Ebenezer Andrews

MATHEMATICS,

COMPILED

FROM THE BEST AUTHORS,

AND INTENDED TO BE THE

Text-Book

OF THE

COURSE OF PRIVATE LECTURES ON THESE SCIENCES

IN THE

UNIVERSITY AT CAMBRIDGE.

SECOND EDITION.

BY SAMUEL WEBBER, D. D. A. A. et S. P. A. SOC.

PRESIDENT OF THE UNIVERSITY AT CAMBRIDGE.

IN TWO VOLUMES—VOL. I.

CAMBRIDGE.

PRINTED AT THE UNIVERSITY PRESS,

BY WILLIAM HILLIARD.

1808.

ADVERTISEMENT.

THE design, in making this Compilation, is to collect suitable exercises to be performed by the Classes at the private Lectures on Mathematics, given in the University. The view, which the CORPORATION had of the great advantages, that the Students might derive from a judicious work of the kind, produced this attempt to promote their improvement. The parts of the most approved writings, selected for the purpose, are copied, with only such alterations, as appeared to be useful. The Authors of the principal part of most of the branches are Dr. HUTTON and Mr. BONNYCASTLE; the *Navigation* is principally from that of Mr. NICHOLSON; and much use has been made of a *Manuscript*, containing certain mathematical exercises to be performed by the Students, MOLE's *Algebra*, the *Works* of EMERSON, CRAKELT's *translation of* MAUDUIT's *Treatise on Spheric Trigonomety*, and KELLY's *Spherics*.

ADVERTISEMENT

Respecting the Second Edition.

In compiling this work, when I was Hollis Professor of Mathematics and Natural Philosophy, such parts of the most approved writings on the several branches of Mathematics, contained in it, as seemed best adapted to the purpose, were selected; and such alterations and additions made, as appeared to be useful, or conducive to the accomplishment of the proposed object. Some variations from the first are to be found in this edition, which, having occurred to observation in using and reviewing the work, were deemed to be advantageous. To diminish the difference of the two volumes in bulk, the first is extended to the end of *Mensuration of Superficies*. Beside this the most material alterations in the first volume are reduction, in a few arithmetical articles, of matter merely practical, and unnecessary for the purpose of illustrating the principles; and the addition of Notes to most of the Geometrical Problems, containing the principles, on which their solutions respectively depend. Notice of the principal alterations in the second volume is prefixed to it. S. W.

Cambridge, *April* 18, 1808.

CONTENTS OF THE FRST VOLUME.

ARITHMETIC.

NOTATION 9
Simple Addition 14
Subtraction 17
Multiplication 19
Division 24
Reduction 34
Compound Addition 40
Subtraction 42
Multiplication 45
Division 48

DUODECIMALS 50

VULGAR FRACTIONS 53
Reduction of Vulgar Fractions 56
Addition of do. 65
Subtraction of do. 67
Multiplication of do. . . . . ibid
Division of do. 68

DECIMAL FRACTIONS 69
Addition of Decimals 70
Subtraction of do. 71
Multiplication of do. 72
Division of do. 74
Reduction of do. . . . . 7[illegible]

FEDERAL MONEY 83

CIRCULATING DECIMALS 86
Reduction of Circulating Decimals 87
Addition of do. 91
Subtraction of do. 93
Multiplication of do. . . . . ibid.
Division of do. 94

PROPORTION IN GENERAL 95
Simple Proportion, or Rule of Three 97
Practice 110
Tare and Trett 112
Compound Proportion 115
Conjoined do. 118
Barter . / Loss and Gain } in Notes 121
Single Fellowship . . . . . . ibid.
Double do. 124
Alligation Medial 126
Alternate 128
Involution 133
Evolution 136
Extraction of the Square Root 138
Cube Root 143
Cube Root by Approximation . 145
Roots of Powers in general . . 148
Do. by Approximation . . . 150
Arithmetical Progression 151
Geometrical do. 156
Simple Interest 164
by Decimals 166
Commission 168
Brokerage . . . . ibid.
Insurance 169
Discount 170
by Decimals 173

Equation of Payments 175
by Decimals . . . 177
Compound Interest 181
by Decimals 182
Annuities at Simple Interest . . - . 185
Compound do. 188
Value of an Annuity forever, at Compound Interest . 191
Value of an Annuity in Reversion, at Compound Interest 193
Single Position 194
Double do. 196
Permutation, Combination, and Composition of Quantities 198
Miscellaneous Questions 210

LOGARITHMS.

Logarithms 216
Computation of Logarithms 218
Description and Use of the Table of Logarithms . 222
Multiplication by Logarithms 226
Division by do. 228
Involution by do. 229
Evolution by do. 230

ALGEBRA.

Definitions and Notation 232
Addition 237
Subtraction 242
Multiplication 244
Division 251
Fractions 259
Involution 273
Evolution 277
Surds 280
Infinite Series 289
Simple Equations 295
Reduction of two, three, or more Simple Equations to one 301

Quadratic Equations 316
Cubic and Higher Equations 330

GEOMETRY.

Definitions. 340
Problems 349
Construction of the Plane Scale (Prob. L.) . . 380

PLANE TRIGONOMETRY.

Definitions 387
Problems 391

MENSURATION OF SUPERFICIES.

Problems relating to
the areas of right lined figures. . . . . . 406—425
certain lines belonging to a circle, its areas,
and the areas of parts of it . . 425—451
the areas of long irregular figures . 451
the circumferences and areas of the
Conic Sections . . . . . 452—end.

ARITHMETIC.

NUMBER is the abstract ratio of one quantity to another of the same kind, taken for unity.

Theoretic Arithmetic is the science of numbers.

Practical Arithmetic is the art of numbering.

In Arithmetic there are five principal or fundamental rules for its operations, namely, Notation, Addition, Subtraction, Multiplication, and Division.

NOTATION.*

Notation teaches how to read any proposed number, expressed in characters, and to write any proposed number in characters.

---

* As it is absolutely necessary to have a perfect knowledge of our excellent method of notation, in order to understand the reasoning made use of in the following Notes, I shall endeavour to explain it in as clear and concise a manner as possible.

1. It may then be observed, that the characters, by which all numbers are expressed, are these ten; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; 0 is called a *cypher*, and the rest, or rather all of them, are called *figures* or *digits*. The names and signification of these characters, and the origin or generation of the numbers they stand for, are as follow; 0 nothing; 1 one, or a single

1. *To read Numbers.*

RULE.

To the simple value of each figure join the name of its place, beginning at the left and reading toward the right.

EXAMPLES.

Read the following numbers.

37	30791	111000111
101	70079	1234567890
1107	3306677	10203040506070̇8090

---

thing, called an unit; 1 and 1 are 2 two; 2 and 1 are 3 three; 3 and 1 are 4 four; 4 and 1 are 5 five; 5 and 1 are 6 six; 6 and 1 are 7 seven; 7 and 1 are 8 eight; 8 and 1 are 9 nine; and 9 and 1 are ten, which has no single character; and thus by continual addition of one, all numbers are generated.

2. Beside the simple value of the figures, as above noted, they have each a local value according to the following law, namely, in a combination of figures, reckoning from right to left, the figure in the first place represents its primitive simple value; that in the second place, ten times its simple value; that in the third place, a hundred times its simple value, and so on; the value of the figure in each place being ten times the value of it in that immediately preceding it.

3. The names of the places are denominated according to their order. The first is called the place of units; the second, that of tens; the third, of hundreds; the fourth, of thousands; the fifth, of ten thousands; the sixth, of hundred thousands; the seventh, of millions, and so on. Thus, in the number 3456789; 9 in the first place signifies only nine; 8 in the second place signifies eight tens, or eighty; 7 in the third place is seven hundred; 6 in the fourth place is six thousand; 5 in the fifth place is fifty thousand; 4 in the sixth place is four hundred thousand; and 3 in the seventh place is three million; and the whole number is read thus, three million, four hundred and fifty six thousand, seven hundred and eighty nine.

11. *To write Numbers.*

RULE.

Write the figures in the same order as their values are expressed in, begining at the left, and writing toward the right; remembering to supply those places of the natural order with cyphers, which are omitted in the question.

---

4. A cypher, though it signifies nothing of itself, yet it occupies a place, and, when set on the right of other figures, increases their value like any other in a tenfold proportion; thus, 5 signifies only five; but 50, five tens or fifty; and 500, five hundred, &c.

5. For the more easy reading of large numbers, they are divided into periods, and half periods, each half period consisting of three figures; the name of the first period being units; that of the second, millions; of the third, billions; of the fourth trillions, &c. Also the first part of any period is the part of units; and the latter part, that of thousands.

The following Table contains a summary of the whole doctrine.

Periods.	Quadril.	Tril.	Billions.	Millions.	Units
Half Per.	th. un.	th. un.	th. un.	th. un.	cxt cxu
Figures.	123,456	789,098	765,432	101,234	567,800

A Synopsis of the Roman Notation.

1=I
2=II As often as any character is repeated, so many
3=III times is its value repeated.
4=IIII or IV A less character before a greater diminishes its value.
5=V
6=VI A less character after a greater increases its value.
7=VII
8=VIII
9=IX

EXAMPLES.

Write in figures the following numbers.

Eighty one. Two hundred and eleven. One thousand and thirty nine. A million and a half. A hundred and four score and five thousand. Eleven thousand million, eleven hundred thousand and eleven. Thirteen billion, six hundred thousand million, four thousand and one.

EXPLANATION OF CHARACTERS.

NOTE. It may be proper to explain here certain *signs*, used in this work.

= SIGNIFIES *equality* ; as 20 shillings = 1 pound signifies, that 20 shillings are equal to one pound.

+ Signifies *plus*, or *addition* ; as, 4 + 2 = 6.

— Signifies *minus*, or *subtraction* ; as, 6—2=4.

× or., *Into*, signifies *multiplication* ; as, 3 × 2 or 3·2=6.

÷ *By*, or) (signifies *division* ; as, 6 ÷ 2 = 3, or 2)6(3.

Division may also be denoted by placing the dividend over a line, and the divisor under it ; thus $\frac{6}{2}$=6÷2=3.

10=X	
50=L	
100=C	
500=D or IↃ	For every Ↄ affixed, this becomes 10 times as many.
1000=M or CIↃ	For every C and Ↄ, put one at each end,
2000=MM	it becomes ten times as much.
5000=IↃↃ : or $\overline{\text{V}}$	A line over any number increases it
6000=$\overline{\text{VI}}$	1000 fold.
10000=$\overline{\text{X}}$ or CCIↃↃ	
50000=IↃↃↃ	
60000=$\overline{\text{LX}}$	
100000=$\overline{\text{C}}$ or CCCIↃↃↃ	
1000000=$\overline{\text{M}}$ or CCCCIↃↃↃↃ	
2000000=$\overline{\text{MM}}$	
&c. &c.	

.. : : .. Signifies *arithmetical proportion*; thus 2 .. 4 :: 6 .. 8; here the meaning is, that $4-2=8-6=2$.

: : : : Signifies *geometrical proportion*; thus 2 : 4 : : 3 : 6, which is to be read, as 2 to 4, so is 3 to 6.

$\div$ Signifies *continual arithmetical proportion*, or *arithmetical progression*; thus 2 .. 4 .. 6 .. 8 $\div$ signifies, that 2, 4, 6, and 8 are in arithmetical progression.

∺ Signifies *continual geometrical proportion*, or *geometrical progression*; thus, 2 : 4 : 8 : 16 ∺ signifies, that 2, 4, 8, 16 are in geometrical progression.

∴ Signifies *therefore*.

$\rceil^2$ Signifies *the second power*, or *square*; thus, $x\rceil^2$ signifies the square of x.

$\rceil^3$ Signifies *the third power*, or *cube*.

$\rceil^n$ Signifies *any power*.

$\surd$, or $\rceil^{\frac{1}{2}}$, Signifies *the square root*; thus $\surd x$, or $x\rceil^{\frac{1}{2}}$ signifies the square root of x.

$^3\surd$, or $\rceil^{\frac{1}{3}}$, Signifies *the cube root*.

$^n\surd$, or $\rceil^{\frac{1}{n}}$, Signifies *any root*.

$\rceil^{\frac{m}{n}}$ Signifies *any root of any power*.

The number, or letter, belonging to the above signs of powers and roots, is called *the index*, or *exponent*.

A line, called a *vinculum*, drawn over several numbers, signifies, that the numbers under it are to be considered *jointly*; thus, $20-\overline{7+8}=5$; but without the vinculum, $20-7+8=21$. The same thing is also sometimes expressed by a parenthesis, inclosing two or more numbers or quantities; thus, $20-(7+8)=5$.

Two or more letters, joined together like those of a word, signify, that the numbers, which they represent, are to be multiplied together; thus, $a\,b=a\times b$; and $abc=a\times b\times c$.

SIMPLE ADDITION.

Simple Addition teaches to collect several numbers of the same denomination into one number, called the *sum.*

RULE.*

1. Place the numbers under each other, so that units may stand under units, tens under tens, &c. and draw a line under them.

---

* This rule, as well as the method of proof, is founded on the known axiom, " the whole is equal to the sum of all its parts." All, that requires explaining, is the method of placing the numbers, and carrying for the tens, both which are evident from the nature of notation. For any other disposition of the numbers would entirely alter their value ; and carrying one for every ten, from an inferior row or column to a superior, is evidently right, since an unit in the latter case is of the same value as ten in the former.

Beside the method here given, there is another very ingenious one of proving addition by casting out the nines.

RULE.

1. Add the figures in the first line, and find how many nines are contained in their sum.

2. Reject the nines and set the remainder in the same line, on the right.

3. Do the same in each of the other lines, and find the sum of the row of excesses. Then the nines of this sum, and of the sum of the given numbers being rejected, if the two excesses be equal, the addition is proved to be rightly performed.

EXAMPLE.

	Excesses.
3782	2
5766	6
8755	7
18303	6

2. Add the figures in the row of units, and find how many tens are contained in their sum.

3. Set the remainder under the line, and carry as many units to the next row, as there are tens, with which proceed as before; and so on till the whole is finished.

This method depends on a property of the number 9, which belongs to no other digit whatever, except 3, namely, that any number divided by 9 leaves the same remainder, as the sum of its figures or digits divided by 9; which may be thus demonstrated.

DEMON. Let there be any number, as 3467; this separated into its several parts becomes $3000+400+60+7$; but $3000=3\times 1000=3\times\overline{999+1}=3\times 999+3$. In like manner $400=4\times 99+4$, and $60=6\times 9+6$. Therefore $3467=3\times 999+3+4\times 99+4+6\times 9+6+7=3\times 999+4\times 99+6\times 9+3+4+6+7$. And $\frac{3467}{9}=\frac{3\times 999+4\times 99+6\times 9}{9}+\frac{3+4+6+7}{9}$. But $3\times 999+4\times 99+6\times 9$ is evidently divisible by 9; therefore 3467 divided by 9 will leave the same remainder, as $3+4+6+7$ divided by 9; and the same will hold for any other number whatever. Q. E. D.

The same may be demonstrated universally thus.

DEMON. Let $N=$ any number whatever, a, b, c, &c. the digits, of which it is composed, and $n=$ as many cyphers as a, the highest digit, is places from unity. Then $N=a$ with n 0s$+b$ with $\overline{n-1}$ 0s$+c$ with $\overline{n-2}$ 0s, &c. by the nature of notation; $=a\times n$ 9s$+a+b\times\overline{n-1}$ 9s$+b+c\times\overline{n-2}$ 9s$+c$, &c. $=a\times n$ 9s$+b\times\overline{n-1}$ 9s$+c\times\overline{n-2}$ 9s, &c. $+a+b+c$, &c. but $a\times n$ 9s$+b\times\overline{n-1}$ 9s $+c\times\overline{n-2}$ 9s, &c. is plainly divisible by 9; therefore N divided by 9 will leave the same remainder, as $a+b+c$, &c. divided by 9. Q. E. D.

In the very same manner, this property may be shown to belong to the number three; but the preference is usually given to the number 9, on account of its being more convenient in practice.

Now from the demonstration here given, the reason of the rule itself is evident; for the excess of nines in each of two or

METHOD OF PROOF.

1. Draw a line between the first and second lines of figures to cut off the first number.

2. Add all the other numbers, and set their sum under the sum of all the numbers.

3. Add the number last found and the number cut off; and if their sum be the same, as that found by the first addition, the sum is right.

EXAMPLES.

(1)	(2)	(3)
23456	22345	34578
78901	67890	3750
23456	8752	87
78901	340	328
23456	350	17
78901	78	327
307071 Sum.	99755 Sum.	39087 Sum.
283615	77410	4509
307071 Proof.	99755 Proof.	39087 Proof

---

more numbers being taken, and the excess of nines also in the sum of these excesses, it is plain, the last excess must be equal to the excess of nines, contained in the sum of all the numbers; the parts being equal to the whole.

This rule was first given by Dr. Wallis in his Arithmetic, published A. D. 1657, and is a very simple, easy method; though it is liable to this inconvenience, that a wrong operation may sometimes appear to be right. For if we change the places of any two figures in the sum, it will still be the same. A true sum will however always appear to be true by this proof; and to make a false one appear true, there must be at least two errors, and these opposite to each other. And if there be more than two errors, they must balance among themselves; but the chance against this particular circumstance is so great, that we may pretty safely trust to this proof.

4. Add 8635, 2194, 7421, 5063, 2196, and 1245 together.
Answer 26754.

5. Add 246034, 298765, 47321, 58653, 64218, 5376, 9821, and 340 together. Ans. 730528.

6. Add 562163, 21964, 56321, 18536, 4340, 279, and 83 together. Ans. 663686.

7. How many days are there in the twelve calendar months? Ans. 365.

8. How many days are there from the 19th day of April, 1774, to the 27th day of November, 1775, both days exclusive? Ans. 586.

SIMPLE SUBTRACTION.

Simple Subtraction teaches to take a less number from a greater of the same denomination, and thereby shows the difference or remainder. The less number, or that which is to be subtracted, is called the *subtrahend;* the other, the *minuend;* and the number, that is found by the operation, the *remainder* or *difference.*

RULE.*

1. Place the less number under the greater, so that units may stand under units, tens under tens, &c. and draw line under them.

---

* Demon. 1. When all the figures of the less number are less than their correspondent figures in the greater, the differences of the figures in the several like places must, taken together, make the true difference sought; because, as the sum of the parts is equal to the whole, so must the sum of the differences of all the similar parts be equal to the difference of the wholes, or given numbers.

2. When any figure of the greater number is less than its correspondent figure in the less, the ten, which is added by the rule, is the value of an unit in the next higher place, by the nature of notation; and the one, that is added to the next place of the less number, is to diminish the correspondent place of the greater accordingly; and therefore the operation in this case is

2. Beginning, at the right, take each figure in the subtrahend from the figure over it, and set the remainder under the line.

3. If the lower figure be greater than that over it, add ten to the upper figure; from which figure, so increased, take the lower, and write the remainder, carrying one to the next figure in the lower line, with which proceed as before; and so on till the whole is finished.

Method of Proof.

Add the remainder to the less number, and if the sum be equal to the greater, the work is right.

EXAMPLES.

	(1)		(2)		(3)
From	3287625	From	5327467	From	1234567
Take	2343756	Take	1008438	Take	345678
Rem.	943869	Remain.	4319029	Remain.	888889
Proof	3287625	Proof	5327467	Proof	1234567

4. From 2637804 take 2376982. Ans. 260822.

5. From 3762162 take 826541. Ans. 2935621.

6. From 78213606 take 27821890 Ans. 50391716.

7. The Arabian method of notation was first known in England about the year 1150; how long was it thence to the year 1776? Ans. 626 years.

---

only taking from one place and adding as much to another, whereby the number is never changed. And by this method the greater number is resolved into such parts, as are each greater than, or equal to the similar parts of the less; and the differences of the corresponding figures, taken together, will evidently make up the difference of the given numbers. Q. E. D.

The truth of the method of proof is evident; for the difference of two numbers, added to the less, is manifestly equal to the greater.

8. Sir Isaac Newton was born in the year 1642, and died in 1727; how old was he at the time of his decease?

Ans. 85 years.

SIMPLE MULTIPLICATION.

Simple Multiplication is a compendious method of addition, and teaches to find the amount of any given number of one denominotion, by repeating it any proposed number of times.

The number, to be multiplied, is called the *multiplicand.*

The number, to multiply, is called the *multiplier.*

The number, found from the operation, is called the *product.*

Both the multiplier and multiplicand are, in general, called *terms* or *factors.*

Multiplication and Division table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	[illegible]	24	30	36	42	48	54	60	66	72
[illegible]	14	21	28	35	42	49	56	63	70	77	84
8	[illegible]	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	[illegible]	[illegible]	144

Use of the table in Multiplication.

Find the multiplier in the first column on the left, and the multiplicand in the first line; and the product is in the common angle of meeting, or against the multiplier, and under the multiplicand.

Use of the table in Division.

Find the divisor in the first column on the left, and the dividend in the same line ; then the quotient will be, over the dividend, the first number of the column.

RULE.*

1. Place the multiplier under the multiplicand, so that units may stand under units, tens under tens, &c. and draw a line under them.

---

* DEMON. 1. When the multiplier is a single digit, it is plain, that we find the product ; for by multiplying every figure, that is, every part of the multiplicand, we multiply the whole ; and writing down the products, that are less than ten, and the excesses above tens respectively in the places of the figures multiplied, and carrying the number of tens in each product to the product of the next place is only gathering together the similar parts of the respective products, and is therefore the same thing, in effect, as writing the multiplicand under itself so often as the multiplier expresses, and adding the several repetitions together ; for the sum of each column is the product of the figures in the place of that column ; and these products, collected together, are evidently equal to the whole required product.

2. If the multiplier consist of more than one digit ; having then found the product of the multiplicand by the first figure of the multiplier, as above, we suppose the multiplier divided into parts, and find, after the same manner, the product of the multiplicand by the second figure of the multiplier ; but as the figure we are multiplying by stands in the place of tens ; the product must be ten times its simple value ; and therefore the first figure of this product must be placed in the place of tens ; or, which is the same thing, directly under the figure we are multiplying by. And proceeding in this manner separately with all the figures of the multiplier, it is evident, that we shall multiply all the parts of the multiplicand by all the parts of the multiplier ; or the whole of the multiplicand by the whole of the multi-

2. Begin at the right, and multiply the whole multiplicand severally by each figure in the multiplier, setting the first figure of every line produced directly under the figure you are multiplying by, and carrying for the tens, as in addition.

3. Add all the lines together, and their sum is the product.

---

plier ; therefore the sum of these several products will be equal to the whole required product. Q. E. D.

The reason of the method of proof depends on this proposition, namely, " that two numbers being multiplied together, either of them may be made the multiplier, or the multiplicand, and the product will be the same." A small attention to the nature of the numbers will make this truth evident ; for $3\times7=21=7\times3$; and, in general, $3\times4\times5\times6$, &c.$=4\times3\times6\times5$, &c. without any regard to the order of the terms ; and this is true of any number of factors whatever.

The following examples are subjoined to make the reason of the rule appear as plain as possible.

(1)

37565		
5		
25	=	5×5
30	=	60×5
25	=	500×5
35	=	7000×5
15	=	30000×5
187825	=	37565×5

(2)

1375435			
4567			
9628045	=	7 times the multiplicand.	
8252610	=	60 times	do.
6877175	=	500 times	do.
5501740	=	4000 times	do.
6281611645	=	4567 times	do.

Beside the preceding method of proof, there is another very convenient and easy one by the help of that peculiar property of the number 9, mentioned in addition ; which is performed thus.

Rule 1. Cast the nines out of the two factors, as in addition, and write the remainder.

2. Multiply the two remainders together, and, if the excess

Method of Proof.

Make the former multiplicand the multiplier, and the multiplier the multiplicand, and proceed as before; and if the product be equal to the former, the product is right.

EXAMPLES.

(1)

Multiply 23456787454
by 7

164197512178 Product.

(2)

Multiply 32745654473
by 234

130982617892
98236963419
65491308946

Product 7662483146682

---

of nines in their product be equal to the excess of nines in the total product, the answer is right.

EXAMPLE.

4215	3=excess of 9's in the multiplicand.
878	5=ditto in the multiplier.
33720	
29505	
33720	
3700770	6=ditto in the product=excess of 9s in 3×5.

DEMONSTRATION OF THE RULE. Let M and N be the number of 9s in the factors to be multiplied, and a and b what remains; then $M+a$ and $N+b$ will be the numbers themselves, and their product is $\overline{M\times N}+\overline{M\times b}+\overline{N\times a}+\overline{a\times b}$; but the first three of these products are each a precise number of 9s, because one of their factors is so; therefore, these being cast away, there remains only $a\times b$; and if the 9s be also cast out of this, the excess is the excess of the 9s in the total product; but a and b are the excesses in the factors themselves, and $a\times b$ their product; therefore the rule is true. Q. E. D.

3. Multiply 32745675474 by 2. Ans. 65491350948.
4. Multiply 84356745674 by 5. Ans. 421783728370.
5. Multiply 3274656461 by 12. Ans. 39295877532.
6. Multiply 273580961 by 23. Ans. 6292362103.
7. Multiply 82164973 by 3027. Ans. 248713373271.
8. Multiply 8496427 by 874359. Ans. 7428927415293.

CONTRACTIONS.

I. *When there are cyphers on the right of one or both the factors.*

RULE.

Proceed as before, neglecting the cyphers, and on the right of the product place as many cyphers as are in both the factors.

EXAMPLES.

1. Multiply 1234500 by 7500.

```
     12345
        75
    ------
     61725
    86415
  --------
  9258750000 the Product.
```

2. Multiply 461200 by 72000. Ans. 33206400000.
3. Multiply 815036000 by 70300. Ans. 57297030800000.

This method is liable to the same inconvenience with that in addition.

Multiplication may also very naturally be proved by division; for the product being divided by either of the factors, the quotient will evidently be the other; but it would have been contrary to good method to give this rule in the text, because the pupil is supposed as yet to be unacquainted with division.

II. *When the multiplier is the product of two or more numbers in the table.*

RULE.*

Multiply continually by those numbers or parts, instead of the whole number at once.

EXAMPLES.

1. Multiply 123456789 by 25.

```
  123456789
          5
 ----------
  617283945
          5
 ----------
 3086419725 the Product.
```

2. Multiply 364111 by 56. Ans. 20390216.
3. Multiply 7128368 by 96. Ans. 684323328.
4. Multiply 123456789 by 1440. Ans. 177777776160.

SIMPLE DIVISION.

Simple Division teaches to find how often one number is contained in another of the same denomination, and thereby performs the work of many subtractions.

The number, to be divided, is called the *dividend.*

The number, to divide, is called the *divisor.*

The number of times, the dividend contains the divisor, is called the *quotient.*

---

* The reason of this method is obvious ; for any number, multiplied by the component parts of another number, must give the same product, as if it were multiplied by that number at once ; thus, in example the second, 7 times the product of 8, multiplied into the given number, makes 56 times that given number, as plainly as 7 times 8 makes 56.

If the dividend contain the divisor any number of times and an excess, that excess is called the *remainder*.

RULE.*

1. On the right and left of the dividend, draw a curved line, and write the divisor on the left, and the quotient, as it rises, on the right.

---

* According to the rule, we resolve the dividend into parts, and find by trial the number of times, the divisor is contained in each of those parts ; the only thing then, which remains to be proved, is, that the several figures of the quotient, taken as one number, according to the order, in which they are placed, is the true quotient of the whole dividend by the divisor, which may be thus demonstrated.

Demon. The complete value of the first part of the dividend is, by the nature of notation, 10, 100, or 1000, &c. times the value of which it is taken in the operation according as there are 1, 2, or 3, &c. figures standing on the right of it ; and consequently the true value of the quotient figure, belonging to that part of the dividend, is also 10, 100, or 1000, &c. times its simple value. But the true value of the quotient figure, belonging to that part of the dividend, found by the rule, is also 10, 100, or 1000, &c. times its simple value ; for the number of figures on the right of it is equal to the number of remaining figures in the dividend. Therefore this first quotient figure, taken in its complete value at the place it stands in, is the true quotient of the divisor in the complete value of the first part of the dividend. For the same reason, all the rest of the figures of the quotient, taken according to their places, are each the true quotient of the divisor, in the complete value of the several parts of the dividend, belonging to each ; because, as the first figure on the right of each succeeding part of the dividend has a less number of figures, by one standing on the right of it, so ought their quotients to have ; and so they are actually ordered ; consequently, all the quotient figures being taken in order as they are placed by the rule, they make one number, which is equal to the sum of the true quotients of all the several parts of the dividend ; and therefore is the true quotient of the whole dividend by the divisor. Q. E. D.

2. Find how many times the divisor may be had in so many figures of the dividend, as are just necessary, and write the number in the quotient.

To leave no obscurity in this demonstration, I shall illustrate it by an example.

EXAMPLE.

Divisor 36)85609 Dividend.

1st part of the dividend	85000	
36 × 2000 =	72000	- - 2000 the 1st quotient.
1st remainder -	13000	
add	600	
2d part of the dividend	13600	
36 × 300 =	10800	- - 300 the 2d quotient.
2d remainder -	2800	
add	00	
3d part of the dividend	2800	
36 × 70 =	2520	- - 70 the 3d quotient.
3d remainder -	280	
add	9	
4th part of the dividend	289	
36 × 8 =	288	- - 8 the 4th quotient.
Last remainder -	1	2378 sum of the quotients, or the answer.

Explanation. It is evident, that the dividend is resolved into these parts, 85000+600+00+9; for the first part of the dividend is considered only as 85, but yet it is truly 85000; and therefore its quotient, instead of 2, is 2000, and the remainder 13000; and so of the rest, as may be seen in the operation.

When there is no remainder after the operation of dividing is finished, the quotient is the absolute and perfect answer to the question; but where there is a remainder, it may be observed, that it gives a part of another unit for the quotient, which is

3. Multiply the divisor by the quotient figure, and set the product under the part of the dividend used.

---

greater as it approaches nearer to the divisor. Thus, if the remainder be a fourth part of the divisor, the part is one fourth, or one fourth of the divisor is contained in the dividend beside the quotient already found; if half the divisor, the part is one half, or one half of the divisor is, in addition to the quotient already found, contained in the dividend; and so on. In order therefore to complete the quotient, put the last remainder at the end of it, above a small line, and the divisor under it.

It is sometimes difficult to find how often the divisor is contained in the numbers of the several steps of the operation; the best way will be to find how often the first figure of the divisor is contained in the first, or two first, figures of the dividend, and the answer, made less by one or two, is generally the figure wanted. Beside, if after subtracting the product of the divisor and quotient from the dividend, the remainder be equal to, or exceed the divisor, the quotient figure must be increased accordingly.

If, when you have brought down a figure on the right of the remainder, it be still less than the divisor, a cypher must be put in the quotient, and another figure brought down, and then proceed as before.

The reason of the method of proof is plain; for since the quotient is the number of times the dividend contains the diviser, the product of the quotient and divisor must evidently be equal to the dividend.

There are several other methods, used to prove division; the best and most useful are the following.

Rule I. Subtract the remainder from the dividend, and divide this remainder by the quotient, and the quotient, found by this division, will be equalto the former devisor, when the work is right.

The reason of this rule is plain from what has been observed above.

Mr. Malcolm, in his Arithmetic, has fallen into an error concerning this method of proof, by making use of particular num-

4. Subtract the last found product from that part of the dividend, under which it stands, and on the right of the remainder bring down the next figure of the dividend; which number divide as before; and proceed in this manner till the whole is finished.

---

bers, instead of a general demonstration. He says, the dividend being divided by the integral quotient, the quotient of this division will be equal to the former divisor, with the same remainder. This is true in some particular cases; but it will not hold, when the remainder is greater than the quotient, as may be easily demonstrated; but one instance will be sufficient; thus 17, divided by 6, gives the integral quotient 2, and remainder 5; but 17, divided by 2, gives the integral quotient 8, and remainder 1. This shows how cautious we ought to be in deducing general rules from particular examples.

Rule II. Add together the remainder, and all the products of the several quotient figures by the divisor, according to the order, in which they stand in the work, and the sum will be equal to the dividend, when the work is right.

The reason of this rule is extremely obvious; for the numbers, that are to be added, are the products of the divisor by each figure of the quotient separately, and each possesses, by its place, its complete value; therefore the sum of the parts, together with the remainder, must be equal to the whole.

Rule III. Subtract the remainder from the dividend, and what remains will be equal to the product of the divisor and quotient; which may be proved by casting out the nines, as was done in multiplication.

This rule has been already demonstrated in multiplication.

To avoid obscurity I shall give an example, proved according to all the different methods.

EXAMPLE.

87)123456789(1419043		123456789
87*	87	43

Method of Proof.

Multiply the quotient by the divisor, and this product, added to the remainder, will be equal to the dividend, when the work is right.

EXAMPLES.

(1)

$5)13545728(2709145\frac{3}{5}$
10
——
35
35
——
45
45
——
7
5
——
22
20
——
28
25
——
3

(2)

365)123456789(338237
1095
——
1395
1095
——
3006
2920
——
867
730
——
1378
1095
——
2839
2555
——
284

364
348*
——
. 165
. . 87*
——
. . 786
. . 783*
——
. . . . 378
. . . . 348*
——
. 309
. . . . . 261*
——
. . . . . . 48*
——
123456789 Proof by Addition.
——

9933301
11352344
48
——
123456789 Proof by Mult.
——

1419043)123456741(87 Proof by Div.
11352344
——
9933301
9933801
——

Proof by casting out the nines.

4 is the excess of 9s in the quotient.
6 ditto - - - - in the divisor.
6 ditto - - - - in 4×6, which is also the excess of 9s in (123456741) the dividend made less by the remainder.

3. Divide 3756789275474 by 2. Ans. 1878394637737.
4. Divide 12345678900 by 7. Ans. $1763668414\frac{2}{7}$.
5. Divide 9876543210 by 8. Ans. $1234567901\frac{2}{8}$.
6. Divide 1357975313 by 9. Ans. $150886145\frac{3}{9}$.
7. Divide 3217684329765 by 17. Ans. $189275548809\frac{12}{17}$.
8. Divide 3211473 by 27. Ans. $118943\frac{12}{27}$.
9. Divide 1406373 by 108. Ans. $13021\frac{105}{108}$.
10. Divide 293839455936 by 8405. Ans. $34960078\frac{346}{8405}$.
11. Divide 4637064283 by 57606. Ans. $80496\frac{11707}{57606}$.

CONTRACTIONS.

I. *To divide by any number with cyphers annexed.*

RULE*.

Cut off the cyphers from the divisor, and the same number of digits from the right of the dividend; then divide, making use of the remaining figures, as usual, and the quotient is the answer; and what remains, written before the figures cut off, is the true remainder.

For illustration, we need only refer to the example; except for the proof by addition; where it may be remarked, that the asterisms show the numbers to be added, and the dotted lines their order.

* The reason of this contraction is easily perceived; for cutting off the same figures from each is the same as dividing each of them by 10, 100, 1000, &c. and it is evident, that as often as the whole divisor is contained in the whole dividend, so often must any part of the divisor be contained in a like part of the dividend. This method is only avoiding a needless repitition of cyphers, which would happen in the common way, as may be seen by working an example at large.

EXAMPLES.

1. Divide 310869017 by 7100.

```
71,00)3108690,17(43784 2617/7100 the quotient.
      284
      ---
       268
       213
       ---
        556
        497
        ---
         599
         568
         ---
          310
          284
          ---
          2617
```

2. Divide 7380964 by 23000. Ans. $320\frac{20964}{23000}$.

3. Divide 29628754963 by 35000. Ans. $846535\frac{20963}{35000}$.

II. *When the divisor is the product of two or more numbers in the table.*

RULE.*

Divide continually by those numbers, instead of the whole divisor at once.

---

* This follows from the second contraction in multiplication, of which it is only the converse; for the third part of the half of any thing is evidently the same as the sixth part of the whole; and so of any other parts. I have omitted saying any thing in the rule about the method of finding the true remainder; for as the learner is supposed, at present, to be unacquainted with the nature of fractions, it would be improper to introduce them in this part of the work, especially as the integral quotient is sufficient to answer most of the purposes of practical division. However, as the quotient is incomplete without this remainder, and in some computations it is necessary it should be known, I shall

EXAMPLES.

1. Divide 31046835 by $56=7\times8$.

```
7)31046835(4435262        8)4435262(554407 the quotient.
  28                        40
  --                        --
   30                        43
   28                        40
   --                        --
    24                        35
    21                        32
    --                        --
     36                        32
     35                        32
     --                        --
      18                        62
      14                        56
      --                        --
       43                        6
       42                       --
       --
       15
       14
       --
        1
```

---

here show the manner of finding it, without any assistance from fractions.

RULE. Multiply the quotient by the divisor, and subtract the product from the dividend, and the result will be the true remainder.

The truth of this is extremely obvious; for if the product of the divisor and quotient, added to the remainder, be equal to the dividend, their product, taken from the dividend, must leave the remainder.

The rule, which is most commonly used, is this.

RULE. Multiply the last remainder by the preceding divisor, or last but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the pro-

2. Divide 7014596 by $72=8\times9$.

8)7014596

9)876824 4

97424 8 the quotient.

3. Divide 5130652 by 132. Ans. $38868\frac{76}{132}$.

4. Divide 83016572 by 240. Ans. $345902\frac{62}{240}$.

---

duct add the next preceding remainder; and so on till you have used all the divisors and remainders.

EXAMPLE.

9)64865 divided by 144

4)7207 2

4)1801 3

450 1

Ans. $450\frac{65}{144}$.

1 the last remainder.
Mult. 4 the preceding divisor.

4
Add 3 the second remainder.

7
Mult. 9 the first divisor.

63
Add 2 the first remainder.

65

To explain this rule from the example, we may observe, that every unit of the first quotient may be looked upon as containing 9 of the units in the given dividend; consequently every unit of it, that remains, will contain the same; therefore this remainder must be multiplied by 9, in order to find the units of the given dividend, which it contains. Again, every unit in the next quotient will contain 4 units of the preceding, or 36 of the given dividend, that is, 9 times 4; therefore what remains must be multiplied by 36; or, which is the same thing, by 9 and 4 continually. Now this is the same as the rule; for instead of finding the remainders separately, they are reduced from the bottom upward step by step, to the first, and the remaining units of the same class taken in as they occur.

III. *To perform division more concisely than by the general rule.*

RULE.*

Multiply the divisor by the quotient figures as before, and subtract each figure of the product when you produce it, always remembering to carry so many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 3104675846 by 833.

833)3104675846(3727101$\frac{713}{833}$ the quotient.
6056
2257
5915
848
1546
713

2. Divide 29137062 by 5317. Ans. 5479$\frac{5219}{5317}$.
3. Divide 62015735 by 7803. Ans. 7947$\frac{5294}{7803}$.
4. Divide 432756284563574 by 873469.
Ans. 495445498$\frac{872012}{873469}$.

REDUCTION.

Reduction is the method of bringing numbers from one name or denomination to another without changing the value.

In order to perform reduction it is necessary to be acquainted with the relative value of the different denominations of coin, weight, and measure, that are used; for which purpose see the following

TABLES of COIN, WEIGHT, and MEASURE.

MONEY.

4 farthings	make 1 penny	£	denotes	pounds
12 pence	1 shilling	ſ or s		shillings.
20 shillings	1 pound	d		pence.

* The reason of this rule is the same as that of the general rule.

TROY WEIGHT.

24 grains	make	1 penny-weight, marked	grs. dwt.
20 dwt.		1 ounce,	oz.
12 oz.		1 pound,	℔ or lb.

By this weight are weighed jewels, gold, silver, corn, bread, and liquors.

APOTHECARIES' WEIGHT.

20 grains make	1 scruple, marked	gr. sc. or ℈
3 sc. *or* ℈	1 dram	dr. or ʒ.
8 dr.	1 ounce	oz. or ℥.
12 oz.	1 pound	℔ or lb.

Apothecaries use this weight in compounding their medicines; but they buy and sell their drugs by Avoirdupois weight. Apothecaries' is the same as Troy weight, having only some different divisions.

AVOIRDUPOIS WEIGHT.

16 drams make	1 ounce, marked	dr. oz.
16 ounces	1 pound	lb.
28 lb.	1 quarter	qr.
4 quarters	1 hundred weight	cwt.
20 cwt.	1 ton	T.

By this weight are weighed all things of a coarse or drossy nature; such as butter, cheese, flesh, grocery wares, and all metals, except gold and silver.*

---

	lb.
* A firkin of butter . . is .	56
A firkin of soap	64
A barrel of pot-ashes . . .	200
A barrel of anchovies	30
A barrel of candles	120
A barrel of soap	256
A barrel of butter	224
A fother of lead is 19½ cwt	
A stone of iron	14
A stone of butcher's meat .	8

	lb.		
A gallon of train oil	7½		
A faggot of steel	120		
A stone of glass	5		
A seam of glass is 24 stone, or	120		
	lb.	oz.	dr.
A peck loaf of bread weighs	17	6	1
A half peck	8	11	
A quartern	4	5	8

DRY MEASURE.

		Marked
2 pints make	1 quart	pts. qts.
2 quarts	1 pottle	pot.
2 pottles	1 gallon	gal.
2 gallons	1 peck	pe.
4 pecks	1 bushel	bu.
2 bushels	1 strike	str.

		Marked
8 bushels	1 quarter	qr.
5 quarters	1 wey *or* load	wey
4 bushels	1 coomb	co.
5 pecks	1 bushel water meas.	
10 coombs	1 wey	
2 weys	1 last	L.

NOTE.—The diameter of a Winchester bushel is $18\frac{1}{2}$ inches, and its depth 8 inches.—And one gallon by dry measure contains $268\frac{4}{5}$ cubic inches.

By this measure, salt, lead, ore, oysters, corn, and other dry goods are measured.

ALE AND BEER MEASURE.

		Marked
2 pints mske	1 quart	pts. qts.
4 quarts	1 gallon	gal.
8 gallons	1 firkin of Ale	fir.
9 gallons	1 firkin of Beer	fir.

		Marked
2 firkins	1 kilderkin	kil.
2 kilderkins	1 barrel	bar.
3 kilderkins	1 hogshead	hhd.
3 barrels	1 butt	butt.

NOTE.—The ale gallon contains 282 cubic inches. In London the ale firkin contains 8 gallons, and the beer firkin 9; other measures being in the same proportion.

56lb. old hay } make a truss.
60lb. new hay }
36 trusses . . . . a load.

4 pecks coal make 1 bushel.
9 bushels . . . . 1 vat or strike.
36 bushels . . . . . . 1 chaldron.
21 chaldrons . . . . 1 score.

7 lb. wool make . . . . 1 clove.
2 cloves . . . . 1 stone.
2 stones . . . . 1 tod.
$6\frac{1}{2}$ tods . . . . 1 wey.
2 weys . . . . 1 sack.
12 sacks . . . . 1 last.

	lb.
A barrel of pork is	220.
A barrel of beef is	220.
A quintal of fish	112.

20 things make . . . . 1 score.
12 . . . . 1 dozen.
12 dozen . . . . 1 gross.
144 dozzen . . 1 greater gross.

Farther,—5760 grains = 1 lb. Troy; 7000 grains = 1 lb. Avoirdupois; therefore the weight of the pound Troy is to that of the pound Avoirdupois, as 5760 to 7000, or as 144 to 175.

WINE MEASURE.

		Marked			Marked
2 pints make	1 quart	pts. qts.	2 hogshead	1 pipe *or* butt	p. *or* b.
4 quarts	1 gallon	gal.	2 pipes	1 tun	T.
42 gallons	1 tierce	tier.	18 gallons	1 runlet	rund.
63 gallons	1 hogshead	hhd.	31½ gallons	1 barrel	bar.
84 gallons	1 puncheon	pun.			

By this measure, brandy, spirits, perry, cider, mead, vinegar, and oil are measured.

NOTE.—231 cubic inches make a gallon, and 10 gallons make an anchor.

CLOTH MEASURE.

		Marked			Marked
2¼ inches make	1 nail	nls.	3 qrs.	1 ell Flemish	Ell Fl.
4 nails	1 quarter	qrs.	5 qrs.	1 ell English	Ell Eng.
4 quarters	1 yard	yds.	6 qrs.	1 ell French	Ell Fr.

LONG MEASURE.

		Marked			Marked
3 barley corns make	1 inch	bar. c. in.	60 geographical miles, *or* 69½ statute miles	1 degree	deg. *or* °
12 inches	1 foot	ft.	360 degrees the circumference of the earth.		
3 feet	1 yard	yd.	*Note.* 4 inches make	1 hand.	
6 feet	1 fathom	fath.	5 feet	1 geometrical pace.	
5½ yards	1 pole	pol.	6 points	1 line.	
40 poles	1 furlong	fur.	12 lines	1 inch.	
8 furlongs	1 mile	mls.			
3 miles	1 league	l.			

TIME.

		Marked			Marked
60 seconds make	1 minute	s. *or* ″ m. *or* ′	4 weeks	1 month	m.
60 minutes	1 hour	h. *or* ᶜ	13 months, 1 day, and 6 hours, *or*		
24 hours	1 day	d.	365 days and 6 hours,	1 Julian year	Y.
7 days	1 week	w.			

NOTE 1. The second may be supposed to be divided into 60 thirds, and these again into 60 fourths, &c.

NOTE 2. April, June, September, and November, have each 30 days; each of the other months has 31, except February, which has 28 in common years, and 29 in leap years.

CIRCULAR MOTION.

60 seconds make	1 minute, marked	″ ′
60 minutes	1 degree	°
30 degrees	1 sign	s
12 signs, *or* 360°	1 circle.	

I. *When the reduction is from a greater name to a less.*

RULE.*

Multiply the highest name or denomination by as many as one makes of the next less, adding to the product the parts of the second name; then multiply this sum by as many as one makes of the next less name, adding to the product the parts of the third name; and so on through all the denominations to the last.

II. *When the reduction is from a less name to a greater.*

RULE.

Divide the given number by as many as make one of the next superior denomination; and this quotient again by as many as make one of the next following; and so on through

* The reason of this rule is exceedingly obvious; for pounds are brought into shillings by multiplying them by 20; shillings into pence by multiplying them by 12; and pence into farthings by multiplying them by 4; and the contrary by division; and this will be true in the reduction of numbers, containing any denominations whatever.

all the denominations to the highest; and this last quotient, together with the several remainders, will be the answer required.

The method of proof is by reversing the question.

EXAMPLES.

1. In 1465l. 14s. 5d. how many farthings?

```
      20                4)1407092
   ------              ---------
   29314               12)351773
      12               ---------
   ------              2,0)2931,4 5
  351773               ------------
       4          Proof 1465l. 14s. 5d.
 -------
 1407092 the answer.
```

2. In 12l. how many farthings? Ans. 11520.

3. In 6169 pence how many pounds? Ans. 25l. 14s. 1d.

4. In 35 guineas how many farthings? Ans. 47840.

5. In 420 quarter guineas how many moidores? Ans. $81\frac{2}{3}$

6. In 231l. 16s. how many ducats at 4s. 9d. each? Ans. 976.

7. In 274 marks, each 17s. 9d. and 87 nobles, each 8s. 11d. how many pounds? Ans. 281l. 19s. 3d.

8. In 1776 quarter guineas how many six pences? Ans. 24864.

9. Reduce 1776 six and thirties to half-crowns sterling. Ans. $25574\frac{2}{5}$.

10. In 50807 moidores how many pieces of coin, each 4s. 6d.? Ans. 406456.

11. In 213210 grains how many lb.? Ans. 37lb. 3dwt. 18gr.

12. In 59lb. 13dwts. 5gr. how many grains? Ans. 340157grs.

13. In 8012131 grains how many lb.? Ans. 1390lb. 11oz. 18dwts. 19grs.

14. In 35 tons, 17cwt. 1qr. 23lb. 7oz. 13dr. how many drams? Ans. 20571005dr.

15. In 37cwt. 2qr. 17lb. how many pounds Troy, a pound Avoirdupois being equal to 14oz. 11dwt. 15½grs. Troy? Ans. 5124lb. 5oz. 10dwt. 11½grs.

16. How many barley corns will reach round the world, supposing it, according to the best calculations, to be 8340 leagues? Ans. 4755801600.

17. In 17 pieces of cloth, each 27 Flemish ells, how many yards? Ans. 344yds. 1qr.

18. How many minutes were there from the birth of CHRIST to the year 1776, allowing the year to consist of 365d. 5h. 48′ 58″? Ans. 934085364′ 48″.

COMPOUND ADDITION.

Compound Addition teaches to collect several numbers of different denominations into one sum.

RULE.*

1. Place the numbers so, that those of the same denomination may stand directly under each other, and draw a line under them.

2. Add the figures in the lowest denomination, and find how many ones of the next higher denomination are contained in their sum.

3. Write the remainder, and carry the ones to the next denomination; with which proceed as before; and so on

---

* The reason of this rule is evident from what has been said in simple addition; for, in addition of money, as 1 in the pence is equal to 4 in the farthings; 1 in the shillings, to 12 in the pence; and 1 in the pounds, to 20 in the shillings; therefore, carrying as directed, is nothing more than providing a method of digesting the money, arising from each column, properly in the scale of denominations; and this reasoning will hold good in the addition of compound numbers of any description whatever.

through all the denominations to the highest, whose sum must be all written ; and this sum, together with the several remainders, is the whole sum required.

The method of proof is the same as in simple addition.

EXAMPLES.

MONEY.

£.	s.	d.	£.	s.	d.	£.	s.	d.
17	13	4	84	17	5½	175	10	10
13	10	2	75	13	4¼	107	13	11¾
10	17	3	51	17	8¾	89	18	10
8	8	7	20	10	10¼	75	12	2¼
3	3	4	17	15	4½	3	3	3¾
	8	8	10	10	11	1		½
54	1	4	261	5	8¼	452	19	2¼
36	8	0	176	8	2¾	277	8	4¼
54	1	4	261	5	8¼	452	19	2¼

TROY WEIGHT.

lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.	lb	oz.	dwt.	gr.
17	3	15	11	14	10	13	20	27	10	17	18
13	2	13	13	13	10	18	21	17	10	13	13
15	3	14	14	14	10	10	10	13	11	13	1
13	10			10	1	2	3	10	1		2
12	1		17	1	4	4	4	4	4	3	3
		13	14		1	19		2			1

AVOIRDUPOIS WEIGHT.

cwt.	qr.	lb.	oz.	dr.
15	2	15	15	15
13	2	17	13	14
12	2	13	14	14
10	1	17	15	
12	1	10		10
10	1	12	1	7

T.	cwt.	qr.	lb.	oz.	dr.
2	17	3	13	8	7
2	13	3	14	8	8
1	16		10		5
2	13			1	7
1	14	1	1	2	2
4	16	1	7	7	5

T.	cwt.	qr.	lb.	oz.	dr.
3	13	2	10	7	7
2	14	1	17	6	6
4	17		14		6
2	13		12	7	7
3	13		10	4	4
5		2	12	8	8

LONG MEASURE.

Mls.	fur.	pol.	yd.	ft.	in.
37	3	14	2	1	5
28	4	17	3	2	10
17	4	4	3	1	2
10	5	6	3	1	7
29	2	2	2		3
30			4		2

Mls.	fur.	pol.	yd.	ft.	in.
28	2	13	1	1	4
39	1	17	2	2	10
28	1	14	2	2	
48	1	17	2	2	7
37	1	29			3
2		20		2	1

Mls.	fur.	pol.	yd.	ft.	in.
28	3	7	2		7
30			1		7
27	6	30	2	2	
7	6	20	2	1	
5	2			2	10
	7	10		2	2

COMPOUND SUBTRACTION.

Compound Subtraction teaches to find the difference of any two numbers of different denominations.

RULE.*

1. Place the less number under the greater so, that those parts, which are of the same denomination, may stand directly under each other, and draw a line under them.

---

* The reason of this rule will readily appear from what has

2. Beginning at the right, take the number in each denomination of the lower line from the number in the same denomination over it, and set the remainders in a line under them.

3. But if the lower number be greater than that above it, increase the upper number by as manny as make one of the next higher denomination, and from this sum take the lower number and set the remainder as before.

4. Carry one for the number borrowed to the next number in the lower line, and subtract as before; and so on till the whole is finished; and all the several remainders, taken together as one number, will be the whole difference required.

The method of proof is the same as in simple subtraction.

EXAMPLES.

MONEY.

	£.	s.	d.	£.	s.	d.	£.	s.	d.
From	275	13	4	454	14	2¾	274	14	2¼
Take	176	16	6	276	17	5½	85	15	7¾
Rem.	98	16	10	177	16	9¼	188	18	6½
Proof	275	13	4	454	14	2¾	374	14	2¼

TROY WEIGHT.

	lb.	oz.	dwt.	gr.	lb.	oz	dwt.	gr.	lb.	oz.	dwt.	gr.
From	7	3	14	11	27	2	10	20	29	3	14	5
Take	3	7	15	20	20	3	5	21	20	7	15	7
Rem.												
Proof												

---

been said in simple subtraction; for the borrowing depends upon the very same principle, and is only different, as the numbers to be subtracted are of different denominations.

AVOIRDUPOIS WEIGHT.

	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.
From	5		17	5	9	22	2	13	4	8	21	1	7	6	13
Take	3	3	21	1	7	20	1	17	6	6	13		8	8	14
Rem.															
Proof															

LONG MEASURE.

	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd.	ft.	in.
Fr.	14	3	17	1	2	1	70	7	13	1	1	2	70	3	10			3
Ta.	10	7	30	2		10	20		14	2	2	7	17	3	11	1	1	7
R.																		
P.																		

TIME.

	m.	w.	d.	h.	′	m.	w.	d.	h.	′	m.	w.	d.	h.	′
From	17	2	5	17	26	37	1		13	1	71				5
Take	10			18	18	15	2		15	14	17		5	5	7
Rem.															
Proof															

COMPOUND MULTIPLICATION.

Compound Multiplication teaches to find the amount of any given number of different denominations by repeating it any proposed number of times.

RULE.*

1. Place the multiplier under the lowest denomination of the multiplicand.

2. Multiply the number of the lowest denomination by the multiplier, and find how many ones of the next higher denomination are contained in the product.

3. Write the excess, and carry the ones to the product of the next higher denomination, with which proceed as before; and so on through all the denominations to the highest, whose product, together with the several excesses, taken as one number, will be the whole amount required.

The method of proof is the same as in simple multiplication.

EXAMPLES OF MONEY.

1. 9lb. of tobacco, at 2s. 8½d. per lb.

2s. 8½d.
9

1l. 4s. 4½d. the answer.

---

* The product of a number, consisting of several parts or denominations, by any simple number whatever will evidently be expressed by taking the product of that simple number and each part by itself, as so many distinct questions; thus, 25l. 12s. 6d. multiplied by 9 will be 225l. 108s. 54d.=(by taking the shillings from the pence, and the pounds from the shillings, and placing them in the shillings and pounds respectively) 230l. 12s. 6d. which is agreeable to the rule; and this will be true, when the multiplicand is any compound number whatever.

2. 3lb. of green tea, at 9s. 6d. per lb. Ans. 1l. 8s. 6d.
3. 5lb. of loaf sugar, at 1s. 3d. per lb. Ans. 6l. 3s.
4. 9cwt. of cheese, at 1l. 11s. 5d. per cwt.
Ans. 14l. 2s. 9d.
5. 12 gallons of brandy, at 9s. 6d. per gallon.
Ans. 5l. 14s.

CASE 1.

If the multiplier exceed 12, multiply succesively by its component parts, instead of the whole number at once, as in simple multiplication.

EXAMPLES.

1. 16cwt. of cheese, at 1l. 18s. 8d. per cwt.

	1l.	18s.	8d.
			4
	7	14	8
			4
£	30	18	8 the answer.

2. 28 yards of broad cloath, at 19s. 4d. per yard.
Ans. 27l. 1s. 4d.
3. 96 quarters of rye, at 1l. 3s. 4d. per quarter.
Ans. 112l.
4. 120 dozen of candles, at 5s. 9d. per doz.
Ans. 34l. 10s.
5. 132 yards of Irish cloth, at 2s. 4d. per yard.
Ans. 15l. 8s.
6. 144 reams of paper, at 13s. 4d. per ream.
Ans. 96l.

CASE. II.

If the multiplier cannot be produced by the multiplication of small numbers, find the product of such numbers nearest to it, either greater or less, then multiply by the component parts as before; and for the odd parts, add or subtract as the case requires.

EXAMPLES.

1. 17 ells of holland, at 7s. 8½d. per ell.

	7s.	8½d.
		4
1	10	10
		4
6	3	4
	7	8½
£6	11	0½ the answer.

2. 23 ells of dowlas, at 1s. 6½d. per ell.
Ans. 1l. 15s. 5½d.

3. 46 bushels of wheat, at 4s. 7¼d. per bushel.
Ans. 10l. 11s. 9½d.

4. 59 yards of tabby, at 7s. 10d. per yard.
Ans. 23l. 2s. 2d.

5. 94 pair of silk stockings, at 12s. 2d. per pair.
Ans. 57l. 3s. 8d.

6. 117cwt. of Malaga raisins, at 1l. 2s. 3d. per cwt.
Ans. 130l. 3s. 3d.

EXAMPLES OF WEIGHTS AND MEASURES.

lb.	oz.	dwt.	gr.	lb.	oz.	dr.	sc.	gr.	cwt.	qr.	lb.	oz.	mls.	fur.	pls.	yd.
21	1	7	13	2	4	2	1	0	27	1	13	12	24	3	20	2
			4					7				12				6

COMPOUND DIVISION.

Compound Division teaches to find how often one number is contained in another of different denominations.

RULE.*

1. Place the numbers as in simple division.
2. Beginning at the left, divide each denomination by the divisor, setting the quotients under their respective dividends.
3. But if there be a remainder after dividing any of the denominations except the least, reduce it to the next lower denomination, and add to it any number, which may be in that denomination; then divide the sum as usual; and so on till the whole is finished.

The method of proof is the same as in simple division.

EXAMPLES OF MONEY.

1. Divide 225l. 2s. 4d. by 2.

2)225l. 2s. 4d.

112l. 11s. 2d. the quotient.

2. Divide 751l. 14s. 7$\frac{1}{4}$d. by 3. Ans. 250l. 11s. 6$\frac{1}{2}$d.
3. Divide 821l. 17s. 9$\frac{3}{4}$d. by 4. Ans. 205l. 9s. 5$\frac{1}{4}$d.

---

* To divide a number, consisting of several denominations, by any simple number whatever is evidently the same as dividing all the parts or members, of which that number is composed, by the same simple number. And this will be true, when any of the parts are not an exact multiple of the divisor: for by conceiving the number, by which it exceeds that multiple, to have its proper value by being placed in the next lower denomination, the dividend will still be divided into parts, and the true quotient found as before; thus 25l. 12s. 3d. divided by 9, will be the same as 18l. 144s. 99d. divided by 9, which is equal to 2l. 16s. 11d. as by the rule; and the method of carrying from one denomination to another is exactly the same.

4. Divide 28l. 2s. 1½d. by 6. Ans. 4l. 13s. 8¼d.
5. Divide 135l. 10s. 7d. by 9. Ans. 15l. 1s. 2d.
6. Divide 227l. 10s. 5d. by 11. Ans. 20l. 13s. 8d.

CASE I.

If the divisor exceed 12, divide continually by its component parts, as in simple division.

EXAMPLES.

1. What is cheese per cwt. if 16cwt. cost 30l. 18s. 8d.?

```
4)30l.  18s.  8d.
------------------
4)7     14    8
------------------
£1      18    8  the answer.
```

2. If 20cwt. of tobacco come to 120l. 10s. what is that per cwt.? Ans. 6l. 6d.
3. Divide 57l. 3s. 7d. by 35. Ans. 1l. 12s. 8d.
4. Divide 85l. 6s. by 72. Ans. 1l. 3s. 8¼d.
5. Divide 31l. 2s. 10½d. by 99. Ans. 6s. 3½d.
6. At 18l. 18s. per cwt. how much per lb.?
Ans. 3s. 4½d.

CASE II.

If the divisor cannot be produced by the multiplication of small numbers, divide by long division.

EXAMPLES.

1. Divide 74l. 13s. 6d. by 17.

```
17)74  13  6 (4  7  10
   68
   --
    6
   20
   ---
  133
  119
  ---
```

```
 14
 12
----
174
17
----
  4
```

2. Divide 23l. 15s. $7\frac{1}{4}$. by 37. Ans. 12s. $10\frac{1}{4}$d.

3. Divide 315l. 3s. $10\frac{1}{2}$d. by 365. Ans. 17s. $3\frac{1}{4}$d.

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 23lb. 7oz. 6dwt. 12gr. by 7.
Ans. 3lb. 4oz. 9dwt. 12.gr.

2. Divide 13lb. 1oz. 2dr. 10gr. by 12.
Ans. 1lb. 1oz. 2sc. 10gr.

3. Divide 1061cwt. 2qrs. by 28.
Ans. 37cwt. 3qrs. 18lb.

4. Divide 375mls. 2 fur. 7pls. 2yds. 1ft. 2in. by 39.
Ans. 9mls. 4fur. 39pls. 2ft. 8in.

5. Divide 120L. 2qrs. 1bu. 2pe. by 74.
Ans. 1L. 6qrs. 1bu. 3pe.

6. Divide 120mo. 2w. 3d. 5h. 20′ by 111.
Ans. 1mo. 2d. 10h. 12′.

DUODECIMALS.

DUODECIMALS are so called because they decrease by twelves, from the place of feet toward the right. Inches are sometimes called *primes*, and are marked thus ′; the next division, after inches, is called parts, or *seconds*, and is marked thus ″; the next is *thirds*, and marked thus ‴; and so on.

Duodecimals are commonly used by workmen and artificers in finding the contents of their work.

Multiplication of Duodecimals; or, Cross Multiplication.

RULE.

1. Under the multiplicand write the same names or denominations of the multiplier, that is, feet under feet, inches under inches, &c.

2. Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and write each result under its respective term, observing to carry an unit for every 12, from each lower denomination to its next superior.

3. In the same manner multiply every term in the multiplicand by the inches in the multiplier, and set the result of each term one place farther toward the right of those in the multiplicand.

4. Proceed in like manner with the seconds and all the rest of the denominations, if there be any more; and the sum of all the lines will be the product required.

Or the denominations of the particular products will be as follow.

Feet by feet give feet.
Feet by primes give primes.
Feet by seconds give seconds,
&c.
Primes by primes give seconds.
Primes by seconds give thirds.
Primes by thirds give fourths,
&c.
Seconds by seconds give fourths.
Seconds by thirds give fifths.
Seconds by fourths give sixths,
&c.
Thirds by thirds give sixths.
Thirds by fourths give sevenths.
Thirds by fifths give eights,
&c.

In general thus;

When feet are concerned, the product is of the same denomination with the term multiplying the feet.

When feet are not concerned, the name of the product will be expressed by the sum of the indices of the two factors, or of the strokes over them.

EXAMPLES.

1. Multiply 10f. 4′ 5″ by 7f. 8′ 6″.

10	4	5		
7	8	6		
72	6	11		
6	10	11	4	
	5	2	2	6
79	11	0	6	6 Answer.

2. Multiply 4f. 7′ by 6f. 4′ 0″. Ans. 29f. 0′ 4″.

3. Multiply 14f. 9′ by 4f. 6′.

Ans. 66f. 4′ 6″.

4. Multiply 4f. 7′ 8″ by 9f. 6′.

Ans. 44f. 0′ 10″.

5. Multiply 7f. 8′ 6″ by 10f. 4′ 5″.

Ans. 79f. 11′ 0″ 6‴ 6^{iv}.

6. Multiply 39f. 10′ 7″ by 18f. 8′ 4″.

Ans. 745f. 6′ 10″ 2‴ 4^{iv}.

7. Multiply 44f. 2′ 9″ 2‴ 4^{iv}. by 2f. 10′ 3″.

Ans. 126f. 2′ 10″ 8‴ 10^{iv}. 11^{v}.

8. Multiply 24f. 10″ 8″ 7‴ 5^{iv}. by 9f. 4′ 6″.

Ans. 233f. 4′ 5″ 9‴ 6^{iv}. 4^{v}. 6^{vi}.

9. Required the content of a floor 48f. 6′ long, and 24f. 3′ broad.

Ans. 1176f. 1′ 6″.

10. What is the content of a marble slab, whose length is 5f. 7′, and breadth 1f. 10′ ?

Ans. 10f. 2′ 10″.

11. Required the content of a ceiling, which is 43f. 3′ long, and 25f. 6′ broad.

Ans. 1102f. 10′ 6″.

12. The length of a room being 20f. its breadth 14f. 6′, and height 10f. 4′ ; how many yards of painting are in it, deducting a fire place of 4f. by 4f. 4′, and two windows, each 6f. by 3f. 2′ ? Ans. $73\frac{2}{27}$ yards.

13. Required the solid content of a wall 53f. 6′ long, 10f. 3′ high, and 2f. thick.

Ans. 1310f. 9′.

VULGAR FRACTIONS.

1. FRACTIONS are expressions for parts of an integer or whole. *Vulgar Fractions* are represented by two numbers, placed one above the other, with a line between them.

2. The number above the line is called the *numerator;* and that below the line, the *denominator.*

The denominator shows how many parts the integer is divided into; and the numerator shows how many of those parts are contained in the fraction.

3 A *proper fraction* is one, whose numerator is less than the denominator; as $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{9}$, &c.

4. An *improper fraction* is one, whose numerator exceeds the denominator; as $\frac{8}{3}$, $\frac{110}{21}$, &c.

5. A *single fraction* is a simple expression for any number of parts of the integer.

6. A *compound fraction* is the fraction of a fraction; as $\frac{1}{2}$ of $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{5}{6}$, &c.

7. A *mixed number* is composed of a whole number and a fraction; as $8\frac{1}{5}$, $17\frac{6}{13}$, &c.

NOTE.—Any whole number may be expressed like a fraction by writing 1 under it; as $\frac{3}{1}$.

8. The *common measure* of two or more numbers is that number, which will divide each of them without a remainder. Thus 3 is the common measure of 12 and 15; and the *greatest* number, that will do this, is called the *greatest common measure.*

9. A number, which can be measured by two or more numbers, is called their *common multiple;* and if it be the *least* number, which can be so measured, it is called their

least common multiple; thus 30, 45, 60, and 75, are multiples of 3 and 5; but their least common multiple is 15.*

PROBLEM I.

To find the greatest common measure of two or more numbers.

RULE.†

1. If there be two numbers only, divide the greater by the less, and this divisor by the remainder, and so on till nothing

---

* A *prime number* is that, which can only be measured by an unit.

That number, which is produced by multiplying several numbers together, is called a *composite number.*

A *perfect number* is equal to the sum of all its aliquot parts.

The following perfect numbers are taken from the Petersburgh acts, and are all, that are known at present.

6
28
496
8128
33550336
8589869056
137438691328
2305843008139952128
2417851639228158837784576
9903520314282971830448816128

There are several other numbers, which have received different denominations, but they are principally of use in Algebra, and the higher parts of mathematics.

† The truth of the rule may be shown from the first example.—For since 54 measures 108, it also measures 108+54, or 162.

Again, since 54 measures 108, and 162, it also measures $5\times\overline{162}+108$, or 918. In the same manner it will be found to measure $2\times918+162$, or 1998, and so on. Therefore 54 measures both 918 and 1998.

remains, always dividing the last divisor by the last remainder; then will the last divisor be the greatest common measure required.

2. When there are more than two numbers, find the greatest common measure of two of them as before; and of that common measure and one of the other numbers; and so on through all the numbers to the last; then will the greatest common measure, last found, be the answer.

3. If 1 happen to be the common measure, the given numbers are prime to each other, and found to be incommeasurable.

EXAMPLES.

1. Required the greatest common measure of 918, 1998, and 522.

```
918)1998(2
    1836
    ----
     162)918(5
         810
         ---
         108)162(1
             108
             ---
              54)108(2
                 108
                 ---
```

So 54 is the greatest common measure of 1998 and 918.

```
Hence 54)522(9
         486
         ---
          36)54(1
             36
             --
             18)36(2
                36
                --
```

Therefore 18 is the answer required.

2. What is the greatest common measure of 612 and 540? Ans. 36.

3. What is the greatest common measure of 720, 336, and 1736? Ans. 8.

It is also the greatest common measure; for suppose there be a greater, then since the greater measures 918 and 1998, it also measures the remainder 162; and since it measures 162 and 918, it also measures the ramainder 108; in the same manner it will be found to measure the remainder 54; that is, the greater measures the less, which is absurd. Therefore 54 is the greatest common measure.

In the very same manner, the demonstration may be applied to 3 or more numbers.

PROBLEM II.

To find the least common multiple of two or more numbers.

RULE.*

1. If there be only two numbers, divide their product by their greatest common measure; and the quotient will be their least common multiple.

2. When there are more than two numbers, find the least common multiple of two of them as before; and of that common multiple and one of the other numbers; and so on through all the numbers to the last; then will the least common multiple, last found, be the answer.

3. If the numbers be prime to each other, their product is their least common multiple.

EXAMPLES.

1. What is the least common multiple of 3, 5, 8, and 10?

```
    3
    5
   ---
   15 the least common multiple of 3 and 5.
    8
   ---
  120 the least common multiple of 3, 5, and 8.
   10                       10)1200(120, hence 10 is the
  ----                                   greatest common
10)1200(120 the answer.                  measure of 10 &
                                         1200.
```

2. What is the least common multiple of 4 and 6?

Ans. 12.

3. What is the least number, that 3, 4, 8, and 12 will measure? Ans. 24.

4. What is the least number, that can be divided by the nine digits without a remainder? Ans. 2520.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions is the bringing them out of one form into another, in order to prepare them for the operations of addition, subtraction, &c.

---

* The truth of this rule may in some measure be seen by an examination of the first example. It may be easily ascertained

CASE I.

To abbreviate or reduce fractions to their lowest terms.

RULE*.

Divide the terms of the given fraction by any number, that will divide them without a remainder, and these quotients

---

that 15 is the least number, that can be divided by 3 and 5 without a remainder ; and that 120 is the least number, that can be divided by 3, 5, and 8 without a remainder ; but this can also be divided by 10 without a remainder ; therefore 120 appears to be the least common multiple of 3, 5, 8, and 10.

* That dividing both the terms of the fraction equally by any number whatever will give another fraction, equal to the former, is evident. And if those divisions be performed as often as can be done, or the common divisor be the greatest possible, the terms of the resulting fraction must be the least possible.

NOTE 1. Any number, ending with an even number or a cypher, is divisible by 2.

2. Any number, ending with 5 or 0, is divisible by 5.

3. If the first place of any number on the right be 0, the whole is divisible by 10.

4. If the first two figures on the right of any number be divisible by 4, the whole is divisible by 4.

5. If the first three figures on the right of any number be divisible by 8, the whole is divisible by 8.

6. If the sum of the digits, constituting any number, be divisible by 3, or 9, the whole is divisible by 3, or 9.

7. All prime numbers, except 2 and 5, have 1, 3, 7, or 9, in the place of units ; and all other numbers are composite.

8. When numbers, with the sign of addition or subtraction between them, are to be divided by any number, each of the numbers must be divided. Thus $\frac{4+8+10}{2}=2+4+5=11$.

9. But if the numbers have the sign of multiplication between them, only one of them must be divided. Thus $\frac{3\times8\times10}{2\times6}$ $=\frac{3\times4\times10}{\times6}=\frac{1\times4\times10}{1\times2}=\frac{1\times2\times10}{1\times1}=\frac{20}{}=20$.

again in the same manner; and so on till it appears, that there is no number greater than 1, which will divide them, and the fraction willl be in its lowest terms.

Or,

Divide both the terms of the fraction by their greatest common measure, and the quotients will be the terms of the fraction required.

EXAMPLES.

1. Reduce $\frac{144}{240}$ to its lowest terms.

$$\frac{144}{240} \overset{(2)}{=} \frac{72}{120} \overset{(2)}{=} \frac{36}{60} \overset{(3)}{=} \frac{12}{20} \overset{(2)}{=} \frac{6}{10} \overset{(2)}{=} \frac{3}{5}$$, the answer.

Or thus:

```
144)240(1
    144
    ---
     96)144(1
         96
         --
         48)96(2
            96
            --
```

Therefore 48 is the greatest common measure, and $48)\frac{144}{240}=\frac{3}{5}$, the same as before.

2. Reduce $\frac{48}{272}$ to its least terms. Ans. $\frac{3}{17}$.
3. Reduce $\frac{192}{576}$ to its lowest terms. Ans. $\frac{1}{3}$.
4. Bring $\frac{825}{960}$ to its lowest terms. Ans. $\frac{55}{64}$.
5. Reduce $\frac{252}{364}$ to its least terms. Ans. $\frac{9}{13}$.
6. Reduce $\frac{5184}{6912}$ to its least terms. Ans. $\frac{3}{4}$.
7. Reduce $\frac{1344}{1536}$ to its lowest terms. Ans. $\frac{7}{8}$.
8. Abbreviate $\frac{6896800}{36700160}$ as much as possible.
Ans. $\frac{43105}{229376}$.

CASE II.

To reduce a mixed number to its equivalent improper fraction.

RULE.*

Multiply the whole number by the denominator of the

* All fractions represent a division of the numerator by the

fraction, and add the numerator to the product, then that sum written above the denominator will form the fraction required.

EXAMPLES.

1. Reduce $27\frac{2}{9}$ to its equivalent improper fraction.

$$\begin{array}{r} 27 \\ 9 \\ \hline 243 \\ 2 \\ \hline 245 \\ \hline 9 \end{array}$$

Or $\frac{\overline{27\times9+2}}{9}=\frac{245}{9}$ the answer.

2. Reduce $183\frac{5}{21}$ to its equivalent improper fraction. Ans. $\frac{3843}{21}$.

3. Reduce $514\frac{5}{16}$ to an improper fraction. Ans. $\frac{8229}{16}$.

4. Reduce $100\frac{19}{59}$ to an improper fraction. Ans. $\frac{5919}{59}$.

5. Reduce $47\frac{3147}{8400}$ to an improper fraction. Ans. $\frac{307947}{8400}$.

CASE III.

To reduce an improper fraction to its equivalent whole or mixed number.

RULE.*

Divide the numerator by the denominator, and the quotient will be the whole or mixed number required.

---

denominator, and are taken altogether as proper and adequate expressions for the quotient. Thus the quotient of 2 divided by 3 is $\frac{2}{3}$; whence the rule is manifest; for if any number be multiplied and divided by the same number, it is evident the quotient must be the same as the quantity first given.

* This rule is plainly the reverse of the former, and has its reason in the nature of common division.

EXAMPLES.

1. Reduce $\frac{981}{16}$ to its equivalent whole or mixed number.

$16)981(61\frac{5}{16}$
96
———
21
16
———
5

Or,

$\frac{981}{16}=981\div16=61\frac{5}{16}$ the answer.

2. Reduce $\frac{56}{8}$ to its equivalent whole or mixed number. Ans. 7.

3. Reduce $\frac{1245}{22}$ to its equivalent whole or mixed number. Ans $56\frac{13}{22}$.

4. Reduce $\frac{3848}{21}$ to its equivalent whole or mixed number. Ans. $183\frac{5}{21}$.

5. Reduce $\frac{621613}{514}$ to its equivalent whole or mixed number. Ans. $1209\frac{187}{514}$.

CASE IV.

To reduce a whole number to an equivalent fraction, having a given denominator.

RULE.*

Multiply the whole number by the given denominator, and place the product over the said denominator, and it will form the fraction required.

EXAMPLES.

1. Reduce 7 to a fraction, whose denominator shall be 9.

$7\times9=63$, and $\frac{63}{9}$ the answer.

And $\frac{63}{9}=63\div9=7$ the proof.

2. Reduce 13 to a fraction, whose denominator shall be 12. Ans. $\frac{156}{12}$.

3. Reduce 100 to a fraction, whose denominator shall be 90. Ans. $\frac{9000}{90}$.

---

* Multiplication and division are here equally used, and consequently the result is the same as the quantity first proposed.

CASE V.

To reduce a compound fraction to an equivalent single one.

RULE.*

Multiply all the numerators together for the numerator, and all the denominators together for the denominator, and they will form the single fraction required.

If part of the compound fraction be a whole or mixed number, it must be reduced to a fraction by one of the former cases.

When it can be done, divide any two terms of the fraction by the same number, and use the quotients instead thereof.

EXAMPLES.

1. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{8}{11}$ to a single fraction.

$$\frac{2\times3\times4\times8}{3\times4\times5\times11}=\frac{192}{660}=\frac{16}{55} \text{ the answer.}$$

Or,

$$\frac{2\times\not{3}\times\not{4}\times8}{\not{3}\times\not{4}\times5\times11}=\frac{16}{55} \text{ as before.}$$

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a single fraction. Ans. $\frac{1}{4}$.
3. Reduce $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{4}{11}$ to a single fraction. Ans. $\frac{5}{33}$.
4. Reduce $\frac{11}{12}$ of $\frac{7}{13}$ of $\frac{8}{19}$ of 10 to a single fraction.
Ans. $\frac{1540}{741}$.

---

* That a compound fraction may be represented by a single one is evident, since a part of a part must be equal to some part of the whole. The truth of the rule for this reduction may be shown as follows.

Let the compound fraction to be reduced be $\frac{2}{3}$ of $\frac{4}{7}$. Then $\frac{1}{3}$ of $\frac{4}{7}=\frac{4}{7}\div\frac{4}{21}$, and consequently $\frac{2}{3}$ of $\frac{4}{7}=\frac{4}{21}\times2=\frac{8}{21}$ the same as by the rule, and the like will be found to be true in all cases.

If the compound fraction consist of more numbers than 2, the first two may be reduced to one, and that one and the third will be the same as the fraction of two numbers; and so on.

CASE VI.

To reduce fractions of different denominators to equivalent fractions, having a common denominator.

RULE 1.*

Multiply each numerator into all the denominators, except its own, for a new numerator; and all the denominators continually for the common denominator.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{4}{7}$ to equivalent fractions, having a common denominator.

