then

$$\frac{1}{D} = \frac{1}{\sqrt{f^2 + a^2 - 2fa\cos\theta}}$$

$$= \frac{1}{f}\frac{1}{\sqrt{1 + \frac{a^2}{f^2} - 2\frac{a}{f}\cos\theta}} \quad \text{if } f > a,$$

$$= \frac{1}{a}\frac{1}{\sqrt{1 + \frac{f^2}{a^2} - 2\frac{f}{a}\cos\theta}} \quad \text{if } f < a,$$

$$\left.\begin{array}{l} = \frac{1}{f}\left\{1 + \frac{a}{f}Q_1 + \frac{a^2}{f^2}Q_2 + \&c.,\right. \\ \text{or } \frac{1}{a}\left\{1 + \frac{f}{a}Q_1 + \frac{f^2}{a^2}Q_2 + \&c.,\right. \end{array}\right\} \text{according as } f > \text{ or } < a.$$

$$\left.\begin{array}{l} = \frac{1}{f}\Sigma\frac{a^i}{f^i}Q_i, \\ \text{or } \frac{1}{a}\Sigma\frac{f^i}{a^i}Q_i. \end{array}\right\} \text{according as } f > \text{ or } < a.$$

Therefore, if $f > a$,

$$2f\frac{d}{df}\frac{1}{D} + \frac{1}{D} = -2\Sigma\{i+1\}\frac{a^i}{f^{i+1}}Q_i + \Sigma\frac{a^i}{f^{i+1}}Q_i$$

$$= -\Sigma(2i+1)\frac{a^i}{f^{i+1}}Q_i.$$

But

$$2f\frac{d}{df}\frac{1}{D} + \frac{1}{D} = \frac{-2f^2 + 2fa\cos\theta + f^2 + a^2 - 2fa\cos\theta}{\{f^2 + a^2 - 2fa\cos\theta\}^{\frac{3}{2}}}$$

$$= -\frac{f^2 - a^2}{D^3}.$$

Therefore

$$\frac{f^2-a^2}{D^3} = \Sigma(2i+1)\frac{a^i}{f^{i+1}}Q_i;$$

and similarly if $f < a$,

$$\frac{a^2-f^2}{D^3} = \Sigma(2i+1)\frac{f^i}{a^{i+1}}Q_i.$$

27.] With the same notation as before we can prove that

$$\iint\frac{d\sigma}{D^3} = \frac{a}{f}\cdot\frac{4\pi a}{f^2-a^2} \text{ when } P \text{ is without } S,$$

and $$\iint\frac{d\sigma}{D^3} = \frac{4\pi a}{a^2-f^2} \text{ when } P \text{ is within } S,$$

the integrations being taken over the surface S.

Let $EOP = \theta$, and let ϕ be the angle between the plane of EOP and a fixed plane through OP; then

$$d\sigma = a^2\sin\theta\, d\theta\, d\phi,$$

$$\iint\frac{d\sigma}{D^3} = 2\pi a^2\int_0^\pi\frac{\sin\theta\, d\theta}{D^3}.$$

Also $$D^2 = a^2-2af\cos\theta+f^2;$$

$$\therefore\quad \sin\theta\, d\theta = \frac{D\,dD}{af};$$

$$\therefore\quad \iint\frac{d\sigma}{D^3} = \frac{2\pi a}{f}\int\frac{dD}{D^2};$$

the limits on the right-hand side being

$f-a$ and $f+a$ when P is external,

and $a-f$ and $a+f$ when P is internal;

$$\therefore\quad \iint\frac{d\sigma}{D^3} = \frac{2\pi a}{f}\left\{\frac{1}{f-a}-\frac{1}{f+a}\right\} \text{ when } P \text{ is external,}$$

$$= \frac{2\pi a}{f}\left\{\frac{1}{a-f}-\frac{1}{a+f}\right\} \text{ when } P \text{ is internal;}$$

or $$\iint\frac{d\sigma}{D^3} = \frac{a}{f}\cdot\frac{4\pi a}{f^2-a^2} \text{ and } \frac{4\pi a}{a^2-f^2}$$

in the respective cases.

Hence

$$\iint\frac{f^2-a^2}{D^3}d\sigma = \frac{4\pi a^2}{f} \text{ and } \iint\frac{a^2-f^2}{D^3}d\sigma = 4\pi a$$

for external and internal positions of P respectively, and for both cases

$$lt_{f=a}\iint \frac{f^2 \sim a^2}{D^3}\,d\sigma = 4\pi a.$$

28.] In the last case let $F(E)$ be any function of the position of E on the surface which does not vanish at the point in which OP cuts the surface, nor become infinite at any other point on the surface, let Q_i be the surface zonal harmonic at E of order i, the common axis being OP, then, if P be made to approach the surface, ultimately shall

$$F(P) = \frac{1}{4\pi a^2}\left\{\iint Q_0 \,.\, F(E)\,d\sigma + 3\iint Q_1\,F(E)\,d\sigma \right.$$
$$\left. + 5\iint Q_2\,F(E)\,d\sigma + \&c.\right\}.$$

For with the notation of the last Article let

$$u = \iint \frac{(f^2 \sim a^2)F(E)\,d\sigma}{D^3},$$

then when P approaches the surface and f is indefinitely nearly equal to a, every element of the integral vanishes except when D is indefinitely small. In this case P is ultimately on the surface, and the integral has the same value as if $F(E)$ were equal to $F(P)$, its value at the point of S with which P ultimately coincides, or

$$u = \iint F(P)\frac{f^2 \sim a^2}{D^3}\,d\sigma = F(P)\iint \frac{f^2 \sim a^2}{D^3}\,d\sigma \text{ when } f = a \text{ ultimately.}$$

Therefore

$$lt_{f=a}\,u = F(P)\,.\,lt_{f=a}\iint \frac{f^2 \sim a^2}{D^3}\,d\sigma$$
$$= 4\pi a\,F(P) \text{ by the last Article.}$$

Suppose that f is originally greater than a, then

$$u = \iint \frac{(f^2 - a^2)\,F(E)}{D^3}\,d\sigma,$$

and $$F(P) = \frac{1}{4\pi a}\,.\,lt_{f=a}\,u = \frac{1}{4\pi a}\,.\,lt_{f=a}\iint \frac{f^2 - a^2}{D^3}\,F(E)\,d\sigma;$$

$$\therefore\ F(P) = -\frac{1}{4\pi a}\,lt_{f=a}\iint\left\{2f\frac{d}{df}\cdot\frac{1}{D} + \frac{1}{D}\right\}F(E)\,d\sigma.$$

And, by Art. 25,

$$\frac{1}{D} = \frac{1}{f} \cdot \left\{ Q_0 + Q_1 \frac{a}{f} + Q_2 \frac{a^2}{f^2} + \&c. \right\}.$$

Performing the differentiations and substituting, we get

$$F(P) = \frac{1}{4\pi a^2} \left\{ \iint Q_0 F(E)\, d\sigma + 3 \iint Q_1 F(E)\, d\sigma + 5 \iint Q_2 F(E)\, d\sigma + \&c. \right\}.$$

29.] If Y_i be any surface spherical harmonic of the order i, and if Q_i be the zonal surface harmonic of the same order and origin referred to any axis OP, and if $d\sigma$ be an element of a spherical surface of radius a described round the origin O as centre, then

$$\iint Y_i Q_i\, d\sigma = \frac{4\pi a^2}{2i+1} (Y_i),$$

where (Y_i) is the value of Y_i at P the pole of Q_i, the integrations being over the spherical surface dS.

Substitute Y_i for $F(E)$ in the last proposition.

Then $F(P)$ is the value of Y_i at P;

$$\therefore \quad Y_i \text{ (at } P) = \frac{1}{4\pi a^2} \left\{ \iint Q_0 Y_i\, d\sigma + 3 \iint Q_1 Y_i\, d\sigma + 5 \iint Q_2 Y_i\, d\sigma + \&c. \right\}.$$

And, by Art. 24, each double integral vanishes except

$$\iint Q_i Y_i,$$

$$\therefore \quad Y_i \text{ (at } P) = \frac{2i+1}{4\pi a^2} \iint Q_i Y_i\, d\sigma,$$

$$\text{or} \quad \iint Q_i Y_i\, d\sigma = \frac{4\pi a^2}{2i+1} (Y_i),$$

if (Y_i) denote the value of Y_i at P.

By putting $Y_i = Q_i$ we obtain

$$\iint Q_i^2\, d\sigma = \frac{4\pi a^2}{2i+1},$$

since, by Art. 25, $Q_i = 1$ at the pole.

30.] If $F(E)$ be a spherical surface harmonic, i.e. $F(E) = Y_i$, then, whether P be on the surface or outside of it,

$$\iint \frac{f^2-a^2}{D^3} F(E)\,d\sigma = 4\pi a \frac{a^{i+1}}{f^{i+1}} F(P).$$

For $$\iint \frac{f^2-a^2}{D^3} F(E)\,d\sigma = \iint (2i+1) \frac{a^i}{f^{i+1}} Q_i Y_i\,d\sigma,$$

by Arts. 24 and 26,

$$= \frac{4\pi a^{i+2}}{f^{i+1}} (Y_i),$$

where (Y_i) denotes the value of Y_i at the common axis of the zonal harmonics, that is, along OP.

Therefore

$$\iint \frac{f^2-a^2}{D^3} F(E)\,d\sigma = 4\pi a \left(\frac{a}{f}\right)^{i+1} (Y_i).$$

31.] Considered as a function of μ derived by the expansion of $\dfrac{1}{\sqrt{1-2\mu e+e^2}}$, the zonal harmonic Q_i is called the *Legendre's coefficient of order* i, and is frequently written P_i.

We can prove the following properties of the coefficients P.

(*a*) As proved above, if $\mu = 1$,

$$\frac{1}{\sqrt{1-2\mu e+e^2}} = \frac{1}{1-e} = 1+e+e^2+\&c.$$

Hence, if $\mu = 1$, $P_i = 1$ for all values of i; if $\mu = -1$,

$$\frac{1}{\sqrt{1-2\mu e+e^2}} = \frac{1}{1+e} = 1-e+e^2-\&c.$$

Hence, if $\mu = -1$, $P_i = +$ or -1 according as i is even or odd.

If $\mu < 1$ $\dfrac{1}{\sqrt{1-2\mu e+e^2}}$ is always finite, and is finite if $e = 1$.

Hence the series $P_1 + P_2 + \ldots$ is a convergent series.

(*b*) It is evident from the formation of P_i as the coefficient of e^i in the expansion of

$$(1-2\mu e+e^2)^{-\frac{1}{2}} \quad \text{or} \quad (1-e\overline{2\mu-e})^{-\frac{1}{2}}$$

that P_i must contain μ^i, μ^{i-2}, μ^{i-4}, &c., but can contain no higher powers of μ than μ^i, and no powers of which the index differs from i by an odd number. Hence if i be even, P_i has the same value for $+\mu$ as for $-\mu$, and if i be odd the same value with opposite sign.

Hence also μ^i can be expressed in terms of P_i, P_{i-2}, &c.

(*c*)
$$\int_{-1}^{1} P_i P_j d\mu = 0 \text{ if } i \neq j,$$
$$= \frac{2}{2i+1} \text{ if } i = j.$$

For since $\mu = \cos\theta$,
$$d\mu = -\sin\theta\, d\theta.$$

Also P_i and P_j are both functions of μ, and therefore of θ. Hence
$$\int_{-1}^{1} P_i P_j d\mu = \int_0^{\pi} P_i P_j \sin\theta\, d\theta$$
$$= \frac{1}{2\pi a^2}\iint P_i P_j d\sigma$$
over the surface of a sphere of radius a; $= 0$ by Art. 24, unless $i = j$.

And if $i = j$,
$$\int_{-1}^{1} P_i^2 d\mu = \frac{1}{2\pi a^2}\iint P_i^2 d\sigma = \frac{2}{2i+1}.$$

(*d*)
$$\int_{-1}^{1} P_i \mu^j d\mu = 0 \text{ if } i > j, \text{ or if } j-i \text{ is odd.}$$

For expanding μ^j in terms of the P's, the integral is resolved into a number of integrals of the form $\int_{-1}^{1} P_i P_j d\mu$, in each of which $i \neq j$, and is therefore zero.

(*e*) To find the value of $\int_0^1 \mu^\kappa P_i d\mu$, where κ is any positive number integral or fractional *.

Let
$$P_i = a\mu^i + \beta\mu^{i-2} + \dots.$$
Then
$$\int_0^1 \mu^\kappa P_i d\mu = \frac{a}{\kappa+i+1} + \frac{\beta}{\kappa+i-1} + \cdots,$$
$$= \frac{K}{(\kappa+i+1)(\kappa+i-1)\dots(\kappa+1)}, \text{ if } i \text{ be even,}$$
$$= \frac{K'}{(\kappa+i+1)(\kappa+i-1)\dots(\kappa+2)}, \text{ if } i \text{ be odd.}$$

* See Todhunter's Functions of Laplace, Lamé, and Bessel, Art. 34, 35.

Let i be even. Then if κ has any of the values $i-2$, $i-4$, &c., or zero, the left-hand member

$$\int_0^1 \mu^\kappa P_i\, d\mu = \frac{1}{2}\int_{-1}^1 \mu^\kappa P_i\, d\mu$$
$$= 0,$$

and therefore $K=0$.

It follows that

$$K = \lambda\,.\,\kappa\,.\,\overline{\kappa-2}\;\overline{\kappa-4}\ldots\overline{\kappa-i+2}.$$

Also λ is the coefficient of the highest power of κ; therefore

$$\lambda = \alpha+\beta+\gamma\ldots$$
$$= P_i(\mu) \text{ when } \mu = 1$$
$$= 1.$$

Hence, if i be even,

$$\int_0^1 \mu^\kappa P_i\, d\mu = \frac{\kappa\,.\,\overline{\kappa-2}\ldots\overline{\kappa-i+2}}{(\kappa+i+1)\,(\kappa+i-1)\ldots(\kappa+1)}.$$

Similarly, if i be odd,

$$\int_0^1 \mu^\kappa P_i\, d\mu = \frac{\overline{\kappa-1}\;\overline{\kappa-3}\ldots\overline{\kappa-i+2}}{(\kappa+i+1)\,(\kappa+i-1)\ldots(\kappa+2)}.$$

If κ be either an integer or a fraction whose denominator when reduced to its lowest terms is odd, then

$$\int_{-1}^1 \mu^\kappa P_i\, d\mu = 2\int_0^1 \mu^\kappa P_i\, d\mu,$$

if $\mu^\kappa P_i$ does not change sign with μ,

$$= 0,$$

if $\mu^\kappa P_i$ does change sign with μ.

(f) Hence any function, $f(\mu)$, which can be expanded in a series of positive powers of μ, whether integral or fractional, can be expanded in a series of the form

$$f(\mu) = A_0 + A_1 P_1 + A_2 P_2 + \ldots.$$

For we have

$$\int_{-1}^1 P_i f(\mu)\, d\mu = A_i \int_{-1}^1 P_i^2 d\mu$$
$$= \frac{2}{2i+1} A_i,$$

or
$$A_i = \frac{2i+1}{2}\int_{-1}^1 P_i f(\mu)\, d\mu,$$

which determines A_i, if $f(\mu)$ is known in terms of positive powers of μ.

It is perhaps necessary to show that the series

$$A_0 + A_1 P_1 + A_2 P_2 + \&c.$$

converges, if $f(\mu)$ can be expanded in a converging series of ascending powers of μ.

For let $c_\kappa \mu^\kappa$ be any term in the expansion of $f(\mu)$. Then the term in A_i derived from this term in $f(\mu)$ is

$$c_\kappa \frac{2i+1}{2} \int_{-1}^{1} P_i \mu^\kappa \, d\mu,$$

and the corresponding term in A_{i+2} is

$$c_\kappa \frac{2i+5}{2} \int_{-1}^{1} P_{i+2} \mu^\kappa \, d\mu;$$

from which it is easily seen from the expressions for $\int_{-1}^{1} P_i \mu^\kappa \, d\mu$ above obtained that, if i be large enough, $A_{i+2} < A_i$.

Now the series $P_1 + P_2 + P_3 \ldots$ converges.

Hence $A_0 + A_1 P_1 + A_2 P_2 + \&c.$ converges.

32.] We have hitherto regarded the coefficients Q or P as functions of μ derived from the expansion of

$$\frac{1}{\sqrt{1 - 2\mu e + e^2}} = 1 + Q_1 e_1 + Q_2 e^2 + \&c.$$

We may however take for initial radius any line OC not coinciding with the common axis, and the direction of the common axis OH of the zonal harmonics may be defined with reference to this line by the usual angular coordinates, namely, $\theta' = \angle HOC$, and ϕ' the angle between the plane HOC and a fixed plane through OC. In this case the angular coordinates defining the direction of OP or r will be θ and ϕ, and the cosine of the angle HOP will be

$$\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

Now Q_i is, as we have seen, a function of $\cos HOP$, and is therefore a function of

$$\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

It is evidently symmetrical with regard to θ and θ'. So that the value of Q_i at P, when OH is the common axis, is the same as the value of Q_i at H, when OP is the common axis, see Art. 25. In this form Q_i is called a Laplace's coefficient.

33.] Of the differential equation which a spherical surface harmonic satisfies.

By definition any spherical harmonic u satisfies the equation

$$\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\} u = 0.$$

If we change the variables to the usual spherical coordinates r, θ, ϕ, where

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta,$$

the equation becomes

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \frac{1}{r^2}\frac{d^2u}{d\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{d^2u}{d\phi^2} + \frac{\cot\theta}{r^2}\frac{du}{d\theta} = 0. \quad \text{......(1)}$$

Let $u = \dfrac{Y_i}{r^{i+1}}$. Then u is a spherical harmonic function of degree $-(i+1)$, and satisfies the above differential equation.

Now $Y_i = r^{i+1}u$, where Y_i is independent of r, therefore $r^{i+1}u$ is independent of r, whence

$$(i+1)\,r^i u + r^{i+1}\frac{du}{dr} = 0,$$

and $$i(i+1)\,r^{i-1}u + 2(i+1)\,r^i\frac{du}{dr} + r^{i+1}\frac{d^2u}{dr^2} = 0,$$

and $$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = \frac{i(i+1)u}{r^2}.$$

Hence the differential equation becomes

$$\frac{d^2u}{d\theta^2} + \cot\theta\frac{du}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2u}{d\phi^2} + i(i+1)u = 0. \quad \text{............(2)}$$

Let us now change the variable from θ to $\cos\theta$, and let $\cos\vartheta = \gamma$. Then

$$\frac{d^2u}{d\theta^2} + \cot\theta\frac{du}{d\theta} = \frac{d}{d\gamma}\left(\sin^2\theta\frac{du}{d\gamma}\right).$$

Substituting in the differential equation, we obtain

$$\frac{d}{d\gamma}\left\{(1-\gamma^2)\frac{du}{d\gamma}\right\} + \frac{1}{1-\gamma^2}\frac{d^2u}{d\phi^2} + i(i+1)u = 0;$$

or restoring $\frac{Y_i}{r^{i+1}}$ for u,

$$\frac{d}{d\gamma}\left\{\overline{1-\gamma^2}\frac{dY_i}{d\gamma}\right\}+\frac{1}{1-\gamma^2}\frac{d^2Y_i}{d\phi^2}+i(i+1)Y_i=0 \quad \text{.........(3)}$$

This is true for any spherical surface harmonic Y_i, and therefore for the zonal harmonic Q_i as a particular case.

In the case of the zonal harmonic, if the common axis be taken for the initial line from which θ is measured, Q_i is, as above mentioned, written P_i, and P_i is independent of ϕ. Hence P_i satisfies the equation

$$\frac{d}{d\gamma}\left\{\overline{1-\gamma^2}\frac{dP_i}{d\gamma}\right\}+i(i+1)P_i=0,$$

or

$$\frac{d}{d\mu}\left\{\overline{1-\mu^2}\frac{dP_i}{d\mu}\right\}+i(i+1)P_i=0. \quad \text{.................(4)}$$

34.] If we differentiate equation (4) of last Article k times, we obtain the equation

$$(1-\mu^2)\frac{d^{k+2}P_i}{d\mu^{k+2}}-2(k+1)\mu\frac{d^{k+1}P_i}{d\mu^{k+1}}+(i+k+1)(i-k)\frac{d^kP}{d\mu^k}=0. \quad (5)$$

From (4) and (5) above it appears that P_i and $\frac{d^kP_i}{d\mu^k}$ respectively satisfy the differential equations

$$\left.\begin{aligned}&(1-\mu^2)\frac{d^2y}{d\mu^2}-2\mu\frac{dy}{d\mu}+i(i+1)y=0,\\&(1-\mu^2)\frac{d^2y}{d\mu^2}-2(k+1)\mu\frac{dy}{d\mu}+(i+k+1)(i-k)y=0.\end{aligned}\right\} \quad \ldots (6)$$

We may also prove that P_i and $\frac{d^kP_i}{d\mu^k}$ are the only solutions of (6) both finite and integral in μ.

For if in the former of equations (6) we write $P_i u$ for y, we obtain a differential equation in u which gives on integration

$$u=A+A'\int\frac{d\mu}{P_i^2(1-\mu^2)},$$

where A and A' are arbitrary constants and the integral commences from some fixed limit;

$$\therefore\quad y=AP_i+A'P_i\int\frac{d\mu}{P_i^2(1-\mu^2)}.$$

If $A'=0$, $y=AP_i$, an integral finite solution in μ.

If $A' \neq 0$, the expression for y contains the term

$$A' P_i \int \frac{d\mu}{P_i^2 (1-\mu^2)},$$

and therefore can be neither finite nor integral.

Hence P_i or AP_i is the only finite integral solution in μ of the former of equations (6). And in the same way it may be proved that $\frac{d^k P_i}{d\mu^k}$ is the only finite integral solution in μ of the second of equations (6).

35.] By means of equation (5) of the last Article we may generalise the proposition of Art. 24 by proving that

$$\int_{-1}^{+1} (1+\mu^2)^k \frac{d^k P_i}{d\mu^i} \cdot \frac{d^k P_j}{d\mu^j} d\mu = 0 \text{ when } i \neq j$$

$$= \frac{2}{2i+1} \cdot (i+k)(i+k-1) \dots (i-k+1) \text{ when } i = j.$$

For if we multiply the left-hand side of (5) of Art. 34 by $(1-\mu^2)^k$, it may be written

$$\frac{d}{d\mu} \cdot \left\{ (1-\mu^2)^{k+1} \frac{d^{k+1} P_i}{d\mu^{k+1}} \right\} + (i+k+1)(i-k)(1-\mu^2)^k \frac{d^k P_i}{d\mu^k} = 0;$$

and changing k into $k-1$ this becomes

$$\frac{d}{d\mu} \cdot \left\{ (1-\mu^2)^k \frac{d^k P_i}{d\mu^k} \right\} + (i+k)(i-k+1)(1-\mu^2)^{k-1} \frac{d^{k-1} P_i}{d\mu^{k-1}} = 0.$$

But integrating by parts, we get

$$\int_{-1}^{+1} \left\{ (1-\mu^2)^k \frac{d^k P_i}{d\mu^k} \frac{d^k P_j}{d\mu^k} \right\} d\mu = -\int_{-1}^{+1} \frac{d^{k-1} P_j}{d\mu^{k-1}} \cdot \frac{d}{d\mu} \left((1-\mu^2)^k \right) \frac{d^k P_i}{d\mu^k},$$

since the integrated terms vanish; and therefore

$$\int_{-1}^{+1} \left\{ (1-\mu^2)^k \frac{d^k P_i}{d\mu^k} \cdot \frac{d^k P_j}{d\mu^k} \right\} d\mu$$

$$= (i+k)(i-k+1) \int_{-1}^{+1} (1-\mu^2)^{k-1} \frac{d^{k-1} P_i}{d\mu^{k-1}} \frac{d^{k-1} P_j}{d\mu^{k-1}} d\mu;$$

and therefore by successive reductions,

$$\int_{-1}^{+1} \left\{ (1-\mu^2)^k \frac{d^k P_i}{d\mu^k} \frac{d^k P_j}{d\mu^k} \right\} d\mu$$

$$= (i+k)(i+k-1) \dots (i-k+1) \int_{-1}^{+1} P_i P_j d\mu;$$

and therefore by Art. 31,

$$\int_{-1}^{+1}(1-\mu^2)^k\frac{d^kP_i}{d\mu^k}\cdot\frac{d^kP_j}{d\mu^k}\,d\mu = 0 \text{ if } i\neq j$$

$$= \frac{2}{2i+1}(i+k)(i+k-1)\dots(i-k+1) \text{ if } i=j.$$

36.] To expand Q_i in a series of cosines of multiples of $(\phi-\phi')$.

Since Q_i is the coefficient of e^i in the expansion of

$$\{1-2e(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'))+e^2\}^{-\frac{1}{2}},$$

it follows that the term in Q_i which involves $\cos k(\phi-\phi')$ must contain $(\sin\theta)^k$ as a factor, or in other words, that the required expansion of Q_i must be of the form

$$q_0+q_1\cos(\phi-\phi')+\&c.+q_k\cos k(\phi-\phi')+\&c.,$$

where $q_k=(\sin\theta)^k f(\cos\theta)$, and the function denoted by f is rational and integral.

If we perform the requisite differentiations on Q_i, substitute in (6) of last Article and equate to zero the coefficients of $\cos k(\phi-\phi')$, we obtain the equation

$$(1-\gamma^2)\frac{d^2f}{d\gamma^2}-2(k+1)\gamma\frac{df}{d\gamma}+(i+k+1)(i-k)f=0,$$

where $\gamma=\cos\theta$.

And since f is a finite integral function of γ, it follows from Art. 34 that

$$f=A_k\frac{d^kP_i}{d\gamma^k},$$

where A_k is independent of γ or of θ.

Now Q_i is a symmetrical function of θ and θ', if therefore we denote $\cos\theta'$ by γ' it will follow that A_k must be of the form

$$a_k\frac{d^kP_i'}{d\gamma'^k},$$

where P_i is the same function of γ' that P_i is of γ, and therefore that

$$Q_i=a_0P_iP_i'+a_1\sin\theta\sin\theta'\frac{dP_i}{d\gamma}\cdot\frac{dP_i'}{d\gamma'}\cos(\phi-\phi')+\&c.$$

$$+a_k\sin\theta^k\sin\theta'^k\frac{d^kP}{d\gamma^k}\cdot\frac{d^kP'}{d\gamma'^k}\cos k(\phi-\phi')+\&c.$$

For most of the applications of Q_i the actual values of the numerical coefficients a_0, a_1, a_2 are not required, they may however be determined without much difficulty as follows.

37.] To prove that

$$a_k = \frac{2}{(i+k)(i+k-1)\dots(i-k+1)}.$$

For

$$Q_i = a_0 P_i P_i' + \&c. + a_k \frac{d^k P_i}{d\gamma^k} \cdot \frac{d^k P_i'}{d\gamma'^k} \sin\theta^k \sin\theta'^k \cos k(\phi - \phi') + \&c.$$

Square both sides and integrate with respect to ϕ from 0 to 2π, remembering that the integrals of all terms containing products of cosines of unequal multiples of $\phi - \phi'$ are zero, and that the integrals of all quantities of the form $(\cos m(\phi - \phi'))^2$ are equal to π and the integral of $a_0{}^2 P_i^2 P_i'^2$ is $2\pi a^2{}_0 P_i^2 P_i'^2$;

$$\therefore \int_0^{2\pi} Q_i^2 \, d\phi = \pi \Big\{ 2(a_0 P_i P_i')^2 + \&c.$$
$$+ \Big(a_k \frac{d^k P_i}{d\gamma^k} \cdot \frac{d^k P_i'}{d\gamma'^k} \sin\theta^k \sin\theta'^k\Big)^2 + \&c. \Big\}.$$

Again, integrate both sides with regard to γ from -1 to $+1$, remembering that

$$\int_{-1}^1 \int_0^{2\pi} (Q_i)^2 \, d\gamma \, d\phi = \frac{4\pi}{2i+1},$$

and we get

$$\frac{4\pi}{2i+1} = \pi \Big\{ 2 a_0{}^2 P_i'^2 \int_{-1}^1 P_i^2 + \&c.$$
$$+ a_k^2 \Big(\sin\theta'^k \frac{d^k P_i'}{d\gamma'^k}\Big)^2 \int_{-1}^1 \Big(\frac{d^k P_i}{d\gamma^k} \sin\theta^k\Big)^2 + \&c. \Big\}. \quad (\alpha)$$

But if $\theta = \theta'$ and $\phi = \phi'$, $Q_i = 1$;

$$\therefore \frac{4\pi}{2i+1} = \frac{4\pi}{2i+1} \Big\{ a_0 P_i'^2 + \&c. + a_k \Big(\frac{d^k P_i'}{d\gamma'^k} \sin\theta'^k\Big)^2 + \&c. \Big\}. \quad (\beta)$$

For in this case Q_i becomes

$$a_0 P_i'^2 + a_1 \Big\{ \sin\theta' \frac{dP_i'}{d\gamma'} \Big\}^2 + \&c. + a_k \Big\{ \sin^k\theta' \frac{d^k P_i'}{d\gamma'^k} \Big\}^2 + \&c.$$

The two expressions on the right-hand sides of (α) and (β) cannot be equal for all values of θ' unless the corresponding terms are separately equal;

$$\therefore \quad a_k^2 (\sin\theta'^k \frac{d^k P_i'}{d\gamma'^k})^2 \int_{-1}^{1} (\frac{d^k P_i}{d\gamma^k} \sin\theta^k)^2 d\gamma = \frac{4}{2i+1} a_k (\frac{d^k P_i'}{d\gamma'^k} \sin\theta'^k)^2;$$

$$\therefore \quad a_k = \frac{4}{2i+1} \cdot \frac{1}{\int_{-1}^{1} (\frac{d^k P_i}{d\gamma^k} \sin\theta^k)^2 d\gamma}.$$

But $$\int_{-1}^{1} (\frac{d^k P_i}{d\gamma^k} \sin\theta^k)^2 d\gamma = \frac{2}{2i+1} (i+k) \dots (i-k+1);$$

$$\therefore \quad a_k = \frac{2}{(i+k)(i+k-1)\dots(i-k+1)}.$$

38.] If Y_i be any surface spherical harmonic of the i^{th} order, then Y_i is a rational and integral function of $\cos\theta$, $\sin\theta$, $\cos\phi$, and $\sin\phi$, for

$$Y_i = \frac{r^{i+1}}{\lfloor i} \cdot \frac{d}{dh_i} \cdot \frac{d}{dh_{i-1}} \cdots \frac{d}{dh_1} \cdot (\frac{1}{r}).$$

Also if l, m, n be direction cosines of the axis h,

$$\frac{d}{dh} = l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz};$$

$$\therefore \quad \frac{d}{dh_i} \cdot \frac{d}{dh_{i-1}} \cdots \frac{d}{dh_1} \frac{1}{r} = \Sigma . (l)^\rho (m)^\sigma (n)^\tau \frac{d^i}{dx^\rho dy^\sigma dz^\tau} \cdot (\frac{1}{r});$$

where l^ρ means the product of ρ, l's, and so of m^σ and n^τ, and where $\rho + \sigma + \tau = i$.

But $$\frac{d^i}{dx^\rho dy^\sigma dz^\tau} (\frac{1}{r}) = \Sigma \frac{A x^m y^n z^p}{r^{2i+1}}, \text{ where } m+n+p = i;$$

also $$x = r\cos\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$$

$$\therefore \quad \Sigma \frac{A x^m y^n z^p}{r^{2i+1}} = \Sigma \frac{A \sin\theta^\lambda \cos\theta^\mu \cos\phi^\nu \sin\phi^\pi}{r^{i+1}};$$

therefore Y_i is of the form stated above.

39.] Y_i is of the form

$$\Sigma_0^i \{a_k \cos k\phi + \beta_k \sin k\phi\} \sin\theta^k \frac{d^k P}{d\gamma^k},$$

where a_k and β_k are numerical constants.

It is clear that we may assume the coefficients of $\cos k\phi$ and $\sin k\phi$ in Y_i to be A_k and B_k, where A_k and B_k are functions of θ to be determined.

Also we may assume $A_k = a_k \sin\theta^k v$, where a_k is constant and v to be determined.

But Y_i and therefore the coefficient of the cosine and sine of every multiple of ϕ satisfies the second of equations (6) of Art. 34;

$$\therefore \quad (1-\gamma^2)\frac{d^2v}{d\gamma^2} - 2(k+1)v\frac{dv}{d\gamma} + (i+k)(i+k+1)v = 0;$$

$$\therefore \quad v = A\frac{d^kP_i}{d\gamma^k} + A'\int \frac{2k+1}{\left(\frac{d^kP}{d\gamma^k}\right)^2(1-\gamma^2)} \cdot \frac{d^kP}{d\gamma^k}.$$

Now $v\sin\theta^k$ is to be a rational and integral function of γ and $\sqrt{1-\gamma^2}$, which clearly cannot be attained so long as the second term in v remains;

$$\therefore \quad v = A\frac{d^kP}{d\gamma^k};$$

$$\therefore \quad A_k = a_k \sin\theta^k \frac{d^kP}{d\gamma^k}.$$

Similarly

$$B_k = \beta_k \sin\theta^k \frac{d^kP}{d\gamma^k},$$

where a_k and β_k are numerical constants;

$$\therefore \quad Y_i = \Sigma\,\{a_k \cos k\phi + \beta_k \sin k\phi\} \sin\theta^k \frac{d^kP}{d\gamma^k},$$

the constants a_k and β_k depending upon the directions of the i-axes.

It has already been proved, Art. 24, that

$$\int_{-1}^{1}\int_0^{2\pi} Y_i Y_j \, d\gamma \, d\phi = 0$$

unless $i = j$, and we may now see that the same result follows from the general form of the function Y_i or Y_j.

For $$Y_i = \Sigma\,(a_k \cos k\phi + \beta_k \sin k\phi) \sin\theta^k \frac{d^kP_i}{d\gamma^k},$$

and $$Y_j = \Sigma\,(a_k' \cos k\phi + \beta_k' \sin k\phi) \sin\theta^k \frac{d^kP_j}{d\gamma^k}.$$

If now we multiply Y_i by Y_j and integrate with regard to ϕ from 0 to 2π, all the terms will vanish except those in which the multiples of ϕ are the same, and the result therefore will be of the form

$$\Sigma A\,(\sin\theta)^{2k}\frac{d^kP_i}{d\gamma^k}\cdot\frac{d^kP_j}{d\gamma^k}.$$

If we again integrate with regard to γ from -1 to $+1$, the result will be of the form

$$\Sigma A \int_{-1}^{1} (\sin\theta)^{2k} \frac{d^k P_i}{d\gamma^k} \cdot \frac{d^k P_j}{d\gamma^k} d\gamma;$$

and by Art. 24 each of these terms is zero unless $i = j$.*

It does not follow that

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_i Y_j \, d\gamma \, d\phi$$

is always finite, inasmuch as the values of the A's may be such that although each term in the integral is finite, their sum may be equal to zero. The values of the A's depend upon the inclinations of the two sets of i-axes of the Y_i and Y_i', and when these axes are so related that

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_i Y_i' \, d\gamma \, d\phi$$

is zero, the two spherical harmonics are said to be *conjugate*.

For example, take two spherical harmonics of the first order Y_1 and Y_1'. If θ' and ϕ' be the polar coordinates determining the axis of Y_1, and θ'' and ϕ'' those for the axis of Y_i', then Y_1 may be easily seen to be

$$\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'),$$

and similarly Y_1' is $\cos\theta \cos\theta'' + \sin\theta \sin\theta'' \cos(\phi - \phi'')$,

and

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_1 Y_1' \, d\gamma \, d\phi = \frac{4\pi}{3} (\cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \cos\phi' \cos\phi'' + \sin\theta' \sin\theta'' \sin\phi' \sin\phi'')$$

$$= \frac{4\pi}{3} (ll' + mm' + nn'),$$

if l, m, n be direction cosines of the axis of Y, l', m', n' those of Y'.

* In a similar manner the proposition of Art. 26

$$\iint Q_i Y_i \, d\sigma = \frac{4\pi a^2}{2i+1} Y_i$$

may be deduced from the form of Y_i proved in this article.

If therefore these axes are at right angles to one another,

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_1\, Y_1'\, d\gamma\, d\phi = 0,$$

or two spherical harmonics of the first order are conjugate when their axes are perpendicular to each other.

For the second and higher orders there is no such simple geometrical relation.

CHAPTER III.

POTENTIAL.

ARTICLE 40.] IF the forces acting on a material system be such that the work done by them upon the system in its motion from an initial to a final position is, whatever those positions may be, a function of the coordinates defining those positions only, and independent of the course taken between them, the system is said to be *Conservative.* The work done by the forces on the system in its motion from any position S to any given position which may be chosen as a position of reference, is defined to be the *potential energy*, or shortly the *potential*, of the system in the position S in relation to the forces in question.

If we denote by U the potential, and by T the kinetic, energy of the system, then, as shown in treatises on dynamics, $T+U$ is constant throughout any motion of the system under the influence of the forces in question. If q be any one of the generalised coordinates defining the position of the system, it follows from definition that $-\frac{dU}{dq}\partial q$ is the work done by the forces on the system as q becomes $q+\partial q$, and therefore the force tending to increase the coordinate q is $-\frac{dU}{dq}\cdot$

If the system be a material particle of unit mass, situated at the point P, we may without inaccuracy speak of the potential as the *potential of the forces* at P.

41.] We are in this chapter concerned only with forces of attraction and repulsion to or from fixed centres, the force varying inversely as the square of the distance from the centre. Now if the central force be any continuous function of the distance, whether varying according to the law of the inverse square or any other law, a potential exists.

For let there be at O a particle of matter of mass m which repels any other particle of mass m' with the force $mm'f(r)$, where $f(r)$ represents any continuous function of r, the distance between m and m'; then it can be shown that if O be fixed, the work done by the force upon m' as m' moves from a point at the distance r_1 from O, to another at the distance r_2 from O, is a function of r_1 and r_2, the initial and final values of r, and of these quantities only, and is independent of the form of the curve described by m' between these initial and final positions, and of the directions from O in which the distances r_1 and r_2 are measured.

For at any instant during the motion let m' be at P, and let Q be a point in the course indefinitely near to P. Let $PQ = ds$, the angle $OPQ = \phi$, $OP = r$, $OQ = r + dr$.

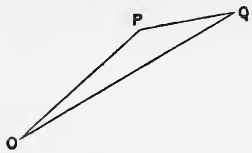

Fig. 5.

In the limit, if Q be taken near enough to P, the force of repulsion may be considered constant, as m' moves from P to Q, and equal to $mm'f(r)$.

Therefore the work done by the force in moving the repelled particle from P to Q is $-mm'f(r)\cos\phi\, ds$, or $mm'f(r)dr$, and is independent of ϕ if dr be given.

Therefore the whole work done by the force in the motion from distance r_1 to distance r_2 from O is

$$mm'\int_{r_1}^{r_2} f(r)\, dr,$$

and depends upon r_1 and r_2, and these quantities only.

We have for simplicity considered m fixed at O, but the proof evidently holds if both m and m' be moveable, and move from a distance r_1 to a distance r_2 apart under the influence of the mutual repulsive force $mm'f(r)$. If the mutual force had been attractive instead of repulsive, in other respects following the same law, the expression for the work done would be the same as that for the repulsive force, but with reversed sign. If in any case on effecting the integrations the expression for the work done prove to be negative, this result must be interpreted as expressing the fact that positive work is done *against*, and not *by*, the force in the motion considered.

In either case, whether the force be repulsive or attractive, the work done is proved to be a function of r_1 and r_2 only, and independent of the course taken between the initial and final positions of m'.

We have thus shown that if $f(r)$ be any continuous function of the distance between the two particles m and m', a potential exists.

At present, as above stated, we are concerned only with the case in which $f(r) = \frac{1}{r^2}$. In that case the work done by the mutual force between m and m', as their distance varies from r_1 to r_2, is, if the force be repulsive, $mm'\int_{r_1}^{r_2}\frac{1}{r^2}dr$, that is

$$mm'\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\},$$

and if the force be attractive

$$-mm'\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\}.$$

42.] We shall now consider two kinds of matter, such that two particles, both of the same kind, *repel* one another with a mutual force varying directly as the masses of the particles, and inversely as the square of the distance between them, and two particles of different kinds *attract* one another according to the same law.

Then the work done by the mutual force between two particles m and m', as they move from a distance r_1 to a distance r_2 apart, is, if the masses be of the same kind, and therefore the force repulsive,

$$mm'\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\};$$

and if they be of different kinds, and the force attractive,

$$-mm'\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\}.$$

If now we agree to regard all particles of one kind of matter as *positive*, and all particles of the other kind as *negative*, we can combine both results under one formula

$$mm'\left\{\frac{1}{r_1}-\frac{1}{r_2}\right\};$$

in which m or m' may have either sign, expressing the work done by the mutual force between m and m' in the motion from distance r_1 to r_2 apart.

Finally, we will take for the position of reference to which potential is measured, the position in which the two particles are at an infinite distance apart, that is, in which r_2 is infinite. Then we shall arrive at the following definition.

The potential of two material particles m and m', distant r from each other, is the work done by the force of mutual repulsion as they move to an infinite distance apart; that is, $mm'\int_r^{r_2}\frac{1}{r^2}\,dr$, when r_2 is infinite, that is $\frac{mm'}{r}$, and is positive or negative according as m and m' are of the same or different kinds of matter.

In physics a body which is within the range of the action of another body is said to be *in the field* of that other body, and when it is so distant from that other body as to be sensibly out of the range of its action it is said to be *out of the field*.

The following definition is therefore equivalent to the one above adopted. *The potential of two material particles distant r apart is the work done by their mutual repulsion as they move from the distance r apart to such a distance as to be out of the field of one another's action, attraction being included as negative repulsion.*

Taking $m' = 1$, we define $\frac{m}{r}$ to be the potential of m at a point distant r from m.

43.] The potential at any point of any mass occupying a finite portion of space is evidently the sum of the potentials at that point of all the particles of which the mass is composed. If m be any particle of this mass, and r the distance of m from P, the potential of the mass at P is $\Sigma\frac{m}{r}$, where the summation extends throughout the mass, or if ρ be the density of the mass at x, y, z, the potential is

$$\iiint\frac{\rho\,dx\,dy\,dz}{r}.$$

Let this potential be denoted by V.

44.] The repulsion at P of a mass at O resolved in any direction is the rate of diminution of the potential of the mass per unit of length in that direction. This is a particular case of the general theorem proved above, that the force tending to increase any coordinate q is $-\frac{dV}{dq}$.

If V be the potential of the particle m, and ds the given direction,

$$-\frac{dV}{ds} = -\frac{dV}{dr} \cdot \frac{dr}{ds}$$

$$= \frac{m}{r^2} \cdot \frac{dr}{ds} = \frac{m}{r^2} \cos(r, ds)$$

$$= \text{the repulsion resolved in } ds.$$

And this proposition being true of every particle of which the mass is composed is evidently true of the whole mass.

Hence, if V be the potential at P of any mass M, the repulsion of the mass in the direction indicated by ds is $-\frac{dV}{ds}$.

45.] *If S be any closed surface, dS an element of its area, N the repulsive force at dS resolved along the normal to dS measured outwards arising from a particle of matter of mass m placed at the point O, then if the integration extend over the whole surface*

$$\iint N\,dS = 4\pi m, \text{ if } m \text{ be within } S;$$

and

$$\iint N\,dS = 0, \text{ if } m \text{ be without } S.$$

Let a line drawn from O in any direction cut the surface S at the point P distant r from O, and let this line make the angle ϕ with the surface S at P.

Let a small cone with solid angle $d\omega$ be described about OP as axis, cutting off from S in the neighbourhood of P the elementary surface dS.

The area of dS is equal to $\frac{r^2\,d\omega}{\sin\phi}$, also the repulsion at P from O is $\frac{m}{r^2}$, and the resolved part N of this repulsion in the direction of the normal to S at P drawn outwards from S is

$$+\frac{m}{r^2}\sin\phi \text{ or } -\frac{m}{r^2}\sin\phi,$$

according as OP is passing out of S from within, or into S from without;

$$\therefore\quad N\,dS = +m\,d\omega, \text{ or } -m\,d\omega$$

in the two cases respectively.

But if O be within S, the line drawn from it in any direction as above must emerge from S one time more than it enters it, and therefore the sum of all the values of NdS for this line $= +m\,d\omega$.

Taking the corresponding sum for all lines drawn from O we get the integral $\iint N\,dS$, and therefore

$$\iint N\,dS = +m\int d\omega = 4\pi m;$$

since the sum of the solid angles about O is 4π.

If O be without S the line drawn from it in any direction must meet S in an even number of points, and therefore the sum of all the values of NdS for every such line must be zero; therefore in this case

$$\iint N\,dS = 0.$$

This proposition is true for any particle within or without S respectively.

Therefore it follows that if any quantity of matter of mass M be distributed in any manner within a closed surface S, and if N be the repulsive force of that matter at any point on S resolved in the direction of the normal at that point drawn outwards, then

$$\iint N\,dS = 4\pi M.$$

And, similarly, that if M be without S, then

$$\iint N\,dS = 0;$$

and writing $-\frac{dV}{d\nu}$ for N, by Art. 44 we have

$$\iint \frac{dV}{d\nu}\,dS = -4\pi M, \quad \text{and} \quad \iint \frac{dV}{d\nu}\,dS = 0,$$

in the two cases respectively.

46.] It follows from Art. 45, that if ρ, the density of matter, be finite in any portion of space, the first differential coefficients of V cannot be discontinuous in that portion of space.

For consider a cylinder whose axis is parallel to x and of length l. Let the proposition be applied to this cylinder. If l be very small compared with the dimensions of the base, we may neglect that portion of the surface integral which relates to the curved surface, and the proposition becomes

$$\iint \frac{dV}{dx}\,dy\,dz = -4\pi \iiint \rho\,dx\,dy\,dz,$$

in which the surface integral is taken over the ends of the cylinder, and the triple integral throughout the interior space. Also in the surface integral $\frac{dV}{dx}$ is the rate of increase of V with the normal measured outwards from the enclosed space, in the case of both ends of the cylinder. If it be measured in the same direction in space for both ends, the surface integral may be written

$$\iint\left\{\left(\frac{dV}{dx}\right)_2 - \left(\frac{dV}{dx}\right)_1\right\}dy\,dz = -4\pi\iiint\rho\,dx\,dy\,dz.$$

Now if ρ be finite, the triple integral ultimately vanishes when l, and therefore the enclosed space, become infinitely small; and therefore the left-hand member also vanishes, and $\left(\frac{dV}{dx}\right)_1$ cannot differ by any finite quantity from $\left(\frac{dV}{dx}\right)_2$, or $\frac{dV}{dx}$ cannot be discontinuous. Therefore also V cannot be discontinuous.

Equations of Poisson and Laplace.

47.] In the equation of Green's theorem let V be the potential of any distribution of matter of which the density ρ is everywhere finite, and therefore such that $\frac{dV}{dx}$, $\frac{dV}{dy}$, and $\frac{dV}{dz}$ are continuous, let S be any closed surface, and let $u' =$ unity. Since $\frac{du'}{dx}$, $\frac{du'}{dy}$, and $\frac{du'}{dz}$ are zero, the equation becomes

$$\iint\frac{dV}{d\nu}\,dS = \iiint\nabla^2 V\,dx\,dy\,dz.$$

But $-\frac{dV}{d\nu}$ is the repulsive force of the matter referred to resolved in the normal to S outwards from the surface element dS. And therefore by Art. 45

$$-\iint\frac{dV}{d\nu}\,dS = \iiint 4\pi\rho\,dx\,dy\,dz.$$

Therefore also

$$-\iiint\nabla^2 V\,dx\,dy\,dz = \iiint 4\pi\rho\,dx\,dy\,dz.$$

Since this equation holds for every possible closed surface, it follows that

$$\nabla^2 V + 4\pi\rho = 0$$

at every point. This is called Poisson's equation.

At a point in free space $\rho = 0$, and the equation becomes

$$\nabla^2 V = 0.$$

This is called Laplace's equation.

It follows as a corollary from Poisson's equation that if V be the potential of any material system at x, y, z,

$$V = -\frac{1}{4\pi}\iiint \frac{\nabla^2 V}{r}\, dx'\, dy'\, dz',$$

where $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ *;

and the integral is throughout all space.

48.] Laplace's equation can be deduced by direct differentiation of $\frac{1}{r}$. For if the density of matter at x', y', z' is ρ, the potential at x, y, z is

$$V = \iiint \frac{\rho\, dx'\, dy'\, dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \iiint \frac{\rho\, dx'\, dy'\, dz'}{r}.$$

Now if O, or x, y, z, be any point not within the mass, the limits of the integration are not altered by any infinitely small change of position of O. Hence we may place the symbol ∇^2 under the integral sign, and obtain

$$\nabla^2 V = \iiint \rho \nabla^2 \frac{1}{r}\, dx'\, dy'\, dz' = 0.$$

But if O be within the mass, we cannot, in forming the triple integral for V, include in integration the point O at which the element function $\frac{1}{r}$ becomes infinite. It is necessary in this

* It may be proved by Green's theorem to be identically true for all functions (V) vanishing at infinity that

$$V = -\frac{1}{4\pi}\iiint \frac{\nabla^2 V}{r}\, dx\, dy\, dz,$$

the integration being extended over all space, and r being the distance from the point at which V is estimated to the element $dx\, dy\, dz$; and this proposition may, of course, be made the foundation of an independent proof of Poisson's equation

$$\nabla^2 V + 4\pi\rho = 0.$$

case to take for the limits of integration some surface inclosing O and infinitely near to it, and to form V as the sum of two separate integrals, one on each side of that surface. Hence any infinitely small change of position of O involves in this case a change in the limits of integration, and we are not at liberty in forming $\nabla^2 V$ to insert ∇^2 under the sign of integration. This is the reason why Laplace's equation fails at a point occupied by matter.

49.] *Definition.* We have hitherto supposed the matter with which we have been concerned to be distributed in such a manner that the density ρ is finite, or in other words that the mass vanishes with the volume of the space in which it is contained. According to this conception the mass of a small volume dv of density ρ, is ρdv, i. e. ρ is the limiting ratio of the mass to the containing volume when that volume is indefinitely diminished. At all parts of space for which this condition is satisfied we have obtained the equation

$$\nabla^2 V + 4\pi\rho = 0,$$

if V be the potential of any distribution at the point at which the density is ρ.

It may, however, happen that ρ becomes indefinitely great at certain points. The distribution may be such that although the volume becomes infinitely small the mass comprised in it may remain finite.

Suppose such a state of things to hold at all the points on a certain surface S, so that the mass of matter comprised between any portion of this surface, an adjacent surface S' infinitely near to it, and a cylindrical surface whose generating lines are the normals to S along its bounding curve, remains finite however close S' is taken to S, then if the mass vanishes with the area of S, inclosed by this bounding curve, we call the distribution *superficial* in distinction from the *volume* distribution hitherto considered.

In this conception of superficial distribution we disregard the distance between S and S' altogether, and we say that the mass corresponding to an element of surface dS is σdS, where σ is the *superficial density*, σ being in other words defined as the limiting ratio of the mass corresponding to, or as we say *on*, the surface dS to the area of dS, when dS is indefinitely small.

Still further there may be points for which not only ρ, but σ also, is infinite, and such that if a line l be drawn through these points, the mass of the superficially distributed matter comprised between this line l, an adjacent indefinitely near and parallel line l', and perpendiculars to l at its extremities remains finite, however near l' be taken to l. In such cases the distribution is said to be *linear*, and neglecting as before the distance between l and l', we say that the quantity of matter corresponding to, or *on* the element ds of l is λds, where λ is the linear density at ds.

50.] *On the modification of Poisson's equation at points of superficial distribution of matter.*

Let dS be an element of the surface, and let us form on dS a cylindrical surface like that mentioned in the definition of the last article.

