Wm Jas Carter R.N.
Sat: 5 Dec: 1908.

MATHEMATICAL TRACTS.

PART II.

BY

F. W. NEWMAN, M.R.A.S.

EMERITUS PROFESSOR OF UNIVERSITY COLLEGE, LONDON;
HONORARY FELLOW OF WORCESTER COLLEGE, OXFORD.

Cambridge:
MACMILLAN AND BOWES.
1889

Cambridge:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

CONTENTS (OF ANTICYCLICS).

* I add a *zero* Index, as in $ק_0$, $ס_0$, $ח_0$, $כ_0$ for Mutilated Functions.

PART II.

ANTICYCLICS.

1. A VOLUME exists of 354 *quarto* pages by Dr C. Gudermann, Professor of Mathematics in Münster. It was published in 1833 by G. Reiner, Berlin, bearing the title, Theory of Potential Cyclic Hyperbolic Functions. These I call simply ANTICYCLIC. The substance of Gudermann's treatise, it seems, appeared previously in volumes 6, 7, 8, 9 of Crelle's Journal. From p. 159 to p. 260 is an elaborate table of the integral $u=\int_0 \frac{d\theta}{\cos\theta}$, tabulated previously by Legendre for use in Elliptic Integrals; but Gudermann's table is tenfold in amplitude. From p. 263 to p. 336 is a second large table, giving the common logarithms of

$$\tfrac{1}{2}(\epsilon^k+\epsilon^{-k}), \quad \tfrac{1}{2}(\epsilon^k-\epsilon^{-k})$$

and of their ratio, to 9 and at last 10 decimals, with k increasing by only ·001 at every step. Perhaps this was primarily intended to aid the valuation of Elliptic series: for he begins his table at $k=2$. If he had begun at $k=1{\cdot}57$ (for $k=\frac{1}{2}\pi$), his task would have been complete. For in Elliptics two constants kk' bear the relation $kk'=\frac{1}{4}\pi^2$. We can work, at pleasure, through either; and one or other must *exceed* $\frac{1}{2}\pi$. Gudermann has certainly achieved a great and arduous task.

2. He was probably first to introduce (*with German types*) [I content myself with *capital* S and C] $\text{Sin}\,x$ for $\frac{1}{2}(\epsilon^x-\epsilon^{-x})$ and $\text{Cos}\,x$ for $\frac{1}{2}(\epsilon^x+\epsilon^{-x})$. *This is the beginning of Anticyclic notation.* The beauty of it is seen in formulas which abound in the higher theory of Elliptics, such as

$$\omega=x+\frac{\sin 2x}{\text{Cos}\,2\rho}+\frac{1}{2}\frac{\sin 4x}{\text{Cos}\,4\rho}+\frac{1}{3}\cdot\frac{\sin 6x}{\text{Cos}\,6\rho}+\&\text{c.}\ldots$$

where ρ is the leading constant, a function of the modulus c; x, the leading variable, is proportional to Legendre's First Integral

$$\int_0 \frac{d\omega}{\sqrt{(1-c^2\sin^2\omega)}},$$

becoming equal to ω at every complete quadrant. An eminent mathematician observed, that while our *theory* of these integrals seems complete, the extreme difficulty of calculating the constants baffles us when we try to *use* the higher scales. Until we have better aid in Anticyclic tables, apparently this difficulty must remain.

3. Analogy drives us on to use Tan for $\frac{\text{Sin}}{\text{Cos}}$, and Cot for the reciprocal. Nor can we any the more refuse Sec Cosec for the reciprocals of Cos and Sin. Thus we have

$$\text{Tan}\, x = \frac{\epsilon^{x} - \epsilon^{-x}}{\epsilon^{x} + \epsilon^{-x}} = \frac{1 - \epsilon^{-2x}}{1 + \epsilon^{-2x}};\ \text{Cot}\, x = \frac{1 + \epsilon^{-2x}}{1 - \epsilon^{-2x}},\ \text{functions } \textit{of } \epsilon^{-2x} \textit{ alone.}$$

$$\text{Hence too } 1 - \text{Tan}\, x = \frac{2\epsilon^{-2x}}{1 + \epsilon^{-2x}};\ \text{Cot}\, x - 1 = \frac{2\epsilon^{-2x}}{1 - \epsilon^{-2x}}.$$

$$\text{These, as well as Cosec}\, x = \frac{2\epsilon^{-x}}{1 - \epsilon^{-2x}} \text{ and Sec}\, x = \frac{2\epsilon^{-x}}{1 + \epsilon^{-2x}}$$

are very simple functions of ϵ^{-x}. We may almost say the same of

$\log_\epsilon \text{Sin}\, x$ and $\log_\epsilon \text{Cos}\, x$; since $\log \text{Sin}\, x = x - \log 2 + \log (1 - \epsilon^{-2x})$

and $\log \text{Cos}\, x = x - \log 2 + \log (1 + \epsilon^{-2x})$.

A complete and trustworthy table of ϵ^{-x} is presupposed in this whole theory. Because Gudermann had *not* such at hand, therefore (perhaps) he began his table at $k = 2$.

Since $\text{Cos}\, x \pm \text{Sin}\, x = \epsilon^{\pm x}$, $\therefore\ \text{Cos}^2 x - \text{Sin}^2 x = 1$.

Dividing the last by $\text{Cos}^2 x$, we obtain $1 - \text{Tan}^2 x = \text{Sec}^2 x$.

Evidently $\text{Cos}\, x$, like $\sec\theta$, varies from 1 to ∞; $\text{Sin}\, x$, like $\tan\theta$, from 0 to ∞: but $\text{Tan}\, x$ and $\text{Sec}\, x$, like positive $\sin\theta$ and $\cos\theta$, from 0 to 1.

Gudermann's Längezahl.

4. I cannot translate this word: it is not in my German Dictionary. The "*Length-Number*" sounds to me nonsensical.

Since $\text{Tan}\, x$ has the same limits as positive *sin* θ, we are led to assume the equation $\text{Tan}\, x = \sin\theta$ tentatively, and are instantly rewarded by a series of important relations. First, from

$$1 - \text{Tan}^2 x = \text{Sec}^2 x,\ \text{it gives Sec}\, x = \cos\theta;$$

whence again $\text{Cos}\, x = \sec\theta$. Also

$$\text{Sin}\, x = \frac{\text{Tan}\, x}{\text{Sec}\, x} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Thus, if from a given x we can pass to θ, a trigonometrical table that furnishes us with the circular functions of θ will make us masters of the Anticyclic functions of x. To pass from x to $\text{Tan}\, x$ will enable us to reach θ.

From our definitions of $\text{Cos}\, x$ and $\text{Sin}\, x$ as $\frac{1}{2}(\epsilon^x \pm \epsilon^{-x})$ we forthwith deduce $d\,.\,\text{Cos}\, x = \text{Sin}\, x dx$ and $d\,.\,\text{Sin}\, x = \text{Cos}\, x dx$. Therefore also we get

$$d\,.\,\text{Tan}\, x = d\,.\,\frac{\text{Sin}\, x}{\text{Cos}\, x} = \frac{\text{Cos}\, x d\, \text{Sin}\, x - \text{Sin}\, x d\, \text{Cos}\, x}{(\text{Cos}\, x)^2} = \frac{\text{Cos}^2 x - \text{Sin}^2 x}{\text{Cos}^2 x}\,.\, dx$$
$$= \text{Sec}^2\, x dx.$$

Hence on differentiating $\text{Tan}\, x = \sin\theta$, you find

$$\text{Sec}^2\, x\,.\,dx = \cos\theta\,.\,d\theta.$$

$$\text{But } \text{Sec}^2 x = \cos^2\theta\,; \text{ whence } dx = \frac{d\theta}{\cos\theta}, \text{ or } x = \int_0 \frac{d\theta}{\cos\theta};$$

since x vanishes with θ.

This is the integral tabulated, first by Legendre; next, more elaborately by Gudermann. To have the mastery over x when θ is given, and conversely, is our first problem.

5. If we take $\rho = \int_0 \sec\theta\,.\,d\theta$ as a Polar curve, with ρ as radius vector, the locus has ρ as an asymptote (logarithmic infinity) when $\theta = 90^\circ$. The curve starts at $\theta = 0$ perpendicular to this asymptote, from which it attains its maximum distance, nearly where $\theta = 60^\circ$. As attempts at admissible nomenclature, I have sometimes called ρ the Range and θ the Elevation.

Legendre has two integrations slightly differing:

$$(a)\quad \int_0 \frac{d\theta}{\cos\theta} = \int_0 \frac{\cos\theta\, d\theta}{\cos^2\theta} = \int_0 \frac{d\sin\theta}{1 - \sin^2\theta} = \tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta};$$

$$(b)\quad \text{Let } \theta = 2\omega \;\therefore\; \int_0 \frac{d\theta}{\cos\theta} = \int_0 \frac{2d\omega}{\cos 2\omega} = \int_0 \frac{2d\omega}{\cos^2\omega - \sin^2\omega}$$
$$= \int_0 \frac{2\sec^2\omega\, d\omega}{1 - \tan^2\omega} = \int_0 \frac{2d\tan\omega}{1 - \tan^2\omega} = \log\frac{1+\tan\omega}{1-\tan\omega}$$
$$= \log\tan(45^\circ + \omega).$$

6. From the last, if $x =$ this integral,

$$\epsilon^x = \tan(45^\circ + \tfrac{1}{2}\theta),\;\; \epsilon^{-x} = \tan(45^\circ - \tfrac{1}{2}\theta)$$

whence, reverted, $\frac{1}{2}\theta = 45^\circ - \tan^{-1}(\epsilon^{-x})$, which avails us, *if we have a good table of* ϵ^{-x} from x as argument.

As I regarded such a table as of first necessity for Anticyclics, I prepared one myself, and was rewarded by the Philosophical Society of Cambridge publishing it in 87 quarto pages, under the zealous and toilful superintendence of Mr Glaisher. By the kind support of Professor Adams, the same society has since published my table of ϵ^x from $x = 0$ to $x = 2$.

If x exceeds 3, the series $\tan^{-1}.\,\epsilon^{-x} = \epsilon^{-x} - \frac{1}{3}\epsilon^{-3x} + \frac{1}{5}\epsilon^{-5x} - \&c.$ converges very rapidly: but it is more convenient to have θ in degrees; and unless x is less than 1, I suppose from a trigonometrical table, with ϵ^{-x} known, $\tan^{-1}.\,\epsilon^{-x}$ can be found in degrees with the needful accuracy. But to meet the case of $x < 1$, Professor J. C. Adams of Cambridge (to whom I sent a table of Tan x, calculated for x less than 1, wishing him to get it tested by differencing), was kind enough to compute it—by help of a new machine, as I understand—independently, from my values of ϵ^{-x}, and had his own results verified by differencing. Thus I am able to present to the reader the table attached, which now rests not on me, but on the authority of the distinguished astronomer.

TABLE OF

$$\text{Tan}\,x = \frac{1 - \epsilon^{-2x}}{1 + \epsilon^{-2x}},\ \text{from } x = {\cdot}01 \text{ to } x = 1,$$

as corrected by Professor J. C. ADAMS.

x	Tan x.	x	Tan x.
·01	·0099 9966 6680	·11	·1095 5847 0215
·02	·0199 9733 3760	·12	·1194 2729 8535
·03	·0299 9100 3239	·13	·1292 7258 3606
·04	·0399 7868 0318	·14	·1390 9244 7878
·05	·0499 5837 4958	·15	·1488 8503 3623
·06	·0599 2810 3529	·16	·1586 4850 4297
·07	·0698 8589 0316	·17	·1683 8104 5870
·08	·0798 2976 9111	·18	·1780 8086 8117
·09	·0897 5778 4747	·19	·1877 4620 5869
·10	·0996 6799 4625	·20	·1973 7532 0225

x	Tan x.	x	Tan x.
·21	·2069 6649 9730	·61	·5441 2709 8854
·22	·2165 1806 1493	·62	·5511 2802 8538
·23	·2260 2835 2279	·63	·5580 5221 5559
·24	·2354 9574 9539	·64	·5648 9955 2846
·25	·2449 1866 2403	·65	·5716 6996 6085
·26	·2542 9553 2627	·66	·5783 6341 3044
·27	·2636 2483 5472	·67	·5849 7988 2881
·28	·2729 0508 0563	·68	·5915 1939 5433
·29	·2821 3481 2670	·69	·5979 8200 0499
·30	·2913 1261 2451	·70	·6043 6777 7117
·31	·3004 3709 7147	·71	·6106 7683 2817
·32	·3095 0692 1213	·72	·6169 0930 2877
·33	·3185 2077 6903	·73	·6230 6534 9572
·34	·3274 7739 4808	·74	·6291 4516 1414
·35	·3363 7554 4337	·75	·6351 4895 2388
·36	·3452 1403 4136	·76	·6410 7696 1186
·37	·3539 9171 2477	·77	·6469 2945 0442
·38	·3627 0746 7578	·78	·6527 0670 5962
·39	·3713 6022 7877	·79	·6584 0903 5955
·40	·3799 4896 2255	·80	·6640 3677 0268
·41	·3884 7268 0216	·81	·6695 9025 9620
·42	·3969 3043 2005	·82	·6750 6987 4838
·43	·4053 2130 8689	·83	·6804 7600 6113
·44	·4136 4444 2187	·84	·6858 0906 2230
·45	·4218 9900 5251	·85	·6910 6946 9833
·46	·4300 8421 1403	·86	·6962 5767 2687
·47	·4381 9931 4833	·87	·7013 7413 0938
·48	·4462 4361 0249	·88	·7064 1932 0397
·49	·4542 1643 2682	·89	·7113 9373 1818
·50	·4621 1715 7260	·90	·7162 9787 0199
·51	·4699 4519 8933	·91	·7211 3225 4078
·52	·4777 0001 2168	·92	·7258 9741 4849
·53	·4853 8109 0606	·93	·7305 9389 6096
·54	·4929 8796 6675	·94	·7352 2225 2916
·55	·5005 2021 1190	·95	·7397 8305 1273
·56	·5079 7743 2898	·96	·7442 7686 7362
·57	·5153 5927 8008	·97	·7487 0428 6969
·58	·5226 6542 9685	·98	·7530 6590 4870
·59	·5298 9560 7528	·99	·7573 6232 4216
·60	·5370 4956 6998	1·00	·7615 9415 5955

When x is given (less than 1), this table shows $\text{Tan}\, x$, and the equation $\sin\theta = \text{Tan}\, x$ determines θ.

Moreover Gudermann shows how to use his table inversely, and obtain θ from x.

Every one acquainted with Elliptic Integrals will see that the assumption there admitted, of

$$\sin\theta = \sqrt{-1}\,.\,\tan\psi,$$

whence $$\cos\theta = \sec\psi,\ \tan\theta = \sqrt{-1}\sin\psi,\ \&c.$$

does but introduce Anticyclics in disguise.

7. Some other elegant relations must be mentioned,

$$\text{Tan}\,\tfrac{1}{2}x = \tan\tfrac{1}{2}\theta.$$

Proof: $$\tan\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{\sec\theta - 1}{\sec\theta + 1}} = \sqrt{\frac{2(\text{Cos}\,x - 1)}{2(\text{Cos}\,x+1)}}$$

$$= \sqrt{\frac{\epsilon^x + \epsilon^{-x} - 2}{\epsilon^x + \epsilon^{-x} + 2}} = \frac{\epsilon^{\frac{1}{2}x} - \epsilon^{-\frac{1}{2}x}}{\epsilon^{\frac{1}{2}x} + \epsilon^{-\frac{1}{2}x}} = \text{Tan}\,\tfrac{1}{2}x.$$

$$x = \int_0 \frac{d\theta}{\cos\theta} = \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x}$$

may of course be developed into

$$\sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + \&c. \;\ldots\ldots\ldots\ldots (a).$$

This development of x in odd powers of $\sin\theta$ suggests the assumption x, or

$$\int_0 \sec\theta d\theta = A_1\sin\theta - \tfrac{1}{3}A_3\sin 3\theta + \tfrac{1}{5}A_5\sin 5\theta - \&c.$$

Differentiate: then

$$\sec\theta = A_1\cos\theta - A_3\cos 3\theta + A_5\cos 5\theta - \&c.$$

Multiply by $2\cos\theta$, and apply the formula

$$2\cos\theta\,.\cos(2n+1)\theta = \cos 2n\theta + \cos(2n+2)\theta;$$

$$\therefore\quad 2 = A_1(1+\cos 2\theta) - A_3(\cos 2\theta + \cos 4\theta) + A_5(\cos 4\theta + \cos 6\theta) - \&c.,$$

which requires $$A_1 = 2 = A_3 = A_5 = A_7\ \&c.$$

Hence $$\tfrac{1}{2}x = \sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta - \tfrac{1}{7}\sin 7\theta + \&c.$$
a series which can be otherwise confirmed. Namely, it is known in the Higher Trigonometry that if r is < 1,

$$\frac{(1+r)\cot.\theta}{1+2r\cos 2\theta + r^2} = \cos\theta - r\cos 3\theta + r^2\cos 5\theta - r^4\cos 7\theta + \&c.$$

With r constant and θ variable, multiply by $d\theta$ and integrate:

$$\therefore \quad (1+r)\int_0 \frac{\cos\theta\,.\,d\theta}{1+2r\cos 2\theta + r^2} = \sin\theta - \tfrac{1}{3}r\sin 3\theta + \tfrac{1}{5}r^2 \sin 5\theta - \&c.$$

This, being true as long as $r<1$, and the series on the right converging even when r reaches 1, will not prove false at the extreme value $r=1$. But when $r=1$, the left member becomes

$$2\int \frac{\cos\theta\,.\,d\theta}{2(1+\cos 2\theta)} \text{ or } 2\int_0 \frac{\cos\theta d\theta}{4\cos^2\theta} \text{ or } \int_0 \frac{d\theta}{2\cos\theta}.$$

Thus if $$x = \int_0 \sec\theta d\theta,$$

we find $\quad \tfrac{1}{2}x = \sin\theta - \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta - \&c.$ as before.

Of course, from slow convergence, this series does not aid computation.