$1\times5\times7=35$ the new numerator for $\frac{1}{2}$.
$3\times2\times7=42$ do. for $\frac{3}{5}$.
$4\times2\times5=40$ do. for $\frac{4}{7}$.
$2\times5\times7=70$ the common denominator.

Therefore the new equivalent fractions are $\frac{35}{70}$, $\frac{42}{70}$, and $\frac{40}{70}$, the answer.

2. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{8}$ to fractions, having a common denominator. Ans. $\frac{144}{288}$, $\frac{192}{288}$, $\frac{240}{288}$, $\frac{252}{288}$.

3. Reduce $\frac{1}{3}$, $\frac{3}{4}$ of $\frac{4}{5}$, $5\frac{1}{2}$, and $\frac{2}{19}$ to a common denominator. Ans. $\frac{190}{570}$, $\frac{342}{570}$, $\frac{3135}{570}$, $\frac{60}{570}$.

4. Reduce $\frac{11}{13}$, $\frac{3}{4}$ of $1\frac{1}{4}$, $\frac{9}{11}$, and $\frac{5}{7}$ to a common denominator. Ans. $\frac{13552}{16016}$, $\frac{15015}{16016}$, $\frac{13104}{16016}$, $\frac{11440}{16016}$.

RULE. II.

To reduce any given fractions to others, which shall have the least common denominator.

1. Find the least common multiple of all the denomina-

---

* By placing the numbers multiplied properly under one another, it will be seen, that the numerator and denominator of every fraction are multiplied by the very same number, and consequently their values are not altered. Thus in the first example;

$$\frac{1}{2}\Big|\frac{\times5\times7}{\times5\times7}\qquad\frac{3}{5}\Big|\frac{\times2\times7}{\times2\times7}\qquad\frac{4}{7}\Big|\frac{\times2\times5}{\times2\times5}$$

In the 2d rule, the common denominator is a multiple of all tors of the given fractions, and it will be the common denominator required.

2. Divide the common denominator by the denominator of each fraction, and multipy the quotient by the numerator, and the product will be the numerator of the fraction required.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ to fractions, having the least common denominator.

2
3
---
6 the least common denominator.

$6 \div 2 \times 1 = 3$ the first numerator; $6 \div 3 \times 2 = 4$ the second numerator; $6 \div 6 \times 5 = 5$ the third numerator.

Whence the required fractions are $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$.

2. Reduce $\frac{7}{12}$ and $\frac{11}{18}$ to fractions, having the least common denominator. Ans. $\frac{21}{36}$, $\frac{22}{36}$.

3. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ to the least common denominator. Ans. $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, $\frac{10}{12}$.

4. Reduce $\frac{2}{5}$, $\frac{4}{6}$, $\frac{5}{9}$, and $\frac{7}{10}$ to the least common denominator. Ans. $\frac{36}{90}$, $\frac{60}{90}$, $\frac{50}{90}$, $\frac{63}{90}$.

5. Reduce $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{16}$, and $\frac{17}{24}$ to equivalent fractions, having the least common denominator.

Ans. $\frac{16}{48}$, $\frac{36}{48}$, $\frac{40}{48}$, $\frac{42}{48}$, $\frac{33}{48}$, $\frac{34}{48}$.

CASE VII.

To find the value of a fraction in the known parts of the integer.

RULE.*

Multiply the numerator by the parts in the next inferior

---

the denominators, and consequently will divide by any of them; it is therefore manifest that proper parts may be taken for all the numerators as required.

* The numerator of a fraction may be considered as a re-

denomination, and divide the product by the denominator; and if any thing remain, multiply it by the next inferior denomination, and divide by the denominator as before; and so on as far as necessary; and the quotients placed after one another, in their order, will be the answer required.

EXAMPLES.

1. What is the value of $\frac{5}{7}$ of a shilling?

```
     5
    12
   ----
  7)60(8d. 2 2/7 q. Ans.
    56
   ----
     4

   ----
    16
    14
   ----
     2
```

2. What is the value of $\frac{3}{8}$ of a pound sterling?

Ans. 7s. 6d.

3. What is the value of $\frac{3}{5}$ of a pound Troy?

Ans. 7oz. 4dwt.

4. What is the value of $\frac{4}{7}$ of a pound avoirdupois?

Ans. 9oz. $2\frac{2}{7}$dr.

5. What is the value of $\frac{7}{9}$ of a cwt.?

Ans. 3qrs. 3lb. 1oz. $12\frac{4}{9}$dr.

6. What is the value of $\frac{3}{17}$ of a mile?

Ans. 1fur. 16pls. 2yds. 1ft. $9\frac{3}{17}$in.

7. What is the value of $\frac{5}{9}$ of an ell English?

Ans. 2qrs. $3\frac{1}{9}$nls.

8. What is the value of $\frac{7}{8}$ of a tun of wine?

Ans. 3hhd. 31gal. 2qts.

9. What is the value of $\frac{7}{13}$ of a day?

Ans. 12h. 55′ $23\frac{1}{13}$″.

---

mainder, and the denominator as a divisor; therefore this rule has its reason in the nature of compound division.

CASE VIII.

To reduce a fraction of one denomination to that of another, retaining the same value.

RULE.*

Make a compound fraction of it, and reduce it to a single one.

EXAMPLES.

1. Reduce $\frac{5}{6}$ of a penny to the fraction of a pound.

$\frac{5}{6}$ of $\frac{1}{12}$ of $\frac{1}{20}=\frac{5}{1440}=\frac{1}{288}$ the answer.

And $\frac{1}{288}$ of $\frac{20}{1}$ of $\frac{12}{1}=\frac{240}{288}=\frac{5}{6}$d. the proof.

2. Reduce $\frac{2}{3}$ of a farthing to the fraction of a pound. Ans. $\frac{1}{1440}$.

3. Reduce $\frac{1}{18}$l. to the fraction of a penny. Ans. $\frac{40}{3}$.

4. Reduce $\frac{4}{5}$ of a dwt. to the fraction of a pound Troy. Ans. $\frac{1}{300}$.

5. Reduce $\frac{6}{7}$ of a pound avoirdupois to the fraction of a cwt. Ans. $\frac{3}{392}$.

6. Reduce $\frac{9}{6552}$ of a hhd. of wine to the fraction of a pint. Ans. $\frac{9}{13}$.

7. Reduce $\frac{3}{13}$ of a month to the fraction of a day. Ans. $\frac{84}{13}$.

8. † Reduce 7s. 3d. to the fraction of a pound. Ans. $\frac{29}{80}$.

9. Express 6fur. 16pls. to the fraction of mile. Ans. $\frac{4}{5}$.

ADDITION OF VULGAR FRACTIONS.

RULE.‡

Reduce compound fractions to single ones; mixed numbers to improper fractions; fractions of different integers

---

* The reason of this practice is explained in the rule for reducing compound fractions to single ones.

The rule might have been distributed into two or three different cases, but the directions here given may very easily be applied to any question, that can be proposed in those cases, and will be more easily understood by an example or two, than by a multiplicity of words.

† Thus 7s. 3d. = 87d. and 1l. = 240d. $\therefore \frac{87}{240}=\frac{29}{80}$ the answer.

‡ Fractions, before they are reduced to a common denomina-

to those of the same; and all of them to a common denominator; then the sum of the numerators, written over the common denominator, will be the sum of the fractions required.

EXAMPLES.

1. Add $3\frac{5}{8}$, $\frac{7}{8}$, $\frac{4}{5}$ of $\frac{7}{8}$, and 7 together.

First $3\frac{5}{8}=\frac{29}{8}$, $\frac{4}{5}$ of $\frac{7}{8}=\frac{7}{10}$, $7=\frac{7}{1}$.

Then the fractions are $\frac{29}{8}$, $\frac{7}{8}$, $\frac{7}{10}$, and $\frac{7}{1}$; ∴

$$\begin{array}{r} 29\times8\times10\times\ 1=2320 \\ 7\times8\times10\times\ 1=\ 560 \\ 7\times8\times\ 8\times\ 1=\ 448 \\ 7\times8\times\ 8\times10=4480 \\ \hline 7808 \end{array}$$

$$\frac{7808}{8\times8\times10\ 1=640.}=12\frac{128}{640}=12\frac{1}{5} \text{ the answer.}$$

2. Add $\frac{5}{8}$, $7\frac{1}{2}$, and $\frac{1}{3}$ of $\frac{3}{4}$ together. Ans. $8\frac{3}{8}$.

3. What is the sum of $\frac{3}{5}$, $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.

4. What is the sum of $\frac{9}{10}$ of $6\frac{7}{8}$, $\frac{4}{7}$ of $\frac{1}{2}$, and $7\frac{1}{2}$?

Ans. $13\ \frac{109}{112}$.

5. Add $\frac{1}{7}$l. $\frac{2}{9}$s. and $\frac{5}{12}$ of a penny together.

Ans. $\frac{3139}{1008}$, or 3s. 1d. $1\frac{10}{21}$.

6. What is the sum of $\frac{2}{7}$ of 15l. $3\frac{3}{7}$l. $\frac{1}{3}$ of $\frac{5}{7}$ of $\frac{3}{5}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{7}$ of a shilling?

Ans 7l. 17s. $5\frac{1}{7}$d.

7. Add $\frac{2}{3}$ of a yard, $\frac{3}{4}$ of a foot, and $\frac{3}{8}$ of a mile together. Ans. 660yds. 2ft. 9in.

8. Add $\frac{1}{3}$ of a week, $\frac{1}{4}$ of a day, and $\frac{1}{2}$ of an hour together. Ans. 2d. $14\frac{1}{2}$h.

---

tor, are entirely dissimilar, and therefore cannot be incorporated with one another; but when they are reduced to a common denominator, and made parts of the same thing, their sum or difference may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any two quantities whatever by the sum or difference of their individuals; whence the reason of the rules, both for addition and subtraction, is manifest.

SUBTRACTION OF VULGAR FRACTIONS.

RULE.

Prepare the fractions as in addition, and the difference of the numerators, written above the common denominator, will give the difference of the fractions required.

EXAMPLES.

1. From $\frac{2}{3}$ take $\frac{2}{9}$ of $\frac{3}{7}$.

   $\frac{2}{9}$ of $\frac{3}{7} = \frac{2}{21}$, and $\frac{2}{3}$;

   $\therefore \frac{14}{21} - \frac{2}{21} = \frac{12}{21} = \frac{4}{7}$ the answer required.

2. From $\frac{97}{100}$ take $\frac{3}{7}$. Ans $\frac{379}{700}$.
3. From $96\frac{1}{3}$ take $14\frac{3}{7}$. Ans. $81\frac{19}{21}$.
4. From $14\frac{1}{4}$ take $\frac{2}{3}$ of 19. Ans. $1\frac{7}{12}$.
5. From $\frac{1}{2}$l. take $\frac{3}{4}$s. Ans. 9s. 3d.
6. From $\frac{3}{5}$oz. take $\frac{7}{8}$dwt. Ans. 11dwt. 3gr.
7. From 7 weeks take $9\frac{7}{10}$ days. Ans. 5w. 4d. 7h. 12'.

MULTIPLICATION OF VULGAR FRACTIONS.

RULE.*

Reduce compound fractions to single ones, and mixed numbers to improper fractions; then the product of the numerators is the numerator; and the product of the denominators, the denominator of the product required.

EXAMPLES.

1. Required the continued product of $2\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{3}$ of $\frac{5}{6}$, and 2.

$$2\tfrac{1}{2} = \tfrac{5}{2},\ \tfrac{1}{3} \text{ of } \tfrac{5}{6} = \frac{1 \times 5}{3 \times 6} = \tfrac{5}{18}, \text{ and } 2 = \tfrac{2}{1};$$

Then $\frac{5}{2} \times \frac{1}{8} \times \frac{5}{18} \times \frac{2}{1} = \frac{5 \times 1 \times 5 \times 2}{2 \times 8 \times 18 \times 1} = \frac{25}{144}$ the answer.

---

* Multiplication by a fraction implies the taking some part or parts of the multiplicand, and therefore may be truly expressed by a compound fraction. Thus $\frac{3}{4}$ multiplied by $\frac{5}{8}$ is the same as $\frac{3}{4}$ of $\frac{5}{8}$; and as the directions of the rule agree with the method already given to reduce these fractions to single ones, it is shown to be right.

2. Multiply $\frac{4}{15}$ by $\frac{5}{24}$. Ans. $\frac{1}{18}$.
3. Multiply $4\frac{1}{2}$ by $\frac{1}{8}$. Ans. $\frac{9}{16}$.
4. Multiply $\frac{1}{2}$ of 7 by $\frac{3}{6}$. Ans. $1\frac{3}{4}$.
5. Multiply $\frac{2}{9}$ of $\frac{3}{5}$ by $\frac{5}{8}$ of $3\frac{2}{7}$. Ans. $\frac{23}{84}$.
6. Multiply $4\frac{1}{2}$, $\frac{3}{4}$ of $\frac{1}{7}$, and $18\frac{4}{5}$, continually together. Ans. $9\frac{9}{140}$.

DIVISION OF VULGAR FRACTIONS.

RULE.*

Prepare the fractions as in multiplication; then invert the divisor, and proceed exactly as in multiplication.

EXAMPLES.

1. Divide $\frac{1}{5}$ of 19 by $\frac{2}{3}$ of $\frac{3}{4}$.

$\frac{1}{5}$ of $19=\frac{1\times19}{5\times1}=\frac{19}{5}$, and $\frac{2}{3}$ of $\frac{3}{4}=\frac{\not2\times\not3}{\not3\times\not4_{2}}=\frac{1}{2}$;

$\therefore \frac{19}{5}\times\frac{2}{1}=\frac{19\times2}{5\times1}=\frac{38}{5}=7\frac{3}{5}$ the quotient required.

2. Divide $\frac{4}{7}$ by $\frac{2}{3}$. Ans. $\frac{6}{7}$.
3. Divide $9\frac{1}{6}$ by $\frac{1}{2}$ of 7. Ans. $2\frac{13}{21}$.
4. Divide $3\frac{1}{6}$ by $9\frac{1}{2}$. Ans. $\frac{1}{3}$.
5. Divide $\frac{7}{8}$ by 4. Ans. $\frac{7}{32}$.
6. Divide $\frac{1}{2}$ of $\frac{2}{3}$ by $\frac{2}{3}$ of $\frac{3}{4}$. Ans. $\frac{2}{3}$.

---

* The reason of the rule may be shown thus. Suppose it were required to divide $\frac{3}{4}$ by $\frac{2}{5}$. Now $\frac{3}{4}\div2$ is manifestly $\frac{1}{2}$ of $\frac{3}{4}$ or $\frac{3}{4\times3}$; but $\frac{2}{5}=\frac{1}{5}$ of 2, $\therefore \frac{1}{5}$ of 2, or $\frac{2}{5}$ must be contained 5 times as often in $\frac{3}{4}$ as 2 is; that is $\frac{3\times5}{4\times2}$=the answer; which is according to the rule; and will be so in all cases.

NOTE.—A fraction is multiplied by an integer, by dividing the denominator by it, or multiplying the numerator. And divided by an integer, by dividing the numerator, or multiplying the denominator.

DECIMAL FRACTIONS.

A DECIMAL is a fraction, whose denominator is an unit, or 1, with as many cyphers annexed, as the numerator has places; and is commonly expressed by writing the numerator only, with a point before it, called the *separatrix*.

Thus,	0·5	is equal to $\frac{5}{10}$	or	$\frac{1}{2}$.
	0·25	$\frac{25}{100}$	or	$\frac{1}{4}$.
	0·75	$\frac{75}{100}$	or	$\frac{3}{4}$.
	1·3	$\frac{13}{10}$	or	$1\frac{3}{10}$.
	24·6	$24\frac{6}{10}$		
	·02	$\frac{2}{100}$	or	$\frac{1}{50}$.
	·0015	$\frac{15}{10000}$	or	$\frac{3}{2000}$.

A *finite* decimal is that, which ends at a certain number of places. But an *infinite* decimal is that, which is understood to be indefinitely continued.

A *repeating* decimal has one figure, or several figures, continually repeated, as far as it is found. As $\cdot 3\dot{3}$, &c. which is a *single repetend*. And $20\cdot 24\dot{2}\dot{4}$, &c. or $20\cdot 246\dot{2}4\dot{6}$, &c. which are *compound repetends*. Repeating decimals are also called *circulates*, or *circulating decimals*. A point is set over a single repetend, and a point over the first and last figures of a compound repetend.

The first place, next after the decimal mark, is 10th parts, the second is 100th parts, the third is 1000th parts, and so on, decreasing toward the right by 10ths, or increasing toward the left by 10ths, the same as whole or integral numbers do. As in the following

SCALE OF NOTATION.

&c.	Millions.	Hundreds of thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.	Tenth parts.	Hundredth parts.	Thousandth parts.	Ten thousandth parts.	Hundred thousandth parts.	Millionth parts.	&c.
	8	8	8	8	8	8	8.	8	8	8	8	8	8	8

Cyphers on the right of decimals do not alter their value.

For ·5 or $\frac{5}{10}$ is $\frac{1}{2}$
And ·50 or $\frac{50}{100}$ is $\frac{1}{2}$
And ·550 or $\frac{500}{1000}$ is $\frac{1}{2}$.

But cyphers before decimal figures, and after the separating point, diminish the value in a tenfold proportion for every cypher.

So ·5 is $\frac{5}{10}$ or $\frac{1}{2}$
But ·05 is $\frac{5}{100}$ or $\frac{1}{20}$
And ·005 is $\frac{5}{1000}$ or $\frac{1}{200}$

And so on.

So that, in any mixed or fractional number, if the separating point be moved one, two, three, &c. places to the right, every figure will be 10, 100, 1000, &c. times greater than before.

But if the point be moved toward the left, then every figure will be diminished in the same manner, or the whole quantity will be divided by 10, 100, 1000, &c.

ADDITION OF DECIMALS.

RULE.

1. Set the numbers under each other according to the value of their places, as in whole numbers, or so that the decimal points may stand each directly under the preceding.

2. Then add as in whole numbers, placing the decimal point in the sum directly under the other points.

EXAMPLES.

(1)

7530
16·201
3·0142
957·13
6·72819
·03014
8513·10353

2. What is the sum of 276, 39·213, 72014·9, 417, 5032, and 2214·298? Ans. 79993·411.

3. What is the sum of ·014, ·9816, ·32, ·15914, ·72913, and ·0047? Ans. 2·20857.

4. What is the sum of 27·148, 918·73, 14016, 294304, ·7138, and 221·7? Ans. 309488·2918.

5. Required the sum of 312·984, 21·3918, 2700·42, 3·153, 27·2, and 581·06. Ans. 3646·2088.

SUBTRACTION OF DECIMALS.

RULE.

1. Set the less number under the greater in the same manner as in addition.

2. Then subtract as in whole numbers, and place the decimal point in the remainder directly under the other points.

EXAMPLES.

(1)

21·481
4·90142
209·90858

2. From ·9173 subtract ·2138. Ans. ·7035.

3. From 2·73 subtract 1·9185. Ans. 0·8115.

4. What is the difference between 91·713 and 407?

Ans. 315·287.

5. What is the difference between 16·37 and 800·135?

Ans. 783·765.

MULTIPLICATION OF DECIMALS.

RULE.*

1. Set down the factors under each other, and multiply them as in whole numbers.

2. And from the product, toward the right point off as many figures for decimals, as there are decimal places in both the factors. But if there be not so many figures in the product as there ought to be decimals, prefix the proper number of cyphers to supply the defect.

EXAMPLES.

(1)

```
  91·78
   ·381
 ------
   9178
  73424
 27534
 ------
34·96818
```

2. What is the product of 520·3 and ·417?

Ans. 216·9651.

3. What is the product of 51·6 and 21? Ans. 1083·6.

4. What is the product of ·217 and ·0431?

Ans. ·0093527.

---

* To prove the truth of the rule, let ·9776 and ·823 be the numbers to be multiplied; now these are equivalent to $\frac{9776}{10000}$ and $\frac{823}{1000}$; whence $\frac{9776}{10000}\times\frac{823}{1000}=\frac{8045648}{10000000}=$·8045648 by the nature of notation, and consisting of as many places, as there are cyphers, that is, of as many places as are in both the numbers; and the same is true of any two numbers whatever.

5. What is the product of ·051 and ·0091?

Ans. ·0004641.

Note. When decimals are to be multiplied by 10, or 100, or 1000, &c. that is, by 1 with any number of cyphers, it is done by only moving the decimal point so many places farther to the right, as there are cyphers in the said multiplier; subjoining cyphers, if there be not so many figures.

EXAMPLES.

1. The product of 51·3 and 10 is	513.
2. The product of 2·714 and 100 is	271·4.
3. The product of ·9163 and 1000 is	916·3.
4. The product of 21·31 and 10000 is	213100.

CONTRACTION.

When the product would contain several more decimals than are necessary for the purpose in hand, the work may be much contracted, and only the proper number of decimals retained.

RULE.

1. Set the unit figure of the multiplier under such decimal place of the multiplicand as you intend the last of your product shall be, writing the other figures of the multiplier in an inverted order.

2. Then in multiplying reject all the figures in the multiplicand, which are on the right of the figure you are multiplying by; setting down the products so that their figures on the right may fall each in a straight line under the preceding; and carrying to such figures on the right from the product of the two preceding figures in the multiplicand thus, namely, 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c. inclusively; and the sum of the lines will be the product to the number of decimals required, and will commonly be the nearest unit in the last figure.

EXAMPLES.

1. Multiply 27·14986 by 92·41035, so as to retain only four places of decimals in the product.

Contracted.	Common way.	
27·14986	27·14986	
53014·29	92·41035	
24434874	13	574930
542997	81	44958
108599	2714	986
2715	108599	44
81	542997	2
14	24434874	
2508·9280	2508·9280	650510

2. Multiply 480·14936 by 2·72416, retaining four decimals in the product.

Ans. 1308·0037.

3. Multiply 73·8429753 by 4·628754, retaining five decimals in the product.

Ans. 341·80097.

4. Multiply 8634·875 by 843·7527, retaining only the integers in the product.

Ans. 7285699.

DIVISION OF DECIMALS.

RULE.*

Divide as in whole numbers; and to know how many decimals to point off in the quotient, observe the following rules.

---

* The reason of pointing off as many decimal places in the quotient, as those in the dividend exceed those in the divisor, will easily appear; for since the number of decimal places in the dividend is equal to those in the divisor and quotient, taken together, by the nature of multiplication; it follows, that the quotient contains as many as the dividend exceeds the divisor.

1. There must be as many decimals in the dividend, as in both the divisor and quotient; therefore point off for decimals in the quotient so many figures, as the decimal places in the dividend exceed those in the divisor.

2. If the figures in the quotient are not so many as the rule requires, supply the defect by prefixing cyphers.

3. If the decimal places in the devisor be more than those in the dividend, add cyphers as decimals to the dividend, till the number of decimals in the dividend be equal to those in the divisor, and the quotient will be integers till all these decimals are used. And, in case of a remainder, after all the figures of the dividend are used, and more figures are wanted in the quotient, annex cyphers to the remainder, to continue the division as far as necessary.

4. The first figure of the quotient will possess the same place of integers or decimals, as that figure of the dividend, which stands over the units place of the first product.

EXAMPLES.

1. Divide 3424·6056 by 43·6.

43·6)3424·6056(78·546
3052
———
3726
3488
———
2380
2180
———
2005
1744
———
2616
2616
———

2. Divide 3877875 by ·675.	Ans. 5745000.
3. Divide ·0081892 by ·347.	Ans. ·0236.
4. Divide 7·13 by ·18.	Ans. 39.

CONTRACTIONS.

I. *If the divisor be an integer with any number of cyphers at the end;* cut them off, and remove the decimal point in the dividend so many places farther to the left, as there were cyphers cut off, prefixing cyphers, if need be; then proceed as before.

EXAMPLES.

1. Divide 953 by 21000.

```
21·000)
      3)·953
      7)·31766
        ·04538, &c.
```

Here I first divide by 3, and then by 7, because 3 times 7 is 21.

2. Divide 41020 by 32000. Ans. 1·281875.

Note. Hence, if the divisor be 1 with cyphers, the quotient will be the same figures with the dividend, having the decimal point so many places farther to the left, as there are cyphers in the divisor.

EXAMPLES.

217·3÷100=2·173. 419 by 10=41·9.
5·16 by 1000=·00516. ·21 by 1000=·00021.

II. *When the number of figures in the divisor is great, the operation may be contracted, and the necessary number of decimal places obtained.*

RULE.

1. Having, by the 4th general rule, found what place of decimals or integers the first figure of the quotient will possess; consider how many figures of the quotient will serve the present purpose; then take the same number of figures on the left of the divisor, and as many of the dividend figures as will contain them less than ten times; by these find the first figure of the quotient.

2. And for each following figure, divide the last remainder by the divisor, wanting one figure to the right more than before, but observing what must be carried to the first product for such omitted figures, as in the contraction of Multiplication; and continue the operation till the divisor is exhausted.

3. When there are not so many figures in the divisor, as are required to be in the quotient, begin the division with all the figures as usual, and continue it till the number of figures in the divisor and those remaining to be found in the quotient be equal; after which use the contraction.

EXAMPLES.

1. Divide 2508·928065051 by 92·41035, so as to have four decimals in the quotient.—In this case, the quotient will contain six figures. Hence

Contraction.

```
92·4103,5)2508·928,065051(27·1498
  .....    1848207
           -------
            660721 .
            646872
            ------
             13849 . .
              9241
              ----
              4608 . . .
              3696
              ----
               912 . . . .
               832
               ---
                80 . . . . .
                74
                --
                 6 . . . . . .
```

Common Way.

```
92·41035)2508·928065051(27·1498
          1848207|0
          -------|--
           660721|06
           646872|45
           ------|---
            13848|615
             9241|035
             ----|----
             4607|5800
             3696|4140
             ----|-----
              911|16605
              831|69315
              ---|------
               79|472901
               73|928280
               --|------
                5|544621
```

2. Divide 721·17562 by 2·257432, so that the quotient may contain three decimals. Ans. 319·467.

3. Divide 12·169825 by 3·14159, so that the quotient may contain five decimals. Ans. 3·87377.

4. Divide 87·076326 by 9·365407, and let the quotient contain seven decimals. Ans. 9·2976559.

REDUCTION OF DECIMALS.

CASE 1.

To reduce a vulgar fraction to its equivalent decimal.

RULE.*

Divide the numerator by the denominator, annexing as many cyphers as are necessary; and the quotient will be the decimal required.

---

* Let the given vulgar fraction, whose decimal expression is required, be $\frac{7}{13}$. Now since every decimal fraction has 10, 100,

EXAMPLES.

1. Reduce $\frac{5}{24}$ to a decimal.

$$4)5\text{'}000000$$
$$6)1\text{'}250000$$
$$\text{'}208333, \&c.$$

2. Required the equivalent decimal expressions for $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. Ans. '25, '5, and '75.
3. What is the decimal of $\frac{3}{8}$? Ans. '375.
4. What is the decimal of $\frac{1}{25}$? Ans. '04.
5. What is the decimal of $\frac{3}{192}$? Ans. '015625.
6. Express $\frac{275}{3842}$ decimally. Ans. '071577, &c.

CASE II.

To reduce numbers of different denominations to their equivalent decimal values.

RULE.*

1. Write the given numbers perpendicularly under each other for dividends, proceeding orderly from the least to the greatest.

2. Opposite to each dividend, on the left, place such a number for a divisor, as will bring it to the next superior name, and draw a line between them.

1000, &c. for its denominator; and, if two fractions be equal, it will be, as the denominator of one is to its numerator, so is the denominator of the other to its numerator; therefore 13 : 7 : : 1000, &c. : $\frac{7\times1000,\&c.}{13}=\frac{70000,\ \&c.}{13}=\cdot5384\frac{8}{13}$, the numerator of the decimal required; and is the same as by the rule.

* The reason of the rule may be explained from the first example; thus, three farthings are $\frac{3}{4}$ of a penny, which brought to a decimal is '75; consequently $9\frac{3}{4}$d. may be expressed 9'75d. but 9·75 is $\frac{975}{100}$ of a penny $=\frac{975}{1200}$ of a shilling, which brought to a decimal is '8125; and therefore 15s. $9\frac{3}{4}$d. may be expressed 15·8125s. In like manner 15'8125s. is $\frac{158125}{10000}$ of a shilling $=\frac{158125}{200000}$ of a pound =, by bringing it to a decimal, ·790625l. as by the rule.

3. Begin with the highest, and write the quotient of each division, as decimal parts, on the right of the dividend next below it; and so on till they are all used, and the last quotient will be the decimal sought.

EXAMPLES.

1. Reduce 15s. $9\frac{3}{4}$d. to the decimal of a pound.

4	3˙
12	9˙75
20	15˙8125

˙790625 the decimal required.

2. Reduce 9s. to the decimal of a pound. Ans. ˙45

3. Reduce 19s. $5\frac{1}{2}$d. to the decimal of a pound. Ans. ˙972916.

4. Reduce 10oz. 18dwt. 16gr. to the decimal of a lb. Troy. Ans. ˙911111, &c.

5. Reduce 2qrs. 14lb. to the decimal of a cwt. Ans. ˙625, &c.

6. Reduce 17yd. 1ft. 6in. to the decimal of a mile. Ans. ˙00994318, &c.

7. Reduce 3qrs. 2nls. to the decimal of a yard. Ans. ˙875.

8. Reduce 1gal. of wine to the decimal of a hhd. Ans. ˙015873.

9. Reduce 3bu. 1pe. to the decimal of a quarter. Ans. ˙40625.

10. Reduce 10w. 2d. to the decimal of a year. Ans. ˙1972602, &c.

CASE III.

To find the decimal of any number of shillings, pence, and farthings by inspection.

RULE.*

Write half the greatest even number of shillings for the first decimal figure, and let the farthings in the given pence

* The invention of the rule is as follows; as shillings are so many 20ths of a pound, half of them must be so many 10ths,

and farthings possess the second and third places; observing to increase the second place by 5, if the shillings be odd; and the third place by 1, when the farthings exceed 12; and by 2, when they exceed 36.

EXAMPLES.

1. Find the decimal of 15s. $8\frac{1}{2}$d. by inspection.

7 $=\frac{1}{2}$ of 14s.
5 for the odd shilling.
34 = farthings in $8\frac{1}{2}$d.
1 for the excess above 12.

·785 = decimal required.

2. Find by inspection the decimal expression of 16s. $4\frac{1}{2}$d. and 13s. $10\frac{1}{2}$d. Ans. ·819 and ·694.

3. Value the following sums by inspection, and find their total, viz. 19s. $11\frac{1}{4}$d. + 6s. 2d. + 12s. $8\frac{3}{4}$d. + 1s. $10\frac{1}{4}$d. + $\frac{3}{4}$d. + $1\frac{1}{4}$d. Ans. 2·042 the total.

CASE IV.

To find the value of any given decimal in terms of the integer.

RULE.

1. Multiply the decimal by the number of parts in the next less denomination, and cut off as many places for a re-

---

and consequently take the place of 10ths in the decimal; but when they are odd, their half will always consist of two figures, the first of which will be half the even number, next less, and the second a 5; and this confirms the rule as far as it respects shillings.

Again, farthings are so many 960ths of a pound; and had it happened, that 1000, instead of 960, had made a pound, it is plain any number of farthings would have made so many thousandths, and might have taken their place in the decimal accordingly. But 960, increased by $\frac{1}{24}$ part of itself, is $=1000$; consequently any number of farthings, increased by their $\frac{1}{24}$ part, will be an exact decimal expression for them. Whence, if the mainder on the right as there are places in the given decimal.

2. Multiply the remainder by the parts in the next inferior denomination, and cut off for a remainder as before.

3. Proceed in this manner through all the parts of the integer, and the several denominations, standing on the left, make the answer.

EXAMPLES.

1. Find the value of ·37623 of a pound.

```
        20
   -------
   7·52460
        12
   -------
   6·29520
         4
   -------
   1·18080      Ans. 7s. 6¼d.
```

2. What is the value of ·625 of a shilling? Ans. $7\frac{1}{2}$d.

3. What is the value of ·8322916l.? Ans. 16s. $7\frac{1}{2}$d.

4. What is the value of ·6725cwt.? Ans. 2qrs. 19lb. 5oz.

5. What is the value of ·67 of a league?
Ans. 2mls. 3pls. 1yd. 3in. 1b. c.

6. What is the value of ·61 of a tun of wine?
Ans. 2hhd. 27gal. 2qt. 1pt.

7. What is the value of ·461 of a chaldron of coals?
Ans. 16bu. 2pe.

8. What is the value of ·42857 of a month?
Ans. 1w. 4d. 23h. 59′ 56″.

CASE V.

To find the value of any decimal of a pound by inspection.

RULE.

Double the first figure or place of tenths for shillings, and if the second be 5 or more than 5 reckon another shil-

---

number of farthings be more than 12, a $\frac{1}{24}$ part is greater than $\frac{1}{2}$, and therefore 1 must be added; and when the number of farthings is more than 37, a $\frac{1}{24}$ part is greater than $1\frac{1}{2}$, for which 2 must be added; and thus the rule is shown to be right, ling; then call the figures in the second and third places, after 5, if contained, is deducted, so many farthings; abating 1, when they are above twelve; and 2, when above 36; and the result is the answer.

EXAMPLES.

1. Find the value of ˙785l. by inspection.

14s. = double 7.
1s. for 5 in the place of tenths.
$8\frac{3}{4}$ = 35 farthings.
$\frac{1}{4}$ for the excess of 12, abated.

15s. $8\frac{1}{2}$d. the answer.

2. Find the value of ˙875l. by inspection. Ans. 17s. 6d.

3. Value the following decimals by inspection, and find their sum, viz. ˙927l. + ˙351l. + ˙203l. + ˙061l. + ˙02l. + ˙009l. Ans. 1l. 11s. $5\frac{1}{4}$d.

FEDERAL MONEY.*

THE denominations of *Federal Money*, as determined by an Act of Congress, Aug. 8, 1786, are in a *decimal* ratio; and therefore may be properly introduced in this place.

* The coins of federal money are two of gold, four of silver, and two of copper. The gold coins are called an *eagle* and *half-eagle*; the silver, a *dollar*, *half-dollar*, *double dime*, and *dime*; and the copper, a *cent* and *half-cent*. The standard for gold and silver is eleven parts fine and one part alloy. The weight of fine gold in the eagle is 246˙268 grains; of fine silver in the dollar, 375˙64 grains; of copper in 100 cents, $2\frac{1}{4}$lb. avoirdupois. The fine gold in the half-eagle is half the weight of that in the eagle; the fine silver in the half-dollar, half the weight of that in the dollar, &c. The denominations less than a dollar are expressive of their values: thus, *mill* is an abbreviation of *mille*, a thousand, for 1000 mills are equal to 1 dollar: *cent*, of *centum*, a hun-

A mill, which is the lowest money of account, is ·001 of a dollar, which is the money unit.

A cent	is	·01	Or 10 mills = 1 cent.	
A dime		·1	marked	m. c.
A dollar		1·	10 cents = 1 dime,	d.
An eagle		10·	10 dimes = 1 dollar,	D.
			10 dollars = eagle,	E.

A number of dollars, as 754, may be read 754 dollars, or 75 eagles, 4 dollars ; and decimal parts of a dollar, as ·365,

---

dred, for 100 cents are equal to 1 dollar ; a *dime* is the French of *tithe*, the tenth part, for 10 dimes are equal to 1 dollar.

The mint-price of uncoined gold, 11 parts being fine and 1 part alloy, is 209 dollars, 7 dimes, and 7 cents per lb. Troy weight ; and the mint-price of uncoined silver, 11 parts being fine and 1 part alloy, is 9 dollars, 9 dimes and 2 cents, per lb. Troy.

In Mr. PIKE's "Complete System of Arithmetic," may be seen "RULES for reducing the Federal Coin, and the Currencies of the several United States ; also English, Irish, Canada, Nova-Scotia, Livres Tournois, and Spanish milled dollars, each to the *par* of all the others." It may be sufficient here to observe respecting the currencies of the several States, that a dollar is equal to 6s. in New-England and Virginia ; 8s. in New-York and North-Carolina ; 7s. 6d. in New-Jersey, Pennsylvania, Delaware, and Maryland ; and 4s. 8d. in South-Carolina and Georgia.

The English standard for gold is 22 carats of fine gold, and 2 carats of copper, which is the same as 11 parts fine and 1 part alloy. The English standard for silver is 18oz. 2dwt. of fine silver, and 18dwt. of copper ; so that the proportion of alloy in their silver is less than in their gold. When either gold or silver is finer or coarser than standard, the variation from standard is estimated by carats and grains of a carat in gold, and by penny-weights in silver. Alloy is used in gold and silver to harden them.

NOTE.—Carat is not any certain weight or quantity, but $\frac{1}{24}$ of any weight or quantity ; and the minters and goldsmiths divide it into 4 equal parts, called *grains* of a carat.

may be read 3 dimes, 6 cents, 5 mills, or 36 cents, 5 mills, or 365 mills; and others in a similar manner.

Addition, *subtraction*, *multiplication*, and *division* of federal money are performed just as in decimal fractions; and consequently with more ease than in any other kind of money.

EXAMPLES.

1. Add 2 dollars, 4 dimes, 6 cents, 4D. 2d., 4d. 9c., 1E. 3D. 5c. 7m., 3c. 9m., 1D. 2d. 8c. 1m., and 2E. 4D. 7d. 8c. 2m. together.

	(2)	(3)
E. D. d. c. m.	E. D. d. c. m.	E. D d. c. m.
2·46	34·123	30·671
4·2	1·178	3·123
·49	78·001	4·567
13·057	1·7	·03
·039	·32	70·308
1·281	61·789	7·17
24·782	6·341	8·231
46·309 Ans.		

	(4)	(5)	(6)
	E. D. d. c. m.	E. D. d. c. m.	D. d. c. m.
From	32·178	70·000	2·652
Subtract	17·289	7·813	·07
Remain.	14·889		

7. Multiply 3D. 4d. 5c. 1m. by 1D. 2d. 3c. 2m.

```
   D.
   3·451
   1·232
  ------
   6902
  10353
  6902
 3451
 -------
4·251632=4·251 632/1000 Ans.
```

$4{\cdot}251632 = 4{\cdot}251\frac{632}{1000}$ Ans.

NOTE. The figures after or on the right of mills are decimals of a mill.

8. Multiply $\overset{D.}{6\cdot347}$ by $\overset{D.}{4\cdot532}$. Ans. $\overset{D.}{28\cdot764604}$.

9. Multiply $\overset{D.}{71\cdot012}$ by $\overset{D.}{3\cdot703}$ Ans. $\overset{D.}{262\cdot957436}$.

10. Multiply $\overset{D.}{806\cdot222\frac{2}{9}}$ by 9. Ans. $\overset{D.}{7256}$.

11. Divide $\overset{D.}{4\cdot251632}$ by $\overset{D.}{1\cdot232}$.

```
1·232)4·251632(3·451 Answer.
      3696
      ----
       5556
       4928
       ----
        6283
        6160
        ----
         1232
         1232
         ----
```

12. Divide 20D. by 2000. Ans. $\overset{D.}{0\cdot01}$.

13. Divide 7256D. by 9. Ans. $\overset{D.}{806\cdot222\frac{2}{9}}$.

CIRCULATING DECIMALS.

It has already been observed, that when an infinite decimal repeats always one figure, it is a *single repetend;* and when more than one, a *compound repetend;* also that a point is set over a single repetend, and a point over the first and last figures of a compound repetend.

It may be farther observed, that when other decimal figures precede a repetend in any number, it is called a *mixed repetend:* as $\cdot2\dot{3}$, or $\cdot104\dot{1}2\dot{3}$; otherwise it is a *pure*, or *simple, repetend:* as $\cdot\dot{3}$ and $\cdot\dot{1}2\dot{3}$.

Similar repetends begin at the same place: as $\cdot\dot{3}$ and $\cdot\dot{6}$, or $1\cdot\dot{3}4\dot{1}$ and $2\cdot\dot{1}5\dot{6}$.

Dissimilar repetends begin at different places: as $.\dot{2}5\dot{3}$ and $.47\dot{5}\dot{2}$.

Conterminous repetends end at the same place: as $.\dot{1}2\dot{5}$ and $.\dot{0}0\dot{9}$.

Similar and *conterminous repetends* begin and end at the same place: as $2.9\dot{1}0\dot{4}$ and $.0\dot{6}1\dot{3}$.

REDUCTION OF CIRCULATING DECIMALS.

CASE I.

To reduce a simple repetend to its equivalent vulgar fraction.

RULE*.

1. Make the given decimal the numerator, and let the denominator be a number, consisting of as many nines as there are recurring places in the repetend.

2. If there be integral figures in the circulate, as many cyphers must be annexed to the numerator, as the highest place of the repetend is distant from the decimal point.

---

* If unity, with cyphers annexed, be divided by 9 *ad infinitum*, the quotient will be 1 continually; i. e. if $\frac{1}{9}$ be reduced to a decimal, it will produce the circulate $.\dot{1}$; and since $.\dot{1}$ is the decimal equivalent to $\frac{1}{9}$, $.\dot{2}$ will $=\frac{2}{9}$, $.\dot{3}=\frac{3}{9}$, and so on till $.\dot{9}=\frac{9}{9}=1$.

Therefore every single repetend is equal to a vulgar fraction, whose numerator is the repeating figure, and denominator 9.

Again, $\frac{1}{99}$, or $\frac{1}{99}$, being reduced to decimals, makes $.010101$, &c. or $.001001$, &c. *ad infinitum* $=.\dot{0}\dot{1}$ or $\dot{0}0\dot{1}$; that is, $\frac{1}{99}=.\dot{0}\dot{1}$, and $\frac{1}{999}=.\dot{0}0\dot{1}$; consequently $\frac{2}{99}=.\dot{0}\dot{2}$, $\frac{3}{99}=.\dot{0}\dot{3}$, &c. and $\frac{2}{999}=.\dot{0}0\dot{2}$, $\frac{3}{999}=.\dot{0}0\dot{3}$, &c. and the same will hold universally.

EXAMPLES.

1. Required the least vulgar fractions equal to $\cdot\dot{6}$ and $\cdot\dot{1}2\dot{3}$.

$\cdot\dot{6}=\frac{6}{9}=\frac{2}{3}$; and $\cdot\dot{1}2\dot{3}=\frac{123}{999}=\frac{41}{333}$ Ans.

2. Reduce $\cdot\dot{3}$ to its equivalent vulgar fraction. Ans. $\frac{1}{3}$.

3. Reduce $\dot{1}6\dot{2}$ to its equivalent vulgar fraction.
Ans. $\frac{1620}{999}$.

4. Required the least vulgar fraction equal to $\cdot\dot{7}6923\dot{0}$.
Ans. $\frac{10}{13}$.

CASE II.

To reduce a mixed repetend to its equivalent vulgar fraction.

RULE.*

1. To as many nines as there are figures in the repetend, annex as many cyphers as there are finite places, for a denominator.

2. Multiply the nines in the said denominator by the finite part, and add the repeating decimal to the product, for the numerator.

3. If the repetend begin in some integral place, the finite value of the circulating part must be added to the finite part.

---

* In like manner for a mixed circulate; consider it as divisible into its finite and circulating parts, and the same principle will be seen to run through them also: thus, the mixed circulate $\cdot 1\dot{6}$ is divisible into the finite decimal $\cdot 1$, and the repetend $\cdot 0\dot{6}$; but $\cdot 1=\frac{1}{10}$, and $\cdot\dot{6}$ would be $=\frac{6}{9}$, provided the circulation began immediately after the place of units; but as it begins after the place of tens, it is $\frac{6}{9}$ of $\frac{1}{10}=\frac{6}{90}$, and so the vulgar fraction $=\cdot 1\dot{6}$ is $\frac{1}{10}+\frac{6}{90}=\frac{9}{90}+\frac{6}{90}=\frac{15}{90}$, and is the same as by the rule.

EXAMPLES.

1. What is the vulgar fraction equivalent to $.13\dot{8}$?
$9\times13+8=125=$ numerator, and $900=$ the denominator; $\therefore$ $.13\dot{8}=\frac{125}{900}=\frac{5}{36}$ the answer.

2. What is the least vulgar fraction equivalent to $.5\dot{3}$?
Ans. $\frac{8}{15}$.

3. What is the least vulgar fraction equal to $.\dot{5}92\dot{5}$?
Ans. $\frac{16}{27}$

4. What is the least vulgar fraction equal to $.008\dot{4}9713\dot{3}$?
Ans. $\frac{83}{9768}$.

5. What is the finite number equivalent to $3\dot{1}.6\dot{2}$?
Ans. $31\frac{23}{37}$.

CASE III.

To make any number of dissimilar repetends similar and conterminous.

RULE.*

Change them into other repetends, which shall each consist of as many figures as the least common multiple of the several numbers of places, found in all the repetends, contains units.

EXAMPLES.

1.	Dissimilar.		Made similar and conterminous.
	$9.\dot{8}1\dot{4}$	=	$9.81\dot{4}8148\dot{1}$

---

* Any given repetend whatever, whether single, compound, pure, or mixed, may be transformed into another repetend, that shall consist of an equal or greater number of figures at pleasure: thus $.\dot{4}$ may be transformed to $.\dot{4}\dot{4}$, or $.\dot{4}4\dot{4}$, or $.4\dot{4}$, &c. Also $.\dot{5}\dot{7}=.\dot{5}75\dot{7}=.57\dot{5}\dot{7}=.5\dot{7}\dot{5}$; and so on; which is too evident to need any farther demonstration.

$$1\cdot5 = 1\cdot50\dot{0}0000\dot{0}$$

$$87\cdot2\dot{6} = 87\cdot26\dot{6}6666\dot{6}$$

$$\cdot08\dot{4} = \cdot08\dot{3}3333\dot{3}$$

$$124\cdot\dot{0}\dot{9} = 124\cdot09\dot{0}9090\dot{9}$$

2. Make $\cdot\dot{3}$, $\cdot2\dot{7}$ and $\cdot04\dot{5}$ similar and conterminous.

3. Make $\cdot\dot{3}2\dot{1}$, $\cdot82\dot{6}\dot{2}$, $\cdot\dot{0}\dot{5}$ and $\cdot09\dot{0}\dot{2}$ similar and conterminous.

4. Make $\cdot521\dot{7}$, $3\cdot\dot{6}4\dot{3}$ and $17\cdot\dot{1}2\dot{3}$ similar and conterminous.

CASE IV.

To find whether the decimal fraction, equal to a given vulgar one, be finite or infinite, and of how many places the repetend will consist.

RULE.*

1. Reduce the given fraction to its least terms, and divide the denominator by 2, 5, or 10, as often as possible.

---

* In dividing $1\cdot\dot{0}000$, &c. by any prime number whatever, except 2 or 5, the figures in the quotient will begin to repeat as soon as the remainder is 1. And since 9999, &c. is less than 10000, &c. by 1, therefore 9999, &c. divided by any number whatever will leave 0 for a remainder, when the repeating figures are at their period. Now whatever number of repeating figures we have, when the dividend is 1, there will be exactly the same number, when the dividend is any other number whatever. For the product of any circulating number, by any other given number, will consist of the same number of repeating figures as before. Thus, let ·507650765076, &c. be a circulate, whose repeating part is 5076. Now every repetend (5076) being equally multiplied, must produce the same product. For though these products will consist of more places, yet the overplus in each,

2. If the whole denominator vanish in dividing by 2, 5, or 10, the decimal will be finite, and will consist of so many places, as you perform divisions.

3. If it do not so vanish, divide 9999, &c. by the result, till nothing remain, and the number of 9s used will show the number of places in the repetend; which will begin after so many places of figures, as there were 10s, 2s, or 5s, used in dividing.

EXAMPLES.

1. Required to find whether the decimal equal to $\frac{210}{1120}$ be finite or infinite; and if infinite, how many places the repetend will consist of.

$\frac{210}{1120} = 2|\frac{3}{16}$ | $\overset{2}{8}$ | $\overset{2}{4}$ | $\overset{2}{2}$ | 1; therefore the decimal is finite, and consists of 4 places.

2. Let $\frac{1}{11}$ be the fraction proposed.
3. Let $\frac{2}{7}$ be the fraction proposed.
4. Let $\frac{13}{407}$ be the fraction proposed.
5. Let $\frac{1}{8544}$ be the fraction proposed.

ADDITION OF CIRCULATING DECIMALS.

RULE.*

1. Make the repetends similar and conterminous, and find their sum as in common addition.

---

being alike, will be carried to the next, by which means each product will be equally increased, and consequently every four places will continue alike. And the same will hold for any other number whatever.

Now hence it appears, that the dividend may be altered at pleasure, and the number of places in the repetend will still be the same: thus $\frac{1}{11} = \cdot\dot{9}\dot{0}$, and $\frac{3}{11}$, or $\frac{1}{11} \times 3 = \cdot\dot{2}\dot{7}$, where the number of places in each is alike, and the same will be true in all cases.

* These rules are both evident from what has been said in reduction.

2. Divide this sum by as many nines as there are places in the repetend, and the remainder is the repetend of the sum; which must be set under the figures added, with cyphers on the left, when it has not so many placcs as the repetends.

3. Carry the quotient of this division to the next column, and proceed with the rest as in finite decimals.

EXAMPLES.

1. Let $3.\dot{6}+78.3\dot{4}7\dot{6}+735.\dot{3}+375+.\dot{2}\dot{7}+187.\dot{4}$ be added together.

Dissimilar.		Similar and conterminous.
$3.\dot{6}$	=	$3.6\dot{6}6666\dot{6}$
$78.3\dot{4}7\dot{6}$	=	$78.3\dot{4}7647\dot{6}$
$735.\dot{3}$	=	$735.3\dot{3}3333\dot{3}$
$375.$	=	$375.0\dot{0}0000\dot{0}$
$.\dot{2}\dot{7}$	=	$0.2\dot{7}2727\dot{2}$
$187.\dot{4}$	=	$187.4\dot{4}4444\dot{4}$

$1380.0\dot{6}4819\dot{3}$ the sum.

In this question, the sum of the repetends is 2648191, which, divided by 999999, gives 2 to carry, and the remainder is 648193.

2. Let $5391.357+72.3\dot{8}+187.2\dot{1}+4.296\dot{5}+217.849\dot{6}$ $+42.17\dot{6}+.52\dot{3}+58.3004\dot{8}$ be added together.

Ans $5974.1037\dot{1}$.

3. Add $9.\dot{8}1\dot{4}+1.5+87.2\dot{6}+.08\dot{3}+124.\dot{0}\dot{9}$ together.

Ans. $222.75\dot{5}7239\dot{0}$.

4. Add $\dot{1}6\dot{2}+134.\dot{0}\dot{9}+2.\dot{9}\dot{3}+97.2\dot{6}+3.\dot{7}6923\dot{0}+99.08\dot{3}+1.5$ $+.\dot{8}1\dot{4}$ together.

Ans. $501.62\dot{6}5107\dot{7}$.

SUBTRACTION OF CIRCULATING DECIMALS.

RULE.

Make the repetends similar and conterminous, and subtract as usual; observing, that, if the repetend of the subtrahend be greater than the repetend of the minuend, then the figure of the remainder on the right must be less by unity, than it would be, if the expressions were finite.

EXAMPLES.

1. From $85{\cdot}\dot{6}\dot{2}$ take $13{\cdot}7\dot{6}43\dot{2}$.

$$85{\cdot}\dot{6}\dot{2} = 85{\cdot}6\dot{2}62\dot{6}$$

$$13{\cdot}7\dot{6}43\dot{2} = 13{\cdot}7\dot{6}43\dot{2}$$

$$71{\cdot}8\dot{6}19\dot{3} \text{ the difference.}$$

2. From $476{\cdot}3\dot{2}$ take $84{\cdot}769\dot{7}$. Ans. $391{\cdot}552\dot{4}$.

3. From $3{\cdot}85\dot{6}\dot{4}$ take ${\cdot}03\dot{8}\dot{2}$. Ans. $3{\cdot}\dot{8}\dot{1}$.

MULTIPLICATION OF CIRCULATING DECIMALS.

RULE.

1. Turn both the terms into their equivalent vulgar fractions, and find the product of those fractions as usual.

2. Turn the vulgar fraction, expressing the product, into an equivalent decimal, and it will be the product required.

EXAMPLES.

1. Multiply ${\cdot}\dot{3}\dot{6}$ by ${\cdot}2\dot{5}$.

$${\cdot}\dot{3}\dot{6} = \tfrac{36}{99} = \tfrac{4}{11}$$

$${\cdot}2\dot{5} = \tfrac{23}{90}$$

$$\tfrac{4}{11} \times \tfrac{23}{90} = \tfrac{92}{990} = {\cdot}0\dot{9}2\dot{9} \text{ the product.}$$

2. Multiply $37{\cdot}2\dot{3}$ by ${\cdot}2\dot{6}$. Ans. $9{\cdot}92\dot{8}$.

3. Multiply $8574{\cdot}\dot{3}$ by $87{\cdot}\dot{5}$. Ans. $750730{\cdot}\dot{5}1\dot{8}$.

4. Multiply $3{\cdot}\dot{9}7\dot{3}$ by 8. Ans. $31{\cdot}\dot{7}9\dot{1}$.

5. Multiply $49640{\cdot}5\dot{4}$ by ${\cdot}7050\dot{3}$. Ans. $34998{\cdot}41\dot{9}900\dot{3}$.

6. Multiply $3{\cdot}1\dot{4}\dot{5}$ by $4{\cdot}\dot{2}9\dot{7}$. Ans. $13{\cdot}516\dot{9}53\dot{3}$.

DIVISION OF CIRCULATING DECIMALS.

RULE.

1. Change both the divisor and dividend into their equivalent vulgar fractions, and find their quotient as usual.

2. Turn the vulgar fraction, expressing the quotient, into its equivalent decimal, and it will be the quotient required.

EXAMPLES.

1. Divide ${\cdot}\dot{3}\dot{6}$ by ${\cdot}2\dot{5}$.

$${\cdot}\dot{3}\dot{6}=\tfrac{36}{99}=\tfrac{4}{11}$$

$${\cdot}2\dot{5}=\tfrac{23}{90}$$

$\tfrac{4}{11}\div\tfrac{23}{90}=\tfrac{4}{11}\times\tfrac{90}{23}=\tfrac{360}{253}=1\tfrac{107}{253}=1{\cdot}\dot{4}22924901185770750988\dot{1}$. the quotient.

2. Divide $319{\cdot}28007\dot{1}1\dot{2}$ by $764{\cdot}\dot{5}$. Ans. ${\cdot}41763\dot{2}\dot{5}$.

3. Divide $234{\cdot}\dot{6}$ by ${\cdot}\dot{7}$. Ans. $301{\cdot}\dot{7}1428\dot{5}$.

4. Divide $13{\cdot}516\dot{9}53\dot{3}$ by $4{\cdot}\dot{2}9\dot{7}$. Ans. $3{\cdot}1\dot{4}\dot{5}$.

PROPORTION IN GENERAL.

NUMBERS are compared together to discover the relations they have to each other.

There must be two numbers to form a comparison; the number, which is compared, being written first, is called the *antecedent;* and that, to which it is compared, the *consequent.* Thus of these numbers 2 : 4 : : 3 : 6, 2 and 3 are called the antecedents; and 4 and 6, the consequents.

Numbers are compared to each other two different ways; one comparison considers the *difference* of the two numbers, and is called *arithmetical relation*, the difference being sometimes named the *arithmetical ratio;* and the other considers their *quotient*, and is termed *geometrical relation*, and the quotient the *geometrical ratio.* So of these numbers 6 and 3, the difference or arithmetical ratio is 6—3 or 3; and the geometrical ratio is $\frac{6}{3}$ or 2.

If two or more couplets of numbers have equal ratios, or differences, the equality is named *proportion;* and their terms similarly posited, that is, either all the greater, or all the less, taken as antecedents, and the rest as consequents, are called *proportionals.* So the two couplets 2, 4, and 6, 8, taken thus, 2, 4, 6, 8, or thus 4, 2, 8, 6, are arithmetical proportionals; and the couplets 2, 4, and 8, 16, taken thus, 2, 4, 8, 16, or thus, 4, 2, 16, 8, are geometrical proportionals.*

Proportion is distinguished into *continued* and *discontinued.*

If, of several couplets of proportionals written in a

---

* In geometrical proportionals a colon is placed between the terms of each couplet, and a double colon between the couplets; in arithemetical proportionals a colon may be turned horizontally between the terms of each couplet, and two colons written between the couplets. Thus the above geometrical proportionals are written thus, 2 : 4 : : 8 : 16, and 4 : 2 : : 16 : 8; the arithmetical, 2 ·· 4 : : 6 ·· 8, and 4 ·· 2 : : 8 ·· 6.

series, the difference or ratio of each consequent and the antecedent of the next following couplet be the same as the common difference or ratio of the couplets, the proportion is said to be *continued*, and the numbers themselves a series of *continued arithmetical* or *geometrical proportionals*. So 2, 4, 6, 8, form an arithmetical progression; for $4-2=6-4=8-6=2$; and 2, 4, 8, 16, a geometrical progression; for $\frac{4}{2}=\frac{8}{4}=\frac{16}{8}=2$.

But if the difference or ratio of the consequent of one couplet and the antecedent of the next couplet be not the same as the common difference or ratio of the couplets, the proportion is said to be *discontinued*. So 4, 2, 8, 6, are in *discontinued arithmetical proportion;* for $4-2=8-6=2$, but $8-2=6$; also 4, 2, 16, 8, are in *discontinued geometrical proportion;* for $\frac{4}{2}=\frac{16}{8}=2$, but $\frac{16}{2}=8$.

Four numbers are *directly proportional*, when the ratio of the first to the second is the same, as that of the third to the fourth. As $2 : 4 :: 3 : 6$. Four numbers are said to be *reciprocally* or *inversely proportional*, when the first is to the second, as the fourth is to the third, and vice versa. Thus, 2, 6, 9, and 3, are reciprocal proportionals; $2 : 6 :: 3 : 9$.

Three or four numbers are said to be in *harmonical proportion*, when, in the former case, the difference of the first and second is to the difference of the second and third, as the first is to the third; and, in the latter, when the difference of the first and second is to the difference of the third and fourth, as the first is to the fourth. Thus, 2, 3, and 6; and 3, 4, 6, and 9, are harmonical proportionals; for $3-2=1 : 6-3=3 :: 2 : 6$; and $4-3=1 : 9-6=3 :: 3 : 9$.

Of four arithmetical proportionals the sum of the extremes is equal to the sum of the means.* Thus of $2 \cdot\cdot 4 :: 6 \cdot\cdot 8$

---

* DEMONSTRATION. Let the four arithmetical proportionals be A, B, C, D, viz. $A \cdot\cdot B :: C \cdot\cdot D$; then, $A-B=C-D$, and $B+D$ being added to both sides of the equation, $A-B+B+D=C-D+B+D$; that is, $A+D$ the sum of the extremes $=C+B$ the sum of the means.—And three A, B, C, may be thus expressed, $A \cdot\cdot B :: B \cdot\cdot C$; therefore $A+C=B+B=2B$. Q. E. D.

the sum of the extremes $(2+8)=$ the sum of the means $(4+6)$ $=10$. Therefore, of three arithmetical proportionals, the sum of the extremes is double the mean.

Of four geometrical proportionals, the product of the extremes is equal to the product of the means.* Thus, of $2 : 4 :: 8 : 16$, the product of the extremes (2×16) is equal to the product of the means $(4\times8)=32$. Therefore of three geometrical proportionals, the product of the extremes is equal to the square of the mean.

Hence it is easily seen, that either extreme of four geometrical proportionals is equal to the product of the means divided by the other extreme ; and that either mean is equal to the product of the extremes divided by the other mean.

SIMPLE PROPORTION, OR RULE OF THREE.

The Rule of Three is that, by which a number is found, having to a given number the same ratio, which is between two other given numbers. For this reason it is sometimes named the *Rule of Proportion.*

It is called the *Rule of Three*, because in each of its questions there are given *three numbers* at least. And because of its excellent and extensive use, it is often named the *Golden Rule.*

RULE.†

1. Write the number, which is of the same kind with the answer or number required.

---

* DEMONSTRATION. Let the proportion be $A : B :: C : D$, and let $\frac{A}{B}=\frac{C}{D}=r$; then $A=Br$, and $C=Dr$; multiply the former of these equations by D, and the latter by B; then $AD=BrD$, and $CB=DrB$, and consequently AD the product of the extremes is equal to BC the product of the means.—And three may be thus expressed, $A : B :: B : C$, therefore $AC=B\times B=B^2$. Q. E. D.

† DEMONSTRATION. The following observations, taken col-

2. Consider whether the answer ought to be greater or less than this number; if greater, write the greater of the

---

lectively, form a demonstration of the rule, and of the reductions mentioned in the notes subsequent to it.

1. There can be comparison or ratio between two numbers, only when they are considered either abstractly, or as applied to things of the same kind, so that one can, in a proper sense, be contained in the other. Thus there can be no comparison between 2 men and 4 days; but there may be between 2 and 4, and between 2 days and 4 days, or 2 men and 4 men. Therefore, the 2 of the 3 given numbers, that are of the same kind, that is, the first and the third, when they are stated according to the rule, are to be compared together, and their ratio is equal to that, required between the remaining or second number and the fourth or answer.

2. Though numbers of the same kind, being either of the same or of different denominations, have a real ratio, yet this ratio is the same as that of the two numbers taken abstractly, only when they are of the same denomination. Thus the ratio of 1l. to 2l. is the same as that of 1 to 2 $=\frac{1}{2}$; 1s. has a real ratio to 2l. but it is not the ratio of 1 to 2; it is the ratio of 1s. to 40s. that is, of 1 to 40 $=\frac{1}{40}$. Therefore, as the first and third numbers have the ratio, that is required between the second and answer, they must, if not of the same denomination, be reduced to it; and then their ratio is that of the abstract numbers.

3. The product of the extremes of four geometrical proportionals is equal to the product of the means; hence, if the product of two numbers be equal to the product of two other numbers, the four numbers are proportionals; and if the product of two numbers be divided by a third, the quotient will be a fourth proportional to those three numbers. Now as the question is resolvable into this, viz. to find a number of the same kind as the second in the statement, and having the same ratio to it, that the greater of the other two has to the less, or the less has to the greater; and as these two, being of the same denomination, may be considered as abstract numbers; it plainly follows, that the fourth number or answer is truly found by multiplying the second by one of the other two, and dividing the product by that, which remains.

two remaining numbers on the right of it for the third, and the other on the left for the first number or term ; but if less

---

4. It is very evident, that, if the answer must be greater than the second number, the greater of the other two numbers must be the multiplier, and may occupy the third place ; but, if less, the less number must be the multiplier.

5. The reduction of the second number is only performed for convenience in the subsequent multiplication and division, and not to produce an abstract number. The reason of the reduction of the quotient, of the remainder after division, and of the product of the second and third terms, when it cannot be divided by the first, is obvious.

6. If the second and third numbers be multiplied together, and the product be divided by the first ; it is evident, that the answer remains the same, whether the number compared with the first be in the second or third place.

Thus is the proposed demonstration completed.

There are four other methods of operation beside the general one given above, any of which, when applicable, performs the work much more concisely. They are these :

1. Divide the second term by the first, multiply the quotient by the third, and the product will be the answer.

2. Divide the third term by the first, multiply the quotient by the second, and the product will be the answer.

3. Divide the first term by the second, divide the third by the quotient, and the last quotient will be the answer.

4. Divide the first term by the third, divide the second by the quotient, and the last quotient will be the answer.

The general rule above given is equivalent to those, which are usually given in the direct and inverse rules of three, and which are here subjoined.

The RULE OF THREE DIRECT teaches, by having three numbers given, to find a fourth, that shall have ths same proportion to the third, as the second has tó the first.

RULE.

1. State the question ; that is, place the numbers so, that the first and third may be the terms of supposition and demand, and the second of the same kind with the answer required.

write the less of the two remaining numbers in the third place, and the other in the first.

---

2. Bring the first and third numbers into the same denomination, and the second into the lowest name mentioned.

3. Multiply the second and third numbers together, and divide the product by the first, and the quotient will be the answer to the question, in the same denomination you left the second number in; which may be brought into any other denomination required.

EXAMPLE.

If 24lb. of raisins cost 6s. 6d. what will 18 frails cost, each weighing net 3qrs. 18lb.?

24lb. : 6s. 6d. : : 18 frails, each 3qrs. 18lb. :

```
          12                       28
         ----                     ----
          78                      102
                                   18
                                 -----
                                  816
                                 102
                                 -----
                                 1836
                                   78
                                 -----
                                14688
                               12852
                              -------  12)
                           24)143208 (    5967
                               232        ----
                                160    2,0)49,7 3
                                168       ----
Ans. 24l. 17s. 3d.            -----   £.24 17 3
```

The rule is founded on this obvious principle, that the magnitude or quantity of any effect varies constantly in proportion to the varying part of the cause: thus the quantity of goods bought is in proportion to the money laid out; the space gone over by an uniform motion is in proportion to the time, &c. The truth of the rule, as applied to ordinary inquiries, may be made very evident by attending only to the principles of compound multi-

3. Multiply the second and third terms together, divide the product by the first, and the quotient will be the answer.

---

plication and division. It is shown in multiplication of money, that the price of one, multiplied by the quantity, is the price of the whole ; and in division, that the price of the whole, divided by the quantity, is the price of one. Now, in all cases of valuing goods, &c. where one is the first term of the proportion, it is plain, that the answer, found by this rule, will be the same as that found by multiplication of money ; and where one is the last term of the proportion, it will be the same as that found by division of money. In like manner, if the first term be any number whatever, it is plain, that the product of the second and third terms will be greater than the true answer required by as much as the price in the second term exceeds the price of one, or as the first term exceeds an unit. Consequently this product divided by the first term will give the true answer required, and is the rule.

There will sometimes be difficulty in separating the parts of complicated questions, where two or more statings are required, and in preparing the question for stating, or after a proportion is wrought; but as there can be no general directions given for the management of these cases, it must be left to the judgment and experience of the learner.

The RULE OF THREE INVERSE teaches, by having three numbers given to find a fourth, that shall have the same proportion to the second, as the first has to the third.

If *more* require *more*, or *less* require *less*, the question belongs to the rule of three direct.

But if *more* require *less*, or *less* require *more*, it belongs to the rule of three inverse.

NOTE. The meaning of these phrases, " if *more* require *more*, *less* require *less*," &c. is to be understood thus: *more* requires *more*, when the third term is greater than the first, and requires the fourth to be greater than the second; *more* requires *less*, when the third term is greater than the first, and requires the fourth to be less than the second; *less* requires *more*, when the third term is less than the first, and requires the fourth to be greater than the second ; and *less* requires *less*, when the third

NOTE 1. It is sometimes most convenient to multiply and divide as in compound multiplication and division; and

---

term is less than the first, and requires the fourth to be less than the second.

RULE.

1. State and reduce the terms as in the rule of three direct.

2. Multiply the first and second terms together, and divide their product by the third, and the quotient is the answer to the question, in the same denomination you left the second number in.

The method of proof, whether the proportion be direct or inverse, is by inverting the question.

EXAMPLE.

What quantity of shalloon, that is three quarters of a yard wide, will line $7\frac{1}{2}$ yards of cloth, that is $1\frac{1}{2}$ yard wide ?

```
1yd. 2qrs. : 7yds. 2qrs. : : 3qrs. :
    4           4
  ----        ----
    6          30
                6
            ------
            3)180
            ------
             4)60
             ----
              15 yards, the answer.
```

The reason of this rule may be explained from the principles of compound multiplication and division, in the same manner as the direct rule. *For example;* If 6 men can do a piece of work in 10 days, in how many days will 12 men do it ?

As 6 *men* : 10 *days* : : 12 *men* : $\frac{6\times 10}{12}=5$ *days*, the answer.

And here the product of the first and second terms, that is, 6 times 10, or 60, is evidently the time, in which one man would perform the work; therefore 12 men will do it in one twelf part of that time, or 5 days; and this reasoning is applicable to any other instance whatever.

sometimes it is expedient to multiply and divide according to the rules of vulgar or decimal fractions. But when neither of these modes is adopted, reduce the compound terms, each to the lowest denomination mentioned in it, and the first and third to the same denomination; then will the answer be of the same denomination with the second term. And the answer may afterward be brought to any denomination required.

Note 2. When there is a remainder after division, reduce it to the denomination next below the last quotient, and divide by the same divisor, so shall the quotient be so many of the said next denomination; proceed thus, as long as there is any remainder, till it is reduced to the lowest denomination, and all the quotients together will be the answer. And when the product of the second and third terms cannot be divided by the first, consider that product as a remainder after division, and proceed to reduce and divide it in the same manner.

Note 3. If the first term and either the second or third can be divided by any number without a remainder, let them be divided, and the quotient used instead of them.

Direct and *inverse* proportion are properly only parts of the same general rule, and are both included in the preceding.

Two or more statings are sometimes necessary, which may always be known from the nature of the question.

The method of proof is by inverting the question.

EXAMPLES.

1. Let it be proposed to find the value of 14oz. 8dwt. of gold, at 3l. 19s. 11d. an ounce.

```
oz.      £   s.  d.      oz. dwt.
 1  :    3  19  11  : :  14   8  :
20      20               20
--      --               ---
20      79               288
        12
        ---
        959
        288
        ----
       7672
      7672
     1918
     -----
2,0)27619,2
     13809 12/20 pence, or
 12)13809d. 2 8/20 q.
  2,0)115,0s. 9d. 2 8/20 q.
Ans. 57l 10s. 9d. 2 8/20 q.
```

Explanation. The three terms being stated by the general rule, as above, the second term is reduced to pence, and the third to penny-weights, these being their lowest denominations, as directed in the first note. The first term is also reduced to dwts. that it may agree with the third, by the same note. The second term is then multiplied by the third, and the product divided by the first, according to the general rule, when the answer comes out 13809 pence, and 12 remaining; which remainder being reduced to farthings, and these divided by the same divisor 20, by the second note, the quotient is 2 farthings, 8 remaining. Lastly, the pence are divided by 12, to reduce them to shillings, and these again by 20 for pounds; when the final sum comes out 57l. 10s. 9d. 2q. for the answer.

2. How much of that in length which is $4\frac{1}{2}$ inches broad, will make a square foot?

Breadth.		Length.		Breadth.	
4˙5	:	12	: :	12	:
		12			

in.

4˙5)144˙0(32=2f. 8in. the answer.
135
———
90
90
———

3. At $10\frac{1}{2}$d. per lb. what is the value of a firkin of butter, containing 56lb. ?

	lb.		d.	q.		lb.	
	1	:	10	2	: :	56	:
56=8×7				8			
		7	0	0			
				7			
	£2	9	0	0 the answer.			

Or thus :

lb.		d.	d.		lb.	
$\frac{1}{1}$	:	$10\frac{1}{2}$	$=\frac{21}{2}$	: :	$\frac{56}{1}$	:

$\frac{21}{2}\times\frac{56}{1}=\frac{1176}{2}$=588d.=49s.=2l. 9s. as before.

Or thus :

lb.		d.		lb.	
1	:	10˙5	: :	56	:
				10˙5	
				280	
				560	

12)588˙0
2,0)4,9
£2 9 as before.

4. If $\frac{3}{7}$ of a yard cost $\frac{7}{12}$ of a pound, what will $\frac{6}{13}$ of an English ell cost ?

First $\frac{3}{5}$ of a yard $=\frac{3}{5}$ of $\frac{4}{1}$ of $\frac{1}{5}=\frac{3\times4\times1}{5\times1\times5}=\frac{12}{25}$ of an ell.

Then $\frac{12}{25}$ ell : $\frac{7}{12}$l. : : $\frac{6}{15}$ ell :

$$\text{And } \frac{7}{12}\times\frac{6}{15}\times\frac{25}{12}=\frac{7\times\overset{}{\not6}\times\overset{5}{\not25}}{\underset{2}{\not12}\times\underset{3}{\not15}\times12}=\frac{35}{72}$$

=9s. 8d. $\frac{2}{3}$ the answer.

5. If $\frac{3}{8}$ of a yard cost $\frac{2}{5}$ of a pound, what will $\frac{1}{4}$ of an English ell cost?

$\frac{3}{8}$ = ˙375

$\frac{2}{5}$ = ˙4l.

$\frac{1}{4}$ ell $=\frac{5}{16}$yd. = ˙3125

˙375yd. : ˙4l. : : ˙3125yd. :

˙3125

˙375)˙12500(˙333, &c.=6s. 8d. the answer.

1125

1250

1125

1250

1125

125

6. What is the value of a cwt. of sugar at $5\frac{1}{2}$d. per lb.? Ans. 2l. 11s. 4d.

7. What is the value of a chaldron of coals at $11\frac{1}{2}$d. per bushel? Ans. 1l. 14s. 6d.

8. What is the value of a pipe of wine at $10\frac{1}{2}$d. per pint? Ans. 44l. 2s.

9. At 3l. 9s. per cwt. what is the value of a pack of wool, weighing 2cwt. 2qrs. 13lb. Ans. 9l. 6d. $\frac{12}{112}$.

10. What is the value of $1\frac{1}{2}$cwt. of coffee at $5\frac{1}{2}$d. per ounce? Ans. 61l. 12s.

11. Bought 3 casks of raisins, each weighing 2cwt. 2qrs. 25lb. what will they come to at 2l. 1s. 8d. per cwt.? Ans. 17l. $4\frac{3}{4}$d. $\frac{32}{112}$.

12. What is the value of 2qrs. 1nl. of velvet at 19s. $8\frac{1}{2}$d. per English ell? Ans. 8s. $10\frac{1}{4}$d. $\frac{14}{20}$.

13. Bought 12 pockets of hops, each weighing 1cwt. 2qrs. 17lb.; what do they come to at 4l. 1s. 4d. per cwt.? Ans. 80l. 12s. $1\frac{1}{2}$d. $\frac{96}{112}$.

14. What is the tax upon 745l. 14s. 8d. at 3s. 6d. in the pound? Ans. 130l. 10s. $0\frac{3}{4}$d. $\frac{48}{240}$.

15. If $\frac{3}{4}$ of a yard of velvet cost 7s. 3d. how many yards can I buy for 13l. 15s. 6d.? Ans. $28\frac{1}{2}$ yards.

16. If an ingot of gold, weighing 9lb. 9oz. 12dwt. be worth 411l. 12s. what is that per grain? Ans. $1\frac{3}{4}$d.

17. How many quarters of corn can I buy for 140 dollars at 4s. per bushel? Ans. 26qrs. 2bu.

18. Bought 4 bales of cloth, each containing 6 pieces, and each piece 27 yards, at 16l. 4s. per piece; what is the value of the whole, and the rate per yard?

Ans. 388l. 16s. at 12s. per yard.

19. If an ounce of silver be worth 5s. 6d. what is the price of a tankard, that weighs 1lb. 10oz. 10dwt. 4gr.?

Ans. 6l. 3s. $9\frac{1}{2}$d. $\frac{96}{480}$.

20. What is the half year's rent of 547 acres of land at 15s. 6d. per acre?

Ans. 211l. 19s. 3d.

21. At 1·75D. per week, how many months' board can I have for 100l.?

Ans. 47m. 2w. $\frac{60}{126}$.

22. Bought 1000 Flemish ells of cloth for 90l. how must I sell it per ell in Boston to gain 10l. by the whole?

Ans. 3s. 4d.

23. Suppose a gentleman's income is 1750 dollars a year, and he spends 19s. 7d. per day, one day with another, how much will he have saved at the year's end?

Ans. 167l. 12s. 1d.

24. What is the value of 172 pigs of lead, each weighing 3cwt. 2qrs. $17\frac{1}{2}$lb. at 8l. 17s. 6d. per fother of $19\frac{1}{2}$cwt.?

Ans. 286l. 4s. $4\frac{1}{2}$d.

25. The rents of a whole parish amount to 1750l. and a rate is granted of 32l. 16s. 6d. what is that in the pound?

Ans. $4\frac{1}{2}$d. $\frac{2880}{420000}$.

26. If keeping for my horse be $11\frac{1}{2}$d. per day, what will be the charge of 11 horses for the year?

Ans. 192l. 7s. $8\frac{1}{2}$d.

27. A person breaking owes in all 1490l. 5s. 10d. and has in money, goods, and recoverable debts, 784l. 17s. 4d. if these things be delivered to his creditors, what will they get in the pound? Ans. 10s. $6\frac{1}{4}$d. $\frac{20993}{35767}$.

28. What must 40s. pay toward a tax, when 652l. 13s. 4d. is assessed at 83l. 12s. 4d.? Ans. 5s. $1\frac{1}{4}$d. $\frac{15376}{15664}$.

29. Bought 3 tuns of oil for 151l. 14s. 85 gallons of which being damaged, I desire to know how I may sell the remainder per gallon, so as neither to gain nor lose by the bargain? Ans. 4s. $6\frac{1}{4}$d. $\frac{25}{671}$.

30. What quantity of water must I add to a pipe of mountain wine, valued at 33l. to reduce the first cost to 4s 6d. per gallon? Ans. $20\frac{2}{3}$ gallons.

31. If 15 ells of stuff, $\frac{3}{4}$ yard wide, cost 37s. 6d. what will 40 ells of the same stuff cost, being yard wide?

Ans. 6l. 13s. 4d.

32. Shipped for Barbadoes 500 pairs of stockings at 3s. 6d. per pair, and 1650 yards of baize at 1s. 3d. per yard, and have received in return 348 gallons of rum at 6s. 8d. per gallon, and 750lb. of indigo at 1s. 4d. per lb. what remains due upon my adventure? Ans. 24l. 12s. 6d.

33. If 100 workmen can finish a piece of work in 12 days, how many are sufficient to do the same in 3 days?

Ans. 400 men.

34. How many yards of matting, 2ft. 6in. broad, will cover a floor, that is 27ft. long, and 20ft. broad?

Ans. 72 yards.