Let ρ be the uniform density of matter within that cylindrical surface. If dS_1 denote any element of that surface, including its bases, we have by Art. 45

$$\iint \frac{dV}{d\nu} dS_1 = -4\pi \iiint \rho\, dx\, dy\, dz.$$

In the limit, when the bases of that cylinder become infinitely near each other, the right-hand member of this equation becomes $-4\pi \iint \sigma dS$. And if $d\nu$, $d\nu'$ be elements of the normal on either side of S, measured in each case from S, the left-hand member becomes

$$\iint \left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right) dS.$$

$$\therefore \iint \left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right) dS = -4\pi \iint \sigma\, dS;$$

or

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} + 4\pi\sigma = 0^*.$$

* The cases of finite and infinite ρ have been considered separately, with the view to their physical interpretations. There is no exception in any case to the equation $\nabla^2 V + 4\pi\rho = 0$, because, $\nabla^2 V$ becomes infinite whenever $\frac{dV}{dx}$, &c. are discontinuous, i.e. when ρ is infinite.

51.] *The mean value over the surface of any sphere of the potential due to any matter entirely without the sphere is equal to the potential at the centre.*

For let a be the radius of the sphere, r the distance of any point in space from the centre, $a^2 d\omega$ an element of the surface. Then denoting by $\bar{V}$ the mean value of V over the sphere, we have

$$\bar{V} = \frac{1}{4\pi a^2}\iint Va^2 d\omega,$$

$$= \frac{1}{4\pi} V d\omega,$$

$$\therefore \frac{d\bar{V}}{dr} = \frac{1}{4\pi}\iint \frac{dV}{dr} d\omega = \frac{1}{4\pi a^2}\iint \frac{dV}{dr} a^2 d\omega,$$

but
$$\iint \frac{dV}{dr} a^2 d\omega = 0 \text{ by Art. 45.}$$

Hence $\dfrac{d\bar{V}}{dr} = 0$ or $\bar{V}$ is independent of the radius of the sphere, and therefore equal to the potential at the centre.

Corollary. The potential of any matter uniformly distributed over the surface of a sphere, at any point outside of the sphere, is the same as if such matter were collected at the centre. Hence also the potential of a uniform solid sphere at any point outside of it is the same as if its mass were collected at the centre.

51 *a.*] *The mean value over the surface of any sphere of the potential, due to any matter entirely within the sphere, is the same as if such matter were collected at the centre.*

For using the same notation as before, and denoting by M the algebraic sum of the matter in question, we have in this case

$$\bar{V} = \frac{1}{4\pi a^2}\iint Va^2 d\omega = \frac{1}{4\pi}\iint V d\omega$$

$$\frac{d\bar{V}}{dr} = \frac{1}{4\pi}\iint \frac{dV}{dr} d\omega = \frac{1}{4\pi a^2}\iint \frac{dV}{dr} a^2 d\omega$$

$$= -\frac{4\pi M}{4\pi a^2}, \text{ by Art. 45,}$$

$$= -\frac{M}{a^2} = -\frac{M}{r^2},$$

and $\overline{V} = \frac{M}{r}$, no constant being required, since $\overline{V}$ vanishes when r is infinite.

52.] The mean potential over the surface of an infinite cylinder due to a uniform distribution of matter along an infinite straight line parallel to the axis and outside of the cylinder is equal to the potential on the axis.

For let a be the radius of the cylinder, l its length, θ the angle between a radius of the cylinder and a fixed plane through the axis, r the distance of any point from the axis. Let V be the potential, $\overline{V}$ its mean value. Evidently V, if r be given, is a function of θ only.

Then we have
$$\overline{V} = \frac{\int la\, d\theta V}{2\pi la},$$
$$\frac{d\overline{V}}{dr} = \frac{\int la\, d\theta \frac{dV}{dr}}{2\pi la}$$
$$= 0, \text{ by Art. 45};$$
because that part of the normal attraction which relates to the ends of the cylinder may be neglected.

It follows that $\overline{V}$ is independent of r, and is therefore equal to the potential on the axis.

It follows also that the potential at any point outside of a cylinder of a uniform distribution of matter over the surface of the cylinder, or throughout its interior, is the same as if all the matter were uniformly distributed along the axis, and therefore that the potential of such a uniform distribution at any point outside of it and distant r from the axis is $\int_{-\infty}^{\infty} \frac{\rho A}{\sqrt{x^2+r^2}}\, dx$, where $\rho\,.\,A dx$ is the quantity of matter corresponding to a length dx of the cylinder.

That is, $\rho A\,.\,(C-2\log r)$, where C is constant.

53.] The potential of any distribution of matter can never be a maximum or minimum at any point in a region not occupied by any portion of that matter. For suppose the potential to be a maximum at any point O, and describe a small sphere about O as centre. Then $\frac{dV}{d\nu}$, the rate of increase of V per unit of length of the normal to the sphere measured outwards, must, if the sphere be small enough, be negative at every point of the surface. Therefore

$$\iint \frac{dV}{d\nu}\, dS \text{ is negative;}$$

therefore

$$\iiint \rho\, dx dy dz \text{ is positive;}$$

or there must be positive matter within the sphere, and as this is true for any sphere, however small, described about O as centre, there must be positive matter at O. Similarly, if V be minimum at O, there must be negative matter at O. V can therefore never have a maximum value except at a point situated in positive matter, and never have a minimum value except at a point situated in negative matter.

53 *a*.] If V be constant throughout any finite region free from attracting matter, it has the same constant value at every point of space which can be reached from that region without passing through attracting matter.

For let the whole of space in which V is constant which can be so reached from the given region be comprised within the closed surface S.

Then on S, V either increases or diminishes continuously outwards. Let a small closed surface S' be described lying partly within S, and partly outside of it, and in the parts where V increases outwards from S. The normal integral $\iint \frac{dV}{d\nu}\, dS'$ applied to such surface is not zero, and therefore the interior space must be occupied by matter. But there is no matter in the portion of the small closed surface within S, therefore there must be matter in the closed surface immediately outside of S.

53 *b*.] If two systems of matter have the same potential throughout any finite portion of space bounded by a surface S, they have the same potential at all points in space which can be reached from that portion without passing through any matter of either system.

For let V and V' be the potentials of the two systems, so that $V = V'$ throughout the space enclosed by S. If possible let V be greater than V' in some region contiguous to S. Then we may describe a closed surface S', partly within and partly without S, such that on the part without S, $\frac{dV}{d\nu}$ is everywhere greater than $\frac{dV'}{d\nu}$. It follows that for such surface

$$\iint \frac{dV}{d\nu}\, dS' \neq \iint \frac{dV'}{d\nu}\, dS'.$$

But unless there be attracting matter belonging to either system within S' both these quantities are zero, and they cannot therefore be unequal.

54.] The propositions of the last article can also be extended to the case where V is given equal to V', not throughout any finite portion of space, but only at all points in a finite straight line, provided that both V and V' be symmetrical about that line as axis.

For we must suppose that there exists some space about the given line which contains no matter of either system. We may describe wholly within that space about the given line as axis a right cylinder of very small section. For that cylinder both $\iint \frac{dV}{d\nu}\, dS$ and $\iint \frac{dV'}{d\nu}\, dS$ must be zero, and therefore by the symmetry about the axis V cannot differ from V' at any point upon, or within, the cylinder. And V being proved equal to V' at all points within the cylinder, the case is reduced to that of Art. 53 *b*.

55.] Let V, instead of denoting a potential, be any spherical solid harmonic, and let S be any closed surface not enclosing the origin. Then by Art. 6

$$\iint \frac{dV}{d\nu}\, dS = \iiint \nabla^2 V\, dx\, dy\, dz,$$

the integrals being taken over and throughout S respectively. Writing $4\pi\rho$ for $-\nabla^2 V$, we obtain the result of Art. 45 as a particular case of the general theorem. Hence the propositions of Arts. 53 and 54 may be extended to the case in which V or V', instead of being a potential, is any spherical solid harmonic. If, for instance, the potential of a given mass be proved equal to a certain spherical solid harmonic U at all points within a certain region, as the finite space S, or the given length of the axis of a symmetrical system, it can be shewn that the potential is equal to U at all points which can be reached from the given region of equality without passing through any matter of the system.

Further, U, instead of being a single spherical solid harmonic, may be an infinite series of such harmonics, and the proposition will still be true for all space which can be reached from the given region of equality without passing through any matter of the system, or through any point where the series U ceases to be convergent.

56.] If the potential due to any distribution of matter on a closed surface S be constant at all points on S, the superficial density, σ, is equal to $-\frac{1}{4\pi}\frac{dV}{d\nu}$ at each point on S, the normal being measured from S on the outside of it.

For since the potential V is constant at each point on S, and satisfies $\nabla^2 V = 0$ at all points within S, it has by Art. 7 the same constant value at all points within S. Hence in Poisson's superficial equation $\frac{dV}{d\nu'} = 0$, and therefore

$$\frac{dV}{d\nu} + 4\pi\sigma = 0, \quad \text{or} \quad \sigma = -\frac{1}{4\pi}\frac{dV}{d\nu}.$$

But whether the potential be constant or not, the algebraic sum of the distribution over S is

$$-\frac{1}{4\pi}\iint\frac{dV}{d\nu}\,dS, \text{ by Art. 45.}$$

57.] *It is always possible to form one, and only one, distribution of matter over a closed surface S, the potential of which shall have any arbitrarily given value at each point of that surface.*

For, as we have proved in Art. 7, there exists one determinate function u which has the given value at each point of S, and satisfies $\nabla^2 u = 0$ at each point in the infinite external space, and vanishes at an infinite distance.

And there exists one determinate function u' which has the given value at each point of S, and satisfies $\nabla^2 u' = 0$ at each point within S.

Then a distribution over S, whose density is

$$-\frac{1}{4\pi}\left\{\frac{du}{d\nu}+\frac{du'}{d\nu'}\right\},$$

the normals being measured from S, $d\nu'$ on the inside, and $d\nu$ on the outside of the surface, is the required distribution.

For let a small sphere S' be described about any external point, Q, as centre. Let $V=\frac{1}{r}$ where r is the distance of any point from Q.

Then, applying Green's theorem to the space outside of S and S', we have with the given meaning of $d\nu$,

$$\iint u\frac{dV}{d\nu}dS+\iint u\frac{dV}{d\nu}dS'+\iiint u\nabla^2 V\,dx\,dy\,dz$$
$$=\iint V\frac{du}{d\nu}dS+\iint V\frac{du}{d\nu}dS'+\iiint V\nabla^2 u\,dx\,dy\,dz.$$

Now $\nabla^2 u = 0$ and $\nabla^2 V = 0$ at all points within the limits of the triple integral, and

$$\iint u\frac{dV}{d\nu}dS' = u_Q\iint\frac{dV}{d\nu}dS' = -4\pi u_Q,$$

if u_Q denote the value of u at Q. Also $\iint V\frac{du}{d\nu}dS'$ vanishes. Therefore the equation becomes

$$\iint u\frac{dV}{d\nu}dS-4\pi u_Q=\iint V\frac{du}{d\nu}dS \quad \ldots\ldots\ldots \quad \text{(A)}$$

Again, applying Green's theorem to the space within S, we have with the given meaning of $d\nu'$

$$\iint u'\frac{dV}{d\nu'}dS+\iiint u'\nabla^2 V\,dx\,dy\,dz$$
$$=\iint V\frac{du'}{d\nu'}dS+\iiint V\nabla^2 u'\,dx\,dy\,dz,$$

or since both $\nabla^2 V = 0$ and $\nabla^2 u' = 0$ everywhere within S,

$$-\iint u'\frac{dV}{d\nu'}dS=-\iint V\frac{du'}{d\nu'}dS. \quad \ldots\ldots\ldots \quad \text{(B)}$$

Now $-\frac{dV}{d\nu}=\frac{dV}{d\nu'}$ if Q be not actually on S, however near to

S it may be, and $u = u'$ on S. Hence, subtracting A from B, we have

$$4\pi u_Q = -\iint V\left\{\frac{du}{d\nu} + \frac{du'}{d\nu'}\right\} dS.$$

But if P be any point on S, V at $P = \dfrac{1}{PQ}$.

Therefore $$u_Q = -\frac{1}{4\pi}\iint \frac{\dfrac{du}{d\nu} + \dfrac{du'}{d\nu'}}{PQ}\, dS.$$

Now the right-hand member is the potential at Q of the supposed distribution whose density is

$$-\frac{1}{4\pi}\left\{\frac{du}{d\nu} + \frac{du'}{d\nu'}\right\}.$$

It follows that this potential is equal to u_Q at every point outside of S, however near to S; and therefore, since the potential is a continuous function, has the value of u, or the given value, at each point on S.

Similarly, if Q be an internal, instead of an external, point, we can prove that the distribution over S whose density is

$$-\frac{1}{4\pi}\left\{\frac{du}{d\nu} + \frac{du'}{d\nu'}\right\}$$

has u' for potential at Q.

And the functions u and u' being both determinate, their differential coefficients $\dfrac{du}{d\nu}$ and $\dfrac{du'}{d\nu'}$ are determinate and of single value.

58.] If $S_1 \ldots S_n$ be any closed surfaces, there exists one and only one distribution of matter over them whose potential u satisfies the following conditions, viz.

$u = C_1$, constant, but arbitrary at all points on S_1,

$u = C_2$, constant, but arbitrary at all points on S_2;

&c., &c. And

$$\iint \frac{du}{d\nu}\, dS_1 = e_1 \text{ over } S_1,$$

$$\iint \frac{du}{d\nu}\, dS_2 = e_2 \text{ over } S_2,$$

and so on, and u vanishes at an infinite distance.

For we have proved in Art. 10 that there exists one determinate function u satisfying the above conditions. It follows that $\frac{du}{d\nu}$ has a single and determinate value at each point of each of the surfaces. Then if we take for density of the distribution at each point $-\frac{1}{4\pi}\frac{du}{d\nu}$, we can prove exactly as before that the potential of the distribution so formed at any point external to the surface is u, and therefore satisfies all the conditions.

59.] The proposition of Art. 57 may be extended to an unclosed surface thus. Let S be an unclosed surface, S' a similar and equal surface so placed as that each point on S' shall be very near to the corresponding point on S. If we now connect the boundaries of S and S' by a diaphragm we obtain a closed surface. Let a distribution be formed on this closed surface having potential V on S, and at each point on S' the same potential as at the corresponding point on S. Let σ and σ' be the densities of this distribution on S and S' respectively. Then ultimately, if S' be made to coincide with S, we obtain $\sigma+\sigma'$ as the density of a distribution on S which has potential V at each point on S.

60.] If two systems of matter, both within a closed surface S, have the same potential at each point on S, then

(*a*) they have the same potential throughout all external space. For let V, V' be the potentials of the two systems respectively. Then $V = V'$ on S,

$\nabla^2 V = 0$ and $\nabla^2 V' = 0$ at all points in the external space,

V and V' are both of lower degree than $-\frac{1}{2}$.

Hence, by Art. 9, V cannot differ from V' at any point in the external space.

(*b*) The algebraic sum of the matter of either system is equal to that of the other. For the algebraic sum of the matter within S whose potential is V is

$$-\frac{1}{4\pi}\iint\frac{dV}{d\nu}\,dS,$$

the normal being measured outwards on the outside of S, by Art. 45. Now, since $V=V'$ at all points external to S,

$$\frac{dV}{d\nu}=\frac{dV'}{d\nu},$$

and therefore

$$-\frac{1}{4\pi}\iint\frac{dV}{d\nu}\,dS = -\frac{1}{4\pi}\iint\frac{dV'}{d\nu}\,dS.$$

(*c*) The two systems have the same centre of inertia. For taking for the plane of y, z any arbitrary plane, and applying Green's theorem to the space within any closed surface S' enclosing S, we have

$$\iint x\,\frac{dV}{d\nu}\,dS' - \iiint x\nabla^2 V\,dx\,dy\,dz = \iint V\,\frac{dx}{d\nu}\,dS',$$

$$\iint x\,\frac{dV'}{d\nu}\,dS' - \iiint x\nabla^2 V'\,dx\,dy\,dz = \iint V'\,\frac{dx}{d\nu}\,dS',$$

and since on S' $V = V'$, and $\dfrac{dV}{d\nu} = \dfrac{dV'}{d\nu}$,

$$\iiint x\nabla^2 V\,dx\,dy\,dz = \iiint x\nabla^2 V'\,dx\,dy\,dz.$$

And therefore if m, m' be the quantities of matter of the two systems respectively within the element of volume $dx\,dy\,dz$,

$$\iiint mx\,dx\,dy\,dz = \iiint m'x\,dx\,dy\,dz,$$

which, as the direction of x is arbitrary, proves the proposition.

(*d*) The two systems have the same principal axes. For

$$\iint xy\,\frac{dV}{d\nu}\,dS' - \iiint xy\nabla^2 V\,dx\,dy\,dz = \iint V\,\frac{d\,(xy)}{d\nu}\,dS',$$

$$\iint xy\,\frac{dV'}{d\nu}\,dS' - \iiint xy\nabla^2 V'\,dx\,dy\,dz = \iint V'\,\frac{d\,(xy)}{d\nu}\,dS',$$

and therefore

$$\iiint xym\,dx\,dy\,dz = \iiint xym'\,dx\,dy\,dz;$$

and if the axis of z be a principal axis of one system, it is a principal axis of the other system.

(*e*) If A, B, C be the principal moments of inertia of one system, those of the other are $A' = A - K$, $B' = B - K$, $C' = C - K$. For

$$\iint x^2\,\frac{dV}{d\nu}\,dS' - \iiint x^2\nabla^2 V\,dx\,dy\,dz = \iint V\,\frac{d\,(x^2)}{d\nu}\,dS'$$
$$- \iiint 2V\,dx\,dy\,dz,$$

$$\iint x^2 \frac{dV'}{d\nu} dS' - \iiint x^2 \nabla^2 V' \, dx\,dy\,dz = \iint V' \frac{d\,(x^2)}{d\nu} dS' - \iiint 2V' \, dx\,dy\,dz,$$

and therefore

$$\iiint x^2 m \, dx\,dy\,dz = \iiint x^2 m' \, dx\,dy\,dz + 2\iiint (V - V')\,dx\,dy\,dz.$$

Similarly

$$\iiint y^2 m \, dx\,dy\,dz = \iiint y^2 m' \, dx\,dy\,dz + 2\iiint (V - V')\,dx\,dy\,dz.$$

Therefore $\quad C' = C - 4\iiint (V - V')\,dx\,dy\,dz = C - K.$

Similarly, $\quad B' = B - K,$

$$A' = A - K.$$

Definition. A body which has the same potential at all points outside of itself, as if its mass were collected at a point O within it, is a *centrobaric body*, and O its *centre.*

It follows from (c) that if a body be centrobaric, its centre is its centre of inertia.

61.] It follows from Art. 59 that a distribution of matter always exists over a surface S which has any given constant potential at each point of S; and therefore that any given quantity of matter can be distributed over S in such a way as to have constant potential at each point of S. Such a distribution is defined to be an *equipotential distribution.*

Definition. If M be the algebraic sum of a distribution of matter over a closed surface whose potential has the constant value V at each point of that surface, $\frac{M}{V}$ is the *capacity* of the surface.

The capacity of a sphere is equal to its radius. For the sphere being charged to potential V, the potential, being constant over the surface, must have the same constant value V at the centre. But if M be the algebraic sum of the distribution, the potential at the centre is $\frac{M}{a}$, where a is the radius.

We have then $\quad \frac{M}{a} = V, \quad \text{or} \quad \frac{M}{V} = a.$

If S be an equipotential surface to a system of matter wholly within it, and V be the potential of the system on S, the capacity of S is $\frac{M}{V}$, where M is the algebraic sum of the matter of the enclosed system. For, by Art. 60, M is also the algebraic sum of a distribution over S which has potential V at every point on it.

62.] If V be the potential of any distribution of matter over a closed surface S, and if σ' be the density of a distribution of matter over S which has the same potential at each point on S as that of unit of matter placed at any point O, then $\iint V\sigma' dS$ is the potential at O of the first distribution.

For let σ be the density of the first distribution, V' the potential of the σ' distribution, r the distance of any point from O. Then on S,

$$V' = \frac{1}{r},$$

and

$$\sigma = -\frac{1}{4\pi}\left\{\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right\},$$

$$\sigma' = -\frac{1}{4\pi}\left\{\frac{dV'}{d\nu} + \frac{dV'}{d\nu'}\right\}.$$

And

$$\iint V\sigma' dS = -\frac{1}{4\pi}\iint\left\{V\frac{dV'}{d\nu} + V\frac{dV'}{d\nu'}\right\}dS$$

$$= -\frac{1}{4\pi}\iint V'\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right)dS$$

$$= \iint \frac{\sigma}{r}\, dS$$

$$= \text{the potential at } O \text{ of the first distribution.}$$

63.] *If S be a closed equipotential surface in any material system, and if ρ, ρ' denote densities of the matter of the system inside and outside of S respectively, and if R be the force due to the whole system at any point on S in the direction of the normal measured outwards, then the potential at any external point due to the internal portion is equal to that of a distribution of matter over S whose density is $\frac{R}{4\pi}$, and the potential at any internal point due to the external portion differs* *from that of a distribution over S whose density is* $-\frac{R}{4\pi}$, *by the potential of the surface.* For if we take for origin any point outside of S, and if V be the potential of the entire system, we have by applying Green's theorem to the space inside of S, with $u = V$, and $u' = \frac{1}{r}$,

$$\iint \frac{1}{r}\frac{dV}{d\nu}dS - \iiint \frac{1}{r}\nabla^2 V\,dx\,dy\,dz$$
$$= \iint V\frac{d}{d\nu}\frac{1}{r}dS - \iiint V\nabla^2\frac{1}{r}dx\,dy\,dz$$
$$= V_s\iint \frac{d}{d\nu}\frac{1}{r}dS - \iiint V.\nabla^2\frac{1}{r}dx\,dy\,dz,$$

where V_s is the constant value of V on S,

$$= 0, \text{ since} \iint \frac{d}{d\nu}\frac{1}{r}dS = 0, \text{ by Art. 45,}$$

and $\nabla^2\frac{1}{r}$ is zero at all points within S.

The equation therefore becomes

$$\iint \frac{1}{r}\frac{dV}{d\nu}dS = \iiint \frac{1}{r}\nabla^2 V\,dx\,dy\,dz.$$

But $$\frac{dV}{d\nu} = -R, \text{ and } \nabla^2 V = -4\pi\rho.$$

Hence $$\iint \frac{1}{r}\frac{R}{4\pi}dS = \iiint \frac{\rho}{r}.\,dx\,dy\,dz,$$

which proves the first part of the proposition.

Secondly, if we take for origin a point inside of S, and apply Green's theorem to the external space, with V and $\frac{1}{r}$ for u and u', we obtain

$$\iint \frac{1}{r}\frac{dV}{d\nu}dS - \iiint \frac{1}{r}\nabla^2 V\,dx\,dy\,dz$$
$$= V\iint \frac{d}{d\nu}\frac{1}{r}dS - \iiint V\nabla^2\frac{1}{r}dx\,dy\,dz$$
$$= 4\pi V_s, \text{ since} \iint \frac{d}{d\nu}\frac{1}{r}dS = 4\pi$$

in this case, and as before $\nabla^2 \frac{1}{r} = 0$. Also in this case, the normal is measured inwards from S, and therefore

$$\frac{dV}{d\nu} = R, \quad \text{also} \quad \nabla^2 V = -4\pi\rho'.$$

Hence the potential at any internal point of the distribution $-\frac{R}{4\pi}$ over S differs by a constant quantity from that of the external portion M', and therefore the force due to the distribution $-\frac{R}{4\pi}$ over S is equal to that due to the external portion. Hence it follows that the force at any external point due to the internal portion is equal to that due to the distribution $\frac{R}{4\pi}$ over S, and the force at any internal point due to the external portion is equal to that of the distribution $-\frac{R}{4\pi}$.

64.] To express the potential at any point P of any distribution of matter in a series of spherical solid harmonics.

Take as origin any point O. Let $OP = f$.

Let M be any point in the distribution.

Let the coordinates of M referred to OP as axis, be r, θ, ϕ, where θ is the angle POM. Let $\mu = -\cos\theta$. Then $\sin\theta d\theta = d\mu$, and an element of volume in the neighbourhood of M is $r^2 d\mu\, d\phi\, dr$. If ρ be the density of the given distribution in this element of volume, its potential at P is

$$\frac{\rho r^2 d\mu\, d\phi\, dr}{PM} = \rho r^2 d\mu\, d\phi\, dr \,.\left\{\frac{1}{f} + Q_1\frac{r}{f^2} + Q_2\frac{r^2}{f^3} + \ldots\right\} \text{ if } r < f,$$

$$\text{or} \quad \rho r^2 d\mu\, d\phi\, dr \,.\left\{\frac{1}{r} + Q_1\frac{f}{r^2} + \ldots\right\} \text{ if } r > f.$$

The potential at P of the whole distribution is then

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{f} \rho\frac{r^2}{f}\, d\mu\, d\phi\, dr + \int_{-1}^{1}\int_{0}^{2\pi}\int_{f}^{\infty} \rho\, r\, d\mu\, d\phi\, dr$$

$$+\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{f} Q_1 \rho\frac{r^3}{f^2}\, d\mu\, d\phi\, dr + \int_{-1}^{1}\int_{0}^{2\pi}\int_{f}^{\infty} Q_1 \rho f\, d\mu\, d\phi\, dr$$

$$+ \&c.,$$

and since Q depends on μ only, this may be put in the form

$$\int_{-1}^{1} d\mu \left\{\int_0^{2\pi} d\phi \int_0^f dr\, \rho \frac{r^2}{f} + \int_0^{2\pi} d\phi \int_f^{\infty} dr\, \rho\, r\right\}$$
$$+ \int_{-1}^{1} d\mu\, Q_1 \left\{\int_0^{2\pi} d\phi \int_0^f dr\, \rho \frac{r^3}{f_2} + \int_0^{2\pi} d\phi \int_f^{\infty} dr\, \rho f\right\}$$
$$+ \&c.,$$

in which the quantities within brackets are known if the given distribution is known.

If we denote these quantities by A_0, A_1, A_2, &c., we have

$$V = \int_{-1}^{1} d\mu \{A_0 + Q_1 A_1 + Q_2 A_2 + \dots\},$$

in which the A's are generally functions of μ.

65.] To find the density of a distribution of matter over a spherical surface, whose potential at any point on that surface shall be equal and opposite to that of a mass e, placed at an external point.

Let O be the centre of the sphere, a its radius, C the point outside of it, $OC = f$.

Let σ be the required density.

It is evident that the density of this distribution on the sphere must be symmetrical about OC, and must therefore be expressible in a series of zonal harmonics with OC as axis. Let this be

$$\sigma = A_0 Q_0 + A_1 Q_1 + \dots + A_i Q_i + \&c.$$

Let E be any point on the surface, E' any other point.

Let us denote by Q_i' the zonal harmonic of order i referred to OE as axis. Then

$$\frac{1}{EE'} = \frac{1}{a}\{1 + Q_1' + Q_2' + \dots\}.$$

And the potential at E due to the distribution is

$$V_E = \frac{1}{a}\left\{A_0 \iint Q_0 Q_0'\, dS + A_1 \iint Q_1 Q_1'\, dS + \&c.\right\},$$

because every term of the form $\iint Q_i Q_j'\, dS$, where $i \neq j$, is zero; that is,

$$V_E = \frac{1}{a} \Sigma \frac{4\pi a^2}{2i+1} A_i \overline{Q_i},$$

$\overline{Q_i}$ being the value of Q_i at E.

But by hypothesis the potential at E of the distribution is to be the same as that of the mass e at C with reversed sign; that is,

$$V_E = -\frac{e}{CE}$$

$$= -\frac{e}{f}\Sigma\frac{a^i}{f^i}\overline{Q}_i.$$

We have therefore

$$\frac{1}{a}\Sigma\frac{4\pi a^2}{2i+1}A_i\overline{Q}_i = -\frac{e}{f}\Sigma\frac{a^i}{f^i}\overline{Q}_i;$$

and equating coefficients of Q_i,

$$A_i = -e\frac{2i+1}{4\pi a}\frac{a^i}{f^{i+1}},$$

and
$$\sigma = \Sigma A_i Q_i$$

$$= -e\,.\,\Sigma\frac{2i+1}{4\pi a}\cdot\frac{a^i}{f^{i+1}}Q_i$$

$$= -\frac{e}{4\pi a}\cdot\frac{f^2-a^2}{D^3}, \text{ where } D = CE \text{ (by Art. 26).}$$

66.] If the density of any distribution of matter over a spherical surface be equal to Y_i, where Y_i is a spherical surface harmonic of order i, the potential at any point within or without the sphere due to this distribution is proportional to the corresponding spherical solid harmonic.

For let O be the centre of the sphere, a its radius, P any external or internal point, $OP = r$, and M a point on the surface. Then at P

$$V = \iint\frac{Y_i}{MP}dS$$

$$= \iint Y_i\Sigma\frac{r^j}{a^{j+1}}Q_j\,dS, \text{ if } P \text{ be internal,}$$

$$= \iint Y_i\Sigma\frac{a^j}{r^{j+1}}Q_j\,dS, \text{ if } P \text{ be external.}$$

But
$$\iint Y_i Q_j\,dS = 0, \text{ unless } i = j,$$

and therefore

$$V = \frac{r^i}{a^{i+1}}\iint Y_i Q_i\,dS, \text{ if } P \text{ be internal,}$$

$$= \frac{a^i}{r^{i+1}}\iint Y_i Q_i\,dS, \text{ if } P \text{ be external.}$$

But
$$\iint Y_i Q_i \, dS = \frac{4\pi a^2}{2i+1} \overline{Y}_i,$$
where $\overline{Y}_i$ is the value of Y_i at the point where OP, produced if necessary, cuts the sphere.

And therefore $r_i \overline{Y}_i$, or $\frac{\overline{Y}_i}{r^{i+1}}$, according as P is internal or external, is the spherical solid harmonic at P corresponding to Y_i. If we denote this by H_i, we have

$$V = \frac{4\pi}{2i+1} \frac{H_i}{a^{i-1}}, \quad \text{or} \quad V = \frac{4\pi}{2i+1} a^{i+2} H_i,$$

in the two cases respectively. The following proposition may easily be deduced from this, but we prefer to prove it independently thus.

67.] If the potential of any material system wholly within a spherical surface S be given at each point of that surface in a series of spherical surface harmonics, then the potential of the same system at any point on the outside of the surface is found by substituting for each surface harmonic the corresponding solid harmonic.

For let the given potential be $\Sigma A_i Y_i$, and let ρ be the density of the superficial distribution on S whose potential at every point of S is equal to $\Sigma A_i Y_i$.

Let P be any point distant f from the centre on the outside of S.

Then the potential at P of the given system is equal to that of the surface distribution.

But, as shewn in Art. 62, if ρ' be the density of a distribution over S whose potential at any point of S is equal to that of unit of matter situated at P, then

$$\iint \rho' \Sigma A_i Y_i \, dS$$

is the potential at P of the superficial distribution whose potential is $\Sigma A_i Y_i$, and therefore of the given system.

Now

$$\rho' = \frac{f^2 - a^2}{4\pi a D^3},$$

where a is the radius of the sphere. Therefore

$$V \text{ at } P = \iint \frac{f^2 - a^2}{4\pi a D^3} \Sigma A_i Y_i dS$$

$$= \Sigma \left(\frac{a}{f}\right)^{i+1} A_i \overline{Y}_i,$$

where $\overline{Y}_i$ is the value of Y_i at P.

The potential of the given system is also equal to $\Sigma \left(\frac{a}{f}\right)^{i+1} A_i Y_i$ for a certain distance within the given spherical surface S.

For V and $\Sigma \left(\frac{a}{f}\right)^{i+1} A_i Y_i$ both satisfy Laplace's equation throughout all external space, and are identical at all points outside of S. They must therefore be identical throughout all space which can be reached from S without passing through attracting matter so long as $\Sigma \left(\frac{a}{f}\right)^{i+1} A_i Y_i$ is a convergent series.

68.] To express in zonal solid harmonics the potential of any material system symmetrical about an axis.

Let us take for origin any point O on the axis. Let r be the distance from O of any point in space.

Then we can first shew that the potential at any point P on the axis, if more distant from the origin than any point in the system, can be expressed in the form $\Sigma B_i \frac{1}{r^{i+1}}$, and if less distant from the origin than any part of the system can be expressed in the form $C + \Sigma A_i r^i$, where the functions B and A are determinate if the given system is known, and are independent of r.

For let O be the origin, P the point on the axis, M any point in space at which there is matter belonging to the system of density ρ,

$$\cos MOP = \mu, \quad MO = a.$$

Then since the system is symmetrical about the axis, we may take for an element of its volume the space between the two cones whose vertices are at O and semivertical angles $\cos^{-1} \mu$ and $\cos^{-1} (\mu + d\mu)$ and whose distance from O is between a and $a + da$.

If ρ be the density of matter within this element its potential at P is

$$2\pi a^2 \rho \, d\mu \, da \, . \frac{1}{MP},$$

that is,

$$2\pi a^2 \rho \, d\mu \, da \, . \frac{1}{r}\left\{1 + Q_1 \frac{a}{r} + \dots\right\} \text{ if } r > a,$$

$$\text{or} \quad 2\pi a^2 \rho \, d\mu \, da \, . \frac{1}{a}\left\{1 + Q_1 \frac{r}{a} + \dots\right\} \text{ if } r < a.$$

Then if a_1, a_2 be the greatest and least distances from O of any matter between the two cones, the potential of all the matter between them is

$$2\pi \, d\mu \, . \int_{a_2}^{r} a^2 \rho \, . \left\{\frac{1}{r} + Q_1 \frac{a}{r^2} + \dots\right\} da$$

$$+ \; 2\pi \, d\mu \int_{r}^{a_1} a^2 \rho \left\{\frac{1}{a} + Q_1 \frac{r}{a^2} + \dots\right\} da,$$

in which the first integral will be omitted when $r < a_2$, and the second will be omitted if $r > a_1$.

Finally, the potential at P of the whole system is found by integrating the above expression according to μ from $\mu = 1$ to $\mu = -1$, remembering that a_1 and a_2 and ρ are generally functions of μ.

Let a'_1 and a'_2 denote the greatest and least values of r for any point in the system. Then the result, if the integrations can be effected, must appear in the form

$$\Sigma B_i \frac{1}{r^{i+1}} \text{ if } OP > a_1',$$

$$\text{and} \quad C + \Sigma A_i r^i \text{ if } OP < a_2';$$

$$\text{and} \quad C + \Sigma A_i r^i + \Sigma B_i \frac{1}{r^{i+1}},$$

$$\text{if} \quad r > a'_2 < a'_1.$$

We can now find the potential of the system at any point R not in the axis and distant r from O, by multiplying each term by the corresponding zonal harmonic referred to OP as axis.

For instance, suppose $r > a'_1$.

Let V be the potential and let

$$V' = \Sigma B_i \frac{Q_i}{r^{i+1}}.$$

Then since on the axis $Q_i = 1$, and

$$V = \Sigma B_i \frac{1}{r^{i+1}},$$

V and V' are identical throughout a finite length of the axis.

Now both V and V' satisfy Laplace's equation at all points not occupied by matter belonging to the system. And therefore since they are identical throughout some finite length on the axis, and are symmetrical about the axis, they must by Arts. 53 and 54 be identical at all points in space which can be reached from that part of the axis without passing either through the system, or through any part of space where $\Sigma B_i \frac{Q_i}{r^{i+1}}$ does not converge.

Similarly, the potential at any point R' in space distant r from O, where $r < a_2'$, is $C + \Sigma A_i Q_i r^i$, provided R' can be reached from the part of the axis whose distance from O is less than a_2' without passing, either through the system, or through any part of space where $\Sigma A_i Q_i r^i$ does not converge.

CHAPTER IV.

DESCRIPTION OF PHENOMENA.

Electrification by Friction.

ARTICLE 69.] EXPERIMENT I*. Let a piece of glass and a piece of resin be rubbed together and then separated; they will attract each other.

If a second piece of glass and a second piece of resin be similarly treated and suspended in the neighbourhood of the former pieces of glass and resin, it may be observed that—

(1) The two pieces of glass repel each other.

(2) Each piece of glass attracts each piece of resin.

(3) The two pieces of resin repel each other.

These phenomena of attraction and repulsion are called *electrical* phenomena, and the bodies which exhibit them are said to be *electrified* or to be *charged with electricity*.

The electrical properties of the two pieces of glass are similar to each other but opposite to those of the two pieces of resin, the glass attracts what the resin repels, and repels what the resin attracts.

Bodies may be electrified in many other ways as well as by friction.

If a body electrified in any manner whatever behaves as the glass does in the experiment above described, that is, if it repels the glass and attracts the resin, it is said to be *vitreously* electrified, and if it attracts the glass and repels the resin, it is said to be *resinously* electrified. All electrified bodies are found to be either vitreously or resinously electrified.

When the electrified state is produced by the friction of dissimilar bodies, as above described, it is found that so long as the

* The description of these experiments is taken almost verbatim from Maxwell's *Electricity*.

rubbed surfaces of the two excited bodies are in contact the combined mass does not exhibit electrical properties, but behaves towards other bodies in its neighbourhood precisely as if no friction had taken place.

The exactly opposite properties of bodies vitreously and resinously electrified respectively, and the fact that they neutralise each other, has given rise to the terms '*positive*' and '*negative*' electrification, the term *positive* being by a perfectly arbitrary, but now universal convention among men of science, applied to the vitreous, and the term *negative* to the resinous electrification.

Electric actions similar to those above described may be observed between a body electrified in any manner and another body not previously electrified when brought into the neighbourhood of the electrified body, but in all such cases it will be found that the body so acted upon itself exhibits evidence of the electrification. This electrification is said to be produced by *induction*, a process which will be illustrated in the second experiment.

No force, either of attraction or repulsion, can be observed between an electrified body and a body manifesting no signs of electrification.

Electrification by Induction.

70.] Experiment II. Let a hollow vessel of metal, furnished with a close-fitting metal lid, be suspended by white silk threads, and let a similar thread be attached to the lid, so that the vessel may be opened or closed without touching it; suppose also that the vessel and lid are perfectly free from electrification.

Let the pieces of glass and resin of Experiment I be suspended in the same manner as the vessel and lid, and be electrified as before.

If then the electrified piece of glass be hung up within the suspended vessel by its thread, without touching the vessel, and the lid closed, the outside of the vessel will be found to be vitreously electrified, and it may be shown that the electrification outside of the vessel, as indicated by the attractive or repulsive forces on electrified bodies in its neighbourhood, is exactly the same in whatever part of the interior the glass be suspended.

If the glass be now taken out of the vessel without touching it, the electrification of the glass will be found to be the same as before it was put in, and that of the vessel will have disappeared.

This electrification of the vessel, which depends on the glass being within it, and which vanishes when the glass is removed, is called Electrification by *Induction.*

If the piece of electrified resin of Experiment I were substituted for the glass within the vessel, exactly opposite effects would be produced. If both the pieces of glass and resin, after the friction of Experiment I, were suspended within the vessel, whether in contact with each other or not, no electrical effects whatever would be manifested.

Similar effects would be produced if the glass were suspended near the vessel on the outside, but in that case we should find an electrification vitreous in one part of the outside of the vessel and resinous in another part. Whereas, as has been just now mentioned, when the glass is inside the vessel the whole of the outside is vitreously electrified. In this case, as in the case of internal suspension, the electrification disappears on removal of the exciting body.

Experiment proves that throughout the inside of the closed vessel there is an electrification of the opposite kind to that of the outside, that is, when the electrified piece of glass is suspended within the vessel, and the latter is therefore vitreously electrified on the outside, as just now explained, the inside will be resinously electrified, and vice versâ when the resin is substituted for the glass.

Experiment proves also that the electrification on the outside is equal in quantity to that of the glass, and the electrification on the inside equal and opposite to that of the glass.

Electrification by Conduction.

71.] Experiment III. The metal vessel being electrified by induction, as in the last experiment, let a second metallic body be suspended by white silk threads near it, and let a metal wire similarly suspended be brought so as to touch simultaneously the electrified vessel and the second body.

The second body will now be found to be vitreously electrified and the vitreous electrification of the vessel will have *diminished.*

The electrical condition has been transferred from the vessel to the second body by means of the wire. The wire is called a *conductor* of electricity, and the second body is said to be *electrified by conduction.*

Conductors and Insulators.

If a glass rod, a stick of resin or gutta-percha, or a white silk thread had been used instead of the metal wire, no transfer of electricity would have taken place. Hence these latter substances are called *non-conductors* of electricity. A non-conducting support or handle employed in electrical apparatus is called an *Insulator*, and the body thus supported is said to be *insulated.* Thus the lid and vessel of Experiment II are insulated.

The metals are good conductors; air, glass, resins, gutta-percha, vulcanite, paraffin, &c., are good insulators; but all substances resist the passage of electricity, and all substances allow it to pass although in exceedingly different degrees. For the present we shall, in speaking of conductors or non-conductors, imagine that the bodies spoken of possess these properties in perfection, a conception exactly similar to that of perfectly fluid or perfectly rigid bodies, although such conceptions cannot be realised in nature.

In Experiment II an electrified body produced electrification in the metal vessel while separated from it by air, a non-conducting medium. Such a medium, considered as transmitting these electrical effects without conduction, is called a *Dielectric* medium, and the action which takes place through it is called, as has been said, *Induction.*

72.] EXPERIMENT IV. In Experiment III the electrified vessel produced electrification in the second metallic body through the medium of the wire. Let us suppose the wire removed and the electrified piece of glass taken out of the vessel without touching it and removed to a sufficient distance. The second body will still exhibit vitreous electrification, but the vessel when the glass is removed will have resinous electrification. If we now bring the wire into contact with both bodies, conduction will take place along the wire, and all electrification will disappear from both bodies, from which we infer that the electrification of the two bodies was equal and opposite.

73.] EXPERIMENT V. In Experiment II it was shown that if a piece of glass, electrified by rubbing it with resin, is hung up in an insulated metal vessel, the electrification observed outside does not depend upon the position of the glass. If we now introduce the piece of resin with which the glass was rubbed into the same vessel without touching it or the vessel, it will be found, as stated in Art. 70, that there is no electrification on the outside of the vessel. From this we conclude that the electrification of the resin is exactly equal and opposite to that of the glass. By putting in any number of electrified bodies, some vitreous and others resinous, and taking account of the amount of electrification of each, we shall find that the whole electrification of the outside of the vessel is that due to the algebraic sum of the electrifications of all the inserted bodies, the signs being used in accordance with the convention already described. We have thus a practical method of adding the electrical effects of several bodies without altering the electrification of any of them.

74.] EXPERIMENT VI. Let a second insulated metallic vessel B be provided, and let the electrified piece of glass of Experiment I be placed in the first vessel A, and the electrified piece of resin in the second vessel B. Let the two vessels be then put in communication by the metal wire, as in Experiment III. All signs of electrification will disappear.

Next, let the wire be removed, and let the pieces of glass and resin be taken out of the vessels without touching them. It will be found that A is electrified resinously and B vitreously.

If now the glass and the vessel A be introduced together (the glass being no longer within A) into a larger insulated vessel C, it will be found that there is no electrification on the outside of C.

This shows that the electrification of A is exactly equal and opposite to that of the piece of glass, and similarly that of *B* may be shown to be equal and opposite to that of the piece of resin.

Thus the vessel *A* has been charged with a quantity of electricity exactly equal and opposite to that of the electrified piece of glass without altering the electrification of the latter, and we may in this way charge any number of vessels with exactly equal quantities of electricity of either kind which we may take as provisional units.

75.] Experiment VII. Let the vessel *B*, charged with a quantity of positive electricity, which we shall call for the present unity, be introduced into the larger insulated vessel *C* without touching it. It will produce a positive electrification on the outside of *C*. Now let *B* be made to touch the inside of *C*. No change of the external electrification of *C* will be observed. If *B* be now taken out of *C* without again touching it and removed to a sufficient distance, it will be found that *B* is completely discharged, and that *C* has become charged with a unit of positive electricity.

We have thus a method of transferring the charge of *B* to *C*.

Let *B* be now recharged with a unit of electricity, introduced into *C* already charged, made to touch the inside of *C* and removed. It will be found that *B* is again completely discharged, so that the charge of *C* is doubled.

If this process be repeated it will be found that however highly *C* is previously charged, and in whatever way *B* is charged when it is first inclosed in *C*, then made to touch *C*, and finally removed without touching *C*, the charge of *B* is completely transferred to *C*, and *B* is entirely free from electrification.

This experiment indicates a method of charging a body with any number of units of electricity. The experiment is also an illustration of a general fact of great importance, namely, that no charge whatever can be maintained in the interior of any conducting mass.

76.] In what has hitherto been said it has been assumed that we possess the means of testing the nature and measuring the amount of electrification on any body, or on any part of a body. This we can do with great accuracy by the aid of instruments called *electroscopes* or *electrometers,* whose modes of action will be more easily understood when the theory of the subject has been somewhat developed; and which are fully described in practical treatises on electricity; for our present purpose it will suffice to describe one of these instruments in its simplest form, called the *gold-leaf* electroscope.

A strip of gold-leaf hangs between two bodies A and B, charged one positively and the other negatively.

If the gold-leaf be placed in conducting contact with the body whose electrification is to be investigated, it will itself become a part of that body for all electrical purposes, and it will incline towards A or B according as its electrification, and therefore the electrification of the body under investigation, is negative or positive.

77.] From the foregoing experiments we conclude that

(1) The total electrification of a body or system of bodies remains always the same except in so far as it receives electrification from, or gives electrification to, other bodies.

In all electrical experiments the electrification of bodies is found to change, but it is always found that this change is due to want of perfect insulation, and that with improved insulation the change diminishes. We may therefore assert that the electrification of a body placed in a perfectly insulating medium would remain perfectly constant.

(2) When one body electrifies another by conduction the total electrification of the two bodies remains the same, that is, the one loses as much positive, or gains as much negative electrification, as the other gains of positive or loses of negative electrification.

For if the two bodies are enclosed in the same hollow conducting vessel no change of the total electrification is observed on their being connected by a wire.

(3) When electrification is produced by friction or by any other known method, equal quantities of positive and negative electrification are produced.

For the electrification of the whole system may be tested in the hollow vessel, or the process of electrification may be carried on within the vessel itself, and however intense the electrification of the parts of the system may be, the electrification of the whole is invariably zero.

The electrification of a body is therefore a physical quantity capable of measurement, and two or more electrifications may be combined experimentally with a result of the same kind as when two quantities are added algebraically.

78.] Experiment VIII. Let there be a needle suspended horizontally by a fine vertical wire or fibre, so as to be capable of vibrating horizontally about the vertical wire as an axis, and let a small pith ball A be attached to one end of the needle. Then the needle will rest in a certain position; in which position, supposing there are no forces at work in the neighbourhood of the apparatus except the force of gravity, the suspending wire or fibre will be perfectly free from any twist or torsion. Let another pith ball B be situated at a certain point in the circumference of the horizontal circle described by A.

Now let the pith balls A and B be each charged with one unit of positive electrification. A repulsive action will arise between A and B so that A will after certain oscillations come to rest at a certain increased distance from B, thus producing a twist in the suspending wire. The opposite untwisting tendency of the wire thus called into play depends upon the torsional rigidity of the wire and the angle through which the needle has been deflected, and can be estimated in any given apparatus with great accuracy. Hence the repulsive force between A and B, assumed to act in the line joining them, can also be determined with corresponding accuracy: let it be called f.

Suppose now that the same experiment is made with another apparatus equal to the former in all respects, but with a suspending wire of different torsional rigidity; and suppose that in this case the position taken up by A with respect to B is observed to be exactly the same as in the former case when the number of units of positive electrification of A is e, and of B is e'.

It will be found that the repulsive force between *A* and *B* is in this case $ee'f$.

Precisely the same positions would be taken up in the two cases respectively, if *A* and *B* had been each negatively electrified, and to the same degree as before. By suitable adjustments of the two cases with *opposite* electrifications upon *A* and *B* each of the same number of units as before, it may be proved that, if the distances between *A* and *B* in their positions of equilibrium are the same as before, the forces between them are attractive and equal to f and $ee'f$ in the two cases respectively.

Hence we infer that the force between two electrified particles at any given distance apart is in all respects represented by the product of the two electrifications upon them, regard being paid to the signs of the electrifications, and the force being considered repulsive when the above-mentioned product is positive.

79.] EXPERIMENT IX. In the experiment of the last Article let the electrifications of *A* and *B* in the second apparatus as well as in the first apparatus be each one unit of positive electrification. It will be found that in the positions of equilibrium the distances between *A* and *B* are not the same in one apparatus as they are in the other. If, however, the forces between *A* and *B* in these positions be estimated as before, and if the distances between *A* and *B* in the two cases be r and r', it will be found that there are repulsive forces between them which are to each other in the ratio of r'^2 to r^2, or inversely as the squares of the distances between them in the two cases.

Combining the results of this and the preceding experiment we arrive at the following general law of action between two electrified particles, viz. that if the number of units of electrification of the particles be e and e' respectively, and the distance between them be r, then there is a force F such that

$$F = \frac{ee'}{r^2} f;$$

where f is the repulsive force between two particles each charged with unit of electrification, and at the distance unity apart, regard being paid to the signs of e and e', and F being considered positive when the force is repulsive, i.e. when e and e' have the same signs.

In conducting Experiments VIII and IX care must be taken that the dimensions of A and B are small as compared with the distance between them, so that they may be regarded as material points. They must also be suspended in air and at a considerable distance from any other bodies on which they might induce electrification (Art. 70), inasmuch as this induced electrification would also act upon A and B and produce a problem of great intricacy.

Electrical Theory.

80.] The most important researches into the laws of electrical phenomena up to the present time have been based upon what is known as the *two fluid theory*. It is conceived that all bodies in nature, whether electrified or not, are charged with, or pervaded by, two fluids to which the names of positive and negative, or vitreous and resinous, electricity are assigned. It is further supposed either that these fluids exist in all bodies in such quantities that no process yet discovered has ever deprived any body, however minute, of all the electricity of either kind, or that the changes in the proportion in which these fluids are combined, required to produce electrical phenomena, are indefinitely small. It is further supposed that in unelectrified bodies these fluids exist in exactly equal quantities, but that it is possible by friction, as in Experiment I, or by other means, to cause one body to give up to another part of its positive or negative electricity, thus causing in either body an excess of one or other kind of electricity.

When the quantity of either fluid is in excess in any body, that body is said to be positively or negatively electrified according to the sign of the predominant fluid, and the amount of electrification is measured by the quantity by which this predominant fluid exceeds the other. The fluid of either kind in any electrified body in excess of that of the opposite kind is called the *Free Electricity* of the body, and the remaining fluids of the body, consisting of equal amounts of fluids of opposite kinds, together constitute what is called the *Latent*, *Combined* or *Fixed Electricity* of the body.

In the simplest form of the theory, although not essential to it, every process of electrification is supposed to consist of a transference of a certain quantity of one of the fluids from any body as A to another as B, together with the transference of an equal amount of the opposite fluid from B to A, so that the total amount of electricity free and latent (without regard to sign) in every body and every particle of every body cannot be changed by any process whatever.

It is further supposed that these fluids are not acted upon by gravitation or any of the forces of ordinary mechanics, nor, so far as our present knowledge goes, by ordinary molecular or chemical forces; but they are supposed to exercise forces upon themselves and each other which are conceived to be proportional to the quantities of the mutually acting fluids, thus giving rise to the conception of *electrical mass*. And it is further supposed that the forces between two particles of fluid of the same kind is repulsive, and proportional to the product of their masses directly, and to the square of the distance between them inversely, that between two particles of fluid of opposite kinds being attractive, but in other respects following the same law. According to this hypothesis the latent or fixed electricity in any body, consisting of equal quantities of opposite kinds, exerts zero force on all electricity. The forces of attraction and repulsion above mentioned manifest themselves only between the free electricities.

If all bodies be divided for the time into two classes, perfect conductors and perfect insulators, it is conceived that either kind of electricity may pass with absolute and perfect freedom from point to point of the former, while the latter offer a complete and absolute bar to any such transference.

On the hypothesis thus described we are able to explain many electrical phenomena. It is of course merely an hypothesis, and of value as supplying formally an explanation of facts; in this respect being exactly on a par with the conception of the luminiferous ether in the undulatory theory of light. The general mathematical treatment of this hypothesis is principally due to Poisson and Green.

There is another hypothesis, known as the one-fluid theory, which is equally successful as a basis of investigation, but it has not been adopted and developed to the same extent as the two-fluid theory.

We shall confine our investigation to the two-fluid theory in the form which is above enunciated.

