J.H.1826

MATHEMATICS

FOR PRACTICAL MEN:

BEING

A COMMON-PLACE BOOK

OF

PRINCIPLES, THEOREMS, RULES, AND TABLES,

IN

VARIOUS DEPARTMENTS

OF

PURE AND MIXED MATHEMATICS,

WITH THEIR

MOST USEFUL APPLICATIONS;

ESPECIALLY TO THE PURSUITS OF

SURVEYORS, ARCHITECTS,

MECHANICS, AND CIVIL ENGINEERS.

BY

OLINTHUS GREGORY, LL. D.

Corresponding Associate of the Academy of Dijon, Honorary Member of the Literary and Philosophical Society of New York; of the New York Historical Society; of the Literary and Philosophical, and the Antiquarian Societies of Newcastle-upon-Tyne; of the Cambridge Philosophical Society; of the Institution of Civil Engineers, &c. &c.; Secretary to the Astronomical Society of London, and Professor of Mathematics in the Royal Military Academy.

"That pains we take in books or arts which treat of things remote from the use of life, is but a busy idleness." Dr. Fuller.

"Only let men awake, and fix their eyes, one while on the nature of things, another while on the application of them to the use and service of mankind." Lord Bacon.

LONDON:
PRINTED FOR BALDWIN, CRADOCK, AND JOY.

1825.

C. Baldwin, Printer,
New Bridge-street, London.

TO

THOMAS TELFORD, Esq. F.R.S.E.

PRESIDENT;

AND TO THE VARIOUS

OFFICERS AND MEMBERS

OF THE

INSTITUTION OF CIVIL ENGINEERS,

MEETING IN

BUCKINGHAM STREET, ADELPHI;

THIS COMPENDIUM

OF

MATHEMATICS FOR PRACTICAL MEN

Is most respectfully dedicated,

By their

Faithful and obliged Servant,

THE AUTHOR.

PREFACE.

The work now presented to the public had its origin in a desire which I felt to draw up an Essay on the principles and applications of the mechanical sciences for the use of the younger members of the Institution of Civil Engineers. The eminent individuals who are deservedly regarded as the main pillars of that useful Institution, stand in need of no such instructions as are in my power to impart: but it seemed expedient to prepare an Essay, comprized within moderate limits, which might furnish scientific instruction for the many young men of ardour and enterprize who have of late years devoted themselves to the interesting and important profession, of whose members that Institution is principally constituted. My first design was to compose a paper which might be read at one or two of the meetings of that Society: but, as often happens in such cases, the embryo thought has grown, during meditation, from an essay to a book; and what was first meant to be a very compendious selection of principles and rules, has, in its execution assumed the appearance of a systematic analysis of principles, theorems, rules, and tables.

Indeed, the circumstances in which the inhabitants of this country are now placed, with regard to the love and acquisition of knowledge, impelled me, almost unconsciously, to such an extension of my original plan, as sprung from a desire to contribute to the instruction of that numerous class, the practical mechanics of this country. Besides the early disadvantages under which many of them have laboured, there is another which results from the activity of their pursuits. Unable, therefore, to go through the details of an extensive systematic course, they must for the most part be satisfied with imperfect views of theories and principles, and take much upon trust: an evil, however, which the establishment of Societies, and the composition of treatises, with an express view to their benefit, will probably soon diminish.

Mr. Brougham, in his "*Practical Observations upon the Education of the People,*" remarks that "a most essential ser-"vice will be rendered to the cause of knowledge by him who "shall devote his time to the composition of elementary trea-

"tises on the Mathematics, sufficiently clear, and yet suffi-"ciently compendious, to exemplify the method of reasoning "employed in that science, and to impart an accurate know-"ledge of the most useful fundamental propositions, with their "application to practical purposes; and treatises upon Na-"tural Philosophy, which may teach the great principles of "physics, and their practical application, to readers who have "but a general knowledge of mathematics, or who are even "wholly ignorant of the science beyond the common rules of "arithmetic." And again, "He who shall prepare a treatise "simply and concisely unfolding the doctrines of Algebra, "Geometry, and Mechanics, and adding examples calculated "to strike the imagination, of their connexion with other "branches of knowledge, and with the arts of common life, "may fairly claim a large share in that rich harvest of dis-"covery and invention which must be reaped by the thousands "of ingenious and active men, thus enabled to bend their "faculties towards objects at once useful and sublime."

I do not attempt to persuade myself that the present volume will be thought adequately to supply the desiderata to which these passages advert: yet, I could not but be gratified, after full two-thirds of it were written, to find that the views which guided me in its execution accorded so far with the judgment of an individual, distinguished as Mr. Brougham was, in early life, for the elegance and profundity of his mathematical researches.

With a view to the elementary instruction of those who have not previously studied mathematics, I have commenced with brief, but, I hope, perspicuous, treatises on *Arithmetic* and *Algebra*; a competent acquaintance with both of these being necessary to ensure that accuracy in computation which every practical man ought to attain, and that ready comprehension of scientific theorems and formulæ which becomes the key to the stores of higher knowledge. As no man sharpens his tool or his weapon, merely that it may be sharp, but that it may be the fitter for use; so no thoughtful man learns arithmetic and algebra for the mere sake of knowing those branches of science, but that he may employ them; and these being possessed as valuable pre-requisites, the course of an author is thereby facilitated: for then, while he endeavours to express even common matters so that the learned shall not be disgusted, he may so express the more abstract and difficult that the comparatively ignorant (and the mere knowledge of *arithmetic* and *algebra* is, in our times comparative ignorance) may practically understand and apply them.

After the first 100 pages, the remaining matter is synoptical. The general topics of geometry, trigonometry, conic sections, curves, perspective, mensuration, statics, dynamics, hydrostatics, hydrodynamics, and pneumatics, are thus treated. The definitions and principles are exhibited in an orderly series; but investigations and demonstrations are only sparingly introduced. This portion of the work is akin in its nature to a syllabus of a Course of Lectures on the departments of science which it treats; with this difference, however, occasioned by the leading object of the publication, that popular illustrations are more frequently introduced, practical applications incessantly borne in mind, and such tables as seemed best calculated to save the labour of architects, mechanics, and civil engineers, inserted under their appropriate heads. Of these latter, several have been collected from former treatises, &c. but not a few have been either computed or contributed expressly for this Common-place Book.

In a work like this, it would be absurd to pretend to originality. The plan, arrangement, and execution, are my own; but the materials have long been regarded, and rightly, as common property. It has been my aim to reduce them into the smallest possible space, consistently with my general object; but, wherever I have found the work in this respect prepared to my hands, I have transcribed it into the following pages, with the usual references to the sources from whence it was taken. They who are conversant with the best writers on subjects of mixed mathematics and natural philosophy, will know that *Smeaton*, *Robison*, *Playfair*, *Young*, *Du Buat*, *Leslie*, *Hachette*, *Bland*, *Tredgold*, &c. are authors who *ought* to be consulted, in the preparation of a volume like this. I hope it will appear that I have duly, yet, at the same time, honourably, availed myself of the advantages which they supply. I have, also, made such selections from my own earlier publications, as were obviously suitable to my present purpose; but not so copiously, I trust, as to diminish the utility of those volumes, or to make me an unfair borrower even from myself.

Besides our junior Civil Engineers, and the numerous Practical Mechanics who are anxious to store their minds with scientific facts and principles; there are others to whom, I flatter myself, the following pages will be found useful. Teachers of mathematics and those departments of natural philosophy, which are introduced into our more respectable seminaries, may probably find this volume to occupy a convenient intermediate station between the merely popular exhibitions of the truths of mechanics, hydrostatics, &c. and the larger treatises in which the whole chain of inquiry and demonstration is carefully presented link by link, and the successive portions firmly connected upon irrefragable principles. While students who have recently terminated a scientific course, whether in our Universities, or other institutions public or private, may, I would fain believe, find in this Common-place Book, an abridged repository of the most valuable principles and theorems, and of hints for their applications to practical purposes.

The only performances with which I am acquainted, that bear any direct analogy to this, are Martin's *Young Student's Memorial Book*, Jones's *Synopsis Palmariorum Matheseos*, and Brunton's *Compendium of Mechanics;* the latter of which I had not seen until the present volume was nearly completed. The first and last mentioned of these, are neat and meritorious productions; but restricted in their utility by the narrow space into which they are compressed. The other, written by the father of the late Sir William Jones, is a truly elegant Introduction to the principles of Mathematics, considering the time in which it was written (1706); but as it is altogether theoretical, and is, moreover, now becoming exceedingly scarce, it by no means supersedes the *necessity*, for such I have been induced to regard it, of a Compendium like that which I now offer to the public.

In its execution I have aimed at no higher reputation than that of being perspicuous, correct, and useful; and if I shall be so fortunate as to have succeeded in those points, I shall be perfectly satisfied.

OLINTHUS GREGORY.

Royal Military Academy,
Woolwich, October 1st, 1825.

ERRATA.

Page 40, line 28, *for* as to the, *read* as the.
79, line 6, *for* $+ x^2$ *read* $+ x^2)$
128, The first diagram is inverted, in some copies,
136, in the note, *for* 1741, *read* 1743.
223, line 9, *for* all along. Supposing, *read* All along supposing.
240, bottom line, *for* sin $i \quad f$, *read* sin $i - f$.
288, line 5 from bottom *for* specified, *read* specific.
299, line 3 from bottom *for* *Dr. Robison*, *read* *Du Buat*.

CONTENTS.

ARITHMETIC.

ALGEBRA.

GEOMETRY.

TRIGONOMETRY.

CONIC SECTIONS.

CURVES USEFUL TO ARCHITECTS, &c.

MENSURATION.

MECHANICS.

STATICS.

DYNAMICS.

HYDROSTATICS.

HYDRODYNAMICS.

PNEUMATICS.

PLATES

(*To be placed at the end.*)

A
COMMON-PLACE BOOK,
&c. &c.

CHAP. I.
ARITHMETIC.
SECTION I. *Definitions and Notation.*

ARITHMETIC is the science of numbers.

We give the name of *number* to the assemblage of many *units*, or of many parts of an assumed unit; *unit* being the quantity which, among all those of the same kind, forms a *whole* which may be regarded as the base or element. Thus, when I speak of *one house*, *one guinea*, I speak of *units*, of which the first is the thing called a house, the second that called a guinea. But, when I say *four houses*, *ten guineas*, *three quarters of a guinea*, I speak of *numbers*, of which the first is the unit *house* repeated *four times*; the second is the unit *guinea* repeated *ten times*; the third is the *fourth* part of the unit *guinea*, repeated *three times*.

In every particular classification of numbers, the unit is a measure taken arbitrarily, or established by usage and convention.

Numbers formed by the repetition of an unbroken unit are called *whole numbers*, or *integers*, as *seven* miles, *thirty* shillings: those which are formed by the assemblage of many parts of a unit, are called *fractional numbers*, or simply *fractions*; as *two thirds* of a yard, *three eighths* of a mile.

When the unit is restricted to a certain thing in particular, as *one man*, *one horse*, *one pound*, the collection of many of those units is called a *concrete* number, as *ten men*, *twenty horses*, *fifty pounds*. But, if the unit does not denote any particular thing, and is expressed simply by *one*, numbers which are constituted of such units are denominated *discrete* or *abstract*, as *five*, *ten*, *thirty*. Hence, it is evident that abstract numbers can only be compared with their unit, as concrete numbers are compared with, or measured by, theirs; but that it is not possible to compare an abstract with a concrete number, or a concrete number of one kind with a concrete number of another; for there can exist no measurable relations but between quantities of the *same* kind.

The series of numbers is indefinite; but only the first nine of them are expressed by different characters, called *figures*: thus,

Names. one, two, three, four, five, six, seven, eight, nine.
Figures. 1, 2, 3, 4, 5, 6, 7, 8, 9.

Besides these, another character is employed, namely 0, called the *cypher* or *zero;* which has no particular value of itself, but by its *position* is made to change the value of any significant figures with which it is connected.

In the system of numeration now generally adopted, and borrowed from the Indians, an infinitude of words and characters is avoided, by a simple yet most ingenious expedient, which is this;—*every figure placed to the left of another assumes ten times the value that it would have if it occupied the place of the latter.*

Thus, to express the number that is the sum of 9 and 1, or ten units, called *ten*, we place a 1 to the *left* of a 0, thus 10. So again the sum of 10 and 1, or *eleven*, is represented by 11; the sum of 11 and 1, or of 10 and 2, called *twelve*, is represented by 12; and so on for *thirteen*, *fourteen*, *fifteen*, &c. denoted respectively by 13, 14, 15, &c. the figure 1 being all along equivalent to *ten*, because it occupies the second rank.

In like manner *twenty*, *twenty one*, *twenty two*, &c. are represented by 20, 21, 22, because the 2 in the second rank is equivalent to twice ten, or *twenty*. And thus we may proceed with respect to the numbers that fall between twenty and three tens or *thirty* 30, four tens or *forty* 40, five tens or *fifty* 50, six tens or *sixty* 60, seven tens or *seventy* 70, eight tens or *eighty* 80, nine tens or *ninety* 90. After 9 are added to the 90 (ninety) numbers can no longer be expressed by *two* figures, but require a third rank to the left hand of the second.

The figure that occupies the third rank, or of *hundredths*, is expressed by the word *hundred*. Thus 369, is read three *hundred* and sixty nine; 428, is read four *hundred* and twenty eight; 837, eight *hundred* and thirty seven: and so on for all numbers that can be represented by three figures.

But if the number be so large that more than three figures are required to express it, then it is customary to divide it into periods of *three* figures each, reckoning from the right hand towards the left, and to distinguish each by a peculiar name.

The second period is called that of *thousands*, the third that of *millions*, the fourth that of *milliards* or *billions*,* the fifth that of *trillions*, and so on; the terms units, tens, and hundreds, being successively applied to the first, second, and third ranks of figures from the right towards the left, in each of these periods.

Thus, 1111, is read one thousand one hundred and eleven.

23456, twenty three thousands, four hundred and fifty six.

421835, four hundred and twenty one thousands, eight hundred and thirty five.

732846915, seven hundred and thirty two millions, eight hundred and forty six thousands, nine hundred and fifteen.

The manner of estimating and expressing numbers we have here described, is conformable to what is denominated the *decimal notation*. But besides this there are other kinds invented by philosophers, and others indeed in common use: as the *duodecimal*, in which every superior name contains *twelve* units of its next inferior name; and the *sexagesimal*, in which *sixty* of an inferior name are equivalent to one of its next superior. The former of these is employed in the measurement and computation of artificers' work; the latter in the division of a circle, and of an hour in time.

To the head of notation we may also refer the explanation of the principal symbols or characters employed to express operations or results in computation. Thus,

The sign + (*plus*) belongs to addition, and indicates that the numbers between which it is placed are to be added together. Thus, 5 + 7 expresses the sum of 5 and 7, or that 5 and 7 are to be added together.

The sign − (*minus*) indicates that the number which is placed after it is to be subtracted from that which precedes it. So, 9 − 3 denotes that 3 is to be taken from 9.

The sign $\backsim$ denotes *difference*, and is placed between two quantities when it is not immediately evident which of them is the greater.

The sign × (*into*) for multiplication, indicates the product of two numbers between which it is placed. Thus 8 × 5 denotes 8 times 5, or 40.

The sign ÷ (*by*) for division, indicates that the number which precedes it is to be divided by that which follows it; and the quotient that results from this operation is often

* It has been customary in England, to give the name of *billions* to millions of millions, of *trillions* to millions of millions of millions, and so on: but the method here given of dividing numbers into periods of three figures instead of six, is universal on the Continent; and, as it seems more simple and uniform than the other, I have adopted it.

represented by placing the first number over the second with a small bar between them. Thus, $15 \div 8$ denotes that 15 is to be divided by 8, and the quotient is expressed thus $\frac{15}{8}$.

The sign $=$, two equal and parallel lines placed horizontally, is that of equality. Thus, $2 + 3 + 4 = 9$, means that the sum of 2, 3, and 4, is equal to 9.

Inequality is represented by two lines so drawn as to form an angle, and placed between two numbers, so that the angular point turns towards the least. Thus, $7 > 4$, and $A > B$, indicate that 7 is *greater than* 4, and the quantity represented by A *greater than* the quantity represented by B: on the other hand $3 < 5$ and $C < D$ indicates that 3 is *less than* 5, and C less than D.

Colons and double colons are placed between quantities to denote their proportionality. So, $3 : 5 :: 9 : 15$, signifies that 3 are to 5 as 9 to 15, or $\frac{3}{5} = \frac{9}{15}$.

The extraction of roots is indicated by the sign $\sqrt{}$, with a figure occasionally placed over it to express the *degree* of the root. Thus $\sqrt{4}$ signifies the square root of 4, $\sqrt[3]{27}$ the cube root of 27, $\sqrt{16}$ the fourth or biquadrate root of 16; and so on.

These characters find their most frequent use in algebra and the higher departments of mathematics: but may, without hesitation, be employed whenever they secure brevity without a sacrifice of perspicuity.

Section II. *Addition of Whole Numbers.*

Addition is the rule by which two or more numbers are collected into one aggregate or *sum*.

Suppose it were required to find the sum of the numbers 3731, 349, 12487, and 54. It is evident that if we computed separately the sums of the units, of the tens, of the hundreds, of the thousands, &c. their combined results would still amount to the same. We should thus have 15 thousands + 14 hundreds + 20 tens + 21 units, or 15000 + 1400 + 200 + 21; operating again upon these, in like manner, rank by rank, we should have 10 thousands + 6 thousands + 6 hundreds + 2 tens + 1, or 16621, which is the sum required.

But the calculation is more commodiously effected by this

Rule.

Place the given numbers under each other, so that units stand under units, tens under tens, hundreds under hundreds, &c.

Add up all the figures in the column of units, and observe for every *ten* in its amount to carry *one* to the place of tens in the second column, putting the overplus figure in the first column.

Proceed in the same manner with the second column, then with the third, and so on till all the columns be added up: the figures thus obtained in the several amounts, indicate, according to the rules of notation, the sum required.

Note. Whether the addition be conducted upwards or downwards, the result will be the same; but the operation is most frequently conducted by adding upwards.

Example. Taking the same numbers as before, and disposing them as the rule directs, we have 4 + 7 + 9 + 1 = 21, of which we put down the 1 in the place of *units*, and carry the two to the tens: then 2 + 5 + 8 + 4 + 3 = 22, of which we put down the left hand 2 in the place of *tens*, and carry the other to the hundreds: then 2 + 4 + 3 + 7 = 16, of which the 6 is put in the place of *hundreds*, and the 1 carried to the thousands. This progress continued will give the same sum as before.

3731
349
12487
54
16621

Other Examples.

57	6475	77756	10376786
762	9830	3388	789632
5389	2764	9763	1589
97615	5937	90257	73
103823	25006	181164	11168080

SECTION III. *Subtraction of Whole Numbers.*

SUBTRACTION is the rule by which one number is taken from another, so as to show the difference, or excess.

The number to be subtracted or taken away is called the *subtrahend*: the number from which it is to be taken, the *minuend*: the quantity resulting, the *remainder*.

RULE.

Write the minuend and the subtrahend in two separate lines, units under units, tens under tens, and so on.

Beginning at the place of units, take each figure in the subtrahend from its corresponding figure in the minuend, and write the difference under those figures in the same rank or place.

But if the figure in the subtrahend be greater than its corresponding figure in the minuend, add ten to the latter, and then take the figure in the subtrahend from the sum, putting down the remainder, as before: and in this case add 1 to the next figure to the *left* in the subtrahend, to compensate for the ten borrowed in the preceding place.

Thus proceed till all the figures are subtracted.

Note. It is customary to place the minuend above the subtrahend; but this is not absolutely necessary. Indeed, it is often convenient in computation to find the difference between a number and a greater that naturally stands beneath it: it is, therefore, expedient to practise the operation in both ways, so that it may, however it occur, be performed without hesitation.

Example . .	Minuend	26565874
	Subtrahend	9853642
	Remainder	16712232

Here the five figures on the right of the subtrahend are each less than the corresponding figures in the minuend, and may therefore be taken from them, one by one. But the sixth figure, viz. 8, cannot be taken from the 5 above it. Yet, as a unit in the *seventh* place is equivalent to 10 in the *sixth*, this unit *borrowed* (for such is the technical word here employed) makes the 5 become 15. Then 8 taken from 15 leaves 7, which is put down; and 1 is added to the 9 in the 7th place of the subtrahend, to compensate or balance the 1 which was borrowed from the 7th place in the minuend. Recourse must be had to a like process whenever a figure in the subtrahend exceeds the corresponding one in the minuend.

Other Examples.

From 8217	From 44444	Take 21498	Take 45624
Take 3456	Take 3456	From 76262	From 80200
Remains 4761	Remains 40988	Remains 54764	Remains 34576

Section IV. *Multiplication of Whole Numbers.*

Multiplication of whole numbers is a rule by which we find what a given number will amount to, when it is repeated as many times as are represented by another number.*

* This definition, though not the most scientific that might be given, is placed here, because others depend, implicitly if not explicitly, on proportion, and therefore cannot, logically, be introduced thus early in the course.

The number to be multiplied, or repeated, is called the *multiplicand*, and may be either an abstract or a concrete number.

The number to be multiplied by is called the *multiplier*, and *must* be an abstract number, because it simply denotes the number of times the multiplicand is to be repeated.

Both multiplicand and multiplier are called *factors*.

The number that results from the multiplication is called the *product*.

Before any operation can be performed in multiplication, the learner must commit to memory the following table of products, from 2 times 2 to 12 times 12.

times	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

It is very advantageous in practice, to have this table carried on, at least intellectually, to 20 times 20. All the products to this extent are easily remembered.

The learner will perceive that in this table 7 times 5 is equal to 5 times 7, or $7 \times 5 = 35 = 5 \times 7$. In like manner that $8 \times 3 = 24 = 3 \times 8$, $4 \times 11 = 44 = 11 \times 4$, and so of other products. This is often made a subject of formal proof, as well as that $3 \times 5 \times 8 = 3 \times 8 \times 5 = 5 \times 3 \times 8 = 5 \times 8 \times 3$, &c. But to attempt the demonstration of things so nearly axiomatical as these, is quite unnecessary.

4827
8
56
16
64
32
38616

Previously to exhibiting the rules, let us take a simple example, and multiply 4827 by 8. Here placing the numbers as in the margin, and multiplying in their order 7 units by 8, 2 tens by eight, 8 hundreds by 8, 4 thousands by 8, the several products are 56 units, 16 tens, 64 hundreds, 32 thousands: these placed in their several ranks, according to the rules of notation, and then added up, give for the sum of the whole, or for the product of 4827 multiplied by 8, the number 38616.

The same example may be worked thus:

$$\left.\begin{array}{r} 8 \times 7 = 56 \\ 8 \times 20 = 160 \\ 8 \times 800 = 6400 \\ 8 \times 4000 = 32000 \\ \hline 38616 \end{array}\right\} \text{which is evidently the same in effect as before.}$$

CASE I. To multiply a number consisting of several figures, by a number not exceeding 12.

Multiply each figure of the multiplicand by the multiplier, beginning at the units; write under each figure the units of the product, and carry on the tens to be added as units to the product following.

Examples.

Multiply	4827	218043	440052	8765400
By	8	9	11	12
Products	38616	1962387	4840572	105184800

CASE II. To perform multiplication when each factor exceeds 12.

Place the factors under each other (usually the smallest at bottom) and so that units stand under units, tens under tens, and so on.

Multiply the multiplicand by the figure which stands in the unit's place of the multiplier, and dispose the product so that its unit's place shall stand under the unit of the multiplicand; then multiply successively by the figure in the place of tens, hundreds, &c. of the multiplier, and place the first figure of each product under that figure of the multiplier which gave the said product.

The sum of these products will be the product required.

Example.

Multiply 8214356 by 132.

Multiplicand	8214356
Multiplier	132
8214356 × 2 = ...	16428712
8214356 × 3 tens = ...	24643068
8214356 × 1 hundred = ...	8214356
8214356 × 132 = ...	1084294992

Other Examples.

Multiply	821436
By	672576
	4928616
	5750052
	4107180
	1642872
	5750052
	4928616
Product	552478139136

Multiply	8210075
By	420306
	49260450
	24630225
	16420150
	32840300
Product	3450743782950

Note. Multiplication may frequently be shortened by separating the multiplier into its component parts or factors, and multiplying by them in succession. Thus, since 132 times any number are equal to 12 times 11 times that number, the first example may be performed in this manner:

Multiply	8214356	Here one line of multiplication, and one of addition, are saved.
By	11	
And this product	90357916	
By	12	
Product as before	1084294992	

So, again, the multiplier of the second example, viz. 672576, divides into three numbers, 600000, 72000, and 576;

where, omitting the cypher, we have 72 = 12 × 6, and 576 = 8 × 72. Hence the operation may be performed thus:—

Multiplicand	821436	
Multiply by	6	in the 6th place.
	4928616	
Previous product × 12 .	59143392	for 72 thousands.
Second product × 8	473147136	for 576 units.
Same product as before	552478139136 :	three lines saved.

Other modes of contraction will appear as we proceed.

SECTION V. *Division of Whole Numbers.*

DIVISION is a rule by which we determine how often one number is contained in another. Or, it is a rule by which, when we know a product and one of the factors which produced it, we can find the other.

The number to be divided is called the *dividend.*

That by which it is divided, the *divisor.*

That which results from the division, the *quotient,* when division and multiplication are regarded as reciprocal operations.

The *dividend* is equivalent to the *product,*
The *divisor* ———— to the *multiplier,*
The *quotient* ———— to the *multiplicand.*

RULE.

Draw a curved line on the right and left of the dividend, and place the divisor on the left.

Find the number of times the divisor is contained in as many of the left-hand figures of the dividend as are just necessary, and place that number on the right.

Multiply the divisor by that number, and place the product under the above-mentioned figures of the dividend.

Subtract the said product from that part of the dividend under which it stands, and bring down the next figure of the dividend to the right of the remainder.

Divide the remainder thus increased, as before; and if at any time it be found less than the divisor, put a cypher in the quotient, bring down the next figure of the dividend, and continue the process till the whole is finished: the quotient figures thus arranged will be that required.

Example.

Divide 743256 by 324.

```
          Dividend.
Divisor 324)743256(2294 Quotient.
            648                        Divisor    324
            ---                        Quotient  2294
             952                                 ----
             648                                 1296
             ---                                2916
             3045                               648
             2916                              648
             ----                             ------
              1296                       Proof 743256
              1296                             ------
              ----
Remain        ....
              ----
```

```
71)29754(419 5/71          131)135076(1031 15/131
   284                         131
   ---                         ---
   135                           407
    71                           393
   ---                           ---
    644                           146
    639                           131
    ---                           ---
      5 Remain.                    15 Remain.
    ---                           ---
```

In these two examples the numbers which remain are placed *over* their respective divisors, and attached to the quotients.—The meaning of this will be explained when we treat of *fractions*.

Note. When the divisor does not exceed 12, the operation may readily be performed in a single line; as will appear very evident if the following example be compared with the two methods of working the first example in multiplication.

Divide 38616 by 8.

```
8)38616(4827            Dividend 38616
  32                     Divisor 8
  --                             -----
   66                   Quotient  4827
   64                             -----
   --
    21
    16
    --
     56
     56
     --
     ..
```

Here 8 in 38 go 4 times and 6 over; these carried as 6 tens to the next 6, make 66: 8 in 66 go 8 times and 2 over; these carried as 2 tens to the next figure 1 make 21: and so of the rest.

In division, also, upon the same principle as in multiplication, the labour may often be abridged by taking component parts of the divisor. Thus, in the first example, the divisor is equal to 4 times 81, or 4 times 9 times 9. Hence the dividend may be divided by 4, 9, and 9, successively, as in the margin, and the result will be the same as before.

Divide	743256
by	4
this quotient	185814
by	9
and this	20646
by	9
Quotient	2294

Since 25 is a fourth part of 100, and 125 the 8th part of 1000, it will be easy to multiply or to divide by either of these numbers in a single line—thus,

To multiply 4827 by 25, put two cyphers on the right, which is equivalent to multiplying by 100; and divide by 4.

4)482700

120675 Answer

To multiply 6218 by 125, put 3 cyphers, which is equivalent to multiplying by 1000; then divide by 8.

8)6218000

777250 Answer.

To divide 582100 by 25, strike off two figures on the right hand, which is equivalent to dividing by 100; then multiply by 4.

5821|00
4

23284

To divide 4567000 by 125, strike off three figures on the right hand, which is equivalent to dividing by 1000; then multiply by 8.

4567|000
8

36536

Proof of the first four Rules of Arithmetic.

Simple as these four rules are, it is not unusual to commit errors in working them: it is, therefore, useful to possess modes of proof.

1. Now, *addition* may be proved by adding downwards, as

well as upwards, and observing whether the two sums agree: or, by dividing the numbers, to be added into two portions, finding the sum of each, and then the sum of those two separate amounts. Thus, in the margin, the sum of the four numbers is 7355, the sum of the two upper ones 5857, of the two lower ones 1498, and *their* sum is 7355, the same as before.

2758	2758
3099	3099
469	——
1029	469
——	1029
7355	——
——	5857
	——
	1498
	——
	7355
	——

2. The proof of *subtraction* is effected by adding the remainder to the subtrahend; if their sum agrees with the minuend the work is right, otherwise not.

3. *Multiplication* and *division* reciprocally prove each other.

There is also another proof for multiplication known technically by the phrase *casting out the nines.* Add together the numbers from left to right in the multiplicand, dropping 9 whenever the sum exceeds 9, and carry on the remainder, dropping the nines as often as the amount is beyond them; and note the last remainder. Do the same with the multiplier and with the product; then if the product of the first two remainders is equal to the last remainder, this is regarded as a test that the work is right. Thus, taking the second example in multiplication, the figures in the multiplicand amount to 6 above two nines, those in the multiplier to 6 above three nines, those in the product to 0 above six nines; the product 6×6 of the two first excesses is 36 or 0 above four nines: the coincidence of the two 0's is the proof. It is plain, however, that the proof will be precisely the same so long as the figures in the product be the same, whatever be their *order:* the proof, therefore, though ingenious, is defective.*

A similar proof applies to *division.*

* The correctness of this proof, with the exception above specified, may be shown algebraically, thus:—put M and N = the number of nines in the multiplicand and multiplier respectively, m and n their excesses; then, $9M + m$ = the multiplicand, and $9N + n$ = the multiplier, and the product of those factors will be $= 81MN + 9Mn + 9Nm + mn$; but the three first terms are each a precise number of nines; because one of the factors in each is so; these therefore, being neglected, there remains mn to be divided by nine; but mn is the product of the two former excesses: therefore the truth of the method is evident. Q. E. D.

TABLES OF COINS, WEIGHTS, AND MEASURES, THUS GIVEN FOR THE USE OF COMPUTORS.

English Money.

Farthings.					qrs.
4 =	1 Penny.				d.
48 =	12 =	1 Shilling.			s.
960 =	240 =	20 =	1 Pound.		£
1008 =	252 =	21 =	1 Guinea.		

Troy Weight.

Grains.				gr.
24 =	1 Penny-weight.			dwt.
480 =	20 =	1 Ounce.		oz.
5760 =	240 =	12 =	1 Pound.	lb.

1 Pound troy = 372·965 French grammes.

Apothecaries Weight.

Grains.					gr.
20 =	1 Scruple.				℈
60 =	3 =	1 Dram.			ʒ
480 =	24 =	8 =	1 Ounce.		℥
5760 =	288 =	96 =	12 =	1 Pound.	lb.

Avoirdupois Weight.

Drams.						dr.
16 =	1 Ounce.					oz.
256 =	16 =	1 Pound.				lb.
7168 =	448 =	28 =	1 Quarter.			qr.
28672 =	1792 =	112 =	4 =	1 Hundred wt.		cwt.
573440 =	35840 =	2240 =	80 =	20 =	1 Ton.	t.

14 oz. 11 dwt. 15½ qrs. Troy = 1 lb. avoirdupois.

Wool Weight.

Pounds.						lb.
7 =	1 Clove.					cl.
14 =	2 =	1 Stone.				st.
28 =	4 =	2 =	1 Tod.			t.
182 =	26 =	13 =	6½ =	1 Wey.		w.
364 =	52 =	26 =	13 =	2 =	1 Sack.	sa.
4368 =	624 =	312 =	156 =	24 =	12 = 1 Last.	la.

Long Measure.

Barley Corns.							b.
3 =	1 Inch.						in.
36 =	12 =	1 Foot.					ft.
108 =	36 =	3 =	1 Yard.				yd.
594 =	198 =	16½ =	5½ =	1 Pole.			p.
23760 =	7920 =	660 =	220 =	40 =	1 Furlong.		f.
190080 =	63360 =	5280 =	1760 =	320 =	8 =	1 Mile.	m.

1 foot English = ·3047909 of a French metre.

Square Measure.

Inches.						in.
144 =	1 Foot.					ft.
1296 =	9 =	1 Yard.				yd.
39204 =	272¼ =	30¼ =	1 Pole.			p.
1568160 =	10890 =	1210 =	40 =	1 Rood.		r.
6272640 =	43560 =	4840 =	160 =	4 =	1 Acre.	ac.

Cloth Measure.

Inches.				in.
2¼ =	1 Nail.			nl.
9 =	4 =	1 Quarter.		qr.
36 =	16 =	4 =	1 Yard.	yd.
47 =	12 =	3 =	1 Flemish Ell.	F. e.
45 =	20 =	5 =	1 English Ell.	E. e.
54 =	24 =	6 =	1 French Ell.	Fr. e.

Wine Measure.

Pints.							pt.
2 =	1 Quart.						qt.
8 =	4 =	1 Gallon.					gall.
336 =	168 =	42 =	1 Tierce.				tr.
504 =	257 =	63 =	1½ =	1 Hogshead.			hhd.
672 =	336 =	84 =	2 =	1⅓ =	1 Puncheon.		pun.
1008 =	504 =	126 =	3 =	2 =	1½ =	1 Pipe.	pip.
2016 =	1008 =	252 =	6 =	4 =	3 =	2 = 1 Tun.	tun.

231 cubic inches = 1 Gallon.

Ale and Beer Measure.

Pints.									
2 =	1 Quart.								qt.
8 =	4 =	1 Gallon.							gall.
72 =	36 =	9 =	1 Firkin.						fir.
144 =	72 =	18 =	2 =	1 Kilderkin.					kil.
288 =	144 =	36 =	4 =	2 =	1 Barrel.				bar.
432 =	216 =	54 =	6 =	3 =	1½ =	1 Hogshead.			hhd.
576 =	288 =	72 =	8 =	4 =	2 =	1⅓ =	1 Puncheon		pun.
864 =	432 =	108 =	12 =	6 =	3 =	2 =	1½ =	1 Butt.	bt.

(Abbreviation for pints: pt.)

282 cubic inches = 1 Gallon.

Dry Measure.

Pints.								
8 =	1 Gallon.							gall.
16 =	2 =	1 Peck.						pec.
64 =	8 =	4 =	1 Bushel.					bus.
256 =	32 =	16 =	4 =	1 Coom.				co.
512 =	64 =	32 =	8 =	2 =	1 Quarter.			qr.
2560 =	320 =	160 =	40 =	10 =	5 =	1 Wey.		w.
5120 =	640 =	320 =	80 =	20 =	10 =	2 =	1 Last.	la.

(Abbreviation for pints: pt.)

2150·42 cubic inches = 1 Bushel.

Time.

Seconds.							
60 =	1 Minute.						min.
3600 =	60 =	1 Hour.					h.
86400 =	1440 =	24 =	1 Day.				d.
604800 =	10080 =	168 =	7 =	1 Week.			w.
2419200 =	40320 =	672 =	28 =	4 =	1 Month.		mon.
31449600 =	524160 =	8736 =	364 =	52 =	13 =	1 Year.	yr.

(Abbreviation for seconds: sec.)

N. B. The odd time is here neglected, the complete revolution being performed in 365 days, 6 hours, 9 minutes, and 11 seconds.

These are the principal weights and measures now in use in England. For their comparison with the measures, &c. proposed by the new act of parliament, to take place May 1st, 1825, see the end of the Arithmetic.

SECTION VI. *Vulgar Fractions.*

The fractions of which we have already spoken in section the 1st, are usually denominated *Vulgar Fractions*, to distinguish them from another kind, hereafter to be mentioned, called *decimal fractions*.

A fraction is an expression for part of a unit, or integer, when it represents a *whole* of any kind. Thus, if a *pound* sterling be the unit, then a *shilling* will be the twentieth part of that unit, and *four pence* will be four twelfths of that twentieth part. These represented according to the usual notation of Vulgar Fractions, will be $\frac{1}{20}$ and $\frac{4}{12}$ of $\frac{1}{20}$, respectively.

The lower number of a fraction thus represented (denoting the number of parts into which the integer is supposed to be divided) is called the *denominator;* and the upper figure (which indicates the number of those parts expressed by the fraction) the *numerator*. Thus, in the fractions $\frac{5}{7}$, $\frac{8}{15}$, 7 and 15 are *denominators*, 5 and 8 *numerators*.

Vulgar fractions are divided into proper, improper, mixed, simple, compound, and complex.

Proper fractions have their numerators less than their denominators, as $\frac{3}{4}$, $\frac{5}{8}$, &c.

Improper fractions have their numerators equal to, or greater than, their denominators, as $\frac{4}{4}$, $\frac{12}{8}$, &c.

Mixed fractions, or numbers, are those compounded of whole numbers and fractions, as $7\frac{5}{8}$, $12\frac{3}{4}$, &c.

Simple fractions are expressions for parts of given units, as $\frac{4}{9}$, $\frac{5}{6}$, &c.

Compound fractions are expressions for the parts of given fractions, as $\frac{3}{4}$ of $\frac{5}{8}$, $\frac{2}{3}$ of $\frac{7}{11}$, &c.

Complex fractions have either one or both terms mixed numbers, as $\frac{5\frac{2}{3}}{24}$, $\frac{12}{14\frac{3}{4}}$, $\frac{6\frac{5}{8}}{12\frac{2}{3}}$, &c.

Any number which will divide two or more numbers without remainder, is called their common measure.

Reduction of Vulgar Fractions.

This consists principally in changing them into a more commodious form for the operations of addition, subtraction, &c.

Case 1. To reduce fractions to their lowest terms.

Divide the numerator and denominator of a fraction by any number that will divide them both, without a remainder; the quotient again, if ' ssible, by any other number: and so on, till 1 is the greate divisor.

Thus, $\frac{1470}{2205} = \frac{294}{441} = \frac{98}{147} = \frac{14}{21} = \frac{2}{3}$, where 5, 3, 7, 7, respectively, are the divisors.

Or, $\frac{1470}{2205} = \frac{2}{3}$, by dividing at once by 735.

Note. This number 735 is called the *greatest* common measure of the terms of the fraction: it is found thus—Divide the greater of the two numbers by the less; the last divisor by the last remainder, and so on till nothing remains: the last divisor is the greatest common measure required.*

Case 2. To reduce an improper fraction to its equivalent whole or mixed number.

Divide the numerator by the denominator, and the quotient will be the answer: as is evident from the nature of division.

Ex. Let $\frac{957}{43}$ and $\frac{5480}{274}$ be reduced to their equivalent whole or mixed numbers.

```
43)957(22 11/43 Answer.        274)5480(20 Answer.
   86                               548
   ——                               ———
    97                                0
    86                              ———
    ——
    11
    ——
```

* The following Theorems are useful for abbreviating Vulgar Fractions:

THEOREMS.

1. If any number terminates on the right hand with a cypher, or a digit divisible by 2, the whole is divisible by 2; for the one which remains in the second place is 10: but 2 measures 10; therefore the whole is divisible by 2.

2. If any number terminates on the right hand with a cypher, or 5, the whole is divisible by 5; for every unit which remains in the second place is 10: but 5 measures every multiple of 10; therefore the whole is divisible by 5.

3. If the two right-hand figures of any number are divisible by 4, the whole is divisible by 4; for every unit which remains in the third place is 100: but 4 measures every multiple of 100; therefore the whole is divisible by 4.

4. If the three right-hand figures of any number are divisible by 8, the whole is divisible by 8; for every unit which remains in the fourth place is 1000: but 8 measures every multiple of 1000; therefore the whole is divisible by 8.

5. If the sum of the digits constituting any number be divisible by 3 or 9, the whole is divisible by 3 or 9.

6. If the sum of the digits constituting any number be divisible by 6, and the right-hand digit by 2, the whole is divisible by 6: for by the data it is divisible both by 2 and 3.

7. If the sum of the 1st, 3d, 5th, &c. digits constituting any number be equal to that of the 2d, 4th, 6th, &c. that number is divisible by 11; for if

Case 3. To reduce a mixed number to its equivalent improper fraction; or a whole number to an equivalent fraction having any assigned denominator.

This is, evidently, the reverse of Case 2; therefore multiply the whole number by the denominator of the fraction, and add the numerator (if there be one) to obtain the numerator of the fraction required.

Ex. Reduce $22\frac{11}{43}$ to an improper fraction, and 20 to a fraction whose denominator shall be 274.

$(22 \times 43) + 11 = 957$ new numerator, and $\frac{957}{43}$ the 1st fraction.

$20 \times 274 = 5480$ new numerator, and $\frac{5480}{274}$ the 2d fraction.

Case 4. To reduce a compound fraction to an equivalent simple one.

Multiply all the numerators together for the numerator, and all the denominators together for the denominator of the simple fraction required.

If part of the compound fraction be a mixed or a whole number, reduce the former to an improper fraction, and make the latter a fraction by placing 1 under the numerator.

When like factors are found in the numerators and denominators, cancel them both.

Ex. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{7}$ of $\frac{7}{9}$ of $\frac{8}{11}$ to a simple fraction.

$$\frac{2 \times 3 \times 5 \times 7 \times 8}{3 \times 4 \times 7 \times 9 \times 11} = \frac{2 \times 5 \times 8}{4 \times 9 \times 11} = \frac{1 \times 5 \times 8}{2 \times 9 \times 11} = \frac{1 \times 5 \times 4}{1 \times 9 \times 11} = \frac{20}{99}$$

Here the 3 and 7 common to numerator and denominator are first cancelled; then the fraction is divided by 2; and then by 2 again.

Ex. Reduce three farthings to the fraction of a pound sterling.

A farthing is the fourth of a penny, a penny the twelfth of a shilling, and a shilling the twentieth of a pound.

Therefore $\frac{3}{4}$ of $\frac{1}{12}$ of $\frac{1}{20} = \frac{3}{960} = \frac{1}{320}$ the answer.

Ex. Simplify the complex fraction $\dfrac{2\frac{2}{3}}{4\frac{4}{5}}$.

Here, reducing the mixed numbers to improper fractions, we have $\dfrac{\frac{8}{3}}{\frac{24}{5}}$: multiplying by 3, to get quit of the denominator of

a, b, c, d, e, m, n, be the digits, constituting any number, its digits, when multiplied by 11, will become

(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)
a,	$a+b$,	$b+c$,	$c+d$,	$d+e$,	$e+m$,	$m+n$,	n;

where the odd terms are = to the even.

the upper fraction, we have $\frac{8}{\frac{72}{5}}$: multiplying by 5, to get quit of the denominator of the lower fraction, we have $\frac{40}{72}$: dividing both terms of this fraction by 8, there results $\frac{5}{9}$ for the simple fraction required.

Case 5. To reduce fractions of different denominators to equivalent fractions having a common denominator.

Multiply each numerator into all the denominators except its own, for new numerators; and all the denominators together for a common denominator.

Ex. Reduce $\frac{2}{3}$, $\frac{6}{7}$, and $\frac{5}{9}$, to equivalent fractions having a common denominator.

$$\left.\begin{array}{r} 2 \times 7 \times 9 = 126 \\ 6 \times 3 \times 9 = 162 \\ 5 \times 3 \times 7 = 105 \end{array}\right\} \text{the numerators.}$$

$$3 \times 7 \times 9 = 189, \text{ the common denominator.}$$

Hence the fractions are $\frac{126}{189}$, $\frac{162}{189}$, $\frac{105}{189}$, or $\frac{42}{63}$, $\frac{54}{63}$, $\frac{35}{63}$, when abbreviated.

Hence, also, it appears that $\frac{2}{3}$ exceed $\frac{5}{9}$, and that $\frac{6}{7}$ exceed $\frac{2}{3}$.

Ex. Reduce $\frac{4}{5}$ of a penny, and $\frac{2}{3}$ of a shilling, each to the fraction of a pound; and then reduce the two to fractions having a common denominator.

$\frac{4}{5}$ of a penny $= \frac{4}{5}$ of $\frac{1}{12}$ of $\frac{1}{20} = \frac{4}{1200} = \frac{1}{300}$ of a pound.

$\frac{2}{3}$ of a shilling $= \frac{2}{3}$ of $\frac{1}{20} = \frac{2}{60} = \frac{1}{30} = \frac{10}{300}$ of a pound.

Hence $\frac{2}{3}$ of a shilling are 10 times as much as $\frac{4}{5}$ of a penny.

Note. Other methods of reduction will occur to the student after tolerable practice, and still more after the principles of algebra are acquired.

Addition and Subtraction of Fractions.

Rule. If the fractions have a common denominator, add or subtract the numerators, and place the sum or difference as a new numerator over the common denominator.

If the fractions have not a common denominator, they must be reduced to that state before the operation is performed.

In addition of mixed numbers, it is usually best to take the sum of the integers, and that of the fractions, separately; and then *their* sum, for the result required.

Examples.

1. Find the sum of $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{3}{4}$.

$\frac{2}{3} + \frac{5}{7} + \frac{3}{4} = \frac{56}{84} + \frac{60}{84} + \frac{63}{84} = \frac{179}{84} = 2\frac{11}{84}$.

2. Take $\frac{3}{4}$ of a shilling from $\frac{1}{12}$ of a pound sterling.

$\frac{3}{4}$ of a shilling $= \frac{3}{4}$ of $\frac{1}{20} = \frac{3}{80}$ of a pound $= \frac{18}{480}$. Also $\frac{1}{12}$ of a pound $= \frac{40}{480}$. Hence $\frac{40}{480} - \frac{18}{480} = \frac{22}{480}$ $= \frac{11}{240} = 11$ pence.

3. Find the difference between $12\frac{5}{6}$ and $8\frac{3}{5}$.

$12\frac{5}{6} - 8\frac{3}{5} = \frac{77}{6} - \frac{43}{5} = \frac{385}{30} - \frac{258}{30} = \frac{127}{30} = 4\frac{7}{30}$.

Multiplication and Division of Fractions.

RULE 1. To multiply a fraction by a whole number, multiply the numerator by that number, and retain the denominator.

2. To divide a fraction by a whole number, multiply the denominator by that number, and retain the numerator.

3. To multiply two or more fractions is the same as to take a fraction of a fraction; and is, therefore, effected by taking the product of the numerators for a new numerator, and of the denominators for a new denominator. (The product is, evidently, smaller than either factor when each is less than unity.)

4. To divide one fraction by another, invert the divisor and proceed as in multiplication. (The quotient is always greater than the dividend when the divisor is less than unity.)

Examples.

1. Multiply $\frac{3}{4}$ by 2, and divide $\frac{6}{7}$ by 5.

$\frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$, and $\frac{6}{7} \div 5 = \frac{6}{7 \times 5} = \frac{6}{35}$, *ans.*

2. Multiply $2\frac{2}{3}$ by $\frac{3}{4}$, and divide $\frac{8}{9}$ by $\frac{4}{5}$.

$2\frac{2}{3} \times \frac{3}{4} = \frac{8}{3} \times \frac{3}{4} = \frac{8}{4} = 2$, and $\frac{8}{9} \div \frac{4}{5} = \frac{8}{9} \times \frac{5}{4} = \frac{10}{9}$ $= 1\frac{1}{9}$, *ans.*

3. Multiply £2 13*s.* 4*d.* by $3\frac{1}{2}$, and divide £4 15*s.* by $3\frac{1}{3}$.

£2 13*s.* 4*d.* $= 2 + \frac{13}{20} + \frac{4}{12}$ of $\frac{1}{20} = 2\frac{2}{3} = \frac{8}{3}$, and $\frac{8}{3} \times 3\frac{1}{2} = \frac{8}{3} \times \frac{7}{2} = \frac{56}{6} = \frac{28}{3} = 9\frac{1}{3} =$ £9 6*s.* 8*d.*

£4 15*s.* $\div 3\frac{1}{3} = 4\frac{3}{4} \div 3\frac{1}{3} = \frac{19}{4} \div \frac{10}{3} = \frac{19}{4} \times \frac{3}{10} = \frac{57}{40}$ $= 1\frac{17}{40} =$ £1 8*s.* 6*d.*

Note. In the multiplication of mixed numbers, it is often less laborious to perform the multiplication of each part separately, and collect their sum, as in the margin, than to reduce the mixed numbers to improper fractions, and reduce their product back again to a mixed number.

	Multiply	$45\frac{3}{4}$
	By	$17\frac{2}{3}$
45×7	=	315
45×1 ten	=	45 .
$\frac{2}{3} \times \frac{3}{4}$	=	. . . $\frac{1}{2}$
$45 \times \frac{2}{3}$	=	. 30
$17 \times \frac{3}{4}$	=	. $12\frac{3}{4}$
	Product	$808\frac{1}{4}$

Section VII. *Decimal Fractions.*

The embarrassment and loss of time occasioned by the computation of quantities expressed in vulgar or ordinary fractions, have inspired the idea of *fixing* the denominator so as to know what it is without actually expressing it. Hence originate two dispositions of numbers, *decimal fractions*, and *complex numbers*. Of the latter, such, for example, as when we express lineal measures in yards, in feet (or thirds of a yard), and inches (or twelfths of a foot), we shall treat after a few pages: we shall now treat of the former.

Decimal fractions or substantively *decimals* are fractions expressed as whole numbers, but whose values decrease from the place of units progressively to the right hand in the same decuple or tenfold proportion as the common scale of whole numbers increase to the left. They are usually separated from the integers by a dot placed between the upper part of the figures. Thus, $22\frac{7}{10}$ expressed according to the decimal notation is 22·7.

Thus, also,	·1	is the same as	$\frac{1}{10}$
	·01	———	$\frac{1}{100}$
	·001	———	$\frac{1}{1000}$
	·0001	———	$\frac{1}{10000}$
	·7	———	$\frac{7}{10}$
	·43	———	$\frac{43}{100}$
	·125	———	$\frac{125}{1000}$
	7·3	———	$7\frac{3}{10}$
	42·85	———	$42\frac{85}{100}$
	57·217	———	$57\frac{217}{1000}$
	&c. &c. &c.		

The value of a decimal fraction is not altered by cyphers on the right hand: for ·500, or $\frac{500}{1000}$, is in value the same as $\frac{5}{10}$, or ·5, that is $\frac{1}{2}$.

When decimals terminate after a certain number of figures, they are called finite, as ·125 = $\frac{125}{1000}$ = $\frac{1}{8}$, ·958 = $\frac{958}{1000}$ = $\frac{479}{500}$.

When one or more figures in the decimal become repeated, it is called a repeating or circulating decimal; as ·333333, &c. = $\frac{1}{3}$, ·66666, &c. = $\frac{2}{3}$, ·428571428571, &c. = $\frac{3}{7}$, and many others.

Rules for the management of this latter kind of decimals are given by several authors; but, in general, it is more simple and commodious to perform the requisite operations by means of the equivalent vulgar fractions, from which circulating decimals are for the most part educed.

Reduction of Decimals.

Reduction of Decimals is a rule by which the known parts of given integers are converted into equivalent decimals, and *vice versa.*

Case 1. To reduce a given vulgar fraction to an equivalent decimal.

Annex cyphers to the numerator, divide by the denominator, and the quotient will be the decimal required.

Example.

1. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{16}$, $\frac{11}{64}$, to equivalent decimals.

2) 1·0
·5 decimal = $\frac{1}{2}$;

4) 3·00
·75 decimal = $\frac{3}{4}$:

16 { 4) 7·0000
4) 1·7500
·4375 decimal = $\frac{7}{16}$;

64 { 8) 11·000
8) 1·375000
·171875 decimal = $\frac{11}{64}$.

2. Reduce $\frac{4}{27}$ and $\frac{11}{63}$ to equivalent decimals.

$$27\begin{cases}3)\ 4{\cdot}000000\\ 9)\ 1{\cdot}333333\end{cases} \qquad 63\begin{cases}7)\ 11{\cdot}0000000\\ 9)\ 1{\cdot}5714285714285\end{cases}$$

$$\cdot148148,\ \&c. \qquad\qquad \cdot1746031746031,\ \&c.$$

decimal $= \frac{4}{27}$: decimal $= \frac{11}{63}$.

These two are evidently circulating decimals, in the former of which the figures 148 become indefinitely repeated, in the latter the figures 174603.

3. Reduce 14*s.* 6*d.* to the decimal of a pound.
First 14*s.* 6*d.* $= \frac{14}{20} + \frac{1}{2}$ of $\frac{1}{20} = \frac{28}{40} + \frac{1}{40} = \frac{29}{40}$.
Then $\frac{29}{40} = \frac{7{\cdot}25}{10} = {\cdot}725$, the decimal required.

4. Reduce $\frac{44}{57}$ to its equivalent decimal.

57) 44·000000 (·77192, &c. decimal $= \frac{44}{57}$.

```
      399
      ---
       410
       399
       ---
        110
         57
        ---
         530
         513
         ---
          170
          114
          ---
           56
```

Note. The above fraction is $= \frac{4}{3} \cdot \frac{11}{19}$, of which the two denominators are both *prime numbers* (that is, divisible by no other number than unity), the entire equivalent decimal is a circulator of 18 places, i. e. one less than the last prime...... ·771929824561403508, 7719, &c. over again *ad infinitum.**

* There are many curious properties of fractions whose denominators are prime numbers, one of which may be here shown in reference to fractions having the denominator 7. The circulating figures of the equivalent decimals are precisely the same, for $\frac{1}{7}$, $\frac{2}{7}$, &c. and in the same order: the circulate merely commences at a different place for each numerator.

$\frac{1}{7}$ = ·14285714, &c.
$\frac{2}{7}$ = ·28571428, &c.
$\frac{3}{7}$ = ·42857142, &c.
$\frac{4}{7}$ = ·57142857, &c.
$\frac{5}{7}$ = ·71428571, &c.
$\frac{6}{7}$ = ·85714 85, &c.

Case 2. Any decimal being given to find its equivalent vulgar fraction: or, to express its value by integers of lower denominations.

When the equivalent vulgar fraction is required, place under the decimal as a denominator a unit with as many cyphers as there are figures in the proposed decimal; and let the fraction so constituted be reduced to its lowest terms.

Or, if the value of the decimal be required in lower denominations, multiply the given decimal by the value of its integer in the next inferior order; and point off, from right to left, as many figures of the product as there were places in the given decimal.

Multiply the decimal last pointed off by the value of its integer, in the next inferior order, pointing off the same number of decimals as before: and thus continue the process to the lowest integer, or until the decimals cut off become all cyphers; then will the several numbers on the left of the separating points, together with the remaining decimal, if any, express the required value of the given decimal.

Examples.

1. Find the vulgar fractions equivalent to ·25 and ·375.

·25 $= \frac{25}{100} = \frac{1}{4}$; and ·375 $= \frac{375}{1000} = \frac{3}{8}$, *answers.*

2. Find the value in shillings, &c. of ·528125 of a £.

```
 ·528125
      20
 -------
10·562500
      12
 -------
 6·7500 = 6¼
```

Ans. 10s. 6¼d.

3. Find the value of ·74375 of an acre.

```
·74375
     4
------
2·97500
    40
------
39·000
```

Ans. 2 roods 39 perches.

Addition and Subtraction of Decimals.

These are performed precisely as in whole numbers, the numbers being so arranged that units stand under units, tens under tens, &c. or, which amounts to the same thing, that the decimal points stand under one another. Thus,

Add together	421·75 32·8165 ·0027 11·	From	2486·173
		Take	14·56789
		Remains	2471·60511
Sum	465·5692	Proof	2486·17300

Multiplication and Division of Decimals.

Here, again, the operations are performed as in integers. Then, in multiplication let the product contain as many decimal places as there are in both multiplier and multiplicand, cyphers being prefixed, if necessary, to make that number: and, in division, point off as many decimals in the quotient as the number in the dividend (including the cyphers supplied, if there be any) exceeds that in the divisor.

Examples.

1. Multiply 43·7 by 3·91, and 2·4542 by ·0053.

43·7 3·91 ——— 437 3933 1311 ——— 170·867	Here 43·7 × 3·91 $= \frac{437}{10} \times \frac{391}{100} = \frac{170867}{1000} = 170\frac{867}{1000}$, as in the decimal operation.	2·4542 ·0053 ——— 73626 122710 ——— ·01300726	Here one cypher is prefixed to make the requisite number of decimals.

2. Divide 172·8 by ·144, and 192 by 5·423.

```
·144) 172·8 (1200· quotient.
      144
      ---
       288
       288
       ---
      ···00
```

```
5·423) 192·000 (35·40475
       16269
       -----
        29310
        27115
        -----
         21950
         21692
         -----
          25800
          21692
          -----
           41080
           37961
           -----
            31190
            27115
            -----
             4075
```

In the first of these examples, the two cyphers brought down, together with the decimal 8, make the number of decimals in the dividend the same as in the divisor, therefore the quotient is entirely of integers. In the second example, 3, the decimal places in the divisor, taken from 8, the decimal places in the dividend (including those brought down) leave 5 for the decimal places in the quotient.

Section VIII. *Complex Fractions used in the Arts and Commerce.*

In the arts and in commerce, it is customary to assume a series of units having a constant relation to each other, so that the units of one denomination become fractions of another. One farthing, for example, is $\frac{1}{4}$ of a penny, 1 penny $\frac{1}{12}$ of a shilling, 1 shilling $\frac{1}{20}$ of a pound, or $\frac{1}{21}$ of a guinea. One lineal inch, again, is $\frac{1}{12}$ of a foot, 1 foot $\frac{1}{3}$ of a yard; and so on, according to the relations expressed in the tables at the end of the fifth section. The arithmetical operations on complex numbers of these kinds are usually effected by simpler rules than those which apply to vulgar fractions generally; of which it will, therefore, be proper here to specify a few.

Reduction.

Here we have two general cases:

Case 1. When the numbers are to be reduced from a higher denomination to a lower.

1. Multiply the number in the higher denomination by as many of the next lower as make an integer, or one, in that higher, and set down the product.

2. To this product add the number, if any, which was in this lower denomination before; and multiply the sum by as many of the next lower denomination as make an integer in the present one.

3. Proceed in the same manner through all the denominations to the lowest, and the number last found will be the value of all the numbers which were in the higher denominations taken together.

Case 2. When the numbers are to be reduced from a lower denomination to a higher.

1. Divide the given number by as many of that denomination as make one of the next higher, and set down what remains.

2. Divide the quotient by as many of this as make one of the next higher denomination, aud set down what remains in like manner as before.

3. Proceed in the same manner through all the denominations to the highest; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

The method of proof is to work the question back again.

Examples.

1. Reduce £14 to shillings, pence, and farthings; and 24316 farthings into pounds, &c.

14	24316
20	÷ 4
280 shillings	6079 pence
12	÷ 12
3360 pence	506 7
4	÷ 20
13440 farthings.	£25 6s. 7d.

2. Reduce 22 Ac. 3 R. 24 P. into perches; and 52187 perches into acres.

22 3 24	52187
4	÷ 40
91 Roods	1304 27
40	÷ 4
3664 Perches.	Ac. 326 0 R. 27 P.

Addition and Subtraction.

Rule. Write, one under the other, the parts which have the same denomination, and operate successively on each of them, beginning with the smallest. If any sum surpass the number of units necessary to form one or more units of the next superior order, put down the excess, and carry on the other. Proceed similarly with regard to what is borrowed in subtraction.

Examples.

	£.	s.	d.		lb.	oz.	dwt.	gr.		lb.	oz.	dwt.	gr.
Add	368	10	3	Add	14	6	12	13	Add	10	8	11	17
	257	10	5		17	5	3	12		42	5	16	12
	88	11	4½		15	0	9	16		12	2	14	18
	33	10	0		2	7	15	20		51	6	0	22
	12	13	5		13	2	10	19		24	9	17	17
	8	8	8½		4	1	5	21		29	4	18	22
Sum	769	4	2	Sum	66	11	18	5	Sum	171	2	0	12

	£.	s.	d.		£.	s.	d.		lb.	oz.	dwt.	gr.
From	16	12	8¾	From	21	13	4¾	From	18	9	10	8
Take	10	11	6¼	Take	18	9	8½	Take	9	10	15	20
Rem.	6	1	2½	Rem.	3	3	8¼	Rem.	8	10	14	12

Multiplication and Division.

1. In compound *multiplication*, place the multiplier under the lowest denomination of the multiplicand.—Multiply the number in the lowest denomination by the multiplier, and find how many integers of the next higher denomination are contained in the product, and write down what remains.—Carry the integers, thus found, to the product of the next higher denomination, with which proceed as before; and so on, through all the denominations to the highest; and this product, together with the several remainders, taken as one number, will be the whole amount required.

If the multiplier exceed 12, multiply successively by its component parts; as in the following examples:

2. In Compound *Division* place the divisor and dividend as in simple division.—Begin at the left hand, or highest denomination of the dividend, which divide by the divisor, and write down the quotient.—If there be any remainder after this division, find how many integers of the next lower denomination it is equal to, and add them to the number, if any, which stands in that denomination.—Divide this number, so found, by the divisor, and write the quotient under its proper denomination.—Proceed in the same manner through all the denominations to the lowest, and the whole quotient, thus found, will be the answer required.

Examples.

1. Multiply £4 17s. 6¼d. by 441, and 3a. 2r. 14p. by 531.

	£	s.	d.
	4	17	6¼
9 × 7 × 7 = 441			9
	43	17	10¼
			7
	307	4	11¾
			7
Ans.	£2150	14	10¼

	a.	r.	p.	
	3	2	14	
			10	
	35	3	20	
			10	
	358	3	0	for 100
			5	
	1793	3	0	for 500
	107	2	20	3 times 10
	3	2	14	1 top line.
Ans.	1904	3	34	

2. Divide £521 18s. 6d. by 432.

$$432 = 12 \times 12 \times 3.$$

Therefore, by short division:

	£	s.	d.	
	521	18	6	
÷ 12				
	43	9	$10\frac{1}{2}$	
÷ 12				
	3	12	$5\frac{3}{4}$	+ $\frac{1}{2}$ a farthing.
÷ 3				
Quotient	£1	4	$1\frac{3}{4}$	+ $\frac{5}{6}$ of a farthing.

By long division.

```
       £   s.  d.  £   s.  d.
432) 521  18   6(1   4   1¾ + ⅚ of a farthing.
     432
     ---
      89
      20
     ----
432)1798(4        360/432 = 30/36 = 5/6.
    1728
    ----
      70
      12
     ---
432) 846(1
     432
     ---
     414
       4
    ----
432)1656(3
    1296
    ----
     360
    ----
```

Duodecimals.

Fractions whose denominators are 12, 144, 1728, &c. are called *duodecimals;* and the division and sub-division of the integer are *understood* without being expressed, as in *decimals*. The method of operating by this class of fractions, is principally in use among artificers, in computing the contents of work, of which the dimensions were taken in *feet*, *inches*, and *twelfths* of an inch.

RULE. Set down the two dimensions to be multiplied together, one under the other, so that feet shall stand under feet, inches under inches, &c. Multiply each term in the multiplicand beginning at the lowest, by the feet in the multiplier, and set the result of each immediately under its corresponding term, observing to carry 1 for every 12, from the inches to the feet. In like manner, multiply all the multiplicand by the inches of the multiplier, and then by the twelfth parts, setting the result of each term *one* place removed to the right hand when the multiplier is inches, and *two* places when the parts become the multiplier. The sum of these partial products will be the answer required.

Or, instead of multiplying by the inches, &c. take such parts of the multiplicand as these are of a foot.

Examples.

1. Multiply 4 f. 7 inc. into 8 f. 4 inc.

4*f*.	7 *i*.		or,		4*f*.	7 *i*.
8	4					8
36	8				36	8
1	6	4		$4 = \frac{1}{3}$	1	$6\frac{1}{3}$
38	2	4			37	$2\frac{1}{3}$

Here, the 2 which stands in the second place does not denote square inches, but rectangles of an inch broad and a foot long, which are to be added to the square inches in the third place, so that $(2 \times 12) + 4 = 28$ are the square inches, and the product is 38 square feet, 28 square inches.

2. Multiply 35 *f.* 4½ *inc.* into 12 *f.* 3¼ *inc:*

35	4	6			or,	35	4½		
12	3	4					12		
424	6	0				424	6		
8	10	1	6		3 = ¼	8	10	1⅛	
	11	9	6	0	⅓ = ⅙ of 3		11	9⅛	
434	3	11	0	0		434	3	11	

Here, again, the product is 435 square feet, $+ (3 \times 12) + 11$ inches, or 434 square feet, 47 square inches. And this manner of estimating the inches must be observed in all cases where two dimensions in feet and inches are thus multiplied together.

SECTION IX. *Powers and Roots.*

A *power* is a quantity produced by multiplying any given number, called the root or radix, a certain number of times continually by itself. The operation of thus raising powers is called *involution*.

$3 = 3$ is the root, or 1st power of 3.
$3 \times 3 = 3^2 = 9$, is the 2d power, or square of 3.
$3 \times 3 \times 3 = 3^3 = 27$, is the 3d power, or cube of 3.
$3 \times 3 \times 3 \times 3 = 3^4 = 81$, do. 4th power, or biquadrate of 3.
&c. &c. &c.

Table of the first Nine Powers of the first Nine Numbers.

1st	2d	3d	4th.	5th.	6th.	7th.	8th.	9th.
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

So again, $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9} =$ square of $\frac{2}{3}$; $\frac{4}{9} \times \frac{2}{3} = \frac{8}{27} =$ cube of $\frac{2}{3}$; $\frac{8}{27} \times \frac{2}{3} = \frac{16}{81}$, biquadrate of $\frac{2}{3}$; and so of others. Where it is evident, that as the powers of integers become successively larger and larger, the powers of pure or proper fractions become successively smaller and smaller.

Evolution.

Evolution, or the extraction of roots, is the reverse of involution.

Any power of a given number may be found exactly; but we cannot, conversely, find every root of a given number exactly. Thus, we know the *square root* of 4 exactly, being 2; but we cannot assign exactly the *cube root* of 4. So, again, though we know the *cube root* of 8, viz. 2, we cannot exactly assign the *square root* of 8. But, of 64 we can assign both the square root and the cube root, the former being 8, the latter 4.

By means of decimals we can in all cases *approximate* the root, to any proposed degree of exactness.

Those roots which only approximate are called surd roots, or *surds*, or *irrational* numbers; as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{9}$, &c. while those which can be found exactly are called *rational*; as $\sqrt{9} = 3$, $\sqrt[3]{125} = 5$, $\sqrt[4]{16} = 2$.

To extract the square root.

RULE. Divide the given number into periods of two figures each, by setting a point over the place of units, another over the place of hundreds, and so on over every second figure, both to the left hand in integers, and to the right hand in decimals.

Find the greatest square in the first period on the left-hand, and set its root on the right-hand of the given number, after the manner of a quotient figure in Division.

Subtract the square thus found from the said period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right-hand figure; and set that quotient figure both in the quotient and divisor.

Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to the next period of the given number, for a new dividend.

Repeat the same process over again, viz. find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last.*

Note. The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following Examples.—Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of cyphers, two in each period.

Examples.

1. Find the square root of 17·3056.

	17·3056 (4·16
	16
81	130
1	81
826	4956
	4956
	....

4·16 the root: in which the number of decimal places is the same as the number of decimal *periods* into which the given number was divided.

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures, of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation, appears from the algebraic form of the square of any number of terms, whether two or three, or more. Thus, $35^2 = 30^2 + 2 \cdot 30 \cdot 5 + 5^2$, or generally $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)b$, the square of two terms; where it appears that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor is $2a + b$, or double the first term increased by the second. And hence the manner of extraction is as in the rule.

2. Find the square root of 2, to six decimals.

2 (1·414213 root.
1

24 | 100
4 | 96

281 | 400
1 | 281

2824 | 11900
4 | 11296

28282 | 60400
2 | 56564

282841 | 383600
1 | 282841

2828423 | 10075900
| 8485269

1590631

3. Find the square root of $\frac{5}{12}$.

$\frac{5}{12}$ = ·416666666, &c.

0·416666 (0·64549, &c.
36

124 | 566
4 | 496

1285 | 7066
5 | 6425

12904 | 64166
4 | 51616

129089 | 1255066
| 1161801

93265

Note. In cases where the square roots of all the integers up to 1000 are tabulated, such an example as the above may be done more easily by a little reduction. Thus $\sqrt{\frac{5}{12}} = \sqrt{(\frac{5}{12} \cdot \frac{12}{12})} = \sqrt{\frac{60}{144}} = \frac{1}{12}\sqrt{60} = \frac{7 \cdot 7459667}{12} = \cdot 645497$, &c.

Cube and higher roots.

The rules usually given in books of arithmetic for the cube and higher roots, are very tedious in practice: on which account it is adviseable to work either by means of approximating rules, or by means of logarithms. The latter is, generally speaking, the best method. We shall merely present here Dr. Hutton's approximating rule for the *cube root*, which may sometimes be serviceable when logarithmic tables are not at hand.

Rule. By trials take the nearest rational cube to the given number, whether it be greater or less, and call it the assumed cube.

Then say, by the Rule of Three, as the sum of the given number and double the assumed cube, is to the sum of the assumed cube and double the given number, so is the root of the assumed cube, to the root required, nearly. Or, as the first sum is to the difference of the given and assumed cube, so is the assumed root, to the difference of the roots, nearly.

Again, by using, in like manner, the cube of the root last found as a new assumed cube, another root will be obtained still nearer. And so on as far as we please; using always the cube of the last found root, for the assumed cube.

Example.

To find the cube root of 21035·8.

Here we soon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the assumed cube. Then

```
    19683        21035·8
        2              2
    -----        -------
    39366        42071·6
    21035·8      19683
    -------      -------
As  60401·8  :   61754·6 :: 27 : 27·6047
                      27
                 -------
                 4322822
                1235092
                --------
60401·8 ) 1667374·2 ( 27·6047 the root nearly.
           459388       Again, assuming 27·6
            36525       and working as before,
              284       the root will be found
               42       27·60491.
```

SECTION X. *Proportion.*

Two magnitudes may be compared under two different points of view; that is to say, either by inquiring what is the *excess* of one above the other, or *how often* one is contained in the other. The result of this comparison is obtained by *subtraction* in the first case, by *division* in the second. The *ratio* of two numbers is indicated by the quotient resulting from dividing one by the other. Thus 3 may be regarded as the ratio of 12 to 4, since $\frac{12}{4}$ or 3 is the quotient of the numbers 12 and 4.

The first of two numbers constituting a ratio is called the *antecedent*, the second the *consequent*.

The *difference* of two numbers is not changed by *adding* one and the same number to each, or by *subtracting* the same number from each.

Thus $12 - 5 = (12 + 2) - (5 + 2) = 14 - 7 = (12 - 2) - (5 - 2) = 10 - 3$.

In like manner, a *ratio* is not changed by either *multiplying* both its terms, or *dividing* both its terms by the same number.

Thus $\frac{14}{4} = (\frac{14}{4} \cdot \frac{3}{3} =) \frac{42}{12} = (\frac{14}{4} \div \frac{2}{2} =) \frac{7}{2}$.

Since surds enter arithmetical calculations by means of their approximate values, a sufficiently precise idea may be obtained of their ratio: thus, $\frac{\sqrt{3}}{\sqrt{2}} = \frac{1 \cdot 73205}{1 \cdot 41421}$. This ratio is often commensurable even with respect to surds: as $\frac{\sqrt{12}}{\sqrt{3}} = \frac{\sqrt{(12 \div 3)}}{\sqrt{(3 \div 3)}} = \frac{\sqrt{4}}{\sqrt{1}} = \frac{2}{1}$.

Equality of differences, or equidifference, is a term used to indicate that the difference between two numbers is the same as the difference between other two, or other two. Such, for example, as $12 - 9 = 8 - 5 = 7 - 4$.

Equality of ratios, or proportion, is similarly employed to denote that the ratio of two numbers is the same as that between two others. Thus 20 and 10, 14 and 7, have 2 for the measure of the ratio: we have therefore a proportion between 20 and 10, 14 and 7, which is thus expressed 20 : 10 :: 14 : 7, and thus read 20 *are to* 10 as 14 *are to* 7. The same proportion may also be represented thus, $\frac{20}{10} = \frac{14}{7}$. Though, by whatever notation it be represented, it is best to read or enumerate it as above. It is true, however, that in all cases when two fractions are equal, the numerator of one of them is to its denominator, as the numerator of the other to its denominator.

In a proportion as 20 : 10 :: 14 : 7, the second and third terms are called the *means*, the first and fourth the *extremes*.

When the two means are equal the proportion is said to be *continued*. Thus 3 : 6 :: 6 : 12 are in continued proportion. This is usually expressed thus ∺ 3 : 6 : 12; and the second term is called the *mean proportional*.

In the case of *equidifference*, as $12 - 9 = 7 - 4$, the *sum* of the extremes (12 and 4) is equal to that of the means (9 and 7). In like manner in a proportion, as 20 : 10 :: 14 : 7, the *product* of the extremes (20 and 7) is equal to that of the means (10 and 14). The converse of this likewise obtains, that if $20 \times 7 = 10 \times 14$, then 20 : 10 :: 14 : 7.

Hence, 1. If there be four numbers, 5, 3, 15, 9, such that the products 5×9 and 3×15 are found equal, we may infer the equality of their ratios, or the proportion $\frac{5}{3} = \frac{15}{9}$, or 5 : 3 :: 15 : 9. So that a proportion may always be constituted with the factors of two equal products.

2. If the means are equal, their product becomes a square; therefore the mean proportional between two numbers is equal to the square root of their product. Thus, between 4 and 9 the mean proportional is $\sqrt{(4 \times 9)} = 6$.

3. If a proportion contain an unknown term, such, for example, as 5 : 3 :: 15 : the unknown quantity; since 5 times

the unknown quantity must be equal to 3×15 or 45, that quantity itself is equal to $45 \div 5$ or 9. Generally, one of the extremes is equal to the product of the means divided by the other extreme: and one of the means is equal to the product of the extremes divided by the other mean.

4. We may, without affecting the correctness of a proportion, subject the several terms which compose it to all the changes which can be made, while the product of the extremes remains equal to that of the means. Thus, for $5 : 3 :: 15 : 9$, which gives $5 \times 9 = 3 \times 15$, we may

I. Change the places of the means, without changing those of the extremes, or change the places of the extremes without changing those of the means: this is denoted by the term *alternando.*

Thus, $5 : 3 :: 15 : 9$
become $5 : 15 :: 3 : 9$
or $9 : 3 :: 15 : 5$
or $9 : 15 :: 3 : 5$

II. Put the extremes in the places of the means; this is called *invertendo.*

$$3 : 5 :: 9 : 15$$

III. Multiply or divide the two antecedents or the two consequents by the same number.

It also appears with regard to proportions, that the sum or the difference of the antecedents is to that of the consequents, as either antecedent is to its consequent.

And, that the sum of the antecedents is to their difference, as to the sum of the consequents is to their difference.

Thus $\frac{5 \pm 15}{3 \pm 9} = \frac{5}{3} = \frac{15}{9}$, and $\frac{5 + 15}{3 + 9} = \frac{5 \backsim 15}{3 \backsim 9}$.

If there be a series of equal ratios represented by $\frac{6}{3} = \frac{10}{5} = \frac{14}{7} = \frac{30}{15}$, we shall have $\frac{6 + 10 + 14 + 30}{3 + 5 + 7 + 15} = \frac{60}{30} = \frac{6}{3} = \frac{10}{5} = \&c.$ Therefore, in a series of equal ratios, the sum of the antecedents, is to the sum of the consequents, as any one antecedent is to its consequent.

If there be two proportions, as $30 : 15 :: 6 : 3$, and $2 : 3 :: 4 : 6$, then multiplying them term by term we shall have $30 \times 2 : 15 \times 3 :: 6 \times 4 : 3 \times 6$, which is evidently a proportion, because $30 \times 2 \times 3 \times 6 = 15 \times 3 \times 6 \times 4 = 1080$.

Thus, also, any powers of quantities in proportion are in proportion; and conversely of the roots.

$$\left.\begin{matrix} 2 : 3 :: 6 : 9 \\ 2 : 3 :: 6 : 9 \\ 2 : 3 :: 6 : 9 \end{matrix}\right\} \text{Hence} \left\{\begin{matrix} 2^2 : 3^2 :: 6^2 : 9^2 \\ \\ 2^3 : 3^3 :: 6^3 : 9^3 \end{matrix}\right.$$

Rule of Three.

When the elements of a problem will form a proportion of which the unknown quantity is the last term, a simple calculation will determine it, and the problem is said to belong to the *Golden Rule*, or *Rule of Three.* The operation is regulated by the foregoing principles of proportion.

Of the three given numbers, two are called the terms of *supposition*, and the other the term of *demand.* Now if the term of *demand* be greater or less than the other term of the same kind, and the question require the term sought to be respectively greater or less than the other, the question belongs to the Rule of Three *direct;* otherwise it belongs to the Rule of Three *inverse.*

For the Rule of Three Direct we have this

RULE. Write the three given terms in the following order: viz. let that which implies or asks the demand be put in the third place, and the other of the same kind in the first: then will the remaining term, which is similar to the fourth or required one, occupy the second place. Having thus disposed the numbers, called stating the question, reduce the first and third terms to one and the same denomination; and if the second term be a compound one, reduce it to the lowest name mentioned. Multiply the second and third terms together, and divide the product by the first, and the quotient will be the answer in the same denomination to which you reduced the second term.

When the second term is a compound one, and the third a composite number, it is generally better to multiply the second term, without any previous reduction, by the component parts of the third, as in compound multiplication, after which divide the compound product by the first term, or, by its factors. (Here, the first and third terms are homogeneous, in a given ratio, the second and fourth in the same.)

For the Inverse Rule.

State the question and reduce the terms as in the direct rule: then multiply the first and second terms together and divide the product by the third, and the quotient will be the answer.

Examples in the Direct Rule.

1. If 3 gallons of brandy cost 19*s.*, what will 126 gallons cost at the same rate?

```
gal.  s.   gal.
 3 : 19 :: 126   :  ?
            19
          ----
          1134
          126
          ----      2|0
       3)2394 (79|8
         21      ----
         --    39l. 18s. Ans.
          29
          27
          --
           24
           24
           --
```

2. How much brandy may be bought for 39*l.* 18*s.*; at the rate of 3 gallons for 19 shillings?

```
 s.   gal.  £.  s.
19 : 3 :: 39  18  :  ?
           20
          ----
          798s.
            3
          ----  gal.
      19)2394(126 Ans.
         19
         --
          49
          38
          --
          114
          114
          ---
```

3. If 21 yards of cloth cost 24*l.* 10*s.* what will 160 yards cost?

```
           £.   s.
Here, 21 : 24  10 :: 160  :  ?
              × 4
            ------- 4×4×10=160
            98   0
              × 4
            -------
            392  0
             × 10
            -------
           3920  0
       ÷ 3
           ----------
           1306 13  4
       ÷ 7
           ----------
         £186 13s. 4d. Ans.
```

4. If by selling cloth at 1*l.* 2*s.* per yard, 10 per cent. is gained, what would be gained if it had been sold at 1*l.* 5*s.* per yard?

```
       £.  s.        £.  s.
Here, 1    2 : 110 :: 1   5  :  ?
      20              20
      --             ----
      22              25
      --            × 110
                    -----
                 ( 2)2750
              22 {   ----
                 (11)1375
                    -----
         Amount  £125
         Deduct  £100
                 ----
Gain per cent.    £25
                 ----
```

5. What is the simple interest of 560*l.* for 5 years, at 4 per cent. per annum?

£. £. £.
Here, 100 : 4 :: 560
4
100) 2240
£22·4
20
80

Interest for 1 year £22 8*s.*

y. £. s. y.
Then 1 : 22 8 :: 5
5

£112 0 *Answer.*

6. If 100 workmen can finish a piece of work in 12 days, how many men working equally hard would have finished it in 3 days?

This example is manifestly in the Rule of Three *inverse.*

d. w. d.
Hence, 12 : 100 :: 3 : ?
12

3) 1200

Answer 400 *workmen.*

A distinct rule is usually given for the working of problems in *Compound Proportion*: but they may generally be solved with greater mental facility by means of separate statings. Thus:

7. If a person travel 300 miles in 10 days of 12 hours each, in how many days of 16 hours each may he travel 600 miles?

First, if the days were of the same length it would be, by direct proportion,

m. d. m.
As 300 : 10 :: 600 : 20 days.

But these would be days of 12 hours each, instead of 16, of which fewer will be required. Hence, by inverse proportion,

h. d.
As 12 : 20 :: 16 : $\frac{12 \times 20}{16} = 15$

So that the answer is 15 days.

8. If a family of 9 persons spend £480 in 8 months, how much will serve a family (living upon the same scale) of 24 persons 16 months?

p. £. p.
First, as 9 : 480 :: 24 : $\frac{480 \times 24}{9}$ = £1280.

But this would only be the expence for 8 months. Hence, again,

m. £. m. £.
As 8 : 1280 :: 16 : 2560, the expence of the 24 persons for 16 months.

Note. The Rule of Three receives its application in questions of *Interest, Discount, Fellowship, Barter,* &c.

Properties of Numbers.

To render these intelligible to the student, we shall here collect a few definitions.

1. An *unit*, or *unity*, is the representation of any thing considered individually, without regard to the parts of which it is composed.

2. An *integer* is either a unit or an assemblage of units: and a *fraction* is any part or parts of a unit.

3. A *multiple* of any number, is that which contains it some exact number of times.

4. One number is said to *measure* another, when it divides it without leaving any remainder.

5. And if a number exactly divides two, or more numbers, it is then called their *common measure.*

6. An *even number*, is that which can be halved, or divided into two equal parts.

7. An *odd number*, is that which cannot be halved, or which differs from an even number by unity.

8. A *prime number*, is that which can only be measured by 1, or unity.

9. One number is said to be *prime* to another when unity is the only number by which they can both be measured.

10. A *composite number*, is that which can be measured by some number greater than unity.

11. A *perfect number*, is that which is equal to the sum of all its aliquot parts: thus $6 = \frac{6}{2} + \frac{6}{3} + \frac{6}{6}$.

Prop. 1. The sum, or difference of any two even numbers, is an even number.

2. The sum, or difference, of any two odd numbers, is even; but the sum of three odd numbers, is odd.

3. The sum of any even number of odd numbers is even; but the sum of any odd number of odd numbers is odd.

4. The sum or difference of an even and an odd number, is odd.

5. The product of an even and an odd number, or of two even numbers, is even.

6. An odd number cannot be divided by an even number, without a remainder.

7. Any power of an even number is even.

8. The product of any two odd numbers is an odd number.

9. The product of any number of odd numbers is odd: and every power of an odd number is odd.

10. If an odd number divides an even number, it will also divide the half of it.

11. If a number consist of many parts, and each of those parts have a common divisor d; then will the whole number taken collectively, be divisible by d.

12. Neither the sum nor the difference of two fractions, which are in their lowest terms, and of which the denominator of the one contains a factor not common to the other, can be equal to an integer number.

13. If a square number be either multiplied or divided by a square, the product or quotient is a square; and conversely, if a square number be either multiplied or divided by a number that is not a square, the product or quotient is not a square.

14. The product arising from two different prime numbers cannot be a square number.

15. The product of no two different numbers prime to each other can make a square, unless each of those numbers be a square.

16. The square root of an integer number, that is not a complete square, can neither be expressed by an integer nor by any rational fraction.

17. The cube root of an integer that is not a complete cube, cannot be expressed by either an integer or a rational fraction.

18. Every prime number greater than 2, is of one of the forms $4n + 1$, or $4n - 1$.

19. Every prime number greater than 3, is of one of the forms $6n + 1$, or $6n - 1$.

20. No algebraical formula can contain prime numbers only.

21. The number of prime numbers is unlimited.

22. The first twenty prime numbers are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, and 67.

23. A square number cannot terminate with an odd number of cyphers.

24. If a square number terminate with a 4, the last figure but one (towards the right hand) will be an even number.

25. If a square number terminate with 5, it will terminate with 25.

26. If a square number terminate with an odd digit, the last figure but one will be even; and if it terminate with any even digit, except 4, the last figure but one will be odd.

27. No square number can terminate with two equal digits, except two *cyphers* or two *fours*.

28. No number whose last, or right-hand, digit is 2, 3, 7, or 8, is a square number.

29. If a cube number be divisible by 7, it is also divisible by the cube of 7.

30. The difference between any integral cube and its root is always divisible by 6.

31. Neither the sum nor the difference of two cubes, can be a cube.

32. A cube number may end with any of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0.

33. If any series of numbers, beginning from 1, be in continued geometrical proportion, the 3d, 5th, 7th, &c. will be squares; the 4th, 7th, 10th, &c. cubes; and the 7th, of course, both a square and a cube.

34. All the powers of any number that end with either 5 or 6, will end with 5 or 6, respectively.

35. Any power, n, of the natural numbers 1, 2, 3, 4, 5, 6, &c. has as many orders of differences as there are units in the common exponent of all the numbers; and the last of those differences is a constant quantity, and equal to the continual product $1 \times 2 \times 3 \times 4 \times \ldots\ldots \times n$, continued till the last factor, or the number of factors be n, the exponent of the powers. Thus,

The 1st powers 1, 2, 3, 4, 5, &c. have but one order of
differences 1 1 1 1 &c. and that difference is 1.

The 2d powers 1, 4, 9, 16, 25, &c. have two orders of
differences 3 5 7 9
2 2 2
of which the last is, constantly, $2 = 1 \times 2$.

The 3d powers 1, 8, 27, 64, 125, &c. have three orders of
differences 7 19 37 61
12 18 24
6 6
of which the last is $6 = 1 \times 2 \times 3$.

In like manner, the 4th, or last, differences of the 4th powers, are each $= 24 = 1 \times 2 \times 3 \times 4$; and the 5th, or last differences of the 5th powers, are each $125 = 1 \times 2 \times 3 \times 4 \times 5$.

36. If unity be divided into any two unequal parts, the sum of one of those parts added to the square of the other, is equal to the sum of the other part added to the square of that. Thus, of the two parts $\frac{1}{5}$ and $\frac{4}{5}$, $\frac{1}{5} + (\frac{4}{5})^2 = \frac{4}{5} + (\frac{1}{5})^2 = \frac{21}{25}$; so, again, of the parts $\frac{2}{5}$ and $\frac{3}{5}$, $\frac{2}{5} + (\frac{3}{5})^2 = \frac{3}{5} + (\frac{2}{5})^2 = \frac{19}{25}$.