35. How many yards of cloth, 3qrs. wide, are equal in measure to 30 yards, 5qrs. wide? Ans. 50 yards.

36. A borrowed of his friend B 250l. for 7 months, promising to do him the like kindness; sometime after B had occasion for 300l. how long may he keep it to receive full amends for the favor?

Ans. 5 months and 25 days.

37. If, when the price of a bushel of wheat is 6s. 3d. the penny loaf weigh 9oz. what ought it to weigh when wheat

is at 8s. $2\frac{1}{2}$d. per bushel? Ans. 6oz. 13dr.

38. If $4\frac{1}{2}$cwt. may be carried 36 miles for 35 shillings, how many pounds can I have carried 20 miles for the same money? Ans. 907lb. $\frac{4}{25}$.

39. How many yards of canvass, that is ell wide, will line 20 yards of say, that is 3qrs. wide? Ans. 12yds.

40. If 30 men can perform a piece of work in 11 days, how many men will accomplish another piece of work, 4 times as big, in a fifth part of the time? Ans. 600.

41. A wall, that is to be built to the height of 27 feet, was raised 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days at the same rate of working? Ans. 36.

42. If $\frac{5}{7}$oz. cost $\frac{11}{12}$l. what will 1oz. cost?

Ans. 1l. 5s. 8d.

43. If $\frac{3}{16}$ of a ship cost 273l. 2s. 6d. what is $\frac{5}{32}$ of her worth? Ans. 227l. 12s. 1d.

44. At $1\frac{1}{2}$l. per cwt. what does $3\frac{1}{3}$lb. come to?

Ans. $10\frac{5}{7}$d.

45. If $\frac{5}{8}$ of a gallon cost $\frac{5}{8}$l. what will $\frac{5}{9}$ of a tun cost?

Ans. 140l.

46. A person, having $\frac{3}{5}$ of a coal mine, sells $\frac{3}{4}$ of his share for 171l. what is the whole mine worth?

Ans. 380l.

47. If, when the days are $13\frac{5}{8}$ hours long, a traveller perform his journey in $35\frac{1}{2}$ days; in how many days will he perform the same journey, when the days are $11\frac{9}{10}$ hours long?

Ans. $40\frac{615}{952}$ days.

48. A regiment of soldiers, consisting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth, that is $1\frac{5}{8}$yd. wide, and to be lined with shalloon, $\frac{7}{8}$yd. wide; how many yards of shalloon will line them?

Ans. 4531yds. 1qr. $2\frac{6}{7}$nl.

PRACTICE.

Practice is a contraction of the rule of three, when the first term happens to be an unit, or one ; and has its name from its daily use among merchants and tradesmen, being an easy and concise method of working most questions, that occur in trade and business.

The method of proof is by the rule of three.

An *aliquot* part of any number is such a part of it, as, being taken a certain number of times, exactly makes that number.

GENERAL RULE.*

1. Suppose the price of the given quanty to be 1l. 1s. or 1d. as is most convenient ; then will the quantity itself be the answer, at the supposed price.

2. Divide the given price into aliquot parts, either of the supposed price, or of one another, and the sum of the quotients, belonging to each, will be the true answer required.

Note 1. When there is any fractional part, or inferior denomination of the quantity, take the same part of the price, that the given fraction, or inferior denomination, is of the unit, of which the price is given, and add it to the price of the whole number.

---

* The rule will be rendered very evident by an explanation of the example. In this example it is plain, that the quantity 526 is the answer at 1l. consequently, as 3s. 4d. is the $\frac{1}{6}$ of 1l. $\frac{1}{6}$ of that quantity, or 87l. 13s. 4d. is the price at 3s. 4d. In like manner, as 4d. is $\frac{1}{10}$ of 3s. 4d. so $\frac{1}{10}$ of 87l. 13s. 4d. or 8l. 15s. 4d. is the answer at 4d. And by reasoning in this way 4l. 7s. 8d. will be shown to be the price at 2d. and 10s. 11$\frac{1}{2}$d. the price at $\frac{1}{4}$.— Now as the sum of all these parts is equal to the whole price (3s. 10$\frac{1}{4}$d.) so the sum of the answers, belonging to each price, will be the answer at the full price required. And the same will be true in any example whatever.

NOTE 2. The rule of practice is nearly superseded by the use of Federal Money.

EXAMPLE.

What is the value of 526 yards of cloth at 3s. 10$\frac{1}{4}$d. per yard?

	526l.			Ans. at 1l.		
3s. 4d. is $\frac{1}{6}$ =	87	13	4	do. at 0	3s.	4d.
4d. is $\frac{1}{10}$ =	8	15	4	do. at		4
2d. is $\frac{1}{2}$ =	4	7	8	do. at		2
$\frac{1}{4}$d. is $\frac{1}{8}$ =	0	10	11$\frac{1}{2}$	do. at		0$\frac{1}{4}$
	101	7	3$\frac{1}{2}$	do. at	3	10$\frac{1}{4}$ the full price.

Ans. 101l. 7s. 3$\frac{1}{2}$d.

By Federal Money.

At $ 0·6423 per yard.
526

38538
12846
32115

$ 337·8498 Answer.

2. 8cwt. 2qrs. 16lb. at 2l. 5s. 6d.

	2l.	5s.	6d.
			8
	18	4	
2qrs. is $\frac{1}{2}$	1	2	9
14lb. is $\frac{1}{4}$		5	8$\frac{1}{4}$
2lb. is $\frac{1}{7}$			9$\frac{3}{4}$

19l. 13s. 3d. the answer.

3. 5275 yards at 2d. Ans. 43l. 19s. 2d.
4. 1776 yards at 3d. Ans. 22l. 4s.
5. 273$\frac{1}{4}$ at 2s. 6d. Ans. 34l. 3s. 1$\frac{1}{2}$d.
6. 937$\frac{1}{2}$ at 3l. 17s. 8d. Ans. 3640l. 12s. 6d.

TARE AND TRETT.

TARE AND TRETT are practical rules for deducting certain allowances, which are made by merchants and tradesmen in selling their goods by weight.

Tare is an allowance, made to the buyer, for the weight of the box, barrel, or bag, &c. which contains the goods bought, and is either at so much per box, &c. at so much per cwt. or at so much in the gross weight.

Trett is an allowance of 4lb. in every 104lb. for waste, dust, &c.

Cloff is an allowance of 2lb. upon every 3cwt.

Gross weight is the whole weight of any sort of goods, together with the box, barrel, or bag, &c. that contains them.

Suttle is the weight, when part of the allowance is deducted from the gross.

Net weight is what remains after all allowances are made.

CASE I.

When the tare is a certain weight per box, barrel, or bag, &c.

RULE.*

Multiply the number of boxes, or barrels, &c. by the tare, and subtract the product from the gross, and the remainder is the net weight required.

EXAMPLES.

1. In 7 frails of raisins, each weighing 5cwt. 2qrs. 5lb. gross, tare 23lb. per frail, how much net?

23×7=1cwt. 1qr. 21lb.

---

* It is manifest, that this, as well as every other case in this rule, is only an application of the rules of proportion and practice

cwt.	qrs.	lb.	
5	2	5	
		7	
38	3	7	gross.
1	1	21	tare.
37	1	14	the answer.

2. In 241 barrels of figs, each 3qrs. 19lb. gross, tare 10lb. per barrel, how many pounds net? Ans. 22413.

3. What is the net weight of 14 hogsheads of tobacco, each 5cwt. 2qrs. 17lb. gross, tare 100lb. per hhd.?

Ans. 66cwt. 2qrs. 14lb.

CASE II.

When the tare is a certain weight per cwt.

RULE.

Divide the gross weight by the aliquot parts of a cwt. contained in the tare, and subtract the quotient from the gross, and the remainder is the net weight.

EXAMPLES.

1. Gross 173cwt. 3qrs. 17lb. tare 16lb. per cwt. how much net?

	cwt.	qrs.	lb.	
	173	3	17	gross.
14lb. is $\frac{1}{8}$	21	2	26	
2lb. is $\frac{1}{7}$	3	0	11	
	24	3	9	
	149	0	8	the answer.

2. What is the net weight of 7 barrels of pot-ash, each weighing 201lb. gross, tare being at 10lb. per cwt.?

Ans. 1281lb. 6oz.

3. In 25 barrels of figs, each 2cwt. 1qr. gross, tare 16lb. per cwt. how much net? Ans. 48cwt. 24lb.

CASE III.

When Trett is allowed with Tare.

RULE.

Divide the suttle weight by 26, and the quotient is the trett, which subtract from the suttle, and the remainder is the net weight.

EXAMPLES.

1. In 9cwt. 2qrs. 17lb. gross, tare 37lb. and trett as usual, how much net?

cwt.	qrs.	lb.
9	2	17 gross.
0	1	9 tare
26)9	1	8 suttle.
	1	11 trett.
8	3	25 the answer.

2. In 7 casks of prunes, each weighing 3cwt. 1qr. 5lb. gross, tare $17\frac{1}{2}$lb. per cwt. and trett as usual, how much net?
Ans. 18cwt. 2qrs. 25lb.

3. What is the net weight of 3 hogsheads of sugar weighing as follows: the first, 4cwt. 5lb. gross, tare 73lb.; the second, 3cwt. 2qrs. gross, tare 56lb. and the third, 2cwt. 3qrs. 17lb. gross, tare 47lb. and allowing trett to each as usual?
Ans. 8cwt. 2qrs. 4lb.

CASE IV.

When tare, trett, and cloff are all allowed.

RULE.

Deduct the tare and trett, as before, and divide the suttle by 168, and the quotient is the cloff, which subtract from the suttle, and the remainder is the net.

EXAMPLES.

1. What is the net weight of a hhd. of tobacco, weighing 15cwt. 3qrs. 20lb. gross, tare 7lb. per cwt. and trett and cloff as usual?

	cwt.	qrs.	lb.	
	15	3	20	gross.
7lb. is $\frac{1}{16}$		3	27	tare.
26)	14	3	21	
		2	8	trett.
168)	14	1	13	suttle.
			9	cloff.
	14	1	4	the answer.

2. In 19 chests of sugar, each containing 13cwt. 1qr. 17lb. gross, tare 13lb. per cwt. and trett and cloff as usual, how much net, and what is the value at 5$\frac{3}{4}$d. per pound?

Ans. 215cwt. 17lb. and value 577l. 6s. 5$\frac{3}{4}$d.

COMPOUND PROPORTION.

Compound Proportion teaches how to resolve such questions, as require two or more statings in Simple Proportion.

In these questions there is always given an odd number of terms, as five, seven, or nine, &c. These are distinguished into *terms of supposition*, and *terms of demand*, the number of the former always exceeding that of the latter by one, which is of the same kind with the term or answer sought.

This rule is often named the *Double Rule of Three*, because its questions are sometimes performed by two operations of the Rule of Three.

RULE* FOR STATING.

1. Write the term of supposition, which is of the same kind with the answer, for the middle term.

2. Take one of the other terms of supposition, and one of the demanding terms of the same kind with it; then place one of them for a first term, and the other for a third, according to the directions given in the rule of three. Do the same with another term of supposition and its correspondent demanding term; and so on, if there be more terms of each kind; writing the terms under each other, which fall on the same side of the middle term.

METHOD OF OPERATION.

1. *By several operations.*—Take the two upper terms and the middle term, in the same order as they stand, for the first stating of the rule of three; then take the fourth number, resulting from the first stating, for the middle term, and the two next terms in the general stating, in the same order as they stand, for the extreme terms of the second stating; and so on, as far as there are any numbers in the general stating, always making the fourth number, resulting from each simple stating, the second term of the next. So shall the last resulting number be the answer required.

2. *By one operation.*—Multiply together all the terms in the first place, and also all the terms in the third place. Then multiply the latter product by the middle term, and divide the result by the former product; and the quotient will be the answer required.

NOTE 1. It is generally best to work by the latter method,

---

* The reason of this rule for stating, and of the methods of operation, may be easily shown from the nature of simple proportion; for every line in this case is a particular stating in that rule. And therefore with respect to the second method, it is evident, that, if all the separate dividends be collected into one dividend, and all the divisors into one divisor, their quotient must be the answer sought.

namely, by one operation. And after the stating, and before the commencement of the operation, if one of the first terms, and either the middle term, or one of the last terms, can be exactly divided by one and the same number, let them be divided, and the quotients used instead of them; which will much shorten the work.

NOTE 2. The first and third terms of each line, if of different denominations, must be reduced to the same denomination.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, provided 16 men can dig 54 yards in 6 days?

GENERAL STATING.

54 yds. or 2, 8 days, or 4 } : 16 men : : { 135 yds. or 5, 6 days, or 3 } :

FIRST METHOD.

$54 \div 27 = 2$
$135 \div 27 = 5$
$8 \div 2 = 4$
$6 \div 2 = 3$

yds. men. yds.
2 : 16 : : 5 :
5
2)80(40 men.
8
0

days. men. days
4 : 40 : : 3 :
3
4)120(30 men, answer.
12
0

SECOND METHOD.

2, 4 } : 16 : : { 5, 3 } :
8 : 16 : : 15 :
15
80
16
8)240(30 men, the answer as before.
24
0

2. If 100l. in one year gain 5l. interest, what will be the interest of 750l. for 7 years? Ans. 262l. 10s.

3. What principal will gain 262l. 10s. in 7 years, at 5l. per cent. per annum? Ans. 750l.

4. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days.

5. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses?

Ans. $102\frac{16}{45}$ days.

6. If 7oz. 5dwts. of bread be bought at $4\frac{3}{4}$d. when corn is at 4s. 2d. per bushel, what weight of it may be bought for 1s. 2d. when the price of the bushel is 5s. 6d.?

Ans. 1lb. 4oz. $3\frac{477}{627}$dwts.

7. If the carriage of 13cwt. 1qr. for 72 miles be 2l. 10s. 6d. what will be the carriage of 7cwt. 3qrs. for 112 miles?

Ans. 2l. 5s. 11d. $1\frac{77}{159}$q.

8. A wall, to be built to the height of 27 feet, was raised to the height of 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 36 men.

9. If a regiment of soldiers, consisting of 939 men, can eat up 351 quarters of wheat in 7 months; how many soldiers will eat up 1464 quarters in 5 months, at that rate?

Ans. $5483\frac{23}{195}$.

10. If 248 men, in 5 days of 11 hours each, dig a trench 230 yards long, 3 wide and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide and 3 deep? Ans. $288\frac{59}{207}$.

CONJOINED PROPORTION.

Conjoined Proportion is when the coins, weights, or measures, of several countries are compared in the same question; or it is the joining together of several ratios, and the inferring of the ratio of the first antecedent and the last consequent from the ratios of the several antecedents and their respective consequents.

NOTE 1. The solution of questions, under this rule, may frequently be much shortened by cancelling equal numbers, when in both the columns, or in the first column and third term, and abbreviating those, that áre commensurable.

NOTE 2. The proof is by so many statements in the single rule of three, as the nature of the question requires.

CASE I.

When it is required to find how many of the last kind of coin, weight, or measure, mentioned in the question, are equal to a given number of the first.

RULE.

1. Multiply continually together the antecedents for the first term, and the consequents for the second, and make the given number the third.

2. Then find the fourth term, or proportional, which will be the answer required.

EXAMPLES.

1. If 10lb. at Boston make 9lb. at Amsterdam; 90lb. at Amsterdam, 112lb. at Thoulouse; how many pounds at Thoulouse are equal to 50lb. at Boston?

```
Ant.      Cons.
 10   :     9
 90   :   112
----      ----
900   :  1008   : :   50 :
           50
        ------
        )50400(56 the answer.
         4500
         ----
          5400
          5400
```

Or by abbreviation.

```
10 :   9 : : 50 10 :   1 : : 50 1 :    1 : : 5
90 : 112        10 : 112*       10 : 112    2 : 112 : : 1 : 56.
                                                   56 the answer.
```

* In performing this example, the first abbreviation is obtain-

2. If 20 braces at Leghorn be equal to 10 vares at Lisbon; 40 vares at Lisbon to 80 braces at Lucca; how many braces at Lucca are equal to 100 braces at Leghorn?

Ans. 100 braces.

CASE II.

When it is required to find how many of the first kind of coin, weight, or measure, mentioned in the question, are equal to a given number of the last.

RULE.

Proceed as in the first case, only make the product of the consequents the first term, and that of the antecedents the second.

EXAMPLES.

1. If 100lb. in America make 95lb. Flemish; and 19lb. Flemish, 25lb. at Bolognia; how many pounds in America are equal to 50lb. at Bolognia?

```
 Cons.     Ant.
   95  :   100
   25  :    19
  ----    ----
  475
 190
 ----
 2375  :  1900 : : 50 :
             50
        -------
        )95000(40lb. the answer.
         9500
        -------
             0
```

---

ed by dividing 90 and 9 by their common measure 9; the second by dividing 10 and 50 by their common measure 10; the third by dividing 10 and 5 by their common measure 5; and the fourth, or answer, by dividing 2 and 112 by their common measure 2.

Or by abbreviation.

95 : 100	5 : 100	5 : 4	
25 : 19 : : 50	25 : 1 : : 50	1 : 1 : : 50	1 : 4 : : 10 :
			4

Ans. 40lb.

2. If 25lb. at Boston be 22lb. at Nuremburg; 88lb. at Nuremburg, 92lb. at Hamburgh; 46lb. at Hamburgh, 49lb. at Lyons; how many pounds at Boston are equal to 98lb. at Lyons? Ans. 100lb.

3. If 6 braces at Leghorn make 3 ells English; 5 ells English, 9 braces at Venice; how many braces at Leghorn will make 45 braces at Venice? Ans. 50 braces.*

FELLOWSHIP.

Fellowship is a general rule, by which merchants, &c. trading in company, with a joint stock, determine each person's particular share of the gain or loss in proportion to his share in the joint stock.

By this rule a bankrupt's estate may be divided among his creditors; as also legacies adjusted, when there is a deficiency of assets or effects.

SINGLE FELLOWSHIP.

Single Fellowship is when different stocks are employed for any certain equal time.

---

* *Barter* is the exchanging of one commodity for another, and directs traders so to proportion their goods, that neither party may sustain loss.

Loss and Gain is a rule, that discovers what is got or lost in the buying or selling of goods; and instructs merchants and traders to raise or lower the price of their goods, so as to gain or lose a certain sum per cent. &c.

Questions in these rules are performed by the rule of Three.

RULE.*

As the whole stock is to the whole gain or loss, so is each man's particular stock to his particular share of the gain or loss.

METHOD OF PROOF.

Add all the shares together, and the sum will be equal to the gain or loss, when the question is right.

EXAMPLES.

1. Two persons trade together; A put into stock \$130 and B \$220, and they gained \$500; what is each person's share thereof?

130
220
———
350 : 500 :: 130 :
500
———
35,0)6500,0(185·71$\frac{15}{35}$
35
———
300
280
———
200
175
———
250
245
———
50
35
———
15

---

* That the gain or loss, in this rule, is in proportion to their stocks is evident: for, as the times the stocks are in trade are equal, if I put in $\frac{1}{2}$ of the whole stock, I ought to have $\frac{1}{2}$ of the whole gain; if my part of the whole stock be $\frac{1}{3}$, my share of the

```
350 : 500 :: 220 :
                500
            ---------
            35,0)11000,0(314·28 20/35
                 105
                 ---
                  50
                  35
                  ---
                  150
                  140
                  ---
                   100
                    70
                   ---
                    300
                    280
                    ---
                     20
```

$185·71 $\frac{15}{35}$ = A's share.
314·28 $\frac{20}{35}$ = B's share.

$500·00 the proof.

2. A and B have gained by trading $182. A put into stock $300 and B $400; what is each person's share of the profit? Ans. A $78 and B $104.

3. Divide $120 between three persons, so that their shares shall be to each other as 1, 2, and 3 respectively.

Ans. $20, $40, and $60.

4. Three persons make a joint stock. A put in $185·66, B $98·50, and C $76·85; they trade and gain $222; what is each person's share of the gain?

Ans. A $104·17 $\frac{83}{36101}$, B $60·57 $\frac{6243}{36101}$, and C $47·25 $\frac{29775}{36101}$.

5. Three merchants A, B, and C freight a ship with 340

---

whole gain or loss ought to be $\frac{1}{3}$ also. And generally, if I put in $\frac{1}{n}$ of the stock, I ought to have $\frac{1}{n}$ part of the whole gain or loss; that is, the same ratio, that the whole stock has to the whole gain or loss, must each person's particular stock have to his particular gain or loss.

tuns of wine; A loaded 110 tuns, B 97, and C the rest. In a storm the seamen were obliged to throw 85 tuns overboard; how much must each sustain of the loss?

Ans. A $27\frac{1}{2}$, B $24\frac{1}{4}$, and C $33\frac{1}{4}$.

6. A ship worth $860 being entirely lost, of which $\frac{1}{8}$ belonged to A, $\frac{1}{4}$ to B, and the rest to C; what loss will each sustain, supposing $500 of her to be insured?

Ans. A $45, B $90, and C $225.

7. A bankrupt is indebted to A $277'33, to B $305'17, to C $152, and to D $105. His estate is worth only $677'50; how must it be divided?

Ans. A $223'81$\frac{2580}{8395}$, B $246'28$\frac{615}{8395}$,
C $122'36$\frac{6930}{8395}$, and D $84'73$\frac{6665}{8395}$.

8. A and B, venturing equal sums of money, clear by joint trade $154. By agreement A was to have 8 per cent. because he spent his time in the execution of the project, and B was to have only 5 per cent.; what was A allowed for his trouble? Ans. $35'53$\frac{11}{13}$.

DOUBLE FELLOWSHIP.

Double Fellowship is when different or equal stocks are employed for different times.

RULE.*

Multiply each man's stock into the time of its continuance, then say,

As the total sum of all the products is to the whole gain or loss,

So is each man's particular product to his particular share of the gain or loss.

EXAMPLES.

1. A and B hold a piece of ground in common, for which they are to pay $36. A put in 23 oxen for 27 days, and B 21 oxen for 35 days; what part of the rent ought each man to pay?

---

* Mr. MALCOM, Mr. WARD, and several other authors have given an analytical investigation of this rule; but the most gen-

$23 \times 27 = 621$
$21 \times 35 = 735$

1356

1356 : 36 :: 621 :

```
           621
           ---
            36
           72
          216
          ----
1356)22356(16·48 912/1356
          1356
          ----
           8796
           8136
           ----
           6600
           5424
           -----
          11760
          10848
          -----
            912
```

1356 : 36 :: 735 :

```
       735
       ---
       180
      108
     252
     ----
1356)26460(19·51 444/1356
     1356
     ----
     12900
     12204
     -----
      6960
      6780
      ----
       1800
       1356
       ----
        444
```

$\$16{\cdot}48\frac{912}{1356}$ = A's share.
$19{\cdot}51\frac{444}{1356}$ = B's share.

$36·00 the proof.

---

eral and elegant method perhaps is that, which Dr. HUTTON has given in his Arithmetic, namely,

When the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship ; and when the stocks are equal, the shares are as the times ; wherefore, when neither are equal, the shares must be as their products.

2. Three graziers hired a piece of land for $60·50. A put in 5 sheep for $4\frac{1}{2}$ months, B put in 8 for 5 months, and C put in 9 for $6\frac{1}{2}$ months; how much must each pay of the rent? Ans. A $11·25, B $20, and C $29·25.

3. Two merchants enter into partnership for 18 months; A put into stock at first $200, and at the end of 8 months he put in $100 more; B put in at first $550, and at the end of 4 months took out $140. Now at the expiration of the time they find they have gained $526; what is each man's just share? Ans. A's $$192·95\frac{70}{1254}$.

B's $333·04\frac{1184}{1254}$.

4. A, with a capital of $1000 began trade January 1, 1776, and meeting with success in business he took in B as a partner, with a capital of $1500 on the first of March following. Three months after that they admit C as a third partner, who brought into stock $2800, and after trading together till the first of the next year, they find the gain, since A commenced business, to be $1776·50. How must this be divided among the partners?

Ans. A's $$457·46\frac{364}{466}$.

B's $571·83\frac{222}{466}$.

C's $747·19\frac{346}{466}$.

ALLIGATION.

Alligation teaches how to mix several simples of different qualities, so that the composition may be of a middle quality; and is commonly distinguished into two principal cases, called *Alligation medial* and *Alligation alternate.*

ALLIGATION MEDIAL.

Alligation medial is the method of finding the rate of the compound, from having the rates and quantities of the several simples given.

RULE.*

Multiply each quantity by its rate; then divide the sum of the products by the sum of the quantities, or the whole composition, and the quotient will be the rate of the compound required.

EXAMPLES.

1. Suppose 15 bushels of wheat at 5s. per bushel, and 12 bushels of rye at 3s. 6d. per bushel were mixed together; how must the compound be sold per bushel without loss or gain?

```
 60          42          15
 15          12          12
———         ———         ———
300         504          27
 60         900
———         ———
900     27)1404)52d.=4s. 4d. the answer
            135
            ———
             54
             54
             ——
```

2. A composition being made of 5lb. of tea at 7s. per pound, 9lb. at 8s. 6d. per pound, and $14\frac{1}{2}$lb. at 5s. 10d. per pound, what is a pound of it worth? Ans. 6s. $10\frac{1}{2}$d.

3. Mixed 4 gallons of wine at 4s. 10d. per gallon, with 7

---

* The truth of this rule is too evident to need a demonstration.

NOTE. If an ounce or any other quantity of pure gold be reduced into 24 equal parts, these parts are called carats; but gold is often mixed with some baser metal, which is called the alloy, and the mixture is said to be of so many carats fine, according to the proportion of pure gold contained in it; thus, if 22 carats of pure gold and 2 of alloy be mixed together, it is said to be 22 carats fine.

If any one of the simples be of little or no value with respect to the rest, its rate is supposed to be nothing, as water mixed with wine, and alloy with gold or silver.

gallons at 5s. 3d. per gallon, and $9\frac{3}{4}$ gallons at 5s. 8d. per gallon; what is a gallon of this composition worth?

Ans. 5s. $4\frac{1}{4}$d.

4. A goldsmith melts 8lb. $5\frac{1}{2}$oz. of gold bullion of 14 carats fine, with 12lb. $8\frac{1}{2}$oz. of 18 carats fine; how many carats fine is this mixture? Ans. $16\frac{204}{508}$ carats.

5. A refiner melts 10lb. of gold of 20 carats fine with 16lb. of 18 carats fine; how much alloy must he put to it to make it 22 carats fine?

Ans. It is not fine enough by $3\frac{6}{26}$ carats, so that no alloy must be put to it, but more gold.

ALLIGATION ALTERNATE.

Alligation alternate is the method of finding what quantity of any number of simples, whose rates are given, will compose a mixture of a given rate; so that it is the reverse of alligation medial, and may be proved by it.

RULE 1.*

1. Write the rates of the simples in a column under each other.

---

* Demonstration. By connecting the less rate to the greater, and placing the differences between them and the mean rate alternately, the quantities resulting are such, that there is precisely as much gained by one quantity as is lost by the other, and therefore the gain and loss upon the whole are equal, and are exactly the proposed rate; and the same will be true of any other two simples, managed according to the rule

In like manner, let the number of simples be what it may, and with how many soever each is linked, since it is always a less with a greater than the mean price, there will be an equal balance of loss and gain between every two, and consequently an equal balance on the whole. Q. E. D.

It is obvious from the rule, that questions of this sort admit of a great variety of answers; for having found one answer, we may find as many more as we please, by only multiplying or dividing each of the quantities found by 2, 3, or 4, &c. the rea-

2. Connect or link with a continued line the rate of each simple, which is less than that of the compound, with one or any number of those, that are greater than the compound; and each greater rate with one or any number of the less.

3. Write the difference between the mixture rate and that of each of the simples opposite to the rates, with which they are respectively linked.

4. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity.

EXAMPLES.

1. A merchant would mix wines at 14s. 19s. 15s. and 22s. per gallon, so that the mixture may be worth 18s. the gallon; what quantity of each must be taken?

18	14	4 at 14s.
	15	1 at 15s.
	19	3 at 19s.
	22	4 at 22s.

Or thus:

18	14	1+4	5 at 14s.
	15	1	1 at 15s.
	19	4+3	7 at 19s.
	22	4	4 at 22s.

2. How much wine at 6s. per gallon and at 4s. per gallon must be mixed together, that the composition may be worth 5s. per gallon?

Ans. 12 gallons, or equal quantities of each.

---

son of which is evident; for, if two quantities of two simples make a balance of loss and gain, with respect to the mean price, so must also the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of these quantities, and so on, *ad infinitum*.

Questions of this kind are called by algebraists *indeterminate* or *unlimited* problems, and by an analytical process theorems may be raised, that will give all the *possible* answers.

3. How much corn at 2s. 6d. 3s. 8d. 4s. and 4s. 8d. per bushel must be mixed together, that the compound may be worth 3s. 10d. per bushel?

Ans. 12 at 2s. 6d. 12 at 3s. 8d. 18 at 4s. and 18 at 4s. 8d.

4. A gold smith has gold of 17, 18, 22, and 24 carats fine; how much must be taken of each to make it 21 carats fine?

Ans. 3 of 17, 1 of 18, 3 of 22, and 4 of 24.

5. It is required to mix brandy at 8s. wine at 7s. cider at 1s. and water at 0 per gallon together, so that the mixture may be worth 5s. per gallon?

Ans. 9 of brandy, 9 of wine, 5 of cider, and 5 of water.

RULE 2.*

When the whole composition is limited to a certain quantity, find an answer as before by linking; then say, as the sum

---

* A great number of questions might be here given relating to the specific gravity of metals, &c. but one of the most curious, with the operation at large, may serve as a sufficient specimen.

Hiero, king of Syracuse, gave orders for a crown to be made entirely of pure gold; but suspecting the workmen had debased it by mixing it with silver or copper, he recommended the discovery of the fraud to the famous Archimedes; and desired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the imposition, procured two other masses, one of pure gold, the other of silver or copper, and each of the same weight with the former; and each being put separately into a vessel full of water, the quantity of water expelled by them determined their specific bulks; from which and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppose the weight of each crown to be 10lb. and that the water expelled by the copper or silver was 92lb. by the gold 52lb. and by the compound crown 64lb. what will be the quantities of gold and alloy in the crown?

The rates of the simples are 92 and 52, and of the compound 64; therefore

64 { 92— 12 of copper,
     52— 28 of gold.

of the quantities, or differences thus determined, is to the given quantity, so is each ingredient, found by linking, to the required quantity of each.

EXAMPLES.

1. How many gallons of water at 0s. per gallon, must be mixed with wine worth 3s. per gallon, so as to fill a vessel of 100 gallons, and that a gallon may be afforded at 2s. 6d. ?

```
     ⎧ 0 ——⎤    6
  30 ⎨     |
     ⎩36 ——⎦   30
              ——
               36
```

```
36 : 100 : : 6 :            36 : 100 : : 30 :
      6                           30
    ———                          ———
 36)600(16                  36)3000(83
    36                          288
    ———                         ———
    240                          120
    216                          108
    ———                          ———
     24                           12
```

Ans. $83\frac{12}{36}$ gallons of wine, and $16\frac{24}{36}$ of water.

2. A grocer has currants at 4d. 6d. 9d. and 11d. per lb. and he would make a mixture of 240lb. so that it may be afforded at 8d. per pound; how much of each sort must he take ?

Ans. 72lb. at 4d. 24 at 6d. 48 at 9d. and 96 at 11d.

3. How much gold of 15, of 17, of 18, and of 22 carats fine must be mixed together to form a composition of 40 ounces of 20 carats fine ?

Ans. 5oz. of 15, of 17, and of 18, and 25 of 22.

And the sum of these is 12+28=40, which should have been but 10; whence, by the rule,

40 : 10 : : 12 : 3lb. of copper, ⎫ the answer
40 : 10 : : 28 : 7lb. of gold, ⎭

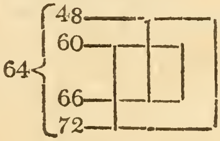
RULE 3.*

When one of the ingredients is limited to a certain quantity; take the difference between each price and the mean rate as before; then,

As the difference of that simple, whose quantity is given, is to the rest of the differences severally, so is the quantity given to the several quantities required.

EXAMPLES.

1. How much wine at 5s. at 5s. 6d. and 6s. the gallon must be mixed with 3 gallons at 4s. per gallon, so that the mixture may be worth 5s. 4d. per gallon?



8+2=10
8+2=10
16+4=20
16+4=20

10 : 10 :: 3 : 3
10 : 20 :: 3 : 6
10 : 20 :: 3 : 6

Ans. 3 gallons at 5s. 6 at 5s. 6d. and 6 at 6s.

2. A grocer would mix teas at 12s. 10s. and 6s. with 20lb. at 4s. per pound; how much of each sort must he take to make the composition worth 8s. per. lb.?

Ans. 20lb. at 4s. 10 at 6s. 10 at 10s. and 20 at 12s.

3. How much gold of 15, of 17, and of 22 carats fine,

---

* In the very same manner questions may be wrought, when several of the ingredients are limited to certain quantities, by finding first for one limit and then for another.

The two last rules can want no demonstration, as they evidently result from the first, the reason of which has been already explained.

must be mixed with 5oz. of 18 carats fine, so that the composition may be 20 carats fine?

Ans. 5oz. of 15 carats fine, 5 of 17, and 25 of 22.

INVOLUTION.

A Power is a number produced by multiplying any given number continually by itself a certain number of times.

Any number is itself called the *first power;* if it be multiplied by itself, the product is called the *second power*, or the *square*; if this be multiplied by the first power again, the product is called the *third power*, or the *cube;* and if this be multiplied by the first power again, the product is called the *fourth power*, or *biquadrate;* and so on; that is, the power is denominated from the number, which exceeds the multiplications by 1.

Thus, 3 is the first power of 3.
$3\times3=9$ is the second power of 3.
$3\times3\times3=27$ is the third power of 3.
$3\times3\times3\times3=81$ is the fourth power of 3.
&c. &c.

And in this manner is calculated the following table of powers.

TABLE *of the first twelve Powers of the* 9 *Digits.*

1st Pow.	1	2	3	4	5	6	7	8	9
2d Pow.	1	4	9	16	25	36	49	64	81
3d Pow.	1	8	27	64	125	216	343	512	729
4th Pow.	1	16	81	256	625	1266	2401	4096	6561
5th Pow.	1	32	243	1024	3125	7776	16807	32768	5904.
6th Pow.	1	64	729	4096	15625	46656	117649	262144	531441
7th Pow.	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th Pow.	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th Pow.	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th Pow.	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
11th Pow.	1	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609
12th Pow.	1	4096	531441	16777216	244140625	2176782336	13841287201	68719476736	282420536481

NOTE 1. The number, which exceeds the multiplications by 1, is called the *index*, or *exponent*, of the power; so the index of the first power is 1, that of the second power is 2, and that of the third is 3, &c.

NOTE 2. Powers are commonly denoted by writing their indices above the first power; so the second power of 3 may be denoted thus 3^2, the third power thus 3^3, the fourth power thus 3^4, &c. and the sixth power of 503 thus 503^6.

Involution is the finding of powers; to do which we have evidently the following

RULE.

Multiply the given number, or first power, continually by itself, till the number of multiplications be 1 less than the index of the power to be found, and the last product will be the power required.*

NOTE. Whence, because fractions are multiplied by taking the products of their numerators and of their denominators, they will be involved by raising each of their terms to the power required. And if a mixed number be proposed, either reduce it to an improper fraction, or reduce the vulgar fraction to a decimal, and proceed by the rule.

---

* NOTE. The raising of powers will be sometimes shortened by working according to this observation, viz. whatever two or more powers are multiplied together, their product is the power, whose index is the sum of the indices of the factors; or if a power be multiplied by itself, the product will be the power, whose index is double of that, which is multiplied: so if I would find the sixth power, I might multiply the given number twice by itself for the third power, then the third power into itself would give the sixth power; or if I would find the seventh power, I might first find the third and fourth, and their product would be the seventh; or lastly, if I would find the eighth power, I might first find the second, then the second into itself would be the fourth, and this into itself would be the eighth.

EXAMPLES.

1. What is the second power of 45 ? Ans. 2025.
2. What is the square of ˙027 ? Ans. ˙000729.
3. What is the third power of 3˙5 ? Ans. 42˙875.
4. What is the fifth power of ˙029 ?
Ans. ˙000000020511149.
5. What is the sixth power of 5˙03 ?
Ans. 16196˙005304479729.
6. What is the second power of $\frac{2}{3}$? Ans. $\frac{4}{9}$.

EVOLUTION.

The Root of any given number, or power, is such a number as, being multiplied by itself a certain number of times, will produce the power; and it is denominated the *first*, *second*, *third*, *fourth*, *&c. root* respectively, as the number of multiplications, made of it to produce the given power, is 0, 1, 2, 3, &c. that is, the name of the root is taken from the number, which exceeds the multiplications by 1, like the name of the power in involution.

Note 1. The *index* of the root, like that of the power in involution, is 1 more than the number of multiplications, necessary to produce the power or given number.

Note 2. Roots are sometimes denoted by writing $\surd$ before the power, with the index of the root against it: so the third root of 50 is $\sqrt[3]{50}$, and the second root of it is $\sqrt{50}$, the index 2 being omitted, which index is always understood, when a root is named or written without one. But if the power be expressed by several numbers with the sign + or —, &c. between them, then a line is drawn from the top of the sign of the root, or radical sign, over all the parts of it: so the third root of 47 — 15 is $\sqrt[3]{47 - 15}$. And sometimes roots are designed like powers, with the reciprocal of the index of the root above the given number. So the second root of 3 is $3^{\frac{1}{2}}$; the second root of 50 is $50^{\frac{1}{2}}$; and the third root of it is $50^{\frac{1}{3}}$; also the third root of 47 — 15 is $\overline{47-15.}|^{\frac{1}{3}}$. And this method of notation has justly prevailed in the modern algebra; because such roots, being considered as fractional powers, need no other directions for any operations to be made with them, than those for integral powers.

Note 3. A number is called a *complete* power of any kind, when its root of the same kind can be accurately extracted; but if not, the number is called an *imperfect* power, and its root a *surd* or *irrational* number: so 4 is a complete power of the second kind, its root being 2; but an imperfect power of the third kind, its root being a surd number.

Evolution is the finding of the roots of numbers either accurately, or in decimals, to any proposed extent.

The power is first to be prepared for extraction, or evolution, by dividing it from the place of units, to the left in integers, and to the right in decimal fractions, into periods, each containing as many places of figures, as are denominated by the index of the root, if the power contain a complete number of such periods: if it do not, the defect will be either on the right, or left, or both; if the defect be on the right, it may be supplied by annexing cyphers, and after this, whole periods of cyphers may be annexed to continue the extraction, if necessary; but if there be a defect on the left, such defective period must remain unaltered, and is accounted the first period of the given number, just the same, as if it were complete.

Now this division may be conveniently made by writing a point over the place of units, and also over the last figure of every period on both sides of it; that is, over every second figure, if it be the second root; over every third, if it be the third root, &c.

Thus, to point this number 21035896·12735;

for the second root, it will be $2\dot{1}0\dot{3}5\dot{8}9\dot{6}\cdot1\dot{2}7\dot{3}5\dot{0}$;

but for the third root $2\dot{1}03\dot{5}89\dot{6}\cdot12\dot{7}35\dot{0}$;

and for the fourth $210\dot{3}589\dot{6}\cdot127\dot{3}500\dot{0}$.

Note. The root will contain just as many places of figures, as there are periods or points in the given power; and they will be integers or decimals respectively, as the periods are so, from which they are found, or to which they correspond; that is, there will be as many integral or decimal figures in the root, as there are periods of integers or decimals in the given number.

TO EXTRACT THE SQUARE ROOT.

RULE.*

1. Having distinguished the given number into periods, find a square number by the table or trial, either equal to, or next less than the first period, and put the root of it on the right of the given number, in the manner of a quotient figure in division, and it will be the first figure of the root required.

---

* In order to show the reason of the rule, it will be proper to premise the following

Lemma. The product of any two numbers can have at most but as many places of figures, as are in both the factors, and at least but one less.

Demonstration. Take two numbers, consisting of any number of places, but let them be the least possible of those places, namely, unity with cyphers, as 1000 and 100; then their product will be 1 with as many cyphers annexed, as are in both the numbers, namely, 100000; but 100000 has one place less than 1000 and 100 together have; and since 1000 and 100 were taken the least possible, the product of any other two numbers, of the same number of places, will be greater than 100000; consequently the product of any two numbers can have at least but one place less than both the factors.

Again, take two numbers of any number of places, that shall be the greatest of these places possible, as 999 and 99. Now 999×99 is less than 999×100; but 999×100 $(=99900)$ contains only as many places of figures, as are in 999 and 99; therefore 999×99, or the product of any other two numbers, consisting of

2. Subtract the assumed square from the first period, and to the remainder bring down the next period for a dividend.

3. Place the double of the root, already found, on the left of the dividend for a divisor.

---

the same number of places, cannot have more places of figures than are in both its factors.

Corollary 1. A square number cannot have more places of figures than double the places of the root, and at least but one less.

Cor. 2. A cube number cannot have more places of figures than triple the places of the root, and at least but two less.

The truth of the rule may be shown algebraically thus :

Let $\mathcal{N}$= the number, whose square root is to be found.

Now it appears from the lemma, that there will be always as many places of figures in the root, as there are points or periods in the given number, and therefore the figures of those places may be represented by letters.

Suppose $\mathcal{N}$ to consist of two periods, and let the figures in the root be represented by a and b.

Then $\overline{a+b} = a^2+2ab+b^2 = \mathcal{N} =$ given number ; and to find the root of $\mathcal{N}$ is the same, as finding the root of $a^2+2ab+b^2$, the method of doing which is as follows :

1st divisor $a)a^2+2ab+b^2(a+b=$ root.
a^2

2d divisor $2a+b)2ab+b^2$
$2ab+b^2$

Again suppose $\mathcal{N}$ to consist of 3 periods, and let the figures of the root be represented by a, b, and c.

Then $\overline{a+b+c}^2 = a^2+2ab+b^2+2ac+2bc+c^2$, and the manner of finding a, b, and c will be, as before : thus,

1st divisor $a)a^2+2ab+b^2+2ac+2bc+c^2\ (a+b+c=$ root.
a^2

2d divisor $2a+b)2ab+b^2$
$2ab+b^3$

4. Consider what figure must be annexed to the divisor, so that if the result be multiplied by it, the product may be equal to, or next less than the dividend, and it will be the second figure of the root.

5. Subtract the said product from the dividend, and to the remainder bring down the next period for a new dividend.

6. Find a divisor as before, by doubling the figures already in the root; and from these find the next figure of the root, as in the last article; and so on through all the periods to the last.

NOTE 1. When the root is to be extracted to a great number of places, the work may be much abbreviated thus: having proceeded in the extraction by the common method till you have found one more than half the required number of figures in the root, the rest may be found by dividing the last remainder by its corresponding divisor, annexing a cypher to every dividual, as in division of decimals; or rather, without annexing cyphers, by omitting continually the first figure of the divisor on the right, after the manner of contraction in division of decimals.

NOTE 2. By means of the square root we readily find the fourth root, or the eighth root, or the sixteenth root, &c. that is, the root of any power, whose index is some power of the number 2; namely by extracting so often the square root, as is denoted by that power of 2; that is, twice for the fourth root, thrice for the eighth root, and so on.

3d divisor $2a+2b+c)2ac+2bc+c^2$

$$2ac+2bc+c^2$$

Now the operation in each of these cases exactly agrees with the rule, and the same will be found to be true, when N consists of any number of periods whatever.

TO EXTRACT THE SQUARE ROOT OF A VULGAR FRACTION.

RULE.

First prepare all vulgar fractions by reducing them to their least terms, both for this and all other roots. Then

1. Take the root of the numerator and that of the denominator for the respective terms of the root required. And this is the best way, if the denominator be a complete power. But if not, then

2. Multiply the numerator and denominator together; take the root of the product: this root, being made the numerator to the denominator of the given fraction, or the denominator to the numerator of it, will form the fractional root repuired.

That is, $\sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}$.

And this rule will serve, whether the root be finite or infinite.

Or 3. Reduce the vulgar fraction to a decimal, and extract its root.

EXAMPLES.

1. Required the square root of 5499025.

```
      5499025(2345 the root.
      4
     ----
  43 | 149
   3 | 129
  ---------
 464 | 2090
   4 | 1856
 ----------
4685 | 23425
     | 23425
 ----------
```

2. Required the square root of 184·2.

```
       184·2000(13·57 the root.
       1
      ----
    23|84
     3|69
    ------
   265|1520
     5|1325
    -------
  2707|19500
      |18949
      ------
        551 remainder
```

3. Required the square root of 2 to 12 places.

```
      2(1·41421356237 + root.
      1
     ---
     24|100
      4| 96
     ------
    281|400
      1|281
    -------
   2824|11900
      4|11296
   ---------
  28282|60400
      2|56564
  -----------
 282841|383600
      1|282841
 -------------
2828423|10075900
      3| 8485269
 ---------------
2828426)1590631(56237+
   ·····1414213
        -------
         176418
         169706
         ------
```

```
6712
5657
----
1055
 849
----
 206
 198
----
   8
```

4. What is the square root of 152399025?

Ans. 12345.

5. What is the square root of ·00032754?

Ans. ·01809.

6. What is the square root of $\frac{5}{12}$? Ans. ·645497.

7. What is the square root of $6\frac{2}{5}$? Ans. 2·5298, &c.

8. What is the square root of 10? Ans. 3·162277, &c.

TO EXTRACT THE CUBE ROOT.

RULE.*

1. Having divided the given number into periods of 3 figures, find the nearest less cube to the first period by the table of powers or trial; set its root in the quotient, and subtract the said cube from the first period; to the remainder bring down the second period, and call this the *resolvend.*

---

* The reason of pointing the given number, as directed in the rule, is obvious from Cor. 2, to the Lemma, used in demonstrating the square root; and the rest of the operation will be best understood from the following analytical process.

Suppose N, the given number, to consist of two periods, and let the figures in the root be denoted by a and b.

Then $\overline{a+b}|^3 = a^3+3a^2b+3ab^2+b^3 = N =$ given number, and to find the cube root of N is the same as to find the cube root of $a^3+3a^2b+3ab^2+b^3$; the method of doing which is as follows:

$$a^3+3a^2b+3ab^2+b^3 \ (a+b = \text{root}.$$
$$a^3$$

2. To three times the square of the root, just found, add three times the root itself, setting this one place more to the right than the former, and call this sum the *divisor*. Then divide the resolvend, wanting the last figure, by the divisor, for the next figure of the root, which annex to the former; calling this last figure e, and the part of the root before found call a.

3. Add together these three products, namely, thrice the square of a multiplied by e, thrice a multiplied by the square of e, and the cube of e, setting each of them one place farther toward the right than the former, and call the sum the *subtrahend;* which must not exceed the resolvend; and if it do, then make the last figure e less, and repeat the operation for finding the subtrahend.

4. From the resolvend take the subtrahend, and to the remainder join the next period of the given number for a new resolvend; to which form a new divisor from the whole root now found; and thence another figure of the root, as before, &c.

EXAMPLES.

1. To extract the cube root of 48228·544.

$3 \times 3^2 = 27$		48228·544(36·4 root.
$3 \times 3 = 09$		27
Divisor	279	21228 resolvend.

$3a^2b+3ab^2+b^3$ resolvend.

$3a^2$

$+3a$

$3a^2 + 3a$ divisor.

$3a^2b$

$+3ab^2$

$+b^3$

$3a^2b+3ab^2+b^3$ subtrahend.

* * *

And in the same manner may the root of a quantity, consisting of any number of periods whatever, be found.

$3\times 3^2\times 6 = 162$
$3\times 3\ \times 6^2 = 324$ add
$6^3 = 216$

$3\times 36^2 = 3888$ | 19656 subtrahend.
$3\times 36 = 108$ |
38988 | 1572544 resolvend.

$3\times 36^2\times 4 = 15552$
$3\times 36\ \times 4^2 = 1728$ add
$4^3 = 64$

1572544 subtrahend.

2. What is the cube root of 1092727? Ans. 103.
3. What is the cube root of 27054036008? Ans. 3002.
4. What is the cube root of ·0001357? Ans. ·05138, &c.
5. What is the cube root of $\frac{4520}{5130}$? Ans. $\frac{2}{3}$.
6. What is the cube root of $\frac{2}{3}$? Ans. ·873 &c.

RULE FOR EXTRACTING THE CUBE ROOT BY APPROXIMATION.*

1. Find by trial a cube near to the given number, and call it the *supposed cube.*

* That this rule converges extremely fast may be easily shown thus:

Let N= given number, a^3= supposed cube, and x= correction.

Then $2a^3+N : 2N+a^3 :: a : a+x$ by the rule, and consequently $\overline{2a^3+N}\times\overline{a+x}=\overline{2N+a^3}\times a$, or $\overline{2a^3+\overline{a+x}^3}\times\overline{a+x}=\overline{2N+a^3}\times a$.

Or $2a^4+2a^3x+a^4+4a^3x+6a^2x^2+4ax^3+x^4=2aN+a^4$, and by transposing the terms, and dividing by $2a$

$N=a^3+3a^2x+3ax^2+x^3+x^3+\frac{x^4}{2a}$, which by neglecting the

2. Then twice the supposed cube added to the given number is to twice the given number added to the supposed cube, as the root of the supposed cube is to the root required nearly. Or as the first sum is to the difference of the given and supposed cube, so is the supposed root to the difference of the roots nearly.

3. By taking the cube of the root thus found for the supposed cube, and repeating the operation, the root will be had to a still greater degree of exactness.

EXAMPLES.

1. It is required to find the cube root of 98003449.

Let 125000000 = supposed cube, whose root is 500;

Then	125000000		98003449			
	2		2			
	250000000		196006898			
	98003449		125000000			
	348003449	:	321006898	: :	500	:
			500			

348003449)160503449000(461=corrected root, or [root nearly.
1392013796
2130206940
2088020694
421862460
348003449
73859011

2. Required the cube root of 21035·8.

Here we soon find that the root lies between 20 and 30,

---

terms $x^3 + \frac{x^4}{2a}$, as being very small, becomes $N = a^3 + 3a^2x + 3ax^2 + x^3 =$ the known cube of $a + x$. Q. E. I.

and then between 27 and 28. Therefore 27 being taken, its cube is 19683 the assumed cube. Then

```
        19683      21035·8
            2            2
       ------     --------
        39366      42071·6
        21035·8    19683
       -------    --------
As  60401·8 : 61754·6  : :  27  :  27·6047
                   27
             --------
              4322822
             1235092
             ---------
             1667374·2
  60401·8)1667374·2(27·6047 the root nearly.
  · · · · · 1208036
            -------
             459338
             422813
            -------
              36525
              36241
             ------
                284
                242
               ----
                 42
```

Again for a second operation, the cube of this root is 21035·318645155823, and the process by the latter method is thus:

```
    21035·318645, &c.
                  2
    ----------------
    42070·637290     21035·8
    210358           21035·318645, &c.
    ------------     -----------------
As 63106·43729  : diff. ·481355 : : 27·6047 : the diff.=
                                        ·000210834
                                      -------------
                                      27·604910834 =
                                        the root required.
```

3. What is the cube root of 157464? Ans. 54.
4. What is the cube root of $\frac{4}{9}$? Ans. ·763, &c.
5. What is the cube root of 117? Ans. 4·89097.

TO EXTRACT THE ROOTS OF POWERS IN GENERAL.

RULE.*

1. Prepare the given number for extraction by pointing off from the units place as the root required directs.

---

* This rule will be sufficiently obvious from the work in the following example.

Extract the cube root of $a^6+6a^5-40a^3+96a-64$.

$$a^6+6a^5-40a^3+96a-64\,(a^2+2a-4$$
$$a^6$$
$$3a^4)\,6a^5\,(+2a$$
$$a^6+6a^5+12a^4+8a^3=\overline{a^2+2a}^{\,3}$$
$$\overline{a^2+2a}^{\,2}\times3=3a^4+12a^3+12a^2)\,-12a^4-48a^3+96a-64\,(-4$$
$$a^6+6a^5-40a^3+96a-64=\overline{a^2+2a-4}^{\,3}$$

* *

When the index of the power, whose root is to be extracted, is a composite number, the following rule will be serviceable:

Take any two or more indices, whose product is the given index, and extract out of the given number a root answering to one of these indices; and then out of this root extract a root, answering to another of the indices, and so on to the last.

Thus, the fourth root = square root of the square root.

The sixth root = square root of the cube root, &c.

The proof of all roots is by involution.

The following theorems may sometimes be found useful in extracting the root of a vulgar fraction; $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{ab}}{b}=\frac{a}{\sqrt{ab}}$;

2. Find the first figure of the root by trial, and subtract its power from the given number.

3. To the remainder bring down the first figure in the next period, and call it the *dividend.*

4. Involve the root to the next inferior power to that, which is given, and multiply it by the number denoting the given power for a *divisor.*

5. Find how many times the divisor may be had in the dividend, and the quotient will be another figure of the root.

6. Involve the whole root to the given power, and subtract it from the given number as before.

7. Bring down the first figure of the next period to the remainder for a new dividend, to which find a new divisor, and so on, till the whole be finished.

EXAMPLES.

1. What is the cube root of 53157376 ?

$$\begin{array}{rl}
 & \dot{5}31\dot{5}7\dot{3}76(376 \\
 & 27=3^3 \\
 & \overline{} \\
3^2\times3=27) & 261 \text{ dividend.} \\
 & \overline{} \\
 & 50653=37^3 \\
 & \overline{} \\
3^2\times3=4107) & 25043 \text{ second dividend.} \\
 & 53157376 \\
 & \overline{} \\
 & 0
\end{array}$$

2. What is the biquadrate root of 19987173376 ?

Ans. 376.

---

or universally, $\overline{\frac{a}{b}}\Big|^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\overline{ab^{n-1}}\big|^{\frac{1}{n}}}{b} = \frac{a}{\overline{ba^{n-1}}\big|^{\frac{1}{n}}}.$

3. Extract the sursolid, or fifth root, of 30768282110671 5625.

Ans. 3145.

4. Extract the square cubed, or sixth root, of 43572838 1009267809889764416.

Ans. 27534.

5. Find the seventh root of 34487717467307513182492 153794673.

Ans. 32017.

6. Find the eighth root of 112101628132047623624649 7942460481.

Ans. 13527.

TO EXTRACT ANY ROOT WHATEVER BY APPROXIMATION.

RULE.

1. Assume the root nearly, and raise it to the same power with the given number, which call the *assumed power*.

2. Then, as the sum of the assumed power multiplied by the index more 1 and the given number multiplied by the index less 1, is to the sum of the given number multiplied by the index more 1 and the assumed power multiplied by the index less 1, so is the assumed root to the required root.

Or, as half the first sum is to the difference between the given and assumed powers, so is the assumed root to the difference between the true and assumed roots; which difference, added or subtracted, gives the true root nearly.

And the operation may be repeated as often as we please by using always the last found root for the assumed root, and its power as aforesaid for the assumed power.

EXAMPLES.

1. Required the fifth root of 21035·8.

Here it appears, that the fifth root is between 7·3 and 7·4. 7·3 being taken, its fifth power is 20730·71593. Hence then

21035·8 = given number.
20730·716 = assumed power.

305·084 = difference.

5 = index.	20730·716	21035·8
5 + 1 = 6	3	2
5 — 1 = 4		
6 ÷ 2 = 3	62192·148	42071·6
4 ÷ 2 = 2	42071·6	
	104263·748 = ½ the first sum.	

104263·7 : 305·084 : : 7·3 : ·0213605

305·084
7·3
915252
2135588

104263·7)2227·1132(·0213604 = difference.

Division	Root
...... 208527	7·3
14184	7·321360 = root,
10426	true to the last figure.
3758	
3128	
630	
626	
4	

2. What is the third root of 2? Ans. 1·259921.
3. What is the sixth root of 21035·8? Ans. 5·254037.
4. What is the seventh root of 21035·8? Ans. 4·145392.
5. What is the ninth root of 21035·8? Ans. 3·022239.

ARITHMETICAL PROGRESSION.

ANY rank of numbers, increasing by a common excess or decreasing by a common difference, is said to be in *Arith-*

metical Progression; such are the numbers 1, 2, 3, 4, 5, &c. 7, 5, 3, 1; and ˙8, ˙6, ˙4, ˙2. When the numbers increase they form an *ascending series;* but when they decrease, they form a *descending series.*

The numbers, which form the series, are called the *terms* of the progression.

Any *three* of the *five* following terms being given, the other two may be readily found.

1. The first term, } commonly called the
2. The last team, } *extremes.*
3. The number of terms.
4. The common difference.
5. The sum of all the terms.

PROBLEM 1.

The first term, the last term, and the number of terms being given, to find the sum of all the terms.

RULE.*

Multiply the sum of the extremes by the number of terms, and half the product will be the answer.

EXAMPLES.

1. The first term of an arithmetical progression is 2, the last term 53, and the number of terms 18; required the sum of the series.

---

* Suppose another series of the same kind with the given one be placed under it in an inverse order; then will the sum of every two corresponding terms be the same as that of the first and last; consequently any one of those sums, multiplied by the number of terms, must give the whole sum of the two series, and half that sum will evidently be the sum of the given series: thus,

Let 1, 2, 3, 4, 5, 6, 7, be the given series;
and 7, 6, 5, 4, 3, 2, 1, the same inverted;
then $8+8+8+8+8+8+8=8\times7=56$ and $1+3+4+5+6+7=\frac{56}{2}=28$. Q. E. D.

```
   53
    2
  ---
   55
   18
  ---
  440
  55
  ---
2)990
  ---
  495
```

Or,

$$\frac{\overline{53+2}\times 18}{2}=495 \text{ the answer.}$$

2. The first term is 1, the last term 21, and the number of terms 11; required the sum of the series. Ans. 121.

3. How many strokes do the clocks of Venice, which go to 24 o'clock, strike in the compass of a day? Ans. 300.

4. If 100 stones be placed in a right line, exactly a yard asunder, and the first a yard from a basket, what length of ground will that man go, who gathers them up singly, returning with them one by one to the basket? Ans. 5 miles and 1300 yards.

PROBLEM II.

The first term, the last term, and the number of terms being given, to find the common difference.

RULE 3.*

Divide the difference of the extremes by the number of terms less 1, and the quotient will be the common difference sought.

---

* The difference of the first and last terms evidently shows the increase of the first term by all the subsequent additions, till it becomes equal to the last; and as the number of those additions is evidently one less than the number of terms, and the increase by every addition equal, it is plain, that the total increase, divided by the number of additions, must give the difference at every one separately; whence the rule is manifest.

EXAMPLES.

1. The extremes are 2 and 53, and the number of terms is 18; required the common difference.

```
   53            18
    2             1
  ----          ----
 17)51(3         17
    51
   ----
```

Or,

$$\frac{53-2}{18-1}=\frac{51}{17}=3 \text{ the answer.}$$

2. If the extremes be 3 and 19, and the number of terms 9, it is required to find the common difference, and the sum of the whole series.

Ans. The difference is 2, and the sum is 99.

3. A man is to travel from London to a certain place in 12 days, and to go but three miles the first day, increasing every day by an equal excess, so that the last day's journey may be 58 miles; required the daily increase, and the distance of the place from London.

Ans. Daily increase 5, distance 366 miles.

PROBLEM III.

Given the first term, the last term, and the common difference, to find the number of terms.

RULE.*

Divide the difference of the extremes by the common difference, and the quotient, increased by 1, is the number of terms required.

---

* By the last problem, the difference of the extremes, divided by the number of terms less 1, gives the common difference; consequently the same, divided by the common difference, must give the number of terms less 1; hence this quotient, augmented by 1, must be the answer to the question.

EXAMPLES.

1. The extremes are 2 and 53, and the common difference 3; what is the number of terms?

```
  53
   2
  ——
3)51
  ——
  17
   1
  ——
  18
```

Or,

$\frac{53-2}{3}+1=18$ the answer.

In any arithmetical progression, the sum of any two of its terms is equal to the sum of any other two terms, taken at an equal distance on contrary sides of the former; or the double of any one term is equal to the sum of any two terms, taken at an equal distance from it on each side.

The sum of any number of terms (n) of the arithmetical series of odd numbers 1, 3, 5, 7, 9, &c. is equal to the square (n^2) of that number.

That is, if 1, 3, 5, 7, 9, &c. be the numbers,

Then will 1, 2^2, 3^2, 4^2, 5^2, &c. be the sums of 1, 2, 3, &c. of those terms.

For, $0+1$ or the sum of 1 term $= 1^2$ or 1

$1+3$ or the sum of 2 terms $= 2^2$ or 4

$4+5$ or the sum of 3 terms $= 3^2$ or 9

$9+7$ or the sum of 4 terms $= 4^2$ or 16, &c.

Whence it is plain, that, let n be any number whatsoever, the sum of n terms will be n^2.

The following table contains a summary of the whole doctrine of arithmetical progression.

CASES OF ARITHMETICAL PROGRESSION.			
Cas.	Giv.	Req.	Solution.
1	adn	l	$\overline{n-1}\times d+a.$
		s	$n\times\overline{a+\overline{n-1}\times\frac{d}{2}}.$

2. If the extremes be 3 and 19, and the common difference 2, what is the number of terms? Ans. 9.

Case	Giv.	Req.	Solution.
2	adl	n	$\frac{l-a}{d}+1.$
		s	$\frac{\overline{l+a}\times\overline{l-a+d}}{2d}.$
3	ads	n	$\frac{\sqrt{\overline{2a-d}\vert^2+8ds}-\overline{2a-d}}{2d}.$
		l	$\frac{\sqrt{\overline{2a-d}\vert^2+8ds}-d}{2}.$
4	als	d	$\frac{\overline{l+a}\times\overline{l-a}}{2s-\overline{l+a}}.$
		n	$\frac{2s}{a+l}.$
5	ans	d	$\frac{2\times\overline{s-an}}{\overline{n-1}\times n}.$
		l	$\frac{2s}{n}-a.$
6	aln	d	$\frac{l-a}{n-1}.$
		s	$\frac{\overline{a+l}\times n}{2}.$

3. A man, going a journey, travelled the first day 5 miles, the last day 35 miles, and increased his journey every day by 3 miles; how many days did he travel?

Ans. 11 days.

Case	Giv.	Req.	Solution.
7	*dnl*	*a*	$l - \overline{n-1} \times d.$
		s	$n \times \overline{l - \overline{n-1} \times \frac{d}{2}}.$
8	*snd*	*a*	$\frac{s}{n} - \frac{d \times \overline{n-1}}{2}.$
		l	$\frac{s}{n} - \frac{d \times \overline{n-1}}{2}.$
9	*dls*	*a*	$\frac{d \pm \sqrt{\overline{2l+d}\rvert^2 - 8ds}}{2}.$
		n	$\frac{2l + d \mp \sqrt{\overline{2l+d}^2 - 8ds}}{2d}.$
10	*lns*	*a*	$\frac{2s}{n} - l.$
		d	$\frac{2 \times \overline{nl - s}}{\overline{n-1} \times n}.$

Here:
- a = least term.
- n = number of terms,
- s = sum of all the terms.
- d = common difference.
- l = greatest term.

GEOMETRICAL PROGRESSION.

Any series of numbers, the terms of which gradually increase or decrease by a constant multiplication or division, is said to be in *Geometrical Progression.* Thus, 4, 8, 16, 32, 64, &c. and 81, 27, 9, 3, 1, &c. are series in geometrical progression, the one increasing by a constant multiplication by 2, and the other decreasing by a constant division by 3.

The number, by which the series is constantly increased or diminished, is called the *ratio.*

PROBLEM I.

Given the first term, the last term, and the ratio, to find the sum of the series.

RULE.*

Multiply the last term by the ratio, and from the product subtract the first term, and the remainder, divided by the ratio less 1, will give the sum of the series.

---

* Demonstration. Take any series whatever, as 1, 3, 9, 27, 81, 243, &c. multiply this by the ratio, and it will produce the series 3, 9, 27, 81, 243, 729, &c. Now let the sum of the proposed series be what it will, it is plain, that the sum of the second series will be as many times the former sum, as is expressed by the ratio; subtract the first series from the second, and it will give 729—1; which is evidently as many times the sum of the first series, as is expressed by the ratio less 1; consequently $\frac{729-1}{3-1}$ = sum of the proposed series, and is the rule; or 729 is the last term multiplied by the ratio, 1 is the first term, and 3—1 is the ratio less one; and the same will hold, let the series be what it will. Q. E. D.

Note 1. Since, in any geometrical series of progression, when it consists of four terms, the product of the extremes is equal to the product of the means; and when it consists of three, the product of the extremes is equal to the square of the mean; it

EXAMPLES.

1. The first term of a series in geometrical progression is 1, the last term is 2187, and the ratio 3; what is the sum of the series?

$$\begin{array}{r} 2187 \\ 3 \\ \hline 6561 \\ 1 \\ \hline 3-1=2)\overline{6560} \\ \hline 3280 \end{array}$$

Or,

$$\frac{3\times2187-1}{3-1}=3280 \text{ the answer.}$$

2. The extremes of a geometrical progression are 1 and 65536, and the ratio 4; what is the sum of the series?

Ans. 87381.

3. The extremes of a geometrical series are 1024 and 59049, and the ratio is $1\frac{1}{2}$; what is the sum of the series?

Ans. 175099.

---

follows, that in any geometrical series, when it consists of an even number of terms, the product of the extremes is equal to the product of ony two means, equally distant from the extremes; and when the number of terms is odd, the product of the extremes is equal to the square of the mean or middle term, or to the product of any two terms, equally distant from them.

NOTE 2. If $a : b :: c : d$ directly,

Then
$a : c :: b : d$ by alternation.
$b : a :: d : c$ by inversion.
$a+b : b :: c+d : d$ by composition.
$a-b : b :: c-d : d$ by division.
$a : a+b :: c : c+d$ by conversion.
$a+b : a-b :: c+d : c-d$ mixedly.

For in each of these proportions the product of the extremes is equal to that of the means.

PROBLEM II.

Given the first term and the ratio, to find any other term assigned.

RULE.*

1. Write a few of the leading terms of the series, and place their indices over them, beginning with a cypher.

2. Add together the most convenient indices to make an index less by 1 than the number, expressing the place of the term sought.

3. Multiply the terms of the geometrical series together, belonging to those indices, and make the product a dividend.

4. Raise the first term to a power, whose index is 1 less than the number of terms multiplied, and make the result a divisor.

5. Divide the dividend by the divisor, and the quotient will be the term sought.

NOTE. When the first term of the series is equal to the ratio, the indices must begin with an unit, and the indices added must make the entire index of the term required; and the product of the different terms, found as before, will give the term required.

EXAMPLES.

1. The first term of a geometrical series is 2, the number of terms 13, and the ratio 2; required the last term.

---

* DEMONSTRATION. In example 1, where the first term is equal to the ratio, the reason of the rule is evident; for as every term is some power of the ratio, and the indices point out the number of factors, it is plain from the nature of multiplication, that the product of any two terms will be another term corresponding with the index, which is the sum of the indices standing over those respective terms.

And in the second example, where the series does not begin with the ratio, it appears, that every term after the two first contains some power of the ratio, multiplied into the first term, and therefore the rule, in this case, is equally evident.

1, 2, 3, 4, 5, indices.
2, 4, 8, 16, 32, leading terms.
Then 4+4+3+2 = index to the 13th term.
And 16×16×8×4=8192 the answer.

The following Table contains all the possible cases of geometrical progression.

CASES OF *GEOMETRICAL PROGRESSION.*			
Case	Giv.	Req.	Solution.
1	*arn*	*l*	ar^{n-1}.
		s	$\frac{r^{n-1}}{r-1}\times a$.
2	*arl*	*s*	$l+\frac{l-a}{r-1}$.
		n	$\frac{L,l-L,a}{L,r}+1$.
3	*ars*	*l*	$\frac{\overline{r-1}\times s+a}{r}$.
		n	$\frac{L,\overline{\overline{r-1}\times s+a}-L,a}{L,r}$.
4	*asl*	*r*	$\frac{s-a}{s-l}$.
		n	$\frac{L,l-L,a}{L,\overline{s-a}-L,\overline{s-l}}+1$.

In this example the indices must begin with 1, and such of them be chosen, as will make up the entire index to the term required.

2. Required the 12th term of a geometrical series, whose first term is 3, and ratio 2.

Case	Giv.	Req.	Solution.
5	*ans*	*r*	$r^n - \frac{rs}{a} = \frac{a-s}{a}$.
		l	$l \times \overline{s-l}\vert^{n-1} = a \times \overline{s-a}\vert^{n-1}$.
6	*anl*	*r*	$\overline{\frac{l}{a}}\Big\vert^{\frac{1}{n-1}}$.
		s	$l + \frac{l-a}{\overline{\frac{l}{a}}\Big\vert^{\frac{1}{n-1}} - 1}$.
7	*rnl*	*a*	$\frac{l}{r^{n-1}}$
		s	$l + \frac{l - \frac{l}{r^{n-1}}}{r-1}$.
8	*rns*	*a*	$\frac{r-1}{r^n-1} \times s$.
		l	$\frac{r^n - r^{n-1}}{r^n-1} \times s$.

0, 1, 2, 3, 4, 5, 6, indices.

3, 6, 12, 24, 48, 96, 192, leading terms.

Then $6+5=$ index to the 12th term.

And $192\times96=18432=$ dividend.

The number of terms multiplied is 2, and $2-1=1$ is the power, to which the term 3 is to be raised; but the first power of 3 is 3, and therefore $18432\div3=6144$ the 12th term required.

3. The first term of a geometrical series is 1, the ratio 2, and the number of terms 23; required the last term.

Ans. 4194304.

QUESTIONS

TO BE SOLVED BY THE TWO PRECEDING PROBLEMS.

1. A person being asked to dispose of a fine horse, said he would sell him on condition of having one farthing for

Case	Giv.	Req.	Solution.
9	rls	a	$s-r\times\overline{s-l}$.
		n	$\frac{L,l-L,\overline{s-r\times\overline{s-l}}}{L,r}+1$.
10	nls	a	$a\times\overline{s-a}\vert^{n-1}=l\times\overline{s-l}\vert^{n-1}$.
		r	$r^n+\frac{s}{l-s}r^{n-1}=\frac{l}{l-s}$.

Here $\begin{cases} a = \text{least term.} \\ l = \text{greatest term.} \\ s = \text{sum of all the terms.} \\ n = \text{number of terms.} \\ r = \text{ratio.} \\ L = \text{Logarithms.} \end{cases}$

the first nail in his shoes, 2 farthings for the second, one penny for the third, and so on, doubling the price of every nail to 32, the number of nails in his four shoes; what would the horse be sold for at that rate?

Ans. 4473924l. 5s. $3\frac{3}{4}$d.

2. A young man, skilled in numbers, agreed with a farmer to work for him eleven years without any other reward than the produce of one wheat corn for the first year, and that produce to be sowed the second year, and so on from year to year till the end of the time, allowing the increase to be in a tenfold proportion; what quantity of wheat is due for such service, and to what does it amount at a dollar per bushel?

Ans. $226056\frac{1}{8}$ bushels, allowing 7680 wheat corns to be a pint; and the amount is $226056\frac{1}{8}$ dollars.

3. What debt will be discharged in a year, or twelve months, by paying $1 the first month, $2 the second, $4 the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans. The debt is $4095 and the last payment $2048.

SIMPLE INTEREST.

INTEREST is the premium, allowed for the loan of money.

The sum, which is lent, is called the *principal.*

The sum of the principal and interest is called the *amount.*

Interest is allowed at so much *per cent. per annum*, which premium *per cent. per annum*, or interest of 100l. for a year, is called the *rate* of interest.

Interest is of two sorts, *simple* and *compound.*

Simple interest is that, which is allowed only for the principal lent.

NOTE. Commission, Brokerage, Insurance, Stocks,*

---

* *Stock* is a general name for public funds, and capitals of trading companies, the shares of which are transferable from one person to another.

and, in general, whatever is at a certain rate, or sum per cent. are calculated like Simple Interest.

RULE.*

1. Multiply the principal by the rate, and divide the product by 100; and the quotient is the answer for one year.

2. Multiply the interest for one year by the given number of years, and the product is the answer for that time.

3. If there be parts of a year, as months or days, work for the months by the aliquot parts of a year, and for the days by Simple Proportion.

EXAMPLES.

1. What is the interest of 450l. for a year, at 5 per cent. per annum?

```
      450l.
         5
     ------
1,00)22'50
        20
     -----
     10'00  Ans. 22l. and
```

$\frac{50}{100}=\frac{5}{10}='5=10s.$

2. What is the interest of 720l. for 3 years, at 5 per cent. per annum?

```
720l.        36
    5         3
-----      -----
36'00      108l. Ans.
```

3. What is the interest of 107l. for 117 days, at $4\frac{3}{4}$ per cent. per annum?

```
107l.    5  1 7 3'2    5  1 7 3'2
  4¾             11             7
-----    ----------    ----------
```

* The rule is evidently an application of Simple Proportion and Practice.

428 | 55 18 1 3˙2 | 35 11 6 2˙4

53 10 | 10

26 15

$11 \times 10 + 7 = 117$

5˙08 5 | 559 1 6 0

20 | 35 11 6 2˙4

365)594 13 0 2˙4(1l. 12s. $7\frac{8}{1825}$d. the answer.

1˙65 | 365

12

229

7˙80 | 20

4

)4593

3˙20 | 365

q. q.

$2˙4 = 2\frac{2}{5} = \frac{3}{5}$d.

943

730

$1\frac{3}{5} = \frac{8}{5}$

213

12

$\frac{\frac{8}{5}}{365} = \frac{8}{1825}$

)2556

2555

$1\frac{3}{5}$

4. What is the interest of $607˙50 for 5 years, at 6 per cent. per annum ? Ans. $182˙25.

5. What is the interest of 213l. from Feb. 12, to June 5, 1796, it being leap year, at $3\frac{1}{2}$ per cent. per annum ?

Ans. 2l. 6s. 6d. $3\frac{501}{1825}$q.

SIMPLE INTEREST BY DECIMALS.

RULE.*

Multiply continually the principal, ratio, and time, and it will give the interest required.

---

* The following theorems will show all the possible cases of simple interest, where p = principal, t = time, r = ratio, and a = amount.

Ratio is the simple interest of 1l. for 1 year, at the rate per cent. agreed on ; thus the ratio

at	3	per cent. is	·03.
	$\frac{1}{2}$		·035.
	4		·04.
	$4\frac{1}{2}$		·045.
	5		·05.
	$\frac{1}{2}$		·055.
	6		·06.

EXAMPLES.

1. What is the interest of 945l. 10s. for 3 years, at 5 per cent. per annum ?

```
 945·5
   ·05
 ------
 47·275
      3
 -------
141·825
     20
 -------
 16·500
     12
 ------
  6·000
```

Ans. 141l. 16s. 6d.

2. What is the interest of 796l. 15s. for 5 years, at $4\frac{1}{2}$ per cent. per annum ?

Ans. 179l. 5s. $4\frac{1}{2}$d.

3. What is the interest of 537l. 15s. from November 11, 1764, to June 5, 1765, at $3\frac{5}{8}$ per cent. ?

Ans. 11l. $\frac{1}{4}$d.

I. $ptr+p=a.$ II. $\frac{a-p}{rp}=t.$

III. $\frac{a}{tr+1}=p.$ IV. $\frac{a-p}{tp}=r.$

COMMISSION.

Commission is an allowance of so much per cent. to a factor or correspondent abroad, for buying and selling goods for his employer.

EXAMPLES.

1. What comes the commission of 500l. 13s. 6d. to at $3\frac{1}{2}$ per cent.?

500l.	13s.	6d.
		$3\frac{1}{2}$
1502	0	6
250	6	9
17·52	7	3
20		
10·47		
12		
5·67		
4		
2·68		

Ans. 17l. 10s. $5\frac{1}{2}$d.

2. My correspondent writes me word, that he has bought goods on my account to the value of 754l. 16s. What does his commission come to at $2\frac{1}{2}$ per cent.?

Ans. 18l. 17s. $4\frac{3}{4}$d.

3. What must I allow my correspondent for disbursing on my account 529l. 18s. 5d. at $2\frac{1}{4}$ per cent.?

Ans. 11l. 18s. $5\frac{1}{2}$d.

BROKERAGE.

Brokerage is an allowance of so much per cent. to a person, called a Broker, for assisting merchants or factors in procuring or diposing of goods.

EXAMPLES.

1. What is the brokerage of 610l. at 5s. or $\frac{1}{4}$ per cent.?

5s. is $\frac{1}{4}$ 610l.

```
     ----
     1˙52  10
       20
    -----
    10˙50
       12
    -----
     6˙00      Ans. 1l. 10s. 6d.
```

2. If I allow my broker $3\frac{3}{4}$ per cent. what may he demand, when he sells goods to the value of 876l. 5s. 10d.?

Ans. 32l. 17s. $2\frac{1}{2}$d.

3. What is the brokerage of 879l. 18s. at $\frac{3}{8}$ per cent.?

Ans. 3l. 5s. $11\frac{3}{4}$d.

INSURANCE.

INSURANCE is a premium of so much per cent. given to certain persons and offices for a security of making good the loss of ships, houses, merchandize, &c. which may happen from storms, fire, &c.

EXAMPLES.

1. What is the insurance of 874l. 13s. 6d. at $13\frac{1}{2}$ per cent.?

```
        874l. 13s. 6d.
                   12
     ----------------
     10496    2    0
       874   13    6
       437    6    9
     ----------------
     118˙08   2    3
         20
     ------
```

1·62
12
———
7·47
4
———
1·88 Ans. 118l. 1s. $7\frac{1}{4}$d.

2. What is the insurance of 900l. at $10\frac{3}{4}$ per cent. ?
Ans. 96l. 15s.

3. What is the insurance of 1200l. at $7\frac{5}{8}$ per cent. ?
Ans. 91l. 10s.

DISCOUNT.