8. We pass to Inverse Anticyclic Functions.

If $$t = \operatorname{Tan} x, \quad x = \operatorname{Tan}^{-1}.\,t.$$

But $$dt = d\operatorname{Tan} x = \operatorname{Sec}^2 x dx = (1-\operatorname{Tan}^2 x)\,dx$$
$$= (1-t^2)\,dx.$$

$$\therefore \quad dx = \frac{dt}{1-t^2} = (1+t^2+t^4+t^6+\&c.)\,dt,$$

whence $$x = t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \&c. \ldots\ldots\ldots\ldots (a).$$

But $t = \sin\theta$ or $x = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \&c.$

The t and $\sin\theta$ are always less than 1.

9. We proceed to Sin^{-1} and Cos^{-1}. Let $u = \operatorname{Sin} x$, $v = \operatorname{Cos} x$.

First, $du = \operatorname{Cos} x dx = \sqrt{1+\operatorname{Sin}^2 x}\,.\,dx = \sqrt{(1+u^2)}\,dx$,

$$dx = \frac{du}{\sqrt{(1+u^2)}}.$$

The development by Bin. Th. is twofold. First, when u is less than 1. $\quad dx = \left\{1 - \tfrac{1}{2}u^2 + \frac{1\,.\,3}{2\,.\,4}\,.\,u^4 - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}u^6 + \&c.\right\} du,$

whence $\quad x$ or $\operatorname{Sin}^{-1} u = u - \tfrac{1}{2}\,.\,\frac{u^3}{3} + \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{u^5}{5} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{u^7}{7} + \&c.$

Next when u is > 1, develop $(u^2+1)^{-\frac{1}{2}}$ in descending order,

$$x = \int u^{-1}\left\{1 - \tfrac{1}{2}u^{-2} + \frac{1.3}{2.4}u^{-4} - \frac{1.3.5}{2.4.6}u^{-6} + \&c.\right\} du$$

$$= \log(\alpha u) + \tfrac{1}{2}\cdot\frac{u^{-2}}{2} + \frac{1.3}{2.4}\cdot\frac{u^{-4}}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{u^{-6}}{6} - \&c. \;\ldots.(b).$$

To find α, the constant of integration, observe that

$$\epsilon^x = \text{Cos}\, x + \text{Sin}\, x$$

$$= \surd(1+u^2) + u;$$

$$\therefore \quad x = \log\{\surd(1+u^2)+u\}.$$

Make u infinite; then $x = \log(2u)$. But x then by (b)

$$= \log(\alpha u), \quad \therefore \quad \alpha = 2.$$

10. Next, from $v = \text{Cos}\, x$, $dv = \text{Sin}\, x dx = \surd(v^2-1)\, dx$,

or $$dx = \frac{dv}{\surd(v^2-1)} = v^{-1}\left\{1 + \tfrac{1}{2}v^{-2} + \frac{1.3}{2.4}v^{-4} + \frac{1.3.5}{2.4.6}v^{-6}\right\} dv;$$

$$\therefore \quad x = \log(\beta v) - \tfrac{1}{2}\cdot\frac{v^{-2}}{2} - \frac{1.3}{2.4}\cdot\frac{v^{-4}}{4} - \frac{1.3.5}{2.4.6}\cdot\frac{v^{-6}}{6} - \&c. \ldots\ldots\ldots(c).$$

To find β, we have

$$\epsilon^x = \text{Cos}\, x + \text{Sin}\, x = v + \surd(v^2-1),$$

$$x = \log(v + \sqrt{v^2-1}).$$

Make v infinite; $\therefore$ $x = \log 2v$. This proves β to be 2, just as α previously. These results are Gudermann's.

11. Recurring to the "Range",

$$x = \int_0 \sec\theta d\theta,$$

observe that from $\quad \epsilon^x = \text{Cos}\, x + \text{Sin}\, x,$

we have $\quad \epsilon^x = \sec\theta + \tan\theta.$

When $x = 1$, let θ have the special value θ_1; then $\epsilon = \sec\theta_1 + \tan\theta_1$.

From above, we infer

$$\tfrac{1}{2}\theta_1 = 45^\circ - \tan^{-1}.(\epsilon^{-1}),$$

where $\quad \epsilon^{-1} = \cdot3678\ 7944\ 1171,$

from which I deduced that θ_1 slightly exceeds $49^\circ 36'$. I since find that Dr James Booth had found

$$\theta_1 = 49^\circ 36' 15'', \text{ and } \tan\theta_1 = 1{\cdot}17520\ 3015.$$

Series which advance by powers of $\tan\theta$ or $\sin\theta$ are not convenient for a continuous table. Especially if all the terms are of one sign, Legendre evades them. We may here notice one such series with alternate signs, *viz.*

$$x = \text{Sin}^{-1}(\tan\theta) = \tan\theta - \tfrac{1}{2}\frac{\tan^3\theta}{3} + \frac{1.3}{2.4}\cdot\frac{\tan^5\theta}{5} - \&\text{c.} \ldots\ldots (d).$$

12. Why Gudermann is not satisfied to work from Legendre's original equation $x = \log\tan(45^\circ + \frac{1}{2}\theta)$ I have not understood; but it seems to belong to liberal knowledge to be acquainted with his series.

For small values of θ Gudermann has $\theta = \frac{1}{2}\pi v$, and v a small fraction. To go back; from

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right), \ \&\text{c.}$$

you get in Trigonometry,

$$\theta\cot\theta = 1 - \Sigma\frac{2\theta^2}{n^2\pi^2 - \theta^2};$$

where $n = 1, 2, 3, 4\ldots$ Next,

$$\frac{1}{\sin\theta} = \frac{1}{\theta} + \frac{2\theta}{\pi^2 - \theta^2} - \frac{2\theta}{2^2\pi^2 - \theta^2} + \frac{2\theta}{3^2\pi^2 - \theta^2} - \&\text{c.}$$

Thence, putting $\theta = \frac{1}{2}\pi - \omega$, resolving

$$\frac{2\theta}{n^2\pi^2 - \theta^2} \text{ into } \frac{1}{n\pi - \theta} - \frac{1}{n\pi + \theta},$$

and *for a moment making* $\frac{1}{2}\pi = p$, you find

$$\frac{1}{\cos\omega} = \frac{2p}{p^2 - \omega^2} - \frac{2.3p}{3^2p^2 - \omega^2} + \frac{2.5p}{5^2p^2 - \omega^2} - \&\text{c.}$$

In this last, restore θ for ω, since the equation is identical, then

$$x = \int_0 \frac{d\theta}{\cos\theta} = \log\frac{p+\theta}{p-\theta} - \log\frac{3p+\theta}{3p-\theta} + \log\frac{5p+\theta}{5p-\theta} - \&\text{c.}$$

If you here develop *every* term on the right, you have a result in powers of θ. But, to improve convergence, leave the first term undeveloped, and where $\theta = pv$ or $\frac{1}{2}\pi v$, you obtain

$$x = \log\frac{1+v}{1-v} - 2\{M_1 v + M_3 v^3 + M_5 v^5 + \&\text{c.}\} \ldots\ldots (e),$$

$$\text{if} \quad M_n = \frac{1}{n}\{3^{-n} - 5^{-n} + 7^{-n} + \&\text{c.}\}$$

in which n is an *odd* integer.

If for other purposes $1 - V_n$ has been tabulated [a task which I had myself assumed] for $n = 2, 3, 4, 5$, &c. where V_n means

$$1^{-n} - 3^{-n} + 5^{-n} - 7^{-n} + \&c.$$

then we have simply M_r in our series

$$= \frac{1 - V_r}{r}.$$

13. For increasing values of θ, Gudermann seems to use the results of (b) or (c) in Art. 9 and 10 above, viz.

$$\left.\begin{aligned} x = \operatorname{Sin}^{-1}(\tan\theta) &= \log(2\tan\theta) + \tfrac{1}{2}\cdot\frac{\cot^2\theta}{2} - \frac{1\,.\,3}{2\,.\,4}\cdot\frac{\cot^4\theta}{4} \\ &\qquad + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{\cot^6\theta}{6} - \&c. \\ x = \operatorname{Cos}^{-1}(\sec\theta) &= \log(2\sec\theta) + \tfrac{1}{2}\cdot\frac{\cos^2\theta}{2} - \frac{1\,.\,3}{2\,.\,4}\cdot\frac{\cos^4\theta}{4} - \&c. \end{aligned}\right\} \ldots(f).$$

Embarrassing wealth of series possibly gave him great power of verification.

Finally, when θ approaches 90^0, put $\theta = \frac{1}{2}\pi - \omega$; then ω is very small,

$$x = \int \frac{-d\omega}{\sin\omega},$$

which further suggests $\omega = u\pi$,

or $\theta = \frac{1}{2}\pi - \pi u = \frac{1}{2}\pi(1 - 2u)$,

and our series will be developable in powers of u.

Write U_n for $1 - 2^{-n} + 3^{-n} - 4^{-n} + \&c.$ and you easily get

$$x = \int \frac{-d\omega}{\sin\omega} = \log\frac{C}{\omega} - U_2\cdot\frac{\omega^2}{\pi^2} - \tfrac{1}{2}U_4\cdot\frac{\omega^4}{\pi^4} - \tfrac{1}{3}U_6\cdot\frac{\omega^6}{\pi^6} - \&c.$$

where $C = 2$, when ω converges to *zero*,

or $$x = \log\frac{2}{u\pi} - U_2\,.\,u^2 - \tfrac{1}{2}U_4u^4 - \tfrac{1}{3}U_6\,.\,u^6 - \&c. \ldots\ldots(g).$$

But, for better convergence, add to the last

$$-\log(1 - u^2) = u^2 + \tfrac{1}{2}u^4 + \tfrac{1}{3}u^6 + \&c.,$$

and observe that

$$\log\frac{2}{u\pi} + \log(1 - u^2) = \log(u^{-1} - u) - \log(\tfrac{1}{2}\pi);$$

$$\therefore\ x = \log(u^{-1} - u) - \log(\tfrac{1}{2}\pi) + (1 - U_2)u^2$$
$$+ \tfrac{1}{2}(1 - U_4)u^4 + \tfrac{1}{3}(1 - U_6)u^6 + \&c. \ldots\ldots (h),$$

where $$\theta = \tfrac{1}{2}\pi(1 - 2u),$$

or $$u = \frac{\tfrac{1}{2}\pi - \theta}{\pi}.$$

Calculation of the Primary Anticyclics.

14. Since Sin x and Cos x increase rapidly, only their logarithms, when x exceeds 2, can well be registered, a task which Gudermann has executed, up to $x = 12$. When x exceeds 12, $\log(1 \pm \epsilon^{-2x})$ in series converges so rapidly, that its first term probably suffices; the two first are ϵ^{-2x} and $\frac{1}{2}\epsilon^{-4x}$. Thus when x is large, log Sin x and log Cos x are sufficiently known. For small values of x, put

$$\left.\begin{array}{l} P = \epsilon^{-2x} + \frac{1}{3}\epsilon^{-6x} + \frac{1}{5}\epsilon^{-10x} + \&c. \\ Q = \frac{1}{2}\epsilon^{-4x} + \frac{1}{4}\epsilon^{-8x} + \frac{1}{6}\epsilon^{-12x} + \&c. \\ \text{Then}\quad -\log(1 - \epsilon^{-2x}) = P + Q;\ \ \log(1 + \epsilon^{-2x}) = P - Q \end{array}\right]$$

and from x given, P and Q are computable by a general table of ϵ^{-p} (such as has been published by the Cambridge Philosophical Society). I find it convenient to write

$\sigma(x)$ as equivalent to $P + Q$, and $\kappa(x)$ for $P - Q$;

whence $$\left.\begin{array}{l} \log \text{Sin}\, x = x - \log 2 - \sigma(x) \\ \log \text{Cos}\, x = x - \log 2 + \kappa(x) \end{array}\right] \text{ and } \log \text{Cot}\, x = 2P.$$

The reciprocals of Sin x and Cos x can be obtained from $\dfrac{2\epsilon^{-x}}{1 \mp \epsilon^{-2x}}$ by long division, by aid of the table of ϵ^{-x}. But when x is not very small, the development rising by powers of ϵ^{-2x} yields a result nearly accurate and less tedious: moreover it will give two results at once. I write ף (x) for Cosec x, i.e. for the reciprocal of Sin x, and ס for Sec x or the reciprocal of Cos x.

When x exceeds 1·37, ף and ס are found most easily by summing

$$\left.\begin{array}{l} M = \epsilon^{-x} + \epsilon^{-5x} + \epsilon^{-9x} + \epsilon^{-13x} + \epsilon^{-17x} + \ldots \\ N = \epsilon^{-3x} + \epsilon^{-7x} + \epsilon^{11x} + \epsilon^{-15x} + \ldots \end{array}\right].$$

Then ף $(x) = 2(M + N)$; ס $(x) = 2(M - N)$.

After calculating this latter part of the table, we may go back to x less than 1·37, and take the sums

$$\left.\begin{aligned} H &= \epsilon^{-x} + \epsilon^{-5x} + \epsilon^{-13x} + \epsilon^{-17x} + \epsilon^{-25x} \\ K &= \epsilon^{-7x} + \epsilon^{-11x} + \epsilon^{-19x} + \epsilon^{-23x} \end{aligned}\right] .$$

Evidently then by the expansion of $\text{ק}(3x)$ and $\text{ס}(3x)$, you find

$$\left.\begin{aligned} \text{ק}(x) &= 2(H+K) + \text{ק}(3x) \\ \text{ס}(x) &= 2(H-K) - \text{ס}(3x) \end{aligned}\right]$$ of which $\text{ק}(3x)$ and $\text{ס}(3x)$ are supposed already in our tables.

Occasional long division is a valuable check on error, especially as to the last figures.

The table of ק and ס ends naturally when the second term $2\epsilon^{-3x}$ is insignificant, so that $\text{ק}(x)$ and $\text{ס}(x)$ are undistinguishable from $2\epsilon^{-x}$.

Since Cot x and Tan x converge towards 1 when x increases, I write כ for Cot $-$ 1 and ת for 1 $-$ Tan; which give $\text{כ}(x) = \dfrac{2\epsilon^{-2x}}{1-\epsilon^{-2x}}$; $\text{ת}(x) = \dfrac{2\epsilon^{-2x}}{1-\epsilon^{-2x}}$.

When an entire table of $\text{ק}(x)$ pre-exists, you can deduce from it entire tables of $\text{כ}(x)$ and $\text{ת}(x)$ by the process of $a \pm b$ for each entry For we have as identity

$$\frac{2y}{1 \mp y} = \frac{2y}{1-y^2} \pm \frac{2y^2}{1-y^2};$$

in which you have merely to assume $y = \epsilon^{-2x}$,

then, with the upper sign, $\text{כ}(x) = \text{ק}(2x) + \text{כ}(2x)$
with the lower, $\text{ת}(x) = \text{ק}(2x) - \text{כ}(2x)$.

Begin with x so large, that $\text{כ}(2x)$ is undistinguishable from $2\epsilon^{-2x}$ that is, when $2\epsilon^{-4x}$ is insignificant; and work backward.

The great ease of this method seems to give primacy to a table of $\text{ק}x$. That of $\text{ס}(x)$ is less serviceable.

Moreover if you repeat the equation

$$\text{כ}(x) - \text{כ}(2x) = \text{ק}(2x)$$

by writing for x, first $2x$, next $4x$, next $8x$, ... and so on to $2^{n-1}x$, then adding all together, you get

$$\text{כ}(x) - \text{כ}(2^n x) = \text{ק}(2x) + \text{ק}(2^2 x) + \text{ק}(2^3 x) + \ldots + \text{ק}(2^n x);$$

when n is large, $\text{כ}(2^n x) = 0$. Practically, when 16 decimals suffice, $\epsilon^{-37} = 0$; $\therefore$ $\text{כ}(18) = 0$.

The last series converges nearly as

$$2\,(\epsilon^{-2x} + \epsilon^{-4x} + \epsilon^{-8x} + \epsilon^{-16x} + \ldots$$

If you calculate $\text{כ}(x)$ by long division, this formula avails to *verify* a table of כ. Like remarks may be made on the companion formula [obtained from $\text{כ}(x) + \text{ח}(x) = 2\text{ק}(2x)$],

$$\text{ח}x = \text{ק}(2x) - \text{ס}(2^2 x) - \text{ק}(2^3 x) - \text{ק}(2^4 x) - \&\text{c}.$$

The Mutilated and the Secondary Functions.

15. Advantage is sometimes found in using the Anticyclics $\text{ק}\ \text{ס}\ \text{כ}\ \text{ח}$ deprived of their first term of development. I call these *Mutilated*, and denote them by $\text{ק}_0\ \text{ס}_0\ \text{כ}_0\ \text{ח}_0$. Then

$$\text{ק}_0(x) = \frac{2\epsilon^{-3x}}{1 - \epsilon^{-2x}}; \quad \text{ס}_0(x) = \frac{2\epsilon^{-3x}}{1 + \epsilon^{-x}};$$

$$\text{כ}_0(x) = \frac{2\epsilon^{-4x}}{1 - \epsilon^{-2x}}; \quad \text{ח}_0(x) = \frac{2\epsilon^{-4x}}{1 + \epsilon^{-2x}};$$

or

$$\text{ק}_0(x) = \text{ק}(x) - 2\epsilon^{-x}; \quad \text{ס}_0(x) = 2\epsilon^{-x} - \text{ס}(x);$$

$$\text{כ}_0(x) = \text{כ}(x) - 2\epsilon^{-2x}; \quad \text{ח}_0(x) = 2\epsilon^{-2x} - \text{ח}(x);$$

If tables are calculated, the Mutilated Functions $\text{ק}_0 x$, $\text{ס}_0 x$ may be made *auxiliary*; thus we may calculate them first, and $\text{ק}(x)$, $\text{ס}x$ from them; then proceed to $\text{כ}(x)$ and $\text{ח}(x)$, and from these deduce $\text{כ}_0(x)$ and $\text{ח}_0(x)$.