81.] It is evident without any further explanation that the two-fluid theory explains the qualitative results of Experiment I given above; and we proceed now to shew that it also explains the quantitative results of Experiments VIII and IX.

For, suppose two bodies A and B, either conductors or non-conductors, to contain m and m' units of mass of positive electricity respectively, and n and n' units of negative electricity.

Suppose that they are situated in an insulating medium, as air, and that their dimensions are very small as compared with the distance between them which we shall call r.

Then, according to the two-fluid theory, the m positive units of A exert upon the m' positive units of B a repulsive force which may be represented by $\frac{mm'}{r^2}$, and the n and n' units exert a repulsive force upon each other, represented on the same scale by $\frac{nn'}{r^2}$, so that on the whole there is a repulsive force between the electrical fluids in A and B represented by $\frac{(mm'+nn')}{r^2}$.

In the same way there is an attractive force between the two electricities represented by $\frac{mn'+m'n}{r^2}$.

Altogether therefore there is a repulsive force between the electricities on A and B represented by $\frac{mm'+nn'-mn'-m'n}{r^2}$; that is, by $\frac{(m-n)(m'-n')}{r^2}$.

But $m-n$ is the number of units of positive electrification on A, and $m'-n'$ is the same for B, so that with the notation used above the force between the electricities in A and B is represented by $\frac{ee'}{r^2}$, and is repulsive when ee' is positive.

The unit of electricity in this measurement is such that the repulsive force between two units of positive electricity at the distance unity apart is unit force.

It appears therefore that the two-fluid theory involves the existence of a force between the electric fluids in two charged bodies in all respects following the law which has been experimentally proved to be obeyed by the mechanical forces between the bodies themselves. But the bodies are either non-conductors or else conductors in an insulating medium, and on either hypothesis the fluids cannot move without the containing bodies accompanying them. Whatever force therefore is proved to exist between the fluids becomes phenomenally a corresponding force between the bodies. We thus see that the results of Experiments I, III, and IV, and of Experiments VIII and IX are explained qualitatively and quantitatively by the two-fluid hypothesis.

The application of the theory to the Induction Experiments II, V, VI, and VII, is not so obvious, and can only be demonstrated after some further development.

CHAPTER V.

ELECTRICAL THEORY.

Article 82.] We proceed now to develop the two-fluid theory as before enunciated, regarding for the present all substances as divided into two classes, namely, (1) perfect insulators, called generally dielectrics, throughout which there is an absolute bar to the motion of the fluids from one particle to another, and (2) perfect conductors, throughout which the fluids are free to move with no resistance whatever from one particle to another. And it is assumed for the present that the repulsion between two masses, e and e', of electricity placed at distance r apart is $\frac{ee'}{r^2}$. The phenomena with which we have at present to deal are those of repulsion and attraction between particles at a distance according to the above law. The investigations of Chap. III are therefore applicable.

It will be understood that we do not assert the actual existence of the fluids, or that direct action at a distance actually takes place. It is proposed merely to show how the phenomena of Electrostatics may be explained on this hypothesis. In like manner the conception of space as divided into perfect conductors and perfect insulators, will have to be materially modified hereafter.

83.] It follows from the above definition of a conductor, that when the electricities are in equilibrium, the resultant force is zero at each point within the conductor. For if there be any force, it must tend to move one kind of electricity at the point in one direction, and the other in the opposite direction, and therefore to separate them. And since the substance of the conductor opposes no resistance to their motion, such separation will in fact take place until equilibrium is attained; that is, until the mutual attraction of the separated electricities, tending to reunite them, becomes equal and opposite to the force which tends to separate them, and so the resultant force becomes zero.

Now it follows from the reasoning of Chap. III that in a field of electric fluid distribution a potential function V exists such that $-\frac{dV}{dx}, -\frac{dV}{dy}, -\frac{dV}{dz}$ are the component forces parallel to the axes of x, y, and z at any point. And since the resultant force is zero, each of these components is zero at every point within the conductor, and therefore V has some constant value throughout the substance of the conductor. This is true whatever the law of force, provided there be a potential.

84.] It follows further from the law of the inverse square, that there can be no free electricity within the substance of the conductor. For whatever closed surface be described wholly within it, the normal force N at every point of that surface is zero. Therefore $\iint N ds = 0$ over the surface. That is, by Art. 45, the algebraic sum of all the free electricity within the surface is zero, and this being true for every closed surface that can be described within the substance of the conductor, it follows that there can be no free electricity, of either volume or superficial density, within the substance of the conductor.

It follows that, in order to insure the constancy of V throughout the conductor, it is sufficient to make it constant at all points on the surface. For we have seen that if V be constant at all points on a closed surface, within which is no attracting matter, it has the same constant value throughout the interior.

85.] Whatever free electricity is formed by the separation of the two kinds of electricity within the conductor, since it cannot exist within the substance of the conductor, and cannot penetrate the surrounding dielectric, must be found upon the surface in the form of a superficial distribution.

And such superficial distribution must be in the aggregate zero for the whole surface; because since the two kinds of electricity are supposed to exist in equal quantities at all points, for every quantity of positive electricity resulting from their separation, there must be an equal quantity of negative electricity, and each must be found somewhere on the surface.

But it is possible to place upon the conductor from external sources a quantity of electricity of either sign. This also, for the same reason, can only exist in the form of a superficial distribution.

It follows then that if a conductor be in equilibrium its electrification is wholly on the surface, and the algebraic sum of all the superficial distribution upon it is equal to that of the electricity placed upon it from external sources.

86.] *Definition.*—The algebraic sum of all the electricity on the surface of a conductor is called *the charge* on the conductor.

If σ be the density of the superficial distribution at any point, $\frac{dV}{d\nu}$ the rate of increase of V per unit of length of the normal measured outwards in direction, immediately outside of the distribution, $\frac{dV}{d\nu'}$ the same thing measured inwards in direction, immediately inside of the distribution, Poisson's equation gives

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} = -4\pi\sigma.$$

But $-\frac{dV}{d\nu'}$, being the force within the substance of the conductor, is in this case zero. We have therefore at every point of the surface

$$\frac{dV}{d\nu} + 4\pi\sigma = 0;$$

and the charge upon the conductor or

$$\iint \sigma\, dS = -\frac{1}{4\pi}\iint \frac{dV}{d\nu}\, dS.$$

87.] We have seen that when an electrical system is in equilibrium, the potential must have a constant value throughout each conductor. Conversely, if the potential have a constant value throughout each conductor, the electricity on fixed conductors is in equilibrium. For the potential being constant throughout the conductor, there can be no tangential or other force to move the superficial distribution along the surface or through the substance of the conductor. And since by the hypothesis concerning the nature of the dielectric medium there can be no motion of electricity in the medium, all the electricity in the field must be at rest. The constancy of the potential throughout each conductor is thus the sufficient and necessary condition of equilibrium.

Hence can be established the following principle.

The Principle of Superposition.

88.] If σ be the density of the superficial distribution on a conductor when in equilibrium in presence of any electrified system E, which may include a charge on the conductor itself, and if σ' be the density on the conductor when in equilibrium in presence of the system E', then if E and E' both be present, the conductor will be in equilibrium when the density is $\sigma + \sigma'$.

For if every conductor of the system had placed upon it for an instant the distribution whose density is $\sigma + \sigma'$, we know that the potential at any point is the sum of the two potentials, one due to the system E and density σ, the other due to the system E' and density σ'. But both of these potentials are constant for each conductor. Therefore their sum is constant, and therefore the supposed instantaneous distribution is in equilibrium and is permanent.

It follows that if all the volume, or superficial or linear densities of electricity, in a system in equilibrium be increased in any given ratio, the system will remain in equilibrium, and the potential at any point will be increased in the same ratio as the densities.

89.] Let us consider the simple case of a single conductor, and a point O outside of it having a fixed charge of positive electricity m. There will form on the surface of the conductor an induced distribution of electricity whose algebraic sum is zero, and of which the negative part is on the side of the conductor nearest to O, and the positive part on the opposite side. The tendency of the charge at O is to make the potential higher on the side of the conductor nearest to O than on the other side. The surface distribution has the opposite tendency. And the surface distribution must be such that these two tendencies shall exactly neutralize one another, and the potential be the same at all points of the conductor.

The actual solution of this problem consists in the determination of a function V, the potential of the system, to satisfy the conditions

(1) V is constant over C;

(2) $\iint \frac{dV}{d\nu} dS = 0$, taken over the surface of C;

(3) $\nabla^2 V = 0$ at all points in space external to C, except where the given external electricity is situated, and there $\nabla^2 V = -4\pi m$.

We have seen in Art. 10 that one determinate function V, always exists satisfying these conditions. If it were determined, V, and therefore $\frac{dV}{d\nu}$, would be known at each point on or outside of C, and a distribution over C whose density is $-\frac{1}{4\pi}\frac{dV}{d\nu}$ satisfies all the conditions.

If the external charge were at another point O' instead of O, the superficial distribution would assume a different form. If there be a charge both at O and at O', then, by the principle of superposition above proved, the density of the distribution at any point on the conductor in this case is the sum of the densities due to the charges at O and at O' separately, and so on for any electrified system outside the conductor. In like manner if there be a charge on the conductor itself, that charge will so distribute itself as to give constant potential at all points on the conductor, and the density of this equipotential distribution together with that due to any external electrification will be the actual superficial density.

90.] The case in which an electrified system is placed *inside* of a closed conducting shell is of special importance. It will be found that in this case we have two systems, separated by the shell, each of which would be in equilibrium separately if the other were removed.

For let C be any such shell, and let there be any electrified system within it, and any other electrified system outside of it. The general reasoning shows as before that the electrification of the conductor is wholly on the surface, that is to say, in this case, partly on the inner and partly on the outer surface of the shell. We can then prove that the algebraic sum of the distribution on the inner surface, together with that of the enclosed system, is always zero. For let a closed surface S be described wholly within the substance of the conductor, and entirely dividing the inner from the outer surface. The potential V is constant at all points within the substance of the shell, and therefore $\frac{dV}{d\nu}$ is zero, at every point of S, and

$$\iint \frac{dV}{d\nu}\, dS = 0.$$

But
$$-\iint \frac{dV}{d\nu}\, dS = 4\pi m,$$

where m is the algebraic sum of all the free electricity within S. It follows that m is zero. But the only free electricity within S is the distribution on the inner surface of the conductor C, and that of the enclosed system. If therefore the algebraic sum of the electricity of the enclosed system be e, that of the distribution on the inner surface is $-e$.

It follows, that unless there be a charge on the conductor, the algebraic sum of the induced distribution on the outer surface is $+e$, since the whole surface distribution on the conductor is zero.

We can next prove the following proposition.

91.] If a hollow conducting shell be in electrical equilibrium under the influence of any enclosed electrified system, and of any external electrified system, then the potential V of the enclosed system and of the induced distribution on the inner surface will be zero at all points on or outside of the inner surface; and the potential V' of the external electrification and of the induced distribution on the outer surface will be constant at all points on or inside of the outer surface of the shell.

For let S be any closed surface within the substance of the shell entirely dividing the inner from the outer surface.

Then S is an equipotential surface, and separates the enclosed system with the induced distribution on the inner surface of the shell from the external system, and the induced distribution on the outer surface. Therefore (by Art. 63) the enclosed system and the induced distribution on the inner surface have, at all points outside of S, the same potential as a distribution over S whose density is $\frac{R}{4\pi}$, where R is the normal force on S due to the whole electrification; that is zero potential, because $R = 0$ on S. Hence the enclosed system and the distribution on the inner surface have together zero potential at all points outside of S. Similarly the external system and induced distribution on the outer surface have together constant potential V' at all points inside of S; and since S may be made to coincide with either the inner or the outer surface of the shell, this proves the proposition.

It follows that if the enclosed system, together with the distribution on the inner surface, were both removed, or allowed to communicate and neutralise each other, the distribution on the outer surface would remain in equilibrium. Its density is therefore independent of the position of the enclosed distribution within the shell. It follows further that any charge placed on the conductor will assume a position of equilibrium on the outer surface without causing any electrification on the inner surface.

Again, if the external electrification and the distribution on the outer surface were removed, that on the inner surface and the enclosed system would remain in equilibrium.

The agreement with experiment of the above proposition, that a charge of electricity upon a hollow conducting shell causes no electrification on its inner surface or on a conductor placed within it, has been employed, as we shall hereafter see, to establish the most conclusive proof of the law of the inverse square in electric action.

92.] In Chap. IV it was shown that the qualitative results of Experiment I, and the qualitative and quantitative results of Experiments I, III, IV, VIII and IX, were completely explained by the two-fluid theory of electricity. We are now in a position to do the same with reference to the results of Experiments II, V, VI, and VII.

For it has been proved (Art. 84), that there can be no free electricity within the substance of conducting bodies, but that in the case of such bodies the charges, if any, are entirely superficial.

It has been also proved (in Arts. 90, 91) that in the case of the electrical equilibrium of a hollow conducting shell in the presence of any given electrical distributions, whether internal or external,—

(1) There is a superficial electrical distribution on the inner surface of the shell equal in amount, but of opposite algebraic sign to, the algebraic sum of the given internal system.

(2) That the given internal system, with the last-mentioned superficial electrification of the inner surface, constitute a system producing electrical equilibrium throughout the surface of the shell and the whole of external space; and that the given external system, with any superficial electrification on the outer surface of the shell, constitute a system producing electrical equilibrium throughout the shell and the whole of the internal space.

It follows therefore that in the case of the closed insulated metal vessel of Experiment II, containing an electrified piece of glass as therein described,—

(1) There will be a superficial electrification on the inner surface, the total amount of which will be resinous, and equal to the vitreous electricity of the glass, but the intensity of which at different points will depend upon the position of the glass.

(2) That inasmuch as the vessel is insulated, and the total charge zero, and as all the electrification must be superficial, there will be a superficial distribution on the external surface equal in amount to, and of the same sign as, the vitreous electricity of the glass.

Since however the external and internal distributions are in equilibrium separately by Art. 91, it follows that the intensity of the external superficial electrification at any point, unlike that of the corresponding internal electrification, will be entirely independent of the position of the glass, and will be determined by the given distributions in the field external to the vessel and the shape of the vessel.

93.] In Experiment VI the external electrifications of the vessels A and B are equal and opposite before the introduction of the wire. When the two vessels are connected by the wire, the two equal and opposite distributions coalesce, producing evidently by that means external equilibrium. The effect on either vessel is the same as if, there being no introduction of the wire, it received an independent charge equal in amount to, and of the same sign as, that of the glass or resin in the other vessel, and therefore equal and opposite to that of the resin or glass within itself. These charges remain when the wire, and afterwards the glass and resin, are removed, as the experiment shows.

94.] The result of Experiment VII also follows at once from the same reasoning. For the external superficial charge on C is the same in whatever part of its interior B be situated, and is equal to that of B in magnitude and of the same sign. If therefore B be made to touch C, the external electrification of the latter will not be affected, but inasmuch as C and B after contact may be regarded as constituting one conducting body, the vessel C with B in contact constitutes a metallic shell with a given internal distribution zero. Hence the internal superficial electrification must be zero, and there is no free electricity within the compound conductor C and B, and therefore the whole of B is discharged.

95.] We have hitherto considered cases of equilibrium in which certain conductors have given charges. It is sometimes required to determine the density of the induced distribution on a conductor or system of conductors placed in a known field of force; as, for instance, when the force before the introduction of the conductors is uniform throughout the field, such as may be conceived to be due to an infinite quantity of electricity placed at an infinite distance from the conductors.

Another class of problems is found when the potentials of certain conductors are given.

When two conductors of known shapes are joined together by any conducting connection, the conductors with their connection of course form one compound conductor, and must be treated as such. In the particular case however of the connection between them being a very thin wire, the total amount of electricity on the surface of the wire must be very small, and generally is inappreciable in its effect upon the field.

As far therefore as the electricity on the connection is concerned, such conductors may be regarded as two separate and independent conductors of known form; the existence however of the connection will ensure that they are of the same potential.

If in the case last mentioned, of two conducting bodies joined by a thin wire, one of them be removed to a great distance from the field, the charge upon the one so removed will at length cease to exercise any appreciable effect, and may be neglected.

If, at the same time, the potential of this removed conductor be maintained at any given value, we may by this contrivance regard the remaining conductor as an insulated conductor at a given potential. In order to effect this object the charge upon the conductor must be capable of variation. In fact, the distant conductor, or some other body connected with it, must be a reservoir containing infinite quantities of either kind of electricity, and so large that the withdrawal of electricity necessary to maintain the given conductor at the required potential has no appreciable effect upon it.

A very common case of such an arrangement occurs when one or more of the conductors of the field are connected by a thin wire with the earth, for this latter is an infinite conductor always at the same potential*, which is taken as zero, the potentials of all bodies being measured by their excess or defect above or below that of the earth. A conductor connected with the earth is said to be *uninsulated*.

96.] It follows from what has gone before that the most general problem of electrical equilibrium, in such a dielectric medium as we have described, is reduced to that of given electrical distributions in the presence of given insulated conductors with given charges, or at given potentials, in a dielectric medium of infinite extent.

The solution of any such problem, that is, the determination of the electric density and potential at any point, involves the determination of a function V, the potential of the system, to satisfy the following conditions:—

* The earth for any distances within the limits of any experiment is at the same potential. But there may be differences in the potential of the earth between distant points, as England and America.

(1) V has some (not given) constant values over each of the surfaces $S_1 \dots S_n$ bounding the conductors on which the charges are given.

(2)
$$-\frac{1}{4\pi}\iint \frac{dV}{d\nu}\, dS_1 \text{ taken over } S_1 = e_1;$$
$$-\frac{1}{4\pi}\iint \frac{dV}{d\nu}\, dS_2 = e_2;$$
&c.;
$$-\frac{1}{4\pi}\iint \frac{dV}{d\nu}\, dS_n = e_n.$$

(3) V has given constant value over each of the surfaces $S_1' \dots S'_m$ bounding the conductors on which the potentials are given.

(4) $\nabla^2 V + 4\pi\rho = 0$ at any point where there is fixed electricity of density ρ, and of course, if such fixed electricity be what is called superficial, this may be put in the form
$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} + 4\pi\sigma = 0.$$

(5) V vanishes at an infinite distance.

It was proved in Art. 10 that one such function always exists, and if it be V, a distribution of electricity over the surfaces of density
$$-\frac{1}{4\pi}\frac{dV}{d\nu}$$
satisfies all the conditions of the problem. Then the equation
$$\frac{dV}{d\nu} + 4\pi\sigma = 0$$
determines the density of electricity at any point of the surface of any conductor, and the problem is completely solved.

97.] It was stated in Art. 91 that the fact of a charge of electricity on a hollow conducting shell causing no electrification on a conductor placed within it furnishes the most conclusive proof of the law of the inverse square in electric action.

By hypothesis there is internal equilibrium when a distribution itself in equilibrium is placed on the outer surface of the shell. Let the outer surface be a sphere. Then by symmetry this distribution must be uniform. Let us take σ for the superficial density at any point, and since there must be a potential function, let it be $\frac{f(r)}{r}$ at the distance r from a particle of unit electricity.

Let P be any point within the shell at the distance p from the centre O. Let the radius of the shell be a, and let θ be the angle between OP and the line drawn from O to any point Q on the surface of the shell. Let dS be an elementary area of that surface in the neighbourhood of Q, and let V be the potential of the whole charge at P. Then

$$V = \iint \sigma \frac{f(r)}{r} . dS$$

$$= 2\pi \int_0^\pi \sigma \frac{f(r)}{r} a^2 \sin\theta \, d\theta.$$

Also $$r^2 = a^2 - 2ap\cos\theta + p^2,$$

$$r\,dr = ap\sin\theta\, d\theta;$$

$$\therefore \quad V = 2\pi\sigma \frac{a}{p} \int_{a-p}^{a+p} f(r)\, dr.$$

But, by hypothesis, V is to be constant for all values of p. Therefore, multiplying by p and differentiating,

$$V = 2\pi\sigma a \{f(a+p) + f(a-p)\};$$

$$\therefore \quad 0 = f'(a+p) - f'(a-p);$$

$$\therefore \quad f'(r) = C;$$

$$\therefore \quad f(r) = Cr + C';$$

$$\frac{f(r)}{r} = C + \frac{C'}{r};$$

and the force $$= -\frac{dV}{dr} = \frac{C'}{r^2}.$$

Hence the inverse square must be the law of force necessary to satisfy the experimental data.

98.] It may be of interest to enquire within what degrees of accuracy the experiments which have been made may be depended upon.

Let there be an insulated conducting spherical shell within and concentric with the given spherical shell, and of radius b. If the law of force were that mentioned, the charge on the smaller sphere would be accurately zero, even with the two spheres in conducting communication; and, conversely, if the charge were accurately zero, the law of force must be that of the inverse square.

If, however, the law of force differed slightly from that of the inverse square, there might be a small charge on the inner shell, and we propose to investigate the amount of this charge with any assumed small deviation from the above-mentioned law.

Let the metallic communication between the surface of the inner sphere and the external surface of the outer sphere be made by a very thin wire, then the electricity on this wire may be neglected, and therefore, by symmetry, the charges on the two spheres must be uniformly distributed. And if the shells be very thin, we may, whatever be the law of force, regard the charges as superficial.

Let E be that on the outer sphere, and E' that on the inner.

Let $f(r) = C + m\phi(r)$ where m is small; i.e. let the law of force be

$$\frac{C}{r^2} + \frac{m}{r^2}\{r\phi'(r) - \phi(r)\},$$

where m is small compared with C.

At any point P the potential from the two charges will be

$$\frac{E'}{2bp}\int_{p-b}^{p+b} f(r)\,dr + \frac{E}{2ap}\int_{a-p}^{a+p} f(r)\,dr,$$

and this must be the same at the two ends of the wire.

Therefore

$$\frac{E'}{2b^2}\int_0^{2b} f(r)\,dr + \frac{E}{2ab}\int_{a-b}^{a+b} f(r)\,dr$$
$$= \frac{E'}{2ab}\int_{a-b}^{a+b} f(r)\,dr + \frac{E}{2a^2}\int_0^{2a} f(r)\,dr.$$

That is,

$$\frac{E'}{2b}\left\{\frac{1}{b}\int_0^{2b} f(r)\,dr - \frac{1}{a}\int_{a-b}^{a+b} f(r)\,dr\right\}$$
$$= \frac{E}{2a}\left\{\frac{1}{a}\int_0^{2a} f(r)\,dr - \frac{1}{b}\int_{a-b}^{a+b} f(r)\,dr\right\}.$$

But $$f(r) = C + m\phi(r).$$

Therefore, substituting and neglecting the products of the small magnitudes E' and m, we get

$$E' = \frac{mEb}{2C(a-b)}\left\{\frac{1}{a}\int_0^{2a}\phi(r)\,dr - \frac{1}{b}\int_{a-b}^{a+b}\phi(r)\,dr\right\}.$$

For example, suppose the law of force to be $\frac{1}{r^{2+q}}$, where q is small. Then

$$V = \frac{1}{1+q}\,\frac{1}{r^{1+q}}$$

and
$$f(r) = \frac{1}{1+q} - \frac{q}{1+q}\log r.$$

Therefore $C = \frac{1}{1+q}$, $m = -\frac{q}{1+q}$, $\phi(r) = \log r$.

Substituting in the expression for E', and remembering that

$$\int \log r\,dr = r\log r - r,$$

we get

$$E' = -\frac{qb}{2(a-b)}E\left\{2\log 2a - \frac{(a+b)}{b}\log(a+b) + \frac{(a-b)}{b}\log(a-b)\right\}^*.$$

This is the theoretical basis of the experiment by which Cavendish demonstrated the law of the inverse square.

The experiment is given in great detail in the second edition of Maxwell's *Electricity and Magnetism*, pp. 76–82; and it appears, from what is there stated, that we may regard it as absolutely demonstrated that the arithmetical value of q cannot exceed $\frac{1}{21600}$.

Lines of Force.

99.] The state of the electric field under any given distribution of charges and arrangement of conductors is completely known when the value of the potential at each point of the field has been determined. It is obvious however that the direct subject of experimental investigation in any case must be the magnitude and direction of the force at any point of the field, and hence has

* See *Senate House Questions*, 1877.

arisen the conception of *lines*, *tubes*, and *fluxes* of force, originally suggested by Faraday and developed by subsequent writers.

Line of Force. Suppose a sphere of indefinitely small radius to be charged with unit mass of positive electricity and placed with its centre at any given point P in an electric field, and suppose the electrical distribution of the rest of the field to be unaffected by the presence of this charged sphere, and suppose further the inertia of the sphere to be always neglected, then the centre of the small sphere would move through the field under the action of the electric forces of the field in a definite line, generally curved, this line is defined as the *line of force in the field through P.*

When the electricity of the field consists of an electrified mass of very small volume, inclosing a point O and therefore all sensibly situated at the point O, the lines of force are clearly straight lines *radiating from* O if the charge at O be positive, and *terminating in* O if the charge at O be negative.

If the point O moved off to an infinite distance, and the charge at O were infinitely increased, the field would become what is called a uniform field, and the lines of force would be parallel straight lines.

So also if the distribution consisted of an infinite plane with a charge of uniform density over its surface, the lines of force would be parallel straight lines normal to the plane and proceeding from or towards that plane, according as the density thereon was positive or negative.

If the distribution were that of uniform density on the surface of an infinite circular cylinder, the lines of force would be in parallel planes perpendicular to the axis of the cylinder, radiating from or converging to the point in which that axis met each of these planes according as the electrification of the cylinder was positive or negative.

For less simple cases of distribution the lines of force are not capable of any such immediate determination; they are generally curved lines, their direction at every point coinciding with the normal to the equipotential surface through that point and proceeding towards the region of lower potential. It follows that no line of force can be drawn between points at the same potential, and that all lines of force in the immediate neighbourhood of an electrical particle, i. e. a very small volume with a charge of infinite density, must radiate from or to the point with which that volume sensibly coincides, according as the density of the charge is positive or negative, because the potential in the immediate neighbourhood of such point is positive or negative infinity in the respective cases.

100.] *Tubes of Force.* A region of space in the field bounded laterally by lines of force, as above described, is called a *tube of force.* See Fig. 6.

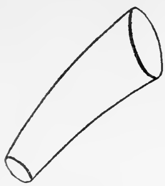

Fig. 6.

When the transverse section of the region is indefinitely small it is called an *elementary tube of force.*

Flux of Force. Suppose any transverse section dS made through any point P in the surface of an elementary tube of force, as in the figure, the angle between the normal to dS and the bounding lines of force being i. If the intensity of the force at dS be F, and the area of the orthogonal section of the tube at the point P be a, the force resolved perpendicular to dS will be $F\cos i$, and if this be denoted by F_n, the product $F_n\,dS$ will be equal to $F\,dS\cos i$, or Fa, and will be the same for every transverse section of the tube in the neighbourhood of dS.

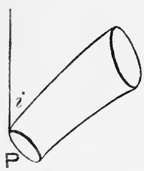

Fig. 7.

This product, from its analogy to the flux of a fluid flowing through a small tube with velocity $u=F$, is called the *flux of force* across dS; the limiting value of the ratio of the flux of force across any elementary area to the area is the intensity of the force in the field at that elementary area and perpendicular to it.

When the distribution arises from a so-called charged particle, the tubes of force are conical surfaces with their vertex at the particle; when in a uniform field they are surfaces limited laterally by parallel straight lines, and so forth.

101.] Let a charge of electricity of either kind, and with mass numerically equal to m, be situated at a given point O. Let a sphere of any radius be described about O as centre. Then the fluxes of force across all equal elementary areas of the sphere's surface will be equal to one another, and will take place from within outwards, or from without inwards, according as the electricity at O is positive or negative, the total flux over the whole sphere being $4\pi m$.

Faraday regarded the charge at O as a *source* from which, or a *sink* towards which, lines of force proceed symmetrically in all directions, and he further regarded the density of these lines of force, or the number contained in each unit of solid angle at O, as proportional to m. The number of lines of force therefore, which, in this view, traverse any surface, corresponds to the flux of force across that surface, and the force in any given direction at a point P in the field is the limiting value of the ratio which the number of lines traversing a small plane at P perpendicular to the given direction bears to the area of that plane when the latter is indefinitely diminished.

If the point O were eccentric, the equality of flux over all equal elementary areas would no longer be maintained, but the flux over the whole surface would, as we know from Art. 45, or as would result at once from the equality of flux over every transverse section at any point of an elementary tube of force, proved in Art. 100, still remain equal to $4\pi m$. We know also from Art. 45, or we might prove at once from Art. 100, that the total flux across a closed surface of any form surrounding O would be $4\pi m$.

If there were any number of sources or sinks within the closed surface, the traversing flux across the whole surface from each such source or sink would be $4\pi m$, where m is the numerical value of the charge at such source or sink, and the flux is outwards or inwards according to the sign.

The total flux in this case across the inclosing surface would be $4\pi\,(\Sigma p \sim \Sigma n)$, where Σp and Σn are the sums of the charges of the sources and sinks respectively, and would be outwards or inwards according as Σp was greater or less than Σn.

If there were any number of sources or sinks in the field external to the aforesaid surface, their existence would not affect the value of the total flux across the whole surface.

102.] Suppose that a tube of force, elementary or otherwise, in any electric field, is limited by transverse surfaces S and S', and that it contains electrical distributions, such that the difference of the sums of the masses of the positive and negative charges is m, then the flux of force across the whole surface of the tube thus closed from within outwards will exceed that from without inwards by the quantity $4\pi m$ if the preponderating included electricity be positive, and the former flux will fall short of the latter by $4\pi m$ if the preponderating electricity be negative.

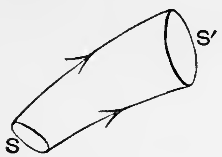

Fig. 8.

But the flux of force across that portion of the tube's surface which contains the lines of force is zero. If therefore the direction of the lines of force be from S to S' (see Fig. 8), the flux of force across S' will exceed or fall short of that across S by the quantity $4\pi m$, according to the sign of the preponderating included electricity.

If F and F' be the forces normal to S and S' at any points in them respectively, and if m be now taken to represent the *algebraical* sum of the included electricity, these statements are expressed by the equation

$$\iint F' dS' - \iint F\, dS = 4\pi m.$$

The portions of any surfaces in an electric field intercepted by the same tube of force are called *corresponding surfaces*, and therefore in proceeding along any tube of force, finite or elementary, the fluxes across corresponding surfaces are continually increased by the quantity $4\pi m$, where m is the algebraic sum of the electricities included in the tube in its passage from any one surface to any other, such increase being a numerical decrease when m is negative. And if there is no such included electricity, or if its algebraic sum is zero, then the fluxes across the corresponding surfaces are all equal to one another.

103.] Suppose that there is in the field a surface S charged with electricity, the density at any point P being σ.

Let dS be an element of S about the point P, and conceive a small cylinder to be drawn with its generating lines passing through the contour of dS and perpendicular to that element.

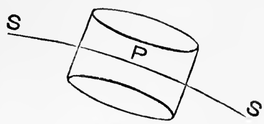

Fig. 9.

The total flux across this cylinder must be equal to the included electricity, i.e. to σdS.

Also, if the length of the cylinder's axis be indefinitely diminished, the flux across the curved surface will become infinitely less than either of the fluxes across the bounding planes, and these fluxes therefore must ultimately differ from one another by $4\pi\sigma dS$, so that if N and N' be the forces in the field normal to dS and on opposite sides of it, we have

$$N'dS - NdS = 4\pi\sigma dS, \quad \text{or} \quad N' - N = 4\pi\sigma.$$

Hence the force normal to an electrified surface changes suddenly in value by the quantity $4\pi\sigma$ in passing from one side of the surface to the other; and we may also prove that the normal force upon the electrified element of the surface itself is the arithmetic mean of the normal forces which would act on that element if placed first on one side and then on the other of the surface. For, considering the elementary cylinder above mentioned, it is clear that the force arising from all the electricity in the field, besides that on the element dS, must be continuous throughout the cylinder, inasmuch as all the electricity from which it arises is without the cylinder, and therefore the normal force throughout the cylinder arising from that external electricity will be ultimately the same as it is at the surface. But the normal force arising from the charge on the included element σdS on points at any equal small distances from the surface and on opposite sides must be equal and opposite, and therefore the sum of the total normal forces on either side of the surface must be equal to twice the normal force of the external electricity throughout the cylinder; or the normal force of the external electricity at the surface must be the arithmetic mean of the total normal forces on opposite sides of the surface; and therefore the normal force on the elementary charge σdS is the arithmetic mean of what the normal forces on the same charge would be if placed on each of the two sides of the surface respectively, for the charge σdS can exert no force upon itself.

It is clear also that the charge σdS can exert no tangential force in one direction rather than another, and therefore the force resolved tangentially must be the same on either side of the surface. If therefore F and F' be the forces on opposite sides of the surface, and if i and i' be the angles between the lines of force and the surface normal, we have

$$F' \cos i' = F \cos i + 4\pi\sigma,$$

$$F' \sin i' = F \sin i\,;$$

and therefore $$\tan i = \tan i' \left(1 + \frac{4\pi\sigma}{F\cos i}\right);$$

or the lines of force on traversing a surface with superficial electric density σ are deflected towards or from the normal according as σ is positive or negative; see Fig. 10.

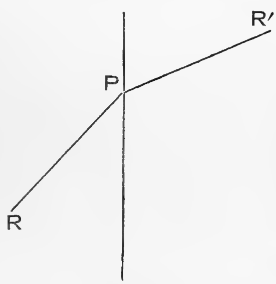

Fig. 10.

It appears also from the foregoing that the force exerted by an element dS of a surface of superficial density σ at points very close to dS is a normal force $2\pi\sigma$, and repulsive or attractive according as σ is + or −.

104.] We may now trace the possible course of an elementary tube of force through an electric field in equilibrium.

The axis of such a tube in passing through any point P must proceed from P towards regions of continually diminishing potential.

It may then pass on to an infinite distance if it encounters no free electricity.

Or it may traverse a charged surface, in which case, if the transit be oblique, it will be bent through a finite angle at the surface in the manner above explained.

If this charged surface be that of a conductor, the line, or rather elementary tube, of force will proceed no further, but it will be, so to speak, quenched in the *sink* afforded by the negative density of the surface at the point or element in which it meets it.

Or it may traverse a region of finite volume density, in which case it suffers no abrupt refraction, and if the density of the region be positive the tube emerges therefrom with augmented flux, if the density be negative the tube may be, as in the case of the conductor, quenched in the sink thus afforded and proceed no further.

If the tube, elementary or finite, has emerged from a positively charged conducting surface and is quenched, as above described, in another conducting surface without traversing any region of electric charge, then the positive charge on that portion of the surface of emersion contained within the tube must be equal in magnitude to the negative charge on the corresponding surface of the surface of reception; or, in the language of Faraday, the number of lines of force emanating from the source is equal to those quenched in the sink.

In other words, the number of lines of force emanating from or converging to an elementary area of any conducting surface is a measure of the positive or negative density of the electrification of that surface.

CHAPTER VI.

APPLICATION TO PARTICULAR CASES.

ARTICLE 105.] IT is proved above that whatever be the given charges or potentials on a system of conductors, combined with any fixed distribution of electricity in space, there exists always one, and only one, mode of distribution upon the conductors consistent with equilibrium.

But the actual solution of the problem, the determination, that is, of the actual density of electricity at a point of any given conductor, is one of great difficulty, and has only been achieved in a few simple and comparatively easy cases.

Case of an infinite conducting plane and an electrified point. Let there be an infinite conducting plane, and a unit of positive electricity fixed at a point O above it. It is required to find the density at any point in the plane in order that the potential of the plane may be everywhere zero.

The potential of the required distribution on the plane must be equal and opposite to that of the unit at O at all points on the plane, and therefore also at all points in space on the opposite side of the plane to O, by Art. 60.

If a unit of negative electricity were placed at O', the optical image of O, formed with respect to the plane as a mirror, its potential at any point of the plane would be equal and opposite to that of the unit at O, and therefore equal to that of the required distribution. It would therefore also be equal to that of the required distribution at all points in space on the same side of the plane as O.

Let V be the potential of the required distribution, and of the unit at O. Then, by Poisson's equation, the density of the distribution at any point P in the plane is

$$\sigma = -\frac{1}{4\pi}\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right),$$

where $d\nu$ is an element of the normal to the plane measured from the plane on the same side of the plane as O, and $d\nu'$ the same thing on the same side as O'.

Now the value of V at any point P on the same side of the plane as O is

$$\frac{1}{OP} - \frac{1}{O'P},$$

and on the opposite side of the plane V is constant because it is constant over the plane, and there is no electrification on that side of the plane.

Therefore $$\frac{dV}{d\nu'} = 0.$$

Also on the plane $$\frac{1}{O'P} = -\frac{1}{OP},$$

and $$-4\pi\sigma = \frac{dV}{d\nu} = \frac{d}{d\nu}\left\{\frac{1}{OP} - \frac{1}{O'P}\right\}$$

$$= 2\frac{d}{d\nu}\frac{1}{OP};$$

therefore $$\sigma = -\frac{1}{2\pi}\frac{d}{d\nu}\frac{1}{OP};$$

and if h be the distance of O from the plane, r the distance of a point P in the plane from the intersection of OO' with the plane,

$$\sigma = \frac{1}{2\pi}\frac{d}{dh}\frac{1}{\sqrt{h^2+r^2}}$$

$$= -\frac{1}{2\pi}\frac{h}{OP^3},$$

which determines the density at any point in the plane.

106.] In certain very simple cases the value of V may be determined by the integration of Laplace's equation. For instance—*Two infinite conducting planes at given potentials.* Let the planes be parallel to the plane of xy. Then, since the density is uniform throughout each plane, V is in this case a function of z only, and Laplace's equation becomes $\frac{d^2V}{dz^2} = 0$, from which V can be found with two arbitrary constants, and the constants are to be determined by the given conditions on the planes.

(1) *Two infinite coaxal cylinders.*

In like manner, if we have two infinite coaxal conducting cylinders at given potentials, the density is uniform throughout the surface of each cylinder, and V is a function of r, the distance from the axis. Laplace's equation is in this case

$$\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0,$$

which admits of integration.

(2) *Two concentric spheres.*

Again, if there be two concentric conducting spheres at given potentials, the density is uniform throughout the surface of each sphere, and V is a function of r, the distance from the centre. Laplace's equation becomes in this case

$$\frac{d^2 V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0,$$

which admits of integration.

In this problem, as in the last, the two arbitrary constants which enter into V in solving the differential equation must be determined with reference to the given conditions on the cylinders or spheres.

107.] *Case of an insulated Conducting Sphere in a Field of Uniform Force.*

Let us take the direction of the force for axis of x. Let X be the force, a the radius of the sphere, V the potential. Then V must satisfy the conditions,

(1) V is constant and $= C$ on the surface of the sphere;

(2) $\nabla^2 V = 0$ at all points outside of it;

(3) $V = -Xx + C$ at a sufficiently great distance from the sphere;

(4) The total electrification on the sphere is zero.

The function

$$V = -Xx.\left\{1 - \frac{a^3}{r^3}\right\} + C,$$

where r is the distance of any point from the centre, satisfies all these conditions.

The density on the sphere is $-\frac{1}{4\pi}\frac{dV}{dr}$, that is, $\frac{3Xx}{4\pi a}\cdot$

It is easily seen that

$$\iint \frac{3Xx}{4\pi a}dS = 0.$$

108.] *Case of an uninsulated Conducting Sphere and another Sphere outside of it uniformly filled with electricity of density ρ.*

This is the same problem as that treated in Chap. III, Art. 65. We give another method of solution.

Let C be the centre, a the radius, of the conducting sphere; and let O be the centre, b the radius, of the other sphere.

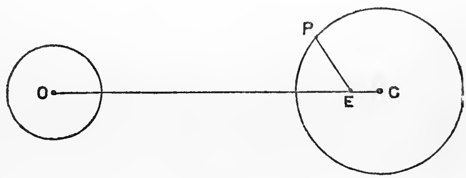

Fig. 11.

Let $OC = f$. Let V be the potential of the whole system.

It is required to find the density of the induced distribution on the conducting sphere which gives zero potential on that sphere, and the general value of V in this case.

V has to satisfy the conditions,

(1) $V = 0$ at all points on the conducting sphere;

(2) $\nabla^2 V = 0$ at all points external to both spheres;

(3) $\nabla^2 V + 4\pi\rho = 0$ within the non-conducting sphere.

Take a point E in CO such that $EC = \frac{a^2}{f}\cdot$

Let $OP = r$, $EP = r'$, where P is any point.

Let V_0 be the potential of the charged sphere at P. Then if $e = \frac{4\pi}{3}b^3\rho$, or the total charge of electricity in the charged sphere, the function

$$V = V_0 - e\frac{a}{fr'}$$

satisfies all the conditions, and must therefore be the required potential.

For outside of the charged sphere $V_0 = \frac{e}{r}$, and therefore the above equation becomes

$$V = e\left\{\frac{1}{r} - \frac{a}{fr'}\right\}.$$

Now by a known property of the sphere, if the point P be on its surface,

$$\frac{EP}{OP} = \frac{r'}{r} = \frac{a}{f}.$$

Therefore for a point on the conducting sphere $V = 0$.

Also for a point outside of both spheres

$$\nabla^2 \frac{1}{r} = 0 \text{ and } \nabla^2 \frac{1}{r'} = 0;$$

therefore $\nabla^2 V = 0$.

For a point inside of the charged sphere

$$\nabla^2 V + 4\pi\rho = \nabla^2 V_0 + 4\pi\rho = 0.$$

The density at any point on the conducting sphere is

$$\sigma = -\frac{e}{4\pi}\frac{d}{d\nu}\left\{\frac{1}{r} - \frac{a}{fr'}\right\}.$$

Also $$r^2 = f^2 + \nu^2 - 2f\nu\cos\theta,$$

where the angle $PCO = \theta$, and ν denotes the distance of a point from C; also

$$r'^2 = \frac{a^4}{f^2} + \nu^2 - \frac{2a^2}{f}\nu\cos\theta;$$

and in the expression for σ, ν is to be made equal to a after differentiation. We have therefore

$$\frac{dr}{d\nu} = \frac{a - f\cos\theta}{r},$$

$$\frac{dr'}{d\nu} = \frac{a - \frac{a^2}{f}\cos\theta,}{r'}$$

$$\sigma = \frac{e}{4\pi}\left\{\frac{a - f\cos\theta}{r^3} - \frac{a}{f}\cdot\frac{a - \frac{a^2}{f}\cos\theta}{r'^3}\right\},$$

but $$r' = \frac{a}{f}r.$$

Therefore $$\sigma = -\frac{e}{4\pi}\frac{f^2 - a^2}{ar^3},$$ as already found.

From the form of the potential function

$$V = \frac{e}{r} - \frac{ae}{fr'}$$

it follows that the potential of the induced electricity on the conducting sphere in the presence of the charge e at the external point O is the same as that of the charge $-\frac{ae}{f}$ at the point E.

The point E is called *the electrical image* of O in the conducting sphere.

109.] *Case of an infinitely long Conducting Cylinder, and a uniform distribution of Electricity throughout the substance of another infinitely long cylinder outside of the former one, and whose axis is parallel to that of the former one.*

In this problem V has to satisfy the following conditions, viz.

(1) $V = 0$ on the surface of the conducting cylinder;

(2) $\nabla^2 V = 0$ outside of both cylinders;

(3) $\nabla^2 V + 4\pi\rho = 0$ inside of the charged cylinder, ρ being the density of the distribution within it.

Let a plane perpendicular to the axis cut the axis of the conducting cylinder in C, that of the charged cylinder in O. (See Fig. 1 of last example.)

Let $OC = f$.

In OC take a point E such that $EC = \frac{a^2}{f}$.

Let r be the distance of any point P from the axis of the cylinder through O, r' its distance from a line parallel to the axis through E. Then $\frac{r'}{r} = \frac{a}{f}$ for every point in the section made by the plane with the conducting cylinder.

Let R be the quantity of electricity contained in unit length of the charged cylinder. Then the potential of the charged cylinder at any point outside of it is

$$V_0 = C - 2R\log r.$$

It will be found that

$$V = V_0 - C + 2R\log\frac{f}{a}r'$$

$$= 2R\log\frac{fr'}{ar}$$

satisfies all the conditions, and is therefore the potential. For in this case r is independent of z, and therefore

$$\nabla^2 \log r = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \log r.$$

Now
$$\frac{d}{dx} \log r = \frac{x}{r^2},$$

$$\frac{d^2}{dx^2} \log r = \frac{1}{r^2} - \frac{2x^2}{r^4}.$$

Similarly
$$\frac{d^2}{dy^2} \log r = \frac{1}{r^2} - \frac{2y^2}{r^4},$$

whence
$$\nabla^2 \log r = 0.$$

On the conducting cylinder $r = \frac{f}{a} r'$; and therefore

$$V = V_0 - C + 2R \log \frac{f}{a} r' = 2R \log \frac{fr'}{ar} = 0.$$

Outside of both cylinders

$$\nabla^2 V_0 = 0, \quad \text{and} \quad \nabla^2 \log \frac{fr'}{ar} = 0, \quad \text{therefore} \quad \nabla^2 V = 0;$$

and within the charged cylinder

$$\nabla^2 V + 4\pi\rho = \nabla^2 V_0 + 4\pi\rho = 0.$$

The density at any point on the conducting cylinder is found from

$$\sigma = + \frac{R}{2\pi} \left\{ \frac{d}{dv} \log r - \frac{d}{dv} \log r' \right\},$$

where
$$r^2 = f^2 + v^2 - 2fv\cos\theta,$$

$$r'^2 = \frac{a^4}{f^2} + v^2 - 2\frac{a^2}{f} v \cos\theta;$$

and v is to be made equal to a after differentiation. The result is

$$\sigma = - \frac{R}{2\pi a} \cdot \frac{f^2 - a^2}{r^2}.$$

110.] *On Electric Images.* We have seen in Art. 108 that if a sphere be at zero potential under the influence of a charged point outside of it, the induced distribution has at all external points the same potential as that due to a certain charge placed at a point within the sphere, and the last-mentioned charged point is defined to be *the image* of the influencing point in the sphere. An infinite plane is for this purpose a particular case of the sphere.

Every electrical system outside of a sphere, inasmuch as it may be regarded as consisting of a number of charged points, is represented by a series of images in the sphere, and these together may be said to form *the image of the external system.* In like manner, if the sphere be at zero potential under the influence of a charged point within it, the induced distribution has the same potential at all internal points as that due to a certain charge at a certain point without the sphere. The external point is called *the image of the internal point.* Every electrified system within the sphere has its image outside of the sphere.

It can easily be shewn that no closed surface except a sphere or infinite plane generally gives rise to an image.

For let S be any uninsulated closed surface, and let E be an external point at which a charge e is placed. If the induced distribution on S have at all points on S the same potential as that of a charge e' at a point F within S, that is, if F be an image of E within S, we must have

$$\frac{EP}{FP} = \frac{e}{e'},$$

P being any point on S. Thus the locus of P is a sphere, that is, S is a sphere.

111.] By the method of electric images many problems relating to the distribution of electricity on spherical or plane surfaces can be solved.

The case of two spheres cutting each other orthogonally (Maxwell's *Electricity and Magnetism*, p. 168).

Let C_1, C_2 be the centres, a_1, a_2 the radii of the spheres.

Let $C_1 C_2 = f$.

Let AB represent the circle of intersection, E the point in which the line $C_1 C_2$ intersects the plane of that circle.

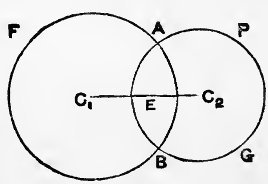

Fig. 12.

Then $C_1 A C_2$, $C_1 B C_2$ are right angles, and

$$f^2 = a_1^2 + a_2^2.$$

Also
$$C_1 E = \frac{a_1^2}{f}, \quad C_2 E = \frac{a_2^2}{f};$$

or E is the image of C_2 in the sphere C_1, and the image of C_1 in the sphere C_2.

If therefore we place at C_1 a quantity of electricity a_1, at C_2 a quantity a_2, and at E a quantity

$$-\frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}},$$

the potential at any point on either sphere will be unity, because if the point be, for instance, on the sphere C_2, the two charges,

$$a_1 \text{ at } C_1 \quad \text{and} \quad -\frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}} \text{ at } E,$$

have together zero potential at each point on that sphere, while the charge a_2 at C_2 has potential unity.

112.] Now let us consider the conductor *FAGB*, formed by the two external segments of the spheres. The aggregate of a distribution upon its surface, which gives unit potential at all points on it, is equal to

$$a_1+a_2-\frac{a_2 a_2}{\sqrt{a_1^2+a_2^2}}, \text{ by Art. 60.}$$

This then is the capacity of the conductor.

Again, since the potential of that distribution is the same at all external points as that of the three charges at C_1, C_2, and E, its density is

$$-\frac{1}{4\pi}\frac{dV}{d\nu},$$

where
$$V=\frac{a_1}{C_1P}+\frac{a_2}{C_2P}-\frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}}\frac{1}{EP}.$$

But if P be on the sphere C_2,

$$\frac{1}{4\pi}\frac{d}{d\nu}\left\{\frac{a_1}{C_1P}-\frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}}\frac{1}{EP}\right\}=-\frac{a_1}{4\pi a_2}\frac{f^2-a_2^2}{C_1P^3}, \text{ by Art. 108,}$$
$$=-\frac{1}{4\pi a_2}\cdot\frac{a_1^3}{C_1P^3};$$

and therefore the density is

$$\frac{1}{4\pi a_2}\left\{1-\frac{a_1^3}{C_1P^3}\right\}.$$

By symmetry the density at a point on the sphere C_1 is

$$\frac{1}{4\pi a_1}\left\{1-\frac{a_2^3}{C_2P^3}\right\}.$$

Instead of the figure formed by the two external segments, we may take the lens formed by the two internal segments, or the meniscus formed by one internal and one external segment, and calculate the superficial density in the same way. We shall consider this problem further when we come to the Theory of Inversion.

113.] Another interesting example is afforded by the following question:

An uninsulated conductor $ADEFB$ consists of an infinite plane with a hemispherical projection DEF, the centre C of the hemisphere being in the plane AB. A mass of electricity m is situated at the point m, in the radius CE produced, where CE is perpendicular to the plane. Then if the points m_1 and m_1' be taken on opposite sides of C such that

$$Cm_1 = Cm_1' = \frac{CE^2}{Cm},$$

and if m' be taken on mC produced such that $Cm' = Cm$, the effect of the induced charge on the conductor under the influence of the mass m at m may be represented by the joint effect of the masses $-m_1$ at m_1, $+m_1$ at m_1' and $-m$ at m', where $m_1 = \frac{CE}{Cm'}m$*.

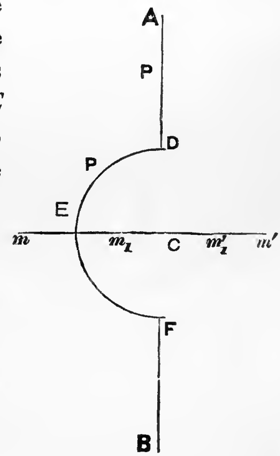

Fig. 13.

For let P be any point on the same side of the conductor as m, and let the distances of P from m, m_1, m_1', and m' be r, r_1, r_1' and r' respectively.

Let
$$V = \frac{m}{r} - \frac{m_1}{r_1} + \frac{m_1}{r_1'} - \frac{m}{r'}.$$

Then at all points on the hemispherical surface we have

$$\frac{m}{r} = \frac{m_1}{r_1} \quad \text{a d} \quad \frac{m}{r'} = \frac{m_1}{r_1'},$$

and therefore $V = 0$ over that surface.

* Or in other words, the induced charge on the composite conductor is equivalent to the image m_1 of the charge m at the electrical image of m in the hemisphere together with the image of m and m_1 in the plane.

Similarly at all points of the plane's surface

$$r = r' \quad \text{and} \quad r_1 = r_1',$$

and therefore $V = 0$ over that surface.

Again, $$\nabla^2 \frac{m}{r} = \nabla^2 \frac{m_1}{r_1} = \nabla^2 \frac{m_1'}{r_1'} = \nabla^2 \frac{m'}{r'} = 0$$

at all points on the same side of the plane as m, where there is no electricity, or where ρ the electrical density is zero, and at m

$$\nabla^2 \frac{m}{r} = -4\pi\rho,$$

where ρ is the density within the small volume of m at m.