Discount is an allowance, made for the payment of any sum of money before it becomes due; and is the difference between that sum, due some time hence, and its present worth.

The *present worth* of any sum, or debt, due some time hence, is such a sum, as, if put to interest, would in that time and at the rate per cent. for which the discount is to be made, amount to the sum or debt then due.

RULE.*

1. As the amount of 100l. for the given rate and time is to 100l. so is the given sum or debt to the present worth.

* That an allowance ought to be made for paying money before it becomes due, which is supposed to bear no interest till after it is due, is very reasonable; for, if I keep the money in my own hands till the debt becomes due, it is plain I may make an advantage of it by putting it out to interest for that time; but if I pay it before it is due, it is giving that benefit to another; therefore we have only to inquire what discount ought to be allowed. And here some debtors will be ready to say, that since by not paying the money till it becomes due, they may employ it at interest, therefore by paying it before due they shall lose that in-

2. Subtract the present worth from the given sum, and the remainder is the discount required.

Or,

As the amount of 100l. for the given rate and time is to the interest of 100l. for that time, so is the given sum or debt to the discount required.

EXAMPLES.

1. What is the discount of 573l. 15s. due 3 years hence, at $4\frac{1}{2}$ per cent.?

---

terest, and for that reason all such interest ought to be discounted: but that is false, for they cannot be said to lose that interest till the time, when the debt shall become due; whereas we are to consider what would properly be lost at present, by paying the debt before it becomes due; and this can, in point of equity or justice, be no other than such a sum, as, being put out to interest till the debt becomes due, would amount to the interest of the debt for the same time. It is beside plain, that the advantage arising from discharging a debt, due some time hence, by a present payment, according to the principles we have mentioned, is exactly the same as employing the whole sum at interest till the time, when the debt becomes due; for, if the discount allowed for present payment be put out to interest for that time, its amount will be the same as the interest of the whole debt for the same time: thus the discount of 105l. due one year hence, reckoning interest at 5 per cent. will be 5l. and 5l. put out to interest at 5 per cent. for one year, will amount to 5l. 5s. which is exactly equal to the interest of 105l. for one year at 5 per cent.

The truth of the rule for working is evident from the nature of simple interest; for since the debt may be considered as the amount of some principal, called here the present worth, at a certain rate per cent. and for the given time, that amount must be in the same proportion, either to its principal or interest, as the amount of any other sum, at the same rate, and for the same time, is to its principal or interest.

```
  4l.  10s.
        3
  ---------
  13   10
 100
  ---------   l.   s.       l.   s.
 113   10  :  13   10  : :  573  15  :
  20          20             20
 ----        ----           -----
2270         270            11475
                              270
                           ------
                           803250
                           22950
                          --------(2,0)
                  227,0)309825,0( 136,4
                        828       -----
                        1472       68  4
                         1105
                          197
                           12
                        ------
                     227)2364(10
                           94
                            4
                          -----
                          376(1
                          149
```

Ans. 68l. 4s. 10¼d.

2. What is the present worth of 150l. payable in ¼ of a year, discount being at 5 per cent.?

Ans. 148l. 2s. 11½d.

3. Bought a quantity of goods for 150l. ready money, and sold them again for 200l. payable at ¾ of a year hence; what was the gain in ready money, supposing discount to be made at 5 per cent.?

Ans. 42l. 15s. 5d.

4. What is the present worth of 120l. payable as follows, viz. 50l. at 3 months, 50l. at 5 months, and the rest at 8 months, discount being at 6 per cent.?

Ans. 117l. 5s. 5¼d.

DISCOUNT BY DECIMALS.

RULE.*

As the amount of 1l. for the given time is to 1l. so is the interest of the debt for the said time to the discount required.

Subtract the discount from the principal, and the remainder will be the present worth.

EXAMPLES.

What is the discount of 573l. 15s. due 3 years hence, at $4\frac{1}{2}$ per cent. per annum?

$\dot{}045\times3+1=1\cdot135$=amount of 1l. for the given time.

And $573\dot{}75\times\dot{}045\times3=77\dot{}45625$ = interest of the debt for the given time.

---

* Let m represent any debt, and n the time of payment; then will the following Tables exhibit all the variety, that can happen with respect to present worth and discount.

Of the *Present Worth* of Money paid before it is due at *Simple Interest*.			
The present worth of any sum m.			
Rate per cent.	For n years.	n months.	n days.
r per cent.	$\frac{100m}{nr+100}$	$\frac{1200m}{nr+1200}$	$\frac{36500m}{nr+36500}$
3 per cent.	$\frac{100m}{3n+100}$	$\frac{400m}{n+400}$	$\frac{36500m}{3n+36500}$
4 per cent.	$\frac{25m}{n+25}$	$\frac{300m}{n+300}$	$\frac{9125m}{n+9125}$
5 per cent.	$\frac{20m}{n+20}$	$\frac{240m}{n+240}$	$\frac{7300m}{n+7300}$

```
1·35  :  1  ::  77·45625  :
     1·135)77·45625(68·243
            6810
            ----
             9356
             9080
             ----
              2762
              2270
              ----
               4925
               4540
               ----
                3850
                3405
                ----
68·243 = 68l. 4s. 10¼d. Ans.     445
```

2. What is the discount of 725l. 16s. for five months, at $3\frac{7}{8}$ per cent. per annum?

Ans. 11l. 10s. $7\frac{3}{4}$d.

Of *Discounts* to be allowed for paying of Money before it becomes due at *Simple Interest*.

The discount of any sum m.			
Rate per cent.	For n years,	n months.	n days.
r per cent.	$\frac{mnr}{nr+100}$	$\frac{mnr}{nr+1200}$	$\frac{mnr}{nr+36500}$
3 per cent.	$\frac{3mn}{3n+100}$	$\frac{mn}{n+400}$	$\frac{3mn}{3n+36500}$
4 per cent.	$\frac{mn}{n+25}$	$\frac{mn}{n+300}$	$\frac{mn}{n+9125}$
5 per cent.	$\frac{mn}{n+20}$	$\frac{mn}{n+240}$	$\frac{mn}{n+7300}$

3. What ready money will discharge a debt of 1377l. 13s. 4d. due 2 years, 3 quarters and 25 days hence, discounting at $4\frac{3}{8}$ per cent. per annum?

Ans. 1226l. 8s. $8\frac{1}{2}$d.

EQUATION OF PAYMENTS.

Equation of Payments is the finding a time to pay at once several debts, due at different times, so that no loss shall be sustained by either party.

RULE.*

Multiply each payment by the time, at which it is due; then divide the sum of the products by the sum of the payments, and the quotient will be the time required.

---

* This rule is founded on a supposition, that the sum of the interests of the several debts, which are payable before the equated time, from their terms to that time, ought to be equal to the sum of the interests of the debts payable after the equated time, from that time to their terms. Among others, who defend this principle, Mr. Cocker endeavours to prove it tobe right by this argument; that what is gained by keeping some of the debts after they are due, is lost by paying others before they are due. But this cannot be the case; for, though by keeping a debt unpaid after it is due there is gained the interest of it for that time, yet by paying a debt before it is due the payer does not lose the interest for that time, but the discount only, which is less than the interest, and therefore the rule is not true.

Although this rule be not accurately true, yet in most questions, that occur in business, the error is so trifling, that it will be much used.

That the rule is universally agreeable to the supposition may be thus demonstrated.

Let
- d = first debt payable, and the distance of its term of payment t.
- D = last debt payable, and the distance of its term T.
- x = distance of the equated time.
- r = rate of interest of 1l. for one year.

EXAMPLES.

1. A owes B 190l. to be paid as follows, viz. 50l. in 6 months, 60l. in 7 months, and 80l. in 10 months; what is the equated time to pay the whole?

$$50\times\ 6=300$$
$$60\times\ 7=420$$
$$80\times 10=800$$

50+60+80=190)1520(8
1520   Ans. 8 months.

2. A owes B 52l. 7s. 6d. to be paid in $4\frac{1}{2}$ months, 80l. 10s. to be paid in $3\frac{1}{2}$ months, and 76l. 2s. 6d. to be paid in 5 months; what is the equated time to pay the whole?

Ans. 4 months, 8 days.

3. A owes B 240l. to be paid in 6 months, but in one month and a half pays him 60l. and in $4\frac{1}{2}$ months after that 80l. more; how much longer than 6 months should B in equity defer the rest? Ans. 2·7 months.

4. A debt is to be paid as follows, viz. $\frac{1}{4}$ at 2 months, $\frac{1}{8}$ at 3 months, $\frac{1}{8}$ at 4 months, $\frac{1}{8}$at 5 months, and the rest at 7 months; what is the equated time to pay the whole?

Ans. 4 months and 18 days.

Then, since x lies between T and t $\left\{\begin{array}{l}\text{The distance of the time } t \text{ and } x \text{ is } = x-t. \\ \text{The distance of the time } T \text{ and } x \text{ is } = T-x\end{array}\right.$

Now the interest of d for the time $x-t$ is $\overline{x-t}\times dr$; and the interest of D for the time $T-x$ is $\overline{T-x}\times Dr$; therefore $\overline{x-t}\times dr=\overline{T-x}\times Dr$ by the supposition; and from this equation x is found$=\frac{DT+dt}{D+d}$, which is the rule. And the same might be shown of any number of payments.

The true rule is given in equation of payments by decimals.

EQUATION OF PAYMENTS BY DECIMALS.

Two debts being due at different times, to find the equated time to pay the whole.

RULE.*

1. To the sum of both payments add the continual product of the first payment, the rate, or interest of 1l. for one year, and the time between the payments, and call this the first number.

---

* No rule in Arithmetic has been the occasion of so many disputes, as that of Equation of Payments. Almost every writer on this subject has endeavoured to show the fallacy of the methods, used by other authors, and to substitute a new one in their stead. But the only true rule seems to be that of Mr. MALCOLM, or one similar to it in its essential principles, derived from the consideration of interest and discount.

The rule, given above, is the same as Mr. MALCOLM's, except that it is not incumbered with the time before any payment is due, that being no necessary part of the operation.

DEMONSTRATION OF THE RULE. Suppose a sum of money to be due immediately, and another sum at the expiration of a certain given time, and it is proposed to find a time to pay the whole at once, so that neither party shall sustain loss.

Now it is plain, that the equated time must fall between those of the two payments; and that what is got by keeping the first debt after it is due, should be equal to what is lost by paying the second debt before it is due.

But the gain, arising from the keeping of a sum of money after it is due, is evidently equal to the interest of the debt for that time.

And the loss, which is sustained by the paying of a sum of money before it is due, is evidently equal to the discount of the debt for that time.

Therefore it is obvious, that the debtor must retain the sum immediately due, or the first payment, till its interest shall be equal to the discount of the second sum for the time it is paid

2. Multiply twice the first payment by the rate, and call this the second payment.

3. Divide the first number by the second, and call the quotient the third number.

---

before due; because, in that case, the gain and the loss will be equal, and consequently neither party can be loser.

Now to find such a time, let a = first payment, b = second, and t = time between the payments; r = rate, or interest of 1l. for one year, and x = equated time after the first payment.

Then arx = interest of a for x time,

and $\frac{btr-brx}{1+tr-rx}$ = discount of b for the time $t-x$.

But $arx=\frac{btr-brx}{1+tr-rx}$ by the question, from which equation x is found $=\frac{a+b+atr}{2ar}\pm\overline{\overline{\frac{a+b+atr}{2ar}}\Big|^{2}-\frac{b}{ar}}\Big|^{\frac{1}{2}}$.

Let $\frac{a+b+atr}{2ar}$ be put equal to n, and $\frac{bt}{ar}=m$.

Then it is evident, that n, or its equal $\overline{n^2}\big|^{\frac{1}{2}}$ is greater than $\overline{n^2-m}\big|^{\frac{1}{2}}$, and therefore x will have two affirmative values, the quantities $n+\overline{n^2-m}\big|^{\frac{1}{2}}$ and $n-\overline{n^2-m}\big|^{\frac{1}{2}}$ being both positive.

But only one of those values will answer the conditions of the question; and, in all cases of this problem, x will be $=n-\overline{n^2-m}\big|^{\frac{1}{2}}$.

For suppose the contrary, and let $x=n+\overline{n^2-m}\big|^{\frac{1}{2}}$.

Then $t-x=t-n-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{\overline{t-n}\big|^{2}}\Big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{t^2-2tn+n^2}\big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{n^2+t^2-2tn}\big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}$.

Now, since $\overline{a+b+atr}\times\frac{1}{2ar}=n$, and $bt\times\frac{1}{ar}=m$, we shall have from the first of these equations $t^2-2tn=-\overline{bt-at}\times\frac{1}{ar}$, and consequently $t-x=\overline{n^2-\overline{bt-at}\times\frac{1}{ar}}\Big|^{\frac{1}{2}}-\overline{n^2-bt\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$.

But $\overline{n^2-bt\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$ is evidently greater than $\overline{n^2-\overline{bt-at}\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$,

4. Call the square of the third number the fourth number.

5. Divide the product of the second payment, and time between the payments, by the product of the first payment and the rate, and call the quotient the fifth number.

6. From the fourth number take the fifth, and call the square root of the difference the sixth number.

7. Then the difference of the third and sixth numbers is the equated time, after the first payment is due.

---

and therefore $\overline{n^2 - bt \times \frac{1}{ar}}\Big|^{\frac{1}{2}} - \overline{n^2 - \overline{bt - at} \times \frac{1}{a}}\Big|^{\frac{1}{2}}$, or its equal $t - x$, must be a negative quantity ; and consequently x will be greater than t, that is, the equated time will fall beyond the second payment, which is absurd. The value of x therefore cannot be $= \frac{a+b+atr}{2ar} + \overline{\overline{\frac{a+b+atr}{2ar}}\Big|^2 - \frac{bt}{ar}}\Big|^{\frac{1}{2}}$, but must in all cases be $= \frac{a+b+atr}{2ar} - \overline{\overline{\frac{a+b+atr}{2ar}}\Big|^2 - \frac{b}{ar}}\Big|^{\frac{1}{2}}$, which is the same as the rule.

From this it appears, that the double sign, made use of by Mr. Malcolm, and every author since, who has given his method, cannot obtain, and that there is no ambiguity in the problem.

In like manner it might be shown, that the directions, usually given for finding the equated time, when there are more than two payments, will not agree with the hypothesis ; but this may be easily seen by working an example at large, and examining the truth of the conclusion.

The equated time for any number of payments may be readily found when the question is proposed in numbers, but it would not be easy to give algebraic theorems for those cases, on account of the variation of the debts and times, and the difficulty of finding between which of the payments the equated time would happen.

Supposing r to be the amount of 1l. for one year, and the other letters as before, then $t - \frac{\frac{\log.\ \overline{ar^t + b}}{s}}{\log.\ r}$ will be a general theorem for the equated time of any two payments, reckoning compound interest, and is found in the same manner as the former.

EXAMPLES.

1. There is 100l. payable one year hence, and 105l. payable 3 years hence; what is the equated time, allowing simple interest at 5 per cent. per annum?

```
      100              100
      ˙05                2
      ———              ———
      5˙00             200
         2             ˙05
     —————            ————
     10˙00            10˙00=2d number.
    100
    105
    ——————
 10)215=1st number.
    ———
    21˙5=3d number.
    21˙5
  ————
   1075
   215
  430
  ——————
  462˙25=4th number.          105
                                2
                             ————
     1st payment × rate =  5)210
                              ———
                               42=5th number.
  462˙25              21˙5
   42                (20˙5=6th number.
  ——————              ———
  . . .                 1= equated time from the first
  420˙25(20˙5              payment, and ∴ 2 years =
  4                        whole equated time.
  —————
405)2025
    2025
    ————
```

2. Suppose 400l. are to be paid at the end of 2 years, and 2100l. at the end of 8 years; what is the equated time for one payment, reckoning 5 per cent. simple interest?

Ans. 7 years.

3. Suppose 300l. are to be paid at one year's end, and 300l. more at the end of $1\frac{1}{2}$ year; it is required to find the time to pay it at one payment, 5 per cent. simple interest being allowed. Ans. 1·248437 year.

COMPOUND INTEREST.

COMPOUND INTEREST is that, which arises from the principal and interest taken together, as it becomes due, at the end of each stated time of payment.

RULE.*

1. Find the amount of the given principal for the time of the first payment by simple interest.

2. Consider this amount as the principal for the second payment, whose amount calculate as before, and so on through all the payments to the last, still accounting the last amount as the principal for the next payment.

EXAMPLES.

1. What is the amount of 320l. 10s. for 4 years, at 5 per cent. per annum, compound interest?

	l.	s.	d.	
$\frac{1}{20}$)	320l.	10s.		1st year's principal.
	16	0	6d.	1st year's interest.
$\frac{1}{20}$)	336	10	6	2d year's principal.
	16	16	$6\frac{1}{4}$	2d year's interest.
$\frac{1}{20}$)	353	7	$0\frac{1}{4}$	3d year's principal.
	17	13	4	3d year's interest.
$\frac{1}{20}$)	371	0	$4\frac{1}{4}$	4th year's principal.
	18	11	0	4th year's interest.
	389	11	$4\frac{1}{4}$	whole amount, or the answer required.

* The reason of this rule is evident from the definition, and the principles of simple interest.

2. What is the compound interest of 760l. 10s. forborn 4 years, at 4 per cent.? Ans. 129l. 3s. $6\frac{1}{4}$d.

3. What is the compound interest of 410l. forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum; interest payable half-yearly? Ans. 48l. 4s. $11\frac{3}{4}$d.

4. Find the several amounts of 50l. payable yearly, half-yearly and quarterly, being forborn 5 years, at 5 per cent. per annum, compound interest.

Ans. 63l. 16s. $3\frac{1}{4}$d., 64l. and 64l. 1s. $9\frac{1}{2}$d.

COMPOUND INTEREST BY DECIMALS.

RULE.*

1. Find the amount of 1l. for one year at the given rate per cent.

---

* DEMONSTRATION. Let r = amount of 1l for one year, and p = principal or given sum; then since r is the amount of 1l. for one year, r^2 will be its amount for two years, r^3 for 3 years, and so on; for when the rate and time are the same, all principal sums are necessarily as their amounts; and consequently as r is the principal for the second year, it will be as $1 : r :: r : r^2$ = amount for the second year, or principal for the third; and again, as $1 : r :: r^2 : r^3$ = amount for the third year, or principal for the fourth, and so on to any number of years. And if the number of years be denoted by t, the amount of 1l. for t years will be r^t. Hence it will appear, that the amount of any other principal sum p for t years is pr^t; for as $1 : r^t :: p : pr^t$, the same as in the rule.

If the rate of interest be determined to any other time than a year, as $\frac{1}{2}$, $\frac{1}{4}$, &c. the rule is the same, and then t will represent that stated time.

Let
- r = amount of 1l, for one year at the given rate per cent.
- p = principal, or sum at interest.
- i = interest.
- t = time.
- m = amount for the time t.

Then the following theorems will exhibit the solutions of all the cases in compound interest.

2. Involve the amount, thus found, to such a power, as is denoted by the number of years.

3. Multiply this power by the principal, or given sum, and the product will be the amount required.

4. Subtract the principal from the amount, and the remainder will be the interest.

I. $pr^t = m$ II. $pr^t - p = i.$

III. $\frac{m}{r^t} = p$ IV. $\left|\frac{m}{p}\right|^{\frac{1}{t}} = r.$

The most convenient way of giving the theorem for the *time*, as well as for all the other cases, will be by logarithms, as follows.

I. $t \times log.\ r + log.\ p = log.\ m.$ II. $log.\ m - t \times log.\ r = log.\ p$

III. $\frac{log.\ m - log.\ p}{log.\ r} = t.$ IV. $\frac{log.\ m - log.\ p}{t} = log.\ r.$

If the compound interest, or amount of any sum be required for the parts of a year, it may be determined as follows.

I. *When the time is any aliquot part of a year.*

RULE.

1. Find the amount of 1l. for one year, as before, and that root of it, which is denoted by the aliquot part, will be the amount sought.

2. Multiply the amount, thus found, by the principal, and it will be the amount of the given sum required.

II. *When the time is not an aliquot part of a year.*

RULE.

1. Reduce the time into days, and the 365th root of the amount of 1l. for one year is the amount for one day.

2. Raise this amount to that power, whose index is equal to the number of days, and it will be the amount of 1l. for the given time.

3. Multiply this amount by the principal, and it will be the amount of the given sum required.

To avoid extracting very high roots, the same may be done by logarithms thus; divide the logarithm of the rate, or amount of 1l. for one year, by the denominator of the given aliquot part, and the quotient will be the logarithm of the root sought.

EXAMPLES.

1. What is the compound interest of 500l. for 4 years, at 5 per cent. per annum?

1·05 = amount of 1l. for one year at 5 per cent.
1·05

525
1050

1·1025
1·1025

55125
22050
110250
11025

1·21550625=4th power of 1·05.
500=principal.

607·75312500=amount.
500

107·753125 = 107l. 15s. $0\frac{3}{4}$d. =interest required.

2. What is the amount of 760l. 10s. for 4 years, at 4 per cent.? Ans. 889l. 13s. $6\frac{1}{2}$d.

3. What is the amount of 721l. for 21 years, at 4 per cent. per annum? Ans. 1642l. 19s. 10d.

4. What is the amount of 217l. forborn $2\frac{1}{4}$ years, at 5 per cent. per annum, supposing the interest payable quarterly? Ans. 242l. 13s. $4\frac{1}{2}$d.

ANNUITIES.

An Annuity is a sum of money payable every year, for a certain number of years, or forever.

When the debtor keeps the annuity in his own hands, beyond the time of payment, it is said to be in *arrears*.

The sum of all the annuities for the time they have been forborn together with the interest due upon each is called the *amount*.

If an annuity be to be bought off, or paid all at once, at the beginning of the first year, the price, which ought to be given for it, is called the *present worth*.

To find the amount of an Annuity at Simple Interest.

RULE.*

1. Find the sum of the natural series of numbers 1, 2, 3, &c. to the number of years less one.

---

* DEMONSTRATION. Whatever the time is, there is due upon the first year's annuity, as many years' interest as the whole number of years less one; and gradually one less upon every succeeding year to the last but one; upon which there is due only one year's interest, and none upon the last; therefore in the whole there is due as many years' interest of the annuity, as the sum of the series 1, 2, 3, 4, &c. to the number of years less one. Consequently one year's interest, multiplied by this sum, must be the whole interest due; to which, if all the annuities be added, the sum is plainly the amount. Q. E. D.

Let r be the ratio, n the annuity, t the time, and a the amount.

Then will the following theorems give the solutions of all the different cases.

I. $\frac{t^2rn-trn}{2}+tn=a.$ II. $\frac{2a-2tn}{t^2n-tn}=r.$

III. $\frac{2a}{t^2r-tr+2t}=n.$ IV. $\left(\frac{2a}{rn}+\frac{d}{4}\right)^{\frac{1}{2}}-\frac{d}{2}=t.$

In the last theorem $d=\frac{2n-rn}{rn}$, and in theorem first, if a sum cannot be found equal to the amount, the problem is impossible in whole years.

NOTE. Some writers look upon this method of finding the amount of an annuity as a species of *compound interest*; the annuity itself, they say, being properly the simple interest, and the capital, whence it arises, the principal.

2. Multiply this sum by one year's interest of the annuity, and the product will be the whole interest due upon the annuity.

3. To this product add the product of the annuity and time, and the sum will be the amount sought.

NOTE. When the annuity is to be paid half-yearly or quarterly; then take, in the former case, $\frac{1}{2}$ the ratio, $\frac{1}{2}$ the annuity, and twice the number of years; and in the latter case, $\frac{1}{4}$ the ratio, $\frac{1}{4}$ the annuity, and 4 times the number of years, and proceed as before.

EXAMPLES.

1. What is the amount of an annuity of 50l. for 7 years, allowing simple interest at 5 per cent.?

$1+2+3+4+5+6=21=3\times 7$

2l. 10s. = 1 year's interest of 50l.
3
―――――
7 10
7
―――――
52 10
350 0 = 50l. × 7
―――――
402l. 10s. = amount required.

2. If a pension of 600l. per annum be forborn 5 years, what will it amount to, allowing 4 per cent. simple interest?

Ans. 3240l.

3. What will an annuity of 250l. amount to in 7 years, to be paid by half-yearly payments, at 6 per cent. per annum, simple interest? Ans. 2091l. 5s.

To find the present worth of an Annuity at Simple Interest.

RULE.*

Find the present worth of each year by itself, discounting from the time it becomes due, and the sum of all these will be the present worth required.

* The reason of this rule is manifest from the nature of discount, for all the annuities may be considered separately, as so

EXAMPLES.

1. What is the present worth of an annuity of 100l. to continue 5 years, at 6 per cent. per annum, simple interest?

106 : 100 : : 100 :	94·3396 =	present worth for first year.
112 : 100 : : 100 :	89·2857 =	2d year.
118 : 100 : : 100 :	84·7457 =	3d year.
124 : 100 : : 100 :	80·6451 =	4th year.
130 : 100 : : 100 :	76·9230 =	5th year.

425·9391 = 425l. 18s. 9¾d. = present worth of the annuity required.

2. What is the present worth of an annuity or pension of 500l. to continue 4 years, at 5 per cent. per annum, simple interest? Ans. 1782l. 3s. 8½d.

---

many single and independent debts, due after 1, 2, 3, &c. years; so that the present worth of each being found, their sum must be the present worth of the whole.

The estimation, however, of annuities at simple interest is highly unreasonable and absurd. One instance only will be sufficient to show the truth of this assertion. The price of an annuity of 50l. to continue 40 years, discounting at 5 per cent. will, by either of the rules, amount to a sum, of which one year's interest only exceeds the annuity. Would it not therefore be highly ridiculous to give, for an annuity to continue only 40 years, a sum, which would yield a greater yearly interest forever.

It is most equitable to allow compound interest.

Let p = present worth, and the other letters as before.

Then
$$\begin{cases} n\times\overline{\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}, \text{ \&c. to } \frac{1}{1+tr}}=p. \\ p\div\overline{\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}, \text{ \&c. to } \frac{1}{1+tr}}=n. \end{cases}$$

The other two theorems for the time and rate cannot be given in general terms.

To find the Amount of an Annuity at Compound Interest.

RULE.*

1. Make 1 the first term of a geometrical progression, and the amount of 1l. for one year, at the given rate per cent. the ratio.

---

* DEMONSTRATION. It is plain, that upon the first year's annuity, there will be due as many years' compound interest, as the given number of years less one, and gradually one year's interest less upon every succeeding year to that preceding the last, which has but one year's interest, and the last bears no interest.

Let r therefore = rate, or amount of 1l. for 1 year; then the series of amounts of 1l. annuity, for several years, from the first to the last, is $1, r, r^2, r^3$, &c. to r^{t-1}. And the sum of this, according to the rule in geometrical progression, will be $\frac{r^t-1}{r-1}$ = amount of 1l. annuity for t years. And all annuities are proportional to their amounts, therefore $1 : \frac{r^t-1}{r-1} :: n : \frac{r^t-1}{r-1}\times n$ = amount of any given annuity n. Q. E. D.

Let r = rate, or amount of 1l. for one year, and the other letters as before, then $\frac{r^t-1}{r-1}\times n=a$, and $\frac{ar-a}{r^t-1}=n$

And from these equations all the cases relating to annuities, or pensions in arrears, may be conveniently exhibited in logarithmic terms, thus:

I. $Log.\, n + Log.\, \overline{r^t-1} - Log.\, \overline{r-1} = Log.\, a.$

II. $Log.\, a. - Log.\, \overline{r^t-1} + Log.\, \overline{r-1} = Log.\, n.$

III. $\frac{Log.\, \overline{ar-a+n} - Log.\, n}{Log.\, r.} = t.$ IV. $r^t - \frac{ar}{n} + \frac{a}{n} - 1 = 0.$

2. Carry the series to as many terms as the number of years, and find its sum.

3. Multiply the sum thus found by the given annuity, and the product will be the amount sought.

EXAMPLES.

1. What is the amount of an annuity of 40l. to continue 5 years, allowing 5 per cent. compound interest?

$$1+1{\cdot}05+\overline{1{\cdot}05}|^2+\overline{1{\cdot}05}|^3+\overline{1{\cdot}05}|^4=5{\cdot}52563125$$

```
5·52563125
        40
----------
221·02525
       20
---------
    0·505
       12
    -----
     6·06
```

Ans. 221l. 6d.

2. If 50l. yearly rent, or annuity, be forborn 7 years, what will it amount to, at 4 per cent. per annum, compound interest? Ans. 394l. 18s. 3½d.

To find the present value of Annuities at Compound Interest.

RULE.*

1. Divide the annuity by the ratio, or the amount of 1l. for one year, and the quotient will be the present worth of the first years annuity.

---

* The reason of this rule is evident from the nature of the question, and what was said on the same subject in the purchasing of annuities at simple interest.

Let p = present worth of the annuity, and the other letters as before, then as the amount $=\frac{r^t-1}{r-1}\times n$, and as the present worth or principal of this, according to the principles of compound interest, is the amount divided by r^t, therefore

2. Divide the annuity by the square of the ratio, and the quotient will be the present worth of the annuity for the second year.

3. Find, in like manner, the present worth of each year by itself, and the sum of all these will be the value of the annuity sought.

$$n \times \frac{r^t - 1}{r^{t+1} - r^t} = p, \text{ and } p \times \frac{r^{t+1} - r^t}{r^t - 1} = n.$$

And from these theorems all the cases, where the purchase of annuities is concerned, may be exhibited in logarithmic terms, as follows.

$$\text{I. } Log.\, n + Log.\, \overline{1 - \frac{1}{r^t}} - Log.\, r - 1 = Log.\, p.$$

$$\text{II. } Log.\, p + Log.\, \overline{r - 1} - Log.\, \overline{1 - \frac{1}{r^t}} = Log.\, n.$$

$$\text{III. } \frac{Log.\, n - Log.\, \overline{n + p - pr}}{Log.\, r} = t. \quad \text{IV. } r^{t+1} - \overline{\frac{n}{p} + 1} \times r^t + \frac{n}{p} = 0.$$

Let t express the number of half years or quarters, n the half year's or quarter's payment, and r the sum of one pound and $\frac{1}{2}$ or $\frac{1}{4}$ year's interest, then all the preceding rules are applicable to half-yearly and quarterly payments, the same as to whole years.

The amount of an annuity may also be found for years and parts of a year thus:

1. Find the amount for the whole years as before.

2. Find the interest of that amount for the given parts of a year.

3. Add this interest to the former account, and it will give the whole amount required.

The present worth of an annuity for years and parts of a year may be found thus:

1. Find the present worth for the whole years as before.

2. Find the present worth of this present worth, discounting for the given parts of a year, and it will be the whole present worth required.

EXAMPLES.

1. What is the present worth of an annuity of 40l. to continue 5 years, discounting at 5 per cent. per annum, compound interest? [year.

ratio = 1˙05)40˙00000(38˙095 = present worth for first

$\overline{\text{ratio}}|^2$ = 1˙1025)40˙00000(36˙281 = do. for 2d year.

$\overline{\text{ratio}}|^3$ = 1˙157525)40˙00000(34˙556 = do. for 3d year.

$\overline{\text{ratio}}|^4$ = 1˙215506)40˙00000(32˙899 = do. for 4th year.

$\overline{\text{ratio}}|^5$ = 1˙276218)40˙00000(31˙342 = do. for 5th year.

173˙173 = 173l. 3s. 5½d. = whole present worth of the annuity required.

2. What is the present worth of an annuity of 21l. 10s. 9½d. to continue 7 years, at 6 per cent. per annum, compound interest? Ans. 120l. 5s.

3. What is 70l. per annum, to continue 59 years, worth in present money, at the rate of 5 per cent. per annum? Ans. 1321˙3021l.

To find the present worth of a Freehold Estate, or an Annuity to continue forever, at Compound Interest.

RULE.*

As the rate per cent. is to 100l. so is the yearly rent to the value required.

---

* The reason of this rule is obvious; for since a year's interest of the price, which is given for it, is the annuity, there can neither more nor less be made of that price than of the annuity, whether it be employed at simple or compound interest.

The same thing may be shown thus: the present worth of an annuity to continue forever is $\frac{n}{r}+\frac{n}{r^2}+\frac{n}{r^3}+\frac{n}{r^4}$, &c. *ad infinitum*, as has been shown before; but the sum of this series, by the rules of geometrical progression, is $\frac{n}{r-1}$; therefore $r-1:1::n:$

EXAMPLES.

1. An estate brings in yearly 79l. 4s. what would it sell for, allowing the purchaser $4\frac{1}{2}$ per cent. compound interest for his money?

$$4{\cdot}5 : 100 :: 79{\cdot}2 :$$

```
                 100
               -------
         4·5)7920·0(1760l. the answer.
             45
             ---
             342
             315
             ---
              270
              270
              ---
```

2. What is the price of a perpetual annuity of 40l. discounting at 5 per cent. compound interest?

Ans. 800l.

3. What is a freehold estate of 75l. a year worth, allowing the buyer 6 per cent. compound interest for his money? Ans. 1250l.

$\frac{n}{r-1}$, which is the rule.

The following theorems show all the varieties of this rule.

I. $\frac{n}{r-1}=p$. II. $\overline{r-1} \times p=n$. III. $\frac{n}{p}+1=r$, or $\frac{n}{p}=r-1$.

The price of a freehold estate, or an annuity to continue for-ever, at simple interest, would be expressed by $\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}+\frac{1}{1+4r}$, &c. *ad infinitum*; but the sum of this series is infinite, or greater than any assignable number, which sufficiently shows the absurdity of using simple interest in these cases.

To find the present worth of an Annuity, or Freehold Estate, in Reversion, at Compound Interest.

RULE.*

1. Find the present worth of the annuity, as if it were to be entered on immediately.

2. Find the present worth of the last present worth, discounting for the time between the purchase and commencement of the annuity, and it will be the answer required.

EXAMPLES.

1. The reversion of a freehold estate of 79l. 4s. per annum, to commence 7 years hence, is to be sold: what is it worth in ready money, allowing the purchaser $4\frac{1}{2}$ per cent. for his money?

4·5 : 100 :: 79·2 :
100

4·5)7920·0(1760 = present worth, if entered on immediately.
45
342
315
270
270
0

and $\overline{1·045|}^7$ = 1·360862)1760·000(1293·297 = 1293l. 5s. $11\frac{1}{4}$d. = present worth of 1760l. for 7 years, or the whole present worth required.

* This rule is sufficiently evident without a demonstration.

Those, who wish to be acquainted with the manner of computing the values of annuities on lives, may consult the writings of Mr. DEMOIVRE, Mr. SIMPSON, and Dr. PRICE, all of whom have handled this subject in a very skilful and masterly manner.

Dr. PRICE'S Treatise on Annuities and Reversionary Payments is an excellent performance, and will be found a very valuable acquisition to those, whose inclinations lead them to studies of this nature.

2. Which is most advantageous, a term of 15 years in an estate of 100l. per annum, or the reversion of such an estate forever, after the expiration of the said 15 years, computing at the rate of 5 per cent. per annum, compound interest ?

Ans. The first term of 15 years is better than the reversion forever afterward, by 75l. 18s. 7½d.

3. Suppose I would add 5 years to a running lease of 15 years to come, the improved rent being 186l. 7s. 6d. per annum ; what ought I to pay down for this favour, discounting at 4 per cent. per annum, compound interest ?

Ans. 460l. 14s. 1¾d.

POSITION.

Position is a method of performing such questions, as cannot be resolved by the common direct rules, and is of two kinds, called *single* and *double*.

SINGLE POSITION.

Single Position teaches to resolve those questions, whose results are proportional to their suppositions.

RULE.*

1. Take any number and perform the same operations with it, as are described to be performed in the question.

---

* Such questions properly belong to this rule, as require the multiplication or division of the number sought by any proposed number ; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times. For in this case the reason of the rule is obvious ; it being then evident, that the results are proportional to the suppositions.

$$\text{Thus}\begin{cases} nx : x :: na : a \\ \frac{x}{n} : x :: \frac{a}{n} : a \\ \frac{x}{n}+\frac{x}{m}, \text{\&c.} : x :: \frac{a}{n}+\frac{a}{m}, \text{\&c.} : a, \text{and so on.} \end{cases}$$

2. Then say, as the result of the operation is to the position, so is the result in the question to the number required.

EXAMPLES.

1. A's age is double that of B, and B's is triple that of C, and the sum of all their ages is 140 : what is each person's age ?

Suppose A's age to be 60
Then will B's = $\frac{60}{2}$ = 30
And C's = $\frac{30}{3}$ = 10

100 sum.

As 100 : 60 : : 140 : $\frac{140 \times 60}{100}$ = 84 = A's age.
Consequently $\frac{84}{2}$ = 42 = B's.
And $\frac{42}{3}$ = 14 = C's.

140 Proof.

2. A certain sum of money is to be divided between 4 persons, in such a manner, that the first shall have $\frac{1}{3}$ of it ; the second $\frac{1}{4}$; the third $\frac{1}{6}$; and the fourth the remainder, which is 28l. : what is the sum ? Ans. 112l.

3. A person, after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, had 60l. left : what had he at first ? Ans. 144l.

4. What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 125 ? Ans. 60.

5. A person bought a chaise, horse, and harness for 60l. ; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness : what did he give for each ?

Ans. 13l. 6s. 8d. for the horse, 6l. 13s. 4d. for the harness, and 40l. for the chaise.

6. A vessel has three cocks, A, B, and C ; A can fill it in 1 hour, B in 2, and C in 3 : in what time will they all fill it together ? Ans. $\frac{6}{11}$ hour.

NOTE. 1 may be made a constant supposition in all questions ; and in most cases it is better than any other number.

DOUBLE POSITION.

Double Position teaches to resolve questions by making two suppositions of false numbers.

RULE.*

1. Take any two convenient numbers, and proceed with each according to the conditions of the question.

2. Find how much the results are different from the result in the question.

3. Multiply each of the errors by the contrary supposition, and find the sum or difference of the products.

4. If the errors be alike, divide the difference of the pro-

---

* The rule is founded on this supposition, that the first error is to the second, as the difference between the true and first supposed number is to the difference between the true and second supposed number: when that is not the case, the exact answer to the question cannot be found by this rule.

That the rule is true, according to the supposition, may be thus demonstrated.

Let A and B be any two numbers, produced from a and b by similar operations: it is required to find the number, from which N is produced by a like operation.

Put $x =$ number required, and let $N - A = r$, and $N - B = s$.

Then according to the supposition, on which the rule is founded, $r : s :: x - a : x - b$, whence, by multiplying means and extremes, $rx - rb = sx - sa$; and, by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s} =$ number sought.

Again, if r and s be both negative, we shall have $-r : -s :: x - a : x - b$, and therefore $-rx + rb = -sx + sa$; and $rx - sx = rb - sa$; whence $x = \frac{rb - sa}{r - s}$, as before.

In like manner, if r or s be negative, we shall have $x = \frac{rb + sa}{r + s}$, by working as before, which is the rule.

Note. It will be often advantageous to make 1 and 0 the suppositions.

ducts by the difference of the errors, and the quotient will be the answer.

5. If the errors be unlike, divide the sum of the products by the sum of the errors, and the quotient will be the answer.

Note. The errors are said to be *alike*, when they are both too great or both too little; and *unlike*, when one is too great and the other too little.

EXAMPLES.

1. A lady bought tabby at 4s. a yard, and Persian at 2s. a yard; the whole number of yards she bought was 8, and the whole price 20s.: how many yards had she of each sort?

Suppose 4 yards of tabby, value	16s.
Then she must have 4 yards of Persian, value	8
Sum of their values	24
So that the first error is +	4
Again, suppose she had 3 yards of tabby at	12s.
Then she must have 5 yards of Persian at	10
Sum of their values	22
So that the second error is +	2

Then 4—2=2= difference of the errors.

Also 4×2=8= product of the first supposition and second error.

And 3×4=12= product of the second supposition by the first error.

And 12—8=4= their difference.

Whence 4÷2=2= yards of tabby, } the answer.
And 8—2=6= yards of Persian, }

2. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly; but B, by spending 50l. per annum more than A, at the end of 4 years finds himself 100l. in debt: what is their income, and what do they spend per annum?

Ans. Their income is 125l. per annum; A spends 100l. and B 150l. per annum.

3. Two persons, A and B, lay out equal sums of money in trade; A gains 126l. and B loses 87l. and A's money is now double that of B: what did each lay out?

Ans. 300l.

4. A laborer was hired for 40 days, on this condition, that he should receive 20d. for every day he wrought, and forfeit 10d. for every day he was idle; now he received at last 2l. 1s. 8d.: how many days did he work, and how many was he idle?

Ans. He wrought 30 days, and was idle 10.

5. A gentleman has two horses of considerable value, and a saddle worth 50l.; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first: what is the value of each horse? Ans. One 30l. and the other 40l.

6. There is a fish, whose head is 9 inches long, and his tail is as long as his head and half as long as his body, and his body is as long as his tail and his head: what is the whole length of the fish? Ans. 6 feet.

PERMUTATION AND COMBINATION.

THE *Permutation of Quantities* is the showing how many different ways the order or position of any given number of things may be changed.

This is also called *Variation*, *Alternation*, or *Changes;* and the only thing to be regarded here is the order they stand in; for no two parcels are to have all their quantities placed in the same situation.

The *Combination of Quantities* is the showing how often a less number of things can be taken out of a greater, and combined together, without considering their places, or the order they stand in.

This is sometimes called *Election*, or *Choice;* and here every parcel must be different from all the rest, and no two are to have precisely the same quantities, or things.

The *Composition of Quantities* is the taking a given number of quantities out of as many equal rows of different quantities, one out of each row, and combining them together.

Here no regard is had to their places; and it differs from combination only, as that admits of but one row, or set of things.

Combination of the same form are those, in which there is the same number of quantities, and the same repetitions: thus, *abcc*, *bbad*, *deef*, &c. are of the same form; but *abbc*, *abbb*, *aacc*, &c. are of different forms.

PROBLEM I.

To find the number of permutations, or changes, that can be made of any given number of things, all different from each other.

RULE.*

Multiply all the terms of the natural series of numbers, from 1 up to the given number, continually together, and the last product will be the answer required.

EXAMPLES.

1. How many changes may be made with these three letters, *abc*?

---

* The reason of the rule may he shown thus: any one thing *a* is capable only of one position, as *a*.

Any two things, *a* and *b*, are only capable of two variations; as *ab*, *ba*; whose number is expressed by 1×2.

If there be 3 things, *a*, *b*, and *c*, then any two of them, leaving out the third, will have 1×2 variations; aud consequently, when the third is taken in, there will be $1 \times 2 \times 3$ variations.

In the same manner, when there are 4 things, every 3, leaving out the fourth, will have $1 \times 2 \times 3$ variations. Then, the fourth being taken in, there will be $1 \times 2 \times 3 \times 4$ variations. And so on, as far as you please.

1
2
—
2
3
—
6

Or $1\times2\times3=6$ the answer.

the changes.

abc
acb
bac
bca
cab
cba

2. How many changes may be rung on 6 bells ?

Ans. 720.

3. For how many days can 7 persons be placed in a different position at dinner ? Ans. 5040 days.

4. How many changes may be rung on 12 bells, and how long would they be in ringing, supposing 10 changes to be rung in 1 minute, and the year to consist of 365 days 5 hours and 49 minutes ?

Ans. 479001600 changes, and 91y. 26d. 22h. 41m.

5. How many changes may be made of the words in the following verse ? *Tot tibi sunt dotes, virgo, quot sydera cœlo.*

Ans. 40320.

PROBLEM II.

Any number of different things being given, to find how many changes can be made out of them, by taking any given number at a time.

RULE.*

Take a series of numbers, beginning at the number of things given, and decreasing by 1 till the number of terms

---

* This rule, expressed in terms, is as follows : $m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}$, &c. to n terms ; where m = number of things given, and n = quantities to be taken at a time.

be equal to the number of things to be taken at a time, and the product of all the terms will be the answer required.

In order to demonstrate the rule, it will be necessary to premise the following

LEMMA.

The number of changes of m things, taken n at a time, is equal to m changes of $m-1$ things, taken $n-1$ at a time.

DEMONSTRATION. Let any 5 quantities, *abcde*, be given.

First, leave out the a, and let v = number of all the variations of every two, *bc*, *bd*, &c. that can be taken out of the 4 remaining quantities, *bcde*.

Now let a be put in the first place of each of them, *abc*, *abd*, &c. and the number of changes will still remain the same; that is, v = number of variations of every 3 out of the 5, *abcde*, when a is first.

In like manner, if b, c, d, e, be successively left out, the number of variations of all the twos will also $= v$; and b, c, d, e, being respectively put in the first place, to make 3 quantities out of 5, there will still be v variations as before.

But there are all the variations, that can happen of 3 things out of 5, when a, b, c, d, e, are successively put first; and therefore the sum of all these is the sum of all the changes of 3 things out of 5.

But the sum of these is so many times v, as is the number of things; that is, $5v$, or mv, = all the changes of three things out of 5. And the same way of reasoning may be applied to any numbers whatever.

DEMONSTRATION OF THE RULE. Let any 7 things, *abcdefg*, be given, and let 3 be the number of quantities to be taken.

Then $m=7$, and $n=3$.

Now it is evident, that the number of changes, that can be made by taking 1 by 1 out of 5 things, will be 5, which let $=v$.

Then, by the lemma, when $m=6$ and $n=2$, the number of changes will $=mv=6\times5$; which let $=v$ a second time.

Again by lemma, when $m=7$ and $n=3$, the number of changes $=mv=7\times6\times5$; that is, $mv=m\times\overline{m-1}\times\overline{m-2}$, continued to 3, or n terms. And the same may be shown for any other numbers.

EXAMPLE.

1. How many changes may be made out of the 3 letters, *abc*, by taking 2 at a time

3
2
—
6

Or $3 \times 2 = 6$ the answer.

The changes.

ab
ba
ac
ca
bc
cb

2. How many words can be made with 5 letters of the alphabet, it being admitted, that a number of consonants may make a word? Ans. 5100480.

PROBLEM III.

Any number of things being given, whereof there are several given things of one sort, several of another, &c. to find how many changes can be made out of them all.

RULE.*

1. Take the series 1, 2, 3, 4, &c. up to the number of things given, and find the product of all the terms.

---

* This rule is expressed in terms thus:

$$\frac{1 \times 2 \times 3 \times 4 \times 5, \text{ \&c. to } m}{1 \times 2 \times 3, \text{ \&c. to } p \times 1 \times 2 \times 3, \text{ \&c. to } q, \text{ \&c.}};$$ where m = number of things given, p = number of things of the first sort, q = number of things of the second sort, &c.

The DEMONSTRATION may be shown as follows.

Any two quantities, a, b, both different, admit of 2 changes; but if the quantities be the same, or ab become aa, there will be but one alteration, which may be expressed by $\frac{1 \times 2}{1 \times 2} = 1$.

2. Take the series 1, 2, 3, 4, &c. up to the number of given things of the first sort, and the series 1, 2, 3, 4, &c. up to the number of given things of the second sort, &c.

3. Divide the product of all the terms of the first series by the joint product of all the terms of the remaining ones, and the quotient will be the answer required.

Any three quantities, *abc*, all different from each other, afford 6 variations; but if the quantities be all alike, or *abc* become *aaa*, then the 6 variations will be reduced to 1, which may be expressed by $\frac{1\times2\times3}{1\times2\times3}=1$. Again, if two of the quantities only be alike, or *abc* become *aac*, then the six variations will be reduced to these 3, *aac*, *caa*, and *aca*, which may be expressed by $\frac{1\times2\times3}{1\times2}=3$.

Any four quantities, *abcd*, all different from each other, will admit of 24 variations; but if the quantities be the same, or *abcd* become *aaaa*, the number of variations will be reduced to one; which is $=\frac{1\times2\times3\times4}{1\times2\times3\times4}=1$. Again, if three of the quantities only be the same, or *abcd* become *aaab*, the number of variations will be reduced to these 4, *aaab*, *aaba*, *abaa*, and *baaa*, which is $=\frac{1\times2\times3\times4}{1\times2\times3}=4$. And thus it may be shown, that, if two of the quantities be alike, or the 4 quantities be *aabc*, the number of variations will be reduced to 12, which may be expressed by $\frac{1\times2\times3\times4}{1\times2}=12$.

And by reasoning in the same manner it will appear, that the number of changes, which can be made of the quantities, *abbccc*, is equal to 60, which may be expressed by $\frac{1\times2\times3\times4\times5\times6}{1\times2\times1\times2\times3}$ $=60$; and so of any other quantities whatever.

EXAMPLES.

1. How many variations may be made of the letters in the word *Bacchanalia?*

$1\times2(=\text{number of } cs)=2$

$1\times2\times3=4(=\text{number of } as)=24$

$1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11(=\text{number of letters in the word})=39916800$

2×24=48)39916800(831600 the answer.
151
76
288

2. How many different numbers can be made of the following figures, 1220005555? Ans. 12600.

3. What is the variety in the succession of the following musical notes, fa, fa, fa, sol, sol, la, mi, fa?

Ans. 3360.

PROBLEM IV.

To find the changes of any given number of things, taken a given number at a time; in which there are several given things of one sort, several of another, &c.

RULE.*

1. Find all the different forms of combination of all the given things, taken as many at a time as in the question.

2. Find the number of changes in any form, and multiply it by the number of combinations in that form.

3. Do the same for every distinct form; and the sum of all the products will give the whole number of changes required.

NOTE. *To find the different forms of combination proceed thus:*

1. Place the things so, that the greatest indices may be first, and the rest in order.

---

* The reason of this rule is plain from what has been shown before, and the nature of the problem.

2. Begin with the first letter, and join it to the second, third, fourth, &c. to the last.

3. Then take the second letter, and join it to the third, fourth, &c. to the last; and so on through the whole, always remembering to reject such combinations as have occurred before; and this will give the combinations of all the twos.

4. Join the first letter to every one of the twos following it; and the second, third, &c. as before; and it will give the combinations of all the threes.

5. Proceed in the same manner to get the combinations of all the fours, &c. and you will at last get all the several forms of combination, and the number in each form.

EXAMPLES.

1. How many changes may be made of every 4 letters, that can be taken out of these 6, *aaabbc*?

No. of forms.	Forms.	No. of combinations.	No. of changes in each form.
1st	a^3b, a^3c	2	$\frac{1\times2\times3\times4=24}{1\times2\times3=6}=4.$
2d	a^2b^2	1	$\frac{1\times2\times3\times4=24}{1\times2\times1\times2=4}=6.$
3d	a^2bc, b^2ac	2	$\frac{1\times2\times3\times4=24}{1\times2=2}=12.$

$$\begin{array}{r} 4\times2=8 \\ 6\times1=6 \\ 12\times2=24 \\ \hline 38 \end{array}$$

38 = the number of changes required.

2. How many changes can be made of every 8 letters out of these 10, *aaaabbccde*? Ans. 22260.

3. How many different numbers can be made out of 1 unit, 2 twos, 3 threes, 4 fours, and 5 fives, taken 5 at a time? Ans. 2111.

PROBLEM V.

To find the number of combinations of any given number of things, all different from one another, taken any given number at a time.

RULE.*

1. Take the series 1, 2, 3, 4, &c. up to the number to be taken at a time, and find the product of all the terms.

---

* This rule, expressed algebraically, is $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c. to n terms; where m is the number of given quantities, and n those to be taken at a time.

Demonstration of the Rule. 1. Let the number of things to be taken at a time be 2, and the things to be combined $= m$.

Now, when m, or the number of things to be combined, is only two, as a and b, it is evident, that there can be only one combination, as ab; but if m be increased by 1, or the letters to be combined be 3, as abc, then it is plain, that the number of combinations will be increased by 2, since with each of the former letters, a and b, the new letter c may be joined. It is evident therefore, that the whole number of combinations, in this case, will be truly expressed by $1+2$.

Again, if m be increased by one letter more, or the whole number of letters be four, as $abcd$; then it will appear, that the whole number of combinations must be increased by 3, since with each of the preceding letters the new letter d may be combined. The combinations therefore, in this case, will be truly expressed by $1+2+3$.

In the same manner it may be shown, that the whole number of combinations of 2, in 5 things, will be $1+2+3+4$; of 2, in 6 things, $1+2+3+4+5$; and of 2, in 7, $1+2+3+4+5+6$, &c.

Whence universally, the number of combinations of m things, taken 2 by 2, is $=1+2+3+4+5+6$, &c. to $\overline{m-1}$ terms.

2. Take a series of as many terms, decreasing by 1 from the given number, out of which the election is to be made, and find the product of all the terms.

3. Divide the last product by the former, and the quotient will be the number sought.

EXAMPLES.

1. How many combinations can be made of 6 letters out of ten?

But the sum of this series is $=\frac{m}{1}+\frac{m-1}{2}$, which is the same as the rule.

2. Let now the number of quantities in each combination be supposed to be three.

Then it is plain, that when $m=3$, or the things to be combined are *abc*, there can be only one combination; but if m be increased by 1, or the things to be combined be 4, as *abcd*, then will the number of combinations be increased by 3; since 3 is the number of combinations of 2 in all the preceding letters *abc*, and with each two of these the new letter *d* may be combined.

The number of combinations therefore, in this case, is $1+3$.

Again, if m be increased by one more, or the number of letters be supposed 5; then the former number of combinations will be increased by 6; that is, by all the combinations of 2 in the 4 precedin letters, *abcd*; since, as before, with each two of these the new letter *e* may be combined.

The number of combinations therefore, in this case, is $1+3+6$.

Whence universally, the number of combinations of m things, taken 3 by 3, is $1+3+6+10$, &c. to $m-2$ terms.

But the sum of this series is $=\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$, which is the same as the rule.

And the same thing will hold, let the number of things, to be taken at a time, be what it may; therefore the number of combinations of m things, taken n at a time, will $=\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c. to n terms. Q. E. D.

$1\times2\times3\times4\times5\times6$(=the number to be taken at a time)=720

$10\times9\times8\times7\times6\times5$(=same number from 10)=151200

```
720)151200(210 the answer.
    1440
    ----
      720
      720
      ---
        0
```

2. How many combinations can be made of 2 letters out of 24 letters of the alphabet? Ans. 276.

3. A general, who had often been successful in war, was asked by his King, what reward he should confer on him for his services; the general only desired a farthing for every file of 10 men in a file, which he could make with a body of 100 men: what is the amount in pounds sterling? Ans. 18031572350l. 9s. 2d.

PROBLEM VI.

To find the number of combinations of any given number of things, by taking any given number at a time; in which there are several things of one sort, several of another, &c.

RULE.

1. Find by trial the number of different forms, which the things, to be taken at a time, will admit of, and the number of combinations in each.

2. Add together all the combinations, thus found, and the sum will be the number required.

EXAMPLES.

1. Let the things proposed be *aaabbc*; it is required to find the number of combinations, that can be made of every three of these quantities.

Forms.	Combinations.
a^3	1
a^2b, a^2c, b^2a, b^2c	4
abc	1
	6 = number of combinations required.

2. Let *aaabbbcc* be proposed; it is required to find the number of combinations of these quantities, taken 4 at a time. Ans. 10.

3. How many combinations are there in *aaaabbccde*, 8 being taken at a time? Ans. 13.

4. How many combinations are there in *aaaaabbbbbcccccdd ddeeeeffffg*, 10 being taken at a time? Ans. 2819.

PROBLEM VII.

To find the compositions of any number, in an equal number of sets, the things themselves being all different.

RULE.*

Multiply the number of things in every set continually together, and the product will be the answer required.

---

* Demonstration. Suppose there are only two sets; then it is plain, that every quantity of one set, being combined with every quantity of the other, will make all the compositions of two things, in these two sets; and the number of these compositions is evidently the product of the number of quantities in one set by that in the other.

Again, suppose there are three sets; then the composition of two, in any two of the sets, being combined with every quantity of the third, will make all the compositions of 3 in the 3 sets. That is, the compositions of 2 in any two of the sets, being multiplied by the number of quantities in the remaining set, will produce the compositions of 3 in the 3 sets; which is evidently the continual product of all the 3 numbers in the 3 sets. And the same manner of reasoning will hold, let the number of sets be what it will. Q.E.D.

The doctrine of permutations, combinations, &c. is of very extensive use in different parts of the mathematics; particularly in the calculation of annuities and chances. The subject might have been pursued to a much greater length; but what has been done already will be found sufficient for most of the purposes, to which things of this nature are applicable.

EXAMPLES.

1. Suppose there are 4 companies, in each of which there are 9 men; it is required to find how many ways 4 men may be chosen, one out of each company.

$$\begin{array}{r} 9 \\ 9 \\ \hline 81 \\ 9 \\ \hline 729 \\ 9 \\ \hline 6561 \end{array}$$

Or, $9\times9\times9\times9=6561$ the answer.

2. Suppose there are 4 companies, in one of which there are 6 men, in another 8, and in each of the other two 9; what are the choices, by a composition of 4 men, one out of each company?

Ans. 3888.

3. How many changes are there in throwing 5 dice?

Ans. 7776.

MISCELLANEOUS QUESTIONS.

1. WHAT difference is there between twice five and twenty, and twice twenty-five? Ans. 20.

2. A was born when B was 21 years of age; how old will A be when B is 47; and what will be the age of B when A is 60? Ans. A 26, B 81.

3. What number, taken from the square of 48, will leave 16 times 54? Ans. 1440.

4. What number, added to the thirty-first part of 3813, will make the sum 200? Ans. 77.

5. The remainder of a division is 325, the quotient 467, and the divisor is 43 more than the sum of both: what is the dividend? Ans. 390270.

6. Two persons depart from the same place at the same time; the one travels 30, the other 35 miles a day: how far are they distant at the end of 7 days, if they travel both the same road; and how far, if they travel in contrary directions? Ans. 35, and 455 miles.

7. A tradesman increased his estate annually by 100l. more than $\frac{1}{4}$ part of it, and at the end of 4 years found, that his estate amounted to 10342l. 3s. 9d. What had he at first? Ans. 4000l.

8. Divide 1200 acres of land among A, B, and C, so that B may have 100 more than A, and C 64 more than B.
Ans. A 312, B 412, and C 476.

9. Divide 1000 crowns; give A 120 more, and B 95 less, than C. Ans. A 445, B 230, C 325.

10. What sum of money will amount to 132l. 16s. 3d. in 15 months, at 5 per cent. per annum, simple interest?
Ans. 125l.

11. A father divided his fortune among his sons, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share 5000l.?
Ans. 11875l.

12. If 1000 men, besieged in a town with provisions for 5 weeks, each man being allowed 16 oz. a day, were reinforced with 500 men more. On hearing, that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time?
Ans. $6\frac{2}{3}$ oz.

13. What number is that, to which if $\frac{2}{7}$ of $\frac{5}{9}$ be added, the sum will be 1? Ans. $\frac{53}{63}$.

14. A father dying left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months; $\frac{3}{7}$ of the remainder lasted him twelve months longer; after which he had only 410l. left. What did his father bequeath him? Ans. 956l. 13s. 4d.

15. A guardian paid his ward 3500l. for 2500l. which he had in his hands 8 years. What rate of interest did he allow him ? Ans. 5 per cent.

16. A person, being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ of the time till midnight. What was the time? Ans. 20 min. past 5.

17. A person, looking on his watch, was asked, what was the time of the day ; he answered, it is between 4 and 5 ; but a more particular answer being required, he said, that the hour and minute hands were then exactly together. What was the time ?

Ans. $21\frac{9}{11}$ minutes past 4.

18. With 12 gallons of Canary, at 6s. 4d. a gallon, I mixed 18 gallons of white wine, at 4s. 10d. a gal. and 12 gallons of cider, at 3s. 1d. a gal. At what rate must I sell a quart of this composition, so as to clear 10 per cent. ?

Ans. 1s. $3\frac{5}{7}$d.

19. What length must be cut off a board, $8\frac{3}{8}$ inches broad, to contain a square foot, or as much as 12 inches in length and 12 in breath ? Ans. $17\frac{13}{67}$ in.

20. What difference is there between the interest of 350l. at 4 per cent. for 8 years, and the discount of the same sum at the same rate and for the same time ?

Ans. 27l. $3\frac{1}{33}$s.

21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life; the children's legacies were found to be 257l. 3s. 4d. different. What money did he leave for the widow?

Ans 635l. 10 $\frac{30}{49}$d.

22. What number is that, from which if you take $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{20}$, the sum will be 10 ?

Ans. $10\frac{191}{2240}$.

23. A man dying left his wife in expectation, that a child would be afterward added to the surviving family ; and making his will ordered, that, if the child were a son, $\frac{2}{3}$ of his estate should belong to him, and the remainder to his mother ; but if it were a daughter, he appointed the mother

$\frac{2}{3}$, and the child the remainder. But it happened, that the addition was both a son and a daughter, by which the widow lost in equity 2400l. more than if there had been only a girl. What would have been her dowry, had she had only a son?

Ans 2100l.

24. A young hare starts 40 yards before a grey-hound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of ten miles an hour, and the dog, on view, makes after her at the rate of 18. How long will the course continue, and what will be the length of it from the place, where the dog set out?

Ans. $60\frac{5}{22}$ seconds, and 530 yards run.

25. A reservoir for water has two cocks to supply it; by the first alone it may be filled in 40 minutes, by the second in 50 minutes, and it has a discharging cock, by which it may, when full, be emtied in 25 minutes. Now supposing, that these three cocks are all left open, that the water comes in, and that the influx and efflux of the water are always alike, in what time would the cistern be filled?

Ans. 3 hours 20 min.

26. A sets out from London for Lincoln precisely at the time, when B at Lincoln sets forward for London, distant 100 miles; after 7 hours they met on the road, and it then appeared, that A had ridden $1\frac{1}{2}$ mile an hour more than B. At what rate an hour did each of them travel?

Ans. A $7\frac{25}{28}$, B $6\frac{11}{28}$ miles.

27. What part of 3d. is a third part of 2d.

Ans. $\frac{2}{9}$.

28. A has by him $1\frac{1}{2}$cwt. of tea, the prime cost of which was 96l. sterling. Now granting interest to be at 5 per cent. it is required to find how he must rate it per pound to B, so that by taking his negotiable note, payable at 3 months, he may clear 20 guineas by the bargain?

Ans. 14s. $1\frac{13}{59}$d. sterling.

29. What annuity is sufficient to pay off 50 millions of pounds in 30 years, at 4 per cent. compound interest?

Ans. 2891505l.

30. There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it; A goes 5 miles a day, B 8, and C 10; when will they all come together again? Ans. 73 days.

31. A man, being asked how many sheep he had in his drove, said, if he had as many more, half as many more, and 7 sheep and a half, he should have 20: how many had he? Ans. 5.

32. A person left 40s. to 4 poor widows, A, B, C, and D; to A he left $\frac{1}{3}$, to B $\frac{1}{4}$, to C $\frac{1}{5}$, and to D $\frac{1}{6}$, desiring the whole might be distributed accordingly: what is the proper share of each?

Ans. A's share 14s. $\frac{16}{38}$d. B's 10s. $6\frac{12}{38}$d. C's 8s. $5\frac{10}{38}$d. D's 7s. $\frac{8}{38}$d.

33. A general, disposing of his army into a square, finds he has 284 soldiers over and above; but increasing each side with one soldier, he wants 25 to fill up the square; how many soldiers had he? Ans. 24000.

34. There is a prize of 212l. 14s. 7d. to be divided among a captain, 4 men, and a boy; the captain is to have a share and a half; the men each a share, and the boy $\frac{1}{3}$ of a share: what ought each person to have?

Ans. The captain 54l. 14s. $\frac{3}{7}$d. each man 36l. 9s. $4\frac{2}{7}$d. and the boy 12l. 3s. $1\frac{3}{7}$d.

35. A cistern, containing 60 gallons of water, has 3 unequal cocks for discharging it; the greatest cock will empty it in one hour, the second in 2 hours, and the third in 3: in what time will it be empty, if they all run together?

Ans. $32\frac{8}{11}$ minutes.

36. In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ pears, $\frac{1}{6}$ plumbs, and 50 of them cherries: how many trees are there in all? Ans. 600.

37. A can do a piece of work alone in ten days, and B in 13; if both be set about it together, in what time will it be finished? Ans. $5\frac{15}{23}$ days.

38. A, B, and C are to share 100000l. in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$, respectively; but C's part being lost by his death, it is required to divide the whole sum properly between the other two.

Ans. A's part is $57142\frac{282}{320}$, and B's $42857\frac{47}{320}$.

LOGARITHMS.

LOGARITHMS are numbers so contrived, and adapted to other numbers, that the sums and differences of the former shall correspond to, and show, the products and quotients of the latter.

Or, logarithms are the numerical exponents of ratios; or a series of numbers in arithmetical progression, answering to another series of numbers in geometrical progression.

Thus	0,	1,	2,	3,	4,	5,	6,	indices, or logarithms.
	1,	2,	4,	8,	16,	32,	64,	geometric progression.

Or	0,	1,	2,	3,	4,	5,	6,	indices, or logar.
	1,	3,	9,	27,	81,	243,	729,	geometric progress.

Or	0,	1,	2,	3,	4,	5,	ind. or log.
	1,	10,	100,	1000,	10000,	100000,	geom. prog.

Where it is evident, that the same indices serve equally for any geometric series; and consequently there may be an endless variety of systems of logarithms to the same common numbers, by only changing the second term 2, 3, or 10, &c. of the geometrical series of whole numbers; and by interpolation the whole system of numbers may be made to enter the geometric series, and receive their proportional logarithms, whether integers or decimals.

It is also apparent from the nature of these series, that if any two indices be added together, their sum will be the index of that number, which is equal to the product of the two terms in the geometric progression, to which those indices belong. Thus, the indices 2 and 3, being added together, make 5; and the numbers 4 and 8, or the terms corresponding to those indices, being multiplied together, make 32, which is the number answering to the index 5.

In like manner, if any one index be subtracted from another, the difference will be the index of that number, which is equal to the quotient of the two terms, to which those indices belong. Thus, the index 6 minus the index 4=2; and the terms corresponding to those indices are 64 and 16, whose quotient =4; which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series, is 2; and if this number be multiplied by 3, the product will be =6; which is the logarithm of 64, or the third power of 4.

And if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus, the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be =3; which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice are such, as are adapted to a geometric series, increasing in a tenfold proportion, as in the last of the above forms; and are those, which are to be found at present in most of the common tables of logarithms.

The distinguishing mark of this system of logarithms is, that the index or logarithm of 10 is 1; that of 100 is 2; that of 1000 is 3, &c. And, in decimals, the logarithm of ·1 is —1; that of ·01 is —2; that of ·001 is —3, &c. the logarithm of 1 being 0 in every system.

Whence it follows, that the logarithm of any number between 1 and 10 must be 0 and some fractional parts; and that of a number between 10 and 100, 1 and some fractional parts; and so on, for any other number whatever.

And since the integral part of a logarithm, thus readily found, shows the highest place of the corresponding number, it is called the *index*, or *characteristic*, and is commonly omitted in the tables; being left to be supplied by the person, who uses them, as occasion requires.

Another definition of logarithms is, that the logarithm of any number is the index of that power of some other number, which is equal to the given number. So if there be $N=r^n$, then n is log. of N; where n may be either positive or negative, or nothing, and the root r any number whatever, according to the different systems of logarithms.

When n is $=0$, then N is $=1$, whatever the value of r is; which shows, that the logarithm of 1 is always 0 in every system of logarithms.

When n is $=1$, then N is $=r$; so that the radix r is always that number, whose logarithm is 1 in every system.

When the radix r is $=2·718281828459$, &c. the indices n are the hyperbolic or Napier's logarithm of the numbers N; so that n is always the hyperbolic logarithm of the number N or $\overline{2·718, \&c.}|^n$.

But when the radix r is $=10$, then the index n becomes the common or Briggs' logarithm of the number N; so that the common logarithm of any number 10^n or N is n the index of that power of 10, which is equal to the said number. Thus, 100, being the second power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm; hence also, if 50 be $=10^{1·69897}$, then is 1·69897 the common logarithm of 50. And in general the following decuple series of terms,

viz. 10^4, 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, or 10000, 1000, 100, 10, 1, ·1, ·01, ·001, ·0001, have 4, 3, 2, 1, 0, —1, —2, —3, —4, for their logarithms, respectively. And from this scale of numbers and logarithms, the same properties easily follow, as beforementioned.

PROBLEM.

To compute the logarithm to any of the natural numbers, 1, 2, 3, 4, 5, *&c.*

RULE.

Let b be the number, whose logarithm is required to be found; and a the number next less than b, so that $b-a=1$, the logarithm of a being known; and let s denote the sum of the two numbers $a+b$. Then

1. Divide the constant decimal ·8685889638, &c. by s, and reserve the quotient; divide the reserved quotient by the square of s, and reserve this quotient; divide this last quotient also by the square of s, and again reserve the quotient; and thus proceed, continually dividing the last quotient by the square of s, as long as division can be made.

2. Then write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers, 1, 3, 5, 7, 9, &c. as long as division can be made; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, and so on.

3. Add all these last quotients together, and the sum will be the logarithm of $b \div a$; therefore to this logarithm add also the given logarithm of the said next less number a, so will the last sum be the logarithm of the number b proposed.

That is, log. of b is log. $a + \frac{n}{s} \times : 1 + \frac{1}{3s^2} + \frac{1}{5s^4} + \frac{1}{7s^6} +$ &c.

where n denotes the constant given decimal ·8685889638, &c.

EXAMPLES.

Example 1. Let it be required to find the logarithm of the number 2.

Here the given number b is 2, and the next less number a is 1, whose logarithm is 0; also the sum $2+1=3=s$, and its square $s^2=9$. Then the operation will be as follows.

3)	·868588964	1)	·289529654(	·289529654
9)	·289529654	3)	32169962(	10723321
9)	32169962	5)	3574440(	714888
9)	3574440	7)	397160(	56737
9)	397160	9)	44129(	4903
9)	44129	11)	4903(	446
9)	4903	13)	545(	42
9)	545	15)	61(	4
9)	61			

Log. of $\frac{2}{1}$	·301029995
Add log. 1	000000000
Log. of 2	·301029995

EXAMPLE 2. To compute the logarithm of the number 3. Here $b=3$, the next less number $a=2$, and the sum $a+b=5=s$, whose square s^2 is 25, to divide by which, always multiply by ·04. Then the operation is as follows.

5)	·868588964	1)	·173717793(	·173717793	
25)	·173717793	3)	6948712(	2316237	
25)	6948712	5)	277948(	55590	
25)	277948	7)	11118(	1588	
25)	11118	9)	445(	50	
25)	445	11)	18(	2	
	18				

Log. of $\frac{3}{2}$ ·176091260
Log. of 2 add ·301029995

Log. of 3 sought ·477121255

Then, because the sum of the logarithms of numbers gives the logarithm of their product, and the difference of the logarithms gives the logarithm of the quotient of the numbers, from the above two logarithms, and the logarithm of 10, which is 1, we may raise a great many logarithms, as in the following examples.

EXAMPLE 3.

Because $2 \times 2=4$, therefore
To logarithm 2 ·301029995$\frac{2}{3}$
Add logarithm 2 ·301029995$\frac{2}{3}$

Sum is logarithm 4 ·602059991$\frac{1}{3}$.

EXAMPLE 4.

Because $2\times3=6$, therefore
To logarithm 2 ·301029995
Add logarithm 3 ·477121255

Sum is logarithm 6 ·778151250.

EXAMPLE 5.

Because $2^3=8$, therefore

Logarithm 2	$\cdot 301029995\frac{2}{3}$
Multiplied by 3	3
Gives logarithm 8	$\cdot 903039987.$

EXAMPLE 6.

Because $3^2=9$, therefore

Logarithm 3	$\cdot 477121254\frac{7}{10}$
Multiplied by 2	2
Gives logarithm 9	$\cdot 954242509.$

EXAMPLE 7.

Because $\frac{10}{2}=5$, therefore

From logarithm 10	$1\cdot 000000000$
Take logarithm 2	$\cdot 301029995\frac{2}{3}$
Remains logarithm 5	$\cdot 698970004\frac{1}{3}$

EXAMPLE 8.

Because $3\times 4=12$, therefore

To logarithm 3	$\cdot 477121255$
Add logarithm 4	$\cdot 602059991$
Sum is logarithm 12	$1\cdot 079181246.$

And thus, computing by this general rule, the logarithms to the other prime numbers 7, 11, 13, 17, 19, 23, &c. and then using composition and division, we may easily find as many logarithms as we please, or may speedily examine any logarithm in the table.*

* Many other ingenious methods of finding the logarithms of numbers, and peculiar artifices for constructing an entire table of them, may be seen in Dr. HUTTON's *Introduction to his Tables*, and Baron MASERES' *Scriptores Logarithmici*.

DESCRIPTION AND USE OF THE TABLE OF LOGARITHMS.

Integral numbers are supposed to form a geometrical series, increasing from unity toward the left; but decimals are supposed to form a like series, decreasing from unity toward the right, and the indices of their logarithms are negative. Thus, + 1 is the logarithm of 10, but—1 is the logarithm of $\frac{1}{10}$, or ˙1; and + 2 is the logarithm of 100, but —2 is the logarithm of $\frac{1}{100}$, or ˙01; and so on.

Hence it appears in general, that all numbers, which consist of the same figures, whether they be integral, or fractional, or mixed, will have the decimal parts of their logarithms the same, differing only in the index, which will be more or less, and positive or negative, according to the place of the first figure of the number. Thus, the logarithm of 2651 being 3˙4234097, the logarithm of $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c. part of it will be as follows.

Numbers.	Logarithms.
2651	3˙4234097
265˙1	2˙4234097
26·51	1˙4234097
2·651	0˙4234097
·2651	—1˙4234097
˙02651	—2˙4234097
·002651	—3˙4234097

Hence it appears, that the index, or characteristic, of any logarithm is always less by 1 than the number of integral figures, which the natural number consists of; or it is equal to the distance of the first or left hand figure from the place of units, or first place of integers, whether on the left, or on the right of it; and this index is constantly to be placed on the left of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and it is to be marked by a short line drawn before, or above it. Thus, a number having 1, 2, 3, 4, 5, &c. integral places, the index of its logarithm is 0, 1, 2, 3, 4, &c. or 1 less than *the number of* those places.

And a decimal fraction, having its first figure in the 1st, 2d, 3d, 4th, &c. place of decimals, has always —1, —2, —3, —4, &c. for the index of its logarithm.

It may also be observed, that though the indices of fractional quantities be negative, yet the decimal parts of their logarithms are always affirmative.

1. *To find, in the Table, the Logarithm to any Number.**

1. *If the number do not exceed* 100000, the decimal part of the logarithm is found, by inspection in the table, standing against the given number, in this manner, viz. in most tables, the first four figures of the given number are in the first column of the page, and the fifth figure in the uppermost line of it; then in the angle of meeting are the last four figures of the logarithm, and the first three figures of the same at the beginning of the same line; to which is to be prefixed the proper index.

So the logarithm of 34·092 is 1·5326525, that is, the decimal 5326525, found in the table, with the index 1 prefixed, because the given number contains two integers.

2. *But if the given number contain more than five figures*, take out the logarithm of the first five figures by inspection in the table as before, as also the next greater logarithm, subtracting one logarithm from the other, and also one of their corresponding numbers from the other. Then say,

As the difference between the two numbers
Is to the difference of their logarithms,
So is the remaining part of the given number
To the proportional part of the logarithm.

Which part being added to the less logarithm, before taken out, the whole logarithm sought is obtained very nearly.

---

* The Tables, considered as the best, are those of Gardiner in 4to. first published in the year 1742; of Dr. Hutton, in 8vo. first printed in 1785; of Taylor, in large 4to. published in 1792; and in France, those of Callet, the second edition published in 1795.

EXAMPLE.

To find the logarithm of the number 34·09264.

The log. of	3409200,	as before, is	5326525,
and log. of	3409300	is	5326652,
the diff.	100	and	127

Then, as 100 : 127 : : 64 : 81, the proportional part.
This added to 5326525, the first logarithm,

gives, with the index, 1·5326606 for the logarithm of 34·09264.

Or, in the best tables, the proportional part may often be taken out by inspection, by means of the small tables of proportional parts, placed in the margin.

If the number consist both of integers and fractions, or be entirely fractional, find the decimal part of the logarithm, as if all its figures were integral; then this, the proper characteristic being prefixed, will give the logarithm required.

And if the given number be a proper fraction, subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which, being that of a decimal fraction, must always have a negative index.

But if it be a mixed number, reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the same manner as before.

EXAMPLES.

1. To find the logarithm of $\frac{37}{94}$.

Logarithm of 37	1·5682017
Logarithm of 94	1·9731279
Diff. log. of $\frac{37}{94}$	—1·5950738

Where the index 1 is negative.

2. To find the logarithm of $17\frac{14}{23}$.

First, $17\frac{14}{23} = \frac{405}{23}$. Then,

Logarithm of 405	2·6074550
Logarithm of 23	1·3617278
Diff. log. of $17\frac{14}{23}$	1·2457272

II. *To find, in the Table, the natural number to any Logarithm.*

This is to be found by the reverse method to the former, namely, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers is to be pointed off, viz. 1 more than the units of the affirmative index. For, in finding the number answering to any given logarithm, the index always shows how far the first figure must be removed from the place of units to the left or in integers, when the index is affirmative; but to the right or in decimals, when it is negative.

EXAMPLES.

So, the number to the logarithm 1·5326525 is 34·092. And the number of the logarithm—1·5326525 is ·34092.

But if the logarithm cannot be exactly found in the table, take out the next greater and the next less, subtracting one of these logarithms from the other, and also one of their natural numbers from the other, and the less logarithm from the logarithm proposed. Then say,

As the first difference, or that of the tabular logarithms,
Is to the difference of their natural numbers,
So is the difference of the given logarithm and the last tabular logarithm
To their corresponding numeral difference.

Which being annexed to the least natural number above taken, the natural number corresponding to the proposed logarithm is obtained.

EXAMPLE.

Find the natural number answering to the given logarithm 1·5326606.