Again, these Mutilated forms facilitate our estimate of what I further call the *Secondary* Functions, which are suggested by Elliptic Integrals. If, as in Legendre's notation, $F(c\omega)$ *mean*

$$\int_0 \frac{d\omega}{\sqrt{(1 - c^2 \sin^2 \omega)}},$$

while

$$\rho = \tfrac{1}{2}\pi \cdot \frac{F(b,\ \tfrac{1}{2}\pi)}{F(c,\ \tfrac{1}{2}\pi)},$$

where

$$b^2 + c^2 = 1;$$

it is convenient also to take

$$\frac{x}{\frac{1}{2}\pi} = \frac{F(c\omega)}{F(c, \frac{1}{2}\pi)},$$

then x is the leading independent variable, and ρ the leading constant in the Higher Theory. It is never *necessary* to suppose ρ *less* than $\frac{1}{2}\pi$; for if ρ' be related to b as ρ to c, we obviously have $\rho\rho' = (\frac{1}{2}\pi)^2$, so that either ρ or ρ' must *exceed* $\frac{1}{2}\pi$; and b is symmetrical with c. The relation of ρ to c is transcendental. For conciseness we may write C [*not* for $F(c, \frac{1}{2}\pi)$ as in Legendre's great Supplement, *but*]

$$\frac{F(c, \frac{1}{2}\pi)}{\frac{1}{2}\pi};$$

and B for the like function of b. The relation of these constants guides us to eight Secondary Anticyclics. In Legendre's notation ρ is not used, but instead he has q equivalent to what here is $\epsilon^{-2\rho}$,

so that $$\text{ק}(\rho) = \frac{2\sqrt{q}}{1-q}, \quad \text{ס}(\rho) = \frac{2\sqrt{q}}{1+q},$$

$$\text{כ}(q) = \frac{2q}{1-q}, \quad \text{ת}(q) = \frac{2q}{1+q}.$$

For conciseness let simple l mean $\log_\epsilon$. Then with the Hebrew letters ל מ פ צ ר ז ח שׂ for functional symbols, we may assume,

Secondary Anticyclics.

1. $\text{ל}(\rho)$ for $l\,\mathrm{Cot}\,\rho + l\,\mathrm{Cot}\,3\rho + l\,\mathrm{Cot}\,5\rho +$ &c. *ad infin.*
2. $\text{מ}(\rho)$ for $l\,\mathrm{Cot}\,\rho - l\,\mathrm{Cot}\,2\rho + l\,\mathrm{Cot}\,3\rho - l\,\mathrm{Cot}\,4\rho +$ &c.
3. $\text{פ}(\rho)$ for $\text{ת}(\rho) + \frac{1}{2}\text{ת}(2\rho) + \frac{1}{3}\text{ת}(3\rho) +$ &c.
4. $\text{צ}(\rho)$ for $\text{ת}(\rho) - \frac{1}{2}\text{ת}(2\rho) + \frac{1}{3}\text{ת}(3\rho) -$ &c.
5. $\text{ר}(\rho)$ for $\text{כ}(\rho) - \text{כ}(3\rho) + \text{כ}(5\rho) - \text{כ}(7\rho) +$ &c.
6. $\text{ז}(\rho)$ for $\frac{1}{2}\text{כ}(2\rho) + \frac{1}{4}\text{כ}(4\rho) + \frac{1}{6}\text{כ}(6\rho) +$ &c.
7. $\text{ח}(\rho)$ for $\text{ת}(\rho) - \text{ת}(3\rho) + \text{ת}(5\rho) -$ &c.
8. $\text{שׂ}(\rho)$ for $\text{ק}(x) - \text{ק}(3x) + \text{ק}(5x) -$ &c.

The routine of the Calculus in Elliptic Integrals then elicits the equations

$$\text{ל}(\rho) = \tfrac{1}{4} . l . \frac{1}{b}; \qquad \text{מ}(\rho) = \tfrac{1}{2} . l . C;$$

$$l . \frac{4}{c} = \rho + 2\text{צ}(\rho); \qquad C = 1 + 2\text{ר}(\rho);$$

$$Cc = 2\text{שׁ}(\rho); \qquad Cb = 1 - 2\text{ת}(\rho).$$

[where C is defined by $\frac{1}{2}\pi . C = F_c$; ρ by the equation $\rho . F_c = \frac{1}{2}\pi . F_b$.]

The function $\text{ז}(\rho)$ arises in the course of the same theory, and is needful in certain transformations. As for $\text{פ}(\rho)$, it is a companion to $\text{צ}(\rho)$ and somewhat aids computation.

16. Each of these Secondary functions admits of transformations into a new series, to which the notation through q leads most easily. Thus for $\text{ל}(\rho)$, observe that

$$\text{Cot}\ \rho = \frac{1 + \epsilon^{-2\rho}}{1 - \epsilon^{-2\rho}} = \frac{1+q}{1-q},$$

and $$\text{Cot} . n\rho = \frac{1 + \epsilon^{-2n\rho}}{1 - \epsilon^{-2n\rho}} = \frac{1+q^n}{1-q^n}.$$

Hence $$l\,\text{Cot} . n\rho = 2\left\{q^n + \tfrac{1}{3}q^{3n} + \tfrac{1}{5}q^{5n} + \&\text{c.}\right\}$$

In the last, put 1, 3, 5, 7, ... for n, then you get

$$\text{ל}(\rho) = \left.\begin{array}{l} 2\left\{q + \tfrac{1}{3}q^3 + \tfrac{1}{5}q^5 + \tfrac{1}{7}q^7 + \&\text{c.}\right\} \\ + 2\left\{q^3 + \tfrac{1}{3}q^9 + \tfrac{1}{5}q^{15} + \tfrac{1}{7}q^{21} + \&\text{c.}\right\} \\ + 2\left\{q^5 + \tfrac{1}{3}q^{15} + \tfrac{1}{5}q^{25} + \tfrac{1}{7}q^{35} + \&\text{c.}\right\} \\ + 2\left\{q^7 + \tfrac{1}{3}q^{21} + \tfrac{1}{5}q^{35} + \tfrac{1}{7}q^{49} + \&\text{c.}\right\} \\ \qquad \&\text{c.} \qquad \&\text{c.} \end{array}\right\}.$$

Add these up in *vertical* columns; using the formula

$$q^m + q^{3m} + q^{5m} + \&\text{c.} = \frac{q^m}{1 - q^{2m}}.$$

Then $$\text{ל}(\rho) = \frac{2q}{1-q^2} + \tfrac{1}{3} . \frac{2q^3}{1-q^6} + \tfrac{1}{5} . \frac{2q^5}{1-q^{10}} + \&\text{c.}$$

$$= \text{ק}(2\rho) + \tfrac{1}{3} . \text{ק}(6\rho) + \tfrac{1}{5} . \text{ק}(10\rho) + \&\text{c.}$$

By a perfectly similar process

$\text{מ}(\rho)$ is changed to $\text{ת}(\rho) + \tfrac{1}{3}\text{ת}(\rho) + \tfrac{1}{5}\text{ת}(5\rho) + \&\text{c.}$;

$\text{ר}(\rho)$ into $\text{ם}(2\rho) + \text{ם}(4\rho) + \text{ם}(6\rho) + \&\text{c.}$;

$צ(\rho)$ into $2\{\kappa(\rho) - \kappa(2\rho) + \kappa(3\rho) - \&c.\}$;

$פ(\rho)$ into $2\{\sigma(\rho) - \sigma(2\rho) + \sigma(3\rho) - + \&c.\}$;

$ז(\rho)$ into $\sigma(2\rho) + \sigma(4\rho) + \sigma(6\rho) + \&c.$;

$ת(\rho)$ into $ס(2\rho) - ס(4\rho) + ס(6\rho) - \&c.$;

$ש(\rho)$ into $ס(\rho) + ס(3\rho) + ס(5\rho) + \ldots$

By reason of this double expression, the convergence of each function in series may be increased by the help of the Mutilated forms.

Thus

(1) From $ל(\rho) = ק(2\rho) + \frac{1}{3}ק(6\rho) + \frac{1}{5}ק(10\rho) + \&c.$

subtract $l \operatorname{Cot} \rho = 2(\epsilon^{-2\rho} + \frac{1}{3}\epsilon^{-6\rho} + \frac{1}{5}\epsilon^{-10\rho} + \&c.)$

Thence $ל(\rho) = l \operatorname{Cot} \rho + ק_0(2\rho) + \frac{1}{3}ק_0(6\rho) + \frac{1}{5}ק_0(10\rho) + \&c.$

converging as $\epsilon^{-6\rho},\ \epsilon^{-18\rho},\ \epsilon^{-30\rho}\ldots$

(2) From $ם(\rho) = ת(\rho) + \frac{1}{3}ת(\rho) + \frac{1}{5}ת(5\rho) + \&c.$

subtract $l \operatorname{Cot} \rho = 2(\epsilon^{-2\rho} + \frac{1}{3}\epsilon^{-6\rho} + \frac{1}{5}\epsilon^{-10\rho} + \ldots \&c.$

Thence $ם(\rho) = l \operatorname{Cot} \rho - ת_0(\rho) - \frac{1}{3}ת_0(3\rho) - \frac{1}{5}ת_0(5\rho) - \&c.$

(3) From $ס(\rho) = ת(\rho) + \frac{1}{2}ת(2\rho) + \frac{1}{3}ת(3\rho) + \frac{1}{4}.\&c.$

subtract $2\sigma(\rho) = 2(\epsilon^{-2\rho} + \frac{1}{2}\epsilon^{-4\rho} + \frac{1}{3}.\epsilon^{-6\rho} + \&c.$

[See Art. 14 for σ.]

Thence $פ(\rho) = 2\sigma(\rho) - ת_0\rho - \frac{1}{2}ת_0(2\rho) - \frac{1}{3}ת_0(3\rho) - \&c.$

(4) From $צ(\rho) = ת(\rho) - \frac{1}{2}ת(2\rho) + \frac{1}{3}ת(3\rho) - \&c.$

subtract $2\kappa(\rho) = 2(\epsilon^{-2\rho} - \frac{1}{2}\epsilon^{-4\rho} + \frac{1}{3}\epsilon^{-6\rho} - \&c.$

Thence $צ(\rho) = 2\kappa(\rho) - ת_0(\rho) + \frac{1}{2}ת_0(2\rho) - \frac{1}{3}ת_0(3\rho) + \frac{1}{4}.\&c.$

(5) From $ר(\rho) = ב(\rho) - ב(3\rho) + ב(5\rho) - \&c.$

subtract $ס(2\rho) = 2(\epsilon^{-2\rho} - \epsilon^{-6\rho} + \epsilon^{-10\rho} - \&c.$

Thence $ר(\rho) = ס(2\rho) + ב_0(\rho) - ב_0(3\rho) + ב_0(5\rho) - \&c.$

(6) From $ז(\rho) = \frac{1}{2}ב(2\rho) + \frac{1}{4}ב(4\rho) + \frac{1}{6}ב(6\rho) - \&c.$

subtract $\sigma(2\rho) = 2\{\frac{1}{2}\epsilon^{-4\rho} + \frac{1}{4}\epsilon^{-8\rho} + \frac{1}{6}\epsilon^{-12\rho} + \ldots\}$

Thence $ז(\rho) = \sigma(2\rho) + \frac{1}{2}ב_0(2\rho) + \frac{1}{4}ב_0(4\rho) + \frac{1}{6}ב_0(6\rho) + \ldots$

The function $ז$ is the logarithm of

$$Q = \{(1-q^2)(1-q^4)(1-q^6)\ldots\}^{-1},$$

a factor known in Elliptics.

(7) From $\text{ח}(\rho) = \text{ת}(\rho) - \text{ת}(3\rho) + \text{ת}(5\rho) - \text{ת}(7\rho) + \&c.$

subtract $\text{ס}(2\rho) = 2\epsilon^{-2\rho} - 2\epsilon^{-6\rho} + 2\epsilon^{-10\rho} - \&c.$

Thence $\text{ח}(\rho) = \text{ס}(2\rho) - \text{ת}_0(\rho) + \text{ת}_0(3\rho) - \text{ת}_0(5\rho) + \&c.$

(8) Finally, from $\text{שׁ}(\rho) = \text{ק}(\rho) - \text{ק}(3\rho) + \text{ק}(5\rho) - \&c.$

subtract $\text{ס}(\rho) = 2\epsilon^{-\rho} - 2\epsilon^{-3\rho} + 2\epsilon^{-5\rho} - \&c.$

Thence $\text{שׁ}(\rho) = \text{ס}(\rho) + \text{ק}_0(\rho) - \text{ק}_0(3\rho) + \text{ק}_0(5\rho) - \&c.$

17. Suppose that $\text{מ}(\rho)$ has been tabulated. From it the pair פ and צ can be deduced by working backwards. The process at each step is only that of $m \pm n$. For by mere inspection of the series we get

$$\left\{\begin{array}{l} \text{פ}(\rho) + \text{צ}(\rho) = 2\text{מ}(\rho) \\ \text{פ}(\rho) - \text{צ}(\rho) = \text{פ}(2\rho) \end{array}\right\};$$

whence further

$$\left\{\begin{array}{l} \text{פ}(\rho) = \text{מ}(\rho) + \tfrac{1}{2}\text{פ}(2\rho) \\ \text{צ}(\rho) = \text{מ}(\rho) - \tfrac{1}{2}\text{פ}(2\rho) \end{array}\right\}.$$

If a whole table is aimed at, we begin when ρ is so large that $\text{פ}(2\rho)$ is undistinguishable from $\text{ת}(\rho)$, or indeed from $2\epsilon^{-2\rho}$. Moreover from the former equation of the last pair, we get by repetition and dividing by 2,

$$\text{פ}(\rho) - 2^{-1}\text{פ}(2\rho) = \text{מ}(\rho); \quad 2^{-1}\text{פ}(2\rho) - 2^{-2}\text{פ}(2^2\rho) = 2^{-1}.\text{מ}(2\rho);$$

$$2^{-2}\text{פ}(2^2\rho) - 2^{-3}\text{פ}(2^3\rho) = 2^{-2}\text{מ}(2^2\rho);$$

up to $2^{-n+1}\text{פ}(2^{n-1}\rho) - 2^{-n}\text{פ}(2^n\rho) = 2^{-n+1}\text{מ}(2^{n-1}\rho);$

of which the sum is

$$\text{פ}(\rho) - 2^{-n}\text{פ}(2^n\rho) = \text{מ}(\rho) + 2^{-1}\text{מ}(2\rho) + 2^{-2}\text{מ}(2^2\rho) + \ldots + 2^{-n+1}\text{מ}(2^{n-1}\rho);$$

Make $n = \infty$, then

$$\text{פ}(\rho) = \text{מ}(\rho) + 2^{-1}\text{מ}(2\rho) + 2^{-2}\text{מ}(2^2\rho) + 2^{-3}\text{מ}(2^3\rho) + \&c.\ \textit{ad infin.}$$

which involves

$$\text{צ}(\rho) = \text{מ}(\rho) - 2^{-1}\text{מ}(2\rho) - 2^{-2}\text{מ}(2^2\rho) - 2^{-3}\text{מ}(2^3\rho) - \&c.$$

with very high convergence, even when $\rho = 1$. Of this pair, צ is the more obviously important in Elliptics.

If you express ר and שׁ in series of ס, mere inspection shews that

$$\text{ר}(\rho) - \text{ר}(2\rho) = \text{שׁ}(2\rho).$$

In this last write 2ρ, 4ρ, $8\rho \ldots 2^{n-1}$ for ρ, and add together the results: then $\text{ר}(\rho) - \text{ר}(2^n\rho) = \text{שׁ}(2\rho) + \text{שׁ}(2^2\rho) + \text{שׁ}(2^3\rho) + \ldots + \text{שׁ}(2^n\rho);$

so that, making $n = \infty$,

$$\text{ר}(\rho) = \text{שׁ}(2\rho) + \text{שׁ}(2^2\rho) + \text{שׁ}(2^3\rho) + \dots \textit{ ad infin.}$$

In passing from the transcendental constant ρ to the constants which dominate in the Lower theory of Elliptics; the most obvious and serviceable relations are

$$\log . \frac{1}{b} = 4\text{ל}(\rho); \qquad \log \frac{4}{c} = \rho + 2\text{צ}(\rho).$$

(The logarithms have ϵ for base.)

$$C = 1 + 2\text{ר}(\rho). \quad \text{Also } \log C = 2\text{מ}(\rho).$$

If $b = \cos\gamma$ and $c = \sin\gamma$, I covet a table, which, from γ given, will show ρ. *Long Division* will give it, from

$$\rho = \tfrac{1}{2}\pi . \frac{B}{C};$$

in fact, Legendre found B by first calculating ρ.

18. Gudermann's great table of log Sin ρ requires the ρ not less than 2, and that 8 *decimals* suffice. If we had 9 decimals, $\epsilon^{-9\rho}$ would be omissible, when $\rho > 2$. Under these conditions our chief functions are easily expressed in powers of $\epsilon^{-\rho}$. For we have

$$\tfrac{1}{2}\text{ק}(\rho) = \epsilon^{-\rho} + \epsilon^{-3\rho} + \epsilon^{-5\rho} + \epsilon^{-7\rho}; \quad \tfrac{1}{2}\text{כ}(\rho) = \epsilon^{-2\rho} + \epsilon^{-4\rho} + \epsilon^{-6\rho} + \epsilon^{-8\rho};$$

and with *even* terms made *negative*, these yield $\tfrac{1}{2}\text{ס}(\rho)$ and $\tfrac{1}{2}\text{ח}(\rho)$.

Next, $$\tfrac{1}{2}\text{ל}(\rho) = \epsilon^{-2\rho} + (1 + \tfrac{1}{3})\,\epsilon^{-6\rho};$$

$$\tfrac{1}{2}\text{צ}(\rho) = (q - q^2 + q - q^4) - \tfrac{1}{2}(q^2 - q^4) + \tfrac{1}{3}(q^3) - \tfrac{1}{4}q^4,$$

(where q means $\epsilon^{-2\rho}$);

$$= \epsilon^{-2\rho} - \tfrac{3}{2}\epsilon^{-4\rho} + \tfrac{4}{3}\epsilon^{-6\rho} - \tfrac{3}{4}\epsilon^{-8\rho}.$$

So $$\tfrac{1}{2}\text{מ}(\rho) = \epsilon^{-2\rho} - \epsilon^{-4\rho} + \tfrac{4}{3}\epsilon^{-6\rho} - \epsilon^{-8\rho}.$$

At most, we find 4 terms; and the last terms drop off, as ρ increases.