Therefore at all points on the aforesaid side of the plane

$$\nabla^2 V + 4\pi\rho = 0.$$

Therefore the function V taken as above satisfies the superficial and solid conditions of the potential of m at m, and the induced charge on the conductor, and must therefore be the potential of m and that induced charge.

In other words, the induced charge produces at all points on the side of m the same effect as the charges $-m_1$, m_1 and $-m$ at the points m_1, m_1' and m' respectively.

From the equation

$$4\pi\sigma + \frac{dV}{d\nu} = 0,$$

we easily find that the superficial density σ of the induced charge is everywhere negative, except at the circle of intersection of the hemisphere and plane, where it is zero, and that at any point P on the hemisphere σ is proportional to

$$\frac{1}{mP^3} - \frac{1}{m'P^3},$$

and at any point P on the plane outside of the hemisphere σ is proportional to

$$\frac{Cm^3}{mP^3} - \frac{CE^3}{m_1P^3}.$$

114.] *On Systems of Successive Images.*

If we have given any two conducting spheres, including in that designation an infinite plane, at zero potential under the influence of an electrified point, the electrical distribution on either sphere will be found to be equivalent in its effects to two infinite series of images, the magnitudes or values of which converge. Hence the density of the actual distribution on either sphere generally admits of being calculated approximately.

For instance, let us consider a sphere and infinite plane not intersecting, and an influencing point in the perpendicular from the centre of the sphere on the plane, between the centre and the plane.

Let A be the centre of the sphere, c its radius, E the influencing point at which the charge e is placed, B and H the points in which AEA' cuts the sphere and plane respectively.

Let $AH = h$, $HB' = HB$, $HA' = \sqrt{h^2 - c^2}$.

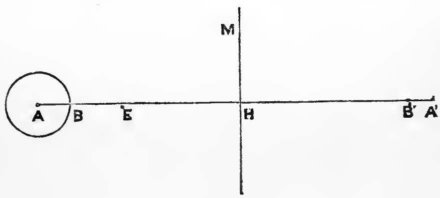

Fig. 14.

The charge at E produces on the sphere a certain distribution, which we may call the primary distribution on the sphere, the effect of which at all points outside of the sphere is the same as that of a charge $-\frac{c}{AE}e$ placed at a point between A and E, whose distance from A is $\frac{c}{AE}$, and its distance from H is $h - \frac{c^2}{AE}$. That produces on the plane a distribution whose density we may denote by ρ_1; and the effect of this distribution over the plane at all points on the left side of the plane is the same as that of its image, namely, a charge $\frac{c}{AE}e$ placed at a point distant $h - \frac{c^2}{AE}$ to the right of the plane.

Let
$$x_1 = h - \frac{c^2}{AE}.$$

From this distribution, or its equivalent image, we derive in the same way a second distribution on the sphere equivalent to a charge

$$-\frac{c}{AE} \cdot \frac{c}{h + x_1} e$$

at a point distant $\frac{c^2}{h+x_1}$ from A, and from this again a second distribution of density ρ_2 on the plane.

We shall then have a series of images to the right of the plane, whose distances from H are x_1, x_2, &c. And

$$x_1 = h - \frac{c^2}{AE},$$

$$x_2 = h - \frac{c^2}{h+x_1}, \text{ \&c., \&c.,}$$

and generally
$$x_{n+1} = h - \frac{c^2}{h+x_n}, \text{ \&c.}$$

It is easily seen that $x_{n+1} > x_n$, and every x is less than $\sqrt{h^2-c^2}$. The successive images continually approach A'.

The charges at these images are successively

$$\frac{c}{AE}e, \quad \frac{c}{AE}\cdot\frac{c}{h+x_1}e, \quad \frac{c}{AE}\cdot\frac{c}{h+x_1}\cdot\frac{c}{h+x_2}e, \text{ \&c.};$$

and the ratio between two successive charges continually approaches $\frac{c}{AA'}$.

Again, the charge at E induces on the plane a primary distribution which is equivalent to the image of E in the plane. This original image is at a point distant from H, $x'_1 = HE$, and the distances from H of the derived images are

$$x'_2 = h - \frac{c^2}{h+x'_1},$$

$$x'_3 = h - \frac{c^2}{h+x'_2},$$

&c., &c.

$$x'_{n+1} = h - \frac{c^2}{h+x'_n},$$

which continually approach HA'. The charges at these images are successively

$-e$ at the first image,

$-\frac{c}{h+x'_1}e$ at the second image,

and so on.

Hence the density of the induced distribution at any point M on the plane, where $HM = r$, is

$$\frac{e}{2\pi}\frac{c}{AE}\left\{\frac{x_1}{(r^2+x_1^2)^{\frac{3}{2}}}+\frac{c}{(h+x_1)}\cdot\frac{x_2}{(r^2+x_2^2)^{\frac{3}{2}}}+\&c.\right\},$$
$$-\frac{e}{2\pi}\left\{\frac{x'_1}{(r^2+x'^2_1)^{\frac{3}{2}}}+\frac{c}{h+x'_1}\cdot\frac{x'_2}{(r^2+x'^2_2)^{\frac{3}{2}}}+\&c.\right\}.$$

Each series converges rapidly, and the terms soon cease to differ sensibly from those of a geometric series whose common ratio is $\frac{c}{AA'}$. Hence the actual density at M can be calculated to any required degree of accuracy.

The integral charge on the plane is the sum of both series of images irrespective of their position. That is

$$e\frac{c}{AE}\left\{1+\frac{c}{h+x_1}+\frac{c^2}{(h+x_1)(h+x_2)}+\&c.\right\},$$
$$-e\left\{1+\frac{c}{h+x'_1}+\frac{c^2}{(h+x'_1)(h+x'_2)}+\&c.\right\}.$$

115.] Another very interesting case is that of two concentric spheres and an electrified point placed between them, treated in Maxwell's *Electricity*.

In that case the distances of the images from the common centre are in geometrical progression. Also the charges are in geometrical progression, and their sum can be accurately determined.

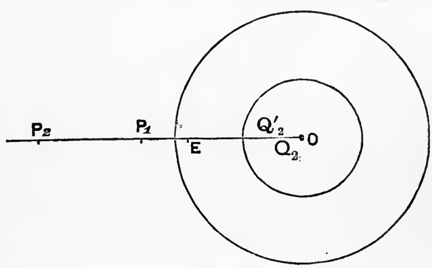

Fig. 15.

Let O be the common centre, a the radius of the inner sphere, b the radius of the outer sphere, E the point where the charge e is placed, $OE = h$; all the images are in the line OE produced.

We have then

an image at P_1, where $OP_1 = \frac{b^2}{h}$,

an image at Q_1, where $OQ_1 = \frac{a^2}{OP_1} = \frac{a^2 h}{b^2}$,

an image at P_2, where $OP_2 = \frac{b^4}{a^2 h}$,

an image at Q_2, where $OQ_2 = \frac{a^2}{OP_2} = \frac{a^4}{b^4}h.$

We see then that the distances from O of the successive images derived from the primary distribution on the outer sphere are

$$\frac{a^2}{b^2}h,\quad \frac{a^4}{b^4}h,\quad \&c.$$

and the charges at those images are

$$\frac{a}{b}e,\quad \frac{a^2}{b^2}e,\quad \&c.$$

Again, if we start with the primary distribution on the inner sphere, represented by $\frac{a}{h}e$ at Q_1', we obtain a second series of images whose distances from O are

$$\frac{a^2}{h},\quad \frac{a^4}{b^2h},\quad \frac{a^6}{b^4h},\quad \&c.$$

and whose values are

$$-\frac{a}{h}e,\quad -\frac{a^2}{bh}e,\quad -\frac{a^3}{b^2h},\quad \&c.$$

Hence the total charge on the inner sphere, or the sum of the images within it, is

$$\left(\frac{a}{b-a} - \frac{ab}{h(b-a)}\right)e,$$

or

$$\frac{h-b}{h}\cdot\frac{a}{b-a}e,$$

and that on the outer sphere is

$$-e - \frac{h-b}{h}\,\frac{a}{b-a}e = -\frac{h-a}{h}\,\frac{b}{b-a}e.$$

116.] Another class of cases is that in which the number of images is finite.

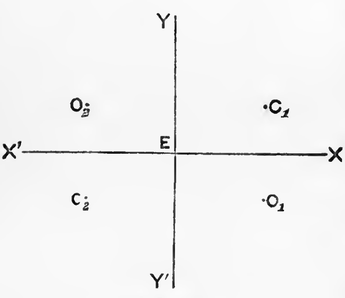

Fig. 16.

For instance, let us consider two infinite conducting planes at right angles to each other, in presence of an electrified point.

Let the projections of the planes on the plane of the paper be XEX', YEY', and let O_1 be an electrified point.

The image of O_1 in XEX' is C_1.
The image of C_1 in YEY' is O_2.
The image of O_2 in XEX' is C_2.
The image of C_2 in YEY' is O_1.

The two planes are at zero potential under the influence of $+e$ at O_1 and O_2, and $-e$ at C_1 and C_2. If we now substitute for C_1 and O_2 the distributions on XEX' which have the same effect with them on the lower side of the plane XEX', and for C_2 the distribution representing it in YEY', the potential will still be zero on the plane YEY' and on the part EX of the plane XEX'.

117.] In like manner instead of two planes intersecting at right angles, we may have n planes intersecting at angles $\frac{\pi}{n}$ (Maxwell's *Electricity*, p. 165). Taking an electrified point O_1, and forming successive images in the planes, we shall have a series of positive points $O_1, O_2, \dots O_n$, and a series of negative points $C_1, C_2, \dots C_n$ placed symmetrically round E, the projection of the common section on the plane of the paper. The potential is zero on every plane.

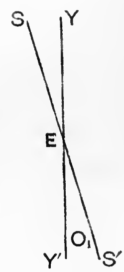

Fig. 17.

If YEY' and SES' be the two planes between which the point O_1 lies, we may substitute for all the points on the left of YEY' their corresponding distribution on YEY', and for all the remaining points except O_1 itself their corresponding distribution on SES'. Then the potential on the portion $Y'ES$ of the system is unaffected, and remains zero.

CHAPTER VII.

THE THEORY OF INVERSION AS APPLIED TO ELECTRICAL PROBLEMS.

ARTICLE 118.] THE solution of some electrical problems involving spherical surfaces, or portions of spherical surfaces including planes, can be effected by the method of inversion. This application of inversion is due originally to Sir W. Thomson.

Taking for origin any point O, and for coordinates the usual spherical coordinates r, θ, ϕ, let us suppose we have found the solution of a given electrical problem, that is, we have found the single function, V, of r, θ, ϕ, which is constant within each conductor of the system, and satisfies the characteristic equations at all external points, and vanishes at an infinite distance, and hence we have found the density at every point on any conductor.

We will then invert the geometrical system as follows :—For any point P of the system distant r from O we will take a point P' in the line OP, and distant r' from O, where $r' = \frac{\kappa^2}{r}$, and κ is a constant line called the *radius of inversion*, and O is called the *centre of inversion*.

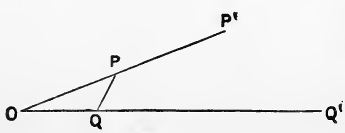

Fig. 18.

If P and Q be any two points in the original system, P' and Q' the points corresponding to them in the inverted system, the triangles POQ, $Q'OP'$ are similar, and therefore

$$P'Q' = \frac{\kappa^2}{OP\,OQ} \cdot PQ,$$

$$\text{or} \quad \frac{1}{P'Q'} = \frac{OP.OQ}{\kappa^2} \frac{1}{PQ}.$$

Again, if Q be very near P, and if $OP = r$,

$$P'Q' = \frac{\kappa^2}{r^2} PQ,$$

$$\frac{1}{P'Q'} = \frac{r^2}{\kappa^2} \frac{1}{PQ}.$$

Therefore any linear element of length dv in the original system acquires the length $\frac{\kappa^2}{r^2} dv$ in the inverted system. It follows that any element of area dA in the original system becomes $\frac{\kappa^4}{r^4} dA$ in the inverted system; and the element of volume dv in the original system becomes $\frac{\kappa^6}{r^6} dv$ in the inverted system.

119.] Every sphere in the original system becomes another sphere in the inverted system.

For let the point E be taken in the line joining O with the centre C of the sphere, such that

$$CE = \frac{a^2}{OC},$$

where a is the radius. Let $OC = f$. Then, if Q be a point on the sphere, $\frac{OQ}{EQ}$ is constant, and $= \frac{f}{a}$.

Therefore if E' and Q' be the points in the inverted system corresponding to E and Q,

$$E'Q' = \frac{\kappa^2}{OE\,OQ} \cdot EQ = \frac{\kappa^2 a}{f^2 - a^2} \quad \text{or} \quad \frac{\kappa^2 a}{a^2 - f^2},$$

according as the centre of inversion is without or within the original sphere, and in either case is constant. Therefore E' is the centre of a new sphere.

If the centre of inversion be without the sphere, and if

$$\kappa^2 = f^2 - a^2,$$

the sphere is unchanged in position. For in that case,

$$OE' = \frac{\kappa^2}{OE} = f.$$

That is, the centre of the new sphere coincides with C, the point which was the centre of the original sphere, and the radius of the new sphere, $\frac{\kappa^2 a}{f^2 - a^2}, = a$.

Again, a plane in the original system whose perpendicular distance from the centre of inversion is p, becomes a sphere of diameter $\frac{\kappa^2}{p}$ passing through the centre of inversion, and whose centre is in p.

Again, a sphere of radius a in the original system passing through the centre of inversion becomes when inverted an infinite plane at right angles to the diameter through the centre of inversion, and distant $\frac{\kappa^2}{2a}$ from that centre. Again, since two intersecting spheres becomes spheres when inverted, their common section becomes a circle. Hence every circle on the original sphere becomes a circle on the inverted sphere.

Again, since the triangles POQ, $Q'OP'$ are similar, if θ be the angle made by the radius r with any elementary line at its extremity, the corresponding angle in the new system is $\pi-\theta$.

Every point which in the original system is within any closed surface S, not enclosing the centre of inversion, will in the inverted system be within the corresponding closed surface S'. And every point without S will in the inverted system be without S'. But if S enclose the centre, all points within S correspond to points without S', and *vice versa*.

Evidently every conductor in the given electrical system will be represented in the inverted system by a certain closed surface.

120.] We will now construct upon the inverted system a new electrical system as follows, viz.

If
$$\rho r^2 \sin\theta\, d\theta\, d\phi\, dr$$
be the quantity of electricity in the space-element
$$r^2 \sin\theta\, d\theta\, d\phi\, dr$$
of the original system, we will place in the corresponding space element of the inverted system the quantity
$$\frac{\kappa}{r}\rho r^2 \sin\theta\, d\theta\, d\phi\, dr.$$
Since as we have seen the element of volume dv in the original system becomes $\frac{\kappa^6}{r^6}dv$ in the inverted system, it follows that the volume density in the new system is $\frac{r^5}{\kappa^5}\rho = \frac{\kappa^5}{r'^5}\rho$.

In like manner if σdA be the quantity of electricity on the surface element dA of the original system, we will place on the corresponding element dA' of the inverted system the quantity

$$\frac{\kappa}{r}\sigma\, dA.$$

This gives a surface density $\frac{r^3}{\kappa^3}\sigma$ in the new system.

We have thus constructed a new electrical system, in which every conductor S of the original system is represented geometrically by a surface S' in the new system, and every quantity of electricity in the original system is represented by a corresponding quantity in the new system.

121.] We now proceed to find the relation between the potential at any point Q of the original system and that at the corresponding point Q', due to the electricity which we have supposed placed on the inverted system.

Let s denote an element of volume at P in the original system, ρs the quantity of electricity in it. Then the potential at Q of the element is $v = \frac{\rho s}{PQ}$.

In the inverted system, ρs at P becomes $\frac{\kappa}{OP}\rho s$ at P'. And its potential at Q' is

$$v' = \frac{\kappa}{OP}\rho s \frac{1}{P'Q'};$$

but $$\frac{1}{P'Q'} = \frac{OP.OQ}{\kappa^2}\frac{1}{PQ};$$

whence $$v' = \frac{OQ}{\kappa}\frac{\rho s}{PQ} = \frac{OQ}{\kappa}v.$$

As this is independent of the position of P and P', it holds true for the whole potential of the original system at Q, and of the inverted system at Q'. That is, if V and V' denote the potentials at Q and Q',

$$V' = \frac{OQ}{\kappa}V = \frac{\kappa}{OQ'}V.$$

122.] It follows that if V be zero for any conductor whose bounding surface is S in the original system, V' is zero throughout the corresponding surface S' in the inverted system. Therefore if the space S' be occupied by a conductor, the assumed distribution of electricity throughout the inverted system will, as regards such conductor, be in equilibrium with zero potential. And if any electrical system consists of conductors all at zero potential in presence of fixed charges of electricity, the inverted system will also be in equilibrium with all its conductors at zero potential.

Again, let the original system be one in which the potential of a distribution over a closed surface S is equal at each point on S to that of any electrification enclosed within S. Then if we invert with respect to an external point, and S becomes S', the potential of the corresponding surface distribution over S' will be equal at each point of S' to that of the corresponding enclosed electrification. If, for instance, the distribution on S have the same potential in all external space as if it were collected at a point C within S, that is, if the original system be a centrobaric shell, the surface distribution over S' will have the same potential in all space outside of S', as if it were collected at C', the point corresponding to C; that is, the new system will be a centrobaric shell too.

If in any system V be not zero for the conductor S, V' is not generally constant over S', and the inverted system will not be in equilibrium with S' for a conductor. But, as we have seen,

$$V' = \frac{\kappa}{OQ'} V.$$

If therefore we place at the centre of inversion a charge $-\kappa V$, the potential of this charge, together with that of the inverted system, will be zero at each point on S'.

If therefore we have given a conductor S, and know the density at every point on its surface of an equipotential distribution giving potential V, we can, by inverting the conductor so electrified with any point O for centre and κ for radius of inversion, find the density of the distribution over S' required to give zero potential in presence of a charge $-\kappa V$ at O; namely, if σ be the density at any point P on the original conductor, the density at the corresponding point P' of the distribution giving zero potential is $\frac{\kappa^3}{r'^3}\sigma$, where $r' = OP'$.

123.] *Example.* A conducting sphere of radius a, uniformly coated with electricity of density σ, has constant potential $4\pi a\sigma$ at each point of the surface. Let us invert it with respect to a point O, distant f from the centre, with κ for radius of inversion. The sphere becomes another sphere of radius $\frac{\kappa^2}{f^2-a^2}a$, if O be external, or $\frac{\kappa^2}{a^2-f^2}a$ if O be internal. And according to the general result above proved, the distribution on the new sphere will be such as together with a charge $-\kappa V$, or $-4\pi\kappa a\sigma$, at O will give zero potential at each point of the inverted sphere. But if dA be an elementary area of the original sphere distant r from O, σdA is the charge upon it in the original system. The charge upon the corresponding area in the new sphere will be $\frac{\kappa}{r}\sigma dA$, and dA becomes $\frac{r^4}{\kappa^4}dA$. Therefore the density at the corresponding point of the new sphere is $\frac{r^3}{\kappa^3}\sigma dA$ or $\frac{\kappa^3}{r'^3}\sigma dA$, that is, it varies inversely as the cube of the distance from O.

Again, O being without the original sphere, let $\kappa^2 = f^2 - a^2$, then the sphere does not change its position. Let the charge at O, or $-4\pi\kappa a\sigma = e$, or

$$\sigma = -\frac{e}{4\pi\kappa a}.$$

Then the density for zero potential is

$$-\frac{e}{4\pi\kappa a}\frac{\kappa^2}{r'^3},$$

or

$$-\frac{e}{4\pi a}\frac{f^2-a^2}{r'^3},$$

as we have already found by different methods.

124.] Again, if at the centre of the original sphere there be placed a quantity of electricity $-4\pi a^2\sigma$, and on the surface the uniform distribution of density σ, the potential at any point on the surface or in external space is zero.

If we invert the system with respect to an external point O distant f from the centre, with $\sqrt{f^2-a^2}$ for radius of inversion, the sphere is unaltered in position, but the original centre C upon which the charge $-4\pi a^2\sigma$ was placed becomes a point C', distant $\frac{a^2}{f}$ from the centre in the line OC, and the charge $-4\pi a^2\sigma$ becomes

$$-\frac{\kappa}{f}.\, 4\pi a^2\sigma \text{ at } C',$$

and the distribution on the inverted sphere whose density is $\frac{\kappa^3}{r'^3}\sigma$ gives, in conjunction with that charge at C', zero potential at each point on the inverted sphere and in external space, without there being any charge at O. That which was a centrobaric shell with centre of gravity C has become a centrobaric shell with centre of gravity C'.

125.] Again, we can sometimes make use of the converse proposition to that of Art. 122.

If, namely, we have given any system in which all the conductors are at zero potential under certain electrifications, and if part of the given electrification consist of an electrified point O at which a given charge is placed, we can, by inverting the system with O for centre, obtain a new electrified system in which the conductors have unit potential.

For let $-\kappa$ be the charge at O. Let us take for centre of inversion a point distant x from O, and κ for radius of inversion.

Then all the conductors when inverted remain at zero potential, and the charge $-\kappa$ at O becomes a charge $-\frac{\kappa^2}{x}$ at a point distant $\frac{\kappa^2}{x}$ from the centre of inversion. Now let x be indefinitely diminished.

If all the conductors, when inverted, are of finite magnitude, the infinite charge $-\frac{\kappa^2}{x}$ at distance $\frac{\kappa^2}{x}$ will, when x is indefinitely diminished, that is, when O is taken for centre of inversion, have potential -1 at each point on the conductors. The remaining electrification of the inverted system will therefore have potential $+1$ throughout the conductors.

126.] As an example of this process, let us take two infinite planes XEX', YEY' at right angles to each other, and four points O_1, C_1, O_2, C_2, as in the figure, forming a rectangle whose diameters intersect in E.

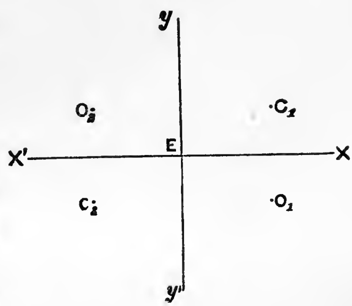

Fig. 19.

O_1 and O_2 having charges, each $-\kappa$, and C_1, C_2 having charges, each $+\kappa$, the potential is zero at each point on either plane.

Let y be the distance irrespective of sign of any one of the four points from the plane XEX', x the distance of any one of them from the plane YEY'.

If we invert the system with O_1 for centre and κ for radius, the two infinite planes become two orthogonally intersecting spheres. The common section of the planes becomes the circle of intersection of the spheres and passes through O_1. The plane XEX' becomes a sphere whose centre is C_1', the point corresponding to C_1 and whose radius is $a_2 = \frac{\kappa^2}{2y}$.

Similarly the plane YEY' becomes a sphere whose centre is C_2' and whose radius is $a_1 = \frac{\kappa^2}{2x}$.

The portion XEY' of the two infinite planes becomes on inversion the figure formed of the two outer segments of the spheres. Similarly $X'EY$ becomes the lens formed of the two inner segments, and XEY, or $X'EY'$, becomes a meniscus formed of the outer segment of one and the inner segment of the other sphere. (See Fig. 19.)

The charge at C_1' is $\frac{\kappa^2}{2y}$ or a_1, and the charge at C_1' is $\frac{\kappa^2}{2}$ or a_2, and the charge at O_2' is

$$-\frac{\kappa^2}{2\sqrt{x^2+y^2}},$$

that is,

$$-\frac{a_1 a_2}{\sqrt{a_1^2+a_2^2}}.$$

We obtain therefore the system already treated in Art. 112.

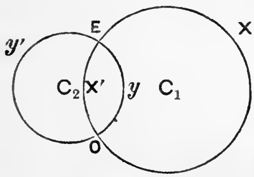

Fig. 20.

Further, if before inversion we substitute for the charges at C_1 and O_2 their equivalent distributions on the plane XEX', and for C_2 its equivalent distributions on YEY', these densities on XEX' and YEY' will in the inverted system give unit potential on XEY', and are the same which we found by a different method in Art. 112.

127.] Again, instead of two infinite planes, let there be $2n$ infinite planes, having a common section E and making with each other the angle $\frac{\pi}{n}$.

Let there be n negative points $O_1 \dots O_n$ each having charge $-\kappa$, and n positive points $C_1 \dots C_n$ each having charge $+\kappa$, all at the same distance from E and placed alternately, so that each negative point is the image of the next positive point in the plane between them. Then all the planes are at zero potential.

Let YEY' and SES' be two adjacent planes. Let the n points on the left of YEY' be replaced by the corresponding distributions on YEY', and the $n-1$ points on the right of YEY', that

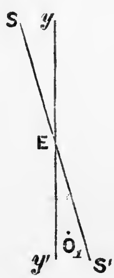

Fig. 21.

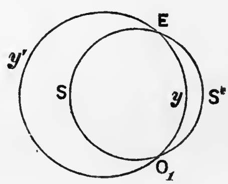

Fig. 22.

is, all the points on that side except O_1, be replaced by their corresponding distribution on SES'. Then the portions $S'EY'$ are at zero potential.

When we invert the system with respect to O_1, $S'EY'$ becomes the figure formed by the outer segments of two spheres intersecting at the angle $\frac{\pi}{n}$.

The density at P', any point on the outer segment of the sphere corresponding to YEY', is $\frac{O_1 P^3}{\kappa^3}\sigma$, that is, $\frac{C_1 P^3}{\kappa^3}\sigma$, where σ is the density at P, the corresponding point to P', of the distribution on YEY', which can be determined without much difficulty.

128.] Returning to the conductor $XEY'O$ of Art. 126, with its surface distribution above determined in Art. 112, let us invert the system, taking for centre of inversion a point on the internal segment of the sphere C_1.

The sphere C_2 becomes then another sphere, and the sphere C_1 an infinite plane cutting the inverted sphere C_2 orthogonally, that is, a diametral plane, and the external segment OXE becomes the portion of that infinite plane which lies within the new sphere C_2. So that the figure $XEY'O$ becomes on inversion the closed surface formed by a hemisphere and its diametral plane. Let P be a point on the outer segment of the sphere C_1, P' the point on the diametral plane which corresponds to P. And σ being the density above found for P, namely

$$\frac{1}{4\pi a_1}\left\{1 - \frac{a_1^{\,3}}{C_2 P^3}\right\},$$

the density at P' required to give zero potential under the influence of a charge unity at a point O within the hemisphere is

$$-\frac{\kappa^2}{OP'^3}\sigma.$$

129.] The construction for finding σ in terms of known quantities on the hemisphere will be as follows.

Let C be the centre of the hemisphere, a its radius, O the point where the unit charge is placed. Then by inverting the system with respect to O, we shall reconstruct the original figure of two orthogonally intersecting spheres, which by its inversion gave rise to the existing system of the hemisphere and plane.

Let C_2 be the image of O referred to the existing sphere.

Then the point corresponding to C_2 becomes on inversion the centre of one of the two original spheres. Inasmuch as the absolute value of κ affects only the scale, and not the proportions, of the inverted figure, we may take $OC_2 = \kappa$.

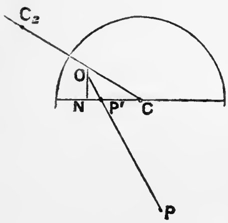

Fig. 23.

Then C_2 becomes the actual centre of the new sphere. Its radius is

$$a_2 = \frac{OC_2^2}{a^2 - OC^2} a.$$

The radius of the other orthogonally intersecting sphere, that namely into which the infinite plane is converted, is

$$a_1 = \frac{OC_2^2}{2ON}.$$

Let P' be a point on the diametral plane, and on inversion let P' become P on the sphere. Then

$$OP = \frac{OC_2^2}{OP'}.$$

Then $$C_2P^2 = \frac{OC_2^2}{OP'^2} C_2P'^2.$$

Thus all the quantities in the expression for σ, namely

$$\frac{1}{4\pi a_1}\left\{1 - \frac{a_2^3}{C_2P^3}\right\},$$

are known in terms of given dimensions in the hemisphere, and therefore the density at P is known.

In like manner we might find the density of the same distribution at a point on the hemispherical surface.

130.] By inversion of the system of sphere and infinite plane, or two concentric spheres, Arts. 114, 115, with the point at which the charge is placed for centre of inversion, we obtain two spheres external to each other at unit potential, and the density at any point on either sphere required to produce this result can be calculated approximately. The subject is fully treated in Maxwell's *Electricity*, Chap. XI.

131.] We shall here give only one more example of the method taken from Sir W. Thomson's papers.

It is proved in treatises on attraction that an ellipsoidal shell between two similar, concentric, and similarly situated ellipsoids has constant potential at all points on its surface, and that if its equation be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the external equipotential surfaces are the confocal ellipsoids whose equation is

$$\frac{x^2}{a^2+h^2}+\frac{y^2}{b^2+h^2}+\frac{z^2}{c^2+h^2}=1.$$

The thickness of such a shell when the ellipsoids nearly coincide is proportional to p, the perpendicular from the centre on the tangent plane at the point considered. It follows that the density of electricity on the surface of a conducting ellipsoid which gives constant potential at all points on that surface, in the absence of any other electrification, is proportional to p.

If the axes a and b of the ellipsoid are equal, and if c be diminished without limit, the ellipsoid becomes ultimately a flat circular disc. And therefore the density of the equipotential distribution of electricity at any point on the surface of such a disc is proportional to the limiting value of p for that point.

Now generally

$$\begin{aligned}\frac{1}{p^2}&=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\\&=\frac{x^2+y^2}{a^4}+\frac{1}{c^2}\frac{z^2}{c^2},\text{ if } a=b,\\&=\frac{x^2+y^2}{a^4}+\frac{1}{c^2}\left\{\frac{a^2-(x^2+y^2)}{a^2}\right\},\end{aligned}$$

and the limiting value of p is therefore proportional to

$$\frac{1}{\sqrt{a^2-(x^2+y^2)}}.$$

Let C be the centre of the disc, P any point on it. Let

$$CP^2=x^2+y^2=r^2.$$

Then the density at P is

$$\frac{\lambda}{\sqrt{a^2-r^2}},$$

where λ is a constant.

To determine λ, we note that the potential at the centre is

$$2\pi \int_0^a \frac{\lambda \, dr}{\sqrt{a^2-r^2}} = \pi^2 \lambda,$$

and if the potential be unity $\lambda = \frac{1}{\pi^2}$. The density of the distribution which gives unit potential is therefore

$$\frac{1}{\pi^2 \sqrt{a^2-r^2}},$$

and the whole charge on the disc is in this case

$$\frac{1}{\pi^2} \int_0^a \frac{2\pi . r\, dr}{\sqrt{a^2-r^2}} = \frac{2a}{\pi}.$$

Therefore $\frac{2a}{\pi}$ is the capacity of the disc. (a)

132.] The potential of the disc so charged at any point M in its axis of figure, for which $CM = h$, is

$$\frac{1}{\pi^2} \int_0^a \frac{2\pi r \, dr}{\sqrt{a^2-r^2}\sqrt{r^2+h^2}} = \frac{2}{\pi} \tan^{-1} \frac{a}{h}$$

$$= \frac{2}{\pi} \beta, \qquad (b)$$

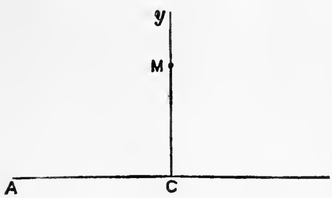

Fig. 24.

if β be the angle CMA, where A is a point in the circumference of the disc.

Again, the equipotential surfaces to the disc are the confocal ellipsoids whose equation is

$$\frac{x^2+y^2}{a^2+h^2} + \frac{z^2}{h^2} = 1,$$

and since the potential of the disc at the point in the axis of z distant h from the origin is

$$\frac{2}{\pi} \tan^{-1} \frac{a}{h},$$

it follows that the potential of the disc at the point x, y, z is

$$\frac{2}{\pi} \tan^{-1} \frac{a}{h},$$

where h is the positive root of

$$\frac{x^2+y^2}{a^2+h^2} + \frac{z^2}{h^2} = 1.$$

To find the potential at any point P in the plane of the disc distant r from the centre we make $z = 0$, and therefore the potential at P is

$$\frac{2}{\pi}\tan^{-1}\frac{a}{\sqrt{r^2-a^2}}. \qquad (c)$$

133.] Let us now invert the disc with respect to a point O in the axis with κ for radius of inversion. The infinite plane now becomes a sphere passing through O, and the disc a spherical bowl, whose rim is a circle at right angles to the axis. The colatitude of that rim measured from the point where the axis cuts the bowl is the vertical angle of the cone at O. Let it be a.

The density on the bowl, according to the method of inversion, is that which would be assumed by the bowl as a conductor in presence of a charge $-\kappa$ at O. And we can find the potential at any point M in the axis due to a spherical bowl under influence of a charge at the extremity of the diameter thus:

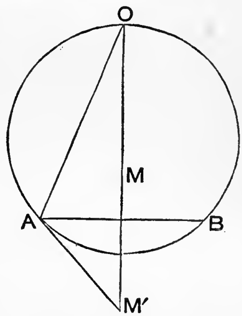

Fig. 25.

For simplicity let κ, the radius of inversion, $= OA$, the distance from O to the rim of the bowl. Then the rim of the bowl coincides with the circumference of the disc which the bowl was before inversion. Find M', the point in the uninverted system corresponding to M, that is, let

$$OM \cdot OM' = OA^2.$$

The potential at M' due to the disc was $\frac{2\beta}{\pi}$, where 2β is the angle of the cone subtended by the disc at M'. And the potential at M is

$$\frac{OA}{OM}\cdot\frac{2\beta}{\pi}.$$

Now the triangles AOM and $M'OA$ are similar. Therefore $\beta = OAM$, if $OM <$ the diameter, or $\pi - OAM$ if $OM >$ the diameter, and the potential at M due to the disc under the influence of a charge OA at O is

$$\frac{OA}{OM}\cdot\frac{2\beta}{\pi}. \qquad (d)$$

Again, the potential of the bowl so influenced at any point P on the remaining segment of the sphere whose colatitude is θ is $\frac{OA}{OP}V$, where V is the potential of the uninverted disc at the point in its plane which on inversion becomes P. And

$$V = \frac{2}{\pi}\tan^{-1}\frac{NA}{\sqrt{NP'^2 - NA^2}}$$

$$= \frac{2}{\pi}\tan^{-1}\frac{\tan\frac{a}{2}}{\sqrt{\tan^2\frac{\theta}{2} - \tan^2\frac{a}{2}}}. \qquad (e)$$

134.] We have thus dealt with the case of a conducting circular disc, which may be regarded as part of an infinite plane of which the infinite external part is non-conducting.

We will now take the converse problem, namely, that of a circular non-conducting disc of radius a, the infinite external portion of the plane of the disc being a conductor. Let it be required to find the density at a point on the conducting plane when that plane is at zero potential under the influence of a charge at a point in the non-conducting disc. In order to solve this problem we will invert the conducting disc when at unit potential as before determined, with respect to a point O in itself distant f from C the centre, and with $\sqrt{a^2 - f^2}$ for radius of inversion. The disc then becomes the infinite external plane, and the infinite plane becomes a disc, the boundary between the two after inversion being a circle of radius a, and whose centre C' is distant f from O on the opposite side to C.

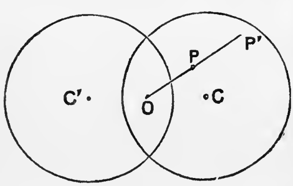

Fig. 26.

Let P be a point in the plane outside of the new circle, P' the point within the original conducting disc which on inversion becomes P.

Then the density at P, when the infinite plane is at zero potential under the influence of a charge $-\kappa$ at O, is

$$\frac{\kappa^3}{OP^3}\cdot\frac{1}{\pi^2}\cdot\frac{1}{\sqrt{a^2 - CP'^2}}.$$

Let $$C'P = r, \quad \angle PC'O = \theta.$$

Then $$a^2 - CP'^2 = \frac{\kappa^2}{OP^2}(r^2-a^2),$$
and the density at P is
$$\frac{\kappa^2}{OP^2}\cdot\frac{1}{\pi^2}\cdot\frac{1}{\sqrt{r^2-a^2}}, \text{ or } \frac{a^2-f^2}{OP^2}\,\frac{1}{\pi^2}\,\frac{1}{\sqrt{r^2-a^2}}. \qquad (f)$$

The density at P due to a uniform ring of electricity of density -1 in the plane of the disc distant from C', $f \ldots f+df$ is
$$2\frac{\sqrt{a^2-f^2}}{\pi^2}\cdot\frac{f.df}{\sqrt{r^2-a^2}}\cdot\int_0^\pi \frac{d\theta}{r^2+f^2-2rf\cos\theta}$$
$$= \frac{2}{\pi}\cdot\frac{\sqrt{a^2-f^2}}{\sqrt{r^2-a^2}}\cdot\frac{f\,df}{r^2-f^2}. \qquad (g)$$

The aggregate of the distribution over the plane due to any electricity m in the disc distant f from the centre is
$$-\frac{m}{\pi^2}\sqrt{a^2-f^2}\int_a^\infty \frac{2\pi r\,dr}{\sqrt{r^2-a^2}(r^2-f^2)} = -m. \qquad (h)$$

135.] If the whole non-conducting disc be covered with electricity of density -1, the density at P in the surrounding plane when at zero potential under that influence is
$$\frac{2}{\pi}\cdot\frac{1}{\sqrt{r^2-a^2}}\int_0^a \frac{\sqrt{a^2-f^2}.f\,df}{r^2-f^2},$$
or
$$\frac{2}{\pi}\left\{\frac{a}{\sqrt{r^2-a^2}} - \tan^{-1}\frac{a}{\sqrt{r^2-a^2}}\right\}.$$

Now let the entire plane, including the non-conducting disc, be covered with a uniform stratum of density $+1$. There will then be zero density on the non-conducting disc, which may therefore be regarded as a circular aperture, and the conducting plane will have constant potential, and the density at any point upon it distant r from the centre of the disc is
$$\frac{2}{\pi}\frac{a}{\sqrt{r^2-a^2}} - \frac{2}{\pi}\tan^{-1}\frac{a}{\sqrt{r^2-a^2}} + 1.$$

This differs from the constant density $+1$ by
$$\frac{2}{\pi}\left\{\frac{a}{\sqrt{r^2-a^2}} - \tan^{-1}\frac{a}{\sqrt{r^2-a^2}}\right\}.$$

If therefore we have an infinite conducting plane with a circular aperture, the total charge that must be placed upon the plane in order to bring it to the same potential as a complete plane would have when coated with density $+1$, is

$$4\int_a^\infty \frac{a\,r\,dr}{\sqrt{r^2-a^2}} - 4\int_a^\infty r\,dr\tan^{-1}\frac{a}{\sqrt{r^2-a^2}} = \pi a^2, \qquad (i)$$

or the same quantity which would be removed from the infinite plane so coated in the act of making the aperture.

136.] We will now proceed to Sir W. Thomson's problem, to find the density of electricity on a spherical bowl, or portion of a sphere cut off by a circle, when at unit potential under the influence of its own charge alone. In order most easily to effect this, let us recur to the non-conducting disc and infinite external conducting plane, and instead of the density of electricity on the disc being uniform,

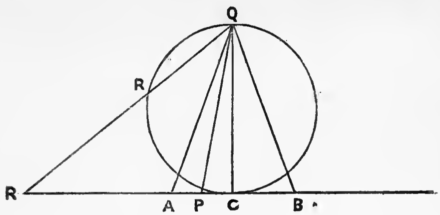

Fig. 27.

let the density at P be $-\dfrac{h^2}{QP^3}$, where Q is a point in the axis of the disc distant h from the centre, and P any point within the disc distant f from the centre. Then

$$\frac{1}{QP^3} = \frac{1}{(h^2+f^2)^{\frac{3}{2}}}.$$

The density at any point in the conducting plane, when at zero potential under the influence of this distribution, is

$$\frac{2}{\pi}\frac{h^2}{\sqrt{r^2-a^2}}\int_0^a \frac{\sqrt{a^2-f^2}\,f\,df}{(h^2+f^2)^{\frac{3}{2}}(r^2-f^2)} \quad \text{by } (g).$$

Let $f = h\cot\dfrac{\alpha}{2}$, $a = h\cot\dfrac{\beta}{2}$, $r = h\cot\dfrac{\theta}{2}$,

where $\dfrac{\pi}{2}-\dfrac{\alpha}{2}$, $\dfrac{\pi}{2}-\dfrac{\beta}{2}$, and $\dfrac{\pi}{2}-\dfrac{\theta}{2}$

are the angles subtended at Q by f, a, and r respectively. The integral then becomes

$$\frac{1}{2}\int_\beta^\pi d\alpha \cdot \frac{\cot\frac{\alpha}{2}\operatorname{cosec}^2\frac{\alpha}{2}\sqrt{\cot^2\frac{\beta}{2}-\cot^2\frac{\alpha}{2}}}{\left(\cot^2\frac{\theta}{2}-\cot^2\frac{\alpha}{2}\right)\operatorname{cosec}^3\frac{\alpha}{2}}.$$

This may be put in the form

$$\frac{1}{2\sqrt{2}}\frac{\sin^2\frac{\theta}{2}}{\sin\frac{\beta}{2}}\int_\beta^\pi d\alpha\,.\,\frac{\sin\alpha\sqrt{\cos\beta-\cos\alpha}}{\cos\theta-\cos\alpha}.$$

Let

$$\sqrt{\cos\beta-\cos\alpha}=x,$$
$$\cos\beta-\cos\alpha=x^2,$$
$$\sin\alpha\, d\alpha=2x\,dx.$$

Then when $\alpha=\pi$, $x=\sqrt{\cos\beta+1}$, and when $\alpha=\beta$, $x=0$.

Then $$\int_\beta^\pi d\alpha\,.\,\frac{\sin\alpha\sqrt{\cos\beta-\cos\alpha}}{\cos\theta-\cos\alpha}=2\int_0^{\sqrt{\cos\beta+1}}dx\,.\,\frac{x^2}{\cos\theta-\cos\beta+x^2}$$

$$=2\left\{\sqrt{\cos\beta+1}-\sqrt{\cos\theta-\cos\beta}\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}\right\}.$$

Hence the density is

$$\frac{2}{\pi}\sin^3\frac{\theta}{2}\left\{\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}-\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}\right\}. \quad (j)$$

137.] Let us now again invert the system, taking Q for centre of inversion, and h for radius. The infinite plane becomes a sphere whose diameter is QC, or h. The infinite conducting plane outside of the disc becomes a spherical bowl, cut off by a circle at right angles to QC, and whose colatitude measured from the pole Q is β. (See Fig. 27.)

The density on the remaining segment of the sphere, which before inversion was

$$-\frac{h^2}{(f^2+h^2)^{\frac{3}{2}}}$$

on the disc, is constant, and equal to $-\frac{1}{h}\cdot$ The potential of the bowl remains zero: and the density upon it at colatitude θ is $\frac{\rho}{\sin^3\frac{\theta}{2}}$, where ρ is the density at the corresponding point of the plane. Hence the density at colatitude θ on a spherical bowl, which makes the potential zero in presence of a uniform charge of density $-\frac{1}{h}$ on the remaining portion of the sphere, is

$$\sigma=\frac{2}{\pi h}\left\{\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}-\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}\right\}. \quad (k)$$

138.] Let us now place round the sphere a concentric and nearly equal sphere with a uniform density of electricity $+\frac{1}{h}$ upon it. When the two spheres ultimately coincide, the potential at any point on the bowl, which was zero, is now 2π. The density upon it is $\sigma+\frac{1}{h}$, and the density on the remaining segment of the sphere is $-\frac{1}{h}+\frac{1}{h}$, or zero.

Therefore $\sigma+\frac{1}{h}$ is the density at colatitude θ of the distribution on a spherical bowl which gives potential 2π in the absence of any other electrification, and therefore $\frac{\sigma}{2\pi}+\frac{1}{2\pi h}$ is the density which gives unit potential under the like circumstances.

139.] The capacity of the bowl is therefore

$$\frac{h^2}{2}\int_0^\beta \left(\sigma+\frac{1}{h}\right)\sin\theta\, d\theta$$

$$=\frac{h^2}{4}\int_0^\beta \frac{2}{\pi h}\left\{\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}-\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}\right\}$$
$$+\frac{h}{4}\int_0^\beta \sin\theta\, d\theta$$

$$=\frac{h}{2\pi}(\beta+\sin\beta).$$

The capacity of the bowl formed by the other segment of the sphere is

$$\frac{h}{2\pi}(\pi-\beta+\sin\beta).$$

Hence we see that if a sphere be divided by a plane into any two parts, the sum of the capacities of the two parts exceeds the capacity of the sphere by the capacity of a circular disc coinciding with the intercepted plane.

If the bowl be hemispherical the capacity is $\frac{a}{\pi}+\frac{a}{2}$, a being the radius, or the arithmetic mean between the capacities of the sphere and disc of the same radius.

140.] Recurring to Art. 138, let us next place at the centre of the sphere of which the bowl forms part, a charge $-\frac{h}{2}$. This will reduce the potential of the bowl to zero.

We may now again invert the system so formed, taking for centre of inversion any point O whether in the spherical surface or not, distant f from the centre. In that case, if O be not on the surface, the sphere becomes a new sphere, and the bowl becomes a new bowl. In the particular case of O being on the original spherical surface the new sphere is an infinite plane, and the new bowl a circular disc upon it.

The centre of the original sphere becomes O', the image of O in the new sphere; and the charge $-\frac{h}{2}$ at the centre becomes a charge $-\frac{f}{\kappa}\frac{h}{2}$ at the image.

If r be the distance from O of a point P on the new bowl, σ the density of the equipotential distribution, as found above, at the corresponding point of the original bowl, then $\frac{\kappa^3}{r^3}\sigma$ is the density at P of the distribution on the new bowl which gives zero potential in presence of $-\frac{f}{\kappa}\frac{h}{2}$ at O'.

We can therefore give the following rule for finding the density at any point on a spherical bowl under the influence of an electrified point O not on the surface of the sphere. First, find O', the image of O in the sphere. Secondly, suppose the system inverted with respect to O', and a new bowl so formed, and let β be the colatitude of the rim of the supposed new bowl, and let θ be the colatitude of P', the point on the supposed bowl corresponding to P on the given bowl. Then we know σ, the density at P' of the equipotential distribution on the supposed bowl, as a function of β and θ. And if r be the distance of P from O', the density at P is proportional to $\frac{\sigma}{r^3}$.

141.] *On the effect of making a small hole in a spherical or infinite plane conductor.*

The above results enable us to estimate some of the effects of making a small circular aperture in a conductor otherwise spherical.

For instance, let $\beta = \pi - \gamma$, where γ is a very small angle. The spherical bowl becomes then a spherical conductor of radius $\frac{h}{2}$, with a small aperture whose radius is $\frac{h}{2}\gamma$.

Its capacity is

$$\frac{h}{2\pi}(\beta + \sin\beta), \quad \text{that is} \quad \frac{h}{2} - \frac{h}{2}\cdot\frac{\gamma - \sin\gamma}{\pi}.$$

The capacity of the complete sphere is $\frac{h}{2}$. We see then that the effect of making an aperture, whose radius subtends at the centre the small angle γ, is to diminish the capacity by

$$\frac{h}{2}\cdot\frac{\gamma - \sin\gamma}{\pi}.$$

If A be the area of the aperture, we may write, neglecting higher powers than γ^3,

$$\frac{h}{2}\cdot\frac{\gamma - \sin\gamma}{\pi} = \lambda\frac{A^{\frac{3}{2}}}{h^2}, \text{ where } \lambda = \frac{2}{3\pi^{\frac{5}{2}}}.$$

Again, the conductor being charged to unit potential, the density at a point whose colatitude is θ (less than β) is

$$\frac{1}{\pi^2 h}\left\{\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}} - \tan^{-1}\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}}\right\} + \frac{1}{2\pi h}.$$

Now $\frac{1}{2\pi h}$ is the uniform density which would give unit potential on the complete sphere. The term

$$\frac{1}{\pi^2 h}\left\{\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}} - \tan^{-1}\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}}\right\}$$

expresses the density due to the existence of the aperture.

The total quantity of the distribution due to the aperture on a ring between the parallels of θ and $\theta + d\theta$ is

$$\frac{h^2}{2}\frac{\sin\theta d\theta}{\pi}\left\{\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}} - \tan^{-1}\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}}\right\}.$$

Now unless θ be very nearly equal to π, not only does

$$\sqrt{\frac{\cos\beta + 1}{\cos\theta - \cos\beta}}$$

itself become very small, but also it tends to vanish in a ratio of equality with

$$\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}.$$

Hence if θ_1 be a value of θ which is, and θ_2 a value which is not, nearly equal to β, it is easily seen that the quantity of the distribution due to the aperture on the ring between θ_1 and θ_2 is very small compared with that on the ring between β and θ_1. The distribution due to the aperture has therefore the same effect as if it were all collected on the aperture.

For instance,

$$\int_\theta^\beta \left\{\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}-\tan^{-1}\sqrt{\frac{\cos\beta+1}{\cos\theta-\cos\beta}}\right\}\sin\theta\, d\theta,$$

$$=\tfrac{1}{3}\int_\theta^\beta \left\{\frac{\cos\beta+1}{\cos\theta-\cos\beta}\right\}^{\frac{3}{2}}\sin\theta\, d\theta, + \&c.$$

and is independent of θ.

The system is therefore equivalent to a complete sphere charged to unit potential, that is, having a uniform density $\frac{1}{2\pi h}$ on its surface, together with the additional charge

$$-\frac{h}{2}\frac{\gamma-\sin\gamma}{\pi}$$

on the aperture. This quantity

$$-\frac{h}{2}\frac{\gamma-\sin\gamma}{\pi}, \quad \text{or} \quad -\lambda\frac{A^{\frac{3}{2}}}{h^2},$$

shall be called the abnormal charge, since it constitutes the difference between the capacities of the perfect and the imperfect sphere.

Let P be any external point distant r from the centre of the sphere, and r' from the centre of the aperture. Then the potential at P of the charged sphere is

$$\frac{h}{2r}-\frac{h}{2}\cdot\frac{\gamma-\sin\gamma}{\pi}\cdot\frac{1}{r'};$$

or is the potential of the perfect sphere, together with that of the abnormal charge

$$-\frac{h}{2}\frac{\gamma-\sin\gamma}{\pi}$$

placed on the aperture.

142.] Let us now invert the charged conductor, taking for centre of inversion a point O.

On the imperfect sphere when charged to unit potential, such point not being very near the aperture. The sphere becomes then an infinite plane with a circular aperture, at zero potential under the influence of unit charge at O. And the potential at any point P on the opposite side of the plane to O, instead of being zero, is that due to a small positive charge upon the aperture.

These results, which are accurately true in the limit as the aperture vanishes, are approximately true for a sphere whenever the aperture subtends a very small angle at the influencing point.

To find the effect of a large aperture it would be necessary to find the potential at any point due to a spherical bowl charged to unit potential, when β is not nearly equal to π. This might be done approximately by the method of Art. 61, or otherwise.

CHAPTER VIII.

CONJUGATE FUNCTIONS AND ELECTRICAL SYSTEMS IN TWO DIMENSIONS.

ARTICLE 143.] Let there be an infinite cylinder whose axis is parallel to the axis of z, and whose section is the element of area $dx\,dy$, cutting the plane of xy in the point x, y.

Let this cylinder be charged with electricity of uniform density ρ, so that ρ is independent of z, but is a function of x and y. In like manner we may conceive an infinite cylindrical surface whose axis is parallel to that of z, having σ for surface density of electricity, constant along any infinite line parallel to the axis, so that σ is independent of z, and a function of x and y only. If an electrical system be made up of such cylinders, the potential, V, is evidently independent of z, and a function of x and y only. Poisson's equation then becomes at a point in the plane of x, y

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + 4\pi\rho = 0\,;$$

and at a curve in the plane of xy coated with electricity we have as usual

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} = -4\pi\sigma,$$

and $d\nu$, the element of the normal to the surface, is in the plane of x, y.