For the demonstrations of these and a variety of other properties of numbers, those who wish to pursue this curious line of inquiry, may consult Legendre "Sur la Theorie des Nombres," the "Disquisitiones Arithmeticæ" of Gauss, or "Barlow's Elementary Investigation of the Theory of Numbers."

Also, for the highly interesting properties of *Circulating Decimals*, and their connection with *prime numbers*, consult the curious works of the late Mr. H. Goodwyn, entitled "A first Centenary," and "A Table of the Circles arising from the Division of a Unit by all the Integers from 1 to 1024."

A useful Numerical Problem to reduce a given fraction, or a given ratio, to the least terms; and as near as may be of the same value.

RULE 1. Let A, B, be the two numbers. Divide the latter B by the former A, and you will have 1 for A; and some number and a fraction annexed, for B, call this C. Place these in the first step.

Then subtract the fractional parts from the denominator, and what remains put after C + 1, with a negative sign. Then throw away the denominator, and place 1 and that last number in the second step. This is the foundation of all the rest.

If the fractional parts in both be nearly equal, add these two steps together; if not, multiply the lesser by such a number as will make the fractional parts, in both, nearly equal, and then add. And this multiplier is found by dividing the greater fraction by the lesser, so far as to get an integer quotient. When you add the steps together, you must subtract the fractional parts from one another, because they have contrary signs.

The process is to be continued on, the same way, adding the last step, or its multiple, to a foregoing step, viz. to that which has the least fraction.

Note. The ratios thus found will be alternately greater and less than the true one, but continually approaching nearer and nearer. And that is the nearest in small numbers, which precedes far larger numbers: and the excess or defect of any one is visible in the operation.

Example 1.

To find the ratio of 10000 to 7854, in small numbers.

	A	B	
1	1	0 + ·7854	
2	1	1 — ·2146,	first ratio.
	·2146) ·7854 (3		
3	3	3 — ·6438	
4	4	3 + ·1416,	2d ratio.
5	5	4 — ·0730,	3d ratio.
6	9	7 + ·0686,	4th ratio.
7	14	11 — ·0044,	5th ratio.
	·0044) ·0686 (15		
8	210	165 — ·0660	
9	219	172 + ·0026,	6th ratio.
10	233	183 — ·0018,	7th ratio.
11	452	355 + ·0008,	8th ratio.
	·0008) ·0018 (2		
12	904	710 + ·0016	
13	1137	893 — ·0002,	9th ratio.
	·0002) ·0008 (4		
14	4548	3572 — ·0008	
15	5000	3927 ± ·0000,	10th ratio.

Explanation.

The ratio of 10000 to 7854 is the same as 1 to 0 + ·7854 or 1 to 1 — ·2146; here 1 and 1 is the first ratio. But 2146 being less than 7854, divide the latter by the former, and you get 3 in the quotient, then multiply 1 and 1 — ·2146 by 3, produces 3 and 3 — ·6438 as in the 3d step. This third step added to the first step produces 4 and 3 for the integers, and subtracting the fractional parts, leaves ·1416. So the 4th step is 4 and 3 + ·1416; and the integers 4 and 3 is the 2d ratio. In this manner it is continued to the end; and the several ratios approximating nearer and nearer, are $\frac{1}{1}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{9}{7}$, $\frac{14}{11}$, $\frac{219}{172}$, $\frac{233}{183}$, $\frac{452}{355}$, $\frac{1137}{893}$, and $\frac{5000}{3927}$. Here $\frac{14}{11}$ is the nearest in small numbers, the defect being only $\frac{44}{10000}$.

Example 2.

To find the ratio of 268·8 to 282 in the least numbers.

$$2688)\,2820\,(1\tfrac{132}{2688} = 2 - \tfrac{2556}{2688}.$$

2688
———
132

1	1	1 + 0132,	first ratio.
2	1	2 − 2556	
	132) 2556 (19		
3	19	19 + 2508	
4	20	21 − 48,	2d ratio.
	48) 132 (2		
5	40	42 − 96	
6	41	43 + 36,	3d ratio.
7	61	64 − 12,	4th ratio.
	12) 36 (3		
8	183	192 − 36	
9	224	235	, 5th ratio.

So the several ratios are $\frac{1}{1}$, $\frac{20}{21}$, $\frac{41}{43}$, $\frac{61}{64}$, $\frac{224}{235}$. And the defect or excess is plain by inspection, *e. g.* $\frac{41}{43}$ differs from the truth only $\frac{36}{2688}$ parts; and $\frac{20}{21}$, but 48 such parts.

Rule 2. Divide the greater number by the less, and the divisor by the remainder, and the last divisor by the last remainder, and so on till 0 remain. Then

1 divided by the first quotient, gives the first ratio.

And the terms of the first ratio multiplied by the second quotient, and 1 added to the denominator, give the second ratio.

And in general, the terms of any ratio, multiplied by the next quotient, and the terms of the foregoing ratio added, give the next succeeding ratio.

Example 3.

Let the numbers be 10000 and 31416, or the ratio $\frac{10000}{31416}$.

```
10000) 31416 ( 3
       30000
       -----
       1416) 10000 ( 7
              9912
              ----
                88) 1416 ( 16
                      88
                     ---
                     536
                     528
                     ---
                       8) 88 ( 11
                          88
                          --
                           0
```

Then $\frac{1}{3}$ = 1st or least ratio.

$\frac{1 \times 7}{3 \times 7 + 1}$ or $\frac{7}{22}$ = 2d ratio.

$\frac{7 \times 16 + 1}{22 \times 16 + 3}$ or $\frac{113}{355}$ = 3d ratio.

$\frac{113 \times 11 + 7}{355 \times 11 + 22}$ or $\frac{1250}{3927}$ = 4th ratio.

Example 4.

The ratio of 268·8 to 282 is required.

```
2688) 2820 ( 1
      2688
      ----
       132) 2688 ( 20
             264
             ---
             48) 132 ( 2
                  96
                  --
                  36) 48 ( 1
                      36
                      --
                      12) 36 ( 3
                          36
                          --
                           0
```

Then $\frac{1}{1}$ = first ratio.

$\frac{1 \times 20}{1 \times 20 + 1}$ or $\frac{20}{21}$ = 2d ratio.

$\frac{20 \times 2 + 1}{21 \times 2 + 1}$ or $\frac{41}{43}$ = 3d ratio.

$\frac{41 \times 1 + 20}{43 \times 1 + 21}$ or $\frac{61}{64}$ = 4th ratio.

$\frac{61 \times 3 + 41}{64 \times 3 + 43}$ or $\frac{224}{235}$ = 5th ratio.

English Measures.

According to the Act of Parliament passed in June 1824, but whose operation is postponed until January 1st, 1826, the chief part of the weights and measures remain as they were: the Act simply prescribing scientific modes of determining them, in case they should be lost.

The pound *troy* contains 5760 grains.
The pound *avoirdupois* contains 7000 grains.
The *imperial gallon* contains 277·274 cubic inches.
The *corn bushel*, eight times the above.

Hence, with respect to Ale, Wine, and Corn, it will be expedient to possess a

TABLE OF FACTORS,

For converting old measures into new, and the contrary.

	By Decimals.			By vulgar fractions nearly.		
	Corn Measure.	Wine Measure.	Ale Measure.	Corn Measure.	Wine Measure.	Ale, Measure.
To convert old measures to new	·96943	·83311	1·01704	$\frac{32}{33}$	$\frac{5}{6}$	$\frac{60}{59}$
To convert new measures to old.	1·03153	1·20032	·98324	$\frac{33}{32}$	$\frac{6}{5}$	$\frac{59}{60}$

N. B. For reducing the *prices*, these numbers must all be reversed.

Ex. 1. Reduce 63 gallons, wine measure, to the equivalent number in imperial measure.

63 × ·83311 = 52·486; or 63 × $\frac{5}{6}$ = $\frac{315}{6}$ = 52½ imperial gallons nearly.

Ex. 2. Reduce 8 bushels imperial measure, to the equivalent number in Winchester measure.

8 × 1·03153 = 8·25224; or 8 × $\frac{33}{32}$ = 8¼ Winchester bushels nearly.

FRENCH WEIGHTS AND MEASURES.

French Measures, Old System.

A point is	·0148025 English inches, or nearly $\frac{2}{135}$.
A line	·088815, or nearly $\frac{8}{90}$.
An inch, or pouce	1·06578, or $\frac{81}{76}$.
A foot	12·78933.
An ell, or aune	46·8947, or 44 French inches, or according to Vega, 43·9.
A sonde	63·9967, or 5 French feet, about $\frac{8}{9}$ English fathom.
A toise, or fathom	76·7360, or 6 French feet; formerly 76·71. Phil. Trans. for 1742.
A perche	230·2080, or 18 French feet.
A perche, mesure royale	22 French feet.
A league	2282 toises, or $\frac{1}{25}$ of a degree.
A square inch	1·13582 English square inches.
An arpent	100 square perches, about $\frac{5}{6}$ acre English, used near Paris.
An arpent mesure royale	about 1¼ English acre.
A cubic inch	1·21068 cubic inches.
A litron	65·34.
A boisseau	1045·44, or 16 litrons.
A minot	2090·875, or 2 boisseaux, nearly an English bushel.
A mine	4181·75, or 2 minots.
A septier	8363·5, or 2 mines, or 6912 inches French, double for oats.
A muid	100362, or 12 septiers.

N. B. The perch, which determines the measure of the acre, varies in different parts of the country; but the arpent of woodland is every where the same, the perch being 22 feet long, and this arpent contains 48400 French square feet, or 6108 English square yards, or one acre, one rood, one perch. The arpent for cultivated land in the vicinity of Paris contains 900 square toises, or 4088 English yards; so that 48 such arpents are equal to 38 English acres nearly.

French Measures, New System.

MEASURES OF LENGTH.

	English inches.
Millemetre	·039371.
Centimetre	·39371.
Decimetre	3·9371.
Metre	39·371, or 3·281 feet, or 1·09364 yards, or nearly 1 yard, 1½ nail, or 443·2959 French lines, or ·513074 toises.
Decametre	393·71, or 10 yards, 2 feet, 97 inches.
Hecatometre	3937·1, or 100 yards, 1 foot, 1 inch.
Chiliometre	39371· or 4 furlongs, 213 yards, 1 foot, 10·2 inches: so that 1 chiliometre is nearly ⅝ of a mile.
Myriometre	393710· or 6 miles, 1 furlong, 136 yards, 6 inches.

N. B. An inch = ·0254 metres; 2441 inches = 62 metres; 10000 feet = 305 metres nearly. See, for a fuller comparison, Mr. H. Goodwyn's "Synoptical Table."

SUPERFICIAL OR SQUARE MEASURE.

Are = a square decametre	3·95 English perches, of 119·6046 square yards.
Decare	1196·0460 square yards.
Hecatare	11960·46 square yards, or 2 acres, 1 rood, 35·4 perches.

MEASURES OF CAPACITY.

	Cubic inches, English.
Millilitre	·06103.
Centilitre	·61028.
Decilitre	6·1028.
Litre, a cubic decimetre	61·028, or 2·113 wine pints.
Decalitre	610·28, or 2·64 wine gallons.
Hecatolitre	6102·8, or 3·5317 cubic feet, or 26·4 wine gallons.
Chiliolitre	61028· or 35·3170 cubic feet, or 1 tun, 12 wine gallons.
Myriolitre	610280· or 353·1700 cubic feet.

SOLID MEASURE.

	Cubic feet, English.
Decistre for fire wood	3·5317.
Stere, a cubic metre	35·3170.
Decastre	353·1700.

N. B. In order to express decimal proportions in this new system, the following terms have been adopted. The term *deca* prefixed denotes 10 times; *heca* 100 times; *chilio* 1000 times; and *myrio* 10,000 times. On the other hand, *deci* expresses the 10th part; *centi* the 100th part; and *milli* the 1000th part: so that *decametre* signifies 10 metres; and *decimetre* the 10th part of a metre, &c. &c. The *metre* is the element of long measures; *are* that of square measures; *stere* that of solid measures: the *litre* is the element of all measures of capacity; and the *gramme*, which is the weight of a cubic centimetre of distilled water, is the element for all weights.

French Weights, Old System.

72 grains	= 1 gros =	$59\frac{9}{128}$ English grains Troy.
8 gros	= 1 ounce =	$472\frac{9}{16}$ English grains Troy.
16 ounces	= 1 pound =	7561 English grains Troy.

Sometimes the gros is divided into 3 deniers, and each denier into 24 grains.

French Weights, New System.

	English Troy grains.
Milligramme	·01544
Centigramme	·15445
Decigramme	1·54457
Gramme	15·44579
Decagramme	154·45793
Hecatogramme	1544·57938
Chiliogramme	15445·79386
Myriogramme	154457·93860

A decagramme is 6 dwt. 10·45 grains Troy, or 2 dr. 1 scr. 14·45 grains Apothecaries weight, or 5·648 dr. Avoirdupois.

The hecatogramme = 3 oz. 8·48 dr. Avoirdupois.
The chiliogramme = 2 lb. 3 oz. 4·87 dr. Avoirdupois.
The myriogramme = 22 lb. 1 oz. 0·73 dr. Avoirdupois.

CHAP. II.

ALGEBRA.

Section I. *Definitions and Notation.*

Algebra is the science of the computation of magnitudes in general, as arithmetic is the particular science of the computation of numbers.

Every figure or arithmetical character has a determinate and individual value; the figure 5, for example, represents always one and the same number, namely, the collection of 5 units, of an order depending upon the position and use of the figure itself. Algebraical characters, on the contrary, must be, in general, independent of all particular signification, and proper to represent all sorts of numbers or quantities, according to the nature of the questions to which we apply them. They should, moreover, be simple and easy to trace, so as to fatigue neither the attention nor the memory. These advantages are obtained by employing the letters of the alphabet a, b, c, &c. to represent any kinds of magnitudes which become the subjects of mathematical research. The consequence is that when we have resolved by a single algebraical computation all the problems of the same kind proposed in the utmost generality of which they are susceptible; the application of the investigation to all particular cases requires no more than arithmetical operations.

It is usual, though by no means absolutely necessary, to represent quantities that are *known* by the commencing letters of the alphabet, as a, b, c, d, &c. and those that are *unknown* by the concluding letters w, x, y, z. But it is often convenient, especially as it assists the memory, to represent any quantity, whether known or unknown, which enters an investigation, by its initial letter; as *sum* by s, *product* by p, *density* by d, *velocity* by v, *time* by t; and so of others.

Now, if s denote the sum of four numbers represented by a, b, c, and d, then, adopting the other symbols explained at the beginning of arithmetic, we should express this algebraically by writing $s = a + b + c + d$.

If the four quantities be all equal, or $s = a + a + a + a$, this evidently reduces to $s = 4 \times a$, or simply $s = 4\,a$, dropping the sign of multiplication, which is here understood. The figure 4 is named the *coefficient*. In the quantities $3\,a$, $5\,a$, $7\,a$, $n\,a$; 3, 5, 7, and n, are respectively the coefficients.

The continual product of three or more quantities is expressed either by interposing the sign of multiplication, as $a \times b \times c \times d$; or by interposing *dots*, which have the same signification, as $a . b . c . d$; or, lastly, by placing the letters in juxta position, as $a\,b\,c\,d$.

When the quantities are equal, their continued multiplication produces powers, as $a\,a$, $a\,a\,a$, $a\,a\,a\,a$, &c. which are usually represented, instead of repeating the letters, by placing a figure a little above the single letter to expound or tell how many equal factors are multiplied together; this figure is called the *exponent*. Thus, instead of $a\,a$, $a\,a\,a$, $a\,a\,a\,a$, we put a^2, a^3, a^4, the figures 2, 3, 4, being the exponents.

Since roots are the reverse of powers, they are expressed by exponents which are the *reciprocals* of those that express the corresponding powers. Thus the square root of a is represented either by $\sqrt{a}$, or by $a^{\frac{1}{2}}$; the cube root of $a + b$, either by $\sqrt[3]{a + b}$, or by $(a + b)^{\frac{1}{3}}$; the fourth root of $a + b - c$, either by $\sqrt[4]{(a + b - c)}$, or by $(a + b - c)^{\frac{1}{4}}$.

We give the name *term* to any quantity separated from another by the sign + or −. A *monomial* has *one* term; a *binomial* has *two* terms, as $a + b$, $a\,c - 4\,a\,b$: when the second term of a binomial is −, it is frequently called a *residual*. A *trinomial* has three terms, as $a + b + c$, $a\,d - 4\,a\,b + 5\,b\,c$. A *quadrinomial* has 4, as $a + b + c - d$. A *multinomial*, or *polynomial*, has many terms.

The signs + and −, which in arithmetic simply indicate the operations of addition and subtraction, are employed more extensively in algebra, to denote, besides addition and subtraction, any two operations or any two states which are as opposed in their nature as addition and subtraction are. And if, in an algebraical process, the sign + is prefixed to a quantity to mark that it exists in a certain state, position, direction, &c. then, whenever the sign − occurs in connection with such quantity, it *must* indicate precisely the contrary state, position, &c. and no intermediate one. This is a matter of pure convention, and not of metaphysical reasoning. Other characters *might* have been contrived to denote this opposition; but they would be superfluous, because the characters + and −, though originally restricted to denote addition and subtraction, may safely be extended to other purposes.

Thus if $+a$ signifies any thing	added above, to the right, forwards,	$-a$ signifies	subtracted below, to the left, backwards.

If $+a$ signifies any assigned	Increase, Gravity, Money due, Motion upwards,	$-a$ signifies a corresponding	Decrease, Levity, Money owing, Motion downward.

And so on in every kind of contrariety. And two such quantities connected together in any case destroy each other's effect, or are equal to nothing, as $+a - a = 0$. Thus, if a man has but 10*l.* and at the same time owes 10*l.* he is worth *nothing*. And, if a vessel which would, otherwise, sail six miles an hour, be carried back six miles an hour by a current, it makes no advance.

Like signs are either all positive (+), or all negative (−). And *unlike* are when some are positive and others negative. If there be no sign before a quantity the sign + is understood.

An *equation* is when two sets of quantities which make an equal aggregate are placed with the sign of equality (=) between them:

As $12 + 5 = 20 - 3$, or $x + y = a + b - c\,d$.

The quantities placed on both sides the sign of equality are called respectively the *members* of the equation.

Section II. *Addition and Subtraction.*

1. Properly speaking, there is not in algebra, either *addition* or *subtraction*, but a *reduction*, namely, the algebraic operation by which several terms are, when it is possible, combined into one term. This, however, can only be effected upon quantities that differ in their coefficients and their signs, while they are formed of the same letters and the same exponents.

Thus, $3a, + 4a, + 7a,$
$5a^3b, - 3a^3b, + 8a^3b,$ } are evidently reducible.

In the first set, the incorporation gives $(3 + 4 + 7)\,a = 14\,a$, in the second $(5 - 3 + 8)\,a^3b = 10\,a^3b$.

2. Generally, taking the similar terms the reduction affects their coefficients, which are to be *added* when their signs are *alike*, *subtracted* when they are *different*: in the first case, give to the result the common sign; in the second, the sign of the quantity having the greatest coefficient.

3. When quantities are presented promiscuously it is best to classify them, previously to the incorporation.

Thus, $3a^2$, $-3bc$, $+2c^2$, $+4d$, $+7a^2$, $+5bc$, $+a^2$, $-2c^2$, $-4bc$, when arranged become as in the margin, and their sum is readily obtained as in the fourth line.

$$\begin{array}{r}
3a^2 + 5bc - 2c^2 \\
7a^2 - 3bc + 4d \\
a^2 - 4bc + 2c^2 \\
\hline
11a^2 - 2bc + 4d \\
\hline
\end{array}$$

4. If it were required to *subtract* a residual, as $b - c$, from a single term, as a; it is evident that the required difference would not be changed if a quantity c were added to both. We should then have to take $b - c + c$, or b, from $a + c$, that is we should have $a + c - b$, for the difference sought, in which, as is manifest, the signs of the letters b and c which were to be subtracted have become changed.

Hence, to subtract a polynomial, change all the signs, and reduce by incorporating the coefficients, where that is possible.

$$\begin{array}{lr}
\text{Thus} & 4ab - 3bc \\
 & -(2ab - 6bc) \\
\hline
\text{become} & 4ab - 3bc \\
 & -2ab + 6bc \\
\hline
\text{Result} & 2ab + 3bc \\
\hline
\end{array}
\qquad
\begin{array}{lr}
\text{And} & 4ab - 3c^2 + bc \\
 & -(ab - c^2 - 2bc) \\
\hline
\text{become} & 4ab - 3c^2 + bc \\
 & -ab + c^2 + 2bc \\
\hline
\text{Result} & 3ab - 2c^2 + 3bc \\
\hline
\end{array}$$

5. In addition and subtraction of *algebraic fractions*, the quantities must be reduced to a common denominator, and occasionally undergo other reductions similar to those in vulgar fractions in arithmetic; and then the sum or the difference of the numerators may be placed over the common denominator, as required.

Thus, $\frac{a}{b} + \frac{c}{d} = \frac{a\,d}{b\,d} + \frac{b\,c}{b\,d} = \frac{a\,d + b\,c}{b\,d}$.

And, $\frac{a}{b\,c} + \frac{b}{a\,c} + \frac{c}{a\,b} + \frac{a\,b}{c} + \frac{a\,c}{b} + \frac{b\,c}{a} = \frac{a^2}{a\,b\,c}$
$+ \frac{b^2}{a\,b\,c} + \frac{c^2}{a\,b\,c} + \frac{a^2\,b^2}{a\,b\,c} + \frac{a^2\,c^2}{a\,b\,c} + \frac{b^2\,c^2}{a\,b\,c} =$
$\frac{a^2 + b^2 + c^2 + a^2\,b^2 + a^2\,c^2 + b^2\,c^2}{a\,b\,c}$.

Also, $\frac{a}{b} - \frac{c}{d} = \frac{a\,d - b\,c}{b\,d}$.

And, $\frac{a + x}{d} - \frac{a - x}{c} = \frac{a\,c + c\,x}{c\,d} - \frac{a\,d - d\,x}{c\,d} =$
$\frac{a\,c + c\,x - a\,d + d\,x}{c\,d} = \frac{a\,(c - d) + x\,(c + d)}{c\,d}$.

And, $\frac{b + x}{b - x} - \frac{b - x}{b + x} =$
$\frac{(b^2 + 2\,b\,x + x^2) - (b^2 - 2\,b\,x + x^2)}{b^2 - x^2} = \frac{4\,b\,x}{b^2 - x^2}$.

SECTION III. *Multiplication.*

1. To multiply *monomials*, multiply their coefficients, add together the exponents which affect the same letters (ascribing the exponent 1 to quantities which have none), then write in order the coefficients and letters thus obtained.

2. To find the product of two *polynomials*, multiply each term of the one into all those of the other, following the rule given for monomials: and observe to take each partial product *negatively* when its factors have contrary signs, and *positively* when they have the same signs. Or, briefly, observe that *like* signs give +, *unlike* signs −.

3. To multiply algebraic *fractions*, take the product of the numerators for the new numerator, and that of the denominators for the new denominator.

Note. The general rule for the signs may be rendered evident from the following definition: multiplication is the finding a magnitude which has to the multiplicand the proportion of the multiplier to unity. Hence, the multiplier must be an abstract number, and, if a simple term, can have neither + nor − prefixed to its notation. Now 1st, $+a \times +m = +ma$, for the quality of a cannot be altered by increasing or diminishing its value in any proportion; therefore the product is of the quality *plus*, and ma by the definition is the product of a and m. Secondly, $-a \times +m = -ma$, for the same reasons as before, *mutatis mutandis*. Thirdly, $+a \times -m$ has no meaning; for m must be an abstract number, therefore here we can have no proof. But $+a \times (m-n) = ma - na$, n being less than m; for a taken as often as there are units in m is $= ma$, by the first case: but a was to have been taken only as often as there are units in $m-n$; therefore a has been taken too often by the units in n; consequently a taken n times, or na, must be subtracted; and of course $ma - na$ is the true product. Fourthly, $-a \times (m-n) = -ma + na$. For $-a \times m = -ma$ (by case 2); but this, as above, is too great by $-na$; therefore $-ma$ with $-na$ subtracted from it is the true product; but this, by the rule of subtraction, is $= -ma + na$.

Examples.

1. $4ab \times 5cd = 4 \,.\, 5 \,.\, ab \,.\, cd = 20abcd$.

2. $8a^2b^3 \times 4a^5b = 8 \,.\, 4 \,.\, a^2 \,.\, a^5 \,.\, b^3 \,.\, b = 32a^{2+5}b^{3+1} = 32a^7b^4$.

3. Multiply $2a + bc - 2b^2$
By $2a - bc + 2b^2$

$$\begin{array}{l} 4a^2 + 2abc - 4ab^2 \\ \quad\quad - 2abc - b^2c^2 + 2b^3c \\ \quad\quad\quad\quad + 4ab^2 + 2b^3c - 4b^4 \\ \hline \text{Product } 4a^2 - b^2c^2 + 4b^3c - 4b^4 \end{array}$$

4. $$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + ab \\ \quad\quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

5. $$\begin{array}{l} a^2 + 2ab + b^2 \\ a + b \\ \hline a^3 + 2a^2b + ab^2 \\ \quad + a^2b + 2ab^2 + b^3 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

$$6 \begin{cases} a + b \\ a - b \end{cases}$$

$$\begin{array}{l} a^2 + a\,b \\ \quad\; - a\,b - b^2 \\ \hline a^2 - b^2 \end{array}$$

7. $\frac{a+b}{c} \times \frac{a-b}{d} = \frac{(a+b)(a-b)}{c \times d} = \frac{a^2-b^2}{c\,d}.$

8. $\frac{2\,x}{a} \times \frac{3\,a\,b}{c} \times \frac{3\,a\,c}{2\,b} = \frac{18\,a^2\,b\,c\,x}{2\,a\,b\,c} = \frac{9\,a\,x}{1} \times \frac{2\,a\,b\,c}{2\,a\,b\,c} = 9\,a\,x.$

Note. From the above examples (4, 5 and 6) we may learn—

1. That the square of the sum of two quantities is equal to the sum of the squares of the two quantities together with twice their product.

2. That the product of the sum and difference of two quantities is equal to the difference of their squares.

3. That the cube of the binomial $a + b$, is $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$.

SECTION IV. *Division.*

1. To divide one *monomial* by another, suppress the letters that are common to both, subtract the exponents which affect the same letters, and divide the coefficients one by another.

2. To divide a *polynomial* by a *monomial*, divide each term of the polynomial by the monomial according to rule 1, and connect the results by their proper signs.

3. To divide two *polynomials* one by the other, arrange them with respect to the same letter, then divide the first terms one by the other, and thence will result one term of the quotient; multiply the divisor by this, and subtract the product from the dividend: proceed with the remainder in the same manner. Observe in the partial divisions the same rules for the determination of the *signs* as in multiplication.

4. To divide algebraic *fractions*, invert the terms of the divisor, and proceed as in multiplication.

Examples.

1. $12\,a^3\,b^2\,c \div 3\,a\,b = \frac{12}{3}\,a^{3-1}\,b^{2-1}\,c = 4\,a^2\,b\,c.$

2. $15\,a^3\,b^5 \div 5\,a^2\,b^2 = \frac{15}{5}\,a^{3-2}\,b^{5-2} = 3\,a\,b^3.$

3. $6\,x^2 + 12\,x\,y - 9\,x\,y\,z \div 3\,x = \frac{6\,x^2}{3\,x} + \frac{12\,x\,y}{3\,x} - \frac{9\,x\,y\,z}{3\,x} = 2\,x + 4\,y - 3\,y\,z.$

4. Divide $x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3$ by $x - z$.

$$x - z)\,x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3\,(x^2 - 2\,x\,z + z^2 \text{ quotient.}$$

$$\begin{array}{l}
x^3 - x^2\,z \\
\hline
-2\,x^2\,z + 3\,x\,z^2 \\
-2\,x^2\,z + 2\,x\,z^2 \\
\hline
x\,z^2 - z^3 \\
x\,z^2 - z^3 \\
\hline
* \quad * \\
\hline
\end{array}$$

5. Divide $a^5 - b^5$ by $a - b$.

$$a - b)\,a^5 - b^5\,(a^4 + a^3\,b + a^2\,b^2 + a\,b^3 + b^4 \text{ quotient.}$$

$$\begin{array}{l}
a^5 - a^4\,b \\
\hline
a^4\,b \\
a^4\,b - a^3\,b^2 \\
\hline
a^3\,b^2 \\
a^3\,b^2 - a^2\,b^3 \\
\hline
a^2\,b^3 \\
a^2\,b^3 - a\,b^4 \\
\hline
a\,b^4 - b^5 \\
a\,b^4 - b^5 \\
\hline
\end{array}$$

Here the second term of the dividend is brought down to stand over the corresponding term in the last product.

6. Divide 1 by $1 - x$.

$$
\begin{array}{r|l}
1 - x & 1 \quad \left(1 + x + x^2 + x^3 + x^4 + \frac{x^5}{1 - x}\right. \\
 & \underline{1 - x} \\
 & \quad x \\
 & \quad \underline{x - x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2 - x^3} \\
 & \qquad\quad x^3 \\
 & \qquad\quad \underline{x^3 - x^4} \\
 & \qquad\qquad x^4 \\
 & \qquad\qquad \underline{x^4 - x^5} \\
 & \qquad\qquad\quad \underline{x^5}
\end{array}
$$

7. $$\frac{2x^2}{a^3 + x^3} \div \frac{x}{a + x} = \frac{2x^2}{a^3 + x^3} \times \frac{a + x}{x} = \frac{2x^2(a + x)}{(a^3 + x^3)\,x} = \frac{2x}{x^2 - ax + a^2}.$$

8. $$\frac{x^4 - b^4}{x^2 - 2bx + b^2} \div \frac{x^2 + bx}{x - b} = \frac{x^4 - b^4}{(x - b)^2} \times \frac{x - b}{x(x + b)} = \frac{x^4 - b^4}{x(x + b)(x - b)} = \frac{x^4 - b^4}{x(x^2 - b^2)} = \frac{x^2 + b^2}{x} = x + \frac{b^2}{x}.$$

9. Divide $96 - 6a^4$ by $6 - 3a$. *Quot.* $16 + 8a + 4a^2 + 2a^3$.

10. Divide $10a^3 + 11a^2b - 19abc - 15a^2c + 3ab^2 + 15bc^2 - 5b^2c$ by $3ab + 5a^2 - 5bc$. *Quot.* $2a + b - 3c$.

11. Divide $x^2 + y^2 + \frac{y^4}{x^2}$ by $x + y + \frac{y^2}{x}$. *Quot.* $x - y + \frac{y^2}{x}$.

SECTION V. *Involution.*

1. *To involve or raise monomials to any proposed power.*

Involve the coefficient to the power required, for a new coefficient. Multiply the index of each letter by the index of the required power. Place each product over its respective letter, and prefix the coefficient found as above: the result will be the power required.

All the powers of an *affirmative* quantity will be $+$; of a *negative* quantity, the *even* powers, as the 2d, 4th, 6th, &c. will be $+$; the *odd* powers, as the 3d, 5th, 7th, &c. will be $-$.

To involve fractions, apply these rules to both numerator and denominator.

Examples.

1. Find the fourth power of $2\,a^2$.
   $2 \times 2 \times 2 \times 2 = 16$, new coefficient.
   $2 \times 4 = 8$, new exponent. Hence $16\,a^8$ the answer.
2. The fifth power of $-3\,y^2$ is $243\,y^{10}$.
3. The fourth power of $-4\,x^3$ is $256\,x^{12}$.
4. The sixth power of $\frac{2\,z}{3\,y^2}$ is $\frac{64\,z^6}{729\,y^{12}}$.

2. *To involve polynomials.*

Multiply the given quantity into itself as many times, wanting one, as there are units in the index of the required power, and the last product will be the power required.

Example.

Cube $x \pm z$ and $2\,x - 3\,z$.

$x \pm z$	$2\,x - 3\,z$
$x \pm z$	$2\,x - 3\,z$
$x^2 \pm x\,z$	$4\,x^2 - 6\,x\,z$
$\quad \pm x\,z + z^2$	$\quad - 6\,x\,z + 9\,z^2$
$x^2 \pm 2\,x\,z + z^2$ *squares*	$4\,x^2 - 12\,x\,z + 9\,z^2$
$x \pm z$	$2\,x - 3\,z$
$x^3 \pm 2\,x^2\,z + x\,z^2$	$8\,x^3 - 24\,x^2\,z + 18\,x\,z^2$
$\quad \pm x^2\,z + 2\,x\,z^2 + z^3$	$\quad - 12\,x^2\,z + 36\,x\,z^2 - 27\,z^3$
$x^3 \pm 3\,x^2\,z + 3\,x\,z^2 + z^3$ *cubes*	$8\,x^3 - 36\,x^2\,z + 54\,x\,z^2 - 27\,z^3$

The operation required by the preceding rules, however simple in its nature, becomes tedious when even a binomial is raised to a high power. In such cases it is usual to employ

Sir Isaac Newton's Rule for involving a Binomial.

1. *To find the terms without the coefficients.* The index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ or coefficients.* The first is always 1, and the second is the index of the power: and in general, if the coefficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.*

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be + ; but if the second term be −, all the odd terms will be +, and the even terms −.

Examples.

1. Let $a + x$ be involved to the fifth power.

The terms without the coefficients will be

$$a^5,\ a^4 x,\ a^3 x^2,\ a^2 x^3,\ a x^4,\ x^5,$$

and the coefficients will be

$$1,\ 5,\ \frac{5 \times 4}{2},\ \frac{10 \times 3}{3},\ \frac{10 \times 2}{4},\ \frac{5 \times 1}{5}.$$

And therefore the fifth power is

$$a^5 + 5a^4 x + 10a^3 x^2 + 10a^2 x^3 + 5a x^4 + x^5.$$

* This rule, expressed in general terms, is as follows:

$$(a + b)^n = a^n + n \,.\, a^{n-1} b + n \,.\, \frac{n-1}{2} a^{n-2} b^2 + n \,.\, \frac{n-1}{2} \,.\, \frac{n-2}{3} a^{n-3} b^3 \ \&c.$$

$$(a - b)^n = a^n - n \,.\, a^{n-1} b + n \,.\, \frac{n-1}{2} a^{n-2} b^2 - n \,.\, \frac{n-1}{2} \,.\, \frac{n-2}{3} a^{n-3} b^3 \ \&c.$$

The same theorem applied to fractional exponents, and with a slight modification, serves for the extraction of roots in infinite series; as will be shown a little farther on.

Here we have, for the sake of perspicuity, exhibited separately, the manner of obtaining the several terms and their respective coefficients. But in *practice* the separation of the two operations is inconvenient. The best way to obtain the *coefficients* is to perform the *division* first, upon either the requisite coefficient or exponent (one or other of which may always be divided without a remainder), and to multiply the quotient into the other. Thus, the result may be obtained at once in a single line, nearly as rapidly as it can be written down.

2. $(x + y)^7 = x^7 + 7\,x^6\,z + 21\,x^5\,z^2 + 35\,x^4\,z^3 + 35\,x^3\,z^4 + 21\,x^2\,z^5 + 7\,x\,z^6 + z^7$.

3. $(x - z)^8 = x^8 - 8\,x^7\,z + 28\,x^6\,z^2 - 56\,x^5\,z^3 + 70\,x^4\,z^4 - 56\,x^3\,z^5 + 28\,x^2\,z^6 - 8\,x\,z^7 + z^8$.

For Trinomials and Quadrinomials. Let two of the terms be regarded as *one*, and the remaining term or terms as the other; and proceed as above.

Example.

Involve $x + y - z$ to the fourth power.

Let x be regarded as one term of the binomial, and $y - z$ as the other; then will $(x + y - z)^4 = \{x + (y - z)\}^4 = x^4 + 4\,x^3\,(y - z) + 6\,x^2\,(y - z)^2 + 4\,x\,(y - z)^3 + (y - z)^4$, where the powers of $y - z$ being expanded by the same rule, and multiplied into their respective factors, we shall at length have $x^4 + 4\,x^3\,y - 4\,x^3\,z + 6\,x^2\,y^2 - 12\,x^2\,y\,z + 6\,x^2\,z^2 + 4\,x\,y^3 - 12\,x\,y^2\,z + 12\,x\,y\,z^2 - 4\,x\,z^3 + y^4 - 4\,y^3\,z + 6\,y^2\,z^2 - 4\,y\,z^3 + z^4$, the fourth power required.

Had $(x + y)$ and $-z$ been taken for the two terms of the binomial, the result would have been the same.

Note. The rule for the involution of multinomials is too complex for this place.

SECTION VI. *Evolution.*

1. *To find the roots of monomials.* Extract the corresponding root of the coefficient for the new coefficient: then multiply the index of the letter or letters by the index of the root, the result will be the exponents of the letter or letters to be placed after the coefficient for the root required.

Examples.

1. Find the fourth root of $81\,a^4\,z^8$.
   First $\sqrt[4]{81} = \sqrt{9} = 3$, new coefficient.
   Then $4 \times \frac{1}{4} = 1$, exponent of a; and $8 \times \frac{1}{4} = 2$, exponent of z.
   Hence $3\,a\,z^2$ is the root required.

2. $\sqrt[5]{(32\,a^5\,x^{10})} = \sqrt[5]{32} \times a^{5 \times \frac{1}{5}} \times x^{10 \times \frac{1}{5}} = 2\,a\,x^2.$

3. $\sqrt[3]{\dfrac{8\,x^3}{27\,z^9}} = \dfrac{\sqrt[3]{8} \times x^{3 \times \frac{1}{3}}}{\sqrt[3]{27} \times z^{9 \times \frac{1}{3}}} = \dfrac{2\,x}{3\,z^3}.$

2. *To find the square root of a polynomial.* Proceed as in the extraction of the square root, in arithmetic.

Examples.

1. Extract the square root of $a^4 + 4\,a^3\,x + 6\,a^2\,x^2 + 4\,a\,x^3 + x^4$.

$$a^4 + 4\,a^3\,x + 6\,a^2\,x^2 + 4\,a\,x^3 + x^4\;(a^2 + 2\,a\,x + x^2 \text{ [root req.]}$$
$$a^4$$
$$\overline{}$$
$$2\,a^2 + 2\,a\,x)\;4\,a^3\,x + 6\,a^2\,x^2$$
$$4\,a^3\,x + 4\,a^2\,x^2$$
$$\overline{}$$
$$2\,a^2 + 4\,a\,x + x^2)\;2\,a^2\,x^2 + 4\,a\,x^3 + x^4$$
$$2\,a^2\,x^2 + 4\,a\,x^3 + x^4$$
$$\overline{}$$
$$*\quad\quad *\quad\quad *$$

2. Extract the square root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

$$
\begin{array}{rl}
 & x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}\,(x^2 - x + \frac{1}{4} \textit{ root.} \\
 & x^4 \\
 & \overline{} \\
2x^2 - x) & -2x^3 + \frac{3}{2}x^2 \\
 & -2x^3 + \phantom{\frac{3}{2}}x^2 \\
 & \overline{\phantom{-2x^3 + \frac{3}{2}x^2}} \\
2x^2 - 2x + \frac{1}{4}) & \frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{16} \\
 & \frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{16} \\
 & \overline{\phantom{\frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{16}}} \\
 & * \quad * \quad *
\end{array}
$$

3. *To find the roots of powers in general.* If they be not the roots of high powers that are required, the following rule may be employed:

Find the root of the first term, and place it in the quotient.—Subtract its power, and bring down the second term for a dividend.—Involve the root, last found, to the next lowest power, and multiply it by the index of the given power for a divisor.—Divide the dividend by the divisor, and the quotient will be the next term of the root.—Involve the whole root, and subtract and divide as before; and so on till the whole is finished.

Examples.

1. Find the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

$$
\begin{array}{rl}
 & x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1\,(x^2 - 2x \\
 & x^6 \qquad\qquad\qquad\qquad\qquad [+ 1, \textit{ root req.} \\
 & \overline{} \\
(x^2)^2 \times 3 = 3x^4) & -6x^5 \\
 & x^6 - 6x^5 + 12x^4 - 8x^3 \\
 & \overline{} \\
 & \quad 3x^4)\,3x^4 \\
 & \overline{} \\
 & x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \\
 & \overline{} \\
 & * \quad * \quad * \quad * \quad * \quad * \quad *
\end{array}
$$

2. Find the 4th root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

$$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 \;(2a - 3x, \textit{ root sought.}$$

$$16a^4$$

$$8a^3 \times 4 = 32a^3) - 96a^3x$$

$$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$$

* * * * *

Note. In the higher roots proceed thus:
For the biquadrate, extract the square root of the square root.
——— sixth root, ——— cube root of the square root.
——— eighth root, ——— sq. rt. of the sq. rt. of the sq. rt.
——— ninth root, ——— the cube root of the cube root.
Examples, however, of such high roots seldom occur in any practical inquiries.

Section VII. *Surds.*

A *Surd* or irrational quantity, is a quantity under a radical sign or fractional index, the root of which cannot be exactly obtained. (See Arith. Sect. 9. *Evolution.*)

Surds, as well as other quantities, may be considered as either simple or compound, the first being *monomials*, as $\sqrt{3}$, $a^{\frac{1}{3}}$, $\sqrt[5]{a\,b^2}$, the others *polynomials*, as $\sqrt{3} + \sqrt{5}$, $\sqrt[3]{a} + \sqrt{b} - \sqrt{c^3}$, &c.

Rational quantities may be expressed in the *form* of surds, and the operation, when effected, often diminishes subsequent labour.

Reduction.

1. *To reduce surds into their simplest expressions.*

1. If the surd be not fractional, but consist of integers or integral factors under the radical sign.

Divide the given power by the greatest power, denoted by the index, contained therein, that measures it without remainder; let the quotient be affected by the radical sign, and have the root of the divisor prefixed as a coefficient, or connected by the sign $\times$.

Examples.

1. $\sqrt{75} = \sqrt{(25 \times 3)} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.
2. $\sqrt[3]{448} = \sqrt[3]{(64 \times 7)} = \sqrt[3]{64} \times \sqrt[3]{7} = 4\sqrt[3]{7}$.
3. $\sqrt[4]{176} = \sqrt[4]{(16 \times 11)} = \sqrt[4]{16} \times \sqrt[4]{11} = 2\sqrt[4]{11}$.
4. $\sqrt{(8x^3 - 12x^2y)} = \sqrt{4x^2(2x - 3y)} = \sqrt{4x^2} \times \sqrt{(2x - 3y)} = 2x\sqrt{(2x - 3y)}$.
5. $\sqrt[3]{108x^3y^4} = \sqrt[3]{(27x^3y^3 \times 4y)} = \sqrt[3]{27x^3y^3} \times \sqrt[3]{4y} = 3xy\sqrt[3]{4y}$.
6. $\sqrt[3]{(56x^3y + 8x^3)} = \sqrt[3]{8x^3(7y + 1)} = \sqrt[3]{8x^3} \times \sqrt[3]{(7y + 1)} = 2x\sqrt[3]{(7y + 1)}$.

2. If the surd be fractional, it may be reduced to an equivalent integral one, thus:

Multiply the numerator of the fraction under the radical sign, by that power of its denominator whose exponent is 1 less than the exponent of the surd. Take the denominator from under the radical sign, and divide the coefficient (whether unity, number, or letter) by it, for a new coefficient to stand before the surd so reduced.

Note. This reduction saves the labour of actually dividing by an approximated root; and will often enable the student to value any surd expressions by means of a table of roots of integers.

Examples.

1. $\sqrt{\frac{1}{3}} = \sqrt{(\frac{1}{3} \cdot \frac{3}{3})} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{9}} \times \sqrt{3} = \frac{1}{3}\sqrt{3}$.
2. $\sqrt{\frac{1}{5}} = \sqrt{(\frac{1}{5} \cdot \frac{5}{5})} = \sqrt{\frac{5}{25}} = \sqrt{\frac{1}{25}} \times \sqrt{5} = \frac{1}{5}\sqrt{5}$.
3. $\sqrt{\frac{1}{n}} = \sqrt{\left(\frac{1}{n} \cdot \frac{n}{n}\right)} = \sqrt{\frac{n}{n^2}} = \sqrt{\frac{1}{n^2}} \times \sqrt{n}$ $= \sqrt{n}$.
4. $\sqrt[3]{\frac{4}{5}} = \sqrt[3]{(\frac{4}{5} \cdot \frac{25}{25})} = \sqrt[3]{\frac{100}{125}} = \sqrt[3]{\frac{1}{125}} \times \sqrt[3]{100} = \frac{1}{5}\sqrt[3]{100}$.
5. $\sqrt[3]{\frac{2a}{5x}} = \sqrt[3]{\left(\frac{2a}{5x} \cdot \frac{25x^2}{25x^2}\right)} = \sqrt[3]{\frac{50ax^2}{125x^3}} = \sqrt[3]{\frac{1}{125x^3}}$ $\times \sqrt[3]{50ax^2} = \frac{1}{5x}\sqrt[3]{50ax^2}$.
6. $\sqrt[3]{\frac{16}{81}} = \sqrt[3]{\frac{8.2}{9.9}} = \sqrt[3]{\frac{8.2}{9.9} \cdot \frac{9}{9}} = \sqrt[3]{\frac{8.18}{729}} \sqrt[3]{\frac{8}{729}}$ $\times \sqrt[3]{18} = \frac{2}{9}\sqrt[3]{18}$.

3. If the denominator of the fraction be a binomial or residual, of which one or both terms are irrational and roots of squares:

Then, multiply this fraction by another which shall have its numerator and denominator alike, and each to contain the same two quantities as the denominator of the given expression, but connected with a different sign.

Note. By means of this rule, since any fraction whose numerator and denominator are the same, is equal to *unity*, the quantity to be reduced assumes a new appearance without changing its value; while the expression becomes freed from the surds in the denominator, because the product of the sum and difference of two quantities is equal to the difference of their squares.

Examples.

1. $\frac{8}{\sqrt{5}-\sqrt{3}} = \frac{8}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{8(\sqrt{5}+\sqrt{3})}{2} =$ $4(\sqrt{5}+\sqrt{3})$.

2. $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{3}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{3(\sqrt{5}-\sqrt{2})}{3} =$ $\sqrt{5}-\sqrt{2}$.

3. $\frac{\sqrt{20}-\sqrt{12}}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{20}-\sqrt{12}}{\sqrt{5}+\sqrt{3}} \cdot \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{100}-2\sqrt{60}+\sqrt{36}}{5-3}$ $= \frac{16-2\sqrt{60}}{2} = 8-\sqrt{60} = 8-2\sqrt{15}$.

4. $\frac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}} = \frac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}} \cdot \frac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt[4]{5}-\sqrt[4]{3}} = \frac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}\sqrt{ab} =$ $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} \cdot \frac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}\sqrt{ab} = (5^{\frac{3}{4}} - 5^{\frac{1}{2}}3^{\frac{1}{4}} + 5^{\frac{1}{4}}3^{\frac{1}{2}} - 3^{\frac{3}{4}})$ $\cdot (\frac{1}{2}\sqrt{ab})$.

Note 2. Upon the same general principle *any* binomial or residual surd, as $\sqrt[n]{A} \pm \sqrt[n]{B}$ may be rendered rational by taking $\sqrt[n]{A^{n-1}} \mp \sqrt[n]{(A^{n-2}B)} + \sqrt[n]{(A^{n-3}B^2)} \mp \sqrt[n]{(A^{n-4}B^3)}$ + &c. for a multiplier: where the upper signs must be taken with the upper, the lower with the lower, and the series continued to n terms.

Thus, the expression $\sqrt[4]{a^3} - \sqrt[4]{b^3}$, multiplied by $\sqrt[4]{a^9} + \sqrt[4]{a^6 b^3} + \sqrt[4]{a^3 b^6} + \sqrt[4]{b^9}$, gives the rational product $a^3 - b^3$.

2. *To reduce surds having different exponents, to equivalent ones that have a common exponent.* Involve the powers reciprocally, according to each other's exponent, for new powers: and let the product of the exponents be the common exponent.

Note. Hence, rational quantities may be reduced to the *form* of any assigned root; and roots with rational coefficients may be so reduced as to be brought entirely under the radical sign.

Examples.

1. $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$, become $a^{\frac{1}{n} \cdot \frac{m}{m}}$ or $a^{\frac{m}{mn}}$ and $b^{\frac{1}{m} \cdot \frac{n}{n}}$ or $b^{\frac{n}{mn}}$.

2. $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$, become $a^{\frac{1}{2} \cdot \frac{3}{3}}$ or $a^{\frac{3}{6}}$ and $b^{\frac{1}{3} \cdot \frac{2}{2}}$ or $b^{\frac{2}{6}}$.

3. $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$, become $3^{\frac{3}{6}}$ and $2^{\frac{2}{6}}$, or $\sqrt[6]{3^3}$ and $\sqrt[6]{2^2}$, or $\sqrt[6]{27}$ and $\sqrt[6]{4}$.

4. $(a + b)^{\frac{1}{3}}$, and $(a - b)^{\frac{1}{4}}$, become $\sqrt[12]{(a + b)^4}$ and $\sqrt[12]{(a - b)^3}$.

5. The rational quantity a^2, becomes $\sqrt{a^4}$, $\sqrt[3]{a^6}$, $\sqrt[4]{a^8}$, or $\sqrt[n]{a^{2n}}$.

6. $4\,a \sqrt[3]{5\,b}$, becomes $\sqrt[3]{(4\,a)^3} \times \sqrt[3]{5\,b}$, $\sqrt[3]{64\,a^3} \times \sqrt[3]{5\,b}$, or $\sqrt[3]{320\,a^3\,b}$.

These and other obvious reductions which will at once suggest themselves being effected, the operations of addition, subtraction, &c. are so easily performed upon such surd quantities as usually occur, that it will merely suffice to present a few examples without detailing rules.

Addition.

Ex. 1. $\sqrt{8} + \sqrt{18} = \sqrt{(4 \,.\, 2)} + \sqrt{(9 \,.\, 2)} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.

2. Add together $\sqrt{54}$, $\sqrt{\frac{1}{6}}$, and $\sqrt{\frac{8}{27}}$.

$\sqrt{54} = \sqrt{(9 \,.\, 6)} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$
$\sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1}{36}} \times \sqrt{6} = \frac{1}{6}\sqrt{6}$
$\sqrt{\frac{8}{27}} = \sqrt{(\frac{4 \cdot 2}{27} \cdot \frac{3}{3})} = \sqrt{\frac{4 \cdot 6}{81}} = \frac{2}{9}\sqrt{6}$

The sum of these is $(3 + \frac{1}{6} + \frac{2}{9})\sqrt{6} = 3\frac{7}{18}\sqrt{6}$.

3. $\sqrt{27\,a^4\,x} + \sqrt{3\,a^2\,x} = \sqrt{(9\,a^4 \,.\, 3\,x)} + (a^2 \,.\, 3\,x) = 3\,a^2\sqrt{3\,x} + a\sqrt{3\,x} = (3\,a^2 + a)\sqrt{3\,x}$.

4. $3\sqrt[3]{a^3\,b} + \sqrt[3]{a^6\,b} = (3\sqrt[3]{a^3} \times \sqrt[3]{b}) + (\sqrt[3]{a^6} \times \sqrt[3]{b}) = 3\,a\sqrt[3]{b} + a^2\sqrt[3]{b} = (3\,a + a^2)\sqrt[3]{b}$.

Subtraction.

Ex. 1. $2\sqrt{50} - \sqrt{18} = 2\sqrt{(25 \cdot 2)} - \sqrt{(9 \cdot 2)} = 2 \cdot 5 \sqrt{2} - 3\sqrt{2} = (10 - 3)\sqrt{2} = 7\sqrt{2}$.

2. $\sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}} = \sqrt{(\frac{3}{5} \cdot \frac{5}{5})} - \sqrt{(\frac{5}{27} \cdot \frac{3}{3})} = \sqrt{\frac{15}{25}} - \sqrt{\frac{15}{81}} = \frac{1}{5}\sqrt{15} - \frac{1}{9}\sqrt{15} = \frac{4}{45}\sqrt{15}$.

3. $\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{9}{32}} = \sqrt[3]{(\frac{2}{3} \cdot \frac{9}{9})} - \sqrt[3]{(\frac{9}{32} \cdot \frac{2}{2})} = \sqrt[3]{\frac{18}{27}} - \sqrt[3]{\frac{18}{64}} = \frac{1}{3}\sqrt[3]{18} - \frac{1}{4}\sqrt[3]{18} = \frac{1}{12}\sqrt[3]{18}$.

4. $\sqrt[3]{250\,a^3 x} - \sqrt[3]{16\,a^3 x} = \sqrt[3]{(125\,a^3 \cdot 2\,x)} - \sqrt[3]{(8\,a^3 \cdot 2\,x)} = 5\,a\sqrt[3]{2\,x} - 2\,a\sqrt[3]{2\,x} = 3\,a\sqrt[3]{2\,x}$.

5. $\sqrt{45\,s^4 x} - \sqrt{20\,s^2 x^3} = \sqrt{(9\,s^4 \cdot 5\,x)} - \sqrt{(4\,s^2 x^2 \cdot 5\,x)} = (3\,s^2 - 2\,s\,x)\sqrt{5\,x}$.

6. $\left(\frac{a}{c}\right)^{\frac{1}{2}} - \left(\frac{c}{a}\right)^{\frac{1}{2}} = \frac{a}{1} \times \left(\frac{1}{ac}\right)^{\frac{1}{2}} - \frac{c^2}{a} \times \left(\frac{1}{ac}\right)^{\frac{1}{2}}$

$$= \frac{a^2 - c^2}{a}\left(\frac{1}{ac}\right)^{\frac{1}{2}}.$$

Multiplication.

Ex. 1. $\sqrt[3]{18} \times 5\sqrt[3]{4} = 5\sqrt[3]{(18 \cdot 4)} = 5\sqrt[3]{(4 \cdot 2 \cdot 9)} = 5\sqrt[3]{(8 \cdot 9)} = 5 \cdot 2\sqrt[3]{9} = 10\sqrt[3]{9}$.

2. $\frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{2}{3} \cdot \frac{3}{4}\sqrt{(\frac{1}{8} \cdot \frac{7}{10})} = \frac{1}{2}\sqrt{\frac{7}{80}} = \frac{1}{2}\sqrt{(\frac{7}{80} \cdot \frac{5}{5})} = \frac{1}{2}\sqrt{\frac{35}{400}} = \frac{1}{2} \cdot \frac{1}{20}\sqrt{35} = \frac{1}{40}\sqrt{35}$.

3. $a^{\frac{2}{3}} \times a^{\frac{1}{4}} = a^{\frac{2}{3} + \frac{1}{4}} = a^{\frac{8}{12} + \frac{3}{12}} = a^{\frac{11}{12}}$.

4. $(x + z)^{\frac{1}{2}} \times (x + z)^{\frac{1}{3}} = (x + z)^{\frac{1}{2} + \frac{1}{3}} = (x + z)^{\frac{5}{6}}$.

5. $(x + \sqrt{y}) \times (x - \sqrt{y}) = x^2 - y$.

6. $(x + \sqrt{y})^{\frac{1}{3}} \times (x - \sqrt{y})^{\frac{1}{3}} = (x^2 - y)^{\frac{1}{3}}$.

7. $z^{\frac{1}{n}} \times z^{\frac{1}{m}} = z^{\frac{1}{n} + \frac{1}{m}} = z^{\frac{m+n}{mn}}$.

8. $\sqrt{d} \times \sqrt[3]{a\,b} = \sqrt[6]{d^3} \times \sqrt[6]{a^2 b^2} = \sqrt[6]{a^2 b^2 d^3}$.

9. $\sqrt{a - \sqrt{b - \sqrt{3}}} \times \sqrt{a + \sqrt{b - \sqrt{3}}} = \sqrt{(a^2 - b + \sqrt{3})}$.

10. $a^m \times a^{-n} = a^m \times \frac{1}{a^n} = a^{m-n}$.

11. $\sqrt{-a} \times \sqrt{-a} = \sqrt{a}\sqrt{-1} \times \sqrt{a}\sqrt{-1} = a \times -1 = -a$.

12. $\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1} = \sqrt{a\,b} \times -1 = -\sqrt{a\,b}$.

Division.

Ex. $4\sqrt[3]{1000} \div 2\sqrt[3]{4} = \frac{4}{2}\sqrt[3]{\frac{1000}{4}} = 2\sqrt[3]{250} = 2\sqrt[3]{(125 \cdot 2)} = 10\sqrt[3]{2}$.

2. $\frac{5}{7}\sqrt[3]{\frac{2}{3}} \div \frac{2}{5}\sqrt[3]{\frac{3}{4}} = \frac{5}{7} \cdot \frac{5}{2}\sqrt[3]{(\frac{2}{3} \cdot \frac{4}{3})} = \frac{25}{14}\sqrt[3]{\frac{8}{9}} = \frac{25}{14}\sqrt[3]{(\frac{8}{9} \cdot \frac{3}{3})} = \frac{25}{14}\sqrt[3]{(\frac{8}{27} \cdot 3)} = \frac{25}{14} \cdot \frac{2}{3}\sqrt[3]{3} = \frac{25}{21}\sqrt[3]{3}$.

3. $x^{\frac{2}{3}} \div x^{\frac{4}{3}} = x^{\frac{2}{3} - \frac{4}{3}} = x^{-\frac{2}{3}} = 1 \div x^{\frac{2}{3}}$.

4. $z^{\frac{1}{n}} \div z^{\frac{1}{m}} = z^{\frac{1}{n} - \frac{1}{m}} = z^{\frac{m-n}{mn}}$.

5. $(x^2 - x\,d - b + d\sqrt{b}) \div (x - \sqrt{b}) = x + \sqrt{b} - d$.

6. $\frac{ac - ad}{2b}\sqrt{(a^2 x - a x^2)} \div \frac{a}{2b}\sqrt{(a - x)} = \frac{ac - ad}{2b} \times \frac{2b}{a} \times \sqrt{\frac{a^2 x - a x^2}{a - x}} = (c - d)\sqrt{a x}$.

Involution.

Ex. 1. $(\frac{2}{3}a^{\frac{1}{3}})^2 = \frac{2}{3} \cdot \frac{2}{3} \cdot a^{\frac{1}{3}+\frac{1}{3}} = \frac{4}{9}a^{\frac{2}{3}} = \frac{4}{9}\sqrt[3]{a^2}$.

2. $(\frac{1}{3}\sqrt{\frac{1}{2}})^3 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \sqrt{(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2})} = \frac{1}{27}\sqrt{\frac{1}{8}} = \frac{1}{27}\sqrt{(\frac{1}{8} \cdot \frac{2}{2})} = \frac{1}{27}\sqrt{(\frac{1}{16} \cdot 2)} = \frac{1}{27} \cdot \frac{1}{4}\sqrt{2} = \frac{1}{108}\sqrt{2}$.

3. $(3 + \sqrt{5})^2 = \{(3 + \sqrt{5})(3 + \sqrt{5})\} = 14 + 6\sqrt{5}$.

4. $(a - \sqrt{b})^3 = a^3 - 3a^2\sqrt{b} + 3ab - b\sqrt{b}$.

Evolution.

Ex. 1. $\sqrt{10^3} = \sqrt{1000} = \sqrt{(100 \cdot 10)} = \sqrt{100} \times \sqrt{10} = 10\sqrt{10}$.

2. $\sqrt[4]{81 a^4 y^6 z} = \sqrt[4]{(81 a^4 y^4 \cdot y^2 z)} = 9 a^2 y^2 \sqrt[4]{y^2 z}$.

3. $\sqrt{(a^2 - 4a\sqrt{b} + b^2)} = a - 2\sqrt{b}$, the operation being performed as in the arithmetical extraction of the square root.

Note. The *square* root of a binomial or residual $a \pm b$, or even of a trinomial or quadrinomial, may often be conveniently extracted thus:—Take $d = \sqrt{(a^2 - b^2)}$; then $\sqrt{(a \pm b)} = \sqrt{\frac{a+d}{2}} \pm \sqrt{\frac{a-d}{2}}$. This is evident: for, if $\sqrt{\frac{a+d}{2}} \pm \sqrt{\frac{a-d}{2}}$ be squared, it will give $a + \sqrt{(a^2 - d^2)}$ or $a + b$, as it ought: and, in like manner, the square of $\sqrt{\frac{a+d}{2}} - \sqrt{\frac{a-d}{2}}$, is $a - \sqrt{(a^2 - d^2)}$, or $a - b$.

Ex. 1. Find the square root of $3 + 2\sqrt{2}$.

Here $a = 3$, $b = 2\sqrt{2}$, $d = \sqrt{(9-8)} = 1$,

And $\sqrt{\frac{a+d}{2}} + \sqrt{\frac{a-d}{2}} = \sqrt{\frac{3+1}{2}} + \sqrt{\frac{3-1}{2}} = \sqrt{\frac{4}{2}} + \sqrt{\frac{2}{2}} = \sqrt{2} + \sqrt{1} = 1 + \sqrt{2}$.

2. Find the square root of $6 - 2\sqrt{5}$.

Here $a = 6$, $b = 2\sqrt{5}$, $d = \sqrt{(36-20)} = \sqrt{16} = 4$,

And $\sqrt{\frac{a+d}{2}} - \sqrt{\frac{a-d}{2}} = \sqrt{\frac{6+4}{2}} - \sqrt{\frac{6-4}{2}} = \sqrt{\frac{10}{2}} - \sqrt{\frac{2}{2}} = \sqrt{5} - 1$.

3. Find the square root of $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$.

Here $a = 6 + \sqrt{8}$, $b = \sqrt{12} + \sqrt{24}$, $d = \sqrt{(6+\sqrt{8})^2 - (\sqrt{12}+\sqrt{24})^2} = \sqrt{(44 + 12\sqrt{8} - 36 - 2\sqrt{12 \cdot 24})} = \sqrt{(44 - 36 + 12\sqrt{8} - 12\sqrt{8})} = \sqrt{8}$.

Conseq. $\frac{a+d}{2} = \frac{6 + 2\sqrt{8}}{2} = 3 + \sqrt{8}$, and $\frac{a-d}{2} = \frac{6 + \sqrt{8} - \sqrt{8}}{2} = 3$.

But (*Ex.* 1), $\sqrt{(3 + 2\sqrt{2})} = \sqrt{(3 + \sqrt{8})} = 1 + \sqrt{2}$.
Therefore the root required is $1 + \sqrt{2} - \sqrt{3}$.*

Section VIII. *Simple Equations.*

An algebraic equation is an expression by which two quantities, called members (whether simple or compound), are indicated to be equal to each other, by means of the sign of equality = placed between them.

In equations consisting of known and unknown quantities, when the unknown quantity is expressed by a simple power, as x, x^2, x^3, &c. they are called *simple equations*, generally;

* For the cube and higher roots of binomials, &c. the reader may consult the treatises on Algebra, by Maclaurin, Emerson, Lacroix, Bonnycastle, and J. R. Young.

and particularly, simple or pure quadratics, cubics, &c. according to the exponent of the unknown quantity. But when the unknown quantity appears in two or more different powers in the same equation, it is named an *adfected* equation. Thus $x^2 = a + 15$, is a simple quadratic equation: $x^2 + a x = b$, an adfected quadratic.

It is the former class of equations that we shall first consider.

The reduction of an equation consists in so managing its terms, that, at the end of the process, the *unknown* quantity may stand alone, and in its first power, on one side of the sign =, and *known* quantities, whether denoted by letters, or figures, on the other. Thus, what was previously unknown is now affirmed to be *equal* to the aggregate of the terms in the second number of the equation.

"In general, *the unknown quantity is disengaged from the known ones, by performing upon both members the* REVERSE OPERATIONS,"* to those indicated by the equation, whatever they may be. Thus,

If any known quantity be added to the unknown quantity, let it be subtracted from both members or sides of the equation.

If any such quantity be subtracted, let it be added.†

If the unknown quantity have a multiplier, let the equation be divided by it.

If it be divided by any quantity, let that become the multiplier.

If any power of the unknown quantity be given, take the corresponding root.

If any root of it be known, find the corresponding power.

If the unknown quantity be found in the terms of a proportion (*Arith.* Sect. 10), let the respective products of the means and extremes constitute an equation; and then apply the general principle, as above.

* This simple direction, comprehending the 7 or 8 particular rules for the reduction of equations given by most writers on algebra, from the time of Newton down to the present day, is due to Dr. Hutton. It is obviously founded upon the mathematical axiom, that equal operations performed upon equal things, produce equal results.

† These two operations constitute what is usually denominated *transposition*.

Examples.

1. Given $x - 3 + 5 = 9$, to find x.

First, by adding 3 to both sides, we have $x + 5 = 9 + 3 = 12$.

Then, by subtracting 5, $x + 5 - 5 = 12 - 5$, or $x = 7$.

Otherwise, in appearance only, not in effect,

By transposing the 3, and changing its sign, $x + 5 = 9 + 3$.

By transposing the 5, and changing its sign, $x = 9 + 3 - 5 = 7$.

2. Given $3x + 5 = 20$, to find x.

First, by transposing the 5, $3x = 20 - 5 = 15$.

by dividing by 3, $x = \frac{15}{3} = 5$.

3. Given $\frac{x}{a} + d = 3b - 2c$, to find x.

First, transposing d, $\frac{x}{a} = 3b - 2c + d$.

Then, multiplying by a, $x = 3ab - 2ac + ad$.

4. Given $\sqrt[3]{(3x + 4)} + 2 = 6$, to find x.

First, transposing the 2, $\sqrt[3]{(3x + 4)} = 6 - 2 = 4$.

Then, cubing, $3x + 4 = 4^3 = 64$.

Then, transposing the 4, $3x = 64 - 4 = 60$.

Lastly, dividing by 3, $x = \frac{60}{3} = 20$.

5. Given $4ax - 5b = 3dx + 4c$, to find x.

First, transposing $5b$ and $3dx$, $4ax - 3dx = 5b + 4c$.

That is, by collecting the coefficients, $(4a - 3d)x = 5b + 4c$.

Therefore, by dividing by $4a - 3d$, $x = \frac{5b + 4c}{4a - 3d}$.

6. Given $\frac{1}{4}x + \frac{1}{5}x - \frac{1}{6}x = 3$, to find x.

Multipltiplying by $120 = 4 \times 5 \times 6$, we have $30x + 24x - 20x = 360$.

That is, collecting the coefficients, $34x = 360$.

Therefore, dividing by 34, $x = \frac{360}{34} = \frac{180}{17} = 10\frac{10}{17}$.

7. Given $\frac{3}{4}x : a :: 5b : 3c$, to find x.

Mult. means and extremes, $\frac{9}{4}cx = 5ab$.

Dividing by $\frac{9}{4}c$, $x = 5ab \div \frac{9}{4}c = \frac{20ab}{9c}$.

8. Given $a + x = \sqrt{a^2 + x\sqrt{(4b^2 + x^2)}}$, to find x.

First, by squaring, we have, $a^2 + 2ax + x^2 = a^2 + x\sqrt{(4b^2 + x^2)}$

Then, striking out a^2 from both sides, $2ax + x^2 = x\sqrt{(4b^2 + x^2)}$

dividing by x, $2a + x = \sqrt{(4b^2 + x^2)}$

squaring, $4a^2 + 4ax + x^2 = 4b^2 + x^2$

striking out x^2, and transposing $4a^2$, $4ax = 4b^2 - 4a^2$

dividing by $4a$, $x = \frac{4b^2 - 4a^2}{4a} = \frac{b^2}{a} - a$.

9. Given $\sqrt{cx - ac} = b + \sqrt{x - a}$, to find x.

First, dividing by $\sqrt{x - a}$, we have $\sqrt{c} = \frac{b}{\sqrt{(x - a)}} + 1$

transposing the 1, $\sqrt{c} - 1$, or $\frac{\sqrt{c} - 1}{1} = \frac{b}{\sqrt{(x - a)}}$

inverting and transposing the fractions, $\frac{\sqrt{(x - a)}}{b} = \frac{1}{\sqrt{c} - 1}$

multiplying by b, $\sqrt{(x - a)} = \frac{b}{\sqrt{c} - 1}$

squaring both sides, $x - a = \frac{b^2}{c - 2\sqrt{c} + 1}$

transposing a, $x = a + \frac{b^2}{c - 2\sqrt{c} + 1}$.

10. Given $13 - \sqrt{3}\,x = \sqrt{13 + 3x}$, to find x.

Ans. $x = 12$.

11. Given $y + \sqrt{4 + y^2} = \frac{8}{\sqrt{(4 + y^2)}}$, to find y.

Ans. $y = \frac{2}{3}\sqrt{3}$.

12. Given $\frac{1}{3}(x+1) + \frac{1}{4}(x+3) = \frac{1}{5}(x+4) + 16$, to find x.

Ans. $x = 41$.

13. Given $\sqrt{\frac{3x}{2}} : \sqrt{(x - 1)} :: 3 : 1$, to find x. *Ans.* $1\frac{1}{5}$.

14. Given $(b^4 + x^4)^{\frac{1}{4}} = (a^2 + x^2)^{\frac{1}{2}}$, to find x.

Ans. $x = \frac{1}{a}\sqrt{\frac{b^4 - a^4}{2}}$.

Extermination.

When two or more unknown quantities occur in the consideration of an algebraical problem, they are determinable by a series of given independent equations. In order, however, that specific and finite solutions may be obtained, this condition must be observed, that there be *given as many independent equations as there are unknown quantities.* For, if the number of independent equations be fewer than the unknown quantities, the question proposed will be susceptible of an indefinite number of solutions:* while, on the other hand, a greater number of independent equations than of unknown quantities, indicates the impossibility, or the absurdity of the thing attempted.

Where two unknown quantities are to be determined from two independent equations, one or other of the following rules may be employed.

1. Find the value of one of the unknown letters in each of the given equations; make those two values equal to one another in a third equation, and from thence deduce the value of the other unknown letter. This substituted for it in either of the former equations, will lead to the determination of the first unknown quantity.

2. Find the value of either of the unknown quantities in one of the equations, and substitute this value for it in the other equation: so will the other unknown quantity become known, and then the first, as before.

3. Or, after due reduction when requisite, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of *the same* unknown quantity in the first equation: then the addition or subtraction of the resulting equations (according as the signs of the unknown quantity whose coefficients are now made equal, are unlike or like) will exterminate that unknown quantity, and lead to the determination of the other by former rules.

Notes. The third rule is usually the most commodious and expeditious in practice.

The same precepts may be applied, *mutatis mutandis*, to equations comprising three, four, or more unknown quantities: and they often serve to depress equations, or reduce them from a higher to a lower degree.

* This, though generally true, has one striking exception, namely, in the case of equations constituted partly of rational quantities and partly of quadratic surds; where two unknown quantities are determinable by one equation, four unknown quantities by two equations; and so on.

Thus, if $x + \sqrt{y} = a + \sqrt{b}$
and $z - \sqrt{v} = c - \sqrt{d}$
Then $x = a$, $y = b$, $z = c$, $v = d$.

Examples.

1. Given $4x^3 + 3y = 41$, and $3x^3 - 4y = 12$, to find x and y.

1st. equa. × by 3, gives $12x^3 + 9y = 123$
2d. equa. × by 4, gives $12x^3 - 16y = 48$.
The difference of these, $25y = 75$, whence $y = 3$.
Then, from equa. 2d, $3x^3 = 12 + 4y = 12 + 12 = 24$
Whence dividing by 3, $x^3 = 8$, or $x = 2$.

Ex. 2. Given $x + y + z = 53$, $x + 2y + 3z = 105$, and $x + 3y + 4z = 134$.

1. $x + y + z = 53$
2. $x + 2y + 3z = 105$
3. $x + 3y + 4z = 134$
4. 1st equa. taken from 2d, gives $y + 2z = 52$
5. 2d equa. taken from 3d, $y + z = 29$
6. 5th equa. taken from 4th, $z = 23$
7. 6th equa. taken from 5th, $y = 6$
8. 5th equa. taken from 1st, $x = 24$.

Ex. 3. Given $x + y = a$, $x + z = b$, $y + z = c$, to find x, y, and z.

1. $x + y = a$
2. $x + z = b$
3. $y + z = c$
4. 1st + 2d + 3d, gives $2x + 2y + 2z = a + b + c$.
5. Half equa. 4th gives $x + y + z = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$.
6. 3d equa. taken from 5th, gives $x = \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c$.
7. 2d equa. taken from 5th, $y = \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c$.
8. 1st equa. taken from 5th, $z = -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$.

Ex. 4. Given $ax + by = c$, and $a'x + b'y = c'$, to find x and y.

Ans. $x = \dfrac{cb' - bc'}{ab' - ba'}$ and $y = \dfrac{ac' - ca'}{ab' - ba'}$.

Ex. 5. Given $ax + by + cz = d$, $a'x + b'y + c'z = d'$, $a''x + b''y + c''z = d''$ to find x, y, and z.

$$Ans.\ x = \frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

Ex. 6. Given $\left.\begin{array}{l} x(x + y + z) = 18 \\ y(x + y + z) = 27 \\ z(x + y + z) = 36 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 2$, $y = 3$, $z = 4$.

Ex. 7. Given $(x + y)\frac{x}{y} = 60$, and $(x + y)\frac{y}{x} = 2\frac{2}{5}$, to find x and y.

Ans. $x = 10$, $y = 2$.

Ex. 8. Given $\left.\begin{array}{l} x + y : x - y :: 8 : 5 \\ \text{and } x + y : 2y :: 8 : 3 \end{array}\right\}$ to find x and y.

Ans. $x = 65$, $y = 15$.

Solution of General Problems.

A general algebraic problem is that in which all the quantities concerned, both known and unknown, are expressed by letters, or other general characters. Not only such problems as have their conditions proposed in general terms are here implied; every particular numeral problem may be made general, by substituting letters for the known quantities concerned in it; when this is done, the problem which was originally proposed in a particular form becomes general.

In solving a problem algebraically some letter of the alphabet must be substituted for an unknown quantity. And if there be more unknown quantities than one, the second, third, &c. must either be expressed by means of their dependence upon the first and one or other of the data conjointly, or by so many distinct letters. Thus, so many separate equations will be obtained, the resolution of which, by some of the foregoing rules, will lead to the determination of the quantities required.

Examples.

1. Given the sum of two magnitudes, and the difference of their squares, to find those magnitudes separately.

Let the given sum be denoted by s, the difference of the squares by D; and let the two magnitudes be represented by x and y respectively.

Then, the first condition of the problem expressed algebraically is $x + y = s$

And the second is $x^2 - y^2 = \text{D}$

Equa. 2 divided by equa. 1, gives $x - y = \frac{\text{D}}{s}$

Equa. 1 added to equa. 3, gives $2x = \frac{\text{D}}{s} + s = \frac{s^2 + \text{D}}{s}$

Equa. 4 divided by 2, gives $x = \frac{s^2 + \text{D}}{2s}$

Equa. 5 taken from equa. 1, gives $y = s - \frac{s^2 + \text{D}}{2s} = \frac{s^2 - \text{D}}{2s}$

To apply this general solution to a particular example, suppose the sum to be 6, and the difference of the squares 12. Then $s = 6$ and $\text{D} = 12$,

and $x = \frac{s^2 + \text{D}}{2s} = \frac{36 + 12}{12} = \frac{48}{12} = 4$

and $y = \frac{s^2 - \text{D}}{2s} = \frac{36 - 12}{12} = \frac{24}{12} = 2.$

Suppose, again, $s = 5$, $\text{D} = 5$:

then $x = \frac{25 + 5}{10} = 3$, and $y = \frac{25 - 5}{10} = 2.$

Ex. 2. Given the product of two numbers, and their quotient, to find the numbers.

Let the given product be represented by p, the quotient by q; and the required numbers by x and y, as before.

Then we have, 1. $x y = p$

and 2. $\frac{x}{y} = q$,

Equa. 2 × by y, gives $x = q y$

Substituting this value of x for it in equa. 1 } $q y^2 = p$

Dividing by q, $y^2 = \frac{p}{q}$

Extracting the square root $y = \sqrt{\frac{p}{q}}$

Then, by substitution $x = q\,y = q\sqrt{\frac{p}{q}} = \sqrt{\frac{p\,q^2}{q}} = \sqrt{p\,q}$.

Suppose the product were 50 and the quotient 2.

Then $y = \sqrt{\frac{p}{q}} = \sqrt{\frac{50}{2}} = \sqrt{25} = 5$, and $x = \sqrt{p\,q} = \sqrt{100} = 10$.

Again, suppose the product 36, and the quotient $2\frac{1}{4}$.

Then $y = \sqrt{\frac{p}{q}} = \sqrt{\frac{36}{2\frac{1}{4}}} = \sqrt{16} = 4$, and $x = \sqrt{p\,q} = \sqrt{81} = 9$.

Ex. 3. Given the sum (s) of two numbers, and the sum of their squares s, to find those numbers.

Ans. $x = \frac{1}{2}s + \frac{1}{2}\sqrt{2\,\text{s} - s^2}$, and $y = \frac{1}{2}s - \frac{1}{2}\sqrt{2\,\text{s} - s^2}$.

Ex. 4. The sum and product of two numbers are equal, and if to either sum or product the sum of the squares be added, the result will be 12. What are the numbers?

Ans. each = 2.

Ex. 5. The square of the greater of two numbers multiplied into the less, produces 75; and the square of the less multiplied into the greater produces 45. What are the numbers?

Ex. 6. A man has six sons whose successive ages differ by four years, and the eldest is thrice as old as the youngest. Required their several ages?—*Ans.* 10, 14, 18, 22, 26, and 30 years.

SECTION IX. *Quadratic Equations.*

When, after due reduction, equations assume the general form $\text{A}\,x^2 + \text{B}\,x + c = 0$; then dividing by A, the coefficient of the first term, there results $x^2 + \frac{\text{B}}{\text{A}}x + \frac{\text{C}}{\text{A}} = 0$, or, making

$p = \frac{\text{B}}{\text{A}}$, $q = \frac{\text{C}}{\text{A}}$, we have $x^2 + p\,x + q = 0 \ldots\ldots (1)$

an equation which may represent all those of the second degree, p and q being known numbers positive or negative.

Let a be a number or quantity which when substituted for x renders $x^2 + p\,x + q = 0$; then $a^2 + p\,a + q = 0$, or $q = - a^2 - p\,a$. Consequently $x^2 + p\,x + q$, is the same thing as $x^2 - a^2 + p\,x - p\,a$, or as $(x + a)\,(x - a) + p\,(x - a)$, or, lastly, as $(x - a)\,(x + a + p)$.

The inquiry, then, is reduced to this, viz. to find all the values of x which shall render the product of the above two factors equal to nothing. This will evidently be the case when either of the factors is $= 0$; but in no other case. Hence, we have $x - a, = 0$, and $x + a + p = 0$, or $x = a$, and $x = - a - p$.*

And hence we may conclude—

1. That every equation of the second degree whose conditions are satisfied by one value a of x, admits also of another value $- a - p$. These values are called the *roots* of the quadratic equation.

2. The sum of the two roots a and $- a - p$ is $- p$; their product is $- a^2 - a\,p$, which as appears above is $= q$. So that the *coefficient*, p, of the *second* term is the *sum* of the roots with a contrary sign; the *known term*, q, is their *product*.

3. It is easy to constitute a quadratic equation whose roots shall be any given quantities b and d. It is evidently $x^2 - (b + d)\,x + b\,d = 0$.

4. The determination of the roots of the proposed equation (1) is equivalent to the finding two numbers whose sum is $-p$, and product q.

5. If the roots b and d are equal, then the factors $x - b$ and $x - d$ are equal; and $x^2 + p\,x + q$ is the square of one of them.

To solve a quadratic equation of the form $x^2 + p\,x + q = 0$, let it be considered that the square of $x + \frac{1}{2}p$ is a trinomial $x^2 + p\,x + \frac{1}{4}p^2$, of which the first two terms agree with the first two terms of the given equation, or with the first member of that equation when q is transposed.

That is, with $x^2 + p\,x = - q$

Let then $\frac{1}{4}p^2$ be added, we have, $x^2 + p\,x + \frac{1}{4}p^2 = \frac{1}{4}p^2 - q$

of which the first number is a complete square.

* If it be affirmed that the given equation admits of another value of x, besides the above, b for instance, it may be proved as before that $x - b$ must be of the number of the factors of $x^2 + p\,x + q$, or of $(x - a)\,(x + a + p)$. But $x - a$ and $x + a + p$ being prime to each other, or having no common factor, their product cannot have any other factor than they. Consequently b must either be equal to a or to $- a - p$; and the number of roots is restricted to two.

Its root is $x + \frac{1}{2} p = \pm \sqrt{(\frac{1}{4} p^2 - q)}$

and consequently $x = - \frac{1}{2} p \pm \sqrt{(\frac{1}{4} p^2 - q)}$

otherwise, from number 2 above, we have $x + x' = - p$

and $x x' = q$

Taking 4 times the second of these equations from the square of the first, there remains $x^2 - 2 x x' + x'^2 = p^2 - 4 q$

Whence, by taking the root, $x - x' = \sqrt{(p^2 - 4 q)}$

Half this *added* to half equa. 1, gives $x = - \frac{1}{2} p + \frac{1}{2} \sqrt{(p^2 - 4 q)} = - \frac{1}{2} p + \sqrt{(\frac{1}{4} p^2 - q)}$

And the same *taken* from half equa. 1, gives $x' = - \frac{1}{2} p - \frac{1}{2} \sqrt{(p^2 - 4 q)} = - \frac{1}{2} p - \sqrt{(\frac{1}{4} p^2 - q)}$ which two values of a evidently agree with the preceding.

It would be easy to analyze the several cases which may arise, according to the different signs and different values, of p and q. But these need not here be traced. It is evident that whether there be given

1. $x^2 + p x = q$
2. $x^2 - p x = q$
3. $x^2 + p x = - q$
4. $x^2 - p x = - q$

The general method of solution is by *completing the square*, that is, adding the square of $\frac{1}{2} p$, to both members of the equation, and then extracting the root.

It may farther be observed that all equations may be solved as quadratics, by completing the square, in which there are two terms involving the unknown quantity or any function of it, and the index of one double that of the other. Thus, $x^6 \pm p x^3 = q$, $x^{2n} \pm p x^n = q$, $x^{\frac{n}{2}} \pm p x^{\frac{n}{4}} = q$, $(x^2 + p x + q)^2 \pm (x^2 + p x + q) = r$, $(x^{2n} - x^n)^2 \pm (x^{2n} - x^n) = q$, &c. are of the same form as quadratics, and admit of a like determination of the unknown quantity. Many equations, also, in which more than one unknown quantity are involved, may be reduced to lower dimensions by completing the square and reducing; such, for example, as $(x^3 + y^3)^2 \pm p (x^3 + y^3) = q$, $\frac{x^2}{y^2} \pm \frac{p x}{y} = q$, and so on.

Note. In some cases a quadratic equation may be conveniently solved without dividing by the coefficient of the square, and thus without introducing fractions. To solve the general equation $a x^2 \pm b x = c$, for example, multiply the whole by $4 a$, whence $4 a^2 x^2 \pm 4 a b x = 4 a c$, adding b^2 to complete the square, $4 a^2 x^2 \pm 4 a b x + b^2 = 4 a c + b^2$ taking the square root, $2 a x \pm b = \pm \sqrt{(4 a c + b^2)}$;

whence $x = \dfrac{\mp b \pm \sqrt{(4ac + b^2)}}{2a}$; which will serve for a general theorem.

Examples.

1. Given $x^2 - 8x + 10 = 19$, to find x.
   transposing the 10, $x^2 - 8x = 19 - 10 = 9$
   completing the squ., $x^2 - 8x + 16 = 9 + 16 = 25$
   extracting the root, $x - 4 = \pm 5$
   consequently $x = 4 \pm 5 = 9$ or -1.

2. Given $\dfrac{10}{x} - \dfrac{14 - 2x}{x^2} = \dfrac{22}{9}$, to find the values of x.
   multiplying by x^2, $10x - 14 + 2x = \dfrac{22x^2}{9}$,
   transposing, $\frac{22}{9}x^2 - 12x = -14$,
   dividing by $\frac{22}{9}$, $x^2 - \frac{54}{11}x = -\frac{63}{11}$,
   complet. squ. $x^2 - \frac{54}{11}x + (\frac{27}{11})^2 = \frac{729}{121} - \frac{63}{11} = \frac{36}{121}$,
   extract. root, $x - \frac{27}{11} = \pm \frac{6}{11}$.
   transposing, $x = \frac{27}{11} \pm \frac{6}{11} = 3$ or $\frac{21}{11}$.

3. Given $x^2 + 2x + 4\sqrt{x^2 + 2x + 1} = 44$, to find x.
   adding 1, we have $(x^2 + 2x + 1) + 4\sqrt{(x^2 + 2x + 1)} = 45$
   complet. squ. $(x^2 + 2x + 1) + 4\sqrt{(x^2 + 2x + 1)} + 4 = 49$
   extract. root, $\sqrt{(x^2 + 2x + 1)} + 2 = \pm 7$
   transposing the 2, $\sqrt{(x^2 + 2x + 1)} = \pm 7 - 2 = 5$ or -9
   that is, $x + 1 = 5$ or -9
   hence $x = 4$ or -10.

4. Given $x^n - 2ax^{\frac{n}{2}} = c$, to find x
   complet. squ. $x^n - 2ax^{\frac{n}{2}} + a^2 = c + a^2$
   extract. root, $x^{\frac{n}{2}} - a = \pm\sqrt{(c + a^2)}$
   transposing, $x^{\frac{n}{2}} = a \pm \sqrt{(c + a^2)}$
   consequently, $x = (a \pm \sqrt{c + a^2})^{\frac{2}{n}}$

5. Given $\frac{x^2}{y^2} + \frac{4x}{y} = 12$, and $x - y = 2$, to find x and y.

complet. squ. in equa 1, $\frac{x^2}{y^2} + 4\frac{x}{y} + 4 = 16$.

Extracting root $\frac{x}{y} + 2 = \pm 4$: whence $\frac{x}{y} = 2$ or -6, and $x = 2y$ or $= -6y$.

Substituting the former value of x in the 2d equa. it becomes $2y - y = 2$, or $y = 2$; whence $x = 4$.

Again, substituting the 2d value of x, in equa. 2, it becomes $-6y - y$ or $-7y = 2$; whence $y = -\frac{2}{7}$, and $x = +\frac{12}{7}$.

6. Given $x^2y^2 - 5 = 4xy$, and $\frac{1}{2}xy = \frac{5}{2}y^3$, to find x and y.

equa. 1, by transposition, becomes $x^2y^2 - 4xy = 5$
completing the square, $x^2y^2 - 4xy + 4 = 9$
extracting the root, $xy - 2 = \pm 3$
whence $xy = 5$ or -1.

Substituting the first of these values for xy in equa. 2, it becomes $\frac{5}{2}y^3 = \frac{5}{2}$: whence $y = 1$ and $x = 5$.