Indeed, if $\epsilon^{-10\rho}$ be the highest term admissible, (i.e. $\epsilon^{-11\rho}$ be negligible,) then, since

$$\tfrac{1}{2}\text{ס}(2\rho) = \frac{\epsilon^{-2\rho}}{1 + \epsilon^{-4\rho}} = \epsilon^{-2\rho} - \epsilon^{-6\rho} + \epsilon^{-10\rho};$$

we deduce $\frac{1}{2}\text{ם}(4\rho) = \text{simply } \epsilon^{-4\rho}$; whence $\frac{1}{2}\text{ר}(\rho)$ or

$$\frac{1}{2}\text{ם}(2\rho) + \frac{1}{2}\text{ם}(4\rho) + \frac{1}{2}\text{ם}(6\rho) + \frac{1}{2}\text{ם}(8\rho) + \frac{1}{2}\text{ם}(10\rho)$$

$$= \{\epsilon^{-2\rho} - \epsilon^{-6\rho} + \epsilon^{-10\rho}\} + \epsilon^{-4\rho} + \epsilon^{-6\rho} + \epsilon^{-8\rho} + \epsilon^{-10\rho}$$

$$= \epsilon^{-2\rho} + \epsilon^{-4\rho} + \epsilon^{-8\rho} + 2\epsilon^{-10\rho};$$

$$\therefore\ C - 1 = 4\{\epsilon^{-2\rho} + \epsilon^{-4\rho} + \epsilon^{-8\rho}\} + 8\epsilon^{-10\rho};$$

a very simple expression for C.

[The great advantage of ρ as the leading constant in Elliptics, is that to change from ρ to 2ρ, 3ρ, 4ρ... changes to the scales whose index is 2, 3, 4...]

Numerical Illustrations.

19. To fix ideas and give confidence to the student, it may be well to set forth examples of calculation under unfavourable conditions. To exact 16 decimals and (*as the case of worst convergence*) $\rho = 1$, is a severe test. I will calculate $\text{ל}(1)$, $\text{מ}(1)$, $\text{ר}(1)$, $\text{שׁ}(1)$ by three different methods, and compare the three results. (The successive entries are taken from my own tables of the Primary Anticyclics, which I complete for high numbers from the table of ϵ^{-x}, when they merge in it.)

I take $\text{ל}(1)$ first from the series

$$l\,\text{Cot}\,\rho + l\,\text{Cot}\,2\rho + l\,\text{Cot}\,3\rho + \text{\&c.},$$

next from $\text{ק}(2\rho) + \frac{1}{3}\text{ק}(6\rho) + \frac{1}{5}\text{ק}(10\rho) + \text{\&c.}$,

lastly from $l\,\text{Cot}\,\rho + \text{ק}_0(2\rho) + \frac{1}{3}\text{ק}_0(6\rho) + \frac{1}{5}\text{ק}_0(10\rho) + \text{\&c.}$;

observing that, for high values of ρ, (indeed $\rho > 6{\cdot}1$,) $l\,\text{Cot}\,\rho$ merges in $2\epsilon^{-2\rho}$; also for $\rho > 12{\cdot}6$, $\text{ק}(\rho)$ merges in $2\epsilon^{-\rho}$.—The like remark need not be repeated. Besides the case of $\rho = 1$, others have been taken *at random*.

ל (1) from first series.

l Cot 1	·2723 4146 8911 8315
3	49 5751 4506 6900
5	...... 9079 9859 5874
7	...... 166 3057 4382
9	...... 3 0459 9594
11	.... 557 8936
13	.... 10 2182
15	 1872
17	 34
ל (1) =	·2773 9147 7363 8089

ל (1) from second series.

פ (2) =	·2757 2056 4771 7832
$\frac{1}{3}$ פ (6) =	16 5251 1604 4931
$\frac{1}{5}$ פ (10) =	...... 1815 9971 9422
$\frac{1}{7}$ פ (14) = $\frac{2}{7}\epsilon^{-14}$	23 7579 6340
$\frac{1}{9}$ פ (18) = $\frac{2}{9}\epsilon^{-18}$	...... 3344 4399
(22)	.... 50 7176
(26)	 7860
(30)	 125
(34)	 2
ל (1) =	·2773 9147 7363 8087

ל (1) by the Mutilated Functions.

l Cot 1 =	·2723 4146 8911 8315
פ$_0$ (2) =	50 4999 8298 5578
$\frac{1}{3}$ פ$_0$ (6) =	1 0153 3822
$\frac{1}{5}$ פ$_0$ (10) =	373
ל (1) =	·2773 9147 7363 8088

20. מ (1) from second series.

ח (1) =	·2384 0584 4044 2351
$\frac{1}{3}$ ח (3) =	16 4841 5437 7566
$\frac{1}{5}$ ח (5) =	...... 1815 9147 4810
$\frac{1}{7}$ ח (7) =	...... 23 7579 4365
$\frac{1}{9}$ ח (9) =	.... 3384 4399
$\frac{1}{11}$ ח (11) = $\frac{2}{11}\epsilon^{-22}$	...... 50 7176
(13) =	 7860
(15) =	 124
(17) =	 2
מ (1) =	·2400 7265 9644 8653

מ (1) from original series.

	Positive terms.
l Cot 1	·2723 4146 8911 8315
(3)	49 5751 &c. ......
⋮	*as under* ל (1) ......
(17)	....
	·2773 9147 7363 8089
	sum of Positive terms.
	Negative terms.
l Cot 2	·0366 3537 4743 6963
(4)	6 7092 5280 9725
(6)	...... 1228 8424 7067
(8)	$= 2\epsilon^{-16}$ 22 5070 3494
(10)	.... 4122 3072
(12)	.... 75 5026
(14)	.... 1 3828
(16)	 254
(18)	 4
	·0373 1881 7718 9433
	sum of Negative terms.

whence

מ (1) = ·2400 7265 9644 8651

מ(1) by the Mutilated Functions.

l Cot 1	·2723	4146	8911	8315
− ח$_0$ (1)	−322	6472	2428	9903
− $\frac{1}{3}$ ח$_0$ (3)	......	−408	6013	3544
− $\frac{1}{5}$ ח$_0$ (5)	......	......	−824	4240
− (7)	$=\frac{2}{7}\epsilon^{-28}$......		.....	−1975
(9)	is insignificant"			
	− ·0322	6880	9266	9662
	sum of *negative* terms.			
∴ מ(1) =	·2400	7265	9644	8653

21. ר(1) from first series.

כ(1)	·3130	3528	5499	3310
−כ(3)	−49	6982	3313	6888
כ(5)	......	9080	3982	0194
−כ(7)	......	−166	3058	8210
כ(9)	......	3	0459	9598
−כ(11)	......	......	−557	8936
כ(13)	......	......	10	2182
−כ(15)	......	......	......	−1872
כ(17)	......	......	......	34
ר(1) =	·3081	5463	3020	9412

ר(1) by second series.

ס(2)	·2658	0222	8834	0797
(4)	366	1899	3473	6866
(6)	49	5747	3893	5604
(8)	6	7092	5180	3024
(10)	......	9079	9859	3378
(12)	......	1228	8424	7062
(14)	......	166	3057	4382
(16)	......	22	5070	3494
(18)	......	3	0459	9594
(20)	......	......	4122	3072
(22)	......	......	557	8936
(24)	......	......	75	5026
(26)	......	......	10	2182
(28)	......	......	1	3828
(30)	......	......	......	1872
(32)	......	......	......	254
(34)	......	......	......	34
(36)	......	......	......	4
ר(1) =	·3081	5463	3020	9409

ר(1) by the Auxiliaries.

ס(2)	·2658	0222	8834	0797
כ$_0$(1)	+423	6471	9026	1059
− כ$_0$(3)	......	−1231	8960	3561
כ$_0$(5)	......	......	+4122	4945
− כ$_0$(7)	......	......	−1	3828
כ$_0$(9)	......	......	......	+4
ר(1) =	·3081	5463	3020	9416

As ρ increases, the advantage of the auxiliaries lessens.

22. To find ש(1). Observe that when

$$\rho > 11,\ \text{ס}(\rho) = 2\epsilon^{-\rho} = \text{ק}(\rho).$$

It is convenient to separate the following terms. Put

$$a = \epsilon^{-13} - \epsilon^{-15} + \epsilon^{-17} - \epsilon^{-19} + \&c.$$

$$b = \epsilon^{-13} + \epsilon^{-15} + \epsilon^{-17} + \epsilon^{-19} + \&c.$$

(13) =	226	0329	4070	(preceded by 5 *zeros*)
(15) =	30	5902	3205	
(17) =	4	1399	3772	
(19) =	......	5602	7964	
(21) =	......	758	2561	
(23) =	......	102	6188	
(25) =	......	13	8879	
(27) =	......	1	8795	
(29) =	......	......	2544	
(31) =	......	......	344	
(33) =	......	......	46	
(35) =	......	......	6	
∴ b =	261	4110	8374	(16 decimals).

Also taking the *even* rows negatively

$$a = 199\ 0891\ 5370.$$

Then ש(1) by its first series shows

ק (1) =	·8509	1812	8239	3215
−(3)	−998	2156	9668	8232
(5)	+134	7650	5830	5877
−(7)	−18	2376	5447	6230
(9)	2	4681	9611	9324
−(11)		−3340	3401	5896
six terms =	·7629	6271	5163	8058
$2a$ =		398	1783	0740
ש (1) =	·7629	6669	6946	8798

Second Method.

ס (1) =	·6480	5427	3663	8854
(3)	993	2793	7419	4324
(5)	134	7528	2221	3057
(7)	18	2376	2414	5974
(9)	2	4681	9604	4144
(11)		3340	3401	5712
six terms =	·7629	6146	8725	2065
$2b$ =		522	8221	6748
ש (1) =	·7629	6669	6946	8813

The latter is in excess by 15 in the two last decimals, which probably results from the number of rows that were added,—18 rows, all positive. Treated by the Mutilated Functions, in which negative rows enter as balance, the result agrees with the first method.

Third Method.

ס(1)	·6480 5427 3663 8854
$ק_0$(1)	+ ·1151 5924 5896 4369
− $ק_0$(3)	− 2 4743 2933 0953
$ק_0$(5)	...... + 61 1832 4168
− $ק_0$(7)	.... − 1516 5140
$ק_0$(9)	.... + 3 7592
− $ק_0$(11)	.... − 92
ש(1)	= ·7629 6669 6946 8798

Accurate agreement in the last figure can only be matter of chance.

23. I proceed to some other trials at random. To find ר(1·5).

First ($2\rho = 3\cdot$).

ס(3)	·0993 2792 7419 4324
(6)	49 5747 3893 5604
(9)	2 4681 9604 4144
(12)	...... 1228 8424 7062
(15)	...... 61 1804 6410
(18)	...... 3 0459 9594
(21)	.... 1516 5128
(24)	.... 75 5026
(27)	.... 3 7590
(30)	 1872
(33)	 92
(36)	 4
ר(1·5)	= ·1045 4515 3202 6850

Next,

ב(1·5)	·1047 9139 2982 5120
− ב(4·5)	− 2 4685 0071 8922
ב(7·5)	...... + 61 1804 8282
− ב(10·5)	...... − 1516 5128
ב(13·5)	.... + 3 7590
− ב(16·5)	.... − 92
ר(1·5)	= ·1045 4515 3202 6850

Otherwise:

ס(3)	= ·0993 2792 7419 4324
+ $ב_0$(1·5)	= + 52 1725 6246 7841
− $ב_0$(4·5)	= ...... − 3 0463 7189
+ $ב_0$(7·5)	= .... + 1872
ר(1·5)	= ·1045 4515 3202 6848

24. To find ר(1·7). ($2\rho = 3\cdot4$).

ס(3·4)	·0666 7228 1989 9218
(6·8)	22 2754 7532 4278
(10·2)	...... 7434 0637 2656
(13·6)	...... 248 0990 1600
(17·0)	...... 8 2794 7544
(20·4)	.... 2763 2652
(23·8)	.... 92 2192
(27·2)	.... 3 0776
(30·6)	 1026
(34·0)	 34
ר(1·7)	= ·0689 7673 6807 1976

ב(1·7)	·0690 5099 7501 2924
− ב(5·1)	...... − 7434 3400 7360
ב(8·5)	...... + 8 2798 7578
− ב(11·9)	.... − 92 2192
ב(15·3)	 1026
ר(1·7)	= ·0689 7673 6807 1976

$$\begin{array}{r|rrrr}
\text{ס}(3\cdot4) & \cdot0666 & 7228 & 1989 & 9218 \\
\text{כ}_0(1\cdot7) & +23 & 0445 & 7580 & 6402 \\
-\text{כ}_0(5\cdot1) & \ldots\ldots & & -2763 & 3678 \\
\text{כ}_0(8\cdot5) & \ldots\ldots & \ldots\ldots & \ldots\ldots & +34 \\
\hline
\text{ד}(1\cdot7) = & \cdot0689 & 7673 & 6807 & 1976
\end{array}$$

25. So far, I have worked by my skeleton tables, which afford 16 decimals. They have borne the test well. When ρ has *two* decimals, I am driven to my longer tables, which yield only 12 decimals for the entries.

I have naturally calculated the secondary Functions by the Mutilated Auxiliars, which give a correct result by fewer terms to add or subtract. I take *at random* cases to corroborate by *other* methods. I chose *small* values of ρ, solely because with them the process is less speedy. My tables give

$$\text{ד}(1\cdot01) = \cdot3012\ 8975\ 9279.$$

To check this I calculate the same through $\dot{\text{ש}}$, thus

$$\begin{array}{rrrr}
\text{ש}(2\cdot02) = & \cdot2654 & 7527 & 3838 \\
(4\cdot04) = & 351 & 9495 & 1578 \\
(8\cdot08) = & 6 & 1934 & 2070 \\
(16\cdot16) = & \ldots\ldots & 19 & 1792 \\
\hline
\text{ד}(1\cdot01) = & \cdot3012 & 8975 & 9278
\end{array}$$

I take at random $\text{ד}(1\cdot07)$. To proceed by $\dot{\text{ש}}$,

$$\begin{array}{rrrrr}
\dot{\text{ש}}(2\cdot14) = & \ldots\ldots\ldots & \cdot2353 & 9987 & 4442 \\
\dot{\text{ש}}(4\cdot28) = & \ldots\ldots\ldots & 276 & 8532 & 6204 \\
\dot{\text{ש}}(8\cdot56) = 2\epsilon^{-8\cdot56} = & & 3 & 8323 & 8588 \\
\dot{\text{ש}}(17\cdot12) = 2\epsilon^{-17\cdot12} = & \ldots\ldots & & 7 & 3436 \\
\hline
\therefore\ \text{ד}(1\cdot07) = & & \cdot2634 & 6851 & 2670
\end{array}$$

The computation by the auxiliary כ_0 is

$$\begin{array}{rrrr}
\text{ס}(2\cdot14) = & \cdot2320 & 9684 & 7809 \\
\text{כ}_0(1\cdot07) = & 313 & 7697 & 7541 \\
-\text{כ}_0(3\cdot21) = & \ldots\ldots & -531 & 3696 \\
\text{כ}_0(5\cdot35) = & \ldots\ldots & \ldots\ldots & 1016 \\
\hline
\therefore\ \text{ד}(1\cdot07) = & \cdot2634 & 6851 & 2670
\end{array}$$

Further, to calculate $\text{ד}(1\cdot11)$.

First,

$$\begin{array}{rrrr}
\text{ס}(2\cdot22) = & \cdot2146 & 8579 & 7187 \\
\text{כ}_0(1\cdot11) = & 264 & 6636 & 5398 \\
-\text{כ}_0(3\cdot33) = & & -328 & 6883 \\
\text{כ}_0(5\cdot55) = 2\epsilon^{-22\cdot20} & & & 457 \\
\hline
\text{ד}(1\cdot11) = & \cdot2411 & 4887 & 6159
\end{array}$$

Otherwise,

$$\begin{array}{rrrr}
\text{כ}(1\cdot11) = & \cdot2436 & 8458 & 3048 \\
-\text{ס}_0(2\cdot22) = & -25 & 3242 & 0463 \\
-\text{ס}_0(4\cdot44) = & & -328 & 2216 \\
-\text{ס}_0(6\cdot66) = & \ldots\ldots & & -4206 \\
-\text{ס}_0(8\cdot88) = & \ldots\ldots & \ldots\ldots & -5 \\
\hline
\text{ד}(1\cdot11) = & \cdot2411 & 4887 & 6158
\end{array}$$

Or again:

ש(2·22) =	·2172	7867	1153
ש(4·44) =	235	9187	7952
ש(8·88) =	2	7828	8332
ש(17·76) =	......	3	8722
ר(1·11) =	·2411	4887	6159

Make some trials on ש.

To find ש(1·34).

First,

ם (1·34) =	·4900	8927	0938
$ק_0$ (1·34) =	385	4896	8756
$-ק_0$ (4·02) =		−1157	6533
$ק_0$ (6·70) =	......		3730
ש (1·34) =	·5286	2666	6891

Or thus,

ק (1·34) =	·5622	4030	5916
$-ם_0$ (1·34) =	−336	0206	6222
$-ם_0$ (4·02) =		−1156	9073
$-ם_0$ (6·70) =	......		−3731
ש (1·34) =	·5286	2666	6890

To find ש(1·5) to 16 decimals.

First,

ם (1·5) =	·4250	9603	4942	2804
$ק_0$ (1·5) =	233	8212	0298	3647
$-ק_0$ (4·5) =	......	−274	2256	5942
$ק_0$ (7·5) =	......	......	338	3796
$-ק_0$ (10·5) =	......	......	......	−418
∴ ש (1·5) =	·4484	7541	3322	3887

Next,

ק (1·5) =	·4696	4244	0595	2243
$-ם_0$ (1·5) =	−211	6428	5354	5792
$-ם_0$ (4·5) =	......	−274	1579	8350
$-ם_0$ (7·5) =	......	......	−338	3796
$-ם_0$ (10·5) =	......	......	......	−418
∴ ש (1·5) =	·4484	7541	3322	3887

as before.

To find ש(1·81). In my table, calculated through $ק_0$, I have ·3277 7800 6407. I now check it by calculating through $ם_0$.

ק (1·81) =	·3363	1570	8424
$-ם_0$ (1·81) =	−85	3753	3605
$-ם_0$ (5·43) =	......	−16	8407
$-ם_0$ (9·05) =	......	......	−3
ש (1·81) =	·3277	7800	6409

nearly as before.