We may treat such a system as in two dimensions only. In the present chapter the charge *at a point* P in the plane of xy will be understood to mean a uniform distribution of electricity along an infinite line or cylinder of small section through P parallel to the axis of z. And in like manner, the density at a point on a line in the plane of xy means the density per unit area of a cylindrical surface parallel to the axis of z drawn through an element of the line at the point in question.

In like manner, a conductor the equation to whose surface is $f(x, y) = 0$ means an infinite cylindrical conductor whose axis is parallel to the axis of z, and whose section with the plane of xy is $f(x, y) = 0$.

If P and Q be points in the plane of x, y, the potential at Q due to a uniform distribution of electricity of density ρ along an infinite line parallel to z drawn through P, is evidently of the form

$$\rho \{C - 2 \log PQ\},$$

and does not vanish if Q be removed to an infinite distance. But if there be another parallel line through R, a point in the same plane of xy with P and Q, on which there is a distribution of density $-\rho$, the potential at Q is

$$2\rho \log \frac{PQ}{RQ},$$

and vanishes if Q be removed to an infinite distance. It will be understood in this chapter that the potential does so vanish, and therefore that the algebraic sum of all the electricity in the system of which we treat is zero.

144.] Let us suppose then that in such a system there are certain conductors whose equations are

$$f_1(x, y) = 0, \quad f_2(x, y) = 0, \text{ \&c.},$$

and given charges are placed upon them; and also certain fixed charges on given points or lines of the system. Let us further suppose that we have by any method obtained the solution of this electrical problem: that is, we have found the single function, V, of x and y, which is constant within all the conductors, and satisfies Poisson's equations

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + 4\pi\rho = 0$$

at every point in free space, and

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} = -4\pi\sigma$$

at every curve charged with electricity; and by consequence we have determined the density at any point on any of the conductors.

The solution so found contains implicitly the solution of a class of problems; all those namely that can be formed from the given one by substituting $\xi(x, y)$ and $\eta(x, y)$ for x and y, $\xi(x, y)$ and $\eta(x, y)$, or shortly ξ and η, being functions of x and y having a certain property.

145.] For let ξ and η be so chosen that

$$\left.\begin{aligned} \frac{d\xi}{dx} &= \frac{d\eta}{dy}, \\ \frac{d\xi}{dy} &= -\frac{d\eta}{dx}; \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

ξ and η are then defined to be CONJUGATE TO x AND y.

It follows immediately that

$$\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} = 0, \qquad \frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} = 0,$$

$$\frac{d\xi}{dx}\frac{d\eta}{dx} + \frac{d\xi}{dy}\frac{d\eta}{dy} = 0,$$

and $$\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 = \mu^2, \text{ suppose.}$$

By the ordinary formula for change of independent variables we know that

$$\frac{dx}{d\xi} = \frac{\dfrac{d\eta}{dy}}{\dfrac{d\xi}{dx}\dfrac{d\eta}{dy} - \dfrac{d\xi}{dy}\dfrac{d\eta}{dx}}, \qquad \frac{dx}{d\eta} = \frac{\dfrac{d\xi}{dy}}{\dfrac{d\xi}{dy}\dfrac{d\eta}{dx} - \dfrac{d\eta}{dy}\dfrac{d\xi}{dx}},$$

$$d\xi\, d\eta = \left(\frac{d\xi}{dx}\frac{d\eta}{dy} - \frac{d\xi}{dy}\frac{d\eta}{dx}\right) dx\, dy.$$

Also $\dfrac{d\eta}{dy} = \dfrac{d\xi}{dx}$, and $\dfrac{d\eta}{dx} = -\dfrac{d\xi}{dy}$ in this case,

$$\therefore \quad \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dx}{d\eta}\right)^2 = \frac{1}{\left(\dfrac{d\xi}{dx}\right)^2 + \left(\dfrac{d\xi}{dy}\right)^2} = \frac{1}{\mu^2},$$

and $$d\xi\, d\eta = \mu^2\, dx\, dy.$$

Again, if V be any given function of x and y, we have by ordinary differentiations,

$$\frac{d^2V}{dx^2} = \frac{d^2V}{d\xi^2}\left(\frac{d\xi}{dx}\right)^2 + 2\frac{d^2V}{d\xi\, d\eta}\frac{d\xi}{dx}\frac{d\eta}{dx} + \frac{d^2V}{d\eta^2}\cdot\left(\frac{d\eta}{dx}\right)^2 + \frac{dV}{d\xi}\cdot\frac{d^2\xi}{dx^2} + \frac{dV}{d\eta}\frac{d^2\eta}{dx^2},$$

$$\frac{d^2V}{dy} = \frac{d^2V}{d\xi^2}\left(\frac{d\xi}{dy}\right)^2 + 2\frac{d^2V}{d\xi\, d\eta}\frac{d\xi}{dy}\frac{d\eta}{dy} + \frac{d^2V}{d\eta^2}\cdot\left(\frac{d\eta}{dy}\right)^2 + \frac{dV}{d\xi}\cdot\frac{d^2\xi}{dy^2} + \frac{dV}{d\eta}\frac{d^2\eta}{dy^2}.$$

Therefore, remembering that

$$\frac{d\xi}{dx}\frac{d\eta}{dx}+\frac{d\xi}{dy}\frac{d\eta}{dy}=0,$$

and that $$\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2=\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{dy}\right)^2=\mu^2,$$

we get $$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=\mu^2\left(\frac{d^2V}{d\xi^2}+\frac{d^2V}{d\eta^2}\right).$$

146.] Let us now take two planes, one in which the position of any point, as P, is determined by the values of the rectangular coordinates x and y, and the other in which the position of a point P' is determined by the values of the rectangular conductors x' and y'; and when x' and y' are connected with x and y by the equations

$$\xi(x', y')=kx,\quad \eta(x', y')=ky,\text{[1]}$$

where k is such a power of the unit of length as may be required to make $\frac{\xi}{k}$ and $\frac{\eta}{k}$ linear, let the point P' in the second plane be called the corresponding point to P in the first plane.

Then to every curve in the first plane of the form $f(x,y)=0$ there will be a corresponding curve $f(\xi,\eta)=0$ in the second plane, and if the former curve be closed, so also will be the latter, and if any point P in the former plane be within or without the closed curve $f(x,y)=0$, the corresponding point P' in the latter plane will also be within or without the closed curve $f(\xi,\eta)=0$.

It follows from the equation

$$\frac{d\xi}{dx'}\frac{d\eta}{dx'}+\frac{d\xi}{dy'}\frac{d\eta}{dy'}=0,$$

that the curves $\xi=a$, $\eta=b$ in the second plane intersect each other at right angles; these curves may be regarded as a species of curvilinear coordinates, the case in which they are linear being that in which the point P' is always so taken in the second plane that its coordinates referred to axes inclined to those of x', y' are

[1] The quantity k will be omitted until we come to the application to special cases, none of the general results obtained in the next few Articles being affected by regarding k as unity.

respectively equal to the coordinates of the corresponding point P, viz. x and y in the first plane.

It follows also that any two curves, as

$$F(x, y) = 0, \quad f(x, y) = 0,$$

in the original plane intersect each other at the same angle as the corresponding curves

$$F(\xi, \eta) = 0, \quad f(\xi, \eta) = 0$$

in the new plane.

For the tangent of the angle which the tangent to $F(x, y) = 0$ at any point P makes with the axis of x is

$$-\frac{\dfrac{d}{dx}F(x, y)}{\dfrac{d}{dy}F(x, y)},$$

and this is equal to
$$-\frac{\dfrac{d}{d\xi}F(\xi, \eta)}{\dfrac{d}{d\eta}F(\xi, \eta)},$$

from the relations between x and ξ, y and η.

But
$$-\frac{\dfrac{d}{d\xi}F(\xi, \eta)}{\dfrac{d}{d\eta}F(\xi, \eta)}$$

is $= \dfrac{d\eta}{d\xi}$ in the curve $F(\xi, \eta) = 0$ at the point P corresponding to P in the second plane, i.e. it is the tangent of the angle between the curve $F(\xi, \eta) = 0$ at P' and the curve $\eta =$ const. through P', since the curves $\eta =$ const., $\xi =$ const., intersect everywhere at right angles.

Also, if dA be any elementary area $dx\,dy$ in the original plane, we have

$$dA = dx dy = d\xi\, d\eta.$$

But from the last article

$$d\xi\, d\eta = \mu^2 dx' dy',$$

and therefore if dA' be the elementary area in the second plane corresponding to dA in the first plane, we have

$$dA' = \frac{dA}{\mu^2}.$$

And in like manner the length of any elementary line in the second plane corresponding to the element dv in the first plane may be proved to be $\frac{dv}{\mu}$.

147.] Suppose now that we have any given electrical system of two dimensions in equilibrium in the original plane of x, y, with conductors whose bounding equations are given by closed curves of the form $f(x, y) = 0$, the algebraic sum of all the electricity being zero. Construct in the new plane of x', y' a system of corresponding curves $f(\xi, \eta) = 0$, and for every linear or superficial charge in the original plane of x, y, place the same linear or superficial charge upon the corresponding lines and areas in the new plane of x', y'; then the electrical system so formed in the plane of x', y' will be a system of two dimensions in equilibrium with conductors bounded by the corresponding closed curves to the original curves in the plane of x, y. And the potential V at any point P in the old plane of x, y will be equal to the potential V' at the corresponding point P' in the new plane of x', y'.

For since the total charges on corresponding superficial areas are the same, but the areas themselves are in the ratio of μ^2 to 1, it follows that if ρ be the surface density at any point in the old plane, and ρ' that at the corresponding point in the new plane, then $\rho' = \mu^2\rho$, and similarly if σ and σ' be corresponding linear densities $\sigma' = \mu\sigma$.

Again, let V' be the same function of ξ and η that V is of x and y. Then from the equations

$$\xi = x, \quad \eta = y,$$

it follows that since V is constant over the closed curves $f(x, y) = 0$ in the plane of x, y, V' is constant over the corresponding closed curves $f(\xi, \eta) = 0$ in the plane of x', y'.

Again, $$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = \frac{d^2V'}{d\xi^2} + \frac{d^2V'}{d\eta^2} = \frac{1}{\mu^2}\left(\frac{d^2V'}{dx'^2} + \frac{d^2V'}{dy'^2}\right).$$

But $$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = -4\pi\rho,$$

if ρ be the electrical density at P in the plane of x, y.

Therefore $$\frac{d^2V'}{dx'^2}+\frac{d^2V'}{dy^2}=-4\pi\mu^2\rho=-4\pi\rho',$$

where ρ' is the electrical density at P' in the plane of x', y'.

Again, if $d\nu$ and $d\nu'$ be elements of the normal to a curve, as $f(x, y) = 0$ in the plane of x, y, and dn and dn' be the elements of the normal at the corresponding point to the corresponding curve $f(\xi, \eta) = 0$ in the plane of x', y', we have, since $V = V'$ and

$$\frac{dn}{d\nu}=\frac{d\nu'}{dn'}=\frac{1}{\mu},$$

by what is proved above,

$$\frac{dV'}{dn}=\mu\frac{dV}{d\nu},\quad \frac{dV'}{dn'}=\mu\frac{dV}{d\nu'}.$$

But if σ be the linear density at P on the curve $f(x, y) = 0$,

$$\frac{dV}{d\nu}+\frac{dV}{d\nu'}=-4\pi\sigma;$$

therefore $$\frac{dV'}{dn}+\frac{dV'}{dn'}=-4\pi\mu\sigma=-4\pi\sigma',$$

σ' being the linear density at the point P' on the curve $f(\xi, \eta) = 0$ in the plane of x', y'.

148.] We see then that the function V', formed as above stated, is constant over each of the conductors $f(\xi, \eta) = 0$ in the new plane, and satisfies the characteristic equations at each point of that plane. If therefore the function V be the potential at any point of the x, y system, the function $V' = V$ will be the potential at the corresponding point P' in the x', y' system when in equilibrium, and we have only to eliminate ξ and η and express V' in terms of x', y' to obtain a complete solution of the problem of an electrical system of two dimensions bounded by conductors of the form $f(\xi, \eta) = 0$.

Of course the procedure might be reversed, and if we had found V' a function of x', y' with conductors bounded by curves $f(\xi, \eta) = 0$, we have only to express V' in terms of ξ and η, and take V, the same function of x and y that V' is of ξ and η, to obtain the solution for conductors bounded by the curves $f(x, y) = 0$.

149.] Further, if ξ and η be conjugate to two other functions α and β, then also α and β are conjugate to x and y, for

$$\frac{d\alpha}{dx} = \frac{d\alpha}{d\xi} \cdot \frac{d\xi}{dx} + \frac{d\alpha}{d\eta} \cdot \frac{d\eta}{dx}.$$

But $\frac{d\alpha}{d\xi} = \frac{d\beta}{d\eta}$, $\frac{d\alpha}{d\eta} = -\frac{d\beta}{d\xi}$, $\frac{d\xi}{dx} = \frac{d\eta}{dy}$, $\frac{d\eta}{dx} = -\frac{d\xi}{dy}$;

and therefore $$\frac{d\alpha}{dx} = \frac{d\beta}{d\eta} \cdot \frac{d\eta}{dy} + \frac{d\beta}{d\xi} \cdot \frac{d\xi}{dy} = \frac{d\beta}{dy},$$

and similarly $$\frac{d\alpha}{dy} = -\frac{d\beta}{dx}.$$

If therefore we take a third plane x'', y'' and determine a point P'' on it such that $\alpha(x'', y'') = \xi$, and $\beta(x'', y'') = \eta$, and the points P, P' and P'' be called corresponding points on the three planes, the solution for a distribution of electricity for conductors bounded by the curves

$$f(x, y) = 0, \quad f(\xi, \eta) = 0, \quad f(\alpha, \beta) = 0,$$

on any one of these planes respectively, leads at once to the corresponding distribution on the two others, and similarly for any number of planes and systems of conductors.

150.] Further, as is easily seen, if the problem were to determine, not the potential due to given charges, but the charge on any conductor necessary to produce given potentials, the solutions for the several members of the class would be connected by the same law as in the case we have already considered. That is if σ be the required density in the known problem, $\mu\sigma$ is the density in the new one.

151.] We proceed to illustrate the above process by an example.

Let there be a conductor in the form of an infinite cylinder of circular section and radius a, having its axis parallel to z, and meeting the plane of x, y in the point C. Let a charge e per unit of length be uniformly distributed along an infinite straight line passing through an external point O and parallel to the axis of the cylinder, then, as proved in Art. 109, the density at any point of the cylinder of the charge necessary to reduce the potential of the cylinder to a constant value in presence of the charge on the line is

$$\sigma = -\frac{e}{2\pi a}\cdot\frac{f^2-a^2}{\rho^2},$$

where f is the distance of the axis of the cylinder, ρ that of the point in question, from O. Also the algebraic sum of this distribution over the whole circle is $-e$.

We now proceed to transform this problem. Expressing the conditions in polar coordinates $\log r$ and θ with C for origin and CO for fixed line, we have in the original system a conductor whose equation is $\log r = \log a$, a constant, and a charge e at the point whose coordinates are

$$\log r = \log f \quad \text{and} \quad \theta = \pm 2i\pi,$$

where i is any integer.

Now x and y are conjugate to $\log r$ and θ. Corresponding therefore to $\log r = \log a$ we shall have the infinite line $x = \kappa \log a$, where κ is unit of length. And corresponding to the charge e at the point

$$\log r = \log f, \quad \theta = \pm 2i\pi,$$

we shall have a charge e at each of a series of points in the line $x = \kappa \log f$, distant from each other $\pm 2\pi\kappa$, whereof one is in the axis of x. The density σ of a distribution of electricity on the line $x = \kappa \log a$ which will give constant potential on that line in presence of the infinite series of charged points on the line $x = \kappa \log f$ is found by substituting $\frac{y}{\kappa}$ for θ in the expression for σ found above, and is therefore

$$-\frac{1}{2\pi}\frac{e}{a}\frac{f^2-a^2}{f^2+a^2-2fa\cos\frac{y}{\kappa}}.$$

It will be remembered that to obtain the actual physical conditions we must understand by the infinite line $x = \kappa \log a$ an infinite conducting plane whose equation is $x = \kappa \log a$, and by any one of the series of charged points a charge uniformly distributed along an infinite line through the point parallel to the axis of z.

152.] The general problem, the solution of which is derivable from the given problem by the substitution of x and y for $\log r$ and θ, is found by choosing for origin, instead of C, the centre of the circle, any point in the plane. Suppose we choose a point D such that $CD = c$, and $\angle CDO = a$, $DO = p$. The equation to the circle referred to polar coordinates $\log r$ and θ with D for origin and DC for fixed line is

$$\log r = \log\{c\cos\theta \pm \sqrt{a^2 - c^2\sin^2\theta}\}.$$

We obtain then by the transformation the density of electricity on the curve

$$x = \kappa\log\left\{c\cos\frac{y}{\kappa} \pm \sqrt{a^2 - c^2\sin^2\frac{y}{\kappa}}\right\}$$

in presence of a charge on a series of points situated in the line $x = \kappa\log p$ at equal distances $2\pi\kappa$ apart, of which one is distant κa from the axis of x.

This includes all the problems the solution of which can be derived from that of the given one by the use of the particular functions x and y as conjugate to $\log r$ and θ. But we may obtain others by the use of different functions.

For instance, $x^2 - y^2$ and $2xy$ are conjugate to x and y, and therefore to $\log r$ and θ. If therefore, taking C again for origin, we write $\frac{x^2-y^2}{\kappa^2}$ for $\log r$, and $\frac{2xy}{\kappa^2}$ for θ in the above problem, we shall obtain the solution for the density on the hyperbola $x^2 - y^2 = \kappa^2\log a$ in presence of charges placed on the hyperbola $x^2 - y^2 = \kappa^2\log f$ at the intersections of that hyperbola with the hyperbolas $xy = \pi\kappa^2$, $xy = 2\pi\kappa^2$, &c.

It is of course understood that the hyperbolas represent infinite cylindrical surfaces parallel to z whose intersections with the plane of xy are the hyperbolas in question. And in like manner the points represent infinite straight lines.

As in the former case, we can generalise this solution by taking for origin any point in the plane of the circle.

153.] A particular case of transformation by conjugate functions is that of inversion in two dimensions. Evidently $\log\frac{\kappa^2}{r}$ and $-\theta$ are conjugate to $\log r$ and θ.

It follows therefore, from the theory of conjugate functions, that if we transform a system in equilibrium, and in which V vanishes at an infinite distance, by substituting the coordinates $\frac{\kappa^2}{r}$ and $-\theta$ for r and θ, and placing on corresponding elements the same charges, the transformed system will be in equilibrium.

154.] We will now prove the same result by a method analogous to that of Chap. VII.

In the plane of x, y let us take O for centre and κ for radius of inversion, and let P, Q be any two points in the plane. Then if P' Q' be corresponding points to P and Q, we have

$$P'Q' = \frac{\kappa^2}{OP\,.\,OQ}PQ.$$

In the present case the potential at P of a charge ρ at Q means the potential at P of a uniform distribution of linear density ρ along an infinite line drawn through Q parallel to the axis of z. The potential is therefore

$$v = \rho\,\{C - 2\log PQ\}.$$

If in the inverted system there be the same quantity of matter placed at Q', according to the method of transformation used with conjugate functions, the potential at P' of the charge at Q' is

$$v' = \rho C - 2\rho\log\frac{\kappa^2}{OP\,OQ} - 2\rho\log PQ$$

$$= v - 2\rho\log\frac{\kappa^2}{OP\,OQ},$$

and therefore, if V, V' denote the potentials at P and P' of the whole system,

$$V' = V - 2\iint\rho\log OP'\,dx\,dy + 2\iint\rho\log OQ\,dx\,dy$$

$$= V + 2\log OP'\iint\rho\,dx\,dy + 2\iint\rho\log OQ\,dx\,dy.$$

Since we assume the potential to vanish at an infinite distance, we must have in this system

$$\iint\rho\,dx\,dy = 0.$$

Hence $$V' = V + 2\iint\rho\log OQ\,dx\,dy,$$

that is, V' exceeds V by the potential at the origin of the original system, that is, by a constant for all positions of P.

Therefore since V is constant over every conductor in the original system, V' is constant over every conductor in the new system. The new system is therefore in equilibrium.

If the original system consist of a conductor S at zero potential under the influence of an electrified point on the opposite side of S to the origin, the potential at the origin is zero, that is

$$2\iint \rho \log OQ\, dx\, dy = 0,$$

and therefore $V' = 0$ at every point of the new conductor.

For instance, if the original system be an infinite cylinder uniformly coated with electricity of density σ, and if there be a distribution of density $-2\pi r\sigma$ along the axis, the potential is zero, and we might by inverting the system with respect to an external point obtain the result obtained synthetically in Chap. VI, Art. 109.

CHAPTER IX.

ON SYSTEMS OF CONDUCTORS.

Article 155.] We proceed to consider further the properties of a system of insulated conductors external to one another, and each charged in any manner. And we will suppose that there is no electrification in the field except the charges on the conductors.

Let $C_1, C_2 \dots C_n$ be the conductors. First let a charge e_1 be placed on C_1, the other conductors being uncharged.

Let the potentials of the several conductors be denoted by $V_1, V_2 \dots V_n$.

By the principle of superposition, if e_1 were increased in any ratio, $V_1, V_2 \dots V_n$ would be increased in the same ratio. It follows that we may express $V_1, V_2 \dots V_n$ in terms of e_1 in the form

$$V_1 = A_{11} e_1, \quad V_2 = A_{12} e_1, \text{ \&c.};$$

where A_{11}, A_{12}, &c. are coefficients depending only on the forms and positions of the conductors.

In like manner if C_2 received a charge e_2, all the others being uncharged, we should have

$$V_1 = A_{21} e_2, \quad V_2 = A_{22} e_2, \text{ \&c.};$$

the coefficients being again dependent on the forms and position of the conductors.

By the principle of superposition, if at the same time C_1 receive a charge e_1, and C_2 a charge e_2, the others remaining uncharged, we shall have

$$\begin{aligned} V_1 &= A_{11} e_1 + A_{21} e_2, \\ V_2 &= A_{12} e_1 + A_{22} e_2, \\ \text{\&c.} &= \text{\&c.}, \\ V_n &= A_{1n} e_1 + A_{2n} e_2. \end{aligned}$$

And, generally, if the conductors all receive charges $e_1, e_2 \dots e_n$, the potentials will be expressed by the linear equations

$$\left.\begin{aligned} V_1 &= A_{11}e_1 + A_{21}e_2 + \dots + A_{n1}e_n, \\ V_2 &= A_{12}e_1 + A_{22}e_2 + \dots + A_{n2}e_n, \\ \&c. &= \&c. \\ V_n &= A_{1n}e_1 + A_{2n}e_2 + \dots + A_{nn}e_n. \end{aligned}\right\} \dots\dots\dots \quad (A)$$

The coefficients A are called the *coefficients of potential.*

156.] Evidently there exist algebraic values of V corresponding to any assigned values of $e_1, e_2 \dots e_n$, though we do not assert that it is practically possible to charge the conductors without limit.

By solving the above linear equations we should obtain a new set expressing the charges in terms of the potentials, namely,

$$\left.\begin{aligned} e_1 &= B_{11}V_1 + B_{21}V_2 + \dots + B_{n1}V_n, \\ e_2 &= B_{12}V_1 + B_{22}V_2 + \dots + B_{n2}V_n, \\ \&c. &= \&c., \\ e_n &= B_{1n}V_1 + B_{2n}V_2 + \dots + B_{nn}V_n; \end{aligned}\right\} \dots\dots\dots \quad (B)$$

in which the coefficients B are functions depending only on the forms and positions of the conductors.

Since the equations (B) must give possible and determinate values of e for any assigned values of $V_1, V_2 \dots V_n$, it follows that there must exist a set of charges corresponding algebraically to any assigned set of values of the potentials.

The *capacity of a conductor* in presence of any other conductors is the charge upon it required to raise it to unit potential, when all the other conductors have potential zero. Thus, if $V_2 \dots V_n$ are all zero, we have from equation (B),

$$e_1 = B_{11}V_1;$$

and if $V_1 = 1$, $e_1 = B_{11}$, so that B_{11} is the capacity of C_1.

The coefficients B_{11}, B_{22}, &c., with repeated suffixes, are called *coefficients of capacity.* The coefficients B_{12}, B_{21}, &c., with distinct suffixes, are called *coefficients of induction.*

Properties of the Coefficients of Potential.

157.] $$A_{12} = A_{21}.$$

For let $V_1, V_2, \dots V_n$ be the potentials of the conductors, when C_1 has the charge e and all the others are uncharged, V the general value of the potential in this case. Then $V_2 = A_{12}e$.

Let $U_1, U_2, \ldots U_n$ be the potentials of the conductors, when C_2 has the charge e, and all the others are uncharged, U the general value of the potential in this case. Then $U_1 = A_{21}e$.

By Green's theorem applied to the infinite space external to all the conductors, in which $\nabla^2 V$ and $\nabla^2 U$ are everywhere zero, we have

$$V_1\iint\frac{dU}{d\nu}dS_1 + V_2\iint\frac{dU}{d\nu}dS_2 + \ldots + V_n\iint\frac{dU}{d\nu}dS_n$$
$$= U_1\iint\frac{dV}{d\nu}dS_1 + U_2\iint\frac{dV}{d\nu}dS_2 + \ldots + U_n\iint\frac{dV}{d\nu}dS_n,$$

in which $\iint dS_1$ denotes integration over C_1, and so on for the other conductors.

But
$$-\frac{1}{4\pi}\iint\frac{dV}{d\nu}dS$$

is the charge on the conductor C in the system whose potential is V. Hence
$$\iint\frac{dU}{d\nu}dS_2 = -4\pi e,$$

and all the other integrals in the first member vanish. Similarly
$$\iint\frac{dV}{d\nu}dS_1 = -4\pi e,$$

and all the other integrals in the second member vanish. The equation therefore becomes

$$V_2 \,.\, 4\pi e = U_1 \,.\, 4\pi e,$$
or
$$A_{12}e^2 = A_{21}e^2,$$
or
$$A_{21} = A_{12}.$$

In other words, the potential of C_1, due to unit charge on C_2 in presence of any conductors, is equal to the potential of C_2, due to unit charge on C_1 under the same circumstances.

158.] The coefficients of potential are all positive, and no one with distinct suffixes, as A_{1r}, is greater than the coefficient with either suffix repeated, as A_{11} or A_{rr}.

For, as proved in Art. 53, the potential can never be a maximum or minimum at any point unoccupied by free electricity. If therefore there be any positive potentials, the highest positive potential must be on some conductor; and if there be any negative, the lowest negative must be on some conductor. If the potentials be all greater or all less than zero, then zero, the potential at an infinite distance, is the least, or the greatest, potential as the case may be.

If any conductor has zero charge, the density of the distribution upon it must be positive on some parts of its surface, negative on other parts. Where the density is positive the lines of force proceed from the surface, and there must be some neighbouring part of space in which the potential is less than that of the conductor. Where the density is negative, there must be some neighbouring part where it is greater than that of the conductor.

Therefore, neither the greatest nor the least potential in the field can be the potential of a conductor with zero charge, neither can it be in free space.

Such greatest or least value must be that of a conductor having an actual charge, and the density on such conductor must be of the same sign throughout its surface, and must be positive for the highest positive potential, negative for the lowest negative potential. Therefore, if all the conductors $C_2 \dots C_n$ be uncharged, and C_1 have positive charge e_1, V_1, i.e. the potential of C_1, must be the greatest potential in the field, and zero must be the least, namely the potential at an infinite distance. We have then from equations A

$$V_1 = A_{11} e_1, \quad V_2 = A_{12} e_1, \text{ \&c.};$$

in which V_1 is positive, and $V_2 \dots V_n$ lie between V_1 and zero, and are therefore all positive. Hence also, A_{11} is greater than $A_{12} \dots$ or A_{1n}, and each of these latter is positive.

159.] *Properties of the Coefficients of Capacity and Induction.*

$$B_{12} = B_{21}.$$

For let V denote the general value of the potential, when C_1 is charged and has potential K, and all the other conductors are uninsulated. And let U denote the general value of the potential when C_2 is charged and has potential K, and all the other conductors are uninsulated. The equation of Art. 157 then becomes

$$K\iint \frac{dU}{d\nu}\,dS_1 = K\iint \frac{dV}{d\nu}\,dS_2,$$

or $$B_{21}\,.\,K^2 = B_{12}\,K^2,$$

or $$B_{21} = B_{12}.$$

In other words, the charge on C_1 if uninsulated when C_2 is raised to unit potential is equal to the charge on C_2 if uninsulated when C_1 is raised to unit potential, all the other conductors being in either case uninsulated.

160.] Each of the coefficients of capacity is positive.

For as we have seen it is possible so to charge the conductors as to make $V_2 \dots V_n$ each zero. Then V_1 must be either the greatest or least potential in the field, viz. the greatest if e_1 be positive, the least if e_1 be negative.

Therefore, we have in this case

$$e_1 = B_{11}\,V_1,$$

and since e_1 and V_1 are of the same sign, B_{11} is positive.

161.] Each of the coefficients of induction is negative.

If $V_2 \dots V_n$ are all zero and e_1 not zero, the density on each of the conductors $C_2 \dots C_n$ must be of the same sign throughout its surface, viz. opposite to that of e_1; for let e_1 be positive, then if the density at any point on any other conductor were positive, there would be a less potential than zero, that is, less than that of any of the conductors, at some point in free space.

But the charge on C_2 is in this case

$$e_2 = B_{12}\,V_1,$$

and since V_1 is positive and e_2 negative, B_{12} is negative.

162.] The sum of the coefficient of capacity and all the coefficients of induction relating to the same conductor is positive.

For let the conductors be so charged as to be all at the same potential V. Then

$$e_1 = \{B_{11} + B_{21} + \dots + B_{n1}\}\,V.$$

Now if V be positive, e_1 must be positive, for if it were negative, there would be a greater potential than V somewhere in free space.

Therefore $B_{11} + B_{21} + \dots + B_{n1}$ is positive.

163.] If two conductors C_1, C_2, originally separate, be connected together by a very thin wire so as to form one new conductor, the capacity of the new conductor is less than the sum of the capacities of the two original conductors. For let e_1 be the charge on C_1 when it is at unit, and all the others at zero potential, that is, e_1 is the capacity of C_1. Let e'_2 be the charge on C_2 in this case.

Similarly, when C_2 is at unit, and all the others at zero potential, let e'_1 and e_2 be the charges on C_1 and C_2 respectively. Then e'_1 and e'_2 are negative, e_1 and e_2 positive.

If C_1 have the charge $e_1+e'_1$, and C the charge $e_2+e'_2$, and every other conductor have the sum of its charges in the two cases, C_1 and C_2 will both be at unit, and all the other conductors at zero potential, and if the connexion between C_1 and C_2 be now made, no alteration takes place in the distribution of electricity.

The charge upon the new conductor, that is, its capacity, is $e_1+e_2+e'_1+e'_2$, which is less than e_1+e_2.

164.] Any conductor of given bounding surface may be either solid or a hollow shell, and all the coefficients of potential, capacity, or induction, whether relating to that, or any other conductor, are the same in either case, and are not affected by the introduction of any conductors whatever inside a hollow conductor.

For let C_r be a hollow conductor. Then, as proved in Art. 91, if there be any electrification whatever the algebraic sum of which is zero within C_r, that electrification together with the induced distribution on the inner face of C_r have zero potential at each point in the substance of, or external to, C_r, and may be removed without affecting the distribution on the outer face of C_r or anywhere external to it. It follows that the potential of any other conductor due to a distribution on the outer surface of C_r is unaffected by the presence or absence of such electrification within C_r, and depends only on the forms and positions of the surfaces bounding C_r and the other conductors. Therefore the coefficients A, and therefore also the coefficients B, depend only on these forms and positions.

165.] *On the Comparison of similar Electrified Systems.*

Let there be given two electrical systems similar in all respects but of different linear dimensions.

Let the linear dimensions be denoted by λ, so that λ has different values in the two systems respectively. If the quantity of electricity per unit of volume be the same in both systems, the potential will vary as λ^2. For it is of the form $\iiint \frac{\rho\, dx\, dy\, dz}{r}$, and in this case ρ is independent of λ, and $dx\ dy\ dz$ and r each proportional to λ. Evidently the force at corresponding points, being the variation of V per unit of length, varies in this case as λ.

If, on the other hand, the quantity of electricity in homologous portions of space, instead of in unit of volume, be given constant, ρ will vary as $\frac{1}{\lambda^3}$, and V as $\frac{1}{\lambda}$, and the force as $\frac{1}{\lambda^2}$.

If the system consist entirely of conductors, and the superficial density, that is the quantity of electricity per unit of surface, be constant, V will vary as λ, and the force will be invariable.

If, on the other hand, the quantity of electricity on homologous portions of surface be invariable, V will vary as $\frac{1}{\lambda}$, and the force as $\frac{1}{\lambda^2}$.

It follows from these considerations that, as between similar systems of conductors of different linear dimensions, the coefficients A vary inversely, and the coefficients B directly, as the linear dimensions.

CHAPTER X.

ENERGY.

On the Intrinsic Energy of an Electrical System.

ARTICLE 166.] *If V be the potential of any electrified system, the work done in constructing the system against the repulsion of its own parts is*

$$\frac{1}{2}\iiint V e\,dx\,dy\,dz$$

taken throughout the system, where $e\,dx\,dy\,dz$ is the quantity of free electricity that exists within the volume element $dx\,dy\,dz$.

For we may suppose the charges in all the volume elements to be originally zero, and to be gradually increased, always preserving the same proportion to one another, till they attain their values in the actual system. The potentials at any instant during this process will be proportional to the charges at that instant. Further, we may suppose the process to take place uniformly throughout any time τ. Then if t be the time that has elapsed since the beginning of the process, the charge in any element of volume may be represented by λt, and the potential by μt,. where λ and μ are constants for the same element. The final values of e and V will be $\lambda\tau$ and $\mu\tau$. The charges which will be added in the small interval of time dt, or $t+dt-t$, will be $\lambda\,dt$, and the work done in bringing these charges to the then existing potentials, represented by μt, will be

$$\iiint \mu\lambda\,t\,dt\,dx\,dy\,dz.$$

The whole work done from beginning to end of the process is

$$\int_0^\tau \iiint \mu\lambda\,t\,dt\,dx\,dy\,dz = \frac{1}{2}\iiint \mu\lambda\,\tau^2\,dx\,dy\,dz = \frac{1}{2}\iiint V e\,dx\,dy\,dz.$$

This quantity is called *the intrinsic energy* of the system. We shall generally denote it by E.

If the system consist only of conductors on which the charges are e_1, e_2, &c., we have $E = \frac{1}{2}\Sigma Ve$, Σ denoting summation for all the conductors.

In like manner if there be two distinct electrical systems and the charges and potentials in one be denoted by e and V, and in the other by e' and V', the work done in constructing the first against the repulsion of the second, supposed existing independently, is $\iiint V'e\,dx\,dy\,dz$, and this must be equal to the work done in constructing the second against the repulsion of the first, that is

$$\iiint Ve'\,dx\,dy\,dz = \iiint V'e\,dx\,dy\,dz,$$

or for a system of conductors

$$\Sigma Ve' = \Sigma V'e.$$

167.] *To prove that*

$$\iiint Ve\,dx\,dy\,dz = \frac{1}{4\pi}\iiint\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz,$$

the integral being taken throughout all space.

Let us consider an infinitely distant closed surface S enclosing the whole electric field. Applying Green's theorem to the space within S, we have

$$\iiint\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz$$
$$= \iint V\frac{dV}{d\nu}dS - \iiint V\nabla^2 V\,dx\,dy\,dz - \iint V\left\{\frac{dV}{d\nu}+\frac{dV}{d\nu'}\right\}dS',$$

in which the first double integral relates to the infinitely distant surface S, and the second to the surfaces S', if any, within S on which there is superficial electrification.

But
$$\iint V\frac{dV}{d\nu}dS = 0,$$

and the two remaining terms on the right-hand side of the equation are together equal to

$$\iiint 4\pi\,.\,Ve\,dx\,dy\,dz.$$

Hence

$$\iiint Ve\,dx\,dy\,dz = \frac{1}{4\pi}\iiint +\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\Big\}dx\,dy\,dz.$$

It was proved in Art. 13 that if V be one of the class of functions which satisfy the following conditions, viz.

(1) $\iint \frac{dV}{d\nu}\, dS_1$ taken over the surface $S_1 = e_1$,

$\iint \frac{dV}{d\nu}\, dS_2$ taken over the surface $S_2 = e_2$,

&c. = &c.,

(2) $\nabla^2 V$ has given value at every point outside of all the surfaces,

(3) V vanishes at an infinite distance,

then the integral

$$Q_V = \iiint \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} dx\, dy\, dz$$

throughout all space outside of all the surfaces has its least value when V is constant over each surface.

We now see the physical meaning of the theorem as applied to an electrified system. For V being the potential, $-\frac{1}{4\pi}\iint \frac{dV}{d\nu} dS$, taken over any surface, is the charge upon that surface, whether the charge be so distributed over it as to make V constant or not.

Now in whatever way the charges be distributed over the surfaces, consistently with the whole charge on each surface being given, $\frac{1}{2}\iiint V e\, dx\, dy\, dz$ is the energy of the system.

$$\text{But } \frac{1}{2}\iiint V e\, dx\, dy\, dz = \frac{1}{8\pi} \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} dx\, dy\, dz$$

$$+ \frac{1}{8\pi} \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} dx'\, dy'\, dz',$$

the first integral on the right-hand side relating to the space outside, and the second to the space inside of the surfaces.

And the proposition shows that the first integral, or $\frac{1}{8\pi} Q_V$, has its least possible value when the charges are so distributed as to make V constant over each surface. And in that case V is constant throughout the space inside of the surfaces, and therefore the second integral is zero.

It follows that the distribution actually assumed on each surface when the electricity is free to distribute itself, that is, when the surface is a conductor, is the one which makes the intrinsic energy the least possible, given the charges on the conductors respectively.

168.] The potential of any conductor, as C_r, due to a quantity of electricity e in the volume element $dx\,dy\,dz$, is evidently of the form $A_{rs}e$, where A_{rs} is a coefficient depending, like the coefficients A already investigated, on the forms and positions of the conductors and the position of the element in question, and the suffix s relates to that element, and r to the conductor.

Similarly the potential at the element due to a charge e on C_r will be $A_{sr}e$, where A_{sr} depends only on the form and position of the conductors and the position of the element.

Also the equality of A_{rs} and A_{sr} follows from that of $\Sigma Ve'$ and $\Sigma V'e$ above proved.

Thus the systems of equations (A) and (B) of Arts. 155, 156 can be extended to any electrified system, whether consisting exclusively of conductors of finite size or not.

Evidently if in the equation $E = \frac{1}{2}\Sigma Ve$ we express every V in terms of the charges by means of equations (A), E will be a quadratic function of the charges with coefficients depending on the forms and positions of the conductors. In this form we shall write it E_e.

Similarly if we express every e in terms of the potentials by means of equations (B), E will be a quadratic function of the potentials. In this form we shall write it E_V.

It follows from the equality of A_{12} and A_{21}, &c., that

$$\frac{dE}{de_1} = V_1, \qquad \frac{dE}{de_2} = V_2, \text{ \&c.};$$

or generally,

$$\frac{dE}{de} = V.$$

For as from $E = \frac{1}{2}\Sigma Ve$ we have

$$\frac{dE}{de_1} = \left(\frac{dE}{de_1}\right) + \left(\frac{dE}{dV_1}\right)\frac{dV_1}{de_1} + \left(\frac{dE}{dV_2}\right)\frac{dV_2}{de_1}, \text{ \&c.}$$

and

$$\left(\frac{dE}{de_1}\right) = \tfrac{1}{2}V_1, \quad \left(\frac{dE}{dV_1}\right) = \tfrac{1}{2}e_1, \quad \text{\&c.},$$

therefore $$\frac{dE}{de_1} = \frac{1}{2}V_1 + \frac{1}{2}A_{11}e_1 + \frac{1}{2}A_{12}e_2 + \dots \&c.$$
$$= \frac{1}{2}V_1 + \frac{1}{2}(A_{11}e_1 + A_{21}e_2 + \dots \&c.)$$
$$= V_1.$$

Similarly $$\frac{dE}{dV_1} = e_1, \ \&c.$$

169.] *On the Mechanical Action between Electrified Bodies.*

We have seen that the intrinsic energy of any electrified system is of the form $\frac{1}{2}\Sigma Ve$, where e is any quantity of electricity forming part of the system, V the potential at the point where that quantity is situated.

If q be any generalised coordinate defining the position of the system, the force tending to produce in the system the displacement dq is

$$-\frac{dE}{dq} \quad \text{or} \quad -\frac{1}{2}\frac{d}{dq}(\Sigma Ve).$$

Now if the charges e are invariable in magnitude, the potentials V are functions of the coordinates q, and therefore the force is

$$-\frac{1}{2}\Sigma e\frac{dV}{dq},$$

in which every V is a function of q in respect of the coefficients A.

If, on the other hand, the potentials be maintained constant notwithstanding displacement, by proper variation of the charges, the force is

$$-\frac{1}{2}\Sigma V\frac{de}{dq},$$

in which every e is a function of q as it is involved in the coefficients B. It remains to find the relation between the forces in these two cases.

170.] *If q be any one of the generalised coordinates defining the position of the conductors, and if R denote the force tending to increase q when the charges are invariable, and R' be that force when all the potentials are maintained constant, then* $R + R' = 0$.

For $$E_e + E_V = 2\Sigma Ve;$$

let e, V, and q all vary.

Then we have

$$\Sigma \frac{dE_e}{de}\delta e + \frac{dE_e}{dq}\delta q + \Sigma \frac{dE_V}{dV}\delta V + \frac{dE_V}{dq}\delta q = \Sigma V\delta e + \Sigma e\delta V.$$

But $$\frac{dE_e}{de} = V,$$

and $$\frac{dE_V}{dV} = e,$$

hence $$\Sigma \frac{dE_e}{de}\delta e = \Sigma V\delta e,$$

and $$\Sigma \frac{dE_V}{dV}\delta V = \Sigma e\delta V;$$

therefore $$\frac{dE_e}{dq}\delta q + \frac{dE_V}{dq}\delta q = 0.$$

Now $\frac{dE_e}{dq}$ is the rate of variation of E with q when the charges are constant, and is therefore the force tending to diminish q under those circumstances; that is, it is $-R$.

Similarly $$\frac{dE_V}{dq} = -R',$$

hence $$R + R' = 0.$$

171.] If any group of conductors previously insulated from one another become connected by very thin wires, so as to form one conductor, the energy of the system is thereby diminished; and the energy lost by it is equal to that of an electrical system in which the superficial density at any point is the difference of the densities at the same point before and after the connection is made: that is, is equal to the energy of the system which must be added to the original in order to produce the new system.

For let V denote the potential of the system after the connection is made, $V + V'$ the original potential. Then V and V' are both constant throughout each conductor of the system. The charge on any conductor which retains its insulation, or $-\frac{1}{4\pi}\iint\frac{dV}{d\nu}dS$, remains unaltered.

Any group of conductors which become connected form one combined conducting surface on which the aggregate charge, or $-\frac{1}{4\pi}\iint\frac{dV}{dv}dS$, is unaltered.

V is then a function which satisfies the conditions (1) $\nabla^2 V = 0$ at all points outside of all the connected conductors, (2) $\iint\frac{dV}{dv}dS$ has given values over each of them, (3) V is constant over each of them, while $V+V'$ satisfies conditions (1) and (2), but is not constant over each of the connected conductors. Therefore, by Art. 13,

$$Q_{V+V'} = Q_V + Q_{V'},$$

where

$$Q_V = \iiint\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz$$

throughout all space outside of the surfaces, and $Q_{V+V'}$ and $Q_{V'}$ have corresponding values.

Now $\frac{1}{8\pi}Q_{V+V'}$ is the original energy, $\frac{1}{8\pi}Q_V$ is the energy after the connection is made, and $\frac{1}{8\pi}Q_{V'}$ is the energy of the system in which the potential is the difference of the two potentials, and therefore, by the principle of superposition, in which the density at any point is the difference between the densities before and after the connection is made.

172.] If any portion of space S, previously not a conductor, become a conductor, or which is the same thing, if a conductor be brought from outside of the field and made to occupy the space S within the field, the energy of the system is thereby diminished; and the energy lost by it is equal to that of the system in which the superficial density at any point on any conductor or on the surface of S is the difference of the densities at the same point before and after S became a conductor; that is to the energy of the system of densities which must be combined with the original to produce the new system.

For let V be the potential of the system when the space S is a conductor, $V+V'$ the potential when S is non-conducting space.

Then V is constant throughout S and throughout each conductor, $V+V'$ is constant over each other conductor but not over S.

Let $\frac{1}{8\pi}Q_V$ be the energy of the system in the former case, $\frac{1}{8\pi}Q_{V+V'}$ the energy in the latter case.

It can then be shewn, as in the last case, that

$$\frac{1}{8\pi}Q_{V+V'} = \frac{1}{8\pi}Q_V + \frac{1}{8\pi}\iiint\left\{\left(\frac{dV'}{dx}\right)^2 + \left(\frac{dV'}{dy}\right)^2 + \left(\frac{dV'}{dz}\right)^2\right\}dx\,dy\,dz$$

throughout all space outside of the original conductors whether within or without S. But the integral of the second term is the energy of the system in which the potential is V', that is of the system which must be combined with the original to produce the new system.

It follows that a conductor without charge is always *attracted* by any electrical system if at a sufficient distance from it.

Hence also any number of uncharged conductors in a field of constant force generally attract each other.

173.] It follows as a corollary to the two last propositions that if a conductor increase in size, the energy of the system is thereby diminished.

For if C be a conductor, S an adjoining space, if S became a conductor insulated from C, the energy would be diminished, and if the new conductor were then connected with C so as to form one conductor with it, the energy would be further diminished. Hence the resultant force on an element of surface of a conductor is, if the charges be constant, in the direction of the normal outwards.

174.] If S be a surface completely enclosing a conductor C, then if S were itself a conductor, its capacity would be greater than that of C.

For let the conductor C be charged to potential unity, all the other conductors being at zero potential. Let M be its capacity, that is, the charge upon it under these circumstances.

Since unity is in this case the highest potential in the field, the potential on S is less than unity at every point. Let it be denoted by V. If σ be the density of a distribution over S which has at every point on S the potential V,

$$\iint \sigma dS = M \quad \text{by Art. 60},$$

and
$$\iint V\sigma dS, < \iint \sigma dS,$$

since $V < 1$, therefore $\iint V\sigma dS < M$.

Next let the same quantity of electricity M be so distributed over S as to have constant potential V' in presence of the other uninsulated conductors. Let σ' be its density in that case.

Then
$$\iint V'\sigma' dS < \iint V\sigma dS \quad \text{by Art. 167},$$

also
$$\iint V'\sigma' dS = V' \iint \sigma' dS = V'M.$$

Therefore, à fortiori, $V'M < M$ or $V' < 1$, and therefore a larger charge would be necessary to bring S to unit potential.

Earnshaw's Theorem.

175.] If an electrified system A be in mechanical equilibrium in presence of another electrified system B, and be capable of movement without touching the latter, the equilibrium cannot be stable.

For let us suppose first that all the electricity in B is fixed in space, and all the electricity in A fixed in the system, so that the system A is capable of movement as a rigid body. And let us further suppose first that it is capable of motion of translation only.

In this case the position of A is completely determined by the position in space of any single point O in it, and is therefore a function of the coordinates of O referred to any system of axes fixed in space.

Let the position of equilibrium be when O coincides with a certain point C in space.

Let V be the potential of the system B. Let $\rho\, dx dy dz$ be the quantity of electricity of the system A in the volume element $dx\,dy\,dz$. Then

$$\frac{1}{2}\iiint \rho V\,dx\,dy\,dz$$

throughout A is the intrinsic energy of the mutual action of the two systems, that is, that part of the whole energy which varies with the position of O.

Let $$\frac{1}{2}\iiint \rho V\,dx\,dy\,dz = E;$$

then $$\frac{dE}{dx} = \frac{1}{2}\iiint \rho\frac{dV}{dx}\,dx\,dy\,dz,$$

$$\frac{d^2E}{dx^2} = \frac{1}{2}\iiint \rho\frac{d^2V}{dx^2}\,dx\,dy\,dz;$$

and as similar equations hold for y and z,

$$\nabla^2 E = \frac{1}{2}\iiint \rho\,\nabla^2 V\,dx\,dy\,dz.$$

Now so long as no part of A coincides with any part of B, $\nabla^2 V = 0$, and therefore $\nabla^2 E = 0$.

Again, since A is capable of motion of translation without any part of it coinciding with any part of B, it must be possible to describe a small closed surface S about C, the position of equilibrium, such that, if O be anywhere within S, $\nabla^2 V = 0$, and $\nabla^2 E = 0$.

But E is a function of x, y, z, the coordinates of O. Therefore, by Green's theorem applied to the surface S and the function E,

$$\iint \frac{dE}{d\nu}\,dS = \iiint \nabla^2 E\,dx\,dy\,dz,$$
$$= 0.$$

Either therefore E must be constant for all positions of O in the neighbourhood of C which are consistent with A not touching B, in which case the equilibrium is neutral; or there must be some part of the surface S such that, if O be on that portion, E is less than when O is at C, that is less than in the position of equilibrium. Therefore any small displacement of A in this direction will bring it into a field of force tending to move it still further in the same direction, and the equilibrium is therefore unstable.

If the system has any more degrees of freedom, as for instance freedom to rotate about an axis, or for the electricity on A or B to change its position, instead of being fixed as we assumed it to be, *à fortiori* the energy will be capable of becoming less in the displaced position, and therefore the equilibrium is unstable.

176.] *On a system of insulated conductors fixed in a field of uniform force and without charge.*

Let us first suppose the uniform force is unit force parallel to the axis of x. Then we may always choose the origin of coordinates, so that the potential of the force at any point may be denoted by $-x$.

The induced distributions on the conductors must be such that the potential due to them at any point in any conductor shall be $C+x$, where C is constant over each conductor but has generally a different value for different conductors.

Let the density of this induced distribution at any point be $\phi_x(x, y, z)$, or shortly ϕ_x. Then $X\phi_x$ is the density of the induced distribution which would be formed if the force were X. And the potential due to the distribution $X\phi_x$ is therefore $CX+xX$.

Definition.—The function $\iint x\,\phi_x dS$ taken over the surface of every conductor of the system is the *electric polarisation* of the system in direction x due to a unit force in direction x acting at every point of the system. It is evidently independent of the position of the origin. For

$$\iint (a+x)\,\phi_x dS = a\iint \phi_x dS + \iint x\,\phi_x dS = \iint x\,\phi_x dS,$$

since $\iint \phi_x dS$ is the algebraic sum of the induced distribution and is therefore zero.

In like manner we may define

$$\iint y\,\phi_x dS, \quad \iint z\,\phi_x dS$$

to be the *electric polarisation* of the system in the directions of y and z respectively due to unit force in direction x.

In like manner if there be forces Y and Z parallel to the other coordinate axes, they will produce in the system induced distributions whose densities are $Y\phi_y$ and $Z\phi_z$ respectively, and the functions

$$\iint y\,\phi_y\,dS, \quad \iint x\,\phi_y\,dS, \quad \iint z\,\phi_y\,dS$$

will denote the polarisation in the directions y, x, and z respectively due to unit force in direction y. Similarly, $\iint z\,\phi_x\,dS$, &c. denote polarisation due to unit force in direction z. And each of the quantities $\iint y\phi_x\,dS$, &c. is independent of the position of the origin if the direction of the axes be given.