Substituting the 2d value in the same equation, it becomes $\frac{5}{2}y^3 = -\frac{1}{2}$: whence $y = -\sqrt[3]{\frac{1}{5}} = -\frac{1}{5}\sqrt[3]{25}$, and $x = -1 \div -\frac{1}{5}\sqrt[3]{25} = \sqrt[3]{5}$.

7. Given $\frac{x^2}{(x^2 - 4)^2} + \frac{6}{x^2 - 4} = \frac{351}{25x^2}$, to find x.

Ans. $x = \pm 3$, or $\pm \sqrt{\frac{39}{11}}$.

8. A man travelled 105 miles at a uniform rate, and then found that if he had not travelled so fast by two miles an hour, he would have been six hours longer in performing the same journey. How many miles did he travel per hour?

Ans. 7 miles per hour.

9. Find two such numbers that the sum, product, and difference of their squares may be equal.

Ans. $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, and $\frac{3}{2} + \frac{1}{2}\sqrt{5}$.

10. A waterman who can row eleven miles an hour with the tide, and two miles an hour against it, rows five miles up a river and back again in three hours: now, supposing the tide to run uniformly the same way during these three hours, it is required to find its velocity?

Ans. 4 $\frac{19}{44}$ miles per hour.

SECTION X. *Equations in General.*

Equations in general may be prepared or constituted by the multiplication of factors, as we have shown in quadratics. Thus, suppose the values of the unknown quantity x in any equation were to be expressed by a, b, c, d, &c. that is, let $x=a$, $x=b$, $x=c$, $x=d$, &c. disjunctively, then will $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, &c. be the simple radical equations of which those of the higher orders are composed. Then, as the product of any two of these gives a *quadratic* equation; so the product of any three of them as $(x-a)(x-b)(x-c)=0$, will give a *cubic* equation, or one of three dimensions. And the product of four of them will constitute a *biquadratic equation*, or one of four dimensions; and so on. Therefore, in general *the highest dimension of the unknown quantity x is equal to the number of simple equations that are multiplied together to produce it.*

When any equation equivalent to this biquadratic $(x-a)(x-b)(x-c)(x-d)=0$ is proposed to be resolved, the whole difficulty consists in finding the simple equations $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, by whose multiplication it is produced; for each of these simple equations gives one of the values of x, and one solution of the proposed equation. For, if any of the values of x deduced from those simple equations be substituted in the proposed equation, in place of x, then all the terms of that equation will vanish, and the whole be found equal to nothing. Because when it is supposed that $x=a$, or $x=b$, or $x=c$, or $x=d$, then the product $(x-a)(x-b)(x-c)(x-d)$ vanishes, because one of the factors is equal to nothing. There are therefore four suppositions that give $(x-a)(x-b)(x-c)(x-d)=0$, according to the proposed equation; that is, there are four *roots* of the proposed equation. And after the same manner *any other* equation admits of as many solutions as there are simple equations multiplied by one another that produce it, or as many as there are units in the highest dimension of the unknown quantity in the proposed equation.

But as there are no other quantities whatsoever besides these four (a, b, c, d,) that, substituted in the proposed product in the place of x, will make that product vanish; therefore, the equation $(x-a)(x-b)(x-c)(x-d)=0$, cannot possibly have more than these four roots, and cannot admit of more solutions than four. If we substitute in that product a quantity neither equal to a, nor b, nor c, nor d, which suppose e, then since neither, $e - a$, $e - b$, $e - c$, nor $e - d$, is equal to nothing; their product cannot be equal to nothing, but must be some real product: and therefore, there is no supposition beside one of the aforesaid four, that gives a just value of x according to the proposed equation. So that it can have no more than these four roots. And after the same manner it appears, that *no equation can have more roots than it contains dimensions of the unknown quantity.*

To make all this still plainer by an example, in numbers, suppose the equation to be resolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that we discover that this equation is the same with the product of $(x-1)(x-2)(x-3)(x-4)$, then we certainly infer that the four values of x are 1, 2, 3, 4; seeing any of these numbers, placed for x, makes that product, and consequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the proposed equation. And it is certain that there can be no other values of x besides these four: for when we substitute any other number for x in those factors $x-1$, $x-2$, $x-3$, $x-4$, none of them vanish, and therefore their product cannot be equal to nothing, according to the equation.

A variety of rules, some of them very ingenious, for the solution of equations, may be found in the best writers on Algebra;* but we shall simply exhibit the easy rule of Trial-and-Error, as it is given by Dr. *Hutton* in the 1st vol. of his "Course of Mathematics."

Rule.

"1. Find, by trial, two numbers, as near the true root as possible, and substitute them in the given equation instead of the unknown quantity; marking the errors which arise from each of them.

"2. Multiply the difference of the two numbers, found by trial, by the least error, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike. Or say, as the difference or sum of the errors is to the difference of the two numbers, so is the least error to the correction of its supposed number.

* See the treatises of Lacroix, Bonnycastle, Wood, J. R. Young, &c.

"3. Add the quotient, last found, to the number belonging to the least error, when that number is too little, but subtract it when too great, and the result will give the true root nearly.

"4. Take this root and the nearest of the two former, or any other that may be found nearer; and, by proceeding in like manner, a root will be had still nearer than before; and so on to any degree of exactness required.

"*Note.* It is best to employ always two assumed numbers that shall differ from each other only by unity in the last figure on the right; because then, the difference, or multiplier, is 1."

Example.

To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of x in it.

Here it is soon found that x lies between 4 and 5. Assume therefore these two numbers, and the operation will be as follows:

1st Sup.		*2d Sup.*
4	x	5
16	x^2	25
64	x^3	125
84	sums	155
−16	errors	+55

the sum of which is 71.

Then as 71 : 1 :: 16 : ·225.
Hence $x = 4·225$ nearly.

Again, suppose 4·2 and 4·3, and repeat the work as follows:

1st Sup.		*2d Sup.*
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	sums	102·297
−4·072	errors	+2·297

the sum of which is 6·369.

As 6·369 : ·1 :: 2·297 : 0·036
This taken from 4·300

leaves x nearly = 4·264

Again, suppose 4·264 and 4·265, and work as follows:

4·264	- -	x	- -	4·265
18·181696	- -	x^2	- -	18·190225
77·526752	- -	x^3	- -	77·581310
99·972448	-	sums	- -	100·036535
−0·027552	-	errors	- -	+0·036535

the sum of which is ·064087.

Then as ·064087 : ·001 :: ·027552 : 0·0004299
To this adding - 4·264

gives x very nearly = 4·2644299

When one of the roots of an equation has been thus found, then take for a dividend the given equation with the known term transposed to the unknown side of the equation; and for a divisor take x minus the root just determined: the quotient will be equal to nothing, and will be a new equation depressed a degree lower than the former. From this a new value of x may be found: and so on, till the equation is reduced to a quadratic, of which the roots may be found by the proper rules.

Section XI. *Progression.*

When a series of terms proceed according to an assignable order, either from less to greater or from greater to less, by continual equal differences or by successive equal products or quotients, they are said to form a *progression.*

If the quantities proceed by successive equi-differences they are said to be in *Arithmetical Progression.* But if they proceed in the same continued proportion, or by equal multiplications or divisions, they are said to be in *Geometrical Progression.*

If the terms of a progression successively increase, it is called an *ascending* progression: if they successively decrease, it is called a *descending* progression.

Thus, 1, 3, 5, 7, 9, &c. form an ascending arithmetical
24, 22, 20, 18, 16, &c. form a descending arithmet.
1, 3, 9, 27, 81, &c. form an ascending geometrical
4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. form a descending geometrical
} Progression.

Arithmetical Progression.

1. Let a be the first term of an arithmetical progression
d the common difference of the terms
z the last term
n the number of terms
s the sum of all the terms.

Then $a, a+d, a+2d, a+3d$, &c. is an ascending progression.
and $a, a-d, a-2d, a-3d$, &c. a descending progression.

Hence, in an ascending progression $a + (n-1)d$, is the last term;
in a descending progression, $a - (n-1)d$, is the last term.

2. Let a series be $a + (a+d) + (a+2d) + (a+3d)$.
The same inverted, $(a+3d) + (a+2d) + (a+d) + a$,
The sum of the two, $(2a+3d) + (2a+3d) + (2a+3d) + (2a+3d) = 2s$.

That is, $(2a+3d) \times 4$, in this case $(a + \overline{a+3d})\,n = 2s$.

Consequently, $s = \frac{1}{2}(a + \overline{a+3d})\,n$, or $= \frac{1}{2}(a+z)\,n$, since z is here $= a + 3d$. The same would be obtained, if the progression were descending; and let the number of terms be what it may.

3. From the equations $z = a + (n-1)d$, $s = \frac{1}{2}n(a+z)$, and $s = \frac{1}{2}n(a + a + (n-1)d)$, we may readily deduce the following theorems applicable to ascending series. When the series is descending, either the signs of the terms affected with d must be changed, or a must be taken for z; and *vice versâ*.

(1.) $a = z - nd + d = \sqrt{(-2sd + z^2 + dz + \frac{1}{4}d^2)} + \frac{1}{2}d = \frac{s}{n} + \frac{1}{2}d - \frac{1}{2}nd = \frac{2s}{n} - z.$

(2.) $d = \frac{z-a}{n-1} = \frac{z^2-a^2}{2s-z-a} = \frac{2zn-2s}{n^2-n} = \frac{2s-2na}{n^2-n}.$

(3.) $z = a + nd - d = \frac{s}{n} + \frac{1}{2}nd - \frac{1}{2}d =$

$\sqrt{(2sd + a^2 - ad + \frac{1}{4}d^2)} - \frac{1}{2}d = \frac{2s}{n} - a.$

(4.) $s = \frac{1}{2}n(a+z) = (a + \frac{1}{2}nd - \frac{1}{2}d)\,n = (z - \frac{1}{2}nd + \frac{1}{2}d)\,n = \frac{z^2 - a^2 + d(z+a)}{2d}.$

(5.) $n = \frac{2s}{a+z} = \frac{z-a}{d} + 1.$

Examples.

1. Required the sum of 20 terms of the progression 1, 3, 5, 7, 9, &c.

Here $a = 1$, $d = 2$, $n = 20$; which substituted in the theorem $s = (a + \frac{1}{2} n d - \frac{1}{2} d) n$, transform it to $s = (1 + 20 - 1) 20 = 20 \times 20 = 400$, the sum required.

Note. In any other case the sum of a series of *odd* numbers beginning with unity, would be $= n^2$, the square of the number of terms.

2. The first term of an arithmetical progression is 5, the last term 41, the sum 299. Required the number of the terms, and the common difference.

$$\text{Here } n = \frac{2s}{a + z} = \frac{598}{46} = 13, \text{ the number of terms,}$$

$$\text{and } d = \frac{z - a}{n - 1} = \frac{41 - 5}{12} = 3, \text{ the common difference.}$$

There are 8 equidifferent numbers: the least is 4, the greatest 32. What are the numbers?

$$\text{Here } d = \frac{z - a}{n - 1} = \frac{32 - 4}{7} = 4, \text{ the common difference.}$$

Whence 4, 8, 12, 16, 20, 24, 28, 32, are the numbers.

4. The first term of an arithmetical progression is 3, the number of terms 50, the sum of the progression 2600. Required the last term and the common difference.

$$\text{Here } z = \frac{2s}{n} - a = \frac{5200}{50} - 3 = 104 - 3 = 101,$$

the last term,

$$\text{and } d = \frac{z - a}{n - 1} = \frac{101 - 3}{49} = 2, \text{ the common difference.}$$

5. The sum of six numbers in arithmetical progression is 48; and if the common difference d be multiplied into the less extreme, the product equals the number of terms.— Required those terms. *Ans.* 3, 5, 7, 9, 11, and 13.

Geometrical Progression.

1. Let a be the first term of a geometrical series;
r the common ratio; z the last term;
n the number; and s the sum of the terms.

Then a, ra, $r^2 a$, $r^3 a$, $r^{n-1} a$, is a geometrical progression, which will be ascending or descending, according as r is an integer or a fraction.

2. Let the prog. $a + ra + r^2 a + r^3 a + r^4 a = s$, be $\times$ by r, it becomes $ra + r^2 a + r^3 a + r^4 a + r^5 a = rs$

The diff. of these is, $-a + r^5 a = rs - s$.

But $r^5 a$ is the last term of the original progression multiplied by r, or in general terms $r^{n-1} a \times r$, that is $r^n a$. Consequently $r^n a - a = rs - s$

Whence $s = \dfrac{r^n a - a}{r - 1} = \dfrac{r^n - 1}{r - 1} a$, the sum of the series.

A similar method will lead to a like expression for s, whatever be the value of n. If r be a fraction, the expression becomes transformed to $s = \dfrac{1 - r^n}{1 - r} a$.

3. Now from these values of z and s the following theorems may be deduced.

(1.) $a = \dfrac{z}{r^{n-1}} = \dfrac{(r-1)s}{r^n - 1} = s + rz - rs$.

(2.) $z = ar^{n-1} = \dfrac{rs - s + a}{r} = \dfrac{s(r^n - r^{n-1})}{r^n - 1}$.

(3.) $s = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{rz - a}{r - 1} = \dfrac{zr^n - z}{r^n - r^{n-1}}$.

(4.) $r = \dfrac{s - a}{s - z} = \left(\dfrac{z}{a}\right)^{\frac{1}{n-1}}$.

And, if the logarithm of $\dfrac{z}{a} = N$, that of $\dfrac{sr - s + a}{a} = M$, and that of $r = R$; then

(5.) $n = \frac{N}{R} + 1 = \frac{M}{R}$.

Also, if when r is a fraction, n is infinite, then is $r_n = 0$, and the expression for s becomes

(6.) $s = \frac{a}{1 - r}$ [This expression is often of use in the summation of infinite series.]

Examples.

1. The least of ten terms in geometrical progression is 1, the ratio 2. Required the greatest term, and the sum.

Here $z = a\,r^{n-1} = 1 \times 2^9 = 512$, the greatest term;

and $s = \frac{rz - a}{r - 1} = \frac{1024 - 1}{1} = 1023$, the sum.

2. Find the sum of 12 terms of the progression 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c.

Here $s = \frac{1 - r^n}{1 - r} = \frac{1 - \frac{1}{531441}}{1 - \frac{1}{3}} = \frac{265720}{177147}$, the sum.

3. Find the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. carried to infinity.

Here theor. 6, that is $s = \frac{a}{1 - r}$, becomes $s = \frac{1}{1 - \frac{1}{2}} = 2$, the sum required.

4. Find the vulgar fraction equivalent to the circulating decimal ·36363636.

This decimal, expressed in the form of a series, is, $\frac{36}{100} + \frac{36}{10000} + \frac{36}{1000000}$ + &c. where $a = \frac{36}{100}$, and $r = \frac{1}{100}$.

Consequently, $s = \frac{a}{1 - r} = \frac{36}{100} \div \frac{99}{100} = \frac{4}{11}$, the fraction sought.

5. Find the sum of the descending infinite series $1 - x + x^2 - x^3 + x^4$, &c.

Here $a = 1$, $r = -x$, and $s = \frac{a}{1 - r} = \frac{1}{1 + x}$, the sum req.

And, by way of proof, it will be found that if 1 be divided by $1 + x$, the quotient will be the above series.

6. Of four numbers in geometrical progression the product of the two least is 8, and of the two greatest 128. What are the numbers? *Ans.* 2, 4, 8, and 16.

SECTION XII. *Logarithms.*

The logarithm of any given number is the index of such a power of some other number, as is equal to the given one.

Let us suppose that the number r greater than unity, is the base of a system of logarithms, and let there be given to it the variable exponent p, in such manner that the expression r^p shall represent all possible numbers, by attributing successively different values to the exponent p. It is manifest,

1. That the logarithm of unity will be always *zero* or nothing, whatever be the base r: for, in general, $r^0 = 1$.

2. That the logarithm of the base r will be 1, since r is the same thing as r^1.

3. That all numbers above 1 will have positive numbers for their logarithms. Thus, supposing $r = 10$, then the number 10000 or 10^4 has for its logarithm the positive number 4.

4. All fractions, or numbers below unity, have negative numbers for their logarithms. Thus, if $r = 10$, then $\frac{1}{10000}$ or ·0001 or 10^{-4} has -4 for its logarithm.

5. Assuming two numbers N and N′, to which correspond respectively the two logarithms p and p', to the same base or root r: we have $N = r^p$, and $N' = r^{p'}$, and consequently $N + N' = r^p \times r^{p'} = r^{p+p'}$. Whence it appears that in every system of logarithms, the logarithm $p + p'$ of a product N N′ composed of two factors, is equal to the sum of the logarithms of those factors.

6. If we have any numbers A, B, C, D, how many soever, we may prove in a similar manner, that (using the initial λ to denote the logarithm) we shall have $\lambda\,(A \cdot B \cdot C \cdot D) = \lambda A + \lambda B + \lambda C + \lambda D$.

7. If $A = B = C = D$, we shall have $\lambda\,(A \times A \times A \times A)$, or, $\lambda A^4 = \lambda A + \lambda A + \lambda A + \lambda A = 4\,\lambda A$: and in general $\lambda A^n = n\,\lambda A$. Thus it appears that the logarithm of any integral positive power, n, of any number A is equal to n times the logarithm of A.

8. We have also $\lambda A^{\frac{n}{p}} = \frac{n}{p}\lambda A$ (n and p being positive integers). For, let $A^{\frac{n}{p}} = K$, and consequently $\lambda A^{\frac{n}{p}} = \lambda K$. From the equation $A^{\frac{n}{p}} = K$, we have, by raising the whole to the power p, $A^n = K^p$, and, of consequence, $n\,\lambda A = p\,\lambda K$, or by division $\frac{n}{p}\lambda A = \lambda K = \lambda A^{\frac{n}{p}}$.

9. From the same principle it follows, that $\lambda \frac{A}{B} = \lambda A - \lambda B$. For, let $\frac{A}{B} = Q$, and consequently $A = B \times Q$: we shall then have $\lambda A = \lambda B + \lambda Q$; whence $\lambda Q = \lambda A - \lambda B$. So that the logarithm of the quotient is equal to the logarithm of the dividend minus that of the divisor; or the logarithm of a fraction is equal to the logarithm of the numerator made less by that of the denominator.

10. Farther, $\lambda A^{-n} = -n \lambda A$. For $A^{-n} = \frac{1}{A^n}$: therefore $\lambda A^{-n} = \lambda 1 - \lambda A^n = 0 - n \lambda A$; which is no other than $-n \lambda A$.

11. Again, $\lambda A^{-\frac{n}{p}} = -\frac{n}{p} \lambda A$. For $A^{-\frac{n}{p}} = 1 \div A^{\frac{n}{p}}$: whence results $\lambda A^{-\frac{n}{p}} = 0 - \frac{n}{p} \lambda A = -\frac{n}{p} \lambda A$.

12. Suppose there be two systems of logarithms whose roots or bases are r and s. Let any number N have p for its logarithm in the first system, and q for its logarithm in the second: we shall have $N = r^p$ and $N = s^q$; which gives $r^p = s^q$, and $s = r^{\frac{p}{q}}$. Therefore, taking the logarithms for the system r, we shall have $\lambda s = \frac{p}{q} \lambda r$; or, if in the system r we have $\lambda r = 1$, then $\lambda s = \frac{p}{q}$, or $q = \frac{p}{\lambda s} = p \times \frac{1}{\lambda s}$. Thus, knowing the logarithm p of any number N, for the system whose base is r, we may obtain the logarithm q of the same number for the system s, by multiplying p by a fraction whose numerator is unity and denominator the logarithm of s taken in the system r.

13. In the system of logarithms first constructed by Baron Napier, the great inventor, $r = 2{\cdot}718281828459$, &c. and the exponents are usually denominated *Napierian*, or *Hyperbolic* logarithms; the latter name being given because of the relation between these logarithms and the lines and asymptotic spaces in the equilateral hyberbola: so that in this system n is always the hyperbolic logarithm of $(2{\cdot}71828, \&c)^n$. But in the system constructed by Mr. Briggs (corresponding with the spaces in a hyperbola whose asymptotes make an angle of 25° 44′ 25″ 28‴), called *common* or *Briggean* logarithms, $r = 10$; so that the common logarithm of any number is the index of that power of 10 which is equal to the said number.

Thus, if $50 = 10^{1\cdot69897}$, and $9023 = 10^{3\cdot955351}$; then is 1·69897 the common logarithm of 50, and 3·955351 the common logarithm of 9023.

14. The rules for the management and application of logarithms being given in the best collections of logarithmic tables, are here omitted. The tables published in England by Dr. *Hutton*, and those published in France by *Callet*, are recommended as the most correct, and best fitted for common use. Tables by Mr. *Galbraith* of Edinburgh, are now printing, which I have no doubt will be correct and useful.

Section XIII. *Computation of Formulæ.*

Since the comprehension, and the numerical computation of formulæ expressed algebraically, are of the utmost consequence to practical men, enabling them to avail themselves advantageously of the theoretical results of men of science, as well as to express in scientific language the results of their own experimental or other researches; it has appeared expedient to present brief treatises of Arithmetic and Algebra. The thorough understanding of these two initiatory departments of science will serve essentially in the application of all that follows in the present volume; and that application may probably be facilitated by a few examples, as below:—

Ex. 1. Let $a = 5$, $b = 12$, $c = 13$, and $s = a + b + c$; then what is the numerical value of the expression, $\sqrt{[\frac{1}{2}s(\frac{1}{2}s - a)(\frac{1}{2}s - b)(\frac{1}{2}s - c)]}$, which denotes the area of the triangle whose sides are 5, 12, and 13?

Here $s = a + b + c = 5 + 12 + 13 = 30$; $\frac{1}{2}s = 15$; $\frac{1}{2}s - a = 15 - 5 = 10$; $\frac{1}{2}s - b = 15 - 12 = 3$; $\frac{1}{2}s - c = 15 - 13 = 2$.

Consequently, by substituting the numerical values of the several quantities between the parentheses for them, we shall have

$$\sqrt{(15 \times 10 \times 3 \times 2)} = \sqrt{900} = 30, \text{ the value required.}$$

Ex. 2. Suppose $g = 32\frac{1}{6}$, $t = 6$: required the value of $\frac{1}{2}gt^2$, an expression denoting the space in feet which a heavy body would fall vertically from quiescence in six seconds, in the latitude of London.

$$\text{Here } \tfrac{1}{2}gt^2 = 16\tfrac{1}{12} \times 6^2 = 96\tfrac{1}{2} \times 6 = 579 \text{ feet.}$$

Ex. 3. Given $D = 6$, $d = 4$, $h = 12$, $\pi = 3\cdot141593$; required the value of $\frac{1}{12}\pi h(D^2 + Dd + d^2)$, a theorem for the solid content of a conic frustum whose diameters of the two ends are D, d, and height h.

Here $D^2 = 36$, $Dd = 6 \times 4 = 24$, $d^2 = 16$, $\frac{\pi}{12} = \cdot2618$ nearly.

Hence $\frac{1}{12}\pi h(D^2 + Dd + d^2) = \cdot2618\ (36 + 24 + 16)\ 12 = \cdot2618 \times 76 \times 12 = 3\cdot141593 \times 76 = 238\cdot761068$.

Ex. 4. Let $a = 1$, $h = 25$, $g = 193$ inches: what is the value of $2a\sqrt{gh}$? This being the expression for the cubic inches of water discharged in a second, from an orifice whose area is a, and depth below the upper surface of water in the vessel, or reservoir, h, both in inches.

Here $2a\sqrt{gh} = 2\sqrt{(25 \times 193)} = 10\sqrt{193} = 10 \times 13\cdot89244 = 138\cdot9244$ cubic inches.

Ex. 5. Suppose the velocity of the wind to be known in miles per hour; required short approximative expressions for the yards per minute, and for the feet per second.

First $1760 \div 60 = \frac{88}{3} = 29\frac{1}{3} = 30$ nearly.

Also $5280 \div (60 \times 60) = \frac{5280}{3600} = \frac{88}{60} = \frac{44}{30} = 1\frac{1}{2}$ nearly.

If, therefore, n denote the number of miles per hour;

$30n$ will express the yards per minute; and $1\frac{1}{2}n$, the feet per second.

These are approximative results: to render them correct, where complete accuracy is required, subtract from each result its 45th part, or the *fifth* part of its *ninth* part.

Thus suppose the wind blows at the rate of 20 miles per hour:

Then $30n = 30 \times 20 = 600$ yards per minute, or more correctly $600 - \frac{600}{45} = 600 - 13\frac{1}{3} = 586\frac{2}{3}$ yards.

Also $1\frac{1}{2}n = 30$ feet per second;

or, correctly $30 - \frac{30}{45} = 30 - \frac{2}{3} = 29\frac{1}{3}$ feet.

Conversely, $\frac{2}{3}$ of the feet per second will indicate the miles per hour, correct within the 45th part, which is to be *added* to obtain the true result.

Ex. 6. To find a theorem by means of which it may be ascertained when a general law exists, and what that law is.

Suppose, for example, it were required to determine the law which prevailed between the resistances of bodies moving in the air and other resisting media, and the velocities with which they move. Let V, v, denote any two velocities, and R, r, the corresponding resistances experienced by a body moving with those velocities: we wish to ascertain what power of V it is to which R is proportional. Let x denote the index or exponent of the power: then will $V^x : v^x :: R : r$, if a law subsist. Div. the consequents by the antecedents, we have $1 : \left(\frac{v}{V}\right)^x :: 1 : \frac{r}{R}$. Consequently $\left(\frac{v}{V}\right)^x = \frac{r}{R}$. This, expressed logarithmically, gives $x \times \log. \frac{v}{V} = \log. \frac{r}{R}$;

$$\text{or } x = \frac{\log. r - \log. R}{\log. v - \log. V}.$$

Hence, the quotient of the differences of the logs. of the resistances, divided by the difference of the corresponding velocities, will express the exponent x required.

This theorem is of very frequent application in reference to the motion of cannon balls, of barges on canals, &c. and may indeed be applied to the planetary motions.

CHAP. III.

PRINCIPLES OF GEOMETRY.

Definitions.

1. Geometry is a department of science, by means of which we demonstrate the properties, affections, and measures of all sorts of magnitude.

2. *Magnitude* is a continued quantity, or any thing that is extended; as a *line*, *surface*, or *solid*.

3. A *point* is that which has no parts: i. e. neither length, breadth, nor thickness.

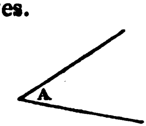
4. A *line* is a length without breadth or thickness.

Cor. The extremes of a line are points.

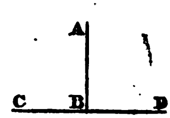
5. A *right line* is that which lies evenly, or in the same direction, between two points. A *curve line* continually changes its direction.

Cor. Hence there can only be one species of right lines, but there is infinite variety in the species of curves.

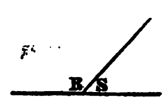
6. An *angle* is the inclination of two lines to one another, meeting in a point, called the angular point. When it is formed by two right lines, it is a plane angle, as A; if by curve lines, it is a curvilineal angle.



7. A *right angle* is that which is made by one right line A B falling upon another C D, and making the angles on each side equal, A B C = A B D; so that A B does not incline more to one side than another: A B is called a *perpendicular*. All other angles are called oblique angles.



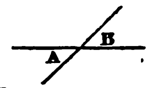
8. An *obtuse angle* is greater than a right angle, as R.



9. An *acute angle* is less than a right angle, as S.

10. *Contiguous angles* are those made by one line falling upon another, and joining to one another, as R, S.

11. *Vertical* or *opposite* angles, are those made on contrary sides of two lines intersecting one another, as A, B.



12. A *surface* has only length and breadth. The extremes or limits of a surface are lines.

13. A *plane* is that surface which lies perfectly even between its extremes; or in which, right lines any way drawn coincide.

14. A *solid* is a magnitude extended every way, or which has length, breadth, and depth.

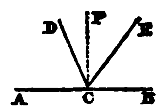
The terms or extremes of a solid, are surfaces.

15. The *square* of a right line is the space included by four right lines equal to it, set perpendicular to one another.

16. The rectangle of two lines is the space included by four lines equal to them, set perpendicular to one another, the opposite ones being equal.

Section I. *Of Angles, and Right Lines, and their Rectangles.*

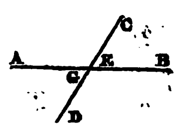
Prop. 1. If to any point C in a right line A B, several other right lines D C, E C are drawn on the same side; all the angles formed at the point C, taken together, are equal to two right angles, A C D + D C E + E C B = two right angles.



Cor. 1. All the angles made about one point in a plane, being taken together, are equal to four right angles.

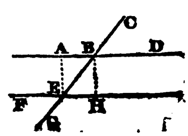
Cor. 2. If all the angles at C, on one side of the line A B, are found to be equal to two right angles; then A C B is a straight line.

2. If two right lines, A B, C D, cut one another; the opposite angles E and G will be equal.



3. A right line, B H, which is perpendicular to one of two parallels, is perpendicular to the other.

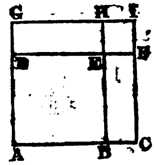
4. If a right line C G, intersects two parallels A D, F H; the alternate angles, A B E, and B E H, will be equal.



Cor. 1. The external angle C B D, is equal to the internal angle on the same side B E H.

Cor. 2. The two internal angles on the same side are equal to two right angles, D B E + B E H = two right angles.

5. Right lines, parallel to the same right line, are parallel to one another.



6. If a right line A C be divided into two parts A B, B C; the square of the whole line is equal to the squares of both the parts, and twice the rectangle of the parts, $AC^2 = AB^2 + BC^2 + 2\,AB \cdot BC$.

7. The square of the difference of two lines A C, B C, is equal to the sum of their squares, wanting twice their rectangle, $AB^2 = AC^2 + BC^2 - 2\,AC \cdot BC$.

8. The rectangle of the sum and difference of two lines, is equal to the difference of their squares.

9. The square of the sum, together with the square of the difference of two lines, is equal to twice the sum of their squares.

Section II. *Of Triangles.*

Definitions.

1. A *triangle* is a plane figure bounded by three right lines, called the sides of the triangle.

2. An *equilateral triangle* is one which has three equal sides.

3. An *equiangular triangle* is one which has three equal angles.

4. An *isosceles triangle* has two sides equal.

5. A *right-angled triangle* is that which has a right angle. The side opposite to the right angle is called the *hypothenuse.*

6. An *oblique triangle* is one having oblique angles.

7. An *obtuse angled triangle* has one obtuse angle.

8. An *acute angled triangle* has three acute angles.

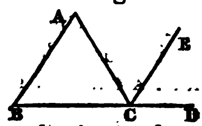
9. A *scalene triangle* has three unequal sides.

10. *Similar triangles* are those whose angles are respectively equal, each to each. And *homologous sides* are those lying between equal angles.

11. The *Base* of a triangle, is the side on which a perpen-

dicular is drawn from the opposite angle called the *vertex*; the two sides, proceeding from the vertex, are called the *legs*.

Prop. 1. In any triangle A B C, if one side B C be produced or drawn out; the external angle A C D will be equal to the two internal opposite angles A, B.



2. In any triangle, the sum of the three angles is equal to two right angles.

Cor. 1. If two angles in one triangle be equal to two angles in another: the third will also be equal to the third.

Cor. 2. If one angle of a triangle be a right angle, the sum of the other two will be equal to a right angle.

3. The angles at the base of an isosceles triangle, are equal.

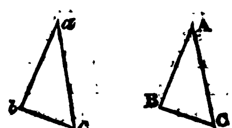
Cor. 1. An equilateral triangle is also equiangular; and the contrary.

Cor. 2. The line which is perpendicular to the base of an isosceles triangle, bisects it and the vertical angle.

4. In any triangle, the greatest side is opposite to the greatest angle, and the least to the least.

5. In any triangle A B C, the sum of any two sides B A, A C, is greater than the third B C, and their difference is less than the third side.

6. If two triangles A B C, *a b c*, have two sides, and the included angle equal in each; these triangles, and their correspondent parts, shall be equal.



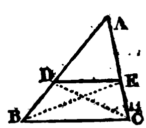
7. If two triangles A B C and *a b c*, have two angles and an included side equal, each to each; the remaining parts shall be equal, and the whole triangles equal.

8. If two triangles have all their sides respectively equal; all the angles will be equal, and the wholes equal.

9. Triangles of equal bases and heights are equal.

10. Triangles of the same height, are in proportion to one another as their bases.

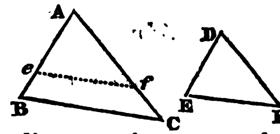
11. If a line D E be drawn parallel to one side B C, of a triangle; the segments of the other sides will be proportional; A D : D B :: A E : E C.



Cor. 1. If the segments be proportional, A D : D B :: A E : E C; then the line D E is parallel to the side B C.

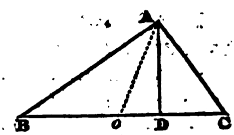
Cor. 2. If several lines be drawn parallel to one side of a triangle, all the segments will be proportional.

Cor. 3. A line drawn parallel to any side of a triangle; cuts off a triangle similar to the whole.



12. In similar triangles, the homologous sides are proportional; AB : AC :: DE : DF.

13. Like triangles are in the duplicate ratio, or as the squares of, their homologous sides.



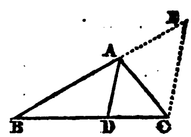
14. In a right-angled triangle BAC, if a perpendicular be let fall from the right angle upon the hypothenuse, it will divide it into two triangles similar to one another and to the whole, ABD, ADC.

Cor. 1. The rectangle of the hypothenuse and either segment is equal to the square of the adjoining side.

15. The distance A *o* of the right angle, from the middle of the hypothenuse is equal to half the hypothenuse.

16. In a right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the two sides.

17. If the square of one side of a triangle be equal to the sum of the squares of the other two sides; then the angle comprehended by them is a right angle.



18. If an angle A, of a triangle BAC be bisected by a right line AD, which cuts the base; the segments of the base will be proportional to the adjoining sides of the triangle; BD : DC :: AB : AC.

19. If the sides be as the segments of the base the line AD, bisects the angle A.

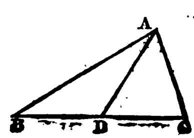
20. Three lines drawn from the three angles of a triangle to the middle of the opposite sides, all meet in one point.

21. Three perpendicular lines erected on the middle of the three sides of any triangle, all meet in one point.

22. The point of intersection of the three perpendiculars, will be equally distant from the three angles: or, it will be the centre of the circumscribing circle.

23. Three perpendiculars drawn from the three angles of a triangle, upon the opposite sides, all meet in one point.

24. Three lines bisecting the three angles of a triangle, all meet in one point.



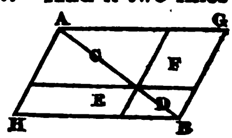
25. If D be any point in the base of a scalene triangle, ABC: then is $AB^2 . DC + AC^2 . BD = AD^2 . BC + BC . BD . DC$.

Section III. *Of Quadrilaterals and Polygons.*

Definitions.

1. A *quadrangle* or *quadrilateral*, is a plane figure bounded by four right lines.

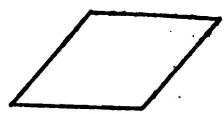
2. A *parallelogram* is a quadrangle whose opposite sides are parallel, as A G B H. The line A B drawn to the opposite corners is called the *diameter* or *diagonal*. And if two lines be drawn parallel to the two sides, through any point of the diagonal; they divide it into several others, and then C, D are called parallelograms about the diameter: and E, F the *complements*: and the figure E D F a *gnomon*.



3. A *rectangle* is a parallelogram whose sides are perpendicular to one another.

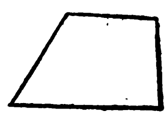
4. A *square* is a rectangle of four equal sides and four equal angles.

5. A *rhombus* is a parallelogram, whose sides are equal, and angles oblique.

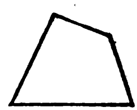


6. *A rhomboid* is a parallelogram, whose sides are unequal, and angles oblique.

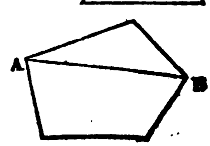
7. A *trapezoid* is a quadrangle, having only two sides parallel.



8. A *trapezium* is a quadrangle, that has no two sides parallel.



9. A *polygon* is a plane figure enclosed by many right lines. If all the sides and angles are equal, it is called a regular polygon, and denominated according to the number of sides or angles, as a *pentagon* 5, a *hexagon* 6, a *heptagon* 7, &c.



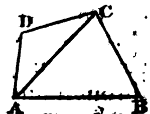
10. The *diagonal* of a quadrangle or polygon is a line drawn between any two opposite corners of the figure, as A B.

11. The *height* of a figure is a line drawn from the top, perpendicular to the base, or opposite side on which it stands.

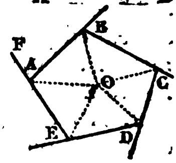
12. *Like* or *similar* figures, are those whose several angles are equal to one another, and the sides about the equal angles, proportional.

13. *Homologous* sides of two like figures are those between two angles, respectively equal.

14. The *perimeter* or circumference of a figure, is the compass of it, or sum of all the lines that enclose it.



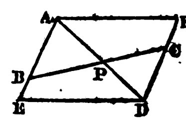
15. The internal angles of a figure are those on the inside, made by the lines that bound the figure, A D C B.



16. The *external angle* of a figure is the angle made by one side of a figure, and the adjoining side drawn out, as B A F.

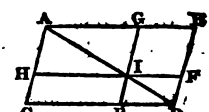
Prop. 1. In any parallelogram the opposite sides and angles are equal; and the diagonal divides it into two equal triangles.

2. The diagonals of a parallelogram intersect each other in the middle point of both.



3. Any line B C passing through the middle of the diagonal of a parallelogram P, divides the area into two equal parts.

4. Any right line B C drawn through the middle point P of the diagonal of a parallelogram, is bisected in that point; B P = P C. (See preceding fig.)



5. In any parallelogram A B D C, the complements C I, and I B are equal.

6. Parallelograms of equal bases and heights are equal.

7. A parallelogram is double a triangle of the same, or an equal base and height.

8. Parallelograms of the same height are to one another as their bases.

9. Parallelograms of equal bases are as their heights.

10. Parallelograms are to one another, as their bases and heights.

11. In any parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of all the sides.

12. The sum of the four internal angles of any quadrilateral figure, is equal to four right angles.

13. If two angles of a quadrangle be right angles, the sum of the other two amounts to two right angles.

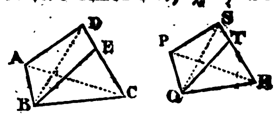
14. The sum of all the internal angles of a polygon is equal to twice as many right angles, abating four, as the polygon has sides.

15. Hence all right-lined figures of the same number of sides, have the sum of all the internal angles equal.

16. The sum of the external angles of any polygon is equal to four right angles.

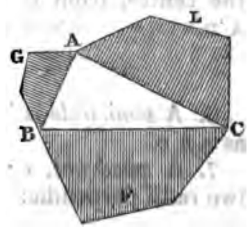
17. All right-lined figures have the sum of their external angles equal.

18. In two similar figures A C, P R; if two lines B E, Q T, be drawn after a like manner, as suppose, to make the angle C B E = R Q T; then these lines have the same proportion, as any two homologous sides of the figure; viz. B E : Q T :: B C : Q R :: A B : P Q :: A D : P S.

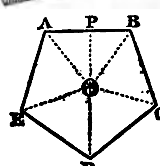


19. All similar figures are to one another as the squares of their homologous sides.

20. Any figure described on the hypothenuse of a right-angled triangle, is equal to two similar figures described the same way upon the two sides: B F C = A L C + A G B.



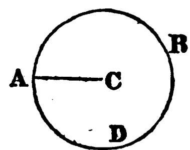
21. Any regular figure A B C D E, is equal to a triangle whose base is the perimeter A B C D E A; and height, the perpendicular O P, drawn from the centre, perpendicular to one side.



22. Only three sorts of regular figures can fill up a plane surface, that is, the whole space round an assumed point; and these are six triangles, four squares, and three hexagons.

Section IV. *Of the Circle, and Inscribed and Circumscribed Figures.*

Definitions.



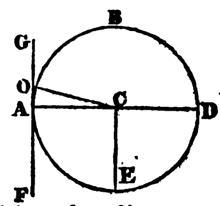
1. A *circle* is a plane figure described by a right line moving about a fixed point, as A C about C: or it is a figure bounded by one line equidistant from a fixed point.

2. The *centre* of a circle is the fixed point about which the line moves, C.

3. The *radius* is the line that describes the circle, C A.

Cor. All the radii of a circle are equal.

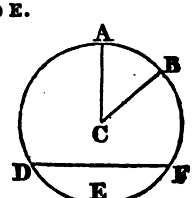
4. The *circumference* is the line described by the extreme end of the moving line, A B D A.



5. The *diameter* is a line drawn through the centre, from one side to the other, A D.

6. A *semicircle* is half the circle, cut off by the diameter, as A B D.

7. A *quadrant*, or quarter of a circle, is the part between two radii perpendicular to one another, as C D E.



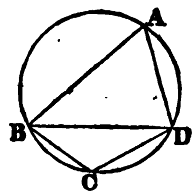
8. An *arch* is any part of the circumference, A B.

9. A *sector* is a part bounded by two radii, and the arch between them, A C B.

10. A *segment* is a part cut off by a right line, or cord, D E F, or D A B F.

11. A *cord*, a right line drawn through the circle, as D F.

12. *Angle at the centre* is that whose angular point is at the centre A C B. (See the last figure.)



13. *Angle at the circumference* is when the angular point is in the circumference, as B A D.

14. *Angle in a segment*, is the angle made by two lines drawn from some point of the arch of that segment to the ends of the base; as B C D is an angle in the segment B C D.

15. *Angle upon a segment*, is the angle made in the opposite segment, whose sides stand upon the base of the first; as B A D, which stands upon the segment B C D.

16. A *tangent* is a line touching a circle, which, produced, does not cut it, as G A F. (Fig. to def. 5.)

17. Circles are said to touch one another, which meet, but do not cut one another.

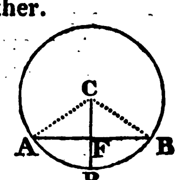
18. *Similar arches*, or *similar sectors*, are those bounded by radii that make the same angle.

19. *Similar segments* are those which contain similar triangles, alike placed.

20. A figure is said to be *inscribed in a circle*, or a circle *circumscribed about a figure*, when all the angular points of the figure are in the circumference of the circle.

21. A circle is said to be *inscribed in a figure*, or a figure *circumscribed about a circle*, when the circle touches all the sides of the figure.

22. *One figure is inscribed in another*, when all the angles of the inscribed figure are in the sides of the other.



Prop. 1. The radius C R, bisects any cord at right angles, which passes not through the centre, as A B.

Cor. 1. If a line bisects a cord at right angles, it passes through the centre of the circle.

Cor. 2. The radius that bisects the cord also bisects the arch.

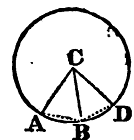
2. In a circle equal cords are equally distant from the centre.

3. If several lines be drawn through a circle, the greatest is the diameter, and those that are nearer the centre are greater than those that are farther off.

4. If from any point three equal right lines can be drawn to the circumference; that point is the centre.

5. No circle can cut another in more than two points.

6. There can only two equal lines be drawn from any exterior point P, to the circumference of a circle.



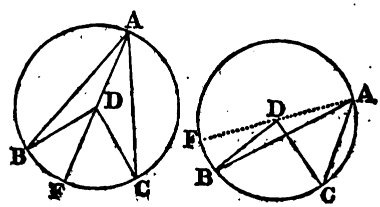
7. In any circle, if several radii be drawn making equal angles, the arches and sectors comprehended thereby will be equal, if A C B = B C D: then, arch A B = arch B D; and sector A C B = B C D.

8. In the same or equal circles, the arches, and also the sectors, are proportional to the angles intercepted by the radii.

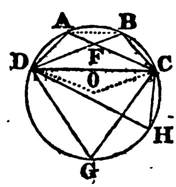
9. The circumferences of circles are to one another as their diameters.

10. A right line, perpendicular to the diameter of a circle, at the extreme point, touches the circle in that point, and lies wholly without the circle.

11. If two circles touch one another, either inwardly or outwardly, the line passing through their centres shall also pass through the point of contact.

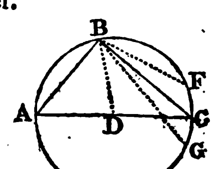


12. In a circle the angle at the centre is double the angle at the circumference, standing upon the same arch; B D C = 2 B A C.



13. All angles in the same segment of a circle are equal, D A C = D B C, and D G C = D H C.

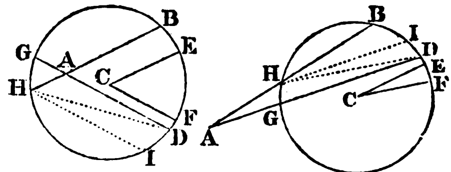
14. If the extremities of two equal arches D A, B C, be joined by right lines, D C, A B; they will be parallel.



15. The angle A B C in a semicircle is a right angle.

16. Angle A B G, in a greater segment A B F G, is less than a right angle; and the angle A B F, in a less segment A B F, is greater than a right angle.

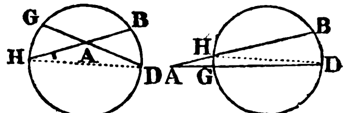
17. If two lines cutting a circle, intersect one another in A; and there be made at the centre, ∠ ECF = BAD;



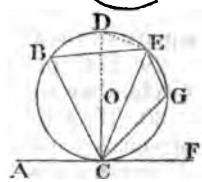
Then arch BD + GH = 2 EF, if A is within the circle; or arch BD − GH = 2 EF, if A is without.

18. If from a point without, two lines touch a circle; the angle made by them is equal to the angle at the centre, standing on half the difference, of these two parts of the circumference.

19. The angle A = ∠ BHD + HDG, when A is within; or A = BHD − HDG, when A is without the circle.

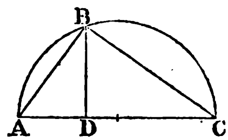


20. In a circle, the angle made at the point of contact between the tangent and any chord, is equal to the angle in the alternate segment; ECF = EBC, and ECA = EGC.



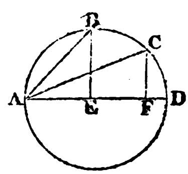
21. A tangent to the middle point of an arch, is parallel to the chord of it.

22. If from any point B in a semicircle, a perpendicular BD be let fall upon the diameter, it will be a mean proportional between the segments of the diameter : AD : DB :: DB : DC.

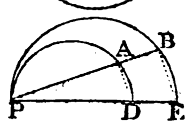


23. The chord is a mean proportional between the adjoining segment and the diameter, from the similarity of the triangles.

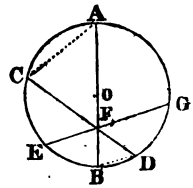
24. In a circle, if the diameter AD be drawn, and from the ends of the chords AB, AC, perpendiculars be drawn upon the diameter; the squares of the chords will be as the segments of the diameter; AE : AF :: $AB^2 : AC^2$.



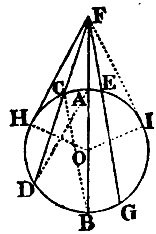
25. If two circles touch one another in P, and the line PDE be drawn through their centres; and any line PAB is drawn through that point to cut the circles, that line will be divided in proportion to the diameters; PA : PB :: PD : PE.



26. If through any point F in the diameter of a circle, any chord C F D be drawn, the rectangle of the segments of the chord is equal to the rectangle of the segments of the diameter; C F . F D = A F : F B = also G F . F E.



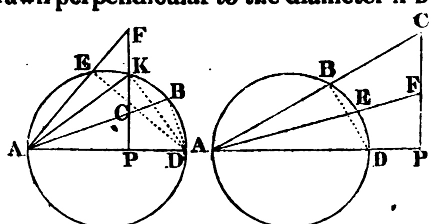
27. If through any point F out of the circle in the diameter B A produced, any line F C D be drawn through the circle: the rectangle of the whole line and the external part is equal to the rectangle of the whole line passing through the centre, and the external part; D F . F C = A F . F B.



28. Let H F be a tangent at H; then the rectangle C F D = square of the tangent F H.

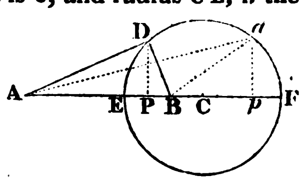
29. If from the same point F, two tangents be drawn to the circle, they will be equal; F H = F I.

30. If a line P F C be drawn perpendicular to the diameter A D of a circle; and any line drawn from A to cut the circle and the perpendicular; then the rectangle of the distances of the sections from A, will be equal to the rectangle of the diameter and the distance of the perpendicular from A; A B × A C = A P × A D.

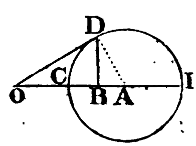


Also, A B × A C = A K².

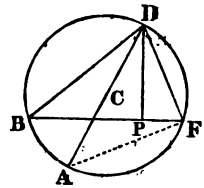
31. In a circle E D F whose centre is C, and radius C E, if the points B, A, be so placed in the diameter produced, that C B, C E, C A be in continual proportion, then two lines B D, A D drawn from these points to any point in the circumference of the circle will always be in the given ratio of B E to A E.



32. In a circle, if a perpendicular D B be let fall from any point D, upon the diameter C I, and the tangent D O drawn from D, then A B, A C, A O, will be continually proportional.

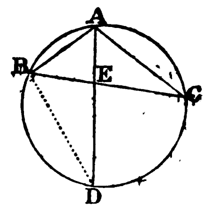


33. If a triangle B D F be inscribed in a circle, and a perpendicular D P let fall from D on the opposite side B F, and the diameter D A drawn; then as the perpendicular is to one side including the angle D, so the other side to the diameter of the circle; D P : D B :: D F : D A.

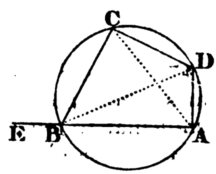


34. The rectangle of the sides of an inscribed triangle is equal to the rectangle of the diameter, and the perpendicular on the third side B D . D F = A D . D P.

35. If a triangle B A C be inscribed in a circle, and the angle A bisected by the right line A E D, then as one side, to the segment of the bisecting line, within the triangle, so the whole bisecting line, to the other side; A B : A E :: A D : A C; and A B . A C = B E . E C + A E².



36. If a quadrilateral A B C D be inscribed in a circle, the sum of two opposite angles is equal to two right angles; A D C + A B C = two right angles.



37. If a quadrangle be inscribed in a circle, the rectangle of the diagonals is equal to the sum of the rectangles of the opposite sides.

38. A circle is equal to a triangle whose base is the circumference of the circle; and height, its radius.

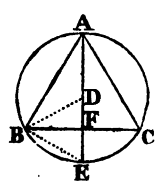
39. The area of a circle is equal to the rectangle of half the circumference and half the diameter.

40. Circles (that is, their areas) are to one another as the squares of their diameters, or as the squares of the radii, or as the squares of the circumferences.

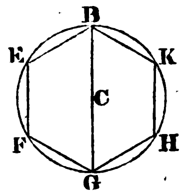
41. Similar polygons inscribed in circles, are to one another as the circles wherein they are inscribed.

42. A circle is to any circumscribed rectilineal figure, as the circle's periphery to the periphery of the figure.

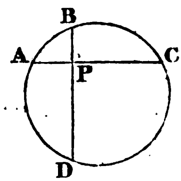
43. If an equilateral triangle A B C be inscribed in a circle; the square of the side thereof is equal to three times the square of the radius: A B² = 3 A D².



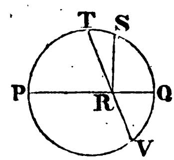
44. A square inscribed in a circle, is equal to twice the square of the radius.



45. The side of a regular hexagon inscribed in a circle, is equal to the radius of the circle; B E = B C.

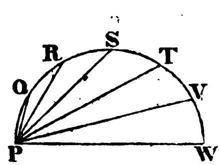


46. If two chords in a circle mutually intersect at right angles, the sum of the squares of the segments of the chords is equal to the square of the diameter of the circle. $AP^2 + PB^2 + PC^2 + PD^2 = \text{diam.}^2$



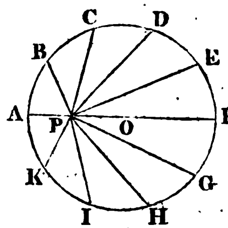
47. If the diameter P Q be divided into two parts at any point R, and if R S be drawn perpendicular to P Q; also R T applied equal to the radius, and T R produced to the circumference at V: then, between the two segments P R, R Q,—

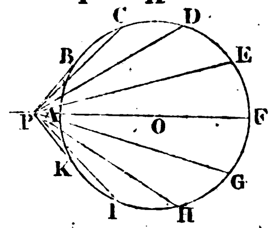
R T is the *arithmetical* mean,
R S is the *geometrical* mean,
R V is the *harmonical* mean.



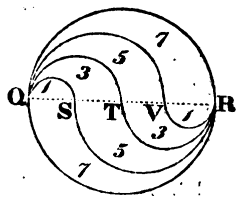
48. If the arcs P Q, Q R, R S, &c. be equal, and there be drawn the chords P Q, P R, P S, &c. then it will be P Q : P R :: P R : P Q + P S :: P S : P R + P T :: P T : P S + P V, &c.

49. The centre of a circle being O, and P a point in the radius, or in the radius produced; if the circumference be divided into as many equal parts A B, B C, C D, &c. as there are units in $2n$, and lines be drawn from P to all the points of division; then shall the continual product of all the alternate lines, viz. P A × P C × P E, &c. be $= r^n - x^n$ when P is within the circle, or $= x^n - r^n$ when P is without the circle; and the product of the rest of the lines, viz. P B × P D + P F, &c. $= r^n + x^n$: where $r =$ A O the radius, and $x =$ O P the distance of P from the centre.





50. A circle may thus be divided into any number of parts that shall be equal to one another both in area and perimeter. Divide the diameter Q R into the same number of equal parts at the points S, T, V, &c.; then, on one side of the diameter describe semicircles on the diameters Q S, Q T, Q V, and on the other side of it describe semicircles on R V, R T, R S; so shall the parts 1 7, 3 5, 5 3, 7 1 be all equal, both in area and perimeter.



Section V. *Of Planes and Solids.*

Definitions.

1. The *common section* of two planes, is the line in which they meet, or cut each other.

2. A line is perpendicular to a plane, when it is perpendicular to every line in that plane which meets it.

3. One plane is perpendicular to another, when every line of the one, which is perpendicular to the line of their common section, is perpendicular to the other.

4. The *inclination* of one plane to another, or the angle they form between them, is the angle contained by two lines, drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

5. *Parallel planes* are such as being produced ever so far both ways, will never meet, or which are every where at an equal perpendicular distance.

6. A *solid angle* is that which is made by three or more plane angles, meeting each other in the same point.

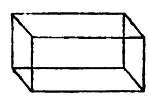
7. *Similar solids*, contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes, alike placed.

8. A *prism* is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms.

9. A prism takes particular names according to the figure of its base or ends, whether triangular, square, rectangular, pentagonal, hexagonal, &c.

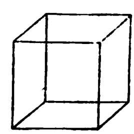
10. A right or *upright prism*, is that which has the planes of the sides perpendicular to the planes of the ends or base.

11. A *parallelopiped*, or parallelopipedon, is a prism bounded by six parallelograms, every opposite two of which are equal, alike, and parallel.

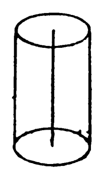


12. A *rectangular parallelopidedon* is that whose bounding planes are all rectangles, which are perpendicular to each other.

13. A *cube* is a square prism, being bounded by six equal square sides or faces, which are perpendicular to each other.

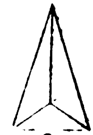


14. A *cylinder* is a round prism having circles for its ends; and is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.



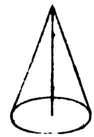
15. The *axis* of a cylinder is the right line joining the centres of the two parallel circles, about which the figure is described.

16. A *pyramid* is a solid whose base is any right-lined plane figure, and its sides triangles, having all their vertices meeting together in a point above the base, called the vertex of the pyramid.



17. A pyramid, like the prism, takes particular names from the figure of the base.

18. A *cone* is a round pyramid having a circular base, and is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.



19. The *axis* of a cone is the right line, joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

20. *Similar cones and cylinders*, are such as have their altitudes and the diameters of their bases proportional.

21. A *sphere* is a solid bounded by one curve surface, which is everywhere equally distant from a certain point within, called the centre. It is conceived to be generated by the rotation of a semicircle about its diameter, which remains fixed.

22. The *axis* of a sphere is the right line about which the semicircle revolves, and the centre is the same as that of the revolving semicircle.

23. The *diameter* of a sphere is any right line passing through the centre, and terminated both ways by the surface.

24. The *altitude* of a solid is the perpendicular drawn from the vertex to the opposite side or base.

Prop. 1. If any prism be cut by a plane parallel to its base, the section will be equal and like to the base.

2. If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.

3. All prisms and cylinders, of equal bases and altitudes, are equal to each other.

4. Rectangular parallelopipedons, of equal altitudes, are to each other as their bases.

5. Rectangular parallelopipedons, of equal bases, are to each other as their altitudes.

6. Because, prisms and cylinders are as their altitudes, when their bases are equal: and, as their bases when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes: hence, also, these products are the proper numeral measures of their quantities or magnitudes.

7. Similar prisms and cylinders are to each other as the cubes of their altitudes, or of any like linear dimensions.

8. In any pyramid a section parallel to the base is similar to the base; and these two planes are to each other as the squares of their distances from the vertex.

9. In a cone, any section parallel to the base is a circle; and this section is to the base as the squares of their distances from the vertex.

10. All pyramids and cones, of equal bases and altitudes, are equal to one another.

11. Every pyramid is the third part of a prism of the same base and altitude.

12. If a sphere be cut by a plane, the section will be a circle.

13. Every sphere is two-thirds of its circumscribing cylinder.

14. A cone hemisphere, and cylinder of the same base and altitude, are to each other as the numbers 1, 2, 3.

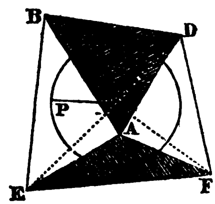
15. All spheres are to each other as the cubes of their diameters; all these being like parts of their circumscribing cylinders.

16. None but three sorts of regular plane figures joined together can make a solid angle; and these are, 3, 4, or 5 triangles, 3 squares, and 3 pentagons.

And therefore there can only be five regular bodies, the *pyramid*, *cube*, *octaedron*, *dodecaedron*, and *icosaedron*.

17. No other but only one sort, of the five regular bodies, joined at their angles, can completely fill a solid space; viz. eight cubes.

18. A sphere is to any circumscribing solid B F, (all whose planes touch the sphere); as the surface of the sphere to the surface of the solid.



19. All bodies circumscribing the same sphere, are to one another as their surfaces.

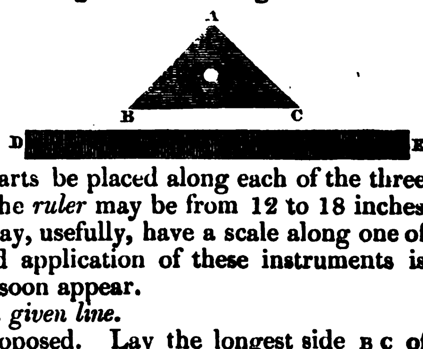
20. The sphere is the greatest or most capacious of all bodies of equal surface.

SECTION VI. *Practical Geometry.*

It is not intended in this place to present a complete collection of Geometrical Problems, but merely a selection of the most useful, especially in reference to the employments of mechanics and engineers.

The instruments well known to be used in geometrical constructions, are the scale and compasses, the semicircular or the circular protractor, the sector, and a parallel ruler. To these a few other useful instruments may be added, which we shall describe as we proceed; speaking first of *the Triangle and Ruler.*

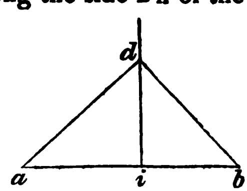
These are, as their names indicate, a *triangle*, that is to say, an isosceles right-angled triangle, and a *ruler*, both made of well seasoned wood, or of ivory, ebony, or metal. Each side A B, A C, of the triangle about the right angle A, being 3, 4, 6, or 8 inches, according to the magnitude of the figures in whose construction it is likely to be employed. About the middle of the triangle there should be a circular orifice, as shown in the figures;



and if a scale of equal parts be placed along each of the three sides, all the better. The *ruler* may be from 12 to 18 inches in length; and it also may, usefully, have a scale along one of its sides. The conjoined application of these instruments is of great utility; as will soon appear.

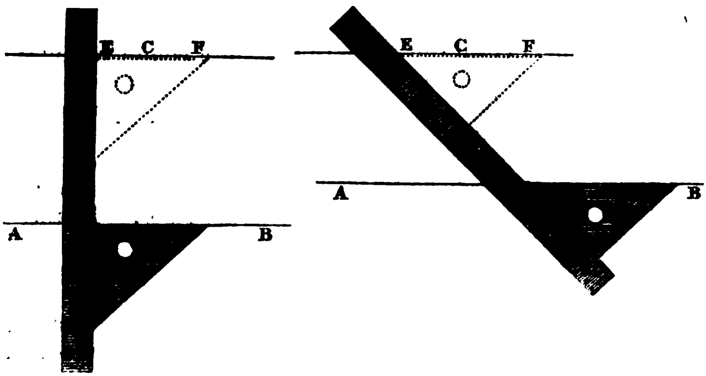
PROB. 1. *To bisect a given line.*

Let *a b* be the line proposed. Lay the longest side B C of the triangle so as to coincide with *a b*, and so that its angle B shall coincide with the point *a*; and along the side B A of the triangle draw a line *a d*. Then, slide the base B C of the triangle along the line *a b*, until C coincides with *b*, and draw in coincidence with the side C A, the line *b d*, intersecting the former in *d*. Next bring the ruler to coincide with *a b*, and in contact with it lay one



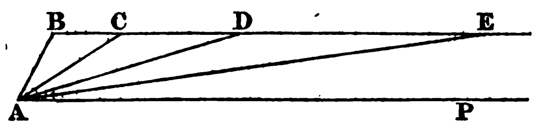
of the legs B A of the triangle; then slide the triangle along the ruler, until the other leg A C passes through the point *d*: draw along A C, so posited, the line *d i*; it will be perpendicular to *a b*, and will bisect it in *i*, the point required.

Prob. 2. Through a given point, C, to draw a line parallel to a given line A B.



Place one of the sides of the triangle in contact with the line A B. Lay the ruler against one of the other sides of the triangle; and keeping it steady, slide along the triangle until the same side which had been made to coincide with part of the line A B touches the point C; then, along that side, draw through C the line E F: it will be parallel to A B as required.

Prob. 3. To bisect a given angle; then to bisect its half; and so on.



Let B A P be the proposed angle. Through any point B draw B E parallel to A P (by the former problem). Upon B E set off, with the compasses, from the scale at the edge of the ruler, B C = B A: join A C; it will bisect the angle B A P.

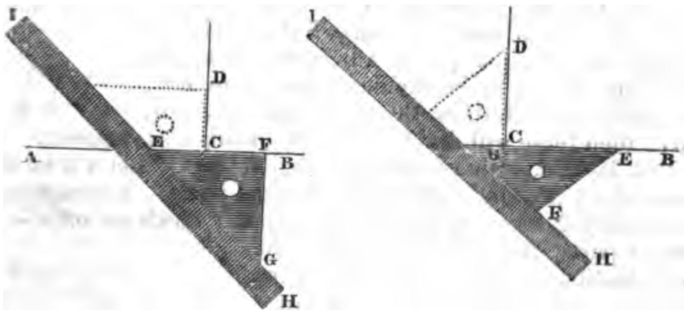
Again, set off, upon B E, from C, C D = C A: join A D; it will bisect C A P, or quadrisect B A P.

Again, set off, upon B E, D E = D A: join E A; so shall E A P be $\frac{1}{8}$ of B A P: and so on.

Prob. 4. To erect a perpendicular at any given point C, in a given line A B.

1st *Method*. Apply one of the legs, E F, of the triangle, upon the line A B. Lay the side of the ruler H I, against the hypothenuse, E G, of the triangle, and, keeping it steady, slide the triangle upwards until the side F G touches the point C. Then

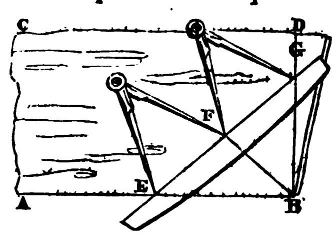
draw C D in contact with that leg, and it will be the perpendicular required.



2d *Method.* Apply the hypothenuse, G H, of the triangle to the line A B. Lay the edge of the ruler, H I, against the leg G E. Keep it steady, and turn the triangle so that the other leg F E may be laid against the ruler. Then slide the triangle upwards until the hypothenuse touches the point C: then in coincidence with it draw C D, and it will be the perpendicular required.

Note. After similar methods may a perpendicular be let fall from a point D above a line A B upon it.

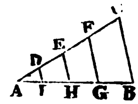
3d *Method,* by a ruler and compasses only: as suppose it were required to cut the end of a plank *square.* Let A B C D be the plank, of which the end B D is required to be squared. The edge A B being quite straight, open the compasses to any convenient distance, and place the point of one leg at B, and the other at any point, as F. Keep one leg at F, and turn the other round till it touches the edge A B at E; keep them firm, and apply the straight edge to E F, as the figure shows; keep the leg still at F, and turn them over into the position F G, G being close to the straight edge, and make a mark at G. Now, if the straight edge be applied to G and B, and G B be drawn, it will be square to the edge A B.



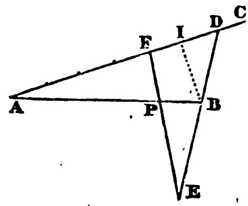
Note. In this construction it is evident that F is the centre, and E G the diameter of a semicircle that passes through B: consequently B is a right angle.

PROB. 5. To divide a given line A B into any proposed number of equal parts.

1st. *Method.* Draw any other line A C, forming any angle with the given line A B; on which set off as many of any equal parts, A D, D E, E F, F C, as the line A B is to be divided into. Join B C; parallel to which draw the other lines F G, E H, D I: then these will divide A B in the manner as required.

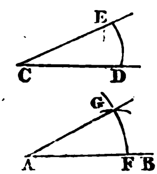


2d *Method*, without drawing parallel lines. Let A B be the line which is to be divided into n equal parts. Through one extremity A draw any right line A C, upon which set off $n + 1$ equal parts, the point D being at the termination of the $(n + 1)$th part. Join D B and produce it until the prolongation B E = B D. Let F be the termination of the $(n - 1)$th part. Join F E, and the right line of junction will cut the given line A B in the point P, such that B P = $\frac{1}{n}$ A B; and of course distances equal to B P set off upon B A, will divide it, as required.*



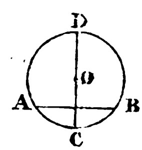
PROB. 6. At a given point A in a given line A B, to make an angle equal to a given angle C.

From the centres A and C, with any one radius, describe the arcs D E, F G. Then, with radius D E, and centre F, describe an arc, cutting F G in G. Through G draw the line A G; and it will form the angle required.



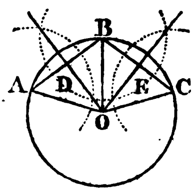
PROB. 7. To find the centre of a circle.

Draw any chord A B, and bisect it perpendicularly with the line C D. Then bisect C D in O the centre required.



* The truth of this method is easily demonstrated. Through I the intermediate point of division, on A C, between F and D, draw I B. Then, because D B = B E, and D I = I F, I B is parallel to F P. Consequently, B P : B A :: I F : I A :: 1 : n, by construction.

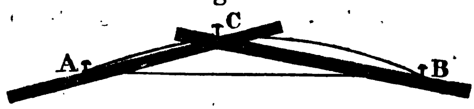
PROB. 8. To describe the circumference of a circle through three given points, A, B, C.



From the middle point B draw chords B A, B C, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the distance of any of the points, as O A, describe a circle, and it will pass through the two other points B, C, as required.

PROB. 9. On a given chord A B to describe an arc of a circle that shall contain any number of degrees; performing the operation without compasses, and without finding the centre of the circle.

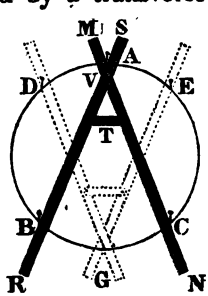
Place two rulers, forming an angle A C B, equal to the supplement of half the given number of degrees, and fix them in C. Place two pins at the extremities of the given chord, and hold a pencil in C; then move the edges of this instrument against the pins, and the pencil will describe the arc required.



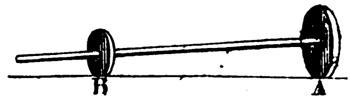
Suppose it is required to describe an arc of 50 degrees on the given chord A B; subtract 25 degrees (which is half the given angle) from 180, and the difference, 155 degrees, will be the supplement. Then form an angle A C B of 155° with the two rulers, and proceed as has been shown above.

PROB. 10. To describe mechanically the circumference of a circle through three given points, A, B, C, when the centre is inaccessible; or the circle too large to be described with compasses.

Place two rulers M N, R S, cross ways, touching the three points A B C. Fix them in V by a pin, and by a transverse piece T. Hold a pencil in A, and describe the arc B A C, by moving the angle R A N, so as to keep the outside edges of the rulers against the pins B C. Remove the instrument R V N, and on the arc described mark two points D, E, so that their distance shall be equal to the length B C. Apply the edges of the instrument against D E, and with a pencil in G describe the arc B C, which will complete the circumference of the circle required.



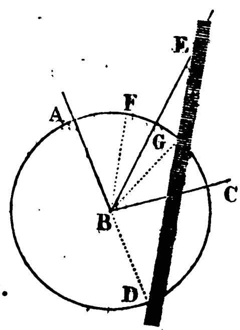
Otherwise. Let an axle of 12 or 15 inches long carry two unequal wheels A and B, of which one, A, shall be fixed, while the other, B, shall be susceptible of motion along the axle, and then placed at any assigned distance, A B, upon the paper or plane on which the circle is to be described. Then will A and B be analogous to the ends of a conic frustrum, the vertex of the complete cone being the centre of the circle which will be described by the rim, or edge, of the wheel A, as it rolls upon the proposed plane. Then, it will be, as the diameter of the wheel A, is to the difference of the diameters of A and B, so is the radius of the circle proposed to be described by A, to the distance, A B, at which the two wheels must be asunder, measured upon the plane on which the circle is to be described.



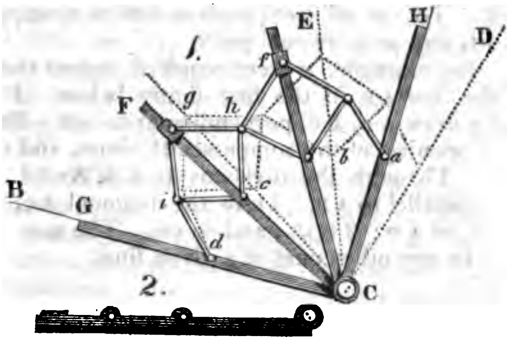
The wheel B will evidently describe, simultaneously, another circle whose radius will be less than that of the former by A B.

PROB. 11. To divide any given angle A B C into three equal parts.

From B, with any radius, describe the circle A C D A. Bisect the angle A B C by B E, and produce A B to D. On the edge of a ruler mark off the length of the radius A B. Lay the ruler on D, and move it till one of the marks on the edge intersects B E, and the other the arc A C in G. Set off the distance C G from G to F: and draw the lines B F, and B G, they will trisect the angle A B C.



Otherwise, by means of *Mr. R. Christie's* ingenious instrument for the mechanical trisection of an angle.



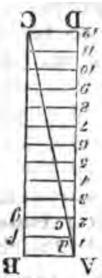
The instrument may be made either of wood or metal. Fig. 1 represents it applied to, and trisecting the angle H C B, and fig. 2 represents it shut up. The pieces H C, E C, F C, and G C are all of the same length, and moveable on the joint C. The joints *a*, *b*, *c*, and *d* are all equally distant from C. The connecting pieces *a e*, *e b*, *b h*, *h c*, *c i*, and *i d* are all equal; and the pieces *e f*, *f h*, *h g*, and *g i* are equal to each other, but longer than the preceding pieces. Two sockets, *f* and *g*, fit, and move up or down on the pieces E C and F C. The pieces are all connected by pivots at the joints, represented by the small letters *a*, *b*, *c*, *d*, &c. and the connecting pieces fit in between the other when the apparatus is shut.

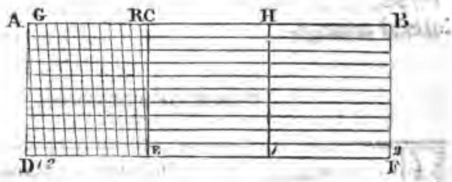
In applying this instrument it is only necessary to lay the centre C on the vertex of the given angle, C G, on one of the sides forming it, and to move H C till it coincides with the other; then each of the angles, H C E, E C F, and F C G, will be a third of the angle A C B. For it is manifest, that the angle H C E cannot be increased without increasing the angle *a e b*, and that *a e b* cannot be increased without diminishing the angle *b e f* and the distance *f b*. But because *b e* is equal to *b h*, *f e* to *f h*, and *f b* common to the two triangles *f e b* and *f h b*; the angle *f h b* must be always equal to the angle *f e b*, and consequently *b h c* to *a e b*; therefore, A C E must, in all positions of the apparatus, continue equal to E C F. In the same manner it might be shown that the angles E C F and F C G will always continue equal. Hence the angle H C B has been trisected by the straight lines E C and F C. If the instrument had been applied to the angle D C B, it would have taken the position represented by the dotted lines.

Note. It is evident that instruments may be made on the same principle to divide an angle into any other number of equal parts.

Prob. 12. To cut off from a given line A B, supposed to be very short, any proportional part.

Suppose, for example, it were required to find the $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, &c. of the line A B in the first figure below. From the ends A and B draw A D, B C, perpendicular to A B. From A to D set off any opening of the compasses 12 times, and the same from B to C. Through the divisions 1, 2, 3, &c. draw lines 1 *f*, 2 *g*, &c. parallel to A B. Draw the diagonal A C, and 1 *d* will be the $\frac{1}{12}$ of A B; 2 *c*, $\frac{2}{12}$, and so on. The same method is applicable to any other part of a given line.





Prob. 13. To make a diagonal scale, say, of feet, inches, and tenths of an inch.

Draw an indefinite line A B, on which set off from A to B the given length for one foot, any required number of times. From the divisions A, C, H, B, draw A D, C E, &c. perpendicular to A B. On A D and B F set off any length ten times; through these divisions draw lines parallel to A B. Divide A C and D E into 12 equal parts, each of which will be one inch. Draw the lines A 1, G 2, &c. and they will form the scale required: viz. each of the larger divisions from E to 1, 1 to 2, &c. will represent a foot; each of the twelve divisions between D and E, an inch; and the several perpendiculars parallel to R C in the triangle E C R, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. of an inch.

Note. If the scale be meant to represent feet, or any other unit, and *tenths* and *hundredths*, then D E must be divided into *ten* instead of *twelve* equal parts.

Prob. 14. Given the side of a regular polygon of any number of sides, to find the radius of the circle in which it may be inscribed.

Multiply the given side of the polygon by the number which stands opposite the given number of sides in the column entitled *radius of circum. circle,* the product will be the radius required.

Thus, suppose the polygon was to be an octagon, and each side 12, then 1·3065628 × 12 = 15·6687536 would be the radius sought. Take 15·67 as a radius from a diagonal scale, describe a circle, and from the same scale, taking off 12, it may be applied as the side of an octagon in that circle.

Prob. 15. Given the radius of a circle to find the side of any regular polygon (sides not exceeding 12) inscribed in it.

Multiply the given radius by the number in the column entitled *factors for sides*, standing opposite the number of the proposed polygon; the product is the side required.

Thus, suppose the radius of the circle to be 5, then 5 × 1·732051 = 8·66025, will be the side of the inscribed equilateral triangle.

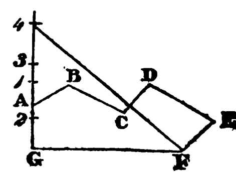
TABLE OF POLYGONS.

No. of sides.	Names.	Multipliers for areas.	Radius of circum. cir.	Factors for sides.
3	Trigon	0·4330127	0·5773503	1·732051
4	Tetragon, or Square	1·0000000	0·7071068	1·414214
5	Pentagon	1·7204774	0·8506508	1·175570
6	Hexagon	2·5980762	1·0000000	1·000000
7	Heptagon	3·6339124	1·1523824	0·867767
8	Octagon	4·8284271	1·3065628	0·765367
9	Nonagon	6·1818242	1·4619022	0·684040
10	Decagon	7·6942088	1·6180340	0·618034
11	Undecagon	9·3656399	1·7747324	0·563465
12	Dodecagon	11·1961524	1·9318517	0·517638

Prob. 16. To reduce a rectilinear figure of 6, 7, or more sides, to a triangle of equal area.

This is a very useful problem, as it saves much labour in computation.

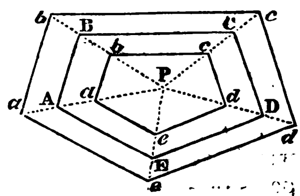
Suppose A B C D E F G to be the proposed space to be reduced to a triangle. Lay a parallel ruler from A to C, and move it until it pass through B, marking the point 1 in which it cuts A G continued. Then lay the ruler through 1 and D, and move it until it pass through C, and mark the point 2 where it cuts A G. Next lay the ruler through 2 and F, move it up till it pass through D, marking the point 3 where it cuts A G continued. Again, lay the



ruler through 3 and F, move it up until it pass through E, and mark 4 the point of intersection with G A produced. Lastly, draw the right line 4 F; so shall the triangle 4 F G be equal in area to the irregular polygon A B C D E F G.

Here B l is parallel to A C; so that if C I were drawn, the triangle A l C would be equal to A C B: and by the mechanical process this reduction is effected. In like manner, the other triangles are referred, one by one, to equal triangles, having their bases on G A or its prolongation. Hence the principle of the reduction is obvious.

PROB. 17. To reduce a simple rectilinear figure to a similar one upon either a smaller or a larger scale.



Pitch upon a point P any where about the given figure A B C D E, either within it, or without it, or in one side or angle; but near the middle is best. From that point P draw lines through all the angles; upon one of which take P *a* to P A in the proposed proportion of the scales, or linear dimensions; then draw *a b* parallel to A B, *b c* to B C, &c; so shall *a b c d e* be the reduced figure sought, either greater or smaller than the original. (*Hutton's Mens.*)

Otherwise, to Reduce a Figure by a Scale.—Measure all the sides and diagonals of the figure, as A B C D E, by a scale; and lay down the same measures respectively from another scale, in the proportion required.

To Reduce a Map, Design, or Figure, by Squares.—Divide the original into a number of little squares, and divide a fresh paper, of the dimensions required, into the same number of other squares, either greater or smaller as required. This done, in every square of the second figure, draw what is found in the corresponding square of the first or original figure.

The cross lines forming these squares, may be drawn with a pencil, and these rubbed out again after the work is finished. But a more ready and convenient way, especially when such reductions are often wanted, would be to keep always at hand frames of squares ready made, of several sizes; for by only just laying them down upon the papers, the corresponding parts may be readily copied. These frames may be made of four stiff or inflexible bars, strung across with horse hairs, or fine catgut.

When figures are rather complex, the reduction to a different scale will be best accomplished by means of such an instrument as Professor Wallace's *Eidograph*, or by means of a *Pantograph*, an instrument which is now considerably improved by simply changing the place of the fulcrum. See the *Mechanics' Oracle*, Part II. page 33.

Chap. IV.

TRIGONOMETRY.

Section I. *Plane Trigonometry.*

1. *Plane Trigonometry* is that branch of mathematics by which we learn how to determine or compute three of the six parts of a plane, or rectilinear triangle, from the other three, when that is possible.

The determination of the mutual relation of the sines, tangents, secants, &c. of the sums, differences, multiples, &c. of arcs or angles; or the investigation of the connected formulæ, is, also, usually classed under plane trigonometry.

2. Let A C B be a rectilinear angle, if about C as a centre, with any radius C A, a circle be described, intersecting C A, C B, in A, B, the arc A B is called the *measure* of the angle A C B. (See the next figure.)

3. The circumference of a circle is supposed to be divided, or to be divisible into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each of these into 60 equal parts, called *seconds;* and so on to the minutest possible subdivisions. Of these, the first is indicated by a small circle, the second by a single accent, the third by a double accent, &c. Thus, $47^\circ\ 18'\ 34''\ 45'''$, denotes 47 degrees, 18 minutes, 34 seconds, and 45 thirds. So many degrees, minutes, seconds, &c. as are contained in any arc, of so many degrees, minutes, seconds, &c. is the angle of which that arc is the measure said to be. Thus, since a quadrant, or quarter of a circle, contains 90 degrees, and a quadrantal arc is the measure of a right angle, a right angle is said to be one of 90 degrees.

4. The *complement of an arc* is its difference from a quadrant; and the *complement of an angle* is its difference from a right angle.

5. The *supplement of an arc* is its difference from a semi-

circle, and the *supplement of an angle* is its difference from two right angles.

6. The *sine* of an arc is a perpendicular let fall from one extremity upon a diameter passing through the other.

7. The *versed sine* of an arc is that part of the diameter which is intercepted between the foot of the sine and the arc.

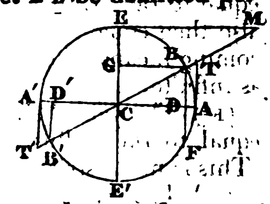
8. The *tangent* of an arc is a right line which touches it in one extremity, and is limited by a right line drawn from the centre of the circle through the other extremity.

9. The *secant* of an arc is the sloping line which thus limits the tangent.

10. These are also, by way of accommodation, said to be the sine, tangent, &c. of the angle measured by the aforesaid arc, to its determinate radius.

11. The *cosine* of an arc or angle, is the sine of the complement of that arc or angle: the *cotangent* of an arc or angle is the tangent of the complement of that arc or angle. The *co-versed sine* and *co-secant* are defined similarly.

To exemplify these definitions by the annexed diagram; let A B be an assumed arc of a circle described with the radius A C, and let A E be a quadrantal arc; let B D be demitted perpendicularly from the extremity B upon the diameter A A′; parallel to it let A T be drawn and limited by C T: let G B and E M be drawn parallel to A A′, the latter being limited by C T or C T produced. Then B E is the *complement* of B A, and angle B C E the



complement of angle B C A; B E A′ is the *supplement* of B A, and angle B C A′ the *supplement* of B C A; B D is the *sine*, D A the *versed sine*, A T the *tangent*, C T the *secant*, G B the *cosine*, G E the *coversed sine*, E M the *cotangent*, and C M the *cosecant*, of the arc A B, or, by convention, of the angle A C B.

Note. These terms are indicated by obvious contractions:

Thus, for sine of the arc A B we use		sin A B,
tangent	ditto	tan A B,
secant	ditto	sec A B,
versed sine .	ditto	versin A B,
cosine	ditto	cos A B,
cotangent ..	ditto	cot A B,
cosecant	ditto	cosec A B,
coversed sine	ditto	coversin A B.

Corollaries from the above Definitions.

12. (A.) Of any arc less than a quadrant, the arc is less than its corresponding tangent; and of any arc whatever, the chord is less than the arc, and the sine less than the chord.

(B.) The sine B D of an arc A B, is half the chord B F of the double arc B A F.

(C.) An arc and its supplement have the same sine, tangent, and secant. (The two latter, however, are affected by different signs, + or −, according as they appertain to arcs less or greater than a quadrant).

(D.) When the arc is evanescent, the sine, tangent, and versed sine, are evanescent also, and the secant becomes equal to the radius, being its minimum limit. As the arc increases from this state, the sines, tangents, secants, and versed sines increase; thus they continue till the arc becomes equal to a quadrant A E, and then the sine is in its maximum state, being equal to radius, thence called the *sine total;* the versed sine is also then equal to the radius; and the secant and tangent becoming incapable of mutually limiting each other, are regarded as infinite.

(E.) The versed sine of an arc, together with its cosine are equal to the radius. Thus, A D + B G = A D + D C = A C. (This is not restricted to arcs less than a quadrant.)

(F.) The radius, tangent, and secant, constitute a right-angled triangle C A T. The cosine, sine, and radius, constitute another right-angled triangle C D B, similar to the former. So again, the cotangent, radius, and cosecant, constitute a third right-angled triangle, M E C, similar to both the preceding. Hence, when the sine and radius are known, the cosine is determined by the property of the right-angled triangle. The same may be said of the determination of the secant, from the tangent and radius, &c. &c. &c.

(G.) Further, since C D : D B :: C A : A T, we see that the tangent is a fourth proportional to the cosine, sine, and radius.

Also, C D : C B :: C A : C T; that is, the secant is a third proportional to the cosine and radius.

Again, C G : G B :: C E : E M; that is, the cotangent is a fourth proportional to the sine, cosine, and radius.

And, B D : B C :: C E : C M; that is, the cosecant is a third proportional to the sine and radius.

(H.) Thus, employing the usual abbreviations, we should have

1. $\cos = \sqrt{(\text{rad}^2 - \sin^2)}$. 2. $\tan = \sqrt{(\sec^2 - \text{rad}^2)}$.
3. $\sec = \sqrt{(\text{rad}^2 + \tan^2)}$. 4. $\text{cosec} = \sqrt{(\text{rad}^2 + \cot^2)}$.
5. $\tan = \frac{\text{rad} \times \sin}{\cos} = \frac{\text{rad}^2}{\cot}$. 6. $\cot = \frac{\text{rad} \times \cos}{\sin} = \frac{\text{rad}^2}{\tan}$.
7. $\sec = \frac{\text{rad}^2}{\cos}$. 8. $\text{cosec} = \frac{\text{rad}^2}{\sin}$.

These, when unity is regarded as the radius of the circle, become

1. $\cos = \sqrt{(1 - \sin^2)}$. 2. $\tan = \sqrt{(\sec^2 - 1)}$.
3. $\sec = \sqrt{(1 + \tan^2)}$. 4. $\text{cosec} = \sqrt{(1 + \cot^2)}$.
5. $\tan = \frac{\sin}{\cos} = \frac{1}{\cot}$. 6. $\cot = \frac{\cos}{\sin} = \frac{1}{\tan}$. 7. $\sec = \frac{1}{\cos}$.
8. $\text{cosec} = \frac{1}{\sin}$.

13. From these, and other properties and theorems, mathematicians have computed the lengths of the sines, tangents, secants, and versed sines, to an assumed radius, that correspond to arcs from 1 second of a degree, through all the gradations of magnitude, up to a quadrant, or 90°. The results of the computations are arranged in tables called *Trigonometrical Tables* for use. The arrangement is generally appropriated to two distinct kinds of these artificial numbers, classed in their regular order upon pages that face each other. On the left-hand pages are placed the sines, tangents, secants, &c. adapted at least to every degree, and *minute*, in the quadrant, computed to the radius 1, and expressed decimally. On the right-hand pages are placed in succession the corresponding *logarithms* of the numbers that denote the several sines, tangents, &c. on the respective opposite pages. Only, that the necessity of using negative indices in the logarithms may be precluded, they are supposed to be the logarithms of sines, tangents, secants, &c. computed to the radius 10000000000. The numbers thus computed and placed on the successive right-hand pages are called *logarithmic* sines, tangents, &c. The numbers of which these are the logarithms, and which are arranged on the left-hand pages, are called *natural* sines, tangents, &c.