If any figure (but the last) in any of the entries here *elicited at random* were erroneous, the error would show itself in the result. No test which I have in these is so complete and absolute as that of the table of ϵ^{-x}, in which I had Mr Glaisher's valued revision. But I have laboured, by double methods and by recomputing after intervals of time, to impart what accuracy I can to my other tables. I am painfully aware that a tired brain will go wrong in the simplest process: but I have a strong faith that these functions, *elicited by the progress of the Calculus*, will live in the mathematics of the future.

I have executed tables of all these functions, (1) skeleton tables in which ρ increases by ·1 at each step, and the entries are carried to 16 decimals; (2) ampler tables, with ρ increasing by ·01 at each step, but the entries having only 12 decimals. Each set is complete in this sense, that they are continued from $\rho = 1$ until the function is merged in the form $2\epsilon^{-n\rho}$, so as no longer to deserve a separate registration. Besides the large table of ϵ^{-x} already published with the skeleton table of ϵ^{-x} to 16 decimals, I have also an intermediate table (unpublished) in which x proceeds by ·01 at each step, and the entries have 12 decimals until $x = 18·50$, after which I give 16 decimals (perhaps with no adequate advantage), and the table is continued until ϵ^{-x} fails to affect the 12th decimal.

Whether any but the skeleton tables, which now follow, will ever see the light, the writer is uncertain. It seems that competent mathematicians are too busy to put forth any judgment on an eccentric undertaking.

Skeleton Anticyclics to 16 *Decimals.*

Summary, here repeated, for compactness.

Gudermann writes Sin, Cos, Tan in *German* type, not easy to imitate. Here *capital* letters contrast *Sin* to *sin*, *Cos* to *cos*, &c. and

$$\text{Cos}\,.\,x \textit{ means } \tfrac{1}{2}(\epsilon^{x}+\epsilon^{-x});\quad \text{Sin}\,x \textit{ means } \tfrac{1}{2}(\epsilon^{x}-\epsilon^{-x}).$$

Conformably $\qquad \text{Tan}\,x \text{ means } \dfrac{\text{Sin}\,x}{\text{Cos}\,x},$

or $\qquad \dfrac{\epsilon^{x}-\epsilon^{-x}}{\epsilon^{x}+\epsilon^{-x}},$

that is, $\qquad \dfrac{1-\epsilon^{-2x}}{1+\epsilon^{-2x}};$

whence further $\qquad \text{Cot}\,x \textit{ means } \dfrac{1+\epsilon^{-2x}}{1-\epsilon^{-2x}},$

and $\qquad \text{Sec}\,x \textit{ means } \dfrac{2}{\epsilon^{x}+\epsilon^{-x}},$

or $\qquad \dfrac{2\epsilon^{-x}}{1+\epsilon^{-2x}},$

and $\qquad \text{Cosec}\,x \text{ means } \dfrac{2\epsilon^{-x}}{1-\epsilon^{-2x}}.$

The possession of a good table for ϵ^{-x} opens the way to a registration of these Anticyclics.

For conciseness it is convenient to write also

$$\text{ק}(x) \text{ for Cosec}\,x,\quad \text{ס}(x) \text{ for Sec}\,x,$$

$$\text{ח}(x) \text{ for } 1-\text{Tan}\,x, \text{ or } \frac{2\epsilon^{-2x}}{1+\epsilon^{-2x}},$$

$$\text{ב}(x) \text{ for Cot}\,(x)-1, \text{ or } \frac{2\epsilon^{-2x}}{1-\epsilon^{-2x}}.$$

I also include as *Mutilated* Anticyclics the functions

$$\text{ק}_0(x)=\text{ק}(x)-2\epsilon^{-x};\qquad \text{ס}_0(x)=2\epsilon^{-x}-\text{ס}(x);$$

$$\text{ח}_0(x)=2\epsilon^{-2x}-\text{ח}(x);\qquad \text{ב}_0(x)=\text{ב}(x)-2\epsilon^{-2x}.$$

Finally, I write κ and σ (as auxiliaries towards log Cos and log Sin) interpreted as

$$\kappa(x)=\log_{\epsilon}(1+\epsilon^{-2x}) \text{ and } -\sigma(x)=\log_{\epsilon}(1-\epsilon^{-2x}).$$

Since Tan x is positive, and less than 1, we may assume

$$\sin\theta = \text{Tan}\, x,$$

and take for θ an arc between *zero* and 90°. Till a better name is found, I call θ the *Elevation* and x its *Range**. We have now

$$\cos\theta = \text{Sec}\, x, \qquad \tan\theta = \text{Sin}\, x, \qquad \sec\theta = \text{Cos}\, x,$$

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\text{Sec}\, x}{1+\text{Sec}\, x} = \frac{2\,\text{Cos}\, x - 2}{2\,\text{Cos}\, x + 2} = \frac{\epsilon^x - 2 + \epsilon^{-x}}{\epsilon^x + 2 - \epsilon^{-x}} = \left(\frac{\text{Sin}\,\frac{1}{2}x}{\text{Cos}\,\frac{1}{2}x}\right)^2;$$

whence $$\tan\tfrac{1}{2}\theta = \text{Tan}\,.\,\tfrac{1}{2}x.$$

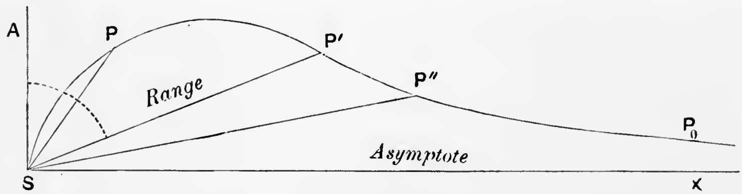
Also $$x = \int_0 \frac{d\theta}{\cos\theta}.$$

Legendre tabulated this integral, and Gudermann has enlarged the table tenfold. He calls x the *Längezahle* of θ, but I cannot translate this. The "Length-number" sounds nonsensical.

N.B. $$\log_\epsilon \text{Cos}\, x = x - \log_\epsilon 2 + \kappa(x),$$

and $$\log_\epsilon \text{Sin}\, x = x - \log_\epsilon 2 - \sigma(x).$$

* Taking a Polar Curve, with ρ radius vector, and $\sin\theta = \text{Tan}\,\rho$, which amounts to $\rho = \int_0 \sec\theta \,.\, d\theta$.



With $\angle ASX = 90°$, $\angle ASP = \theta$, $SP = \rho$, $\text{Tan}\,\frac{1}{2}\rho = \tan\frac{1}{2}\theta$,

then with $\theta = 90°$, $\text{Tan}\,\frac{1}{2}\rho = 1$, $\dfrac{\epsilon^\rho - 1}{\epsilon^\rho + 1} = 1$, $\epsilon^\rho = \infty$, $\rho = \log\infty$.

For small values of x.

Primary Anticyclics.

x	$\frac{1}{x} - \text{Cosec } x$	x	$1 - \text{Sec } x$
·9	·1369 4286 3331 1060	·9	·3022 0535 8899 6676
·8	·1240 0826 0211 5182	·8	·2523 0008 1762 5802
·7	·1103 2533 7105 1180	·7	·2032 9454 0007 1249
·6	·0959 5375 7731 5912	·6	·1564 4931 2378 1164
·5	·0809 6524 8668 2201	·5	·1131 8118 9128 4733
·4	·0654 4287 8392 7115	·4	·0749 9254 8094 2449
·3	·0494 7993 6590 5823	·3	·0433 7208 8099 7516
·2	·0331 7843 1185 4831	·2	·0196 7200 2355 2746
·1	·0166 4724 2703 8901	·1	·0049 7925 1046 7735
	approximating to $\frac{1}{6}x$.		approximating to $\frac{1}{2}x^2$.

x	$\text{Cot } x - \frac{1}{x}$	x	$\text{Tan } x$
·9	·2849 5614 1918 9029	·9	·7162 9787 0199 0224
·8	·2559 4070 2043 7062	·8	·6640 3677 0267 8494
·7	·2260 5020 7231 2008	·7	·6043 6777 7117 1635
·6	·1953 5885 4719 9995	·6	·5370 4956 6998 0353
·5	·1639 5341 3738 6538	·5	·4621 1715 7260 0098
·4	·1319 3244 1832 1884	·4	·3799 4896 2255 2245
·3	·0994 0509 6988 4084	·3	·2913 1261 2451 5909
·2	·0664 8956 3439 4728	·2	·1973 7532 0224 9047
·1	·0333 1113 2253 9896	·1	·0996 6799 4624 9558
	approximating to $\frac{1}{3}x$.		approximating to x.

If $H_1\, H_2\, H_3 \ldots$ are Euler's coefficients,—in

$$x \cot x = 1 - 2\,(H_1 x^2 + H_2 x^4 + H_3 x^6 + \&\text{c.})$$

we have

$$\text{Cot } x - \frac{1}{x} \text{ [or, say } K(x)] = 2\,(H_1 x - H_2 x^3 + H_3 x^5 - \&\text{c.}\ldots).$$

Thence
$$\frac{1}{x} - \text{Cosec } x = K(x) - K(\tfrac{1}{2}x);$$

$$\text{Tan } x = 2K(2x) - K(x).$$

ק and ס to Sixteen Decimals.

x		x	
ק(1·) ס(1·)	·8509 1812 8239 3215 ·6480 5427 3663 8854	2·4	·1829 4146 8590 0977 ·1799 5492 3081 6373
1·1	·7487 0055 3378 0705 ·5993 3406 0570 7929	2·5	·1652 8366 9855 0954 ·1630 7123 1929 9781
1·2	·6624 8797 7194 3154 ·5522 8615 4278 2047	2·6	·1493 7117 2122 2848 ·1477 3218 2327 8366
1·3	·5887 9553 7472 7589 ·5073 7875 0740 6021	2·7	·1350 2085 8114 1130 ·1338 0667 6793 1016
1·4	·5251 2692 9342 7329 ·4649 2199 2408 9817	2·8	·1220 7152 9128 8169 ·1211 7204 7532 4136
1·5	·4696 4244 0595 2243 ·4250 9603 4942 2804	2·9	·1103 8062 3493 2692 ·1097 1427 4141 5019
1·6	·4209 5196 5887 9284 ·3879 7818 9874 4896	3·0	·0998 2156 9668 8225 ·0993 2792 7419 4331
1·7	·3779 8152 7668 3616 ·3535 6734 9501 4020	3·1	902 8162 5082 9535 899 1592 6650 8797
1·8	·3398 8469 1415 4933 ·3218 0486 9506 5875	3·2	816 6009 0874 6531 813 8917 5180 7533
1·9	·3059 8229 8640 1752 ·2925 9173 5483 7630	3·3	·0738 6682 0864 6194 ·0736 6612 1764 9807
2·0	·2757 2056 4771 7832 ·2658 0222 8834 0795	3·4	·0668 2096 3449 0966 ·0666 7228 1989 9217
2·1	·2486 4137 7381 2334 ·2412 9450 6201 8549	3·5	·0604 4989 0009 1559 ·0603 3974 4120 1677
2·2	·2243 6087 1403 8413 ·2189 1857 8920 1682	3·6	·0546 8827 4384 1248 ·0546 0667 6324 9982
2·3	·2025 5372 4210 8383 ·1985 2217 5149 3391	3·7	·0494 7729 6074 5175 ·0494 1684 6756 6524

ק(x) and ס(x) to Sixteen Decimals.

x		x	
3·8	·0447 6394 5893 2198 ·0447 1916 3942 6338	5·2	·0110 3346 4617 2459 3279 3086 2335
3·9	·0405 0041 7329 2519 ·0404 6724 2047 0393	5·3	·0099 8343 6561 2227 8293 9078 8133
4·0	·0366 4357 0325 8654 1899 3473 6865	5·4	·0090 3334 6161 0009 3297 7616 9677
4·1	·0331 5445 6793 4411 3624 9814 2153	5·5	·0081 7367 9391 2755 7340 6367 1407
4·2	·0299 9789 9188 2770 8441 1126 6582	5·6	·0073 9582 8564 9758 9562 6303 7217
4·3	·0271 4211 5045 0343 3212 2843 3915	5·7	·0066 9200 5835 1925 9185 5996 3701
4·4	·0245 5838 1566 5102 5097 9161 5505	5·8	·0060 5516 4992 9252 5505 3989 5956
4·5	·0222 2073 5333 0788 1525 1496 6496	5·9	·0054 7893 0754 4899 7884 8521 2007
4·6	·0201 0570 2957 4680 0164 0431 5416	6·0	·0049 5753 4813 4793 5747 3893 5605
4·7	·0181 9205 9124 4828 8904 9531 2660	6·1	·0044 8575 8004 3780 8571 2873 7920
4·8	·0164 6060 8954 2863 5837 9392 7991	6·2	·0040 5887 7989 4405 5884 4555 8801
4·9	·0148 9399 2037 5291 9234 0337 7575	6·3	·0036 7262 1938 1950 7259 7170 0042
5·0	·0134 7650 5830 5889 7528 2221 3045	6·4	·0033 2312 3720 7367 2310 5372 0099
5·1	·0121 9394 6383 9042 9303 9911 8518	6·5	·0030 0688 5182 5061 0687 1589 4349

ק(x) and ס(x) to Sixteen Decimals.

x		x	
6·6	·0027 2074 1110 1024 2073 1040 1076	8·0	67092 5331 3076 5180 3024
6·7	·0024 6182 7535 3703 0075 3347	8·1	60707 8332 0915 8220 2239
6·8	·0022 2755 3058 9582 2754 7532 4278	8·2	54930 7181 3811 7098 5075
6·9	·0020 1557 2905 1762 1556 8811 0218	8·3	49703 3684 9129 3623 5189
7·0	·0018 2376 5447 6224 2414 5980	8·4	44973 4671 0985 4625 6169
7·1	·0016 5021 0969 9924 5020 8723 0728	8·5	40693 6754 8681 6721 1745
7·2	·0014 9317 2449 0332 0784 4744	8·6	36821 1599 8157 1574 8545
7·3	·0013 5107 8166 9557 6933 8201	8·7	33317 1631 2211 1612 7295
7·4	·0012 2250 5979 0238 5065 4946	8·8	30146 6157 0402 6143 3415
7·5	·0011 0616 9078 6753 8401 9160	8·9	27277 7858 0382 7847 8898
7·6	·0010 0090 3117 5590 2616 2034	9·0	24681 9611 9323 9604 4143
7·7	·0009 0565 4551 4802 4180 0670	9·1	22333 1619 7650 1614 1954
7·8	81947 0095 5344 81946 9820 3848	9·2	20207 8805 7372 8801 6112
7·9	74148 7182 8362 6979 0002	9·3	18284 8464 4847 8461 4279

When x reaches 10, the equations ק$(x) = 2\epsilon^{-x} =$ ס(x) are true to 12 decimals.

ק(x) and ס(x) to Sixteen Decimals.

x		x		x	
9·4	16544 8132 2454 8129 9810	10·5	5507 2898 7412 6576	11·6	1833 2175 4741 4709
9·5	14970 3660 6142 3658 9366	10·6	4983 2019 4940 4320	11·7	1658 7638 3227 3203
9·6	13545 7473 6031 7472 3603	10·7	4508 9875 8494 8034	11·8	1500 9115 8309 8293
9·7	12256 6990 5668 6989 6460	10·8	4079 9006 8393 8053	11·9	1358 0809 6154 6142
9·8	11090 3199 2054 3198 5234	10·9	3691 6468 0042 6467 9790	12·0	1228 8424 7070 7062
9·9	10034 9364 3649 9363 8597	11·0	3340 3401 5897 5713	12·1	1111 9026 4836 4829
10·0	9079 9859 7122 3378	11·1	3022 4647 6465 6329	12·2	1006 0911 2145 2140
10·1	8215 9110 5892 3120	11·2	2734 8392 1364 1264	12·3	910 3488 9263 9259
10·2	7434 0637 4708 2656	11·3	2474 5848 5274 5198	12·4	823 7177 4152 4149
10·3	6726 6190 4474 2954	11·4	2239 0969 6880 6824	12·5	745 3306 3442 3440
10·4	6086 4966 0732 4965 9604	11·5	2026 0187 1993 1953	12·6	674 4030 4683 4682

When x exceeds 12·6, we have ק$(x)=$ ס$(x)=2\epsilon^{-x}$ true to 16 decimals. They are true to 12 decimals, even when x reaches 10.