If ξ, η, ζ denote the total polarisation parallel to x, y, z respectively, due to the three forces, we shall have

$$\xi = X\iint x\,\phi_x\,dS + Y\iint x\,\phi_y\,dS + Z\iint x\,\phi_z\,dS,$$

$$\eta = X\iint y\,\phi_x\,dS + Y\iint y\,\phi_y\,dS + Z\iint y\,\phi_z\,dS,$$

$$\zeta = X\iint z\,\phi_x\,dS + Y\iint z\,\phi_y\,dS + Z\iint z\,\phi_z\,dS.$$

177.] We can now prove that

$$\iint x\,\phi_y\,dS = \iint y\phi_x\,dS.$$

For the potential of the system of densities denoted by ϕ_x is, as we have seen, $C+x$, C being, as before mentioned, a constant for each conductor, but having generally different values for different conductors. Therefore

$$\iint C\,\phi_y\,dS + \iint x\,\phi_y\,dS$$

is the work done in constructing the distribution whose density is ϕ_y (which we may call the system ϕ_y), against the repulsion of the system ϕ_x previously existing. But since for each conductor the algebraic sum of the induced distribution ϕ_y is zero, and C is constant,

$$\iint C\phi_y\,dS = 0,$$

and $\iint x\,\phi_y\,dS$ expresses the amount of work above mentioned. In like manner $\iint y\,\phi_x\,dS$ is the work done in constructing the system ϕ_x against the repulsion of the system ϕ_y previously existing. But by the conservation of energy these two quantities of work must be equal. Therefore

$$\iint x\,\phi_y\,dS = \iint y\,\phi_x\,dS.$$

Similarly,

$$\iint x\,\phi_z\,dS = \iint z\,\phi_x\,dS,$$

$$\iint y\,\phi_z\,dS = \iint z\,\phi_y\,dS.$$

178.] If the direction cosines of any line referred to any system of rectangle axes be α, β, γ, the polarisation parallel to this line expressed in terms of these coordinates becomes

$$\iint (\alpha x + \beta y + \gamma z)(X\phi_x + Y\phi_y + Z\phi_z)\,dS;$$

that is,

$$\alpha\left\{X\iint x\,\phi_x\,dS + Y\iint x\,\phi_y\,dS + Z\iint x\,\phi_z\,dS\right\}$$
$$+\beta\left\{X\iint y\,\phi_x\,dS + Y\iint y\,\phi_y\,dS + Z\iint y\,\phi_z\,dS\right\}$$
$$+\gamma\left\{X\iint z\,\phi_x\,dS + Y\iint z\,\phi_y\,dS + Z\iint z\,\phi_z\,dS\right\};$$

that is, $\alpha\xi + \beta\eta + \gamma\zeta$. The polarisations ξ, η, ζ are in fact vector quantities, and admit of composition and resolution.

Evidently, if α, β, γ be proportional to ξ, η, ζ, and

$$\rho = \alpha\xi + \beta\eta + \gamma\zeta,$$

then

$$\xi = \alpha\rho,\quad \eta = \beta\rho,\quad \zeta = \gamma\rho,$$

and the polarisation in any direction α', β', γ', is

$$\alpha'\xi + \beta'\eta + \gamma'\zeta,$$

that is

$$(\alpha\alpha' + \beta\beta' + \gamma\gamma')\,\rho,$$

and is therefore zero for any direction at right angles to the direction of the resultant of ξ, η, ζ.

179.] Let it now be required to find a direction relative to the system such that if the resultant force be in that direction, the resultant polarisation shall be in the same direction.

In that case ξ, η, ζ must be proportional to X, Y, Z. That is,

$$\xi = \epsilon X,$$
$$\eta = \epsilon Y,$$
$$\zeta = \epsilon Z,$$

where ϵ is a quantity to be determined.

Hence we have

$$\left(\iint x\,\phi_x\,dS - \epsilon\right)X + \iint x\,\phi_y\,dS\,.\,Y + \iint x\,\phi_z\,dS\,.\,Z = 0,$$
$$\iint y\,\phi_x\,dS\,.\,X + \left(\iint y\,\phi_y\,dS - \epsilon\right)Y + \iint y\,\phi_z\,dS\,.\,Z = 0,$$
$$\iint z\,\phi_x\,dS\,.\,X + \iint z\,\phi_y\,dS\,.\,Y + \left(\iint z\,\phi_z\,dS - \epsilon\right)Z = 0.$$

These equations are of the same form as those employed in Thomson and Tait's *Natural Philosophy*, 2nd Edition, p. 127, for determining the principal axes of a strain. As there shown, the system leads to a cubic equation in ϵ, of which, when

$$\iint y\,\phi_x\,dS = \iint x\,\phi_y\,dS, \text{ \&c.},$$

the three roots are always real. The equation is the same as that treated of in Todhunter's *Theory of Equations*, 2nd Edition, p. 108. As shown by Thomson and Tait, each of the three values of ϵ corresponds to a fixed line in the system, and the three lines corresponding to the three roots are mutually at right angles.

There exist, therefore, for every system of conductors three directions at right angles to each other, and fixed with reference to the system, such that a uniform force in any one of these directions produces no polarisation in either of the others. We might define these directions as the *principal axes of polarisation* of the system of conductors in question. If we take these three lines for axes we shall have evidently

$$\iint x\,\phi_y\,dS = 0, \quad \iint y\,\phi_z\,dS = 0, \text{ \&c.}$$

Let us denote

$$\iint x\,\phi_x\,dS \text{ by } Q_x, \quad \iint y\,\phi_y\,dS \text{ by } Q_y, \quad \text{and} \quad \iint z\,\phi_z\,dS \text{ by } Q_z\,;$$

then the polarisation in any line whose direction cosines referred to these axes are α, β, γ, due to unit force in that line is

$$\alpha^2 Q_x + \beta^2 Q_y + \gamma^2 Q_z.$$

180.] *Of the energy of the polarisation of a system of conductors placed in a field of uniform force.*

Let the system be referred to its principal axes. Let X, Y, Z be the forces parallel to these axes respectively. Let $X\phi_x$ be the density at any point on any conductor which would be produced by X alone acting, $Y\phi_y$ and $Z\phi_z$ the same for Y and Z. Then $X\phi_x + Y\phi_y + Z\phi_z$ is the density when all three forces act.

The potential of the three forces is

$$-(Xx + Yy + Zz).$$

The work done in constructing the system against the external forces is then

$$-\iint (Xx + Yy + Zz)(X\phi_x + Y\phi_y + Z\phi_z)\, dS$$

taken over the surface of every conductor. The work done against the mutual forces of the system itself is

$$+\frac{1}{2}\iint (Xx + Yy + Zz)(X\phi_x + Y\phi_y + Z\phi_z)\, dS.$$

The whole work is therefore

$$-\frac{1}{2}\iint \{Xx + Yy + Zz\}\{X\phi_x + Y\phi_y + Z\phi_z\}\, dS$$

taken over the surface of every conductor, and the work done against the mutual forces of the separated electricities is the same expression with the positive sign.

But the axes being principal axes,

$$\iint x\phi_y\, dS = 0, \quad \iint x\phi_z\, dS = 0, \text{ \&c.}$$

Therefore the energy is

$$-\frac{1}{2}\left\{X^2\iint x\phi_x dS + Y^2\iint y\phi_y dS + Z^2\iint z\phi_z\, dS\right\},$$

$$\text{or} \quad -\frac{1}{2}\{X^2 Q_x + Y^2 Q_y + Z^2 Q_z\}.$$

181.] If the conductors be free to move in any manner, either by translation or rotation, they will endeavour to place themselves in such positions as that the above expression within brackets shall be maximum, given the resultant external force.

If the system consists of a single rigid conductor Q_x, Q_y and Q_z will not be altered by any motion of the conductor. If placed in a field of uniform force and free to move about an axis, its position of stable equilibrium will be that in which the axis of greatest polarisation coincides with the direction of the force. If the system consist of many conductors, and they be not so distant that their mutual influence may be neglected, they will still tend either by rotation or translation to assume a position in which the axis of greatest polarisation shall coincide with the direction of the force; but Q_x, Q_y, and Q_z will in this case generally vary with change of position of the conductors.

182.] We have hitherto treated ϕ_x, ϕ_y, &c. as the densities of the distribution produced on a conductor in a field of uniform force. If we extend our definition, and define $\iint x\phi\, dS$ to be the polarisation in direction x of an uncharged conductor placed in any electric field, ϕ being the density of the induced distribution, we can prove the following proposition:—

The electric polarisation of any spherical conductor due to any distribution of electricity entirely without it is equal in magnitude to the resultant force at the centre of the sphere due to that distribution multiplied by the cube of the radius, and is in the same direction with that resultant.

For let X be the x-component of force at the centre of the sphere due to the external system. Then $-X$ is the x-component of force at the centre due to the induced electrification.

But the last-named component is

$$-\iint \frac{x}{r^3}\,\sigma\, dS,$$

if σ be the density of the induced distribution.

Hence $$\iint x\,\sigma\, dS = r^3 X.$$

Similarly $$\iint y\,\sigma\, dS = r^3 Y,$$

$$\iint z\,\sigma\, dS = r^3 Z,$$

if Y and Z be the components of force parallel to y and z, and this establishes the proposition.

If v be the volume of the sphere,

$$\iint x\,\sigma\,dS = \frac{3\,v}{4\,\pi}X,$$

$$\iint y\,\sigma\,dS = \frac{3\,v}{4\,\pi}Y,$$

$$\iint z\,\sigma\,dS = \frac{3\,v}{4\,\pi}Z.$$

CHAPTER XI.

SPECIFIC INDUCTIVE CAPACITY.

ARTICLE 183.] HITHERTO, in accordance with the plan laid down in Chapter IV, and with the view of giving the two-fluid theory as complete a development as possible in its most abstract form, all space has been regarded as made up, so far as electrical properties are concerned, of two kinds; conducting space, through which the fluids pass from point to point under the influence of any electromotive force however small, and non-conducting or insulating space, through which no force however large can cause such passage of the fluids to take place.

Conducting spaces are necessarily occupied by actual substances, prominent among which are all metals. Non-conducting spaces may be occupied by actual substances, called non-conductors, insulators, or dielectrics (the last term having reference to their transmission of electric action as distinguished from the passage of the electric fluids), such as dry air and other gases, wood, shell lac, sulphur, &c., or these spaces may be vacuum. A perfect vacuum was at one time regarded as a perfect insulator.

184.] Faraday was the first to point out, as the result of experiments performed by him, that there is a considerable diversity of constitution in dielectric media, in reference to their electric properties; that, in fact, while different substances possess the common property of refusing passage to the electric fluids, they are nevertheless endowed with very different electrical properties in certain other respects. If, for instance, there be, as in Faraday's experiments, two concentric conducting spherical surfaces, and the space between them be filled with any dielectric substance, the potential of the inner sphere due to any charge e on itself (which according to the simple theory hitherto investigated should be $\frac{e}{r}$, where r is the radius) will be found to depend on the nature of the dielectric medium in the space between the spheres. And therefore the charge on the sphere necessary to produce a given potential, or the *capacity* of the sphere, is greater for some of such substances than for others. The dielectric, in Faraday's language, has *inductive capacity*. It is less for air and the permanent gases than for any solid dielectrics, and rather less for vacuum than for air.

185.] In order to explain this phenomenon, Faraday adopts the hypothesis that any dielectric medium consists of a great number of very small conducting bodies interspersed in, and separated by, a completely insulating medium impervious to the passage of electricity. In his own words, 'If the space round a charged 'globe were filled with a mixture of an insulating dielectric, as 'oil of turpentine or air, and small globular conductors, as shot, 'the latter being at a little distance from each other, so as to be 'insulated, then these in their condition and action exactly 'resemble what I consider to be the condition and action of the 'particles of the insulating dielectric itself. If the globe were 'charged, these little conductors would all be polar; if the globe 'were discharged, they would all return to their normal state, to 'be polarised again upon the recharging of the globe.'

The properties of such a medium closely resemble, as far as their mechanical action is concerned, those of a magnetic mass, as conceived by Poisson, each of Faraday's 'shot' being in fact when polarised equivalent to a little magnet, except that in dealing with magnetic masses the polarisation is usually understood to be in parallelopipeds instead of in spherical particles, and Poisson's investigations are therefore applicable. (See *Mémoires de l'Institut* for 1823 and 1824.)

The mathematical theory has also been treated by Mossotti with especial reference to Faraday's theory, but by a different method from that here employed.

186.] In accordance with this hypothesis of Faraday's, we will consider the dielectric as consisting of a great number of very small conducting bodies, *not necessarily spherical*, and separated by a perfectly insulating medium. A first object is to find the form assumed by Poisson's equation when the conductors become infinitely small.

In this medium take any parallelopiped whose edges, h, k, l, are parallel to the coordinate axes. Let these edges be very small compared with the general dimensions of the electric field, but yet infinitely great compared with the dimensions of any of the little conductors in question. The face kl of the parallelopiped will intersect a great number of these little conductors.

If the medium be subjected to any electromotive forces, there will be on each conductor an induced distribution of electricity whose superficial density we shall denote by ϕ. Then for each conductor $\iint \phi\, ds = 0$.

Let X be the average value over the face kl of the force parallel to x, and therefore normal to kl, due to the whole electric field, including the induced distributions on all the conductors whether intersected by kl or not. We will first assume that the corresponding forces Y, Z are zero.

In this case the average value per unit area of kl of the algebraic sum of the induced distribution on the intersected conductors which lies to the right, or positive, side of kl is, by the principle of superposition, proportional to X. Let it be QX, Q being the value which it would have if X were the unit of force.

If the forces Y and Z are not zero, then the quantity of the induced distribution on any individual conductor lying to the right of kl will generally depend on Y and Z as well as on X. But if the conductors be in all manner of orientation indifferently, the quantity of free electricity to the right of kl, due to Y and Z, will disappear on taking the average; because for any conductor, if the total density of the induced distribution arising from Y or Z for any position of the conductor be calculated, then on turning the conductor through two right angles about an axis parallel to x, the corresponding density in the new position will be equal to that in the former position, but of opposite sign.

We shall for the present confine ourselves to the case in which the conductors *are* orientated indifferently in all directions, and we shall define a medium in which this is the case to be an *isotropic* medium, and any other to be a *heterotropic* medium.

It is evident that in an isotropic medium the quantity of the induced electricity in the conductors intersected by the faces hk or hl of the parallelopiped which lies to the positive side of those faces respectively, due to unit force in direction y or z, is the same as the corresponding quantity for the face kl. That is, it is Q.

187.] The total electricity included within the parallelopiped will consist of (1) $\rho\, hkl$, the quantity of the given electrical distribution, which is supposed to exist independently of the condition of the medium within hkl, whether it be vacuum or dielectric; (2) the sum of the induced superficial distribution on those parts of the conductors intersected by the faces of the parallelopiped which lie within its volume. The part of (2) arising from the two kl faces will be

$$klQX \quad \text{and} \quad -klQX - hkl\frac{d}{dx}(QX)$$

respectively, and their sum is therefore

$$-hkl\frac{d}{dx}(QX).$$

In like manner the part of (2) arising from the faces hk, hl of the parallelopiped are

$$-hkl\frac{d}{dy}(QY) \quad \text{and} \quad -hkl\frac{d}{dz}(QZ)$$

respectively.

We obtain therefore for the whole electricity within the parallelopiped, which we will call E,

$$E = hkl\left\{\rho - \frac{d}{dx}(QX) - \frac{d}{dy}(QY) - \frac{d}{dz}(QZ)\right\}. \quad \ldots \ldots (1)$$

Now let N be the normal force at any point on the surface of the parallelopiped measured outwards. Then on the two kl faces the average value of N is

$$-X \quad \text{and} \quad +X + h\frac{dX}{dx}$$

respectively. Therefore for these two faces

$$\iint N\,ds = +\frac{dX}{dx}\,hkl.$$

Similarly, for the other two faces we shall have

$$\iint N\,ds = +\frac{dY}{dy}\,hkl,$$

and

$$\iint N\,ds = +\frac{dZ}{dz}\,hkl.$$

Therefore for the whole surface of the parallelopiped

$$\iint N\,ds = \left\{\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right\}hkl.$$

But

$$\iint N\,ds = 4\pi E.$$

Therefore

$$hkl\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right) = 4\pi E.$$

But from (1)

$$4\pi E = hkl\left\{4\pi\rho - \frac{d}{dx}(4\pi QX) - \frac{d}{dy}(4\pi QY) - \frac{d}{dz}(4\pi QZ)\right\}.$$

Hence we obtain

$$\frac{d}{dx}\{\overline{1+4\pi Q}X\} + \frac{d}{dy}\{\overline{1+4\pi Q}Y\} + \frac{d}{dz}\{\overline{1+4\pi Q}Z\} + 4\pi\rho = 0;$$

or, if we write

$$1+4\pi Q = K,$$

and if V be the mean potential, so that

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz},$$

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) + 4\pi\rho = 0. \quad \ldots\ldots (1)$$

This is the form assumed by Poisson's equation in such an isotropic medium as now under consideration. K evidently depends on the form, number, and position of the conductors, that is, on the nature of the substance. It is called *the dielectric constant.*

188.] Again, if $\rho+\rho'$ be the whole free electricity in the element of volume $dx\,dy\,dz$, including both that of the general electric field and that of the induced distributions, evidently

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}+4\pi(\rho+\rho')=0.$$

Hence we have

$$\frac{d}{dx}(\overline{K-1}\,\frac{dV}{dx})+\frac{d}{dy}(\overline{K-1}\,\frac{dV}{dy})+\frac{d}{dz}(\overline{K-1}\,\frac{dV}{dz})=4\pi\rho',$$

and $\rho' hkl$ is the sum of those portions of the induced distributions on the conductors intersected by the faces of the parallelopiped which lie within its volume.

189.] If over any surfaces there be superficial electricity of the given electric distribution, the equation

$$\frac{d}{dx}(K\,\frac{dV}{dx})+\frac{d}{dy}(K\,\frac{dV}{dy})+\frac{d}{dz}(K\,\frac{dV}{dz})+4\pi\rho=0$$

becomes, as in the cases previously investigated,

$$(K\,\frac{dV}{d\nu})_1+(K\,\frac{dV}{d\nu})_2+4\pi\sigma=0, \quad \ldots\ldots\ldots \quad (2)$$

or $$(\frac{dV}{d\nu})_1+(\frac{dV}{d\nu})_2+\left\{(K-1)\,\frac{dV}{d\nu}\right\}_1+\left\{(K-1)\,\frac{dV}{d\nu}\right\}_2+4\pi\sigma=0;$$

where the suffixes relate to the media on either side of the surface of separation.

Let (1) and (2) denote two media bounded by a plane surface AB, such that K, Q, and X have the suffixes 1 and 2 in these media respectively. Let C_1, C_2 be two parallel planes on either side of AB. Then, by the preceding, the superficial induced electrification in the space between C_1 and AB, and C_2 and AB, per unit area of the planes, is $+Q_1X_1$ and $-Q_2X_2$ respectively, and the total induced electrification in the space between C_1 and C_2 is $Q_1X_1-Q_2X_2$ per unit area of AB.

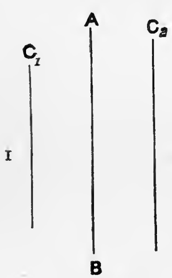

Fig. 28.

If the two planes C_1 and C_2 be made to approach each other till they become infinitely near, this gives a superficial electrification σ' over AB, the surface of separation of the media (1) and (2), arising from induction on the conductors, such that

$$\sigma' = -\{Q_1 X_1 - Q_2 X_2\},$$

$$\text{or} \quad \sigma' = \frac{1}{4\pi}\left\{\overline{K_1 - 1}\,\frac{dV}{dv_1} + \overline{K_2 - 1}\,\frac{dV}{dv_2}\right\},$$

if the normals be reckoned inwards from the surface in case of each medium.

Also if the electricity per unit volume arising from such induction be called ρ', we have seen that

$$\rho' = \frac{1}{4\pi}\left\{\frac{d}{dx}(\overline{K-1}\,\frac{dV}{dx}) + \frac{d}{dy}(\overline{K-1}\,\frac{dV}{dy}) + \frac{d}{dz}(\overline{K-1}\,\frac{dV}{dz})\right\}.$$

Therefore our equations may be written

$$\nabla^2 V + 4\pi(\rho + \rho') = 0,$$

$$\text{and} \quad \frac{dV}{dv_1} + \frac{dV}{dv_2} + 4\pi(\sigma + \sigma') = 0;$$

ρ' and σ' are called by Maxwell the *apparent electricities* solid and superficial respectively, and Faraday's hypothesis of the dielectric medium supplies us therefore with a physical meaning for these quantities.

It follows from the equation (2) that at the surface of separation of any two isotropic media, in which the constant K has the values K_1 and K_2 respectively, if there be no real electrification on that surface, that is, if $\sigma = 0$, the normal forces on either side of the surface are to each other in a constant ratio, namely

$$\frac{\left(\frac{dV}{dv}\right)_1}{\left(\frac{dV}{dv}\right)_2} = \frac{K_2}{K_1}.$$

190.] Now let ϕ be the superficial electrification at any point of the surface of any one of the small conductors in the neighbourhood of any point P, (x, y, z) in the dielectric, and let $\Sigma\iint x\phi\, dS$, or the sum of the integrals $\iint x\phi\, dS$ taken over the surfaces of all these conductors within unit volume, be denoted by σ_x, assuming that the distribution of electricity on each conductor and the distribution of the conductors themselves is constant throughout that volume and the same as it is at P; thus σ_x is a physical property of the dielectric at P analogous to the pressure p referred to unit of surface at any points in a fluid mass, and other similar quantities. Let a be the average distance between two planes parallel to y, z, touching any small conductor, i.e. the average breadth of a conductor parallel to x, and let n be the average number of such small conductors in unit of volume. It follows that the number of conductors in an elementary parallelopiped $dx\,dy\,dz$ is $n\,dx\,dy\,dz$, and the number intersecting the $dy\,dz$ face must be $\dfrac{n\,dx\,dy\,dz}{\dfrac{dx}{a}}$ or $n\,a\,dy\,dz$.

Now if x_1 be the x coordinate of the left-hand plane parallel to yz touching any conductor, the average amount of the electricity lying to the right of any other plane parallel to yz intersecting the same conductor must be $\iint \frac{x-x_1}{a}\phi\,dS$, the integration being over the surface of the conductor, i.e. it is $\frac{1}{a}\iint x\,\phi\,dS$, since $\iint \phi\,dS = 0$.

Therefore the amount of electricity on the small conductors intersected by the left-hand $dy\,dz$ face of the parallelopiped $dx\,dy\,dz$ and lying to the right of that face must be

$$n\,a\,dy\,dz \iint \frac{x\phi\,dS}{a}; \text{ or } n\iint x\phi\,dS\,.\,dy\,dz.$$

But $n\iint x\phi\,dS$ is the quantity above designated by σ_x.

Therefore σ_x is the amount of electricity per unit surface on the right hand of a plane at P parallel to yz situated on small conductors intersected by that plane; it is the same quantity as is denoted by QX in Art. 186; similarly for σ_y and σ_z.

The quantities σ_x, σ_y, σ_z are components of a vector σ, the value of any one of them when the corresponding axis is taken in the direction of σ. (See Chap. X, Art. 178).

From Arts. 186 and 188 it follows that ρ', the density at any point P of induced electricity belonging to the conducting surfaces, is

$$-\left(\frac{d\sigma_x}{dx}+\frac{d\sigma_y}{dy}+\frac{d\sigma_z}{dz}\right),$$

and that if σ_x, σ_y, σ_z be discontinuous over any surface S through P whose normal has direction cosines l, m, n, then there will be a superficial electrification arising from the induced electricity on the small conductors over the surface S, the density of which at P is

$$l(\sigma_x-\sigma'_x)+m(\sigma_y-\sigma'_y)+n(\sigma_z-\sigma'_z),$$

where σ_x and σ'_x are the values of σ_x and σ'_x on opposite sides of S at P.

If V' be the potential at any point of the field arising from the induced charges on the small conductors and from these alone, then from Art. 189 V' must satisfy the equations

$$\nabla^2 V'+4\pi\rho'=0,$$

$$\frac{dV'}{d\nu}+\frac{dV'}{d\nu'}+4\pi\sigma'=0.$$

And since the solution of these equations is determinate and unique, we must have

$$V'=\iiint\frac{\rho'\,dx\,dy\,dz}{r}+\iint\frac{\sigma'\,dS}{r},$$

where the integration extends over the whole field, a result in fact which is at once obvious from the physical meanings of ρ' and σ'.

191.] We know that the average force in direction x over any finite plane parallel to yz due to an electrical distribution whose algebraic sum is m placed on the positive side of the plane, and at a distance from it very small compared with the dimensions of the plane, is $-2\pi m$.

If therefore any such plane be situated in the medium under consideration, the average force upon it arising from intersected conductors will be $-2\pi\sigma_x$ from those on the right-hand side, and again $-2\pi\sigma_x$ from those on the left-hand side, or $-4\pi\sigma_x$ on the whole; and if we limit ourselves to the consideration of the little conductors situated between two planes very close to and parallel to the supposed plane on opposite sides of it, the average force will be $-4\pi\sigma_x$ parallel to x, since the value of m for non-intersected conductors is zero *.

It follows therefore that if in the dielectric medium we take any two planes very close together, and each perpendicular to σ, the resultant of σ_x, σ_y, σ_z, at any point lying between them, the force at that point arising from the induced charges on the

* The proposition in the text is an important one and may be proved rigorously as follows:—

Let there be an infinite number of polarised molecules between two infinite planes parallel to yz, and not intersected by the planes. Let the equations of the planes be $x = a_1$ and $x = a_2$. Let V_1 and V_2 be the values of the potential from the polarised molecules upon the respective planes; and let $dy_1\,dz_1$, $dy_2 dz_2$ denote elements of their surfaces, and $d\nu_1$ and $d\nu_2$ elements of their normals measured outwards from the space between the planes.

Applying Green's theorem to the space between the planes, using x and V for functions, we have

$$\iint V_1 \frac{dx}{d\nu_1} dy_1 dz_1 + \iint V_2 \frac{dx}{d\nu_2} dy_2 dz_2$$
$$= a_1 \iint \frac{dV}{d\nu_1} dy_1 dz_1 + a_2 \iint \frac{dV}{d\nu_2} dy_2 dz_2 - \iiint x \nabla^2 V\, dx\, dy dz,$$

the last term $\iiint x \nabla^2 V\, dx\, dy\, dz$ including $\iint x \left(\frac{dV}{d\nu} + \frac{dV}{d\nu_1}\right) dS$ for surfaces of superficial density or discontinuous $\frac{dV}{dx}$, &c.

Since the distribution within the planes is algebraically zero, $\iint \frac{dV}{d\nu_1} dy_1\, dz_1$ and $\iint \frac{dV}{d\nu_2} dy_2\, dz_2$ are separately zero. Also $\frac{dx}{d\nu_1}$ and $\frac{dx}{d\nu_2}$ are -1 and $+1$ respectively, and $\iiint x \nabla^2 V\, dx dy\, dz = -4\pi \iiint x \phi\, dx dy\, dz$ if ϕ be the density at any point within the planes.

Therefore $\iint V_2\, dy_2\, dz_2 - \iint V_1\, dy_1\, dz_1 = 4\pi \iiint x\phi\, dx\, dy\, dz.$

But if F be the mean force in direction of x, and C the area of either plane,

$$\iint V_2\, dy_2\, dz_2 - \iint V_1\, dy_1\, dz_1 = -FC(a_2 - a_1) = -F\Omega,$$

where Ω is the volume between the planes.

Therefore $F = -4\pi \dfrac{\iiint x\phi\, dx\, dy\, dz}{\Omega}.$

These conclusions are equally true when the planes are of any magnitude, provided the distance between them is infinitely small compared with their linear dimensions, in which case $\dfrac{\iiint x\phi\, dx\, dy\, dz}{\Omega}$ becomes the same quantity as that above denoted by σ_x and the proposition is proved.

small conductors situated between these planes and not intersected by them will be in the direction of σ and equal to $-4\pi\sigma$, and the forces from these small conductors thus included between the planes parallel to x, y, and z will be $-4\pi\sigma_x$, $4\pi\sigma_y$, $-4\pi\sigma_z$ respectively, the same as the force would have been from the included conductors if the parallel planes including the point had been perpendicular to x, y, or z respectively.

If instead of taking a region between two parallel planes whose distance is very small compared with their linear dimensions we had considered a space inclosed within any small sphere in the medium, then we might prove that the force at the centre of the sphere arising from all the small conductors entirely included within it has for its components $-\frac{4\pi}{3}\sigma_x$, $-\frac{4\pi}{3}\sigma_y$, and $-\frac{4\pi}{3}\sigma_z$ respectively.

If instead of a medium of small conductors with electricity on their surfaces distributed according to the law of induced electricity under the action of a constant force with components X, Y, Z we were considering a medium composed of small discrete molecules, each separately containing an electrical distribution according to any law, subject only to the condition that the algebraic sum of the distribution in or upon every such molecule is zero, and if we denoted by ϕ the electrical density at any point in any molecule, and by σ_x, σ_y, and σ_z the sums

$$\Sigma\iiint x\phi\, dv,\quad \Sigma\iiint y\phi\, dv,\quad \Sigma\iiint z\phi\, dv$$

for unit of volume in the neighbourhood of any point $P, (x, y, z)$ in the medium, the triple integrals $\iiint x\,\phi\, dv$ &c. being replaced by $\iint x\phi\, dS$, &c., where ϕ is superficial, we should still have σ_x, σ_y, σ_z the components of a vector, inasmuch as by changing to any other rectangular axes ξ, η, ζ, such that the direction-cosines of ξ are l, m, n, we get

$$\sigma_\xi = \Sigma\iiint (lx+my+nz)\,\phi\, dv$$
$$= l\sigma_x + m\sigma_y + n\sigma_z;$$

and the results arrived at in the preceding articles as to the density, solid ρ' or superficial σ', of the electricity at any point, and the value of the potential V', would hold good of this medium.

192.] Comparing two electrified systems in all respects similar, except that in the one the dielectric constant has the uniform value K_1, and in the other the uniform value K_2, we see from the form of Poisson's equation above obtained, that, if V has at each point in either system the same value as at the corresponding point in the other system, all the volume and superficial densities must as between the two systems vary directly as K, the dielectric constant. Conversely, if the densities be the same at all corresponding points, the potential at corresponding points will vary inversely as K.

The effect of substituting a uniform dielectric medium with dielectric constant K, for air, in which the constant is unity, in any electrified system, is therefore the same as if all the charges on the conductors were reduced in the ratio $1 : K$, or as if the repulsion between the masses e and e' at distance r were $\frac{ee'}{Kr^2}$, instead of $\frac{ee'}{r^2}$. It follows that the charges necessary to produce given potentials, that is the capacity of the system, must in similar systems vary directly as K. For this reason K is called the *specific inductive capacity*, or briefly the *inductive capacity* of the medium.

The phenomenon observed by Faraday, using two concentric conducting spheres separated by a homogeneous isotropic medium, is a particular case of the general result of this article.

193.] The conception and treatment of lines, tubes, and fluxes of force developed in Arts. 96–103 of Chap. V are equally applicable to an isotropic dielectric with any value of K, either uniform or variable, and might, indeed, have been applied to establish the equations above obtained in such a medium. If we integrate each term of the equation above proved, viz.

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right)+\frac{d}{dx}\left(K\frac{dV}{dy}\right)+\frac{d}{dz}\left(K\frac{dV}{dz}\right)+4\pi\rho=0,$$

over a space inclosed within any closed surface S, we get

$$\iint K\frac{dV}{dx}dy\,dz+\iint K\frac{dV}{dy}dx\,dz+\iint K\frac{dV}{dz}dx\,dy$$
$$+4\pi\iiint \rho\,dx\,dy\,dz=0\,;$$

and if l, m, n be the direction cosines of the normal to any element dS of the surface, this becomes

$$\iint K\left(l\frac{dV}{dx}+m\frac{dV}{dy}+n\frac{dV}{dz}\right)dS+4\pi\iiint \rho\,dx\,dy\,dz=0,$$

or
$$\iint KF_n\,dS=4\pi m;$$

where F_n is the normal force measured outwards at each point of S, and m is the algebraic sum of the included electricities.

If, therefore, as in Art. 102, any tube of force be limited by the transverse surfaces S and S', and if F and F' be the normal forces at points on S and S' respectively, and if K and K' be the inductive capacities at those points, then the equation becomes

$$\iint K'F'dS'-\iint KFdS=4\pi m,$$

or as it may be written

$$\iint F'dS'-\iint FdS=4\pi\left\{m-\iint F'Q'dS'+\iint FQ\,dS\right\},$$

an equation expressing the same physical property as that of Art. 102, inasmuch as

$$\iint FQdS-\iint F'Q'dS'$$

is the addition to the electricity included in the limited tube of force arising from the polarisation of the small conductors in the dielectric medium.

In a medium with a continuously varying specific inductive capacity and finite volume densities, the force F obviously varies continuously both in direction and magnitude.

If however there be an abrupt transition from one dielectric medium to another at any surface S, then, whether there be an *actual* charge on S or not, there is, as we have seen, a charge over S arising from the polarisation of the small conductors, called generally the *apparent* electrification, although both from theory and experiment it is proved to have an existence as real as what is called by distinction the *actual* charge; if σ' be the density of this charge we have seen that

$$\sigma' = \frac{1}{4\pi} \cdot \{(K_1 - 1) F_1 - (K_2 - 1) F_2\},$$

where F_1 and F_2 are the forces normal to S at the point in the two media respectively, each supposed to act from medium K_1 towards medium K_2. If therefore there be no *actual* electrification on S, we have

$$F_2 - F_1 = 4\pi\sigma' = (K_1 - 1) F_1 - (K_2 - 1) F_2,$$

or

$$K_1 F_1 - K_2 F_2 = 0,$$

as above shewn.

And if i and i' be the inclination of the lines of force to the normal to S before and after the transit over S,

$$\tan i = \left(1 + \frac{4\pi\sigma'}{F\cos i}\right) \tan i'$$

$$= \frac{K_1}{K_2} \tan i',$$

as above proved.

All the properties of tubes of force, elementary or otherwise, proved in Chap. V, for a dielectric of uniform K hold good in the case of a medium with varying K, if we substitute for F (the force at any point) the quantity KF. This quantity KF is sometimes called the *induction*.

Should there be an *actual* charge σ over S, then the ordinary equations would give

$$\tan i = \left(1 + \frac{4\pi(\sigma + \sigma')}{F\cos i}\right) . \tan i',$$

σ' being determined as before.

194.] As an illustration of the application of the preceding results, let us consider the state of the electric field when two media of different but uniform inductive capacities are separated by an infinite plane surface, and an electric charge is situated at a given point in one of them.

Suppose the plane of the paper to be perpendicular to the plane of separation; let YEY' (see Fig. 29) be the line of intersection of these planes, and let the charge m be situated at the point m in the medium whose inductive capacity is k_1, that of the other medium being k_2.

Let mEm' be drawn perpendicular to the bounding plane, and let m' be so taken that $m'E = mE = a$, suppose.

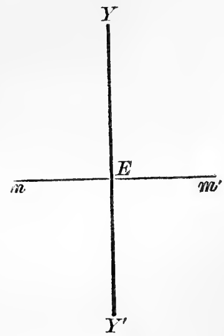

Fig. 29.

Let the distances of any point P from m and m' be called r and r' respectively.

If V and V' be the potentials on the left and right of the plane YEY' respectively, then V and V' must satisfy the following conditions, k_1 and k_2 being uniform in each medium:

(1) $V = V'$ on the plane and each vanishes at infinity,

(2) $\nabla^2 V + \frac{4\pi\rho}{k_1} = 0,$

(3) $\nabla^2 V' + \frac{4\pi\rho}{k_2} = 0,$

(4) $k_1 \frac{dV}{d\nu} + k_2 \frac{dV'}{d\nu'} = 0.$

Now $r = r'$ over the plane and the differential coefficients of the functions $\frac{1}{r}$ and $\frac{1}{r'}$ along the normal to the plane are proportional to $\frac{1}{r^3}$ and $\frac{1}{r'^3}$ respectively.

It follows therefore that the conditions (1) and (4) may be satisfied by assuming

$$V = \frac{A}{r} + \frac{B}{r'}, \quad \text{and} \quad V' = \frac{C}{r} + \frac{D}{r'},$$

and properly determining A, B, C, and D.

Since ρ is zero to the left of YEY' except in the neighbourhood of m, the condition (2) requires that A should be equal to $\frac{m}{k_1}$, and since ρ is zero everywhere to the right of YEY', the condition (3) requires that D should be zero, and therefore

$$V = \frac{m}{k_1 r} + \frac{B}{r'}, \quad V' = \frac{C}{r},$$

(1) gives
$$\frac{m}{k_1} + B = C,$$

(4) gives
$$k_1 \left(\frac{m}{k_1} - B\right) = k_2 C;$$

whence we get

$$B = \frac{m}{k_1} \cdot \frac{k_1 - k_2}{k_1 + k_2}, \quad C = \frac{2m}{k_1 + k_2},$$

$$V = \frac{m}{k_1} \cdot \left\{ \frac{1}{r} + \frac{k_1 - k_2}{k_1 + k_2} \cdot \frac{1}{r'} \right\},$$

$$V' = \frac{2m}{k_1 + k_2} \cdot \frac{1}{r},$$

and the problem is completely determined.

If σ' be the superficial electrification of polarisation, or so called apparent electrification, over the plane, then

$$\sigma' = \frac{1}{4\pi} \left\{ (k_1 - 1)\left(\frac{dV}{dv}\right)_1 + (k_2 - 1)\left(\frac{dV'}{dv}\right)_2 \right\}$$

$$= \frac{m}{4\pi} \left\{ \frac{k_1 - 1}{k_1} \left(1 - \frac{k_1 - k_2}{k_1 + k_2}\right) - 2\frac{k_2 - 1}{k_2 + k_1} \right\} \frac{a}{r^3}$$

$$= \frac{2ma}{4\pi r^3} \cdot \frac{k_2(k_1 - 1) - k_1(k_2 - 1)}{k_1(k_1 + k_2)} = \frac{k_1 - k_2}{k_1(k_1 + k_2)} \cdot \frac{ma}{2\pi r^3}.$$

If $k_1 = 1$ and $k_2 = \infty$, we have

$$V = m\left(\frac{1}{r} - \frac{1}{r'}\right), \quad V' = 0, \quad \sigma' = -\frac{ma}{2\pi r^3},$$

agreeing, as they should do, with the results obtained for an infinite conducting plane in presence of a charged point, in Art. 105.

195.] As an instance of a similar treatment let us consider the case of a sphere composed of a dielectric medium of specific inductive capacity k, brought into a field of uniform force F in air.

Before the introduction of the sphere the force throughout the field was F in a given fixed direction, which we will take for the direction of the axis of x.

If the origin be measured from any point, as for instance the point O with which the centre of the dielectric sphere is made to coincide, then, before the introduction of the latter, the potential of the field was $-Fx+C$.

Let a be the radius of the sphere, and let V and V' be the potentials outside and inside of the sphere, then the conditions to be satisfied are:

(1) $V=V'$ at the sphere's surface, and V becomes $-Fx+C$ at infinity;

(2) $\nabla^2 V = \nabla^2 V' = 0$ everywhere;

(3) $k\dfrac{dV'}{d\nu_1}+\dfrac{dV}{d\nu}=0$ at the surface.

Now we know from Art. 107 that a potential V of either of the forms Ax, or $\dfrac{Bx}{r^3}$, gives a normal force at any point on the sphere's surface proportional to x, and satisfies the condition $\nabla^2 V=0$, provided that in the case of the form $\dfrac{Bx}{r^3}$ being chosen, the point is not infinitely near to the centre.

If therefore we make V of the form

$$-Fx\left(1-\frac{a^3}{r^3}\right)+A\frac{a^3x}{r^3}+C,$$

and make V' of the form $Bx+C$, we shall have condition (2) satisfied identically, and shall be able to satisfy conditions (1) and (3) by properly determining A and B.

(1) gives $V'=Bx+C=V=Ax+C$ at the surface, or $B=A$.

(3) gives $-k\dfrac{Bx}{a}=\dfrac{3Fx}{a}+\dfrac{2Ax}{a}$, or $B=A=-\dfrac{3F}{k+2}$.

And therefore the potential V throughout the region external to the sphere is given by the equation

$$V=-Fx+F\frac{k-1}{k+2}\cdot\frac{a^3x}{r^3}+C;$$

and the potential V' within the sphere is given by the equation

$$V'=-\frac{3Fx}{k+2}+C.$$

The density σ' of so-called apparent electrification at the surface is given by the equation

$$\sigma' = \frac{1}{4\pi}\cdot(k-1)\frac{dV'}{d\nu'},$$

that is,
$$\sigma' = \frac{3F}{4\pi}\cdot\frac{k-1}{k+2}\frac{x}{a}.$$

If k become infinitely great, σ' becomes $\frac{3F}{4\pi}\cdot\frac{x}{a}$, as already determined for a conducting sphere in a uniform field.

The potentials V and V' obtained above are obviously those outside and inside of the surface due to the superficial electrification

$$\frac{3F}{4\pi}\cdot\frac{k-1}{k+2}\cdot\frac{x}{a},$$

together with the potential of the field $-Fx+C$. If we subtract this latter from V and V' respectively, it appears that the potentials of σ' within and without the surface respectively are

$$\frac{k-1}{k+2}Fx \quad\text{and}\quad \frac{k-1}{k+2}F\frac{a^3x}{r^3} \quad\text{respectively.}$$

That is to say, a superficial electrification μx over a sphere's surface produces potentials

$$\frac{4\pi}{3}\mu a x, \quad\text{and}\quad \frac{4\pi}{3}\mu\frac{a^4x}{r^3},$$

within and without the sphere respectively, and therefore gives a uniform field of force within the sphere.

195 *a*.] We proceed next to consider the more general case of a medium not necessarily isotropic.

In that case the quantity of electricity of the induced distributions on the little conductors intersected by the kl face of the parallelopiped above mentioned, which lies to the right of that face, depends generally on the forces Y and Z, as well as X. By the principle of superposition it must consist of three portions proportional to X, Y, and Z respectively. Let it be denoted by

$$\sigma_x = Q_{xx}X + Q_{xy}Y + Q_{xz}Z.$$

In like manner the quantity of electricity of the induced distributions on the conductors intersected by the hl and hk

faces of the parallelopiped, which lies on the positive side of these faces respectively, may be denoted by

$$\sigma_y = Q_{yx}X + Q_{yy}Y + Q_{yz}Z,$$

and

$$\sigma_z = Q_{zx}X + Q_{zy}Y + Q_{zz}Z,$$

where $\sigma_x = \Sigma \iint x\phi\, dS,\ \sigma_y = \Sigma \iint y\phi\, dS,\ \sigma_z = \Sigma \iint z\phi\, dS,$

ϕ being the electric density at any point of a small conductor, the double integration extending over the surface of each conductor and the summation Σ extending over the conductors in unit volume, as described above, Art. 190. We shall generally omit Σ for the sake of brevity in cases where there is not likely to be any doubt as to the meaning.

196.] Again, the induced electrification ϕ at any point on any conductor consists, by the principle of superposition, of three portions proportional to X, Y, and Z respectively.

We will denote by $X\phi_x$ the density of the electrification due to the force X, $Y\phi_y$ that due to Y, and $Z\phi_z$ that due to Z, so that the whole density at any point on the surface of any conductor will be

$$\phi = X\phi_x + Y\phi_y + Z\phi_z.$$

We have then

$$\sigma_x = Q_{xx}X + Q_{xy}Y + Q_{xz}Z$$

$$= X\iint x\,\phi_x dS + Y\iint x\,\phi_y\, dS + Z\iint x\,\phi_z\, dS,$$

the integrals being over all conductors in unit volume.

It follows that

$$Q_{xx} = \iint x\,\phi_x\, dS,$$

$$Q_{xy} = \iint x\,\phi_y\, dS,$$

$$Q_{xz} = \iint x\,\phi_z\, dS.$$

Similarly we shall obtain,

$$\sigma_y = Q_{yx}X + Q_{yy}Y + Q_{yz}Z$$

$$= X\iint y\,\phi_x\, dS + Y\iint y\,\phi_y\, dS + Z\iint y\,\phi_z\, dS,$$

$$\sigma_z = Q_{zx}X + Q_{zy}Y + Q_{zz}Z$$

$$= X\iint z\,\phi_x\, dS + Y\iint z\,\phi_y\, dS + Z\iint z\,\phi_z\, dS;$$

and therefore

$$Q_{yy} = \iint y\,\phi_y\,dS,$$

$$Q_{yx} = \iint y\,\phi_x\,dS,$$

$$Q_{yz} = \iint y\,\phi_z\,dS,$$

$$\&\text{c.} = \&\text{c.}$$

The properties of the coefficients Q_{xx}, Q_{xy}, &c. or $\iint x\,\phi_x dS$, $\iint x\,\phi_y dS$, &c. were investigated in Chap. X; and as there proved, $Q_{xy} = Q_{yx}$, &c.

If l, m, n be the direction-cosines to the normal to any plane drawn in the medium, the quantity of electricity on the little conductors intersected by unit area of that plane which lies on the positive side of it is $l\sigma_x + m\sigma_y + n\sigma_z$ by Art. 191.

If the medium be isotropic, as above defined, the forces Y and Z will not affect the quantity of electricity to the right of the plane kl. In fact, in an isotropic medium

$$\iint x\,\phi_y\,dS = 0, \quad \iint y\,\phi_z\,dS = 0, \quad \text{and} \quad \iint z\,\phi_x\,dS = 0.$$

And $$\iint x\,\phi_x\,dS = \iint y\,\phi_y\,dS = \iint z\,\phi_z\,dS.$$

In that case

$$\sigma_x = X \iint x\,\phi_x\,dS,$$

and $$K = 1 + 4\pi Q$$

$$= 1 + 4\pi \iint x\,\phi\,dS.$$

197.] If the medium be not isotropic, the integrals $\iint x\,\phi_y\,dS$, &c. are not generally zero for all directions of the coordinate axes. But it was shewn in Art. 179 that for any system of conductors in a field of constant force there exist three directions at right angles to each other, such that if these be taken for axes, each of the integrals $\iint x\phi_y\,dS$, &c. is zero.

We may therefore describe a small sphere about any point in our medium, and find three perpendicular directions such that, if these be taken for axes, each of these integrals vanishes, if taken throughout the sphere. And we may call these three directions the *principal axes of electric polarisation* at the point in question.

With these directions for axes we have

$$\sigma_x = X\iint x\,\phi_x\,dS,$$

$$\sigma_y = Y\iint y\,\phi_y\,dS,$$

$$\sigma_z = Z\iint z\,\phi_z\,dS.$$

But it will not be generally true that

$$\iint x\,\phi_x dS = \iint y\,\phi_y\,dS = \iint z\,\phi_z\,dS.$$

σ_x, σ_y, and σ_z being in this case proportional to X, Y, and Z respectively, we will write

$$\sigma_x = Q_x X, \quad \sigma_y = Q_y Y, \quad \sigma_z = Q_z Z;$$

and

$$X' = X + 4\pi Q_x X,$$

$$Y' = Y + 4\pi Q_y Y,$$

$$Z' = Z + 4\pi Q_z Z.$$

And we may write

$$K_x = 1 + 4\pi Q_x = 1 + 4\pi\iint x\,\phi_x\,dS,$$

$$K_y = 1 + 4\pi Q_y = 1 + 4\pi\iint y\,\phi_y\,dS,$$

$$K_z = 1 + 4\pi Q_z = 1 + 4\pi\iint z\,\phi_z\,dS,$$

These ratios K_x, K_y, K_z have a determinate value at every point in a heterotropic medium, but may vary from point to point. Also the directions of the principal axes may vary from point to point.

If K_x, K_y, and K_z be constant, and the directions of the principal axes also constant, the medium, whether isotropic or not, is said to be *homogeneous*. If otherwise, *heterogeneous*. An isotropic medium is therefore homogeneous if K be constant.

If K_x, K_y, K_z vary continuously, the same reasoning by which in an isotropic medium we obtained the equation

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right)+\frac{d}{dy}\left(K\frac{dV}{dy}\right)+\frac{d}{dz}\left(K\frac{dV}{dz}\right)+4\pi\rho=0$$

leads, in case of a heterotropic medium if the coordinate axes be the principal axes, to

$$\frac{d}{dx}\left(K_x\frac{dV}{dx}\right)+\frac{d}{dy}\left(K_y\frac{dV}{dy}\right)+\frac{d}{dz}\left(K_z\frac{dV}{dz}\right)+4\pi\rho=0.$$

Also if K_{1x}, K_{1y}, K_{1z} and K_{2x}, K_{2y}, K_{2z} be the constants for two heterotropic media separated by a plane whose direction cosines are l, m, n, we have at the surface of separation

$$l\left\{K_{1x}\left(\frac{dV}{dx}\right)_1+K_{2x}\left(\frac{dV}{dx}\right)_2\right\}+m\left\{K_{1y}\left(\frac{dV}{dy}\right)_1+K_{2y}\left(\frac{dV}{dy}\right)_2\right\}$$
$$+n\left\{K_{1z}\left(\frac{dV}{dz}\right)_1+K_{2z}\left(\frac{dV}{dz}\right)_2\right\}+4\pi\sigma=0,$$

corresponding to equation (2) of Art. 189.

Let X' be the average force which would exist on the plane kl if all the intersected conductors were removed. Then, it follows from Art. 191, that X' is connected with X by the equations

$$X'=X+4\pi\sigma_x,$$
$$=X+4\pi Q_x X \text{ if the axes be the principal axes,}$$
$$=K_x X,$$

so that

$$\frac{X'}{X}=K_x,$$

or K_x is the ratio between the average force which would exist on the plane kl if all the intersected conductors were removed and the average force which does exist over that plane in the medium.

198.] As shown in Art. 180, the energy of the medium per unit of volume is

$$-\frac{1}{2}\left\{X^2\iint x\,\phi_x dS+Y^2\iint y\,\phi_y dS+Z^2\iint z\,\phi_z dS\right\};$$

and the little conductors, if each free to rotate on any axis, will so use their freedom as to make the expression within brackets the greatest possible, by placing themselves in suitable positions; that is, they will endeavour so to place themselves as that the axis of greatest polarisation shall coincide with the resultant force; so that, for instance, if

$$\iint x\phi_x dS > \iint y\phi_y dS > \iint z\phi_z dS,$$

the conductors will so place themselves as that the principal axis x shall coincide with the direction of the force.

If they be perfectly free to move, this object will be effected for any direction of the resultant force; and as in that case there will be no polarisation in any direction at right angles to the force, the expressions

$$\iint x\,\phi_y\,dS,\quad \iint x\,\phi_z\,dS,\quad \iint y\,\phi_z\,dS$$

are zero. Such a medium will then have all the properties of an isotropic medium.

But unless the conductors be perfectly free to move, or are spheres, the medium will in general be heterotropic.

199.] It appears from the preceding that the numerical value of the dielectric constant K in any isotropic medium must depend upon the form and density of distribution of the small conductors within the medium.

Suppose now that these are spherical, and that λ is the fraction of any volume within the medium which is occupied by the whole of the small conductors within that volume.

Suppose also that the *average* force within the medium in the neighbourhood of any point is X, parallel to the axis of x.

Since the force within each conductor is zero, it follows that the average force in non-conducting space in the neighbourhood of the point in question must be $\dfrac{X}{1-\lambda}$.

Now the electrical distribution on the surface of each sphere must be such that the force arising from it within the sphere, together with that from all the other electricity in the field, shall be zero throughout that sphere.

If the spherical distribution were very rare it is clear that the force arising from all the electrical distributions except that in any one sphere must be sensibly constant throughout that sphere. In other words, the sphere is situated in a field of constant force $\frac{X}{1-\lambda}$ parallel to the axis of x.

Therefore the polarisation of any particular sphere must be $\frac{3v}{4\pi}\frac{X}{1-\lambda}$, where v is the volume of the sphere; see Art. 182. Hence the polarisation per unit of volume, which we denoted above by QX, is $\frac{3\lambda}{4\pi}\frac{X}{1-\lambda}$, and therefore

$$K = 1 + 4\pi Q = 1 + \frac{3\lambda}{1-\lambda},$$

or, as λ is supposed very small, $K = 1 + 3\lambda$. This amounts in fact to regarding each sphere as polarised independently of the rest. If λ be not very small, so that we have to consider the mutual influences of the spheres, the reasoning is precarious and cannot be insisted upon.

200.] We may also construct a composite medium, portions of which shall consist of a dielectric whose constant is K_1, and other portions consist of a dielectric whose constant is K_2. If such a medium be uniform throughout any space, that is, if the distribution *inter se* of the two dielectrics be the same for unit of volume in any part of the space in question, the problem presents itself for consideration, what is the average force in such a composite medium due to the induced distributions within it; or, as we may otherwise express it, what must be the value of K, in order that a uniform medium with K for dielectric constant may have the same effect as the composite medium. The solution of such a problem depends on the manner in which the two dielectrics are distributed *inter se*. If, for instance, the medium K_2 is in separate masses bounded by closed surfaces dispersed through the medium K_1, the solution of the problem will depend on the shape as well as the number and magnitude of the separate masses.