II. *General Properties and Mutual Relations.*

1. The chord of any arc is a mean proportional between the versed sine of that arc and the diameter of the circle.

2. As radius, to the cosine of any arc; so is twice the sine of that arc, to the sine of double the arc.

3. The secant of any arc is equal to the sum of its tangent, and the tangent of half its complement.

4. The sum of the tangent and secant of any arc, is equal to the tangent of an arc exceeding that by half its complement. Or, the sum of the tangent and secant of an arc is equal to the tangent of 45° *plus* half the arc.

5. The chord of 60° is equal to the radius of the circle; the versed sine and cosine of 60° are each equal to half the radius, and the secant of 60° is equal to double the radius.

6. The tangent of 45° is equal to the radius.

7. The square of the sine of half any arc or angle is equal to a rectangle under half the radius and the versed sine of the whole; and the square of its cosine, equal to a rectangle under half the radius and the versed sine of the supplement of the whole arc or angle.

8. The rectangle under the radius and the sine of the sum or of the difference of two arcs is equal to the sum or the difference of the rectangles under their alternate sines and cosines.

9. The rectangle under the radius and the cosine of the sum or the difference of two arcs, is equal to the difference or the sum of the rectangles under their respective cosines and sines.

10. As the difference or sum of the square of the radius and the rectangle under the tangents of two arcs, is to the square of the radius; so is the sum or difference of their tangents, to the tangent of the sum or difference of the arcs.

11. As the sum of the sines of two unequal arcs, is to their difference; so is the tangent of half the sum of those two arcs to the tangent of half their difference.

12. Of any three equidifferent arcs, it will be as radius, to the cosine of their common difference, so is the sine of the mean arc, to half the sum of the sines of the extremes; and, as radius to the sine of the common difference, so is the cosine of the mean arc to half the difference of the sines of the two extremes.

(A.) If the sine of the mean of three equidifferent arcs (radius being unity) be multiplied into twice the cosine of the common difference, and the sine of either extreme be deducted from the product, the remainder will be the sine of the other extreme.

(B.) The sine of any arc above 60°, is equal to the sine of another arc as much below 60°, together with the sine of its excess above 60°.

Remark. From this latter proposition, the sines below 60° being known, those of arcs above 60° are determinable by addition only.

13. In any right-angled triangle, the hypothenuse is to one of the legs, as the radius to the sine of the angle opposite to that leg; and one of the legs is to the other, as the radius to the tangent of the angle opposite to the latter.

14. In any plane triangle, as one of the sides is to another, so is the sine of the angle opposite to the former to the sine of the angle opposite to the latter.

15. In any plane triangle it will be, as the sum of the sides about the vertical angle, is to their difference, so is the tangent of half the sum of the angles at the base, to the tangent of half their difference.

16. In any plane triangle it will be, as the cosine of the difference of the angles at the base, is to the cosine of half their sum, so is the sum of the sides about the vertical angles to the third side. Also, as the sine of half the difference of the angles at the base, is to the sine of half their sum, so is the difference of the sides about the vertical angle to the third side, or base.*

17. In any plane triangle it will be, as the base, to the sum of the two other sides, so is the difference of those sides, to the difference of the segments of the base made by a perpendicular let fall from the vertical angle.

18. In any plane triangle it will be, as twice the rectangle under any two sides, is to the difference of the sum of the squares of those two sides and the square of the base, so is the radius to the cosine of the angle contained by the two sides.

Cor. When unity is assumed as radius, then if AC, AB, BC, are the sides of a triangle, this prop. gives $\cos C = \frac{AC^2 + BC^2 - AB^2}{2CB \cdot CA}$: and similar expressions for the other angles.

* These propositions were first given by *Thacker* in his *Mathematical Miscellany*, published in 1743; their practical utility has been recently shown by *Professor Wallace*, in the *Edinburgh Philosophical Transactions*.

19. As the sum of the tangents of any two unequal angles, is to their difference, so is the sine of the sum of those angles, to the sine of their difference.

20. As the sine of the difference of any two unequal angles, is to the difference of their sines, so is the sum of those sines, to the sine of the sum of the angles.

These and other propositions are the foundation of various formulæ, for which the reader who wishes to pursue the enquiry may consult the best treatises on Trigonometry.

III. *Solution of the Cases of Plane Triangles.*

Although the three sides and three angles of a plane triangle, when combined three and three, constitute twenty varieties, yet they furnish only *three* distinct cases in which separate rules are required.

CASE I.

When a side and an angle are two of the given parts.

The solution may be effected by prop. 14 of the preceding section, wherein it is affirmed that the sides of plane triangles are respectively proportional to the sines of their opposite angles.

In practice, if a *side* be required, begin the proportion with a sine, and say,

As the sine of the given angle,
To its opposite side;
So is the sine of either of the other angles,
To its opposite side.

If an *angle* be required, begin the proportion with a side, and say,

As one of the given sides,
Is to the sine of its opposite angle;
So is the other given side,
To the sine of its opposite angle.

The third angle becomes known by taking the sum of the two former from 180°.

Note 1. Since sines are *lines*, there can be no impropriety in comparing them with the sides of triangles; and the rule is better remembered by young mathematicians, than when the sines and sides are compared each to each.

Note 2. It is usually, though not always, best to work the proportions in trigonometry by means of the logarithms, taking the logarithm of the *first* term from the sum of the logarithms of the *second* and *third*, to obtain the logarithm of the *fourth* term. Or, adding the *arithmetical complement* of the logarithm of the first term to the logarithms of the other two, to obtain that of the fourth.

Case II.

When two sides and the included angle are given,

The solution may be effected by means of props. 15 and 16 of the preceding section.

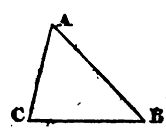
Thus: take the given angle from 180°, the remainder will be the sum of the other two angles.

Then say,—As the sum of the given sides,
Is to their difference;
So is the tangent of half the sum of the remaining angles,
To the tangent of half their difference.

Then, secondly say,—As the cosine of half the said difference,
Is to the cosine of half the sum of the angles;
So is the sum of the given sides,
To the third, or required side.

Or, As the sine of half the diff. of the angles,
Is to the sine of half their sum;
So is the difference of the given sides,
To the third side.

Example. In the triangle ABC are given AC = 450, BC = 540, and the included angle C = 80°; to find the third side, and the two remaining angles.



Here BC + AC = 990, BC − AC = 90, 180° − C = 100° = A + B.

Hence, BC + AC	990	Log. =	2·9956352
To BC − AC	90	Log. =	1·9542425
So is tan $\frac{1}{2}$(A + B)	50°	Log. =	10·0761865
So tan $\frac{1}{2}$(A − B)	6° 11′	Log. =	9·0347938

Cos $\frac{1}{2}$ (A − B)	6° 11′	Log. =	9·9974660
Cos $\frac{1}{2}$ (A + B)	50°	Log. =	9·8080675
So is BC + AC	990	Log. =	2·9956352
To AB	640·08	Log. =	2·8062367

Also $\frac{1}{2}$ (A + B) + $\frac{1}{2}$ (A − B) = 56° 11 = A; and $\frac{1}{2}$ (A + B) − $\frac{1}{2}$ (A − B) = 43° 49′ = B.

Here, much time will be saved in the work by taking cos $\frac{1}{2}$ (A + B) from the tables, at the same time with tan $\frac{1}{2}$ (A + B); and cos $\frac{1}{2}$ (A − B) as soon as tan $\frac{1}{2}$ (A − B) is found. Observe, also, that the log. of BC + AC is the same in the second operation as in the first. Thus the tables need only be opened in *five* places for both operations.

Another solution to Case II.

Supposing C to be the given angle, and CA, CB, the given sides; then the third side may be found by this theorem: viz.

$$AB = \sqrt{(AC^2 + BC^2 - 2\,AC \,.\, CB \,.\, \cos C)}.$$

Thus, taking AC = 450, BC = 540, C = 80°, its cos ·1736482

$$AB = \sqrt{(450^2 + 540^2 - 2 \,.\, 450 \,.\, 540 \times \cdot 1736482)}$$
$$= \sqrt{[90^2 (5^2 + 6^2 - 2 \,.\, 5 \,.\, 6 \times \cdot 1736482)]}$$
$$= 90 \sqrt{50 \cdot 58118} = 90 \times 7 \cdot 112 = 640 \cdot 08, \text{ as before.}$$

Case III.

10. When the three sides of a plane triangle are given, to find the angles.

1st *Method.* Assume the longest of the three sides as base, then say, conformably with prop. 16,

As the base,
To the sum of the two other sides;
So is the difference of those sides,
To the difference of the segments of the base.

Half the base added to the said difference, gives the greater segment, and made less by it gives the less; and thus, by means of the perpendicular from the vertical angle, divides the original triangle into two, each of which falls under the first case.

2d *Method.* Find any one of the angles by means of prop. 18, of the preceding section; and the remaining angles either by a repetition of the same rule, or by the relation of sides to the sines of their opposite angles.

Thus, $\cos C = \frac{AC^2 + BC^2 - AB^2}{2AC \cdot BC}$; $\cos B = \frac{AB^2 + BC^2 - AC^2}{2AB \cdot BC}$

and $\cos A = \frac{BA^2 + AC^2 - BC^2}{2AB \cdot AC}$.

Right-angled Plane Triangles.

1. Right-angled triangles may, as well as others, be solved by means of the rule to the respective case under which any specified example falls: and it will then be found, since a right angle is always one of the data, that the rule usually becomes simplified in its application.

2. When two of the sides are given, the third may be found by means of the property in *Plane Geom. Triangles*, prop. 16.

$$\text{Hypoth.} = \sqrt{(\text{base}^2 + \text{perp.}^2)}$$

$$\text{Base} = \sqrt{(\text{hyp.}^2 - \text{perp.}^2)} = \sqrt{(\text{hyp.} + \text{perp.}) \cdot (\text{hyp.} - \text{perp.})}$$

$$\text{Perp.} = \sqrt{(\text{hyp.}^2 - \text{base}^2)} = \sqrt{(\text{hyp.} + \text{base}) \cdot (\text{hyp.} - \text{base})}.$$

3. There is another method for right-angled triangles, known by the phrase *making any side radius*; which is this.

"*To find a side.* Call any one of the sides radius, and write upon it the word *radius*; observe whether the other sides become sines, tangents, or secants, and write those words upon them accordingly. Call the word written upon each side the *name* of each side: then say,

As the *name* of the given side,
Is to the given side;
So is the *name* of the required side,
To the required side."

"*To find an angle.* Call either of the *given* sides radius, and write upon it the word *radius*; observe whether the other sides become sines, tangents, or secants, and write those words on them accordingly. Call the word written upon each side the *name* of that side. Then say,

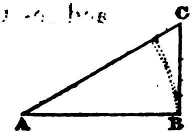
As the side made radius,
Is to radius;
So is the other given side,
To the *name* of that side,

which determines the opposite angle."

4. When the numbers which measure the sides of the triangle are either under 12, or resolvable into factors which are each less than 12, the solution may be obtained, conformably

with this rule, easier without logarithms than with them. For,

Let A B C be a right-angled triangle, in which A B the base is assumed to be radius; B C is the tangent of A, and A C its secant, to that radius; or, dividing each of these by the base, we shall have the tangent and secant of A, respectively, to radius 1. Tracing in like manner, the consequences of assuming B C, and A C, each for radius, we shall readily obtain these expressions.



1. $\frac{\text{perp.}}{\text{base}}$ = tan angle at base.

2. $\frac{\text{base}}{\text{perp.}}$ = tan angle at vertex.

3. $\frac{\text{hyp.}}{\text{base}}$ = sec angle at base.

4. $\frac{\text{hyp.}}{\text{perp.}}$ = sec angle at vertex.

5. $\frac{\text{perp.}}{\text{hyp.}}$ = sin angle at base.

6. $\frac{\text{base}}{\text{hyp.}}$ = sin angle at vertex.

SECTION II. *On the Heights and Distances of Objects.*

The instruments employed to measure angles are quadrants, sextants, theodolites, &c. the use of either of which may be sooner learnt from an examination of the instruments themselves than of any description independently of them. For military men and for civil engineers, a good pocket sextant, and an accurate micrometer (such as Cavallo's), attached to a telescope, are highly useful. For measuring small distances, as bases, 50 feet and 100 feet chains, and a portable box of graduated tape, will be necessary.

We shall here present a selection of such examples as are most likely to occur.

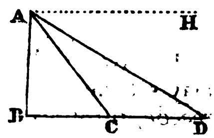
EXAMPLE I.

In order to find the distance between two trees A and B, which could not be directly measured because of a pool which occupied much of the intermediate space, I measured the distance of each of them from a third object C, viz. A C = 588, B C = 672, and then at the point C took the angle A C B between the two trees = 55° 40′. Required their distance.

This is an example to case 2 of plane triangles, in which two sides, and the included angle are given. The work, therefore, may exercise the student: the answer is 593·8.

Example II.

Wanting to know the distance between two inaccessible objects, which lay in a direct line from the bottom of a tower on whose top I stood, I took the angles of *depression* of the two objects, viz. of the most remote $25\frac{1}{2}$°, of the nearest 57°. What is the distance between them, the height of the tower being 120 feet?



The figure being constructed, as in the margin, A B = 120 feet, the altitude of the tower, and A H the horizontal line drawn through its top; there are given,

H A D = 25° 30′, hence B A D = B A H − H A D = 64° 30′.
H A C = 57° 0′, hence B A C = B A H − H A C = 33° 0′.

Hence the following calculation, by means of the *natural* tangents. For, if A B be regarded as radius, B D and B C will be the tangents of the respective angles B A D, B A C, and C D the difference of those tangents. It is, therefore, equal to the product of the difference of the natural tangents of those angles into the height A B.

Thus, nat. tan $64\frac{1}{2}$° =	2·0965436
nat. tan 33° =	0·6494076
difference	1·4471360
multiplied by height . .	120
gives distance C D . . .	173·6563200

Example III.

Standing at a measureable distance, on a horizontal plane, from the bottom of a tower, I took the angle of elevation of the top; it is required from thence to determine the height of the tower.

In this case, there would be given A B and the angle A (see the figure in *Right-angled Triangles*), to find B C = A B × tan A.

By logarithms, when the numbers are large, it will be, log. B C = log. A B + log. tan A.

Note. If angle	A = 11°	19′ then	B C = $\frac{1}{5}$ A B	very nearly.
	A = 16	42	B C = $\frac{3}{10}$ A B	....
	A = 21	48	B C = $\frac{2}{5}$ A B	....
	A = 26	34	B C = $\frac{1}{2}$ A B	....
	A = 30	58	B C = $\frac{3}{5}$ A B	....
	A = 35	0	B C = $\frac{7}{10}$ A B	....
	A = 38	40	B C = $\frac{4}{5}$ A B	....
	A = 45°	....	B C = A B,	*exactly.*

To save the time of computation, therefore, the observer may set the instrument to one of these angles, and advance or recede, till it accords with the angle of elevation of the object: its height above the horizontal level of the observer's eye, will at once be known, by taking the appropriate fraction of the distance A B.

Example IV.

"Wanting to know the height of a church steeple, to the bottom of which I could not measure on account of a high wall between me and the church, I fixed upon two stations at the distance of 93 feet from each other, on a horizontal line from the bottom of the steeple, and at each of them took the angle of elevation of the top of the steeple, that is, at the nearest station 55° 54′, at the other 33° 20′. Required the height of the steeple.

Recurring to the figure of Example II, we have given the distance C D, and the angles of elevation at C and D. The quickest operation is by means of the natural tangents, and the theorem $\text{A B} = \dfrac{\text{C D}}{\cot \text{D} - \cot \text{C}}$.

$$\text{Thus } \cot D = \cot 33^\circ\ 20' = 1{\cdot}5204261$$
$$\cot C = \cot 55\ \ 54 = {\cdot}6770509$$

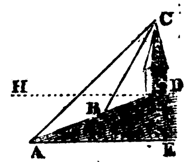
$$\text{Their difference} = {\cdot}8433752$$

$$\text{Hence, } AB = \frac{93}{{\cdot}8433752} = 110{\cdot}27 \text{ feet.}$$

Example V.

Wishing to know the height of an obelisk standing at the top of a regularly sloping hill, I first measured from its bottom a distance of 36 feet, and there found the angle formed by the inclined plane and a line from the centre of the instrument to the top of the obelisk 41°; but after measuring on downward in the same sloping direction 54 feet farther, I found the angle formed in like manner to be only 23° 45′. What was the height of the obelisk, and what the angle made by the sloping ground with the horizon?

The figure being constructed, as in the margin, there are given in the triangle ACB, all the angles and the side AB, to find BC. It will be obtained by this proportion, as sin C (= 17° 15′ = B − A): AB (= 54) :: sin A (= 23° 45′) : BC = 73·3392. Then, in the triangle DBC are known BC as above, BD = 36, CBD = 41°; to find the other angles, and the side CD. Thus, first, as CB + BD : CB − BD :: tan ½ (D + C) = ½ (139°) : tan ½ (D − C) = 42° 24½′. Hence 69° 30′ + 42° 24½′ = 112° 54½° = CDB, and 69° 30′ − 42° 24½′ = 26° 5½′ = BCD. Then, sin BCD : BD :: sin CBD : CD = 51·86, height of the obelisk.

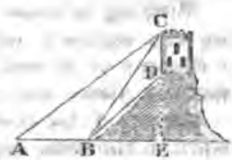


The angle of inclination DAE = HDA = CDB − 90° = 22° 54½′.

Remark. If the line BD cannot be measured, then the angle DAE of the sloping ground must be taken, as well as the angles CAB, and CBD. In that case DAE + 90° will be equal to CDB: so that after CB is found from the triangle ACB, CD may be found in the triangle CBD, by means of the relation between sides and the sines of their opposite angles.

Example VI.

Being on a horizontal plane, and wanting to ascertain the height of a tower standing on the top of an inaccessible hill, I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°, then measuring in a direct line 180 feet farther from the hill, I took in the same vertical plane the angle of elevation of the top of the tower 33° 45′. Required from hence the height of the tower.

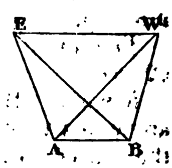


The figure being constructed, as in the margin, there are given, A B = 180, C A B = 33° 45′, A C B = C B E − C A E = 17° 15′, C B D = 11°, B D C = 180° − (90° − D B E) = 130°. And C D may be found from the expression C D rad² = A B sin A sin C B D cosec A C B sec D B E.

Or, using the logarithms, it will be log. A B + log sin A + log sin B + log cosec A C B + log. sec D B E − 40 (in the index) = log C D; in the case proposed = log. of 83·9983 feet.

Example VII.

In order to determine the distance between two inaccessible objects E and W, on a horizontal plane, we measured a convenient base A B of 536 yards, and at the extremities A and B took the following angles, viz. B A W = 40° 16′, W A E = 57° 40′, A B E = 42° 22′, E B W = 71° 7′ Required the distance E W.

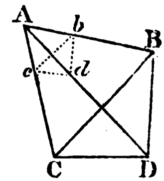


First, in the triangle A B E are given all the angles, and the side A B, to find B E. So, again, in the triangle A B W, are given all the angles and A B to find B W. Lastly, in the triangle B E W are given the two sides E B, B W, and the included angle E B W, to find E W = 959·52 yards.

Remark. In like manner the distances taken two and two, between any number of remote objects posited around a convenient station line, may be ascertained.

EXAMPLE VIII.

Suppose that in carrying on an extensive survey, the distance between two spires A and B has been found equal to 6594 yards, and that C and D are two eminences conveniently situated for extending the triangles, but not admitting of the determination of their distance by actual admeasurement: to ascertain it, therefore, we took at C and D the following angles, viz.



$\begin{cases} ACB = 85^\circ\, 46' \\ BCD = 23^\circ\, 56' \end{cases}$ $\begin{cases} ADC = 31^\circ\, 48' \\ ADB = 68^\circ\ \ 2' \end{cases}$

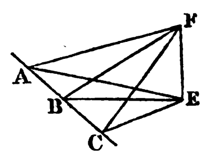
Required CD from these data.

In order to solve this problem, construct a similar quadrilateral A $c\ d\ b$, assuming $c\ d$ equal to 1, 10, or any other convenient number: compute A b from the given angles, according to the method of the preceding example. Then, since the quadrilaterals A $c\,d\,b$, A C D B, are similar, it will be, as A b : $c\,d$ :: AB : CD; and CD is found = 4694 yards.

EXAMPLE IX.

Given the angles of elevation of any distant object, taken at three places in a horizontal right line, which does not pass through the point directly below the object; and the respective distances between the stations; to find the height of the object, and its distance from either station.

Let ABC be the horizontal plane, FE the perpendicular height of the object above that plane, A, B, C, the three places of observation, FAE, FBE, FCE, the angles of elevation, and AB, BC, the given distances. Then, since the triangles AEF, BEF, CEF, are all right angled at E, the distances AE, BE, CE, will manifestly be as the cotangents of the angles of elevation at A, B, and C



Put AB = D, BC = d, EF = x, and then express algebraically the theorem given in Geom. Triangles, 25, which in this case becomes,

$$AE^2 . BC + CE^2 . AB = BE^2 . AC + AC . AB . BC.$$

The resulting equation is

$$d x^2 \cot^2 A + D x^2 \cot^2 C = (D + d) x^2 \cot^2 B + (D + d) D d.$$

From which is readily found

$$x = \sqrt{\frac{(D + d) D d}{d \cot^2 A + D \cot^2 C - (D + d) \cot^2 B}}.$$

Thus E F becoming known, the distances A E, B E, C E, are found, by multiplying the cotangents of A, B, and C, respectively, by E F.

Remark. When $D = d$, or $D + d = 2D = 2d$, that is, when the point B is midway between A and C, the algebraic expression becomes,

$$x = d \div \sqrt{(\tfrac{1}{2} \cot^2 A + \tfrac{1}{2} \cot^2 C - \cot^2 B)},$$

which is tolerably well suited for logarithmic computation. The rule may, in that case, be thus expressed.

Double the log. cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering to twice the log. cotangent of the middle angle of elevation: then half the log. of this remainder subtracted from the log. of the measured distance between the first and second, or the second and third stations, will be the log. of the height of the object.

The distance from either station will be found as above.

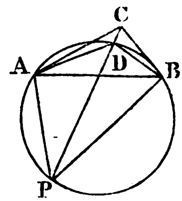
Note. The case explained in this example, is one that is highly useful, and of frequent occurrence. An analogous one is, when the angles of elevation of a remote object are taken from the three angles of a triangle on a horizontal plane, the sides of that triangle being known, or measurable: but the above admits of a simpler computation, and may usually be employed.

Example X.

From a convenient station P, where could be seen three objects A, B, and C, whose distances from each other were known (viz. A B = 800, A C = 600, B C = 400 yards), I took the horizontal angles A P C = 33° 45′, B P C = 22° 30′. It is hence required to determine the respective distances of my station from each object.

Here it will be necessary, as preparatory to the computation, to describe the manner of

Construction. Draw the given triangle A B C from any convenient scale. From the point A draw a line A D to make with A B an angle equal to 22° 30′ and from B a line B D to make an angle D B A = 33° 45′. Let a circle be described to pass through their intersection D, and through the points A and B. Through C and D draw a right line to meet the circle again in P: so shall P be the point required. For, drawing P A, P B, the angle A P D is evidently = A B D, since it stands on the same arc A D: and for a like reason B P D = B A D. So that P is the point where the angles have the assigned value.



The result of a careful construction of this kind, upon a good sized scale, will give the values of P A, P C, P B, true to within the 200dth part of each.

Manner of Computation. In the triangle A B C where the sides are known, find the angles. In the triangle A B D, where all the angles are known, and the sides A B, find one of the other sides A D. Take B A D from B A C, the remainder, D A C is the angle included between two known sides AD, AC; from which the angles A D C and A C D may be found, by chap. iii. case 2. The angle C A P = 180° − (A P C + A C D). Also, B C P = B C A − A C D; and P B C = A B C + P B A = A B C + sup. A D C. Hence, the three required distances are found by these proportions. As sin A P C : A C :: sin P A C : P C, and :: sin P C A : P A; and lastly, as sin B P C : B C :: sin B C P : B P. The results of the computation are, P A = 709·33, P C = 1042·66, P B = 934 yards.

*** The computation of problems of this kind, however, may be a little shortened by means of an analytical investigation. Those who wish to pursue this department of trigonometry may consult the treatises by *Bonnycastle*, *Gregory*, *and Woodhouse*.

Note. If C had been *below* A B, the general principles of construction and computation would be the same; and the modification in the process very obvious.

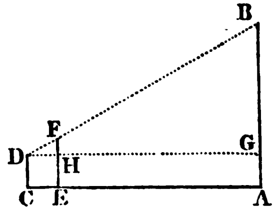
II. *Determination of Heights and Distances by approximate mechanical methods.*

1. *For Heights.*

1. *By shadows*, when the sun shines. Set up vertically a staff of known length, and measure the length of its shadow

upon a horizontal or other plane; measure also the length of the shadow of the object whose height is required. Then it will be, as the length of the shadow of the staff, is to the length of the staff itself; so is the length of the shadow of the object, to the object's height.

2. *By two rods or staves set up vertically.* Let two staves, one say of 6 feet, the other of 4 feet long, be placed upon horizontal circular or square feet, on which each may stand steadily. Let A B be the object, as a tower or steeple, whose altitude is required, and A C the horizontal plane passing through its base. Let C D and E F, the two rods, be placed with their bases in one and the same line C A, passing through A the foot of the object; and let them be moved nearer to, or farther from, each other, until the summit B of the object is seen in the same line as D and F, the tops of the rods. Then by the principle of similar triangles, it will be, as D H (= C E) : F H :: D G (= C A) : B G; to which add A G = C D, for the whole height A B.



3. *By Reflection.* Place a vessel of water upon the ground, and recede from it, until you see the top of the object reflected from the smooth surface of the liquid. Then, since by a principle in optics, the angles of incidence and reflection are equal, it will be as your distance measured horizontally from the point at which the reflection is made, is to the height of your eye above the reflecting surface; so is the horizontal distance of the foot of the object from the vessel, to its altitude above the said surface.*

4. *By means of a portable barometer and thermometer.* Observe the altitude, B, of the mercurial column, in inches, tenths, and hundredths, at the *bottom* of the hill, or other object whose altitude is required; observe, also, the altitude, *b*, of the mercurial column at the *top* of the object; observe the temperatures on Fahrenheit's thermometer, at the times of the two barometrical observations, and take the mean between them. Then $55000 \times \frac{B - b}{B + b}$ = height of the hill, in feet, for the temperature of 55° on Fahrenheit. Add $\frac{1}{440}$ of this result for

* *Leonard Digges*, in his curious work the *Pantometria*, published in 1571, first proposed a method for the determination of altitudes by means of a geometrical square and plummet, which has been described by various later authors, as Ozanam, Donn, Hutton, &c. But, as it does not seem preferable to the methods above given, I have not repeated it here.

every degree which the mean temperature exceeds 55°; subtract as much for every degree below 55°.

This will be a good approximation when the height of the hill is below 2000; and it is easily remembered, because 55°, the assumed temperature, agree with 55, the effective figures in the coefficient; while the effective figures in the denominator of the correcting fraction are two *fours*.

*** Where great accuracy is required logarithmic rules become necessary, of which various are exhibited in treatises on Pneumatics.

5. *By an extension of the principle of pa.* 143. Set the sextant, or other instrument, to the angle 45°, and find the point C (pa. 142.) on the horizontal plane, where the object A B has that elevation: then set the instrument to 26° 34′, and recede from C, in direction B C D, till the object has that elevation. *The distance* C D *between the two stations will be* = A B

So, again, if C = 40°, D = 24° 31½′, C D will be = A B.
or, if C = 35°, D = 22° 33′, C D = A B.
or, if C = 30°, D = 20° 6′, C D = A B.
or, generally, if cot D − cot C = rad. C D = A B.

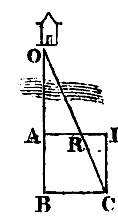
6. *For deviation from level.* Let E represent the elevation of the tangent line to the earth above the true level, in feet and parts of a foot, D the distance in miles: then $E = \frac{2}{3} D^2$.

This gives 8 inches for a distance of one mile; and is a near approximation when the distance does not exceed 2 or 3 miles.

2. *For Distances.*

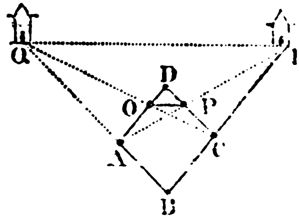
1. *By means of a rhombus set off upon a horizontal plane.*

Suppose O the object and O B the required distance. With a line or measuring tape, whose length is equal to the side of the intended rhombus, say 50 or 100 feet, lay down one side B A in the direction B O towards the object, and B C another side in any convenient direction (for whether B be a right angle, or not, is of no consequence); and put up rods or arrows at A and C. Then fasten two ends of two such lines at A and C, and extend them until the two other ends just meet together at D; let them lie thus stretched upon the ground, and they will form the two other sides of the rhombus A D, C D. Fix a mark or arrow at R, directly between C and O, upon the line A D; and measure R D, R A upon the tape. Then it will be as R D : D C :: C B : B O, the required distance.



Otherwise. To find the length of the inaccessible line Q R.

At some convenient point B, lay down the rhombus B A D C, so that two of its sides B A, B C, are directed to the extremities of the line Q R. Mark the intersections, O and P, of A R, C Q, with the sides of the rhombus (as in the former method): then the triangle O D P will be similar to the triangle R B Q; and the inaccessible distance R Q will be found $= \frac{OP \times BA^2}{OD \times DP}$.*



Thus, if B A = B C, &c. = 100 f. O D = 9 f. 5 in., D P = 11 f. 10 in. O P = 13 f. 7 in. then $QR = \frac{10000 \times 13\frac{7}{12}}{9\frac{5}{12} \times 11\frac{10}{12}} =$ 1219 feet.

2. *By means of a micrometer attached to a telescope.*

Portable instruments for the purpose of measuring extremely small angles, have been invented by Martin, Cavallo, Dollond, Brewster, and others. In employing them for the determination of distances, all that is necessary in the practice is to measure the angle subtended by an object of known dimensions, placed either vertically or horizontally, at the remoter extremity of the line whose length we wish to ascertain. Thus, if there be a house, or other erection, built with bricks, of the usual size; then four courses in *height* are equal to *a foot*, and four in *length* equal to *a yard*: and distances measured by means of these will be tolerably accurate, if care be taken with regard to the angle subtended by the horizontal object, to stand directly in front of it. A man, a carriage wheel, a window, a door, &c. at the remoter extremity of the distance we wish to ascertain, may serve for an approximation. But in all cases where it is possible, let a foot, a yard, or a six-feet measure, be placed vertically, at one end of the line to be measured, while the observer with his micrometer stand at the other. Then, if h be the height of the object,

either $\frac{1}{2} h \times \cot \frac{1}{2}$ angle subtended

or $h \times \cot$ angle

* For $PD : DA :: AB : BR = \frac{AB^2}{PD}$;

and $OD : OP :: BR : RQ = \frac{AB^2 . OP}{OD . DP}$.

will give the distance, according as the eye of the observer is horizontally opposite to the *middle*, or to *one extremity* of the object whose angle is taken.

When a table of natural tangents is not at hand, a *very* near approximation for all angles less than *half a degree*, and a tolerably near one up to angles of *a degree*, will be furnished by the following rules.

1. If the distant object whose angle is taken be 1 foot in length, then

$$\left.\begin{array}{l}3437{\cdot}73 \div \text{the angle in minutes}\\ \text{or } 206264{\cdot} \div \text{the angle in seconds}\end{array}\right\}\text{ will give the distance in feet.}$$

2. If the remote object by 3, 6, 9, &c. feet in length, multiply the former result by 3, 6, 9, &c. respectively.

Ex. 1. What is the distance of a man 6 feet high, when he subtends an angle of 30 seconds?

$206264 \times 6 \div 30 = 206264 \div 5 = 41252{\cdot}8$ feet $=$ $13750{\cdot}9$ yards, the distance required.

Ex. 2. In order to ascertain the length of a street, I put up a foot measure at one end of it, and standing at the other found that measure to subtend an angle of 2 minutes: required the length of the street.

$$3437{\cdot}73 \div 2 = 1718{\cdot}86 \text{ feet} = 572{\cdot}95 \text{ yards.}$$

3. *By means of the velocity of sound.*

Let a gun be fired at the remoter extremity of the required distance, and observe by means of a chronometer that measures tenths of seconds, the interval that elapses between the flash and the report: then estimate the distance for one second by the following rule, and multiply that distance by the observed interval of time; the product will give the whole distance required.

At the temperature of frèezing, 33°, the velocity of sound is 1100 feet per second.

$$\left.\begin{array}{l}\text{For lower temperatures deduct}\\ \text{For higher temperatures add}\end{array}\right\}\text{half a foot.}$$

$$\left.\begin{array}{l}\text{From the } 1100\\ \text{to the } 1100\end{array}\right\}\text{for every degree of difference from } 33^\circ$$

on Fahr. therm.; the result will show the velocity of sound, very nearly, at all such temperatures.

Thus, at the temperature of 50°, the velocity of sound is,

$$1100 \times \tfrac{1}{2}(50-33) = 1108\tfrac{1}{2} \text{ feet.}$$

At temperature 60°, it is $1100 + \frac{1}{2}(60-33) = 1113\frac{1}{2}$ feet.

CHAP. V.

CONIC SECTIONS.

1. *Conic Sections* are the figures made by a plane cutting a cone.

2. According to the different positions of the cutting plane there arise five different figures or sections, viz. a triangle, a circle, an ellipsis, an hyperbola, and a parabola: of which the three last are peculiarly called *Conic Sections.*

3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will be a triangle.

4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle.

5. The section is an ellipse when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.

6. The section is a parabola, when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base.

7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.

8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former.

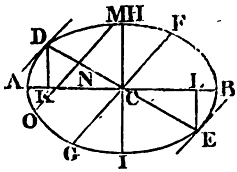
9. The vertices of any section, are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section; as A and B, in the figs. below.

Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

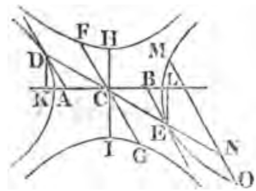
10. The major axis, or transverse diameter, of a conic section, is the line or distance A B between the vertices.

Hence the axis of a parabola is infinite in length, A b being only a part of it.

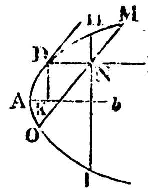
Ellipse.



Hyperbolas.



Parabola.



11. The centre C is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve; but of an hyperbola, without.

12. A diameter is any right line, as A B or D E, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. Hence also every diameter of the ellipse and hyperbola has two vertices: but of the parabola, only one; unless we consider the other as at an infinite distance.

13. The conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, F G, parallel to the tangent at D, is the conjugate to D E; and H I, parallel to the tangent at A, is the conjugate to A B.

Hence the conjugate H I, of the axis A B, is perpendicular to it, and is often called the minor axis.

14. An *ordinate* to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So D K and E L are ordinates to the axis A B; and M N and N O ordinates to the diameter D E.—Hence the ordinates of the axes are perpendicular to it; but of other diameters, the ordinates are oblique to them.

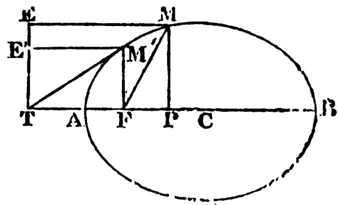
15. An *absciss* is a part of any diameter, contained between its vertex and an ordinate to it; as A K or B K, and D N or E N. Hence, in the ellipse and hyperbola, every ordinate has two abscisses; but in the parabola, only one; the other vertex of the diameter being infinitely distant.

16. The *parameter* of any diameter, is a third proportional to that diameter and its conjugate.

17. The *focus* is the point in the axis where the ordinate is equal to half the parameter: as K and L, where D K or E L is

equal to the semiparameter.—Hence, the ellipse and hyperbola have each two foci, but the parabola only one. The foci, or burning points, were so called, because all rays are united or reflected into one of them, which proceed from the other focus, and are reflected from the curve.

18. The *directrix* is a line drawn perpendicular to the axis of a conic section, through an assignable point in the prolongation of that axis; such that lines drawn from that directrix parallel to the axis to meet the curve, shall be to lines drawn from the points of intersection to the focus, in a constant ratio for the same curve. Thus, if

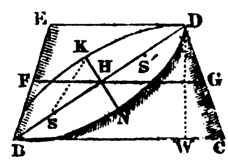


E M : M F :: E′ M′ : M′ F, then T E′ E is the directrix. The curve will be a *parabola*, and *ellipse*, or an *hyperbola*, according as F M is equal to, less than, or greater than, M E.

19. An *asymptote* is a right line towards which a certain curve line approaches continually nearer and nearer, yet so as never to meet, except both be produced indefinitely. The *hyperbola* has two asymptotes.

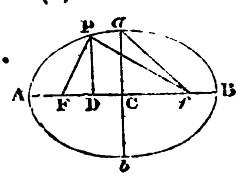
Section I. *Properties of the Ellipse.*

1. If in the annexed diagram the ellipse B K D N, be cut from the frustum of the right cone, the diameter of whose ends are E D, B C. Then, if B D be the transverse, or major axis, K N the conjugate, or minor axis, and S, S′, the foci, we shall have



$$BD^2 = DC^2 + ED \cdot BC \;\ldots\ldots (1)$$
$$KN^2 = ED \cdot BC \;\ldots\ldots\ldots\ldots (2)$$
$$SS' = EB = DC \;\ldots\ldots\ldots (3)$$

2. If A B, $a\,b$, be the two axes, C the centre, F, f, the foci, P any point in the curve, P D, an ordinate: also, if $AB = 2\,t$, $ab = 2\,c$, $CD = x'$, $AD = x$, $DP = y$. $FP = z$, angle $PFD = \varphi$, $\sqrt{(t^2 - c^2)} = d$; then,



$$y^2 = \frac{c^2}{t^2}(2\,t\,x - x^2) \;\ldots\ldots\; (\text{I})$$

$$y^2 = \frac{c^2}{t^2}(t^2 - x'^2) \;\ldots\ldots\ldots\; (\text{II})$$

$$z = \frac{c^2}{t - d\cos\varphi} \;\ldots\ldots\ldots\ldots\; (\text{III})$$

The first of these is the equation of the curve when the absciss are reckoned from the extremity A of the transverse axis: the second is the equation when the abscissæ are reckoned from the centre C: and the third is called the *polar equation*, and is principally used in the investigations of astronomy.

Ex. Suppose A B = 20, $a\,b = 12$, A D = 4. Required the numeral value of P D.

Here $y^2 = \frac{c^2}{t^2}(2\,t\,x - x^2) = \frac{36}{100}\left(20 \times 4) - 16\right)$

$= \frac{36}{100} \,.\, 64 = \frac{6^2 \,.\, 8^2}{10^2}.$

Consequently $y = \frac{6 \times 8}{10} = 4 \,.\, 8 =$ P D.

Or, taking Equa. II. where C D $= x' = 10 - 4 = 6$, and t and c as before, we have

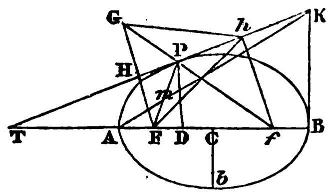
$y^2 = \frac{c^2}{t^2}(t^2 - x'^2) = \frac{36}{100}(100 - 36) = \frac{6^2 \,.\, 8^2}{10^2}$, as before.

3. In the same figure, we have

A C^2 : A C^2 :: A D . D B : D P^2 ...... (4)

also F P + Pf = A B (5); and Ff^2 = A B^2 $- a\,b^2$ .... (6)

4. Let T K be a tangent to the ellipse at any point K, and let T be the point where that tangent meets the prolongation of the axis: let also F H, $f\,h$ be perpendiculars from the foci, F, f, upon the tangent, and let G H = F H: then



∠ F P T = ∠ f P K .... (7) C D : C A :: C A : C T .... (8)

H and h fall in circumf. of circle whose diam. is A B (9)

If m be in the middle of P D, then A m produced will meet the two tangents T K, B K, in their point of intersection K (10)

If D the foot of the ordinate pass through F the focus, then the point, T, of intersection of the tangent and the prolongation of the axis, will be the point T of the *directrix* (Def. 18) ...... (11)

$\text{FH} . fh = \text{C}b^2$ ...... (12) $\text{FH}^2 = \text{C}b^2 . \frac{\text{FP}}{f\text{P}}$ .... (13)

5. If an ordinate be drawn to *any* diameter of an ellipse, then will the rectangle of the abscissæ be to the square of the ordinate in a given ratio ...... (14)

6. All the parallelograms that may be circumscribed about an ellipse are equal to one another; and every such parallelogram is equal to the rectangle of the two axes .... (15)

7. The sum of the squares of every pair of conjugate diameters is equal to the same constant quantity; viz. the sum of the squares of the two axes ...... (16)

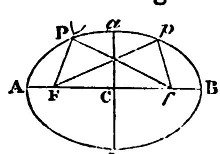
8. *Def. The Radius of Curvature* of a conic section or other curve, is the radius of that circle which is precisely of the same curvature as the curve itself, at any assigned point, or the radius of the circle which *fits* the curve and coincides with it, at a small distance on each side of the point of contact. The circle itself is called the osculatory circle, or the *equicurve* circle; and if the curve be of incessantly varying curvature, each point has a distinct equicurve circle, the radius of which is perpendicular to the tangent at the point of contact.

9. Let P C be the radius of curvature at any point P in an ellipse or hyperbola whose major axis is A B, minor axis $a\,b$, and foci, F, f, then is $\text{PC} = \frac{(\text{PF} . \text{P}f)^{\frac{3}{2}}}{\frac{1}{4}\text{AB} . ab}$ .... (17)

The radius of curvature is *greatest* at the extremities of the minor axis, when it is $= \frac{\text{AB}^2}{ab}$ ...... (18)

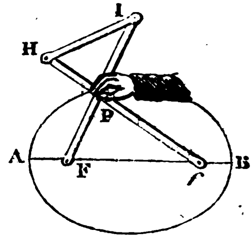
The radius of curvature is *least* at the extremities of the major axis, when it is $= \frac{ab^2}{\text{AB}}$ .... (19)

10. PROB. *To construct an ellipse whose two axes are given.*



Find the distance F f, from the value of Ff^2, given in equa. 6, or from $\text{F}f = \sqrt{\text{AB}^2 - ab^2}$. Then, let a fine thread F P f F, in length $= \text{F}f + \text{AB}$ be put round two pins fixed at the points F, f: then, if a pencil be put within the cord, and the whole become tightened so as to make three right lines F P, P f, f F, the point P may be carried on, the cord slipping round the fixed pins F, f, so as to describe and complete the ellipse A P p B b A.

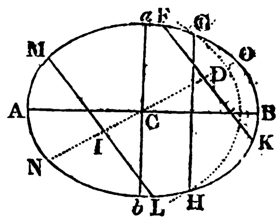
Otherwise. Let there be provided three rulers, of which the two F I, f H, are of the same length as the transverse axis A B, and the third H I, equal in length to F f, the focal distance. Then connecting these rulers so as to move freely about F, f, and about H I, their intersection P will always be in the curve of the ellipse: so that, if there be slits running along the two rulers, and the apparatus turned freely about the foci, a pencil put through the slits at their point of intersection will describe the curve.



*** There are various other methods, as by the elliptic compasses, the trammels, &c. But the first of the above methods, is as accurate and easy as can well be desired.

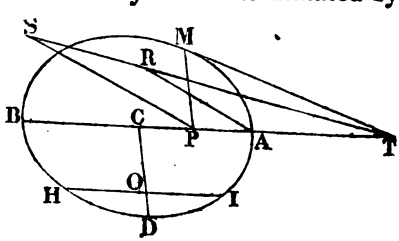
11. Prob. *To find the two axes of any proposed ellipse.*

Draw any two parallel lines across the ellipse, as M L, F K: bisect them in the points I and D, through which draw the right line N I D O, and bisect it in C. From C as a centre, with any adequate radius, describe an arch of a circle to cut the ellipse in the points G, H. Join G, H, and parallel to the line G H, draw through C the minor axis $a b$; perpendicular to which through C, draw A B, it will be the major axis.



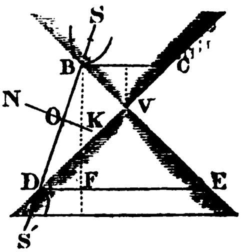
12. Prob. *From any given point out of an ellipse to draw a tangent to it.*

Let T be the given point: through it, and the centre C draw the diameter A B; and parallel to it any line H I terminated by the curve. Bisect H I in O; and C O produced will be the conjugate to A B. Draw any line T S = T B, and make T R = T C. Draw R A, and parallel to it S P cutting A B in P. Through P, draw P M parallel to C D, and join T M, it will be the tangent required.



Section II. *Properties of the Hyperbola.*

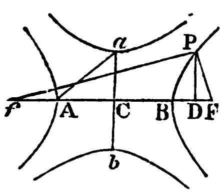
1. If, in the annexed diagram, the conjugate hyperbolas whose vertices are D, B, are cut from the two opposite right cones whose common summit is V, and B C, D E, be the diameters of the two circular bases of the two cones. Then D B, K N being the axes, and S, S′, the foci, we shall have



$$DB^2 = DC^2 - DE \cdot BC \;\ldots\ldots (1)$$
$$KN^2 = DE \cdot BC \;\ldots\ldots\ldots\ldots (2)$$
$$S\,S' = EB = DC \;\ldots\ldots\ldots (3)$$

If these three properties be compared with the corresponding ones for the ellipse, they will be found to agree, with the simple difference of the signs + and − of the connecting quantities in the first property. This at once indicates a general analogy between the properties of the two curves.

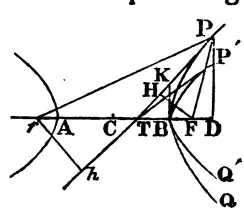
2. Hence, putting $AC = CB = t$, $aC = Cb = c$, $CF = d$, $AD = x$, $CD = x'$, $DP = y$, angle $PFD = \phi$, $z = FP$, we have

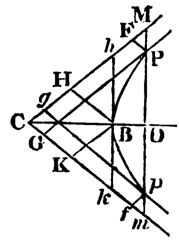


$$y^2 = \frac{c^2}{t^2}\,(2\,t\,x + x^2) \;\ldots\ldots (\mathrm{I})$$
$$y^2 = \frac{c^2}{t^2}\,(x'^2 - t^2) \;\ldots\ldots\ldots (\mathrm{II})$$
$$z = \frac{c^2}{t \times d\cos\phi} \;\ldots\ldots\ldots (\mathrm{III})$$

for the three most useful forms of the equation to the hyperbola, agreeing with those to the hyperbola, except in the signs.

3. And hence it follows, taking this and the preceding marginal figures to correspond with those in arts. 3 and 4 Ellipse; that the properties indicated by the parenthetical figures (4), (5), (6), (7), (8), (9), (11), (12), (13), (14), (15), and (16), hold in the hyperbola; simply changing + to − in (5), − to + in (6), *circumscribed* to *inscribed between* the four hyperbolas in (15), and *sum* to *difference* in (16). Those properties, therefore, need not be here repeated.

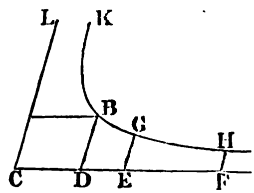




4. Besides the above, however, there are several curious properties which relate to the *asymptotes* of the hyperbola; some of the most useful being these: viz.
M C, m C, being asymptotes, P D p a double ordinate, B H, P F, parallel to C m; &c.

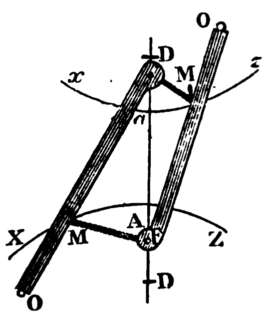
parallelogram C H B K = parallelogram C F P G
= parallelogram C $f\,p\,g$ . . . .(17)
M P . P p = $m\,p$. p P = C B^2 (18); . . M P = $p\,m$ . .(19)
triangle C P T = triangle C B K (20) former diagram.

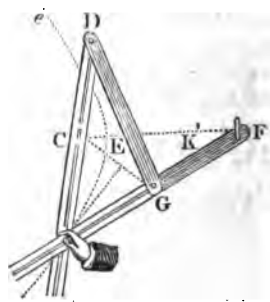
5. Also, if the abscissæ C D, C E, C F, &c. of any hyperbola, be taken on one of the asymptotes in an increasing geometrical progression, the ordinates D B, E G, F H, &c. parallel to the other asymptote are in decreasing geometrical progression having the same ratio . . . .(21).



6. And, when the distances C E, C F, &c. are in geometrical progression, the asymptotic spaces D E G B, D F H B, &c. will be in arithmetical progression, and will, therefore, be analogous to the logarithms of the former. The nature of the *system* of logarithms will depend upon the value of the angle made by the two asymptotes. In *Napier's* logarithms L C F is a right angle: in the common logarithms L C F is 25° 44′ 27″ $\frac{1}{4}$.

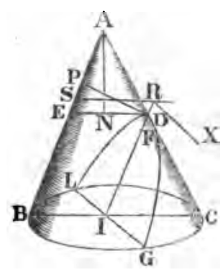
7. *To describe hyperbolas.*





Let one end of a long ruler f M O be fastened at the point f, by a pin on a plane, so as to turn freely about that point as a centre. Then take a thread F M O, shorter than the ruler, and fix one end of it in F, and the other to the end O of the ruler. Then if the ruler *f* M O be turned about the fixed point *f*, at the same time keeping the thread O M F always tight, and its part M O close to the side of the ruler, by means of the pin M; the curve line A X described by the motion of the pin M is one part of an hyperbola. And if the ruler be turned, and move on the other side of the fixed point F, the other part A Z of the same hyperbola may be described after the same manner. —But if the end of the ruler be fixed in F, and that of the thread in *f*, the opposite hyperbola *x a z* may be described. *Otherwise: also by continued motion.* Let C and F be the two foci, and E and K the two vertices of the hyperbola. (See the last fig. above.) Take three rulers C D, D G, G F, so that C D = G F = E K, and D G = C F; the rulers C D and G F being of an indefinite length beyond C and G, and having slits in them for a pin to move in; and the rulers having holes in them at C and F, to fasten them to the foci C and F by means of pins, and at the points D and G they are to be joined by the ruler D G. Then, if a pin be put in the slits, viz. at the common intersection of the rulers C D and G F, and moved along, causing the two rulers G F, C D, to turn about the foci C and F, that pin will describe the portion E *e* of an hyperbola.

Section III. *Properties of the Parabola.*



1. Let the right cone A B C in the marginal figure, have a parabolic section L D G, whose focus is F, vertex D, base L G; from D let fall the perpendicular D P upon the side A B of the cone; let E P be bisected in S: also, let a plane be cut through S parallel to B C, and continued to meet the plane of the parabola; in R X.

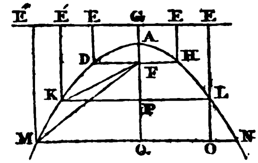
Then .... R X is the directrix of the parabola..(1)

$$DF = \frac{ED^2}{4\,AE} \quad \ldots\ldots\ldots\ldots (2)$$

$$DF = \frac{IG^2}{4\,ED} \quad \ldots\ldots\ldots\ldots (3)$$

$$2\,EP = 4\,DF = \text{parameter} \quad \ldots\ldots (4)$$

2. Let p = parameter of a parabola, x = A P any absciss, y = P K, the corresponding ordinate, z = F K, d = A F, ϕ = angle K F P, F being the focus: then



$$y^2 = p x \text{ .... (I)}$$

$$z = \frac{2d}{1 \mp \cos\phi} \text{...(II)}$$

the equations to the parabola: in the latter of which, or the polar equation, the sign + obtains when P is between A and F, and − when P is below F.

$$\text{Rad. of curvature at K} = \frac{(4x + p)^{\frac{3}{2}}}{2\sqrt{p}} \text{ ...... (III)}$$

at the vertex A, x vanishes, and we have
rad. of curv. at vertex $= \frac{1}{2} p$ .... (IV)

3. In the same figure, where E E G E is the directrix, the following properties obtain: viz.

A F = A G, F D = D E, E K = K E′, F M = M E″ &c. ..(5)

A P : A Q :: P L² : Q N², or $\frac{AP}{AQ} = \frac{PL^2}{QN^2}$............ (6)

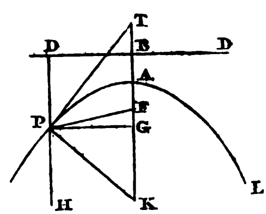
where A P and A Q are any abscissæ, and P L, Q N, their corresponding ordinates.

F K = A P + A F, F M = A Q + A F .... (7)

A F = $\frac{1}{2}$ F H = $\frac{1}{4}$ D H and D H = parameter .... (8)

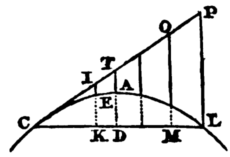
As p : Q N + P L :: Q N − P L : A Q − A P }
or, as p : M O :: O N : O L } .... (9)

4. Again, let P T be the tangent to a parabola at any point P, and let H P D be drawn through P parallel to the axis A K; let P K be perpendicular to T P: then is G T the subtangent, P K the normal, G K the subnormal; and the following properties obtain: viz.



angle F P T = angle F T P = angle T P D .... (10)
angle K P H = angle K P F .... (11)
F P = F T ....(12) G A = A T .... (13)
subtangent G T = 2 A G .... (14)
subnormal G K = 2 A F = $\frac{1}{2}$ param. a constant quan. (15)

5. In the marginal figure also, where C Q is a tangent to the parabola at the point C, and I K, O M, Q L, &c. parallel to the axis A D.



Then I E : E K :: C K : K L .... (16)
and a similar property obtains, whether C L be perpendicular or oblique to T D.

The external parts of the parallels I E, T A, O N, P L, &c. are always proportional to the squares of their intercepted parts of the tangent; that is,

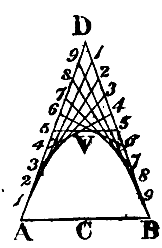
the external parts I E, T A, O N, P L,
are proportional to CI^2, CT^2, CO^2, CP^2, ... (17)
or to the squares CK^2, CD^2, CM^2, CL^2,

And as this property is common to every position of the tangent, if the lines I E, T A, O N, &c. be appended to the points I, T, O, &c. of the tangent, and moveable about them, and of such lengths that their extremities E, A, N, &c. be in the curve of a parabola in any one position of the tangent; then making the tangent revolve about the point C, the extremities E, A, N, &c. will always form the curve of some parabola, in every position of the tangent.

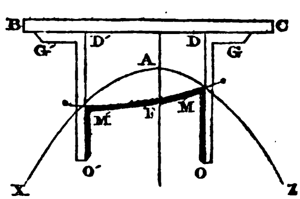
The same properties too that have been shown of the axis, and its abscisses and ordinates, &c. are true of those of any other diameter.

6. Prob. *To construct a Parabola.*

Construct an isosceles triangle A B D, whose base A B shall be the same as that of the proposed parabola, and its altitude C D *twice* the altitude C V of the parabola. Divide each side A D, D B, into 10, 12, 16, or 20, equal parts [16 is a good number, because it can be obtained by continual bisections], and suppose them numbered 1, 2, 3, &c. from A to D, and 1, 2, 3, &c. from D to B. Then draw right lines 1, 1; 2, 2; 3, 3; 4, 4; &c. and their mutual intersection will beautifully approximate to the curve of the parabola A V B.



Otherwise: by continued motion.—Let the ruler, or directrix B C, be laid upon a plane with the square G D O, in such manner that one of its sides D G lies along the edge of that ruler; and if the thread F M O equal in length to D O, the other side of the square, have one end fixed in the extremity of the ruler at

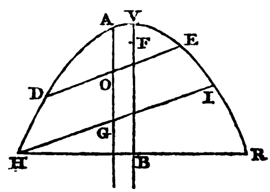


O, and the other end in some point F: then slide the side of the square D G along the ruler B C, and at the same time keep the thread continually tight by means of the pin M, with its part M O close to the side of the square D O; so shall the curve A M X, which the pin describes by this motion, be one part of a parabola.

And if the square be turned over, and moved on the other side of the fixed point F, the other part of the same parabola A M Z will be described.

7. PROB. *Any right line being given in a parabola, to find the corresponding diameter; also, the axis, parameter, and focus.*

Draw H I parallel to the given line D E. Bisect D E, H I, in O and G, through which draw A O G for the diameter.



Draw H R perp. to A G and bisect it in B; and draw V B parallel to A G, for the axis.

Make V B : H B :: H B : parameter of the axis.

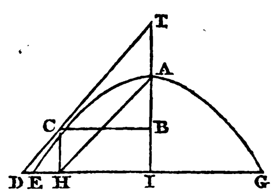
Then $\frac{1}{4}$ the parameter set from M to F gives the focus.

8. PROB. *To draw a tangent to a parabola.*

If the point of contact C be given, draw the ordinate C B, and produce the axis until A T = A B: then join T C, which will be the tangent.

Or if the point be given in the axis produced: take A B = A T, and draw the ordinate B C, which will give C the point of contact; to which draw the line T C as before.

If D be any other point, neither in the curve nor in the axis produced, through which the tangent is to pass: draw D E G perpendicular to the axis, and take D H a mean proportional between D E and D G, and draw H C parallel to the axis, so shall C be the point of contact, through which and the given point D the tangent D C T is to be drawn.

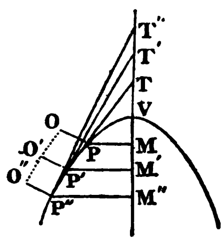


When the tangent is to make a given angle with the ordinate at the point of contact: take the absciss A I equal to half the parameter, or to double the focal distance, and draw the ordinate I E: also draw A H to make with A I the angle A H I equal to the given angle; then draw H C parallel to the axis, and it will cut the curve in C the point of contact, where a line drawn to make the given angle with C B will be the tangent required.

SECTION IV. *General application to architecture.*

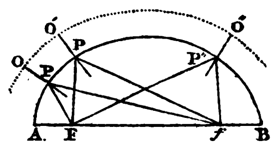
PROB. 1. *To find by construction, the position of the joints of the voussoirs, to a parabolic arch.*

In the practice of arcuation, the voussoirs or arch-stones are so cut, that their joints are perpendicular to the arch or to its tangent, at the points where they respectively fall. Hence, if A V B be the proposed parabola, P, P′, P″, &c. the points at which the positions of the joints are to be determined: draw the ordinates P M, P′ M′, P″ M″, and on the prolongation of the axis set off V T = V M, V T′ = V M′, V T″ = V M″, &c. Join T P, T′ P′, T″ P″, &c. and perpendicular to them respectively the lines P O, P′ O′, P″ O″, &c. they will determine the positions of the joints required.



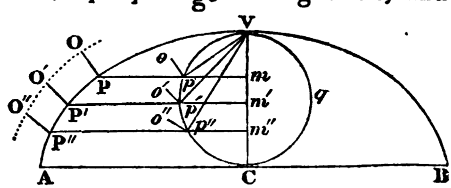
PROB. 2. *To find the same for an elliptical arch.*

Let A B be the span of the arch, and A P P′ P″ B the arch itself, of which F and *f* are the foci. Draw lines F P, *f* P, from the foci to each of the points P: bisect the respective angles F P *f*, F P′ *f*, F P″ *f*, by the lines P O, P′ O′, P″ O″; they will show the positions of the joints at the points P, P′, P″.



PROB. 3. *To find the same for a cycloidal arch.**

Let A V B be the cycloid, C *p* V *q* its generating circle, and P, P′, P″, points in the arch where joints will fall. Draw the ordinates P *m*, P′ *m′*, P″ *m″*, each parallel to the base A B of the cycloid, and cutting the circle in the points *p*, *p′*, *p″*. Join V *p*, V *p′*, V *p″*, and perpendicular to each the lines *p o*, *p′ o′*, *p″ o″*; parallel to each of which respectively, draw P O, P′ O′, P″ O″; they will mark the positions of the joints at the several points proposed.



* This problem is introduced here, as belonging to the subject of arcuation, although it depends upon a property of the cycloid described hereafter, viz. that the tangent to any point P of a cycloid is parallel to the corresponding chord V *p* of the generating circle.

CHAP. VI.

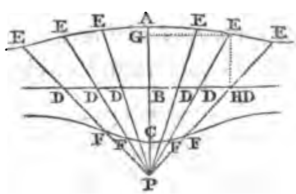
CURVES,

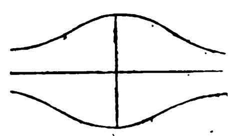
A knowledge of which is required by Architects and Engineers.

SECTION I. *The Conchoid.*

Conchoid or *Conchiles*, is the name given to a curve by its inventor *Nicomedes*, about 200 years before the Christian era.

The conchoid is thus constructed: A P and B D being two lines intersecting at right angles: from P draw a number of other lines P F D E, &c. on which take always D E = D F = A B or B C; so shall the curve line drawn through all the points E, E, E, be the first conchoid, or that of Nicomedes; and the curve drawn through all the other points F, F, F, is called the second conchoid; though in reality, they are both but parts of the same curve, having the same pole P, and four infinite legs, to which the line D B D is a common asymptote.

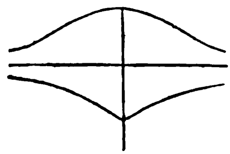




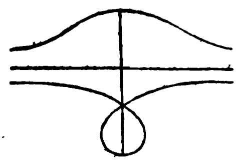
The inventor, Nicomedes, contrived an instrument for describing his conchoid by a mechanical motion: thus, in the ruler D D is a channel or groove cut, so that a smooth nail firmly fixed in the moveable ruler C A, in the point D, may slide freely within it: into the ruler A P is fixed another nail at P, for the moveable ruler A P to slide upon. If therefore the ruler A P be so moved as that the nail D passes along the canal D D, the style, or point in A, will describe the first conchoid.

Conchoids of all possible varieties may also be constructed with great facility by Mr. Jopling's apparatus for curves, now well known.

Let A B = B C = D E = D F = a, P B = b, B G = E H = x, and G E = B H = y: then the equation to the first conchoid will be $x^2 (b + x)^2 + x^2 y^2 = a^2 (b + x)^2$, or $x^4 + 2 b x^3 + b^2 x^2 + x^2 y^2 = a^2 b^2 + 2 a^2 b x + a^2 x^2$; and, changing only the sign of x, as being negative in the other curve, the equation to the 2d conchoid will be $x^2 (b - x)^2 + x^2 y^2 = a^2 (b - x)^2$, or $x^4 - 2 b x^3 + b^2 x^2 + x^2 y^2 = a^2 b^2 - 2 a^2 b x + a^2 x^2$.



Of the whole conchoid, expressed by these two equations, or rather one equation only, with different signs, there are three cases or species; as first, when B C is less than B P, the conchoid will be as in the 2d fig. above; when B C is equal to B P, the conchoid will be as in the 3d fig.; and when B C is greater than B P, the conchoid will be as in the 4th or last fig.



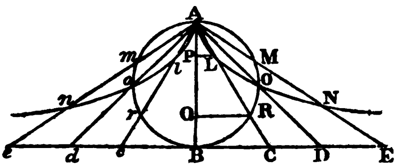
Newton approves of the use of the conchoid for trisecting angles, or finding two mean proportionals, or for constructing other solid problems. But the principal modern use of this curve, and of the apparatus by which it is constructed, is to sketch the contour of the section that shall represent the diminution of columns in architecture.

The fixed point P is called the *pole* of the conchoid; D D D D the *directrix*: it is an asymptote to both the superior and the inferior conchoid. In the last figure the inferior conchoid is also *nodated*.

SECTION II. *The Cissoid, or Cyssoid.*

The *cissoid* is a curve invented by an ancient Greek geometer and engineer named *Diocles*, for the purpose of finding two continued mean proportionals between two given lines. This curve admits of an easy mechanical construction; and is described very beautifully by means of Mr. Jopling's apparatus.

At the extremity B of the diameter A B, of a given circle A O B o, erect the indefinite perpendicular e B E, and from the other extremity A draw any number of right lines A C, A D, A E, &c. cutting the circle in the points R, O, M, &c.; then, if C L be taken = A R, D O = A O, E N = A M, &c. the curve passing through the points A, L, O, N, &c. will be the cissoid.



1. Here the circle A O B *o* is called the generating circle; and A B is called the axis of the curves A L O N, &c. A *l o n*, &c. which meet in a cusp at A, and, passing through the middle points O, *o*, of the two semicircles, tend continually towards the directrix *e* B E, which is their common asymptote.

2. If A O and A *o* are quadrants, the curve passes through *o* and O, or it bisects each semicircle.

3. Letting fall perpendiculars L P, R Q, from any corresponding points L, R: then is A P = B Q, and A L = C R.

4. A P : P B :: P L² : A P². So that, if the diameter A B of the circle $= a$, the absciss A P $= x$, the ordinate P L $= y$; then is $x : a - x :: y^2 : x^2$, or $x^3 = (a - x)\, y^2$, which is the equation to the curve.

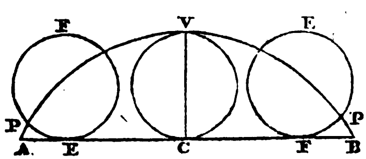
5. The right line *e* B E is an asymptote to the curve.

6. Arch A M of the circle = arch B R, and arch A *m* = B *r*.

7. The whole infinitely long cissoidal space, contained between the asymptote *e* B E and the curves N O L A, &c. A L *o n*, &c. is equal to three times the area of the generating circle A O B *o*.

Section III. *The Cycloid.*

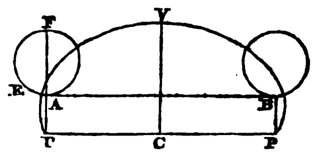
The *cycloid*, or *trochoid*, is an elegant mechanical curve first noticed by *Descartes*, and an account of it published by *Mersenne* in 1615. It is, in fact, the curve described by a nail in the rim of a carriage-wheel while it makes one revolution on a flat horizontal plane.



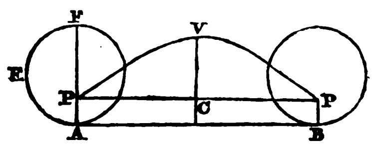
1. Thus, if a circle E P F, keeping always in the same plane, be made to roll along the right line A B, until a fixed point P, in its circumference, which at first touched the line at A, touches it again after a complete revolution at B; the curve A P V P B described by the motion of the point P is called a cycloid.

2. The circle E P F is called the generating circle; and the right line A B, on which it revolves, is called the base of the cycloid.

Also, the right line, or diameter, C V, of the circle, which bisects the base A B at right angles, is called the axis of the cycloid; and the point V, where it meets the curve, is the vertex of the cycloid.

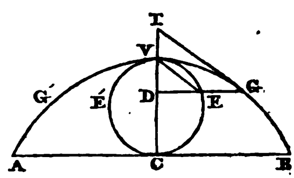


3. If P be a point in the fixed diameter A F produced, and the circle A E F be made to roll along the line A B as before, so that the point A, which first touches it at one extremity, shall touch it again at B, the curve P V P, described by the point P, is called the *curtate* cycloid.



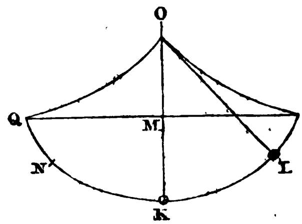
4. And, if the point P be anywhere in the unproduced diameter A P, and the circle A E F be made to roll along A B from A to B; the curve P V P is, in that case, called the *inflected* or *prolate* cycloid.

The following are the chief properties of the common cycloid.



1. The circular arc V E = the line E G between the circle and cycloid, parallel to A B.
2. The semicircumf. V E C = the semibase C B.
3. The arc V G = 2 the corresponding chord V E.
4. The semicycloidal arc V G B = 2 diam. V C.
5. The tangent T G is parallel to the chord V E.
6. The radius of curvature at V = 2 C V.
7. The area of the cycloid A V B C A is triple the circle C E V; and consequently that circle and the spaces V E C B G, V E′ C A G′, are equal to one another.

8. A body falls through any arc L K of a cycloid reversed, in the same time, whether that arc be great or small; that is, from any point L, to the lowest point K, which is the vertex reversed: and that time is to the time of falling perpendicularly through the axis M K, as the semicircumference of a circle is to its diameter, or as 3·141593 to 2. And hence it follows that if a pendulum be made to vibrate in the arc L K N of a cycloid, all the vibrations will be performed in the same time.

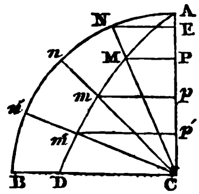


2. The evolute of a cycloid, is another equal cycloid, so that if two equal semicycloids O P, O Q, be joined at O, so that O M be = M K the diameter of the generating circle, and the string of a pendulum hung up at O, having its length = O K or = the curve O P; then, by plying the string round the curve O P, to which it is equal, if the ball be let go, it will describe, and vibrate in the other cycloid P K Q; where O P = Q K, and O Q = P K.

10. The cycloid is the curve of swiftest descent: or a heavy body will fall from one given point to another, by the way of the arc of a cycloid passing through those two points, in a less time, than by any other rout. Hence this curve is at once interesting to men of science and to practical mechanics.

Section IV. *The Quadratrix.*

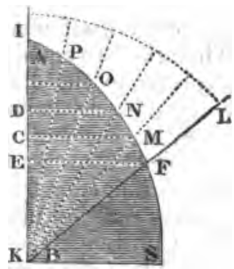
The *quadratrix* is a species of curve by means of which the quadrature of the circle and other curves is determined mechanically. For the quadrature of the circle, curves of this class were invented by *Dinostrates* and *Tschirnhausen*, and for that of the hyberbola by Mr. *Perks*. We shall simply describe in this place, the quadratrix of *Tschirnhausen;* and that in order to show its use in the division of an arc or angle.



To construct this quadratrix, divide the quadrantal arc A B into any number of equal parts, A N, N *n*, *n n'*, *n* B; and the radius A C into the same number of equal parts A P, P *p*, *p p'*, *p'* C. Draw radii C N, C *n*, &c. to the points of division upon the arch; and let lines P M, *p m*, &c. drawn perpendicularly to A C from the several points of division upon it, meet the radii in M, *m*, *m'*, &c. respectively. The curve A M *m m'* D that passes through the points of intersection M, *m*, &c. is the quadratrix of Tschirnhausen.

The figure A C D *m' m* M A thus constructed may be cut out from a thin plate of brass, horn, or pasteboard, and employed in the division of a circular arc.

Thus, suppose the arc I L or the angle I K L is to be divided into five equal parts. Apply the side A B of the quadratrix upon I K, the point B corresponding with the angle K. Draw a line along the curve A S, cutting K L in F. Remove the instrument, and from F let fall the perpendicular F E upon I K. Divide E I into five equal parts by prob. 5, Practical Geometry, and through the points of division draw C M, D N, &c. parallel to E F. Then from their intersections, M, N, O, P, draw the the lines K M, K N, K O, K P, and they will divide the angle I K L into five equal parts, as required.



Note 1. If instead of dividing the arc into equal parts it were proposed to divide it into a certain number of parts having given ratios to each other; it would only be necessary to divide E I into parts having the given ratio, and proceed in other respects as above.

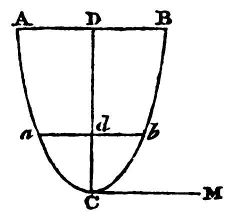
Note 2. If the arc or angle to be divided exceed 90 degrees, bisect it, divide that bisected arc or angle into the proposed number of parts, and take *two* of them for *one* of the required divisions of the whole arc.

Section V. *The Catenary, and its application.*

The *catenary* is a mechanical curve, being that which is assumed by a chain or cord of uniform substance and texture, when it is hung upon two points or pins of suspension (whether those points be in a horizontal plane or not), and left to adjust itself in equilibrio in a vertical plane.

This curve is of great interest to practical men on account of its connection with bridges of suspension, or chain bridges. Its consideration cannot, therefore, with propriety be omitted, although it involves mechanical propositions which will be announced subsequently.

Let A, B, be the points of suspension of such a cord, A a C b B the cord itself when hanging at rest in a vertical position. Then the two equal and symmetrical portions A a C, C b B, both exposed to the force of gravity upon every particle, balance each other precisely at C. And, if one half, as C b B were taken away, the other half A a C would immediately adjust itself in the vertical position under the point A were it not prevented. Suppose it to be prevented by a force acting horizontally at C, and equal to the weight of a portion of the cord or chain equal in length to C M; then is C M the measure of the *tension* at the vertex of the curve: it is also regarded as the *parameter* of the catenary. Whether the portion A a C hang from A, or a shorter portion as a C hang from a, the tension at C is evidently the same: for in the latter case the resistance of the pin at a, accomplishes the same as the tension of the line at a when the whole A a C hangs from A.*



Let the line C M which measures the tension at the vertex be $= p$, let C $d = x$, $a\,d = d\,b = y$, C a = C $b = z$, C D $= h$, A B $= d$, C a A = C b B $= \frac{1}{2}\,l$. Then

* This may easily be determined experimentally, by letting the cord hang very freely over a pulley at C, and lengthening or shortening the portion there suspended, until it keeps A a C, in its due position; then is the portion so hanging beyond the pulley equal in length to C M.

1. $y = p \times$ hyp. log. $\dfrac{p + x + \sqrt{2px + x^2}}{p} = p \times$ hyp. log. $\dfrac{p + x + z}{p} = p \times$ hyp. log. $\dfrac{z + p}{z - p} = p$ M log. tan $(45° + \frac{1}{2}$ S$)$.

2. If the angle S of suspension made between the tangent to the curve at A or B, and the horizon be 45°; then $d : l :: 1 : 1{\cdot}1346$.

3. When $l = 2\,d$, then $h = {\cdot}7966\,d$, and S $= 77° 3'$.

4. When the angle S of suspension is 56° 28′, then p, x, y, and z, are as 1, 0·81, 1·1995, and 1·5089 respectively. In this case t, the tension at the point of suspension, is a *minimum* with respect to y.

5. Generally $\dfrac{d}{l} = -$ tan S M L tan $\frac{1}{2}$ S.

Where M $= 2{\cdot}30255851$, Napier's logarithm of 10.

Or, log. $\dfrac{d}{l} =$ log. tan S + log. (log. cot $\frac{1}{2}$ S − 10) + ·3622157 − 10. This last formula serving to compute an approximative result.

6. The distance of the centre of gravity of the whole curve $2\,z$, from the vertex $= \frac{1}{2}\left(x + \dfrac{p\,y}{z} - p\right)$.

7. The radius of curvature $\dfrac{(p + x)^2}{p} = \dfrac{t^2}{p}$: this at the vertex, is rad. curv. $= p$.

8. When S and p are given; then
$z = p$ tan S .... $t = p$ sec S.
$x = p\,(\text{sec S} - 1) = \dfrac{p\ \text{versin S}}{\text{cos S}}$
and $y = p$ M log. tan $(45° + \frac{1}{2}$ S$)$.

9. When S and z, or $\frac{1}{2}\,l$, are given: then
$p = z$ cot S ...... $t = z$ cosec S
$x = z$ cosec S versin S
$y =$ M z cot S log. tan $(45° + \frac{1}{2}$ S$)$.

10. When S and y are given: then
$p = y \div$ M log. tan $(45° + \frac{1}{2}$ S$)$
$t = y \div$ M cos S log. tan $(45° + \frac{1}{2}$ S$)$
$z = y$ tan S $\div$ M log. tan $(45° + \frac{1}{2}$ S$)$
$x = y$ versin S $\div$ M cos S log. tan $(45° + \frac{1}{2}$ S$)$.

11. When x and y are given; then

$$\frac{\text{log. tan }(45^\circ + \frac{1}{2}\text{s})}{\text{sec s versin s}} = \frac{y}{\text{M}x};$$

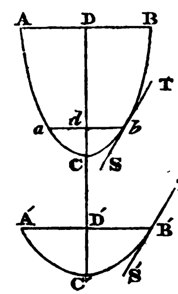
from which s may be found by an approximative process; also

$$p = \frac{x}{\text{sec s} - 1} \; \ldots . \; t = \frac{x}{\text{versin s}} \; \ldots . \; z = \frac{x \text{ sin s}}{\text{versin s}}$$

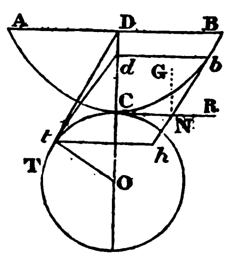
in all these cases t is determined in *length of chain or cord* of which the catenary is actually constructed.

12. *To draw the catenary mechanically.*—If the distance A B between the points of suspension, and the depth D C of the lowermost point, be given, (see the preceding figure,) hang one extremity of a fine uniform chain or cord at one of the points A, and, letting the chain or cord adjust itself as a festoon in a vertical plane, lengthen or shorten it as it is held near the other end, over a pin at B, until, when at rest, it just reaches the point C: so shall the cord form the catenary; and a pencil passing along the cord, from A by a, C, b, to B, will mark the curve upon a vertical board brought into contact with it.

13. All catenaries that make equal angles with their ordinates at their points of suspension are similar, and have x to y a constant ratio: and of any two which do not make equal angles, but have x to y in different ratios, a portion may be cut from one curve similar to the other. Thus, let A C B and A′ C′ B′, be the two curves, of which A′ C′ B′ is the flattest. Suppose them placed upon one axis D C C′, and the tangent T′ S′, to the lower curve, at B′ the point of suspension to be drawn. Then, parallel to T′ S′ draw another line T S to touch the other curve in b. Through b draw b a parallel to B′ A′. So shall the portion a C b of the upper catenary be similar to the lower catenary A′ C′ B′.



14. With reference to the practical uses of the catenary, we may now blend the geometrical and the mechanical consideration of its properties. Taking any portion C b of the catenary, from the lowest point C; its weight may be regarded as supported by tensions acting in the tangential directions C N, b N. The strains at C and b may be conceived as acting at the point of intersection N;



above which, therefore, in the vertical direction N G, the weight of the portion C b may be conceived to act at its centre of gravity G.

Hence, strain at C : weight of C b :: sin G N b : sin b N C
:: cos b N R : sin b N R
:: radius : tan b N R
:: radius : tan d b N.

Hence, the horizontal tension at C being constant, the weight, and consequently the length of any portion c b of the uniform chain must be proportional to the tangent of the inclination of the catenary to the horizon at the extremity b of the said portion. *This may be regarded as the characteristic property of the catenary.*

15. In like manner,

hor. strain at C : oblique strain at b :: sin b N G : sin C N G
:: cos b N R : radius
:: rad : sec b N R.

Therefore, the strain exerted tangentially, at any point b, is proportional to the *secant* of the inclination at that point. These properties evidently accord with the preceding equations.

16. Let, then, C O, in the axis produced downwards, be equal to the parameter or the measure of the horizontal strain at C; and upon O as a centre with radius C O describe a circle. A tangent $d\,t$ drawn to this circle from d, will be parallel to the tangent b N h of the curve at the point b to which $d\,b$ is the ordinate. That tangent $d\,t$ (to the circle) will also be equal in length to the corresponding portion b C of the curve: while the tension at b will be expressed by a length of the chain equal to the secant O d. So again, if D T be a tangent to the circle drawn from D, it will be equal in length to B b C, and parallel to the tangent to the catenary at B; while the secant O D will measure the oblique tension at B; evidently exceeding the constant horizontal tension or strain at C, by the absciss C D.

17. When the parameter of the catenary, or the line which measures the tension at the lowest point, is equal to the depression D C; if each of these be supposed equal to 1, then A B = 2·6339, the length of chain A C B = 3·4641; the strain at the points of suspension A and B will each be 2, that at the lowest point being 1; and the chain at A and B will make an angle of 60° with the horizon.

18. If the strain at C be equal to the weight of the chain, and each denoted by 1: then A B = ·96242, D C = ·1180340, the tension at A or B = 1·118, the angle of suspension at those

points nearly 26° 34′; the width of the curve is 8·1536 times, and the length 8·4719 times, the depression D C.

19. If the strain or tension at the lowest point be double the weight of the chain: then if the parameter be 1, A B will be ·49493, C D = ·03078, the strain at A or B 1·03078, the angle of suspension about 14° 2′, the width or span 16·0816 times, and the length of chain 16·2462 times the depression.

The magnitudes of the lines, angles, and strains in many other cases may be seen in the table below. The whole theory may be verified experimentally, by means of spring steelyards applied to a chain of given length and weight, placed in various positions; according to the method suggested subsequently when treating of the parallelogram of forces, in *Mechanics*.

20. Taking A B = d, C D = h, length A C B = l, strain at C or parameter = p, then, in all cases where the depression is small compared with the length of the chain, *Professor Leslie* shows,* that

$$p = \frac{d^2}{8h} + \tfrac{1}{6}h \quad \ldots\ \text{strain at A or B} = \frac{d^2}{8h} + \tfrac{7}{6}h$$

$$\text{or } p = \frac{l^2}{8h} - \tfrac{1}{2}h \quad \ldots\ \text{strain at A or B} = \frac{l^2}{8h} + \tfrac{1}{2}h$$

$$l = d + \frac{8h^2}{3d}$$

In this case, too, the strains at C and A or B are nearly in the inverse ratio of the depression.

* Elements of Natural Philosophy, pa. 63.

Table of Relations of Catenarian Curves, the Parameter being denoted by 1.

Angle of suspension.	D C	D B	C B	tension at A or B	D B + D C
1° 0′	·00015	·01745	·01745	1·0001	114·586
2 0	·00061	·03491	·03492	1·0006	57·279
3 0	·00137	·05238	·05241	1·0014	38·171
4 0	·00244	·06987	·06993	1·0024	28·613
5 0	·00382	·08738	·08749	1·0038	22.874
6 0	·00551	·10491	·10510	1·0055	19·046
7 0	·00751	·12248	·12278	1·0075	16·309
8 0	·00983	·14008	·14054	1·0098	14·254
9 0	·01247	·15773	·15838	1·0125	12·654
10 0	·01543	·17542	·17633	1·0154	11·372
11 0	·01872	·19318	·19438	1·0187	10·820
12 0	·02234	·21099	·21256	1·0223	9·444
13 0	·02630	·22887	·23087	1·0263	8·701
14 0	·03061	·24681	·24933	1·0306	8·062
15 0	·03528	·26484	·26795	1·0353	7·508
16 0	·04030	·28296	·28675	1·0403	7·021
17 0	·04569	·30116	·30573	1·0457	6·591
18 0	·05146	·31946	·32492	1·0515	6·208
19 0	·05762	·33786	·34433	1·0576	5·863
20 0	·06418	·35637	·36397	1·0642	5·553
21 0	·07114	·37502	·38386	1·0711	5·271
22 0	·07853	·39376	·40403	1·0786	5·014
23 0	·08636	·41267	·42447	1·0864	4·778
24 0	·09484	·43169	·44523	1·0946	4·562
25 0	·10338	·45087	·46631	1·1034	4·361
26 0	·11260	·47021	·48773	1·1126	4·176
28 0	·13257	·50940	·53171	1·1326	3·843
30 0	·15470	·54930	·57735	1·1547	3·551
32 4	·18004	·5912	·62649	1·1800	3·284
34 16	·21008	·6371	·68180	1·2100	3·034
36 52	·24995	·6932	·74991	1·2499	2·773
39 11	·29011	·7443	·81510	1·2901	2·567
41 44	·34004	·8029	·89201	1·3400	2·362
44 0	·39016	·8566	·96569	1·3902	2·196
46 1	·43999	·9066	1·0861	1·4400	2·060
48 11	·49981	·9623	1·1178	1·4998	1·925
50 8	·56005	1·0142	1·1974	1·5800	1·811
52 9	·62973	1·0706	1·2869	1·6297	1·699
54 13	·71021	1·1304	1·3874	1·7102	1·592
56 28	·81021	1.1995	1·5089	1·8102	1·481
58 3	·88972	1.2510	1·6034	1·8897	1·416
60 0	1·0000	1·3169	1·7321	2·0000	1·317
64 6	1·2894	1·4702	2·0594	2·2894	1·140
67 28	1·6095	1·6135	2·4102	2·6095	1·002
67 32	1·6168	1·6164	2·4182	2·6168	0·9998

21. The preceding table is abridged from a very extensive one given by *Mr. Ware* in his "Tracts on Vaults on Bridges." Two examples will serve to illustrate its use.

Ex. 1. Suppose that the span of a proposed suspension bridge is 560 feet, and the depression in the middle $25\frac{7}{8}$ feet; what will be the length of the chain, the angle of suspension at the extremities, and the ratio of the horizontal pressure at the lowest point, and the oblique pressures at the points of suspension with the entire weight of the chain?

Here DB ÷ DC = 280 ÷ 25·875 = 10·82, a number which is to be found in the table.

Opposite to that number, we find 11° for the angle of suspension, DB = ·19318, CB = ·19438, tension at A or B = 1·0187, the constant tension at the vertex being 1 (fig. p. 172).

Consequently, ·19318 : ·19438 :: 560 : 563·48 length of the chain.

Also, horizon. pressure at C —— are as 1·0000
oblique pressure at A or B . . . . . 1·0187
entire weight of chain . . . . . . and ·38876

Ex. 2. Suppose that while the span remains 560, the depression is increased to 51.

Here DB ÷ DC = 280 ÷ 51 = 5·49. This number is not to be found exactly in the table. The nearest is 5·55 in the last column, agreeing with 20°, the angle of suspension.

Now, 5·55 − 5·49 = ·06, and 5·55 − 5·27 = ·28, the former difference being nearly one-fifth of the latter. Hence, adding to each number, in the line agreeing with 20°, one fifth of the difference between that and the corresponding number in the next line, we shall have

Angle of suspension = 20° 12′, DC = ·06556, DB = ·36010, CD = ·36797, tension at A = 1·10656.

Hence ·36010 : ·36797 :: 560 : 572·24, length of chain.

Also, horizontal pressure at C —— are as 1·0000
oblique pressure at A or B —— 1·10656
entire weight of chain 73594

Comparing this with the former case, it will be seen that the tensions at C and A, in reference to the weight of the chain, are diminished nearly in the inverse ratio of the two values of DC; thus confirming the remark in art. 20.

In practical cases with regard to bridges of suspension, it will be easy, when the weight of the material and its cohesive strength are known, to find the relative strength of any proposed structure.

CHAP. VII.

Professor Farish's Isometrical Perspective.