Here the next greater and next less tabular logarithms, with their corresponding numbers, &c. are as below.

Next greater	5326652 its num.	3409300 ; giv. log.	5326606
Next less	5326525 its num.	3409200 ; next less	5326525
Differences	127	100	81

Then, as 127 : 100 : : 81 : 64 nearly, the numeral difference.

Therefore 34·09264 is the number sought, two integers being marked off, because the index of the given logarithm is 1. Had the index been negative, thus, —1·5326606, its corresponding number would have been ·3409264, wholly decimal.

Or, the proportional numeral difference may be found, in the best tables, by inspection of the small tables of proportional parts, placed in the margin.

MULTIPLICATION BY LOGARITHMS.

RULE.

Take out the logarithms of the factors from the table, then add them together, and their sum will be the logarithm of the product required. Then, by means of the table, take out the natural number answering to the sum, for the product sought.

NOTE 1. In every operation, what is carried from the decimal part of a logarithm to its index is affirmative ; and is therefore to be added to the index, when it is affirmative ; but subtracted, when it is negative.

NOTE 2. When the indices have like signs, that is, both + or both —, they are to be added, and the sum has the common sign ; but when they have unlike signs, that is, one + and the other —, their difference, with the sign of the greater, is to be taken for the index of the sum.

EXAMPLES.

1. To multiply 23·14 by 5·062.

	Numbers.	Logarithms.
	23·14	1·3643634
	5·062	0·7043221
Product	117·1347	2·0686855

2. To multiply 2·581926 by 3·457291.

	Numbers.	Logarithms.
	2·581926	0·4119438
	3·457291	0·5387359
Product	8·92647	0·9506797

3. To multiply 3·902, 597·16, and ·0314728 all together.

	Numbers.	Logarithms.
	3·902	0·5912873
	597·16	2·7760907
	·0314728	—2·4979353
Product	73·33533	1·8653133

Here the —2 cancels the 2, and the 1, to be carried from the decimals, is set down..

4. To multiply 3·586, 2·1046, 0·8372, and 0·0294 all together.

	Numbers.	Logarithms.
	3·586	0·5546103
	2·1046	0·3231696
	0·8372	—1·9228292
	0·0294	—2·4683473
Product	0·1857618	—1·2689564

Here the 2, to be carried, cancels the —2, and there remains the —1 to be set down.

DIVISION BY LOGARITHMS.

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required.

NOTE. If 1 be to be carried to the index of the subtrahend, apply it according to the sign of the index; then change the sign of the index to —, if it be +, or to +, if it be —; and proceed according to the second note under the last rule.

EXAMPLES.

1. To divide 24163 by 4567.

	Num.	Log.
Dividend	24163	4·3831509
Divisor	4567	3·6596310
Quotient	5·290782	0·7235199

2. To divide 37·149 by 523·76.

	Num.	Log.
Dividend	37·149	1·5699471
Divisor	523·76	2·7191323
Quotient	·07092752	—2·8508148

3. Divide ·06314 by ·007241.

	Num.	Log.
Dividend	·06314	—2·8003046
Divisor	·007241	—3·8597985
Quotient	8·719792	0·9405061

Here 1, carried from the decimals to the —3, makes it become —2, which, taken from the other —2, leaves 0 remaining.

4. To divide ·7438 by 12·9476.

	Num.	Log.
Dividend	·7438	—1·8714562
Divisor	12·9476	1·1121893
Quotient	·05744694	—2·7592669

Here the 1, taken from the —1, makes it become —2, to be set down.

INVOLUTION BY LOGARITHMS.

RULE.

Multiply the logarithm of the given number by the index of the power, and the number answering to the product will be the power required.

NOTE. A negative index, multiplied by an affirmative number, gives a negative product; and as the number, carried from the decimal part, is affirmative, their difference with the sign of the greater is, in that case, the index of the product.

EXAMPLES.

1. To square the number 2·5791.

	Num.	Log.
Root	2·5791	0·4114682
	The index	2
Power	6·651756	0·8229364

2. To find the cube of 3·07146.

	Num.	Log.
Root	3·07146	0·4873449
	The Index	3
Power	28·97575	1·4620347

3. To raise ˙09163 to the 4th power.

Num.		Log.
Root ˙09163		—2˙9620377
	The Index	4
Power ˙0000704938		—5˙8481508

Here 4 times the negative index being —8, and 3 to be carried, the difference —5 is the index of the product.

4. To raise 1˙0045 to the 365th root.

Num.		Log.
Root 1˙0045		0˙0019499
	The Index	365
		97495
		116994
		58497
Power 5˙148888		˙7117135

EVOLUTION BY LOGARITHMS.

RULE.

Divide the logarithm of the given number by the index of the power, and the number answering to the quotient will be the root required.

NOTE. When the index of the logarithm is negative, and cannot be divided by the divisor without a remainder, increase the index by a number, that will render it exactly divisible, and carry the units borrowed, as so many tens, to the first decimal place; and divide the rest as usual.

EXAMPLES.

1. To find the square root of 365.

Num.	Log.
Power 365	2)2˙5622929
Root 19˙10498	1˙2811465

2. To find the 3d root of 12345.

Num.	Log.
Power 12345	3)4·0914911
Root 23·11162	1·3638304

3. To find the 10th root of 2.

Num.	Log.
Power 2	10)0·3010300
Root 1·071773	0·0301030

4. To find the 365th root of 1·045.

Num.	Log.
Power 1·045	365)0·0191163
Root 1·000121	0·0000524

5. To find the second root of ·093.

Num.	Log.
Power ·093	2)—2·9684829
Root ·304959	—1·4842415

Here the divisor 2 is contained exactly once in the negative index —2, and therefore the index of the quotient is —1.

6. To find the third root of ·00048.

Num.	Log.
Power ·00048	3)—4·6812412
Root ·07829735	—2·8937471

Here the divisor 3 not being exactly contained in —4, 4 is augmented by 2, to make up 6, in which the divisor is contained just 2 times; then the 2, thus borrowed, being carried to the decimal figure 6, makes 26, which, divided by 3, gives 8, &c. For —4=—6+2.

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is the art of computing by symbols. It is sometimes also called ANALYSIS; and is a general kind of arithmetic, or universal way of computation.

2. In Algebra, the *given*, or *known quantities* are usually denoted by the first letters of the alphabet, as a, b, c, d, &c. and the *unknown*, or *required quantities*, by the last letters, as x, y, z.

NOTE. The signs, or characters, explained at the beginning of Arithmetic, have the same signification in Algebra.

3. Those quantities, before which the sign $+$ is placed, are called *positive*, or *affirmative;* and those, before which the sign $-$ is placed, *negative*.

And it is to be observed, that the sign of a negative quantity is never omitted, nor the sign of an affirmative one, except it be a single quantity, or the first in a series of quantities, then the sign $+$ is frequently omitted: thus a signifies the same as $+a$, and the series $a+b-c+d$ the same as $+a+b-c+d$; so that, if any single quantity, or if the first term in any number of terms, have not a sign before it, then it is always understood to be affirmative.

4. *Like signs* are either all positive, or all negative; but *signs* are *unlike*, when some are positive and others negative.

5. *Single*, or *simple* quantities consist of one term only, as a, b, x.

In multiplying simple quantities, we frequently omit the sign $\times$, and join the letters; thus, ab signifies the same as $a\times b$; and abc, the same as $a\times b\times c$. And these products, viz. $a\times b$, or ab, and abc, are called single or simple quantities, as well as the factors, viz. a, b, c, from which they are produced, and the same is to be observed of the products, arising from the multiplication of any number of simple quantities.

6. If an algebraical quantity consist of two terms, it is called a *binomial*, as $a+b$; if of three terms, a *trinomial*, as $a+b+c$; and if of four terms, a *quadrinomial*, as $a+b+c+d$; and if there be more terms, it is called a *multinomial*, or *polynomial;* all of which are *compound quantities*.

When a compound quantity is to be expressed as multiplied by a simple one, then we place the sign of multiplication between them, and draw a line over the compound quantity only; but when compound quantities are to be represented as multiplied together, then we draw a line over each of them, and connect them with a proper sign. Thus, $\overline{a+b}\times c$ denotes, that the compound quantity $a+b$ is multiplied by the simple quantity c; so that if a were 10, b 6, and c 4, then would $\overline{a+b}\times c$ be $\overline{10+6}\times 4$, or 16 into 4, which is 64; and $\overline{a+b}\times\overline{c+d}$ expresses the product of the compound quantities $a+b$ and $c+d$ multiplied together.

7. When we would express, that one quantity, as a, is greater than another, as b, we write $a \sqsubset b$, or $a > b$; and if we would express, that a is less than b, we write $a \sqsupset b$, or $a < b$.

8. When we would express the difference between two quantities, as a and b, while it is unknown which is the greater of the two, we write them thus, $a \backsim b$, which denotes the difference of a and b.

9. *Powers* of the same quantities or factors are the products of their multiplication: thus $a\times a$, or aa, denotes the *square*, or *second power* of the quantity, represented a; $a\times a\times a$, or aaa, expresses the *cube*, or *third power;* and $a\times a\times a\times a$, or $aaaa$, denotes the *biquadrate*, or *fourth power* of a, &c.

And it is to be observed, that the quantity a is the root of all these powers. Suppose $a=5$, then will $aa=a\times a=5\times 5=25=$ the square of 5; $aaa=a\times a\times a=5\times 5\times 5=125=$ the cube of 5; and $aaaa = a\times a\times a\times a = 5\times 5\times 5\times 5 = 625 =$ the fourth power of 5.

10. Powers are likewise represented by placing above the root, to the right, a figure expressing the number of factors, that produce them. Thus, instead of *aa*, we write a^2; instead of *aaa*, we write a^3; instead of *aaaa*, we write a^4, &c.

11. These figures, which express the number of factors, that produce powers, are called their *indices*, or *exponents;* thus, 2 is the index or exponennt of a^2; 3 is that of x^3; 4 is that of x^4, &c.

But the exponent of the first power, though generally omitted, is unity, or 1; thus a^1 signifies the same as a, namely, the first power of a; $a\times a$, the same as $a^1\times a^1$, or a^{1+1}, that is, a^2, and $a^2\times a$ is the same as $a^2\times a^1$, or a^{2+1}, or a^3.

12. In expressing powers of compound quantities, we usually draw a line over the given quantity, and at the end of the line place the exponent of the power. Thus, $\overline{a+b}|^{2}$ denotes the square or second power of $a+b$, considered as one quantity; $\overline{a+b}|^{3}$ the third power; $\overline{a+b}|^{4}$ the fourth power, &c.

And it may be observed, that the quantity $a+b$, called the first power of $a+b$, is the root of all these powers. Let $a=4$ and $b=2$, then will $a+b$ become $4+2$, or 6; and $\overline{a+b}|^{2} = \overline{4+2}|^{2} = 6^2 = 6\times 6 = 36$, the square of 6; also $\overline{a+b}|^{3} = \overline{4+2}|^{3} = 6^3 = 6\times 6\times 6 = 216$, the cube of 6.

13. The division of algebraic quantities is very frequently expressed by writing down the divisor under the dividend with a line between them, in the manner of a vulgar fraction: thus, $\frac{a}{c}$ represents the quantity arising by dividing a by c; so that if a be 144 and c 4, then will $\frac{a}{c}$ be $\frac{144}{4}$, or 36. And $\frac{a+b}{a-c}$ denotes the quantity arising by dividing $a+b$ by

$a-c$; suppose $a=12$, $b=6$, and $c=9$, then will $\frac{a+b}{a-c}$ become $\frac{12+6}{12-9}$ or $\frac{18}{3}=6$.

14. These literal expressions, namely, $\frac{a}{c}$ and $\frac{a+b}{a-c}$, are called *algebraic fractions;* whereof the upper parts are called the *numerators*, and the lower the *denominators*: thus, a is the numerator of the fraction $\frac{a}{c}$, and c is its denominator; $a+b$ is the numerator of $\frac{a+b}{a-c}$, and $a-c$ is its denominator.

15. Quantities, to which the radical sign is applied, are called *radical quantities*, or *surds;* whereof those consisting of one term only, as $\sqrt{a}$ and $\sqrt{a\,x}$, are called *simple surds;* and those consisting of several terms, as $\sqrt{ab+cd}$ and $\sqrt[4]{a^2-b^2+bc}$, *compound surds.*

16. When any quantity is to be taken more than once, the number is to be prefixed, which shows how many times it is to be taken, and the number so prefixed is called the *numeral coefficient:* thus, $2a$ signifies twice a, or a taken twice, and the numeral coefficient is 2; $3x^2$ signifies, that the quantity x^2 is multiplied by 3, and the numeral coefficient is 3; also $5\sqrt{x^2+a^2}$ denotes, that the quantity $\sqrt{x^2+a^2}$ is multiplied by 5, or taken 5 times.

When no number is prefixed, an unit or 1 is always understood to be the coefficient: thus, 1 is the coefficient of a or of x; for a signifies the same as $1a$, and x the same as $1x$, since any quantity, multiplied by unity, is still the same.

Moreover, if a and d be given quantities, and x^2 and y required ones; then ax^2 denotes, that x^2 is to be taken a times, or as many times as there are units in a; and dy shows, that y is to be taken d times; so that the coefficient of ax^2 is a, and that of dy is d. Suppose $a=6$ and $d=4$, then will ax^2 $=6x^2$, and $dy=4y$. Again, $\frac{1}{2}x$, or $\frac{1x}{2}$, denotes half of the quantity x, and the coefficient of $\frac{1}{2}x$ is $\frac{1}{2}$; so likewise $\frac{3}{4}x$, or $\frac{3x}{4}$, signifies $\frac{3}{4}$ of x, and the coefficient of $\frac{3}{4}x$ is $\frac{3}{4}$.

17. *Like quantities* are those, that are represented by the same letters under the same powers, or which differ only in their coefficients: thus, $3a$, $5a$, and a are like quantities, and the same is to be understood of the radicals $\sqrt{x^2+a^2}$ and $7\sqrt{x^2+a^2}$. But *unlike quantities* are those, which are expressed by different letters, or by the same letters under different powers: thus $2ab$, a^2b, $2abc$, $5ab^2$, $4x^2$, y, y^2, and z^2 are all unlike quantities.

18. The *double* or *ambiguous sign* $\pm$ signifies *plus or minus* the quantity, which immediately follows it, and being placed between two quantities, it denotes their sum, or difference. Thus, $\frac{1}{2}a\pm\sqrt{\frac{a^2}{4}-b}$ shows, that the quantity $\sqrt{\frac{a^2}{4}-b}$ is to be added to, or subtracted from $\frac{1}{2}a$.

19. A *general exponent* is one, that is denoted by a letter instead of a figure: thus, the quantity x^m has a general exponent, namely, m, which universally denotes the mth power of the root x. Suppose $m=2$, then will $x^m=x^2$; if $m=3$, then will $x^m=x^3$; if $m=4$, then will $x^m=x^4$, &c. In like manner, $\overline{a-b}|^m$ expresses the mth power of $a-b$.

20. This root, namely, $a-b$, is called a *residual root*, because its value is no more than the residue, remainder, or difference of its terms a and b. It is likewise called a binomial, as well as $a+b$, because it is composed of two parts, connected together by the sign $-$.

21. A fraction, which expresses the root of a quantity, is also called an *index*, or *exponent*; the numerator shows the

power, and the denominator the root: thus $a^{\frac{1}{2}}$ signifies the same as $\sqrt{a}$; and $\overline{a+ab}|^{\frac{1}{3}}$, the same as $\sqrt[3]{a+ab}$; likewise $a^{\frac{2}{3}}$ denotes the square of the cube root of the quantity a. Suppose $a=64$, then will $a^{\frac{2}{3}}=64^{\frac{2}{3}}=4^2=16$; for the cube root of 64 is 4, and the square of 4 is 16.

Again $\overline{a+b}|^{\frac{5}{4}}$ expresses the fifth power of the biquadratic root of $a+b$. Suppose $a=9$ and $b=7$, then will $\overline{a+b}|^{\frac{5}{4}}=\overline{9+7}|^{\frac{5}{4}}=\overline{16}|^{\frac{5}{4}}=2^5=32$; for the biquadratic root of 16 is 2, and the fifth power of 2 is 32.

Also $a^{\frac{1}{n}}$ signifies the nth root of a. If $n=4$, then will $a^{\frac{1}{n}}=a^{\frac{1}{4}}$; if $n=5$, then will $a^{\frac{1}{n}}=a^{\frac{1}{5}}$, &c.

Moreover $\overline{a+b}|^{\frac{m}{n}}$ denotes the mth power of the nth root of $a+b$. If $m=3$ and $n=2$, then will $\overline{a+b}|^{\frac{m}{n}}=\overline{a+b}|^{\frac{3}{2}}$, namely, the cube of the square root of the quantity $a+b$; and as $a^{\frac{1}{n}}$ equals $\sqrt[n]{a^1}$, or $\sqrt[n]{a}$, so $\overline{a+b}|^{\frac{m}{n}}=\sqrt[n]{\overline{a+b}|^{m}}$, namely, the nth root of the mth power of $a+b$. So that the mth power of the nth root, and the nth root of the mth power of a quantity are the very same in effect, though differently expressed.

22. An *exponential quantity* is a power, whose exponent is a variable quantity, as x^x. Suppose $x=2$, then will $x^x=2^2=4$; if $x=3$, then will $x^x=3^3=27$.

ADDITION.

Addition, in Algebra, is connecting the quantities together by their proper signs, and uniting in simple terms such as are similar.

In addition there are three cases.

CASE I.

When like quantities have like signs.

RULE.*

Add the coefficients together, to their sum join the common letters, and perfix the common sign when necessary.

---

* The reasons, on which these operations are founded, will readily appear from a little reflection on the nature of the quantities to be added, or collected together. For with regard to the first example, where the quantities are $3a$ and $5a$, whatever a represents in one term, it will represent the same thing in the other; so that 3 times any thing, and 5 times the same thing, collected together, must needs make 8 times that thing. As if a denote a shilling, then $3a$ is 3 shillings, and $5a$ is 5 shillings, and their sum is 8 shillings. In like manner $-2ab$ and $-7ab$, or -2 times any thing and -7 times the same thing, make -9 times that thing.

As to the second case, in which the quantities are like, but the signs unlike; the reason of its operation will easily appear by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations, denoted by their signs $+$ and $-$, or of addition and subtraction; which being of contrary or opposite natures, one coefficient must be subtracted from the other, to obtain the incorporated or united mass.

As to the third case, where the quantities are unlike, it is plain, that such quantities cannot be united into one, or otherwise added than by means of their signs. Thus, for example, if a be supposed to represent a crown, and b a shilling; then the sum of a and b can be neither $2a$ nor $2b$, that is, neither 2 crowns nor 2 shillings, but only 1 crown plus 1 shilling, that is, $a+b$.

In this rule the word *addition* is not very properly used, being much too scanty to express the operation here performed. The business of this operation is to incorporate into one mass, or algebraic expression, different algebraic quantities, as far as an actual incorporation or union is possible; and to retain the algebraic marks for doing it in cases, where an union is not possible. When we have several quantities, some affirmative and others negative, and the relation of these quantities can be dis-

EXAMPLES.

	1.	2.	3.	4.
Add		$-7b$	$4xy^2$	$7ax-y$
	$3a$	$-b$	$7xy^2$	$8ax-3y$
	$5a$	$-2b$	$5xy^2$	$6ax-2y$
				$4ax-3y$
Sum	$8a$	$-10b$	$16xy^2$	$ax-y$
				$26ax-10y$

5.	6.	7.
$7a-6b$	$3x^{\frac{1}{2}}-xy$	$3xy-x+2ab$
$4a-3b$	$2x^{\frac{1}{2}}-3xy$	$2xy-3x+2ab$
$2a-8b$	$4x^{\frac{1}{2}}-8xy$	$2xy-4x+8ab$
$a-b$	$x^{\frac{1}{2}}-2xy$	$5xy-3x+ab$
$3a-2b$		

CASE II.

When like quantities have unlike signs.

RULE.

Subtract the less coefficient from the greater, to the remainder prefix the sign of the greater, and annex their common letters or quantities.

---

covered, in whole or in part; such incorporation of two or more quantities into one is plainly effected by the foregoing rules.

It may seem a paradox, that what is called addition in algebra should sometimes mean addition, and sometimes subtraction. But the paradox wholly arises from the scantiness of the name, given to the algebraic process, or from employing an old term in a new and more enlarged sense. Instead of addition, call it *incorporation*, *union*, or *striking a balance*, and the paradox vanishes.

EXAMPLES.

	1.	2.	3.*	4.	5.
To	$+6a$	$-7b$	$+2c$	$-cd$	$+3\sqrt{a^2+b^2}$
Add	$-2a$	$+6b$	$-2c$	$+3cd$	$-8\sqrt{a^2+b^2}$
Sum	$+4a$	$-b$	*	$+2cd$	$-5\sqrt{a^2+b^2}$

	6.	7.	8.	9.	10.
To	$+2a$	$-6b$	$+7c$	$-3cd$	$+8\sqrt{a^2+b^2}$
Add	$-6a$	$+7b$	$-4c$	$+cd$	$-3\sqrt{a^2+b^2}$
Sum	$-4a$	$+b$	$+3c$	$-2cd$	$+5\sqrt{a^2+b^2}$

NOTE. *When many like quantities are to be added together, whereof some are affirmative and others negative;* reduce them first to two terms, by adding all the affirmative quantities together, and all the negative ones; and then add the two terms acccording to the rule. Thus,

11. Add $4a^2+7a^2-3a^2+12a^2-8a^2+a^2-5a^2$ together.

First, $4a^2+7a^2+12a^2+a^2=24a^2$, the sum of the affirmative quantities,

And $-3a^2-8a^2-5a^2=-16a^2$, the sum of the negative.

Then $24a^2-16a^2=8a^2$, the sum of the whole.

12. Add $5ax^2-4ax^2+10ax^2-8ax^2-6ax^2$ together.

First, $5ax^2+10ax^2=15ax^2$,

And $-4ax^2-8ax^2-6ax^2=-18ax^2$;

Therefore the sum of these quantities is $+15ax^2-18ax^2=-3ax^2$.

* In example 3, the coefficients of the two quantities, viz. $+2c$ and $-2c$, are equal to each other, therefore they destroy one another, and so their sum makes 0, or *, which is frequently used, in algebra, to signify a vacant place.

13.	14.
$-6\sqrt{ax}$	$-2y+2ax^{\frac{1}{2}}$
$+2\sqrt{ax}$	$+y+ax^{\frac{1}{2}}$
$-6\sqrt{ax}$	$-7y-3ax^{\frac{1}{2}}$
$+10\sqrt{ax}$	$+5y+3ax^{\frac{1}{2}}$

CASE III.

When the quantities are unlike.

RULE.

Set them down in a line, with their signs and coefficients prefixed.

EXAMPLE.

To	$3a$	$-8xy$	$+2\sqrt{a^2-b^2}$
Add	$2b$	$+5y$	-9

Sum $3a+2b-8xy+5y+2\sqrt{a^2-b^2}-9$

OTHER EXAMPLES IN ADDITION.

1.	$3a^2$	$-9b^3$	$+9b^3$	$+xy$
	$4a^2$	$+7\sqrt{ab}$	$-5d$	$-2y^3$
	a^2	$-12b^3$	$-4\sqrt{ab}$	$+10$
	$7a^2$	$+5b^3$	$+6c^2$	-6

Sum $15a^2-7b^3+3\sqrt{ab}+6c^2-5d+xy-2y^3+4$*

* Here the first column is composed of like quantities, which are added together by case 1. The terms $-9b^3$ and $+9b^3$ de-

2.	3.	4.
$3x^2y$	$2\sqrt{x}-8y$	$a^2-\ 8+x^{\frac{1}{2}}-2$
$-2xy^2$	$3\sqrt{xy}+10x$	$a\ -10+a^2-x$
$-3y^2x$	$2x+\sqrt{x+y}$	$x^2-a^2+8\ -4$
$-8x^2y$	$-8+\sqrt{xy}$	$10-a\ -x^2-y$

SUBTRACTION.

RULE.*

Change each + into —, and each — into +, in the subtrahend, or suppose them to be thus changed; then proceed as in addition, and the sum will be the true remainder.

---

stroy one another; and the sum of $-12b^3$ and $+5b^3$ is $-7b^3$, by case 2. The sum of $+7\sqrt{ab}$ and $-4\sqrt{ab}$ is $+3\sqrt{ab}$. In like manner, + 10 and —6 together make +4; and the rest of the terms being unlike, they are set down with their respective signs and coefficients prefixed, conformably to case 3.

* This rule is founded on the consideration, that addition and subtraction are opposite to each other in their nature and operation, as are the signs + and —, by which they are expressed and represented. And since to unite a negative with a positive quantity of the same kind has the effect of diminishing it, or subducting an equal positive quantity from it; therefore to subtract a positive, which is the opposite of uniting or adding, is to add the equal negative quantity. In like manner to subtract a negative quantity is the same in effect, as to add or unite an equal positive quantity. So that, by changing the sign of a quantity from + to —, or from — to +, its nature is changed from a subductive to an additive quantity; and any quantity is in effect subtracted by barely changing its sign.

EXAMPLES.

	1.	2.	3.	4	5.
From	$5a$	$3a$	$+4b$	$-c$	$+8a^2$
Take	$2a$	$5a$	$-2b$	$+8c$	$-8a^2$
Rem.	$3a$	$-2a$	$+6b$	$-9c$	$+16a^2$

	6.	7.	8.	9.
From	$-8bc$	$+8c^2$	$-4d^3$	$-7x^2$
Take	$+8bc$	$+8c^2$	$+2d^3$	$-5x^2$
Rem.	$-16bc$	*	$-6d^3$	$-2x^2$

	10.	11.	12.
From	$-12xy$	$5ax^2$	$5a^2+3bx+14$
Take	$-12xy$	$2ax^2+4$	$a^2+2bx-10$
Rem.	*	$3ax^2-4$*	$4a^2+bx+24$

---

* The ten foregoing examples of simple quantities being obvious, we pass by them; but shall illustrate the eleventh example, in order to the ready understanding of those, which follow. In the eleventh example, the compound quantity $2ax+4$ being taken from the simple quantity $5ax^2$, the remainder is $3ax^2-4$, and it is plain, that the more there is taken from any number or quantity, the less will be left; and the less there is taken, the more will be left. Now, if only $2ax^2$ were taken from $5ax^2$, the remainder would be $3ax^2$; and consequently, if $2ax^2+4$, which is greater than $2ax$ by 4, be taken from $5ax^2$, the remainder will be less than $3ax^2$ by 4, that is, there will remain $3ax^2-4$, as above. For by changing the sign of the quantity $2ax^2+4$, and adding it to $5ax^2$, the sum is $5ax^2-2ax^2-4$; but here the term $-2ax^2$ destroys so much of $5ax^2$ as is equal to itself, and so $5ax^2-2ax^2-4$ becomes equal to $3ax^2-4$, by the general rule for subtraction

	13.	14.	15.
From	$9\sqrt{ax}-5a$	$6\sqrt{a^2+b^2}$	$2\sqrt{x+x^2}$
Take	$6\sqrt{ax}$	$9\sqrt{a^2+b^2}-5a$	$\sqrt{x+y^2}$
Rem.	$3\sqrt{ax}-5a$	$-3\sqrt{a^2+b^2}+5a$	$\sqrt{x+x^2-y^2}$

16.	17.	18.
$5x^2y-8$	$4\sqrt{xy}-x\sqrt{xy}$	$5x^2+\sqrt{x}-8-4b$
$-3x^2y+1$	$2\sqrt{xy}+2+xy$	$6x^2-10+4b-x^{\frac{1}{2}}$

19.	20.	21.
$3xy-20$	$4x^3-3\times\overline{a+b}$	$xy^3+10a\,\overline{x-10}$
$4xy-30$	$3x^2-3\times\overline{a+b}$	$x^2y^2+2a\sqrt{xy+10}$

MULTIPLICATION.

In multiplication of algebraic quantities there is one general rule for the signs; namely, when the signs of the factors are both affirmative or both negative, the product is affirmative; but if one of the factors be affirmative and the other negative, then the product is negative.*

* That like signs make +, and unlike signs —, in the product, may be shown thus.

1. *When + a is to be multiplied by + b;* it implies, that $+a$ is to be taken as many times, as there are units in b; and since the sum of any number of affirmative terms is affirmative, it follows, that $+a\times+b$ makes $+ab$.

2. *When two quantities are to be multiplied together;* the result will be exactly the same, in whatever order they are placed; for a times b is the same as b times a; and therefore, when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$, it is the same thing as taking $-a$ as many times as there are units in $+b$; and since

CASE I.

When both the factors are simple quantities.

RULE.

Multiply the coefficients of the two terms together, to the product annex all the letters of the terms, and prefix the proper sign.

EXAMPLES.

	1.	2.	3.	4.	5.
ultiply	a	$-3b$	$4ab$	$-5cd$	$-a$
by	b	$-2c$	3	$-4x$	b
Product	ab	$+6bc$	$12ab$	$+20cdx$	$-ab$

	6.	7.	8.
Multiply	$3b$	$-4ab$	$5cd$
by	$-2c$	3	$-4x$
Product	$-6bc$	$-12ab$	$-20cdx$

---

the sum of any number of negative terms is negative, it follows, that $-a\times+b$, or $+a\times-b$, makes or produces $-ab$.

3. *When* $-a$ *is to be multiplied by* $-b$ *;* here $-a$ is to be subtracted as often as there are units in b ; but subtracting negatives is the same as adding affirmatives, by the demonstration of the rule for subtraction; consequently the quotient is b times a, or $+ab$.

Otherwise. Since $a-a=0$, therefore $\overline{a-a}\times-b$ is also $=0$, because 0, multiplied by any quantity, is still 0 ; and since the first term of the product, or $a\times-b$, $=-ab$, by the second case ; therefore the last term of the product, or $-a\times-b$, must be $+ab$, to make the sum $=0$, or $-ab+ab=0$; that is, $-a\times-b=+ab$.

	9.	10.	11.
Multiply	$-ax$	$+5xy$	$-7xyz$
by	$-7b$	-3	$-6ax$
Product			

NOTE 1. *To multiply any power by another of the same root;* add the exponent of the multiplier to that of the multiplicand, and the sum will be the exponent of their product.

Thus, the product of a^5, multiplied into a^3, is a^{5+3}, or a^8.

That of x^n into x is x^{n+1}.

That of x^n into x^2 is x^{n+2}.

That of x^m into x^n is x^{m+n}.

And that of cy^{n+2r} into y^{n-r} is $cy^{n+2r+n-r}$, or cy^{2n+r}.

Again, the product of $\overline{a+x}|^r$, multiplied into $a+x$, is $\overline{a+x}|^{r+1}$.

And that of $\overline{x+y}|^n$ into $\overline{x+y}|^r$ is $\overline{x+y}|^{n+r}$.

This rule is equally applicable, when the exponents of any roots of the same quantity are fractional.

Thus, the product of $a^{\frac{1}{2}}$, multiplied into $a^{\frac{1}{2}}$, is $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{2}{2}} = a^1 = a$.

In like manner, $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^{\frac{3}{3}} = x^1 = x$.

Hence it appears, that, if a surd square root be multiplied into itself, the product will be rational; and if a surd cube root be multiplied into itself, and that product into the same root, the product is rational. And in general, when the sum of the numerators of the exponents is divisible by the common denominator, without a remainder, the product will be rational.

Thus, $a^{\frac{5}{4}} \times a^{\frac{3}{4}} = a^{\frac{5}{4}+\frac{3}{4}} = a^{\frac{5+3}{4}} = a^{\frac{8}{4}} = a^2$.

Here the quantity $a^{\frac{8}{4}}$ is reduced to a^2, by actually dividing 8, the numerator of the exponent, by its denominator 4; and the sum of the exponents, considered merely as vulgar fractions, is $\frac{5}{4}+\frac{3}{4}=\frac{8}{4}=2$.

When the sum of the numerators and the denominator of the exponents admit of a common divisor greater than unity, then the exponent of the product may always be reduced, like a vulgar fraction, to lower terms, retaining still the same value.

Thus, $x^{\frac{4}{9}} \times x^{\frac{2}{9}} = x^{\frac{6}{9}} = x^{\frac{2}{3}}$.

Compound surds of the same quantity are multiplied in the same manner as simple ones.

Thus, $\overline{a+x}|^{\frac{1}{2}} \times \overline{a+x}|^{\frac{1}{2}} = \overline{a+x}|^{\frac{2}{2}} = \overline{a+x}|^{1} = a+x$; $\overline{a^2+x^2}|^{\frac{2}{3}} \times \overline{a^2+x^2}|^{\frac{1}{3}} = \overline{a^2+x^2}|^{\frac{3}{3}} = a^2+x^2$.

So likewise $\sqrt[8]{a+x} \times \sqrt[8]{a+x} = \sqrt[4]{a+x} = \overline{a+x}^{\frac{1}{4}}$.

And $\sqrt[4]{a+x} \times \sqrt[4]{a+x} = \sqrt{a+x} = \overline{a+x}^{\frac{1}{2}}$.

And $\sqrt{a+x} \times \sqrt{a+x} = a+x$.

These examples show the grounds, on which the products of surds become rational.

Note 2. Different quantities under the same radical sign are multiplied together like rational quantities, only the product, if it do become rational, must stand under the same radical sign.

Thus, $\sqrt{7} \times \sqrt{3} = \sqrt{7 \times 3} = \sqrt{21}$.

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

$\sqrt[3]{7cx} \times \sqrt[3]{2y} = \sqrt[3]{14cxy}$.

And $\sqrt[n]{4d} \times \sqrt[n]{2cd} = \sqrt[n]{8cd^2} = \overline{8cd^2}|^{\frac{1}{n}}$.

It may not be improper to observe, that unequal surds have sometimes a rational product.

As $\sqrt{32} \times \sqrt{2} = \sqrt{64} = 8$.

$\sqrt{12ab} \times \sqrt{3ab} = \sqrt{36a^2b^2} = 6ab$.

$\sqrt[3]{x^2y} \times \sqrt[3]{xy^2} = \sqrt[3]{x^3y^3} = xy$.

And $\sqrt[n]{\overline{a+x}|^{n-r}} + \sqrt[n]{\overline{a+x}|^{r}} = \sqrt[n]{\overline{a+x}|^{n-r+r}} = \sqrt[n]{\overline{a+x}|^{n}} = a+x$.

CASE II.

When one of the factors is a compound quantity.

RULE.

Multiply every term of the multiplicand by the multiplier.

EXAMPLES.

	1.	2.
Multiply	$5a+bc$	$3\sqrt{ab}-4b^2+7c\sqrt{ax}$
by	$3c$	$2b$
Product	$15ac+3bc^2$	$6b\sqrt{ab}-8b^3+14bc\sqrt{ax}$

3.

Multiply	$2x-4\sqrt{n}+5d\sqrt{x^2-y^2}-6b\sqrt{c}$
by	$2a\sqrt{c}$
Product	$6ax\sqrt{c}-8a\sqrt{cn}+10ad\sqrt{cx^2-cy^2}-12abc.$

	4.	5.
Multiply	$3y-8+2xy$	$2x^2+x$
by	xy	$2xy$
Product		

	6.	7.
Multiply	$12x^2-4y^2$	$2y^2-8x^2-yx$
by	$-2x^2$	$3xy^2$
Product		

CASE III.

When both the factors are compound quantities.

RULE.

Multiply each term of the multiplicand by each term of the multiplier; then add all the products together, and the sum will be the product required.

EXAMPLES.

	1.	2.
Multiply	$a+b$	$a+b$
by	$a+b$	$a-b$
	a^2+ab	a^2+ab
	$+ab+b^2$	$-ab-b^2$
Product	$a^2+2ab+b^2$ *	a^2 * $-b^2$

* In the first example we multiply $a+b$, the multiplicand, into a, the first term of the multiplier, and the product is a^2+ab; then we multiply the multiplicand into b, the second term of the multiplier, and the product is $ab+b^2$. The sum of these two products is $a^2+2ab+b^2$, as above, and is the square of $a+b$. In the first example, the like terms of the product, viz. ab and ab together make $2ab$; but in the second example, the terms $+ab$ and $-ab$, having contrary signs, destroy each other, and the product is a^2-b^2, the difference of the squares of a and b. Hence it appears, that the sum and difference of two quantities, multiplied together, produce the difference of their squares. And by the next following example you may observe, that the square of the difference of two quantities, as a and b, is equal to $a^2-2ab+b^2$, the sum of their squares minus twice their product.

3.

Multiply $a-b$
by $a-b$

$$a^2-ab$$
$$-ab+b^2$$

Product $a^2-2ab+b^2$

4.

Multiply $a-b$
by $c-d$

$$ac-bc$$
$$-ad+bd$$

Product $ac-bc-ad+bd$

5.

Multiply $\sqrt{a}+\sqrt{b}$
by $\sqrt{a}+\sqrt{b}$

$$a+\sqrt{ab}$$
$$+\sqrt{ab}+b$$

Product $a+2\sqrt{ab}+b$

6.

Multiply $\sqrt{a}+\sqrt{b}$
by $\sqrt{a}-\sqrt{b}$

$$a+\sqrt{ab}$$
$$-\sqrt{ab}-b$$

Product $a \quad * \quad -b$

7.

Multiply $7x-4$
by $2y-3$

$$14xy-8y$$
$$-21x+12$$

Product $14xy-21x-8y+12$

8.

Multiply $x^2+10xy+7$
by $x^2-6xy+4$

$$x^4+10x^3y+7x^2$$
$$-6x^3y-60x^2y^2-42xy$$
$$+4x^2 \qquad +40xy+28$$

Product $x^4+4x^3y-60x^2y^2+11x^2-2xy+28.$

9. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.

Ans. x^4-y^4.

10. Multiply x^2+xy+y^2 by x^2-xy+y^2.

Ans. $x^4+x^2y^2+y^4$.

11. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$.

Ans. $3x^4+4x^3y-4x^2-4x^2y^2+16xy-15$.

12. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.

Ans. $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

DIVISION.

Division in Algebra, as well as in Arithmetic, is the converse of multiplication, and is performed by beginning at the left and dividing all the parts of the dividend by the divisor, when it can be done; or by setting them down like a vulgar fraction, the dividend over the divisor, and then reducing the fraction to its lowest terms.

In division the rule for the signs is the same as in multiplication, viz. if the signs of the divisor and dividend be alike, that is, both + or both —, then the sign of the quotient must be +; but if they be unlike, the sign of the quotient must be —.*

* Because the divisor, multiplied by the quotient, must produce the dividend. Therefore,

1. *When both the terms are* +; the quotient must be +, because + in the divisor × + in the quotient produces + in the dividend.

2. *When the terms are both* —; the quotient is also +, because — in the divisor × + in the quotient produces — in the dividend.

3. *When one term is + and the other* —; the quotient must be —, because + in the divisor × — in the quotient produces — in the dividend; or — in the divisor × — in the quotient gives + in the dividend.

So that the rule is general; like signs give +, and unlike signs give —, in the quotient.

CASE I.

When the divisor and dividend are both simple quantities.

RULE.

1. Place the dividend above a line, and the divisor under it, in the form of a vulgar fraction.

2. Expunge those letters, that are common to the dividend and divisor, and divide the coefficients of all the terms by any number, that will divide them without a remainder, and the result will be the quotient required.

EXAMPLES.

	1.	2.	3.
Divide	$18x^2$	$-12ab$	abc
by	$9x$	3	bcd
Quotient	$\frac{18x^2}{9x}=2x$	$-\frac{12ab}{3}=-4ab$	$\frac{abc}{bcd}=\frac{a}{d}$

	4.	5.	6.
Divide	ab	-15	$7abcx^6$
by	$2b$	$3a$	$5ax^4$
Quotient	$\frac{ab}{2b}=\frac{a}{2}=\frac{1}{2}a.$	$\frac{15}{3a}=\frac{5}{a}$	$\frac{7abcx^6}{5ax^4}=\frac{7bcx^2}{5}$

7. Divide $16x^2$ by $8x$. Ans. $2x$.
8. Divide $12a^2x^2$ by $3a^2x$. Ans. $4x$.
9. Divide $-15ay^2$ by $3ay$. Ans. $5y$.
10. Divide $-18ax^2y$ by $-8axz$. Ans. $\frac{9xy}{4z}$.

It may not be amiss to observe, that when any quantity is divided by itself, the quotient will be unity, or 1; because any thing contains itself once: thus $x \div x$ gives 1, and $\sqrt{2ab}$ divided by $\sqrt{2ab}$ gives 1.

NOTE 1. *To divide any power by another of the same root;* subtract the exponent of the divisor from that of the dividend, and the remainder will be the exponent of the quotient.

Thus, the quotient of a^8 divided by a^3 is a^{8-3}, or a^5.

That of x^n by x is x^{n-1}.

That of x^n by x^2 is x^{n-2}.

That of x^{m+n} by x^n is x^m.

And that of x^n by x^r is x^{n-r}.

But it is to be observed, that when the exponent of the divisor is greater than that of the dividend, the quotient will have a negative exponent.

Thus, the quotient of x^5 divided by x^7 is x^{5-7}, or x^{-2}.

And that of ax^2 by x^5 is ax^{-3}.

And these quotients, viz. x^{-2} and ax^{-3}, are respectively equal to $\frac{1}{x^2}$ and $\frac{a}{x^3}$; for x^5 being actually divided by x^7 gives $\frac{x^5}{x^7}=\frac{1}{x^2}$; and ax^2 divided by x^5 gives $\frac{ax^2}{x^5}=\frac{a}{x^3}$, as above.

In like manner, ax^n divided by cx^{2n} gives $\frac{ax^n}{cx^{2n}}=\frac{a}{cx^n}$.

And the quotient of $\overline{a^2+x^2}|^m$ divided by $\overline{a^2+x^2}|^n$ is $\overline{a^2+x^2}|^{m-n}$.

Moreover, $a^{\frac{3}{2}}$ divided by $a^{\frac{1}{2}}$ gives $a^{\frac{3-1}{2}}=a^{\frac{2}{2}}=a$.

$\overline{a+x}|^{\frac{5}{9}}$ divided by $\overline{a+x}|^{\frac{2}{9}}$ gives $\overline{a+x}|^{\frac{5-2}{9}}=\overline{a+x}|^{\frac{3}{9}}=\overline{a+x}|^{\frac{1}{3}}$.

And $\overline{ab+x^2}|^{\frac{m}{n}}$ divided by $\overline{ab+x^2}|^{\frac{r}{n}}$ gives $\overline{ab+x^2}|^{\frac{m-r}{n}}$.

SCHOLIUM.

When fractional exponents of the powers of the same root have not the same denominator, they may be brought to a common denominator, like vulgar fractions, and then their numerators may be added, or subtracted, as before.

Thus, the quotient of $\overline{ac+x}|^{\frac{1}{2}}$ divided by $\overline{ac+x}|^{\frac{1}{4}}$ is $\overline{ac+x}|^{\frac{2}{4}-\frac{1}{4}}=\overline{ac+x}|^{\frac{2-1}{4}}=\overline{ac+x}|^{\frac{1}{4}}$.

NOTE 2. Surd quantities under the same radical sign are divided, one by the other, like rational quantities, only the quotient, if it do not become rational, must stand under the same radical sign.

Thus, the quotient of $\sqrt{21}$ divided by $\sqrt{3}$ is $\sqrt{7}$.

That of $\sqrt{ab}$ by $\sqrt{a}$ is $\sqrt{b}$.

That of $\sqrt[3]{16c}$ by $\sqrt[3]{2c}$ is $\sqrt[3]{8}$, or 2.

That of $\sqrt[n]{\overline{ax}|^{r}}$ by $\sqrt[n]{\overline{ax}|^{r}}$ is 1.

And that of $\overline{12a^2x^3y^5}|^{\frac{m}{n}}$ by $\overline{3a^2x^2y^3}|^{\frac{m}{n}}$ is $\overline{4xy^2}|^{\frac{m}{n}}$.

CASE II.

When the divisor is a simple quantity and the dividend a compound quantity.

RULE.

Divide every term of the dividend by the divisor, as in the first case.

EXAMPLES.

1. $3c)15ac+3bc(5a+b$ quotient.

2. $4ab)8ab\sqrt{x}-12a^3b^2+4ab(2\sqrt{x}-3a^2b+1$ quotient.

3. Divide $3x^2-15+6x+3a$ by $3x$.

Ans. $x-\frac{5}{x}+2+\frac{a}{x}$.

4. Divide $3abc+12abx-9a^2b$ by $3ab$.

Ans. $c+4x-3a$.

5. Divide $10a^2x-15x^2-5x$ by $5x$.

Ans. $2a^2-3x-1$.

CASE III.

When the divisor and dividend are both compound quantities.

RULE.

1. Range the terms according to the powers of some letter in both of them, placing the highest power of it first, and the rest in order.

2. Divide the first term of the dividend by the first term of the divisor, and place the result in the quotient.

3. Multiply the whole divisor by the quotient term, and subtract the product from the dividend.

4. To the remainder bring down as many terms of the dividend as are requisite for the next operation; call the sum a *dividual*, and divide as before; and so on, as in Arithmetic.

EXAMPLES.

1. Let it be required to divide $a^3-3a^2x-3ax^2+x^3$ by $a+x$.

$$a+x)a^3-3a^2x-3ax^2+x^3(a^2-4ax+x^2 *$$
$$a^3+a^2x$$

* The process may be explained thus.

First, a^3 divided by a gives a^2 for the first term of the quotient, by which we multiply the whole divisor, viz. $a+x$, and

$$-4a^2x-3ax^2 \text{ first dividual.}$$
$$-4a^2x-4ax^2$$

$$+ax^2+x^3 \text{ second dividual.}$$
$$+ax^2+x^3$$

$$* \quad *$$

2. Divide $a^5+a^4x-a^3x^2-7a^2x^3-6x^5$ by a^2-x^2.

$$a^2-x^2)a^5+a^4x-a^3x^2-7a^2x^3-6x^5(a^3+a^2x-6x^3.$$
$$a^5 \quad -a^3x^2$$

$$+a^4x \quad -7a^2x^3$$
$$+a^4x \quad -a^2x^3$$

$$-6a^2x^3-6x^5$$
$$-6a^2x^3-6x^5$$

$$* \quad *$$

3. Divide $a-b$ by $\surd a-\surd b$.

$$\surd a-\surd b)a-b(\surd a+\surd b*$$
$$a-\surd ab$$

the product is a^3+a^2x, which, being taken from the two first terms of the dividend, leaves $-4a^2x$; to this remainder we bring down $-3ax^2$, the next term of the dividend, and the sum is $-4a^2x-3ax^2$, the first dividual; now dividing $-4a^2x$, the first term of this dividual, by a, the first term of the divisor, there comes out $-4ax$, a negative quantity, which we also put in the quotient; and the whole divisor being multiplied by it, the product is $-4a^2x-4ax^3$, which being taken from the first dividual, the remainder is $+ax^2$; to which we bring down x^3, the last term of the dividend, and the sum is $+ax^2+x^3$, the second dividual; and $+ax^2$, the first term of the second dividual, divided by a, the first term of the divisor, gives x^2 for the last term of the quotient; by which we multiply the whole divisor, and the product is $+ax^2+x^3$, which being taken from the second dividual leaves nothing; and the quotient required is $a^2-4ax+x^2$.

* Here a, the first term of the dividend, being divided by

$$+\sqrt{ab}-b$$
$$+\sqrt{ab}-b$$

4. Divide $a^3-3a^2c+4ac^2-2c^3$ by $a^2-2ac+c^2$.

$$a^2-2ac+c^2)\ a^3-3a^2c+4ac^2-2c^3\left(a-c+\frac{ac^2-c^3}{a^2-2ac+c^2}\right.$$

$$a^3-2a^2c+ac^2$$

$$-a^2c+3ac^2-2c^3$$
$$-a^2c+2ac^2-\ c^3$$

Remains $+\ ac^2-\ c^3$

Here it is obvious, that the division cannot terminate without a remainder; therefore we write the divisor under the remainder with a line between them, and add the fraction to $a-c$, the other two terms, to complete the quotient.

But when the dividend does not precisely contain the divisor, then we generally express the whole quotient as a frac-

---

$\sqrt{a}$, the first term of the divisor, gives $\sqrt{a}$ for the first term of the quotient. For $a=a^1$, and $\sqrt{a}=a^{\frac{1}{2}}$, and the difference of the exponents is $1-\frac{1}{2}$, or $\frac{1}{2}$; therefore a^1, divided by $a^{\frac{1}{2}}$, gives $a^{1-\frac{1}{2}}=a^{\frac{1}{2}}=\sqrt{a}$, as above. Or it may be considered thus; ask what quantity being multiplied by $\sqrt{a}$ will give a, and the answer is $\sqrt{a}$; then the divisor being multiplied by $\sqrt{a}$, the product is $a-\sqrt{ab}$; but there being no term in the dividend, that corresponds to $-\sqrt{ab}$, the second term of this product, we subtract $a-\sqrt{ab}$ from $a-b$, the dividend, and the sign of the quantity $-\sqrt{ab}$ being changed, the remainder is $+\sqrt{ab}-b$. Now $+\sqrt{ab}$, the first term of this remainder, divided by $\sqrt{a}$, the first term of the divisor, gives $-\sqrt{b}$ for the second term of the quotient, by which we multiply the divisor, and the product, viz. $+\sqrt{ab}-b$, being subtracted from the aforesaid remainder, nothing remains; and the quotient is $\sqrt{a}+\sqrt{b}$.

tion, having reduced it to its lowest terms, or rejected the letters and factors, that are found in every term of the dividend and divisor.

5. Thus, $a^2bx+acx^2+ax^3$, divided by $adx+anx$, gives $\frac{a^2bx+acx^2+ax^3}{adx+anx}$, or $\frac{ab+cx+x^2}{d+n}$.

Here the quotient $\frac{a^2bx+acx^2+ax^3}{adx+anx}$ is reduced to $\frac{ab+cx+x^2}{d+n}$, by dividing every term of its numerator and denominator by ax.

6. And $a+ab+d^2$, divided by $a^2-ac+a^2c^2$, gives $\frac{a+ab+d^2}{a^2-ac+a^2c^2}$.

Here the quotient cannot be reduced to lower terms, because the factor a is not to be found in the term d^2.

But it is to be observed, that though a fraction cannot be reduced to lower terms by a simple divisor, yet it may sometimes be so reduced by a compound one; as will appear in the reduction of fractions.

7. Divide a^3+x^3 by $a+x$. Ans. a^2-ax+x^2.

8. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.
Ans. $a^2-2ay+y^2$.

9. Divide $6x^4-96$ by $3x-6$.
Ans. $2x^3+4x^2+8x+16$.

10. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. Ans. $a^3-3a^2x+3ax^2-x^3$.

FRACTIONS.

ALGEBRAIC FRACTIONS have the same names and rules of operation, as fractions in Arithmetic.

PROBLEM I.

To find the greatest common measure of the terms of a fraction.

RULE.

1. Range the quantities according to the dimensions of some letter, as is shown in division.

2. Divide the greater term by the less, and the last divisor by the last remainder, and so on till nothing remain; then the divisor last used will be the common measure required.

NOTE. All the letters or figures, which are common to each term of any divisor, must be rejected before such divisor is used in the operation.

EXAMPLES.

1. To find the greatest common measure of $\frac{cx+x^2}{ca^2+a^2x}$.

$$cx+x^2)ca^2+a^2x$$
$$\text{or} \quad c+x)ca^2+a^2x(a^2$$
$$ca^2+a^2x$$

Therefore the greatest common measure is $c+x$.

2. To find the greatest common measure of $\frac{x^3-b^2x}{x^2+2bx+b^2}$.

$$x^2+2bx+b^2)x^3-b^2x(x$$
$$x^3+2bx^2+b^2x$$

$$-2bx^2-2b^2x)x^2+2bx+b^2$$

$$\text{or} \qquad x+b\ )x^2+2bx+b^2(x+b$$

$$\begin{array}{l} x^2+bx \\ \hline \quad bx+b^2 \\ \quad bx+b^2 \\ \hline \end{array}$$

Therefore $x+b$ is the greatest common measure.

3. To find the greatest common measure of $\dfrac{x^2-1}{xy+y}$.

Ans. $x+1$.

4. To find the greatest common measure of $\dfrac{x^4-b^4}{x^5+b^2x^3}$.

Ans. x^2+b^2.

PROBLEM II.

To reduce a fraction to its lowest terms.

RULE.

1. Find the greatest common measure, as in the last problem.

2. Divide both the terms of the fraction by the common measure thus found, and it will be reduced to its lowest terms.

EXAMPLES.

1. Reduce $\dfrac{cx+x^2}{ca^2+a^2x}$ to its lowest terms.

$$cx+x^2)ca^2+a^2x$$

$$\text{or} \quad c+x\ )ca^2+a^2x(a^2$$

$$\begin{array}{l} ca^2+a^2x \\ \hline \end{array}$$

Therefore $c+x$ is the greatest common measure; and $c+x)\dfrac{cx+x^2}{ca^2+a^2x}(\dfrac{x}{a^2}$ is the fraction required.

2. Having $\frac{x^3-b^2x}{x^2+2bx+b^2}$ given, it is required to reduce it to its least terms.

$$x^2+2bx+b^2)\,x^3-b^2x\,(x$$
$$x^3+2bx^2+b^2x$$
$$\overline{-2bx^2-2b^2x)\,x^2+2bx+b^2}$$
$$\text{or}\quad x+b\ \ )\,x^2+2bx+b^2\,(x+b$$
$$x^2+\ bx$$
$$\overline{bx+b^2}$$
$$bx+b^2$$

Therefore $x+b$ is the greatest common measure, and $x+b)\,\frac{x^3-b^2x}{x^2+2bx+b^2}\left(\frac{x^2-bx}{x+o}\right.$ is the fraction required.

3. Reduce $\frac{x^4-b^4}{x^5+o^2x^3}$ to its lowest terms. Ans. $\frac{x^2-b^2}{x^3}$.

4. Reduce $\frac{x^2-y^2}{x^4-y^4}$ to its lowest terms. Ans. $\frac{1}{x^2+y^2}$.

5. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{a-x}$.

PROBLEM III.

To reduce a mixed quantity to an improper fraction.

RULE.

Multiply the integer by the denominator of the fraction, and to the product add the numerator; then the denominator being placed under this sum will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{5}{7}$ to an improper fraction.

$$3\frac{5}{7}=\frac{3\times 7+5}{7}=\frac{21+5}{7}=\frac{26}{7}\text{ the answer.}$$

2. Reduce $a-\frac{b}{c}$ to an improper fraction.

$$a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c} \text{ the answer.}$$

3. Reduce $x+\frac{x^2}{a}$ to an improper fraction.

$$x+\frac{x^2}{a}=\frac{x\times a+x^2}{a}=\frac{ax+x^2}{a} \text{ the answer.}$$

4. Reduce $8\frac{6}{7}$ to an improper fraction. Ans. $\frac{62}{7}$.

5. Reduce $1-\frac{2x}{a}$ to an improper fraction.

Ans. $\frac{a-2x}{a}$.

6. Reduce $x-\frac{ax+x^2}{2a}$ to an improper fraction.

7. Reduce $10+\frac{2x-8}{3x}$ to an improper fraction.

PROBLEM IV.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator for the integral part; and place the remainder, if any, over the denominator for the fractional part; the two joined together will be the mixed quantity required.

EXAMPLES.

1. To reduce $\frac{17}{5}$ to a mixed quantity.

$\frac{17}{5}=17\div5=3\frac{2}{5}$ the answer required.

2. Reduce $\frac{ax+a^2}{x}$ to a whole or mixed quantity.

$$\frac{ax+a^2}{x}=\overline{ax+a^2}\div x=a+\frac{a^2}{x} \text{ answer.}$$

3. Reduce $\frac{ab-a^2}{b}$ to a whole or mixed quantity.

$$\frac{ab-a^2}{b}=\overline{ab-a^2}\div b=a-\frac{a^2}{b} \text{ answer.}$$

4. Reduce $\frac{ay+2y^2}{a+y}$ to a whole or mixed quantity.

$$\frac{ay+2y^2}{a+y}=\overline{ay+2y^2}\div\overline{a+y}=y+\frac{y^2}{a+y} \text{ answer.}$$

5. Let $\frac{3ab-b^2}{a}$ be reduced to a whole or mixed quantity.

Ans. $3b-\frac{b^2}{a}$.

6. Let $\frac{a^2+x^2}{a-x}$ be reduced to a whole or mixed quantity.

Ans. $a+x+\frac{2x^2}{a-x}$.

7. Let $\frac{x^3-y^3}{x-y}$ be reduced to a whole or mixed quantity.

Ans. x^2+xy+y^2.

PROBLEM. V.

To reduce fractions to a common denominator.

RULE.

Multiply each numerator into all the denominators severally, except its own, for the new numerators ; and all the denominators together for the common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to a common denominator.

$$\left.\begin{array}{l} a\times c=ac \\ b\times b=b^2 \end{array}\right\} \text{ the new numerators.}$$

$b\times c=bc$ the common denominator.

Therefore $\frac{a}{b}$ and $\frac{b}{c}=\frac{ac}{bc}$ and $\frac{b^2}{bc}$ respectively the fractions required.

2. Reduce $\frac{a}{b}, \frac{b}{c}$, and $\frac{c}{d}$ to a common denominator.

$$\left.\begin{array}{l} a\times c\times d=acd \\ b\times b\times d=b^2d \\ c\times b\times c=c^2b \end{array}\right\}\text{ the numerators.}$$

$$b\times c\times d=bcd \text{ the common denominator.}$$

Therefore $\frac{a}{b}, \frac{b}{c}$, and $\frac{c}{d}=\frac{acd}{bcd}, \frac{b^2d}{bcd}$, and $\frac{c^2b}{bcd}$ respectively the fractions required.

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. Ans. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions, having a common denominator.

Ans. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}, \frac{2b}{3c}$, and d to fractions, having a common denominator.

Ans. $\frac{9cx}{6ac}, \frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

6. Reduce $\frac{3}{4}, \frac{2x}{3}$, and $a+\frac{2x}{a}$ to fractions, having a common denominator.

Ans. $\frac{9a}{12a}, \frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

PROBLEM VI.

To add fractional quantities together.

RULE.

1. Reduce the fractions to a common denominator.*

2. Add all the numerators together, and under the sum write the common denominator, and it will give the sum of the fractions required.

EXAMPLES.

1. Having $\frac{x}{2}$ and $\frac{x}{3}$ given, to find their sum.

Here $\left.\begin{matrix} x\times3=3x \\ x\times2=2x \end{matrix}\right\}$ the numerators.

And $2\times3=6$ the common denominator.

Therefore $\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}$ is the sum required.

2. Having $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ given, to find their sum.

Here $\left.\begin{matrix} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{matrix}\right\}$ the numerators.

And $b\times d\times f=bdf$ the common denominator.

Therefore $\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}$, the sum required.

3. Let $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$ be added together.

$\left.\begin{matrix} 3x^2\times c=3cx^2 \\ 2ax\times b=2abx \end{matrix}\right\}$ the numerators.

* In the addition of mixed quantities, it is best to bring the fractional parts only to a common denominator, and to affix their sum to the sum of the integers, interposing the proper sign.

And $b \times c = bc$ the common denominator.

Therefore $a - \frac{3x^2}{b} + b + \frac{2ax}{c} = a - \frac{3cx^2}{bc} + b + \frac{2abx}{bc} = a + b + \frac{2abx - 3cx^2}{bc}$ the sum required.

4. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together. Ans. $\frac{15x+2bx}{10b}$.

5. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$ together. Ans. $x + \frac{x}{12}$ or $\frac{13}{12}x$.

6. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. Ans. $\frac{19x-14}{21}$.

7. Add $x + \frac{x-2}{3}$ to $3x + \frac{2x-3}{4}$. Ans. $4x + \frac{10x-17}{12}$.

PROBLEM VII.

To subtract one fractional quantity from another.

RULE.

1. Reduce the fractions to a common denominator, as in addition.*

2. Subtract one numerator from the other, and under their difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. To find the difference of $\frac{x}{3}$ and $\frac{2x}{11}$.

Here $\left.\begin{array}{l} x \times 11 = 11x \\ 2x \times 3 = 6x \end{array}\right\}$ the numerators

---

* The same rule may be observed for mixed quantities in subtraction, as in addition.

And $3 \times 11 = 33$ the common denominator.

Therefore $\frac{11x}{33} - \frac{6x}{33} = \frac{5x}{33}$ is the difference required.

2. To find the difference of $\frac{x-a}{3b}$ and $\frac{2a-4x}{5c}$.

Here $\overline{x-a} \times 5c = 5cx - 5ac$ } the numerators.
$\overline{2a-4x} \times 3b = 6ab - 12bx$ }

And $3b \times 5c = 15bc$ the common denominator.

Then $\frac{5cx-5ac}{15bc} - \frac{6ab-12bx}{15bc} = \frac{5cx-5ac-6ab+12bx}{15bc}$ is the difference required.

3. Required the difference of $5y$ and $\frac{3y}{8}$. Ans. $\frac{37y}{8}$.

4. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$. Ans. $\frac{13x}{63}$.

5. Subtract $\frac{c}{d}$ from $\frac{x+a}{b}$. Ans. $\frac{dx+ad-bc}{bd}$.

6. Take $\frac{2x+7}{8}$ from $\frac{3x+a}{5b}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$.

7. Take $x - \frac{x-a}{c}$ from $3x + \frac{x}{b}$.

Ans. $2x + \frac{cx-bx+ab}{bc}$.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.*

Multiply the numerators together for a new numerator, and the denominators for a new denominator; and it will give the product required.

EXAMPLES.

1. Required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$ the product required.

2. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

Here $\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product required.

3. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

Here $\frac{x \times \overline{x+a}}{a \times \overline{a+c}} = \frac{x^2+ax}{a^2+ac}$ the product required.

4. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. Ans. $\frac{9ax}{2b}$.

---

* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to both, the quotients may be used instead of them.

2. When a fraction is to be multiplied by an integer, the product is found by multiplying the numerator by it; and if the integer be the same with the denominator, the numerator may be taken for the product.

3. When a fraction is to be multiplied by any quantity, it is the same thing, whether the numerator be multiplied by it, or the denominator divided by it.

5. Required the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Ans. $\frac{ab+bx}{x}$.

6. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{cb^2+bc^2}$.

7. Required the product of x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

Ans. $\frac{x^3-x}{a^2+ab}$.

PROBLEM IX.

To divide one fractional quantity by another.

RULE.*

Multiply the denominator of the divisor by the numerator of the dividend for a new numerator, and the numerator of the divisor by the denominator of the dividend for a new denominator.

Or, invert the terms of the divisor, and then multiply by it, exactly as in multiplication.

---

* 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the denominator.

2. When a fraction is to be divided by any quantity, it is the same thing, whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions first proposed.

EXAMPLES.

1. Divide $\frac{x}{3}$ by $\frac{2x}{9}$.

Here $\frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\frac{1}{2}$ is the quotient required.

2. Divide $\frac{2a}{b}$ by $\frac{4c}{d}$.

Here $\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc}$ is the quotient required.

3. Divide $\frac{x+a}{2x—2b}$ by $\frac{x+b}{5x+a}$.

$\frac{x+a}{2x—2b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{2x^2—2b^2}$ the quotient required.

4. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

$\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2 \times \overline{x+a}}{\overline{a^3+x^3} \times x} = \frac{2x}{x^2—ax+a^2}$ is the quotient required.

5. Divide $\frac{4x^2}{7}$ by $5x$. Ans. $\frac{4x}{35}$.

6. Divide $\frac{x+1}{6}$ by $\frac{2x}{3}$. Ans. $\frac{x+1}{4x}$.

7. Divide $\frac{x}{x—1}$ by $\frac{x}{2}$. Ans. $\frac{2}{x—1}$.

8. Divide $\frac{x^4—b^4}{x^2—2bx+b^2}$ by $\frac{x^2+bx}{x—b}$. Ans. $x+\frac{b^2}{x}$.

INVOLUTION.

Involution is the continual multiplication of a quantity into itself, and the products thence arising are called the *powers* of that quantity, and the quantity itself is called the *root*. Or it is the method of finding the square, cube, biquadrate, &c. of any given quantity.

RULE.*

Multiply the quantity into itself, till the quantity be taken for a factor as many times as there are units in the index, and the last product will be the power required.

Or,

Multiply the index of the quantity by the index of the power, and the result will be the power required.

EXAMPLES.

a	root	a^2	root
a^2	= square	a^4	= square
a^3	= cube	a^6	= cube
a^4	= 4th power	a^8	= 4th power
a^5	= 5th power.	a^{10}	= 5th power.

$-3a$	root	$-2ax^2$	root
$+9a^2$	= square	$+4a^2x^4$	= square
$-27a^3$	= cube	$-8a^3x^6$	= cube
$+81a^4$	= 4th power	$+16a^4x^8$	= 4th power
$-243a^5$	= 5th power.	$-32a^5x^{10}$	= 5th power.

$x+a$=root
$x+a$

---

* Any power of the product of two or more quantities is equal to the same powers of the factors, multiplied together.

And any power of a fraction is equal to the same power of the numerator, divided by the same power of the denominator.

$$\begin{array}{l} x^2+ax \\ \quad +ax+a^2 \\ \hline x^2+2ax+a^2=\text{square} \\ x+a \\ \hline x^3+2ax^2+a^2x \\ \quad +ax^2+2a^2x+a^3 \\ \hline x^3+3ax^2+3a^2x+a^3=\text{cube} \\ x+a \\ \hline x^4+3ax^3+3a^2x^2+a^3x \\ \quad +ax^3+3a^2x^2+3a^3x+a^4 \\ \hline x^4+4ax^3+6a^2x^2+4a^3x+a^4=\text{4th power.} \\ \hline \end{array}$$

The third power of x^2 is $x^{2\times 3}$, or x^6.

The fourth power of $2a^3b^2$ is $2^4\times a^{12}b^8$, or $16a^{12}b^8$.

The mth power of $a^n b$ is $a^{mn} b^m$.

The second power of $\overline{ax}|^{\frac{1}{2}}$ is $\overline{ax}|^{\frac{1}{2}\times 2}$, or $\overline{ax}|^{\frac{2}{2}}$, that is, ax.

The nth power of $ax^{\frac{1}{n}}$ is $ax^{\frac{n}{n}}$, or ax.

And the mth power of $\overline{a^2+x^2}|^{\frac{n}{3m}}$ is $\overline{a^2+x^2}|^{\frac{mn}{3m}}$, or $\overline{a^2+x^2}|^{\frac{n}{3}}$, namely, the nth power of the cube root of a^2+x^2.

NOTE. All the odd powers, raised from a negative root, are negative, and all the even powers are positive.

Thus, the second power of $-a$ is $-a\times -a=+a^2$, by the rule for the signs in multiplication.

The third power of $-a$ is $+a^2\times -a=-a^3$.

The fourth power is $-a^3\times -a=+a^4$.

The fifth power of $-a$ is $+a^4\times -a=-a^5$, &c.

EXAMPLES FOR PRACTICE.

1. Required the cube of $-8x^2y^3$. Ans. $-512x^6y^9$.

2. Required the biquadrate of $-\frac{2a^2x}{3b^2}$.

Ans. $\frac{16a^8x^4}{81b^8}$.

3. Required the 5th power of $a-x$.

Ans. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

Sir ISAAC NEWTON'S Rule

*For raising a bonomial or residual quantity to any power whatever.**

1. *To find the terms without the coefficients.* The index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unicæ or coefficients.* The first is always 1, and the second is the index of the power; and in general, if the coefficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.

NOTE. The whole number of terms will be one more than the index of the given power; and, when both terms

---

* This rule, expressed in general terms, is as follows:

$\overline{a+b}^n = a^n + n \cdot a^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3$, &c.

$\overline{a-b}^n = a^n - n \cdot a^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3$, &c.

NOTE. The sum of the coefficients, in every power, is equal to the number 2, raised to that power. Thus, $1+1=2$, for the first power; $1+2+1=4=2^2$, for the square; $1+3+3+1=8=2^3$, for the cube, or third power; and so on.

of the root are +, all the terms of the power will be + ; but if the second term be —, then all the odd terms will be +, and the even terms —.

EXAMPLES.

1. Let $a+x$ be involved to the fifth power.

The terms without the coefficients will be

$a^5, a^4x, a^3x^2, a^2x^3, ax^4, x^5$;

and the coefficients will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{5};$$

or 1, 5, 10, 10, 5, 1 ;

And therefore the 5th power is

$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.

2. Let $x-a$ be involved to the sixth power.

The terms without the coefficients will be

$x^6, x^5a, x^4a^2, x^3a^3, x^2a^4, xa^5, a^6$;

and the coefficients will be

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6};$$

or 1, 6, 15, 20, 15, 6, 1 ;

And therefore the 6th power of $x-a$ is

$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6$.

3. Find the 4th power of $x-a$.

Ans. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

4. Find the 7th power of $x+a$.

Ans. $x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7$.

EVOLUTION.

EVOLUTION is the reverse of Involution, and teaches to find the roots of any given powers.

CASE I.

To find the roots of simple quantities.

RULE.*

Extract the root of the coefficient for the numerical part, and divide the indices of the letters by the index of the power, and it will give the root required.

EXAMPLES.

1. The square root of $9x^2 = 3x^{\frac{2}{2}} = 3x$.

2. The cube root of $8x^3 = 2x^{\frac{3}{3}} = 2x$.

3. The square root of $3a^2x^6 = a^{\frac{2}{2}}x^{\frac{6}{2}}\sqrt{3} = ax^3\sqrt{3}$.

4. The cube root of $-125a^3x^6 = -5a^{\frac{3}{3}}x^{\frac{6}{3}} = -5ax^2$.

5. The biquadrate root of $16a^4x^8 = 2a^{\frac{4}{4}}x^{\frac{8}{4}} = 2ax^2$.

---

* Any even root of an affirmative quantity may be either + or —: thus, the square root of $+a^2$ is either $+a$, or $-a$; for $+a \times +a = +a^2$, and $-a \times -a = +a^2$ also.

And an odd root of any quantity will have the same sign as the quantity itself: thus, the cube root of $+a^3$ is $+a$; and the cube root of $-a^3$ is $-a$; for $+a \times +a \times +a = +a^3$; and $-a \times -a \times -a = -a^3$.

Any even root of a negative quantity is impossible; for neither $+a \times +a$, nor $-a \times -a$, can produce $-a^2$.

Any root of a product is equal to the product of the like roots of all the factors. And any root of a fraction is equal to the like root of the numerator, divided by the like root of the denominator.

CASE II.

To find the square root of a compound quantity.

RULE.

1. Range the quantities according to the dimensions of some letter, and set the root of the first term in the place of a quotient, for the first term of the root required.

2. Subtract the square of this root from the first term, and bring down the two next terms to the remainder for a dividend.

3. Divide the dividend by double the root, and set the quotient for the next term of the root.

4. Multiply the divisor and the last term of the root by that term, and subtract the product from the dividend; and so on, as in Arithmetic.

EXAMPLES.

1. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2$$
$$4a^4$$

$$4a^2+3ax)12a^3x+13a^2x^2$$
$$12a^3x+9a^2x^2$$

$$4a^2+6ax+x^2)4a^2x^2+6ax^3+x^4$$
$$4a^2x^2+6ax^3+x^4$$

$$*$$

2. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$x^4-4x^3+6x^2-4x+1(x^2-2x+1$$
$$x^4$$

$$2x^2-2x)-4x^3+6x^2$$
$$-4x^3+4x^2$$

$$2x^2-4x+1)2x^2-4x+1$$
$$2x^2-4x+1$$
$$*$$

3. Required the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$. Ans. $a^2+2ax+x^2$.

4. Required the square root of $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.

Ans. $x^2-x+\frac{1}{4}$.

5. Required the square root of a^2+x^2.

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$, &c.

CASE III.

To find the roots of powers in general.

RULE.

1. Find the root of the first term, and set it in the place of a quotient.

2. Subtract the power, and bring down the second term for a dividend.

3. Involve the root, already found, to the next inferior power, and multiply it by the index of the given power for a divisor.

4. Divide the dividend by the divisor, and the quotient will be the next term of the root.

5. Involve the whole root to the given power, and subtract it from the given quantity; then bring down the next term, and proceed as before; and so on, till all the terms of the root be found.

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2.$

a^4

$2a^2)-2a^3x$

$a^4-2a^3x+a^2x^2$

$2a^2)2a^2x^2$

$a^4-2a^3x+3a^2x^2-2ax^3+x^4.$

$*$

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64.$

$x^6+6x^5-40x^3+96x-64(x^2+2x-4$

x^6

$3x^4)6x^5$

$x^6+6x^5+12x^4+8x^3$

$3x^4)-12x^4$

$x^6+6x^5-40x^3+96x-64$

$*$

3. Required the square root of $a^2+2ab+2ac+b^2+2bc+c^2$.
Ans. $a+b+c$.

4. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
Ans. x^2-2x+1.

5. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.
Ans. $2a-3x$,

SURDS.

SURDS are quantities, which are not exact roots, being usually expressed by the powers with fractional indices, or by means of the radical sign $\sqrt{}$.

Thus, $2^{\frac{1}{2}}$, or $\sqrt{2}$, which denotes the square root of 2.

And $3^{\frac{2}{3}}$, or $\sqrt[3]{3^2}$, signifies the cube root of the square of 3; where the numerator shows the power, to which the quantity is to be raised, and the denominator its root.

PROBLEM I.

To reduce a rational quantity to the form of a surd.

RULE.

Raise the quantity to a power equivalent to that, denoted by the index of the surd; then over this new quantity place the radical sign, and it will be the form required.

EXAMPLES.

1. To reduce 3 to the form of the square root.

First $3\times3=2^3=9$; then $\sqrt{9}$ is the answer.

2. To reduce $2x^2$ to the form of the cube root.

First, $2x^2\times2x^2\times2x^2=\overline{2x^2}|^3=8x^6$;

Then $\sqrt[3]{8x^6}$, or $\overline{8x^6}|^{\frac{1}{3}}$, is the answer.

3. Reduce 5 to the form of the cube root.

Ans. $\overline{125}|^{\frac{1}{3}}$, or $\sqrt[3]{125}$.

4. Reduce $\frac{1}{2}xy$ to the form of the square root.

Ans. $\sqrt{\frac{1}{4}x^2y^2}$.

5. Reduce 2 to the form of the 5th root.

Ans. $\overline{32}|^{\frac{1}{5}}$.

PROBLEM II.

To reduce quantities of different indices to other equivalent ones, that shall have a common index.

RULE.

1. Divide the indices of the quantities by the given index, and the quotients will be the new indices for those quantities.

2. Over the said quantities with their new indices place the given index, and they will make the equivalent quantities required.

NOTE. A common index may also be found by reducing the indices of the quantities to a common denominator, and involving each of them to the power, denoted by its numerator.

EXAMPLES.

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities, having the common index $\frac{1}{2}$.

$\frac{1}{4}\div\frac{1}{2}=\frac{1}{4}\times\frac{2}{1}=\frac{2}{4}=\frac{1}{2}$ the first index.

$\frac{1}{6}\div\frac{1}{2}=\frac{1}{6}\times\frac{2}{1}=\frac{2}{6}=\frac{1}{3}$ the second index.

Therefore $\overline{15^{\frac{1}{2}}}|^{\frac{1}{2}}$ and $\overline{9^{\frac{1}{3}}}|^{\frac{1}{2}}$ are the quantities required.

2. Reduce a^2 and $x^{\frac{1}{4}}$ to the same common index $\frac{1}{3}$.

$\frac{2}{1}\div\frac{1}{3}=\frac{2}{1}\times\frac{3}{1}=\frac{6}{1}$ the first index.

$\frac{1}{4}\div\frac{1}{3}=\frac{1}{4}\times\frac{3}{1}=\frac{3}{4}$ the second index.

Therefore $\overline{a^6}|^{\frac{1}{3}}$ and $\overline{x^{\frac{3}{4}}}|^{\frac{1}{3}}$ are the quantities required.

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Ans. $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Ans. $\overline{a^4}|^{\frac{1}{8}}$ and $\overline{b^2}|^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the same radical sign.

Ans. $\sqrt[mn]{a^m}$ and $\sqrt[mn]{b^n}$.

PROBLEM III.

To reduce surds to their most simple terms.

RULE.*

Resolve the given surd into two factors, one of which shall be the greatest power of the corresponding denomination, contained in the surd, that can be one factor, and set its root before the other factor, with the proper radical sign between them.

EXAMPLES.

1. To reduce $\sqrt{48}$ to its most simple terms.

$\sqrt{48}=\sqrt{16\times3}=\sqrt{16}\times\sqrt{3}=4\times\sqrt{3}=4\sqrt{3}$ the answer.

2. Required to reduce $\sqrt[3]{108}$ to its most simple terms.

$\sqrt[3]{108}=\sqrt[3]{27\times4}=\sqrt[3]{27}\times\sqrt[3]{4}=3\times\sqrt[3]{4}=3\sqrt[3]{4}$ the answer.

3. Reduce $\sqrt{125}$ to its most simple terms.

Ans. $5\sqrt{5}$.

4. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms.

Ans. $\frac{5}{21}\sqrt{6}$.

5. Reduce $\sqrt[3]{243}$ to its most simple terms.

Ans. $3\sqrt[3]{9}$.

6. Reduce $\sqrt[3]{\frac{16}{81}}$ to its most simple terms.

Ans. $\frac{2}{3}\sqrt[3]{\frac{2}{3}}$.

7. Reduce $\sqrt{98a^2x}$ to its most simple terms.

Ans. $7a\sqrt{2x}$.

PROBLEM IV.

To add surd quantities together.

RULE.

1. Reduce such quantities, as have unlike indices, to other equivalent ones, having a common index.

* When the given surd contains no exact power, it is already in its most simple terms.

2. Reduce the fractions to a common denominator, and the quantities to their most simple terms.

3. Then, if the surd part be the same in all of them, annex it to the sum of the rational parts with the sign of multiplication, and it will give the total sum required.