We might endeavour to determine the density of an induced distribution on those surfaces which, if they were filled with the dielectric K_1, would cause the normal forces on opposite sides of the surfaces to bear to each other the ratio $\frac{K_1}{K_2}$. If by that, or by any other method, we could find the value of $\iint x\phi\, ds$ for the composite medium, the value of K is at once known to be

$$1+4\pi\iint x\phi\, ds.$$

The problem is substantially the same as that of the determination of the dielectric constant in a single medium. If, for instance, the dielectric K_2 be contained in spheres, and they be so distant from each other that their mutual influence may be neglected, and the whole system be regarded as placed in a field of uniform force X, it will be found that the density of the induced distribution upon them which causes the normal forces within and without the spheres to have the required ratio is *proportional* to $\frac{3X}{4\pi}\cos\theta$, X being the external force, and θ, as before, the angle between the radius of any point on a sphere and the direction of x.

Let the density be

$$\sigma = n\times\frac{3X}{4\pi}\cos\theta,$$

where n is a ratio to be determined. This distribution gives a force $-nX$ within any sphere. Consequently the normal force within any sphere is

$$(1-n)X\cos\theta.$$

The normal force outside of a sphere is

$$(1-n+3n)X.\cos\theta.$$

We have then by the condition respecting the forces

$$K_1(1+2n) = K_2(1-n),$$

and
$$n=\frac{K_2-K_1}{2K_1+K_2},$$

from which K

$$=1+\frac{3\lambda}{1-3\lambda}\cdot\frac{K_2-K_1}{2K_1+K_2},$$

or, λ being very small,

$$=1+3\lambda.\frac{K_2-K_1}{2K_1+K_2}.$$

CHAPTER XII.

THE ELECTRIC CURRENT.

ARTICLE 201.] HITHERTO we have been engaged in the development of the so-called two-fluid theory of electricity in its application to Electrostatics, or the conditions, on that theory, of the permanence of any electric distribution, one essential condition being that the potential shall have the same value at every point in a conductor. The results arrived at are so far in agreement with experiment as to justify the acceptance of this theory as a formal explanation of electrical phenomena.

We now proceed to consider how far the theory can be adapted to the explanation of observed phenomena in another class of cases, those namely in which different regions of the same conducting substance are maintained by any means at unequal potentials.

Suppose, for example, that two balls of any given metal and at the same temperature, originally at different potentials, are held in insulating supports, and connected together by a wire of the same metal; then it is found that after an interval of time inappreciably small, the potentials are reduced to an equality at all points of the conductor thus formed of the balls and wire, and that the total charge on the ball of higher potential has been diminished, and that on the ball of lower potential has been increased. With the conception and language of the two-fluid theory there has been in this short interval *a flow* of positive electricity in the one direction along the wire, or of negative electricity in the opposite direction, or both such flows have taken place simultaneously.

If a magnetic needle be suspended near to the wire, a slight transitory deflection of this needle may be observed during the process of equalisation of potentials, and it might be possible with a sufficient length of wire and apparatus of sufficient delicacy to detect a slight rise of temperature in the wire.

202.] Methods exist whereby the inequality of potentials in different parts of a conductor may be restored as fast as it is destroyed, and in such cases certain properties are manifested in the conductor and its neighbourhood so long as this inequality is maintained.

For instance, if the conductor be very small in two of its dimensions in comparison with the third, in ordinary language a wire, the deflection of the needle is no longer transitory but persistent, so long as the inequality of potentials is maintained, the amount of such deflection depending upon the amount of the inequality, and the dimensions and constitution of the wire; heat also continues to be generated in the wire at a rate depending upon the same circumstances.

Also if the wire be severed at any point, and the severed ends connected with a composite conducting liquid, thus forming a heterogeneous conductor of wire and liquid, chemical decomposition of the liquid will ensue at a rate dependent on the difference of potentials, and the nature of the wire and liquid.

According to the two-fluid theory, there must be under the given circumstances a permanent flow of one or both electricities between the unequal potential regions, of a like nature to the transitory flow spoken of above, and the wire is, in the language of that theory, spoken of as the seat of an electric *current*. Of course the existence of such a current is as purely hypothetical as that of the electric fluids themselves. A transference of some kind there must be, for it is indicated by the respective gain and loss of electrification in the two connected conductors, but whether that transference be a material transfer as implied by the two-fluid theory, or a formal transfer like a wave, or the transmission of force as in the case of a tension or thrust, we are not in a position to determine. The *current*, as it is designated, must be regarded as a phenomenon by itself, called into existence under certain conditions, and subject to laws to be investigated by independent observation.

203.] The question indeed might arise, how far are we warranted in regarding *current* phenomena as indicating the absence of electrical equilibrium?

When parts of a conductor are maintained at permanently unequal but constant potentials, a certain state of the field ensues, which is also permanent, and it might be said that we have in such a case a system in equilibrium although not satisfying the conditions required by the two-fluid theory. To this it can only be replied, that in the case of electrostatical equilibrium we have a system permanent of itself; whereas in a constant current the permanence always necessitates an expenditure of energy from some external source. The former case resembles the mechanical equilibrium of a heavy body on a horizontal plane. The permanence of the latter case resembles that of a heavy body dragged uniformly up an inclined plane, and requiring at each point of its course the expenditure of external work.

Laws of the Steady Current in a Single Metal at Uniform Temperature.

204.] (1) *The intensity of the current is the same at every point.*

We have mentioned certain physical manifestations accompanying the current, viz. thermal, chemical, and magnetic. These are capable of measurement; and it is reasonable to regard these measurable effects as exhibited in the neighbourhood of different portions of the current as giving a measure of the intensity of the current in those portions. It is found experimentally that in the case of a steady current these effects are the same throughout. Wherever the magnetic needle is suspended—assuming its distance from the wire and other circumstances to be the same—the same deflection results. If the wire be of equal section in every part, then equal portions are heated at the same rate, and in whatever portion of the wire the liquid conductor above described is introduced, chemical action also takes place at the same rate.

This law is evidently consistent with the two-fluid theory, according to which we regard the current as a flow of either fluid across any transverse section of the conducting wire.

(2) *Ohm's Law.*

This law, which is universally accepted, asserts that—

If a uniform current be maintained in a homogeneous wire whose surface is completely enveloped by insulating matter, the intensity of the current in the wire is directly proportional to the electromotive force (i.e. the difference of potentials at its extremities), and inversely proportional to the resistance of the wire; the mathematical expression of the law being $I = \frac{E}{R}$, *where I is the current intensity, E is the electromotive force, and R the resistance.*

The quantity here called the *resistance* depends upon the length and transverse section of the wire and upon the material of which it is composed. For wires of the same substance it is proportional to the length directly and the transverse section inversely, and Ohm's law asserts that if through a wire W the electromotive force E produces a current I, and through another wire W' the electromotive force E' produces a current I', then the fractions $\frac{E}{I}$ and $\frac{E'}{I'}$ will always bear the same ratio to one another so long as the same wires W and W' are employed. If R be the resistance, $\frac{1}{R}$ is called the *conductivity*, and if this be denoted by K, then Ohm's law may be expressed in the form $I = KE$.

205.] If the insulation of the wire is perfect, so that no transference of electricity can take place across its surface, the direction of transference at each point must, in the permanent state, be parallel to the axis of the wire at that point, but this direction must be also perpendicular to the equipotential surface through that point[1]. The wire is in fact a tube of force, and if it be of uniform section the resistance through each element of length ds is proportional to ds, and therefore we have $i \propto \frac{dV}{ds}$, or $\frac{dV}{ds}$ is

[1] This coincidence of direction of the electromotive force at any point and of the current through that point does not hold good in the case of anisotropic substances. At present and hereafter, in the absence of special notice to the contrary, it must be understood that we are confining ourselves to the consideration of isotropic substances.

constant at each point. Hence, by Art. 102, there can be no free electricity at any point within the substance of the wire.

This condition is satisfied according to the most generally accepted theory by regarding the electric current as consisting of equal quantities of positive and negative electricity flowing in opposite directions.

In this case the potential V, at any point distant s from a fixed point in the wire's axis, must be given by the equation $V = Rs + C$, where R is the constant resistance per unit length; and if the axis be a straight line parallel to x we have $V = Rx + C$, i.e. the potential is that of a field of uniform force.

If the wire be not of uniform section, then the resistance along a portion of length ds along the axis is, by Ohm's law, proportional to ds directly, and to dS the transverse section inversely; hence we have $i \propto \frac{dV}{ds} dS$, or $\frac{dV}{ds} dS$ is constant throughout, proving (by Art. 102) that there is still no free electricity within the substance of the wire.

206.] The statement of Ohm's law may be generalised as follows for all forms of homogeneous isotropic conductors with all their dimensions finite. For, from what has been said above, it follows that the current flows from one elementary region to another of such conductors along elementary tubes of force. If we regard such tubes as wires in which the current obeys Ohm's law, this leads us to the equation for the intensity of the current over the elementary area dS of the equipotential surface through any point

$$i \propto \frac{dV}{ds} dS, \quad \text{or} \quad \frac{dV}{ds} dS \text{ is constant.}$$

An equation which, as before stated, proves that there is no free electricity within the substance of any conductor through which a permanent current is passing. If the conductivity be variable and be denoted by K, this becomes

$$i = -K \frac{dV}{ds} dS.$$

207.] It may here be interesting to prove the following proposition, as an illustration of the resistance of a conductor, namely—If one electrode of a conductor be in communication with the earth, and the other with a conducting sphere charged originally with any amount of electricity, then the resistance of the conductor is the reciprocal of the velocity with which the radius of the sphere must diminish, in order that the potential of the sphere may remain constant, notwithstanding the loss of electricity through the conductor[1].

Let the initial radius of the sphere be a, and the mass of its initial charge be M; then the original potential V will be $\frac{M}{a}$.

If dM and da be the simultaneous small decrements of M and a in the small time dt, then, since by hypothesis V is constant, we must have

$$V = \frac{M}{a} = \frac{dM}{da}.$$

But if R be the Resistance of the conductor, we have

$$dM = \frac{V}{R}\,dt.$$

Multiplying these equations together we get

$$1 = \frac{1}{R}\frac{dt}{da},$$

$$\text{or} \quad \frac{da}{dt} = \frac{1}{R}.$$

208.] *On the Value of the Resistance in particular Cases.*

(a) A series of wires of the same material but different transverse sections joined together in series end to end.

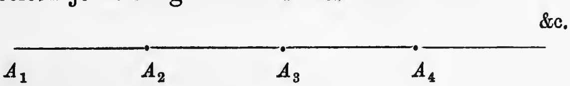

Fig. 30.

Let the wires, n in number, be A_1A_2, A_2A_3, A_3A_4, &c. Let the resistances in these wires be R_1, R_2, R_3, &c. Let the potentials at the points A_1, A_2, A_3, &c. be V_1, V_2, V_3, &c. And let

[1] The proposition is taken from Mascart and Joubert's *Electricity and Magnetism*.

the intensity of the current, which must be the same in each wire, be i.

Then, by the continuity of the current and Ohm's law, we have

$$i = \frac{V_1 - V_2}{R_1} = \frac{V_2 - V_3}{R_2} \; \&\text{c.} = \frac{V_n - V_{n+1}}{R_n} = \frac{V_1 - V_{n+1}}{R_1 + R_2 \ldots + R_n}.$$

If therefore R be the resistance in this case,

$$R = R_1 + R_2 \ldots + R_n.$$

(β) The resistance in the case of a multiple arc conductor.

The conductor is called a multiple arc when it is formed, as in the figure, of a number of separate wires branching off from A,

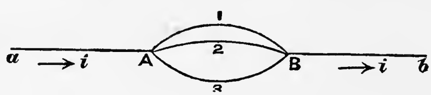

Fig. 31.

the extremity of one wire aA, and converging to the extremity of another wire Bb at B. Let V_a, V_b be the potentials at A and B respectively.

In this case the electromotive force in each separate transit from A to B is the same, viz. $V_a - V_b$. If R_1, R_2, &c. be the resistances of the wires, i_1, i_2, &c. the currents flowing through them, and i the current in Aa, we have

$$i_1 R_1 = i_2 R_2 = \&\text{c.} = i_n R_n = V_a - V_b,$$

$$i_1 + i_2 + \&\text{c.} + i_n = i;$$

$$\therefore \; (V_a - V_b)\left(\frac{1}{R_1} + \frac{1}{R_2} + \&\text{c.} + \frac{1}{R_n}\right) = i = \frac{V_a - V_b}{R};$$

if R be the resistance sought, and therefore

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \&\text{c.} + \frac{1}{R_n}.$$

The current flowing in any particular wire, as for instance i_1, is equal to $\dfrac{iR}{R_1}$.

If we denote the conductivities by $K_1 \ldots K_2 \ldots K_n$, we have

$$V_a - V_b = \frac{i}{K_1 + K_2 + \ldots + K_n} = \frac{i}{\Sigma K}.$$

It follows from (a) that the resistance in a wire of given uniform section varies directly as the length of the wire, and it follows from (β) that the resistance in a conductor consisting of any number (n) of similar and equal wires placed side by side is $\frac{1}{n}$th of the resistance of any one of the wires taken singly, hence the resistance of a wire of given substance and given length varies inversely as the area of the transverse section, or generally the resistance in a wire of given material varies as the length directly and the transverse section inversely, as already stated.

209.] If we have a homogeneous wire of which the area of a transverse section at distance s from one end is $f(s)$, the resistance per unit of length at the same point is $\frac{1}{f(s)}$, and the resistance of the portion s of the wire is $\int_0^s \frac{ds}{f(s)}$. Hence, if V_0 be the potential at the end in question, the potential V_a at distance s, when a current i flows from that end, is found from

$$V_0 - V = i\int_0^s \frac{ds}{f(s)}.$$

Currents of much greater complexity may occur in practice and are of great importance. We shall later on investigate a more general case of a system of wires traversed by electric currents.

210.] When the conductor is not a wire, or collection of wires, but a continuous conducting substance, we have seen that Ohm's law may be expressed by the equation

$$i = -K\frac{dV}{ds}dS;$$

where i is the intensity of current which traverses an equipotential elementary area dS in the neighbourhood of any point at which the potential is V, ds is an element of the line of force through the point, and K is a constant at all points in the substance depending on the nature of its material.

When any given regions of such a conductor are kept at uniform given unequal potentials, a permanent current state is soon established; the given equipotential regions are in such a case generally termed *electrodes*, and sometimes *sources* or *sinks* of electricity, according to the direction of the current flow from or towards them.

When these electrodes are two in number, one source and one sink, we may, as in the case of a wire or wires, determine a value of the ratio of electromotive force to current intensity which will remain constant so long as the substance and position of the electrodes is constant, and this ratio is spoken of as the resistance of the system; the electromotive force is the difference of the constant potentials of the source and sink, and the current intensity is measured by the rate of transference from source to sink per unit of time.

211.] As a particular example let us take an infinitely extended and very thin conducting plate, bounded by parallel planes and pierced by two cylinders P and Q which are maintained at given constant potentials.

If the mean plane of the plate be that of x, y, and V be the potential at any point, the conditions that there shall be no free electricity within the plate, and that the equipotential surfaces are all normal to the plate, lead to the equations

$$\nabla^2 V = 0, \qquad \frac{dV}{dz} = 0.$$

Hence the problem may be treated as one in two dimensions only, and the electrodes may be regarded as circles with radii equal to those of the cylinders; let these radii be a and b, and let the constant potentials be V_p and V_q respectively.

The equation in V, or

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0,$$

may be satisfied by assuming

$$V = C + A_1 \log r_1 + A_2 \log r_2 + \text{\&c.}, \text{ or } C + \Sigma A \log r,$$

where the quantities r_1, r_2, &c. are the distances of the point x, y from any assumed fixed points, and these points must be so taken that V is equal to V_p and V_q respectively at the circumferences of the circular electrodes.

Let O_p and O_q be two points within the circles P and Q such that each is the image of the other in its own circle, and let the potential V at any point be taken equal to $C - A\log\frac{r_1}{r_2}$, where r_1 and r_2 are the distances of the point from O_p and O_q respectively, then all the required conditions will be fulfilled, provided C and A be taken to satisfy the conditions

$$V_p = C - A\log\frac{r_1}{r_2} \text{ at the circumference of } P,$$

$$V_q = C - A\log\frac{r_1}{r_2} \text{ at the circumference of } Q,$$

inasmuch as $\frac{r_1}{r_2}$ is constant over each of these circumferences.

Since V is constant whenever $\frac{r_1}{r_2}$ is so, it follows that the equipotential curves are circles each one of which is conjugate to the centres of P and Q. The orthogonal trajectories of such circles, or the lines of current flow, are circular arcs each passing through these centres, and, if $\frac{dV}{da}$ and $\frac{dV}{db}$ be found at the circumferences of these circles respectively, we can find the whole current in unit time in terms of $V_p - V_q$ in the form of $\frac{V_p - V_q}{R}$; the quantity R is then called the resistance of the system, its reciprocal being the conductivity.

In the particular case of the radii a and b being equal, and each very small compared with f, the distance between the centres, we find from the above equations

$$A = \tfrac{1}{2}\frac{V_p - V_q}{\log\frac{f}{a}}.$$

Let i be the current in unit time over the arc ds of the circumference of the P electrode, then, since r_2 is sensibly constant and the direction ν of the line of flow is along the radius of P,

$$i = -K\frac{dV}{d\nu}ds = -K\frac{dV}{dr_1}ds = \frac{AK}{a}ds,$$

where K is the conductivity of a unit length of a prism of the conductor of unit breadth. Therefore the total current in unit time over the P circumference is $2\pi a i$, or $2\pi K A$.

It is clear that K is proportional to the thickness δ of the plate, and if for it we write $K\delta$, the current per unit time will be

$$2\pi K\delta \,.\, A,$$

where K is now the conductivity through a cube of the substance whose edge is the unit of length.

Writing for A the value already found $\dfrac{V_p - V_q}{2\log\frac{f}{a}}$, we get the current per unit time equal to

$$\pi K\delta \frac{V_p - V_q}{\log\frac{f}{a}},$$

and the resistance of the system is $\dfrac{\log\frac{f}{a}}{\pi K\delta}$.

On Systems of Linear Conductors.

212.] A conductor, two of whose dimensions are very small compared with the third, as for instance a wire, is called *a linear conductor.*

We have had occasion to consider certain properties of linear conductors. Firstly, we have seen that if such a conductor be divided into several parts through which a current flows consecutively, as AB, BC, &c., the resistance of the whole is the sum of the separate resistances of the several parts. Hence, in case of a homogeneous conductor at uniform temperature, if the potentials at the ends are known we can determine the potential at any intermediate point when a current is flowing.

For instance, let APB be a wire the potentials of whose extremities are V_a and V_b. Let P be an intermediate point, and let the resistance of the portion AP be r_{ap}, and that of PB be r_{pb}. Then if i be the current,

$$V_a - V_p = r_{ap}\, i,$$
$$V_p - V_b = r_{pb}\, i.$$

Hence

$$V_a - V_p = \frac{r_{ap}}{r_{ap} + r_{pb}} (V_a - V_b),$$

which determines V_p.

Similarly, if i be given, but the potentials are not given, we can determine the differences of potential $V_a - V_p$ and $V_a - V_b$.

Again, in case of two or more wires connected in multiple arc, we have shown that if V_a, V_b be the potentials of the extremities the currents in the several wires are respectively $K_1(V_a - V_b)$, $K_2(V_a - V_b)$, &c., where K_1, K_2, &c. are the conductivities of the wires. And we can therefore determine all the currents if V_a and V_b are given, or the difference of potentials $V_a - V_b$, if the sum of the currents is given.

It is assumed that the wires are all of the same metal, and at uniform temperature.

213.] The points of junction of the wires are called *the electrodes*. In the above simple case we have only two electrodes. But we may conceive a system of wires meeting in more than one point.

For instance, to take a case a little more complicated, let there be two wires APB, AQB, and the two intermediate points P and Q connected by a third wire.

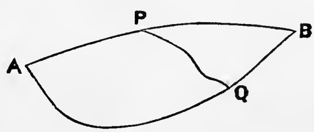

Fig. 32.

If the potentials at A and B are given, we may determine those at P and Q, as follows.

Let K_{ap}, K_{pb}, K_{pq} be the conductivities of the three wires AP, PB, PQ. Then, since the sum of the currents flowing from P must be zero, we have

$$K_{ap}(V_a - V_p) + K_{bp}(V_b - V_p) + K_{pq}(V_q - V_p) = 0.$$

Similarly,

$$K_{aq}(V_a - V_q) + K_{bq}(V_b - V_q) + K_{pq}(V_p - V_q) = 0,$$

from which, the conductivities K being known, the two unknown potentials, V_p and V_q, can be determined; and thence the currents are known.

If instead of the potentials, the current C, entering the system at A and leaving it at B, be given, we have three linear equations to determine the differences of potential $V_a - V_b$, $V_a - V_p$, and $V_a - V_q$, namely,

$$C = K_{ap}(V_a - V_p) + K_{aq}(V_a - V_q),$$
$$0 = K_{ap}(V_p - V_a) + K_{bp}(V_p - V_b) + K_{pq}(V_p - V_q),$$
$$0 = K_{aq}(V_q - V_a) + K_{bq}(V_q - V_b) + K_{pq}(V_q - V_p).$$

The points P and Q will generally be at different potentials, and a current will pass along PQ or QP.

214.] The case in which P and Q happen to be at the same potential is of special importance. In that case no current passes in PQ, and the potentials at every point in either wire are the same as if there were no metallic connexion between P and Q. That is,

$$V_p = V_a - \frac{r_{ap}}{r_{ap} + r_{pb}}(V_a - V_b),$$

$$V_q = V_a - \frac{r_{aq}}{r_{aq} + r_{qb}}(V_a - V_b).$$

A current will pass in one or other direction along PQ, unless P and Q are at the same potential, that is, unless

$$\frac{r_{ap}}{r_{ap} + r_{pb}} = \frac{r_{aq}}{r_{aq} + r_{qb}}.$$

This principle is made use of in instruments for measuring resistance. Suppose, for instance, AX is a wire whose resistance r_{ax} is required. Let BX be a conductor whose resistance r_{xb} is known. Place AX and XB so as to form one conductor AXB. Let AEB be a uniform wire, E a point in it.

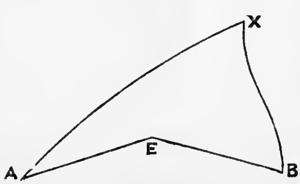

Fig. 33.

If E and X be joined by a wire, a current will pass along it in one or other direction, unless the potential at X is the same as at E.

We increase or diminish the distance of E from A until a needle suspended near EX shows no deflection when an electric current is made to pass from A to B. Then we know that the potential at X is the same as that at E; and therefore

$$r_{ax} = r_{bx}\frac{AE}{EB},$$

which determines r_{ax}. This is the principle of the instrument known as Wheatstone's Bridge.

215.] In a more general case, there may be n points, or electrodes, connected each to each by wires of known conductivities.

Let $V_1 \dots V_n$ be the potentials at the several electrodes, $c_1, c_2, \dots c_n$ the currents which enter the system from without at these electrodes respectively, taken as negative when a positive current leaves the system. Then the current in AB is

$$K_{ab}(V_a - V_b);$$

and we have for the electrodes P and Q

$$\left.\begin{array}{l} c_p = K_{ap}(V_p - V_a) + K_{bp}(V_p - V_b) + \&c. \\ c_q = K_{aq}(V_q - V_a) + K_{bq}(V_q - V_b) + \&c. \end{array}\right\}, \quad \dots \dots \dots (A)$$

and so on for each electrode.

Now since no electricity can be generated or destroyed within the system, the sum of the currents entering the system at all the electrodes must be zero. That is,

$$c_1 + c_2 + \dots + c_n = 0.$$

Therefore only $n-1$ of the c's are independent.

Also, since we are only concerned with the differences of the potentials, there are $n-1$ independent quantities of the form $V_a - V_b$.

In all we have $\overline{n-1}$ independent linear equations of the form A subsisting between the $2n-2$ independent quantities.

If therefore any $n-1$ of the quantities be given, the equations suffice to determine the others. For example, if the entering currents c be given at any $n-1$ of the electrodes, we can determine all the differences of potential. And if all the differences of potential are given we can determine the currents.

If we differentiate the equation A for any electrode, as P, we obtain

$$-dc_p = K_{ap}\,dV_a + K_{bp}\,dV_b + \&c.$$

Similarly, differentiating the equation for A we obtain

$$-dc_a = K_{ap}\,dV_p + \&c.$$

Since $K_{ap} = K_{pa}$, it follows that the potential at P due to the introduction of unit current at A is equal to the potential at A due to the introduction of unit current at P, and so on.

On the Generation of Heat by Electric Currents.

216.] Suppose a uniform current of intensity I to be existing in a linear conductor AB of resistance R, with terminal potentials V_A and V_B.

There is a transference, per unit time, of electricity I from the extremity A to the extremity of B.

Now if e_1, e_2, &c. be the charges upon a system of conductors A_1, A_2, &c., and if V_1, V_2, &c. be the corresponding potentials and W the electric energy of the system, we have proved that

$$\frac{dW}{de} = V.$$

In the case now under consideration, the charge at the extremity A of the conductor, where the potential is V_A, is diminished by Idt in the time dt, and that at the extremity B, where the potential is V_B, is correspondingly increased by the same quantity. Hence, since V_A is greater than V_B, there is by the process a diminution of the electric energy of the system in time dt equal to

$$(V_A - V_B)\, I dt.$$

But by Ohm's law, we have

$$I = \frac{V_A - V_B}{R}.$$

Therefore the diminution of electric energy, owing to the existence of the current, in the time dt is

$$dW = \frac{(V_A - V_B)^2}{R} dt = RI^2 dt.$$

This is the work done *by* the electrical forces in the field in time dt in the passage of the current I through the conductor, and this work done, or electric energy lost, must reappear in heat evolved in the conductor AB in the same time.

If therefore J represent the Joule heat equivalent, the heat evolved per unit time will be

$$\frac{RI^2}{J}.$$

Joule was the first to prove by direct experiment that the rate of evolution of heat in any wire through which a current passes is proportional to the square of the intensity of the current, and we now see that this result follows directly from Ohm's law and the principle of the conservation of energy.

217.] If the current, C, having been generated in a system, be allowed to decay by the resistance R, the value of the current at time t after the commencement is $C\epsilon^{-Rt}$. Hence the total quantity of heat generated when the current has ceased is

$$RC^2\int_0^\infty \epsilon^{-2Rt}\,dt = \tfrac{1}{2}C^2.$$

For this reason $\frac{1}{2}C^2$ is sometimes called the energy of the current.

It is supposed here that the current during this process is uninfluenced by any other current, or by any magnetic field, as we shall see later that electric currents in the same field exert mutual action on each other.

On the Generation of Heat in a System of Linear Conductors.

218.] In the simple case of a number of wires in multiple arc, we have seen that $R_1C_1 = R_2C_2 =$ &c., where C_1, C_2, &c. are the currents, and R_1, R_2, &c. the resistances in the respective wires.

If the total current $C_1 + C_2 +$ &c., or ΣC, be given, this is the distribution of the current among the several wires which makes the heat generated per unit of time a minimum. For $\Sigma R^2 C$ is the heat generated, and the condition that this should be minimum given ΣC is that $R_1C_1 = R_2C_2 =$ &c.

The same property can be proved (Maxwell's *Electricity and Magnetism*, 283) for the more general system, provided there be no internal electromotive forces.

For let C_a, C_b, &c. be the given currents entering a system of linear conductors at the electrodes A, B, &c. Let C_{ps} be the current in any wire PS determined according to Ohm's law by the process above described, so that

$$C_{ps} = (V_p - V_s)\,K_{ps},$$

or

$$V_p - V_s = R_{ps}\,C_{ps}.$$

Let us next suppose the same total current constrained to flow through the system according to any other mode of distribution, without however altering the sum of the currents flowing from or to any electrode. Let, for instance, the current in PS be $C_{ps}+X_{ps}$ instead of C_{ps}.

Then the heat generated in the distribution according to Ohm's law is ΣRC^2.

And the heat generated in the constrained distribution is

$$\Sigma R(C+X)^2,$$

or

$$\Sigma\{RC^2+RX^2+2CRX\}.$$

But for each electrode A, B, &c. the sum of the entering currents is unaltered.

Hence, for any electrode as A,

$$C_{a_1}+C_{a_2}+\ldots = C_{a_1}+X_{a_1}+C_{a_2}+X_{a_2}+\&c.;$$

or

$$\Sigma X_a = 0.$$

Hence

$$2\Sigma RCX = 0.$$

Therefore the heat generated per unit of time in the constrained system is $\Sigma RC^2+\Sigma RX^2$, and exceeds that generated in the original system by the essentially positive quantity ΣRX^2.

Electromotive Force of Contact.

219.] Up to this point we have introduced the restriction that the conductors with which we are concerned shall be of the same substance throughout. The reason of this restriction, which in strictness is equally required in electrostatic investigations, will now be considered.

Volta believed that when two different metals were placed in contact, the potential of one of them was always higher than that of the other, and this without any disturbance of electric equilibrium. In fact, that instead of the condition of electric equilibrium being $V=$ Constant throughout all continuous conducting space, the condition should really be, when such conducting space is composed of substances of different materials, $V=C_1$, $V=C_2$, $V=C_3$, &c., in the regions occupied by these substances respectively; the values of the constants C_1, C_2, C_3, &c. being dependent upon the nature of the substances, and the electric distribution in the field; subject only to this restriction, that in every case of electrostatic equilibrium of a compound conductor the difference $C_r - C_s$ of the constant potentials of any two given substances should always be the same at the same temperature.

220.] For instance, if a zinc wire and a copper wire were held by insulating supports, and brought into contact at one end of each, the potential of each wire would be the same throughout, but that of the zinc would exceed that of the copper by a quantity always the same for the same temperature. If platinum were substituted for copper a similar result would be observed, but the difference of potentials (the temperature being the same as before) would be less. If platinum and copper were similarly connected, the platinum would stand at the higher potential, and the constancy of temperature being still maintained, it would be found that the excess of potential of zinc over copper in the first case, supposed above, was equal to the sum of the excesses of the potentials of zinc over platinum and platinum over copper in the two last cases. This difference of potentials is generally called the electromotive contact forces of the two metals, and is for metals A and B denoted by A/B.

It is considered as positive if the metal of higher contact potential is placed before the line and negative if the reverse, so that $A/B + B/A = 0$, and if there were three metals A, B, and C whose electromotive contact forces at any temperature were A/B for A and B and B/C for B and C, then the electromotive contact force for A and C at the same temperature would be $A/B + B/C$; or in other words, for the same temperature we have the equations

$$A/B + B/C = A/C,$$

and

$$A/B + B/C + C/A = 0.$$

If the metal A in contact with B at any temperature stand at a higher potential than B, it is said to be electropositive with regard to B, and B to be electronegative with regard to A.

221.] Volta with his followers regarded all metals as having certain specific affinities for the positive fluid, so that in cases of contact the electropositive metal of the pair becomes charged positively with reference to the other metal. A similar effect is on this view supposed to attend the contact of all conducting bodies, whether metallic or non-metallic, solid or liquid, *in the absence of chemical action.* In the case of composite liquid conductors chemical decomposition ensues on contact, and the electromotive contact force is on such decomposition diminished, or reduced to zero; the contact difference of potentials being, on this view, dependent upon the absence of chemical action.

According to the views entertained by other physicists, the difference of potential at contact is dependent upon the medium in which the touching bodies are situated, and is in all cases *the result* of chemical action in that medium. The latter hypothesis is to a certain extent at least borne out by experiment*, and the subject cannot be regarded as being yet thoroughly decided. Meanwhile we may, without waiting for a solution of the difficulty, develop the laws of this electromotive force of contact, so far as they have been experimentally determined.

222.] Ohm's law as originally enunciated contemplates a wire of homogeneous substance throughout. The laws of current intensity and of evolution of heat in the case of wires in series require modification when these wires are not of the same materials:

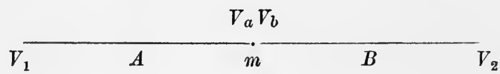

Fig. 34.

For example, let there be two wires of metals A and B touching at m. Let the potentials of A at the free end and at m be V_1 and V_a, and let those of B at the corresponding points be V_2 and V_b. Let R_a and R_b be the resistances of the A and B wires respectively, and let i be the current intensity. Then by Ohm's law

$$i = \frac{V_1 - V_a}{R_a} = \frac{V_b - V_2}{R_b} = \frac{V_1 - V_2 + V_b - V_a}{R_a + R_b} = \frac{V_1 - V_2 + B/A}{R},$$

if V_1 be greater than V_2, and if R be total resistance as

* See a Paper by Exner, *Phil. Mag.* vol. x. p. 280, and works there cited. A third view, suggested by Professor Oliver J. Lodge, is that each metal in the absence of contact with another metal is at lower potential than the surrounding air by an amount depending on the heat developed in its oxidation, that on contact the potentials of the two metals become equal, the more oxidisable metal receiving a positive and the other a negative charge.

previously defined, the term B/A being, as above explained, positive or negative according as B is electropositive or electronegative with regard to A.

If there had been any number of wires of metals A_1, A_2, A_3, &c., the equation would have been

$$i = \frac{V_1 - V_2 + \Sigma . (A_{r+1}/A_r)}{R}.$$

So that Ohm's law might still be enunciated for such an arrangement, provided the external electromotive force $V_1 - V_2$ were increased by the electromotive contact forces at the respective junctions, regard being paid to the signs of these forces, and the resistance being the sum of the resistances in the respective wires.

223.] In the multiple arc arrangement with initial and final wires of metals A and B, and connecting wires of metals m_1, m_2, &c., if i be the current intensity in A or B, and i_r that in the wire m_r, with resistance R_r, V_1 and V_2 the potentials of A and B, and V_r and V_r' of m_r, at the junctions, we have

$$i_r = \frac{V_r - V_r'}{R_r}$$

$$= \frac{V_1 + m_r/A - V_2 - m_r/B}{R_r} = \frac{V_1 - V_2 + m_r/A + B/m_r}{R_r}$$

$$= \frac{V_1 - V_2 + B/A}{R_r};$$

and therefore

$$i = \Sigma i_r = \Sigma \left(\frac{1}{R}\right)(V_1 - V_2 + B/A).$$

So that in this case also the same expression results as in the homogeneous multiple arc already investigated, provided the external electromotive force be increased by B/A.

224.] The expression for the energy dissipated in the case of wires in series also requires to be modified when the wires are not of the same metal throughout.

If as in the last article there be two wires of metals A and B, and the notation of that article be retained, we have

The total loss of electric energy per unit time as the current passes from the free extremity of A to that of $B = (V_1 - V_2)i$.

Therefore the whole heat generated must be

$$\frac{(V_1 - V_2)\,i}{J}.$$

But by the equation above obtained

$$V_1 - V_2 = Ri - B/A.$$

Therefore the heat generated is

$$\frac{1}{J}(Ri^2 - B/A\,.\,i);$$

that is to say, if B/A be positive the heat generated in the compound wire of resistance R by the passage of the current of intensity i is less than $\frac{Ri^2}{J}$, or what it would have been had the wire been homogeneous, by the quantity $\frac{B/A\,.\,i}{J}$, and is greater than $\frac{Ri^2}{J}$ by $\frac{A/B\,.\,i}{J}$ if A/B be positive: that is to say, when a current in passing through a circuit of heterogeneous metal wires traverses a junction from an electronegative to an electropositive metal there is absorption of heat at the junction, and on the contrary, there is evolution of heat in the passage from an electropositive to an electronegative metal.

225.] This absorption and evolution of heat at metal junctions was first observed by Peltier, and the phenomenon is called after his name; it is physically analogous to the absorption and evolution of heat accompanying chemical dissociation and combination respectively, the electricity at the junction being raised to a higher, or sinking to a lower potential in the respective cases, just as the chemical potential of the dissociated or combined elements is raised or depressed. The actual amount of heating or cooling as experimentally observed is always less than the theory requires, and in some cases is of the opposite sign; indicating, apparently, that the whole electromotive contact force of Volta is not to be sought in the mere metallic contact, but in the action of the surrounding medium.

CHAPTER XIII.

OF VOLTAIC AND THERMOELECTRIC CURRENTS.

ARTICLE 226.] IF any number of wires of different metals M_1, M_2, M_3, M_1 are joined together in series, and are kept at the

Fig. 35.

same temperature throughout, the wire of metal M_1 beginning and ending the series, it follows from the laws of contact action above stated that each wire is at the same potential throughout its length, and that the beginning and ending M_1 wires are also at the same potentials, inasmuch as the sum of the electromotive contact forces $M_1/M_2 + M_2/M_3 + M_3/M_1$ is zero; hence if a circuit be formed by joining the free ends of the M_1 wires no current will ensue. If however we substitute for the M_1 wire a composite liquid conductor L, and thus complete the circuit, the electromotive contact forces L/M_2 and L/M_3 are modified, the liquid L being at the same time decomposed.

According to the extreme views of the Volta contact theory, the last-mentioned electromotive forces disappear with the decomposition, the liquid L and the metals M_3 and M_2 at their points of immersion in that liquid are reduced to the same potential, the electromotive contact forces M_2/M_3, &c. of the metallic junctions are no longer compensated by the forces L/M_2 and L/M_3, and a current ensues through the wires and liquid.

Suppose, for instance, the liquid be dilute sulphuric acid and the metals be plates of zinc and copper partially immersed and having their unimmersed ends attached to platinum wires, so long as these platinum wires are not united to each other, the zinc, the copper, and the liquid stand, according to this theory, at the same potential (V suppose), but the platinum wires attached to the zinc and copper plates are at the potentials $V - Z/P$ and

$V+P/C$ respectively. If now the platinum wires be united, electric equilibrium can no longer be maintained, inasmuch as the two portions of the same platinum wire are now at potentials differing from each other by $Z/P+P/C$ or Z/C. Hence a flow of electricity must take place through the platinum wire from the copper to the zinc plate, raising the potential of the zinc and depressing that of the copper.

The inequality of the potentials thus produced in these immersed plates is again destroyed by the action of the liquid, which is at the same time decomposed, oxide of zinc being formed at the zinc plate, which is dissolved as soon as formed, and hydrogen being given off at the copper plate, and thus a permanent current ensues in the closed circuit of copper, platinum, zinc, liquid, copper, and in the direction indicated by the order of these words. Such an arrangement is called a *Voltaic current*, the vessel containing the liquid and plates is called a *Voltaic cell*, the decomposable liquid is called an *electrolyte*, and its decomposition on the passage of the current is called *electrolysis*. The intermediate platinum wire is in no respect essential to the process, which would have equally taken place if the copper and zinc plates had been in immediate external contact with each other.

227.] According to the theory of the Voltaic circuit, above explained in outline, the potential rises discontinuously at the metallic junction or junctions outside the cell, and falls continuously throughout the rest of the circuit; the whole electromotive force of the current is sought for in the contact force at the junctions, the function of the chemical action in the cell being limited to the continued equalisation of potentials within the cell as fast as the equality is destroyed by the electric flow. According to the chemical theory of the circuit, which is now more generally accepted, a discontinuous change of potential takes place at the junctions between the metals and the liquid, those being the points at which, as we shall see presently, energy for the maintenance of the current is in fact evolved or absorbed. The true theory of the cell is not finally settled, only it is known that the chemical decomposition is an essential part of the phenomenon. It is possible however to develop certain fundamental laws of the action, which are essentially the same whatever be the metals constituting the plates, and whatever be the liquid in the cell, provided it be capable of electrolysis.

228.] The plates by which the current enters and leaves the cell are called *electrodes*, that by which it enters is called the *anode*, and that by which it leaves is called the *cathode*, the two elements into which the electrolyte is decomposed are called the *ions*, the element appearing at the anode is called the *anion*, and that appearing at the cathode is called the *cation*.

Let the metal electrodes be called P and N respectively, and the two constituents, or ions, into which the liquid is resolved be called π and ν respectively. On the passage of the current the ion π will appear at the electrode N, and the ion ν at the electrode P. Then it is found that—

(1) The ratio of the masses in which the two constituents π and ν appear at the electrodes is that of their combining weights.

(2) The absolute mass of each ion so deposited per unit of time is proportional to the strength of the current, or in other words, for each unit of positive electricity transmitted a certain mass of each ion is deposited at the corresponding electrode. This is called *the electrochemical equivalent* of that ion.

(3) So long as the electrolyte is the same, the ions into which it is decomposed are the same, whatever the metals constituting the electrodes; and the same ions appear at the anode and cathode respectively. One or both of the ions may be compounds, and the same constituent which in one electrolyte becomes an anion, may in another electrolyte become a cation.

(4) The source whence the energy, requisite for the maintenance of the current, is derived, is the arrangement of the elements of the electrolyte and the immersed plates in a combination of lower chemical potential energy than that which existed anterior to the current.

229.] The action of the typical cell described above of zinc and copper plates in diluted sulphuric acid may be supposed to be as follows. The chemical arrangement before the circuit was completed was $Zn\ \overline{H_2SO_4} \ldots H_2\overline{SO_4}\,.\,Cu;$

and during the existence of the current it is

$$\overline{Zn\,SO_4},\ \ldots\ H_2\,Cu.$$

The zinc first combines with the oxygen of the water ($H_2\,O$), and the zinc oxide is then replaced by the zinc sulphate $\overline{Zn\,SO_4}$, which being soluble leaves the zinc plate free for further action. The potential chemical energy of $Zn\,SO_4$ is less than that of H_2O, or, as more practically expressed, the heat evolved by the combinations $Zn\,O$ and $Zn\,O, SO_3$ is greater than that required for the decomposition of H_2O, the difference furnishing the current energy.

230.] A feeble current might have been obtained with water only in the cell, the chemical arrangements before and after the completion of the circuit being

$$Zn\,\overline{H_2\,O}\ \ldots\ H_2\,O\,Cu,$$

and

$$Zn\,O\ \ldots\ H_2\,Cu$$

respectively.

But in this case, since the oxide $Zn\,O$ is insoluble in water, the zinc plate would soon, by its oxidation, become unfit for action, and the current would cease. We may however use this ideal case as an example.

In this case the ions π and ν are O and H_2 respectively. Taking unity as the combining number for hydrogen, that of oxygen is 8, and that of zinc is 32·53. Therefore one gramme of zinc takes in combination with oxygen the place of $\frac{1}{32{\cdot}53}$ grammes of hydrogen, each combining with $\frac{8}{32{\cdot}53}$, or ·246 gramme of oxygen. The heat evolved by the combination of one gramme of zinc with the oxygen is 1310 units*. The heat which would be evolved on the combination of $\frac{1}{32{\cdot}53}$ gramme of hydrogen with the oxygen, and which is therefore absorbed on their dissociation, is 1060 units. Therefore for every gramme of zinc

* The object in this and the two following articles being illustration only, the absolute numerical values are of less importance. The system of units and the numerical values are those employed in Hospitalier's *Formulaire pratique de l'Electricien*, English Edition, p. 214.

oxidised the excess of heat evolved over that absorbed is (1310—1060) units; that is, 250 units.

Again, for every unit of current ·00034 gramme of zinc is oxidised. In other words, ·00034 is the electrochemical equivalent of zinc. Therefore for every unit of current the excess of heat evolved over that absorbed is $\cdot 00034 \times 250$. And this is equivalent to an amount of mechanical work

$$J \times \frac{34 \times 250}{100000},$$

where J is Joule's factor.

Now if F be the electromotive force of the cell, i the current, the amount of heat evolved is Fi. And therefore the amount of heat evolved by unit current is F. That is,

$$F = J \times \frac{34 \times 250}{100000}.$$

231.] It is usual, as above said, to employ instead of water dilute sulphuric acid, the formula for which is $H_2 O\, S O_3$. In this case we may suppose that the $H_2 O$ is decomposed, and in the first place oxide of zinc, ZnO, is formed, and the hydrogen H_2 is set free. Then the $Zn\, O$ combines with the SO_3, forming $Zn . SO_4$. The heat evolved by this last-mentioned combination must be added to the 250 units above mentioned.

One gramme of zinc combines with ·246 of a gramme of oxygen H_2O being decomposed with the evolution of 250 units of heat. And 1·246 grammes of oxide of zinc combine with SO_3 with the evolution of 360 units. Adding together 360 and 250, we obtain 610 units as the total heat evolved.

232.] In the cell known as Daniell's cell the electrodes are zinc and copper, but there are two liquid electrolytes, one of them saturated solution of sulphate of copper in contact with the copper, and the other dilute sulphuric acid in contact with the zinc, the mixing of the liquids being prevented by a porous diaphragm which does not interfere with the electrolytic conduction, that is to say, the liberated ions pass through the diaphragm but the liquids do not. The following may be supposed to be the action of such a cell.

The electrolysis of the $H_2 SO_4$ in contact with the zinc gives rise to a chemical action identical with that of the last case, but the hydrogen H_2 does not as in that case remain free. It passes through the diaphragm and displaces an equivalent of copper in the sulphate of copper $Cu\,SO_4$, giving as a result $H_2 SO_4$, and depositing the copper on the copper plate.

In estimating the electromotive force of this battery the dissociation and combination of the water $H_2 O$ counteract each other, and the resulting force is the difference between the heat of combination of zinc with SO_4 and that of copper with the same element.

The heat of combination $Zn\,SO_4$ we have already found to be 1670 units, being the sum of $Zn.O$ (1310 units) and $ZO.SO_3$ (360 units), and the heat of combination $Cu\,SO_4$ is 881 units. The difference, i. e. the thermal measure of the chemical action, is 789 units. The product of this 789 by $\frac{34}{10000}$, the electrochemical equivalent of zinc, gives the thermal measure of the chemical action for each unit of electricity transmitted, and this result again, multiplied by Joule's factor, gives the electromotive force of a Daniell's cell in the ordinary mechanical units.

233.] The electromotive force of a cell in which unit work in centimetre-gramme-second measure is done for each unit of electricity transmitted is taken for the unit of electromotive force. It is called a *Volt*. A Daniell's cell gives in practice about 1·079 Volts. The unit resistance is an *Ohm*, and may be defined to be 48·5 metres of copper wire of one millimetre thickness. The unit current is called an *Ampère*, and is the current generated by an electromotive force of one Volt in a conductor whose resistance is one Ohm.

234.] The electromotive force of a cell may be expressed in general terms as follows. See Fleeming Jenkin's *Electricity*.

Let the effect of the unit current be to decompose a constituent into ϵ grammes of one ion π, and ϵ' grammes of the other ion ν. Then ϵ and ϵ' are the electrochemical equivalents of the two substances of π and ν.

Let θ be the quantity of heat absorbed in the combination of unit mass of π with the corresponding mass of ν, and let θ' be the heat absorbed in the same combination for unit mass of ν. Then $\theta \epsilon = \theta' \epsilon'$ is the heat evolved in the circuit for every unit of current.

And in mechanical units $J\theta\epsilon$ is the electromotive force of the cell. If the chemical action be more complex, as in the case of the Daniell's cell, it still remains true that the heat evolved is proportional to ϵ or ϵ', and θ is to be found as the algebraic sum of the heat evolved and absorbed by all the chemical changes from which the ions result.

235.] By increasing the dimensions of the cell we do not increase the electromotive force of the circuit, but we diminish the resistance within the cell, and we therefore increase the intensity of the current, especially in cases where the external portion of the circuit is of a small resistance, and where therefore the resistance of the cell becomes appreciable; and the same result follows when several cells act together, the zinc plates being severally connected, and likewise the copper plates, for this arrangement is in its electrical effects the same as if all the zinc and all the copper plates were severally combined into one zinc and one copper plate with areas respectively equal to the aggregate areas of the zinc and copper plates in the separate cells.

If however the zinc of one cell be united with the copper of the next, and so on in order, the cells are said to be *in series*, the electromotive force is the sum of the separate electromotive forces of the separate cells, and the arrangement is called a voltaic or galvanic battery.

If the discontinuous rise of potential in case of a circuit formed of a single cell be E, then in case of a circuit formed of two or more cells in series it will be repeated as many times as there are cells. According to the Volta theory, this rise of potential takes place at the junction between the copper of the first cell and the zinc of the second, and so on.

According to the chemical theory the change takes place in each cell between the metals and the liquid—according to either view the electromotive force of n cells in series is n times that of a single cell.

236.] If we have a number of cells of different kinds connected in series, the electromotive force of the series will be the algebraic sum of the electromotive forces of the separate cells.

We might for instance place a single Daniell's cell between two more powerful batteries, connecting the zinc plate of the single cell with the terminal zinc plate of one battery, and the copper plate of the single cell with the terminal copper plate of another battery. Then in calculating the electromotive force of the system, we must take that of the single cell as negative.

In that case the current is forced through the single cell against its own electromotive force.

237.] A cell in which no chemical actions can take place on the passage of the current, evolving more heat than is absorbed, cannot maintain a current. But it may be possible by connecting its poles with another battery to force a current through it; and this current may have the effect of decomposing the liquid of the first cell, work being done in it by the external battery against the chemical forces of the cell itself. Such a cell is called an electrolytic cell.

238.] Cases exist in which the ions formed in an electrolytic cell do not escape, but enter into new combinations within the cell. Such new combinations, since work has been done against the chemical forces in forming them, are of higher chemical potential than the original combinations which they replace.

They may be capable of decomposition and restoration to their original condition under the influence of a reverse electric current, in which case heat will be evolved, and the cell in its new state will be a Voltaic cell capable of maintaining a current. Such a cell is called a secondary cell, or an accumulator, because the work done in producing the first chemical changes, or as it is called charging the cell, is stored up in it, and may be made available, as required, to maintain an electric current.

Such is in its essential features the theory of the Planté battery and other allied forms, in which, when charged, one plate is of lead and the other consists of peroxide of lead, and the liquid used is generally dilute sulphuric acid. The cell when so charged maintains a current from lead to peroxide through the liquid, and from peroxide to lead outside, the chemical change being the conversion of the peroxide into protoxide of lead. Then by forcing a current through the cell in the reverse direction the protoxide is again converted into peroxide.

239.] Clausius has suggested a theory of electrolysis, supposing that the molecules of all bodies are in a state of constant agitation; that in solid bodies each molecule never passes beyond a certain distance from its mean position; but that in fluids a molecule, after moving a certain distance from its original position, is just as likely to move further from it as to move back again. Hence the molecules of a fluid apparently at rest are continually changing their positions, and passing irregularly from one part of the fluid to another. In a compound fluid he supposes that not only the compound molecules move about in this way, but that in the collisions that occur between the compound molecules, the molecules, or rather submolecules of which they are composed, are often separated and change partners, so that the same individual submolecule is at one time associated with one submolecule of the opposite kind, and at another time with another. This process Clausius supposes to go on in the liquid at all times, but when an electromotive force acts on the liquid the motions of the submolecules, which before were indifferently in all directions, are now influenced by the electromotive force, so that the positively charged submolecules have a greater tendency towards the cathode than towards the anode, and the negatively charged submolecules have a greater tendency to move in the opposite direction. Hence the submolecules of the cation will during their intervals of freedom struggle towards the cathode, but will be continually checked in their course by pairing for a time with submolecules of the anion, which are also struggling through the crowd but in the opposite direction.

240.] Whether this view of the process of electrolysis be or be not accepted as corresponding to a physical reality, it gives us a clear picture of the process, and is in accordance with the principal known facts. By means of certain assumptions more or less plausible we may extend the hypothesis to the explanation of the process of electrical conduction, at any rate through a liquid, as to do this it is only necessary to suppose that each submolecule when acting as an ion is charged with a definite amount of electricity, in accordance with the statement made above that the amount of electricity transferred during electrolysis is the same for the same number of liberated ions, the charge of the cation being positive and that of the anion negative, by which conception the conduction current becomes assimilated to a convection current, or, perhaps more correctly, to the transfer of motion along a row of equal and perfectly elastic balls in contact. Many difficulties present themselves in the way of this hypothesis, as for instance the fact that certain ions are anions in some electrolytes and cations in others. We do not stop to consider these difficulties in detail, because the whole hypothesis, while useful in furnishing a mental picture of these processes, is not essential to the enunciation and mathematical development of the laws by which they are regulated.