In the course of Lectures which I deliver in the University of Cambridge, I exhibit models of almost all the more important machines which are in use in the manufactures of Britain.

The number of these is so large, that had each of them been permanent and separate, on a scale requisite to make them work, and to explain them to my audience, I should, independently of other objections, have found it difficult to have procured a warehouse large enough to contain them. I procured therefore an apparatus, consisting of what may be called a system of the first principles of machinery; that is, the separate parts, of which machines consist. These are made chiefly of metal, so strong, that they may be sufficient to perform even heavy work: and so adapted to each other, that they may be put together at pleasure, in every form, which the particular occasion requires.

Those parts are various: such as, loose brass wheels, the teeth of which all fit into one another: axes, of various lengths, on any part of which the wheel required may be fixed: bars, clamps, and frames; and whatever else might be necessary to build up the particular machines which are wanted for one lecture. These models may be taken down, and the parts built up again in a different form for the lecture of the following day. As these machines, thus constructed for a temporary purpose, have no permanent existence in themselves, it became necessary to make an accurate representation of them on paper, by which my assistants might know how to put them together without the necessity of my continual superintendance. This might have been done, by giving three orthographic plans of each; one on the horizontal plane, and two on vertical planes at right angles to each other. But such a method, though in some degree in use among artists, would be liable to great objections. It would be unintelligible to an inexperienced eye; and even to an artist, it shows but very imperfectly that which is most essential, the connexion of the different parts of the engine with one another; though it has the advantage of exhibiting the lines parallel to the planes, on which the orthographic projections are taken on a perfect scale.

This will be easily understood, by supposing a cube to be the object represented. The ground plan would be a square representing both the upper and lower surfaces. And the two elevations would also be squares on two vertical planes, parallel to the other sides of the cube. The artist would have exhibited to him three squares; and he would have to discover how to put them together in the form of a cube, from the circumstance of there being two elevations and a ground plan. This method, therefore, giving so little assistance on so essential a point, I thought unsatisfactory.

The taking a picture on the principles of common perspective, was the next expedient that suggested itself. And this might be adapted to the exhibition of a model, by taking a kind of bird's-eye view of the object, and having the plane of the picture, not as is most common in a drawing, perpendicular to the horizon, but to a line, drawn from the eye, to some principal part of the object. For example: in taking the picture of a cube, the eye might be placed in a distant point on the line which is formed by producing the diagonal of the cube. But to this common perspective, there are great objections. The lines, which in the cube itself are all equal, in the representation are unequal. So that it exhibits nothing like a scale. And to compute the proportions of the original from the representation, would be exceedingly difficult, and, for any useful purpose, impracticable: there is equal difficulty too, in computing the angles which represent the right angles of the cube. Neither does the representation appear correct, unless the eye of the person, who looks at it, be placed exactly in the point of sight. It is true that, as we are continually in the habit of looking at such perspective drawings, we get the habit of correcting, or rather overlooking the apparent errors which arise from the eye being out of the point of sight, and are therefore not struck with the appearance of incorrectness, which if we were unaccustomed to it, we should feel at once.

The kind of perspective which is the subject of this paper, though liable in a slight degree to the last-mentioned inconvenience, till the eye becomes used to it, I found much better adapted to the exhibition of machinery. I therefore determined to adopt it, and set myself to investigate its principles, and to consider how it might most easily be brought into practice.

It is preferable to the common perspective on many accounts, for such purposes. It is much easier and simpler in its principles. It is also, by the help of a common drawing-table, and two rulers,* incomparably more easy, and, consequently, more accurate in its application; insomuch, that there is no difficulty in giving an almost perfectly correct representation of any object adapted to this perspective, to which the artist has access, if he has a very simple knowledge of its principles, and a little practice.

It farther represents the straight lines which lie in the three

* It is unnecessary to describe the drawing-table any further than by observing that it ought to be so contrived, as to keep the paper steady on which the drawing is to be made.

Here should be a ruler in the form of the letter T to slide on one side of the drawing-table. The ruler should be kept, by small prominences on the under side, from being in immediate contact with the paper, to prevent its blotting the fresh drawn lines as it slides over them. And a second ruler, by means of a groove near one end on its under side, should be made to slide on the first. The groove should be wider than the breadth of the first ruler, and so fitted, that the second may at pleasure be put into either of the two positions represented in the plate, fig. 1, so as to contain with the former ruler, in either position, an angle of 60 degrees. The groove should be of such a size, that when its shoulders *a* and *d* are in contact with, and rest against the edges of the first ruler, the edge of the second ruler should coincide with *d e*, the side of an equilateral triangle described on *d g*, a portion of the edge of the first ruler; and when the shoulders *b* and *c* rest against the edges of the first ruler, the edge of the second should lie along *g e*, the other side of the equilateral triangle. The second ruler should have a little foot at *k* for the same purpose as the prominences on the first ruler, and both of them should have their edges divided into inches, and tenths, or eighths of inches.

It would be convenient if the second ruler had also another groove *r s*, so formed that when the shoulders *r* and *s* are in contact with the edges of the first ruler, the second should be at right angles to it. For representing circles in their proper positions, the writer made use of the inner edge of rims cut out from cards, into isometrical ellipses as represented in the figure; of these he had a series of different sizes, corresponding to his wheels. Such a series might be cut by help of the concentric ellipses in fig. 5, but he thinks that it would be an easier way to make use of that set of concentric ellipses as they stand, by putting them in the proper place under the picture, if the paper on which the drawing is made, be thin enough for the lines to be traced through, as by the help of them the several concentric circles will go to the representation of one which might be drawn at once. It is difficult to execute them separately with sufficient accuracy to make them correspond. For this purpose a separate plate of fig. 5 should be had, and one edge of the paper on the drawing-table should be loose to admit of the concentric ellipses being slid under it to the proper place, as described, p. 185.

principal directions, all on the same scale. The right angles contained by such lines are always represented either by angles of 60 degrees, or the supplement of 60 degrees. And this, though it might look like an objection, will appear to be none on the first sight of a drawing on these principles, by any person who has ever looked at a picture. For, he cannot for a moment have a doubt, that the angle represented is a right angle, on inspection.

And we may observe further, that an angle of 60 degrees is the easiest to draw of any angle in nature. It may be instantly found by any person who has a pair of compasses, and understands the first proposition of Euclid. The representation, also, of circles and wheels, and of the manner in which they act on one another is very simple and intelligible. The principles of this perspective which, from the peculiar circumstance of its exhibiting the lines in the three principal dimensions on the same scale, I denominate "*Isometrical*," will be understood from the following detail:

Suppose a cube to be the object to be represented. The eye placed in the diagonal of the cube produced. The paper, on which the drawing is to be made to be perpendicular to that diagonal, between the eye and the object, at a due proportional distance from each, according to the scale required. Let the distance of the eye, and consequently that of the paper, be indefinitely increased, so that the size of the object may be inconsiderable in respect of it.

It is manifest, that all the lines drawn from any points of the object to the eye may be considered as perpendicular to the picture, which becomes, therefore, a species of orthographic projection. It is manifest, the projection will have for its outline an equiangular and equilateral hexagon, with two vertical sides, and an angle at the top and bottom. The other three lines will be radii drawn from the centre to the lowest angle, and to the two alternate angles; and all these lines and sides will be equal to each other both in the object and representation: and if any other lines parallel to any of the three radii should exist in the object, and be represented in the picture, their representations will bear to one another, and to the rest of the sides of the cube, the same proportion which the lines represented bear to one another in the object.

If any one of them, therefore, be so taken as to bear any required proportion to *its* object, *e. g.* 1 to 8, as in my representations of my models, the others also will bear the same proportion to *their* objects; that is, the lines parallel to the three radii will be reduced to a scale.

I omit the demonstration of this, and some other points, partly for the sake of brevity, and partly because a geometrician will find no difficulty in demonstrating them himself, from the nature of orthographic projection; and a person, who is not a geometrician, would have no interest in reading a demonstration.

For the same reason, it is unnecessary to show that the three angles at the centre are equal to one another, and each equal to 120 degrees, twice the angle of an equilateral triangle; and the angle contained between any radius and side is 60 degrees, the supplement of the above, and equal to the angle of an equilateral triangle. All this follows immediately from Euclid, B. IV. Prop. 15, on the inscription of a hexagon in a circle.

In models, and machines, most of the lines are actually in the three directions parallel to the sides of a cube, properly placed on the object. And the eye of the artist should be supposed to be placed at an indefinite distance, as before explained, in a diagonal of the cube produced.

Definitions.

The last mentioned line may be called the *line of sight.*

Let a certain point be assumed in the object, as for example c, fig. 2, Pl. I. and be represented in the picture, to be called, the *regulating point*. Through that point on the picture may be drawn a vertical line, c E, fig. 2, and two others, c B, c G, containing with it, and with one another, angles of 120°, to be called the *isometrical lines*, to be distinguished from one another by the names of the *vertical*, the *dexter*, and the *sinister* lines. And the two latter may be called by a common name—the *horizontal isometrical* lines. Any other lines, parallel to them, may be called respectively by the same names. The plane passing through the dexter, and vertical lines, may be called the *dexter isometrical plane;* that passing through the vertical, and sinister lines, the *sinister plane;* and that through the dexter and sinister lines, the *horizontal plane*.

By the use of the simple apparatus described above in the note, the representation of these lines in the objects may be drawn on the picture, and measured to a scale, with the utmost facility, the point at the extremity being first found, or assumed. The position of any point in the picture may be easily found, by measuring its three distances, namely, first its perpendicular distance from the *regulating horizontal plane* (that is, the horizontal plane passing through the regulating point), secondly, the perpendicular distance of that point where the perpendicular meets the horizontal plane, from the regulating dexter line; and thirdly, of the point, where that perpendicular meets the dexter line from the regulating point; and then taking those distances reduced to the scale, first, along the dexter line, secondly, along the sinister line, and thirdly, along the vertical line, in the picture. These three may be called the *dexter distance* of the point, its *sinister distance*, and its *altitude*. And it is manifest they need not be taken in this order, but in any other that may be more convenient to the artist, there being six ways in which this operation may be varied.

If any point in the same isometrical plane, with the point required to be found, is already represented in the picture, that point may be assumed as a new regulating point, and the point required found by taking two distances; and if the new assumed regulating point is in the same isometrical line with the point, it is found by taking only one distance. And this last simple operation will be found in practice all that is necessary for the determination of most of the points required. Thus any parallelopided, or any frame work, or other object with rafters, or lines lying in the isometrical directions, may be most easily and accurately exhibited on any scale required. But if it be necessary to represent lines in other directions, they will not be on the same scale, but may be exhibited, if straight lines, by finding the extremities as above, and drawing the line from one to the other; or sometimes more readily in practice by help of an ellipse, as hereafter described.

If a curved line be required, several points may be found sufficient to guide the artist to that degree of exactness which is required.

The method of exhibiting the representations of any machines, or objects, the lines of which lie, as they generally do, in the isometrical directions; that is, parallel to the three directions of the lines of the cube, is as has been already shown; and likewise the mode of representing any other straight lines, by finding their extremities; or curved lines, by finding a number of points.

But in representing machines and models, there are not only isometrical lines, but also many wheels working into each other, to be represented. These, for the most part, lie in the isometrical planes. And it is fortunate that the picture of a circle in any one of these planes is always an ellipse of the same form, whether the plane be horizontal, dexter, or sinister;

yet they are easily distinguished from each other by the position in which they are placed on their axle, which is an isometrical line, always coinciding with the minor axis of the ellipse.

This will be obvious from considering the picture of a cube with a circle inscribed in each of its planes, fig. 3, and considering these circles as wheels on an axle. The two other lines (or spokes of the wheel) in the ellipse, which are drawn respectively through the opposite points of contact of the circle with the circumscribing figure, are isometrical lines also; for the points of contact bisect the sides of the circumscribing parallelogram, and therefore the lines are parallel to the other sides. They give likewise the true diameter of the wheels, reduced to the scale required. It further appears from the nature of orthographic projection, that the major axis of the ellipse is to the minor axis, as the longer to the shorter diagonal of the circumscribing parallelogram, that is (since the shorter diagonal divides it into two equilateral triangles), as the square root of three to one; as appears from Euclid, Lib. I. Prop. 47: and since the sum of the squares of the conjugate diameters in an ellipse is always the same, if we put $\sqrt{1}$ for the minor axis, the $\sqrt{3}$ for the major, and i for the isometrical diameter, we shall have $2i^2 = 1 + 3, = 4$, and $i = \sqrt{2}$.

Therefore the minor axis, the isometrical diameter, and the major axis, may be represented respectively by $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, or nearly by 1, 1·4142, 1·7321; or more simply, though not so nearly, by 28, 40, 49.

These lines may be geometrically exhibited by the following construction:

Let A B, fig. 4, be equal to B D, and the angle at B, a right angle. In B A produced, take B α = to A D draw α D, and produce both it, and α B. Then will B D, B α, and α D, be respectively to one another, as $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$ by Euclid I. 47. Therefore if $\alpha\ \beta$ be taken equal to the isometrical diameter of the ellipse required, $\beta\ \delta$ drawn perpendicular to it will be the minor axis, and $\alpha\ \delta$ the major axis. The ellipse itself, therefore, may be drawn by an elliptic compass, as that instrument may be properly set, if the major and minor axes are known. If it is to represent a wheel on an axle, care must be taken to make the minor axis lie along that axle. In the absence of the instrument it may be drawn from the concentric ellipses, fig. 5, which may be placed under the paper, in the position above described, and seen through it; if the paper be not too thick, and in this method the smaller concentric circles of the wheel may be described at the same time, as they may be seen through the paper, or if they should not be exactly of the right size, it would be easy to describe them by hand between the two nearest concentric ellipses; and thus also the height of the cogs of a wheel in the different parts of it may be exhibited longer and narrower towards the extremities of the minor axis. Their width may be determined from the divisions of the ellipse. In most cases this may be done with sufficient accuracy from the circumference of the ellipse being divided into eight equal divisions of the circle, by the two axes, and two isometrical diameters, each of which parts may be subdivided by the skill of the artist; and not only the face of the wheel in front may be thus exhibited, but the parts of the back circles also, which are in sight, may be exhibited by pushing back the system of concentric ellipses on the minor axis or axle through a distance representing the breadth of the wheel, and then tracing both the exterior and the interior circles of the wheel, and of the bush on which it is fixed, as far as they are visible. Care should be taken to represent the top of the teeth, or cogs, by isometrical lines, parallel to the axle, in a face-wheel, or tending to a proper point in the axle in a bevil-wheel. And nearly in the same way may the floats of a water-wheel be correctly represented. If a series of concentric ellipses, such as are given, fig. 5, be not at hand, it will still be easy for an artist to draw the ellipses with sufficient accuracy for most purposes, by drawing through the proper point in the axle, the major, and minor axes, and the two isometrical diameters, thus making eight points in the circumference to guide him.

If in any case it should become necessary to represent a circle, which does not lie in an isometrical plane, we may observe that the major axis will be the same in whatever plane it lies: and it will be the picture of that diameter, which is the intersection of the circle with the plane parallel to the picture, passing through its centre. And the major axis will bear to the minor axis the proportion of radius to the sine of the inclination of the line of sight to the plane of the circle. We may observe further, that the diameters of the ellipse, which are to the major axis, as $\sqrt{2}$ to $\sqrt{3}$, when such exist, are isometrical lines.*

* We may remark, that if a cone be described, having its vertex at c which lies in the line of sight, fig. 2, and passing through the three radii ca,

And the representation of every other line parallel, and equal to any diameter of the circle, may be exhibited by drawing it equal and parallel to the corresponding diameter in the ellipse. If it should be desired to divide the circumference of an ellipse into degrees, or any number of parts representing given divisions of the circle, it may be done by the following method.

Let an ellipse be drawn, fig. 6, and on its major axis, A G, a circle described, with its circumference divided into degrees or parts in any desired proportion, at B, C, D, E, F, &c. from which points draw perpendiculars to the major axis. They will cut the periphery of the ellipse in corresponding points. It would be difficult, however, in this way, to mark, with sufficient accuracy, the degrees, which lie near the extremities of the major axis. But the defect may be supplied by transferring those degrees in a similar way, from a graduated circle, described on the minor axis. In this manner an isometrical ellipse may be formed into an isometrical circular instrument, or an isometrical compass, which may show bearings or measure angles on the picture, in the same manner as a real compass or circular instrument would do in nature.

It may be often useful to have a scale to measure distances, not only in the isometrical directions, but in others also. And this may be done by a series of similar concentric ellipses, as in fig. 7, dividing the isometrical diameters into equal portions. The other diameters will be so divided as to serve for a scale for all lines parallel to them respectively.

Thus, in the isometrical squares, exhibited in fig. 2, distances measured on the longer diagonal, or its parallels, would be measured by the divisions on the major axis, those depending on the shorter diagonal by the divisions on the minor axis.

To describe a cylinder lying in an isometrical direction, the circles at its extremities should be represented by the proper isometrical ellipses, and two lines touching both should be

C B, C G, all the straight lines in the superficies of that cone passing through C, and all other lines parallel to any of them, are isometrical, as well as those parallel to the three principal isometrical lines, C B, C E, C G; and no other lines but these can be on the same scale. But though these multiply the number of isometrical lines infinitely, it is of little practical use, because it is only those which are parallel to the three principal lines, that can be easily distinguished at sight, to be isometrical.

We may further remark, that if a line be drawn through the point C parallel to any given line whatever, and that line be made to revolve round the line of sight, at the same angular distance from it, so as to describe the surface of a cone, all other lines parallel to it, in any of its positions, will be isometrical, as they repect one another.

drawn: and in a similar way, a cone, or frustum of a cone, may be described. A globe is represented by a circle, whose radius is the semi-major axis of the ellipse representing a great circle.

It would not be difficult to devise rules for the representation of many other forms which might occur in objects to be represented. But the above cases are sufficient to include almost every thing which occurs in the representation of models, of machines, of philosophical instruments, and, indeed, of almost any regular production of art.

Buildings may be exhibited by this perspective as correctly, in point of measurement, as by plans and elevations, under the advantage of having the full effect of a picture.

A bridge, or any circular, or gothic arch, consisting of portions of circles lying in isometrical planes, may be represented by portions of isometrical ellipses, which will easily be adapted and drawn upon the principles already explained, by which wheels are exhibited on their axles. The centres of those circles must be found with which the centres of the ellipses must be made to coincide, their minor axes lying along the lines drawn from those centres perpendicular to the planes of the circles. The shaft of a pillar consists of a frustum of a cone and a cylinder united; or perhaps of a cylinder alone, or a congeries of cylinders: and we have already shown the method of exhibiting these, as well as their bases. And on the same principles, the position and size of the volutes and ornaments of the capital may be found, and such guiding points as will make it easy to trace their forms. Thus the different courts and edifices of a cathedral, a college, or a palace, may be correctly depicted; and even the rooms and internal structure, though less in the form of a picture, may be exhibited in such a way as to enable an architect, or his employer, to contemplate their situation, their ornaments, furniture, or any other circumstance belonging to their appearance, and to mark down exactly what he would have done, in such a way as could hardly be misunderstood by an attentive agent, though at a distance.

But in thus exhibiting buildings as transparent, and their interior laid open, there is a danger of being confused by a multiplicity of lines, which is a difficulty in a building containing many rooms, that would need some address to get over. It is better adapted to exhibit the inside of a single room; of a cathedral, for instance, the aisles and transepts of which would not cause any great perplexity.

In the same manner a plan of a city might be given, which would not only represent its streets and squares, as well (by the help of the scale above described fig. 7,) as a common plan, but also a picture of its churches and public buildings, and even its private houses, if such were the design contemplated by the artist, as they would almost all become visible when looked down upon from the commanding height which this perspective supposes. And such a single exhibition, if well executed, might give a better idea of a distant capital than a volume of description.

In the instances which have been given, most of the lines are isometrical. But the art is applicable to many cases, where there are few, or none such. It may be necessary, in many of them, to draw isometrical lines, or isometrical ellipses, by way of a guide, to determine the position of certain lines and points to enable the artist to describe with accuracy what he has in view. And there is scarce any form so anomalous as to preclude the artist from taking advantage of these methods of ascertaining such lines or points in it as will give him much assistance in representing it with precision. If the intention be merely to make a picture, the guiding lines may be obliterated as soon as they have served the purpose designed, or they may be retained in some cases, and their lengths or diameters noted down in figures, if it be wished, to give ready information. And often, if the artist wishes to provide materials to enable him at his leisure to give accurate descriptions or exact drawings, the rudest exhibition of such lines may completely serve his purpose, provided he notes down on the spot such measurements with accuracy, however unexact the lines may be on which they are recorded. In many cases it may be expedient to take liberties with this perspective, or with the picture, which will make it suit the purpose designed. And this will produce no confusion provided those liberties are explained: for instance, it may often be expedient to make the scale in the vertical direction larger, sometimes very considerably so, than in the horizontal. It may in some cases be necessary to represent on paper what is hid in nature. What has been said on the internal structure of buildings is an instance of this as well as what we shall observe on the exhibition of subterraneous objects. We shall proceed to give some examples of these observations.

To give such a representation of an Etruscan vase, as would enable an artist to model it exactly, would be exceedingly easy. Let a vertical line be drawn to represent the axis of the vase, fig. 8, and let points be taken in that axis, corresponding to the centres of the principal circles of the vase; through which the horizontal isometrical lines may be drawn, representing the radii of those circles, by the help of which the isometrical ellipses representing them are easily drawn. These will become a complete guide to the artist. He may assist himself by looking at the object along the line of sight, and then, if he has any skill in drawing, he will find no difficulty in tracing the outline from one of these to the other, with sufficient correctness. If he is unskilled in the art, of course he must be at the trouble of finding a larger number of ellipses to guide him. And in a similar manner, any solid formed by the revolution of a plane figure round one of its sides may be represented.

The laying down the timbers of a ship, or making a picture of one, shall be another example.

Let a vertical isometrical plane be conceived to pass through its keel, and to be intersected by the perpendicular planes passing through the ribs, and by planes parallel to the decks. The isometrical lines, which are the intersections of these, may be measured in the ship, and represented with their proper measures noted down in the picture, which will afford the means of representing the ribs, and laying them down in their proper places.

If this should be designed for the purpose of constructing a ship from a given model, it might be sufficient to represent the ribs only on one side; those on the other side being the exact counterparts. If the purpose should be to make use of these lines for a drawing, they need be marked but very faintly, and the artist will have little difficulty when guided by them to fill up the representation by hand.

A regular fortification, which we will suppose to have eight bastions, will afford another example.

A person not conversant in such a subject, is in general puzzled with plans and sections, and has very little idea of what is meant to be conveyed.

But he would easily understand it if he should see every thing exhibited in a correct picture, especially where he has the view of his object varied, as in a fortification, such as has been proposed. Let an isometrical ellipse be drawn expressing the internal circumference of the place; and another concentric one, which marks the salient angles of the fortification on the principles already explained. Draw other guiding lines to every necessary point; the lines of the fortification may be easily transferred from a common plan to the isometrical by the help of the scale of concentric ellipses described above, fig. 7, which will serve also to lay down the length of the bastions and curtains, &c. in whatever direction they lie. Find the elevations of every part on the isometrical scale; and thus the body of the place, the ditches, counterscarp, covered way, glacis, ravellins, and all the outworks, will be represented to the eye as they appear in reality, and in every varied position, with the advantage of having all the admeasurements laid down with geometrical precision.

If the artist should think the vertical lines in such an exhibition too small to give a correct idea of all the minute elevations, there would be no harm in his increasing the scale in that dimension in any desired proportions.

The face of a hilly or mountainous country like Switzerland, or the district of the lakes in the north part of England, will afford another example.

Isometrical horizonal lines may be drawn representing lines in the level from which the height of the mountains is to be reckoned, so that vertical lines drawn from the summits of the mountains may meet them, on which the heights may be marked; (as well as recorded in figures, if required). And the mountains themselves may be drawn in their topographical situation. Their bearings may be marked by the help of the isometrical compass described in page 187. It would be easy to transfer them from a common map to the isometrical plan; and thus the face of the country might be represented just as it would appear from the commanding height which the isometrical perspective supposes.

Yet, as the slopes of hills and mountains are seldom so steep as the line of sight, it might sometimes suit the purpose to represent the height of elevations as twice or three times the reality, in order that mountains might project an outline on the plane behind; otherwise, the summit might be projected on the mountain itself, which would, in a degree, destroy the effect of a picture.

This art might be advantageously employed also for tracing what is below the surface of the earth, as well as what is above it. It may be applied to geological purposes, and give, not only the order of the strata, but their variations and their geographical situations. And for this purpose it might be useful to increase the vertical scale, in a great proportion, above the horizontal. It would be easy to mark the dip, or rise of the strata, as well as of the earth above them: to represent their various disruptions, to show the situation and extent of fissures and metallic veins, to mark the boundaries where the upper strata have been swallowed up, or cease to appear, or where the under strata push up towards the day. It would be easy to mark the variations in the thickness of the strata in different places, and to record the result of experiments made at any point, by boring or sinking shafts, which might be done by drawing a vertical line downward, so as to represent the thickness of the laminæ, which might be marked by different colours. By such a method, the geologist might obtain a map of the country, which might exhibit, at one view, the general results of all the experiments and inquiries that had been made relative to that science. And the owner of an estate might record in a small compass, all that is known respecting its minerals, and be able from a comprehensive view of them all, to judge of the probability of success in sinking a shaft, or driving a level. He might also make good use of this perspective in tracing his shafts and drifts in all their windings, elevations, and depressions, and comparing them with the surface above, marking also the veins and strata in which they run. For if the artist knows what is beneath the surface, he has no difficulty in representing it as transparent. He must be careful however not to perplex himself by lines too much multiplied, and take advantage of his being able to paint the lines with different colours, for the purposes of distinction; and he must use a considerable address in throwing out such lines as would be of little use, and retaining such as will produce the effect of a picture, which should be well preserved in order to make the exhibition easily intelligible.

If he should wish to make a drawing of minerals or crystals, this perspective would be well suited to the purpose.

The point, however, on which the writer of this paper can speak with the greatest confidence is on the representation of machines and philosophical instruments; having been himself so much in the habit of practically applying to them the principles that have been detailed: and this he has exemplified in the plates.

The correct exhibition of objects would be much facilitated by the use of this perspective, even in the hands of a person who is but little acquainted with the art of drawing; and the information given by such drawings is much more definite and precise than that obtained by the usual methods, and better fitted to direct a workman in execution.*

* The author has transcribed this interesting paper from the first volume of the Transactions of the Cambridge Philosophical Society. The method is peculiarly deserving of the attention of mechanics and engineers.

CHAP. VIII.

MENSURATION OF SUPERFICIES AND SOLIDS.

SECTION I. *Mensuration of Superficies.*

The following rules will serve to find the areas or superficial contents of the figures whose names respectively precede them.

1. *Rectangle, Square, Rhombus, or Rhomboid.*—Multiply the base into the height, for the area.

2. *Triangle.* Half base into the height. Or, continual product of two sides, and half the natural sine of their included angle.

Or, when three sides, as AB, AC, BC, are given, their half sum being s; then

area $= \sqrt{[s \times (s - AB) \times (s - AC) \times (s - BC)]}$.

3. *Trapezium.* Base into $\frac{1}{2}$ sum of the perpendiculars.

4. *Trapezoid.* Multiply half the sum of the parallel sides into the perpendicular distance between them.

5. *Irregular Polygon.* Divide it into trapeziums, or trapezoids, and triangles, and find their areas separately: their sum is the area of the polygon.

6. *Regular Polygon.* Multiply the square of the side given into the proper "multiplier for areas," printed in the table in Prob. 15, *Practical Geometry*, the product will be the area.

7. *Circle.* diameter : circumf :: 113 : 355
or, diameter : circumf :: 1 : 3·141593
circumf : diameter :: 1 : ·318309
area = diameter squared × ·785398
area = circumference squared × ·079577
area = $\frac{1}{2}$ diameter × $\frac{1}{2}$ circumf.

8. *Circular arc.* Radius of the circle × ·017453 × degrees in the arc = its length.

9. *Circular sector.* $\frac{1}{2}$ arc × radius = area.

10. *Circular segment.* Multiply the square of the radius by

either half the *difference* of the arc (of the segment) and its sine, or by half their *sum*, according as the segment is less or greater than a semicircle: the product will be the area.

11. *Parabola.* $\frac{2}{3}$ of the product of base and height = area.

12. *Ellipse.* Transverse axis × conjugate axis × ·785398 = area.

Section II. *Mensuration of Solids.*

1. *Prism.* 1. *Superficies.* Multiply the perimeter of one end by the length or height of the solid; the product will be the surface of the sides. To this add the areas of the two ends: the sum is the whole surface.

2. *Solidity* or *Capacity* = area of the base × the height.

Note. The same rules serve for the surface and capacity of a cylinder.

2. *Pyramid, or Cone.* 1. Surface = $\frac{1}{2}$ perimeter of the base × slant height.

2. *Capacity* = area of base × $\frac{1}{3}$ height.

3. *Frustum of Pyramid.* 1. Surface = $\frac{1}{2}$ sum of the perimeters of the two ends × slant height.

2. *Capacity.* Add a diameter or a side of the greater base to one of the less; from the square of the sum subtract the product of the said two diameters or sides: multiply the remainder by a third of the height; and this last product by ·785398 for circles, or by the proper multiplier for polygons; the last product will be the capacity.

That is, capacity = $[(D + d)^2 - Dd]\ \frac{1}{3}\ m\ h$.

4. *Sphere.* 1. *Surface* = diameter squared × 3·141593

2. *Capacity* = diameter cubed × ·5236

or = circumference cubed × ·016887.

5. *Spheric segment.* 1. *Surface* = circumf. sphere × height of segment.

2. *Capacity* = ·5236 $h^2 (3d - 2h)$; where d = diam. h = height.

= ·5236 $h^2 (3r^2 + h^2)$; where r = rad. of the segment's base.

5. *Paraboloid.* *Capacity* = half base × height.

This is a figure produced by the rotation of a parabola upon its axis.

6. *Spheroid.* This is a solid generated by the revolution of an ellipse about one of its axes. To find its capacity multiply

the square of the revolving axis by the fixed axis, and that product by ·5236.

7. *Regular* or *platonic bodies*, are bodies comprehended by like, equal, and regular plane figures, and whose solid angles are all equal.

There are only five regular solids, viz.

The tetraedron, or regular triangular pyramid, having 4 triangular faces;

The hexaedron, or cube, having 6 square faces;

The octaedron, having 8 triangular faces;

The dodecaedron, having 12 pentagonal faces;

The icosaedron, having 20 triangular faces.

PROB. I. *To find either the surface or the solid content of any of the regular bodies.*—Multiply the proper tabular area or surface (taken from the following table) by the square of the linear edge of the solid, for the superficies. And

Multiply the tabular solidity in the last column of the table by the cube of the linear edge for the solid content.

Surfaces and Solidities of Regular Bodies, the side being unity or 1.

No. of sides.	Name.	Surface.	Solidity.
4	Tetraedron	1·7320508	0·1178513
6	Hexaedron	6·0000000	1·0000000
8	Octaedron	3·4641016	0·4714045
12	Dodecaedron	20·6457288	7·6631189
20	Icosaedron	8·6602540	2·1816950

2. The diameter of a sphere being given to find the side of any of the platonic bodies, that may be either inscribed in the sphere, or circumscribed about the sphere, or that is equal to the sphere.

Multiply the given diameter of the sphere by the proper or corresponding number, in the following table, answering to the thing sought, and the product will be the side of the platonic body required.

The diam. of a sphere being 1; the side of a	That may be inscribed in the sphere, is	That may be circumscribed about the square, is	That is equal to the sphere, is
Tetraedron	0·816497	2·44948	1·64417
Hexaedron	0·577350	1·00000	0·88610
Octaedron	0·707107	1·22474	1·03576
Dodecaedron	0·525731	0·66158	0·62153
Icosaedron	0·356822	0·44903	0·40883

3. The side of any of the five platonic bodies being given to find the diameter of a sphere, that may either be inscribed in that body, or circumscribed about it, or that is equal to it. As the respective number in the table above, under the title *inscribed*, *circumscribed*, or *equal*, is to 1, so is the side of the given platonic body to the diameter of its inscribed, circumscribed, or equal sphere.

4. The side of any one of the five platonic bodies being given to find the side of the other four bodies, that may be equal in solidity to that of the given body.—As the number under the title *equal* in the last column of the table above, against the given platonic body, is to the number under the same title, against the body whose side is sought, so is the side of the given platonic body to the side of the body sought.

Besides these there are *thirteen* demiregular bodies, called *Solids of Archimedes*. They are described in the Supplement to *Lidonne's Tables de Tous les Diviseurs des Nombres*, &c. Paris, 1808.

Section III. *Approximate Rules.*

1. When the area of a field is known in *square yards*, to reduce them to acres, instead of dividing by 4840, multiply by ·0002$\frac{1}{16}$, which is much easier.

Thus, suppose the area is found to be 56870 yards.

Then	56870
	·0002
Take $\frac{1}{16}$ of this,	11·3740
Or $\frac{1}{3}$ of $\frac{1}{10}$ .... viz.	3791
The sum is	11·7531 acres: the *true* answer [is 11·75 acres.

2. *For regular polygons.* Let s be the side; then

area of trigon $= \frac{1}{3} s^2 + \frac{1}{10} s^2$

— pentagon $= s^2 + \frac{8}{10} s^2 - \frac{8}{100} s^2$

— hexagon $= 2 s^2 + \frac{6}{10} s^2$

— heptagon $= \frac{3}{100}$ of square of 11 s

— octagon $= \frac{1}{10}$ of square of 7 s $- \frac{7 s}{5}$

— nonagon $= 6 s^2 + \frac{2}{10} s^2 - \frac{2}{100} s^2$

— decagon $= 7 s^2 + \frac{7}{10} s^2$

— undecagon $= 9 s^2 + \frac{4}{100}$ of $9 s^2$

— dodecagon $= 11 s^2 + \frac{1}{5} s^2$.

Where a table of polygons is at hand it is best to employ it. In other cases, one or other of these approximations may occur to a well exercised memory.

Ex. 1. The side of a pentagon is 20. Required the area.

$s^2 = 400$
$\frac{8}{10} s^2 = 320$ . . . . 8 times $\frac{1}{10} s^2$

From their sum = 720
Take $\frac{8}{100} s^2 = 32$ . . . . $\frac{1}{10}$ of $\frac{8}{10} s^2$

Area required = 688 : the *true* area is 688·19.

Ex. 2. The side of an octagon is 20. Required the area.

Square of 7 s = 19600 = square of 140

From $\frac{1}{10}$ of ditto = 1960
Take . . . . . . $\frac{7 s}{5}$ = 28

Area of the octagon = 1932 : the *true* area is 1931·37.

Ex. 3. The side of a nonagon is 20. Required the area.

$s^2 = 400$, $6 s^2 = 2400$
Add $\frac{2}{10} s^2 =$ 80

From the sum 2480
Take $\frac{2}{100} s^2 =$ 8 $= \frac{1}{10}$ of $\frac{2}{10} s^2$.

Area of the nonagon = 2472 : the true area is 2472·72.

Ex. 4. The side of a dodecagon is 20. Required the area.

$$s^2 = 400,\ 11\,s^2 = 4400$$
$$\text{Add } \tfrac{1}{5}\,s^2 = \quad 80$$

Area of the dodecagon = 4480: the true area is 4478·46.

3. *Length of circular arc.* From 8 times the chord of half the arc, subtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc, nearly.

Note. The chord of half the arc is equal to the square root of the sum of the squares of the versed sine or height, and half the chord of the entire arc.

Or, apply a fine flexible string to the arc, then stretch it out straight, and measure it.

4. *Arc of a quadrant*, nearly equal to $1\frac{1}{9}$ times the chord of the quadrant.

This is true within about the 4000th part.

5. *Periphery of an ellipse.* Multiply the square root of half the sum of the squares of the two axes by 3·141593, and the product will be the periphery nearly.

Diminish this by its 200th part, and the result will be still more correct.

6. *Area of a circle* = nearly to $\frac{11}{14}$ of the square of the diameter. Or, multiply D^2 by 11, and divide by 2 and by 7.

This is true within the 2500th part: or, add to $\frac{7}{9}D^2$, $\frac{1}{100}$ of $\frac{7}{9}D^2$; the sum will be the area true to within the 5000th part.

Area of a circle = nearly to $\frac{7}{88}$ of the square of the circumference. Or, multiply c^2 by 7 and divide by 8 and 11.

True within the 2600th part.

7. *Circular segment.* To the chord of the whole arc, add $\frac{4}{3}$ of the chord of half the arc; multiply the sum by the versed sine or height of the segment; and $\frac{4}{10}$ of the product will be the area nearly.

$$\text{Or, } \left(c + \tfrac{4}{3}\sqrt{\tfrac{1}{4}c^2 + v^2}\right)\tfrac{4}{10}\,v = \text{area.}$$

8. *To find the content of irregular plane figures from an accurate plan.*

If the plan be not upon paper, or fine drawing pasteboard of uniform texture, let it be transferred upon such. Then cut out the figure separately close upon its boundaries; and cut out from the same paper *a square* of known dimensions according to the scale employed in drawing the plan. Weigh the two separately in an accurate balance, and the ratio of the weights will be the same as that of the superficial contents.

If great accuracy be required, cut the plan into 4 portions, called 1, 2, 3, 4. First, weigh 1 and 2 together, 3 and 4 together, and take their sum. Then weigh 1 and 3 together, 2 and 4 together, and take their sum. Lastly, weigh 1 and 4 together, 2 and 3 together, and take their sum. The mean of the four aggregate weights thus obtained, compared with the weight of the standard square, will give the ratio of their surfaces very nearly.

*** I have employed in this operation a balance which turns with the 100th part of a grain. The results are proportionally accurate.

9. *Prism.* L = length, B = breadth, D = depth, all in inches: then $\frac{9}{140000}$ L B D = content in yards, nearly.

If L, B, D, be in feet, as suppose the dimensions of a corn bin; then ·8 L B D = the content in Winchester bushels. This is about one bushel in 200 in defect.

Ex. Suppose L = 125 inches, B = 25, D = 24.

Then 125 × 25 = $\frac{12500}{4}$ = 3125;

and 3125 × 24 = 6250 × 12 = 75000

$\frac{9}{140000}$ of 75000

$$\begin{array}{r} 75000 \\ 9 \\ \hline 2\,)\,675000 \\ \hline 7000\,)\,337500 \\ \hline 4{\cdot}8214 \end{array}$$

4·8214 yards: the true answer is 4·8225

For *wrought iron square bars,* allow 100 inches in length of an *inch square* bar to *a quarter of a cwt.* in weight; and so in proportion. This is easily remembered, because the word *hundred* occurs twice.

An inch square *cast iron* bar would require 9 feet in length for a quarter of a cwt.

Or, take $\frac{1}{144}$ of the product of the breadth and thickness, each in eighths of an inch, the result is the weight of one foot in length, in avoirdupois pounds.

Or, one foot in length of an inch square bar weighs $3\frac{1}{3}$ pounds.

Bricks of the usual size require 384 to a *cubic yard.* A *rod* of brick work, brick and a half thick, requires 4356 bricks.

10. *Cylinder.* One-tenth diameter squared, ($\frac{1}{10}d^2$, d being taken in inches) gives the content in ale-gallons of a yard in length.

This rule gives a result defective only by the 376th part.

11. *Timber measuring.* Let L denote the length of a tree

in feet and decimals, and G the mean girth, taken in inches: then the following rules given by Mr. Andrews may be employed.

Rule 1.—No allowance for bark.

$\frac{L\ G^2}{2304}$ = cubic feet, customary, and $\frac{L\ G^2}{1807}$ = cubic feet, true content.

Rule 2.—To allow $\frac{1}{8}$th for bark.

$\frac{L\ G^2}{3009}$ = cubic feet, customary, and $\frac{L\ G^2}{2360}$ = cubic feet, true content.

Rule 3.—To allow $\frac{1}{10}$th for bark.

$\frac{L\ G^2}{2845}$ = cubic feet, customary, and $\frac{L\ G^2}{2231}$ = cubic feet, true content.

Rule 4.—To allow $\frac{1}{12}$th for bark.

$\frac{L\ G^2}{2742}$ = cubic feet, customary, and $\frac{L\ G^2}{2150}$ = cubic feet, true content.

EXAMPLE by *Rule* 1.—No allowance for bark.

A tree 40 feet long, and 60 inches whole girth or circumference.

$\frac{40 \times 60^2}{2304} = 62\frac{1}{2}$ cubic ft. customary, and $\frac{40 \times 60^2}{1807} = 79\frac{2}{3}$ cubic ft. true content.

Ex. by *Rule* 2.—A tree 50 feet long, and 49 inches circumference.

$\frac{50 \times 49^2}{3009} = 40$ cubic ft. customary, and $\frac{50 \times 49^2}{2360} = 50\frac{3}{4}$ cubic ft. true content.

If G as well as L be in feet, then $\cdot 08\ L\ G^2$ = content, nearly.

*** When the difference between the girths at the two ends is considerable, it is best to find the content of the tree as though it were a conic frustum, and make the usual allowances afterwards.

12. *Sphere.* $\frac{11}{21}$ of the cube of the diameter = capacity. Use the component factors, 3 and 7, in dividing by 21.

This rule gives a result true to its 2600th part.

Or, multiply the cube of the circumference by $\cdot 0169$, for the capacity.

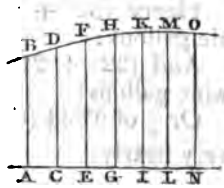
13. *To find the capacity, or solid content of an irregular body.* —Procure a prismatic or cylindric vessel that will hold it. Put in the body, and then pour in water to cover it, marking the height to which the water reaches. Then take out the body, and observe accurately how much the water has descended in consequence. The capacity of the prism or cylinder thus left dry by the water, will be evidently equal to that of the body.

If the vessel will not hold water, sand may be employed, though not with quite so much accuracy.

In this manner, too, a *portion* of a body may be measured without detaching it from the rest, by simply immersing that portion.

Even an *irregular* vessel may be employed for this purpose. In which case it should be placed in a larger vessel, and then filled with water. Then submerge the body whose magnitude you wish to determine, the quantity of water that has run over, and is caught in the exterior vessel, will be the measure. It may be weighed, and its cubic measure estimated by allowing 1000 avoirdupois ounces to the cubic foot. (See farther, *Hydrostatics* and *Specific Gravity.*)

14. *To find the contents of surfaces and solids not reducible to any known figure, by the equidistant ordinate method.*—The general rule is included in this proposition: viz. If any right line A N be divided into any *even* number of equal parts, A C, C E, E G, &c. and at the points of division be erected perpendicular ordinates A B, C D, E F, &c. terminated by any curve B H O: then, if A be put for the sum of the first and last ordinates, A B, N O, B for the sum of the *even* ordinates, C D, G H, L M, &c. viz. the second, fourth, sixth, &c. and C for the sum of all the rest, E F, I K, &c. viz. the third, fifth, &c. or the *odd* ordinates, excepting the first and last: then, the common distance A C, C E, &c. of the ordinates being multiplied into the sum arising from the addition of A, four times B, and twice C, *one third* of the product will be the area A B O N, very nearly.



$$\text{That is, } \frac{A + 4B + 2C}{3} \cdot D = \text{area, } D \text{ being} = AC = CE, \&c.$$

The same theorem will equally serve for the contents of all solids, by using the sections perpendicular to the axis instead of the ordinates. The proposition is quite accurate, for all parabolic and right lined areas, as well as for all solids generated by the revolutions of conic sections or right lines about axes, and for pyramids and their frustrums. For other areas and solidities it is an excellent approximation.

The greater the number of ordinates, or of sections, that are taken, the more accurately will the area or the capacity be determined. But in a great majority of cases *five* equidistant ordinates, or sections, will lead to a very accurate result.

In *cask-gauging*, indeed, three sections will be usually sufficient. Thus, taking the bung and head diameters, and a diameter mid-way between them; the sum of the squares of the bung and head diameters, and of the square of double the middle diameter, multiplied into the length of the cask, and then into ·785398, will give *six times* the content of the cask, very nearly. Or, if H, B, and M, represent the head, bung, and middle diameters respectively, and L the length, all in inches; then $(H^2 + 4M^2 + B^2) \times L \times \cdot 1309$ = content of the cask in inches.

A similar method may be advantageously adopted in all cases of ullaging either standing or lying casks, by taking the areas at the top, bottom, and middle, of the liquor. See *Hutton's Mensuration*, part iv.

EXAMPLE. The bung diameter of a cask is 32 inches, the head diameter 24, the middle 30·2, the length 40. Required the content in ale gallons of 282 cubic inches, and in wine gallons of 231.

The former multiplier, divided by 282 and 231 respectively, gives $\cdot 0004\frac{7}{11}$ and $\cdot 0005\frac{2}{3}$, for the proper multipliers.

Hence $(32^2 + 24^2 + 4 \,.\, 30{\cdot}2^2) \times 40 \times \cdot 0004\frac{7}{11} = 97{\cdot}44$ ale gallons:

And $(32^2 + 24^2 + 4 \,.\, 30{\cdot}2^2) \times 40 \times \cdot 0005\frac{2}{3} = 118{\cdot}98$ wine gallons:

Or $\frac{11}{9}$ of 97·44 (the ale gallons) gives 119, the wine gallons, very nearly.

MECHANICS.

1. *Mechanics* is the science of equilibrium and of motion.

2. Every cause which tends to move a body, or to stop it when in motion, or to change the direction of its motion, is called a *force* or *power*.

3. When the forces that act upon a body, destroy or annihilate each other's operation, so that the body remains quiescent, there is said to be an *equilibrium*.

4. *Statics* has for its object the equilibrium of forces applied to solid bodies.

5. *Dynamics* relates to the circumstances of the *motion* of solid bodies.

6. *Hydrostatics* is devoted to the equilibrium of fluids.

7. *Hydrodynamics* relates to their motion, and connected circumstances.

8. The properties and operation of elastic fluids are often treated distinctly, under the head of *Pneumatics*.

9. A single force, which would give to a physical point, or to a body, the same motion both in velocity and direction as several forces acting simultaneously, is called the *resultant* of those forces, while they are called the *constituents* or the *composants* of the single resulting force.

10. The action of a force is the same in whichever point of its direction it is applied; unless the *manner* of its action be changed.

11. *Vis inertiæ*, or power of inactivity, is defined by Newton to be a power implanted in all matter, by which it resists any change attempted to be made in its state, that is, by which it requires force to alter its state, either of rest or motion.

CHAP. IX.

STATICS.

SECTION I. *Parallelogram of Forces.*

1. The resultant of any two forces whatever, which act upon a physical point, and which are represented by lines, taken in their respective directions from that point, has for its magnitude and direction the diagonal of the parallelogram constructed upon those forces.

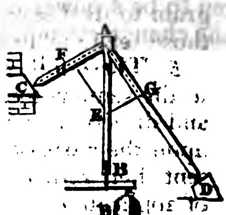
2. Two forces and their resultant may each be represented by the sine of the angle formed by the directions of the two others.

3. If F, f, represent two forces, and A the angle made by their respective directions; also, if R be the resultant, and a the angle which it makes with the direction of the force F: then

$$R = \sqrt{(F^2 + f^2 \pm 2\,F f \cos A)} \quad \ldots\ldots (1)$$

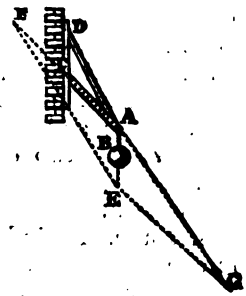
$$\tan a = \frac{f \sin a}{F + f \cos A} \quad \ldots\ldots\ldots\ldots (2)$$

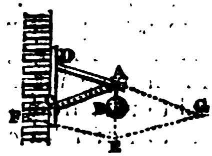
4. These propositions are of extensive, nay, of almost universal application, in the constructions of architects, mechanics, and civil engineers. Two simple examples may here suffice; others will occur as we proceed.



Ex. 1. Suppose that a weight B is attached by a stirrup to the foot of a king-post A B, which is attached to two rafters A C, A D, in the respective positions shown in the marginal diagram. Then if A E be set off upon A B, in numerical value equal to the vertical strain upon A B, and the parallelogram A F E G be completed, A F measured upon the same scale will show the strain upon the rafter A C, and A G the strain upon A D.

Ex. 2. Let it be proposed to compare the strains upon the tie-beams A D, and the struts A C, when they sustain equal weights B, in the two different positions indicated by the figures. Let A E in one figure, be equal to the corresponding vertical line A E in the other, and in each represent the numerical value of the weight B, that hangs from A. Through E in both figures, draw lines parallel to D A, A C, respectively, and let them meet A C, and D A produced in F and G: then A F E G in each figure is the *parallelogram of forces* by which the several strains are to be measured. A G represents the tension upon the tie-beam A D, and A F the strain upon the strut A C. Both these lines are evidently shorter in the lower figure than they are in the upper, A E being of the same length in both: therefore the first figure exhibits the most disadvantageous position of the beams.





It is evident also, that while C A tends upwards and D A downwards, the greater the angle D A C, the less is its supplement C A G, and the less the sides F A, A G, of the parallelogram.

The truth of the general proposition relative to the parallelogram of forces admits of obvious experimental confirmation by means of two spring steelyards. Let any known weight, as 10, 15, &c. pounds, represented by B in the figure to *Ex.* 1, hang from a cord E B, and from a knot at E, let two other cords E F, E G, proceed, and hang from spring steelyards at F and G: then it will be found *universally* that the weights sustained by those steelyards, will be to the weight B hanging vertically, as the respective sides E F and E G of the parallelogram to its diagonal E A. It will, hence, be easy to exhibit those relations in all desirable varieties.

5. Three forces not situated in the same plane, but applied to the same point, have a resultant represented in magnitude and direction by the diagonal of the parallelopiped constructed upon the parts of the directions of those forces which represent their respective magnitudes, and drawn from their point of application.

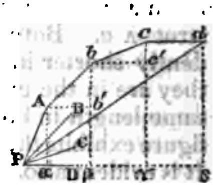
6. We may always decompose or resolve a force into three others, parallel respectively to three given lines. Each composant may be found by multiplying the force which we would

decompose by the cosine of the angle which its direction makes with the axis to which that composant is parallel.

7. If any number of forces be kept in equilibrio by their mutual actions; they may all be reduced to two equal and opposite ones.—For, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane; and so on, till at last they be all reduced to the action of only two opposite forces, which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

8. Prop. To find the resultant of several forces concurring in one point, and acting in one plane.

1st. *Graphically*. Let, for example, four forces, A, B, C, D, act upon the point P, in magnitudes and directions represented by the lines P A, P B, P C, P D.



From the point A draw A b parallel and equal to P B; from b draw $b\,c$ parallel and equal to P C; from c draw $c\,d$ parallel and equal to P D; and so on, till all the forces have thus been brought into the construction. Then join P d, which will represent both the magnitude and the direction of the required resultant.

This is, in effect, the same thing as finding the resultant of two of the forces A and B; then blending that resultant with a third force C; their resultant with a fourth force D; and so on. A careful construction upon a well chosen scale, will give a resultant true to within its 250th part.

2d. *By computation*. Drawing the lines A a, A b', &c. respectively parallel and perpendicular to the last force P D; we have

$$d\delta = \mathrm{A}a + bb' + cc' = \mathrm{A}\sin \mathrm{APD} + \mathrm{B}\sin \mathrm{BPD} + \mathrm{C}\sin \mathrm{CPD}$$

$$\mathrm{P}\delta = \mathrm{P}a + a\beta + \beta\gamma + \gamma\delta = \mathrm{A}\cos \mathrm{APD} + \mathrm{B}\cos \mathrm{BPD} + \mathrm{C}\cos \mathrm{CPD} + \mathrm{D}$$

$$\tan d\mathrm{P}\delta = \frac{d\delta}{\mathrm{P}\delta} \ldots\ldots\ldots, \mathrm{P}d = \sqrt{(\mathrm{P}\delta^2 + d\delta^2)} = \mathrm{P}\delta \sec d\mathrm{P}\delta.$$

The numerical computation is best effected by means of a table of natural sines.

9. The resultant of two parallel forces acting in the same way, is equal to their sum; and the distances of the direction of that resultant from those of the composants are reciprocally proportional to those composants.

10. The resultant of several parallel forces, whether confined to one plane or not, is equal to the sum of those forces, giving to the forces which act in one sense the sign +, and to those which act contrarily the sign −.

Section II. *Centre of Gravity.*

1. *Gravity* is the force in virtue of which bodies left to themselves fall to the earth in directions perpendicular to its surface.

2. We may distinguish between the effect of *gravity* and that of *weight*, by observing that the former is the power of transmitting to every particle of matter a certain velocity which is absolutely independent on the number of material particles; while the latter is the effort which must be exercised to prevent a given mass from obeying the law of gravity. *Weight*, therefore, *depends upon the mass*; but *gravity* has no dependence at all upon it.

3. The centre of gravity of any body or system of bodies is that point about which the body or system, acted upon only by the force of gravity, will balance itself in all positions: or it is a point which, when supported, the body or system will be supported, however it may be situated in other respects.

The centre of gravity of a body is not always *within* the body itself: thus the centre of gravity of a ring is not in the substance of the ring, but in the axis of its circumscribing cylinder; and the centre of gravity of a hollow staff, or of a bone, is not in the matter of which it is constituted, but somewhere in its imaginary axis; every body, however, has a centre of gravity, and so has every system of bodies.

4. Varying the position of the body, will not cause any change in the centre of gravity; since any such mutation will be nothing more than changing the directions of the forces, without their ceasing to be parallel; and if the forces do not continue the same, in consequence of the body being supposed at different distances from the earth, still the forces upon all the moleculæ vary proportionally, and their centre remains unchanged.

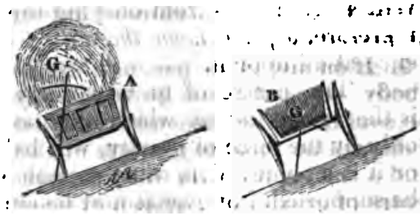
5. When a heavy body is suspended by any other point than its centre of gravity, it will not rest unless that centre is in the same vertical line with the point of suspension: for in all other positions the force which is intended to ensure the equilibrium, will not be directly opposite to the resultant of the parallel forces of gravity upon the several particles of the body, and of course the equilibrium will not be obtained. (See art. 9, on *Pendulums.*)

6. If a heavy body be sustained by two or more forces, their directions must meet either at the centre of gravity of that body, or in the vertical line which passes through it.

7. When a body stands upon a plane, if a vertical line passing through the centre of gravity fall within the base on which the body stands, it will not fall over; but if that verti-

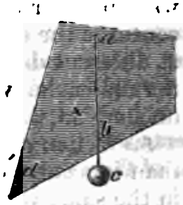
cal line passes without the base, the body will fall, unless it be prevented by a prop or a cord. When the vertical line falls upon the extremity of the base, the body *may* stand, but the equilibrium may be disturbed by a very trifling force; and the nearer this line passes to any edge of the base the more easily may the body be thrown over; the nearer it falls to the middle of the base, the more firmly the body stands.

Upon this principle it is that *leaning towers* have been built at *Pisa*, and various other places; the vertical line of direction from the centre of gravity falling within the base. And, from the same principle it may be seen that a waggon loaded with heavy materials, as B, may stand with perfect safety, on the side of a convex road, the vertical line from the centre of gravity falling between the wheels; while a waggon A with a *high* load as of hay or of wool-packs shall fall over, because the vertical line of direction falls without the wheels.



8. To find the centre of gravity mechanically, it is only requisite to dispose the body successively, in two positions of equilibrium, by the aid of two forces in vertical directions, applied in succession to two different points of the body; the point of intersection of these two directions will show the centre.

This may be exemplified by particularising a few methods. If the body have plane sides, as a piece of board, hang it up by any point, then a plumb-line suspended from the same point will pass through the centre of gravity; therefore mark that line upon it: and after suspending the body by another point, apply the plummet to find another such line; then will their intersection show the centre of gravity.



Or thus: hang the body by two strings from the same point fixed to different parts of the body; then a plummet hung from the same tack will fall on the centre of gravity.

Another method: Lay the body on the edge of a triangular prism, or such like, moving it to and fro till the parts on both sides are in equilibrio, and mark a line upon it close by the edge of the prism: balance it again in another position, and mark the fresh line by the edge of the prism; the vertical line passing through the intersection of these lines, will likewise pass through the centre of gravity. The same thing may be effected by laying the body on a table, till it is just ready to fall off, and then marking a line upon it by the edge of the table: this done in two positions of the body, will in like manner point out the position of the centre of gravity.

When it is proposed to find the centre of gravity of the arch of a bridge, or any other structure, let it be laid down accurately to scale upon pasteboard; and the figure being carefully cut out, *its* centre of gravity may be ascertained by the preceding process.

9. If on any plane passing through the centre of gravity of a body, perpendiculars be let fall from each of its moleculæ, the sum of all the perpendicular distances on one side of the plane will be equal to the sum of all those on the other side. And a similar property obtains with regard to the common centre of gravity of a system of bodies.

10. The position, distance, and motion of the centre of gravity of any body is a medium of the positions, distances, and motions of all the particles in the body.

11. The common centre of gravity, or of position, of two bodies divides the right line drawn between the respective centres of the two bodies in the inverse ratio of their masses.

12. The centre of gravity of three or more bodies may, hence, be found, by considering the first and second as a single body equal to their sum and placed in their common centre of gravity, determining the centre of gravity of this imaginary body, and a third. These three again being conceived united at their common centre, we may proceed, in like manner, to a fourth; and so on, *ad libitum*.

Or, if B, B′, B″, &c. denote the masses of any bodies, D, D′, D″, &c. the perpendicular distances of their respective centres of gravity from any line or plane: then, the distance, Δ, of their common centre of gravity from any line or plane, is found by this theorem: viz. $\Delta = \dfrac{BD + B'D' + B''D'' \&c.}{B + B' + B'' \&c.}$

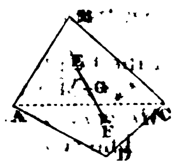
13. If the particles or bodies of any system be moving uniformly and rectilineally, with any velocities and directions whatever, the centre of gravity is either at rest, or moves uniformly in a right line.

Hence, if a rotatory motion be given to a body and it be then left to move freely, the axis of rotation will pass through the centre of gravity: for, that centre, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Here too it may be remarked, that *a force applied at* *the centre of gravity of a body, cannot produce a rotatory motion.*

14. The centre of gravity of a right line, or of a parallelogram, prism, or cylinder, is in its middle point; as is also that of a circle, or of its circumference, or of a sphere, or of a regular polygon; the centre of gravity of a triangle is somewhere in a line drawn from any angle to the middle of the opposite side; that of an ellipse, a parabola, a cone, a conoid, a spheroid, &c. somewhere in its axis. And the same of all symmetrical figures.

15. The centre of gravity of a triangle is the point of intersection of lines drawn from the three angles to the middles of the sides respectively opposite: it divides each of those lines into two portions in the ratio of 2 to 1.



16. *In a Trapezium.* Divide the figure into two triangles by the diagonal A C, and find the centres of gravity E and F of these triangles; join E F, and find the common centre G of these two by this proportion, A B C : A D C :: F G : E G, or A B C D : A D C :: E F : E G. Or, divide the figure into two triangles by a diagonal B D: find their centres of gravity; the line which joins them will intersect E F in G, the centre of gravity of the trapezium.

17. In like manner, for any other plane figure, whatever be the number of sides, divide it into several triangles, and find the centre of gravity of each; then connect two centres together, and find their common centre as above; then connect this and the centre of a third, and find the common centre of these; and so on, always connecting the last found common centre to another centre, till the whole are included in this process; so shall the last common centre be that which is required.

18. The centre of gravity of a circular arc is distant from the centre a fourth proportional to the arc, the radius, and the chord of the arc.

19. In a circular sector, the distance from the centre of the circle is $\frac{2\,c\,r}{3\,a}$; where a denotes the arc, c its chord, and r the radius.

20. The centres of gravity of the surface of a cylinder, of a cone, and of a conic frustum, are respectively at the same distances from the origin as are the centres of gravity, of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

21. The centre of gravity of the surface of a spheric segment, is at the middle of its versed sine or height.

22. The centre of gravity of the convex surface of a spherical zone, is in the middle of that portion of the axis of the sphere which is intercepted by the two bases of the zone.

23. In a cone, as well as any other pyramid, the distance from the vertex is $\frac{3}{4}$ of the axis.

24. In a conic frustum, the distance on the axis from the centre of the less end, is $\frac{1}{4}h \cdot \frac{3R^2 + 2Rr + r^2}{R^2 + Rr + r^2}$: where h the height, R and r the radii of the greater and less ends.

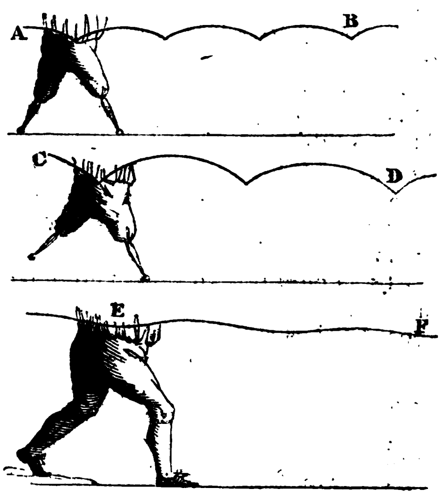
25. The same theorem will serve for the frustum of any regular pyramid, taking R and r for the sides of the two ends.

26. In the paraboloid, the distance from the vertex is $\frac{2}{3}$ axis.

27. In the frustum of the paraboloid, the distance on the axis from the centre of the less end, is $\frac{1}{3}h \cdot \frac{2R^2 + r^2}{R^2 + r^2}$: where h the height, R and r the radii of the greater and less ends.

*** Many other results are given in the first volume of my *Mechanics*. The preceding are selected as the most useful.

The centre of gravity of the human body is always near the same place, viz. in the *pelvis*, between the hips, the *ossa pubis*, and the lower part of the back-bone. Elevating the arms or the legs will elevate the centre of gravity *a little:* still, it is always so placed that the limbs may move freely round it, and this centre moves much less than if it were in any other part of the body. If a man *walked* upon wooden legs, the centre of gravity of his body would describe portions of circles, as A B. If a man with two wooden legs were to *run*, the centre of gravity would describe portions of parabolas, as C D. But the flexibility of the joints and muscles of the human legs, serves to take away the angles from these curves, and give a softer undulation, as E F.



SECTION III. *Mechanical Powers.*

1. The simple machines of which the more complex machines are constituted, and which, indeed, are often employed separately, are called *Mechanical Powers.*

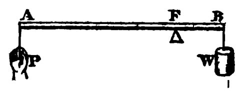
2. Of these we usually reckon six: viz. the *lever*, the *wheel and axle*, the *pulley*, the *inclined plane*, the *wedge*, and the *screw*. To these, however, is sometimes added the *funicular machine*, being that which is formed by the action of several powers, at different points of a flexible cord.

3. *Weight* and *Power*, when regarded as opposed to each other, signify the body to be moved, or the resistance to be overcome, and the body or force by which that is accomplished. They are usually represented by their initial letters, W and P.

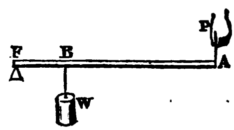
Levers.

1. A *lever* is an inflexible bar, whether straight or bent, and supposed capable of turning upon a fixed, unyielding point, called a *fulcrum*.

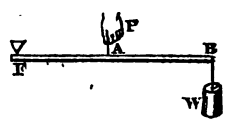
2. When the *fulcrum* is between the power and the weight, the lever is said to be of the *first* kind.



When the *weight* is between the power and the prop, the lever is of the *second* kind.



When the *power* is between the weight and the prop, or fulcrum, the lever is of the *third* kind.



The *hammer lever*, or the operation of a hammer in drawing a nail, is sometimes considered as a fourth kind.

3. In all these cases, when there is an equilibrium, it is indicated by this general property, that the product of the weight into the distance at which it acts, is equal to the product of the power into the distance at which it acts: *the distances being estimated in directions perpendicular to those* in which the weight and power act respectively. Thus, in each of the three preceding figures,

$$P \cdot AF = W \cdot BF,$$

or the power and weight are reciprocally as the distances at which they act.

And if, in the first figure, for example, the arm A F were

4 times F B, 4 lbs. hanging at B would be balanced by 1 lb. at A. If A F were 5 times F B, 1 lb. at A would balance 5 lbs. at B; and so on.

4. If several weights hang upon a lever, some on one side of the fulcrum, some on the other, then there will be an equilibrium, when the sum of the products of the weights into their respective distances on one side, is equal to the several products of weights and distances on the other side.

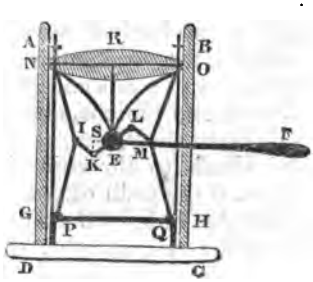
5. When the *weight* of the lever is to be taken into the account, proceed just as though it were a separate weight suspended at its centre of gravity.

6. When two or three more levers act one upon another in succession, then the entire mechanical advantage which they supply, is found by taking the product of their separate advantages. Thus, if the arms of three levers, acting thus in connexion, are as 3 to 1, 4 to 1, and 5 to 1, then the joint advantage is that of $3 \times 4 \times 5$ to 1, or 60 to 1: so that 1 pound would, through their intervention, balance 60.

7. In the *first* kind of lever the pressure upon the fulcrum $= P + W$: in the other two it is $= P \sim W$.

8. Upon the foregoing principles depends the nature of scales and beams for weighing all bodies. For, if the distances be equal, then will the weights be equal also; which gives the construction of the common scales. And the Roman statera, or steel-yard, is also a lever, but of unequal arms or distances, so contrived that one weight only may serve to weigh a great many, by sliding it backwards and forwards to different distances upon the longer arm of the lever. In the common *balance*, or scales, if the weight of an article when ascertained in one scale is not the same as its weight in the other, *the square root of the product of those two weights will give the true weight.*

9. From numerous examples of the power and use of the lever, one which shows its manner of application in the printing-presses of the late *Earl Stanhope* may be advantageously introduced.



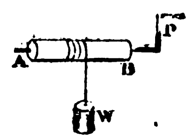
In the adjoining figure, let A B C D be the general frame of the press, connected by the cross pieces N O, D C. E is a centre connected with the frame by the bars E N, E R, E O. To this centre are affixed a bar K L, and a lever E F, to which the hand is applied when the press is used.

There are also several other pieces connected by joints at N, G, I, K, L, M, O, H, which are so adjusted to each other that when the hand is applied to the lever E F at F, by pressing it downwards K L is brought into a horizontal line or parallel to G H or D C, in which situation N I G, O M H, also form each a straight line. It is evident that the nearer these different pieces, as above mentioned, are to a straight line, the greater is the lever E F, in proportion to the perpendicular K S at the other end of the lever E K, formed by a perpendicular from K falling on F E produced. Consequently a small force applied at F will be sufficient to produce a *very great effect* at K, when I K, K E are nearly in a straight line, and so on, for the other pieces above mentioned.

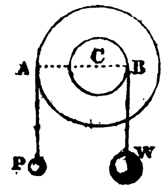
Hence the force applied by hand at F must be very considerable in forcing down G H, which slides on iron cylindrical bars, or in pressing any substance placed in the aperture P Q, between the bar or plate and the frame D C.

Wheel and Axle.

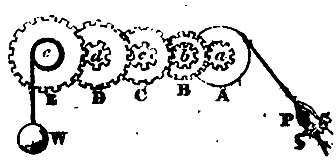
1. The nature of this machine is suggested by its name. To it may be referred all turning or wheel-machines of different radii; as well-rollers and handles, cranes, capstans, windlasses, &c.



2. The mechanical property is the same as in the lever: viz. P . A C = W . B C: and the reason is evident, because the wheel and axle is only a kind of perpetual lever.



3. When a series of wheels and axles act upon each other, so as to transmit and accumulate a mechanical advantage, whether the communication be by means of cords and belts, or of teeth and pinions, the weight will be to the power, as the continual product of the radii of the wheels to the continual product of the radii of the axles. Thus, if the radii of the axles, *a*, *b*, *c*, *d*, *e*, be each 3 inches, while the radii of the wheels A, B, C, D, E, be 9, 6, 9, 10, and 12 inches respectively: then W : P :: 9 × 6 × 9 × 10 × 12 :



$3 \times 3 \times 3 \times 3 \times 3 :: 240 : 1$. A computation, however, in which the effect of friction is disregarded.

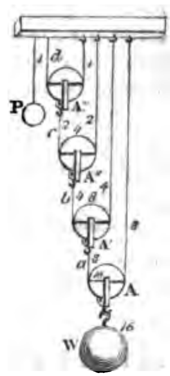
4. A train of wheels and pinions may also serve for the augmentation of velocities. Thus, in the preceding example, whatever motion be given to the circumference of the axle *e*, the rim of the wheel A will move 240 times as fast.

And if a series of 6 wheels and axles, each having their diameters in the ratio of 10 to 1, were employed to accumulate velocity, the *produced* would be to the *producing* velocity, as 10^6 to 1, that is, as 1000000 to 1.

Pulley.

1. A pulley is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

2. If a power sustain a weight by means of a fixed pulley: the power and weight are equal.

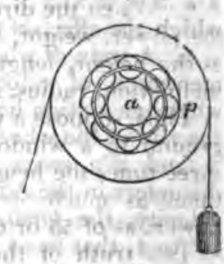


3. When a power sustains a weight of a system of moveable pulleys, each embraced by a cord attached on one part to a fixed point, and on the other to the centre of the pulley next above it, as in the margin: then if the cords are parallel to each other, each pulley gives a mechanical advantage of two to one; and the whole system an advantage denoted by that power of 2 which is equal to the number of pulleys. Here P : W :: 1 : 2^4 :: 1 : 16.

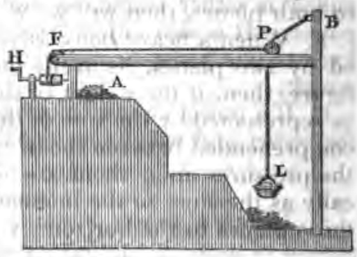
4. When there are three, four, or any other number of pulleys in a fixed block, and an equal number in a moveable block, capable of ascending and descending, the system is called a *muffle*; and the weight is to the power, as 1 to twice the number of pulleys in each block.

5. The friction of a system of pulleys, or even of a single pulley, is very great, according to the common mode of construction. But it may be reduced considerably by means of

Mr. Garnett's patent friction rollers, which produce a great saving of labour and expense, as well as in the wear of the machine, both when applied to pulleys, and to the axles of wheel carriages. His general principle is this: between the axle and nave, or centre pin and box, a hollow space is left, to be filled up by solid equal rollers nearly touching each other. These are furnished with axles inserted into a circular ring at each end, by which their relative distances are preserved; and they are kept parallel by means of wires fastened to the rings between the rollers, and which are rivetted to them. The above contrivance is exhibited in the annexed figure.



6. A useful combination of the wheel and axle, a fixed and a moveable pulley, is exhibited in the marginal diagram. The load, as of stones or bricks to build a wall, is raised from L to A, thus: a rope B P L is fixed at one end to a hook B, and passes over a pulley at P. That pulley, P, is drawn along horizontally from P to F by means of a man who turns the handle H, and thus winds up the cord P F H upon the roller. As the distance P B lengthens, the portion P L shortens; and the length of rope is so adjusted, that when the pulley P is brought to be above A, the basket L has reached that place.



Inclined Plane.

1. A body which touches a plane only in one point, can only remain in equilibrio so long as the forces which act upon it are reducible to a single force which shall act in a direction perpendicular to the plane at the point of contact.

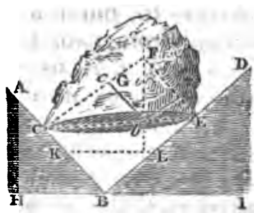
2. When a power sustains a heavy body in equilibrio upon an inclined plane, then the power, the weight, and the pressure upon the plane, will be respectively, as the sine of the plane's inclination, the cosine of the angle which the direction of the power makes with the plane, and the cosine of the angle which the direction of the power makes with the horizon.

3. When the direction of the power is parallel to the plane, the power, weight, and pressure on the plane, are respectively as the height, length, and base of the plane; or as the sine of inclination, radius, and cosine of inclination.

Thus, suppose a plank 12 feet long were laid aslant from the ground to a window 4 feet high: then, since the length is three times the height, a power of 50lbs. would sustain three times as much, or 150lbs. upon the plank; and a greater power, as of 55 or 60lbs. would cause that weight to ascend.

The truth of these propositions may be confirmed most readily, by attaching the weight to a cord tied to a spring steel-yard, by which the relations between the entire weight, and that supported by a cord either parallel to the inclined plane or in any other direction, may at once be measured.

4. If two weights W, W′, sustain each other upon two inclined planes A C, C B, which have a common altitude C D, by means of a cord which runs freely over a pulley and is parallel to both planes, then will W : W′ :: A C : C B.



5. When a heavy body is supported by two planes, as in the marginal figure, then, if the weight of the body be represented by the sine of the angle comprehended between the two planes, the pressures upon them are reciprocally as the sines of the inclinations of those planes to the horizon: viz.

The weight	are as	sin A B D
the pressure on A B		sin D B I
the pressure on B D		sin A B H

Thus, suppose the angle A B H was 30°, D B I 60°, and consequently A B D 90°: since the natural sines of 90°, 60°, and 30°, are 1, ·866, and $\frac{1}{2}$ respectively, or nearly as 100, 86·6, and 50; if the heavy body weigh 100lbs. the pressure upon A B would be 86·6lbs. and upon B D 50lbs.

This proposition is of very extensive utility, comprehending the pressure of arches on their piers, of buttresses against walls, or upon the ground, &c. because the circumstance of one of the planes becoming either horizontal, or vertical, will not affect the general relation above exhibited.

Wedge.

1. A *wedge* is a triangular prism, or a solid conceived to be

generated by the motion of a plane triangle parallel to itself upon a straight line which passes through one of its angular points. The wedge is called *isosceles*, *rectangular*, or *scalene*, according as the generating triangle is isosceles, right-angled, or scalene. It is very frequently used in cleaving wood, as represented in the figure, and often in raising great weights.

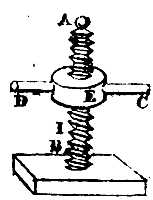
2. When a resisting body is sustained against the face of a wedge, by a force acting at right angles to its direction; in the case of equilibrium, the power is to the resistance as the sine of the semi-angle of the wedge, to the sine of the angle which the direction of the resistance makes with the face of the wedge; and the sustaining force will be as the cosine of the latter angle.

3. When the resistance is made against the face of a wedge by a body which is not sustained, but will adhere to the place to which it is applied without sliding, the power is to the resistance, in the case of equilibrium, as the cosine of the difference between the semi-angle of the wedge and the angle which the direction of the resistance makes with the face of the wedge to radius.

4. When the resisting body is neither sustained nor adheres to the point to which it is applied, but slides freely along the face of the wedge, the power is to the resistance as the product of the sines of the semiangle of the wedge and the angle in which the resistance is inclined to its face to the square of radius.

Screw.

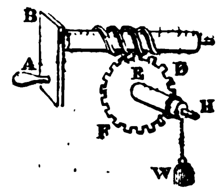
1. The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.



2. The energy of a power applied to turn a screw round, is to the force with which it presses upward or downward, setting aside the friction, as the distance between two threads, is

to the circumference where the power is applied: viz., as circumf. of D C to dist. B I.

3. The *endless screw*, or *perpetual screw*, is one which works in, and turns a dented wheel D F, without a concave or female screw; being so called because it may be turned for ever, without coming to an end. From the scheme it is evident that while the screw turns once round, the wheel only advances the distance of one tooth.



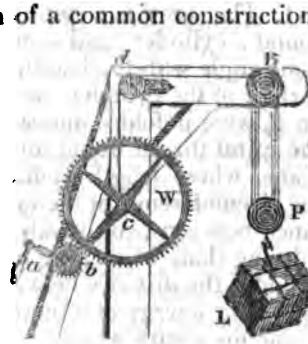
4. If the power applied to the lever, or handle of an endless screw, A B, be to the weight, in a ratio compounded of the periphery of the axis of the wheel E H, to the periphery described by the power in turning the handle, and of the revolutions of the wheel D F to the revolutions of the screw C B, the power will balance the weight. Hence,

5. As the motion of the wheel is very slow, a small power may raise a very great weight by means of an endless screw. And therefore the chief use of such a screw is, either where a great weight is to be raised through a little space; or where only a slow gentle motion is wanted. For which reason it is very serviceable in clocks and watches.

The screw is of admirable use in the mechanism of micrometers, and in the adjustments of astronomical and other instruments of a refined construction.

6. The mechanical advantage of a compound machine may be determined by analyzing its parts, finding the mechanical advantage of each part severally, and then blending or compounding all the ratios. Thus, if m to 1, n to 1, r to 1, and s to 1, show the separate advantages; then $m\,n\,r\,s$ to 1, will measure the advantage of the system.

7. The marginal representation of a common construction of a *crane* to raise heavy loads, will serve to illustrate this. By human energy at the handle *a*, the pinion *b* is turned; that give motion to the wheel, W, round whose axle, *c*, a cord is coiled; that cord passes over the fixed pulley, *d*, and thence over the fixed triple block, B, and the moveable triple block, P, below which the load, L, hangs. Now, if the radius of the handle be 6 times that of the pinion, the radius of the wheel W, 10 times that of its axle, and a power equivalent to 30lbs. be exerted at *a*;



then, since a triple moveable pulley gives a mechanical advantage of 6 to 1, we shall have

$$30 \times 6 \times 10 \times 6 = 10800\text{lbs.}$$

and such would be the load, L, that might be raised by a power of 30lbs. applied at *a*, were it not for the loss occasioned by friction.

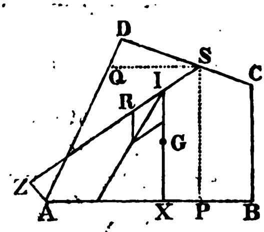
SECTION IV. *General application of the principles of Statics to the equilibrium of Structures.*

Every structure is exposed to the operation of a system of forces; so that the examination of its stability involves the application of the general conditions of equilibrium.

Now, no part of a structure can be dislocated, except it be either by a progressive, or a rotatory motion. For either this part is displaced, without changing its form, in which case it is as a system of invariable form, incapable of receiving any instantaneous motion, which is not either progressive or rotatory; or else it happens to be displaced, changing at the same time its form: and this, considering the cohesion of tenacity, cannot take place, without the breaking of that part in its weakest section; which generates a progressive motion, if the force acts perpendicularly to the section; and a rotatory motion, if it acts obliquely.

We shall here consider the most useful cases; indicating by the word *stress*, that force which tends to give motion to the structure; by *resistance*, that which tends to hinder it.

Equilibrium of Piers.



1. Taking the marginal figure for the vertical section of a pier, we may reason upon that section instead of the pier itself, if it be of uniform structure.

Let G be the place of the centre of gravity, S R Z the direction in which the stress acts, meeting X I, the vertical line through the centre of gravity, in I. Then, considering the stress as resolvable into two forces, one P, vertical, the other, Q, horizontal; the pier (regarding it as one body) can only give way either by a progressive motion from B towards A, or by a rotatory motion about A.

2. The progressive motion is resisted by friction. If W denote the weight of the pier, P the stress estimated vertically, and Q its horizontal effort, then the pressure on the base = W + P, and its friction = f (W + P), which is the amount of

the resistance to progressive motion. So that to ensure stability in this respect we must have

$$f(W + P) > Q \ldots\ldots\ldots\ldots (1)$$

While, to ensure stability in regard to rotation, we must have

$$W . AX + P . AB > Q . ES \ldots\ldots\ldots\ldots (2)$$

3. The second condition may be ascertained by a graphical process, thus:

From the point A, let fall, on the direction of the stress, the perpendicular AZ. Then, S being put for the whole stress, W . AX > S . AZ.

Or, suppose the two forces M and S to be applied at I, and complete the parallelogram, having sides which represent these forces. Then must the diagonal produced, meet the base on this side of A, towards B, to ensure stability.

4. If, as is very frequently the case, the vertical section of the pier is a rectangle, and s represent the specific gravity of the material of which it is constituted; then the condition of the two kinds of equilibrium will be denoted by these two equations: viz.

$$f . CB . AB . s = Q \ldots (3) \ldots\ldots AB^2 s = 2Q \ldots (4)$$

Example. Suppose a rectangular wall 39·4 feet high, and of a material whose specific gravity is 2000, is to sustain a horizontal strain of 9900 lbs. avoirdupois at its summit on the unit of length, 1 foot: what must be the thickness that there may be an equilibrium, taking $f = \frac{3}{4}$.

Here, that the wall may not be displaced *horizontally*, we must have

$$AB > Q \div f . s . CB > 9900 \div \tfrac{3}{4} . \frac{2000}{16} . 39{\cdot}4$$

$$> 3300 \div \frac{2000 \times 39{\cdot}4}{64} > \frac{3300 \times 32}{39400}$$

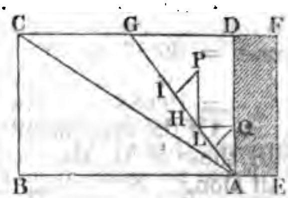
$$> \frac{52800}{39400} > 1{\cdot}34 \text{ feet.}$$

And, 2dly, that it may not be overturned, we must have

$$AB \sqrt{\frac{2Q}{s}} > \sqrt{\frac{19800}{125}} > \sqrt{158{\cdot}4} > 12{\cdot}58 \text{ feet.}$$

Here, as the thickness required to prevent overturning is much the greatest, the computation in reference to the other kind of equilibrium may usually be avoided.

Pressure of Earth against Walls.



1. Let D A E F be the vertical section of a wall, behind which is posited a bank or terrace of earth, of which a prism whose section is represented by D A G would detach itself and fall down, were it not prevented by the wall. Then A G is denominated the *line of rupture* or the *natural slope*, or *natural declivity*. In sandy or loose earth, the angle B A G seldom exceeds 30°; in stronger earth it becomes 37°; and in some favourable cases more than 45°.

2. Now, the prism whose vertical section is D A G, has a tendency to descend along the inclined plane G A by reason of the force of gravity, g; but it is retained in its place, 1st. by the force, q, opposed it by the wall, and 2dly, by its cohesive attachment to the face A G, and by its friction upon the same surface. Each of those forces may be resolved into one, which is perpendicular to G A, and which is inoperative as to this inquiry, and into another whose action is parallel to G A. The lines P I and I H, represent these composants of p, *that* force being represented by the vertical line P H, drawn from the centre of gravity P of the prism. The direction of the force q is represented by the horizontal line Q H, and its composants by the lines Q L, H L. The force that gives the triangle its tendency to descend is I H; and the force opposed to this is L H together with the effects of cohesion and friction. Thus,

$$\text{IH} = \text{LH} + \text{cohesion} + \text{friction}.$$

It is evident, therefore, that the solution to this inquiry must be, in great measure, experimental.

3. It has been found, however, theoretically, by M. Prony,* and confirmed experimentally, that the angle formed with the vertical by the prism of earth that exerts the greatest horizontal stress against a wall, is *half* the angle which the natural slope of the earth makes with the vertical: and this curious result greatly simplifies the whole inquiry.

The state of equilibrium is expressed by this equation: viz.

$$\tfrac{1}{2}\,\text{AD}\,.\,\text{AE}^2\,.\,S = \tfrac{1}{6}\,\text{AD}^3\,.\,s\,.\,\tan^2 \tfrac{1}{2}\,\text{DAG}.$$

S and s representing the specific gravities of the wall and earth respectively.

Example. The wall to be 39·37 feet high, of brick, specific

* See a demonstration at p. 358, vol. ii. of my edition of Dr. Hutton's Course of Mathematics.

gravity 2000, and the terrace of strong earth specific gravity 1428, natural slope 53° from vertex.

Then the above equation becomes

$$\tfrac{1}{2}x^2 \times 2000 \times 39{\cdot}37 = \tfrac{1}{6} \times 39{\cdot}37^3 \times 1428 \times \tan^2 26\tfrac{1}{2}$$

$$\text{or } x = 39{\cdot}37 \tan 26^\circ\tfrac{1}{2}\sqrt{\frac{1428}{3 \times 2000}} = \frac{39{\cdot}37}{2}\sqrt{\frac{1428}{6000}}$$

$= 19{\cdot}685 \times {\cdot}4878 = 9{\cdot}6$ feet, thickness of the wall.

4. Of the experimental results the best which we have seen are those of M. Mayniel, from which the following are selected all along. Supposing the upper surface of the earth and of the wall which supports it, to be both in one horizontal plane.

1st. Both theory and experiment indicate that the resultant H Q of the thrust of a bank, behind a vertical wall, is at a distance A Q from the bottom of the wall $= \frac{1}{3}$ A D, the height.

2dly. That the friction is *half* the pressure, in vegetable earths, *four-tenths* in sand.

3dly. The cohesion which vegetable earths acquire, when cut in turfs, and well laid, course by course, diminishes their thrust by full *two-thirds*.

4thly. The line of rupture behind a wall which supports a bank of vegetable earth is found at a distance D G from the interior face of the wall equal to ·618 h, h being the height of the wall.

5thly. When the bank is of sand, then D G = ·677 h.

6thly. When the bank is of vegetable earth mixed with small gravel, then D G = ·646 h.

7thly. If it be of rubbles, then D G = ·414 h.

8thly. If it be of vegetable earth mixed with large gravel, then D G = ·618 h.

Thickness of Walls, both faces vertical.

1. Wall brick, weight of cubic foot = 109 lbs. avoird. bank vegetable earth, carefully laid, course by course, D F = ·16 h.

2. Wall unhewn stones, 135 lbs. per cubic foot, earth as before, D F = ·15 h.

3. Wall brick, earth clay well rammed, D F = ·17 h.

4. Wall unhewn stones, earth as above, D F = ·16 h.

5. Wall of hewn free stone, 170 lbs. to the cubic foot, bank vegetable earth, D F = ·13 h; if the bank be clay D F = ·14 h.

6. Bank of earth mixed with large gravel,

Wall of bricks	D F = ·19 h
unhewn stone	D F = ·17 h
hewn free stone	D F = ·16 h

7. Bank of sand

Wall of bricks	....	DF = ·33 h
unhewn stone		DF = ·30 h
hewn free stone		DF = ·26 h.

When the earth of the bank or terrace is liable to be much saturated with water, the proportional thicknesses of wall must be at least doubled.*

8. *For walls with an interior slope*, or a slope towards the bank, let the base of the slope be $\frac{1}{n}$ of the height, and let S and s, as before, be the specific gravities of the wall and of the earth; then

$$DF = h\sqrt{\frac{1}{3n^2} + m\frac{s}{S}} - \frac{h}{n};$$

where m = ·0424, for vegetable or clayey earth, mixed with large gravel; m = ·0464, if the earth be mixed with small gravel; m = ·1528, for sand; and m = ·166, for semi-fluid earths.