But if the surd part be not the same in all the quantities, they can only be added by the signs + and —.

EXAMPLES.

1. It is required to add $\sqrt{27}$ and $\sqrt{48}$ together.

First, $\sqrt{27}=\sqrt{9\times3}=3\sqrt{3}$;

And $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$;

Then, $3\sqrt{3}+4\sqrt{3}=\overline{3+4}\times\sqrt{3}=7\sqrt{3}=$ sum required.

2. It is required to add $\sqrt[3]{500}$ and $\sqrt[3]{108}$ together.

First, $\sqrt[3]{500}=\sqrt[3]{125\times4}=5\sqrt[3]{4}$;

And $\sqrt[3]{108}=\sqrt[3]{27\times4}=3\sqrt[3]{4}$;

Then, $5\sqrt[3]{4}+3\sqrt[3]{4}=\overline{5+3}\times\sqrt[3]{4}=8\sqrt[3]{4}=$ sum required.

3. Required the sum of $\sqrt{72}$ and $\sqrt{128}$.

Ans. $14\sqrt{2}$.

4. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$.

5. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.

Ans. $\frac{19}{30}\sqrt{6}$.

6. Required the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$. Ans. $5\sqrt[3]{5}$.

7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$ Ans. $\frac{3}{4}\sqrt[3]{2}$.

PROBLEM V.

To subtract, or find the difference of surd quantities.

RULE.

Prepare the quantities as for addition, and the difference of the rational parts, annexed to the common surd, will give the difference of the surds required.

But if the quantities have no common surd, they can only be subtracted by means of the sign —.

EXAMPLES.

1. Required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

First, $\sqrt{448}=\sqrt{64\times7}=8\sqrt{7}$;

And $\sqrt{112}=\sqrt{16\times7}=4\sqrt{7}$;

Then $8\sqrt{7}-4\sqrt{7}=4\sqrt{7}$ the difference required.

2. Required to find the difference of $192^{\frac{1}{3}}$ and $24^{\frac{1}{3}}$.

First, $192^{\frac{1}{3}}=\overline{64\times3}|^{\frac{1}{3}}=4\times3^{\frac{1}{3}}$;

And $24^{\frac{1}{3}}=\overline{8\times3}|^{\frac{1}{3}}=2\times3^{\frac{1}{3}}$;

Then, $4\times3^{\frac{1}{3}}-2\times3^{\frac{1}{3}}=2\times3^{\frac{1}{3}}$ the difference required.

3. Required the difference of $2\sqrt{50}$ and $\sqrt{18}$.

Ans. $7\sqrt{2}$.

4. Required the difference of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$.

Ans. $2\times5^{\frac{1}{3}}$.

5. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{4}{45}\sqrt{15}$.

6. Required the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.

Ans. $\frac{1}{12}\sqrt[3]{18}$.

7. Find the difference of $\sqrt{08a^4x}$ and $\sqrt{20a^2x^3}$.

Ans. $\overline{4a^2-2ax}\times\sqrt{5x}$.

PROBLEM VI.

To multiply surd quantities together.

RULE.

1. Reduce the surds to the same index.

2. Multiply the rational quantities together, and the surds together.

3. Then the latter product, annexed to the former, will give the whole product required ; which must be reduced to its most simple terms.

EXAMPLES.

1. Required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here, $3\times2\times\sqrt{8}\times\sqrt{6}=6\sqrt{8\times6}=6\sqrt{48}=6\sqrt{16\times3}=6\times4\times\sqrt{3}=24\sqrt{3}$, the product required.

2. Required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

Here, $\frac{1}{2}\times\frac{3}{4}\sqrt[3]{\frac{2}{3}}\times\sqrt[3]{\frac{5}{6}}=\frac{3}{8}\times\sqrt[3]{\frac{10}{18}}=\frac{3}{8}\times\sqrt[3]{\frac{15}{27}}=\frac{3}{8}\times\frac{1}{3}\times\sqrt[3]{15}=\frac{3}{24}\sqrt[3]{15}=\frac{1}{8}\sqrt[3]{15}$, the product required.

3. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

4. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$.

Ans. $\sqrt[3]{4}$.

5. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

6. Required the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

Ans. $10\sqrt[3]{9}$.

7. Required the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$.

Ans. $\overline{a^3}|^{\frac{1}{3}}$ or a.

PROBLEM VII.

To divide one surd quantity by another.

RULE.

1. Reduce the surds to the same index.

2. Then take the quotient of the rational quantities, and to it annex the quotient of the surds, and it will give the whole quotient required; which must be reduced to its most simple terms.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$8\div2\times\sqrt{108\div6}=4\sqrt{18}=4\sqrt{9\times2}=4\times3\sqrt{2}=12\sqrt{2}$ the quotient required.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

$8\div4=2$, and $512^{\frac{1}{3}}\div2^{\frac{1}{3}}=256^{\frac{1}{3}}=4\times4^{\frac{1}{3}}$;

Therefore $2\times4\times4^{\frac{1}{3}}=8\times4^{\frac{1}{3}}=8\sqrt[3]{4}$, is the quotient required

3. Let $6\sqrt{100}$ be divided by $3\sqrt{2}$. Ans. $10\sqrt{2}$.
4. Let $4\sqrt[3]{1000}$ be divided by $2\sqrt[3]{4}$. Ans. $10\sqrt[3]{2}$.
5. Let $\frac{3}{4}\sqrt{\frac{1}{135}}$ be divided by $\frac{2}{3}\sqrt{\frac{1}{5}}$. Ans. $\frac{1}{8}\sqrt{3}$.
6. Let $\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ be divided by $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$. Ans. $\frac{25}{21}\sqrt[3]{3}$.
7. Let $\frac{2}{3}\sqrt{a}$, or $\frac{2}{3}a^{\frac{1}{2}}$, be divided by $\frac{3}{4}a^{\frac{1}{3}}$. Ans. $\frac{8}{13}a^{\frac{1}{6}}$.

PROBLEM. VIII.

To involve, or raise, surd quantities to any power.

RULE.

Multiply the index of the quantity by the index of the power, to which it is to be raised, and annex the result to the power of the rational parts, and it will give the power required.

EXAMPLES.

1. It is required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$.

First, $\overline{\frac{2}{3}}|^{2}=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$;

And $\overline{a^{\frac{1}{3}}}|^{2}=a^{\frac{1}{3}\times2}=a^{\frac{2}{3}}=\overline{a^{2}}|^{\frac{1}{3}}$;

Therefore $\overline{\frac{2}{3}a^{\frac{1}{3}}}|^{2}=\frac{4}{9}.\overline{a^{2}}|^{\frac{1}{3}}=\frac{4}{9}\sqrt[3]{a^{2}}$, the square required.

2. It is required to find the cube of $\frac{5}{7}\sqrt{7}$.

First, $\overline{\frac{5}{7}}|^{3}=\frac{5}{7}\times\frac{5}{7}\times\frac{5}{7}=\frac{125}{343}$;

$\overline{7^{\frac{1}{2}}}|^{3}=7^{\frac{3}{2}}=\overline{7^{3}}|^{\frac{1}{2}}$;

Therefore $\overline{\frac{5}{7}\sqrt{7}}|^{3}=\frac{125}{343}.\overline{7^{3}}|^{\frac{1}{2}}=\frac{125}{343}.\overline{343}|^{\frac{1}{2}}$, the cube required.

3. Required the square of $3\sqrt[3]{3}$. Ans. $9\sqrt[3]{9}$.

4. Required the cube of $2^{\frac{1}{2}}$, or $\sqrt{2}$. Ans. $2\sqrt{2}$.

5. Required the 4th power of $\frac{1}{6}\sqrt{6}$. Ans. $\frac{1}{36}$.

6. It is required to find the nth power of $a^{\frac{1}{m}}$.

Ans. $\overline{a^n}|^{\frac{1}{m}}$.

PROBLEM IX.

To extract the roots of surd quantities.

RULE.*

Divide the index of the given quantity by the index of the root to be extracted; then annex the result to the root of the rational part, and it will give the root required.

EXAMPLES.

1. It is required to find the square root of $9\sqrt[3]{3}$.

First, $\sqrt{9}=3$;

And $\overline{3^{\frac{1}{3}}}|^{\frac{1}{2}} = 3^{\frac{1}{3}\div 2} = 3^{\frac{1}{6}}$;

Therefore $\overline{9\sqrt[3]{3}}|^{\frac{1}{2}} = 3\times 3^{\frac{1}{5}}$ is the square root required.

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

First $\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$;

---

* The square root of a binomial or residual surd, $A+B$, or $A-B$, may be found thus: take $\sqrt{A - B^2} = D$;

Then $\sqrt{A+B} = \sqrt{\frac{A+D}{2}} + \sqrt{\frac{A-D}{2}}$,

And $\sqrt{A-B} = \sqrt{\frac{A+D}{2}} - \sqrt{\frac{A-D}{2}}$.

Thus, the square root of $8+2\sqrt{7} = 1+\sqrt{7}$;

And the square root of $3-\sqrt{8} = \sqrt{2}-1$;

But for the cube, or any higher root, no general rule is given.

And $\overline{\sqrt{2}|}^{\frac{1}{3}} = 2^{\frac{1}{2} \div 3} = 2^{\frac{1}{6}}$;

Therefore $\frac{1}{8}\overline{\sqrt{2}|}^{\frac{1}{3}} = \frac{1}{2} \times 2^{\frac{1}{6}}$ is the cube root required.

3. Required the square root of 10^3. Ans. $10\sqrt{10}$.

4. Required the cube root of $\frac{8}{27}a^3$. Ans. $\frac{2}{3}a$.

5. Required the 4th root of $3x^2$. Ans. $3^{\frac{1}{4}} \times x^{\frac{1}{2}}$.

INFINITE SERIES.

An infinite series is formed from a fraction, having a compound denominator, or by extracting the root of a surd quantity; and is such as, being continued, would run on infinitely, in the manner of some decimal fractions.

But by obtaining a few of the first terms, the law of the progression will be manifest, so that the series may be continued without the continuance of the operation, by which the first terms are found.

PROBLEM I.

To reduce fractional quantities to infinite series.

RULE.

Divide the numerator by the denominator; and the operation, continued as far as may be thought necessary, will give the series required.

EXAMPLES.

1. Reduce $\frac{1}{1-x}$ to an infinite series.

$1-x)\ 1\ (1+x+x^2+x^3+x^4+$, &c. $=\frac{1}{1-x}$, and is the answer.

$$
\begin{array}{l}
1-x \\
\hline
+x \\
+x-x^2 \\
\hline
\quad +x^2 \\
\quad +x^2-x^3 \\
\hline
\qquad +x^3 \\
\qquad +x^3-x^4 \\
\hline
\qquad\quad +x^4 \\
\qquad\quad +x^4-x^5 \\
\hline
\qquad\qquad +x^5, \&c.
\end{array}
$$

Here it is easy to see how the succeeding terms of the quotient may be obtained without any farther division. This law of the series being discovered, the series may be continued to any required extent by the application of it.

2. Reduce $\frac{1}{1+x}$ to an infinite series.

$\frac{1}{1+x}=1-x+x^2-x^3+x^4-$, &c. the answer.

Here the exponent of x also increases continually by 1 from the second term of the quotient; but the signs of the terms are alternately + and —.

3. Reduce $\frac{c}{a+x}$ to an infinite series.

$a+x)\ c\ (\frac{c}{a}-\frac{cx}{a^2}+\frac{cx^2}{a^3}-\frac{cx^3}{a^4}+$, &c.*$=\frac{c}{a+x}$, and is the answer.

* Here we divide c by a, the first term of the divisor, and the quotient is $\frac{c}{a}$, by which we multiply $a+x$, the whole divisor, and the product is $\frac{ac}{a}+\frac{cx}{a}$ or $c+\frac{cx}{a}$, which being subtracted

4. Reduce $\frac{c}{a-x}$ to an infinite series.

Ans. $\frac{c}{a} \times : 1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}$, &c.

---

from the dividend c, there remains $-\frac{cx}{a}$; this remainder, being divided by a, the first term of the divisor, gives $\frac{cx}{a^2}$ for the second term of the quotient, by which we also multiply $a+x$, the divisor, and the product is $-\frac{acx}{a^2}-\frac{cx^2}{a^3}$, or $-\frac{cx}{a}-\frac{cx^2}{a}$, which, being taken from $-\frac{cx}{a}$, leaves $+\frac{cx^2}{a^2}$.

The rest of the quotient is found in the same manner; and four terms being obtained, as above, the law of continuation becomes obvious; but a few of the first terms of the series are generally near enough the truth for most purposes.

And in order to have a true series, the greatest term of the divisor, and of the dividend, if it consist of more than one term, must always stand first.

Thus in the last example; if x be greater than a, then x must be the first term of the divisor, and the quotient will be $\frac{c}{x+a}=$ $\frac{c}{x}-\frac{ac}{x^2}+\frac{a^2c}{x^3}-\frac{a^3c}{x^4}+$, &c. the true series; but if x be less than a, then this series is false, and the farther it is continued, the more it will diverge from the truth.

For let $a=2$, $c=1$, and $x=1$; then if the division be performed with a, as the first term of the divisor, you will have $\frac{c}{a+x}=$ $\frac{1}{2+1}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+$, &c. $=\frac{1}{3}$.

5. Reduce $\frac{1+x}{1-x}$ to an infinite series.

Ans. $1+2x+2x^2+2x^3+2x^4$, &c.

6. Reduce $\frac{a^2}{\overline{a+x}|^2}$ to an infinite series.

Ans. $1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}$, &c.

But if x be placed first in the divisor, then will $\frac{c}{x+a}=\frac{1}{3}=$

$\frac{1}{1+2}=1-2+4-8+16-$, &c.

Now it is obvious, that the first series continually converges to the truth; for the first term thereof, viz. $\frac{1}{2}$, exceeds the truth by $\frac{1}{2}-\frac{1}{3}$, or $\frac{1}{6}$; two terms are deficient by $\frac{1}{12}$; three terms exceed it by $\frac{1}{24}$; four terms are deficient by $\frac{1}{48}$; five terms will exceed the truth by $\frac{1}{96}$, &c. So that each succeeding term of the series brings the quotient continually nearer and nearer to the truth by one half of its last preceding difference; and consequently the series will approximate to the truth nearer than any assigned number or quantity whatever; and it will converge so much the swifter, as the divisor is greater than the dividend.

But the second series perpetually diverges from the truth; for the first term of the quotient exceeds the truth by $1-\frac{1}{3}$, or $\frac{2}{3}$; two terms thereof are deficient by $\frac{4}{3}$; three terms exceed it by $\frac{8}{3}$; four terms are deficient by $\frac{16}{3}$; five terms exceed the truth by $\frac{32}{3}$, &c. which show the absurdity of this series. For the same reason x must be less than unity in the second example; if x were there equal to unity, then the quotient would be alternately 1, and nothing, instead of $\frac{1}{2}$; and it is evident, that x is less than unity in the first example, otherwise the quotient would not have been affirmative; for if x be greater than unity, then $1-x$, the divisor, is negative, and unlike signs in division give negative quotients. From the whole of which it appears, that the greatest term of the divisor must always stand first.

7. Reduce $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ to an infinite series.

Ans. $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$, &c.

PROBLEM II.

To reduce a compound surd to an infinite series.

RULE.*

Extract the root as in Arithmetic, and the operation, continued as far as may be thought necessary, will give the series required.

EXAMPLES.

1. Required the square root of a^2+x^2 in an infinite series.

$a^2+x^2(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$, &c. †

---

* This rule is chiefly of use in extracting the square root; the operation being too tedious, when it is applied to the higher powers.

† Here the square root of the first term, a^2, is a, the first term of the root, which, being squared and taken from the given surd a^2+x^2, leaves x^2; this remainder, divided by $2a$, twice the first term of the root, gives $\frac{x^2}{2a}$ for the second term of the root, which, added to $2a$, gives $2a+\frac{x^2}{2a}$ for the first compound divisor, which, being multiplied by $\frac{x^2}{2a}$, and the product $x^2+\frac{x^4}{4a^2}$ taken from the first remainder x^2, there remains $-\frac{x^4}{4a^2}$; this re-

$$a^2$$

$$2a+\frac{x^2}{2a})\,x^2$$

$$x^2+\frac{x^4}{4a^2}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3})-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}, \text{ \&c.}$$

---

2. Required the square root of a^2-x^2 in an infinite series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$, &c.

3. Convert $\sqrt{1+1}$ into an infinite series.

Ans. $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$, &c.

---

mainder, divided by $2a$, gives $-\frac{x^4}{8a^3}$ for the third term of the root, which must be added to the double of $a+\frac{x^2}{2a}$, the two first terms of the root, for the next compound divisor. And by proceeding thus, the series may be continued as far as is desired.

NOTE. In order to have a true series, the greatest term of the proposed surd must be always placed first.

4. Let $\sqrt{x-x^2}$ be converted into an infinite series.

Ans. $x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2}-\frac{x^{\frac{5}{2}}}{8}-\frac{x^{\frac{7}{2}}}{16}-\frac{5x^{\frac{9}{2}}}{128}$, &c.

5. Let $\sqrt[3]{1-x^3}$ be converted into an infinite series.

Ans. $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$, &c.

SIMPLE EQUATIONS.

An Equation is when two equal quantities, differently expressed, are compared together by means of the sign = placed between them.

Thus, $12-5=7$ is an equation, expressing the equality of the quantities $12-5$ and 7.

A simple equation is that, which contains only one unknown quantity, in its simple form, or not raised to any power.

Thus, $x-a+b=c$ is a simple equation, containing only the unknown quantity x.

Reduction of equations is the method of finding the value of the unknown quantity. It consists in ordering the equation so, that the unknown quantity may stand alone on one side of the equation without a coefficient, and all the rest, or the known quantities, on the other side.

RULE 1.*

Any quantity may be transposed from one side of the equation to the other, by changing its sign.

---

* These are founded on the general principle of performing equal operations on equal quantities, when it is evident, that the results must still be equal; whether by equal additions, or subtractions, or multiplications, or divisions, or roots, or powers.

Thus, if $x+3=7$, then will $x=7-3=4$.

And, if $x-4+6=8$, then will $x=8+4-6=6$.

Also, if $x-a+b=c-d$, then will $x=c-d+a-b$.

And, in like manner, if $4x-8=3x+20$, then will $4x-3x=20+8$, or $x=28$.

RULE 2.

If the unknown term be multiplied by any quantity, that quantity may be taken away by dividing all the other terms of the equation by it.

Thus, if $ax=ab-a$, then will $x=b-1$.

And if $2x+4=16$, then will $x+2=8$, and $x=8-2=6$.

In like manner, if $ax+2ba=3c^2$, then will $x+2b=\frac{3c^2}{a}$, and $x=\frac{3c^2}{a}-2b$.

RULE 3.

If the unknown term be divided by any quantity, that quantity may be taken away by multiplying all the other terms of the equation by it.

Thus, if $\frac{x}{2}=5+3$, then will $x=10+6=16$.

And, if $\frac{x}{a}=b+c-d$, then will $x=ab+ac-ad$.

In like manner, if $\frac{2x}{3}-2=6+4$, then will $2x-6=18+12$, and $2x=18+12+6=36$, or $x=\frac{36}{2}=18$.

RULE 4.

The unknown quantity in any equation may be made free from surds by transposing the rest of the terms according to the rule, and then involving each side to such a power, as is denoted by the index of the said surd.

Thus, if $\sqrt{x}-2=6$, then will $\sqrt{x}=6+2=8$, and $x=8^2=64$.

And, if $\sqrt{4x+16}=12$, then will $4x+16=144$, and $4x=144-16=128$, or $x=\frac{128}{4}=32$.

In like manner, if $\sqrt[3]{2x+3}+4=8$, then will $\sqrt[3]{2x+3}=8-4=4$,

And $2x+3=4^3=64$, and $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

RULE 5.

If that side of the equation, which contains the unknown quantity, be a complete power, it may be reduced by extracting the root of the said power from both sides of the equation.

Thus, if $x^2+6x+9=25$, then will $x+3=\sqrt{25}=5$, or $x=5-3=2$.

And, if $3x^2-9=21+3$, then will $3x^2=21+3+9=33$, and $x^2=\frac{33}{3}=11$, or $x=\sqrt{11}$.

In like manner, if $\frac{2x^2}{3}+10=20$, then will $2x^2+30=60$, and $x^2+15=30$, or $x^2=30-15=15$, or $x=\sqrt{15}$.

RULE 6.

Any analogy, or proportion, may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x : 16 :: 5 : 10$, then will $3x\times10=16\times5$, and $30x=80$, or $x=\frac{80}{30}=2\frac{2}{3}$.

And, if $\frac{2x}{3} : a :: b : c$, then will $\frac{2cx}{3}=ab$, and $2cx=3ab$, or $x=\frac{3ab}{2c}$.

In like manner, if $12-x : \frac{x}{2} :: 4 : 1$, then will $12-x=\frac{4x}{2}=2x$, and $2x+x=12$, or $x=\frac{12}{3}=4$.

RULE 7.

If any quantity be found on both sides of the equation with the same sign, it may be taken away from them both; and if every term in an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $4x+a=b+a$, then will $4x=b$, and $x=\frac{b}{4}$.

And, if $3ax+5ab=8ac$, then will $3x+5b=8c$, and $x=\frac{8c-5b}{3}$.

In like manner, if $\frac{2x}{3}-\frac{8}{3}=\frac{16}{3}-\frac{8}{3}$, then will $2x=16$, and $x=8$.

MISCELLANEOUS EXAMPLES.

1. Given $5x-15=2x+6$; to find the value of x.
   First, $5x-2x=6+15$
   Then $3x=21$
   And $x=\frac{21}{3}=7$.
2. Given $40-6x-16=120-14x$; to find x.
   First, $14x-6x=120-40+16$
   Then $8x=96$
   And, therefore, $x=\frac{96}{8}=12$.
3. Let $5ax-3b=2dx+c$ be given; to find x.
   First, $5ax-2dx=c+3b$
   Or $\overline{5a-2d}\times x=c+3b$
   And, therefore, $x=\frac{c+3b}{5a-2d}$.

4. Let $3x^2-10x=8x+x^2$ be given; to find x.

First, $3x-10=8+x$

And then $3x-x=8+10$

Therefore $2x=18$, and $x=\frac{18}{2}=9$.

5. Given $6ax^3-12abx^2=3ax^3+6ax^2$; to find x.

First, dividing the whole by $3ax^2$, we shall have

$$2x-4b=x+2$$

And then $2x-x=2+4b$

Whence $x=2+4b$.

6. Let $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$ be given; to find x.

First, $x-\frac{2x}{3}+\frac{2x}{4}=20$

And then $3x-2x+\frac{6x}{4}=60$

And $12x-8x+6x=240$

Therefore $10x=240$

And $x=\frac{240}{10}=24$.

7. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$; to find x.

First, $x-3+\frac{2x}{3}=40-x-19$

And then $3x-9+2x=120-3x-57$

And therefore $3x+2x+3x=120-57+9$

That is, $8x=72$, or $x=\frac{72}{8}=9$.

8. Let $\sqrt{\frac{2}{3}x}+5=7$ be given; to find x.

First, $\sqrt{\frac{2}{3}x}=7-5=2$

And then $\frac{2}{3}x=2^2=4$

And $2x=12$, or $x=\frac{12}{2}=6$.

9. Let $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ be given; to find x.

First, $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$

And then $x\sqrt{a^2+x^2}=a^2-x^2$

And $x^2\times\overline{a^2+x^2}=\overline{a^2-x^2}|^2=a^4-2a^2x^2+x^4$

Or $a^2x^2+x^4=a^4-2a^2x^2+x^4$

Whence $a^2x^2+2a^2x^2=a^4$

Or $3a^2x^2=a^4$

And consequently $x^2=\frac{a^4}{3a^2}$,

And $x=\sqrt{\frac{a^4}{3a^2}}=a\sqrt{\tfrac{1}{3}}$.

EXAMPLES FOR PRACTICE.

1. Given $x+18=3x-5$; to find x. Ans. $x=11\frac{1}{2}$.

2. Given $3y-a+b=cd$; to find y. Ans. $y=\frac{cd+a-b}{3}$.

3. Given $6-2x+10=20-3x-2$; to find x. Ans. $x=2$.

4. Given $3ax+\frac{a}{2}-3=bx-a$; to find x. Ans. $x=\frac{6-3a}{6a-2b}$.

5. Given $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$; to find x. Ans. $x=\frac{abcd}{bc+ac+ab}$.

6 Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$; to find x. Ans. $x=\frac{6}{7}$.

7. Given $\sqrt{12+x}=2+\sqrt{x}$; to find x. Ans. $x=4$.

8. Given $\sqrt{a^2+x^2}=\overline{b^4+x^4}|^{\frac{1}{4}}$; to find x. Ans. $x=\sqrt{\frac{b^4-a^4}{2a^2}}$.

9. Given $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$; to find x.

Ans. $x=\frac{b^2}{4a}-a$.

REDUCTION OF TWO, THREE, OR MORE, SIMPLE EQUATIONS, CONTAINING TWO, THREE, OR MORE, UNKNOWN QUANTITIES.

PROBLEM I.

To exterminate two unknown quantities, or to reduce the two simple equations, containing them, to one.

RULE 1.

1. Observe which of the unknown quantities is the least involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values thus found be made equal to each other, and there will arise a new equation with only one unknown quantity in it, whose value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} 2x+3y=23 \\ 5x-2y=10 \end{matrix}\right\}$; to find x and y.

From the first equation $x=\frac{23-3y}{2}$,

And from the second $x=\frac{10+2y}{5}$,

And consequetnly $\frac{23-3y}{2}=\frac{10+2y}{5}$,

Or $115-15y=20+4y$,
Or $19y=115-20=95$,
And $y=\frac{95}{19}=5$,

Whence $x=\frac{23-15}{2}=4.$

2. Given $\left\{\begin{matrix} x+y=a \\ x-y=b \end{matrix}\right\}$; to find x and y.

From the first equation $x=a-y$,
And from the second, $x=b+y$,
Therefore $a-y=b+y$, or $2y=a-b$,

And consequently, $y=\frac{a-b}{2}$,

And $x=a-y=a-\frac{a-b}{2}=\frac{a+b}{2}$.

3. Given $\left\{\begin{matrix} \frac{x}{2}+\frac{y}{3}=7 \\ \frac{x}{3}+\frac{y}{2}=8 \end{matrix}\right\}$; to find x and y.

From the first equation $x=14-\frac{2y}{3}$,

And from the second, $x=24-\frac{3y}{2}$,

Therefore, $14-\frac{2y}{3}=24-\frac{3y}{2}$,

And $42-2y=72-\frac{9y}{2}$,

Or $84-4y=144-9y$;
Whence $5y=144-84=60$,
And $y=\frac{60}{5}=12$,

And $x=14-\frac{2y}{3}=14-\frac{24}{3}=6.$

4. Given $4x+y=34$, and $4y+x=16$; to find x and y.

Ans. $x=8$, and $y=2$.

5. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$; to find x and y.

Ans. $x=\frac{1}{2}$, and $y=\frac{1}{3}$.

6. Given $x+y=s$, and $x^2-y^2=d$; to find x and y.

Ans. $x=\frac{s^2+d}{2s}$, and $y=\frac{s^2-d}{2s}$.

RULE 2.

1. Consider which of the unknown quantities you would first exterminate, and let its value be found in that equation, where it is least involved.

2. Substitute the value thus found for its equal in the other equation, and there will arise a new equation with only one unknown quantity, whose value may be found as before.

EXAMPLES.

1. Given $\left\{\begin{matrix} x+2y=17 \\ 3x-\ y=\ 2 \end{matrix}\right\}$; to find x and y.

From the first equation $x=17-2y$,

And this value, substituted for x in the second, gives

$\overline{17-2y}\times 3-y=2$,

Or $51-6y-y=2$, or $51-7y=2$;

That is, $7y=51-2=49$;

Whence $y=\frac{49}{7}=7$, and $x=17-2y=17-14=3$.

2. Given $\left\{\begin{matrix} x+y=13 \\ x-y=\ 3 \end{matrix}\right\}$; to find x and y.

From the first equation $x=13-y$,

And this value, being substituted for x in the second,

Gives $13-y-y=3$, or $13-2y=3$;

That is, $2y=13-3=10$,

Or $y=\frac{10}{2}=5$, and $x=13-y=13-5=8$.

3. Given $\left\{\begin{matrix} a : b :: x : y \\ x^2 + y^2 = c \end{matrix}\right\}$; to find x and y.

The first analogy, turned into an equation,

Is $bx=ay$, or $x=\frac{ay}{b}$,

And this value of x, substituted in the second, Gives $\overline{\frac{ay}{b}}\Big|^2+y^2=c$, or $\frac{a^2y^2}{b^2}+y^2=c$,

Or $a^2y^2+b^2y^2=cb^2$, or $y^2=\frac{cb^2}{a^2+b^2}$,

And therefore, $y=\overline{\frac{cb^2}{a^2+b^2}}\Big|^{\frac{1}{2}}$, and $x=\overline{\frac{ca^2}{a^2+b^2}}\Big|^{\frac{1}{2}}$.

4. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$; to find x and y.

Ans. $x=7$, and $y=14$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}$ $+27$; to find x and y. Ans. $x=60$, and $y=40$.

6. Given $a:b::x:y$, and $x^3-y^3=d$; to find x and y.

Ans. $\overline{\frac{da^3}{a^3-b^3}}\Big|^{\frac{1}{3}}=x$, and $\overline{\frac{db^3}{a^3-b^3}}\Big|^{\frac{1}{3}}=y$.

RULE 3.

Let the given equations be multiplied or divided by such numbers, or quantities, as will make the term, which contains one of the unknown quantities, to be the same in both equations; and then by adding or subtracting the equations, according as is required, there will arise a new equation with only one unknown quantity, as before.

EXAMPLES.

1. Given $\left\{\begin{matrix}3x+5y=40\\x+2y=14\end{matrix}\right\}$; to find x and y.

First, multiply the second equation by 3,
And we shall have $3x+6y=42$;
Then, from this last equation subtract the first,
And it will give $6y-5y=42-40$, or $y=2$,
And therefore, $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l}5x-3y=9\\2x+5y=16\end{array}\right\}$; to find x and y.

Let the first equation be multiplied by 2, and the second by 5,

And we shall have $10x-6y=18$
$10x+25y=80$;

And if the former of these be subtracted from the latter,
It will give $31y=62$, or $y=\frac{62}{31}=2$,

And consequently, $x=\frac{9+3y}{5}$, by the first equation,

Or $x=\frac{9+6}{5}=\frac{15}{5}=3$.

ANOTHER METHOD.

Multiply the first equation by 5, and the second by 3,
And we shall have $25x-15y=45$
$6x+15y=48$;

Now, let these equations be added together,
And it will give $31x=93$, or $x=\frac{93}{31}=3$,

And consequently, $y=\frac{16-2x}{5}$, by the second equation,

Or $y=\frac{16-6}{5}=\frac{10}{5}=2$, as before.

MISCELLANEOUS EXAMPLES.

1. Given $\frac{x+2}{3}+8y=31$, and $\frac{y+5}{4}+10x=192$; to find x and y. Ans. $x=19$, and $y=3$.

2. Given $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$; to find x and y. Ans. $x=5$, and $y=2$.

3. Given $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$; to find x and y. Ans. $x=6$, and $y=8$.

4. Given $ax+by=c$, and $dx+ey=f$; to find x and y.

Ans. $x=\frac{ce-bf}{ae-db}$, and $y=\frac{af-dc}{ae-db}$.

PROBLEM II.

To exterminate three unknown quantities, or to reduce the three simple equations, containing them, to one.

RULE.

1. Let x, y, and z, be the three unknown quantities to be exterminated.

2. Find the value of x from each of the three given equations.

3. Compare the first value of x with the second, and an equation will arise involving only y and z.

4. In like manner, compare the first value of x with the third, and another equation will arise involving only y and z.

5. Find the values of y and z from these two equations, according to the former rules, and x, y, and z will be exterminated as required.

NOTE. Any number of unknown quantities may be exterminated in nearly the same manner, but there are often much shorter methods for performing the operation, which will be best learnt from practice.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$; to find x, y, and z.

From the first equation $x=29-y-z$.

From the second $x=62-2y-3z$.

From the third $x=20-\frac{2y}{3}-\frac{z}{2}$.

Whence $29-y-z=62-2y-3z$,

And $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$;

But, from the first of these equations, $y=62-29-2z=33-2z$,

And from the second $y=27-\frac{3z}{2}$;

Therefore $33-2z=27-\frac{3z}{2}$, or $z=12$,

And $y=62-29-2z=62-29-24=9$,
And $x=29-y-z=29-12-9=8$.

2. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38\end{array}\right\}$; to find x, y, and z.

First, the given equations, cleared of fractions, become

$$12x+8y+6z=1488$$
$$20x+15y+12z=2820$$
$$30x+24y+20z=4560$$

Then, if the second of these equations be subtracted from double the first, and three times the third from five times the second, we shall have

$$4x+y=156$$
$$10x+3y=420$$

And again, if the second of these be subtracted from three times the first, it will give

$12x-10x=468-420$, or $x=\frac{48}{2}=24$;
Therefore $y=156-4x=60$,

And $z=\frac{1488-8y-12x}{6}=120$.

3. Given $x+y+z=31$, $x+y-z=25$, and $x-y-z=9$; to find x, y, and z. Ans. $x=20$, $y=8$, and $z=3$.

4. Given $x+y=a$, $x+z=b$, and $y+z=c$; to find x, y, and z.

5. Given $\left\{\begin{array}{l}ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p\end{array}\right\}$; to find x, y, and z.

A COLLECTION OF QUESTIONS, PRODUCING SIMPLE EQUATIONS.

1. To find two such numbers, that their sum shall be 40, and their difference 16.

Let x denote the less of the two numbers required,

Then will $x+16=$ the greater,

And $x+x+16=40$ by the question;

That is, $2x=40-16=24$,

Or $x=\frac{24}{2}=12=$ less number,

And $x+16=12+16=28=$ greater number required.

2. What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let x equal number required,

Then will its $\frac{1}{3}$ part be $\frac{x}{3}$, and its $\frac{1}{4}$ part $\frac{x}{4}$;

And therefore $\frac{x}{3}-\frac{x}{4}=16$ by the question,

That is, $x-\frac{3x}{4}=48$, or $4x-3x=192$;

Whence $x=192$ the number required.

3. Divide 1000l. between A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x=$B's share of the given sum,

Then will $x+72=$A's share,

And $x+172=$C's share,

And the sum of all their shares $x+x+72+x+172$,

Or $3x+244=1000$ by the question;

That is, $3x=1000-244=756$,

Or $x=\frac{756}{3}=252$l. = B's share,

And $x+72=252+72=324$l. = A's share,

And $x+172=252+172=424$l. = C's share.

252l.
324l.
424l.

1000l. the proof.

4. A prize of 1000l. is to be divided between two persons, whose shares therein are in the proportion of 7 to 9; required the share of each.

Let x equal first person's share,
Then will $1000-x$ equal second person's share,
And $x : 1000-x :: 7 : 9$, by the question,
That is, $9x=\overline{1000-x}\times 7=7000-7x$
Or $16x=7000$,

Whence $x=\frac{7000}{16}=437$l. 10s. = first share,

And $1000-x=1000-437$l. 10s. = 562l. 10s. second share.

5. The paving of a square at 2s. a yard cost as much as the inclosing of it at 5s. a yard; required the side of the square.

Let x equal side of the square sought,
Then $4x=$ yards of inclosure,
And $x^2=$ yards of pavement;
Whence $4x\times 5=20x$ equal price of inclosing,
And $x^2\times 2=2x^2$ equal price of paving.
But $2x^2=20x$ by the question,
Therefore, $x^2=10x$, and $x=10=$ length of the side required.

6. A labourer engaged to serve for 40 days upon these conditions, that for every day he worked he should receive 20d. but for every day he played, or was absent, he was to forfeit 8d.; now at the end of the time he had to receive 1l. 11s. 8d. The question is to find how many days he worked, and how many he was idle.

Let x be the number of days he worked,
Then will $40-x$ be the number of days he was idle;
Also $x\times 20=20x=$ sum earned,
And $\overline{40-x}\times 8=320-8x=$ sum forfeited,
Whence $20x-\overline{320-8x}=380$d. (=1l. 11s. 8d.) by the question, that is, $20x-320+8x=380$,
Or $28x=380+320=700$,
And $x=\frac{700}{28}=25=$ number of days he worked.

And $40-x=40-25=15=$ number of days he was idle.

7. Out of a cask of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full: how much did it hold?

Let it be supposed to have held x gallons,

Then it would have leaked $\frac{x}{3}$ gallons,

And consequently there had been taken away $21+\frac{x}{3}$ gal.

But $21+\frac{x}{3}=\frac{x}{2}$, by the question,

That is, $63+x=\frac{3x}{2}$,

Or $126+2x=3x$;

Hence $3x-2x=126$,

Or $x=126=$ number of gallons required.

8. What fraction is that, to the numerator of which if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be represented by $\frac{x}{y}$,

Then will $\frac{x+1}{y}=\frac{1}{3}$,

And $\frac{x}{y+1}=\frac{1}{4}$,

Or $3x+3=y$,

And $4x=y+1$,

Hence $4x-3x-3=y+1-y$,

That is, $x-3=1$,

Or $x=4$, and $y=3x+3=12+3=15$;

So that $\frac{4}{15}=$ fraction required.

9. A market woman bought a certain number of eggs, at two a penny, and as many at three a penny, and sold them

all again at the rate of 5 for 2d. and, by so doing, lost 4d. What number of eggs had she?

Let x = number of eggs of each sort,

Then will $\frac{x}{2}$ = price of the first sort,

And $\frac{x}{3}$ = price of the second sort.

But $5 : 2 :: 2x$ (the whole number of eggs) : $\frac{4x}{5}$;

Therefore $\frac{4x}{5}$ price of both sorts together, at 5 for 2d.

And $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$ by the question;

That is, $x+\frac{2x}{3}-\frac{8x}{5}=8$;

Or $3x+2x-\frac{24x}{5}=24$,

Or $15x+10x-24x=120$;

Whence $x=120$ = number of eggs of each sort required.

10. A can do a piece of work alone in ten days, and B in thirteen; if both be set about it together, in what time will it be finished?

Let the time sought be denoted by x,

Then 10 days : 1 work :: x days : $\frac{x}{10}$,

And 13 days : 1 work :: x days : $\frac{x}{13}$,

Hence $\frac{x}{10}$ = part done by A in x days;

And $\frac{x}{13}$ = part done by B in x days.

Consequently, $\frac{x}{10}+\frac{x}{13}=1$;

That is, $\frac{13x}{10}+x=13$, or $13x+10x=130$;

And therefore $23x=130$, or $x=\frac{130}{23}=5\frac{15}{23}$ days, the time required.

11. If one agent A alone can produce an effect e in the time a, and another agent B alone in the time b; in what time will they both together produce the same effect?

Let the time sought be denoted by x,

Then $a : e :: x : \frac{ex}{a}=$ part of the effect produced by A,

And $b : e :: x : \frac{ex}{b}=$ part of the effect produced by B,

Whence $\frac{ex}{a}+\frac{ex}{b}=e$ by the question;

Or $\frac{x}{a}+\frac{x}{b}=1$;

That is, $x+\frac{ax}{b}=a$;

Or $bx+ax=ba$;

And consequently, $x=\frac{ba}{b+a}=$ time required.

QUESTIONS FOR PRACTICE.

1. What two numbers are those, whose difference is 7, and sum 33? Ans. 13 and 20.

2. To divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15. Ans. 54 and 21.

3. In a mixture of wine and cider, $\frac{1}{2}$ of the whole plus 25 gallons was wine, $\frac{1}{3}$ part minus 5 gallons was cider; how many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

4. A bill of 120l. sterling was paid in guineas and moidores, and the number of pieces of both sorts used was just 100; how many were there of each? Ans. 50 of each.

5. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? Ans. 10 days.

6. At a certain election 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each? Ans. 233 for one, and 142 for the other.

7. There is a fish, whose tail weighs 9lb. his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? Ans. 72lb.

8. What number is that, from which if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? Ans. 65.

9. A post is one fourth in the mud, one third in the water, and 10 feet above the water; what is its whole length? Ans. 24 feet.

10. After paying away one fourth and one fifth of my money, I found 66 guineas left in my bag; what was in it at first? Ans. 120 guineas.

11. A's age is double that of B, and B's is triple that of C, and the sum of all their ages is 140; what is the age of each? Ans. A's = 84, B's = 42, and C's = 14.

12. Two persons, A and B, lay out equal sums of money in trade; A gains 126l. and B loses 87l. and A's money is now double that of B; what did each lay out? Ans. 300l.

13. A person bought a chaise, horse, and harness for 60l.; the horse came to twice the price of the harness, and the

chaise to twice the price of the horse and the harness ; what did he give for each ?

Ans. 13l. 6s. 8d. for the horse, 6l. 13s. 4d. for the harness, and 40l. for the chaise.

14. Two persons, A and B, have both the same income ; A saves one fifth of his yearly, but B, by spending 50l. per annum more than A, at the end of 4 years finds himself 100l. in debt ; what is their income ? Ans. 125l.

15. A gentleman has two horses, and a saddle worth 50l. Now if the saddle be put on the back of the first horse, it will make his value double that of the second ; but if it be put on the back of the second, it will make his value triple that of the first ; what is the value of each horse ?

Ans. One 30l. and the other 40l.

16. To divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other.

Ans. The parts are 8, 12, and 16.

17. A footman agreed to serve his master for 8l. a year, and a livery, but was turned away at the end of 7 months, and received only 2l. 13s. 4d. and his livery ; what was its value ? Ans. 4l. 16s.

18. A gentleman was desirous of giving 3d. a piece to some poor beggars, but found, that he had not money enough in his pocket by 8d. he therefore gave them each 2d. and had then 3d. remaining ; required the number of beggars.

Ans. 11.

19. A hare is 50 leaps before a grey hound, and takes 4 leaps to the grey hound's 3 ; but two of the grey hound's leaps are as much as 3 of the hare's ; how many leaps must the grey hound take to catch the hare ? Ans. 300.

20. A person at play lost $\frac{1}{4}$ of his money, and then won 3 shillings ; after which he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings ; lastly, he lost $\frac{1}{7}$ of what he then had,

and, this done, found he had but 12s. remaining ; what had he at first ? Ans. 20s.

21. To divide the number 90 into 4 such parts, that, if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall be all equal to each other.
Ans. The parts are 18, 22, 10, and 40, respectively.

22. The hour and minute hands of a clock are exactly together at 12 o'clock ; when are they next together ?
Ans. 1 hour, $5\frac{5}{11}$ min.

23. When will the hour, minute, and second hands of a clock be all together next after 12 o'clock ?
Ans. Only at 12 o'clock.

24. There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it ; A goes 5 miles a day, B 8, and C 10 ; when will they all come together again ? Ans. 73 days.

25. If A can do a piece of work alone in 10 days, and A and B together in 7 days ; in what time can B do it alone ?
Ans. $23\frac{1}{3}$ days.

26. If three agents, A, B, and C, can produce the effects *a*, *b*, *c*, in the times *e*, *f*, *g*, respectively ; in what time would they jointly produce the effect *d* ?

Ans. $\overline{\frac{e}{a}+\frac{f}{b}+\frac{g}{c}}\times d=$ time.

27. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days ; how many days will it take each person to perform the same work alone ?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

QUADRATIC EQUATIONS.

A SIMPLE QUADRATIC EQUATION is that, which involves the square of the unknown quantity only.

An AFFECTED QUADRATIC EQUATION is that, which involves the square of the unknown quantity, together with the product, that arises from multiplying it by some known quantity.

Thus, $ax^2=b$ is a simple quadratic equation,

And $ax^2+bx=c$ is an affected quadratic equation.

The rule for a simple quadratic equation has been given already.

All affected quadratic equations fall under the three following forms.

1. $x^2+ax=b$
2. $x^2-ax=b$
3. $x^2-ax=-b$.

The rule for finding the value of x, in each of these equations, is as follows.

RULE.*

1. Transpose all the terms, that involve the unknown quantity, to one side of the equation, and the known terms to the other side, and let them be ranged according to their dimensions.

---

* The square root of any quantity may be either + or —, and therefore all quadratic equations admit of two solutions. Thus, the square root of $+n^2$ is $+n$, or $-n$; for either $+n\times+n$, or $-n\times-n$ is equal to $+n^2$. So in the first form, where $x+\frac{a}{2}$ is found $=\sqrt{b+\frac{a^2}{4}}$, the root may be either $+\sqrt{b+\frac{a^2}{4}}$ or $-\sqrt{b+\frac{a^2}{4}}$, since either of them being multiplied by itself will produce $b+\frac{a^2}{4}$. And this ambiguity is expressed by wri-

2. When the square of the unknown quantity has any coefficient prefixed to it, let all the rest of the terms be divided by that coefficient.

---

ting the uncertain sign $\pm$ before $\sqrt{b+\frac{a^2}{4}}$; thus $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$.

In the first form, where $x=\pm\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$, the first value of x, viz. $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ is always affirmative; for since $\frac{a^2}{4}+b$ is greater than $\frac{a^2}{4}$, the greatest square must necessarily have the greatest square root; $\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ will, therefore, always be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and consequently $+\sqrt{b+\frac{a^2}{2}}-\frac{a}{2}$ will always be affirmative.

The second value, viz. $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$, will always be negative, because it is composed of two negative terms. Therefore, when $x^2+ax=b$, we shall have $x=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the affirmative value of x, and $x=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$ for the negative value of x.

In the second form, where $x=\pm\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$, the first value, viz. $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always affirmative, since it is composed of two affirmative tems. The second value, viz. $x=$

3. Add the square of half the coefficient of the second term to both sides of the equation, and that side, which involves the unknown quantity, will then be a complete square.

$-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$, will always be negative; for since $b+\frac{a^2}{4}$ is greater than $\frac{a^2}{4}$, the square root of $b+\frac{a^2}{4}$ ($\sqrt{b+\frac{a^2}{4}}$) will be greater than $\sqrt{\frac{a^2}{4}}$, or its equal $\frac{a}{2}$; and consequently $-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ is always a negative quantity. Therefore, when $x^2-ax=b$, we shall have $x=+\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the affirmative value of x, and $x=-\sqrt{b+\frac{a^2}{4}}+\frac{a}{2}$ for the negative value of x.

In the third form, where $x=\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$, both the values of x will be positive, supposing $\frac{a^2}{4}$ is greater than b. For the first value, viz. $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$, is evidently affirmative, being composed of two affirmative terms. The second value, viz. $x=-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$, is also affirmative; for since $\frac{a^2}{4}$ is greater than $\frac{a^2}{4}-b$, therefore $\sqrt{\frac{a^2}{4}}$ or $\frac{a}{2}$ is greater than $\sqrt{\frac{a^2}{4}-b}$, and consequently $-\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ will always be an affirmative quantity. Therefore, when $x^2-ax=-b$, we shall have $x=+\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$ for the affirmative value of x, and $-\sqrt{\frac{a^2}{4}-b}$

4. Extract the square root from both sides of the equation, and the value of the unknown quantity will be determined, as required.

NOTE 1. The square root of one side of the equation is always equal to the unknown quantity, with half the coefficient of the second term subjoined to it.

NOTE 2. All equations, wherein there are two terms involving the unknown quantity, and the index of one is just double that of the other, are solved like quadratics by completing the square.

Thus, $x^4+ax^2=b$, or $x^n+ax^{\frac{n}{2}}=b$, are the same as quadratics, and the value of the unknown quantity may be determined accordingly.

EXAMPLES.

1. Given $x^2+4x=140$; to find x.

First, $x^2+4x+4=140+4=144$ by completing the square;

$+\frac{a}{2}$ for the negative value of x.

But in this third form, if b be greater than $\frac{a^2}{4}$, the solution of the proposed question will be impossible. For, since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But if b be greater than $\frac{a^2}{4}$, then $\frac{a^2}{4}-b$ is a negative quantity; and consequently $\sqrt{\frac{a^2}{4}-b}$ is impossible, or only imaginary, when $\frac{a^2}{4}$ is less than b; and therefore in that case $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}$ is also impossible or imaginary.

Then $\sqrt{x^2+4x+4}=\sqrt{144}$ by extracting the root;
Or $x+2=12$,
And therefore $x=12-2=10$.

2. Given $x^2-6x+8=80$; to find x.
First, $x^2-6x=80-8=72$ by transposition;
Then $x^2-6x+9=7+92=81$ by completing the square;
And $x-3=\sqrt{81}=9$ by extracting the root;
Therefore $x=9+3=12$.

3. Given $2x^2+8x-20=70$; to find x.
First, $2x^2+8x=70+20=90$ by transposition.
Then $x^2+4x=45$ by dividing by 2,
And $x^2+4x+4=49$ by completing the square;
Whence $x+2=\sqrt{49}=7$ by extracting the root,
And consequently $x=7-2=5$.

4. Given $3x^2-3x+6=5\frac{1}{3}$; to find x.
Here $x^2-x+2=1\frac{7}{9}$ by dividing by 3,
And $x^2-x=1\frac{7}{9}-2$ by transposition;
Also $x^2-x+\frac{1}{4}=1\frac{7}{9}-2+\frac{1}{4}=\frac{1}{36}$ by completing the square,
And $x-\frac{1}{2}=\sqrt{\frac{1}{36}}=\frac{1}{6}$ by evolution;
Therefore $x=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$.

5. Given $\dfrac{x^2}{2}-\dfrac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$; to find x.

Here $\dfrac{x^2}{2}-\dfrac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ by transposition,

And $x^2-\dfrac{2x}{3}=44\frac{1}{3}$ by dividing by $\frac{1}{2}$;

Whence $x^2-\dfrac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=44\frac{4}{9}$ by completing the square,
And $x-\frac{1}{3}=\sqrt{44\frac{4}{9}}=6\frac{2}{3}$,
Therefore $x=6\frac{2}{3}+\frac{1}{3}=7$.

6. Given $ax^2+bx=c$; to find x.

First, $x^2+\dfrac{b}{a}x=\dfrac{c}{a}$ by division;

Then $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$ by completing the square;

And $x+\frac{b}{2a}=\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}=\sqrt{\frac{4ac+b^2}{4a}}$ by evolution;

Therefore $x=\pm\sqrt{\frac{4ac+b^2}{4a^2}}-\frac{b}{2a}$.

7. Given $ax^2-bx+c=d$; to find x.

Here, $ax^2-bx=d-c$ by transposition,

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$ by division;

Also $x^2-\frac{b}{a}x+\frac{b^2a}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}$ by completing the square;

And $x-\frac{b}{2a}=\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$ by evolution;

Therefore $x=\frac{b}{2a}\pm\sqrt{\frac{d-c}{a}+\frac{b^2}{4a^2}}$.

8. Given $x^4+2ax^2=b$; to find x.

Here, $x^4+2ax^2+a^2=b+a^2$ by completing the square.

And $x^2+a=\sqrt{b+a^2}$ by evolution;

Whence $x^2=\sqrt{b+a}-a$,

And consequently, $x=\sqrt{:\sqrt{b+a}-a}$.

9. Given $ax^n-bx^{\frac{n}{2}}-c=-d$; to find x.

First, $ax^n-bx^{\frac{n}{2}}=c-d$ by transposition,

And $x^n-\frac{b}{a}x^{\frac{n}{2}}=\frac{c-d}{a}$ by division;

Also, $x^n-\frac{b}{a}x^{\frac{n}{2}}+\frac{b^2}{4a^2}=\frac{c-d}{a}+\frac{b^2}{4a^2}$ by completing the square, And $x^{\frac{n}{2}}-\frac{b}{2a}=\sqrt{\frac{c-d}{a}+\frac{b^2}{4a^2}}$ by evolution;

Therefore, $x^{\frac{n}{2}}=\frac{b}{2a}\pm\sqrt{\frac{c-d}{a}+\frac{b^2}{4a^2}}$,

And consequently, $x=\left|\frac{b}{2a}\pm\sqrt{\frac{c-d}{a}+\frac{b^2}{4a^2}}\right|^{\frac{2}{n}}$.

EXAMPLES FOR PRACTICE.

1. Given $x^2-8x+10=19$; to find x. Ans. $x=9$.
2. Given $x^2-x-40=170$; to find x. Ans. $x=15$.
3. Given $3x^2+2x-9=76$; to find x. Ans. $x=5$.
4. Given $\frac{x^2}{2}-\frac{x}{3}+7\frac{1}{8}=20$; to find x.

   Ans. $x=5{\cdot}4039$, &c.
5. Given $x^2+x=a$; to find x.

   Ans. $x=\sqrt{a+\frac{1}{4}}-\frac{1}{2}$.
6. Given $x^2+ax-b=c$; to find x.

   Ans. $x=\sqrt{c+b+\frac{a^2}{4}}-\frac{a}{2}$.
7. Given $x^2-ax=-b$; to find x.

   Ans. $x=\pm\sqrt{\frac{a^2}{4}-b}+\frac{a}{2}$.
8. Given $\frac{ax^2}{b}-\frac{cx}{d}+\frac{e}{f}=\frac{g}{h}$; to find x.

   Ans. $x=\pm\sqrt{\frac{b^2c^2}{4a^2d^2}+\frac{bg}{ah}-\frac{be}{af}}+\frac{bc}{2ad}$.
9. Given $2x^4-x^2+104=600$; to find x.

   Ans. $x=4$.

10. Given $3x^n - 2x^{\frac{n}{2}} - \frac{h}{9} = \frac{r}{9}$; to find x.

Ans. $x = \overline{\pm\sqrt{\frac{r+h+3}{27}} + \frac{1}{3}}\Big|^{\frac{2}{n}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers, whose difference is 8 and product 240.

Let x equal the less number,

Then will $x+8$ equal the greater,

And $x \times \overline{x+8} = x^2 + 8x = 240$ by the question;

Whence $x^2 + 8x + 16 = 240 + 16 = 256$ by completing the square;

Also $x+4 = \sqrt{256} = 16$ by evolution;

And therefore $x = 16 - 4 = 12 =$ less number, and $12 + 8 = 20 =$ greater.

2. To divide the number 60 into two such parts, that their product may be 864.

Let $x =$ greater part,

Then will $60 - x =$ less,

And $x \times \overline{60 - x} = 60x - x^2 = 864$ by the question,

That is, $x^2 - 60x = -864$;

Whence $x^2 - 60x + 900 = -864 + 900 = 36$ by completing the square;

Also $x - 30 = \sqrt{36} = 6$ by extracting the root;

And therefore $x = 6 + 30 = 36 =$ greater part,

And $60 - x = 6 - 36 = 24 =$ less.

3. Given the sum of two numbers $= 10$ (a), and the sum of their squares $= 58$ (b); to find those numbers.

Let $x =$ greater of those numbers,

Then will $a - x =$ less;

And $x^2+\overline{a-x}|^2=2x^2+a^2-2ax=b$ by the question,

Or $x^2+\frac{a^2}{2}-ax=\frac{b}{2}$ by division,

Or $x^2-ax=\frac{b}{2}-\frac{a^2}{2}=\frac{b-a^2}{2}$ by transposition;

Whence $x^2-ax+\frac{a^2}{4}=\frac{b-a^2}{2}+\frac{a^2}{4}=\frac{2b-a^2}{4}$ by completing the square;

Also $x-\frac{a}{2}=\sqrt{\frac{2b-a^2}{4}}$ by extracting the root;

And therefore $x=\pm\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ greater number,

And $a-\frac{a}{2}\mp\sqrt{\frac{2b-a^2}{4}}=\mp\sqrt{\frac{2b-a^2}{4}}+\frac{a}{2}=$ less.

Hence these two theorems, being put into numbers, give 7 and 3 for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much per cent. as the cloth cost me; what was the price of the cloth?

Let $x=$ pounds the cloth cost,

Then $24-x=$ the whole gain;

But $100 : x :: x : 24-x$ by the question,

Or $x^2=100\times\overline{24-x}=2400-100x$;

That is, $x^2+100x=2400$;

Whence $x^2+100x+2500=2400+2500=4900$ by completing the square,

And $x+50=\sqrt{4900}=70$ by extracting the roots,

Consequently, $x=70-5=20=$ price of the cloth.

5. A person bought a number of oxen for 80l. and if he had bought 4 more for the same money, he would have paid 1l. less for each; how many did he buy?

Suppose he bought x oxen,

Then $\frac{80}{x}=$ price of each,

And $\frac{80}{x+4}=$ price of each, if $x+4$ had cost 80l.

But $\frac{80}{x}=\frac{80}{x+4}+1$ by the question,

Or $80=\frac{80x}{x+4}+x$,

Or $80x+320=80x+x^2+4x$,

That is, $x^2+4x=320$;

Whence $x^2+4x+4=320+4=324$ by completing the square;

And $x+2=\sqrt{324}=18$ by evolution;

Consequently $x=18-2=16=$ number of oxen required.

6. What two numbers are those, whose sum, product, and difference of their squares, are all equal to each other?

Let $x=$ greater number,

And $y=$ less;

Then $\left\{\begin{array}{l} x+y=xy \\ x+y=x^2-y^2 \end{array}\right\}$ by the question,

And $1=\frac{x^2-y^2}{x+y}=x-y$, or $x=y+1$ from the second equation;

Also $\overline{y+1}+y=\overline{y+1}\times y$ from the first equation,

Or $2y+1=y^2+y$,

That is, $y^2-y=1$,

Whence $y^2-y+\frac{1}{4}=1\frac{1}{4}$ by completing the square;

Also $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$ by evolution;

Consequently $y=\frac{\sqrt{5}}{2}+\frac{1}{2}=\frac{\sqrt{5}+1}{2}$,

And $x=y+1=\frac{\sqrt{5}+3}{2}$.

7. There are 4 numbers in arithmetical progression, whereof the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x=$ less extreme.

And $y=$ common difference;

Then x, $x+y$, $x+2y$, $x+3y$ will be the four numbers,

And $\left\{\begin{array}{l} x \times \overline{x+3y}=x^2+3xy=45 \\ \overline{x+y} \times \overline{x+2y}=x^2+3xy+2y^2=77 \end{array}\right\}$ by the question.

Whence $2y^2=77-45=32$, and $y^2=\frac{32}{2}=16$ by subtraction and division,

Or $y=\sqrt{16}=4$ by evolution,

Therefore $x^2+3xy=x^2+12x=45$ by the first equation,

Also $x^2+12x+36=45+36=81$ by completing the square,

And $x+6=\sqrt{81}=9$ by the extracting of roots,

Consequently $x=9-6=3$,

And the numbers are 3, 7, 11, and 15.

8. To find three numbers in geometrical progression, whose sum shall be 14, and the sum of their squares 84.

Let x, y, and z be the numbers sought;

Then $xz=y^2$ by the nature of proportion,

And $\left\{\begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array}\right\}$ by the question,

But $x+z=14-y$ by the second equation,

And $x^2+2xz+z^2=196-28y+y^2$ by squaring both sides,

Or $x^2+z^2+2y^2=196-28y+y^2$ by putting $2y^2$ for its equal $2xz$;

That is, $x^2+z^2+y^2=196-28y$ by subtraction,

Or $196-28y=84$ by equality;

Hence $y=\frac{196-84}{28}=4$ by transposition and division.

Again, $xz=y^2=16$, or $x=\frac{16}{z}$ by the first equation,

And $x+y+z=\frac{16}{z}+4+z=14$ by the 2d equation,

Or $16+4z+z^2=14z$, or $z^2-10z=-16$,

Whence $z^2-10z+25=25-16=9$ by completing the square;

And $z-5=\sqrt{9}=3$, or $z=3+5=8$;

Consequently $x=14-y-z=14-4-8=2$, and the numbers are 2, 4, 8.

9. The sum (s) and the product (p) of any two numbers being given; to find the sum of the squares, cubes, biquadrates, &c. of those numbers.

Let the two numbers be denoted by x and y;

Then will $\left\{\begin{matrix} x+y=s \\ xy=p \end{matrix}\right\}$ by the question,

But $\overline{x+y}|^2=x^2+2xy+y^2=s^2$ by involution,

And $x^2+2xy+y^2-2xy=s^2-2p$ by subtraction,

That is, $x^2+y^2=s^2-2p=$ sum of the squares.

Again, $\overline{x^2+y^2}\times\overline{x+y}=\overline{s^2-2p}\times s$ by multiplication,

Or $x^3+xy\times\overline{x+y}+y^3=s^3-2sp$,

Or $x^3+sp+y^3=s^3-2sp$ by substituting sp for its equal $xy\times\overline{x+y}$;

And therefore $x^3+y^3=s^3-3sp=$ sum of the cubes.

In like manner, $\overline{x^3+y^3}\times\overline{x+y}=\overline{s^3-3sp}\times s$ by multiplication,

Or $x^4+xy\times\overline{x^2+y^2}+y^4=s^4-3s^2p$,

Or $x^4+p\times\overline{s^2-2p}+y^4=s^4-3s^2p$ by substituting $p\times\overline{s^2-2p}$ for its equal $xy\times\overline{x^2+y^2}$;

And consequently, $x^4+y^4=s^4-3s^2p-p\times\overline{s^2-2p}=s^4-4s^2p+2p^2=$ sum of the biquadrates, or fourth powers; and so on, for any power whatever.

10. The sum (a) and the sum of the squares (b) of four numbers in geometrical progression being given; to find those numbers.

Let x and y denote the two means,

Then will $\frac{x^2}{y}$ and $\frac{y^2}{x}$ be the two extremes, by the nature of proportion.

Also, let the sum of the two means $=s$, and their product $=p$.

And then will the sum of the two extremes $=a-s$ by the question,

And their product $=p$ by the nature of proportion.

$$\text{Hence}\left\{\begin{array}{l} x^2+y^2=s^2-2p \\ \frac{x^4}{y^2}+\frac{y^4}{x^2}=\overline{a-s}|^2-2p \end{array}\right\}\text{ by the last problem,}$$

And $x^2+y^2+\frac{x^4}{y^2}+\frac{y^4}{x^2}=s^2+\overline{a-s}|^2-4p=b$ by the question.

Again, $\frac{x^2}{y}+\frac{y^2}{x}=a-s$ by the question,

Or $x^3+y^3=xy\times\overline{a-s}=p\times\overline{a-s}$.

But $x^3+y^3=s^3-3sp$ by the last problem,

And therefore $p\times\overline{a-s}=s^3-3sp$ by equality,

Or $pa-ps+3ps=pa+2ps=s^3$,

Or $p=\frac{s^3}{a+2s}$;

Whence $s^2+\overline{a-s}|^2-4p=s^2+\overline{a-s}|^2-\frac{4s^3}{a+2s}=b$ by substitution,

Or $s^2+\frac{b}{a}s=\frac{a^2-b}{2}$ by reduction.

And $s=\sqrt{\frac{a^2-b}{2}+\frac{b^2}{4a^2}}-\frac{b}{2a}$ by completing the square, and extracting the root.

And from this value of s all the rest of the quantities p, x, and y may be readily determined.

QUESTIONS FOR PRACTICE.

1. What two numbers are those, whose sum is 20, and their product 36? Ans 2 and 18.

2. To divide the number 60 into two such parts, that their product may be to the sum of their squares in the ratio of 2 to 5. Ans. 20 and 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117; what are those numbers?

Ans. 2 and 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was settled, two of them sneaked off, and then those, who remained, had 10s. a piece more to pay than before; how many were there in the company? Ans. 7.

5. A grazier bought as many sheep as cost him 60l. and after reserving 15 out of the number, he sold the remainder for 54l. and gained 2s. a head by them; how many sheep did he buy? Ans. 75.

6. There are two numbers, whose difference is 15, and half their product is equal to the cube of the less number; what are those numbers? Ans. 3 and 18.

7. A person bought cloth for 33l. 15s. which he sold again at 2l. 8s. per piece, and gained by the bargain as much as one piece cost him; required the number of pieces.

Ans. 15.

8. What number is that, which being divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? Ans. 24.

9. What two numbers are those, whose sum multiplied by the greater is equal to 77; and whose difference multiplied by the less is equal to 12? Ans. 4 and 7.

10. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers?

Ans. 3 and 5.

11. The sum of two numbers is 7, and the sum of their fourth powers is 641; what are the numbers?

Ans. 2 and 5.

12. The sum of two numbers is 6, and the sum of their fifth powers is 1056; what are the numbers?

Ans. 2 and 4.

13. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers?

Ans. 8, 12, 16, and 20.

14. To find four numbers in geometrical progression, whose sum is 15, and the sum of their squares 85.

Ans. 1, 2, 4, and 8.

15. Given $\left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} + \left(x^2 - \frac{a^4}{x^2}\right)^{\frac{1}{2}} = 2a$; to find the value of x.

Ans. $x = \left(\frac{1}{2}a^2 + \sqrt{\frac{5a^4}{4}}\right)^{\frac{1}{2}}$.

CUBIC AND HIGHER EQUATIONS.

A CUBIC EQUATION, or equation of the third degree or power, is one, that contains the third power of the unknown quantity: as $x^3 - ax^2 + bx = c$.

A *biquadratic*, or double quadratic, is an equation, that contains the fourth power of the unknown quantity: as $x^4 - ax^3 + bx^2 - cx = d$.

An *equation of the fifth power*, or degree, is one, that contains the fifth power of the unknown quantity: as $x^5 - ax^4 + bx^3 - cx^2 + dx = e$.

An *equation of the sixth power*, or degree, is one, that contains the sixth power of the unknown quantity: as $x^6 - ax^5 + bx^4 - cx^3 + dx^2 - ex = f$.

And so on, for all other higher powers. Where it is to be noted, however, that all the powers, or terms in the equation, are supposed to be freed from surds, or fractional exponents.

There are various particular rules for the resolution of cubic and higher equations; but they may be all easily resolved by the following rule of Double Position.

RULE.*

1. Find by trial two numbers, as near the true root as possible, and substitute them separately in the given equation, instead of the unknown quantity; marking the errors, which arise from each of them.

2 Multiply the difference of the two numbers, found by trial, by the least error, and divide the product by the difference of the errors, when they are alike, but by their sum, when they are unlike. Or say, as the difference or sum of the errors is to the difference of the two numbers, so is the least error to the correction of its supposed number.

3. Add the quotient last found to the number belonging to the least error, when that number is too little, but subtract it, when too great; and the result will give the true root *nearly*.

4. Take this root and the nearest of the two former, or any other, that may be found nearer; and, by proceeding in like manner as above, a root will be had still nearer than before; and so on, to any degree of exactness required. Each new operation commonly doubles the number of true figures in the root.

NOTE 1. It is best to employ always two assumed numbers, that shall differ from each other only by unity in the last figure on the right; because then the difference, or multiplier, is only 1.

EXAMPLES.

1. To find the root of the cubic equation $x^3+x^2+x=100$, or the value of x in it.

Here it is soon found, that x lies between 4 and 5. Assume,

---

* This rule may be used for solving the questions of Double Position, as well as that given in the Arithmetic, and is preferable for the present purpose. Its truth is easily deduced from the same supposition.

For by the supposition, $r : s :: x-a : x-b$, therefore, by division, $r-s : s :: b-a : x-b$; which is the rule.

therefore, these two numbers, and the operation will be as follows.

1st supposition.		2d supposition.
4	x	5
16	x^2	25
64	x^3	125
84	sums	155
—16	errors	+55

The sum of which is 71.

Then, as 71 : 1 :: 16 : ·225.

Hence x=4·225 nearly.

Again, suppose 4·2 and 4·3, and repeat the work as follows.

1st supposition.		2d supposition.
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	sums	102·297
—4·072	errors	+2·297

The sum of which is 6·369.

As 6·369 : ·1 :: 2·297 : 0·036

This taken from 4·300

Leaves x nearly = 4·264

Again, suppose 4·264, and 4·265, and work as follows.

1st supposition.		2d supposition.
4·264	x	4·265
18·181696	x^2	18·190225
77·526752	x^3	77·581310
99·972448	sums	100·036535
—0·027552	errors	+0·036535

The sum of which is ·064087.

Then, as ·064087 : ·001 :: 4·264 : 0·0 04299

To this adding 4·264

We have x very nearly =4·2644299

2. To find the root of the equation $x^3-15x^2+63x=50$, or the value of x in it.

Here it soon appears, that x is very little above 1.

Suppose, therefore, 1·0 and 1·1, and work as follows.

1·0	x	1·1
63·0	$63x$	69·3
—15	$-15x^2$	18·15
1	x^3	1·331
49	sums	52·481
—1	errors	+2·481

3·481 sum of the errors.

As 3·481 : ·1 :: 1 : ·029 correct.

1·00

Hence x=1·029 nearly.

Again, suppose the two numbers 1·03 and 1·02, and work as follows.

1·03	x	1·02
64·89	$63x$	64·26
—15·9135	$-15x^2$	—15·6060
1·092727	x^3	1·061208
50·069227	sums	49·715208
+·069227	errors	—·284792
—·284792		

As ·354019 : ·01 :: ·069227 :

·0019555

This taken from 1·03

Leaves x nearly = 1·02804

NOTE 2. Every equation has as many roots as it contains dimensions, or as there are units in the index of its highest power. That is, a simple equation has only one value or root; but a quadratic equation has two values or roots; a cubic equation has three roots; a biquadratic equation has four roots, and so on.

And when one of the roots of an equation has been found by approximation, as before, the rest may be found as follows:—Take for a dividend the given equation, with the known term transposed, its sign being changed, to the unknown side of the equation; and for the divisor take x minus the root just found. Divide the said dividend by the divisor, and the quotient will be the equation depressed a degree lower than the given one.

Find a root of this new equation by approximation, as before, and it will be a second root of the original equation. Then, by means of this root, depress the second equation one degree lower, and thence find a third root, and so on, till the equation be reduced to a quadratic; then the two roots of this being found, by the method of completing the square, they will make up the remainder of the roots. Thus, in the foregoing equation, having found one root to be 1·02804, connect it by minus with x for a divisor, and take the given equation with the known term transposed for a dividend: thus,

$$x-1{\cdot}02804)x^3-15x^2+63x-50(x^2-13{\cdot}97196x+48{\cdot}63627=0.$$

Then the two roots of this quadratic equation, or $x^2-13{\cdot}97196x=-48{\cdot}63627$, by completing the square, are 6·57653 and 7·39543, which are also the other two roots of the given cubic equation. So that all the three roots of that equation, viz. $x^3-15x^2+63x=50$,

are	1·02804
and	6·57653
and	7·39543
Sum	15·00000

And the sum of all the roots is found to be 15, being equal to the coefficient of the second term of the equation, which the sum of the roots always ought to be, when they are right.

NOTE 3. It is also a particular advantage of the foregoing rule, that it is not necessary to prepare the equation, as for other rules, by reducing it to the usual final form and state of equations. Because the rule may be applied at once to an unreduced equation, though it be ever so much embarrassed by surd and compound quantities. As in the following example.

3. Let it be required to find the root x of the equation $\sqrt{144x^2 - \overline{x^2+20}|^2} + \sqrt{196x^2 - \overline{x^2+24}|^2} = 114$, or the value of x in it.

By a few trials it is soon found, that the value of x is but little above 7. Suppose therefore first, that $x=7$, and then that $x=8$.

First, when $x=7$.		Second, when $x=8$.
47·906	$\sqrt{144x^2 - \overline{x^2+20}\|^2}$	46·476
65·384	$\sqrt{196x^4 - \overline{x^2+24}\|^2}$	69·283
113·290	the sums of these	115·759
114·000	the true number	114·000
—0·710	the two errors	+1·759
+1·759		

As 2·469 : 1 :: 0·710 : 0·2 nearly.
7·0

$x=7·2$ nearly.

Suppose again $x=7·2$, and, because it turns out too great, suppose also $x=7·1$.

Suppose $x=7{\cdot}2$.		Suppose $x=7{\cdot}1$.
47·990	$\sqrt{144x^2 - \overline{x^2+24}\vert^2}$	47·973
66·402	$\sqrt{196x^2 - \overline{x^2+24}\vert^2}$	65·904
114·392	the sums of these	113·877
114·000	the true number	114·000
+0·392	the errors	—0·123
0·123		
·515		

·515 : ·123 : : ·1 : ·024 the correction.
7·100

Therefore $x=7{\cdot}124$ nearly, the root required.

NOTE 4. The same rule also, among other more difficult forms of equations, succeeds very well in what are called *exponential equations*, or those, which have an unknown quantity for the exponent of the power ; as in the following example.

4. To find the value of x in the exponential equation $x^x = 100$.

For the more easy resolution of this kind of equations, it is convenient to take the logarithms of them, and then compute the terms by means of a table of logarithms. Thus, the logarithms of the two sides of the present equation are, $x\times$ log. of $x=2$, the log. of 100. Then by a few trials it is soon perceived, that the value of x is somewhere between the two numbers 3 and 4, and indeed nearly in the middle between them, but rather nearer the latter than the former. By taking therefore first $x=3{\cdot}5$, and then $x=3{\cdot}6$, and working with the logarithms, the operation will be as follows.

First, suppose $x=3{\cdot}5$.

Logarithm of 3·5 =	0·5440680
Then 3·5 × log. 3·5 =	1·904238
The true number	2·000000
Error, too little,	—·095762

Second, suppose $x = 3{\cdot}6$.

Logarithm of 3·6=	0·5563025
Then 3·6× log. 3·6 =	2·002689
The true number	2·000000
Error, too great,	+·002689

—·095762
+·002689

·098451 sum of the errors. Then,

As ·098451 : ·1 : : ·002689 : 0·00273

Which correction, taken from 3·60000

Leaves 3·59727$=x$ nearly.

On trial, this is found to be very little too small.

Take therefore again $x=3{\cdot}59727$, and next $x=3{\cdot}59728$, and repeat the operation as follows.

First, suppose $x = 3{\cdot}59727$.

Logarithm of 3·59727 is	0·5559731
3·59727× log. of 3·59727 =	1·9999854
The true number	2·0000000
Error, too little,	—0·0000146

Second, suppose $x=3{\cdot}59728$.

Logarithm of 3·59728 is	0·5559743
3·59728 × log. of 3·59728 =	1·9999953
The true number	2·0000000
Error, too little,	—0·0000047

—0·0000146
—0·0000047

0·0000099 difference of the errors. Then,

As ·0000099 : ·00001 : : ·0000047 : 0·00000474747

Which correction, added to 3·59728000000

Gives nearly the value of $x =$ 3·59728474747

5. To find the value of x in the equation $x^3+10x^2+5x=2600$.
Ans. $x=11\cdot00673$.

6. To find the value of x in the equation $x^3-2x=5$.
Ans. $2\cdot094551$.

7. To find the value of x in the equation $x^3+2x^2-23x=70$.
Ans. $x=5\cdot1349$.

8. To find the value of x in the equation $x^3-17x^2+54x=350$.
Ans. $x=14\cdot95407$.

9. To find the value of x in the equation $x^4-3x^2-75x=10000$.
Ans. $x=10\cdot2615$.

10. To find the value of x in the equation $2x^4-16x^3+40x^2-30x=-1$.
Ans. $x=1\cdot284724$.

11. To find the value of x in the equation $x^5+2x^4+3x^3+4x^2+5x=54321$.
Ans. $x=8\cdot414455$.

12. To find the value of x in the equation $x^x=123456789$.
Ans. $x=8\cdot6400268$.

GEOMETRY.*

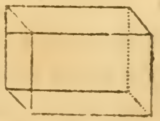
DEFINITIONS.

1. A POINT is that, which has position, but not magnitude.

2. A *line* is length, without breadth or thickness.

3. A *surface*, or *superficies*, is an extension, or a figure, of two dimensions, length and breadth, but without thickness.

4. A *body*, or *solid*, is a figure of three dimensions, namely, length, breadth, and thickness.



Hence surfaces are the extremities of solids; lines the extremities of surfaces; and points the extremities of lines.

5. Lines are either right, or curved, or mixed of these two.

---

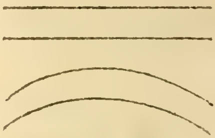
* A TUTOR teaches, in Harvard College, PLAYFAIR's "Elements of Geometry; containing the first six Books of EUCLID, with two Books on the Geometry of Solids." Of this work Mr. F. Nichols of Philadelphia has given a good American Edition.

6. A *right line*, or *straight line*, lies all in the same direction between its extremities, and is the shortest distance between two points.

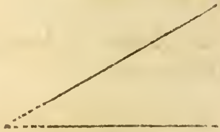
7. A *curve* continually changes its direction between its extreme points.

8. Lines are either parallel, oblique, perpendicular, or tangential.

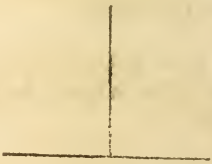
9. *Parallel lines* are always at the same distance, and never meet, though ever so far produced.



10. *Oblique right lines* change their distance, and would meet, if produced, on the side of the least distance.

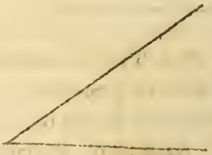


11. One line is *perpendicular* to another, when it inclines not more on one side than on the other.



12. One line is *tangential*, or *a tangent*, to another, when it touches it without cutting, if both be produced.

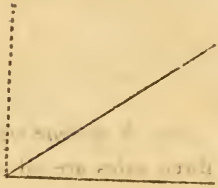
13. An *angle* is the inclination, or opening, of two lines, having different directions, and meeting in a point.



14. Angles are right or oblique, acute or obtuse.

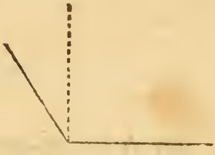
15. A *right angle* is that, which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.

16. An *oblique angle* is that, which is made by two oblique lines, and is either less or greater than a right angle.



17. An *acute angle* is less than a right angle.

18. An *obtuse angle* is greater than a right angle.



19. Superficies are either plane or curved.

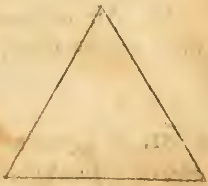
20. A *plane superficies*, or a *plane*, is that, with which a right line may, every way, coincide. But if not, it is *curved.*

21. *Plane figures* are bounded either by right lines or curves.