$כ(x) = \text{Cot}\, x - 1$, and $ת(x) = 1 - \text{Tan}\, x$, to Sixteen Decimals.

x		x	
1·0	·3130 3528 5499 3313 ·2384 0584 4044 2351	2·5	·0135 6730 9812 6083 ·0133 8570 1848 5695
1·1	·2492 2076 4568 3124 ·1995 0097 8239 3703	2·6	·0110 9433 1435 5912 ·0109 7259 7798 9007
1·2	·1995 3754 4192 3508 ·1663 4539 2987 8447	2·7	·0090 7414 6000 1196 ·0089 9254 6321 8822
1·3	·1604 6550 3557 8761 ·1382 7684 0686 6936	2·8	·0074 2317 7331 0795 ·0073 6847 9798 8721
1·4	·1294 9470 6459 8964 ·1146 4835 1797 7375	2·9	·0060 7349 7336 4337 3683 2649 4167
1·5	·1047 9139 2982 5114 ·0948 5174 6355 1336	3·0	·0049 6982 3313 6889 4524 6313 2697
1·6	·0849 8873 6155 7778 ·0783 3144 5593 5284	3·1	·0040 6711 5200 7812 5064 0778 0998
1·7	·0690 5099 7501 2924 ·0645 9092 9396 9008	3·2	·0033 2864 5281 1247 1760 2160 3487
1·8	·0561 8256 1614 5180 ·0531 9398 7153 7316	3·3	·0027 2444 2319 3478 1703 9900 8570
1·9	·0457 6534 9914 1786 ·0437 6254 1872 2610	3·4	·0022 3003 4052 1958 2507 2065 7206
2·0	·0373 1472 0727 5481 ·0359 7241 9924 1831	3·5	·0018 2542 8506 4434 2210 2388 8014
2·1	·0304 4773 4990 0075 ·0295 4806 3386 5465	3·6	·0014 9428 7230 3932 9205 7667 6732
2·2	·0248 5989 3164 4710 ·0242 5686 9968 5494	3·7	·0012 2325 3239 1790 2175 8718 8686
2·3	·0203 0780 2181 1268 ·0199 0360 3733 8092	3·8	·0010 0140 4020 9588 0040 2214 1592
2·4	·0165 9607 5602 2530 ·0163 2514 2306 3198	3·9	·0008 1980 5861 0968 1913 4329 9720

כ (x) and ת (x) to Sixteen Decimals.

x		x	
4·0	67115 0401 6824 67070 0260 9328	5·4	4079 9839 1187 8174 5599
4·1	54945 8050 5595 54915 6312 2027	5·5	3340 3959 4833 2843 6961
4·2	449 83 5801 7305 63 3540 4665	5·6	2734 8766 1037 8018 1692
4·3	368 27 9389 7045 14 3809 9269	5·7	2239 1220 3658 0719 0102
4·4	301 51 1597 9608 42 0716 1195	5·8	1833 2343 5085 2007 4397
4·5	246 85 0071 8921 78 9151 9725	5·9	1500 9228 4677 9003 1941
4·6	202 09 9223 6588 05 8387 8156	6·0	1228 8500 2096 8348 2044
4·7	165 46 1818 7874 43 4445 7034	6·1	1006 0961 8256 0860 6036
4·8	135 46 6647 9665 44 8299 2397	6·2	823 7211 3407 7143 4895
4·9	110 90 9348 9652 89 7049 4456	6·3	674 4053 2094 4007 7274
5·0	9080 3982 0194 9079 5737 4050	6·4	552 1560 3880 1529 9004
5·1	7434 3400 7360 7433 7854 2056	6·5	452 0669 0322 0648 5958
5·2	6086 6818 3452 3113 8012	6·6	370 1209 2454 1195 5466
5·3	4983 3261 1090 0777 8790	6·7	303 0292 8156 0283 6328

ב(x) and ת(x) to Sixteen Decimals.

x		x	
6·8	248 0993 2376 0987 0824	8·1	18 4272 0336 4271 9996
6·9	203 1265 0054 1260 8794	8·2	15 0869 1784 1556
7·0	166 3058 6771 3056 1994	8·3	12 3521 2342 2190
7·1	136 1597 1958 1595 3418	8·4	10 1130 6320 6220
7·2	111 4781 3600 4780 1172	8·5	8 2798 7578 7510
7·3	91 2705 6900 2704 8572	8·6	6 7789 8888 8844
7·4	74 7260 1552 7259 5968	8·7	5 5501 6664 6632
7·5	61 1804 8282 4538	8·8	4 5440 9206 9188
7·6	50 0903 3998 1490	8·9	3 7203 8792 8780
7·7	41 0104 9992 8310	9·0	3 0459 9598 9590
7·8	33 5765 5624 4496	9·1	2 4938 5058 5054
7·9	27 4901 5834 5078	9·2	2 0417 9216 9212
8·0	22 5070 3748 3230	9·3	1 6716 7803 7801

When x is as large as 9·4, ב$(x) = 2\epsilon^{-2x} =$ ת(x) without error in 16^{th} decimal.

MUTILATED Anticyclics.

$ק_0(x)$ or $ק(x) - 2\epsilon^{-x}$ to Sixteen Decimals.

x		x	
1·0	·1151 5924 5896 4369	4·5	274 2256 5942
1·1	829 5838 5981 9114	4·6	203 1468 2009
1·2	600 9955 3369 9112	4·7	150 4921 6432
1·3	437 3195 1404 7337	4·8	111 4856 2462
1·4	319 3300 1459 5200	4·9	82 5895 6803
1·5	233 8212 0298 3647	5·0	61 1832 4168
1·6	171 5892 9898 6175	5·1	45 3252 8730
1·7	126 1447 9562 8923	5·2	33 5775 7245
1·8	92 8691 4972 3250	5·3	24 8747 4022
1·9	68 4506 0194 9051	5·4	18 4275 7757
2·0	50 4999 8298 5578	5·5	13 6514 3475
2·1	37 2852 0875 2596	5·6	10 1132 0098
2·2	27 5455 4679 1739	5·7	7 4920 2501
2·3	20 3603 6765 3296	5·8	5 5502 1736
2·4	15 0556 2011 2726	5·9	4 1116 9533
2·5	11 1367 2607 3008	6·0	3 0460 1465
2·6	8 2401 5693 6173	6·1	2 2565 4064
2·7	6 0983 2634 6135	6·2	1 6716 8490
2·8	4 5140 3878 3811	6·3	1 2384 1372
2·9	3 3418 3380 4546	6·4	9174 3887
3·0	2 4743 2933 0953	6·5	6796 7510
3·1	1 8322 0295 8379	6·6	5035 0066
3·2	1 3568 2917 9206	6·7	3730 0251
3·3	1 0048 6062 1394	6·8	2763 2686
3·4	7442 3528 4445	6·9	2047 0792
3·5	5512 3164 5188	7·0	1516 5140
3·6	4082 9489 5399	7·1	1123 4606
3·7	3024 3133 8386	7·2	832 2799
3·8	2240 2180 8886	7·3	616 5679
3·9	1659 4437 6430	7·4	456 7647
4·0	1229 2548 3370		
4·1	910 5989 9186		
4·2	674 5547 3214		
4·3	499 7020 6524		
4·4	370 1760 3735		

Beyond this value of x, $ק_0(x) = 2\epsilon^{-3x} = \mathbf{D}_0(x)$.

Mutilated $\mathfrak{D}_0(x) = 2\epsilon^{-x} - \mathfrak{D}(x)$. Sixteen Decimals.

x		x	
1·0	·0877 0460 8678 9992	4·5	274 1579 8350
1·1	·0664 0810 6825 6662	4·6	203 1057 7255
1·2	·0501 0226 9546 1995	4·7	150 4672 1257
1·3	·0376 8483 5327 4231	4·8	111 4705 2409
1·4	·0282 7193 5474 2312	4·9	82 5807 0912
1·5	211 6428 5354 5792	5·0	61 1776 8664
1·6	158 1484 6114 8212	5·1	45 3219 1795
1·7	117 9969 8604 0672	5·2	33 5755 2880
1·8	87 9290 6936 5856	5·3	24 8735 0071
1·9	65 4550 2961 5071	5·4	19 0568 2576
2·0	48 6833 7639 1459	5·5	13 5509 7874
2·1	36 1835 0304 1089	5·6	10 1129 2441
2·2	26 8773 7804 4996	5·7	7 4918 5724
2·3	19 9551 2296 2683	5·8	5 5501 1560
2·4	14 8098 3497 1877	5·9	4 1116 3360
2·5	10 9876 5317 8195	6·0	3 0459 7722
2·6	8 1497 4100 8311	6·1	2 2565 1796
2·7	6 0434 8886 3979	6·2	1 6716 7114
2·8	4 4807 7718 0223	6·3	1 2384 0536
2·9	3 3216 5917 3125	6·4	9174 3380
3·0	2 4620 9316 2948	6·5	6796 5202
3·1	1 8247 8136 2359	6·6	5034 9882
3·2	1 3523 2775 9791	6·7	3730 0122
3·3	1 0021 3037 4993	6·8	2763 2618
3·4	7425 7930 7304	6·9	2067 0752
3·5	5502 2724 4693	7·0	1516 5110
3·6	4076 8569 5869	7·1	1123 4590
3·7	3020 6184 0264	7·2	832 2789
3·8	2237 9769 6974	7·3	616 5676
3·9	1658 0844 5695	7·4	456 7645
4·0	1228 4306 7818		I have had few cases before me to *test* this Table.
4·1	910 0989 3072		
4·2	674 2514 2972		
4·3	499 5181 0103		
4·4	370 0644 5864		

When x exceeds 7·4, $\mathfrak{D}_0(x) = 2\epsilon^{-3x}$.

$\beth_0(x)$ or $\beth x - 2\epsilon^{-2x}$ to Sixteen Decimals.

x		x	
1·0	·0423 6741 9026 1059	4·0	22 5145 8774
1·1	·0276 1444 7843 6447	4·1	15 0190 6131
1·2	·0181 0163 7613 5258	4·2	10 1153 3727
1·3	·0119 1834 7129 2084	4·3	6 7802 3695
1·4	·0078 7458 1209 4605	4·4	4 5447 7703
1·5	52 1725 6246 7841	4·5	3 0463 7189
1·6	34 6432 8199 0454	4·6	2 0419 9846
1·7	23 0445 7580 6403	4·7	1 3687 6733
1·8	15 3511 6719 9330	4·8	9174 9847
1·9	10 2380 6201 8424	4·9	6150 1019
2·0	6 8344 2950 0805	5·0	4122 4945
2·1	4 5658 1349 0520	5·1	2763 3678
2·2	3 0521 3358 3347	5·2	1852 3284
2·3	2 0413 0691 8597	5·3	1241 6460
2·4	1 3658 1504 2126	5·4	832 2961
2·5	9141 5814 4375	5·5	557 9027
2·6	6120 2594 0697	5·6	373 9722
2·7	4098 4114 8945	5·7	250 6806
2·8	2744 9898 1145	5·8	168 0379
2·9	1838 7845 6820	5·9	112 6377
3·0	1231 8960 3561	6·0	75 5030
3·1	825 3928 1896	6·1	50 6112
3·2	553 0734 7769		
3·3	370 6244 2520		
3·4	248 3756 5062		
3·5	166 4575 3342		
3·6	111 5613 6399		
3·7	74 7716 9199		
3·8	50 1154 0776		
3·9	33 5903 1373		

For values of x, higher than 6·1, $\beth_0(x) = 2\epsilon^{-4x}$.

ח$_0x = 2\epsilon^{-2x} -$ חx to Sixteen Decimals.

x		x	
1·0	·0322 6472 2428 9903	4·0	22 4994 8722
1·1	·0221 0533 8485 2974	4·1	15 0827 7437
1·2	·0150 9051 3590 9803	4·2	10 1106 8913
1·3	·0102 7031 5741 9741	4·3	6 7777 4081
1·4	·0069 7177 3452 6984	4·4	4 5434 0713
1·5	47 2239 0380 5935	4·5	3 0456 2007
1·6	31 9296 2363 2042	4·6	2 0415 8586
1·7	21 5561 0523 7513	4·7	1 3685 4107
1·8	14 5345 7740 8533	4·8	9173 7421
1·9	9 7900 1840 0702	4·9	6149 4187
2·0	6 5885 7853 2853	5·0	4122 1200
2·1	4 4309 0254 4092	5·1	2763 1626
2·2	2 9780 9837 5877	5·2	1852 2156
2·3	2 0066 7755 4579	5·3	1241 5840
2·4	1 3435 1791 7202	5·4	832 2623
2·5	9091 2149 6013	5·5	557 8843
2·6	6053 1042 6207	5·6	373 9622
2·7	4061 5563 3431	5·7	250 6750
2·8	2724 7634 0939	5·8	168 0329
2·9	1827 6841 3348	5·9	112 6359
3·0	1225 8040 0631	6·0	75 5022
3·1	822 0494 4916	6·1	50 6107
3·2	551 2385 9995		
3·3	369 6174 2388		
3·4	247 8229 9690		
3·5	166 1542 3078		
3·6	111 3949 0801		
3·7	74 6803 3905		
3·8	50 0652 7220		
3·9	33 5627 9875		

For higher values of x, ח$_0x = 2\epsilon^{-4x} =$ ב$_0(x)$.

$\sigma(x)$ means $-\log_\epsilon(1-\epsilon^{-2x})$; $\kappa(x)$ means $\log_\epsilon(1+\epsilon^{-2x})$.

σ and κ to Sixteen Decimals.

x			x	
1·0	σ κ	·1454 1345 7868 8591 ·1269 2801 1042 9725	2·3	101 0269 6562 3132 100 0165 2055 6520
1·1	σ κ	·1174 3664 8812 5736 ·1050 8331 9768 4102	2·4	82 6379 8368 4526 81 9606 7338 2682
1·2		950 9995 0522 4007 868 3615 2153 9481	2·5	67 6074 9449 4885 67 1534 8489 1181
1·3		771 7652 8823 4132 716 4469 1967 6700	2·6	55 3183 6855 7432 55 0140 3909 6574
1·4		627 3754 4004 4562 590 3282 6287 9514	2·7	45 2681 1510 7333 45 0641 1799 2495
1·5		510 6918 0942 7015 485 8735 1573 7420	2·8	37 0471 7716 5048 36 9104 3426 9466
1·6		416 1627 2352 8589 399 5333 3162 4303	2·9	30 3214 7060 5769 30 2298 0930 8316
1·7		339 4286 6281 1794 328 2847 0424 8652	3·0	24 8182 9368 9595 7568 5137 7315
1·8		277 0395 7650 5573 269 5709 3008 2051	3·1	20 3149 2721 0270 2737 4123 8382
1·9		226 2479 3155 9337 221 2421 6454 8791	3·2	16 6293 9190 4285 6017 8414 0455
2·0		184 8544 6825 8866 181 4992 7917 8096	3·3	13 6129 4178 1702 5944 3575 2599
2·1		151 0914 7282 5446 148 8425 4671 9180	3·4	11 1439 5856 3140 1315 5360 4646
2·2		123 5332 9044 1634 122 0258 4607 6962	3·5	9 1229 7982 8390 1146 6453 7742

σ and κ to Sixteen Decimals.

x		x	
3·6	$7\ 46^{86\ 4642\ 3523}_{30\ 7251\ 8276}$	5·1	$371^{7\ 1009\ 5175}_{6\ 9627\ 8849}$
3·7	$6\ 11^{43\ 9472\ 2609}_{06\ 6202\ 2535}$	5·2	$3043^{2946\ 0858}_{2019\ 9498}$
3·8	$5\ 00^{57\ 6701\ 0545}_{32\ 6249\ 3860}$	5·3	$2491^{6320\ 1404}_{5699\ 3329}$
3·9	$4\ 09^{81\ 8943\ 2925}_{65\ 1060\ 5254}$	5·4	$2039^{9711\ 4839}_{9295\ 3442}$
4·0	$3\ 35^{51\ 8908\ 0768}_{40\ 6372\ 8957}$	5·5	$1670^{1840\ 2651}_{1561\ 3183}$
4·1	$2\ 74^{69\ 1294\ 1714}_{61\ 5859\ 5851}$	5·6	$1367^{4289\ 5583}_{4102\ 5747}$
4·2	$2\ 24^{89\ 2610\ 6263}_{84\ 2045\ 3116}$	5·7	$1119\ 5^{547\ 5125}_{422\ 1746}$
4·3	$1\ 84^{12\ 2743\ 2196}_{08\ 8848\ 2758}$	5·8	$916\ 6^{129\ 7451}_{045\ 7278}$
4·4	$1\ 507^{4\ 4436\ 6671}_{2\ 1716\ 2070}$	5·9	$750\ 45^{86\ 0744}_{29\ 7560}$
4·5	$1\ 234^{1\ 7419\ 7031}_{0\ 2189\ 7232}$	6·0	$614\ 4^{231\ 2289}_{193\ 4777}$
4·6	$1\ 010^{4\ 4506\ 6613}_{3\ 4297\ 7006}$	6·1	$503\ 04^{68\ 2598}_{42\ 9544}$
4·7	$8272^{7487\ 3808}_{0644\ 1198}$	6·2	$411\ 85^{97\ 1889}_{80\ 2261}$
4·8	$677^{3\ 1030\ 1852}_{2\ 6443\ 0045}$	6·3	$337\ 20^{20\ 9193}_{09\ 5489}$
4·9	$5545^{3136\ 9290}_{0062\ 0401}$	6·4	$276\ 07^{76\ 3829}_{68\ 7611}$
5·0	$45^{40\ 0960\ 3705}_{39\ 8899\ 2169}$	6·5	$226\ 03^{31\ 9615}_{26\ 8525}$

σ and κ to Sixteen Decimals.

x		x	
6·6	$185\ {}^{0602\ 9103}_{0599\ 4857}$	7·9	$13\ 7450\ {}^{7822}_{7634}$
6·7	$151\ 51^{45}_{42}\ {}^{2599}_{9643}$	8·0	$11\ 2535\ {}^{1810}_{1684}$
6·8	$124\ 049^{5}_{4}\ {}^{8494}_{3106}$	8·1	$9\ 2136\ {}^{0125}_{0041}$
6·9	$101\ 563^{1}_{0}\ {}^{9869}_{9555}$	8·2	$7\ 5434\ 58^{63}_{07}$
7·0	$83\ 152^{9}_{8}\ {}^{0648}_{3734}$	8·3	$6\ 1760\ 61^{52}_{14}$
7·1	$68\ 079^{8}_{7}\ {}^{3661}_{9027}$	8·4	$5\ 0565\ 31^{43}_{23}$
7·2	$50\ 7390\ {}^{5246}_{2140}$	8·5	$4\ 1399\ 37^{80}_{64}$
7·3	$45\ 6352\ {}^{7409}_{5327}$	8·6	$3\ 3894\ 94^{38}_{28}$
7·4	$37\ 36^{30}_{29}\ {}^{0078}_{8682}$	8·7	$2\ 7750\ 83^{28}_{20}$
7·5	$30\ 5902\ {}^{3673}_{2737}$	8·8	$2\ 2720\ 4^{601}_{597}$
7·6	$25\ 0451\ {}^{6685}_{6059}$	8·9	$1\ 8601\ 939^{3}_{1}$
7·7	$20\ 5052\ {}^{4786}_{4366}$	9·0	$1\ 5229\ 979^{8}_{6}$
7·8	$16\ 7882\ {}^{7671}_{7389}$	9·1	$1\ 2469\ 252^{9}_{8}$

For higher values of x, $\sigma(x) = \epsilon^{-2x} = \kappa(x)$.

Remember that all these are natural, not common logarithms.