Example. Suppose the height of a wall to be 20 feet, and $\frac{1}{20}$ of the height for the base of the *talus* or slope; suppose, also, the specific gravities of the wall and of the bank to be 2600, and 1400, and the earth semi-fluid; what, then, must be the thickness of the wall at the crown?

Here the theorem will become,

$$DF = 20\sqrt{\frac{1}{1200} + {\cdot}166 \cdot \tfrac{14}{26}} - \tfrac{20}{20}$$

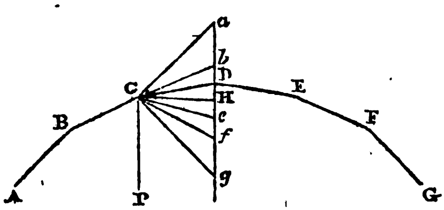
$$= 20\sqrt{{\cdot}0008333 + {\cdot}0894} - 1 = (20 \times {\cdot}3) - 1$$

$= 6 - 1 = 5$ feet: while the thickness of the wall at *bottom* will be 6 feet.

Equilibrium of Polygons.

1. Let there be any number of lines, or bars, or beams, AB, BC, CD, DE, &c. all in the same vertical plane, connected together and freely moveable about the joints or angles A, B, C, D, E, &c. and kept in equilibrio by weights laid on the angles: It is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.

* When weights of French cubic feet are given in *kilogrammes*, $\frac{20}{11}$ of them will be the corresponding weight of an English cubic foot in pounds avoirdupois.



Through any point, as D, draw a vertical line a D H g &c.; to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz. C a parallel to A B, C b parallel to B C, C e to D E, C f to E F, C g to F G, &c; also C H parallel to the horizon, or perpendicular to the vertical line adg, in which also all these parallels terminate.

Then will all those lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, and of all the three kinds, viz. vertical, horizontal, and oblique. That is, the oblique forces or thrusts in direction of the bars .... A B, B C, C D, D E, E F, F G, are proportional to their parallels C a, C b, C D, C e, C f, C g; and the vertical weights on the angles B, C, D, E, F, &c. are as the parts of the vertical .... ab, b D, D e, ef, fg, and the weight of the whole frame A B C D E F G, is proportional to the sum of all the verticals, or to ag; also the horizontal thrust at every angle, is every where the same constant quantity, and is expressed by the constant horizontal line C H.

Corol. 1. It is remarkable, that the lengths of the bars A B, B C, &c. do not affect or alter the proportions of any of these loads or thrusts; since all the lines C a, C b, ab, &c. remain the same, whatever be the lengths of A B, B C, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given the position of D C, and the weights or loads laid on the angles D, C, B; set these on the vertical, D H, D b, ba, then C b, C a give the directions or positions of C B, B A, as well as the quantity or proportion C H of the constant horizontal thrust.

Corol. 2. If C H be made radius; then it is evident that H a is the tangent, and C a the secant of the elevation of C a or A B above the horizon; also H b is the tangent and C b the secant of the elevation of C b or C B; also H D and C D the tangent and secant of the elevation of C D; also H e and C e the tangent and secant of the elevation of C e or D E; also H f and C f the tangent and secant of the elevation of E F; and so on; also the parts of the vertical ab, b D, ef, fg, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts in the directions of the bars, are to one another, directly in proportion as the secants of their angles of elevation above the horizontal directions; or, which is the same thing, reciprocally proportional to the cosines of the same elevations, or reciprocally proportional to the sines of the vertical angles, *a*, *b*, D, *e*, *f*, *g*, &c. made by the vertical line with the several directions of the bars; because the secants of any angles are always reciprocally in proportion to their cosines.

2. The weight or load laid on each angle, is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle, is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical ag, is to the line C H, so is the weight of the whole assemblage of bars, to the horizontal thrust.

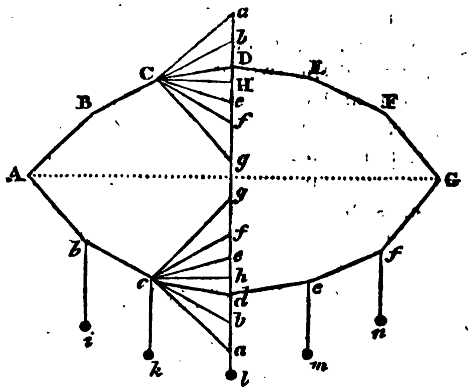
4. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it.

Scholium. This proposition is very fruitful in its practical consequences, and contains the whole theory of centerings, and indeed of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

In all the useful cases, a model of the structure may be made, and the relations of the pressures at any angle, whether horizontal, vertical, or in the directions of the beams, may be determined by a spring steelyard applied successively in the several directions, after the manner described in Art. 4. Sect. 1. *Statics.*

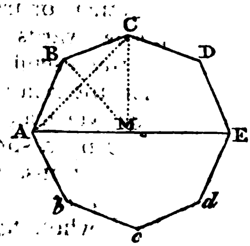
2. If the whole figure in the preceding problem be inverted, or turned round the horizontal line A G as an axis, till it be

completely reversed, or in the same vertical plane below the first position, each angle D, d, &c. being in the same plumb line; and if weights i, k, l, m, n, which are respectively equal to the weights laid on the angles B, C, D, E, F, of the first figure, be now suspended by threads from the corresponding angles b, c, d, e, f, of the lower figure; those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical or horizontal or oblique, will be exactly equal to the corresponding forces of weight or pressure or thrust in the like directions of the first figure.



This, again, is a proposition most fertile in its consequences, especially to the practical mechanic, saving the labour of tedious calculations, but making the results of experiment equally accurate. It may thus be applied to the practical determination of arches for bridges, with a proposed road-way; and to that of the position of the rafters in a curb or mansard roof.

Thus, suppose it were required to make such a roof, with a given width A E, and of four proposed rafters A B, B C, C D, D E. Here, take four pieces that are equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend this from two pins A, E, fixed in a horizontal line, and the chain of the pieces will arrange itself in such a festoon or form, A b c d E, that all its parts will come to rest in equilibrio. Then, by inverting the



figure, it will exhibit the form and frame of a curb roof A B C D E, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual pulls or tensions of the hanging festoon A b c d E.

4. If the mansard be constituted of four equal rafters; then, if angle C A E $= m$, angle C A B $= x$; it is demonstrable that $2 \sin 2x = \sin 2m$. So that if the span A E, and height M C, be given, it will be easy to *compute* the lengths A B, B C, &c.

Example. Suppose A E $= 24$ feet, M C $= 12$.

$$\text{Then } \frac{MC}{MA} = 1 = \tan 45^\circ \text{ angle } CAM = m.$$

$$\therefore \sin 2m = \sin 90^\circ = 1, \text{ and } \sin 2x = \tfrac{1}{2}$$

$$\therefore 2x = 30^\circ, \text{ and } x = 15^\circ = CAB$$

$$\text{Hence } MAB = 45^\circ + 15^\circ = 60^\circ$$

$$\text{and } MBA = \tfrac{1}{2}(180^\circ - 2 \times 15^\circ) = 90^\circ - 15^\circ = 75^\circ$$

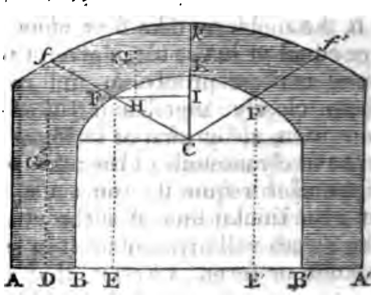
$$\text{and } AMB = 180^\circ - (75^\circ + 60^\circ) = 45^\circ.$$

Lastly, $\sin 75^\circ : \sin 45 :: AM = 12 : AB = 8{\cdot}7846$ feet.

Note. In this example, since A M = M C, as well as A B = B C, it is evident that M B bisects the right angle A M C; yet it seemed preferable to trace the steps of a general solution.

Stability of Arches.

1. If the effect of the force of gravity upon the ponderating matter of an arch and pier, be considered apart from the operation of the cements which unite the stones, &c. the investigation is difficult to practical men, and it furnishes results that require much skill and care in their application. But, in an arch whose component parts are united with a very powerful cement, those parts do not give way in vertical columns, but by the separation of the entire mass, including arch and piers, into three, or, at most, into four parts. In this case, too, the conditions of equilibrium are easily expressed and easily applied.



Let f F, f' F', be the joints of rupture, or places at which the arch would most naturally separate, whether it yield in two pieces or in one. Let G be the centre of gravity of the semi-arch f F K k, and G' that of the pier A B F f. Let F I be drawn parallel to the horizon, and G H be demitted perpendicularly upon it; also let $G'D$ be a perpendicular passing through G', and FE drawn from F parallel to it. Then

2. Prop. If the arch $fFF'f'$ tend to fall vertically in one piece, removing the sections fF, $f'F'$; if A be the weight of the semi-arch $fFKk$, and P that of the pier up to the joint fF, the equilibrium will be determined by these two equations: viz.

$$f \cdot P = A\left(\frac{CI}{FI} - f\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$P \cdot \frac{AD}{FE} = A\left(\frac{CI}{FI} - \frac{AE}{FE}\right) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

where f is the measure of the friction, or the tangent of the angle of repose of the material, and the first equation is that of the equilibrium of the horizontal thrusts, while the latter indicates the equilibrium of rotation about the exterior angle A of the pier.

3. Prop. If each of the two semi-arches Fk, kF', tend to turn about the vertex k of the arch, removing the points F, F', the equilibrium of horizontal translation, and of rotation, will be respectively determined by the following equations: viz.

$$f \cdot P = A\left(\frac{FH}{kI} - f\right) \quad \ldots\ldots\ldots\ldots\ldots (3)$$

$$P \cdot \frac{AD}{FE} = A\left(\frac{FH}{kI} - \frac{AE}{FE}\right) \quad \ldots\ldots (4)$$

4. Hence it will be easy to examine the stability of any cemented arch, upon the hypothesis of these two propositions. Assume different points, such as F in the arch, for which let the numerical values of the equations (1) and (2), or (3) and (4) be computed. To ensure stability, the first members of the respective equations must exceed the second; those parts will be weakest where the excess is least.

If the figure be drawn on smooth drawing pasteboard, upon a good sized scale, the places of the centres of gravity may be found experimentally, as well as the relative weights of the semi-arch and piers, and the measures of the several lines from the scale employed in the construction.

If the dimensions of the arch were given, and the thickness of the pier required; the same equations would serve; and different thicknesses of the pier might be assumed, until the first members of the equations come out largest.

The same rules are applicable to domes, simply taking the ungulas instead of the profiles.

Models.

From an experiment made to ascertain the firmness of the model of a machine, or of an edifice, certain precautions are necessary before we can infer the firmness of the structure itself.

The classes of forces must be distinguished; as, whether they tend to *draw* asunder the parts to *break* them transversely, or *crush* them by compression. To the first class belongs the stretching suffered by key-stones, or bonds of vaults, &c. to the second, the load which tends to bend or break horizontal or inclined beams; to the third the weight which presses vertically upon walls and columns.

PROB. 1. If the side of a model be to the corresponding side of the structure, as 1 to n, the stress which tends to *draw* asunder, or to *break transversely* the parts, increases from the smaller to the greater scale as 1 to n^3; while the resistance of those ruptures increases only as 1 to n^2.

The structure, therefore, will have so much less firmness than the model, as n is greater.

If W be the greatest weight which one of the beams of the model can bear, and w the weight or stress which it actually sustains, then the limit of n will be $n = \frac{W}{w}$.

PROP. 2. The side of the model being to the corresponding side of the structure as 1 to n, the stress which tends to crush the parts by *compression*, increases from the smaller to the greater scale, as 1 to n^3, while the resistance increases only in the ratio of 1 to n.

Hence, if W were the greatest load which a modular wall or column could carry, and w the weight with which it is actually loaded; then the greatest limit of increased dimensions would be found from the expression $n = \sqrt{\frac{W}{w}}$.

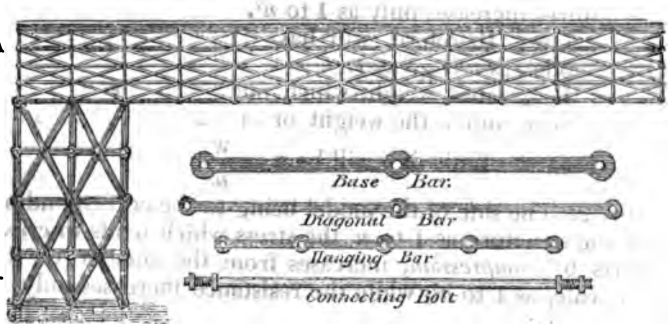
If, retaining the length or height $n\,h$, and the breadth $n\,b$, we wished to give to the solid such a thickness $x\,t$, as that it should not break in consequence of its increased dimensions, we should have $x = n^2 \sqrt{\frac{w}{W}}$.

In the case of a pilaster with a square base, or of a cylindrical column, if the dimension of the model were d, and of the largest pillar, which should not crush with its own weight when n times as high, $x\,d$, we should have

$$x = n\sqrt[3]{\frac{n^2 . w}{\mathrm{W}}}.$$

These theorems will often find their application in the profession of an architect or an engineer, whether civil or military.

3. Suppose, for an example, it were required to ascertain the strength of Mr. Smart's "*Patent mathematical chain-bridge*," from experiments made with a model. In this ingenious construction, the truss-work is carried across from pier to pier, so that the road-way from A to B, and thence entirely across, shall be in a horizontal plane, and all the base bars, diagonal bars, hanging bars, and connecting bolts, shall retain their own respective magnitudes throughout the structure. The annexed representation of half the bridge so exhibits the construction as to supersede the necessity of a minute verbal description.

A  B

Now, let l represent the horizontal length of the model, (say 12 feet,) from interior to exterior of the two piers, w its weight (say 30 pounds), W the weight it will just sustain at its middle point B before it breaks (say 350lbs). Let $n\,l$ the length of a bridge actually constructed of the same material as the model, and all its dimensions similar: then, its weight will be $n^3 w$, and its resisting power to that of the model, as n^2 to 1, being $= n^2 (\mathrm{W} + \frac{1}{2} w.)$ Hence $n^2(\mathrm{W} + \frac{1}{2} w) - \frac{1}{2} n^3 w = n^2 . \mathrm{W} - \frac{1}{2} n^2 (n - 1)\, w$, the load which the bridge itself wouldbear at the middle point.

Suppose $n = 20$, or the bridge 240 feet long, and entirely similar to the model; then we shall have $(400 \times 350) - 200 (20 - 1) 30 = 140000 - 114000 = 26000$lbs. $= 11$ tons $12\frac{1}{7}$ cwt. the load it would just sustain in the middle point of its extent.

Note. This bridge is, in fact, a *suspension* bridge, and would require brace or tie-chains at each pier.

CHAP. X.

DYNAMICS.

1. THE *mass* of a body is the quantity of matter of which it is composed.

The knowledge of the mass of a body is given to us by that property of matter which we call *inertia;* and which being greater or less as the mass is greater or less, we regard as an index of the mass itself.

2. *Density* is a word by which we indicate the comparative closeness or otherwise of the particles of bodies. Those bodies which have the greatest number of particles, or the greatest quantity of matter, in a given magnitude, we call *most dense;* those which have the least quantity of matter, *least dense.* Density and weight are regarded as correlatives; so that the *heaviest* bodies of a given size, are the *most* dense, the *lightest* bodies, the *least* dense.

Thus *lead* is more dense than freestone; *freestone* more dense than oak; *oak* more dense than *cork.*

3. When *bodies* are impelled by certain *forces,* they receive certain *velocities,* and move over certain *spaces,* in certain *times.* So that *body, force, velocity, space, time,* are the subjects of investigation in Dynamics; and in mathematical theorems, they are usually represented by the initial letters, b, f, v, s, t: or, if two or more bodies, &c. are compared, two or more corresponding letters B, b, b', V, v, v', &c. are employed in the formulæ. Gravity, which is a separate force incessantly acting, is represented by g; and *momentum,* or *quantity* of *motion* by m, this being the effect produced by a body in motion.

Force is distinguished into motive and accelerative, or retardive.

4. Motive force, otherwise called momentum, or force of percussion, is the absolute force of a body in motion, &c.; and is expressed by the product of the weight or mass of matter in the body multiplied by the velocity with which it moves. But

5. Accelerative force, or retardive force, is that which respects the velocity of the motion only, accelerating or retarding it; and it is denoted by the quotient of the motive

force divided by the mass or weight of the body. So, if a body of 2lbs. weight be acted upon by a motive force of 40, the accelerating force is 20; but, if the same force of 40 act upon another body of 4lbs. weight, the accelerating force is then only 10; that is, it is only *half* the former, and will produce only half the velocity.

Section I. *Uniform Motions.*

1. The space described by a body moving uniformly, is represented by the product of the velocity into the time: and in comparing two, we say

$$S : s :: TV : tv.$$

2. In regard to momenta, m varies as bv, or

$$M : m :: BV : bv.$$

Example. Two bodies, one of 10, the other of 5 pounds, are acted upon by the same momentum, or receive the same quantity of motion 30. They move uniformly, the first for 8 seconds, the second for 6: required the spaces described by both.

Here $\frac{30}{10} = 3 = V$, and $\frac{30}{5} = 6 = v$

Then $TV = 3 \times 8 = 24 = S$; and $tv = 6 \times 6 = 36 = s$.

Thus the spaces are 24 and 36 respectively.

Section II. *Motion uniformly accelerated.*

1. Motion uniformly accelerated, is that of a material point or body subjected to the continual action of a constant force.

2. In this motion, the velocity acquired at the end of any time whatever is equal to the product of the accelerating force into the time; and the space described is equal to the product of half the accelerating force into the square of the time.

3. The spaces described in successive seconds of time are as the *odd* numbers, 1, 3, 5, 7, 9, &c.

4. *Gravity* is a constant force, whose effect upon a body falling freely in a vertical line is represented by g; and the motion of such body is uniformly accelerated.

5. The following theorems are applicable to all cases of motion uniformly accelerated by any constant force f.

$$s = \tfrac{1}{2} t v = \tfrac{1}{2} g f t^2 = \frac{v^2}{2 g f}$$

$$v = \frac{2 s}{t} = g f t = \sqrt{2 g f s}$$

$$t = \frac{2 s}{v} = \frac{v}{g f} = \sqrt{\frac{s}{\tfrac{1}{2} g f}}$$

$$f = \frac{v}{g t} = \frac{2 s}{g t^2} = \frac{v^2}{2 g s}$$

Hence, in all motions of this nature, as soon as the ratio of the force, f, to the force of gravity, g, is known, the circumstances of space, time, velocity, &c. may be computed; or conversely, knowing the space described in a given time, or the velocity acquired at the end of such time, the value of f may be obtained.

6. When the force of gravity acts freely, as when a body falls in a vertical line, f is omitted in the theorems, and we have

$$s = \tfrac{1}{2} g t^2 = \frac{v^2}{2 g} = \tfrac{1}{2} t v$$

$$v = g t = \frac{2 s}{t} = \sqrt{2 g s}$$

$$t = \frac{v}{g} = \frac{2 s}{v} = \sqrt{\frac{2 s}{g}}$$

$$g = \frac{v}{t} = \frac{2 s}{t^2} = \frac{v^2}{2 s}$$

7. Now, it has been ascertained by very accurate experiments that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet; therefore, if $\frac{1}{2} g$ denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or g the velocity generated in that time; then, if the first series of natural numbers be seconds of time,

namely, the times in seconds	1″,	2″,	3″,	4″,	&c.
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

of which spaces the common difference is $32\frac{1}{6}$ feet, the natural and obvious measure of g, the force of gravity.

8. If, instead of a heavy body being allowed to fall freely it be *propelled* vertically upwards or downwards with a given velocity, v, then

$$s = tv \mp \tfrac{1}{2} g t^2;$$

an expression in which the upper sign − must be taken when the projection is *upwards*, the lower sign + when the projection is downwards.

When only an approximate result is required with reference to bodies falling vertically, 32 may be put for g, instead of $32\frac{1}{6}$: there would then result, in motions from quiescence,

$$s = 16t^2 = \frac{v^2}{64} = \tfrac{1}{2} tv$$

$$t = \frac{v}{32} = \tfrac{1}{4} \sqrt{s} = \frac{2s}{v}.$$

$$v = 8 \sqrt{s} = \frac{2s}{t} = 32t.$$

Thus, if the space descended were 64 feet, we should have $v = 8 \times 8 = 64$, and $t = \frac{8}{4} = 2$ seconds.

If the space descended were 400; then $v = 8 \times 20 = 160$, and $t = \frac{20}{4} = 5$.

9. The force of gravity differs a little at different latitudes; the law of the variation is not as yet *precisely* ascertained; but the following theorems are known to represent it very nearly. That is, if g denote the force of gravity at latitude 45°, g' the force at any other place: then

$$g' = g(1 \cdot - \cdot 002837 \cos 2 \text{ lat.})$$
$$g' = g(1 + \cdot 002837), \text{ at the poles.}$$
$$g' = g(1 - \cdot 002837), \text{ at the equator.}$$

10. *Motion over a fixed pulley.* In this case let the two weights which are connected by the cord that goes over the pulley, be denoted by W and w: then $\frac{W - w}{W + w} = f$, in the formulæ of art. 5; so that

$$s = \frac{W - w}{W + w} \cdot \tfrac{1}{2} g t^2.$$

Or, if the resistance of the friction and inertia of the pulley be represented by r; then

$$s = \frac{W - w}{W + w + r} \cdot \tfrac{1}{2} g t^2.$$

Example. Suppose the two weights to be 5 and 3lbs. respectively, what will be the space descended in 4 seconds?

Here $\frac{\text{W} - w}{\text{W} + w} \cdot \frac{1}{2} g t^2 = \frac{5 - 3}{5 + 3} \cdot 16\frac{1}{12} \cdot 16 = \frac{1}{4} \cdot 16\frac{1}{12} \cdot 16 =$ $16\frac{1}{12} \cdot 4 = 64\frac{1}{3}$ feet.

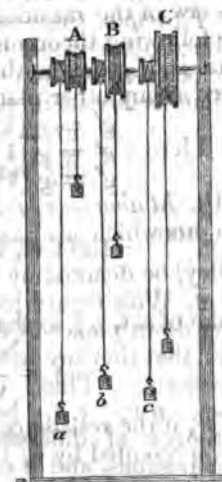
Example II. But, suppose that in an actual experiment with two weights of 5 and 3lbs. over a pulley, the heavier weight descended only 50 feet in four seconds.

Then $\frac{\text{W} - w}{\text{W} + w + r} \cdot \frac{1}{2} g t^2 = 50$ feet: and, as W, w, g, and t, are the same in both examples,

we have $\text{W} + w + r : \text{W} + w :: 64\frac{1}{3} : 50$
or, dividendo $r : \text{W} + w :: 14\frac{1}{3} : 50$
that is, $r : 5 + 3 :: 14\frac{1}{3} : 50$
whence $r = \frac{8 \times 14\frac{1}{3}}{50} = \frac{114\frac{2}{3}}{50} = 2{\cdot}2933$lbs.

the measure of the resistance and the inertia.

Note. Similar principles are applicable in a variety of other cases: and by varying the relations of W, w, and r, the force may have any assigned ratio to that of gravity; which is, indeed, the foundation of Mr. *Atwood's* elegant apparatus for experiments on accelerating forces. The inquisitive reader may see an account of it in the 2d vol. of my *Mechanics*, or in almost any of the general dictionaries of arts and sciences.



11. If instead of pulleys, small wheels and axles, as in the marginal figure, be employed, to raise weights by the preponderance of equal weights: then, if the diameters of wheel and axle A be as 3 to 2; those of wheel and axle B, as 5 to 2; those of C, as 8 to 2; it will be found that the weight b will be elevated more rapidly than either a or c: the proportion of 5 to 2, (or, correctly, of $1 + \sqrt{2}$ to 1) being in that respect the most favourable.

Motion on inclined Planes.

1. When bodies move down inclined planes, the accelerating force (independently of the modification occasioned by the position of the *centre of gyration*) is expressed by $\frac{h}{l}$, the quotient of the height of the plane divided by its length, or by what is equivalent, the sine of the inclination of the plane, that is to say, sin i. In this case, therefore, the formulæ become

$$1 \, . \; s = \tfrac{1}{2} g t^2 \sin i = \frac{v^2}{2 g \sin i} = \tfrac{1}{2} t v$$

$$2 \, . \; v = g t \sin i = \sqrt{(2 g s \sin i)} = \frac{2 s}{t}$$

$$3 \, . \; t = \sqrt{\frac{2 s}{g \sin i}} = \frac{2 s}{v}.$$

Farther, if v be the velocity with which a body is projected up or down a plane, then

$$4 \, . \; v = \mathrm{v} \mp g t \sin i$$

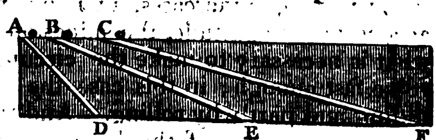
$$5 \, . \; s = \mathrm{v} t \mp \tfrac{1}{2} g t \sin i = \frac{\mathrm{v}^2 - v^2}{2 g \sin i}.$$

Making $\mathrm{v} = 0$, in equa. 4, and the latter member of equation 5, the first will give the *time* at which the body will cease to rise, the latter the *space*.

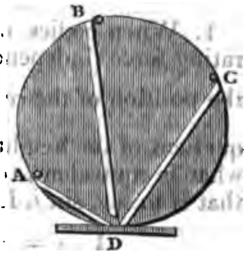
Example. Suppose a body be projected up a smooth inclined plane whose height is 12 and length 193 feet, with a velocity of 20 feet per second, how high will it rise up the plane before its motion is extinguished?

Here $s = \frac{\mathrm{v}^2 - v^2}{2 g \sin i}$, becomes $s = \frac{400 - 0}{64\frac{1}{3} \cdot \frac{12}{193}} = \frac{400}{\frac{193}{3} \cdot \frac{12}{193}}$ $= \frac{400}{4} = 100$ feet, the space required.

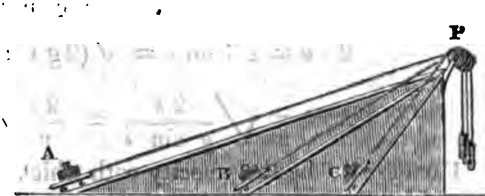
2. With regard to the *velocities* acquired by bodies in falling down planes of the same height, this proposition holds; viz. that they are all equal, estimated in their respective directions. Thus, if A D, B E, C F, be grooves of different inclinations, and A C, D F, horizontal lines, the balls A, B, C, after



descending through those planes will have equal velocities when they arrive at D, E, F, respectively.



3. Also, all the chords, such as A D, B D, C D, that terminate either in the upper or the lower extremity of the vertical diameter of a circle, will be described in the same time by heavy bodies A, B, C, running down them; and that time will be equal to the time of vertical descent through the diameter.



4. If three weights, as A, B, C, be drawn up three planes of different inclinations, by three equal weights hanging from cords over pulleys at P, then if the *length* of the middle plane be *twice* its *height*, the body B will be drawn up that plane, *quicker* than either of the other weights A or C.

Or, generally, to ensure an ascent up a plane in the least time, the *length* of the plane must be to its *height*, as *twice the weight*, to the *power* employed.

5. If it be proposed to construct a roof over a building of a given width, so that the rain shall run quickest off it, then each side of the roof must be inclined 45° to the horizon, or the angle at the ridge must be a right angle.

6. The force by which spheres, cylinders, &c. are caused to revolve as they move down an inclined plane (instead of sliding) is the adhesion of their surfaces occasioned by the pressure against the plane: this pressure is part of the body's weight; for the weight being resolved into its components, one in the direction of the plane, the other perpendicular to it, the latter is the force of the pressure; and, while the same body rolls down the plane, will be expressed by the cosine of the plane's elevation. Hence, since the cosine decreases while the arc or angle increases, after the angle of elevation arrives at a certain magnitude, the adhesion may become less than what is necessary to make the circumference of the body revolve fast enough; in this case the body descends partly by sliding and partly by rolling. And the same may happen in smaller elevations, if the body and plane are very smooth.

But at all elevations the body may be made to roll by the uncoiling of a thread or ribband wound about it.

If W denote the weight of a body, s the space described by a body falling freely, or sliding freely down an inclined plane, then the spaces described by rotation in the same time by the following bodies, will be in these proportions.

1. A hollow cylinder, or cylindrical surface, S $= \frac{1}{2} s$ tension of the cord in the first case $= \frac{1}{2}$ W.

2. In a solid cylinder, S $= \frac{2}{3} s$, tens. $= \frac{1}{3}$ W.

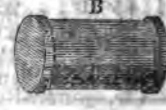
3. In a spheric surface, or thin spherical shell, S $= \frac{3}{5} s$, tens. $= \frac{2}{5}$ W.

4. In a solid sphere, S $= \frac{5}{7} s$, tens. $= \frac{2}{7}$ W.

If two cylinders be taken of equal size and weight, and with equal protuberances upon which to roll, as in the marginal figures: then, if lead be coiled uniformly over the curve surface of B, and an equal quantity of lead be placed uniformly from one end to the other near the axis in the cylinder A, that cylinder will roll down any inclined plane *quicker* than the other cylinder B. The reason is that *each particle of matter in a rolling body, resists motion in proportion to the* SQUARE *of its distance from the axis of motion;* and the particles of lead which most resist motion are placed at a *greater* distance from the axis in the cylinder B than in A.

A

B



7. The friction between the surface of any body and a plane, may be easily ascertained by gradually elevating the plane until the body upon it *just begins to slide.* The friction of the body is to its weight as the height of the plane to its base, or as the tangent of the inclination of the plane to the radius. Thus, if a piece of stone in weight 8 pounds, just begins to slide when the height of the plane is 2 feet, and its base $2\frac{1}{2}$; then the friction will be $\frac{4}{5}$ the weight, or $\frac{4}{5}$ of 8 lbs. $= 6\frac{2}{5}$ lbs.

8. After motion has commenced upon an inclined plane, the friction is usually much diminished. It may easily be ascertained experimentally, by comparing the time occupied by a body in *sliding* down a plane of given height and length, or given inclination, with that which the simple theorem for t, would give. For, if f be the value of the friction in terms of the pressure, the theorem for t' will be

$$t' = \sqrt{\frac{2s}{g(\sin i - f)}}, \text{ instead of } t = \sqrt{\frac{2s}{g \sin i}}. \text{ Hence}$$

$$t'^2 : t^2 :: \sin i : \sin i - f$$

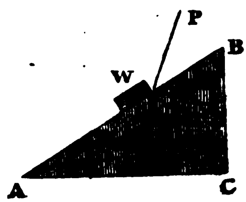
Example. Suppose that a body slides down a plane in length 30 feet, height 10, in $2\frac{3}{5}$ seconds, what is the value of the friction.

$$\text{Here, } t = \sqrt{\frac{2s}{g \sin i}} = \sqrt{\frac{60}{32\frac{1}{6} \times \frac{1}{3}}} = 2{\cdot}366 \text{ nearly.}$$

$$\text{Hence } (2{\cdot}6)^2 : (2{\cdot}366)^2 :: \tfrac{1}{3} : {\cdot}27603 = \sin i - f$$

Consequently, ·33333 — ·27603 = ·0573 value of the friction, the weight being unity.

9. When a weight is to be moved either up an inclined plane, or along an horizontal plane, the angle of traction P W B, that the weight may be drawn with least effort, will vary with the value of f. The magnitude of that angle P W B for several values of f are exhibited below.



f	P W B	f	P W B	f	P W B	f	P W B	f	P W B	f	P W B
1	45° 0′	$\frac{1}{2}$	26° 34′	$\frac{1}{3}$	18° 26′	$\frac{1}{4}$	14° 2′	$\frac{1}{5}$	11° 19′	$\frac{1}{6}$	9 28
$\frac{4}{5}$	38 40	$\frac{4}{9}$	23 58	$\frac{4}{13}$	16 54	$\frac{4}{17}$	13 15	$\frac{4}{21}$	10 47	$\frac{1}{7}$	8 8
$\frac{2}{3}$	33 41	$\frac{2}{5}$	21 48	$\frac{2}{7}$	15 57	$\frac{2}{9}$	12 32	$\frac{2}{11}$	10 18	$\frac{1}{8}$	7 8
$\frac{4}{7}$	29 45	$\frac{4}{11}$	19 59	$\frac{4}{15}$	14 56	$\frac{4}{19}$	11 53	$\frac{4}{23}$	9 52	$\frac{1}{9}$	6 20

10. If, instead of seeking the line of traction so that the moving force should be a *minimum*, we required the position such that the suspending force to keep a load from descending should be a minimum, or a given force should oppose motion with the greatest energy; then the angles in the preceding table will be still applicable, only the angle in any assigned case must be taken below, as B W p. This will serve in the building and fastening walls, banks of earth, fortifications, &c. and in arranging the position of *land-ties*, &c.

SECTION III.—*Motions about a Centre or Axis.*

Pendulum, simple and compound; Centres of Oscillation, Percussion, and Gyration.

DEF. 1. *The centre of oscillation* is that point in the axis of suspension of a vibrating body in which, if all the matter of the system were collected, any force applied there would generate the same angular velocity in a given time as the same force at the centre of gravity, the parts of the system revolving in their respective places.

Or, since the force of gravity upon the whole body may be considered as a single force (equivalent to the weight of the body) applied at its centre of gravity, the *centre of oscillation* is that point in a vibrating body into which, if the whole were concentrated and attached to the same axis of motion, it would then vibrate in the same time the body does in its natural state.

COR. From the first definition it follows that the centre of oscillation is situated in a right line passing through the centre of gravity, and perpendicular to the axis of motion.

DEF. 2. The *centre of gyration* is that point in which, if all the matter contained in a revolving system were collected, the same angular velocity will be generated in the same time by a given force acting at any place as would be generated by the same force acting similarly in the body or system itself.

When the axis of motion passes through the centre of gravity, then is this centre called the *principal* centre of gyration.

The distance of the centre of gyration from the point of suspension, or the axis of motion, is a mean proportional between the distances of the centres of oscillation and gravity from the same point or axis.

If S represent the point of *suspension*, G the place of the centre of *gravity*, O that of the centre of *oscillation*, and R that of the centre of *gyration*. Then,

$$SR = \sqrt{SO \cdot SG} \quad \ldots\ldots\ldots (1)$$

and $SO \cdot SG$ = a constant quantity for the same body and the same plane of vibration.

DEF. 3. *The Centre of Percussion* is that point in a body revolving about an axis, at which, if it struck an immoveable obstacle, all its motion would be destroyed, or it would not incline either way.

When an oscillating body vibrates with a given angular velocity, and strikes an obstacle, the effect of the impact will be the greatest if it be made at the centre of percussion.

For, in this case the obstacle receives the whole revolving motion of the body; whereas, if the blow be struck in any other point, a part of the motion of the pendulum will be employed in endeavouring to continue the rotation.

If a body revolving on an axis strike an immoveable obstacle at the centre of percussion, the point of suspension will not be affected by the stroke.

We can ascertain this property of the point O when we give a smart blow with a stick. If we give it a motion round the joint of the wrist only, and, holding it at one extremity, strike smartly with a point considerably nearer or more remote than $\frac{2}{3}$ of its length, we feel a painful wrench in the hand: but if we strike with that point which is precisely at $\frac{2}{3}$ of the length, no such disagreeable strain will be felt. If we strike the blow with one end of the stick, we must make its centre of motion at $\frac{1}{3}$ of its length from the other end; and then the wrench will be avoided.

PROP. The distance of the centre of percussion from the axis of motion is equal to the distance of the centre of oscillation from the same: supposing that the centre of percussion is required in a plane passing through the axis of motion and centre of gravity.

DEF. 4. A *Simple Pendulum*, theoretically considered, is a single weight, regarded as a point, or as a very small globe hanging at the lower extremity of an inflexible right line, void of weight, and suspended from a fixed point or centre, about which it oscillates.

DEF. 5. A *Compound Pendulum* is one that consists of several weights moveable about one common centre of motion, but so connected together as to retain the same distance both from one another and from the centre about which they vibrate.

Or any body, as a cone, a cylinder, or of any shape, regular or irregular, so suspended as to be capable of vibrating, may be regarded as a compound pendulum; and the distance of its *centre of oscillation* from any assumed point of suspension, is considered as the length of an equivalent simple pendulum.

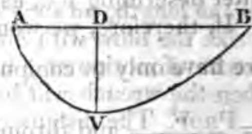
Any such vibrating body will have as many *centres of oscillation* as you give it points of suspension: but when any one of those centres of oscillation is determined, either by theory or experiment, the rest are easily found by means of the property that $SO \cdot SG$ is a constant product, or of the same value for the same body.

DEF. 6. When a body either revolves about an axis, or oscillates, the sum of the products of each of the material elements, or particles of that body, into the squares of their respective distances from the axis of rotation, is called the *momentum of inertia* of that body. (See art. 6, page 240).

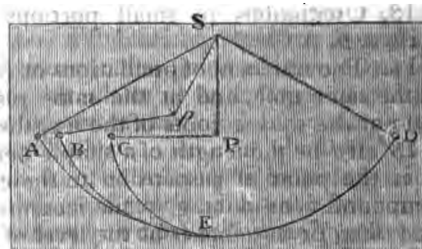
A point, or very small body, on descending along the successive sides of a polygon in a vertical plane, loses at each angle a part of its actual velocity equal to the product of that velocity into the versed sine of the angle made by the side which it has just quitted, and the prolongation of the side upon which it is just entering.

Therefore that loss is indefinitely small in *curves*.

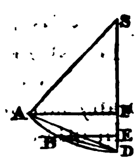
7. A heavy body which descends along a curve posited in a vertical plane, by the force of gravity, has in any point whatever, the same velocity as it would have if it had fallen through a vertical line equal to that between the top and the bottom of the arc run over: and when it has arrived at the bottom of any such curve, if there be another branch either similar or dissimilar, rising on the opposite side, the body will rise along that branch (apart from the consideration of friction) until it has reached the horizontal plane from which it set out. Thus, after having descended from A to V, it will have the same velocity as that acquired by falling through D V, and it will ascend up the opposite branch until it arrives at B.



8. If the body describe a curve by a pendulous motion, the same property will be shown, free from the effects of friction. Thus, let a ball hang by a flexible cord S D from a pin at S: then, after it has descended through the arc D E, it will pass through an equal and similar arc E A, going up to A in the same horizontal line with D, and ascending from E to A in an interval of time equal to that which it descended from D to E. But, if a pin projecting from P or *p* stop the cord in its course, the ball will still rise to B or to C, in the same horizontal line with A and D; but will describe the ascending portions of the curve in shorter intervals of time than the descending branch.



9. When a pendulum is drawn from its vertical position, it will be accelerated in the direction of the tangent of the curve it would describe, by a force which is as the sine of its angular distance from the vertical position. Thus, the accelerating force at A, would be to the accelerating force at B, as A F to B E. (See art. 5, on the *Centre of Gravity*.)



This admits of an easy experimental proof.

10. If the same pendulous body descend through different arcs, its velocity at the lowest point will be proportional to the chords of the whole arcs described. Thus, the velocity at D

after passing through A B D, will be to the velocity at D after descending through the portion B D only, as A D to B D.

11. Farther, velocity after describing A B D, is to velocity after describing B D, as $\sqrt{\text{F D}}$ to $\sqrt{\text{E D}}$.

If, therefore, we would impart to a body a given velocity v, we have only to compute the height F D, such that $\text{F D} = \frac{v^2}{2g} = \frac{v^2}{64\frac{1}{3}\text{ feet}}$, and through the point F draw the horizontal line F A; then, letting the body descend as a pendulum through the arch A B D, when arrived at D it will have acquired the proposed velocity.

This is extremely useful in experiments on the *collision* of bodies.

12. The oscillations of pendulums in any arcs of a cycloid are *isochronal*, or performed in equal times.

13. Oscillations in small portions of a circular arc are *isochronal*.

14. The numbers of oscillations of two different pendulums, in the same time, and at the same place, are in the inverse ratio of the square roots of the lengths of these pendulums.

15. If l be the length of a simple pendulum, or the distance from the point of suspension to the centre of oscillation in a compound pendulum, g = the measure of the force of gravity ($32\frac{1}{6}$ feet, or 386 inches at the level of St. Paul's* in the latitude of London), t = the time of one oscillation in an indefinitely small circular arc, and $\pi = 3{\cdot}141593$: then

$$t = \pi\sqrt{\frac{l}{g}}$$

16. Conformably with this we have

$39\frac{1}{8}$ inches,	length of the second	pendulum in lat. of London.
$9\frac{25}{32}$ inches,	—— half second	
$4\frac{25}{72}$ inches,	—— third of second	
$2\frac{57}{128}$ inches,	—— quarter second	

17. We have also $l = {\cdot}20264 \times \frac{1}{2}g$

and $\frac{1}{2}g = 4{\cdot}9348\ l$

in any latitude, and at any altitude.

In other words, whatever be the force of gravity, the length

* At the level of *the sea*, in the latitude of London, g is 386·289 inches, and the corresponding length of the second pendulum is 39·1393 inches, agreeing with the determination of *Major Kater*. Conformably with this result are the numbers in the table following art. 30 of this section, computed at the expense of Messrs. *Bramahs* and Mr. *Donkin*, and obligingly communicated by them for this work.

of a second pendulum, and the space descended freely by a falling body in 1 second, *are in a constant ratio.*

18. If l' be the length of a pendulum, g' the force of gravity, and t' the time of oscillation at any other place, then

$$t : t' :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}.$$

If the force of gravity be the same,

$$t : t' :: \sqrt{l} : \sqrt{l'}.$$

If the same pendulum be actuated by different gravitating forces, we have

$$t : t' :: \sqrt{\frac{1}{g}} : \sqrt{\frac{1}{g'}} :: \sqrt{g'} : \sqrt{g}.$$

When pendulums oscillate in equal times in different places, we have

$$g : g' :: l : l'.$$

For the variations of gravity in different latitudes, see art. 9, pa. 236.

18. If the arcs are not indefinitely short, let v denote the versed sine of the semi-arc of vibration; then

$$t = \pi\sqrt{\frac{l}{g}}\left(1 + \tfrac{1}{8}v + \tfrac{9}{384}v^2 + \&c.\right)$$

In which, when the semi-arc of vibration does not exceed 4 or 5 degrees, the third term of the series may be omitted.

If the time of an oscillation in an indefinitely small arc be 1 second, the augmentation of the time will be

for a semi-arc of 30°	0·01675
of 15°	0·00426
of 10°	0·00190
of 5°	0·00012
of 2½°	0·00003

So that for oscillations of 2½° on each side of the vertical, the augmentation would not occasion more than 2″ difference in a day.

19. If D denote the degrees in the semi-arc of an oscillating pendulum, the time lost in each second by vibrating in a circle instead of the cycloid, is $\frac{D^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or 24 × 60 × 60 seconds, is $\frac{5}{3}D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the

pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees: and so on.

20. If a clock keep true time very nearly, the variation in the length of the pendulum necessary to correct the error will be equal to twice the product of the length of the pendulum, and the error in time divided by the time of observation in which that error is accumulated.

If the pendulum be one that should beat seconds, and t' the daily variation be given in minutes, and n be the number of threads in an inch of the screw which raises and depresses the bob of the pendulum, then $\lambda = \frac{\pm 2 \times 39\frac{1}{8} \times n t'}{24 \times 60} =$ $\cdot 05434\, n t' = \frac{3}{37}\, n t'$ nearly, for the number of threads which the bob must be raised or lowered, to make the pendulum vibrate truly.

21. For civil and military engineers, and other practical men, it is highly useful to have a *portable pendulum*, made of painted tape with a brass bob at the end, so that the whole, except the bob, may be rolled up within a box, and the whole enclosed in a shagreen case. The tape is marked 200, 190, 180, 170, 160, &c. 80, 75, 70, 65, 60, at points, which being assumed respectively as points of suspension, the pendulum will make 200, 190, &c. down to 60 vibrations in a minute. Such a portable pendulum may be readily employed in experiments relative to falling bodies, the velocity of sound, &c. The pendulum and its box may go in a waistcoat pocket.

22. If the *momentum of inertia* (Def. 6) of a pendulum whether simple or compound, be divided by the product of the pendulum's weight or mass into the distance of its centre of gravity from the point of suspension (or axis of motion), the quotient will express the distance of the centre of oscillation from the same point (or axis).

23. Whatever the number of separate masses or bodies which constitute a pendulum, it may be considered as a single pendulum whose centre of gravity is at the distance d from the axis of suspension, or of rotation: then if κ^2 denote the momentum of inertia of that body divided by its mass, the distance SO from the axis of rotation to the centre of oscillation or the length of an equivalent simple pendulum, will be

$$SO = \frac{d^2 + \kappa^2}{d}.$$

24. To find the distance of the centre of oscillation from the point or axis of suspension, *experimentally*. Count the number, n, of oscillations of the body in a very short arc in a minute; then

$$s\,o = \frac{140850}{n^2}$$

Thus, if a body so oscillating made 50 vibrations in a minute: then $s\,o = \frac{140850}{2500} = 56{\cdot}34$ inches.

Or, $s\,o = 39\frac{1}{8}\,t^2$, in inches, t being the time of one oscillation in a very small arc.

If the arc be of finite appreciable magnitude, the time of oscillation must be reduced in the ratio 8 + versin of semiarc to 8, before the rule is applied.

25. From the foregoing principles are derived the following expressions for the distances of the centres of oscillation for the several figures, suspended by their vertices and vibrating flatwise, viz.

(1) Right line or very thin cylinder, $s\,o = \frac{2}{3}$ of its length.

(2.) Isosceles triangle, $s\,o = \frac{3}{4}$ of its altitude.

(3.) Circle, $s\,o = \frac{5}{4}$ radius.

(4.) Common parabola, $s\,o = \frac{5}{7}$ of its altitude.

(5.) Any parabola, $s\,o = \frac{2m + 1}{3m + 1} \times$ its altitude.

Bodies vibrating laterally or sideways, or in their own plane.

(6.) In a circle, $s\,o = \frac{3}{4}$ of diameter.

(7.) In a rectangle suspended by one angle, $s\,o = \frac{2}{3}$ of diagonal.

(8.) Parabola suspended by its vertex, $s\,o = \frac{5}{7}$ axis $+ \frac{1}{3}$ parameter.

(9.) Parabola suspended by middle of its base, $s\,o = \frac{4}{7}$ axis $+ \frac{1}{2}$ parameter.

(10.) In a sector of a circle, $s\,o = \frac{3\text{ arc} \times \text{radius}}{4\text{ chord}}$.

(11.) In a cone, $s\,o = \frac{4}{5}\text{ axis} + \frac{(\text{radius of base})^2}{5\text{ axis}}$.

(12.) In a sphere, $s\,o = \text{rad.} + d + \frac{2\text{ rad.}^2}{5(d + \text{rad.})}$

Where d is the length of the thread by which it is suspended.

(13.) If the weight of the thread is to be taken into the account, we have the following distance between the centre of the ball and that of oscillation, where B is the weight of the ball, d the distance between the point of suspension and its centre, r the radius of the ball, and w the weight of the thread or wire,

$$\text{G O} = \frac{(\frac{1}{3} w + \frac{2}{5} \text{B}) 4 r^2 - \frac{1}{6} w (2 d r + d^2)}{(\frac{1}{2} w + \text{B}) d - r w}$$; or, if B be expressed in terms of w considered as a unit, then $\text{G O} = \frac{\frac{1}{6} d}{\text{B} + \frac{1}{2}}$.

(14.) If two weights W, W′ be fixed at the extremities of a rod of given length W W′, S being the centre of motion between W and W′; then if d = S W, D = S W′, and m the weight of an unit in length of the rod, we shall have S O =

$$\frac{m \text{D}^3 + 3 \text{W}' \text{D}^2 + m d^3 + 3 \text{W} d^2}{m \text{D}^2 + 2 \text{W}' \text{D} - m d^2 - 2 \text{W} d}$$; the radii of the balls being supposed very small in comparison with the length of the rod.

(15.) In the bob of a clock pendulum, supposing it two equal spheric segments joined at their bases, if the radii of those bases be each = ρ, the height of each segment v, and d the distance from the point of suspension to G the centre of the bob, then

is $\text{G O} = \frac{1}{3} d \cdot \frac{\rho^4 + \frac{1}{2} \rho^2 v^2 + \frac{1}{10} v^4}{\rho^2 + \frac{1}{3} v^2}$; which shows the distance of the centre of oscillation below the centre of the bob.

If r the radius of the sphere be known, the latter expression becomes $\text{G O} = \frac{\frac{2}{3} l^2 v \frac{1}{2} r v - \frac{1}{10} v^3}{d (r - \frac{1}{3} v)}$.

(16.) Let the length of a rectangle be denoted by l, its breadth by $2 w$, the distance (along the middle of the rectangle) from one end to the point of suspension by d, then the distance S O, from the point of suspension to the centre of oscillation, will be

$$\text{S O} = \frac{d^2 - d l + \frac{1}{3} l^2 + \frac{1}{3} w^2}{\frac{1}{2} l - d} = \frac{1}{2} l - d + \frac{\frac{1}{12} l^2 + \frac{1}{3} w^2}{\frac{1}{2} l - d},$$

whether the figure be a mere geometrical rectangle, or a prismatic metallic plate of uniform density.

It follows from this theorem that a plate of 1 foot long, and $\frac{1}{4}$ of a foot broad, suspended at a fourth of a foot from either end, would vibrate as a half second pendulum.

Also, that a plate a foot long, $\frac{1}{16}$ of a foot wide, and suspended at $\frac{1}{4}$ of a foot from the middle, would vibrate 36,469 times in 5 hours.

And hence the length of a foot may be determined experimentally by vibrations.

(17.) If a thin rod, say of a foot in length, have a ball of an inch diameter at each end, A and B, and a moveable point of suspension, S; then *the time of oscillation of such a pendulum may be made as long as we please*, by bringing the point of suspension nearer and nearer to the middle of the rod.

Or, if the point of suspension be fixed, the distance S O (and consequently the time of oscillations which is as $\sqrt{\text{S O}}$) may be varied by placing A nearer or farther from S. *And this is the principle of the* METRONOME, *by which musicians sometimes regulate their time.*

(18.) If the weight of the connecting rod be evanescent with regard to the weight of the balls A and B; then if R = radius of the larger ball, r that of the smaller, D and d the distances of their respective centres from S; we shall have

$$\text{S O} = \frac{R^3 (5 D^2 + 2 R^2) + r^3 (5 d^2 + 2 r^2)}{5 (D R^3 - D r^3)}$$

When R and r are equal, this becomes

$$\text{S O} = (D + d) + \tfrac{4}{5} \cdot \frac{r^2}{D - d}.$$

(19.) If the minor and major axes of an ellipse (or of an elliptical plate of wood or metal) be as 1 to $\sqrt{3}$, or as 1000 to 1732; then if it be suspended at one extremity of the minor axis, the centre of oscillation will be at the other extremity of that axis, or its oscillations will be performed in the same time as those of a simple pendulum whose length is equal to the minor axis.

The same ellipse also possesses this curious and useful property: viz. That any segment or any zone of the ellipse cut off by lines parallel to the major axis, whether it be taken near the upper part of the minor axis, near the middle, or near the bottom of the same, will vibrate in the same time as the whole ellipse, the point of suspension being at an extremity of the minor axis.

26. It is evident from art. 17, that pendulums in different latitudes require to be of different lengths, in order that they may perform their vibrations in the same time; but besides this there is another irregularity in the motion of a pendulum in the same place, arising from the different degrees of temperature. Heat expanding, and cold contracting the rod of

the pendulum, a certain small variation must necessarily follow in the time of its vibration; to remedy which various methods have been invented for constructing what are commonly called *compensation pendulums,* or such as shall always preserve the same distance between the centre of oscillation and the point of suspension; but of these we shall describe two or three.

Compound or Compensation PENDULUMS have received different denominations, from their form and materials, as the *gridiron pendulum*, *mercurial pendulum*, &c.

27. The *Gridiron* PENDULUM consists of five rods of steel, and four of brass, placed in an alternate order, the middle rod being of steel, by which the pendulum ball is suspended; these rods of brass and steel are placed in an alternate order, and so connected with each other at their ends, that while the expansion of the steel rods has a tendency to lengthen the pendulum, the expansion of the brass rods acting upwards tends to shorten it. And thus, when the lengths of the brass and steel rods are duly proportioned, their expansions and contractions will exactly balance and correct each other, and so preserve the pendulum invariably of the same length.

Sometimes 3, 7, or 9, rods are employed in the construction of the gridiron pendulum; and zinc, silver, and other metals, may be used instead of brass and steel.

28. The *mercurial pendulum* was invented by Mr. Graham, an eminent clockmaker, about the year 1715. Its rod was made of brass, and branched towards its lower end, so as to embrace a cylindric glass vessel 13 or 14 inches long, and about 2 inches diameter; which being filled about 12 inches deep with mercury, forms the weight or ball of the pendulum. If upon trial the expansion of the rod be found too great for that of the mercury, more mercury must be poured into the vessel: if the expansion of the mercury exceeds that of the rod, so as to occasion the clock to go fast with heat, some mercury must be taken out of the vessel, so as to shorten the column. And thus may the expansion and contraction of the quicksilver in the glass be made exactly to balance the expansion and contraction of the pendulum rod, so as to preserve the distance of the centre of oscillation from the point of suspension invariably the same.

This kind of pendulum fell entirely into disuse soon after Graham's time; but it has lately been re-adopted with considerable success by practical astronomers. A very instructive paper on its principles, construction, and use, has been published by Mr. F. Baily, in vol. i. part 2, *Memoirs of the Astronomical Society of London.*

29. Reid's *Compensation* PENDULUM, is a recent invention of Mr. Adam Reid of Woolwich, the construction of which is as follows: A N is a rod of wire, and Z Z a hollow tube of zinc, which slips on the wire, being stopped from falling off by a nut N, on which it rests; and on the upper part of this cylinder of zinc rests the heavy ball B; now the length of the tube Z Z being so adjusted to the length of the rod A N, that the expansions of the two bodies shall be equal with equal degrees of temperature; that is, by making the length of the zinc tube to that of the wire, as the expansion of wire is to that of zinc, it is obvious that the ball B will in all cases preserve the same distance from A; for just so much as it would descend by the expansion of the wire downwards, so much will it ascend by the expansion of the zinc upwards, and consequently its vibrations will in all temperatures be equal in equal times.

30. *Drummond's Compensation Pendulum.*

This was proposed by an artist in Lancashire about 70 years ago. A bar of the same metal with the rod of the pendulum, and of the same thickness and length, is placed against the back part of the clock-case: from the top of this a part projects, to which the upper part of the pendulum is connected by two fine pliable chains or silken strings, which just below pass between two plates of brass whose lower edges will always terminate the length of the pendulum at the upper end. These plates are supported on a foot fixed to the back of the case. This bar rests upon an immoveable base on the lower part of the case, and is braced into a proper groove, which admits of no motion any way but that of expansion and contraction in length by heat and cold. In this construction, since the two bars are of equal magnitude and like constitution, their expansions and contractions will always be equal and in opposite directions; so that one will serve to correct and annihilate the effects of the other.

An extensive and valuable table of the *expansions* of different substances is given by Mr. Baily in the paper referred to above.

Table of Lengths and Vibrations of Pendulums.

[See note at foot of page 245.]

Length inches.	Time of vibration.	No. Vibr. per sec.	Length inches.	Time of vibration.	No. Vibr. per sec.
1·0	0·1598	6·256	5·3	0·3679	2·717
1·1	0·1676	5·965	5·4	0·3713	2·692
1·2	0·1751	5·711	5·5	0·3748	2·667
1·3	0·1822	5·487	5·6	0·3782	2·643
1·4	0·1891	5·287	5·7	0·3816	2·620
1·5	0·1957	5·108	5·8	0·3849	2·597
1·6	0·2021	4·945	5·9	0·3882	2·575
1·7	0·2084	4·798	6·0	0·3915	2·554
1·8	0·2144	4·663	6·1	0·3947	2·533
1·9	0·2203	4·538	6·2	0·3980	2·512
2·0	0·2260	4·423	6·3	0·4012	2·492
2·1	0·2316	4·317	6·4	0·4043	2·472
2·2	0·2370	4·217			
2·3	0·2424	4·125	6·5	0·4075	2·453
2·4	0·2476	4·038	6·6	0·4106	2·435
2·5	0·2527	3·956	6·7	0·4137	2·416
2·6	0·2577	3·879	6·8	0·4168	2·399
2·7	0·2626	3·807	6·9	0·4198	2·381
2·8	0·2674	3·738	7·0	0·4229	2·364
2·9	0·2721	3·673	7·1	0·4259	2·347
3·0	0·2768	3·612	7·2	0·4289	2·331
3·1	0·2814	3·553	7·3	0·4318	2·315
3·2	0·2859	3·497	7·4	0·4348	2·399
3·3	0·2903	3·443	7·5	0·4377	2·284
3·4	0·2947	3·392	7·6	0·4406	2·269
3·5	0·2990	3·344	7·7	0·4435	2·254
3·6	0·3032	3·297	7·8	0·4464	2·240
3·7	0·3074	3·252	7·9	0·4492	2·225
3·8	0·3115	3·209	8·0	0·4521	2·211
3·9	0·3157	3·167	8·1	0·4549	2·198
4·0	0·3196	3·128	8·2	0·4577	2·184
4·1	0·3236	3·089	8·3	0·4605	2·171
4·2	0·3275	3·052	8·4	0·4632	2·158
4·3	0·3314	3·016	8·5	0·4660	2·145
4·4	0·3352	2·982	8·6	0·4687	2·133
4·5	0·3390	2·949	8·7	0·4714	2·121
4·6	0·3428	2·916	8·8	0·4741	2·108
4·7	0·3465	2·885	8·9	0·4768	2·097
4·8	0·3502	2·855	9·0	0·4795	2·085
4·9	0·3538	2·826	9·1	0·4821	2·073
5·0	0·3574	2·797	9·2	0·4848	2·062
5·1	0·3609	2·770	9·3	0·4874	2·051
5·2	0·3644	2·743	9·4	0·4900	2·040

Length inches.	Time of vibration.	No. Vibr. per sec.	Length inches.	Time of vibration.	No. Vibr. per sec.
9·5	0·4926	2·029	14·3	0·6044	1·654
9·6	0·4952	2·019	14·4	0·6065	1·648
9·7	0·4978	2·008	14·5	0·6086	1·642
9·8	0·5003	1·998	14·6	0·6107	1·637
9·9	0·5029	1·988	14·7	0·6128	1·631
10·0	0·5054	1·978	14·8	0·6149	1·626
10·1	0·5079	1·968	14·9	0·6170	1·620
10·2	0·5105	1·958	15·0	0·6191	1·615
10·3	0·5130	1·949	15·1	0·6211	1·609
10·4	0·5155	1·939	15·2	0·6231	1·604
10·5	0·5179	1·930	15·3	0·6252	1·599
10·6	0·5204	1·921	15·4	0·6272	1·594
10·7	0·5228	1·912	15·5	0·6293	1·589
10·8	0·5253	1·903	15·6	0·6313	1·584
10·9	0·5277	1·894	15·7	0·6333	1·579
11·0	0·5301	1·886	15·8	0·6353	1·574
11·1	0·5325	1·877	15·9	0·6373	1·569
11·2	0·5349	1·869	16·0	0·6393	1·564
11·3	0·5373	1·861	16·1	0·6413	1·559
11·4	0·5396	1·853	16·2	0·6433	1·554
11·5	0·5420	1·845	16·3	0·6453	1·549
11·6	0·5444	1·837	16·4	0·6473	1·544
11·7	0·5467	1·829	16·5	0·6493	1·539
11·8	0·5490	1·821	16·6	0·6532	1·535
11·9	0·5514	1·813	16·7	0·6551	1·531
12·0	0·5537	1·806	16·8	0·6571	1·526
12·1	0·5560	1·798	16·9	0·6590	1·521
12·2	0·5583	1·791	17·0	0·6609	1·517
12·3	0·5605	1·783	17·1	0·6629	1·512
12·4	0·5628	1·776	17·2	0·6648	1·508
12·5	0·5651	1·769	17·3	0·6648	1·504
12·6	0·5673	1·762	17·4	0·6667	1·499
12·7	0·5696	1·755	17·5	0·6686	1·495
12·8	0·5718	1·748	17·6	0·6705	1·491
12·9	0·5741	1·741	17·7	0·6724	1·487
13·0	0·5763	1·735	17·8	0·6743	1·482
13·1	0·5785	1·728	17·9	0·6762	1·478
13·2	0·5807	1·721	18·0	0·6781	1·474
13·3	0·5829	1·715	18·1	0·6800	1·470
13·4	0·5851	1·709	18·2	0·6819	1·466
13·5	0·5873	1·703	18·3	0·6837	1·462
13·6	0·5894	1·696	18·4	0·6856	1·458
13·7	0·5916	1·690	18·5	0·6875	1·454
13·8	0·5938	1·684	18·6	0·6893	1·450
13·9	0·5959	1·678	18·7	0·6912	1·446
14·0	0·5980	1·672	18·8	0·6930	1·442
14·1	0·6001	1·666	18·9	0·6949	1·439
14·2	0·6023	1·660	19·0	0·6967	1·435

Length inches.	Time of vibration.	No. Vibr. per sec.	Length inches.	Time of vibration.	No. Vibr. per sec.
19·1	0·6985	1·431	23·9	0·7814	1·279
19·2	0·7003	1·427	24·0	0·7830	1·277
19·3	0·7022	1·424	24·1	0·7847	1·274
19·4	0·7040	1·420	24·2	0·7863	1·271
19·5	0·7058	1·416	24·3	0·7879	1·269
19·6	0·7076	1·413	24·4	0·7895	1·266
19·7	0·7094	1·409	24·5	0·7911	1·263
19·8	0·7112	1·405	24·6	0·7927	1·261
19·9	0·7130	1·402	24·7	0·7944	1·259
20·0	0·7148	1·398	24·8	0·7960	1·256
20·1	0·7166	1·395	24·9	0·7976	1·253
20·2	0·7184	1·391	25·0	0·7992	1·251
20·3	0·7201	1·388	25·1	0·8008	1·248
20·4	0·7219	1·384	25·2	0·8024	1·246
20·5	0·7237	1·381	25·3	0·8040	1·243
20·6	0·7254	1·378	25·4	0·8056	1·241
20·7	0·7272	1·375	25·5	0·8071	1·238
20·8	0·7289	1·371	25·6	0·8087	1·236
			25·7	0·8103	1·234
20·9	0·7307	1·368	25·8	0·8119	1·231
21·0	0·7324	1·365	25·9	0·8134	1·229
21·1	0·7342	1·361	26·0	0·8150	1·226
21·2	0·7359	1·358	26·1	0·8166	1·224
21·3	0·7377	1·355	26·2	0·8181	1·222
21·4	0·7394	1·352	26·3	0·8197	1·219
21·5	0·7411	1·349	26·4	0·8212	1·217
21·6	0·7428	1·346	26·5	0·8228	1·215
21·7	0·7446	1·343	26·6	0·8244	1·213
21·8	0·7463	1·339	26·7	0·8259	1·211
21·9	0·7480	1·336	26·8	0·8275	1·208
22·0	0·7497	1·333	26·9	0·8290	1·206
22·1	0·7514	1·330	27·0	0·8305	1·204
22·2	0·7531	1·327	27·1	0·8321	1·201
22·3	0·7548	1·324	27·2	0·8336	1·199
22·4	0·7565	1·321	27·3	0·8351	1·197
22·5	0·7582	1·318	27·4	0·8367	1·195
22·6	0·7598	1·315	27·5	0·8382	1·193
22·7	0·7615	1·313	27·6	0·8397	1·191
22·8	0·7632	1·310	27·7	0·8412	1·189
22·9	0·7649	1·307	27·8	0·8427	1·186
23·0	0·7665	1·304	27·9	0·8443	1·184
23·1	0·7682	1·301	28·0	0·8458	1·182
23·2	0·7699	1·298	28·1	0·8473	1·180
23·3	0·7715	1·296	28·2	0·8488	1·178
23·4	0·7732	1·293	28·3	0·8503	1·176
23·5	0·7748	1·290	28·4	0·8518	1·173
23·6	0·7765	1·287	28·5	0·8533	1·171
23·7	0·7781	1·285	28·6	0·8548	1·169
23·8	0·7798	1·282	28·7	0·8563	1·167

Length inches.	Time of vibration.	No. Vibr. per sec.	Length inches.	Time of vibration.	No. Vibr. per sec.
28·8	0·8578	1·165	34·2	0·9347	1·069
28·9	0·8593	1·163	34·3	0·9361	1·068
29·0	0·8607	1·161	34·4	0·9375	1·066
29·1	0·8622	1·159	34·5	0·9389	1·065
29·2	0·8637	1·157	34·6	0·9402	1·063
29·3	0·8652	1·155	34·7	0·9415	1·062
29·4	0·8667	1·154	34·8	0·9429	1·060
29·5	0·8682	1·152	34·9	0·9443	1·059
29·6	0·8696	1·150	35·0	0·9456	1·057
29·7	0·8711	1·148	35·1	0·9470	1·055
29·8	0·8726	1·146	35·2	0·9483	1·054
29·9	0·8741	1·144	35·3	0·9497	1·052
30·0	0·8755	1·142	35·4	0·9510	1·051
30·1	0·8769	1·140	35·5	0·9523	1·050
30·2	0·8784	1·138	35·6	0·9537	1·048
30·3	0·8798	1·136	35·7	0·9550	1·047
30·4	0·8813	1·135	35·8	0·9563	1·045
30·5	0·8827	1·133	35·9	0·9577	1·044
30·6	0·8842	1·131	36·0	0·9590	1·042
30·7	0·8856	1·129	36·1	0·9603	1·041
30·8	0·8870	1·127	36·2	0·9617	1·039
30·9	0·8885	1·125	36·3	0·9630	1·038
31·0	0·8899	1·123	36·4	0·9643	1·036
31·1	0·8914	1·121	36·5	0·9657	1·035
31·2	0·8928	1·120	36·6	0·9670	1·034
31·3	0·8942	1·118	36·7	0·9683	1·032
31·4	0·8956	1·116	36·8	0·9696	1·031
31·5	0·8971	1·114	36·9	0·9709	1·029
31·6	0·8985	1·112	37·0	0·9722	1·028
31·7	0·8999	1·111	37·1	0·9736	1·027
31·8	0·9013	1·109	37·2	0·9749	1·025
31·9	0·9027	1·107	37·3	0·9762	1·024
32·0	0·9042	1·105	37·4	0·9775	1·023
32·1	0·9056	1·104	37·5	0·9788	1·021
32·2	0·9070	1·102	37·6	0·9801	1·020
32·3	0·9084	1·100	37·7	0·9814	1·018
32·4	0·9098	1·099	37·8	0·9827	1·017
32·5	0·9112	1·097	37·9	0·9840	1·016
32·6	0·9126	1·095	38·0	0·9853	1·014
32·7	0·9140	1·094	38·1	0·9866	1·013
32·8	0·9154	1·092	38·2	0·9879	1·012
32·9	0·9168	1·090	38·3	0·9892	1·010
33·0	0·9182	1·089	38·4	0·9905	1·009
33·1	0·9196	1·087	38·5	0·9918	1·008
33·2	0·9210	1·085	38·6	0·9931	1·006
33·3	0·9224	1·084	38·7	0·9943	1·005
33·4	0·9237	1·082	38·8	0·9956	1·004
33·5	0·9251	1·080	38·9	0·9969	1·003
33·6	0·9265	1·079	39·0	0·9982	1·001
33·7	0·9279	1·077	39·1	0·9995	1·000
33·8	0·9293	1·076	39·2	1·0001	0·9993
33·9	0·9306	1·074			
34·0	0·9320	1·072			
34·1	0·9334	1·071			

Centre of Gyration, Principles of Rotation.

1. The distance of R the centre of gyration, from C the centre or axis of motion, in some of the most useful cases, is exhibited below:

In a circular wheel of uniform thickness $CR = \text{rad.} \sqrt{\tfrac{1}{2}}$.

In the periphery of a circle revolving about the diam .... $CR = \text{rad.} \sqrt{\tfrac{1}{2}}$.

In the plane of a circle ....ditto .... $CR = \tfrac{1}{2}\,\text{rad.}$

In the surface of a sphere ditto .... $CR = \text{rad.} \sqrt{\tfrac{2}{3}}$.

In a solid sphere ....ditto .... $CR = \text{rad.} \sqrt{\tfrac{2}{5}}$ $= \tfrac{7}{11}\, r$ nearly.

In a plane ring formed of circles whose radii are R, r, revolving about centre .... $CR = \sqrt{\dfrac{R^4 - r^4}{2R^2 - 2r^2}}$

In a cone revolving about its vertex .. $CR = \tfrac{1}{2}\sqrt{\tfrac{12}{5}a^2 + \tfrac{3}{5}r^2}$

In a cone ....its axis ..... $CR = r\sqrt{\tfrac{3}{10}}$

In a straight lever whose arms are R and r, $CR = \sqrt{\dfrac{R^3 + r^3}{3(R + r)}}$.

2. If the matter in any gyrating body were actually to be placed as if in the centre of gyration, it ought either to be disposed in the circumference of a circle whose radius is CR, or at two points R, R′, diametrically opposite, and at distances from the centre each = CR.

3. By means of the theory of the centre of gyration, and the values of $CR = \varrho$, thence deduced, the phænomena of rotation on a fixed axis become connected with those of accelerating forces: for then, if a weight or other moving power P act at a radius r to give rotation to a body, weight W, and dist. of centre of gyration from axis of motion $= \varrho$, we shall have for the accelerating force, the expression

$$f = \frac{Pr^2}{Pr^2 + W\varrho^2};$$

and consequently for the space descended by the actuating weight or power P, in a given time t, we shall have the usual formula

$$s = \tfrac{1}{2} f g t^2, \ldots\ t = \sqrt{\frac{2s}{fg}}, \text{ \&c.}$$

introducing the above value of f.

4. In the more complex cases, the distance of the centre of gyration from the axis of motion may best be computed from an experiment. Let motion be given to the system,

turning upon a horizontal axis, by a weight P acting by a cord over a pulley or wheel of radius r upon the same axis, and let s be the space through which the weight P descends in the time t, the proposed body whose weight is W turning upon the same axis with the same angular velocity: then

$$\text{CR} = \varrho = \sqrt{\frac{g \text{P} t^2 r^2 - 2 s \text{P} r^2}{2 s \text{W}}}.$$

Example. A body which weighs 100 lbs. turns upon a horizontal axis, motion being communicated to it by a weight of 10 lbs. hanging from a very light wheel of 1 foot diameter. The weight descends 2 feet in 3 seconds. Required the distance of the centre or circle of gyration from the axis of motion.

Here, I take $g = 32$, instead of $32\frac{1}{6}$, and obtain an approximative result. Whence

$$\text{CR} \sqrt{\frac{32 \times 10 \times 9 \times \frac{1}{4} - 4 \times 10 \times \frac{1}{4}}{4 \times 100}} = \sqrt{\frac{720 - 10}{400}}$$

$$= \frac{1}{20} \sqrt{710} = \frac{26 \cdot 646}{20} = 1 \cdot 3323 \text{ f. the answer.}$$

5. When the impulse communicated to a body is in a line passing through its centre of gravity, all the points of the body move forward with the same velocity, and in lines parallel to the direction of the impulse communicated. But when the direction of that impulse does not pass through the centre of gravity, the body acquires a rotation on an axis, and also a progressive motion, by which its centre of gravity is carried forward in the same straight line, and with the same velocity, as if the direction of the impulse had passed through the centre of gravity.

The progressive and rotatory motion are independent of one another, each being the same as if the other had no existence.

6. When a body revolves on an axis, and a force is impressed, tending to make it revolve on another, it will revolve on neither, but on a line in the same plane with them, dividing the angle which they contain, so that the sines of the parts are in the inverse ratio of the angular velocities with which the body would have revolved about the said axes, separately.

7. A body may begin to revolve on any line as an axis, that passes through its centre of gravity, but it will not continue to revolve permanently about that axis, unless the opposite rotatory forces exactly balance one another.

This admits of a simple experimental illustration. Suspend a thin circular plate of wood or metal by a cord tied to its edge, from a hook to which a rapid rotation can be given. The plate will at first turn upon an axis which is in the continuation of the cord of rotation. As the velocity augments, the plane will soon quit that axis, and revolve permanently upon a vertical axis passing through its centre of gravity, itself having assumed a horizontal position.

The same will happen if a *ring* be suspended, and receive rotation in like manner.

And if a flexible *chain* of small links be united at its two ends, tied to a cord and receive rotation, it will soon adjust itself so as to form a ring and spin round in a horizontal plane.

Also, if a flattened spheroid be suspended from *any* point, however remote from its minor axis, and have a rapid rotation given it, it will ultimately turn upon its shorter axis posited vertically.

This evidently serves to confirm the motion of the earth upon its shorter axis.

8. In every body, however irregular, there are three axes of permanent rotation, at right angles to one another. These are called *the principal axes of rotation:* they have this remarkable property that the momentum of inertia with regard to any of them is either a *maximum* or a *minimum*.

Central Forces.

Def. 1. *Centripetal force* is a force which tends constantly to sollicit or to impel a body towards a certain fixed point or centre.

2. *Centrifugal force* is that by which it would recede from such a centre, were it not prevented by the centripetal force.

3. These two forces are jointly called *central* forces.

4. When a body describes a *circle* by means of a force directed to its centre, its actual velocity is every where equal to that which it would acquire in falling by the same uniform force through half the radius.

5. This velocity is the same as that which a second body would acquire by falling through half the radius, whilst the first describes a portion of the circumference equal to the whole radius.

6. In equal circles the forces are as the squares of the times inversely.

7. If the times are equal, the velocities are as the radii, and the forces are also as the radii.

8. In general, the forces are as the distances or radii of the circles directly, and the squares of the times inversely.

9. The squares of the times are as the distances directly, and the forces inversely.

10. Hence, if the forces are inversely as the squares of the distances, the squares of the times are as the cubes of the distances. That is,

$$\text{if } F : f :: d^2 : D^2, \text{ then } T^2 : t^2 :: D^3 : d^3.$$

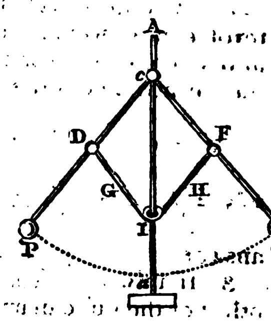
11. The right line that joins a revolving body and its centre of attraction, called the radius *vector*, always describes equal areas in equal times, and the velocity of the body is inversely as the perpendicular drawn from the centre of attraction to the tangent of the curve at the place of the revolving body.

12. If a body revolve in an elliptic orbit by a force directed to one of the foci, the force is inversely as the square of the distance: and the mean distances and the periodic times have the same relation as in art. 10. *This comprehends the case of the planetary motions.*

13. If the force which retains a body in a curve, increase in the simple ratio as the distance increases, the body will still describe an ellipse; but the force will in this case be directed to the centre of the ellipse; and the body in each revolution, will twice approach towards it, and again twice recede from that point.

14. On the principles of central forces depend the operation of a conical pendulum applied as a *governor* or regulator to steam-engines, water-mills, &c.

This contrivance will be readily comprehended from the marginal figure, where A *a* is a vertical shaft capable of turning freely upon the sole *a*. C D, C F, are two bars which move freely upon the centre C, and carry at their lower extremities two equal weights P, Q: the bars C D, C F, are united, by a proper articulation, to the bars G, H, which latter are attached to a ring, I, capable of sliding up and down the vertical shaft, A *a*. When this shaft and connected apparatus are made to revolve, in virtue of the centrifugal force the balls P, Q, fly out more and more from A *a*, as the rotatory velocity increases: if, on the contrary, the rotatory velocity slackens, the balls descend and approach A *a*. The ring I *ascends* in the former case, *descends* in the latter: and a lever connected with I may



be made to correct appropriately, the energy of the moving power. Thus, in the steam-engine, the ring may be made to act on the valve by which the steam is admitted into the cylinder; to augment its opening when the motion is slackening, and reciprocally diminish it when the motion is accelerated.

The construction is, often, so modified that the flying out of the balls causes the ring I to be depressed, and *vice versa;* but the general principle is the same.

Here, if the vertical distance of P or Q below c, be denoted by d, the time of one rotation of the regulator by t, and 3·141593 by π, the theory of central forces gives

$$t = 2\pi \sqrt{\frac{d}{32\frac{1}{6}}} = 1·10784 \sqrt{d}$$

Hence, the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of the base be what it may. Also, when ICQ = ICP = 45°, the centrifugal force of each ball is equal to its weight.

Inquiries connected with Rotation and Central Forces.

1. Suppose the diameter of a grindstone to be 44 inches, and its weight half a ton; suppose also that it makes 326 revolutions in a minute. What will be the centrifugal force, or its tendency to burst?

Here $F = \frac{2\rho\pi^2 w}{\frac{1}{2} g t^2} = \frac{\frac{44}{12} \times 3·141593\, w}{16 \times (\frac{60}{326})^2} = 47·22\, w =$ 23·6 tons, the measure of the required tendency.

2. If a fly wheel 12 feet diameter, and 3 tons weight, revolve in 8 seconds; and another of the same weight revolves in 6 seconds: what must be the diameter of the last, when their centrifugal force is the same?

By art. 8, Central Forces, $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$. Therefore, since F is $= f$, $\frac{D}{T^2} = \frac{d}{t^2}$, or $d = \frac{D t^2}{T^2} = \frac{12 \times 36}{64} = 6\frac{3}{4}$ feet, the answer.

3. If a fly of 12 feet diameter revolve in 8 seconds, and another of the same diameter in 6 seconds: what is the ratio of their weights when their central forces are equal?

By art. 6, Central Forces, the forces are as the squares of the times inversely when the weights are equal: therefore when the weights are unequal, they must be directly as the squares of the times, that the central forces may be equal.

Hence $w : \text{W} :: 36 : 64 :: 1 : 1\frac{7}{9}$

That is, the weight of the more rapidly to that of the more slowly revolving fly, must be as 1 to $1\frac{7}{9}$, in the case proposed.

4. If a fly 2 tons weight and 16 feet diameter, is sufficient to regulate an engine when it revolves in 4 seconds; what must be the weight of another fly of 12 feet diameter revolving in 2 seconds, so that it may have the same power upon the engine?

Here, by art. 8, central forces, we must have $\frac{\text{W D}}{\text{T}^2} = \frac{w\,d}{t^2}$;

therefore $w = \frac{\text{W D}\,t^2}{d\,\text{T}^2} = \frac{40 \text{ cwt.} \times 16 \times 4}{12 \times 16} = \frac{160}{12} = 13\frac{1}{3}$ cwt. the weight of the smaller fly.

Note. A fly should always be made to move rapidly. If it be intended for a mere regulator, it should be near the *first mover.* If it be intended to accumulate force in the *working point*, it must not be far separated from it.

5. Given the radius R of a wheel, and the radius r, of its axle, the weight of both, w, and the distance of the centre of gyration from the axis of motion, ϱ; also a given power P acting at the circumference of the wheel; to find the weight W raised by a cord folding about the axle, so that its momentum shall be a maximum.

Here W =

$$\frac{\sqrt{(\text{R}^4\text{P}^2 + 2\text{R}^2\text{P}\varrho^2 w + \varrho^4 w^2 + \text{P}w\text{R}r\varrho^2 + \text{P}^2\text{R}^3 r)} - \text{R}^2\text{P} - \varrho^2 w}{r^2}.$$

Cor. 1. When $\text{R} = r$, as in the case of the single fixed pulley, then $\text{W} = \sqrt{(2\text{P}^2\text{R}^3 + 2\text{R}\text{P}\varrho^2 w + \frac{\varrho^4}{\text{R}} w^2 + \text{P}w\text{R}\varrho^2)} - \frac{\varrho^2}{\text{R}^2} w - \text{P}$.

Cor. 2. When the pulley is a cylinder of uniform matter $\varrho^2 = \frac{1}{2}\text{R}^2$, and the express. becomes $\text{W} = \sqrt{[\text{R}^3(2\text{P}^2 + \frac{3}{2}\text{P}w + \frac{1}{4}w^2)]} - \frac{1}{2}w - \text{P}$.

6. Let a given power P be applied to the circumference of a wheel, its radius R, to raise a weight W at its axle, whose radius is r, it is required to find the ratio of R and r when W is raised with the greatest momentum; the characters W and ϱ denoting the same as in the last proposition.

Here $r = \frac{\text{R}\sqrt{[\text{P}^2\text{W}^2 + \text{P}^3(q + \text{W})]} - \text{P W}}{\text{P}(q + \text{W})}$.

Cor. When the inertia of the machine is evanescent, with respect to that of P + W, then is $r = R\sqrt{(1 + \frac{P}{W})} - 1$.

7. In any machine whose motion accelerates, the weight will be moved with the greatest velocity, when the velocity of the power is to that of the weight, as $1 + P\sqrt{(1 + \frac{P}{W})}$ to 1; the inertia of the machine being disregarded.

8. If in any machine whose motion accelerates, the descent of one weight causes another to ascend, and the descending weight be given, the operation being supposed continually repeated, the effect will be greatest in a given time when the ascending weight is to the descending weight, as 1 to 1·618, in the case of equal heights; and in other cases when it is to the exact counterpoise in a ratio which is always between 1 to $1\frac{1}{2}$ and 1 to 2.

Percussion or Collision.

1. DEFS. In the ordinary theory of percussion, or collision, bodies are regarded as either *hard*, *soft*, or *elastic*. A *hard* body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A *soft* body is that whose parts yield to any stroke or impression, without restoring themselves again, the shape of the body remaining altered. An *elastic* body is that whose parts yield to any stroke, but presently restore themselves again, so that the body regains the same figure as before the stroke. When bodies which have been subjected to a stroke or a pressure return only in part to their original form, the *elasticity* is then *imperfect*: but if they restore themselves entirely to their primitive shape, and employ just as much time in the restoration as was occupied in the compression, then is the elasticity *perfect*.

It has been customary to treat only of the collision of bodies perfectly hard or perfectly elastic: but as there do not exist in nature any bodies (which we know) of either the one or the other of these kinds, the usual theories are of but little service in practical mechanics, except as they may suggest an extension to the actual circumstances of nature and art.

2. The general principle, for determining the motions of bodies from percussion, and which belongs equally to both elastic and non-elastic bodies, is this: viz. that there exists in the bodies the same momentum, or quantity of motion,

estimated in any one and the same direction, both before the stroke and after it. And this principle is the immediate result of the law of nature or motion, that reaction is equal to action, and in a contrary direction; from whence it happens, that whatever motion is communicated to one body by the action of another, exactly the same motion does this latter lose in the same direction, or exactly the same does the former communicate to the latter in the contrary direction.

From this general principle too it results, that no alteration takes place in the common centre of gravity of bodies by their actions upon one another; but that the said common centre of gravity perseveres in the same state, whether of rest or of uniform motion, both before and after the impact.

3. If the impact of two perfectly hard bodies be direct, they will, after impact, either remain at rest, or move on uniformly together with different velocities, according to the circumstances under which they met.

Let B and b represent two perfectly hard bodies, and let the velocity of B be represented by V, and that of b by v, which may be taken either positive or negative, according as b moves in the same direction as B, or contrary to that direction, and it will be zero when b is at rest. This notation being understood, all the circumstances of the motion of the two bodies, after collision, will be expressed by the formula:

$$\text{velocity} = \frac{\mathrm{BV} + bv}{\mathrm{B} + b},$$

which being accommodated to the three circumstances under which v may enter become

$$\text{velocity} = \frac{\mathrm{BV} + bv}{\mathrm{B} + b} \left\{ \begin{array}{l} \text{when both bodies moved} \\ \text{in the same direction} \end{array} \right.$$

$$\text{velocity} = \frac{\mathrm{BV} - bv}{\mathrm{B} + b} \left\{ \begin{array}{l} \text{when the bodies moved in} \\ \text{contrary directions} \end{array} \right.$$

$$\text{velocity} = \frac{\mathrm{B} + b}{\mathrm{BV}} \left\{ \begin{array}{l} \text{when the body } b \text{ was} \\ \text{at rest.} \end{array} \right.$$

These formulæ arise from the supposition of the bodies being perfectly hard, and consequently that the two after impact move on uniformly together as one mass. In cases of perfectly elastic bodies, other formulæ have place, which express the motion of each body separately; as in the following proposition.

4. If the impact of two perfectly elastic bodies be direct, their *relative* velocities will be the same both before and after impact, or they will recede from each other with the same

velocity with which they met; that is, they will be equally distant, in equal times, both before and after their collision, although the absolute velocity of each may be changed. The circumstances attending this change of motion in the two bodies, using the above notation, are expressed in the two following formulæ;

$$\frac{2bv + (\text{B} - b)\text{V}}{\text{B} + b} \text{ the velocity of B}$$

$$\frac{2\text{BV} + (\text{B} - b)v}{\text{B} + b} \text{ the velocity of } b$$

which needs no modification, when the motion of b is in the same direction with that of B.

5. In the other case of b's motion, the general formulæ become

$$\frac{-2bv + (\text{B} - b)\text{V}}{\text{B} + b} \text{ the velocity of B}$$

$$\frac{2\text{BV} - (\text{B} - b)v}{\text{B} + b} \text{ the velocity of } b$$

when b moves in a contrary direction to that of B, which arise from taking v negative. And

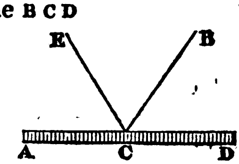
$$\frac{(\text{B} - b)\text{V}}{\text{B} + b} \text{ the velocity of B}$$

$$\frac{2\text{BV}}{\text{B} + b} \text{ the velocity of } b$$

when b was at rest before impact, that is, when $v = 0$.

6. If a perfectly hard body B, impinge obliquely upon a perfectly hard and immoveable plane A D, it will after collision move along the plane in the direction C A.

And its velocity before impact
Is to its velocity after impact
As radius
Is to the cosine of the angle B C D



But if the body be elastic it will rebound from the plane in the direction C E, with the same velocity, and at the same angle with which it met it, that is, the angle A C E will be equal to the angle B C D.

7. In the case of direct impact, if B be the striking body, b the body struck, V and v their respective velocities be-

fore impact, U and u their velocities afterwards; then the two following are general formulæ: viz.

$$\mathrm{U} = \mathrm{V} - n\left(\frac{\mathrm{V} - v}{\mathrm{B} + b}\right) b$$

$$u = v + n\left(\frac{\mathrm{V} - v}{\mathrm{B} + b}\right) \mathrm{B}$$

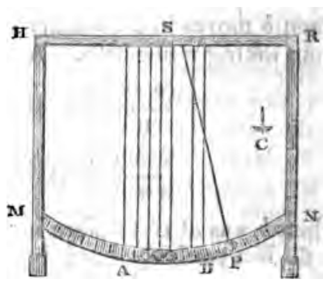
In these, if $n = 1$, they serve for non-elastic bodies; if $n = 2$, for bodies perfectly elastic. If the bodies be imperfectly elastic n has some intermediate value.