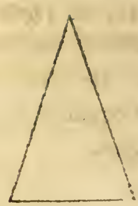
22. Plane figures, bounded by right lines, have names according to the number of their sides, or angles; for they have as many sides as angles; the least number being three.

23. A figure of three sides and angles is called a *triangle.* And it receives particular denominations from the relations of its sides and angles.

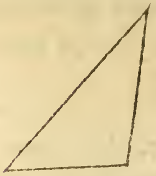
24. An *equilateral triangle* is that, whose three sides are equal.



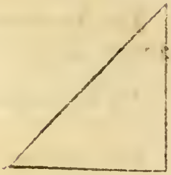
25. An *isosceles triangle* is that, which has two equal sides.



26. A *scalene triangle* is that, whose three sides are all unequal.



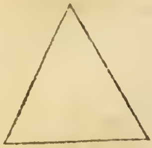
27. A *right-angled triangle* is that, which has one right angle.



28. Other triangles are *oblique-angled*, and are either obtuse-angled or acute-angled.

29. An *obtuse-angled triangle* has one obtuse angle.

30. An *acute-angled triangle* has all its three angles acute.



31. A figure of four sides and angles is called a *quadrangle*, or a *quadrilateral.*

32. A *parallelogram* is a quadrilateral, which has both pair of its opposite sides parallel. And it takes the following particular names.

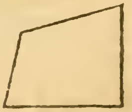
33. A *rectangle* is a parallelogram, having all its angles right.

34. A *square* is an equilateral rectangle, having all its sides equal, and all its angles right.

35. A *rhomboid* is an oblique-angled parallelogram.

36. A *rhombus* is an equilateral rhomboid, having all its sides equal, but its angles oblique.

37. A *trapezium* is a quadrilateral, which has not both pair of its opposite sides parallel.



38. A *trapezoid* has only one pair of opposite sides parallel.

39. A *diagonal* is a right line, joining any two opposite angles of a quadrilateral.

40. Plane figures, having more than four sides, are, in general, called *polygons ;* and they receive other particular names, according to the number of their sides or angles.

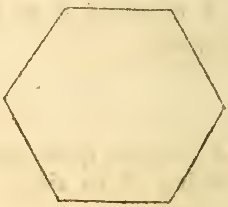
41. A *pentagon* is a polygon of five sides ; a *hexagon* has six sides ; a *heptagon*, seven ; an *octagon*, eight ; a *nonagon*, nine ; a *decagon*, ten ; an *undecagon*, eleven ; and a *dodecagon*, twelve.

42. A *regular polygon* has all its sides and all its angles equal.—If they be not both equal, the polygon is irregular.

43. An *equilateral triangle* is also a regular figure of three sides, and the *square* is one of four ; the former being also called a *trigon*, and the latter a *tetragon*.

Pentagon.

Hexagon.



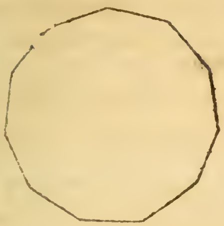
Heptagon.

Octagon.

Nonagon.

Decagon.

Undecagon.



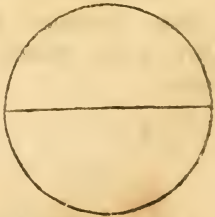
Dodecagon.

44. A *circle* is a plane figure, bounded by a curve line, called the *circumference*, which is every where equidistant from a certain point within, called the *centre*.

NOTE. The circumference itself is often called a circle.

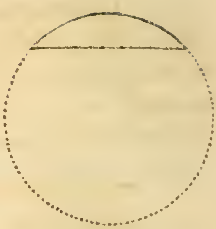
45. The *radius* of a circle is a right line, drawn from the centre to the circumference.

46. The *diameter* of a circle is a right line, drawn through the centre, and terminating in the circumference on both sides.



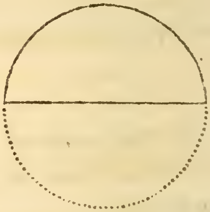
47. An *arc* of a circle is any part of the circumference.

48. A *chord* is a right line, joining the extremities of an arc.



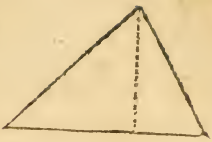
49. A *segment* is any part of a circle, bounded by an arc and its chord.

50. A *semicircle* is half the circle, or a segment cut off by a diameter.



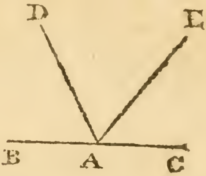
51. A *sector* is any part of a circle, bounded by an arc, and two radii, drawn to its extremities.

52. A *quadrant*, or quarter of a circle, is a sector, having a quarter of the circumference for its arc, and its two radii are perpendicular to each other.



53. The *height*, or *altitude*, of a figure is a perpendicular, let fall from an angle, or its vertex, to the opposite side, called the *base*.

54. In a right-angled triangle, the side opposite to the right angle is called the *hypotenuse;* and the other two the *legs*, or *sides*, or sometimes the *base* and *perpendicular*.



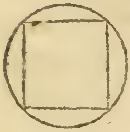
55. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that, which stands at the angular point, *is read in the middle*.

56. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees* ; and each degree into 60 *minutes*, each minute into 60 *seconds*, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

57. The measure of a right-lined angle is an arc of any circle, contained between the two lines, which form that angle, the angular point being the centre; and it is estimated by the number of degrees, contained in that arc. Hence a right angle is an angle of 90 degrees.

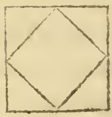
58. *Identical figures* are such, as have all the sides and all the angles of one respectively equal to all the sides and all the angles of the other, each to each ; so that, if one figure were applied to, or laid upon, the other, all the sides of it would exactly fall upon and cover all the sides of the other ; the two becoming coincident.

59. An *angle in a segment* is that, which is contained by two lines, drawn from any point in the arc of the segment to the extremities of the arc.



60. A *right-lined figure is inscribed in a circle*, or *the circle circumscribes it*, when all the angular points of the figure are in the circumference of the circle.

61. A *right-lined figure circumscribes a circle*, or *the circle is inscribed in it*, when all the sides of the figure touch the circumference of the circle.



62. *One right-lined figure is inscribed in another*, or *the latter circumscribes the former*, when all the angular points of the former are placed in the sides of the latter.

63. *Similar figures* are those, that have all the angles of one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

64. *The perimeter of a figure* is the sum of all its sides, taken together.

65. A *proposition* is something, which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

66. A *problem* is something proposed to be done.

67. A *theorem* is something proposed to be demonstrated.

68. A *lemma* is something, which is premised or previously demonstrated, in order to render what follows more easy.

69. A *corollary* is a consequent truth, gained immediately from some preceding truth, or demonstration.

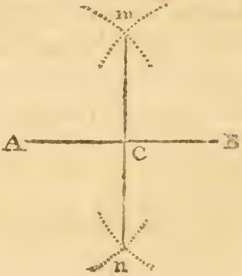
70. A *scholium* is a remark, or observation, made upon something preceding it.

PROBLEMS.

PROBLEM. I.

To divide a given line A B *into two equal parts.*

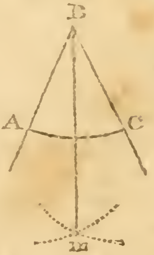
From the centres A and B, with any radius greater than half A B, describe arcs, cutting each other in *m* and *n*. Draw the line *m* C *n*, and it will cut the given line into two equal parts in the middle point C.*



PROBLEM II.

To divide a given angle A B C *into two equal parts.*

From the centre B, with any radius, describe the arc A C. From A and C, with one and the same radius, describe arcs, intersecting in *m*. Draw the line B *m*, and it will bisect the angle, as required.†



* Suppose right lines to be drawn from A to *m*, *m* to B, B to *n*, and *n* to A; then A *m* B *n* is a parrallelogram, and its diagonals A B, *m n*, mutually bisect each other. Therefore A B is bisected in C.

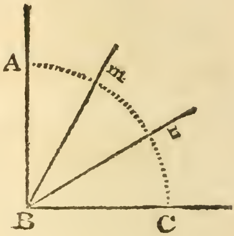
† Suppose right lines to be drawn from A to *m* and from C to

NOTE. By this operation the arc A C is bisected; and in a similar manner may any given arc of a circle be bisected.

PROBLEM III.

To divide a right angle A B C *into three equal parts.*

From the centre B, with any radius, describe the arc A C. From the centre A, with the same radius, cross the arc A C in *n*; and with the centre C, and the same radius, cut the arc A C in *m*. Then through the points *m* and *n* draw B *m* and B *n*, and they will trisect the angle, as required.*



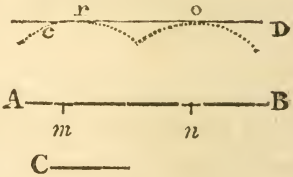
PROBLEM IV.

To draw a line parallel to a given line AB.

CASE I.

When the parallel line is to be at a given distance C.

From any two points *m* and *n*, in the line AB, with a radius equal to C, describe the arcs *r* and *o*. Draw C D to touch these arcs, without cutting them, and it will be the parallel required.†



---

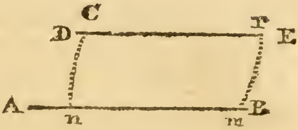
m; then the sides of the triangle B A *m* are respectively equal to the sides of the triangle B C *m*. Therefore the angle A B *m* = the angle C B *m*.

* The number of degrees in a right angle is 90; and the radius of a circle being equal to the chord of 60 degrees, the arc C*m* = A*n* = 60 degrees. Therefore *mn* = 30 degrees = A*m* = *n*C = $\frac{1}{3}$ of a right angle.

† For the points, where the right line CD touches the arcs

CASE 2.

When the parallel line is to pass through a given point C.



From any point m, in the line AB, with the radius mC, describe the arc Cn. From the centre C, with the same radius, describe the arc mr. Take the arc Cn in the compasses, and apply it from m to r. Through C and r draw DE, the parallel required.*

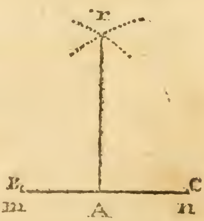
NOTE. In practice, parallel lines are more easily drawn with a Parallel Rule.

PROBLEM V.

To erect a perpendicular from a given point A *in a given line* BC.

CASE I.

When the point is near the middle of the line.



On each side of the point A, take any two equal distances Am, An. From the centres m and n, with any radius greater than Am or An, describe two arcs, intersecting in r. Through A and r draw the line Ar, and it will be the perpendicular required.†

---

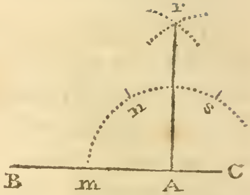
r and o, are equally distant from the line AB. Therefore all the other points, through which CD passes, are equally distant from AB, that is, CD is parallel to AB.

* For if Cm be joined by a right line, it is evident, that the angle nmC = the angle m Cr. Therefore DE is parallel to AB.

† Suppose right lines drawn from r to m, and r to n; then the

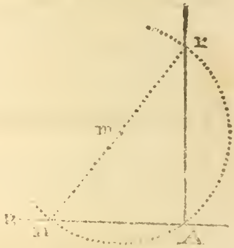
CASE 2.

When the point is near the end of the line.



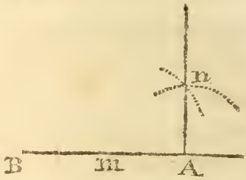
With the centre A, and any radius, describe the arc *m n s*. From the point *m*, with the same radius, turn the compasses twice over on the arc, at *n* and *s*. Again, with the centres *n* and *s*, describe arcs, intersecting in *r*. Then draw A*r*, and it will be the perpendicular required.*

ANOTHER METHOD.



From any point *m*, as a centre, with the radius or distance *m* A, describe an arc, cutting the given line in *n* and A. Through *n* and *m* draw a right line cutting the arc in *r*. Lastly, draw A *r*, and it will be the perpendicular required.†

ANOTHER METHOD.



From any plane scale of equal parts, set off A*m* equal to 4 parts. With centre A, and radius of three parts, describe an arc. And with centre *m*, and radius of 5 parts, cross it at *n*. Draw A*n* for the perpendicular required.

---

sides of the triangle *mr*A are respectively equal to the sides of the triangle *n r* A. Therefore the angles at A are equal to each other, and *r* A is perpendicular to BC.

* Right lines being drawn from A to *n*, *n* to *r*, *r* to *s*, and *s* to A, the sides A*n*, *nr*, of the triangle A*nr*, are respectively equal to the sides A*s*, *sr*, of the triangle A*sr*, and A*r* is common to both triangles. Consequently the angle *n* A*r* = the angle *s* A*r*. And the angle *m* A*n* = the angle CA*s*. Therefore the angles at A are right angles, and A*r* is perpendicular to BC

† For the angle BA*r*, being in a semicircle, is a right angle, or A *r* is perpendicular to BA.

Or any other numbers in the same proportion, as 3, 4, 5, will answer the same purpose.*

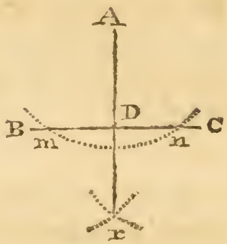
PROBLEM VI.

From a given point A, *out of a given line* BC, *to let fall a perpendicular.*

CASE I.

When the point is nearly opposite the middle of the line.

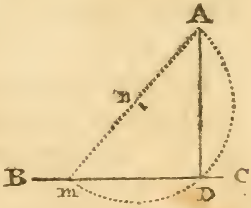
With the centre A, and any radius, describe an arc, cutting BC in *m* and *n*. With the centres *m* and *n*, and the same, or any other radius, describe arcs, intersecting in *r*. Draw AD*r* for the perpendicular required†



CASE 2.

When the point is nearly opposite the end of the line.

From A draw any line A *m* to meet BC, in any point *m*. Bisect A *m* at *n*, and with the centre *n*, and radius A *n* or *m n*, describe an arc, cutting BC in D. Draw AD, the perpendicular required.§



---

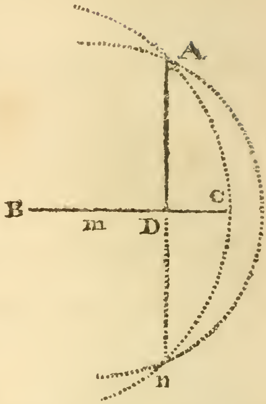
* If a right line be drawn from *m* to *n*; then is A *m n* a right-angled triangle. For the square of *mn* is equal to the square of *m* A, added to the square of A *n*. Therefore A *n* is perpendicular to AB.

† For if right lines be drawn from *m* to A, and A to *n*; then, the angle *m* A *n* being bisected by AD, according to Problem 11, in the triangle *m* AD, the side *m* A and angle *m* AD are respectively equal to the side A *n* and angle *n* AD in the triangle *n* AD, and AD is common to both. Therefore the angles at D are equal, and AD is perpendicular to BC.

§ BDA, being an angle in a semicircle, is a right angle, or AD is perpendicular to BD.

ANOTHER METHOD.

From B or any point in BC, as a centre, describe an arc through the point A. From any other centre *m* in B C, describe another arc through A, cutting the former arc again in *n*. Through A and *n* draw the line A D *n* ; and A D will be the perpendicular required.*

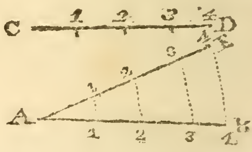


NOTE. Perpendiculars may be more readily raised and let fall, in practice, by means of a square or other fit instrument.

PROBLEM VII.

To divide a given line A B *in the same proportion, as another line* C D *is divided.*

From A draw any line A E equal to C D, and upon it transfer the divisions of the line C D. Join B E, and parallel to it draw the lines 1 1, 2 2, 3 3, &c. and they will divide A B, as required.†



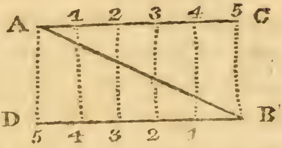
* Right lines being drawn from B to A, *m* to A, B to *n*, and *m* to *n* ; then the sides of the triangle B A *m* being respectively equal to the sides of the triangle B *n m*, the angles of the former are respectively equal to the corresponding angles of the latter ; consequently in the triangle A *m* D the angle A *m* D is equal to the angle D *m n* in the triangle D *m n* ; A *m* is equal to *m n*, and *m* D is common to the two triangles. Therefore the angles at D are right angles, and A D is perpendicular to B D.

† For A 1 1, A 2 2, A 3 3, &c. are similar triangles ; there-

PROBLEM VIII.

To divide a given line A B *into any proposed number of equal parts.*

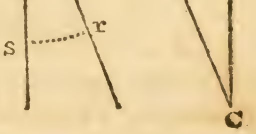
From A draw any line A C at random, and from B draw B D parallel to it. On each of these lines, beginning at A and B, set off as many equal parts, of any length, as A B is to be divided into. Join the opposite points of division by the lines A 5, 1 4, 2 3, &c. and they will divide A B as required.*



PROBLEM IX.

At a given point A, *in a given line* A B, *to make an angle, equal to a given angle* C.

With the centre C, and any radius, describe an arc *mn*.—With centre A, and the same radius, describe the arc *rs*.—Take the distance *mn* in the compasses, and apply it from *r* to *s*. Then a line, drawn through A and *s*, will make the angle A equal to the angle C, as required.

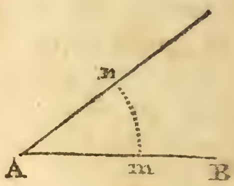


fore as A 1 on A E to A 1 on A B, so is A 2 on A E to A 2 on A B, so is A 3 on A E to A 3 on A B, &c. so is 1 2 on A E to 1 2 on A B, so is 2 3 on A E to 2 3 on A B, &c. and as A 1 to 1 2 on A E, so is A 1 to 1 2 on A B, &c. Therefore A B is divided in the same ratio as A E or C D.

* By the Note under the last Problem A B is divided in the same proportion as A C or B D, and into the same number

PROBLEM X.

At a given point A, *in a given line* A B, *to make an angle of any proposed number of degrees.*



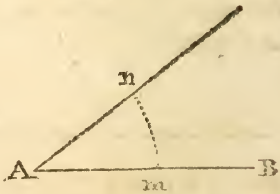
With the centre A, and radius equal to 60 degrees, taken from a scale of chords, describe an arc cutting A B in *m*. Then take in the compasses the proposed number of degrees from the same scale of chords, and apply them from *m* to *n*. Through the point *n* draw A *n*, and it will make the angle A of the number of degrees proposed.

Or the angle may be made with any divided arc, or instrument, by applying the centre to the point A, and its radius along A B; then make a mark *n*, at the proposed number of degrees, through which draw the line A *n* as before.

NOTE. Angles of more than 90 degrees are usually taken off at twice.

PROBLEM XI.

To measure a given angle A.



Describe the arc *m n* with the chord of 60 degrees, as in the last Problem. Take the arc *m n* in the compasses, and that extent, applied to the chords, will show the degrees in the given angle.

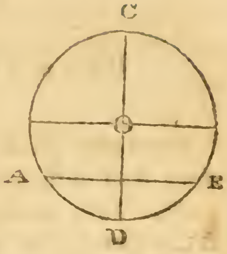
---

of parts; but A B and C D are each divided into the required number of equal parts. Therefore A B is divided into the required number of equal parts.

PROBLEM XII.

To find the centre of a circle.

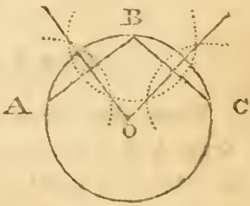
Draw any chord A B; and bisect it perpendicularly with C D, which will be a diameter. Bisect C D in the point O, which will be the centre.*



PROBLEM XIII.

To describe the circumference of a circle through three given points A, B, C, *which are not in a right line.*

From the middle point B draw right lines to the other two points. Bisect these right lines perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, to the distance O A, or O B, or O C, describe the circle.†



Note. In the same manner may the centre of an arc of a circle be found.

---

* As the right line C D bisects the right line A B in the given circle perpendicularly, it passes through the centre, or is a diameter, and the centre is the middle point of it.

† For, by the Note under the last Problem, the centre must be in each of the bisecting lines. Therefore it must be in the point of their intersection.

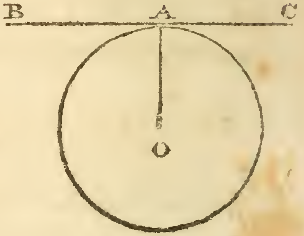
PROBLEM XIV.

Through a given point A *to draw a tangent to a given circle.*

CASE I.

When A *is in the circumference of the circle.*

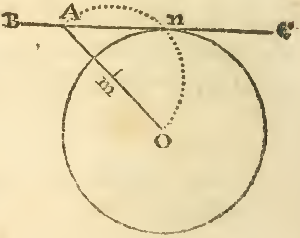
From the given point A, draw A O to the centre of the circle. Then through A draw B C perpendicular to A O, and it will be the tangent required.*



CASE 2.

When A *is out of the circumference.*

From the given point A draw A O to the centre, which bisect in the point *m*. With the centre *m*, and radius *m* A or *m* O, describe an arc, cutting the given circle in *n*. Through the points A and *n* draw the tangent B C.†



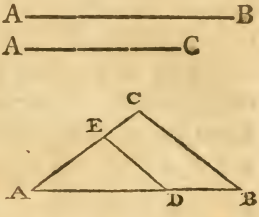
---

* For a tangent is perpendicular to a radius, drawn to the point of contact.

† For if a right line be drawn from *n* to O, the angle A *n* O, being an angle in a semicircle, is a right angle. Therefore A *n* is a tangent to the circle in the point *n*.

PROBLEM XV.

To find a third proportional to two given Lines AB, AC.

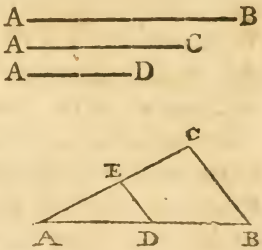


Place the two given lines, AB, AC, making any angle at A, and join BC. In AB take AD equal to AC, and draw DE parallel to BC. So shall AE be the third proportional to AB and AC.

That is, AB : AC : : AC = AD : AE.*

PROBLEM XVI.

To find a fourth proportional to three given lines AB, AC, AD.



Place two of them, AB, AC, so as to make any angle at A, and join B C. Place AD on AB, and draw DE parallel to BC. So shall AE be the fourth proportional required.

That is, AB : AC : : AD : AE.†

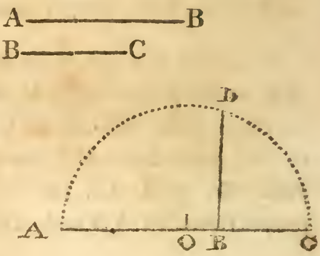
---

* For ADE and ABC are similar triangles, and consequently their like or corresponding sides are proportional.

† ABC and ADE being similar triangles, their like sides are proportional.

PROBLEM XVII.

To find a mean proportional between two given lines AB, BC.

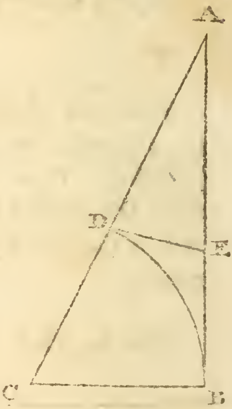


Join AB and BC in one straight line AC, and bisect it in the point O. With the centre O, and radius OA or OC, describe a semicircle. Erect the perpendicular BD, and it will be the mean proportional required.

That is, AB : BD :: BD : BC.*

PROBLEM XVIII.

To divide a line AB *in extreme and mean ratio.*



Raise BC perpendicular to AB, and equal to half AB. Join AC. With centre C, and radius CB, cross AC in D. Lastly, with centre A, and radius AD, cross AB in E, which will divide the line AB in extreme and mean ratio, namely, so that the whole line is to the greater part, as the greater part is to the less part.

That is, AB : AE :: AE : EB.†

* Right lines being drawn from A to D, and D to C; then ABD, and BCD, are similar triangles, and their like sides proportional.

† $AB^2 + BC^2 = AC^2$;
but $AC^2 = AD^2 + 2\,AD \cdot DC + DC^2$, and $AD^2 = AE^2$,
$2\,AD \cdot DC = AE \cdot AB$, and $DC^2 = CB^2 = \frac{1}{4}AB^2$;
hence $AB^2 + \frac{1}{4}AB^2 = AE^2 + AE \cdot AB + \frac{1}{4}AB^2$;

PROBLEM XIX.

To inscribe an isosceles triangle in a given circle, that shall have each of the angles at the base double the angle at the vertex.

Draw any diameter AB of the given circle; and divide the radius CB, in the point D, in extreme and mean ratio, by the last Problem. From the point B apply the chords BE, BF, each equal to CD; then join AE, AF, EF, and AEF will be the triangle required.*

---

and $AB^2 = AE^2 + AE . AB$;
but $AE = AB - BE$,
hence $AB^2 = AE^2 + AB^2 - AB . BE$;
∵ $AE^2 = AB \ BE$,
and AB : AE :: AE : BE.

* As BE = BF, the angle BAE = the angle BAF, and AE = AF.

∵ AEF is an isosceles triangle.

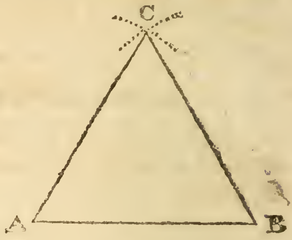
Suppose right lines to be drawn from C to E and D to E.

Then, CB : CD :: CD : DB,
and CB : BE :: BE : BD;
therefore the triangles BDE, BCE, are similar,
and BE = ED = DC;
Hence the angle ABE = the angle EDB.

But if G represent the intersection of AB and EF, the triangles ABE and AEG are similar, and the angle ABE = the angle AEF;
consequently the angle AEF = the angle EDB;
but EDB = ECD + CED = 2 ECD,
and ECD = EAB + AEC = 2EAB,
and 2ECD = 4EAB;
∵ AEF = 4EAB = 2EAF = AFE.

PROBLEM XX.

To make an equilateral triangle on a given line AB.

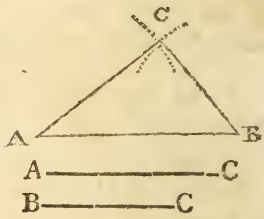


From the centres A and B, with the radius AB, describe arcs, intersecting in C. Draw AC and BC, and it is done.

NOTE. An isosceles triangle may be made in the same manner, by taking for the radius the given length of one of the equal sides.

PROBLEM XXI.

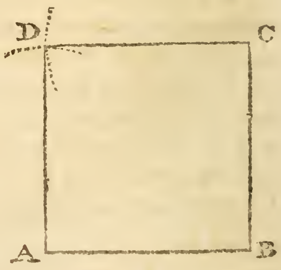
To make a triangle with three given lines AB, AC, BC.



With the centre A and radius A C, describe an arc. With the centre B and radius B C, describe another arc, cutting the former in C. Draw A C and B C, and ABC is the triangle required.

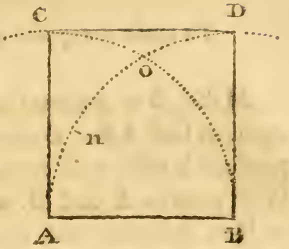
PROBLEM XXII.

To make a square upon a given line AB.



Draw BC perpendicular and equal to AB. From A and C, with the radius AB, describe arcs, intersecting in D. Draw AD and CD, and it is done.

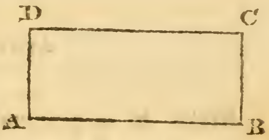
ANOTHER WAY.



On the centres A and B, with the radius A B, describe arcs crossing at O. Bisect A O in *n*. With centre O, and radius O *n*, cross the two arcs in C and D. Then draw AC, BD, CD.*

PROBLEM XXIII.

To describe a rectangle, or a parallelogram, of a given length and breadth.



Place BC perpendicular to AB. With centre A, and radius BC, describe an arc. With centre C, and radius AB, describe another arc, cutting the former in D. Draw AD and CD, and it is done.

B——C

NOTE. In the same manner is described any oblique parallelogram, except in drawing BC so as to make the given oblique angle with AB, instead of a right one.

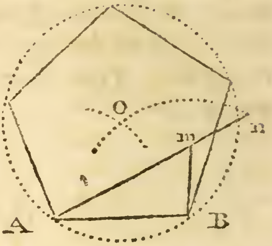
---

* If right lines be drawn from A to O, and B to O, ABO is an equilateral triangle, each angle of which is $\frac{2}{3}$ of a right angle. But the arc AO measures the angle ABO; therefore A *n* = *n* O = O D = $\frac{1}{3}$ of 90° = 30°, and the angle ABD is a right angle. For the same reason BAC is a right angle. AB = BD = AC. Because BC = AD, CD is parallel to AB, and, being included between the same parallels, CD is also equal to AB. Therefore ABCD is a square.

PROBLEM XXIV.

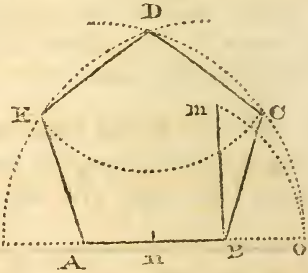
To make a regular pentagon on a given line AB.

Make B *m* perpendicular and equal to half AB. Draw A*m*, and produce it till *m n* be equal to B *m*. With centres A and B, and radius B *n*, describe arcs intersecting in O, which will be the centre of the circumscribing circle. Then with the centre O, and the same radius, describe the circle; and about the circumference of it apply AB the proper number of times.



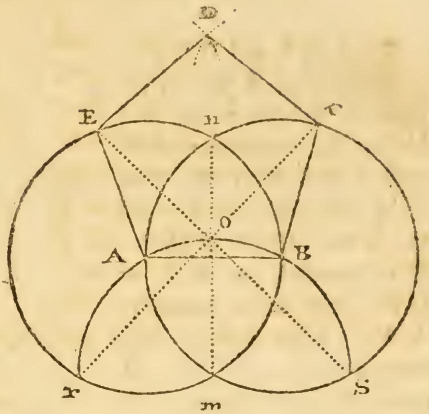
ANOTHER METHOD.

Make B *m* perpendicular and equal to A B. Bisect A B in *n*; then with the centre *n*, and radius *n m*, cross A B produced in O. With the centres A and B, and radius A O, describe arcs intersecting in D, the opposite angle of the pentagon. Lastly, with the centre D, and radius A B, cross those arcs again in C and E, the other two angles of the figure. Then draw the lines from angle to angle, to complete the figure.



A THIRD METHOD, NEARLY TRUE.

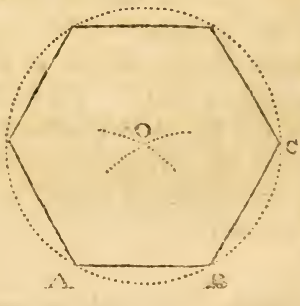
On the centres A and B, with the radius A B, describe two circles intersecting in *m* and *n*. With the same radius, and the centre *m*, describe *r* AO B S, and draw *m n* cutting it in O. Draw *r* O C and S O E, which will give two angles of the pentagon. Lastly, with radius A B, and centres C and E, describe arcs intersecting in D, the other angle of the pentagon nearly.



PROBLEM XXV.

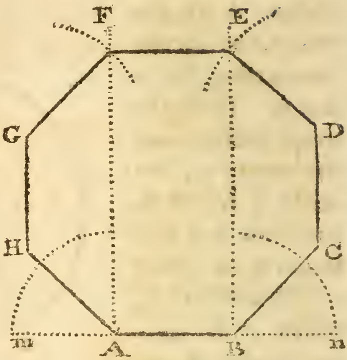
To make a hexagon on a given line A B.

With the radius A B, and the centres A and B, describe arcs intersecting in O. With the same radius, and centre O, describe a circle, which will circumscribe the hexagon. Then apply the line A B six times round the circumference, marking out the angular points, which connect with right lines.



PROBLEM XXVI.

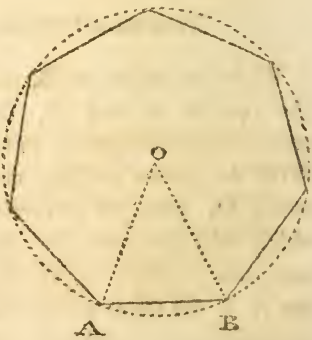
To make an octagon on a given line A B.



Erect A F and B E perpendicular to A B. Produce A B both ways, and bisect the angles *m* A F and *n* B E with the lines A H and B C, each equal to A B. Draw C D and H G parallel to A F or B E, and each equal to A B. With radius A B, and centres G and D, cross A F and B E, in F and E. Then join G F, F E, E D, and it is done.

PROBLEM XXVII.

To make any regular polygon on a given line A B.



Draw A O and B O, making the angles A and B each equal to half the angle of the polygon. With the centre O, and radius O A, describe a circle. Then apply the line A B continually round the circumference the proper number of times, and it is done.

NOTE. The angle of any polygon, that is, the angle formed by any two of its contiguous sides, of which the angles O A B and O B A are each one half, is found thus: divide the whole 360 degrees by the number of sides, and the quotient will be the angle at the centre O; then subtract that from 180 degrees, and the remainder will be the angle of the polygon, and is double of O A B, or of O B A. And thus you will find the following table, containing the degrees in the angle O at the centre, and the angle of the polygon, for all the regular figures from 3 to 12 sides.

No. of sides.	Name of the Polygon.	Angle O at the centre.	Angle of the Polygon.	Angle O A B or O B A
3	Trigon	120°	60°	30°
4	Tetragon	90	90	45
5	Pentagon	72	108	54
6	Hexagon	60	120	60
7	Heptagon	$51\frac{3}{7}$	$128\frac{4}{7}$	$64\frac{2}{7}$
8	Octagon	45	135	$67\frac{1}{2}$
9	Nonagon	40	140	70
10	Decagon	36	144	72
11	Undecagon	$32\frac{8}{11}$	$147\frac{3}{11}$	$73\frac{7}{11}$
12	Dodecagon	30	150	75

PROBLEM XXVIII.

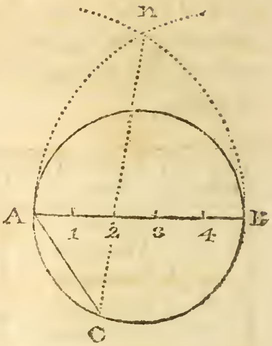
In a given circle to inscribe any regular polygon; or to divide the circumference into any number of equal parts.

[See the last Figure.]

At the centre O make an angle equal to the angle at the centre of the polygon, as contained in the third column of the above table of polygons. Then the distance A B will be one side of the polygon; which, being carried round the circumference the proper number of times, will complete the figure. Or, the arc AB will be one of the equal parts of the circumference.

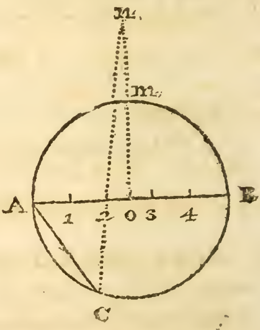
ANOTHER METHOD, NEARLY TRUE.

Draw the diameter A B, which divide into as many equal parts as the figure has sides. With the radius A B, and centres A and B, describe arcs crossing at *n*; whence draw *n* C through the second division on the diameter; so shall A C be a side of the polygon nearly.



ANOTHER METHOD, STILL NEARER.

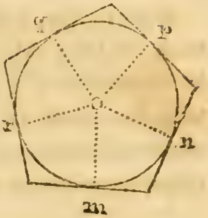
Divide the diameter AB into as many equal parts as the figure has sides, as before. From the centre O raise the perpendicular O*m*, which produce till *m n* be three-fourths of the radius O *m*. From *n* draw *n* C through the second division of the diameter, and the line AC will be the side of the polygon still nearer than before; or the arc A C, one of the equal parts, into which the circumference is to be divided.



PROBLEM XXIX.

About a given circle to circumscribe any polygon.

Find the points m, n, p, &c. as in the last problem, to which draw radii mO, nO, &c. to the centre of the circle. Then through these points m, n, &c. and perpendicular to these radii, draw the sides of the polygon.



PROBLEM XXX.

To find the centre of a given polygon, or the centre of its inscribed or circumscribed circle.

Bisect any two sides with the perpendiculars mO, nO ; and their intersection will be the centre. Then, with the centre O, and the distance Om, describe the inscribed circle ; or with the distance to one of the angles, as A, describe the circumscribing circle.*

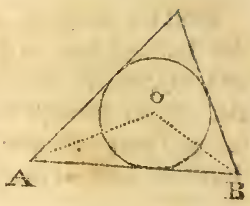
NOTE. This method will also circumscribe a circle about any given oblique triangle.

---

* O m being perpendicular to the chord A m of the circumscribing circle, or the tangent of the inscribed circle at the point of contact, must, if continued far enough, pass through the centre. The same may be said of O n. Therefore their intersection at O is the centre.

PROBLEM XXXI.

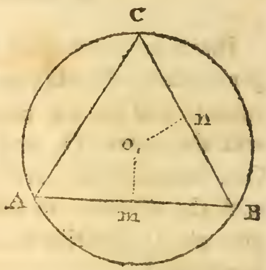
In any given triangle to inscribe a circle.



Bisect any two of the angles with the lines AO, BO, and O will be the centre of the circle. Then, with the centre O, and radius the nearest distance to any one of the sides, describe the circle.*

PROBLEM XXXII.

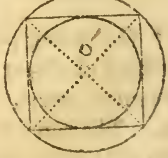
About any given triangle to circumscribe a circle.



Bisect any two of the sides A B, B C, with the perpendiculars *m*O, *n*O. With the centre O, and distance to any one of the angles, describe the circle.

PROBLEM XXXIII.

In, or about, a given square to describe a circle.



Draw the two diagonals of the square, and their intersection O will be the centre of both the circles. Then, with that centre, and the nearest distance to one side for radius, describe the inner circle ; and with the distance to one angle for radius, describe the outer circle.†

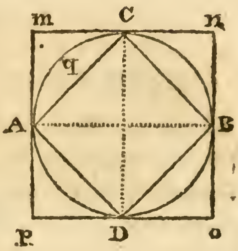
---

* For if perpendiculars be let fall from O on each of the sides, it may easily be shown, that these perpendiculars are equal. And consequently O is the centre of the required circle.

† The diagonals of a square mutually bisect each other.

PROBLEM XXXIV.

In, or about a given circle to describe a square, or an octagon.

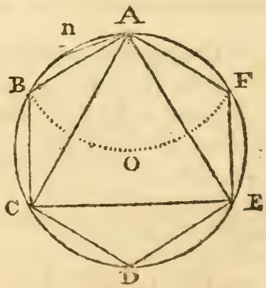


Draw two diameters A B, C D, perpendicular to each other. Then connect their extremities, and they will give the inscribed square A C B D. Also through their extremities draw tangents, each parallel to the other diameter, and they will form the outer square *m n o p*.

Note. If any quadrant, as A C, be bisected in *q*, it will give one eighth of the circumference, or the side of the octagon.

PROBLEM XXXV.

In a given circle to inscribe a trigon, a hexagon, or a dodecagon.

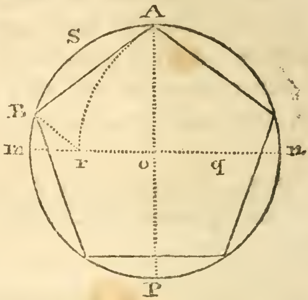


The radius is the side of the hexagon. Therefore from any point A in the circumference, with the distance of the radius, describe the arc B O F. Then is A B the side of the hexagon; and therefore, being carried round six times, it will form the hexagon, or divide the circumference into six equal parts, each containing 60 degrees. The second of these, C, will give A C, the side of the trigon, or equilateral triangle, and the arc A C one third of the circumference, or 120 degrees. Also the half of A B, or A *n*, is one twelfth of the circumference, or 30 degrees, and gives the side of the dodecagon.

NOTE. If tangents to the circle be drawn through all the angular points of any inscribed figure, they will form the sides of a like circumscribing figure.

PROBLEM XXXVI.

In a given circle to inscribe a pentagon, or a decagon.



Draw the two diameters A P, $m\,n$, perpendicular to each other, and bisect the radius O n in q. With the centre q, and radius q A, describe the arc A r; and with the centre A, and radius A r, describe the arc rB. Then is A B one fifth of the circumference; and A B, carried round five times, will form the pentagon. Also the arc A B, bisected in S, will give A S, the tenth part of the circumference, or the side, of the decagon.*

---

* If a regular pentagon be inscribed in a circle, the square of the radius is to the square of its side, as 2 to $5-\sqrt{5}$.

Suppose a right line drawn from A to r, and A to q. Then $Ar^2 = Aq^2 + rq^2 - 2\,rq \times oq = 2\,Aq^2 - 2\,Aq \times oq = 2\,Aq^2 - Aq \times Ao$;

but $Aq^2 = Ao^2 + oq^2 = \frac{5}{4}Ao^2$,

hence $Aq = \frac{1}{2}Ao\sqrt{5}$,

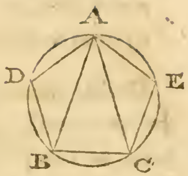
$$\therefore Ar^2 = \frac{5}{2}Ao^2 - \frac{Ao^2}{2}\sqrt{5} = Ao^2 \times \frac{5-\sqrt{5}}{2},$$

and consequently, $Ar = AB$ is the side of the pentagon.

As the square of the side of a regular pentagon, inscribed in a circle, is equal to the sum of the squares of the radius and of the side of a regular decagon, inscribed in the same circle, and $Ar^2 = Ao^2 + or^2$,

$\therefore ro =$ the side of the decagon.

ANOTHER METHOD.



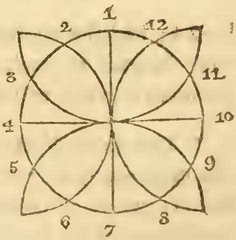
Inscribe the isosceles triangle ABC, having each of the angles ABC, ACB, double the angle BAC. Then bisect the two arcs ADB, AEC, in the points D, E; and draw the chords AD, DE, AE, EC; so shall ADBCE be the inscribed pentagon required.* And the decagon is thence obtained as before.

NOTE. Tangents, being drawn through the angular points, will form the circumscribing pentagon or decagon.

PROBLEM XXXVII.

To divide the circumference of a given circle into twelve equal parts, each being 30 degrees.

Or to inscribe a dodecagon by another method.



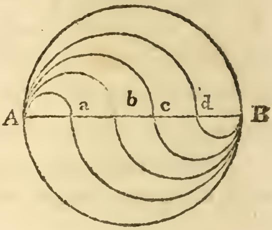
Draw two diameters 1 7 and 4 10 perpendicular to each other. Then, with the radius of the circle, and the four extremities 1, 4, 7, 10, as centres, describe arcs through the centre of the circle; and they will cut the circumference in the points required, dividing it into 12 equal parts at the points marked with the numbers.†

* The angle ACB at the circumference, standing on the arc ADB, is double the angle BAC; consequently the arc ADB = double the arc BC. For the same reason the arc AEC = double the arc BC. Therefore the chords AD, DB, BC, CE, EA are equal to each other; and ADBCE is the required pentagon.

† The radius being equal to the chord of 60°, the arc 1 3, in the quadrant 1 4, = 2 4 = 60°. Therefore the arc 1 2 = 2 3

PROBLEM XXXVIII.

To divide a given circle into any proposed number of parts by equal lines, so that those parts shall be mutually equal, both in area and perimeter.



Divide the diameter A B into the proposed number of equal parts at the points *a*, *b*, *c*, &c. Then on A *a*, A *b*, A *c*, &c. as diameters, describe semicircles on one side of the diameter A B; and on B *d*, B *c*, B *b*, &c. describe semicircles on the other side of the diameter. So shall the corresponding joining semicircles divide the given circle in the manner proposed. And in like manner we may proceed, when the spaces are to be in any given proportion. As to the perimeters, they are always equal, whatever may be the proportion of the spaces.*

= 3 4; and the chords of these equal arcs are equal. The same may be said of each of the other quadrants. Therefore the problem is truly solved.

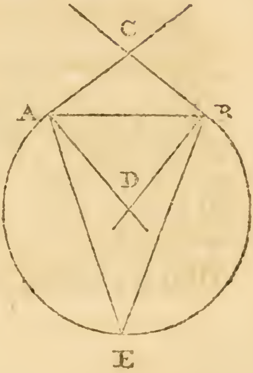
* The several diameters being in arithmetical progression, the common difference being equal to the least of them, and the diameters of circles being as their circumferences, the circumferences are also in arithmetical progression. But in such a progression the sum of the extremes is equal to the sum of each two terms, equally distant from them; therefore the sum of the circumferences on AC and CB is equal to the sum of those on AD and DB, and of those on AE and EB, &c. and each sum equal to the semicircumference of the the given circle on the diameter AB. Therefore all the parts have equal perimeters, and each is equal to the circumference of the given circle.

Again the same diameters being as the members 1, 2, 3, 4, &c. and the areas of circles being as the squares of their diameters, the semicircles will be as the numbers 1, 4, 9, 16, &c. and conse-

PROBLEM XXXIX.

On a given Line A B *to describe the Segment of a Circle, capable of containing a given Angle.*

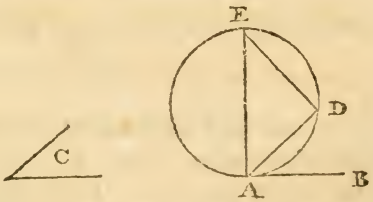
Draw AC and BC, making the angles BAC and ABC each equal to the given angle. Draw A D perpendicular to A C, and B D perpendicular to B C. With centre D, and radius D A, or DB, describe the segment AEB. Then any angle, as E, made in that segment, will be equal to the given angle.*



PROBLEM XL.

To cut off a segment from a given circle, that shall contain a given angle C.

Draw any tangent AB to the given circle; and a chord A D, to make the angle DAB equal to the given angle C;



---

quently the differences between all the adjacent semicircles are as the terms of the arithmetical progression 1, 3, 5, 7, &c. in which the sums of the extremes, and of every two equidistant means, constitute the several equal parts of the circle.

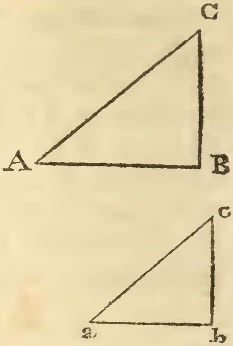
* Let fall a perpendicular from D upon A B, and it will bisect the angle D. Half the angle D is equal to the complement of the angle DBA = the angle ABC = the given angle. But half

then DEA will be the segment required, any angle E, made in it, being equal to the given angle C.*

PROBLEM XLI.

To make a triangle similar to a given triangle ABC.

Let $a\,b$ be one side of the required triangle. Make the angle a equal to the angle A, and the angle b equal to the angle B; then the triangle $a\,b\,c$ will be similar to ABC, as proposed.

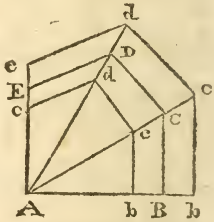


NOTE. If $a\,b$ be equal to AB, the triangles will also be equal, as well as similar.

PROBLEM XLII.

To make a figure similar to any other given figure ABCDE.

From any angle A draw diagonals to the other angles. Take Ab a side of the figure required. Then draw $b\,c$ parallel to BC, and $c\,d$ to C D, and $d\,e$ to D E, &c.*

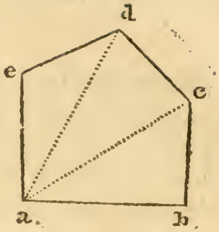


the angle D = angle AEB = any other angle in that segment of the circle. Therefore the required segment is described.

* For the angle BAD is equal to the angle DEA in the alternate segment.

OTHERWISE.

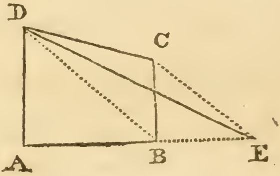
Make the angles at *a*, *b*, *e*, &c. respectively equal to the angles at A, B, E, and the lines will intersect in the angles of the figure required.



PROBLEM XLIII.

To make a triangle equal to a given trapezium A B C D.

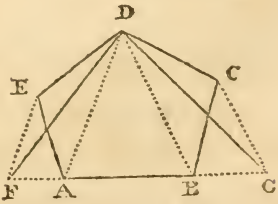
Draw the diagonal DB, and CE parallel to it, meeting AB produced in E. Join D E; so shall the triangle ADE be equal to the trapezium A B C D.*



PROBLEM XLIV.

To make a triangle equal to the figure ABCDEA

Draw the diagonals DA, DB, and the lines EF, CG, parallel to them, meeting the base AB, both ways produced, in F and G. Join DF, DG; and DFG will be the triangle required.

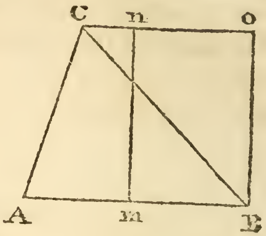


NOTE. Nearly in the same manner may a triangle be made equal to any right-lined figure whatever.

* For the triangles DBE and DBC, being on the same base and between the same parallels, are equal.

PROBLEM XLV.

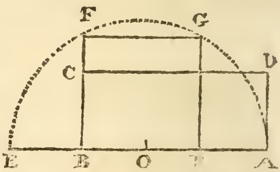
To make a rectangle, or a parallelogram, equal to a given triangle ABC.



Bisect the base AB in *m*. Through C draw C *n* O parallel to AB. Through *m* and B draw *m n* and B O parallel to each other, and either perpendicular to AB, or making any angle with it. And the rectangle or parallelogram *m n* O B will be equal to the triangle, as required.*

PROBLEM XLVI.

To make a square equal to a given rectangle ABCD.



Produce one side AB, till B E be equal to the other side BC. Bisect AE in O; on which as a centre, with radius AO, describe a semicircle, and produce BC to meet it at F. On BF make the square BFGH, and it will be equal to the rectangle ABCD, as required.†

* For a parallelogram on half the base and between the same parallels is equal to a triangle.

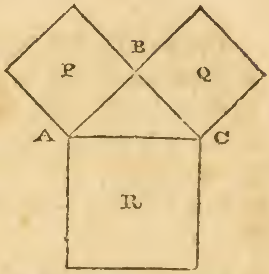
† For EB=BC : BF :: BF : BA, hence $BC \times BA = BF^2$

∵ The given parallelogram and square, that is found, are equal.

PROBLEM XLVII.

To make a square equal to two given squares P *and* Q.

Set two sides AB, BC, of the given squares perpendicular to each other. Join their extremities AC; so shall the square R, constructed on AC, be equal to the two P and Q taken together.*

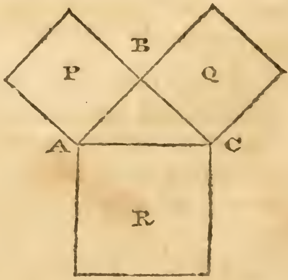


Note. Circles, or any other similar figures, are added in the same manner. For if AB and BC be the diameters of two circles, AC will be the diameter of a circle equal to the other two. And if AB and BC be the like sides of any two similar figures, then AC will be the like side of another similar figure equal to the two former, and upon which the third figure may be constructed, by Problem XLII.

PROBLEM XLVIII.

To make a square equal to the difference of two given squares P, R.

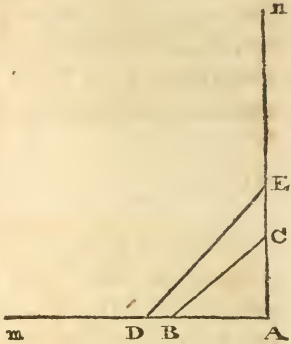
On the side AC of the greater square, as a diameter, describe a semicircle; in which apply AB, the side of the less square. Join BC, and it will be the side of a square, equal to the difference between the two P and R, as required.



* For in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

PROBLEM XLIX.

To make a square equal to the sum of any number of squares taken together.



Draw two indefinite lines A *m*, A *n*, perpendicular to each other at the point A. On one of these set off AB the side of one of the given squares, and on the other AC the side of another of them. Join BC, and it will be the side of a square equal to the two together. Then take AD equal to BC, and AE equal to the side of the third given square. So shall DE be the side of a square equal to the sum of the three given squares. And so on continually, always setting more sides of the given squares on the line A*n*, and the sides of the successive sums on the other line A*m*.

NOTE. And thus any number of any kind of figures may be added together.

PROBLEM L.

To construct the lines of the plane scale.

The divisions on the plane scale are of two kinds; one kind having relation merely to right lines, and the other to the circle and its properties. The former are called *lines*, or *scales*, *of equal parts*, and are either *simple* or *diagonal*.

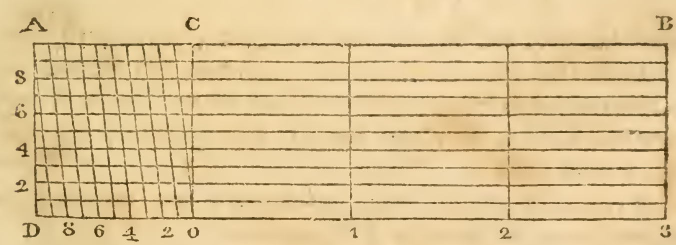
By *the lines of the plane scale*, we here mean the following lines, most of which commonly, and all of them sometimes, are drawn on a Plane Scale.

1. A LINE or SCALE of EQUAL PARTS, marked E. P.

2.	*Chords*	Cho.
3.	*Rhumbs*	Rhu.
4.	*Sines*	Sin.
5.	*Tangents*	Tan.
6.	*Secants*	Sec.
7.	*Semitangents*	S. T.
8.	*Longitude*	Lon.
9.	*Latitudes*	Lat.
10.	*Hours*	Ho.
11.	*Inclination of Meridians* .	In. Mer.

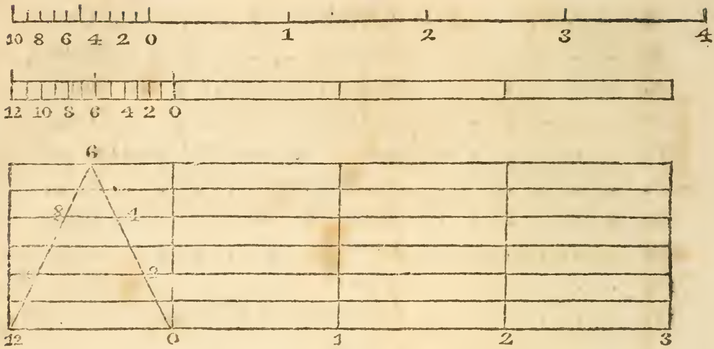
1. *To construct plane diagonal scales.*

Draw any line, as A B, of any convenient length. Divide it into 11 equal parts.* Complete these into rectangles of a convenient height, by drawing parallel and perpendicular lines. Divide the altitude into 10 equal parts, if it be for a decimal scale for common numbers, or into 12 equal parts, if it be for feet and inches ; and through these points of division draw as many parrallel lines, the whole length of the scale. Then divide the length of the first division A C into 10 equal parts, both above and below ; and connect these points of division by diagonal lines, and the scale is finished, after being numbered as you please.



* Only 4 parts are here drawn for want of room.

PLANE SCALES FOR TWO FIGURES.



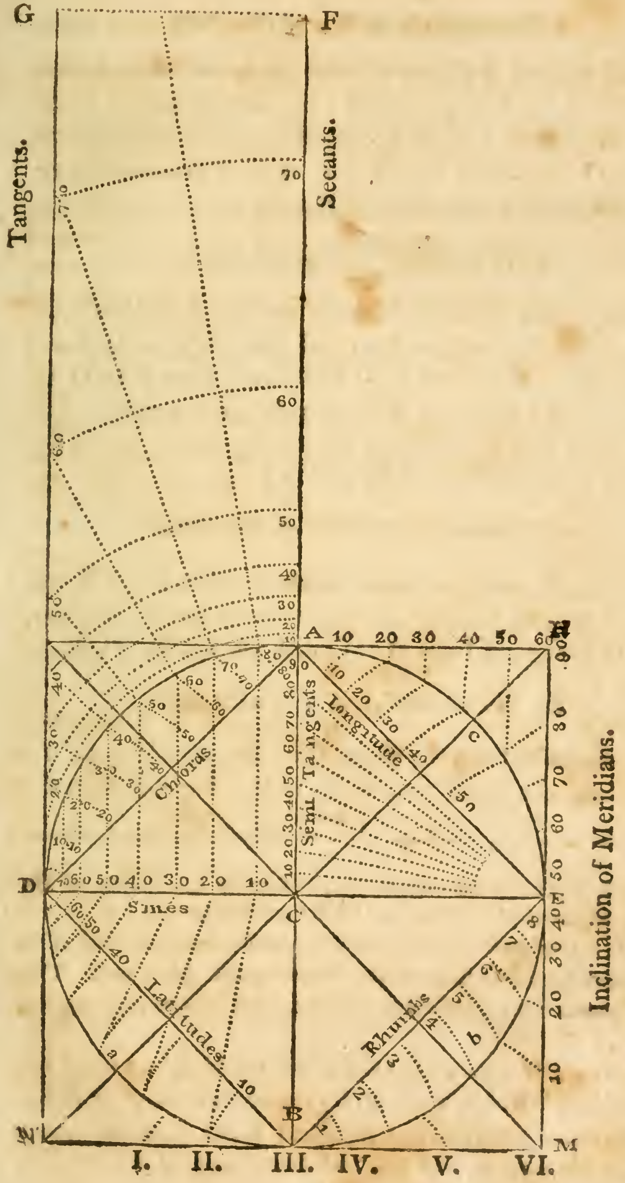
Of the preceding three forms of scales for two figures the first is a decimal scale, for taking off common numbers consisting of two figures. The other two are duodecimal scales, and serve for feet and inches.

In order to construct the other lines, describe a circumference with any convenient radius, and draw the diameters A B, D E, at right angles to each other ; continue B A at pleasure toward F ; through D draw D G parrallel to B F ; and draw the chords B D, B E, A D, A E. Circumscribe the circle with the square HMN, whose sides HM, MN, shall be parallel to AB, ED.

2. *To construct the line of chords.*

Divide the arc A D into 90 equal parts ; mark the 10th divisions with the figures 10, 20, 30, 40, 50, 60, 70, 80, 90 ; on D, as a centre, with the compasses, transfer the several divisions of the quadrantal arc to the chord A D, which, marked with the figures corresponding, will be a line of chords.

NOTE. In the construction of this and the following scales, only the primary divisions are drawn ; the intermediate ones are omitted, that the figure may not appear too much crowded.


G
F
Tangents.
Secants.
A
B
C
D
E
M
N
Chords
Semi Tangents
Longitude
Sines
Latitudes
Rhumbs
Inclination of Meridians.
I.
II.
III.
IV.
V.
VI.

3. *To construct the line of rhumbs.**

Divide the arc B E into 8 equal parts, which mark with the figures 1, 2, 3, 4, 5, 6, 7, 8 ; and divide each of those parts into quarters ; on B, as a centre, transfer the divisions of the arc to the chord BE, which, marked with the corresponding figures, will be a line of rhumbs.

4. *To construct the line of sines.*†

Through each of the divisions of the arc AD draw right lines parallel to the radius AC ; and CD will be divided into a line of sines, which are to be numbered from C to D for the right sines ; and from D to C for the versed sines. The versed sines may be continued to 180 degrees, by laying the divisions of the radius CD from C to E.

5. *To construct the line of tangents.*‡

A rule on C, and the several divisions of the arc AD, will intersect the line DG, which will become a line of tangents, and is to be figured from D to G with 10, 20, 30, 40, &c.

6. *To construct the line of secants.*§

The distances from the centre C to the divisions on the line of tangents, being transferred to the line CF from the

---

* *Rhumbs* here are chords, answering to the points of the Mariners' Compass, which are 32 in the circle.

† The *sine of an arc* is a right line, drawn from one end of an arc perpendicular to the radius, drawn to the other end. The *versed sine* is the part of the radius, included between the arc and its sine.

‡ The *tangent of an arc* is a right line, touching that arc at one end, and terminated by a secant, drawn through the other end.

§ The *secant of an arc* is a right line drawn from the centre through one end of the arc, and terminated by the tangent, drawn from the other end.

centre C, will give the divisions of the line of secants; which must be numbered from A toward F with 10, 20, 30, &c.

7. *To construct the line of semitangents, or the tangents of half the arcs.*

A rule on E, and the several divisions of the arc AD, will intersect the radius CA, in the divisions of the semi or half tangents; mark these with the corresponding figures of the arc AD.

The semitangents on the plane scales are generally continued as far as the length of the rule, on which they are laid, will admit; the divisions beyond 90° are found by dividing the arc AE like the arc AD, then laying a rule by E and these divisions of the arc AE, the divisions of the semitangents above 90 degrees will be obtained on the line CA continued.

8. *To construct the line of longitude.*

Divide AH into 60 equal parts; through each of these divisions parallels to the radius AC will intersect the arc AE in as many points; from E, as a centre, the divisions of the arc EA, being transferred to the chord EA, will give the divisions of the line of longitude.

The points thus found on the quadrantal arc, taken from A to E, belong to the sines of the equally increasing sexagenary parts of the radius; and those arcs, reckoned from E, belong to the cosines of those sexagenary parts.

9. *To construct the line of latitudes.*

A rule on A, and the several divisions of the sines on CD, will intersect the arc BD, in as many points; on B, as a centre, transfer the intersections of the arc BD, to the right line BD; number the divisions from B to D with 10, 20, 30, &c. to 90; and BD will be a line of latitudes.

10. *To construct the line of hours.*

Bisect the quadrantal arcs BD, BE, in *a*, *b*; divide the quadrantal arc *a b* into 6 equal parts, which gives 15 degrees for each hour; and each of these into 4 others, which will give the quarters. A rule on C, and the several divisions of the arc *ab*, will intersect the line MN in the hour, &c. points, which are to be marked as in the figure.

11. *To construct the line of inclination of meridians.*

Bisect the arc EA in *c*; divide the quadrantal arc *bc* into 90 equal parts; lay a rule on C and the several divisions of the arc *bc*, and the intersections of the line HM will be the divisions of a line of inclination of meridians.

The use of these several lines will appear in the subsequent parts of the work.

PLANE TRIGONOMETRY.

PLANE TRIGONOMETRY teaches the relations and calculations of the sides and angles of plane triangles.

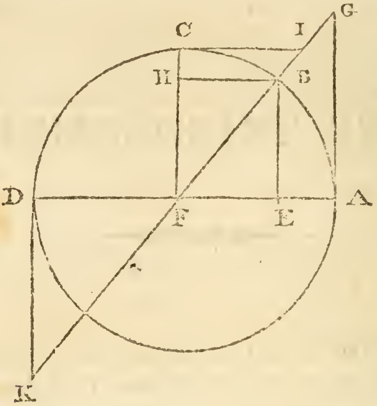
The angles of triangles are measured by the number of degrees, contained in the arc cut off by the legs of the angle, and whose centre is the angular point. A right angle is therefore an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180°. Wherefore, in a right-angled triangle, one acute angle being subtracted from 90°, the remainder will be the other; and the sum of any two angles of a triangle, being taken from 180°, will leave the third angle.

Degrees are marked at the top of the figure with a small °, minutes with ′, seconds with ″, and so on. Thus, 57° 30′ 12″, that is, 57 degrees, 30 minutes, and 12 seconds.

The *complement* of an arc is the difference between that arc and a quadrant. So BC=40° is the complement of AB =50°.

The *supplement* of an arc is what it wants of a semicircle. So BCD=130° is the supplement of AB=50°.

The *sine* of an arc is the line, drawn from one end of the arc perpendicularly upon the diameter, drawn through the other end of the arc. So BE is the sine of AB or of BCD.



The *versed sine* of an arc is the part of the diameter between the sine and the beginning of the arc. So AE is the versed sine of AB, and DE the versed sine of BCD.

The *tangent* of an arc is the line, drawn perpendicularly from one end of the diameter passing through one end of the arc, and terminated by the line, drawn from the centre through the other end of the arc. So AG or DK is the tangent of AB, or of BCD.

The *secant* of an arc is the line, drawn from the centre through the end of the arc, and terminated by the tangent. So FG or FK is the secant of AB, or of BCD.

The *cosine*, *cotangent*, or *cosecant*, of an arc is the sine, tangent, or secant of the complement of that arc. So BH is the cosine, CI the cotangent, and FI the cosecant of AB.

From the definitions it is evident, that the sine, tangent, and secant, are common to two arcs, which are the supplements of each other. So the sine, tangent, or secant of 50° is also the sine, tangent, or secant of 130°.

The sine, tangent, or secant, of an angle is the sine, tangent, or secant, of the arc, or the degrees, by which the angle is measured.

The sine, tangent, and secant of every degree and minute in a quadrant are calculated to the radius 1, and ranged in tables for use. But because trigonometrical operations with these natural sines, tangents, and secants require tedious multiplications and divisions, the logarithms of them are taken, and ranged in tables also ; and the logarithmic sines, tangents, and secants are commonly used, as they require only additions and subtractions, instead of the multiplications and divisions.

There are usually three methods of resolving triangles, or the cases of trigonometry ; namely, *Geometrical Construction*, *Arithmetical Computation*, and *Instrumental Operation*.

In the first method; let the triangle be constructed by making the parts of the given magnitudes, namely, the sides from a scale of equal parts, and the angles from a scale of chords, or other instrument. Then measure the required parts by the same scale.

In the second method; having stated the terms of the proportion according to the rule, resolve it like all other proportions, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers, whether they be sides, or sines, tangents, or secants, of angles. Or, in working with logarithms, add the logarithms of the second and third terms together, and from the sum subtract the logarithm of the first term; then the number answering to the remainder will be the fourth term required.

To work a stating instrumentally; as, for example, by the logarithmic lines on one side of the two foot scales.—Extend the compasses from the first term to the second, or third, which happens to be of the same kind with it; then that extent will reach from the other term to the fourth, taking both extents toward the same side.

Note. For the sides of triangles the line of numbers, marked Num. is used; and for the angles, the line of sines, or of tangents, marked Sin. or Tan. according as the proportion respects sines or tangents. If the extent upon the tangents reach beyond the line, set it so far back as it reaches over.

In a triangle there must be given three parts, one of which, at least, must be a side; because the same angles are common to an infinite number of triangles.

In plane trigonometry, there are only three cases, or varieties, viz.

1. *When two of the three given parts are a side and its opposite angle.*

2. *When two sides and their included angle are given.*

3. *When the three sides are given.*

PROBLEM I.

Given three such parts, that an angle and its opposite side shall be two of them; to find the rest.

In any plane triangle, the sides are proportional to the sines of their opposite angles.* That is,

As one side :
Is to another side : :
So is sin. angle opp. the former :
To sin. angle opp. the latter.

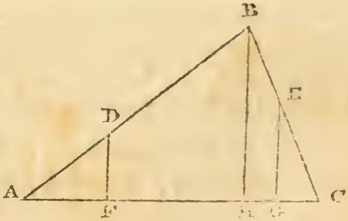
NOTE 1. To find an angle, begin the proportion with a side opposite to a given angle; and to find a side, begin with an angle opposite to a given side.

NOTE 2. An angle, found by this rule, is always ambiguous, except it be a right angle, or except, that the magnitude of the given angle prevent the ambiguity; because the sine answers to two angles, which are the supplements of each other; and accordingly the construction gives two triangles with the same given parts; and when there is no re-

---

* DEMONSTRATION.

Let ABC be any triangle: in AB assume any point D, take CE=AD, and upon AC demit the perpendiculars DF, EG, BH; then will DF and EG be the sines of the angles A, C, to the general radius AD or CE. Now from similar triangles we shall have

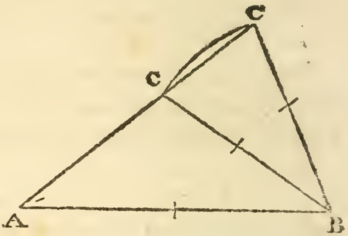


$$\left\{\begin{array}{l} AB : BH :: AD : DF \\ CB : BH :: AD\ (CE) : EG \end{array}\right\}$$ and hence, of equality,

$$AB : BC :: EG : DF.$$

striction or limitation, included in the proposition, either of them may be taken. The degrees, in the table, answering to the sine, is the acute angle; and if the angle be obtuse, take those degrees from 180°, and the remainder will be the obtuse angle. When the given angle is obtuse, or right, there can be no ambiguity; for then neither of the other angles can be obtuse, and the construction will produce but one triangle.

EXAMPLE 1.



In the plane triangle ABC,

$$\text{Given}\left\{\begin{array}{ll} AB & 345 \\ BC & 232 \\ \angle A & 37^\circ\ 20' ; \end{array}\right. \left.\begin{array}{l} \\ \\ \end{array}\right\}\text{yards}$$

Required the other parts.

GEOMETRICALLY.

1. Draw the line AB=345 from some convenient scale of equal parts.
2. Make the angle A=37° 20′.
3. With the centre B, and radius 232, taken from the same scale of equal parts, cross AC in C.
4. Draw BC, and the triangle is constructed.

Then the angles B and C, measured by the scale of chords, and the side AC, measured by the scale of equal parts, will be found to be as follows: viz.

∠B 27°	∠C $115\frac{1}{2}$	AC 174
Or $78\frac{1}{4}$	Or $64\frac{1}{2}$	Or $374\frac{1}{2}$

ARITHMETICALLY.

As side BC	232		2·3654880
To side BA	345		2·5378191
So sine ∠A 37° 20′			9·7827958
To sine ∠C 115° 36′	or	64° 24′	9·9551269
∠A 37° 20		37 20	
Subtract 152 56	or	101 44	
From 180 00		180 00	
Leaves 27 04		78 16 the ∠B.	

Then

As sine ∠A	37° 20′	9·7827958
To sine ∠B	27 04 / 78 16	9·6580371 / 9·9908291
So side BC	232	2·3654880
To side AC	174·07 / 374·56	2·2407293 / 2·5735213

INSTRUMENTALLY.

In the first proportion, extend from 232 to 345 upon the line of numbers; that extent will reach, upon the sines, from $37^\circ\frac{1}{3}$ to $64^\circ\frac{1}{2}$ the angle C.

In the second proportion, extend from $37^\circ\frac{1}{3}$ to 27° or $78^\circ\frac{1}{4}$ upon the sines; that extent will teach, upon the numbers, from 232 to 174 or $374\frac{1}{2}$, for the side AC.

EXAMPLE 2.

In the plane triangle ABC,

Given AB 162, AC 270 chains; ∠B 90°; Required the other parts.

Ans. BC 216 chains. ∠C 36° 52′ 12″. ∠A 53 07 48

EXAMPLE 3.

Iu the plane triangle ABC,

Given: AB 365 poles; ∠B 24° 45′; ∠A 57 12;
Required the other parts.

Ans.: ∠C 98° 03′; AC 154·33 poles; BC 309·86 poles.

EXAMPLE 4.

In the plane triangle ABC,

Given: AB 53 miles; ∠A 121° 14′; ∠C 29 23;
Required the other parts.

Ans.: ∠B 29° 23′; AC 53 miles. BC 92·36 miles.

EXAMPLE 5.

In the plane triangle ABC

Given: AB 365 poles; AC 154·33 poles; ∠C 98° 03′;
Required the other parts.

Ans.: ∠B 24° 45′; ∠A 57 12; BC 309·86 poles.

EXAMPLE 6.

In the plane triangle ABC,

Given: AC 120 ft.; BC 112 ft.; ∠A 57° 27′;
Req. the other parts.

Ans.:
∠B: 64° 34′ 21″ or 115 25 39
∠C: 57 58 39 or 7 07 21
AB: 112·65 or 16·47 feet.

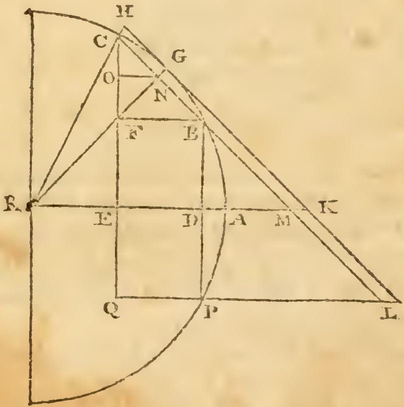
PROBLEM II.

Given two sides and the angle included by them; to find the rest.

In a plane triangle,
As the sum of any two sides :
Is to their difference : :
So is the tangent of half the sum of their opposite angles :
To the tangent of half their difference.*

---

* DEMONSTRATION.



By the first problem, the sides are as the sines of their opposite angles, and consequently the sum of the sides will be to the difference of the sides, as the sum of the sines is to the difference of the sines of the said opposite angles.

Wherefore we have only to prove, that the sum of the sines is to the difference of the sines of two arcs, as the tangent of half the sum of those arcs is to the tangent of half their difference: in order to which, let BD, CE, be the sines of the arcs AB, AC; produce BD to the circumference at P, and produce CE till EQ be = DP; to the middle point G of the arc BC draw the tangent HGK, and draw CNBML parallel to it; join RH, RG, and draw ON, FB, and QL parallel to RAMK.

Now it is evident, that CQ is the sum, and CF the difference, of the sines; and that GK is the tangent of half the sum AG, Then the half difference, added to the half sum of the angles, gives the greater; and subtracted leaves the less angle.

Then all the angles being known, find the unknown side by the first problem.

NOTE 1. When, in this case, the triangle is right angled, the longest side will be found by extracting the square root of the sum of the squares of the other two sides; and then the angles will be found by the first problem.

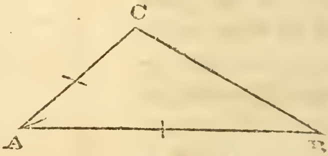
NOTE 2. That instead of the tangent of the half sum, we may use the cotangent of half the given angle, which is the same thing.

EXAMPLE 1.

In the plane triangle ABC,

Given $\begin{cases} \text{AB } 345 \text{ yards,} \\ \text{AC } 174{\cdot}07 \text{ yd.} \\ \angle A \;\; 37^\circ\, 20'; \end{cases}$

Required the other parts.



---

and GH the tangent of half the difference CG, of the two arcs AB, AC; also NM is $= \frac{1}{2}$CL, for BN=NC, and BM=ML: then, CQ : CF : : ($\frac{1}{2}$CQ or) OE : ($\frac{1}{2}$CF or) OC,
and OE : OC : : NM : NC, because ON, in the triangle CEM, is parallel to EM,
and NM : NC : : GK : GH, because CM, in the triangle RHK, is parallel to HK;
therefore, CQ : CF : : GK : GH. Q. E. D.

And that the half sum, increased and diminished by the half difference, gives the greater and less angle respectively, is evident from the figure. And that two quantities of any kind may be found, by the same rule, from their sum and difference, may be proved thus. Let CN represent the less and NL the greater of any two quantities; and let B be the middle of the right line CL. Then it is evident, that BL=BC is the half sum, and BN the half difference, as also, that LB+BN=NL the greater quantity, and CB—BN=NC the less.

GEOMETRICALY.

1. Draw AB equal to 345, from a scale of equal parts.

2. Make the angle A equal to 37° 20′.

3. Make AC equal to 174·07, by the scale of equal parts.

4. Join B, C, and it is done.

Then the parts being measured, we have the ∠C=115°½, the ∠B=27°, and BC= 232 yards.

ARITHMETICALLY.

As sum of sides	AB+AC	519·07	2·7152259
To diff. of sides	AB—AC	170·93	2·2328183
So tang. $\frac{\angle C + \angle B}{2}$		71° 20′	10·4712979
To tang. $\frac{\angle C - \angle B}{2}$		44 16	9·9888903
	Their sum	115 36 ∠C	
	Their diff.	27 04 ∠B	

Then,

As sine ∠C 115° 36′ or 64° 24′		9·9551259
To sine ∠A	37 20	9·7827958
So side AB 345		2·5378191
To side BC 232		2·3654890

INSTRUMENTALLY.

In the first proportion, extend from 519 to 171 on the line of numbers; that extent will reach, upon the tangents, from 71°⅓, (the contrary way, because the tangents are set back from 45°) a little beyond 45, which, being set so far back from 45, falls upon 44°¼, the fourth term.

In the second proportion, extend from 64°⅓ to 37°⅓ on the sines; that extent will reach, on the numbers, from 345 to 232, the fourth term required.

EXAMPLE 2.

In the plane triangle ABC,

Given AB 53, BC 92·36 miles, ∠B 29°23′; Required the other parts. Ans. ∠A 121° 14′, ∠C 29 23, AC 53 miles.

EXAMPLE 3.

In the plane triangle ABC,

Given AC 120, BC 112 poles, ∠C 57° 58′ 39″; Required the other parts. Ans. ∠A 57° 27′ 00″, ∠B 64 34 21, AB 112·65 poles.

PROBLEM III.

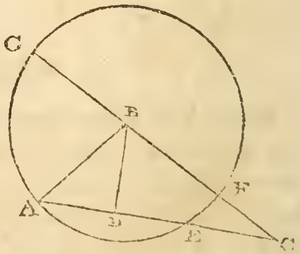
Given the three sides; to find the angles.

In any plane triangle, having let fall a perpendicular from the greatest angle to the opposite side or base, dividing it into two segments, and the whole triangle into two right-angled triangles; it will be

As the base, or sum of the segments :
Is to the sum of the other two sides : :
So is the difference of those sides :
To the difference of the segments of the base.*

---

* DEMONSTRATION.



From one end B of the least side AB, of the triangle ABC, as a centre, and radius AB, describe a circle, cutting the other two sides in E and F; produce CB to the circle at G, and let fall the perpendicular BD. Then is GB=BF=AB, and (by 3 III. Eucl.) AD=DE, and consequently EC=CD—DA the difference of the segments, FC=GB—BA the difference of the sides, and GC=CB+BA the sum of the sides. But (by Cor. to 36 III. Eucl.) the rectan-

Then half the difference, being added to and subtracted from half their sum, will give the greater and less segment.

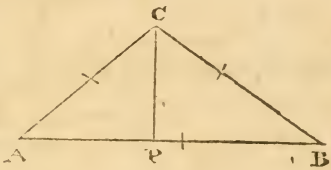
Hence, in each of the right-angled triangles, will be known two sides, and the angle opposite to one of them; and consequently the other angles will be found by the first problem.