$\log_\epsilon$ Cot x to *Sixteen* Decimals. *Primary* Anticyclics with *Natural* logarithms.

x		x	
1·0	·2723 4146 8911 8315	4·0	6 7092 5280 9725
1·1	·2225 1996 8580 9836	4·1	5 4930 7153 7565
1·2	·1819 3610 2676 3487	4·2	4 4973 4655 9379
1·3	·1488 2122 0791 0832	4·3	3 6821 1591 4954
1·4	·1217 7037 0292 4077	4·4	3 0146 6152 8741
1·5	996 5653 2516 4435	4·5	2 4681 9609 4264
1·6	815 6960 5515 2891	4·6	2 0207 8804 3619
1·7	667 7133 6706 0447	4·7	1 6544 8131 4906
1·8	546 6105 0658 7625	4·8	1 3545 7473 1887
1·9	447 4900 9610 8129	4·9	1 1090 3198 9781
2·0	366 3537 4743 6963	5·0	9079 9859 5874
2·1	299 9340 1954 4626	5·1	7434 0637 4024
2·2	245 5591 3651 8595	5·2	6086 4966 0356
2·3	201 0434 8617 9652	5·3	4983 2019 4733
2·4	164 5986 5706 7208	5·4	4079 9006 8281
2·5	134 7609 7938 6066	5·5	3340 3401 4355
2·6	110 3324 0765 4006	5·6	2734 8392 1331
2·7	90 3322 3309 9827	5·7	2239 0969 6861
2·8	73 9576 1143 4507	5·8	1833 2175 4729
2·9	60 5512 7991 4083	5·9	1500 9115 8305
3·0	49 5751 4506 6900	6·0	1228 8424 7067
3·1	40 5886 6844 8652	6·1	and upwards $= 2\epsilon^{-2x}$.
3·2	33 2311 7604 4747		
3·3	27 2073 7753 4301		
3·4	22 2755 1216 7787		
3·5	18 2376 4436 6132		
3·6	14 9317 1894 1799		
3·7	12 2250 5674 5141		
3·8	10 0090 2950 4405		
3·9	8 1947 0003 8179		

To adapt Legendre's Elliptic scale for a rapid calculation of

$$\int_0 \frac{d\omega}{\sqrt{(1-c^2 \sin^2 \omega)}},$$

when ω is given, and the constant

$$\rho = \tfrac{1}{2}\pi \cdot \frac{F(b, \tfrac{1}{2}\pi)}{F(c, \tfrac{1}{2}\pi)};$$

[of which a good table *might be calculated with argument* γ $(c = \sin\gamma)$ *from Legendre's own work;* where, if $c, c_1, c_2 \dots c_n$ are formed on *Lagrange's* scale, $\rho = 2^{-n} \cdot \log_\epsilon \cdot \dfrac{4}{c_n}$, with any large value for n, but $n = 4$, suffices at worst]: it next is requisite for the use of the equation

$$\tan \tfrac{1}{2}(\omega_1 - \omega) = \Delta(c, \beta) \cdot \tan\omega$$

to calculate $\Delta(c, \beta)$ in *Legendre's* scale, with $F(c, \beta) = \tfrac{2}{3}F(c, \tfrac{1}{2}\pi)$ for the definition of β. Put $\phi(\rho)$ as equivalent to $-\log_{10}\Delta(c, \beta)$; where $\Delta(c, \beta) = \sqrt{(1-c^2\sin^2\beta)}$, then

$$-\tfrac{1}{3}\log_\epsilon \cdot \Delta(c, \beta) = \wp(2\rho) + \tfrac{1}{5}\wp(10\rho) + \tfrac{1}{7}\wp(14\rho) + \tfrac{1}{11}\wp(22\rho) + \&c.,$$

in which each term is of the form

$$\frac{1}{2n-1} \cdot \wp(\overline{4n-2} \cdot \rho):$$

but every term in which $(2n-1)$ divides by 3 is excluded.

Then, if ρ and ω are given, the following table of $\phi(\rho)$ enables you to calculate $F(c, \omega)$ much more rapidly than by Lagrange's scale.

Values of $\phi(\rho) = -\log_{10}\sqrt{(1-c^2\sin^2\beta)}$ in Legendre's Elliptic scale.

ρ		ρ	
1·0	3592 5572 9411 6719	4·0	8 7413 7507 5280
1·1	2923 2484 2664 4562	4·1	7 1568 3233 2307
1·2	2383 5463 1860 7711	4·2	5 8595 1857 9358
1·3	1946 1441 0045 9939	4·3	4 7973 6797 9182
1·4	1590 4540 8993 1846	4·4	3 9277 5265 4496
1·5	1300 5603 0350 5642	4·5	3 2157 7186 4632
1·6	1063 9363 9293 4314	4·6	2 6318 5130 7237
1·7	0870 5994 8696 7943	4·7	2 1555 9632 6260
1·8	0712 5245 5310 8973	4·8	1 7648 5299 9547
1·9	0583 2220 7250 4299	4·9	1 4449 3942 3209
2·0	0477 4230 2244 3056	5·0	1 1830 1634 0904
2·1	0390 8376 6653 7070	5·1	9685 7005 9051
2·2	0319 9670 8941 8197	5·2	7929 9956 7201
2·3	0261 9538 7599 8236	5·3	6492 5313 2650
2·4	0214 4625 4932 5163	5·4	5315 6350 6124
2·5	0175 5651 6359 9217	5·5	4352 0738 9615
2·6	0142 8531 8618 4305	5·6	3563 1767 3812
2·7	0117 6939 6969 2738	5·7	2917 2823 7398
2·8	0096 3590 2604 9745	5·8	2388 4687 9488
2·9	0078 8917 4230 4561	5·9	1955 5128 5509
3·0	0064 6209 0040 0540	6·0	1601 0385 1246
3·1	0052 8824 4940 6155	6·1	1301 8194 6701
3·2	0043 2964 2883 7997	6·2	1073 2082 0936
3·3	0035 5301 5415 2412	6·3	878 6685 6544
3·4	0029 0224 2004 9576	6·4	719 3926 7729
3·5	0022 4736 5465 9947		
3·6	0019 4542 9665 4372		
3·7	0017 2307 1146 9322		
3·8	0013 0406 0102 6271		
3·9	0010 6767 4021 7259		

For higher values of ρ, $\phi(\rho) = 6\epsilon^{-2\rho}$, multiplied by *modulus* of common logarithms. This at least shows a new *possible method.*

Carefully as I have worked at this table for $\phi(\rho)$ I must confess that I myself distrust it, because I have no check on error, and am sadly aware how a tired brain may blunder.

Secondary Anticyclics.

Summary, repeated for compactness.

Calling attention to the *capitals* in Sin, Cos, I use ק ס for reciprocals of Sin and Cos: so כ for Cot − 1, and ת for 1 − Tan. But Elliptic Integrals suggest other combinations not unimportant, to denote which I use other Hebrew letters.

In Elliptics we have $b^2 + c^2 = 1$; I put $\frac{1}{2}\pi \,.\, C$ [*not*, as Legendre, mere C,] for

$$\int_0^{\frac{1}{2}\pi} \frac{d\omega}{\sqrt{(1 - c^2 \sin^2 \omega)}};$$

and by B I mean the same function of b which C is of c. It is convenient to call c *modulus*, b *submodulus; C* the *modular*, B the *submodular*, and to assume $\rho = \frac{1}{2}\pi \,.\, \frac{B}{C}$ for our chief constant.

Then it is allowable to assume eight *Secondary* Anticyclics defined by eight Series, with l for Nap. log.

Put

ל (ρ) for $\log \text{Cot}\, \rho + l \,\text{Cot}\, 3\rho + l \,\text{Cot}\, 5\rho + \text{\&c.}$ Hebrew *Lamda.*

מ (ρ) for $l \,\text{Cot}\, \rho - l \,\text{Cot}\, 2\rho + l \,\text{Cot}\, 3\rho - l \,\text{Cot}\, 4\rho + \text{\&c.}$ Hebrew *Māim.*

ר (ρ) for ס (2ρ) + ס (4ρ) + ס (6ρ) + ס (8ρ) + &c. Hebrew *Reish.*

ח (ρ) for ס (2ρ) − ס (4ρ) + ס (6ρ) − &c. Hebrew *Khai.*

ז (ρ) for $\frac{1}{2}$כ (2ρ) + $\frac{1}{4}$כ (4ρ) + $\frac{1}{6}$ (6ρ) + &c. Hebrew *Zāin.*

צ (ρ) for ת (ρ) − $\frac{1}{2}$ת (2ρ) + $\frac{1}{3}$ת (3ρ) − $\frac{1}{4}$ת (4ρ) + &c. Hebrew *Tsaddi.*

פ (ρ) for ת (ρ) + $\frac{1}{3}$ת (2ρ) + $\frac{1}{3}$ת (3ρ) + &c. Hebrew *Pai.*

שׁ (ρ) for ס (ρ) + ס (3ρ) + ס (5ρ) + &c. Hebrew *Shin.*

Then in Elliptics it is known that

$$\text{ל}(\rho) = \tfrac{1}{4}\log\left(\frac{1}{b}\right);\quad \text{מ}(\rho) = \tfrac{1}{2}\log C;$$

$$\text{ר}(\rho) = \tfrac{1}{2}(C-1);\quad \text{ת}(\rho) = \tfrac{1}{2}(1 - Cb);$$

also $$\text{ז}(\rho) = \log Q,$$

if Q^{-1} stand for $$(1-q^2)(1-q^4)(1-q^6)(1-q^8)\ldots$$

where $$q = \epsilon^{-2\rho}.$$

Next $$\text{צ}(\rho) = \tfrac{1}{2}\left(\log\frac{4}{c} - \rho\right) \text{ and } \text{שׂ}(\rho) = \tfrac{1}{2}Cc.$$

Perhaps it is well to add a 9th function,

$$\text{נ}(\rho) = l\,\text{Cot}\,\rho + l\,\text{Cot}\,2\rho + l\,\text{Cot}\,3\rho + \&\text{c.}\quad \text{(Hebrew } \textit{Nun.}\text{)}$$

Of these, ל מ ת ר ז and צ are of the most obvious use in Elliptics, but to compute pairs may be as easy as to compute a single series, and the use of some of these may in the future be greater than we yet know. Their remarkable relations are elsewhere shown.

I have calculated פ and צ in pairs from the equations with $\text{מ}(\rho)$ previously known; working backward from the known fact,

$$\begin{cases} \text{פ}(\rho) = \text{מ}(\rho) + \tfrac{1}{2}\text{פ}(2\rho) \\ \text{צ}(\rho) = \text{מ}(\rho) - \tfrac{1}{2}\text{פ}(2\rho) \end{cases}$$

that when ρ is as large as 4·7, $2\epsilon^{-4\rho}$ is negligible, which permits us in in the last equations to substitute at first $\epsilon^{-4\rho}$ for their last term. Obviously, $\text{ת}(\rho) = \text{ר}(\rho) - 2\,\text{ר}(2\rho)$.

$$ל\rho = l\,\mathrm{Cot}\,\rho + l\,\mathrm{Cot}\,3\rho + l\,\mathrm{Cot}\,5\rho + \&c. = ק(2\rho) + \tfrac{1}{3}ק(6\rho) + \tfrac{1}{5}ק(10\rho) + \&c.$$

ρ	ל(ρ) to 16 decimals.	ρ	ל(ρ) to 16 decimals.
1·0	·2773 9147 7363 8088	4·0	6 7092 5356 4751
1·1	·2252 7452 4938 4952	4·1	5 4930 7195 1933
1·2	·1834 4166 4965 0481	4·2	4 4973 4678 6788
1·3	·1496 4523 6530 5588	4·3	3 6821 1603 9760
1·4	·1222 2177 4178 3692	4·4	3 0146 6159 7235
1·5	·0999 0396 5450 7918	4·5	2 4681 9613 1854
1·6	817 0528 8433 4168	4·6	2 0207 8806 4249
1·7	668 4576 0234 5234	4·7	1 6544 8132 6238
1·8	547 0188 0148 3081	4·8	1 3545 7473 8101
1·9	447 7141 1791 7024	4·9	1 1090 3199 3191
2·0	366 4766 7292 0334	5·0	9079 9859 7738
2·1	300 0014 7501 7840	5·1	7434 0637 5050
2·2	245 5961 5412 2330	5·2	6086 4966 0920
2·3	201 0638 0086 1661	5·3	4983 2019 5043
2·4	164 6098 0562 9670	5·4	4079 9006 8451
2·5	134 7670 9771 0234	5·5	3340 3401 4447
2·6	110 3357 6541 1251	5·6	2734 8392 1381
2·7	90 3340 7585 7584	5·7	2239 0969 6889
2·8	70 9586 2275 4605	5·8	1833 2175 4745
2·9	60 5518 3493 5819	5·9	1500 9115 8313
3·0	49 5754 4966 8365	6·0	1228 8424 7071
3·1	40 5888 3561 7142	6·1	1006 0910 6144
3·2	33 2312 6778 8627	6·2	823 7177 4153
3·3	27 2074 2788 4367	6·3	674 4030 4684
3·4	22 2755 3980 0473		
3·5	18 2376 5953 1272		
3·6	14 9317 2726 4598		
3·7	12 2250 6132 2788		
3·8	10 0090 3201 1183		
3·9	8 1947 0141 3927		

When ρ exceeds 6·3, ל$(\rho) = 2\epsilon^{-2\rho}$, correct to sixteen decimals.

In Elliptics ל$(\rho) = \frac{1}{4}\log_\epsilon\left(\frac{1}{b}\right)$.

$$\text{מ}(\rho) = \tfrac{1}{2}\log C = l\operatorname{Cot}\rho - l\operatorname{Cot}2\rho + l\operatorname{Cot}3\rho - l\operatorname{Cot}4\rho + \&c.$$

ρ	$\text{מ}(\rho)$ to 16 decimals.	ρ	$\text{מ}(\rho)$ to 16 decimals.
1·0	·2400 7265 9644 8655	4·0	6 7070 0286 1003
1·1	·2004 1339 7926 0180	4·1	5 4915 6326 0128
1·2	·1668 4521 7753 8973	4·2	4 4963 3549 0466
1·3	·1385 5079 3171 0697	4·3	3 6814 3814 0873
1·4	·1147 9856 3137 1366	4·4	3 0142 0718 8028
1·5	949 3413 1983 7457	4·5	2 4678 9153 2257
1·6	783 7664 0094 1658	4·6	2 0205 8388 5033
1·7	646 1572 5261 2385	4·7	1 6543 4446 0799
1·8	532 0759 2640 4883	4·8	1 3544 8299 4466
1·9	437 7000 7687 1844	4·9	1 1089 7049 5594
2·0	359 7651 6865 2436	5·0	9079 5737 4674
2·1	295 5031 1692 4731	5·1	7433 7874 2398
2·2	242 5810 3811 9869	5·2	6086 3113 8200
2·3	199 0428 0861 8196	5·3	4983 0777 8893
2·4	163 2551 3914 8935	5·4	4079 8174 5658
2·5	133 8590 5788 9429	5·5	3340 2843 5510
2·6	109 7270 9722 7611	5·6	2734 8018 1709
2·7	89 9260 7746 6340	5·7	2239 0719 0111
2·8	73 6851 3509 3551	5·8	1833 2007 4400
2·9	60 3685 1150 0730	5·9	1500 9003 1946
3·0	49 4525 6466 6268	6·0	1228 8349 2045
3·1	40 5064 6350 3736	6·1	1006 0860 6033
3·2	33 1760 5218 4752	6·2	823 7143 3896
3·3	27 1704 1579 1913	6·3	674 4007 7274
3·4	22 2507 2986 8097		
3·5	18 2210 2894 3054		
3·6	14 9205 7945 0998		
3·7	12 2175 8871 1236		
3·8	10 0040 2297 7185		
3·9	8 1913 4375 8304		

When ρ exceeds 6·3, $\text{מ}(\rho) = \text{ח}(\rho)$, which is given in a table above.

Also, $\text{מ}(\rho) = \text{ל}(\rho) - \text{ל}(2\rho) - \text{ל}(2^2\rho) - \text{ל}(2^3\rho) - \&c.$

If we begin calculation from highest value of ρ we may deduce both מ and נ from a table of $\text{ל}(\rho)$ by the formulas

$$\left.\begin{aligned} \text{נ}(\rho) &= \text{ל}(\rho) + \text{נ}(2\rho) \\ \text{מ}(\rho) &= \text{ל}(\rho) - \text{נ}(2\rho) \end{aligned}\right\},$$

but then errors accumulate.

$$\daleth(\rho) = \tfrac{1}{2}(C-1) = D(2\rho) + D(4\rho) + D(6\rho) + \&c.$$

ρ	$\daleth(\rho)$ to 16 decimals.	ρ	$\daleth(\rho)$ to 16 decimals.
1·0	·3081 5463 3020 9416	4·0	6 7115 0326 1798
1·1	·2465 2932 1077 3797	4·1	5 4945 8009 1205
1·2	·1980 5544 5157 0211	4·2	4 4983 5778 9895
1·3	·1596 5019 3564 1257	4·3	3 6827 9377 2241
1·4	·1290 4652 7589 8850	4·4	3 0151 1591 1118
1·5	·1045 4515 3202 6148	4·5	2 4685 0067 9332
1·6	848 5349 4204 8395	4·6	2 0209 9221 5958
1·7	689 7643 6807 1977	4·7	1 6546 1817 6544
1·8	561 4179 2212 6355	4·8	1 3546 6647 3450
1·9	457 4296 9893 7954	4·9	1 1090 9348 6242
2·0	373 0243 6348 2641	5·0	9080 3981 8322
2·1	304 4099 2452 9691	5·1	7434 3400 6334
2·2	248 5619 2513 0357	5·2	6086 6818 2888
2·3	203 0577 1121 3383	5·3	4983 3261 0780
2·4	165 9496 0896 3902	5·4	4079 9839 1018
2·5	135 6669 8035 5560	5·5	3340 3959 4740
2·6	109 7159 0492 2469	5·6	2734 8766 0986
2·7	90 7396 1731 8453	5·7	2239 1220 3642
2·8	74 2307 6201 8311	5·8	1833 2343 5076
2·9	60 7344 1835 2760	5·9	1500 9228 4670
3·0	49 6979 2853 9161	6·0	1228 8500 2092
3·1	40 6709 8484 0696	6·1	1006 0961 8254
3·2	33 2863 6106 7867		
3·3	27 2443 7284 3596		
3·4	22 3003 1288 9340		
3·5	18 2542 6989 9316		
3·6	14 9428 6398 1143		
3·7	12 2325 2780 4145		
3·8	10 0140 3770 2810		
3·9	8 1980 5723 5221		

When ρ reaches 6·2, $\daleth(\rho) = 2(\epsilon^{-2\rho} + \epsilon^{-4\rho})$. Indeed when ρ is > 3, $\beth_0(3\rho) = 0$; therefore $\daleth(\rho) = D(2\rho) + \beth_0(\rho)$.