When the body struck is at rest, the preceding equations become

$$\mathrm{U} = \mathrm{V} - \frac{n\,\mathrm{V}\,b}{\mathrm{B} + b} \;\ldots\ldots\; u = \frac{n\,\mathrm{V}\,\mathrm{B}}{\mathrm{B} + b} \;\ldots\ldots\; n = \frac{u(\mathrm{B} + b)}{\mathrm{B}\,\mathrm{V}}$$

from which the value of n may be determined experimentally.

8. In the usual apparatus for experiments on collision, balls of different sizes and of various substances, are hung from different points of suspension on a horizontal bar. H R . M N is an arc of a circle whose centre is S; and its graduations, 1, 2, 3, 4, 5, &c. indicate the lengths of *cords*, as measured from the lowest point. Any ball, therefore, as P, may be drawn from the vertical, and made to strike another ball hanging at the lowest point, with any assigned velocities: the height to which the ball struck ascends on the side A M furnishes a measure of its velocity; and from that the value of n may be found from the last equation. Balls not required in an individual experiment, may be put behind the frame, as shown at A and B.



The cup C may be attached to a cord, and carry a ball of clay, &c. when required.

Example. Suppose that a ball weighing 4 ounces, strikes another ball of the same substance weighing 3 ounces, with a velocity of 10, and communicates to it a velocity of $8\frac{1}{4}$: what, in that case will be the value of n?

Here $n = \frac{u(\mathrm{B} + b)}{\mathrm{B}\,\mathrm{V}} = \frac{8\frac{1}{4} \times 7}{4 \times 10} = \frac{57\frac{3}{4}}{40} = 1 \cdot 44375$ the index of the degree of elasticity; *perfect* elasticity being indicated by 2.

Principles of Chronometers.

1. Clockwork, regulated by a simple balance, is inadequate to the accurate mensuration of time.

2. Clockwork, regulated by a pendulum vibrating in the arch of a circle, is of itself inadequate to the accurate mensuration of time.

1st. Because the vibrations in greater and smaller arches are not performed in equal times. 2dly. Because the length of the pendulum is varied by heat and cold.

3. Clockwork, regulated by a pendulum vibrating in the arch of a cycloid, is inadequate to the accurate mensuration of time.

The isochronism of the vibrations of a cycloidal pendulum in greater and smaller arches, is true only on the hypothesis, that the pendulum moves in a non-resisting medium, and that the whole mass of the pendulum is concentrated in a point, both of which positions are false. For these reasons the application of the cycloid in practice has been entirely relinquished.

4. Modern time-keepers owe almost the whole of their superiority over those formerly made to two things; 1st, the application of a thermometer; 2dly, the particular construction of the escapement.

5. Metals expand by heat and contract by cold. This is proved experimentally by the pyrometer. Metallic bars of the same kind are found to expand in proportion to their length. Metals of different kinds expand in different proportions; thus the expansion of iron and steel are as 3, copper $4\frac{1}{2}$, brass 5, tin 6, lead 7. Hence pendulum rods, expanding and contracting by the successive changes of temperature, affect the going of the clocks to which they are applied.

Various have been the contrivances to correct the errors of pendulums from their contraction and expansion by heat and cold; the principal of these are described under the subject of pendulums (pa. 251).

6. The balance of a watch is analogous to the pendulum in its properties and use.

The simple balance is a circular annulus, equally heavy in all its parts, and concentrical with the pivots of the axis on which it is mounted. This balance is moved by a spiral spring called the balance-spring, the invention of the ingenious Mr. Hook.

7. The pendulum requires a less maintaining power than the balance.

Hence the natural isochronism of the pendulum is less disturbed by the relatively small inequalities of the maintaining power.

8. The spring's elastic force which impels the circumference of the balance, is directly as the tension of the spring; that is, the weights necessary to counterpoise a spiral spring's elastic force, when the balance is wound to different distances from the quiescent point, are in the direct ratio of the arcs through which it is wound.

9. The vibrations of a balance, whether through great or small arches, are performed in the same time.

For the accelerating force is directly as the distance from the point of quiescence; hence therefore, the motion of the balance is analogous to that of a pendulum, vibrating in cycloidal arches.

10. The time of the vibration of a balance is the same as if a quantity of matter, whose inertia is equal to that by which the mass contained in the balance opposes the communication of motion to the circumference, described a cycloid whose length is equal to the arc of vibration described by the circumference, the accelerating force being equal to that of the balance.

Because in both cases the spaces described would be equal, as also the accelerating forces in corresponding points, and therefore the times of description.

11. If 1 denote the accelerating force of gravity, L the length of a pendulum vibrating seconds in a cycloid, a the semi-arc of vibration of the balance, T the time of vibration, and F the accelerating force of the balance, then will T $=$

$$\sqrt{\frac{a}{L \times F}}.$$

12. Let $\frac{1}{2}g$ be the space which a body falling freely from a state of rest describes in 1″, and $p = 3{\cdot}141593$ the circumference of a circle whose diameter is unity, then will

$$T = \sqrt{\frac{p^2 a}{g F}}.$$

In this expression for the time of vibration, the letter a denotes the length of the semi-arc of vibration; if this arc should be expressed by a number of degrees c°, and r be the radius of the balance, then a will be $= \frac{p\,r\,c^\circ}{180^\circ}$; and this quantity being substituted for a, the time of a vibration will be T $=$

$\sqrt{\frac{p\,a\,c^{\circ}}{g\,\text{F} \times 180^{\circ}}}$; let the given arc be 90°, in this case $\text{T} = \sqrt{\frac{p\,c\,r}{2\,g\,\text{F}}}$.

13. If the spring's elastic force, when wound through the given angle or arc $a = 90^{\circ}$ from the quiescent position, be $=$ P; the weight of the balance, and the parts which vibrate with it $=$ W, the distance of the centre of gyration from the axis of motion $= \varrho$, then will $\text{T} = \sqrt{\frac{\text{W}\,p^3\,\varrho^2}{2\,\text{P}\,r\,g}}$.

These are expressions for the time of a vibration, whatever may be the figure of the balance, the other conditions remaining the same as above stated. If the balance be an annulus or a cylindrical plate, $\varrho = \frac{r}{\sqrt{2}}$, and the time of vibration $\text{T} = \sqrt{\frac{\text{W}\,p^3\,r}{4\,\text{P}\,g}}$.

14. The times of vibration of different balances are in a ratio compounded of the direct subduplicate ratios of their weights and semidiameters, and the inverse subduplicate ratio of the tensions of the springs or of the weights which counterpoise them, when wound through a given angle.

15. The times of vibration of different balances are in a ratio compounded of the direct simple ratio of the radii, and direct subduplicate ratio of their weights, and the inverse subduplicate ratio of the absolute forces of the springs at a given tension.

16. Hence the absolute force of the balance spring, the diameter and weight of the balance being the same, is inversely as the square of the time of one vibration.

17. The absolute force or strength of the balance spring, the time of one vibration, and the weight of the balance being the same, is as the square of the diameter and the balance.

18. The weight of the balance, the strength of the spring and time of vibration being the same, is inversely as the square of the diameter.

Hence a large balance vibrating in the same time, with the same spring, will be much lighter than a small one.

19. If the rim of the balance be always of the same breadth and thickness, so that the weight shall be as the radius, the strength of the spring must be as the cube of the diameter of

the balance, that the time of vibration may continue the same.

20. If a balance be made with two balls joined by a rod, and the weights and distances of these balls from their common centre of motion be unequal, but such that each separately would vibrate in the same time; the centre of gravity of these balls will not coincide with their centre of motion, nor will they poise each other.

21. The momentum of the balance is increased better by increasing its diameter than its weight.

22. A stronger balance-spring is preferable to a weaker. Because the force of this spring upon the balance remaining the same, whilst the disturbing force varies, the errors arising from the variation will be less, as the fixed force is greater.

23. The longer a detached balance continues its motion the better.

Because, 1st. The friction in this case is less, and therefore the natural isochronism of the vibration is less disturbed. 2dly. When applied to the watch, it requires a less maintaining power, and therefore the variations in the intensity of the maintaining power will be less. 3dly. The maintaining power being less, the friction of the wheel-work will be less, and therefore the motion more regular. 4thly. The pressure on the escapement will be less, and therefore the oscillations of the balance less disturbed.

24. The greater is the number of vibrations performed by a balance in a given time, the less susceptible is it of external agitations.

25. Slow vibrations are preferable to quick vibrations; but there is a limit; for if the vibrations be too slow, the watch will be liable to stop.

If we regarded only the effect of external agitations, balances that vibrate quick should be preferred to such as vibrate slow; but they are attended with two inconveniences, greater than that which we would avoid. 1st. In two balances of the same weight and diameter, the friction on the pivots increases with the number of vibrations. 2dly. It appears by experience that the motion of the same detached balance continues longer, when its vibrations are slow, than when they are quick.

26. A balance should describe as large arches as possible, as suppose 240°, 260°, 300°, or an entire circle.

First, because the momentum of the balance is thus increased; and therefore the inequalities in the force of the maintaining power bear a less proportion to it, and of consequence will have less influence. 2dly. The balance is less susceptible of external agitations. 3dly. A given variation in the extent of the vibrations produces a less variation in the going of the machine. But care must be taken, that in these great vibrations, the spring shall neither touch any obstacle, nor its spires touch each other in contracting.

27. The times of vibration in larger arches are sometimes shorter, sometimes longer than in less arches.

28. A uniform spiral spring may be rendered perfectly isochronal, by adjusting its length and number of spires.

This is the opinion of Mr. Berthoud. His reasoning seems to be this: if the spring forming a spiral of a certain species be so disposed, that when wound through different angles, the accelerating elastic forces of the spires, from the centre towards the circumference, increase faster than they ought to do in order to render the vibrations isochronal, it may be otherwise so disposed, namely, by making the spires approach more nearly to equality with each other in succession, that the law shall vary in such a manner, as absolute isochronism requires. But as the fundamental property of springs, namely, that *as the tension is, so is the force*, is determined by experiment, so must this property likewise be ascertained in the same manner. Accordingly Berthoud tells us, that having attached to a balance a spiral of very large folds, making but three turns, and whose diameter was 15 lines, the angles through which it was wound being successively 5°, 10°, 15°, 20°, 25°, 30°, 35°, 40°, 45°, 60°, 120°, the counterpoising weights in grains were 10½, 21, 32, 42, 54, 65, 76, 88, 99, 134, 278. The same spring forming very small spires, making five turns in eight lines diameter, the angles through which it was wound being the same as before, the counterpoising weights were 11, 22, 33, 45, 56, 67, 78, 89, 100, 133, 250 grains. These experiments, he tells us, were made with great care; and they show that the same spiral, its length continuing unchanged, when folded in large and small spires, has a sufficient difference in its progression to vary its isochronism: when folded in large spires, according to the first experiment, the vibrations in larger arcs are accelerated; and by the second experiment, when folded in narrow spires, they are rendered slower.

29. A spiral spring may be rendered isochronal by a proper adjustment of its strength and thickness in different parts.

30. A spiral spring which is not isochronal, may be rendered such by the addition of two auxiliary springs, whose points of quiescence are properly adjusted.

This was the ingenious invention of Mr. Mudge; the theory of which construction is delivered in the Phil. Trans. for the year 1794, by Mr. Atwood.

31. The influence of the maintaining power on the balance, in restoring the motion which it loses by friction, or otherwise, may be either constant or interrupted.

This depends on the escapement; when the action of the maintaining power is constant, the escapement is called either the *recoil* or the *dead-beat;* when it is interrupted, the escapement is said to be detached.

32. By *escapement* is understood the means by which the action of the wheels is applied to maintain the vibration of the balance; and it consists of the balance-wheel and pallets.

33. Pallets are small plates or levers attached to the axis or verge of the balance, which receive the impulse of the balance-wheel produced by the maintaining power, and thus continually renew the motion which the balance loses by friction, or other resistance.

In a *recoil escapement*, when one tooth of the balance-wheel drops off the first pallet, the other acting tooth falls on the inclined plane of the other pallet, which meeting it obliquely, causes the balance-wheel to recoil, from which circumstance this escapement derives its name.

In the *dead-beat escapement*, when one tooth of the balance-wheel drops off the inclined plane of the first pallet, the other acting tooth immediately falls upon the convex surface of the other pallet, which surface being concentrical with the axis of the balance, the wheel continues at rest until, by the motion of the pallet or cylinder, the inclined plane of the tooth comes to act upon the face of this latter pallet or edge of the cylinder, which then, by its pressure on that edge, throws the cylinder round, and thus gives motion to the balance; then instantly entering the cavity of the cylinder, it falls upon the concave surface, and for the same reason as before continues at rest, until the balance spring drives the cylinder round in a contrary direction to what it did before, so as that the inclined plane of the tooth may act on the second edge of the cylinder; which pressure throws the cylinder round in the contrary direction, and the tooth gets out of the cavity, and at that instant the subsequent tooth falls upon the convex surface, and so on. From the quiescence of the balance-wheel during the interval of time that elapses between the falling of the acting tooth on the surface and its pressure on the edge of the cylinder, this escapement is called the *dead-beat*.

In the *detached escapement* the motion of the maintaining power is suspended during almost the whole time of vibration; just at the end of the return of the balance it unlocks the wheel-work, and a tooth of the balance-wheel, immediately acting on the pallet, restores the motion which the balance had lost; and having given its impulse, the wheel-work is instantly locked again, and the balance performs its vibration freely and disengaged from all other parts of the machine.

34. In the escapement of recoil, the vibrations are quicker than if the balance or pendulum vibrated freely.

For the recoil shortens the ascending part of the vibration by contracting the extent of the arc; and the re-action of the wheel accelerates the descending part of the vibration.

35. In the dead-beat escapement, the vibrations are slower, than when they are performed in a detached state.

For the pressure of the tooth on the surface of the cylinder, retards that part of the vibration which is performed while the cylinder, by the motion of the balance spring, revolves so far as to bring the tooth to the edge of the cylinder: and if the maintaining power be increased, the pressure of the tooth on the cylinder may become so great, as entirely to stop the motion. When the tooth has communicated its impulse to the edge of the cylinder, it moves almost freely; and as the tooth does not yet press with its entire force on the next surface, the cylinder will indeed describe a larger arc, and therefore on that account the time may be shortened; but when it has consumed all the impulse of the wheel, it returns by the sole force of elasticity; now the pressure of the tooth causes a friction which diminishes the tendency to return to the point of rest, so that the balance performs its vibrations slower.

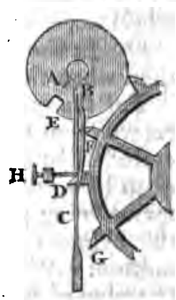
36. In the escapement of recoil, if the maintaining power be increased, the vibrations will be performed in larger arches, but in less time.

Because the greater pressure of the crown wheel on the pallet will cause the balance to vibrate through larger arches; and the time, on this account will be less increased, than it will be diminished by the acceleration of the balance by that pressure, and the diminution of the time of recoil.

37. In the escapement of the cylinder or dead-beat, an increase of the maintaining power renders the vibrations larger, and at the same time slower.

Because the greater pressure of the tooth on the edge of the cylinder throws it round through a greater arch; and its increased pressure on both surfaces of the cylinder retards its motion.

The upper part of the first marginal diagram exhibits the anchor recoil; the lower, Graham's dead-beat escapement.



The second diagram represents Mr. Arnold's watch escapement. The pin A, projecting from the verge or axis of the balance, moving towards B, carries before it the spring B, and with it the stiffer spring C, so as to set at liberty the tooth D, which rests on a pallet projecting from the spring. The angle E of the principal pallet has then just passed the tooth F, and is impelled by it until the tooth G arrives at the detent. In the return of the balance, the pin A passes easily by the detent, by forcing back the spring B. The screw H serves to adjust the position of the detent, which presses against it.

38. The escapement can render those vibrations only isochronal, whose inequality proceeds from the maintaining power, and not such as are produced by external agitations.

39. The effect of external agitations on the balance may be counteracted by the double escapement.

In this escapement, two equal balances are so connected, that they vibrate through equal angles, but in contrary directions; by which means, the one must always be accelerated as much as the other is retarded by any external agitation. But as Mr. Cummins observes, when balances are connected by means of teeth, there arises a resistance which, however small, when applied in this most delicate part, will tend to diminish the momentum of the balances.

40. That escapement is best in which the duration of the action of the balance-wheel on the pallets is least with respect to the time of vibration.

Hence the detached escapement is the best, which appears to have been the invention of the ingenious artist, Mr. Thomas Mudge, who made a watch on this construction for the late King of Spain, Ferdinand VI. in the year 1755.

41. The time of the vibration of the balance is increased by heat, and diminished by cold.

First, because the length of the spiral spring is increased by heat, and therefore its force diminished, and the contrary by cold. 2dly. The diameter of the balance is increased by heat, and therefore also the time of vibration; and the contrary by cold.

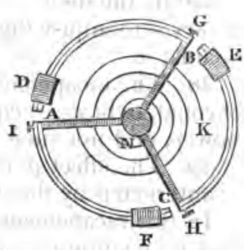
42. That balance is the most perfect which, without the compensation of a thermometer, is most subject to the influence of heat and cold.

Because the obstructions from oil and friction act as a compensation to the expansion or contraction of the spring and balance; therefore that balance which is most affected, is freest from the influence of oil and friction.

43. The errors in the going of a watch, arising from the change of temperature, may be corrected by varying the length of the balance spring.

Nevertheless, as it is extremely difficult to form an isochronal spiral, any variation in its length is dangerous, because we shall thus probably lose that point which determines its isochronism.

44. The errors in the going of a watch, occasioned by the variation of temperature, may be corrected by varying the diameter of the balance.



This may be effected by dividing the rim of the balance into two or more separate parts, G D, I F, H E, each of which is composed of two plates of metal of different expansibility, riveted together, the least expansible being nearest the centre N, and carrying at one end D, F, E, a weight; whilst the other is connected either with the rim of the balance, or one of its radii. Now if the temperature increase, the exterior plate expanding more than the interior, the compound will become more concave towards the centre; and consequently the end which carries the weight will approach the centre of the balance, and on that account the vibrations will be rendered quicker. At the root of each thermometer, there is a screw G, I, H, by which the diameter of the balance may be increased or diminished, so as to alter the time kept by the chronometer, without interfering with the adjustment for heat and cold: and if the mag-

nitude and position of the weights be properly regulated, they will correct the error arising from the variation of the diameter of the balance caused by the variation of temperature. (*M. Young's Analysis.*) The reader who wishes to acquire *practical* knowledge on this subject, may advantageously consult *Hatton's Introduction to the Mechanical part of Clock and Watch Work.*

[illegible]

CHAP. XI.

HYDROSTATICS.

1. *Hydrostatics* comprises the doctrine of the pressure and the equilibrium of non-elastic fluids, as water, mercury, &c. and that of the weight and pressure of solids immersed in them.

2. *Def.* a *fluid* is a body whose parts are very minute, yield to any force impressed upon it (however small), and by so yielding are easily moved among themselves.

Some attempt to give mechanical ideas of a fluid body, by comparing it to a heap of sand: but the impossibility of giving fluidity by any kind of mechanical comminution will appear by considering two of the circumstances necessary to constitute a fluid body: 1. That the parts, notwithstanding any compression, *may* be moved in relation to each other, with the smallest conceivable force, or will give no *sensible resistance* to motion within the mass in any direction. 2. That the parts shall gravitate to each other, whereby there is a constant tendency to arrange themselves about a common centre, and form a spherical body; which as the parts do not resist motion, is easily executed in small bodies. Hence the appearance of drops always takes place when a fluid is in proper circumstances. It is obvious that a body of sand can by no means conform to these circumstances.

Different fluids have different degrees of fluidity, according to the facility with which the particles may be moved amongst each other. Water and mercury are classed among the most perfect fluids. Many fluids have a very sensible degree of tenacity, and are therefore called *viscous* or *imperfect* fluids.

3. Def. Fluids may be divided into *compressible* and *incompressible*, or *elastic* and *non-elastic* fluids. A *compressible* or *elastic* fluid is one whose apparent magnitude is diminished as the pressure upon it is increased, and increased by a diminution of pressure. Such is air, and the different vapours. An *incompressible* or *non-elastic* fluid is one whose dimensions are not, at least as to sense, affected by any augmentation of pressure. Water, mercury, wine, &c. are generally ranged under this class.

It has been of late years proposed to limit the application of the term *fluids* to those which are *elastic*, and to apply the word *liquid* to such as are *non-elastic*. But it is an unnecessary refinement.

4. Def. The *specific gravity* of any solid or fluid body is the absolute weight of a known volume of that substance, namely, of that which we take for unity in measuring the capacities of bodies.

Comparing this definition with that of *density* (Dynamics, Def. 2), it will appear that the two terms express the same thing under different aspects.

Section I. *Pressure of Non-elastic Fluids.*

1. Fluids press equally in all directions, upwards, downwards, aslant, or laterally.

This constitutes one essential difference between fluids and solids, solids pressing only *downwards*, or in the direction of gravity.

2. The upper surface of a gravitating fluid at rest is horizontal.

3. The pressure of a fluid on every particle of the vessel containing it, or of any other surface, real or imaginary, in contact with it, is equal to the weight of a column of the fluid, whose base is equal to that particle, and whose height is equal to its depth below the upper surface of the fluid.

4. If, therefore, any portion of the upper part of a fluid be replaced by a part of the vessel, the pressure against this from below will be the same which before supported the weight of the fluid removed, and every part remaining in equilibrium, the pressure on the bottom will be the same as it would if the vessel were a prism or a cylinder.

5. Hence, the smallest given quantity of a fluid may be made to produce a pressure capable of sustaining any proposed weight, either by diminishing the diameter of the column and increasing its height, or by increasing the surface which supports the weight.

6. The pressure of a fluid on any surface, whether vertical, oblique, or horizontal, is equal to the weight of a column of the fluid whose base is equal to the surface pressed, and height equal to the distance of the centre of gravity of that surface below the upper horizontal surface of the fluid.

7. Fluids of different specific gravities that do not mix, will counterbalance each other in a bent tube, when their heights above the surface of junction are inversely as their specific gravities.

A portion of fluid will be quiescent in a bent tube, when the upper surface in both branches of the tube is in the same horizontal plane, or is equidistant from the earth's centre. And water poured down one branch of such a tube (whether it be of uniform bore throughout, or not) will rise to its own level in the other branch.

Thus, water may be conveyed by pipes from a spring on the side of a hill, to a reservoir of equal height on another hill.

8. The ascent of a body in a fluid of greater specific gravity than itself, arises from the pressure of the fluid upwards against the under surface of the body.

9. DEF. The *centre of pressure* is that point of a surface against which any fluid presses, to which if a force equal to the whole pressure were applied it would keep the surface at rest, or balance its tendency to turn or move in any direction.

10. If a plane surface which is pressed by a fluid be produced to the horizontal surface of it, and their common intersection be regarded as the axis of suspension, *the centres of percussion and of pressure will be at the same distance from the axis.*

11. The centre of pressure of a parallelogram, whose upper side is in the plane of the horizontal level of the liquid, is at ⅔ of the line (measuring downwards) that joins the middles of the two horizontal sides of the parallelogram.

12. If the base of a triangular plane coincides with the upper surface of the water, then *the centre of pressure is at the middle of the line drawn from the middle of the base to the vertex of the triangle.* But, if the vertex of the triangle be in the upper surface of the water, while its base is horizontal, *the centre of pressure is at ¾ of the line drawn from the vertex to bisect the base.*

Illustrations and Applications.

1. If several glass tubes of different shapes and sizes be put into a larger glass vessel containing water, the tubes being all open at top; then the water will be seen to rise to the same height in each of them, as is marked by the upper surface *a c*, of the liquid in the larger vessel.

2. If three vessels of equal bases, one cylindrical, the second

considerably larger at top than at bottom; the third considerably less at top than at bottom, and with the sides of the two latter either regularly or irregularly sloped, have their bottoms moveable, but kept close by the action of a weight upon a lever; then it will be found, that when the same weight acts at the same distance upon the lever, water must be poured in to the same height in each vessel before its pressure will force open the bottom.

3. Let a glass tube, open at both ends (whether cylindrical or not, does not signify) have a piece of bladder tied over one end, so as to be capable of hanging below that end, or of rising up within it, when pressed from the outside. Pour into this tube some water tinged red, so as to stand at the depth of 7 or 8 inches; and then immerse the tube with its coloured water vertically into a larger glass vessel nearly full of colourless water, the bladder being downwards serving as a flexible bottom to the tube. Then, it will be observed that when the depth of the water in the tube exceeds that in the larger vessel, the bladder will be forced *below* the tube, by the excess of the interior over the exterior pressure: but when the exterior water is deeper than the interior, the bladder will be thrust up *within* the tube, by the excess of exterior pressure: and when the water in the tube and that in the larger vessel, have their upper surfaces in the same horizontal plane, then the bladder will adjust itself into a *flat* position just at the bottom of the tube. The success of this experiment does not depend upon the actual depth of the water in the tube, but upon the relation between the depths of that and the exterior water; and proves that in all cases, the deeper water has the greater pressure at its bottom, tending equally upward and downward.

4. The *hydrostatical paradox*, as it is usually denominated, results from the principle that any quantity of a non-elastic fluid, however small, may be made to balance another quantity; or any weight, as large as we please. It may be illustrated by the *hydrostatic bellows*, consisting of two thick boards D C, F E, each about 16 or 18 inches diameter, more or less, covered or connected firmly with leather round the edges, to open and shut like a common bellows, but without valves; only a pipe A B, about 3 feet high is fixed into the bellows at *e*. Now let water be poured into the pipe at A and it will run into the bellows, gradually separating the boards, by raising

the upper one. Then if several weights, as three hundred weights, be laid upon the upper board, by pouring the water in at the pipe till it be full, it will sustain all the weights, though the water in the pipe should not weigh a quarter of a pound: for the pipe or tube may be as small as we please, provided it be but long enough, the whole effect depending upon the height, and not at all on the width of the pipe: for the proportion is always this,

As the area of the orifice of the pipe
is to the area of the bellows board,
so is the weight of water in the pipe, above D C,
to the weight it will sustain on the board.

5. In lieu of the bellows part of the apparatus, the leather of which would be incapable of resisting any very considerable pressure, the late *Mr. Joseph Bramah*, used a very strong metal cylinder, in which a piston moved in a perfectly air and water tight manner, by passing through leather collars, and as a substitute for the high column of water, he adopted a very small forcing pump to which any power can be applied; and thus the pressing column becomes indefinitely long, although the whole apparatus is very compact, and takes but little room. The marginal figure is a section of one of these presses, in which *t* is the piston of the large cylinder, formed of a solid piece of metal turned truly cylindrical, and carrying the lower board *v* of the press upon it: *u* is the piston of the small forcing pump, being also a cylinder of solid metal moved up and down by the handle or lever *w*. The whole lower part of the press is sometimes made to stand in a case *x x*, containing more than sufficient water as at *y*, to fill both the cylinders; and the suction pipe of the forcing pump *u* dipping into this water will be constantly supplied. Whenever, therefore, the handle *w* is moved upwards, the water will rise through the conical metal valve *z*, opening upwards into the bottom of the pump *u*; and when the handle is depressed that water will be forced through another similar valve *a*, opening in an opposite direction in the pipe of communication between the pump and the great cylinder *b*, which will now receive the water by which the piston rod *t* will be elevated at each stroke of the pump *u*. Another small conical valve *c* is applied by means of a screw to an orifice in the lower part of the large cylinder, the use of which is to release the pressure whenever it may be necessary; for, on opening this valve, any water which was previously

contained in the large cylinder *b*, will run off into the reservoir *y* by the passage *d*, and the piston *t* will descend; so that the same water may be used over and over again. The power of such a machine is enormously great; for, supposing the hand to be applied at the end of the handle *w*, with a force of only 10 pounds, and that this handle or lever be so constructed as to multiply that force but 5 times, then the force with which the piston *u* descends will be equal to 50 pounds: let us next suppose that the magnitude of the piston *t* is such, that the area of its horizontal section shall contain a similar area of the smaller piston *u* 50 times, then 50 multiplied by 50 gives 2500 pounds, for the force with which the piston *t* and the presser *v* will rise. A man can, however, exert ten times this force for a short time, and could therefore raise 25,000 pounds; and would do more if a greater disproportion existed between the two pistons *t* and *u*, and the lever *w* were made more favourable to the exertion of his strength.

This machine not only acts as a press, but is capable of many other useful applications, such as a jack for raising heavy loads, or even buildings; to the purpose of drawing up trees by their roots, or the piles used in bridge-building.

To find the thickness of the metal in Bramah's press, to resist certain pressures, Mr. *Barlow* gives this theorem, $t = \frac{pr}{c-p}$; where p = pressure in lbs. per square inch, r = radius of the cylinder, t = its thickness, and c = 18000 lbs. the cohesive power of a square inch of cast iron.

Ex. Suppose it were required to determine the thickness of metal in two presses, each of 6 inches radius, in one of which the pressure may extend to 4278 pounds, in the other to 8556 pounds, per square inch.

Here, in the first case,

$$t = \frac{4278 \times 6}{18000 - 4278} = 1{\cdot}87 \text{ inches, thickness.}$$

In the second,

$$t = \frac{8556 \times 6}{18000 - 8556} = 5{\cdot}43 \text{ inches, thickness.}$$

The usual rules, explained below (art. 10) would make the latter thickness double the former: extensive experiments are necessary to tell which method deserves the preference.

6. If *b* the breadth, and *d* the depth of a rectangular gate, or other surface, exposed to the pressure of water from top to bottom; then the entire pressure is equal to the weight of a prism of water whose content is $\frac{1}{2} b d^2$. Or, if *b* and *d* be in

feet, then the whole pressure $= 31\frac{1}{4}\, b\, d^2$, in lbs. or nearly $= \frac{3}{11}\, b\, d^2$, in cwts.

7. If the gate be in form of a trapezoid, widest at top, then, if B and b be the breadths at the top and bottom respectively, and d the depth.

$$\text{whole pressure in lbs.} = 31\tfrac{1}{4}\,[\tfrac{1}{3}(B - b) + b]\,d^2$$

$$\therefore \text{whole pressure in cwts.} = \tfrac{3}{11}\,[\tfrac{1}{3}(B - b) + b]\,d^2, \text{ nearly.}$$

8. The weight of a cubic foot of rain or river water, is nearly equal to $\frac{6}{11}$ cwt.

The pressure on a square inch, at the depth of THIR-*ty* feet is very nearly THIR-*teen* pounds.

Pressure on a *square foot*, nearly a ton at the depth of thirty-six feet. [The true depth is 35·84 feet.]

The weight of an *ale-gallon* of rain water is nearly $10\frac{1}{4}$ lbs, that of an *imperial* gallon 10 lbs.

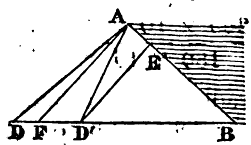
The weight of a cubic foot of *sea-water* is nearly $\frac{4}{7}$ of a cwt.

These are all useful approximations.

Thus, the pressure of rain water upon a square *inch* at the depth of 3000 feet, is 1300 lbs.

And the pressure upon a square *foot* at the depth of 108 feet is nearly three tons.

9. In the structure of dykes or embankments, both faces or slopes should be planes, and the exterior and interior slopes should make an angle of not less than 90°. For, if A D′ be the exterior slope, and the angle D′ A B be acute, E D′ perpendicular to A B is the direction of the pressure upon it; and the portion D′ A E will probably be torn off. But when D A is the exterior face, making with A B an obtuse angle, the direction of the pressure falls *within* the base, and therefore augments its stability.



10. The strength of a circular bason confining water, requires the consideration of other principles.

The perpendicular pressure against the wall depends merely on the altitude of the fluid, without being affected by the volume. But, as Professor *Leslie* remarks, the longitudinal effort of the thrust, or its tendency to open the joints of the masonry, is measured by the radius of the circle. To resist that action in very wide basons, the range or course of stones along the inside of the wall, must be proportionally thicker. On the other hand, if any opposing surface present some convexity to the pressure of water, the resulting longitudinal strain will be exerted in closing the joints and consolidating the building. Such reversed incurvation is, therefore, often adopted in the construction of dams.

In like manner, the thickness of *pipes* to convey water, must vary in proportion to $\frac{h\,d}{c}$, where h is the height of the head of water, d the diameter of the pipe, and c the measure of the cohesion of a bar of the same material as the pipe, and an inch square.

A pipe of *cast iron*, 15 inches diameter, and $\frac{3}{4}$ of an inch thick, will be strong enough for a head of 600 feet.

A pipe of *oak* of the same diameter, and 2 inches thick, would sustain a head of 180 feet.

Where the cohesion is the same, t varies as $h\,d$: or as $\text{HD} : \text{T} :: h\,d : t$, in the comparison of two cases.*

Example. What, then, must be the respective thicknesses of pipes of cast iron and oak, each 10 inches diameter, to carry water from a head of 360 feet?

Here, 1st. for *cast iron:*

$$\text{HD}\ (= 600 \times 15) : \text{T}\ (= \tfrac{3}{4}) :: h\,d\ (= 360 \times 10) : t = \frac{360 \times 10 \times 3}{600 \times 15 \times 4} = \frac{10800}{36000} = \tfrac{3}{10} \text{ of an inch.}$$

2dly. for *oak.*

$$\text{HD}\ (= 180 \times 15) : \text{T}\ (= 2) :: h\,d\ (= 360 \times 10) : t = \frac{360 \times 10 \times 2}{180 \times 15} = \tfrac{40}{15} = \tfrac{8}{3} = 2\tfrac{2}{3} \text{ inches.}$$

SECTION II. *Floating Bodies.*

1. If any body float on a fluid, it displaces a quantity of the fluid equal to itself in weight.

2. Also, the centres of gravity of the body and of the fluid displaced, must, when the body is at rest, be in the same vertical line.

3. If a vessel contain two fluids that will not mix (as water and mercury), and a solid of some intermediate specific gravity be immersed under the surface of the lighter fluid and float on the heavier; the part of the solid immersed in the

* To ascertain whether or not a pipe is strong enough to sustain a proposed pressure, it is a good custom amongst practical men to employ a *safety valve*, usually of an inch in diameter, and load it with the proposed weight, and a *surplus* determined by practice. Then, if the proposed pressure be applied interiorly, by a forcing pump, or in any other way, if the pipe remain *sound* in all its parts after the safety-valve has yielded, such pipe is regarded as sufficiently strong.

The *actual* pressures upon a pipe of any proposed diameter and head, may evidently be determined by a similar method.

heavier fluid, is *to* the whole solid as the difference between the specific gravities of the solid and the lighter fluid, is to the difference between the specific gravities of the two fluids.

4. The buoyancy of casks, or the load which they will carry without sinking, may be estimated by reckoning 10 lbs. averdupois to the ale gallon, or $8\frac{1}{3}$ lbs. to the wine gallon.

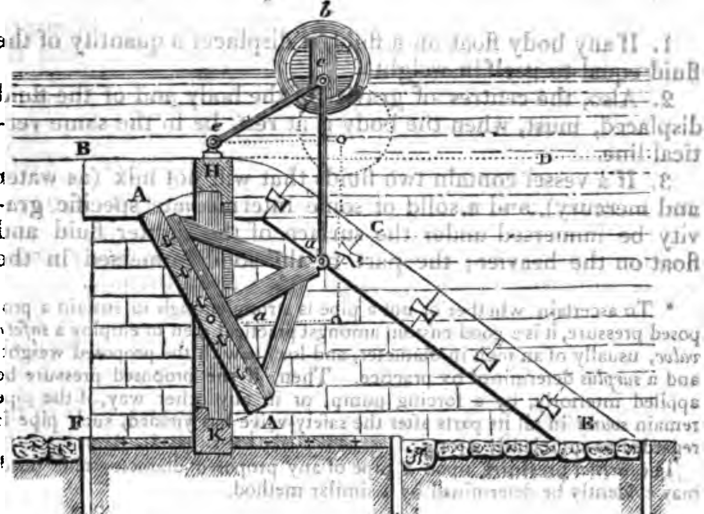
5. The buoyancy of *pontoons* may be estimated at about *half a hundred-weight* for each cubic foot.

Thus a pontoon which contained 96 cubic feet, would sustain a load of 48 cwt. before it would sink.

N. B. This is an approximation, in which the difference between $\frac{6}{11}$ and $\frac{1}{2}$, that is, $\frac{1}{22}$ of the whole weight is allowed for that of the pontoon itself.

6. The principles of buoyancy are very ingeniously applied in Mr. Farey's *self-acting flood-gate.* In the case of common sluices to a mill-dam, when a sudden flood occurs, unless the miller gets up in the night to open the gate or gates, the neighbouring lands may become inundated; and, on the contrary, unless he be present to shut up when the flood subsides, the mill-dam may be emptied and the water lost which he would need the next day. To prevent either of these occurrences, Mr. *John Farey*, whose talent and ingenuity are well known, has proposed a *self-acting flood-gate*, the following description of which has been given in the *Mechanics' Weekly Journal.*

A A represents a vertical section of a gate poised upon a horizontal axis passing rather *above* the centre of pressure of the gate, so as to give it a tendency to shut close. *a a* is a



lever, fixed perpendicular to the gate, and connected by an iron rod with a cask, *b*, floating upon the surface of the water, when it rises to the line, B, D, which is assumed as a level of the wear or mill-dam, B, C, E, F, in which the flood-gate is placed: by this arrangement, it will be seen that when the water rises above the dam, it floats the cask, opens the gate, and allows the water to escape until its surface subsides to the proper level at B, D; the cask now acts by its weight, when unsupported by the water, to close the gate and prevent leakage. The gate should be fitted into a frame of timber, H, K, which is set in the masonry of the dam. The upper beam, H, of the frame being just level with the crown of the dam, so that the water runs over the top of the gate at the same time that it passes through it: to prevent the current disturbing the cask, it is connected by a small rod, *e*, at each end, to the upper beam, H, of the frame, and jointed in such a manner as to admit of motion in a vertical direction.

Any ingenious mechanic will so understand the construction from this brief account, as to be able to apply it to practice when needed.

7. By means of the same principle of buoyancy it is, that a hollow ball of copper attached to a metallic lever of about a foot long, is made to rise with the liquid in a water-tub, and thus to close the cock and stop the supply from the pipe, just before the time when the water would otherwise run over the top of the vessel.

8. This property, again, has been successfully employed in pulling up old piles in a river where the tide ebbs and flows. A barge of considerable dimensions is brought over a pile as the water begins to rise: a strong chain which has been previously fixed to the pile by a ring, &c. is made to gird the barge and is then fastened. As the tide rises the vessel rises too, and by means of its buoyant force draws up the pile with it.

In an actual case, a barge 50 feet long, 12 feet wide, 6 deep, and drawing two feet of water, was employed. Here, $50 \times 12 \times (6-2) \times \frac{4}{7} = \frac{50 \times 12 \times 16}{7} = 192 \times 7\frac{1}{7} = 1344 + 27\frac{3}{7} = 1371\frac{3}{7}$ cwt. $= 66\frac{1}{2}$ tons nearly, the measure of the force with which the barge acted upon the pile.

SECTION III. *Specific Gravities.*

1. If a body float on a fluid, the part immersed is to the whole body, as the specific gravity of the body to the specific gravity of the fluid.

Hence, if the body be a square or a triangular prism, and it be laid upon the fluid, the ratio of that portion of one end which is immersed, to the whole surface of that end, will serve to determine the specific gravity of the body.

2. If the same body float upon two fluids in succession, the parts immersed will be inversely as the specific gravities of those fluids.

3. The weight which a body loses when wholly immersed in a fluid is equal to the weight of an equal bulk of the fluid.

When we say that a body loses part of its weight in a fluid, we do not mean that its *absolute* weight is less than it was before; but that it is partly supported by the re-action of the fluid under it, so that it requires a less power to sustain or to balance it.

4. A body immersed in a fluid ascends or descends with a force equal to the difference between its own weight and the weight of an equal bulk of fluid; the resistance or viscocity of the fluid not being considered.

5. *To find the specific gravity of a fluid, or of a solid.*—On one arm of a balance suspend a globe of lead by a fine thread, and to the other fasten an equal weight, which may just balance it in the open air. Immerse the globe into the fluid, and observe what weight balances it then, and consequently what weight is lost, which is proportional to the specific gravity as above. And thus the proportion of the specific gravity of one fluid to another is determined by immersing the globe successively in all the fluids, and observing the weights lost in each, which will be the proportions of the specific gravities of the fluids sought.

This same operation determines also the specific gravity of the solid immerged, whether it be a globe or of any other shape or bulk, supposing that of the fluid known. For the specific gravity of the fluid is to that of the solid, as the weight lost is to the whole weight.

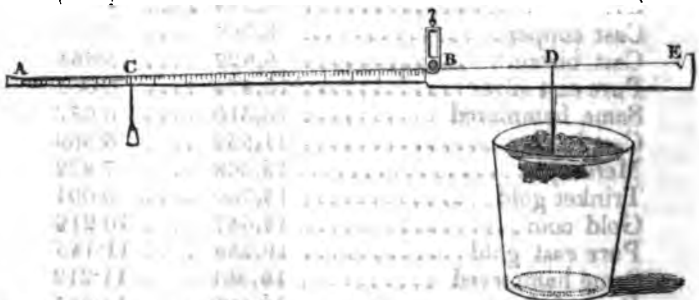
Hence also may be found the specific gravity of a body that is lighter than the fluid, as follows:

6. *To find the specific gravity of a solid that is lighter than the fluid, as water, in which it is put.*—Annex to the lighter body another that is much heavier than the fluid, so as the compound mass may sink in the fluid. Weigh the heavier body and the compound mass separately, both in water and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater.

Then, As this last remainder,
Is to the weight of the light body in air,
So is the specific gravity of the fluid,
To the specific gravity of that body.

7. The specific gravities of bodies of equal weight, are reciprocally proportional to the quantities of weight lost in the same fluid. And hence is found the ratio of the specific gravities of solids, by weighing in the same fluids, masses of them that weigh equally in air, and noting the weights lost by each.

8. Instead of a hydrostatic *balance*, a hydrostatic *steelyard* is now frequently employed. It is contrived to balance exactly by making the shorter end wider, and with an enlargement at the extremity. The shorter arm is undivided, but the longer arm is divided into short equal divisions: thus, if that longer arm be 8 inches long, it may be divided into 400 parts, the divisions *commencing* at A. Then, in using this



instrument, any convenient weight is suspended by a hook from a notch at the end of the scale A. The body whose specific gravity is to be determined, is suspended from the other arm by a horse-hair, and slid to and fro till an equilibrium is produced. Then, without altering its situation at D in the beam, it is immersed in water, and balanced a second time by sliding the counterpoise from A, say to C.

Here, evidently, weight in water : weight in air :: B C : B A; and loss of weight in water : weight in air :: A C : A B.

$$\text{Conseq.}\ \frac{\text{weight in air}}{\text{loss}} = \frac{\text{A B}}{\text{A C}} = \text{specified gravity.}$$

With such an instrument nicely balanced upon a convenient pedestal, I find that the specific gravities of solids are ascertainable both with greater facility and correctness than with any hydrostatic *balance* which I have seen.*

* We owe this contrivance to Dr. *Coates*, of Philadelphia.

Table of Specific Gravities.

METALS.	Spec. Grav.	Weight cub. inch. in avoird. oz.
Arsenic	5,763	3·335
Cast antimony	6,702	3·878
Cast zinc	7,190	4·161
Cast iron	7,207	4·165
Cast tin	7,291	4·219
Bar iron	7,788	4·507
Cast nickel	7,807	4·513
Cast cobalt	7,811	4·520
Hard steel	7,816	4·523
Soft steel	7,833	4·533
Cast brass	8,395	4·858
Cast copper	8,788	5·085
Cast bismuth	9,822	5·684
Pure cast silver	10,474	6·061
Same hammered	10,510	6·082
Cast lead	11,352	6·569
Mercury	13,568	7·872
Trinket gold	15,709	9·091
Gold coin	17,647	10·212
Pure cast gold	19,258	11·145
Same hammered	19,361	11·212
Pure platinum	19,500	11·285
Same hammered	20,336	11·777
Platinum wire	21,041	12·176
Platinum laminated, or beat into leaves	22,069	12·763

All metals become more dense or heavy by hammering.

STONES, EARTHS, &c.	Spec. Grav.	Weight cub. foot avoird. lbs.
Ambergris	926	
Amber	1,078	
Phosphorus	1,714	
Brick	2,000	125·00
Sulphur	2,033	127·06

STONES, &c. continued.	Spec. Grav.	Weight cub foot avoird. lbs
Opal	2,114	
Gypsum, opaque	2,168	135·50
Stone, paving	2,416	151·00
Stone, common	2,520	157·50
Flint and Spar	2,594	162·12
Crystal	2,653	
Granite, red Egyptian	2,654	165·87
Glass, green	2,642	
—— white	2,892	
—— bottle	2,733	
Pebble	2,664	166·50
Slate	2,672	167·00
Pearl	2,684	
Alabaster	2,730	
Marble	2,742	171·38
Porphyry	2,765	172·81
Emerald	2,775	
Chrysolite, jewellers'	2,782	
Chalk	2,784	174·00
Jasper	2,816	
Basaltes (Giants' causey)	2,864	179·00
Hone, white razor	2,876	179·75
Limestone	3,179	198·68
Diamond	3,521	
Beryl	3,549	
Sapphire	3,994	
Topaz	4,011	
Garnet	4,189	
Ruby	4,283	

RESINS, GUMS, &c.	Spec. Grav.
Wax	897
Tallow	945
Bees' wax	965
Camphor	989
Honey	1456
Bone of an Ox	1659
Ivory	1822

LIQUIDS.	Spec. Grav.
Air, at the earth's surface, about	1¼
Sulphuric Ether	716
Alcohol absolute	792
Liquid Bitumen	848
Oil of Turpentine	870
Muriatic Ether	874
Olive Oil	915
Burgundy Wine	991
Wine of Bourdeaux	994
Distilled Water	1,000
Sea Water	1,028
Milk	1,030
Beer	1,034
Nitric Acid	1,218
Water from the Dead Sea	1,240
Nitrous Acid	1,550
Sulphuric Acid	1,841

WOODS.	Spec. Grav.	Weight cub. foot avoird. lbs.
Cork	240	15·00
Poplar	383	23·94
Larch	544	34·00
Elm, and N. E. Fir	556	34·75
Mahogany, Honduras	560	35·00
Poon	579	36·18
Willow	585	36·56
Cedar	596	37·25
Pitch Pine	660	41·25
Pear tree	661	41·31
Walnut	671	41·94
Mar Forest Fir	694	43·37
Elder tree	695	43·44
Beech	696	43·50
Orange wood	705	44·06
Cherry tree	715	44·68
Teak	745	46·56
Maple and Riga Fir	750	46·87
Ash and Dan. Oak	760	47·50

WOODS continued.	Spec. Grav.	Weight cub. foot avoird. lbs.
Yew, Dutch	788	49·25
Apple tree	793	49·56
Alder	800	50·00
Yew, Spanish	807	50·44
Mahogany, Spanish	852	53·25
Oak, Canadian	872	54·50
Box, French	912	57·00
Logwood	913	57·06
Oak, English	970	51·87
Ditto 60 years old	1170	73·12
Ebony	1331	83·18
Lignum Vitæ	1333	83·31

9. Since a cubic foot of water at the temperature of 40° Fahrenheit, weighs 1000 ounces avoirdupois, or $62\frac{1}{2}$ lbs. the numbers in the preceding tables under the head Spec. Grav. exhibit very nearly the respective weights in avoirdupois ounces of a cubic foot of the several substances.

We have also given in another column, the weight in *ounces* of a *cubic inch* of each of the several *metals:* and, with regard to different kinds of *wood* and *stone*, the weight in avoirdupois pounds of a *cubic foot* of each. These, of course, are *medium* specific gravities and weights; for there are variations, sometimes indeed considerable ones, between the specific gravities of different specimens of the same kind of substance.

These additional columns will evidently facilitate the labour of finding *the magnitude of a body from its weight*, or *the weight of a body from its magnitude.* In this respect, too, the following particulars will often be of utility.

10. (1) 430·25 cubic inches of cast iron weigh 1 cwt.
397·60 .... bar iron
368·88 .... cast brass
352·41 .... cast copper
272·8 .... cast lead

(2) 14·835 cubic feet of paving stone weigh a ton
14·222 .... common stone
13·505 .... granite
13·070 .... marble
12·874 .... chalk
11·273 .... limestone
64·460 .... Elm
64·000 .... Honduras mahogany
51·650 .... Mar Forest fir
51·494 .... beech

47·762 cubic feet of Riga fir
47·158 .... ash, and Dantzic oak
42·066 .... Spanish mahogany
36·205 .... English oak

11. *Prob.* to find the internal diameter of a uniform capillary or other small tube.

Let the tube be weighed when empty, and again when filled with mercury, and let w be the difference of those weights in troy grains, and l the length of the tube in inches. Then the diameter required, $d = {\cdot}019252 \sqrt{\frac{w}{l}}$

Thus, if the difference of weights were 500 grains, and the length of the tube were 20 inches: we should have $d = {\cdot}019252 \sqrt{\frac{500}{20}} = {\cdot}019252 \times 5 = {\cdot}09626$ of an inch.

12. *Prob.* To find the weight of a leaden pipe.

If l be the length in feet, d the interior diameter, and t the thickness, both in inches and parts of an inch, w the weight in hundred weights: then $w = {\cdot}1382\, l\, t\, (d + t)$.

For a *cast iron pipe*, the theorem is

$$w = {\cdot}0876\, l\, t\, (d + t);$$

or nearly $\frac{7}{11}$ of the former expression.

Ex. Let the internal diameter of a leaden pipe be 4 inches, the thickness $\frac{1}{4}$ of an inch; required the weight of 12 feet in length.

Here $\cdot 1382 \times 12 \times \frac{1}{4} \times 4\frac{1}{4} = {\cdot}1382 \times 12\frac{3}{4} = 1{\cdot}762$ cwts.

13. *Prob.* To find the weight of the ring or rim of a cast iron fly-wheel.

Supposing this ring or annulus to be similar to a portion of a pipe of large diameter, the expression for the weight will be similar to the above; but it may be advantageous to change the notation. Let then D be the interior diameter of the fly in inches, d half the difference of the exterior and interior diameters, T the thickness from side to side of the fly, and W its weight in hundred weights: then $W = {\cdot}0073\, T d\, (D + d)$.

Ex. Let $D = 100$ inches, $d = 5$ inches, or the exterior diameter $= 110$, and T the thickness $= 4$ inches.

Then, ${\cdot}0073\, T d\, (D + d) = {\cdot}0073 \times 4 \times 5 \times 105 = {\cdot}0073 \times 2100 = 15{\cdot}33$ cwt. the required weight of the cylindrical rim.

Note. The reader will observe that this process is much shorter than those usually employed, even with the aid of tables of circles already computed.

*** See, farther, on kindred subjects, the approximate rules in mensuration, pa. 199, &c.

CHAP. XII.

HYDRODYNAMICS.

1. *Hydrodynamics* is that part of mechanical science which relates to the motion of non-elastic fluids, and the forces with which they act upon bodies.

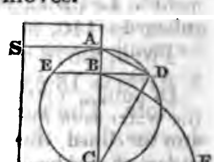
This branch of mechanics is the most difficult, and the least advanced: whatever we know of it is almost entirely due to the researches of the moderns.

Could we know with certainty the mass, the figure, and the number of particles of a fluid in motion, the laws of its motion might be determined by the resolution of this problem, viz. to find the motion of a proposed system of small free bodies acting one upon the other in obedience to some given exterior force. We are, however, very far from being in possession of the data requisite for the solution of this problem. We shall, therefore, simply present a few of the most usually received theoretical deductions; and then proceed to state those rules which have flown from a judicious application of theory to experiment.

SECTION I. *Motion and Effluence of Liquids.*

1. A jet of water, issuing from an orifice of a proper form, and directed upwards, rises, under favourable circumstances, nearly to the height of the head of water in the reservoir; and since the particles of such a stream are but little influenced by the neighbouring ones, they may be considered as independent bodies, moving initially with the velocity which would be acquired in falling from the height of the reservoir. And the velocity of the jet will be the same whatever may be its direction.

2. Hence if a jet issue horizontally from any part of the side of a vessel standing on a horizontal plane, and a circle be described having the whole height of the fluid for its diameter, the fluid will reach the plane at a distance from the vessel, equal to that chord of the circle in which the jet initially moves.



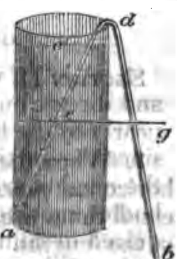
Thus, if A S be the upper surface of the fluid in the vessel, B the place of the orifice, C F the horizontal plane on which the fluid spouts, then C F is equal to E D, the horizontal chord of the circle whose diameter is A C, passing through B.

3. When a cylindrical or prismatic vessel empties itself by a small orifice, the velocity at the surface is uniformly retarded;

and in the time of emptying itself, twice the quantity would be discharged if it were kept full by a new supply.

4. But the quantity discharged is by no means equal to what would fill the whole orifice, with this velocity. If the aperture is made simply in a thin plate, the lateral motion of the particles towards it tends to obstruct the direct motion, and to contract the stream which has left the orifice, nearly in the ratio of two to three. So that in order to find the quantity discharged, the section of the orifice must be supposed to be diminished from 100 to 62 for a simple aperture, to 82 for a pipe of which the length is twice the diameter, and in other ratios according to circumstances.

5. When a syphon, or bent tube, is filled with a fluid, and its orifices immersed in the fluids of different vessels, if both surfaces of the fluids are in the same level, the whole remains at rest; but if otherwise, the longer column of fluid in the syphon preponderates, and the pressure of the atmosphere forces up the fluid from the higher vessel, until the equilibrium is restored; and the motion is the more rapid as the difference of levels is greater: provided that the greatest height of the tube above the upper surface be not more than a counterpoise to the pressure of the atmosphere.



6. If the lower vessel be allowed to empty itself, the syphon will continue running as long as it is supplied from the upper, and the faster, as it descends the further below the upper vessel. In the same manner the discharge of a pipe, descending from the side or bottom of a given vessel, must be increased almost without limit by lengthening it.*

* An improvement in the construction of the syphon has been lately proposed in the *Glasgow Mechanics' Magazine*, and by *M. Buntem* at Paris. It might be very advantageously used if constructed on a large scale, for lowering the water in mill dams or canals. The improvement in the present syphon is, that the exhausting pipe is enlarged to the same diameter as that of the syphon, and its mouth is widened out to something of a funnel shape, as in the figure.

In putting it into action, the short arm is immersed in the water as in the usual manner, the bottom of the long arm is closed, the exhausting pipe is then filled with water by the funnel-shaped mouth. On the bottom of the long arm being opened, the water flows out, exhausts the air from the syphon, when the water, which is wished to be emptied, flows out in a continual stream.

7. If a notch or sluice in form of a rectangle be cut in the vertical side of a vessel full of water, or any other fluid, the quantity flowing through it will be $\frac{2}{3}$ of the quantity which would flow through an equal orifice placed horizontally at the whole depth, in the same time, the vessel being kept constantly full.

8. If a short pipe elevated in any direction from an aperture in a conduit, throw the water in a parabolic curve to the distance or range R, on a board, or other horizontal plane passing through the orifice, and the greatest height of the spouting fluid above that plane, be H, then the height of the *head* of water above that conduit pipe, may be found *nearly*: viz. by taking 1st, $2 \cot E = \frac{R}{2H}$; and 2dly, the altitude of the head $A = \frac{1}{2} R \times \operatorname{cosec} 2E$.

Example. Suppose that $R = 40$ feet, and $H = 18$ feet. Then $\frac{R}{2H} = \frac{40}{36} = 1 \cdot 1111111 = 2 \cot 60° \; 57'$: and $A = \frac{1}{2} R \times \operatorname{cosec} 2E = 20 \times \operatorname{cosec} 121° \; 54' = 20 \times 1 \cdot 177896 = 23 \cdot 55792$ feet, height required.

Note. This result of theory will usually be found about $\frac{1}{5}$ of that which is furnished by experiment.

SECTION II. *Motion of Water in Conduit Pipes and Open Canals, over Weirs, &c.*

1. When the water from a reservoir is conveyed in long horizontal pipes of the same aperture, the discharges made in equal times are nearly in the inverse ratio of the square roots of the lengths.

It is supposed that the lengths of the pipes to which this rule is applied are not very unequal. It is an approximation not deduced from principle, but derived immediately from experiment. [Bossut, tom. 11, § 647, 648. At § 673, he has given a table of the actual discharges of water-pipes, as far as the length of 2340 toises, or 14,950 feet English.]

2. Water running in open canals, or in rivers, is accelerated in consequence of its depth, and of the declivity on which it runs, till the resistance, increasing with the velocity, becomes equal to the acceleration, when the motion of the stream becomes uniform.

It is evident, that the amount of the resisting forces can hardly be determined by principles already known, and therefore nothing remains but to ascertain, by experiment, the velocity corresponding to different declivities, and different depths of water, and to try, by multiplying and extending these experiments, to find out the law which is common to them all.

The Chevalier Du Buat has been successful in this research, and has given a formula for computing the velocity of running water, whether in close pipes, open canals, or rivers, which, though it may be called *empirical*, is extremely useful in practice. *Principes d'Hydraulique.* Professor Robison has given an abridged account of this book, in his excellent article on Rivers and Water-works, in the *Encyclopædia Britannica.*

Let V be the velocity of the stream, measured by the inches it moves over in a second; R a constant quantity, viz. the quotient obtained by dividing the area of the transverse section of the stream, expressed in square inches, by the boundary or perimeter of that section, *minus* the superficial breadth of the stream expressed in linear inches.

The mean velocity is that with which, if all the particles were to move, the discharge would be the same with the actual discharge.

The line R is called by Du Buat the *radius*, and by Dr. Robison the *hydraulic mean depth.* As its affinity to the radius of a circle seems greater than to the depth of a river, we shall call it, with the former, the *radius of the section.*

Lastly, let S be the denominator of a fraction which expresses the slope, the numerator being unity, that is, let it be the quotient obtained by dividing the length of the stream, supposing it extended in a straight line, by the difference of level of its two extremities; or, which is nearly the same, let it be the co-tangent of the inclination or slope.

3. The above denominations being understood, and the section, as well as the velocity, being supposed uniform, V in English feet,

$$= \frac{307\sqrt{(R - \frac{1}{10})}}{S^{\frac{1}{2}} - \frac{1}{2}\log(S + \frac{16}{10})} - \frac{3}{10}\sqrt{(R - \frac{1}{10})};$$

$$\text{or } V = \sqrt{R - \frac{1}{10}}\left(\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\log(S + \frac{16}{10})} - \frac{3}{10}\right).$$

When R and S are very great,

$$V = R^{\frac{1}{2}}\left(\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\log S} - \frac{3}{10}\right), \text{ nearly.}$$

The logarithms understood here are the hyperbolic, and are found by multiplying the common logarithms by 2·3025851.

The slope remaining the same, the velocities are as $\sqrt{R - \frac{1}{10}}$.

The velocities of two great rivers that have the same declivity, are as the square roots of the radii of their sections.

If R is so small, that $\sqrt{R - \frac{1}{10}} = 0$, or $R = \frac{1}{10}$, the velocity will be nothing; which is agreeable to experience; for in a cylindric tube $R = \frac{1}{2}$ the radius; the radius, therefore, equal two-tenths; so that the tube is nearly capillary, and the fluid will not flow through it.

The velocity may also become nothing by the declivity becoming so small, that

$$\frac{307}{s^{\frac{1}{2}} - \frac{1}{2}\log\left(s + \frac{16}{10}\right)} - \frac{3}{10} = 0; \text{ but}$$

if $\frac{1}{s}$ is less than $\frac{1}{500000}$, or than $\frac{1}{10}$th of an inch to an English mile, the water will have sensible motion.

4. In a river, the greatest velocity is at the surface, and in the middle of the stream, from which it diminishes toward the bottom and the sides, where it is least. It has been found by experiment, that if from the square root of the velocity in the middle of the stream, expressed in inches per second, unity be subtracted, the square of the remainder is the velocity at the bottom.

Hence, if the former velocity be $= v$, the velocity at the bottom $= v - 2\sqrt{v} + 1$.

5. The mean velocity, or that with which, were the whole stream to move, the discharge would be the same with the real discharge, is equal to half the sum of the greatest and least velocities, as computed in the last proposition.

The mean velocity is, therefore, $= v - \sqrt{v} + \frac{1}{2}$.

This is also proved by the experiments of Du Buat.

6. Suppose that a river having a rectangular bed, is increased by the junction of another river equal to itself, the declivity remaining the same; required the increase of depth and velocity.

Let the breadth of the river $= b$, the depth before the junction d, and after it, x; and, in like manner, v and v' the mean velocities before and after; then $\frac{b\,d}{b + 2\,d}$ is the radius before, and $\frac{b\,x}{b + 2\,x}$ the radius after, so $v = \frac{307\,R^{\frac{1}{2}}}{s^{\frac{1}{2}}}$, supposing the breadth of the river to be such, that we may reject the small quantity subtracted from R, in art. 3; and in like manner,

$v' = \frac{307 R'^{\frac{1}{2}}}{s^{\frac{1}{2}}}$. Then, substituting for R and R′, we have

$$v = \frac{307}{s^{\frac{1}{2}}} \times \sqrt{\frac{b\,d}{v + 2d}}, \text{ and}$$

$$v' = \frac{307}{s^{\frac{1}{2}}} \times \sqrt{\frac{v\,x}{v + 2x}}.$$ Multiplying these into the areas of the sections $b\,d$ and $b\,x$, we have the discharges, viz. $b\,d\,v = \frac{307}{s^{\frac{1}{2}}} \times \frac{b\,d\,\sqrt{b\,d}}{\sqrt{b+2d}}$; and $b\,x\,v' = \frac{307}{s^{\frac{1}{2}}} \times \frac{b\,x\,\sqrt{b\,x}}{\sqrt{b+2x}}$.

Now the last of these is double of the former; therefore $\frac{b\,x\,\sqrt{b\,x}}{\sqrt{b+2x}} = \frac{2\,b\,d\,\sqrt{b\,d}}{\sqrt{b+2d}}$, or, $\frac{x^3}{b+2x} = \frac{4\,d^3}{b+2d}$, and

$x^3 - \frac{8\,d^3}{b+2d}\,x = \frac{4\,b\,d^3}{b+2d}$, a cubic equation which can always be resolved by Cardan's rule, or by the approximating method given at pa. 90.

As an example, let $b = 10$ feet, and $d = 1$, then $x^3 - \frac{2}{3}\,x = \frac{10}{3}$, and $x = 1{\cdot}4882$, which is the depth of the increased river. Hence we have $1{\cdot}488 \times v' = 2\,v$, and $1{\cdot}488 : 2 :: v : v'$, or $v : v' ::$ 37 to 50 nearly.

When the water in a river receives a permanent increase, the depth and the velocity, as in the example above, are the first things that are augmented. The increase of the velocity increases the action on the sides and bottom, in consequence of which the width is augmented, and sometimes also, but more rarely, the depth. The velocity is thus diminished, till the tenacity of the soil, or the hardness of the rock, afford a sufficient resistance to the force of the water. The bed of the river then changes only by insensible degrees, and, in the ordinary language of hydraulics, is said to be permanent, though in strictness this epithet is not applicable to the course of any river.

7. When the sections of a river vary, the quantity of water remaining the same, the mean velocities are inversely as the areas of the sections.

This must happen, in order to preserve the same quantity of discharge. (*Playfair's Outlines.*)

8. The following table, abridged from *Dr. Robison*, serves at once to compare the surface, bottom, and mean velocities in rivers, according to the principles of art. 4, 5.

Velocities of Rivers.

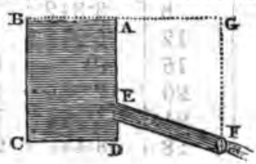
VELOCITY IN INCHES.			VELOCITY IN INCHES.		
Surface.	Bottom.	Mean.	Surface.	Bottom.	Mean.
4	1·	2·5	56	42·016	49·008
8	3·342	5·67	60	45·509	52·754
12	6·071	9·036	64	49·	56·5
16	9·	12·5	68	52·505	60·252
20	12·055	16·027	72	56·025	64·012
24	15·194	19·597	76	59·568	67·784
28	18·421	23·210	80	63·107	71·553
32	21·678	26·839	84	66·651	75·325
36	25·	30·5	88	70·224	79·112
40	28·345	34·172	92	73·788	82·894
44	31·742	37·871	96	77·370	86·685
48	35·151	41·570	100	81·	90·5
52	38·564	45·282			

9. The knowledge of the velocity at the bottom is of the greatest use for enabling us to judge of the action of the stream on its bed.

Every kind of soil has a certain velocity consistent with the stability of the channel. A greater velocity would enable the waters to tear it up, and a smaller velocity would permit the deposition of more moveable materials from above. It is not enough, then, for the stability of a river, that the accelerating forces are so adjusted to the size and figure of its channel that the current may be in train: it must also be in equilibrio with the tenacity of the channel.

We learn from the observations of Du Buat, and others, that a velocity of three inches per second at the bottom will just begin to work upon the fine clay fit for pottery, and however firm and compact it may be, it will tear it up. Yet no beds are more stable than clay when the velocities do not exceed this; for the water soon takes away the impalpable particles of the superficial clay, leaving the particles of sand sticking by their lower half in the rest of the clay, which they now protect, making a very permanent bottom, if the stream does not bring down gravel or coarse sand, which will rub off this very thin crust, and allow another layer to be worn off; a velocity of six inches will lift fine sand; eight inches will lift sand as coarse as linseed; twelve inches will sweep along fine gravel; twenty-four inches will roll along rounded pebbles an inch diameter; and it requires three feet per second at the bottom to sweep along shivery angular stones of the size of an egg. (*Robison on Rivers.*)

10. Mr. *Eytelwein*, a German mathematician, has devoted much time to inquiries in hydrodynamics. In his investigations he has paid attention to the mutual cohesion of the liquid moleculæ, their adherence to the sides of the vessel in which the water moves, and to the contraction experienced by the liquid vein when it issues from the vessel under certain circumstances. He obtains formulæ of the utmost generality, and then applies them to the motion of water; 1st, in a cylindric tube; 2dly, in an open canal.



11. Let d be the diameter of the cylindric tube E F, h the total height F G of the head of water in the reservoir above the orifice F, and l the length E F of the tube, all in inches: then the velocity in inches with which the fluid will issue from the orifice F will be

$$v = 23\tfrac{1}{3}\sqrt{\frac{57\,h\,d}{l + 57\,d}}\text{: (English measure.)}$$

this multiplied into the area of the orifice will give the quantity per second.

12. *For open canals.* Let v be the mean velocity of the current in feet (English), a area of the vertical section of the stream, p perimeter of the section, or sum of the bottom and two sides, l length of the bed of the canal corresponding to the fall h, all in feet: then

$$v = -\,0{\cdot}109 + \sqrt{9582\,\frac{a\,h}{p\,l} + 0{\cdot}0111}.$$

The experiments of M. *Bidone*, of Turin, on the motion of water in canals, agree within the 80th part of the results of computations from the preceding formulæ.

13. The following table also exhibits *Eytelwein's* coefficients for orifices of different kinds; their accuracy has, in many cases, been amply confirmed.

No.	Nature of the Orifices employed.	Ratio between the theoretical and real discharges.	Coefficients for finding the velocities in Eng. feet.
1	For the whole velocity due to the height	1 to 1·00	8·04
2	For wide openings whose bottom is on a level with that of the reservoir	1 to 0·961	7·7
3	For sluices with walls in a line with the orifice	1 to 0·961	7·7
4	For bridges with pointed piers ..	1 to 0·961	7·7
5	For narrow openings whose bottom is on a level with that of the reservoir	1 to 0·861	6·9
6	For smaller openings in a sluice with side walls	1 to 0·861	6·9
7	For abrupt projections and square piers of bridges	1 to 0·861	6·9
8	For openings in sluices without side walls.....................	1 to 0·635	5·1

14. When a pipe is bent in one or more places, then if the squares of the sines of the several changes of direction be added into one sum s, the velocity v will, according to *Langsdorf*, be found by the theorem $v = \sqrt{\frac{548\,d\,h}{d + \frac{1}{35}l + \frac{1}{6}d\,s}}$; l, h, d, and v, being all in English inches.

Table, abridged from one by Mr. Smeaton, for showing the height of head necessary to overcome the friction of water in horizontal pipes.

Velocities per second of water in the pipes.										Bore of the Pipes.
Feet in. 0 6	Feet in. 1 0	Feet in. 1 6	Ft. in. 2 0	Ft. in. 2 6	Feet in. 3 0	Feet in. 3 6	Feet in. 4 0	Feet in. 4 6	Feet in. 5 0	
0 4·5	1 4·7	2 11·0	4 9·7	7 1·7	10 1·0	13 8·0	17 10·0	22 6·7	28 0·2	½ inch.
0 3·0	0 11·1	1 11·3	3 2·5	4 9·2	6 6·6	9 1·3	11 10·6	15 0·5	18 8·1	¾ inch.
0 2·2	0 8·4	1 5·5	2 4·9	3 6·9	5 5·5	6 10·0	8 11·0	11 3·4	14 0·0	1 inch.
0 1·8	0 6·7	1 2·0	1 11·1	2 10·3	4 0·4	5 5·6	7 1·6	9 0·3	11 2·5	1¼ inch.
0 1·5	0 5·6	0 11·7	1 7·2	2 4·6	3 4·3	4 6·7	5 11·3	7 6·2	9 4·1	1½ inch.
0 1·3	0 4·8	0 10·0	1 4·5	2 0·5	2 10·6	3 10·9	5 1·1	6 5·4	8 0·1	1¾ inch.
0 1·1	0 4·2	0 8·7	1 2·4	1 9·4	2 6·2	3 5·0	4 5·5	5 7·7	7 0·0	2 inch.
0 1·0	0 3·7	0 7·8	1 0·8	1 7·0	2 2·9	3 0·4	3 11·6	5 0·1	6 2·7	2¼ inch.
0 0·9	0 3·3	0 7·0	0 11·5	1 5·1	2 0·2	2 8·8	3 6·8	4 6·1	5 7·2	2½ inch.
0 0·7	0 2·8	0 5·0	0 9·6	1 2·3	1 8·2	2 3·3	2 11·7	3 9·1	4 8·0	3 inch.
0 0·6	0 2·4	0 5·0	0 8·2	1 0·2	1 5·3	1 11·4	2 6·6	3 2·7	4 0·0	3½ inch.
0 0·6	0 2·1	0 4·4	0 7·2	0 10·7	1 3·1	1 8·5	2 2·7	2 9·8	3 6·0	4 inch.
0 0·5	0 1·9	0 3·9	0 6·4	0 9·5	1 1·4	1 6·2	1 11·8	2 6·1	3 1·4	4½ inch.
0 0·4	0 1·7	0 3·5	0 5·8	0 8·6	1 0·1	1 4·4	1 9·4	2 3·1	2 9·6	5 inch.
0 0·4	0 1·4	0 2·9	0 4·8	0 7·1	0 10·1	1 1·7	1 5·8	1 10·6	2 4·0	6 inch.
0 0·3	0 1·2	0 2·5	0 4·1	0 6·1	0 8·6	0 11·7	1 3·3	1 7·3	2 0·0	7 inch.
0 0·3	0 1·0	0 2·2	0 3·6	0 5·4	0 7·6	0 10·2	1 1·4	1 4·9	1 9·0	8 inch.
0 0·25	0 0·9	0 1·9	0 3·2	0 4·8	0 6·7	0 9·1	0 11·9	1 3·0	1 6·7	9 inch.
0 0·2	0 0·8	0 1·7	0 2·9	0 4·3	0 6·0	0 8·2	0 10·7	1 1·5	1 4·8	10 inch.
0 0·2	0 0·8	0 1·6	0 2·6	0 3·9	0 5·5	0 7·5	0 9·7	1 0·3	1 3·3	11 inch.
0 0·19	0 0·7	0 1·5	0 2·4	0 3·6	0 5·0	0 6·8	0 8·9	0 11·3	1 2·0	12 inch.

Look for the velocity of water in the pipe in the upper row, and in the column below it, and opposite to the given diameter of the pipe standing in the last column, will be found the perpendicular height of a column or head, in feet, inches, and tenths, requisite to overcome the friction of such pipe for 100 feet in length, and obtain the given velocity.

Table containing the quantity of Water discharged over an inch vertical section of a Weir.

Depth of the upper edge of the waste-board below the surface in English inches.	Cubic feet of water discharged in a minute by an inch of the wasteboard according to Du Buat's formulæ.	Cubic feet of water discharged in a minute by an inch of the waste-board, according to experiments made in Scotland.	Gallons of 282 inches, corresponding with results in col. 3.
1	0·403	0·428	2·621
2	1·140	1·211	7·417
3	2·095	2·226	13·634
4	3·225	3·427	20·990
5	4·507	4·789	29·332
6	5·925	6·295	38·557
7	7·466	7·933	48·589
8	9·122	9·692	59·864
9	10·884	11·564	70·826
10	12·748	13·535	83·164
11	14·707	15·632	95·746
12	16·758	17·805	109·055
13	18·895	20·076	122·965
14	21·117	22·437	137·427
15	23·419	24·883	152·408
16	25·800	27·413	167·905
17	28·258	30·024	183·897
18	30·786	32·710	200·350

To the above table, originally due to *Du Buat*, is added a third column containing the quantities of water discharged, as inferred from experiments made in Scotland, and examined by Dr. Robison, who found that they in general gave a discharge $\frac{1}{16}$ greater than that which is deduced from Du Buat's formulæ. We would recommend it therefore to the engineer to employ the third column in his practice, or the fourth if he wish for the result in gallons.

If they be odd quarters of an inch, look in the table for as many inches as the depth contains quarters, and take the eighth part of the answer. Thus, for $3\frac{3}{4}$ inches, take the eighth part of 24·888, which corresponds to 15 inches. This is 3·110.

15. The quantity discharged increases more rapidly than the width: to obtain a correct measure of it, if n be the width or length of the wasteboard in inches, take $(n + \frac{1}{40} n)$ times the quantity for one inch of wasteboard of the given depth, from the preceding table.

In the preceding table it is supposed that the water from which the discharge is made is perfectly stagnant; but if it should happen to reach the opening with any velocity, we have only to multiply the area of the section by the velocity of the stream.

16. When the quantity of water Q discharged over a weir is known, the depth of the edge of the wasteboard, or H, may be approximated from the following formulæ, l length of wasteboard

$$H = \left(\frac{Q}{11{\cdot}4172\, l}\right)^{\frac{2}{3}} = \left(\frac{Q}{11\frac{1}{2}\, l}\right)^{\frac{2}{3}} \text{ nearly.}$$

Reciprocally, $Q = 11\frac{1}{2}\, l \,.\, H^{\frac{3}{2}}$ nearly: or, more accurately by adding the correction in article 15.

17. The quantities discharged for any given width, are as the $\frac{3}{2}$ power of the depth, or as $H^{\frac{3}{2}}$.

Hence, to extend the use of the table to greater depths, we have only for

Twice any depth, take	Q ×	2·828
3 times	Q ×	5·196
4 times	Q ×	8·000
5 times	Q ×	11·180
6 times	Q ×	14·697
7 times	Q ×	18·520
8 times	Q ×	22·627
9 times	Q ×	27·000
10 times	Q ×	31·623

and the results will be *nearly* true.

To make them still more correct, where great accuracy is required, add to them their *thousandth* par

Examples of the use of the Tables and Rules.

Ex. 1. Let the depth be 10 inches below the upper surface of the water, and the width 8 inches. How many cubic feet of water will be discharged in a minute?

	cub. feet.
By table Q to depth 10, width 1 =	13·535
Multiply this by n = ...	8
	106·280
Add $\frac{1}{20}$ of this product	5·314
Discharge in one minute =	111·594

Ex. 2. Let the depth be 9 feet, and the width 1 foot. Required the cubic feet discharged in a minute.

By table Q to depth 12 inch. width 1 inch =	17·805
Factor for 9 times depth is 27 = 3 × 9	3
	53·415
	9
Quantity for 1 inch width	480·735
Multiply by n = ...	12
	5768·802
Add $\frac{1}{20}$ of the product	286·441
Total quantity in cubic feet =	6055·261

Ex. 3. Let a square orifice of 6 inches each side be placed in a sluice-gate with its top 4 feet below the upper surface of the water: how much will it discharge in a minute?

Here the quantity discharged by a slit in depth 48 inches, must be taken from one in depth 54 inches.

	cub. feet.
For 54, multiply Q at 6 by 3^{3} or 27	169·965
For 48, ——— Q at 12 by $4^{\frac{3}{2}}$ or 8	142·440
Difference	27·525

27·525 × (6 + $\frac{6}{20}$) = 173·4 cubic feet, quantity discharged.

Note. In an example like this, it is a good approximation, *to multiply continually together, the area of the orifice, the number 336,* and the square root of the depth in feet of the middle of the orifice.*

Thus, in the preceding example, it will be $\frac{1}{2} \times \frac{1}{2} \times 336 \times \sqrt{4\cdot25} = \frac{1}{4} \times 336 \times 2\cdot062 = 173\cdot2$ cubic feet.

The less the height of the orifice compared with its depth under the water, the nearer will the result thus obtained approach to the truth.

If the height of the orifice be such as to require consideration, the principle of Art. 7 of the preceding section may be blended with this rule.

Thus, applying this rule to *Ex.* 2, we shall have area $\times \sqrt{\text{depth}} \times 336 \times \frac{2}{3} = 9 \times 3 \times 224 = 6048$, for the cubic feet discharged. This is less than the former result by about its 900th part. It is, therefore, a good approximation, considering its simplicity: it may in many cases supersede the necessity of recurrence to tables.

18. Prob. Given the vertical section of a river, or other stream, to determine the swell occasioned by the piers of a bridge, or the sides of a cleaning sluice which contract the passage by a given quantity, for a short length only of the channel; the velocity of the stream being also known. Let v be the velocity of the stream, independently of the effect of the bridge, r the section of the river, and a the amount of the sections between the piers; let $2g$, instead of being taken at $64\frac{1}{3}$, be reckoned $58\cdot6$, to accord with the effect of experimental contractions through arches of bridges, &c. and let s be the slope of the bed of the river, or the sine of its angle with the horizon; then *Du Buat* (tom. i. p. 225), gives for the swell or rise of the stream in feet, which will be occasioned by the obstruction $\left(\frac{v^2}{58\cdot6} + s\right) \cdot \left(\left[\frac{r}{a}\right]^2 - 1\right)$: a theorem, by means of which we may approximate to the said swell in any proposed case.

The value of s will, of course, be different in different cases; but if we assume $\frac{1}{20}$, or $\cdot05$, as a mean value, it will enable us to compute and tabulate results, which, though they cannot be presented as perfectly correct, may be regarded as exhibiting a medium between those that will usually occur; and will serve to anticipate the consequences of floods of certain velocities, when constrained to pass through bridges which more or less contract the stream.

* $336 = 5\cdot6 \times 60$.

*Table of the Rise of Water occasioned by Piers of Bridges, and other contractions.**

Veloc. of current in feet, per sec.	Amount of obstructions, compared with the vertical section of the River.									
	1-10th.	2-10ths.	3-10ths.	4-10ths.	5-10ths.	6-10ths.	7-10ths.	8-10ths.	9-10ths.	
	Proportional Rise of Water, in feet and decimals.									
	feet	feet	feet	feet	feet	feet	feet	feet	feet	
1	0·0157	0·0377	0·0698	0·1192	0·2012	0·3521	0·6780	1·6094	6·6389	
2	0·0277	0·0665	0·1231	0·2102	0·3548	0·6208	1·1955	2·8378	11·7058	Ordinary floods.
3	0·0477	0·1144	0·2118	0·3618	0·6107	1·0687	2·0580	4·8850	20·1504	
4	0·0760	0·1822	0·3372	0·5759	0·9719	1·7008	3·2755	7·7750	32·0720	
5	0·1165	0·2793	0·5168	0·8782	1·4895	2·6066	5·0202	11·9160	49·1535	Violent floods.
6	0·1558	0·3736	0·6912	1·1807	1·9925	3·4868	6·7154	15·9398	65·7518	
7	0·2078	0·4983	0·9221	1·5750	2·6578	4·6511	8·9578	21·2626	87·7080	
8	0·2678	0·6423	1·1884	2·0299	3·4255	5·9947	11·5454	27·4042	113·0422	Unusually violent floods.
9	0·3359	0·8054	1·4903	2·5566	4·2956	7·5172	14·4777	34·3646	141·7541	
10	0·4119	0·9877	1·8276	3·1218	5·2680	9·2190	17·7551	42·1440	173·844	

19. It will be evident from an inspection of this table, that even in the case of ordinary floods, old bridges with piers and starlings, occupying 6 or 7 tenths of the section of the river, will produce a swell of 2, 3, or more feet, often overflowing the river's banks and occasioning much mischief. Also, that in violent floods, an obstruction amounting to 7 tenths of the channel will cause a rise of 7 or 8 feet, probably choking up the arches, and occasioning the destruction of the bridge. Greater velocities and greater contractions produce a rapid augmentation of danger and mischief; as the table obviously shows.

20. The same principles and tabulated results serve to estimate the fall from the higher to the lower side of a bridge, on account of an ebbing tide, &c. Thus, for London Bridge, where the breadth of the Thames is 926 feet, and the sum of the water ways at *low water* only 236 feet; the amount of obstructions is 690 feet, about 7½ tenths of the entire section: so that a velocity of 3½ feet per second would give a fall of nearly 4¾ feet, agreeing with the actual result.

At Rochester Bridge, before the opening of the middle arches, the piers and starlings presented an obstruction of seven-tenths, and at the time of greatest fall, the velocity

* A similar table was computed by Mr. Wright of Durham, more than 50 years ago, and inserted in the first edition of Dr. Hutton's treatise on Bridges; but it is not constructed upon a correct theory.

100 yards above bridge exceeded 6 feet per second. This, from the table, would occasion a fall of more than 6·7 feet; and the recorded results vary from 6¾ to 7 feet.

At Westminster Bridge, where the obstructions are about one-sixth of the whole channel, when the velocity is $2\frac{1}{2}$ feet, the fall but little exceeds half an inch: a result which the table would lead us to expect.

SECTION III. *Contrivances to measure the velocity of running waters.*

1. For these purposes, various contrivances have been proposed, of which two or three may be here described.

Suppose it be the velocity of the water in a river that is required; or, indeed, both the velocity and the quantity which flows down it in a given time. Observe a place where the banks of the river are steep and nearly parallel, so as to make a kind of trough for the water to run through, and by taking the depth at various places in crossing make a true section of the river. Stretch a string at right angles over it, and at a small distance another parallel to the first. Then take an apple, an orange, or other small ball, just so much lighter than water as to swim in it, or a pint or quart bottle partly filled with water, and throw it into the water above the strings. Observe when it comes under the first string, by means of a quarter second pendulum, a stop watch, or any other proper instrument; and observe likewise when it arrives at the second string. By this means the velocity of the upper surface, which in practice may *frequently* be taken for that of the whole, will be obtained. And the section of the river at the second string must be ascertained by taking various depths, as before. If this section be the same as the former, it may be taken for the mean section: if not, add both together, and take half the sum for the mean section. Then the area of the mean section in square feet being multiplied by the distance between the strings in feet, will give the contents of the water in solid feet, which passed from one string to the other during the time of observation; and this by the rule of three may be adapted to any other portion of time. The operation may often be greatly abridged by taking notice of the arrival of the floating body opposite two stations on the shore, especially when it is not convenient to stretch a string across. An arch of a bridge is a good station for an experiment of this kind, because it affords a very regular section and two fixed points of observation: and in some instances the sea practice of heaving the log may be advantageous. Where a time-piece is not at hand, the observer may easily construct a quarter second or other pendulum, by means of the rules and table relating to pendulums, in the Dynamics.

2. M. Pitot invented a stream measurer of a simple construction, by means of which the velocity of any part of a stream may readily be found. This instrument is composed of two long tubes of glass open at both ends: one of these tubes is cylindrical throughout; the other has one of its extremities bent into nearly a right angle, and gradually enlarges like a funnel, or the mouth of a trumpet: these tubes are both fixed in grooves in a triangular prism of wood; so that their lower extremities are both on the same level, standing thus one by the side of the other, and tolerably well preserved from accidents. The frame in which these tubes stand is graduated, close by the side of them, into divisions of inches and lines.

To use this instrument, plunge it perpendicularly into the water, in such manner that the opening of the funnel at the bottom of one of the tubes shall be completely opposed to the direction of the current, and the water pass freely through the funnel up into the tube. Then observe to what height the water rises in each tube, and note the difference of the sides; for this difference will be the height due to the velocity of the stream. It is manifest, that the water in the cylindrical tube will be raised to the same height as the surface of the stream, by the hydrostatic pressure: while the water entering from the current by the funnel into the other tube, will be compelled to rise above that surface by a space at which it will be sustained by the impulse of the moving fluid: that is, the momentum of the stream will be in equilibrio with the column of water sustained in one tube above the surface of that in the other. In estimating the velocity by means of this instrument, we must have recourse to theory as corrected by experiments. Thus, if h, the height of the column sustained by the stream, or the difference of heights in the two tubes, be in feet, we shall have $v = 6{\cdot}5\ \sqrt{h}$, nearly, the velocity, per second, of the stream; if h be in inches, then $v = 22{\cdot}47\ \sqrt{h}$, nearly: or farther experiments made with the instrument itself may a little modify these coefficients.

It will be easy to put the funnel into the most rapid part of the stream, if it be moved about to different places until the difference of altitude in the two tubes becomes the greatest. In some cases it will happen, that the immersion of the instrument will produce a little eddy in the water, and thus disturb the accuracy of the observation; but keeping the instrument immersed only a few seconds will correct this. The wind would also affect the accuracy of the experiments; it is, therefore, advisable to make them where there is little or no wind. By means of this instrument a great number of curious and useful observations may easily be made: the velocity of water at various depths in a canal or river may be found with tolerable accuracy, and a mean of the whole drawn, or they may be applied to the correcting of the theory of waters running down gentle slopes. The observations may likewise be applied to ascertain whether the augmentations of the velocities are in proportion to the increase of water passing along the same canal, or what other relations subsist between them, &c.

Where great accuracy is not required, the tube, with the funnel at bottom, will alone be sufficient; as the surface of the water will be indicated with tolerable precision, by that part of the prismatic frame for the tube which has been moistened by the immersion: and the *velocities* may be marked against those altitudes in the tube which indicate them.