241.] In practice, Voltaic cells, especially single fluid cells, are liable to certain defects, the chief of these being irregularity of electromotive force arising from the accumulation of the ions at the electrodes, thereby causing what is termed electrolytic polarisation, or an electromotive force opposed to the current. That such an accumulation of the ions would engender this opposing force is obvious, at any rate on the hypothesis of Clausius, because, having parted with their electric charges at their respective electrodes, there is no longer any action tending to keep them in this position, and they necessarily tend to recombine. This opposing or negative electromotive force is not so obvious in its effects in a Voltaic cell, because in such a cell there is at all times a preponderating positive force, but it may be clearly exhibited in an electrolytic or resisting cell. If the electrodes of such a cell be platinum plates and the contained liquid be water, then so long as the current is maintained oxygen is given off at the anode and hydrogen at the cathode. If the current be suspended and the platinum plates externally connected by a wire, a current will pass through this wire in the reverse direction, that is to say from the anode to the cathode, and the liberated gases in the cell will recombine.

The special defect arising from polarisation is the irregularity of current which it produces. If the current be suspended for any experimental purpose and then again renewed, it will start with greater intensity than is ultimately maintained.

Of Thermoelectric Circuits.

242.] If a circuit be formed of wires of two or more metals at the same temperature, the contact differences of potential are consistent with each wire being at uniform potential throughout its length, and therefore produce no current.

But the contact difference of potential is a function of the temperature at the point of contact. If therefore the junctions be at unequal temperatures, it is not generally possible that each wire should have constant potential throughout its length. We therefore expect that a current will ensue.

243.] It has been shown by Magnus that in an unequally heated *complete* circuit of a single metal no current is produced by the inequality of temperature.

On the other hand, Sir W. Thomson has shown that generally there is an electromotive force from the hot to the cold parts of the same metal, or from cold to hot, according to the metal and the temperature, but that in a complete circuit the total electromotive force is zero. As in order to prevent a current from flowing from copper to zinc in contact, it is necessary that the potential of the zinc should exceed that of the copper by the quantity Z/C; so in order to prevent a current from flowing from an element of the zinc at temperature $t+dt$ to an adjoining element at temperature t, it is necessary that the potential of the second element should exceed that of the first by a certain quantity σdt, or, if the potential be constant, there is an electromotive force σdt. This quantity σ is for any given metal a function of the temperature, and may be positive or negative, but has generally different values for different metals. It was originally called by Thomson the *specific heat of electricity* for the metal in question. According to this law, if V_a and V_b be the potentials at the ends of a wire unequally heated, the electromotive force in it is

$$V_a - V_b + \int_b^a \sigma dt.$$

Since σ is for any given metal a function of the temperature alone, it is evident that for any closed circuit of one metal, however the temperature vary, $\int \sigma\, dt = 0$, or there is no electromotive force, which agrees with the law of Magnus. This difference of potential, due to difference of temperature, is frequently called 'the Thomson effect,' and σ the coefficient of the Thomson effect.

Professor Tait has shown experimentally that throughout ordinary temperatures, and probably at all temperatures, σ is proportional to the absolute temperature. It is positive for some metals, negative for others, and is nearly zero for lead.

It follows from the above statements that in a circuit of two metals with unequally heated junctions we have to consider two causes, each of which may produce a current, viz. the unequal contact differences of potential at the junctions, and the electromotive force due to variations of temperature in the same metal.

244.] It is found that in general an electric current flows round the circuit, accompanied with equalisation of the unequal temperatures unless these be artificially maintained. If R be the resistance of the circuit, i the current, the electromotive force is Ri. Such a circuit is called a *thermoelectric circuit*, or *thermoelectric couple*. The electromotive force is found to obey the following experimental laws.

I. If the temperatures of the junctions be t_p and t_q, and if $A\,{}^p/_q\,B$ be the electromotive force when A and B are the two metals, and $B\,{}^p/_q\,C$ when B and C are the two metals, then

$$A\,{}^p/_q\,B + B\,{}^p/_q\,C = A\,{}^p/_q\,C.$$

This was proved by Becquerel.

II. If a thermoelectric couple be formed with given metals A and B, and if its electromotive force with junction temperatures p and q' be $A\,{}^p/_{q'}\,B$, and with junction temperatures q' and q be $A\,{}^{q'}/_q\,B$, then the electromotive force of the couple when the junction temperatures are p and q will be

$$A\,{}^p/_{q'}\,B + A\,{}^{q'}/_q\,B\,;$$

that is, $$A\,{}^p/_q\,B = A\,{}^p/_{q'}\,B + A\,{}^{q'}/_q\,B.$$

This also is due to Becquerel.

III. The direction of the current, that is whether it be from A to B, or from B to A, at the hot junction depends on the mean temperature of the junctions.

When the mean temperature of the junctions for a given pair of metals is below a certain temperature T, dependent upon these metals, the current sets in one direction through the hot junction, and when the mean temperature is above T the current sets in the opposite direction, or the electromotive force is reversed. This was discovered by Seebeck.

The temperature T is called the *neutral temperature* for the pair of metals employed. In an iron and copper couple this neutral temperature is, according to Sir W. Thomson, about 280°C. When the mean temperature of the junctions is below this, the current sets from copper to iron through the hot junction, and when it is above this the current sets from iron to copper through that junction.

IV. For any constant temperature of the cold junction, the electromotive force is the same when that of the hot junction is $T+x$, as when it is $T-x$, and is a maximum when it is T. This was established by Gaugain, and results from Tait's experiments. It may be expressed thus: The electromotive force of the couple between temperatures t and t_0 is proportional to

$$(t-t_0)\left\{T-\tfrac{1}{2}\overline{t+t_0}\right\}.$$

245.] The following is a mathematical explanation of these phenomena:—

If the difference of temperature between the two junctions be very small, as dt, the electromotive force of the couple must be proportional to it, and for the metals A and B may be denoted by $\phi_{ab}\,dt$, where ϕ_{ab} is for the given metals a function of the mean temperature of the junctions, and is taken as positive when the current sets from A to B at the hot junction. It is called the *thermoelectric power* of the two metals at temperature t.

It follows from II. that if the temperatures of the junctions be t_0 and t_1, where t_1-t_0 is finite, the electromotive force is $\int_{t_0}^{t_1}\phi_{ab}\,dt$, which for the given metals is a function of t_0 and t_1.

Again, if we take any particular metal for a standard, and denote it by the suffix c, it follows from I. that the electromotive force for the couple in which the metals are A and B, and the temperatures of the junctions t_0 and t_1, is

$$\int_{t_0}^{t_1} \phi_{ac}\, dt - \int_{t_0}^{t_1} \phi_{bc}\, dt.$$

If the reference to the standard be understood, we may call ϕ_a the thermoelectric power of the metal A. And in that case the thermoelectric power of the couple formed of the metals A and B with junctions at temperatures t_0 and t_1 is

$$\int_{t_0}^{t_1} (\phi_a - \phi_b)\, dt.$$

It is usual to take lead as the standard metal.

The functions ϕ_a and ϕ_b may be positive or negative, and for the same metal may be positive at some temperatures and negative at others.

It is deduced from the experiments of Professor Tait that for each metal $\frac{d\phi_a}{dt}$ has a constant value independent of the temperature; that is, $\phi_a = at - \beta$, where t is the absolute temperature, and a, β are constants for the same metal.

Hence if t_0 and t_1 be the lower and upper temperatures of a circuit of metals A and A',

$$\phi_{aa'} = \int_{t_0}^{t_1} \{(a-a')\,t - (\beta-\beta')\}\, dt$$

$$= (a-a')\left\{\frac{t_1^2 - t_0^2}{2}\right\} - (\beta-\beta')\, t.$$

Also if T be the neutral temperature at which $\phi_{aa'} = 0$,

$$(a-a')\, T = \beta - \beta'.$$

Hence

$$\phi_{aa'} = \{a-a'\}\,(t_1 - t_0)\left\{\frac{t_1+t_0}{2} - T\right\},$$

and is proportional to

$$(t_1 - t_0)\left\{T - \frac{t_1+t_0}{2}\right\},$$

as stated in IV.

246.] Adopting a method originally suggested by Thomson, we may represent the thermoelectric powers of different metals at different temperatures by a diagram. Let the abscissa represent absolute temperature, and for any given metal let the ordinate represent its thermoelectric power, that is, the thermoelectric power of a couple composed of that metal and lead, with the temperatures of the junctions infinitely near that denoted by the abscissa.

It follows then from the constancy of $\frac{d\phi}{dt}$ that the locus of ϕ is a right line inclined to the axis of x at the angle $\tan^{-1}\frac{d\phi}{dt}$, and that for any given abscissa, as that corresponding to 50° C., the difference between the ordinates of any two metals represents the thermoelectric power of a circuit of the two metals at that temperature. In the annexed diagram we see that for temperatures below 50° C. lead is positive to iron and negative to copper; from 50° to 284° C. copper is positive to iron and negative to lead; from 284° to 330° iron is positive to copper and negative to lead; above 330° lead is positive to copper and negative to iron. Generally, if for any two metals NM be the difference of the ordinates at temperature t, and $M'N'$ at temperature t', and if E be the neutral point, the thermoelectric power of the couples with the junctions at t and t' is graphically represented by the area $MEN - M'EN'$, whether $M'N'$ be at temperature below or above E.

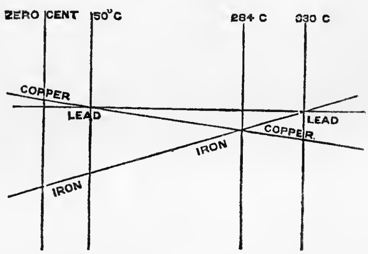

Fig. 36.

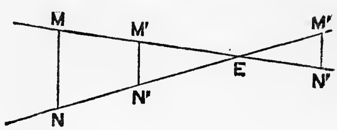

Fig. 37.

So long as the lower temperature represented by MN is unaltered, the difference between MEN and $M'EN'$ has its greatest value when the higher temperature is at E, the neutral point. It becomes zero when the mean temperature of the junctions is the neutral temperature.

Further, if $M'N'$ and $M''N''$ be taken at equal distances from

E on either side of it, $MEN - M'EN' = MEN - M''EN''$. These results agree with IV.

247.] Next, let us consider a circuit of three metals AB, BC, and CA, the junction A being at temperature t_1, B at temperature t_2, and C at temperature t_3.

We may imagine three lead wires AD_1B, BD_2C, and CD_3A connecting the junctions, and forming three distinct circuits.

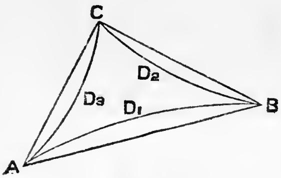

Fig. 38.

The electromotive force of the circuit ABC is the sum of the electromotive forces of the three circuits ABD_1A, BCD_2B, CAD_3C, together with that of the circuit composed of the three lead wires $AD_1BD_2CD_3A$.

But, by the law of Magnus, the electromotive force of the latter circuit is zero.

Hence the electromotive force of the circuit ABC is

$$\int_{t_1}^{t_2} \phi_a dt + \int_{t_2}^{t_3} \phi_b dt + \int_{t_3}^{t_1} \phi_c dt.$$

In like manner we can express the electromotive force due to any circuit of different metals with unequally heated junctions.

248.] We may suppose further a circuit composed of alternate wires of two metals only, A and B, and each alternate junction at the lower temperature t_1, and every other junction at the higher temperature t_2.

If there be n pairs, the total electromotive force of such a circuit is, by the last article,

$$n\int_{t_1}^{t_2} \phi_{ab} dt.$$

The pairs are said to be *joined in series*. By this means the electromotive force of a thermoelectric couple can be multiplied at pleasure. Such an arrangement is called a *thermoelectric pile.*

Of the energy of the current in a Thermoelectric Circuit.

249.] Energy, as above shown, is necessary to maintain the current. In the case of thermoelectric circuits, now considered, no energy is supplied from without, nor are there, so far as we know, any chemical actions between the metals, or between them and the surrounding medium, from which the requisite energy can be obtained.

We infer that the energy required for maintenance of the current is supplied by the conversion of part of the heat of the metals into another form of energy, namely, that of the electric current. This might conceivably be employed to do external work. But if not, it will be reconverted into heat by the resistance of the circuit.

As in the working of a heat engine, the *entropy* of the system must be diminished by the process, that is, there must be equalisation of temperature.

It is found that at the neutral temperature for any two metals a current passing the junction has no heating or cooling effect. The Peltier effect changes sign at that point.

But if a couple be formed with the hot junction at the neutral temperature, the cold junction is nevertheless heated, although the heat cannot be derived from the cooling of the hot junction.

It is evident, therefore, that the current itself must have a heating or cooling effect. For instance, in an iron and copper circuit, with the hot junctions at the neutral temperature, either a current in iron from hot to cold must cool the iron, or a current in copper from cold to hot must cool the copper, or both these effects take place. And it may be inferred that the heat so gained or lost is compensated by a change in the potential of the current. It was this consideration that led Sir W. Thomson to the discovery of the electromotive force in unequally heated portions of the same metal.

250.] The method adopted by some writers (Mascart and Joubert, *Leçons sur l'Electricité et le Magnetisme;* Briot, *Théorie Mécanique de la Chaleur*) is as follows. It is assumed that the heat generated, as unit current passes from potential V_a to potential V_b, is always $V_a - V_b$, whether the fall of potential be gradual as in a single metal, or abrupt as at the junction of two metals.

That being the case, the electromotive force of a couple formed of metals A and B whose hot and cold junctions are at t_1 and t_0 respectively, must be

$$H_1 - H_0 + \int_{t_0}^{t_1} (\sigma_a - \sigma_b)\, dt,$$

where H_1 and H_0 are the Volta contact differences of potential at the junctions.

When $t_1 - t_0$ becomes infinitely small, this becomes

$$\left\{ \frac{dH}{dt} + \sigma_a - \sigma_b \right\} dt;$$

that is,

$$\phi_{ab} = \frac{dH}{dt} + \sigma_a - \sigma_b.$$

Further, if the current be infinitely small, we may regard such a circuit as a reversible Carnot cycle. Then, if δQ be the heat absorbed at temperature t, taken as negative when heat is evolved,

$$\int \frac{\delta Q}{t}\, dt = 0$$

for the entire cycle.

When $t_1 - t_0$ becomes infinitely small, this becomes

$$\frac{H_1}{t_1} - \frac{H_0}{t_0} + \frac{\sigma}{t_0}(t_1 - t_0) = 0,$$

or

$$\frac{d}{dt}\left(\frac{H}{t}\right) + \frac{\sigma}{t} = 0;$$

or

$$H = t\left(\frac{dH}{dt}\right) + \sigma$$

$$= t\phi_a.$$

And the contact difference between two metals is zero at their neutral temperature.

251.] We are now in a position to treat a more general case of a system of linear conductors than that considered in Art. 215, in which the wires were supposed to be all of the same metal and at the same temperature. In that case, the potential of all the wires which meet in any electrode as P is the same at the common extremity P, and may be designated, in the case of each wire, by the common symbol V_p. When the wires are not of the same metal, we may suppose that instead of being in immediate contact with each other at the electrode P, each is in contact with a small wire or disk of some standard metal at that point. If V_p were the potential of this connecting metal at P, then the potential of any other wire as PA of metal (a), suppose, at the extremity P would be $V_p + \chi(at_p)$, where $\chi(at_p)$ represents the contact electromotive force from the metal (a) to the standard metal at temperature t_p; similarly if V_q were the potential of the connecting metal at the electrode Q, where the temperature is t_q, the potential at the extremity Q of the wire PQ would be $V_q + \chi(at_q)$. In estimating the currents therefore in terms of the potentials we may regard the potentials of the common extremities of all wires at any electrode as equal provided we increase the electromotive force in any wire as PA by the quantity

$$\chi(at_p) - \chi(at_q).$$

The Thomson effect treated of in Art. 243 will produce a similar increase of electromotive force of the form $-\int_{t_q}^{t_p} \sigma\, dt$, which may be expressed in the form

$$\psi(at_q) - \psi(at_p).$$

If therefore E_{pq} be the electromotive force arising from a battery, if any, in the course of the wire PQ, the expression for the current in that wire will be

$$K_{pq}\{\chi(at_p) - \chi(at_q) + \psi(at_q) - \psi(at_p) + E_{pq}\},$$

and similarly for each of the remaining wires.

Of course the wire PQ may itself be composed of dissimilar metals, or may consist of two wires communicating with the liquids of an interposed battery, in which cases the requisite corrections are obvious.

CHAPTER XIV.

POLARISATION OF THE DIELECTRIC.

Article 252]. In the preceding chapters we have endeavoured to explain electrostatical phenomena by the method of Poisson and Green as the result of direct attraction and repulsion at a distance, according to the law of the inverse square between the positive and negative electricities, or electric fluids. As explained at the outset, in Chaps. IV and V, we do not assert the actual existence of these fluids. We assert merely that the electrostatical relations between conductors are as they would be if the two fluids existed, and conductors and dielectrics had the properties attributed to them in those Chapters.

Faraday and Maxwell made an important step in advance. They assume all non-conducting space to be pervaded by a medium, and refer the force observed to exist at any point in the electric field, not to the direct action of distant bodies, but to the state of the medium itself at the point considered.

Faraday was led by his experimental researches to believe in the existence of certain stresses in the dielectric medium in presence of electrified bodies. Maxwell shews that if the dielectric medium consist of molecules with equal and opposite charges of electricity on their opposite sides, or, as we expressed it in Chapters X and XI, polarised, these stresses would in fact exist. See Maxwell's *Electricity*, Second Edition, Chap. V.

There would be at every point in the medium a tension along the lines of force, combined with a pressure at right-angles to them, and by such tensions and pressures all the observed phenomena may be accounted for without assuming the direct action of distant bodies on one another. It is true, as Maxwell says, that some action must be supposed between neighbouring molecules, and that we are no more able to account for that than for action between distant bodies. And if only electrostatical phenomena were concerned, it would be perhaps of little importance whether we attributed them to direct action of distant bodies or to a medium, so long at least as the electric fluids and the medium were equally hypothetical, and had no other duties to perform than to account for the phenomena in question.

The advantage of Maxwell's hypothesis is that it connects the phenomena of electricity and magnetism with those of light and radiant heat, both being referred to the vibrations of *the same* medium. There is, in fact, in the phenomena of light, independent evidence of the existence of Maxwell's medium, whereas there is no independent evidence of the existence of the two fluids. The medium therefore has better title to be regarded as a vera causa than the two fluids have.

No treatment of the subject can, in the present state of knowledge, be more complete than Maxwell's own in Chapters II and V of his work, and it is necessary to study those chapters in order properly to understand his views. The whole subject of statical electricity has also been treated very fully from Maxwell's point of view in the article 'Electricity' in the Encyclopaedia Britannica, Ninth Edition, by Professor Chrystal. It may, however, be of some advantage to obtain the same results from a slightly different starting-point.

253.] In Chap. XI we had occasion to treat of a particular case of a polarised medium, a medium, namely, in which are interspersed little conductors polarised under the influence of given forces. If the induced distribution on the surface of any conductor be denoted by ϕ, the quantity $\iint x\phi\, dS$, taken over all the conductors in unit of volume, was defined to be the polarisation in direction x per unit of volume.

We will now adopt a rather more general definition of polarisation. Let us conceive a region containing an infinite number of molecules, conductors or not, each containing within it, or on its surface, a quantity of positive, and an equal quantity of negative, electricity. Let P be a point in that region, and about P let there be taken a unit of volume, containing a very great number of the molecules in question. Let us further suppose that throughout this unit of volume the distribution of the molecules in space, as well as the distribution of electricity in individual molecules, may be regarded as constant, and the same as in the immediate neighbourhood of P. Let $\phi\, dx\, dy\, dz$ be the quantity of electricity of the molecular distributions within the element of volume $dx\, dy\, dz$. Then $\iiint \phi\, dx\, dy\, dz$ throughout the unit of volume is zero; and we will define $\iiint x\phi\, dx\, dy\, dz$ taken throughout the unit of volume to be the *polarisation in direction* x at P.

Let $\iiint x\phi\, dx\, dy\, dz = \sigma_x$; and let σ_y, σ_z have corresponding meanings for the axis of y and z.

If a plane of unit area be drawn through P parallel to the plane of yz, it will intersect certain of the molecules. And the reasoning of Chap. XI (Art. 190) shews, that the quantity of electricity belonging to these intersected molecules which lies on the positive side of that unit of area is σ_x. Similarly if the plane were parallel to xz, or xy, the quantity of electricity of the intersected molecules on the positive side of the unit of area would be σ_y or σ_z in the respective cases.

If the direction-cosines of the normal to the plane were l, m, n, the quantity of electricity of the intersected molecules lying on the positive side of the unit of area would be $l\sigma_x + m\sigma_y + n\sigma_z$. For, by definition, the polarisation in the direction denoted by l, m, n is

$$\sigma = \iiint (lx + my + nz)\, \phi\, dx\, dy\, dz,$$

that is,

$$\sigma = l \iiint x\phi\, dx\, dy\, dz + m \iiint y\phi\, dx\, dy\, dz + n \iiint z\phi\, dx\, dy\, dz;$$

that is,

$$\sigma = l\sigma_x + m\sigma_y + n\sigma_z.$$

Hence σ_x, σ_y, and σ_z are components of a vector.

If the distribution be continuous, so that σ_x, σ_y, and σ_z do not change abruptly at the point considered, the same reasoning as employed in Chap. XI shews that the amount of the distribution within the elementary parallelopiped $dx\,dy\,dz$ is $\rho\,dx\,dy\,dz$, where

$$\rho = -\left\{\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz}\right\}.$$

Should the values of σ_x, σ_y, and σ_z change abruptly at the point in question, there will be over the unit of area a superficial or quasi-superficial distribution $\sigma_x - \sigma'_x$, where σ_x and σ'_x are the values of σ_x on opposite sides of the plane, with similar expressions for the planes parallel to those of xz and xy.

Also, as we have seen in Art. 190, the potential V, of such a polarised distribution at any point x, y, z, is determined by the equation

$$V = \iint \frac{\sigma\,dS}{r} + \iiint \frac{\rho\,dx'\,dy'\,dz'}{r},$$

where r is the distance of the point x, y, z from the superficial element dS, or the solid element $dx'dy'dz'$, as the case may be.

Hence it appears that such a system of polarised molecules as we are supposing gives rise to localised distributions with solid and superficial densities of determinate values throughout given regions and having the same potential at every point of the field as would result from such localised distributions.

Conversely, if we had an electric field with given localised charges, we might substitute for it a system of polarised molecules in an infinite variety of ways, the physical properties of which, so far as we are concerned with them, would be in all respects identical with those of the given localised charges.

For if ρ and σ were the densities, solid or superficial, at any point in the supposed system of given charges, and if the polarisation and arrangement of the molecules were such that (σ_x, σ_y, and σ_z being as above defined),

$$\left.\begin{array}{c} \dfrac{d\sigma_x}{dx} + \dfrac{d\sigma_y}{dy} + \dfrac{d\sigma_z}{dz} = -\rho \\ \text{or} \quad l(\sigma_x - \sigma'_x) + m(\sigma_y - \sigma'_y) + n(\sigma_z - \sigma'_z) = \sigma \end{array}\right\} \quad \ldots\ldots \text{(A)}$$

at each point of the field, then we should have the same density at each point as is given by the localised charges, and the potential V at each point would also be the same as in the case of the localised charges, being determined by the equation

$$V = \iint \frac{\sigma\,dS}{r} + \iiint \frac{\rho\,dx'\,dy'\,dz'}{r}.$$

As the values of σ_x, σ_y, σ_z are subjected to only one equation of condition (A) these quantities may clearly be chosen in an infinite variety of ways.

Among all the possible values of σ_x, σ_y, and σ_z we shall consider only

$$\sigma_x = \frac{1}{4\pi} \cdot \frac{dV}{dx}, \quad \sigma_y = \frac{1}{4\pi} \cdot \frac{dV}{dy}, \quad \sigma_z = \frac{1}{4\pi} \cdot \frac{dV}{dz}.*$$

These relations will satisfy (A) identically, since by Chap. III

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz} = \frac{1}{4\pi} \cdot \nabla^2 V = -\rho,$$

for all points where $\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$ vary continuously.

And also

$$\begin{aligned} & l(\sigma_x - \sigma'_x) + m(\sigma_y - \sigma'_y) + n(\sigma_z - \sigma'_z) \\ &= \frac{1}{4\pi}\left\{l\left(\frac{dV}{dx} - \frac{dV'}{dx}\right) + m\left(\frac{dV}{dy} - \frac{dV'}{dy}\right) + n\left(\frac{dV}{dz} - \frac{dV'}{dz}\right)\right\} \\ &= \frac{1}{4\pi} \cdot \left(\frac{dV}{dv} - \frac{dV'}{dv}\right) \\ &= \sigma \end{aligned}$$

over surfaces of discontinuous values of these coefficients.

It appears then that such a system of polarised molecules not only produces at all points in space the same potential as the system of volume and superficial distributions for which it was substituted, but also causes the distributions themselves to reappear. It can be shewn also that the energy is the same in the two cases. For the polarised medium is in a state of constraint, because the separated electricities are not allowed to coalesce and neutralise each other. Work has been done upon it in producing this state. In ordinary experiments the constrained state of the dielectric is produced by the introduction of charged bodies, and the work is the work done in

* With the distribution of polarisation assumed in the text, if a small cylindrical region be described in the medium whose generating lines are parallel to the force at the point and infinitely smaller than the linear dimensions of the bounding planes, the force at any point within the cylinder is that arising entirely from the polarisation of the molecules completely included within the cylinder, and the total force from all the rest of the molecules is zero. If the polarisation were magnetic, this result would be expressed by saying that the law of magnetisation is such that the *magnetic induction* at every point is zero.

charging them, but according to this theory the energy resides, not in the charged bodies, but in the dielectric.

The energy in unit volume of the polarised system is $\frac{1}{8\pi}R^2$, see Chap. V, that is,

$$\frac{1}{8\pi}\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}.$$

The energy of the entire system estimated in the same way is

$$\frac{1}{8\pi}\iiint\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}dx\,dy\,dz$$

throughout the whole of dielectric space.

But this is also the expression for the energy of the originally given system according to the ordinary theory, as shewn in Chap. X. The two systems are therefore for all purposes equivalent.

We may conceive that the molecules of all dielectrics are capable of assuming such polarisation as required for this hypothesis. If, as we have hitherto supposed, vacuum be a perfect dielectric, it becomes necessary for the hypothesis to conceive it as permeated by a non-material ether, the molecules of which are capable of such electric polarisation. And if the existence of such an ether be assumed, it may be that in case of other dielectrics, the electric polarisation resides in the ether rather than in the molecules of the substance.

We may further suppose that the essential property of conductors, as distinguished from dielectric media or insulators, is that their molecules are incapable of sustaining electric polarisations, or that the substances of conductors are impermeable by the supposed ether, and therefore that no electric force and no free electricity can exist within them.

We might thus construct a theory of electrostatics founded on the polarisation of the dielectric, just as the ordinary theory is founded on the property of conductors.

In the ordinary theory the electromotive force at any point is the space differential of a function V, which is constant throughout any conductor, and satisfies the condition $\nabla^2 V + 4\pi\rho = 0$ at all points where there is free electricity of density ρ. Assuming that no case of electrostatic equilibrium has yet been discovered, which can be proved to be inconsistent with the ordinary theory, it follows that the supposed dielectric polarisation must, when there is equilibrium, be the space differential of a function V, which is constant over and within every closed surface bounding the dielectric, and satisfies $\nabla^2 V + 4\pi\rho = 0$ at all points in the dielectric.

The Stresses in the Dielectric.

254.] If any closed surface S separate one portion E_1 of an electrified system from the other portion E_2, as, for instance, if the whole of E_1 be inside, and the whole of E_2 outside of S, then this hypothesis suggests an explanation of the phenomena without assuming any direct action between E_2 and E_1. For if the polarisation be given in magnitude and direction at each point in S, $\frac{dV}{d\nu}$ is given at each point on S. Then we know that if the form and charge of every conductor within S be given, and if all fixed electrification within S be given, V has single and determinate value at all points within S.

It follows that all electrical phenomena within S, which in the ordinary theory are due to the action of E_2, are on the polarisation hypothesis deducible from the given polarisation, that is the given value of $\frac{dV}{d\nu}$, at each point on S.

We might then always substitute for the external system E_2 a certain polarisation on S, without affecting the equilibrium of E_1. An example of this substitution has already been given (Art. 58) for the case where S is an equipotential surface. For then a distribution over S whose density is $-\frac{R}{4\pi}$ exerts the same force as the external system at any point within S.

If S be not equipotential we obtain a corresponding result as follows*:—

Let V_1 be the potential of the external, V_2 that of the internal system, and $V = V_1 + V_2$ the whole potential.

The whole force in direction x exerted by the external on the internal system is

$$\frac{1}{4\pi}\iiint \frac{dV_1}{dx}\nabla^2 V_2\, dx\, dy\, dz$$

throughout the space within S.

* This investigation is taken from Maxwell's Treatise, Second Edition, Chap. V.

But $$\iiint \frac{dV_2}{dx} \nabla^2 V_2 \, dx\,dy\,dz = 0,$$

and within S $$\nabla^2 V_2 = \nabla^2 V.$$

Hence the whole force is

$$\frac{1}{4\pi} \iiint \frac{dV}{dx} \nabla^2 V \, dx\,dy\,dz.$$

The object is to express this in the form of a surface integral over S.

If we can find three functions X, Y, Z, such that

$$\frac{dV}{dx} \nabla^2 V = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz},$$

then evidently, by Green's theorem,

$$\iiint \frac{dV}{dx} \nabla^2 V \, dx\,dy\,dz = \iint (lX + mY + nZ)\, dS$$

over the surface S. This is the required surface integral.

Let us assume

$$\left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dy}\right)^2 - \left(\frac{dV}{dz}\right)^2 = 8\pi p_{xx},$$

$$\left(\frac{dV}{dy}\right)^2 - \left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dz}\right)^2 = 8\pi p_{yy},$$

$$\left(\frac{dV}{dz}\right)^2 - \left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dy}\right)^2 = 8\pi p_{zz},$$

$$\frac{dV}{dy}\frac{dV}{dz} = 4\pi p_{yz} = 4\pi p_{zy},$$

$$\frac{dV}{dz}\frac{dV}{dx} = 4\pi p_{zx} = 4\pi p_{xz},$$

$$\frac{dV}{dx}\frac{dV}{dy} = 4\pi p_{xy} = 4\pi p_{yx}.$$

Then $$X = p_{xx},$$
$$Y = p_{xy},$$
$$Z = p_{xz}$$

satisfy the condition.

The quantities p_{xx}, p_{yy}, &c. are the six components of the stress on the surface S due to the polarisation of the dielectric.

If S be an equipotential surface, we have

$$\frac{dV}{dx} = -Rl, \quad \frac{dV}{dy} = -Rm, \quad \frac{dV}{dz} = -Rn,$$

where R is the normal force, and therefore

$$p_{xx} = \frac{1}{8\pi} R^2 (l^2 - m^2 - n^2),$$

$$p_{yz} = \frac{1}{4\pi} R^2 mn,$$

$$p_{xz} = \frac{1}{4\pi} R^2 ln,$$

$$p_{xy} = \frac{1}{4\pi} R^2 lm.$$

The x-component of stress is then

$$lp_{xx} + mp_{xy} + np_{xz},$$

that is $\frac{1}{8\pi} R^2 l$. Similarly the y- and z-components are

$$\frac{1}{8\pi} R^2 m, \qquad \frac{1}{8\pi} R^2 n.$$

That is, the stress is normal to S, and is equal to that of the force R acting on the surface electrified to a density $\frac{1}{2}\frac{R}{4\pi}$.

If S be at right angles to an equipotential surface, we find the stress in any element of it thus, in this case,

$$l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz} = 0. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Now $\quad 8\pi \{lp_{xx} + mp_{xy} + np_{xz}\}$

$$= \cdot \left\{\left(\frac{dV}{dx}\right)^2 - \left(\frac{dV}{dy}\right)^2 - \left(\frac{dV}{dz}\right)^2\right\} + 2m\frac{dV}{dx}\frac{dV}{dy} + 2n\frac{dV}{dx}\frac{dV}{dz}. \quad .. \quad (2)$$

Multiplying (1) by $2\frac{dV}{dx}$ and subtracting from (2), we obtain

$$8\pi \{lp_{xx} + mp_{xy} + np_{xz}\} = -lR^2.$$

Hence the components of tension per unit area are

$$-\frac{1}{8\pi} R^2 l, \quad -\frac{1}{8\pi} R^2 m, \quad -\frac{1}{8\pi} R^2 n.$$

If therefore these stresses exist at every point of the surface S, no matter how they arise, they produce on the interior system E_1 exactly the same effect as, according to the theory of action at a distance, would be produced by the attraction and repulsions due to the external system E_2.

255.] According to the theory of dielectric polarisation as explained in Art. 253, the so-called charge on a conductor is to be regarded as the terminal polarisation of the dielectric; as belonging in fact not to the molecules of the conductor, but to the adjacent molecules of the dielectric. (Maxwell's *Electricity*, Art. 111.)

So long as we are dealing with a system at rest and in statical equilibrium, it is indifferent for all purposes of calculation whether we regard the charge as belonging to the dielectric or to the conductor.

It is however possible to induce in any conductor or other solid body the state which in the ordinary theory is called a charge of electricity; and it is possible to move the body in this state from place to place through air without destroying its charge. It should seem therefore that although the electric force at any point in air may be due to the polarised state of the medium at the point, and not to direct action of the charged body, and although the polarised particles be always those of the dielectric, yet the ultimate cause of the phenomena may be in the body and not in the dielectric. And this appears to be Faraday's view, where he says (1298), 'Induction appears to consist in a certain polarised state of the particles into which they are thrown by *the 'electrified body sustaining the action.'*

Certain experiments have been appealed to as shewing that the electrification, whatever it be, is in the dielectric and not in the conductor. If, for instance, a plate of glass be placed between and touching two oppositely charged metallic plates, and these be then removed, it will be found that they exhibit scarcely any trace of electrification. If they be replaced and connected by a wire, a current passes of the same or nearly the same strength as if no removal had taken place. See Jamin, *Cours de Physique*, Leçon 26.

A similar result was obtained by Franklin with a Leyden jar, the metallic coatings of which were moveable.

256.] Up to this point we have not dispensed with the two-fluid theory and the law of the inverse square in electric action, because it is only by the use of that theory that we have proved the properties of our medium. All that we have done is to introduce a somewhat different conception of an electric field, and the distributions of which it is composed.

If any advance is to be made, it must be in the steps of Faraday and Maxwell as follows:—

We observe that in the polarised medium the relation between the force at any point and the polarisation at the point is given by the equations $X = -4\pi\sigma_x$, &c.

These equations are of the same form as those which express the relation between the force existing at any point in an elastic body in equilibrium and the molecular displacement at the point.

In treating of elastic bodies we regard these relations as ultimate facts based on experiment. We might then regard the corresponding equations for the dielectric as ultimate facts, without resorting to the two fluid theory for their explanation. We might regard the dielectric as an elastic medium capable of being thrown into a state of strain, and presenting when in that state the phenomena which we call electric force and electric distribution.

257.] According to the theory in this form, no action is exerted by the electricity in any part of the dielectric on that in any other part, unless the two are contiguous. We might thus dispense with the notion of action at a distance, on which the ordinary theory is founded. Another characteristic of the ordinary theory is the instantaneous nature of the actions with which it deals. For, according to that theory, if any change take place in electrical distributions in any one part of space, the corresponding change takes place at the same instant in every other part however distant. The substitution of the medium for the direct action between distant bodies, suggests that these corresponding electrical changes may not take place at the same instant, but that electrical influence may be propagated from molecule to molecule through the medium with a certain velocity. And herein lies the strength of the theory. For, as Maxwell discovered, if electrical effects are propagated with finite velocity through an insulating medium, such velocity is the same as that of light, *or* so nearly the same as to leave no room for doubt that the two classes of phenomena are physically connected.

Again, an elastic medium, if thrown by any forces into a state of strain, does not on removal of those forces immediately recover its original condition. There is a time of relaxation. Certain phenomena, such as the residual charge of a Leyden jar (see Maxwell's *Electricity*, Chap. X), lend countenance to the supposition that a dielectric medium influenced by electric forces does not immediately, on the removal of those forces, recover its original condition.

258.] We proceed to consider the meaning of the term *electric displacement* as used by Maxwell, for which purpose we must revert to the conception of the two-fluid theory.

If through any point in the medium of polarised particles a plane be drawn perpendicular to the direction of the resultant force R at that point, the density σ, per unit area of that plane, of the electricity on the particles intersected by that plane and on the positive side of it according to R's direction is, as we have seen, determined by the equation

$$\sigma = -\frac{R}{4\pi}.$$

In Maxwell's view, any field of electromotive force in the dielectric is accompanied by a strained state of the particles of the dielectric or of the pervading ether, a *displacement* or transfer of positive electricity equal to $\frac{R}{4\pi}$ per unit area of surface of the particles taking place from each particle to the adjacent particle on the positive side, along with an equal displacement of negative electricity to the adjacent particle on the negative side. The + or − electricities do not coalesce or neutralise each other within each particle, but a polarised state is set up throughout the field, each particle being in a strained state owing to the separations of the electricities within it, the result being graphically represented thus,

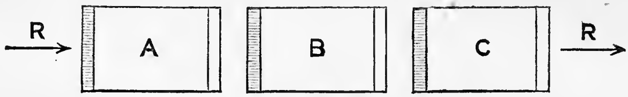

Fig. 39.

the shaded sides of the particles A, B, C, &c. indicating positive, and the unshaded sides negative, electrification. According to this view the *displacement* is the process by which the polarised state of the particles has been brought about. We shall generally denote the polarisation by σ, and the displacement by f. It is easily seen that in a dielectric medium $f = -\sigma$.

259.] If the field were one of uniform force parallel to a line from left to right across the plane of the paper, the total displacement or transfer of electricity across all planes perpendicular to that line would be the same and equal to $\frac{R}{4\pi}$ per unit of area.

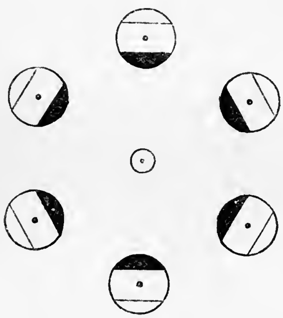

Fig. 40.

If the field were such as corresponds to what is called an electrified point O, i.e. a charge within a very small volume about O, the particles would be polarised as in the figure, the displacement (supposing no other charge in the field) taking place concentrically from within outwards, and the quantities σ_x, σ_y, and σ_z being so determined that

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz} = 0$$

at all points without the small region, and that

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz} = -\rho$$

within that region where ρ is determined to give the requisite charge at O.

The law of resultant polarisation exterior to O may in this case be determined, and the consequent law of force, if it be assumed that the resultant polarisation σ is symmetrical about the point with which O sensibly coincides.

For if this be assumed we must have $\sigma = \phi(r)$ and in the direction of r the distance of each point from O.

Therefore $$\sigma_x = \phi(r)\frac{x}{r}, \quad \sigma_y = \phi(r)\frac{y}{r}, \quad \sigma_z = \phi(r)\frac{z}{r}.$$

Therefore since $$\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz} = 0,$$

we have $$\frac{3\phi(r)}{r} - \phi(r)\frac{(x^2+y^2+z^2)}{r^3} + \phi'(r)\frac{x^2+y^2+z^2}{r^2} = 0,$$

or $$\frac{2\phi(r)}{r} + \phi'(r) = 0,$$

or $$\phi(r) = \frac{C}{r^2}.$$

260.] If in any polarised field we describe a closed surface S, and find the integral $\iint \frac{R\cos\theta\, dS}{4\pi}$ over that surface where θ is the angle between R and the normal to S at each point, we know that the result is M, where M is the total quantity of the electricity situated within S; that is to say, the whole quantity of the electricity lying without the surface S on all the molecules intersected by S would be $-M$, and the whole quantity of the electricity displaced across S, to the adjacent external particles, would be $+M$, in other words the total quantity of electricity within any closed surface whatever is unalterable. The electricity behaves in all respects like an incompressible fluid pervading all space, and the introduction of any quantity into any closed region is accompanied by an efflux of a corresponding quantity from that region.

We have so far supposed the whole region to be dielectric or non-conducting, but the introduction of conducting substances does not affect the result. The special property of conductors is that their molecules are incapable of polarisation, or that the substances of conductors are impermeable by the other whose molecules may be thus polarised.

Suppose now that our closed surface S was intersected by a conductor C, and let us replace S by another closed surface made up of the portion of S external to C, and another surface $S'QS'$ very nearly coinciding with that of the conductor and external to it, the dotted line indicating the continuation of S within C.

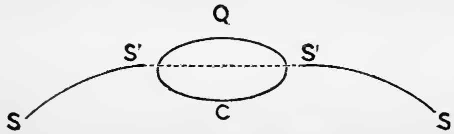

Fig. 41.

The integral $\iint \frac{R \cos\theta \, dS}{4\pi}$ taken over this new closed surface will, as before, be equal to all the original mass of the included electricity, since the charge on the conductor must be zero on the whole; and since there is no polarisation within C we have as before the total quantity of electricity transferred across the original S by displacement equal to the mass within it, the only difference being that instead of such transference being throughout molecular, as it is in the dielectric, it is a transference in mass across the conductor.

This transference by displacement differs from that by conduction, inasmuch as when the force ceases the state of strain and the displacement cease also, and all things return to their original condition, there being no permanent transfer.

261.] Recurring again to the simple illustration of the field of uniform parallel force, suppose the force, remaining uniform, to vary from time to time, then the state of the molecules A, B, C, &c. in Fig. 1 also varies, the shading becoming darker as the force increases, and lighter as it diminishes.

If the displacement at any instant were f, it is clear that this variation of the force and consequently of f would produce a transference of electricity across any plane perpendicular to the force in all respects analogous to a current of intensity $\frac{df}{dt}$ along the lines of force.

For example, suppose the field to be that of the dielectric between the plane armatures of a condenser, and suppose these armatures to be connected by a wire. Then a discharge would take place through the wire from left to right; the effect of this discharge would be to diminish the displacement within the dielectric from left to right, or to produce a counteracting displacement from right to left, inasmuch as the positive charge of the left-hand and the negative charge of the right-hand armature would diminish at the rate u per unit time, if u were the current through the wire.

Therefore we should have $\frac{df}{dt}+u$ equal to zero, or a closed current would flow through the whole apparatus of dielectric, armatures, and wire. And this is what is supposed to take place in every case of transference by conduction.

262.] Hitherto, throughout this chapter, we have treated our dielectric as being what may be called a pure dielectric with specific inductive capacity unity. In the case of impure dielectrics like those treated of in Chap. XI, we may either, as in what has preceded, retain the conceptions of the two fluids with distant action, or adopt Maxwell's more simple conception of a displacement connected with the force by a law regarded as an ultimate fact (Art. 256).

On the former hypothesis we may, as is done in Chap. XI, assume the intermixture of small conductors.

Fig. 42.

In the figure annexed let the plane of the paper be supposed parallel to the axis of x, and let the line AB be the intersection with that plane of a plane drawn through any point P in the medium perpendicular to that axis, and let the dotted line be the intersection with the same plane of the paper of a surface as nearly as possible coincident with the aforesaid plane perpendicular to x, but so drawn as not to intersect any small conductors, this surface will not differ sensibly from the plane. If σ_x be the density per unit area of this surface of the electricity upon the polarised dielectric molecules intersected by it and lying to the right or positive side of the surface, and if σ_y and σ_z be corresponding quantities for planes through P parallel to xz and xy respectively, it follows from what has been already proved that the density ρ of actual charge in the medium at P is determined by the equation

$$\frac{d\sigma_x}{dx} + \frac{d\sigma_y}{dy} + \frac{d\sigma_z}{dz} = -\rho.$$

But if K be the specific inductive capacity, we know from Chap. XI that

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) = -4\pi\rho,$$

whence we may, as in the preceding case, choose as our solution for σ_x, σ_y, and σ_z, and their resultant σ, the equations

$$\sigma_x = \frac{K}{4\pi}\cdot\frac{dV}{dx},\quad \sigma_y = \frac{K}{4\pi}\cdot\frac{dV}{dy},\quad \sigma_z = \frac{K}{4\pi}\cdot\frac{dV}{dz}\quad \text{and}\quad \sigma = -\frac{K}{4\pi}\cdot R.$$

The polarisation σ, as before, measures the displacement at any point, and it follows from Art. 193, Chap. XI, that the total displacement over any closed surface is equal to the total quantity of electricity within the surface, as in the case of pure dielectric media.

According to Maxwell's point of view, we should ignore the analysis of the action on the inverse square hypothesis altogether, and regard the equation

$$\sigma = -\frac{K}{4\pi}.R \quad \text{or} \quad f = \frac{K}{4\pi}.R,$$

where f is the displacement, as an ultimate fact expressing the relation between force and displacement in any isotropic medium, with the requisite modifications for heterotropic media, in which

$$\sigma_x = -\frac{K_x}{4\pi}.X,\quad \sigma_y = -\frac{K_y}{4\pi}.Y,\quad \sigma_z = -\frac{K_z}{4\pi}.Z,$$

when the axes are principal axes; and since K_x, K_y, K_z are generally not equal, the resultant displacement f will not in this case be necessarily coincident with the resultant force R.

The equation $f = \frac{K}{4\pi}.R$, if K be capable of assuming all values between 1 and $+\infty$, expresses the relation between force and displacement in all bodies, the lower limit (1) corresponding to air or rather to vacuum, and the higher limit $(+\infty)$ to conductors.

263.] In ideally perfect insulators, there is, as we have said, no *permanent* transfer arising from displacement; the force ceasing, the polarisation and displacement also cease, and any passage of electricity across a plane is succeeded on the cessation of the force by an equal rebound or retransfer of electricity across the same plane backwards. Such perfect insulators do not exist in nature. Recurring, for instance, to our uniform force illustration, when this force or the corresponding polarisation of the A, B, C, &c. particles reaches a certain intensity, the particles become incapable of retaining their state of strain, and the $+$ and $-$ electricities in each particle intermix. Across any plane perpendicular to R there is transfer of positive electricity from left to right, and of negative from right to left; in fact a temporary current, and each particle returns to its unstrained state. If however R were maintained constant, there must be a renewed displacement to the same extent as before, so that we have what is equivalent to a permanent transfer of electricity, a current from left to right, the intensity of the current u being the rate at which the electricity is transferred in each particle across any transverse section, and which is connected with the force R by the equation $u = \frac{R}{r}$ if r be the resistance within each particle. If the force R remained constant the polarisation σ or the corresponding and equal displacement f must be renewed as fast as it is destroyed, so as always to satisfy the equation $\sigma = -\frac{R}{4\pi}$; in other words, there must be a continually recurring displacement or transfer from particle to particle equal per unit of time to the quantity u *.

Again, suppose that the diminution of polarisation or transfer current u was absolutely impossible, i.e. that the insulation was perfect but that the force varied, producing therefore a variable

* The actual *historical* displacement or transfer at any time must be distinguished from the *instantaneous* displacement or polarisation; this latter is the transfer or displacement which the state of the field requires in accordance with the above theory, and is what would take place if there were no conduction; the former, in case of conduction, is the sum of the continually renewed instantaneous displacements, required for the polarisation of the field which have taken place up to the instant considered.

polarisation σ or displacement f. The result is equivalent to a transfer of positive electricity from left to right at the rate per unit of time of $\frac{df}{dt}$.

If both the conduction current u and the variable force, and consequently variable f coexisted, the resultant effect would be equivalent to the current of intensity $u+\frac{df}{dt}$.

264.] We have already considered the case of a condenser with plane armatures connected by a wire, and have seen that the discharge current u in the wire is accompanied by an equal and opposite current $\frac{df}{dt}$ in the dielectric, but in point of fact the process which, in this case, takes place almost instantaneously is in effect, though much more slowly, always going on throughout the dielectric. For no substance is absolutely and completely non-conducting, the molecular constraint is continually giving way, there is a continual passage of electricity from the positive to the negative armature causing a diminution of force and polarisation throughout the medium. If u be the conduction current, r the resistance, X the force, and f the displacement at any instant, we have, as shewn in Art. 261,

$$u+\frac{df}{dt}=0,$$

where $u=\frac{X}{r}$, and $f=\frac{K}{4\pi}, R.$

Therefore
$$\frac{df}{dt}+\frac{4\pi}{Kr}.f=0,$$

$$f=f_0 e^{-\frac{4\pi}{Kr}t},$$

and
$$X=X_0 e^{-\frac{4\pi}{Kr}t},$$

f_0 and X_0 being the initial values of f and X.

These equations express the law of decay of the efficiency of condensers.

265.] According to Maxwell's doctrine, as we have already said, all electric currents flow in closed circuits.

Let us recur to the case of the charged particle of Art. 259, which we have hitherto regarded as at rest within the medium, and suppose the charge to be unity.

If it move from one position to another we have in effect a current of electricity from O to O'. But from another point of view the effect is the same as if the particle in the first position were annihilated, and another similar particle placed in the second position; that is, as if a particle with unit positive charge were placed in the second position, and a particle with unit negative charge superadded to the positively charged particle in the first position.

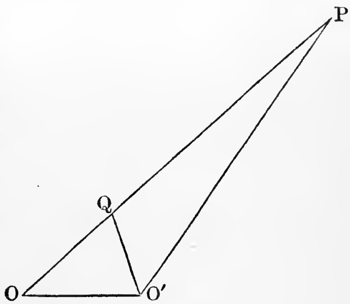

Fig. 43.

If O denote the first, O' the second position, P any point in space, the displacement at P due to the placing of a negative particle or annihilation of a positive particle at O is a displacement $\dfrac{1}{4\pi\, OP^2}$ in direction PO. The displacement at P due to the positive particle at O' is a displacement $\dfrac{1}{4\pi\, O'P^2}$ in $O'P$.

If O' be infinitely near to O, and $OO' = a$, we can find the equation to the resultant as follows:

Let $\angle POO' = \theta$. Let $OQ = 3a\cos\theta$. Then if

$$PO = r,\ PO' = r - a\cos\theta, \text{ and } PQ = r - 3a\cos\theta,$$

and
$$\frac{PQ}{PO'} = \frac{r-3a\cos\theta}{r-a\cos\theta} = \frac{r-2a\cos\theta}{r} = \frac{r^2-2ar\cos\theta}{r^2} = \frac{PO'^2}{PO^2}.$$

Therefore the resultant is parallel to $O'Q$. Its equation is therefore

$$\frac{dy}{dx} = -\frac{3a\cos\theta\sin\theta}{a-3a\cos^2\theta}, \quad \text{and} \quad \frac{y}{x} = \tan\theta.$$

Therefore $\dfrac{dy}{dx} = \dfrac{3xy}{2x^2-y^2}$.

The solution of which is

$$x^2 + y^2 = c^{\frac{2}{3}} y^{\frac{4}{3}},$$

where c is a variable parameter.

This is the equation of a system of closed curves having OO' for a common tangent.

It thus appears that if any quantity of positive electricity flows from O to O' (for our moving particle is equivalent for the purpose to a flow of positive electricity), we have a flow or current of electricity at every point in space, in direction forming closed curves with the line OO'. From which it would seem, as we have already said, that there is no real change of position of all the positive electricity in space. Or, in other words, either kind of electricity behaves like an incompressible fluid, and the quantity of it within any finite space cannot be increased or diminished.

If, for instance, the charge on the moving particle be unity, and it move from O to O', that is a distance a in unit of time, the current in OO' is a. If also P be a point in the plane bisecting OO' at right-angles, and r be measured from O, the displacement current from right to left through a ring of the plane between the distances r and $r + dr$ from O is

$$\frac{a^2}{4\pi} \frac{2\pi r\,dr}{r^3}.$$

The whole displacement current from right to left through the plane is therefore

$$\frac{a^2}{4\pi} \int_{\frac{a}{2}}^{\infty} \frac{2\pi r\,dr}{r^3} = a,$$

and is therefore equal to the current from left to right in OO'.

Hence, according to Maxwell's view, all electric currents in nature flow in closed circuits. This theory will be found to lead to important consequences when we come to deal with the mutual action of electric currents.

THE END.