Among Jacobian Elliptic functions we have

$$Q^{-1} = (1 - q^2)(1 - q^4)(1 - q^6)(1 - q^8)\ldots$$

moreover $$q = \epsilon^{-2\rho}.$$

Put then $$\log Q = \text{ז}(\rho),$$

whence if $$\sigma(\rho) = -\log(1 - \epsilon^{-2\rho}),$$

$$\text{ז}(\rho) = \sigma(2\rho) + \sigma(4\rho) + \sigma(6\rho) + \&\text{c.} = \tfrac{1}{2}\beth(2\rho) + \tfrac{1}{4}\beth(4\rho) + \tfrac{1}{6}\beth(6\rho) + \&\text{c.}$$

ρ	ז (ρ) or log Q to 16 decimals.	(ρ)	ז (ρ)
1·0	·0188 2722 4599 9831	3·0	614 4268 9795
1·1	·0125 0594 7085 8422	3·1	411 8614 1517
1·2	83 3209 1414 3569	3·2	276 0784 0048
1·3	55 6243 8615 8715	3·3	185 0606 3350
1·4	37 1844 2759 0558	3·4	124 0497 3882
1·5	24 8798 8868 0132	3·5	83 1529 7562
1·6	16 6570 4561 6278	3·6	55 7390 8353
1·7	11 1563 7735 3365	3·7	37 3630 1474
1·8	7 4742 2449 3276	3·8	25 0451 7312
1·9	5 0082 7278 1246	3·9	16 7882 7953
2·0	3 3563 1481 0218	4·0	11 2535 1937
2·1	2 2494 3187 3138	4·1	7 5434 5920
2·2	1 5076 7160 5524	4·2	5 0565 3168
2·3	1 0105 4716 6536	4·3	3 3894 9449
2·4	6773 5617 6775	4·4	2 2720 4606
2·5	4540 3022 6177	4·5	1 5299 9800
2·6	3043 3872 2501	4·6	1 0208 9608
2·7	2040 0127 6319		
2·8	1367 4476 5444		
2·9	916 6213 7641		

When ρ exceeds 4·6, ז$(\rho) = \epsilon^{-4\rho}$.

$$\sqcap(\rho) = \mathsf{D}(2\rho) - \mathsf{D}(4\rho) + \mathsf{D}(6\rho) - \mathsf{D}(8\rho) + \&c.$$
$$= \mathsf{D}(2\rho) - \sqcap_0(\rho) + \sqcap_0(3\rho) - \sqcap_0(5\rho) + \&c.$$

ρ	$\sqcap(\rho)$ to 16 decimals.	ρ	$\sqcap(\rho)$ to 16 decimals.
1·0	·2335 4976 0324 4134	4·0	6 7070 0185 4302
1·1	·1968 1693 6051 3083	4·1	5 4915 6270 7637
1·2	·1648 6552 3364 2407	4·2	4 4963 3517 7255
1·3	·1325 1679 7493 0137	4·3	3 6814 3797 4465
1·4	·1142 0037 5186 2228	4·4	3 0142 0709 2706
1·5	·0946 0556 7494 7826	4·5	2 4678 9148 0136
1·6	781 9622 1991 2661	4·6	2 0205 8385 7526
1·7	645 1637 4229 3297	4·7	1 6543 4444 5703
1·8	539 5321 9416 4069	4·8	1 3544 8298 6182
1·9	437 4016 2353 2334	4·9	1 1089 7049 1047
2·0	359 6013 5695 9045	5·0	9079 5737 2178
2·1	295 4132 0894 9901	5·1	7433 7874 1030
2·2	242 5316 9330 8121	5·2	6086 3113 7448
2·3	199 0157 2678 1467	5·3	4983 0777 8480
2·4	163 2402 7601 7002	5·4	4079 8174 5430
2·5	133 8509 0071 8916	5·5	3340 2843 6870
2·6	108 3985 6855 6693	5·6	2734 8018 1642
2·7	89 9236 2053 6417	5·7	2239 0719 0074
2·8	73 6837 8669 6339	5·8	1833 2007 4380
2·9	60 3677 7148 2608	5·9	1500 9003 1934
3·0	49 4421 5853 4977	6·0	1228 8349 2010
3·1	40 5062 4061 3882	6·1	1006 0860 6033
3·2	33 1759 2986 0107	6·2	823 7143 4893
3·3	27 1703 4865 8688	6·3	674 4007 7272
3·4	22 2506 9302 4588		
3·5	18 2210 0872 5774		
3·6	14 9205 6835 3943		
3·7	12 2175 8260 1041		
3·8	10 0040 1963 4814		
3·9	8 1913 4192 3973		

When ρ reaches 6·2, $\sqcap(\rho) = 2\,(\epsilon^{-2\rho} - \epsilon^{-4\rho})$.

ש̇ (ρ) = ם (ρ) + ם (3ρ) + ם (5ρ) + ם (7ρ) + &c. may be developed in powers of $\epsilon^{-\rho}$; then $\tfrac{1}{2}$ש̇ $(\rho) = \epsilon^{-\rho} + 2\epsilon^{-5\rho} + \epsilon^{-9\rho} + 2^{-13\rho} + 2\epsilon^{-17\rho} + 2\epsilon^{-21\rho}$ if the rest may be neglected.

ρ	ש̇ (ρ) to 16 decimals.	ρ	ש̇ (ρ) to 16 decimals.
1·0	·7629 6669 6946 8796	4·0	366 3128 6022 0232
1·1	·6821 9209 6819 4580	4·1	331 4535 5804 1340
1·2	·6123 4490 8595 9986	4·2	299 9115 6673 9794
1·3	·5510 9411 4501 6602	4·3	271 3711 9864 0438
1·4	·4968 4824 9837 3553	4·4	245 5468 0921 9239
1·5	·4484 7541 3322 3887	4·5	222 1799 3753 2438
1·6	·4051 3600 4992 3565	4·6	201 0367 1899 7424
1·7	·3661 8137 5828 2675	4·7	181 9055 4452 3572
1·8	·3310 9160 0206 9033	4·8	164 5949 4249 0452
1·9	·2994 3672 0759 2514	4·9	148 9316 6233 4377
2·0	·2708 5219 6672 6783	5·0	134 7589 4053 7225
2·1	·2450 2301 4693 0188	5·1	121 9349 3164 8248
2·2	·2216 7312 8564 3448	5·2	110 3312 8861 9581
2·3	·2005 5820 9867 4902	5·3	99 8318 7826 2156
2·4	·1814 6048 4260 6303	5·4	90 3316 1892 7435
2·5	·1641 8490 4198 8994	5·5	81 7354 2881 4883
2·6	·1485 5619 7883 8791	5·6	73 9572 7435 7314
2·7	·1344 1650 9371 7815	5·7	66 9193 0916 6203
2·8	·1216 2345 1388 0546	5·8	60 5510 9491 7692
2·9	·1100 4845 7512 7106	5·9	54 7888 9638 1541
3·0	·0995 7536 0348 7687	6·0	49 5750 4353 7069
3·1	900 9914 6945 1892		
3·2	815 2485 8098 0526		
3·3	737 6660 7826 8674		
3·4	667 4670 5518 2637		
3·5	603 9486 7284 6448		
3·6	546 4750 5814 5211		
3·7	494 4708 9890 4844		
3·8	447 4156 6123 5194		
3·9	404 8383 6484 6812		

Observe also that

ש̇ (ρ) = ם (ρ)
+ ק$_0(\rho)$ − ק$_0(3\rho)$
+ ק$_0(5\rho)$ − ק$_0(7\rho)$
+ ק$_0(9\rho)$ − &c.

Since $\tfrac{1}{2}Cc$ = ש̇ (ρ),

and $\tfrac{1}{2}C$ = ד $(\rho) + \tfrac{1}{2}$;

∴ c = ש̇ (ρ) / ($\tfrac{1}{2}$ + ד (ρ)).

When ρ is > 6, ש̇ $(\rho) = 2\epsilon^{-\rho}$ true to sixteen decimals.

If any diligent reader seek to test these small tables, (which the compiler naturally desires,) he may sometimes complain of inability to continue them beyond the highest value of ρ. That all may, on this scale, be complete within these covers, a skeleton table of $\epsilon^{-\rho}$ is here added, which has already, under the title of ϵ^{-x}, appeared in the *Cambridge Philosophical Transactions*, Vol. III. Part III. To obtain 16 decimals, in working for other results, 18 decimals are here given, though the two last cannot be trusted.

ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$
·1	·9048 3741 80359 59545	3·1	450 4920 23935 57806
·2	·8187 3075 30779 81848	3·2	407 6220 39783 66216
·3	·7408 1822 06817 17871	3·3	368 8316 74012 40006
·4	·6703 2004 60356 39307	3·4	333 7326 99603 26081
·5	·6065 3065 97126 33423	3·5	301 0738 34223 18502
·6	·5488 1163 60940 26441	3·6	273 2372 24472 92561
·7	·4965 8530 37014 09523	3·7	247 2352 64703 39390
·8	·4493 2896 41172 21599	3·8	223 7077 18561 65595
·9	·4065 6965 97405 99120	3·9	202 4191 14458 04390
1·0	·3678 7944 11714 42321	4·0	183 1563 88887 34179
1·1	·3328 7108 36980 79553	4·1	165 7267 54017 61246
1·2	·3011 9421 19122 02096	4·2	149 9557 68204 77705
1·3	·2725 3179 30340 12603	4·3	135 6855 90122 00932
1·4	·2465 9696 39416 06475	4·4	122 7733 99030 68440
1·5	·2231 3016 01484 29829	4·5	111 0899 65382 42306
1·6	·2018 9651 79946 55407	4·6	100 5183 57446 33583
1·7	·1826 8352 40527 34648	4·7	90 9527 71016 95819
1·8	·1652 9888 82215 86535	4·8	82 2974 70490 20030
1·9	·1495 6861 92226 35054	4·9	74 4658 30709 24342
2·0	·1353 3528 32366 12691	5·0	67 3794 69990 85467
2·1	·1224 5642 82529 81909	5·1	60 9674 65655 15637
2·2	·1108 0315 83623 33881	5·2	55 1656 44207 60774
2·3	·1002 5884 37228 03731	5·3	49 9159 39069 10218
2·4	907 1795 32894 12500	5·4	45 1658 09426 12670
2·5	820 8499 86238 98791	5·5	40 8677 14384 64068
2·6	742 7357 83143 33876	5·6	36 9786 37164 82931
2·7	672 0551 27397 49761	5·7	33 4596 54574 71272
2·8	608 1006 26252 17961	5·8	30 2755 47453 75813
2·9	550 2322 00564 07225	5·9	27 3944 48187 68370
3·0	497 8706 83678 63943	6·0	24 7875 21766 66358

ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$
6·1	22 4286 77194 85802	9·6	6772 87364 90855
6·2	20 2943 06362 95735	9·7	6128 34950 53224
6·3	18 3630 47770 28910	9·8	5545 15994 32180
6·4	16 6155 72731 73937	9·9	5017 46820 56176
6·5	15 0343 91929 77572	10·0	4539 99297 62485
6·6	13 6036 80375 47893	10·1	4107 95552 25302
6·7	12 3091 19026 73481	10·2	3717 03186 84128
6·8	11 1377 51478 44802	10·3	3363 30951 85721
6·9	10 0778 54290 48510	10·4	3043 24830 08403
7·0	9 1188 19655 54515	10·5	2753 64493 49746
7·1	8 2510 49232 65905	10·6	2491 60097 31501
7·2	7 0658 58083 76681	10·7	2254 49379 13206
7·3	6 7553 87751 93846	10·8	2039 95034 11166
7·4	6 1125 27611 29574	10·9	1845 82339 95777
7·5	5 5308 43701 47832	11·0	1670 17007 90246
7·6	5 0045 14334 40611	11·1	1511 23238 19857
7·7	4 5282 71828 86790	11·2	1367 41960 65685
7·8	4 0973 49789 79781	11·3	1237 29242 61791
7·9	3 7074 35404 59080	11·4	1119 54848 42595
8·0	3 3546 26279 02501	11·5	1013 00935 98631
8·1	3 0353 91380 78857	11·6	916 60877 36245
8·2	2 7465 35699 72135	11·7	829 38191 60755
8·3	2 4851 68271 07947	11·8	750 45579 15075
8·4	2 2486 73241 78844	11·9	679 04048 07381
8·5	2 0346 83690 10644	12·0	614 42123 53327
8·6	1 8410 57936 67577	12·1	555 95132 41665
8·7	1 6658 58109 87632	12·2	503 04556 07114
8·8	1 5073 30750 95474	12·3	455 17444 63084
8·9	1 3638 89264 82008	12·4	411 85887 07538
9·0	1 2340 08040 86675	12·5	372 66531 72085
9·1	1 1166 58084 90111	12·6	337 20152 34153
9·2	1 0103 94018 37091	12·7	305 11255 58050
9·3	9142 42314 78171	12·8	276 07725 72053
9·4	8272 40655 56631	12·9	249 80503 25884
9·5	7485 18298 87702	13·0	226 03294 06997

ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$
13·1	204 52306 24491	16·6	6 17606 13351
13·2	185 06011 97553	16·7	5 58833 13920
13·3	167 44932 09446	16·8	5 05653 13478
13·4	151 51441 12156	16·9	4 57533 87708
13·5	137 09590 86393	17·0	4 13993 77202
13·6	124 04950 79965	17·1	3 74597 05575
13·7	112 24463 65241	17·2	3 38949 43271
13·8	101 56314 71020	17·3	3 06694 12954
13·9	91 89813 57913	17·4	2 77508 32429
14·0	83 15287 19119	17·5	2 51099 91571
14·1	75 23982 99227	17·6	2 27204 59942
14·2	68 07981 34408	17·7	2 05583 22310
14·3	61 60116 26191	17·8	1 86091 39278
14·4	55 73903 69323	17·9	1 68317 30706
14·5	50 43476 62588	18·0	1 52299 79752
14·6	45 63526 36810	18·1	1 37806 55555
14·7	41 29249 41607	18·2	1 24692 52791
14·8	37 36299 38007	18·3	1 12826 46525
14·9	33 80743 48400	18·4	1 02089 60750
15·0	30 59023 20519	18·5	92374 49702
15·1	27 67918 65864	18·6	83583 90136
15·2	25 04516 37241	18·7	75629 84148
15·3	22 66180 12790	18·8	68432 71049
15·4	20 50524 57575	18·9	61920 47706
15·5	18 55391 36271	19·0	56027 96459
15·6	16 78827 53003	19·1	50696 19869
15·7	15 19065 96759	19·2	45871 81754
15·8	13 74507 72802	19·3	41506 53683
15·9	12 43706 02371	19·4	37556 66761
16·0	11 25351 74726	19·5	33982 67815
16·1	10 18260 36937	19·6	30748 79877
16·2	9 21360 08336	19·7	27822 66367
16·3	8 33681 07883	19·8	25174 98715
16·4	7 54345 83479	19·9	22779 27037
16·5	6 82560 33757	20·0	20611 53619

ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$
20·1	18650 08918	23·6	563 18394	27·1	17 00667
20·2	16875 29854	23·7	509 58993	27·2	15 38828
20·3	15269 40156	23·8	461 09586	27·3	13 92387
20·4	13816 32570	23·9	417 21690	27·4	12 59884
20·5	12501 52863	24·0	377 51347	27·5	11 39991
20·6	11311 85098	24·1	341 58831	27·6	10 31506
20·7	10235 38612	24·2	309 08189	27·7	9 33346
20·8	9261 36038	24·3	279 66885	27·8	8 44526
20·9	8380 02554	24·4	253 05484	27·9	7 64157
21·0	7582 56070	24·5	228 97350	28·0	6 91435
21·1	6860 98471	24·6	207 18380	28·1	6 25638
21·2	6208 07569	24·7	187 46766	28·2	5 66101
21·3	5617 29937	24·8	169 62776	28·3	5 12231
21·4	5082 74242	24·9	153 48556	28·4	4 63485
21·5	4599 05558	25·0	138 87944	28·5	4 19376
21·6	4161 39757	25·1	125 66332	28·6	3 79466
21·7	3765 38823	25·2	113 70489	28·7	3 43356
21·8	3407 06418	25·3	102 88446	28·8	3 10681
21·9	3082 83916	25·4	93 09369	28·9	2 81116
22·0	2789 46822	25·5	84 23462	29·0	2 54364
22·1	2524 01519	25·6	76 21864	29·1	2 30158
22·2	2283 82340	25·7	68 96548	29·2	2 08255
22·3	2066 48887	25·8	62 40260	29·3	1 88442
22·4	1869 83647	25·9	56 46419	29·4	1 70511
22·5	1691 89802	26·0	51 09089	29·5	1 54280
22·6	1530 89264	26·1	46 22895	29·6	1 39598
22·7	1385 20895	26·2	41 82968	29·7	1 26313
22·8	1253 38887	26·3	37 84905	29·8	1 14293
22·9	1134 11313	26·4	34 27424	29·9	1 03418
23·0	1026 18800	26·5	30 98820	30·0	93576
23·1	928 53333	26·6	28 03927	30·1	84671
23·2	840 17171	26·7	25 37102	30·2	76612
23·3	760 21882	26·8	22 95663	30·3	69323
23·4	687 87436	26·9	20 72200	30·4	62725
23·5	622 41450	27·0	18 79528	30·5	56757

ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$	ρ	$\epsilon^{-\rho}$
30·6	51356	33·1	4215	35·1	571
30·7	46469	33·2	3812	35·2	517
30·8	42047	33·3	2450	35·3	467
30·9	38044	33·4	3122	35·4	423
31·0	34424	33·5	2825	35·5	383
31·1	31149	33·6	2556	35·6	346
31·2	28184	33·7	2313	35·7	313
31·3	25502	33·8	2093	35·8	283
31·4	23075	33·9	1894	35·9	256
31·5	20878	34·0	1715	36·0	232
31·6	18891	34·1	1552	36·1	210
31·7	17094	34·2	1404	36·2	190
31·8	15466	34·3	1270	36·3	172
31·9	13995	34·4	1150	36·4	156
32·0	12662	34·5	1040	36·5	141
32·1	11460	34·6	941	36·6	128
32·2	10366	34·7	852	36·7	116
32·3	9381	34·8	771	36·8	105
32·4	8487	34·9	698	36·9	95
32·5	7680	35·0	631	37·0	86
32·6	6949				
32·7	6288				
32·8	5689				
32·9	5149				
33·0	4658				

THE END.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SONS, AT THE UNIVERSITY PRESS.