NOTE. In the above proportions, if half the difference of the sides be taken for the third term, then the fourth term will be half the difference of the segments. Which will commonly be more convenient to use than the whole difference.

EXAMPLE 1.

In the plane triangle ABC,

Given $\begin{cases} AB & 345 \\ AC & 174\cdot07 \\ BC & 232 \end{cases}$ yd.



Required the angles.

GEOMETRICALLY.

1. Draw AB equal to 345, by a scale of equal parts.

2. With centres A and B, and radii 174·07 and 232, taken from the same scale, describe arcs intersecting in C.

3. Draw AC and BC, and it is done.

Then, by measuring the angles, they appear to be nearly of the following dimensions, viz. $\angle A=37°\frac{1}{3}$, $\angle B=27°$, and $\angle C=115°\frac{1}{2}$.

ARITHMETICALLY.

Having let fall the perpendicular CP, it will be, As AB $=345$: $BC+CA=406\cdot07$: : $BC-CA=57\cdot93$:

---

gle $CA\times CE=CG\times CF$, or CA : CG : : CF : CE, that is, AC : CB+BA : : CB—BA : CD—DA. Q. E. D.

And that the half-sum of two quantities, increased and diminished by their half-difference, gives the greater and less quantities respectively, was proved in the last problem.

$$\frac{406{\cdot}07\times57{\cdot}93}{345}=68{\cdot}18=BP-PA.$$

Hence, $\frac{345+68{\cdot}18}{2}=206{\cdot}59=BP.$

And $\frac{345-68{\cdot}18}{2}=138{\cdot}41=AP.$

Then, in the triangle APC, right angled at P,

As AC	174·07	2·2407239
To AP	138·41	2·1411675
So S. ∠P	90° 00′	10·0000000
To S.∠ACP	52 40	9·9004436

Which, being taken from 90 00

Leaves 37 20 ∠A.

And in the triangle BPC,

As BC	232	2·3654880
To BP	206·59	2·3151093
So S.∠P	90° 00′	10·0000000
To S. ∠PCB	62 56	9·9496213

Taken from 90 00

Leaves 27 04∠B

Also 52 40∠ACP
Added to 62 56∠BCP

Makes 115 36∠ACB.

Whence the ∠A=37 20′, the ∠B=27° 04′, and the ∠C= 115° 36′.

INSTRUMENTALLY.

In the first proportion, extend from 345 to 406, on the line of numbers; that extent will reach, upon the same line, from 58 to 68·2, the difference of the segments of the base.

In the second proportion, extend from 174 to $138\frac{1}{2}$ on the numbers; that will reach, on the sines, from 90° to $52^{\circ}\frac{2}{3}$.

In the third proportion, extend from 232 to $206\frac{1}{2}$, and that extent will reach from 90° to 63°.

EXAMPLE 2.

In the plane triangle ABC,

Given	AB 162 AC 270 BC 216;	Ans.	∠A 53° 07′ 43″ ∠B 90 00 00 ∠C 36 52 12
Required the angles.			

EXAMPLE 3.

In the plane triangle ABC,

Given	AB 112·65 AC 120 BC 112;	Ans.	∠A 57° 27′ 00″ ∠B 64 34 21 ∠C 57 58 39
Required the angles.			

EXAMPLE 4.

In the plane triangle ABC,

Given	AB 53 AC 53 BC 92·36;	Ans.	∠A 121° 14′ ∠B 29 23 ∠C 29 23
Required the angles.			

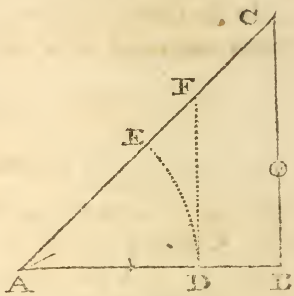
NOTE 1. These three problems include all the cases of plane triangles, as well right-angled as oblique-angled. There are some other theorems, suited to some particular forms of

triangles, which are often more expeditious in practice than the preceding general methods. One of which, as the case, for which it serves, so often occurs, is here given.

PROBLEM.

Given the angles and a leg of a right-angled triangle; to find the other leg and the hypotenuse.

As the radius :
Is to the given leg AB : :
So is tang. of angle A :
To the opposite leg BC : :
And so is secant of the ∠A :
To the hypotenuse AC.*



EXAMPLE.

In the plane triangle ABC, right-angled at B,

Given $\left\{ \begin{array}{ll} AB & 162 \\ \angle A & 53^\circ\ 07'\ 48'' \end{array} \right\}$ Required AC and BC.

GEOMETRICALLY.

Make AB=162, and the angle A=53° 07′ 48″; then raise the perpendicular BC meeting AC in C. So shall AC measure 270, and BC 216.

---

* DEMONSTRATION.

With the centre A and any radius AD, describe an arc DE, and erect the perpendicular DF; which, it is evident, will be the tangent, and AF the secant of the arc DE, or angle A, to the radius AD. And in similar triangles ADF, ABC, it will be AD : AB : : DF : BC : : AF : AC. Q. E. D.

ARITHMETICALLY.

As radius 90°	10·0000000
: AB 162	2·2095150
: : tang. of 53° 7′ 48″	10·1249371
: BC 215·9992	2·3344521

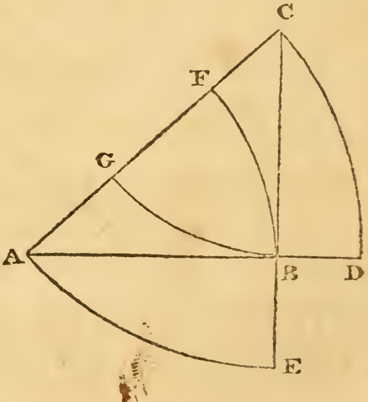
And

As radius 90°	10·0000000
: AB 162	2·2095150
: : secant of 53° 7′ 48″	10·2218477
: AC 269·9993	2·4313627

INSTRUMENTALLY.

The extent from 45° to 53° 08′, upon the tangents, will reach from 162 to 216 upon the numbers.

NOTE 2. It is common to add another method for right-angled triangles, which is this. ABC being the triangle, make a leg AB radius, that is, with centre A and radius AB, describe an arc BF: then it is evident, that the other leg BC represents the tangent, and the hypotenuse AC the secant of of the angle A or arc BF.



In like manner, if the leg BC be made radius, then the leg AB will represent the tangent, and AC the secant of the arc BG, or the angle C.

But if the hypotenuse be made radius, then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C ; and the leg BC, the sine of the arc CD or angle A.

And then the general rule for all these cases is this ; the sides are to each other, as the parts, which they represent.

MENSURATION

OF

SUPERFICIES.

THE *area* of any figure is the measure of its surface, or the space contained within the bounds of the surface, without any regard to thickness.

The area is estimated by the number of squares contained in the surface, the side of those squares being either an inch, a foot, a yard, &c. And hence the area is said to be so many square inches, or square feet, or square yards, &c.

Our ordinary lineal measures, or measures of length, are as in the first of the following tables; and the annexed table of square measures is taken from it by squaring the several numbers.

LINEAL MEASURES.

12 inches	1 foot
3 feet	1 yard
6 feet	1 fathom
$16\frac{1}{2}$ feet, or $5\frac{1}{2}$ yards	1 pole or rod
40 poles	1 furlong
8 furlongs	1 mile.

SQUARE MEASURES.

144 inches	1 foot
9 feet	1 yard
36 feet	1 fathom
272¼ feet, or 30¼ yards	1 pole or rod
1600 poles	1 furlong
64 furlongs	1 mile.

PROBLEM I.

To find the area of a parallelogram ; whether it be a square, a rectangle, a rhombus, or a rhomboid.

RULE.*

Multiply the length by the breadth, or a perpendicular height, and the product will be the area.

---

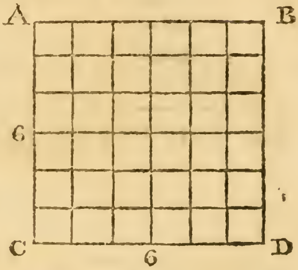
* Take any rectangle ABCD, and divide each of ts sides into as many equal parts as is expressed by the number of times they contain the linear measuring unit, and let all the opposite points of division be connected by right lines. Then, it is evident, that these lines divide the rectangle into a number of squares, each equal to the superficial measuring unit, and that the number of these squares, or the area of the figure, is equal to the number of linear measuring units in the length as often repeated, as there are linear measuring anits in the breadth or height, that is, equal to the length multiplied by the height, *which is the rule.*

And since a rectangle is equal to an oblique parallelogram standing upon the same base, and between the same parallels ; (Euc. I. 35) therefore the rule is true for any parallelogram in general. Q. E. D.

EXAMPLES.

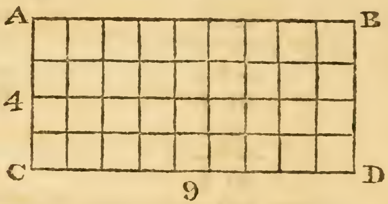
1. To find the area of a square, whose side is 6 inches, or 6 feet, &c.

6
6
36
Answer 36



2. To find the area of a rectangle, whose length is 9, and breadth 4 inches, or feet, &c.

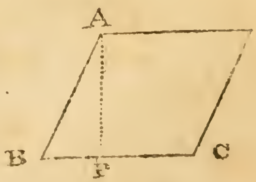
9
4
36
Answer 36



RULE 2. If any two sides of a parallelogram be multiplied together, and the product again by the natural sine of their included angle, the last product will give the area of the parallelogram.

DEMONSTRATION.

For having drawn the perpendicular AP, the area, by the first rule, is AP×BC; but as rad. 1 (s. ∠P) : s. ∠B : : AB : AP = s. ∠B×AB; therefore AP × BC=BC × s. ∠B × AB is the area.



3. To find the area of a rhombus, whose length is 6·20 chains, and perpendicular height 5·45.

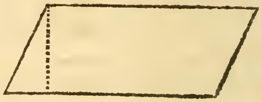
```
  5·45
  6·20
 ------
 10900
3270
 ------
```

Ans. 33·7900 square chains.

4. To find the area of the rhomboid, whose length is 12 feet 3 inches, and breadth 5 feet 4 inches.

```
ft. in.
12  3
 5  4
------
61  3
 4  1
------
65  4
```

Answer $65\frac{1}{3}$ square feet.



5. To find the area of a rectangular board, whose length is 12·5 feet, and breadth 9 inches.

Ans. $9\frac{3}{8}$ ft.

6. To find the square yards of painting in a rhomboid, whose length is 37 feet, and breadth $5\frac{1}{4}$ feet.

Ans. $21\frac{7}{12}$ square yards.

PROBLEM II.

To find the area of a triangle.

RULE 1.*

Multiply the base by the perpendicular height, and half the product will be the area.

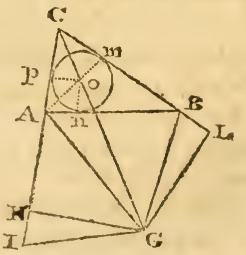
RULE 2.†

When the three sides only are given;—add the three sides together, and take half the sum; from the half sum subtract each side separately; multiply the half sum and the three remainders continually together; and the square root of the last product will be the area of the triangle.

---

* A triangle is half a parallelogram of the same base and altitude,) Euc. I. 41) and therefore the truth of the rule is evident.

† DEMONSTRATION.



Take any triangle ABC, and let *nmp* be its inscribed circle, whose centre is O; join AO and CO, and let fall the perpendiculars O*n*, O*m*, and O*p*; in CA produced take AH =B*n*, and erect the perpendicular HG, meeting CO produced in G; make BL and HI each equal to A*n*, and join GL, GI, GB, and GA.

Now, as C*p*=C*m*, A*p*=A*n*, and B*m*=B*n*, it is evident, that CH, or CL, will = half the perimeter of the triangle, and that HA, A*p*, and *p*C will be the differences between the half perimeter and each side respectively. And since CH=CL, CG common, and the angle HCG = angle LCG, therefore GL= GH, and the angle GLC= angle GHC = a right angle. Also,

EXAMPLES.

1. Required the area of the triangle, whose base is 6·25 chains, and perpendicular height 5·20 chains.

---

as GH=GL, HI=LB, and the angles H and L are right angles, therefore GI=GB. In like manner, as GI=GB, AI=AB, and GA common, therefore the angle GAI = angle GAB.

But the points A, n, O, p, fall in the circumference of a circle, therefore the angle IAB= angle pOn; (Euc. III. 22) and consequently their halves HAG and AOp are also equal to each other, and the triangle AHG similar to the triangle ApO. And, as the triangles CpO and CHG are also similar, we shall have HG : pO : : HC : Cp and pA : HG : : pO : AH; whence HG $\times p$A : pO$\times$HG : : HC$\times p$O : C$p\times$AH or pA : pO : : HC $\times p$O : C$p\times$AH, or pA$\times$CH : pO$\times$CH : : HC$\times p$O : Cp $\times$AH, which is the same as the rule; for if this be expressed algebraically, it will be $\sqrt{CH\times pA\times Cp\times AH}$, which is the rule, $=\sqrt{CH^2\times pO^2}=CH\times pO=$ the area. Q. E. D.

Cor. 1. If the triangle be right-angled, the rectangle of the half perimeter, and the difference between the half perimeter and the hypotenuse will be the area; because when CAB is a right angle, BL will be equal to pO.

Cor. 2. If the triangle be equilateral, $\frac{1}{4}\sqrt{3}$, multiplied by the square of the side, will be the area; because, in that case, the perimeter is three times the side, and the three differences are all equal to each other.

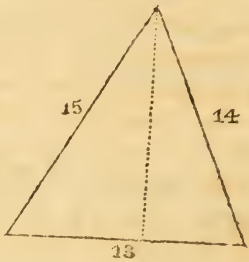
Rule 3. Any two sides of a triangle being multiplied together, and the product again by half the natural sine of their included angle, will give the area of the triangle.

Demonstration.

This follows from Rule 2, mentioned in the note under Prob. I. for a triangle is half of a parallelogram of the same base and height.

```
      6·25
      5·20
    -------
     12500
     3125
    -------
  2)32·5000
  ---------
  16·25 square chains, the answer.
```

2. To find the number of square yards in the triangle, whose three sides are 13, 14, 15 feet.




```
          13
          14
          15
         ----
        2)42
         ----
½ sum    21    21    21    21
         13    14    15     6
        ----  ----  ----  ----
Rem.      8     7     6   126
                            7       . . 9)
                          ----    7056( 84 feet
                           882    64      9⅓ sq. yds.
                             8    --------
                          ----  164|656
                          7056    4|656
                                -------
```

Ans. $9\frac{1}{3}$ square yards.

3. How many square yards are contained in a right-angled triangle, whose base is 40 feet, and perpendicular 30 feet?

Ans. $66\frac{2}{3}$ square yards.

4. How many square yards are contained in the triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet?

Ans. $68\frac{53}{72}$, or 68·7361.

5. To find the area of the triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches.

Ans. 108 feet 5 inches 8''.

PROBLEM III.

To find one side of a right-angled triangle, having the other two sides given.

The square of the hypotenuse is equal to the sum of the squares of the two legs. Therefore,

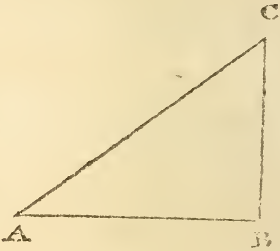
1. *To find the hypotenuse;* add the squares of the two legs together, and extract the square root of the sum.

2. *To find one leg;* subtract the square of the other leg from the square of the hypotenuse, and extract the root of the difference.*

EXAMPLES.

1. Required the hypotenuse of a right-angled triangle, whose base is 40, and perpendicular 30.

```
  40      30
  40      30
 ----    ----
1600     900
 900
----
2500(50 the hypot. AC.
25
---
 00
```



2. What is the perpendicular of a right-angled triangle, whose base AB is 56, and hypotenuse AC 65?

---

* By Euc. 47, I. $AB^2+BC^2=AC^2$, or $AC^2-AB^2=BC^2$, and therefore $\sqrt{AB^2+BC^2}=AC$, or $\sqrt{AC^2-AB^2}=BC$, and is *the same as the rule.*

56	65
56	65
336	325
280	390
3136	4225
	3136
	1089(33 the perpendicular BC.
	9
	63|189
	3|189

3. Required the length of a scaling ladder, to reach the top of a wall, whose height is 28 feet, the breadth of the ditch before it being 45 feet. Ans. 53 feet.

4. To find the length of a shoar, which, strutting 12 feet from the upright of a building, may support a jamb 20 feet from the ground. Ans. 23·32380 feet.

5. A line of 320 feet will reach from the top of a precipice, standing close by the side of a brook, to the opposite bank; required the breadth of the brook, the height of the precipice being 103 feet. Ans. 302·9703 feet.

6. A ladder of 50 feet long, being placed in a street, reached a window 28 feet from the ground on one side; and by turning the ladder over, without removing the foot, it touched a moulding 36 feet high on the other side: required the breadth of the street.

Ans. 76·1233335 feet.

PROBLEM IV.

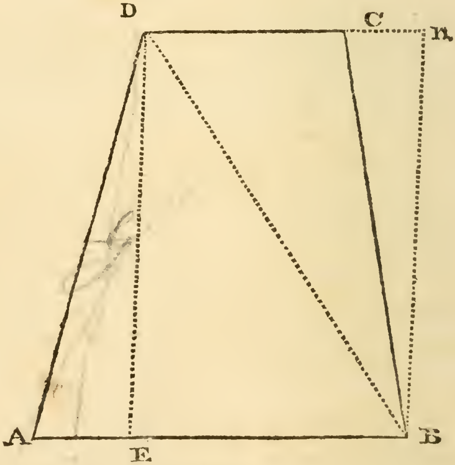
To find the area of a trapezoid.

RULE.*

Multiply the sum of the two parallel sides by the perpendicular distance between them, and half the product will be the area.

EXAMPLES.

1. In a trapezoid, the parallel sides are AB 7·5, and DC 12·25, and the perpendicular distance DE or Bn is 15·4 chains : required the area.




```
  12·25
   7·5
 ------
  19·75
   15·4
 ------
   7900
  9875
 1975
 ------
2)304·150
 ------
152·075 square chains, the answer.
```

---

* DEMONSTRATION.

The Δ ABD is $=\frac{AB \times DE}{2}$, and the Δ BCD$=\frac{DC \times Bn}{2}$, or, (because Bn=DE)$=\frac{DC \times DE}{2}$, Δ $\therefore$ ABD+Δ BCD, or the

2. How many square feet are in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 1 foot 3 inches, and at the less end 11 inches?

Ans. $13\frac{13}{24}$ feet.

3. Required the area of a trapezoid, the parallel sides being 21 feet 3 inches and 18 feet 6 inches, and the distance between them 8 feet 5 inches.

Ans. 167 feet 3 inches 4″ 6‴.

PROBLEM V.

To find the area of a trapezium.

CASE I.

For any trapezium.

RULE.

Divide it into two triangles by a diagonal; then find the areas of these triangles, and add them together.

Or, if two perpendiculars be let fall on the diagonal from the other two opposite angles, the sum of these perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.*

---

whole trapezoid, is $= \frac{AB\times DE}{2} + \frac{DC\times DE}{2} = \frac{\overline{AB+DC}\times DE}{2}$. Q. E. D.

* DEMONSTRATION.

The area of the triangle BDE is $= \frac{BE\times DF}{2}$; and the area of the triangle BAE is $= \frac{BE\times AC}{2}$; and therefore the sum of their areas, or the area of the whole trapezium, is $= \frac{BE\times DF}{2} + \frac{BE\times AC}{2} = \frac{DF+AC}{2}\times BE$.

Q. E. D.

CASE 2.

When the trapezium can be inscribed in a circle.

RULE*.

Add all the four sides together, and take half the sum, and subtract each side separately from the half sum; then multiply the four remainders continually together, and the square root of the last product will be the area of the trapezium.

---

* A trapezium may be inscribed in a circle, when the sum of any two opposite angles in it is equal to two right angles, or 180°. In order to facilitate the demonstration of the rule, it is thought expedient to premise the following

LEMMA.

In a trapezium of the above description, if any two adjacent sides be multiplied together, and also the other two; and if the sum of these two products be multiplied by the sine of the angle, included by either of the pairs of sides multiplied together; then half of the last product is the area.

Thus, $\frac{(AD\times DC+AB\times BC)\times s.\angle D \text{ or } s.\angle B}{2}=$ the area.

DEMONSTRATION.

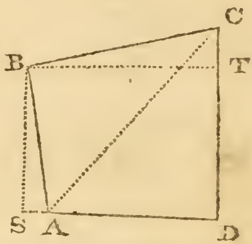
For this product is equal to the two triangles ADC, ABC, according to Rule 3, mentioned in the notes under Prob. II, since the sines of the opposite angles D, B, of a trapezium, inscribed in a circle, are equal to each other.

EXAMPLES.

1. To find the area of the trapezium ABDE, the diagonal BE being 42, the perpendicular DF 18, and the perpendicular AC 16.

Demonstration of the Rule.

For, upon the sides AD, DC, let fall the perpendiculars BS, BT; put $s=$ the sine of the angle A, or of the angle C; and a, b, c, d, $=$ the four sides AB, BC, CD, DA, respectively.



Then $1 : s :: \begin{cases} a : as = \text{BS}, \\ b : bs = \text{BT}, \end{cases}$

And $1 : \sqrt{1-ss} :: \begin{cases} a : a\sqrt{1-ss} = \text{AS}, \\ b : b\sqrt{1-ss} = \text{CT}; \end{cases}$

Hence $\text{SD} = d + a\sqrt{1-ss}$, and $\text{DT} = c - b\sqrt{1-ss}$.

But, by right-angled triangles, $\text{BS}^2 + \text{SD}^2 = \text{DT}^2 + \text{TB}^2$, that is, $dd + 2ad\sqrt{1-ss} + aa = cc - 2bc\sqrt{1-ss} + bb$,

And hence $\sqrt{1-ss} = \dfrac{bb+cc-aa-dd}{2ad+2bc}$,

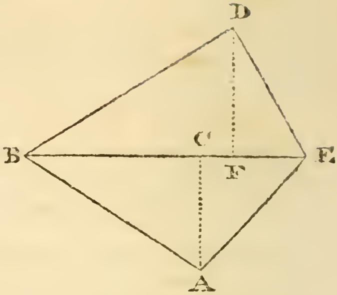
And $s = \dfrac{\sqrt{(2ad+2bc)^2 - (bb+cc-aa-dd)^2}}{2ad+2bc}$;

Then, by lemma, $\frac{1}{4}\sqrt{(2ad+2bc)^2 - (bb+cc-aa-dd)^2} =$ the area, and this expression of the area $= \frac{1}{4}\sqrt{(\overline{b+c}^2 - \overline{a-d}^2) \times (\overline{a+d}^2 - \overline{b-c}^2)} =$

$\frac{1}{4}\sqrt{(a+b+c-d)\times(a+b-c+d)\times(a-b+c+d)\times(-a+b+c+d)} =$

$\sqrt{s-a \times s-b \times s-c \times s-d}$, if s be half the sum of the four sides. Q. E. D.

18
16
———
34
42
———
68
136
———
2)1428
———
714 the answer.



2. In the trapezium AB[redacted], the side AB[redacted] is 15, [redacted] 13, [redacted] 14, AD 12, and the diagonal AC is 16 : required the area.

Cor. 1. The expression marked ‡ is a theorem, which may be useful on many occasions.

Cor. 2. Hence may be deduced Rule 2, for the triangle ; for, if we here suppose one of the sides, as d, to be nothing, or to decrease till it vanish, the rule will become $\sqrt{\overline{s-a}\times\overline{s-b}\times\overline{s-c}\times s}$, the same as in the triangle.

Cor. 3. If, in the trapezium, $a=d$, and $b=c$, the rule will be ab.

Cor. 4. When all the four sides are equal, the rule becomes $\sqrt{a\times a\times a\times a}=a^2$.

AC	16			AC	16		
AB	15			CD	14		
BC	13			AD	12		
	[illegible]44				2)42		
½ sum	22	22	22		21	21	21
	16	15	13		16	14	12
	6	7	9		5	7	0
	7				7		
	42				35		
	9				9		
	378				315		
	22				21		
	756				315		
	756				630		
	8316(91·1921				6615(81·3326		

The triangle ABC	91·1921	
The triangle ADC	81·3326	
The trapezium ABCD	172·5247	the answer.

3. If a trapezium can be inscribed in a circle, and have its four sides 24, 26, 28, 30 ; required its area.

24				
26				
28				
30				
2)108				
54	54	54	54	half sum.
24	26	28	30	

30	28	26	24
28		24	
840		104	
		52	
		624	
		840	
		24960	
		4992	

524160(723·9889488 the answer.

4. How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall upon it 28 and 33·5 feet? Ans. $222\frac{1}{12}$ yards.

5. What is the area of a trapezium, whose diagonal is $108\frac{1}{2}$ feet, and the perpendiculars $56\frac{1}{4}$ and $60\frac{3}{4}$ feet?

Ans. $6347\frac{1}{4}$ feet.

6. What is the area of a trapezium, inscribed in a circle, the four sides being 12, 13, 14, 15?

Ans. 180·9972372.

PROBLEM VI.

To find the area of an irregular polygon.

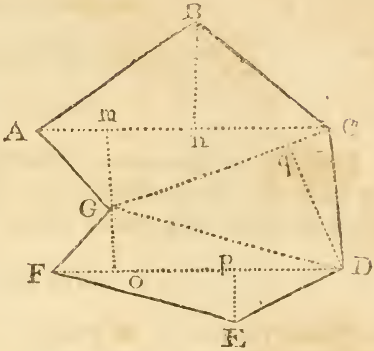
RULE.

Draw diagonals, dividing the figure into trapeziums and triangles. Then find the areas of all these separately, and their sum will be the content of the whole irregular figure.

EXAMPLES.

1. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars, viz.

AC	5˙5
FD	5˙2
GC	4˙4
Gm	1˙3
Bn	1˙8
GO	1˙2
Ep	0˙8
Dq	2˙3



1st. for trapezium ABCG.	2d for trapezium GDEF.	3d for triangle GCD.
1˙3	1˙2	4˙4
1˙8	0˙8	2˙3
3˙1	2˙0	132
5˙5	5˙2	88
155	10˙4	10˙12
155		

17˙05 double ABCG.
10˙40 double GDEF.
10˙12 double GCD.

2)37˙57 double the whole.
18˙785 the answer.

PROBLEM VII.

To find the area of a regular polygon.

RULE 1.*

Multiply the perimeter of the figure, or sum of its sides, by the perpendicular falling from its centre upon one of its sides, and half the product will be the area.

RULE 2.†

Square the side of the polygon; multiply that square by the multiplier, set opposite to its name in the following table, and the product will be the area.

---

* Demonstration. Every regular polygon is composed of as many equal triangles as it has sides, consequently the area of one of those triangles, being multiplied by the number of sides, must give the area of the whole figure; but the area of either of the triangles is equal to the rectangle of the perpendicular and half the base, and therefore the rectangle of the perpendicular and half the sum of the sides is equal to the area of the whole polygon; thus, $OP \times \frac{AB}{2}$ is $=$ area of the ΔAOB, and $OP \times \frac{5AB}{2} =$ area of the polygon ABCDE. Q. E. D.

† Demonstration. The multipliers in the table are the areas of the polygons, to which they belong, when the side is unity or 1.

Now, as all regular polygons, of the same number of sides, are similar to each other, and as similar figures are as the squares of their like sides, (Euc. VI. 20) therefore 1^2 : multiplier in the table : : square of the side of any polygon : area of the polygon; or, which is the same thing, the square of the side of any polygon $\times$ its tabular number is $=$ area of the polygon.

Q. E. D.

No. of sides.	Names.	Multipliers.
3	Trigon, or equil. Δ	0·4330127
4	Tetragon, or square	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

The table is formed by trigonometry, thus: as radius $= 1$: tang. $\angle$ OBP : : BP$(\frac{1}{2})$: PO$=\frac{BP\times \text{tang.} \angle OBP}{\text{radius}}=\frac{1}{2}$ tang. $\angle$OBP; then OP$\times$BP$=\frac{1}{4}$ tang. $\angle$OBP$=$ area of the Δ AOB; and $\frac{1}{4}$ tang. OBP$\times$ number of sides $=$ tabular number, or the area of the polygon.

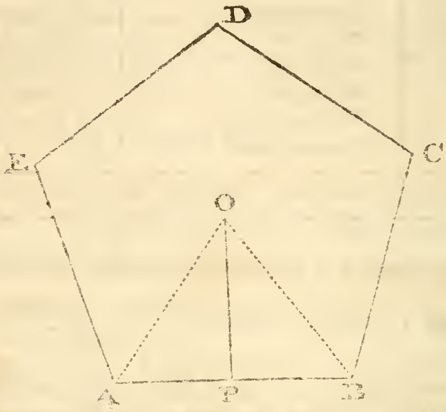
The angle OBP, together with its tangent, for any polygon of not more than 12 sides, is shown in the following

TABLE.

Numb. of sides	Names.	Angle OBP.	Tangents.
3	Trigon	30°	·57735 + $=\frac{1}{3}\sqrt{3}$
4	Tetragon	45°	1·00000 $=1\times 1$
5	Pentagon	54°	1·37638 + $=\sqrt{1+\frac{2}{5}\sqrt{5}}$
6	Hexagon	60°	1·73205 + $=\sqrt{3}$
7	Heptagon	$64°\frac{2}{7}$	2·07652 +
8	Octagon	$67°\frac{1}{2}$	2·41421 + $=1+\sqrt{2}$
9	Nonagon	70°	2·74747 +
10	Decagon	72°	3·07768 + $=\sqrt{5+2\sqrt{5}}$
11	Undecagon	$73°\frac{7}{11}$	3·40568 +
12	Dodecagon	75°	3·73205 + $=2+\sqrt{3}$

EXAMPLES.

1. Required the area of the regular pentagon, whose side AB is 25 feet, and perpendicular OP 17·2047737.



BY THE FIRST RULE.

17·204774	perpendicular.
125	perimeter.
86023870	
34409548	
17204774	
2)2150·596750	
1075·298375	answer.

BY THE SECOND RULE.

First 25	Then 1·7204774
25	625

125	86023870
50	34409548
——	103228644
625	————
——	1075·2983750 answer.

2. To find the area of the hexagon, whose side is 20.
Ans. 1039·2304.

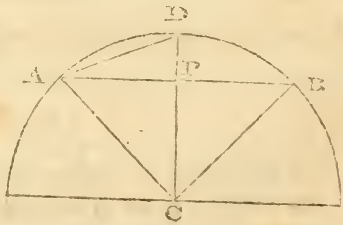
3. To find the area of the trigon, or equilateral triangle, whose side is 20. Ans. 173·2052.

4. Required the area of an octagon, whose side is 20.
Ans. 1931·37084.

5. What is the area of a decagon, whose side is 20?
Ans. 3077·68352.

PROBLEM VIII.

In a circular arc, having any two of these given, to find the rest; namely, the chord AB *of the arc, the height, or versed sine* DP, *the chord* AD *of half the arc, and the diameter, or the radius* AC *or* CD.



Here will always be given two sides of the right-angled triangles APC, APD; and therefore the other parts will easily be found from the property in problem III, namely, that the square of the longest side is equal to the squares of the two shorter added together.

Thus, *if there be given the radius and the chord* AB, *or its half* AP. Then $\sqrt{AC^2 - AP^2} = CP$; and $CD - CP = PD$; and $\sqrt{AP^2 + PD^2} = AD$.

Again, *when the radius and height* PD *are given.* Then CD—DP=CP; and $\sqrt{CA^2-CP^2}=AP$.

And *when* AP *and* PD *are given.* Then as DP : PA :: PA : $CD+CP=\frac{PA^2}{PD}$, and $2CD=\frac{PA^2}{PD}+PD$.

EXAMPLES.

1. Suppose the radius AC or CD to be 10, and the half chord AP 8.

Then $\sqrt{AC^2-AP^2}=\sqrt{100-64}=\sqrt{36}=6=CP$; and $CD-CP=10-6=4=PD$; and $\sqrt{AP^2+PD^2}=\sqrt{64+16}=\sqrt{80}=8{\cdot}94427191=AD$.

2. If the radius be 10, and PD 4.

Then $CD-DP=10-4=6=CP$; and $\sqrt{CA^2-CP^2}=\sqrt{100-36}=\sqrt{64}=8=AP$.

3. When AP is 8, and DP 4.

Then $\frac{PA^2}{PD}=\frac{64}{4}=16=CD+CP$.

And $2CD=16+PD=20$, or $CD=10$.

PROBLEM IX.

To find the diameter and circumference of a circle, either from the other.

RULE 1.*

As 7 is to 22, so is the diameter to the circumference.
As 22 is to 7, so is the circumference to the diameter.

---

* The ratio of the diameter of a circle to its circumference has never yet been exactly attained. Nor can a square, or any other right-lined figure, be found, that shall be equal to a given circle. This is the famous problem, called the *the squaring of the circle*, which has exercised the abilities of the greatest mathematicians for ages, and has been the occasion of so many endless disputes.

RULE 2.

As 113 is to 355, so is the diameter to the circumference.

As 355 is to 113, so is the circumference to the diameter.

Several persons of considerable eminence have, at different times, pretended, that they had discovered the exact quadrature ; but their errors have soon been detected, and it is now generally looked upon as a thing impossible to be done.

But though the relation between the diameter and circumference cannot be expressed in known measure, it may yet be approximated to any assigned degree of exactness. And thus that incomparable Geometer, the great Archimedes, about two thousand years ago, had discovered this ratio to be nearly as 7 is to 22, *which is the same as our first rule.*

By inscribing and circumscribing polygons of 96 sides, he found the ratio to be less than $3\frac{1}{7}$, but greater than $3\frac{10}{71}$ to 1 ; and thence inferred the ratio above mentioned, as may be seen in his book *de dimensione circuli.* And in this manner was the problem more anciently performed by *Philo Gedarensis*, and by *Apollonius Pargæus*, in a work, not come to our hands, called *Ocyteoboos*, as we are informed by *Eutocius*, in his commentary on Archimedes.

The ratio of *Victa* and *Metius* is that of 113 to 355, which is something more exact than the former, *and is the same as the second rule.*

But the first, who ascertained this ratio to any great degree of exactness, was *Van Ceulen*, a Dutchman, in his book *de Circulo et Adscriptis.* He found, that if the diameter of a circle was 1, the circumference would be 3·14159265358979323846264338 3279502884 *nearly.* And this is exactly true to 36 places of decimals, and was effected by means of the continual bisection of an arc of a circle, which was so exceedingly troublesome and laborious, that it must have cost him incredible pains. It is said to have been thought so curious a performance, that the numbers were cut on his tomb stone, in *St. Peter's church yard* at *Leyden.* This last number was not only confirmed, but extended to double the number of places, by the ingenious Mr. *Abraham Sharp.*

RULE 3.

As 1 is to 3·1416, so is the diameter to the circumference.

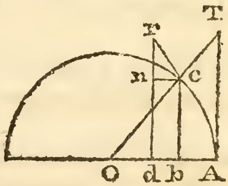
As 3·1416 is to 1, so is the circumference to the diameter.

But since the invention of *Fluxions*, and the *Summation of Infinite Series*, there have been several methods found out for doing the same thing with less labor and trouble, and far more expedition. Mr. *John Machin*, Professor of Astronomy in *Gresham College*, has, by these means, given a quadrature of the circle, which is true to 100 places of decimals ; and M. *De Lagny* and M. *Euler* have carried it still farther. And these last expressions are so extremely near the truth, that, except the ratio could be completely obtained, we need not wish for a greater degree of accuracy.

The method of obtaining this proportion from the doctrine of fluxions may be shewn as follows :—

Take A*c* any arc of a circle, and let *cr* be an indefinitely small tangent at the point *c*.

Then draw the lines as in the figure, and put OA$=r$, A$b=x$, $bc=y$, AT$=t$, and A$c=z$; and for the fluxion of a simple quantity put a point over it.



Now, since the triangles *rcn*, *ch*O, and TAO are similar, we shall have $bc\ (y)\ :\ c\mathrm{O}\ (r)\ ::\ cn\ (\dot{x})\ :\ cr = \dot{z} = \frac{r\dot{x}}{y} =$

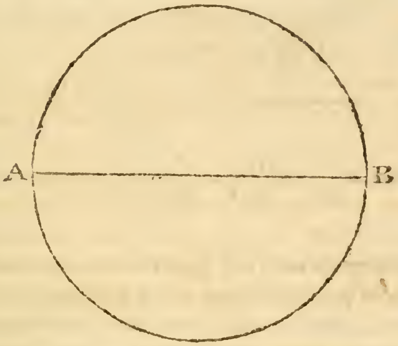
$\frac{r\dot{x}}{\sqrt{2rx-x^2}}$ =fluxion of the arc, in terms of the versed sine.

And also, O$b(\sqrt{r^2-y^2})$: Oc (r) : : $nr(\dot{y})$: $cr = \dot{z} =$

$\frac{r\dot{y}}{\sqrt{r^2-y^2}}$ =fluxion of the arc, in terms of the sine.

EXAMPLES.

1. To find the circumference of a circle, whose diameter AB is 10.



And, in like manner, AT (t) : OT ($\sqrt{r^2+t^2}$) : : cn ($\dot{x}$) : $cr=\dot{z}=\frac{\dot{x}\sqrt{r^2+t^2}}{t}$; but OT ($\sqrt{r^2+t^2}$) : OA ($r$) : : O$c$ (r) : O$b=\frac{r^2}{\sqrt{r^2+t^2}}$, and therefore A$b$ (x)$=r-\frac{r^2}{\sqrt{r^2+t^2}}$, whose fluxion is $\frac{r^2t\dot{t}}{(r^2+t^2)^{\frac{3}{2}}}=\dot{x}$; and consequently $\frac{\dot{x}\sqrt{r^2+t^2}}{t}$ $=\frac{\sqrt{r^2+t^2}}{t}\times\frac{r^2t\dot{t}}{(r^2+t^2)^{\frac{3}{2}}}=\frac{r^2\dot{t}}{r^2+t^2}=$ fluxion of the arc in terms of the tangent.

Now, from any of the three forms of fluxions here found, their fluents, or the value of the arc itself, will become known.

But the third form, expressed in terms of the tangent, will be the most convenient, because it is entirely free from radical quantities; and therefore, if $\frac{r^2\dot{t}}{r^2+t^2}$ be converted into an infinite series, we shall have $z=\frac{r^2\dot{t}}{r^2+t^2}=\dot{t}-\frac{t^2\dot{t}}{r^2}+\frac{t^4\dot{t}}{r^4}-\frac{t^6\dot{t}}{r^6}$, &c. and

BY RULE 1.

$7 : 22 :: 10 : 31{\cdot}42857$

$$\begin{array}{r} 22 \\ 10 \\ \hline 7)220 \\ 31\frac{3}{7} \end{array}$$

Or $31{\cdot}42857$, answer.

---

its fluent $=z=t-\dfrac{t^3}{3r^2}+\dfrac{t^5}{5r^4}-\dfrac{t^7}{7r^6}+\dfrac{t^9}{9r^8}$, &c. = length of the arc Ac.

Then, by taking Ac=to any given arc, whose tangent can be found in terms of the radius, the series will become known ; and being repeated as often as Ac is contained in the whole circumference, we shall have the length of the circumference in terms of the diameter.

Thus, if the radius be 1, and Ac be $\frac{1}{8}$ part of the circumference, or 45°, its tangent will be equal to the radius, and the series will become $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, &c. = arc of 45°, and $8\times : 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, &c. = whole circumference.

This series is the simplest form, that can possibly be obtained, but in order to get another, that will converge faster, we must take a smaller arc ; as, for instance, suppose that of 30° or $\frac{1}{12}$ part of the circumference.

Then, since the tangent of 30°, to radius 1, is $\sqrt{\frac{1}{3}}$, the general series will become $\sqrt{\frac{1}{3}}\times : 1-\dfrac{1}{3\cdot3}+\dfrac{1}{5\cdot3^2}-\dfrac{1}{7\cdot3^3}+\dfrac{1}{9\cdot3^4}$, &c. = arc of 30°, and $12\sqrt{\frac{1}{3}}\times : 1-\dfrac{1}{3\cdot3}+\dfrac{1}{5\cdot3^2}-\dfrac{1}{7\cdot3^3}+\dfrac{1}{9\cdot3^4}$, &c. = whole circumference, *and so for any other arc whatever.*

Those, who would wish to see the methods of *Machin*, *Euler*, &c. may consult Dr. Hutton's Mensuration, and a paper of his in the Philosophical Transactions, upon this subject.

BY RULE 2.

$113 : 355 :: 10 : 31\frac{47}{113}$

$\quad\quad\quad 10$

113)3550(31·41593, the answer.
160
470
180
670
1050
330

BY RULE 3.

1 : 3·1416 :: 10 : 31·416, the answer.

2. To find the diameter, when the circumference is 50.

BY RULE 1.

$$22 : 7 :: 50 : \frac{7 \times 25}{11} = \frac{175}{11} = 15\frac{10}{11} = 15\cdot\dot{9}\dot{0}90, \text{ ans.}$$

BY RULE 2.

$355 : 113 :: 50 : 15\frac{65}{71}$

$\quad\quad\quad 50$

355|5650
71|1130(15·9155
420
650
110
390
350

BY RULE 3.

3'1416 : 1 :: 50 : 15'9156

```
              50
         ----------
3'1416)50'000(15'9156
 .....  18584
         2876
           49
           18
            2
```

3. If the diameter of the earth be 7958 miles, as it is very nearly, what is the circumference, supposing it to be exactly round?

Ans. 25000'8528 miles.

4. To find the diameter of the globe of the earth, supposing its circumference to be 25000 miles.

Ans. $7957\frac{3}{4}$ nearly.

PROBLEM X.

To find the length of any arc of a circle.

RULE 1.

As 180 is to the number of the degrees in the arc,
So is 3'1416 times the radius to its length.
Or, As 3 is to the number of degrees in the arc,
So is '05236 times the radius to its length.

EXAMPLES.

1. To find the length of an arc ADB of 30 degrees, the radius being 9 feet.

[See figure under problem VIII. page 425.]

				3·1416		
				9		
As 180	:	30		———		
Or 6	:	1	::	282744	:	4·7124
Or 3	:	30	::	·05236×9	:	4·7124
				36		
				4·7124 the answer.		

RULE*.

From 8 times the chord of half the arc subtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc nearly.

---

* DEMONSTRATION. Let the radius CA$=r$, and sine AP $=s$. Then will the chord AD $=\sqrt{s^2+PD^2}=\sqrt{s^2+\overline{r-CP}|^2}$ $=\sqrt{s^2+\overline{r-\sqrt{r^2-s^2}}|^2}=s+\frac{s^3}{8r^2}+\frac{7s^5}{128r^4}$, &c.

And therefore 8 times the chord AD $=8s+\frac{s^3}{r^2}+\frac{7s^5}{16r^4}$, &c.

And consequently 8 times the chord AD — chord AB$(2s)=$ $6s+\frac{s^3}{r^2}+\frac{7s^5}{16r^4}$, &c. whose $\frac{1}{3}$ part is $2s+\frac{s^3}{3r^2}+\frac{7s^5}{48r^4}$, &c.

But the length of the arc AD, whose sine is s, is known to be $s+\frac{s^3}{6r^2}+\frac{3s^5}{40r^4}$, &c. and therefore the arc AB $=2s+\frac{s^3}{3r^2}+\frac{6s^5}{40r^4}$, &c. which differs from $2s+\frac{s^3}{6r^2}+\frac{7s^5}{48r^4}$, &c. only by a small quantity, and shows the rule to be very near the truth.

Q. E. D.

2. The chord AB of the whole arc being 4·65874. and the chord AD of the half arc 2·34947 ; required the length of the arc.

$$\begin{array}{r} 2\cdot34947 \\ \\ \hline 18\cdot79576 \\ 4\cdot65874 \\ \hline 3)14\cdot13702 \\ \text{Answer}\quad 4\cdot71234 \end{array}$$

3. Required the length of an arc of 12 degrees 10 minutes, the radius being 10 feet. Ans. $2\cdot1234\frac{8}{9}$.

4. To find the length of an arc, whose chord is 6, and the chord of its half is $3\frac{1}{2}$. Ans. $7\frac{1}{3}$.

5. Required the length of the arc, whose chord is 8, and height PD 3. Ans. $10\frac{2}{3}$.

Cor. When the chord of the whole arc is given, the rule will be $\dfrac{8\sqrt{s^2+\overline{r-\sqrt{r^2-s^2}}^{\,2}}-2s}{3}$.

A great number of approximating rules might be given for finding the arc of a circle ; but the two, given in the text, and the three following ones will be found sufficient.

RULE 1.

·01745, &c. × rad. × number of degrees in any arc = the length of that arc.

RULE 2.

$$4CD\times\sqrt{\frac{3PD}{6CD-PD}} = \text{arc ADB } \textit{nearly}.$$

RULE 3.

$$10CD\times\sqrt{\frac{5PD}{10CD-3PD}}+4\sqrt{2CD\times PD}\times\frac{2}{9} = \text{arc ADB}$$

extremely near.

6. Required the length of the arc, whose chord is 6, the radius being 9. Ans. 6·11706.

PROBLEM XI.

To find the area of a circle.

The area of a circle may be found from the diameter and circumference together, or from either of them alone, by the following rules.

RULE 1.*

Multiply half the circumference by half the diameter.

Or, Take $\frac{1}{4}$ of the product of the whole circumference and diameter.

RULE 2.†

Multiply the square of the diameter by ·7854.

---

* DEMONSTRATION. A circle may be considered as a regular polygon of an infinite number of sides, the circumference being equal to the perimeter, and the radius to the perpendicular. But the area of a regular polygon is equal to half the perimeter multiplied by the perpendicular, and consequently the area of a circle is equal to half the circumference multiplied by the radius, or half the diameter. Q. E. D.

† DEMONSTRATION. All circles are to each other as the squares of their diameters. (Euc. XII. 2.)

But the area of a circle, whose diameter is 1, is ·7854, &c. (by Rule 1.) Therefore $1^2 : d^2 :: {\cdot}7854, \&c. : \frac{{\cdot}7854, \&c. \times d^2}{1^2}$ $= {\cdot}7854, \&c. \times d^2 =$ area of a circle, whose diameter is d.

Q. E. D.

The following propositions are those of *Metius* and *Archimedes*.

RULE 3.

Multiply the square of the circumference by ·07958.

RULE 4.

As 14 is to 11, so is the square of the diameter to the area.

RULE 5.

As 88 is to 7, so is the square of the circumference to the area.

As 452 : 355 : : square of the diameter : area.
As 14 : 11 : : square of the diameter : area.

If the circumference be given, instead of the diameter, the area may be found as follows.

The square of the circumference ×·07958 = area.
As 88 : 7 : : square of the circumference : area.
As 1420 : 113 : : square of the circumference : area.

And if d be the diameter, c the circumference, a the area, and p=3·14159, &c. then

$$1.\ d=\frac{c}{p}=\frac{4a}{c}=2\sqrt{\frac{a}{p}}.$$

$$2.\ c=pd=\frac{4a}{d}=2\sqrt{pa}.$$

$$3.\ a=\frac{pd^2}{4}=\frac{c^2}{4p}=\frac{dc}{4}.$$

The following table will also show most of the usual problems, relating to the circle and its equal or inscribed square.

1. Diameter × ·8862 = side of an equal square.
2. Circumference ·2821 = side of an equal square.
3. Diameter × ·7071 = side of the inscribed square.

EXAMPLES.

1. To find the area of a circle, whose diameter is 10, and circumference 31·4159265.

By Rule 1.

```
     31·4159265
             10
     ----------
   4)314·159265
Area  78·539816
```

By Rule 2.

```
·7854
  100
-----
78·5400
```

By Rule 3.

```
986·96044  square circum.
  85970    inverted.
---------
  6908723
   888264
    49348
     7896
---------
Area 78·54231
```

By Rule 4.

```
      100
       11
    -----
14 | 1100
 7 |  550
 2 | 78·57
```

4. Circumference × ·2251 = side of the inscribed square.
5. Area × ·6366 = side of the inscribed square.
6. Side of a square × 1·4142 = diameter of its circumscribed circle.
7. Side of a square × 4·443 = circumference of its circumscribed circle.
8. Side of a square × 1·128 = diameter of an equal circle.
9. Side of a square × 3·545 = circumference of an equal circle.

By Rule 5.

31·4159265 circumference.
562951413 inverted.

94247779
3141593
1256637
31416
15708
2827
63
19
2

---

88 : 7 : : 986·96044
7

---

8 | 6908·72308
11 | 863·59038
78·50821

2. Required the area of the circle, whose diameter is 7, and circumference 22.

Ans. $38\frac{1}{2}$.

3. What is the area of a circle, whose diameter is 1, and circumference 3·1416 ?

Ans. ·7854.

4. What is the area of a circle, whose diameter is 7 ?

Ans. 38·4846.

5. How many square yards are in a circle, whose diameter is $3\frac{1}{2}$ feet ?

Ans. 1·069.

6. How many square feet does a circle contain, the circumference being 10·9956 yards ?

Ans. 86·19266.

PROBLEM XII.

To find the area of a sector of a circle.

RULE 1.*

Multiply the radius, or half the diameter, by half the arc of the sector for the area. Or, take $\frac{1}{4}$ of the product of the diameter and arc of the sector.

NOTE. The arc may be found by Problem X.

RULE 2.†

As 360 is to the degrees in the arc of the sector, so is the whole area of the circle to the area of the sector.

NOTE. For a semicircle, take one half ; for a quadrant, one quarter, &c. of the whole circle.

EXAMPLES.

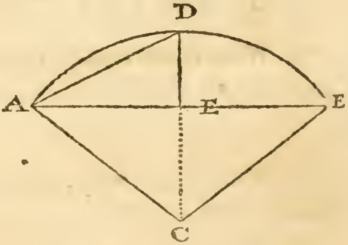
1. What is the area of the sector CAB, the radius being 10, and the chord AB 16 ?

$100=AC^2$
$64=AE^2$

$36(6=CE$
$10=CD$

$4=DE$

$16=DE^2$
$64=AE^2$



---

* The rule for finding the area of the sector is evidently the same, as that for finding the area of the whole circle.

† DEMONSTRATION. Let $r=$ radius, $d=$ number of degrees in the arc of the sector, and $A=$ its area.

Then will $4r^2\times{\cdot}7854=r^2\times3{\cdot}1416=$ area of the whole circle, and $2r\times3{\cdot}1416=$its circumference.

8̇0(8˙9442719=AD
8

71˙5541752
16

3)55˙5541752
2)18˙5180584=arc ABD
9·2590297=half arc
10=radius

92˙590297 answer.

2. Required the area of the sector, whose arc contains 18 degrees, the diameter being 3 feet.

˙7854
9

7˙0686 the area of the whole circle.

Then, as 360 : 18
Or, as 20 : 1 : : 7˙0686 : ˙35343 answer.

3. What is the area of the sector, whose radius is 10, and arc 20?

Ans. 100.

4. What is the area of the sector, whose radius is 9, and the chord of its arc 6?

Ans. 27˙52678.

Also 360 : $2r \times 3^{\cdot}1416$: : d : $\frac{2dr \times 3^{\cdot}1416}{360}$ = length of the arc of the sector. But $\frac{2dr \times 3^{\cdot}1416}{360} \times \frac{1}{2} \times r = \frac{dr^2 \times 3^{\cdot}1416}{360} = A$, by the last rule. And consequently 360 : d : : $r^2 \times 3^{\cdot}1416$: A.

Q. E. D.

5. Required the area of a sector, whose radius is 25, its arc containing 147 degrees 29 minutes. Ans. 804·4017.

6. To find the area of a quadrant and a semicircle, to the radius 13.

Ans. 132·7326 and 265·4652.

PROBLEM XIII.

To find the area of a segment of a circle.

RULE I.

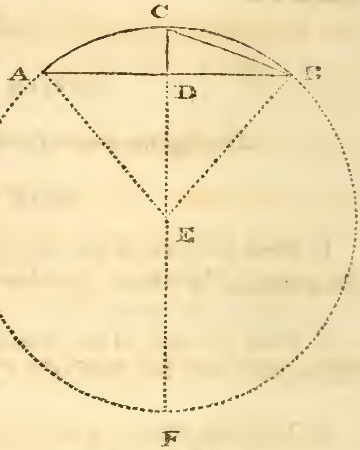
1. Find the area of the sector, having the same arc with the segment, by the last problem.

2. Find the area of the triangle, formed by the chord of the segment and the radii of the sector.

3. Then the sum of these two will be the answer, when the segment is greater than a semicircle; but the difference will be the answer, when it is less than a semicircle.

EXAMPLE.

Required the area of the segment ACBA, its chord AB being 12, and the radius EA or CE 10.



100 AE^2
36 AD^2

64 DE^2

Its root 8 DE
From 10 CE

2 CD

4 CD^2
36 AD^2

40 chord AC^2

Its root 6·324555 chord AC
8

6 AD
8 DE

48 area of △EAB.

50·596440
12·

3)38·59644
2)12·86548 the arc ACB.
6·43274 $\frac{1}{2}$ arc.
10 radius.

64·3274 area of sector EACB.
48·0000 area of triangle EAB.

Ans. 16·3274 area of segment ACBA.

RULE 2.*

1. To the chord of the whole arc add $\frac{4}{3}$ of the chord of half the arc, or add the latter chord and $\frac{1}{3}$ of it more.

2. Multiply the sum by the versed sine or height of the segment, and $\frac{4}{10}$ of the product will be the area of the segment.

---

* DEMONSTRATION. Let $d=$ the diameter CF, and $v=$ the versed sine CD; then $\sqrt{dv-v^2}=$BD, and the fluxion of the half segment $=\dot{v}\sqrt{dv}\times : 1-\frac{v}{2d}-\frac{v^2}{8d^2}-\frac{3v^3}{48d^3}$ &c. and its fluent, or the value of the half segment, $=v\sqrt{dv}\times : \frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$ &c. Consequently its double $2v\sqrt{dv}\times : \frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$, &c. is the value of the whole segment.

Now suppose the segment $=2\text{CD}\times\overline{m\times\text{BD}+n\times\text{BC}}=$ $2v\times\overline{m\sqrt{dv-v^2}+n\sqrt{dv}}=2v\sqrt{dv}\times\overline{m\sqrt{1-\frac{v}{d}}+n}=$ $2v\sqrt{dv}\times : \frac{+m}{+n}-\frac{mv}{2d}-\frac{mv^2}{8d^2}$, &c.

Then let the coefficients of the corresponding terms be equated, and we shall have $m+n=\frac{2}{3}$, and $\frac{m}{2}=\frac{1}{5}$; whence $m=\frac{2}{5}$, and $n=\frac{2}{3}-m=\frac{2}{3}-\frac{2}{5}=\frac{4}{15}$.

Then, by substituting these values of m and n in the assumed quantity, we shall have $2\text{CD}\times\overline{m\times\text{BD}+n\times\text{BC}}=2\text{CD}\times\overline{\frac{2}{5}\text{BD}+\frac{4}{15}\text{BC}}=\frac{4}{10}\text{CD}\times\overline{2\text{BD}+\frac{4}{3}\text{BC}}$, *which is the rule.*

EXAMPLE.

Take the same example, in which the radius is 10, and the chord AB 12.

Then, as before, are found CD=2, and the chord of the half arc AC.

	6·324555
Hence $\frac{1}{3}$ is	2·108185
AB	12·
	20·432740
CD	2
	40·86548
	·4
Ans.	16·346192, area nearly.

RULE 3.*

1. Divide the height of the segment by the diameter, and find the quotient in the column of heights or versed sines, in the table of the areas of the segments of a circle.

2. Take out the corresponding area in the next column on the right, and multiply it by the square of the diameter, for the answer.

---

* The table, to which this rule refers, is formed of the areas of the segments of a circle, whose diameter is 1; and which is supposed to be divided by perpendicular chords into 1000 equal parts, and is at the end of MENSURATION.

The reason of the rule itself depends upon this property.— That the versed sines of similar segments are as the diameters of the circles, to which they belong, and the areas of those segments as the squares of the diameters; which may be thus proved:

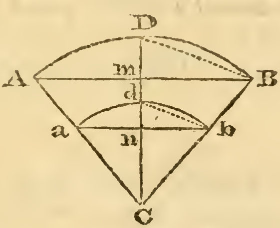
EXAMPLE.

The example being the same as before, we have CD equal to 2, and the diameter 20.

Then 20) 2 (˙1
And to ˙1 answer ˙040875
Square of diam. 400

Ans. 16˙3500.

Let ADBA and *adba* be any two similar segments, cut off from the similar sectors ADBCA and *adb*C*a* by the chords AB and *ab*; and let fall the perpendicular CD.



Then, by similar triangles, DB : *db* : : BC : *b*C,
and DB : *db* : : D*m* : *dn*;
whence, by equality, BC : *b*C : : D*m* : *dn*,
or 2BC : 2*b*C : : D*m* : *dn*.

Again, since similar segments are as the squares of their chords, it will be AB^2 : *ab* : : ADBA : *adba*;
but $AB^2 : ab^2 :: CB^2 : Cb^2$,
and therefore, by equality, ADBA : *adba* : : $CB^2 : Cb^2$,
or ADBA : *adba* : : $4CB^2 : 4Cb^2$. Q. E. D.

Now, if d be put equal to any diameter, and v the versed sine, it will be $d : v :: 1$ (diameter in the table) : $\frac{v}{d}$ = versed sine of a similar segment in the table, whose area let be called a.

OTHER EXAMPLES.

2. What is the area of the segment, whose height is 2, and chord 20 ?

Ans. 26·878787.

3. What is the area of the segment, whose height is 18, and diameter of the circle 50 ?

Ans. 636·375.

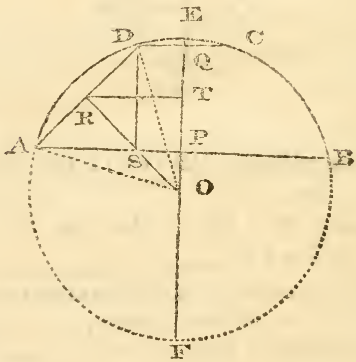
4. Required the area of the segment, whose chord is 16, the diameter being 20.

Ans. 44·7292.

PROBLEM XIV.

*To find the area of a circular zone** ADCBA.

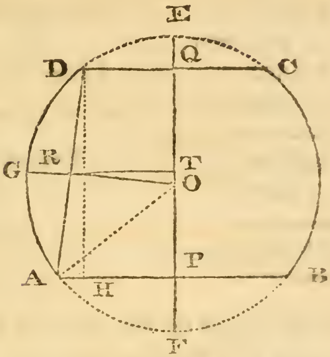
FIGURE 1.



Then $1^2 : d^2 :: a : ad^2 =$ area of the segment, whose height is v, and diameter d, as in the rule.

* The space included between any two parallel chords and their intercepted arcs.

FIGURE 2.



RULE 1.

Find the areas of the two segments AEB, DEC, and their difference will be the zone ADCB.

RULE 2.

To the area of the trapezoid ARDQP add the area of the small segment ADR ; and double the sum for the area of the zone ADCB.

EXAMPLES.

1. What is the area of the zone, less than a semicircle, figure 1, having the greater chord 16, the less chord 6, and the diameter of the circle 20 ?

Here $OQ=\sqrt{OD^2-DQ^2}=\sqrt{100-9}=\sqrt{91}=9{\cdot}539392$.

And $OP=\sqrt{OA^2-AP^2}=\sqrt{100-64}=\sqrt{36}=6$.

10	OE	10	OE
6	OP	9·539392	OQ
20)4	PE	20)0·460608	QE
·2	tab. vers.	·023	tab. vers.
Answers to	·111823	Answers to	·004618
	400		400
Greater seg.	44·7292	Less seg.	1·8472
Less seg.	1·8472		
Answer	42·882		

2. If the greater chord be 96, the less 60, and the distance between them 26; required the area.

Here, if R be the middle of the chord AD, and O the centre of the circle; then, in figure 1, and by the first rule,

DS=26

AS=AP—DQ=48—30=18

RT=$\frac{1}{2}$AP+$\frac{1}{2}$DQ=24+15=39

DS : AS : : RT : OT=$\frac{39\times18}{26}$=27

TP=TQ=$\frac{1}{2}$DS=13

OP=OT—TP=27—13=14

OQ=OT+TQ=27+13=40

OE=OA=$\sqrt{AP^2+OP^2}=\sqrt{48^2+14^2}$=50

EQ=OE—OQ=50—40=10

EP=OE—OP=50—14=36

100)36(·36, its tab. seg. ·25455

100)10(·1, its tab. seg. ·04088

Difference	·21367
Square Diameter	10000
Answer	2136·7

3. If the greater chord be 40, the less 30, and the distance between them 35; required the area of the zone in figure 2, and by the second rule.

Here, as before, we have

$TR=\frac{1}{2}AP+\frac{1}{2}DQ=10+7\frac{1}{2}=17\frac{1}{2}$

$PQ : AP—DQ :: RT : OT=\frac{17\frac{1}{2}\times5}{35}=\frac{5}{2}$

$OP=TP—OT=\frac{1}{2}PQ—OT=15$

$OG=OA=\sqrt{\overline{OP^2+AP^2}}=\sqrt{\overline{15^2+20^2}}=25$

$OR=\sqrt{\overline{OT^2+TR^2}}=\sqrt{\frac{5^2}{4}+\frac{35^2}{4}}=\frac{5}{2}\sqrt{50}=$

17·677669

GR=OG—OR=25—17·677669=7·322331

50)7·322331

·1464466, its tab. seg. ·071349
square diam. 2500

Area of seg. AGDR	178·37
But PQ×TR=trap. ADQP	612·5
Sum	790·87
	2
Whole zone ADCB	1581·74

4. If one end be 48, the other 30, and the breadth or distance 13; what is the area of the zone?

Ans. 534·4249.

PROBLEM XV.

To find the area of a circular ring, or space included between two concentric circles.

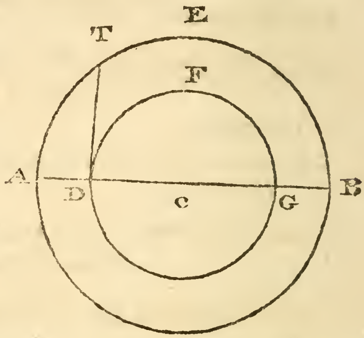
RULE. *

The difference between the two circles will be the ring. Or, multiply the sum of the diameters by their difference, and multiply the product by ·7854 for the answer.

* DEMONSTRATION. The area of the circle AIEBA= $AB^2 \times$ ·7854, and the area of the small circle GFD is = $GD^2 \times$ ·7854; therefore the area of the ring = $AB^2 \times$ ·7854—$GD^2 \times$ ·7854=$\overline{AB+GD} \times \overline{AB-GD} \times$ ·7854. Q. E. D.

EXAMPLES.

1. The diameters of the two concentric circles being AB 10 and DG 6; required the area of the ring contained between their circumferences AEBA and DFGD.



	10
	6
Sum	16
Diff.	4
	64

'7854
64

31416
47124

50'2656 answer.

2. The diameters of two concentric circles being 20 and 10; required the area of the ring between their circumferences.

Ans. 235'62.

3. What is the area of the ring, the diameters of whose bounding circles are 6 and 4?

Ans. 15'708.

Cor. If DI be a perpendicular at the point D, then will the area of the ring be equal to that of a circle, whose radius is DI.

RULE 2.

Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.

PROBLEM XVI.

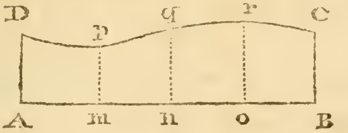
To measure long irregular figures.

RULE.

Take the breadth in several places at equal distances. Add all the breadths together, and divide the sum by the number of them, for the mean breadth, which multiply by the length, for the area.

EXAMPLES.

1. The breadths of an irregular figure, at five equidistant places being AD 8·1, *mp* 7·4, *nq* 9·2, *or* 10·1, BC 8·6; and the length AB 39; required the area.




```
    8·1
    7·4
    9·2
   10·1
    8·6
 ------
 5)43·4
 ------
   8·68
     39
 ------
   7812
  2604
 ------
 338·52 answer.
 ------
```

2. The length of an irregular figure being 84, and the breadths at 6 places 17·4, 20·6, 14·2, 16·5, 20·1, 24·3; what is the area?

Ans. 1583·4.

PROBLEM XVII.

*To find the circumference of an ellipse.**

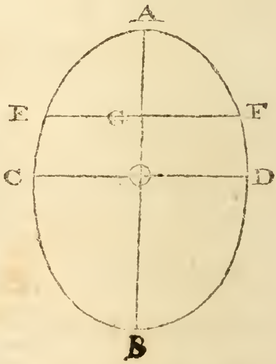
RULE†.

Add the two axes together, and multiply the sum by 1·5708, for the circumference nearly.

EXAMPLES.

1. Required the circumference of an ellipse, whose two axes are 70 and 50.

70	
50	
120	sum
1·5708	
188·4960	circum. nearly.



* For definitions of *ellipse*, *parabola*, and *hyperbola*, see CONIC SECTIONS.

† It will be evident, that this rule is very near the truth, if it be considered, that this arithmetical mean between the axes exceeds their geometrical mean; and, that the geometrical mean is the diameter of a circle, equal in area to the ellipse; which circle is of less ambit than the ellipse, or any other figure of the same area.

2. What is the periphery of an ellipse, whose two axes are 24 and 20?

Ans. 69·1152.

PROBLEM XVIII.

To find the area of an ellipse.

RULE.*

Multiply the transverse by the conjugate, and the product, multiplied by ·7854, will be the area.

Or, Multiply ·7854 by one axe, and the product by the other.

EXAMPLES.

1. To find the area of an ellipse, whose two axes are 70 and 50.

```
   ·7854
      50
 -------
 39·2700
      70
 -------
2748·9000 answer.
```

2. What is the area of an ellipse, whose two axes are 24 and 18? Ans. 339·2928.

---

* The demonstration of this rule is contained in that of the next problem.

PROBLEM XIX.

To find the area of an elliptic segment.

RULE.*

Divide the height of the segment by the axis of the ellipse, of which it is a part; and find, in the table of circular segments, a circular segment having the same versed sine as this quotient. Then multiply continually together this segment and the two axes, for the area required.

EXAMPLES.

1. What is the area of an elliptic segment EAF, whose height AP is 20; the transverse AB being 70, and the conjugate CD 50?

70)20(·285$\frac{5}{7}$ the tab. vers.
The corresponding seg. is

---

* Demonstration. Let the transverse diameter AB $=a$, the conjugate CD $=c$, AG $=x$, and EG $=y$; then, by the property of the curve, we shall have $y=\frac{c}{a}\sqrt{ax-x^2}$, and the fluxion of the area EAF $=(y\dot{x})=\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$. But $\dot{x}\sqrt{ax-x^2}$ is known to express the fluxion of the corresponding circular segment, whose versed sine is x, and diameter a. Let the fluent of this expression, therefore, be denoted by A, and then the fluent of $\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ will be $=\frac{c}{a}\times$ A, whence the rule is formed.

Q. E. I.

Cor. The ellipse is equal to a circle, whose diameter is a mean proportional between the two axes, and hence the rule is formed for the whole ellipse.

```
  ˙185166
       70
 --------
12˙961620
       50
---------
648˙081000
----------
```

2. What is the area of an elliptic segment, cut off parallel to the shorter axis, the height being 10, and axes 25 and 35?

Ans. 162˙0210.

3. What is the area of an elliptic segment, cut off parallel to the longer axis, the height being 5, and the axis 25 and 35?

Ans. 97˙8458.

PROBLEM XX.

To find the length of a parabolic curve.

RULE.*

To the square of the ordinate add $\frac{4}{3}$ of the square of the absciss, extract the square root of the sum, and double it for the length of the curve, cut off by the double ordinate, nearly.

---

* DEMONSTRATION. Let $x=$ any abscissa, $y=$ its ordinate, $a=\frac{1}{2}$ the parameter of the axe, and $q=\frac{y}{a}$. Then it is shown by the writers on fluxions, that

$$aq\sqrt{1+q^2}+a\times \text{ hyp. log. of } q+\sqrt{1+q^2}$$

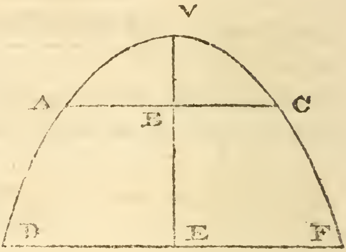
$$=2y\times : 1+\frac{q^2}{2\cdot3} - \frac{q^4}{2\cdot4\cdot5}+\frac{3q}{2\cdot4\cdot6\cdot7}-\frac{3\cdot5q^8}{2\cdot4\cdot6\cdot8\cdot9}, \text{&c.} = c$$

= length of the curve.

$$\text{But } \sqrt{1+\tfrac{1}{3}q^2}=1+\frac{q^2}{2\cdot3}-\frac{q^4}{2\cdot4\cdot9}+\frac{3q^6}{2\cdot4\cdot6\cdot27}, \text{&c.}$$

EXAMPLES.

1. The absciss VB being 2, and the ordinate AB 6; required the length of the curve AVC.



2=VB
2
4=VB²
4
3)16
5·3333
36 =AB²
41·3333(6·4291 root
36 2
124|533 12·8582= arc AVC nearly.
4|496
1282|3733
2|2564
1284)1169(91
·· 1156
13

2. What is the length of the parabolic curve, whose absciss is 3, and ordinate 8?

Ans. 17·4356.

Therefore $\frac{c}{2y} = \sqrt{1+\frac{1}{3}q^2}$ *nearly*. And consequently

$c=2y\sqrt{1+\frac{1}{3}q^2}=\sqrt{y^2+\frac{4}{3}x^2}$, the same as the rule. Q. E. D.

The following rule is a still nearer approximation:

$9\sqrt{y^2+\frac{4}{3}x^2} - 4 \times \frac{y^2+\frac{2}{3}x^2}{y} \times \frac{2}{5}$ = length of the arc *extremely near.*

PROBLEM XXI.

To find the area of a parabola.

RULE.*

Multiply the base by the height, and $\frac{2}{3}$ of the product will be the area.

EXAMPLES.

1. Required the area of the parabola AVCA, the absciss VB being 2, and the ordinate AB 6.

```
  12
   2
 ---
  24
   2
 ---
3)48
  16 answer.
```

2. What is the area of a parabola, whose absciss is 10, and ordinate 8. Ans. $106\frac{2}{3}$

---

* Demonstration. Let $BV=x$, $AB=y$, and the parameter $=p$.

Then $px=y^2$ or $\overline{px}|^{\frac{1}{2}} = y$ by the nature of the curve.

Whence the fluxion of the area $(=y\dot{x}) = \overline{px}|^{\frac{1}{2}}\dot{x}$, and its fluent $=\frac{2}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}=\frac{2}{3}p^{\frac{1}{2}}x^{\frac{1}{2}}\times x$.

But because $y=p^{\frac{1}{2}}x^{\frac{1}{2}}$, therefore $\frac{2}{3}p^{\frac{1}{2}}x^{\frac{1}{2}}\times x=\frac{2}{3}xy=$ area of the parabola, and is the *same as the rule.*

Cor. Every parabola is $=\frac{2}{3}$ of its circumscribing parallelogram.

PROBLEM XXII.

To find the area of a parabolic frustum.

RULE.*

Multiply the difference of the cubes of the two ends of the frustum by double its altitude, and divide the product by triple the difference of their squares, for the area.

EXAMPLES.

1. Required the area of the parabolic frustum ACFD, AC being 6, DF 10, and the altitude BE 4.

Ends.	Squares.	Cubes.
10	100	1000
6	36	216
	64 diff.	784
	3	8
	192)	6272($32\frac{128}{192} = 32\frac{2}{3}$ answer.
		576
		512
		384
		128

2. What is the area of a parabolic frustum, whose two ends are 6 and 10, and its altitude 3?

Ans. $24\frac{1}{2}$.

---

* DEMONSTRATION. Let D=DF, d=AC, and a=EB.

Then, by the nature of the curve,

$$D^2 - d^2 : a :: D^2 : \frac{aD^2}{D^2 - d^2} = VH,$$

and $$D^2 - d^2 : a :: d^2 : \frac{aD^2}{D^2 - d^2} = VF.$$

PROBLEM XXIII.

To find the length of a hyperbolic curve.

RULE.*

1. To 19 times the transverse add 21 times the parameter of the axe, and multiply the sum by the quotient of the abscissa divided by the transverse.

2. To 9 times the transverse add 21 times the parameter, and multiply the sum by the quotient of the abscissa divided by the transverse.

And therefore $\frac{2}{3}\times\frac{aD^2}{D^2-d^2}-\frac{2}{3}\times\frac{ad^2}{D^2-d^2}=\frac{2}{3}a\times\frac{D^3-d^3}{D^2-d^2}$ = area of the frustum. Q. E. D.

* DEMONSTRATION. Let $t=$ semitransverse axe, $c=$ semiconjugate, $x=$ ordinate, and $y=$ abscissa. Then will

$$y\times : 1+\frac{a^2}{6c^4}y^2-\frac{a^4+4a^2c^2}{40c^8}y^4+\frac{a^6+4a^4c^2+8a^2c^4}{112c^{12}}y^6, \&c. =$$

length of the arc, as is shown by the writers on fluxions.

But $x=\frac{a\sqrt{c^2+y^2}}{c}-a$, and $\frac{2c}{t}=$ parameter $=p$, by the nature of the curve. Consequently the rule is

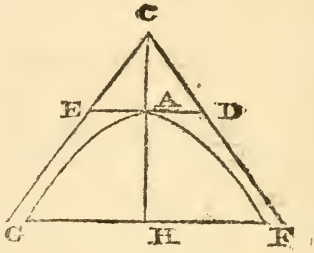
$$=\frac{15p+\frac{19t+21p}{t}x}{15p+\frac{9t+21p}{t}x}\times y=\frac{\frac{30c}{t}+\frac{19t+42\frac{c}{t}}{t}\times\overline{\frac{a\sqrt{c^2+y^2}}{c}-a}}{\frac{30c}{t}+\frac{9t+42\frac{c}{t}}{t}\times\overline{\frac{a\sqrt{c^2+y^2}}{c}-a}}\times y.$$

And if this be thrown into a series, it will be found to agree very nearly with the three first terms of the former, and therefore the rule is an approximation.

3. To each of the products, thus found, add 15 times the parameter, and divide the former by the latter; then, this quotient, being multiplied by the ordinate, will give the length of the arc *nearly*.

EXAMPLES.

1. In the hyperbola GAF, the transverse diameter twice AC is 80, the conjugate ED 60, the ordinate GH 10, and the abscissa AH 2·1637; required the length of the arc AG.



8·0)2·1637
———
·02704

80 : 60 : : 60 :
60
———
8·0)360·0
———
45=parameter.
21
———
45
90
———
945, and 675=15 times 45

80= twice AC.	80= twice AC.
19	9
720	720
80	945
1520	1·665
945	·02704
2465	6660
·02704	116550
	3330
9860	45·02160
172550	675
4930	
66·65360	720·02160)741·65360(1·03004
675	72002160
741·65360	216320000
	216006480
	313520000
	288008640
	25511360

1·03004
10=GH

10·30040= length of the arc required.

2. What is the length of the whole curve to the ordinate 10, the transverse and conjugate axes being 80 and 60?

Ans. 20·601.

PLOBLEM XXIV.

To find the area of a hyperbola.

RULE.*

1. To the product of the transverse and abscissa add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.

2. Add 4 times the square root of the product of the transverse and abscissa to the last found product, and divide the sum by 75.

---

* DEMONSTRATION. Let, $t=$ transverse diameter, $c=$ conjugate, $x=$ abscissa, $y=$ ordinate, and $z=\frac{x}{2t+x}$. Then it is well known, that $4xy\times:\frac{1}{3}-\frac{1}{1\cdot 3\cdot 5}z-\frac{1}{3\cdot 5\cdot 7}z^2-\frac{1}{5\cdot 7\cdot 9}z^3$, &c. = area of the hyperbola.

But $\frac{ty}{\sqrt{tx+x^2}}=c=$ conjugate axe, by the nature of the hyperbola. Consequently the expression for the rule

$$=\frac{4cx}{t}\times\frac{21\sqrt{tx+\frac{5}{7}x^2}+4\sqrt{tx}}{75}=4xy\times\frac{21\sqrt{tx+\frac{5}{7}x}+4\sqrt{tx}}{\sqrt{tx+x^2}}.$$

And this, thrown into a series, will very nearly agree with the former, which shows the rule to be an approximation. Q. E. I.

RULE 2.

If 2Y, $2y=$ bases, V, and v their distances from the centre, and the other letters as before, then will

3. Divide 4 times the product of the conjugate and abscissa by the transverse; and this last quotient, multipliedby the former, will give the area required, *nearly*.

EXAMPLES.

1. In the hyperbola AVC, the transverse is 30, the conjugate is 18, and the abscissa or height is 10; what is the area?

30
10
———
300
71·428571 = $\frac{5}{7}$ of the squ. of the absc.
———
371·428571(19·272
1   21
———   ———
29)271   19272
261   38544
———   ———
382)1042   404·712
764
———
3847)27885
26929
———
38542)95671
77084
———
18587

$VY - vy - \frac{tc}{4} \times$ hyp. log. of $\frac{tY+cV}{ty+cv}$ = area of the frustrum of the hyperbola.

30
10

. .
300 (17·3205
1   4

27)200   69·2820
189

343)1100
1029

3462)7100
6924

346405)1760000
1732025

27975

404·712
69·282

75)473·994(6·3199
450

239
225

149
75

744
675

690
675

15

$18 = ED$
$10 = AH$

180
4

3·0)72·0

24
6·3199
24

252796
126398

151·6776 = area required.

END OF VOLUME I.

[illegible]