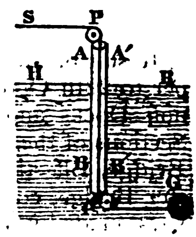
3. Another good and simple method of measuring the velocity of water in a canal, river, &c. is that described by the Abbé Mann, in his Treatise on Rivers; it is this:—Take a cylindrical piece of dry light wood, and of a length something less than the depth of the water in the river; about one end of it let there be suspended as many small weights as may keep the cylinder in a vertical or upright position, with its head just above water. To the centre of this end fix a small straight rod, precisely in the direction of the cylinder's axis: in order that, when the instrument is suspended in the water, the deviations of the rod from a perpendicularity to the surface of it, may indicate which end of the cylinder goes foremost, by which may be discovered the different velocities of the water at different depths; for when the rod inclines forward, according to the direction of the current, it is a proof that the surface of the water has the greatest velocity; but when it reclines backward, it shows that the swiftest current is at the bottom; and when it remains perpendicular, it is a sign that the velocities at the top and bottom are equal.

This instrument, being placed in the current of a river or canal, receives all the percussions of the water throughout the whole depth, and will have an equal velocity with that of the whole current from the surface to the bottom at the place where it is put in, and by that means may be found, both with exactness and ease, the mean velocity of that part of the river for any determinate distance and time.

But to obtain the mean velocity of the whole section of the river, the instrument must be put successively both in the middle and towards the sides, because the velocities at those places are often very different from each other. Having by this means found the several velocities, from the spaces run over in certain times, the arithmetical mean proportional of all these trials, which is found by dividing the common sum of them all by the number of the trials, will be the mean velocity of the river or canal. And if this medium velocity be multiplied by the area of the transverse section of the waters at any place, the product will be the quantity running through that place in a second of time.

The cylinder may be easily guided into that part which we want to measure by means of two threads or small cords, which two persons, one on each side of the canal or river, must hold and direct; taking care at the same time neither to retard nor accelerate the motion of the instrument.

4. Let A A′ B B′ be a hollow cylinder, open at both ends, and let it be capable of being fixed by the side of a platform or of a boat so that its lower extremity B B′ may be placed at any proposed depth below H R the upper surface of the stream. Let P, P′, be pulleys, fixed at opposite sides of the top and bottom of the tube. To G a globe of specific gravity nearly the same as that of water, let a cord G P′ P S be attached, passing freely over the pulleys P′, P, and having sufficient length towards S to allow of its running off to any convenient distance. Then, the bottom of the tube being immersed to any proposed depth, let the globe G, be exposed to the free operation of the stream; and as it is carried along with it, it will in 1, 2, 5, or 10 seconds, or any other interval of time, draw off from a fixed point, as S, a portion of cord; from which and the time elapsed, the velocity at the assigned depth will become known.



Mr. Saumarez invented, in 1720, an instrument called the *Marine Surveyor*, for the double purpose of measuring a ship's way, and ascertaining the velocities of streams. It is described in the *Phil. Trans.* vol. 33; and in the succeeding volume, a curious example of its use is given, in "tables showing the strength and gradual increase and decrease of the tides of flood and ebb in the River Thames, as observed in Lambeth

Reach." They are too extensive to be inserted here; but are truly interesting, and may be seen in *Phil. Trans. Abridged*, vol. vii. p. 133.

Section IV. *Effects of London Bridge on the Tides, &c.*

In the first volume of Dr. Hutton's Tracts, 8vo. there are inserted some curious papers drawn up by Mr. Robertson of Christ's Hospital and others, on London Bridge, and on the probable consequences, in reference to the tides, of erecting a new bridge across the Thames, viz. Blackfriars. Such documents are not only interesting as matters of scientific history, but become valuable in process of time; as the comparison of facts with theoretic predictions is subservient to the correction of the theory itself. With a similar object in view, I here introduce abridged accounts of some valuable facts with regard to the motion and level of the tides in the Thames at London, collected in 1820 and 1821, when the project of a new bridge of five arches, instead of the old bridge, originally of 20 arches and a very contracted water-way, was under consideration.

LONDON BRIDGE.

Result of the Levels of Tides observed from 23d September to 25th October 1820, between the Entrance of the London Docks and Westminster Bridge. Also the Transverse Sections of the River Thames at London Bridge, Southwark Bridge, Blackfriars Bridge, Waterloo Bridge, and Westminster Bridge, collected from four drawings of the above surveys, made by Mr. Francis Giles, under the direction of Mr. James Montague, pursuant to an order of the Select Committee of Bridge House Lands, of the 8th September, 1820.

London Docks to London Bridge.

The high water of spring tides at the entrance of the London Docks, averaged a level of 1.5 inch higher and 10 minutes earlier time, than at the lower side of London Bridge.

The low water of ditto at ditto averaged a level of 8 inches lower, and 9 minutes earlier time, than at ditto.

The high water of neap tides at ditto averaged a level of one level, and 8 minutes earlier time, than at ditto.

The low water of ditto at ditto averaged a level of 2 inches lower, and 14 minutes earlier time, than at ditto.

London Bridge.

High water of the highest spring tides occurs at three or four o'clock.—High water of the lowest neap tides occurs at eight or nine o'clock.

Spring tides flow four or five hours, and ebb seven to eight and a half hours.—Neap tides flow five to five and a half hours, and ebb six and a half to eight hours.

The high water of spring tides produced an average fall through London Bridge of 8 inches, but the greatest fall at high water was 1 foot 1 inch.—October 24th.

The low water of ditto, through ditto, of 4 feet 4 inches, but the greatest fall at low water was 5 feet 7 inches.—September 27th.

The high water of neap tides through ditto of 5 inches.

The low water of ditto, through ditto, 2 feet 1 inch, but the least fall at low water was 1 foot 1 inch.—October 16th.

The flood of spring tides of October 21st and 23d, produced slack water through the bridge in about 40 minutes after low water below bridge, from which time a head gradually increased (below bridge) to 1 foot 10 inches at half flood, and then regularly decreased to about 8 inches at high water.—The first flow of these tides, nevertheless, began above bridge about 20 minutes after low water time below bridge, although the water was then about two feet 6 inches higher above than below bridge; the time of low water below bridge averaged 10 minutes earlier than above bridge.

The ebb of these tides produced slack water at the bridge about 30 minutes after high water, and then gradually sunk to their greatest fall at low water.—The time of high water of October 21st and 23d, was the same below as above bridge; but the average time of high water spring tides is 9 minutes earlier below than above bridge.

The flood of neap tide, October 30th, produced slack water through the bridge, in about two hours after low water time below bridge, (when there was some land flood in the river) from which time a head gradually increased (below bridge) to 1 foot 3 inches at two-thirds flood, and then regularly decreased to 4 inches at high water.—The first flow of this tide, nevertheless, began above bridge about 1 hour after low water time below bridge, although the water was then 1 foot higher above than below bridge; but the average time of low water below bridge is 32 minutes earlier than above bridge.

The ebb of this tide produced slack water at the bridge about 15 minutes after high water above bridge, and then gradually sunk to its greatest fall at low water.—The time of high water of October 30th, was 15 minutes earlier below than above bridge, and the average time of high water neap tides is 15 minutes earlier below than above bridge.

London Bridge to Westminster Bridge.

The high water line from the upper side of London Bridge to Westminster Bridge is generally level, unless influenced by winds and land floods.

The time of high water is about 10 minutes earlier at London than Westminster Bridge.

	in.	pts.		min.	
The mean low water line has a fall of	4	0	from Westminster to Waterloo Bridge, time	7	later at Westminster than at Waterloo Bridge.
Ditto	4	3	from Waterloo to Blackfriars Bridge,	6	Ditto.
Ditto	3	2	from Blackfriars to Southwark Bridge,	5	Ditto.
Ditto	0	5	from Southwark to London Bridge,	4	Ditto.
Total foot 1	0	0		22	

Areas of the Transverse Sections in the River Thames at London Bridge.

	Under the Datum. ft. in.	Feet.	Abate for Water-works. Feet.	Feet.
At an extraordinary high water or level of 2 feet above the average spring tide high water mark, at the Hermitage entrance to the London Docks, as settled by the Corporation of the Trinity House, August, 1800		8,130	1,000	7,130
At the Trinity High Water mark, or Datum		7,360		
At an average Spring Tide High Water below Bridge, or Level of	0 6·5	7,122	850	6,272
At Ditto Spring Tide High Water above Bridge, or Level of	1 2·0	6,837	810	6,027
At Ditto Neap Tide High Water above Bridge, or Level of	4 3·0	5,293	600	4,693
At Ditto Spring and Neap Tide Low Water above Bridge, or Level of	14 5·0	1,488		1,488
At Ditto Neap Tide Low Water below Bridge, or Level of	16 5·0	1,030		1,030
At Ditto Spring Tide Low Water below Bridge, or Level of	18 9·0	540		540

At Southwark, Blackfriars, Waterloo, and Westminster Bridges.

	Southwark Bridge.	Blackfriars Bridge.	Waterloo Bridge.	Westminster Bridge.
At the above level of extraordinary high water	15,260	15,460	19,822	16,750
At the Trinity high water mark, or datum	13,940	14,117	17,707	15,198
At an average Spring tide high water, or level of 1 foot 2 inches under the ditto	13,170	12,975	16,447	14,015
At ditto Neap tide high water, or level of 4 feet 3 inches under the ditto	11,135	10,590	13,116	11,380
At ditto Spring and Neap tide low water	5,012	3,724	3,382	3,720

London, 12th March, 1821.—(Published in a letter addressed to G. H. Sumner, Esq. M. P. by a scientific Architect.)

Gradation of the Ebbing and Flowing of the Tide at London Bridge, taken above and below, on the 29th of July, 1821; being the day of the new moon.

ABOVE BRIDGE.

Low Water at Pepper Alley, 50 Minutes past Nine o'clock in the Morning.

FLOOD TIDE.

	Ft.	In.
Depth of water when Flood commenced	6	0
1st hour rise	2	11
2nd hour	3	0
3rd hour	2	10
4th hour	2	8
45 minutes	1	0
4 hours and 45 minutes	18	5

High Water at Pepper Alley, 35 Minutes past Two o'clock in the Afternoon.

EBB TIDE.

	Ft.	In.
1st hour fall	2	1
2nd hour	2	7
3rd hour	2	0
4th hour	1	9
5th hour	1	5
6th hour	1	2
7th hour	1	0
55 minutes	0	11
Depth at low water	5	6
7 hours and 55 minutes	18	5

BELOW BRIDGE.

Low Water at Coxes' Quay, 30 Minutes past Nine o'clock in the Morning.

FLOOD TIDE.

	Ft.	In.
Depth of water when Flood commenced	1	3
1st hour rise	5	9
2nd hour	5	4
3rd hour	2	9
4th hour	2	5
48 minutes	1	4
4 hours and 48 minutes	18	10

High Water at Coxes' Quay, 18 Minutes past Two o'clock in the Afternoon.

EBB TIDE.

	Ft.	In.
1st hour fall	2	1
2nd hour	4	4
3rd hour	3	1
4th hour	2	7
5th hour	2	3
6th hour	1	9
7th hour	1	6
59 minutes	0	11
Depth left	0	4
7 hours and 59 minutes	18	10

*** The object of this statement was to show, that the old bridge tended to *retain* the water above bridge and assist the navigation up the river.

Difference between the levels of High and Low Water Spring Tides, between Rotherhithe and Battersea in the year 1820.

	f.	i.
Rotherhithe, Old Horse Ferry	21	10
London Old Bridge	18	2
Blackfriars	14	9
Westminster	12	6
Vauxhall	12	2
Battersea	11	6

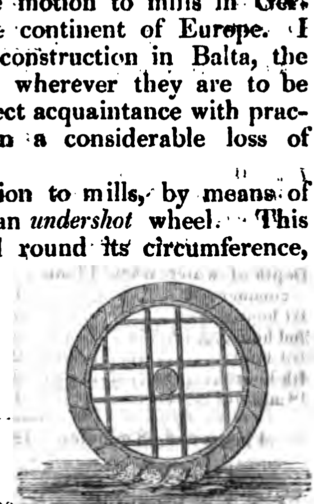
From Battersea Bridge to London Bridge, 5 miles; from London Bridge to Old Horse Ferry 1¼ miles. From London Bridge to the Nore, 44 miles.

SECTION V. *Watermills.*

1. The impulse of a current of water, and sometimes its weight and impulse jointly, are applied to give motion to mills for grinding corn and for various other purposes. Sometimes the impulse is applied obliquely to float-boards in a manner which may be comprehended at once by reference to a *smoke-jack*. In that, the smoke ascends, strikes the vanes obliquely, and communicates a rotatory motion. Imagine the whole mechanism to be inverted, and water to fall upon the vanes, rotation would evidently be produced; and that with greater or less energy in proportion to the quantity of water and the height from which it falls.

Water-wheels of this kind give motion to mills in Germany, and some other parts of the continent of Europe. I have, also, seen mills of the same construction in Balta, the northernmost Shetland Isle. But, wherever they are to be found, they indicate a very imperfect acquaintance with practical mechanics; as they occasion a considerable loss of power.

2. Water frequently gives motion to mills, by means of what is technically denominated an *undershot* wheel. This has a number of planes disposed round its circumference, nearly in the direction of its radii, these *floatboards* (as they are called) dipping into the stream, are carried round by it; as shown in the marginal diagram. The axle of the wheel, of course, by the intervention of proper wheels and pinions, turns the machinery intended to be moved. Where the stream is large



and unconfined, the pressure on each float-board is that corresponding to the head due to the relative velocities (or difference between the velocities of stream and float-board): this pressure is, therefore, a *maximum* when the wheel is at *rest;* but the *work performed* is then *nothing*. On the other hand, the *pressure* is nothing when the velocity of the wheel equals that of the stream. Consequently, there is a certain intermediate velocity, which causes the work performed to be a maximum.

The weight equal to the pressure is $Q(\sqrt{H} - \sqrt{h})^2$, Q being the quantity of water passing in a second, H the height due to V the velocity of the water, and h that due to U the velocity of the float-board. Considering this as a mass attached to the wheel, its moving force is obtained by multiplying it into U: And as $\sqrt{H} - \sqrt{h}$ varies as $V - U$, this moving force varies as $(V - U)^2 U$ which is a *max.* when $U = \frac{1}{3} V$. In this case, then, the rim of the wheel moves with $\frac{1}{3}$ of the velocity of the stream; and the effect which it produces is

$$Q \times (\tfrac{2}{3} V)^2 \times \tfrac{1}{3} V = \tfrac{4}{27} Q V^3:$$

So that the *undershot wheel*, according to the usual theory, performs work $= \frac{4}{27}$ of the moving force.

Friction, and the resistance of fluids, modify these results: but Smeaton and others have found that the maximum work is always obtained when U is between $\frac{1}{2}$ V and $\frac{1}{3}$ V.

3. Where the floats are not totally immersed, the water is heaped upon them; and in this case the pressure is that due to 2 H.

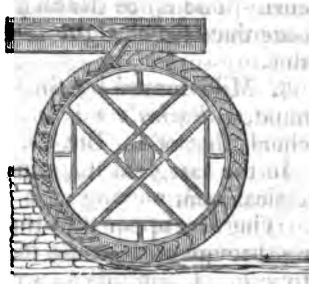
4. When the float-boards move in a circular sweep close fitted to them, or, in general, when the stream cannot escape without acquiring the same velocity as the wheel, the circumstances on which the investigation turns become analogous to what happens in the collision of non-elastic bodies. The stream has the velocity v before the stroke which is reduced to U, and the quantity of motion corresponding to the difference, or to $V - U$, is transferred to the wheel; this turns with the velocity U; and therefore the effect of the wheel is as

$$\left(\frac{V - U}{V}\right) U, \text{ or } \frac{VU - U^2}{V};$$ which is a maximum when $V = 2U$; being then $\frac{1}{4}$ of the moving power.

Hence *appears the utility of constraining the water to move in a narrow channel.*

5. The under-shot wheel is used where a large quantity of water can be obtained with a moderate fall. But where the

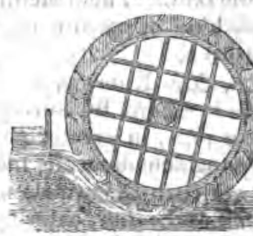
fall is considerable the *overshot* is almost always employed. Its circumference is formed into angular buckets, into which the water is delivered either at the top or within 60° of it: 52° $\frac{3}{4}$ is the most advantageous distance. In that case, if r = the full radius of the wheel, H the whole, and h the effective height of the fall, $h = r\,(1 + \sin 37°\frac{1}{4}) = 1{\cdot}605r$, and $r = {\cdot}623\,h$. If the friction be about $\frac{2}{3}$ of the moving power,



v, the velocity of the circumf. of the wheel to produce a maximum effect, will $= 2{\cdot}67\sqrt{H}$. Here, too, a fall of $\frac{1}{9}$ H will give the water its due velocity of impact upon the wheel: and $122{\cdot}176\, s\, H^{\frac{3}{2}}$ lbs = the mechanical effect, s being the section, in feet, of the stream that supplies the buckets.

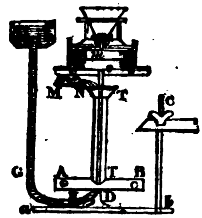
Mr. Smeaton's experiments led him to conclude that overshot wheels do most work when their circumferences move at the rate of 3 feet in a second, and that when they move considerably slower than this, they become unsteady and irregular in their motion. This determination is, however, to be understood with some latitude. He mentions a wheel 24 feet in diameter, that seemed to produce nearly its full effect though the circumference moved at the rate of 6 feet in a second; and another of the diameter of 33 feet, of which the circumference had only a velocity of 2 feet in a second, without any considerable loss of power. The first wheel turned round in 12^s. 6, the latter in 51^s. 9.

6. Where the fall is too small for an overshot wheel it is most adviseable to employ a *breast-wheel* (such as exhibited in the margin), which partakes of its properties: its floatboards meeting at an angle, so as to be assimilated to buckets, and the water being considerably confined within them by means of an arched channel fitting moderately close, but not so as to produce unnecessary friction. But when the circumstances do not admit of a breast-wheel, then recourse must be had to the undershot. For such a wheel it is best, that the floatboards be so placed as



to be perpendicular to the surface of the water at the time they rise out of it; that only one half of each should ever be below the surface, and that from 3 to 5 should be immersed at once. The Abbe Mann proposed that there should not be more than six or eight floatboards on the whole circumference.

7. Mills moved by the *re-action* of water are usually denominated *Barker's mills;* sometimes, however, *Parent's;* at others, *Segner's.* But the invention is doubtless Dr. Barker's.



In the marginal diagram, where C D is a vertical axis, moving on a pivot at D, and carrying the upper millstone *m*, after passing through an opening in the fixed millstone C. Upon this axis is fixed a vertical tube T T communicating with a horizontal tube A B, at the extremities of which A, B, are two apertures in opposite directions. When water from the mill-course M N is introduced into the tube T T, it flows out of the apertures A, B, and by the reaction or counter-pressure of the issuing water the arm A B, and consequently the whole machine, is put in motion. The bridge-tree *a b* is elevated or depressed by turning the nut *c* at the end of the lever *c b*.

In order to understand how this motion is produced, let us suppose both the apertures shut, and the tube T T filled with water up to T. The apertures A B which are shut up, will be pressed outwards by a force equal to the weight of a column of water whose height is T T, and whose area is the area of the apertures. Every part of the tube A B sustains a similar pressure; but as these pressures are balanced by equal and opposite pressures, the arm A B is at rest. By opening the aperture at A, however, the pressure at that place is removed, and consequently the arm is carried round by a pressure equal to that of a column T T, acting upon an area equal to that of the aperture A. The same thing happens on the arm T B; and these two pressures drive the arm A B round in the same direction. This machine may evidently be applied to drive any kind of machinery, by fixing a wheel upon the vertical axis C D.

8. Mr. Rumsey, an American, and Mr. Segner, improved this machine, by conveying the water from the reservoir, not by a pipe, in greater part of which the spindle turns, but by a pipe which descends from a reservoir as F, until it reaches lower than the arms A B, and then turns up by a curvilinear neck and collar, entering between the arms at the lower part, as shown in the figure. This greatly diminishes the friction.

9. Professor Playfair has correctly remarked that the

moving force becomes greater, after the machine has begun to move; for the water in the horizontal arms acquires a centrifugal force, by which its pressure against the sides is increased. When the machine works to the greatest advantage, the centre of the perforations should move with the velocity $\frac{r}{r'}\sqrt{h\,g}$, where r is the radius of the horizontal arm, measured from the axis of motion to the centre of the perforation, and r' the radius of the perpendicular tube, g being put for the force of gravity, or $32\frac{1}{6}$ feet.

As $2\,\pi\,r$ is the circumference described by the centre of each perforation, $\frac{2\,\pi\,r'}{\sqrt{h\,g}}$ is the time of a revolution in seconds.

The quantity $\frac{r}{r'}\sqrt{h\,g}$ is also the velocity of the effluent water; therefore, when the machine is working to the greatest advantage, the velocity with which water issues is equal to that with which it is carried horizontally in an opposite direction; so that, on coming out, it falls perpendicularly down.

10. The following dimensions have been successfully adopted: viz. radius of arms from the centre of pivot to the centre of the discharging holes, 46 inches; inside diameter of the arms, 3 inches; diameter of the supplying pipe, 2 inches; height of the working head of water 21 feet above the points of discharge. When the machine was not loaded, and had but one orifice open, it made 115 turns in a minute. This agrees to a velocity of 46 feet in a second, for the orifice, greater than the full velocity due to the head of water by between 9 and 10 feet: the difference is due to the effect of the centrifugal force.

The theory of this machine is yet imperfect; but there can be no doubt of its utility in cases where the stream is small with a considerable fall.

CHAP. XIII.

PNEUMATICS.

SECTION I. *Equilibrium of air and elastic fluids.*

1. THE fundamental propositions that belong to hydrostatics are common to the compressible and the incompressible fluids: and need not, therefore, be repeated here.

2. *Atmospheric air* is the best known of the elastic fluids, and has been defined an elastic fluid, having weight, and resisting compression with forces that are directly as its density, or inversely as the spaces within which the same quantity of it is contained.

The correctness of this definition is confirmed by experiment.

The weight of air is known from the Torricellian experiment, or that of the barometer. The air presses on the orifice of the inverted tube with a force just equal to the weight of the column of mercury sustained in it.

A bottle, weighed when filled with air, is found heavier than after the air is extracted. The pressure of the atmosphere is at a mean about 14lbs. on every square inch of the earth's surface. Hence the total pressure on the convex surface of the earth = 10,686,000,000 hundreds of millions of pounds.

The elastic force of the air is proved, by simply inverting a vessel full of air in water: the resistance it offers to farther immersion, and the height to which the water ascends within it, in proportion as it is farther immersed, are proofs of the elasticity of the air contained in it.

When air is confined in a bent tube, and loaded with different weights of mercury, the spaces it is compressed into are found to be inversely as those weights. But those weights are the measures of the elasticity; therefore the elasticities are inversely as the spaces which the air occupies.

The densities are also inversely as those spaces; therefore the elasticity of air is directly as its density. This law was first proved by Mariotte's experiments.

In all this, the temperature is supposed to remain unchanged.—These properties seem to be common to all elastic fluids.

Air resists compression equally in all directions. No limit can be assigned to the space which a given quantity of air would occupy if all compression were removed.

2. In ascending from the surface of the earth, the density of the air necessarily diminishes: for each stratum of air is compressed only by the weight of those above it; the upper strata are therefore less compressed, and of course less dense than those below them.

3. Supposing the same temperature to be diffused through the atmosphere, if the heights from the surface be taken increasing in arithmetical progression, the densities of the strata of air will decrease in geometrical progression. Also, since the densities are as the compressing forces, that is, as the columns of mercury in the barometer, the heights from the surface being taken in arithmetical progression, the columns of mercury in the barometer at those heights will decrease in geometrical progression.

As logarithms have, relatively to the numbers which they represent, the same property, therefore if b be the column of mercury in the barometer at the surface, and β at any height h above the surface, taking m for a constant coefficient, to be determined by experiment, $h = m(\log b - \log \beta)$, or $h = m \log \frac{b}{\beta}$: where m may be determined by finding trigonometrically the value of h in any case, where b and β have been already ascertained.

4. If b be the height of the mercury in the barometer at the lowest station, β at the highest, t and t' the temperatures of the air at those stations, f the fixed temperature at which no correction is required for the temperature of the air; and if q and q' be the temperatures of the quicksilver in the two barometers, and n the expansion of a column of quicksilver, of which the length is 1, for 1° of heat; h being the perpendicular height (in fathoms) of the one station above the other,

$$h = 10000\left(1 + \cdot 00244\left(\frac{t + t'}{2} - f\right)\right) \log. \frac{b}{\beta \times (1 + n(q - q'))},$$

n being nearly $= \frac{1}{10000}$.

If the centigrade thermometer is used, because the beginning of the scale agrees with the temperature f, so that $f = 0$, the formula is more simple; and if the expansion for air and mercury be both adapted to the degrees of this scale.

$$h = 10000\,(1 + \cdot 00441\left(\frac{t + t'}{2}\right)\log.\frac{b}{\beta(1 + \cdot 00018\,(q - q')}.$$

5. The temperature of the air diminishes on ascending into the atmosphere, both on account of the greater distance from the earth, the principal source of its heat, and the greater power of absorbing heat that air acquires, by being less compressed.

6. Professor Leslie, in the notes on his *Elements of Geometry*, p. 495. edit. 2d, has given a formula for determining the temperature of any stratum of air when the height of the mercury in the barometer is given. The column of mercury at the lower of two stations being b, and at the upper β, the diminution of heat, in degrees of the centigrade, is $\left(\frac{b}{\beta} - \frac{\beta}{b}\right)25.$— This seems to agree well with observation.

7. If the atmosphere were reduced to a body of the same density which it has at the surface of the earth, and of the same temperature, the height to which it would extend, is in fathoms, equal to $4343\,(1 + \cdot 00441\,\frac{t + t'}{2})$, or, taking the expansion according to Laplace $= 4343\,(1 + \frac{4 \times t}{1000})$.

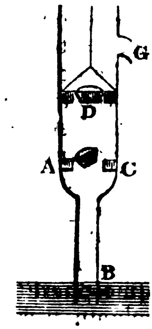
Hence if b be the height of the mercury in the barometer, reduced to the temperature t, the specific gravity of mercury is to that of air, as b to $4343\,(1 + \frac{4\,t}{1000})$, or the specific gravity of air $= \frac{b}{72\,(4343)\,(1 + \frac{4\,t}{1000})}$.

The divisor 72, is introduced in consequence of b being expressed in inches.—(*Playfair's Outlines.*)

Section II. *Pumps.*

1. *Def.* The term *Pump* is generally applied to a machine for raising water by means of the air's pressure.

2. The *common suction-pump* consists of two hollow cylin-



ders, which have the same axis, and are joined in A C. The lower is partly immersed, perpendicularly in a spring or reservoir, and is called the *suction-tube;* the upper the *body of the pump*. At A C is a fixed sucker containing a valve which opens upwards, and is less than 34 feet from the surface of the water. In the body of the pump is a piston D made air-tight, moveable by a rod and handle, and containing a valve opening upwards. And a spout G is placed at a distance greater or less, as convenience may require, above the greatest elevation of D.

3. To explain the action of this pump.

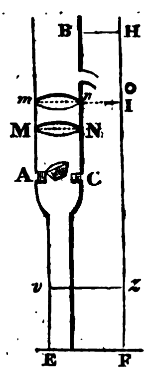
Suppose the moveable piston D at its lowest depression, the cylinders free from water, and the air in its natural state. On raising this piston, the pressure of the air above it keeping its valve closed, the air in the lower cylinder A B forces open the valve at A C, and occupies a larger space, viz., between B the surface of the water and D; its elastic force therefore being diminished, and no longer able to sustain the pressure of the external air, this latter forces up a portion of the water into the cylinder A B to restore the equilibrium. This continues till the piston has reached its greatest elevation, when the valve at A C closes. In its subsequent descent, the air below D becoming condensed, keeps the valve at A C closed, and escapes by forcing open that at D, till the piston has reached its greatest depression. In the following turns a similar effect is produced, till at length the water rising in the cylinder forces open the valve at A C, and enters the body of the pump; when, by the descent of D, the valve in A C is kept closed, and the water rises through that in D, which on re-ascending carries it forward, and throws it out at the spout G.

4. *Cor.* 1. The greatest height to which the water can be raised in the common pump by a single sucker is when the column is in equilibrio with the weight of the atmosphere, that is, between 32 and 36 feet.

5. *Cor.* 2. The quantity of water discharged in a given time is determined by considering that at each stroke of the piston a quantity is discharged equal to a cylinder whose base is a section of the pump, and altitude the play of the piston.

6. To determine the force necessary to overcome the resist-

ance experienced by the piston in ascending. Let h = the height H F of the surface of the water in the body of the pump above E F the level of the reservoir; and a^2 = the area of the section M N. Let h' = the height of the column of water equivalent to the pressure of the atmosphere; and suppose the piston in ascending to arrive at any position $m\,n$ which corresponds to the height I F. It is evident that the piston is acted upon downwards by the pressure of the atmosphere $= a^2 h'$, and by the pressure of the column B m $= a^2 \times$ H I; therefore the whole tendency of the piston to descend $= a^2 (h' + \text{H I})$.



But the piston is acted upon upwards by the pressure of the air on the external surface E F of the reservoir $= a^2 h'$; part of which is destroyed by the weight of the column of water having for its base $m\,n$, and height F I;

$$\therefore \text{ the whole action upwards } = a^2 \times (h' - \text{F I});$$

$$\text{whence F} = a^2 . (h' + \text{H I}) - a^2 . (h' - \text{F I})$$

$$= a^2 . \text{F H} = a^2 h,$$

that is, the piston throughout its ascent *is opposed by a force equal to the weight of a column of water having the same base as the piston, and an altitude equal to that of the surface of the water in the body of the pump above that in the reservoir.* In order therefore to produce the upward motion of the piston, a force must be employed equal to that determined above, together with the weight of the piston and rod, and the resistance which the piston may experience in consequence of the friction against the inner surface of the tube.*

When the piston begins to descend, it will descend by its own weight; the only resistance it meets with being friction, and a slight impact against the water.

7. *Cor.* 1. If the water has not reached the piston, let its

* Suppose the body of the pump to be 6 inches in diameter, and the greatest height to which the water is raised to be 30 feet: suppose, also, the weight of the piston and its rod to be 10lbs, and the friction one-fifth of the whole weight. Then, by the rule at pa. 190, $\frac{1}{10}$ of the square of the diameter gives the ale gallons in a yard in length of the cylinder, and an ale gallon, pa. 283, weighs $10\frac{1}{5}$ lbs. Therefore $(6^2 \times 10) + \frac{1}{50}(6^2 \times 10) = 360 + 7{\cdot}4 = 367{\cdot}4$ lbs. weight of the opposing column of water. And $367{\cdot}4 + 10 + \frac{1}{5}(377{\cdot}4) = 452{\cdot}9$ lbs. whole opposing pressure.

If the piston rod be moved by a lever whose arms are as 10 to 1, this pressure will be balanced by a force of 45·29 lbs, and overcome by any greater force.

level be in $v\,x$. The under surface of the piston will be pressed by the internal rarefied air. But this air, together with the column of water E v, is in equilibrio with the pressure of the atmosphere $a^2\,h'$; and $\therefore$ its pressure $= a^2 \,.\, (h' - \text{E}\,v)$. And the pressure downwards $= a^2\,h'$;

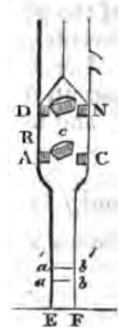
$$\therefore \text{F} = a^2 \times \text{E}\,v.$$

Hence the force requisite to keep the piston in equilibrio increases as the water rises, and becomes constant and $= a^2\,h$ as soon as the water reaches the constant level B H.

8. *Cor.* 2. If the weight of the piston be taken into the account, let this weight be equal to that of a column of water whose base is $m\,n$ and height p, $= a^2\,p$;

$$\therefore \text{F} = a^2 \,.\, (\text{E}\,v + p).$$

9. To determine the height to which the water will rise after one motion of the piston; the fixed sucker being placed at the junction of the suction-tube and body of the pump: supposing that after every elevation of the piston there is an equilibrium between the pressure of the atmosphere on the surface of the water in the reservoir, and the elastic force of the rarefied air between the piston and surface of the column of water in the tube, together with the weight of that column.



Let $a\,b$ be the surface of the water in the suction-tube, after the first stroke of the piston: if the piston were for an instant stationary at D, the pressure of the atmosphere would balance E b, and the elastic force of the air in N a.

Let A E the height of the suction-tube $= a$,

D R the play of the piston $= b$,

$h =$ the height of a column of water equivalent to the pressure of the atmosphere,

$y =$ the height of a column equivalent to the pressure of the air in N a,

$x =$ E a,

and R and $r =$ the radii of the body and the suction-tube.

Then $x + y = h$,

$$\text{and } y = h \,.\, \frac{\text{A F}}{\text{N}\,a} = h \,.\, \frac{\pi\,r^2\,a}{\pi\,\text{R}^2\,b + \pi\,r^2 \,.\, (a - x)}$$

$$= \frac{h\,r^2\,a}{\text{R}^2\,b + r^2\,a - r^2\,x};$$

$$\text{whence } h = x + \frac{h\,r^2\,a}{\text{R}^2\,b + r^2\,a - r^2\,x};$$

$$\therefore h\,\text{R}^2 b + h r^2 a - h r^2 x = \text{R}^2 b x + r^2 a x - r^2 x^2 + h r^2 a,$$

$$\text{or } x^2 - (h + \frac{\text{R}^2}{r^2} . b + a) . x = - h . \frac{\text{R}^2}{r^2} . b,$$

$$\text{and } x^2 - p\,x = - h\,m\,b,$$

$$\text{if } m \text{ be } = \frac{\text{R}^2}{r^2}, \text{ and } p = h + m\,b + a;$$

$$\therefore x = \tfrac{1}{2} . \{p \pm \sqrt{p^2 - 4\,h\,m\,b}\},$$

$$\text{and } y = \tfrac{1}{2} . \{2\,h - p \mp \sqrt{p^2 - 4\,h\,m\,b}\},$$

only one of which values will be applicable, viz. that which answers to the lower sign; since x and y must be less than h; and if the upper sign be used, x will be found greater than h.

10. Having given the height of the water raised, and that due to the pressure of the air in the pump after the first ascent of the piston; to determine them for the second, third, &c. ascents.

Let E a' represent the height of the water after the second ascent, and let it $= x_1$,

and let y_1 = the height due to the elastic force of the air;

$$\text{then } x_1 + y_1 = h;$$

and $y_1 = y . \frac{\text{A}\,b}{\text{N}\,a'}$, since the air which occupied C a now occupies N a';

$$\therefore y_1 = \frac{y\,r^2 . (a - x)}{\text{R}^2 b + r^2 . (a - x_1)} = \frac{y . (a - x)}{m\,b + a - x_1};$$

$$\text{whence } h = x_1 + \frac{y . (a - x)}{m\,b + a - x_1},$$

$$\text{and } \therefore x_1 = \tfrac{1}{2} . \{p - \sqrt{p^2 - 4\,h\,m\,b - 4\,x . (h + a - x)}\},$$

$$\text{and } y_1 = \tfrac{1}{2} . \{2\,h - p + \sqrt{p^2 - 4\,h\,m\,b - 4\,x . (h + a - x)}\}.$$

From these are deduced values of x_2, y_2, x_3, y_3, &c.

$$x_2 = \tfrac{1}{2} . \{p - \sqrt{p^2 - 4\,h\,m\,b - 4\,x_1 . (h + a - x_1)}\},$$

$$y_2 = \tfrac{1}{2} . \{2\,h - p + \sqrt{p^2 - 4\,h\,m\,b - 4\,x_1 . (h + a - x_1)}\},$$

and so on. Whence if x_n be taken to represent the height of the water after $(n + 1)$ ascents,

$$x_n = \tfrac{1}{2} . \{p - \sqrt{p^2 - 4\,h\,m\,b - 4\,x_{n-1} . (h + a - x_{n-1})}\},$$

$$\text{and } y_n = \tfrac{1}{2} . \{2\,h - p + \sqrt{p^2 - 4\,h\,m\,b - 4\,x_{n-1} . (h + a - x_{n-1})}\}.$$

11. *Cor.* 1. Hence may be determined the height to which the water can rise after any given number of ascents of the piston, and the elastic force of the air in the suction-tube.

12. *Cor.* 2. Knowing the elevation due to each particular stroke, the differences of those elevations, and the successive differences in the elastic force of the remaining air, may be known.

13. *Cor.* 3. If the weight of the valve c be not considered, it is evident that after a certain number of strokes a vacuum will be produced in the suction-tube, provided it be equal to, or not greater than the height due to the pressure of the atmosphere, that is, if a be not greater than h.

For in this case, $x_n = x_{n-1}$,

and $\therefore x_{n-1} = \frac{1}{2}\{p - \sqrt{p^2 - 4hmb - 4x_{n-1}.(h + a - x_{n-1})}\}$,

whence $x_{n-1} = h$, the greatest height of the column of water in the tube. If therefore the length of the suction-tube do not exceed the height due to the pressure of the atmosphere, the water will continue to ascend in it after every stroke of the piston, till at length it will pass into the body of the pump.

But if the altitude of A F be greater than h, the water will continue to ascend without ever reaching its maximum height. For in this case, an actual vacuum cannot be produced; and as $x_n + y_n = h$, and y_n can never become $= 0$; $\therefore x_n$ can never $= h$*. But the successive values of y continually decreasing, the corresponding values of x will continually increase.

14. *Cor.* 4. If the weight of the valve c be taken into the account, a column of water must be added equal to the additional pressure to be overcome. Let $l =$ the height of this column, then

$$x + y + l = h;$$
$$\text{and } \therefore x + y = h - l = h'.$$

If therefore this value of h' be substituted for h, the preceding equations are applicable.

15. *Cor.* 5. In the preceding cases, the moveable piston has been supposed to descend to A C. If it does not, it may happen that the water may not reach A C, though A C be less than 34 feet from the surface of the water in the reservoir.

After the first elevation of the moveable piston to its greatest altitude, c being closed, the elastic force of the air between D M

* Hence it appears, that it is not *strictly* true, that water will ascend in the suction-tube to a height equal that of a column equivalent to the pressure of the atmosphere. This is a limit to which it approximates, but does not reach in a finite time.

and A C is $(h - x)$, and its magnitude πb R^2. If in descending, the piston describes a space b' less than b, so as to stop at a distance $b - b'$ from A C, this magnitude becomes $(b - b')$. π R^2; $\therefore$ the elastic force is $(h - x) \cdot \frac{b}{b - b'}$. Now in order that the pressure upwards may open the valve, this must exceed the elastic force of the atmosphere;

$$\therefore (h - x) \cdot \frac{b}{b - b'} > h,$$

$$\text{or } (h - x) \cdot b > h \cdot (b - b');$$

$$\therefore b x < h b', \text{ or } \frac{x}{h} < \frac{b'}{b}.$$

If $\therefore \frac{b'}{b}$ be less than $\frac{x}{h}$, the valve D N will not open; there will therefore be the same quantity of air between A C and the sucker: which, when the piston has reached its highest elevation, will have the same elastic force as that between A C and a' b'; and therefore c being equally pressed on both sides, will remain unmoved, and the water will not ascend.

16. If the fixed sucker be placed at the surface of the water; to determine the ascent of the water in the suction-tube.

Let E a, E a' be the successive heights to which the water rises; then after the first ascent of the piston,

$$x + y = h,$$

$$\text{and } y = \frac{h a}{m b + a - x};$$

$$\text{whence } x = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b}\}$$

$$\text{and } y = \tfrac{1}{2} \cdot \{2 h - p + \sqrt{p^2 - 4 h m b}\},$$

which equations are the same as were determined for the first ascent of the piston (9). Therefore, in the same manner as before,

$$\text{we shall have } x_n = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b - 4 h x_{n-1}}\},$$

$$y_n = \tfrac{1}{2} \cdot \{2 h - p + \sqrt{p^2 - 4 h m b - 4 h x_{n-1}}\}.$$

17. *Cor.* 1. If the water be supposed to stop after $(n + 1)$ ascents of the piston, then $x_n = x_{n-1}$;

$$\text{and } \therefore x_{n-1} = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b - 4 h x_{n-1}}\},$$

$$\text{whence } x_{n-1} = \tfrac{1}{2} \cdot \{a + m b \pm \sqrt{(a + m b)^2 - 4 h m b}\}.$$

Hence, therefore, there are two altitudes at which the water may stop in its ascent, if $(a + m b)^2$ is equal to or greater than $4 h m b$. In the former case the two values of x_{n-1} are equal, that is, there will be only one altitude $= \frac{1}{2} . (a + m b)$, at which the water will stop. In the latter case there are two which may be ascertained.

If $4 h m b$ be greater than $(a + m b)^2$, the water will not stop.

Ex. 1. If $h = 32$ feet, $a = 20$, $b = 4$, and $m = 1$, or the suction-tube and body of the pump be of the same diameter,

$$x_{n-1} = \tfrac{1}{2} . \{20 + 4 \pm \sqrt{(24)^2 - 4 . 1 . 32 . 4}\} = \tfrac{1}{2} . \{24 \pm \sqrt{64}\}$$
$$= 16 \text{ or } 8.$$

Ex. 2. If $h = 32$ feet, $a = 25$, $b = 2$, and $m = 4$,

$x_{n-1} = \frac{1}{2} . \{25 + 8 \pm \sqrt{(33)^2 - 4 . 32 . 4 . 2}\} = \frac{1}{2} . \{33 \pm \sqrt{65}\} = 41{\cdot}8062$ or $24{\cdot}1938$.

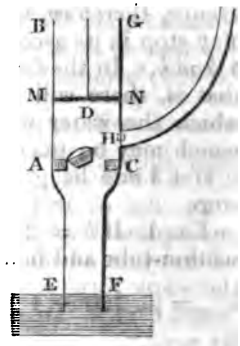
18. *Cor.* 2. If $m = 1$, or the tubes have the same diameter,

$$x_{n-1} = \tfrac{1}{2} . \{a + b \pm \sqrt{(a + b)^2 - 4 h b}\},$$

which is imaginary, if $(a + b)^2$ is less than $4 h b$, or b greater than $\frac{(a + b)^2}{4 h}$.

In order, therefore, that this pump may produce its effect, *the play of the piston must be greater than the square of its greatest altitude above the surface of the water in the reservoir divided by four times the height due to the pressure of the atmosphere.*

19. The *lifting-pump* consists of a hollow cylinder, the body of which is immersed in the reservoir. It is furnished with a moveable piston, which, entering below, lifts the water up, and is moveable by means of a frame which is made to ascend and descend by a handle. The piston is furnished with a valve opening upwards. A little below the surface of the water is a fixed sucker with a valve opening upwards. This is an inconvenient construction, upon the peculiarities of which we need not dwell.



20. The *forcing-pump* consists of a suction-tube A E F C, partly immersed in the reservoir, and of the body of the pump A B G C, and of the ascending tube H M. The body is furnished with a moveable solid sucker or plunger D, made air-tight. And at A C and H are fixed suckers with valves opening upwards.

21. To explain the action of this pump.

Suppose the plunger D at its greatest depression; the valves closed, and the air in its natural state. Upon the ascent of D, the air in A C D occupying a greater space, its elasticity will be diminished, and consequently the greater elasticity of the air in A F will open the valve at A C, whilst that at H is kept closed by the elasticity of the external air; water therefore will rise in the suction-tube. On the descent of D from its greatest elevation, the elasticity of the air in the body of the pump will keep the valve A C closed, and open that at H, whence air will escape. By subsequent ascents of the piston, the air will be expelled, and water rise into the body. The descending piston will then press the water through the valve at H, which will close, and prevent its return into the body of the pump. D therefore ascending again, the space left void will be filled by water pressing through the valve A C: and this upon the next ascent of D is forced into the ascending tube; and thus by the ascents and descents of D, water may be raised to the required height.

22. *Cor.* In this pump D must not ascend higher than about 32 feet from the surface of the water in the reservoir.

23. To determine the force necessary to overcome the resistance experienced by the piston.

Let h = the height of a column of water equivalent to the pressure of the atmosphere, and E B the height to which the water is forced. Let M N be any position of the piston D whose area = A, and the weight of the piston and its appendages = P. Let X = the force necessary to push the piston upwards during the suction, friction not being considered, and Y = that employed to force it down.

When the piston ascends, and H is closed

$$\text{X} = \text{P} + \text{A}h - \text{A}\,.\,(h - \text{ME})$$
$$= \text{P} + \text{A}\,.\,\text{ME}.$$

Let the sucker be in the same position in its descent, and therefore A C closed, and H open,

$$Y = Ak + A \cdot MB - (Ah + P)$$
$$= A \cdot MB - P.$$

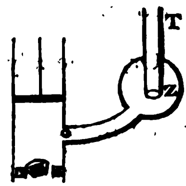
Hence $X + Y = A \cdot EB$; or the whole force exerted, in the case of equilibrium is equal to the weight of a column of water whose base is equal to that of the piston, and altitude the distance between the surface of the water and the point to which it is to be raised.

24. *Cor.* 1. In this pump the effort is divided into two parts, one opposed to the suction, and the other to the forcing; whereby an advantage is gained over the other pumps where the whole force is exerted at once whilst the water is raised.

25. *Cor.* 2. In order to have the force applied uniform, let $X = Y$;

$$\therefore P + A \cdot ME = A \cdot MB - P;$$
$$\therefore P = \tfrac{1}{2} A \cdot (MB - ME).$$

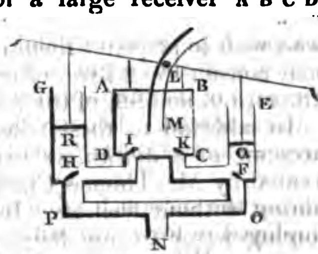
The piston therefore must play in such a manner that MB may be greater than ME.



26. *Cor.* 3. In the common forcing pump, the stream is intermitting; for there is no force impelling it during the return of the sucker.

One mode of remedying this, is by making an interruption in the ascending tube, which is surrounded by an air-vessel T; in which, when the water has risen above Z, the air above it is compressed, and by its elasticity forces the water up through Z; the orifice of which is narrower than that of the tube, and therefore the quantity of water introduced during the descent of the piston will supply its discharge for the whole time of the stroke, producing a continued stream.

27. The *fire-engine* consists of a large receiver A B C D, called the air-vessel, into which water is driven by two forcing-pumps E F, G H, (whose pistons are Q and R), communicating with its lower extremities at I and K, through two valves opening inwards. From the receiver proceeds a tube M L through which the water is thrown, and directed to any point by means of a pipe moveable about the extremity L. The pumps are worked by a lever, so that whilst one piston descends the other ascends. The pumps communicate with a reservoir of water at N.



28. To explain the action of this engine.

The tube N being immersed in the reservoir, and the piston

R drawn up, the pump Q N becomes filled; and the descent of the piston R will, as in the forcing pump (21), keep the valve H close, and cause the water to pass into the air-vessel by the valve I, whilst by the weight of the water in the air-vessel, the valve K will be kept shut. In the same manner, when R ascends, Q descending will force the water through K into the air-vessel. By this means the air above the surface of the water becoming greatly compressed, will by its elasticity force the water to ascend through M L, and to issue with a great velocity from the pipe.*

29. When the air vessel is half full of water, the air being then compressed into half its natural space, will have an elastic force equal to twice the pressure of the atmosphere: therefore, when the stop-cock is turned, the air within pressing on the subjacent water with twice the force of the external air, will cause the water to spout from the engine to the height of $(2 - 1)\ 33$, or 33 feet; except so far as it is diminished by friction.

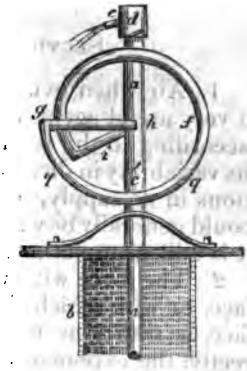
Or, generally, if $\frac{n-1}{n}$ denote the fractional height of the water in the air barrel, then $\frac{1}{n}$ will denote the height of the space occupied by the compressed air, n times the pressure of the atmosphere its elastic force, and $(n - 1)\ 33$, the height in feet to which the water may be projected.

Thus, if $\frac{3}{4}$ of the air barrel be the height of the water, the elastic force of the air will be 4 times the pressure of the atmosphere, and $(4 - 1)\ 33 = 99$ feet, the height to which the water may then be thrown by the engine.

30. The modifications in the constructions of pumps with a view to their practical applications are very numerous. Those who wish to acquaint themselves with some of the most useful, may consult the 2d vol. of my *Treatise of Mechanics*, and Nos. 13, 41, 69, and 93, of the *Mechanics' Magazine*.

In addition to these, there may now be presented a short account of a *quicksilver pump*, which has been recently invented by Mr. Thomas Clark, of Edinburgh, and which works almost without friction. It has great power in drawing and forcing water to any height, and is extremely simple in its construction.

* The preceding part of this section is taken from *Bland's Hydrostatics*; a very elegant and valuable work, which I beg most cordially to recommend to those who wish to obtain a comprehensive knowledge of the theory of this department of science.



a a is the main pipe inserted into the well *b*; a valve is situated at *c*, and another at *d*, both opening upwards; a piece of iron tube is then bent into a circular form, as at *f*, again turned off at *g* in an angular direction, so as to pass through a stuffing box at *h*, and from thence bent outwards as at *i*, connecting itself with the ring A quantity of quicksilver is then put into the ring filling it from *q* to *q*, and the ring being made to vibrate upon its axis *h*, a vacuum is soon effected in the main pipe by the recession of the mercury from *g* to *q*, thereby causing the water to rise and fill the vacuum: upon the motion being reversed, the quicksilver slides back to *g*, forces up the water and expels it at the spout *e*.

"Mr. Clark calculates that a pump of this description with a ring twelve feet in diameter, will raise water the same height as the common lifting pump, and force it one hundred and fifty feet higher without any friction." (*Mechanics' Register*, and *Jamieson's Edinburgh Journal.*)

31. It is usual to class with pumps, the machine known by the name of *Archimedes's screw*, or the *water-snail*. This consists either of a pipe wound spirally round a cylinder, or of one or more spiral excavations formed by means of spiral projections from an internal cylinder, covered by an external cylindrical case, so as to be water tight. The cylinder which carries the spiral is placed aslant, so as to be inclined to the horizon in an angle of from 30° to 45°, and capable of turning upon pivots in the direction of its axis posited at each extremity. The lower end of the spiral canal being immersed in the river or reservoir from which water is to be raised, the water descends at first in the said canal solely by its gravity; but the cylinder being turned, by human or other energy, the water moves on in the canal, and at length it issues at the upper extremity of the tube.

Several circumstances tend to make this instrument imperfect and inefficacious in its operation. The adjustments necessary to ensure a maximum of effectual work, are often difficult to accomplish. It seldom happens, therefore, that the measure of the work done exceeds *a third* of the power employed: so that this apparatus, notwithstanding its apparent ingenuity and simplicity is very sparingly introduced by our civil engineers.

SECTION III. *Wind and Windmills.*

1. Air when in continuous motion in one direction, becomes a very useful agent of machinery, of greater or less energy, according to the velocity with which it moves. Were it not for its variability in direction and force, and the consequent fluctuations in its supply, scarcely any more appropriate first mover could generally be wished for. And even with all its irregularity, it is still so useful as to require a separate consideration.

2. The force with which air strikes against a moving surface, or with which the wind strikes against a quiescent surface, is nearly as the square of the velocity; or, more correctly, the exponent of the velocity, determined according to the rule given at pa. 101, varies between 2·08 and 2·05; so that in most practical cases, the exponent 2, or that of the square, may be employed without fear of error. If I be the angle of incidence, s^2 the surface struck in feet, and v the velocity of the wind, in feet, per second; then for the force in avoirdupois pounds, either of the two following approximations may be used; viz. $f = \frac{v^2 s^2 \sin^2 I}{440}$

$$\text{or } f = \cdot 002288\, v^2 s^2 \sin^2 I.$$

Of these, the first is usually the easiest in operation, requiring only two lines of short division, viz. by 40 and by 11.

If the incidence be perpendicular, $\sin^2 I = 1$, and these become,

$$f = \frac{v^2 s^2}{440} = .002288\ v^2\ s^2.$$

3. The table in the margin exhibits the force of the wind when blowing perpendicularly upon a surface of one foot square, at the several velocities announced. The velocity of 80 miles per hour is that by which the aeronaut Garnerin was carried in his balloon from Ranelagh to Colchester, in June 1802. It was a strong and boisterous wind; but did not assume the character of a *hurricane*, although a wind with that velocity is so characterised in Rouse's table. In Mr. Green's aerial voyage from Leeds, in September 1823, he travelled 43 miles in 18 minutes, although his balloon rose to the height of more than 4000 yards.

Velocity of the Wind.		Perpendicular force on one sq. foot, in avoirdupois pounds.
Miles in one hour.	= feet in one second.	
1	1·47	·005
2	2·93	·020
3	4·40	·044
4	5·87	·079
5	7·33	·123
10	14·67	·492
15	22·00	1·107
20	29·34	1·968
25	36·67	3·075
30	44·01	4·429
35	51·34	6·027
40	58·68	7·873
45	66·01	9·963
50	73·35	12·300
60	88·02	17·715
80	117·36	31·490
100	146·70	49·200

Borda found by experiment in the year 1762, that the force of the wind is very nearly as the square of the velocity, but he assigned that force to be greater than what Rouse found (as expressed in the above table) in the ratio of 111 to 100. Borda ascertained also, as was natural to expect, that, upon different surfaces with the same velocity, the force increased more rapidly than the surface. M. Valz, applying the method of the minimum squares, to Borda's results, ascertained that the whole might be represented by the formula

$$y = 0\cdot001289\, x^2 + 0\cdot000050541\, x^3$$

and nearly as correctly by $y = 0\cdot00108\, x^{2\cdot 2}$

x^2 representing the surface in square inches (French), and y the force corresponding to the velocity of 10 feet per second expressed in French pounds. (See pa. 54.)

4. In the application of wind to mills, whatever varieties there may be in their internal structure, there are certain rules and maxims with regard to the position, shape, and magnitude of the sails, which will bring them into the best state for the action of the wind, and the production of useful effect. These have been considered much at large by Mr. Smeaton: for this purpose he constructed a machine, of which a particular description is given in the Philosophical Transactions, vol. 51. By means of a determinate weight it carried round an axis with an horizontal arm, upon which were four small moveable sails. Thus the sails met with a constant and equable blast of air; and as they moved round, a string with a weight affixed to it was wound about their axis, and thus showed what kind of size or construction of sails answered the purpose best. With this machine a great number of experiments were made; the results of which were as follow:

(1.) The sails set at the angle with the axis, proposed as the best by M. Parent and others, *viz.* 55°, was found to be the worst proportion of any that was tried.

(2.) When the angle of the sails with the axis was increased from 72° to 75°, the power was augmented in the proportion of 31 to 45; and this is the angle most commonly in use when the sails are planes.

(3.) Were nothing more requisite than to cause the sails to acquire a certain degree of velocity by the wind, the position recommended by M. Parent would be the best. But if the sails are intended with given dimensions to produce the greatest effects possible in a given time, we must, if planes are made use of, confine our angle within the limits of 72 and 75 degrees.

(4.) The variation of a degree or two, when the angle is near the best, is but of little consequence.

(5.) When the wind falls upon concave sails it is an advantage to the power of the whole, though each part separately taken should not be disposed of to the best advantage.

(6.) From several experiments on a large scale, Mr. Smeaton has found the following angles to answer as well as any. The radius is supposed to be divided into six parts; and 1/6th, reckoning from the centre, is called 1, the extremity being denoted 6.

N°	Angle with that axis.	Angle with the plane of motion.
1	72°	18°
2	71	19
3	72	18 middle
4	74	16
5	77½	12½
6	83	7 extremity.

(7.) Having thus obtained the best method of *weathering* the sails, *i. e.* the most advantageous manner in which they can be placed, our author's next care was to try what advantage could be derived from an increase of surface upon the same radius. The result was, that a broader sail requires a larger angle; and when the sail is broader at the extremity than near the centre, the figure is more advantageous than that of a parallelogram. The figure and proportion of enlarged sails, which our author determines to be most advantageous on a large scale, is that where the extreme bar is one-third of the radius or whip (as the workmen call it), and is divided by the whip in the proportion of 3 to 5. The triangular or loading sail is covered with board from the point downward of its height, the rest as usual with cloth. The angles above mentioned are likewise the most proper for enlarged sails; it being found in practice, that the sails should rather be too little than too much exposed to the direct action of the wind.

Some have imagined, that the more sail the greater would be the power of the windmill, and have therefore proposed to fill up the whole area; and by making each sail a sector of an ellipsis, according to M. Parent's method, to intercept the whole cylinder of wind, in order to produce the greatest effect possible. From our author's experiments, however, it appeared, that when the surface of all the sails exceeded seven-eighths of the area, the effect was rather diminished than augmented. Hence he concludes, that when the whole cylinder of wind is intercepted, it cannot then produce the greatest effect for want of proper interstices to escape.

"It is certainly desirable (says Mr. Smeaton), that the sails of windmills should be as short as possible; but it is equally desirable, that the quantity of cloth should be the least that may be, to avoid damage by sudden squalls of wind. The best structure, therefore, for large mills, is that where the quantity of cloth is the greatest in a given circle that can be: on this condition, that the effect holds out in proportion to the quantity of cloth; for otherwise the effect can be augmented in a given degree by a lesser increase of cloth upon a larger radius than would be required if the cloth was increased upon the same radius."

(8.) The ratios between the velocities of windmill sails unloaded, and when loaded to their maximum, turned out very different in different experiments; but the most common proportion was as 3 to 2. In general it happened that where the power was greatest, whether by an enlargement of the surface of the sails or an increased velocity of the wind, the second term of the ratio was diminished.

(9.) The ratios between the least load that would stop the sails and the maximum with which they would turn, were confined betwixt that of 10 to 8 and 10 to 9; being at a medium about 10 to 8·3, and 10 to 9, or about 6 to 5; though on the whole it appeared, that where the angle of the sails or quantity of cloth was greatest, the second term of the ratio was less.

(10.) The velocity of windmill sails, whether unloaded or loaded, so as to produce a maximum, is nearly as the velocity of the wind, their shape and position being the same. On this subject Mr. Ferguson remarks, that it is almost incredible to think with what velocity the tips of the sails move when acted upon by a moderate wind. He has several times counted the number of revolutions made by the sails in 10 or 15 minutes; and, from the length of the arms from tip to tip, has computed, that if an hoop of the same size was to run upon plain ground with an equal velocity, it would go upwards of 30 miles in an hour.

(11.) The load at the maximum is nearly, but somewhat less than, as the square of the velocity of the wind; the shape and position of the sails being the same.

(12.) The effects of the same sails at a maximum are nearly, but somewhat less than, as the cubes of the velocity of the wind.

(13.) The load of the same sails at a maximum is nearly as the squares, and the effect as the cubes of their number of turns in a given time.

(14.) When sails are loaded so as to produce a maximum at a given velocity, and the velocity of the wind increases, the load continuing the same; then the increase of effect, when the increase of the velocity of the wind is small, will be nearly as the squares of these velocities: but when the velocity of the wind is double, the effects will be nearly as 10 to 27½; and when the velocities compared are more than double of that where the given load produces a maximum, the effects increase nearly in a simple ratio of the velocity of the wind. Hence our author concludes, that windmills, such as the different species for draining water, &c. lose much of their effect by acting against one invariable opposition.

(15.) In sails of a similar figure and position, the number of turns in a given time will be reciprocally as the radius or length of the sail.

(16.) The load at a maximum that sails of a similar figure and position will overcome, at a given distance from the centre of motion, will be as the cube of the radius.

(17.) The effects of sails of similar position and figure are as the square of the radius. Hence augmenting the length of the sail without augmenting the quantity of cloth, does not increase the power; because what is gained by length of the lever is lost by the slowness of the motion. Hence also, if the sails are increased in length, the breadth remaining the same the effect will be as the radius.

(18.) The velocity of the extremities of the Dutch sails, as well as of the enlarged sails, either unloaded or even when loaded to a maximum, is considerably greater than that of the wind itself. This appears plainly from the observations of Mr. Ferguson, already related, concerning the velocity of sails.

(19.) From many observations of the comparative effects of sails of various kinds, Mr. Smeaton concludes, that the enlarged sails are superior to those of the Dutch construction.

(20.) He also makes several just remarks upon those windmills which are acted upon by the direct impulse of the wind against sails fixed to a vertical shaft: his objections have, we believe, been justified in every instance by the inferior efficacy of these horizontal mills.

"The disadvantage of horizontal windmills (says he) does not consist in this, that each sail, when directly opposed to the wind, is capable of a less power than an oblique one of the same dimensions; but that in an horizontal windmill little more than one sail can be acting at once: whereas in the common windmill, all the four act together; and therefore, supposing each vane of an horizontal windmill to be of the same size with that of a vertical one, it is manifest that the power of a vertical mill will be four times as great as that of an horizontal one, let the number of vanes be what they will. This disadvantage arises from the nature of the thing; but if we consider the further disadvantage that arises from the difficulty of getting the sails back again against the wind, &c. we need not wonder if this kind of mill is in reality found to have not above one-eighth or one-tenth of the power of the common sort; as has appeared in some attempts of this kind."

Coulomb's Experiments.

5. M. Coulomb, whose experiments have tended to the elucidation of many parts of practical mechanics, devoted some time to the subject of windmills. The results of his labours were published in the Memoirs of the Paris Academy for 1781. The mills to which he directed his attention, were in the vicinity of Lille, and were, in fact, oil mills. From the outer extremity of one sail to the corresponding extremity of the opposite sail, was 70 feet, the breadth of each sail $6\frac{1}{2}$ feet, of which the sail-cloth when extended occupies $5\frac{1}{2}$ feet, being attached on one side to a very light plank; the line of junction of the plank and of the sail-cloth, forms, on the side struck by the wind, an angle sensibly concave at the commencement of the sail, but diminishes gradually all along so as to vanish at the remoter extremity. The angle with the axis, at seven feet from the shaft, is 60°, and it increases continually so as to amount to nearly 84° at the extremity. The shaft upon which the sails turn, is inclined to the horizon, in different angles in different mills, from 8 to 15 degrees.

Coulomb infers from his experiments,

(1.) That the ratio between the space described by the wind in a second, and the number of turns of a sail in a minute, is nearly constant, whatever be the velocity of the wind; the said ratio being about 10 to 6, or 5 to 8.

(2.) That with a wind whose velocity is $21\frac{1}{3}$ feet (English) per second, the quantity of action produced by the impulsion of the wind is equivalent to a weight of 1080 pounds avoirdupois raised 270 feet in a minute; the *useful* effect being equivalent to a weight of 1080 pounds raised 232 feet in the same time; whence it results that the quantity of effect absorbed by the stroke of the stampers, the friction, &c. is nearly a sixth part of the quantity of action.

(3.) Suppose one of these mills to work 8 hours in a day, Coulomb regards its daily useful effect as equivalent to that of

11 horses working at a walking wheel, in a path of the usual radius.

(4.) It is observable that in most windmills the velocity at the extremity of the sails is greater than that of the wind. In some cases, indeed, these velocities have been found in about the ratio of 5 to 2. Now, it is evident that the impulsion of a fluid against any surface whatever, can only produce pressure, or mechanical effect, when the velocity of the surface exposed to the impulse is less than that of the fluid; and that the pressure will be nothing when the velocity of the surface is equal to or greater than that of the fluid. Indeed, in the latter case the pressure may operate *against* the motion of the sails, and be injurious. It is desirable, therefore, in order to derive from a windmill all the effect of which it is susceptible, so to adjust the number of the turns that the velocity of the extremity of the sails shall be *less*, or, at most, *equal* to that of the wind.

It would be highly expedient to make comparative experiments on windmills, with a view to the determination of that velocity of the extremity of the sails which corresponds with the maximum of effect.

6. If v denote the velocity of the wind in feet per second, t the time of one revolution of the sails, A the angle of inclination of the sails to the axis, and D the distance from the shaft or axle of rotation to the point which is not at all acted on by the wind, or beyond which the sail-cloth ought to be folded up; then theoretical considerations supply the following theorem: viz.

$$D = \cdot 1092\, t\, v \tan. A.$$

Ex. Suppose $v = 30$ feet per second, $t = 2\cdot 25$ seconds, and $A = 75°$; then

$$D = \cdot 1092 \times 30 \times 2\cdot 25 \times 3\cdot 73205 = 27\cdot 509 \text{ feet}.$$

This result agrees nearly with one of Coulomb's experiments, in which the velocity of the wind was 30 feet English per second, the sails made 17 turns in a minute, and they were obliged to fold off more than 6 feet from the extremity of each sail, of 34 feet long, to obtain a maximum of effect. The angle A at that distance from the tip of the sail was 75° or 76°.

Section IV. *Steam and Steam Engines.*

The whole power of the steam-engine depends on the employment of elastic vapour produced from water at high temperatures.

Steam, in fact, is highly rarefied water, the particles of which are expanded by the absorption of caloric, or the matter of heat. Water rises in vapour at all temperatures, though this is usually supposed to take place only at the boiling point; when, however, the evaporation occurs below 212°. (Fahr.) it is confined to the surface of the fluid acted upon: but, at that heat, 212°, steam is formed at the *bottom* of the water, and ascends through it, carrying off the heat in a latent form, and, therefore, preventing the elevation of temperature of the water itself. At the common pressure of the atmosphere, one cubic inch of water produces about 1760 cubic inches (or nearly a cubic foot) of aqueous vapour; but the boiling point varies considerably under different pressures, and these anomalies materially affect the density of the vapour produced. Thus, in a vacuum water boils at about 70°; under common atmospheric pressure at 212°; and when pressed by a column of mercury 5 inches in height, water does not boil until it is heated to 217°; each inch of mercury producing by its pressure, a rise of about 1° in the thermometer.

According to the elaborate experiments of Dr. Ure of Glasgow, the elastic force of this vapour at 212° is equivalent to the pressure of a column of mercury 30 inches high, or equal to about 15 lbs. avoirdupois on a square inch.

At Temp.	212°	30 inch. mercury	15 lbs. per sq. inch.
	226·3	40	20
	238·5	50·2	25·15
	248·0	60	30
	257·5	69·8	34·9
	273·7	91·2	45·6
	285·2	112·2	56·1
	312	166	83

And Mr. Woolf has ascertained that at these temperatures, omitting the last, a cubic foot of steam will expand to about 5, 10, 20, 30, and 40 times its volume respectively; its elastic force, when thus dilated, being in each case equal to the ordinary pressure of the atmosphere.

One pound of Newcastle coals converts 7 pounds of boiling water into steam; and the time required to convert a given quantity of boiling water into steam, is 6 times that required to raise it from the freezing to the boiling point.

It is found, also, that if a bushel of coals per hour applied to a well constructed boiler, produces steam of the expansive force of 15 lbs. per square inch, it will tend to expand itself with a velocity of 1840 feet per second; then 2 bushels of coals, burnt under the same boiler, are capable of giving to the vapour an expansive force of 120 lbs. per square inch, and a velocity of expansion of 6800 feet per second. A bushel and half of coals would, with the same boiler, carry steam to the pressure of 50 lbs. on a square inch; which is as high as is regarded consistent with safety.

From these data it will be evident that when steam is merely employed to displace the air in a close vessel, and afterwards produce a vacuum by condensation, no more heat is necessary than what will raise the water employed to 212°: but, if, on the contrary, steam capable of giving high pressures is required, a considerable increase of heat, as to 260°, 280°, will be necessary; and, of course, an augmentation of fuel, though not one that is strictly proportional, will be required. This, however, is a consideration upon which we cannot here enlarge.

We proceed to speak of the actual construction of the machine.

The principles and manner of operation of the steam-engines of Savery, Newcomen and Cawley, and of Watt, may be understood from the following brief explanations and remarks.

1. Let there be a sucking pipe with a valve opening upwards at the top, communicating with a close vessel of water, not more than thirty-three feet above the level of the reservoir, and the steam of boiling water be thrown on the surface of the water in the vessel, it will force it to a height as much greater than thirty-three feet as the elastic force of the steam is greater than that of air; and if the steam be condensed by the injection of cold water, and a vacuum thus formed, the vessel will be filled from the reservoir by the pressure of the atmosphere; and the steam being admitted as before, this water will also be forced up; and so on successively.

Such is the principle of the first steam-engine, said by the English to be invented by the Marquis of Worcester; while the French ascribe it to Papin: though we believe the fact is that Brancas, an Italian, applied the force of steam ejected from a large eolopile as an impelling power for a stamping-engine so early as 1629. Brancas's was, in fact, an immense blow-pipe, turning a wheel. The hint so obscurely exhibited in the Marquis of Worcester's century of inventions was carried into effect by Captain Savery.

2. If the steam be admitted into the bottom of a hollow cylinder, to which a solid piston is adapted, the piston will be forced upwards by the difference between the elastic forces of steam and common air; and the steam being then condensed, the piston will descend by the pressure of the atmosphere, and so on successively. This is the principle of the steam-engine first contrived by Messieurs Newcomen and Cawley, of Dartmouth. This is sometimes called the atmospherical engine,

and is commonly a forcing-pump, having its rod fixed to one end of a lever, which is worked by the weight of the atmosphere upon a piston at the other end, a temporary vacuum being made below it by suddenly condensing the steam, that had been admitted into the cylinder in which this piston works, by a jet of cold water thrown into it. A partial vacuum being thus made, the weight of the atmosphere presses down the piston, and raises the other end of the straight lever, together with the water, from the well. Then immediately a hole is uncovered in the bottom of the cylinder, by which a fresh quantity of hot steam rushes in from a boiler of water below it, which proving a counterbalance for the atmosphere above the piston, the weight of the pump-rods, at the other end of the lever, carries that end down, and raises the piston of the steam-cylinder. The steam hole is then immediately shut, and a cock opened for injecting the cold water into the cylinder of steam, which condenses it to water again, and thus making a vacuum below the piston, the atmosphere again presses it down and raises the pump-rods, as before; and so on continually.

3. When the cylinder is full of steam, if a valve be opened, by which the steam is allowed to escape into another vessel, where a jet of cold water is introduced, the condensation is much more complete, and the heat of the cylinder being preserved, the steam possesses its full elasticity.

This improvement was made by Mr. Watt, and completely changed the character of the steam-engine. In the old engines the power was diminished to half its real value, so that the moving force, instead of reaching 15 lbs. on each square inch of the area of the piston, was reduced to about 8 lbs. In this engine of Mr. Watt's the moving force is not less than 12 lbs. upon each square inch of the piston.

4. A farther improvement has been made on this engine, by injecting the steam into the cylinder, alternately above and below the piston, so that the whole motion is produced by the elasticity of the steam, and has no dependance on the weight of the atmosphere.

This improvement is also due to Mr. Watt, and could not have been made without the previous contrivance of condensing the steam in a separate vessel. It is particularly accommodated to the production of a rotatory motion by means of a steam-engine. Three years before Mr. Watt introduced this improvement, viz. in 1778, Mr. Washborough, of Bristol, took out his patent for converting a reciprocating into a rotatory motion; and in 1781, Mr. J. Steed effected the same thing, for the first time, by means of what is now called a *crank*.

From that time Hornblower, Cartwright, Murray, Bramah, Trevithick, Maudslay, Woolf, and others, have, in rapid succession, introduced a series of improvements which have rendered steam-engines as efficacious and perfect as can well be conceived.

5. Another improvement due to Mr. Watt, is that of the *Expansion Engine*, invented about 1769. The principle of this invention, as Mr. Partington correctly remarks, consists in shutting off the farther entrance of steam from the boiler when the piston has been pressed down in the cylinder, for a certain proportion of its total descent, leaving the remainder to be accomplished by the expansive force of the steam already produced.* To regulate the time of closing the valve, and as such the precise amount of steam admitted, Mr. Watt employed a plug-frame with moveable pins, which may be so placed that the steam-valve will shut when the piston has descended one-half, one-third, one-fourth, &c. By the application of this principle, the piston is made to descend uniformly, the pressure on it continually diminishing as the steam becomes more and more rare, and the accelerating force is consequently diminished.

6. The principle of the *high-pressure* steam-engine depends also on the power of steam to expand itself very considerably beyond its original bulk, by the addition of a given quantity of caloric, thus acquiring a considerable elastic force (equivalent to from 40 to 60 lbs. on each square inch) which, in this case, is employed to give motion to a piston. One of the greatest advantages attendant on employing the repellant force of steam, as in this form of the engine, consists in an evident saving of the water usually employed in condensation; and this, in loco-motive engines, for propelling carriages, is an object of considerable importance. The first description of an engine of this kind, which we have met with, is in Leupold's Theatre of Machines, published in Germany, in 1724. The apparatus consists of two cylinders placed at a moderate distance asunder; each of them provided with a piston made to fit air-tight, and connected with a forcing pump. When steam of considerable elasticity is admitted at the bottom of the first cylinder, it is forced upwards, carrying with it the lever of the pump; at the same time that the steam or air is expelled from the other. On this operation being repeated, or rather, reversed, the steam is allowed to enter the second cylinder, which is also connected with the boiler, while the steam

* See Brewster's Robison's Mechanical Philosophy, vol. ii. p. 196.

in the first cylinder is allowed to escape into the air. Thus, it may be remarked, that the process of condensation forms no part of the principle of the "high pressure" engine; and that even the expansion of gunpowder might be employed to produce a similar effect.

7. We have already adverted to Mr. Woolf's discovery, that a quantity of steam having the force of 5, 6, 7, or more pounds on every square inch of the boiler, may be allowed to expand itself to an equal number of times its own volume when it would still have a pressure equal to that of the atmosphere, provided that the cylinder in which the expansion takes place have the same temperature as the stream possessed before it began to increase.

The most œconomical mode of employing this principle consists in the application of two cylinders and pistons of unequal size to a high pressure boiler; the smaller of which should have a communication both at its top and bottom with the steam vessel. A communication being also formed between the top of the smaller cylinder, and the bottom of the larger cylinder; and *vice versa*. When the engine is set to work, steam of a high temperature is admitted from the boiler to act by its elastic force on one side of the smaller piston, while the steam which had last moved it has a communication with the larger or condensing cylinder. If both pistons be placed at the tops of their respective cylinders, and steam of a pressure equal to 40lbs on the square inch be admitted, the smaller piston will be pressed down, while the steam below it, instead of being allowed to escape into the atmosphere, or pass into the condensing vessel, as in the common engine, is made to enter the larger cylinder above its piston, which will make its downward stroke at the same time as that in the smaller cylinder; and during this process, the steam which last filled the larger cylinder, will be passing into the condenser to form a vacuum during the downward stroke.

To perform the upward stroke it is merely necessary to reverse the action of the respective cylinders; and it will be effected by the pressure of the steam in the top of the small cylinder, acting beneath the piston in the great cylinder; thus alternately admitting the steam to the different sides of the smaller piston, while the steam last admitted into the smaller cylinder passes regularly to the different sides of the larger piston, the communication between the condenser and steam boiler being reversed at each stroke.

Mr. Partington states that a double cylinder expansion engine of this kind was constructed for Wheal Vor mine in Cornwall, in 1815. In this, the great cylinder is 53 inches in diameter, and has a nine feet stroke; the small cylinder being in content about one-fifth of the great one. The engine works 6 pumps, which at every stroke raise a load of water of 37,982 lbs. weight, 7½ feet high. This produces a pressure of 14·1 lbs. per square inch on the surface of the great piston, while its average performance has been estimated at 46,000,000 lbs. raised one foot high with each bushel of coals.

8. To render the preceding remarks and descriptions more intelligible, we will now give fuller descriptions of a few engines:—Plate II. fig. 1, exhibits a vertical section of the different parts of a high pressure steam-engine, constructed upon a principle in which simplicity and power are blended as far as possible; and in which the parts are arranged in such a manner as seemed best calculated to facilitate the comprehension of these machines to such as have not already had an opportunity of examining them carefully. The construction is due to Oliver Evans; and every thing not essentially necessary in a popular account of the operation, is omitted in the drawing.

A, section of the boiler, perpendicular to its axis. It consists of two cylinders, one within the other, (the cylindric form being most capable of resisting a great expansive force). The fire is made to burn in the interior cylinder which serves for the furnace, and the water is contained between the two cylinders. The smoke and heated air, after having traversed the interior cylinder in the whole direction of its length, passes out at the other extremity, and is thence conducted under the alimentary or subsidiary boiler, B, where it heats the water which is intended to supply in the great boiler, the place of that which has been carried off in the state of vapour, and thus given motion to the machine. C, the feeding pump, which at every stroke of the piston, and corresponding motion of the great beam, raises a small quantity of cold water and forces it into the supplementary boiler. The object of this part of the apparatus may be explained in few words. If the waste of water in the great boiler, by its conversion into steam, were supplied at once from the cold water pump, the temperature of the liquid in the great boiler would be diminished, and the production of steam proportionally checked. But when water passes from the pump to the great boiler, through the intervention of the subsidiary boiler in which it becomes heated, it enters the great boiler without diminishing the temperature of the water which it contains, and, of course, without checking the operation of the machine. The skill and judgment of the engineer are evinced in proportioning the magnitudes of the two boilers, and of the feeding pump, so that the supply may be neither defective nor super-abundant, but just sufficient.

The steam rises through the tube, and if the injection sucker, D, is opened to permit the entrance of steam into the machine, the valves E and F being opened, the steam will push the piston G to the inferior extremity of the cylinder; as is shown in the diagram. The steam which occupied the cylinder, being driven before the piston, escapes through the valve F. As soon as the piston G arrives at the bottom of the cylinder, the valves E and F close, and the other valves H and I open; the steam enters at H, and causes the piston to rise, while the weaker steam, which is contained in the upper part of the cylinder, escapes through the valve I, presenting scarcely any re-action to the ascent of the piston. These four pistons are moved by two wheels K and L, carrying cams or lifters on their surfaces, which press against four levers, to which stems or handles are attached, and which thus cause them to open and shut precisely at the adequate intervals. [These levers are not represented in the figure, but are omitted to prevent confusion.]

The motion of the piston G, of course communicates a reciprocating motion to the beam M N; and the crank and connecting bar M O give a rotatory motion, the regularity of which is maintained by means of the fly-wheel Q R. After similar operations the valves E, F, H, I, alternately opening and shutting, the motion of the machine is continued. The teethed wheels S, T, of equal diameters, give motion to the wheels and lifters L, K; and the beam M N communicates the requisite motion to the forcing piston of the feeding pump C. The motion of the whole being thus continued; the wheel V, if it have 66 teeth, playing into the wheel U of 23 teeth, would give, say to the mill-stone W, 100 revolutions for every 35 strokes of the piston G: viz. 100 revolutions in a minute if the piston make 35 strokes in that interval of time.

The toothed wheel V may be applied to any other work. Instead of the wheel, indeed, there may be employed a crank to move a pump, or a saw; and the machine may, by modifying the power of the steam, give from 10 to 100 strokes of the piston per minute, according to the purpose for which it is constructed. If the steam cylinder be 8 inches in diameter it will work a mill-stone of 5 feet diameter; or perform any work which requires equal mechanical energy.

The steam, on quitting the machine, escapes by the tube xx, and becomes dissipated in the air above the top of the building: or, if the constructor please, it may pass into the condenser, when a condenser is used; or it may be made to pass along the subsidiary boiler to heat the water which it contains.

y, is the safety valve, kept in its place by a lever, graduated like the steelyard, to weigh and thus balance the effort of the steam. This valve rises and permits the excess of steam to escape when its elastic force becomes more considerable than is requisite for the work of the machine. The weight is, in general, so adjusted upon the lever, that the valve shall open when the pressure of the steam upon a square inch of the interior of the boiler, becomes about *half* that for which its thickness and strength of material are fitted.

To prevent a recurrence of those accidents which have sometimes occurred aboard steam-boats, and which first drew the attention of the legislature to this important part of the engine, it appears advisable to enclose the safety valve in an iron box, and thus place it beyond the controul of the engine-man.

The whole of the machine now described is very simple in its construction, and easy to execute by ordinary mechanics or mill-wrights. The orifices for the passage of the steam are simple metallic plates, each pierced with a hole, on which the valve intended to close it readily falls: they are very easily cast at any common foundry.

9. Manufactories of steam-engines are now numerous, especially in those districts where machinery is much called for. The engines of Maudslay are much celebrated, especially in situations where it is of consequence to erect an engine in small space, as in steam-boats, &c. Next to Maudslay's the engines of Messrs. Fenton, Murray and Wood, of Leeds, have, perhaps, attained the highest celebrity. As the construction of their engines serves well for the elucidation of the manner of operation, we here present a description of one of them, from a very useful work, *Smith's Panorama of Science and Art.*

In Plate III, A represents the boiler, nearly three quarters full of water: the bottom is considerably, and the sides a little concave, that it may receive more fully the force of the flame circulating round it. Boilers are usually of an oblong form, and are furnished with a part that takes off, in order that a person may get in to clean them when needful; they have also a valve, called the safety-valve, opening upwards, which is loaded so that the steam escapes when it is stronger than the engine requires, and, if retained, would hazard the bursting of the boiler. It is not uncommon to have two boilers, one of which is a reserve, that the engine may not be stopped, when the other requires repair.

B, is an apparatus for regulating the fire, and giving action to a bell, which regulates the quantity of coals and time of firing.

C, the steam-pipe from the boiler A to the valve I.

D, the steam-cylinder, generally called only "the cylinder;" it is connected at the top and bottom with the valve I.

E, the piston, which, by its connecting rod *e*, gives motion to the beam F, the other end of which, by another connecting rod, gives motion to the heavy fly-wheel G, by means of a crank. Thus, after the engine has begun to work, its power is accumulated in the fly-wheel, and may be disposed of at the pleasure of the mechanist.

H, an eccentric circle* on the axle of the fly-wheel G; it gives motion by its levers, in a manner easily understood by inspection, to the valve I.

I, a coffer-slide valve, which requires no packing to make it steam-tight, as there is always a vacuum under it: it answers the purpose of the four valves used in double-power engines, and from the simplicity of its construction, when well made at first, is not liable to get out of order.

K, the steam-admission valve and lever, connected with a governor not shown in the figure, which regulates the speed of the engine. See pa. 260.

L, the cylinder of the discharging pump, for extracting the water and uncondensed vapour from the condenser M.

N, a small cistern, filled with water. Into this cistern enters a pipe from the condenser M, the top of which pipe is covered by a valve, which is called the blow-valve, or sometimes the snifting-valve. Through this valve, the air contained in the cylinder D, and passages from it, is discharged, previously to the engine being set in motion.

O, the eduction-pipe, which conducts the steam from the valve I to the condenser M.

P, the pump which supplies with water the cistern SS, in which the condenser and discharging pump stand.

QQ, iron columns, of which the engine has four, although only two are shown; they stand upon one entire plate seen edgeway, on which the principal parts of the engine are fixed;

* Mechanics distinguish by the appellation of "eccentric circle," a wheel, the axle of which is purposely made to pass, not through its real centre, but on one side of it.

by this means the beam and its accompaniments are supported without being connected with any part of the building, except the recess below the floor on which they stand.

RR, the recess below the floor, for containing the cistern of the discharging-pump, condenser, &c. This arrangement enables those engines to be fixed up and tried at the manufactory before they are sent of, which renders the refixing easy and certain. Engines are made according to this plan, from the power of one to twelve horses.

Before the engine is set to work, the cylinder D, the condenser M, and the passages between them, are filled with common air, which it is necessary to extract. To effect this, by opening the valves, a communication is made between the steam-pipe C, the space below the piston in the cylinder D, the eduction-pipe O, and the condenser M. The steam will not at first enter the cylinder D, or will only enter it a little way, because it is resisted by the air; but the air in the eduction-pipe O, and the condenser M, it forcibly drives before it, and this part of the air makes its exit through the valve and water in the cistern N. The steam-admission valve is now closed, and the steam already admitted is converted into water, partly by the coldness of the condenser M, but principally by a jet of cold water which enters it through a cock opening into it from the well SS, in which the condenser is immersed. When this steam is condensed, all the space it occupied would be a vacuum, did not the air in the cylinder D expand, and fill all the space that the original quantity of it filled; but by the repetition of the means for extracting a part of the air, the remainder is blown out, and the cylinder becomes filled with steam alone. Suppose then the cylinder beneath the piston to be filled with steam, and the further admission of steam to that part of it be cut off, while the communication between it and the condenser remains open, it is obvious that there will soon be a vacuum in the cylinder, because as fast as the steam reaches the condenser, it is converted into water by the coldness of that vessel and the jet playing within it. At this moment, therefore, the steam is admitted above the piston, which it immediately presses down. As soon as the piston reaches to the bottom of the cylinder, the steam is admitted to the under side of it, and as the communication from the upper side of the piston to the condenser is opened, while the further admission of steam to that side during the upper stroke, is prevented, the steam which had pressed the piston down passes into the condenser, and is converted into water.

The motion of the piston E, by this alternate admission and extraction of the steam on each side of it, is thus necessarily continued, and the distance of its upward and downward range is called the length of its stroke. It communicates its reciprocating motion, by the connecting rod e, to the great beam F, and thence, by another connecting-rod and a crank, to the fly-wheel G.

To explain the rapid accumulation of power with an increase of the size of the engine, it must be observed, that the force of the steam generally used, is somewhat greater than the pressure of the atmosphere; but supposing it to be no greater, as the atmospheric pressure is fifteen pounds on each square inch, a piston 16 inches in diameter, containing 201 square inches of surface, will alternately be raised and depressed by a force equivalent to a weight of 3015 pounds. Here no allowance is made for friction, but after the requisite deduction on this account, which may be reckoned at one third, the disposable part of the engine, derived from each stroke, will still be very great.

The condenser M, and the discharging-pump L, communicate by means of a horizontal pipe containing a valve y opening towards the pump; the piston, l, of this pump, also contains two valves, and the cistern T, at the top of the pump-cylinder, contains two other valves, which, like those of the piston l, open upwards. When the piston E of the cylinder is depressed, the piston l, of the discharging-pump, it will be obvious to inspection, is depressed likewise, and its valves open, while the valve y closes; hence the water from the condensed steam, as well as the injection-water, and any permanently elastic vapour or gas, which may be present, having passed through the valve y, passes through the piston l; and when that piston is drawn up, its valves close and prevent their return, as in ordinary pump-work. The water and gas that have thus got above the piston, as the latter rises, open the valves at the bottom of the cistern T, in which the water remains till it is full, but the gas passes into the atmosphere. As the water in the cistern T is in a very hot state, it is sometimes, for the purpose of economizing fuel, pumped up and returned to the boiler, the pump-rod being attached to the great beam. The utility of the discharging-pump L, will now be appreciated, and it must be perceived how much more materially it contributes to the perfection of the vacuum in the cylinder D, than if the water from the condenser merely ran off by a pipe.

The steam constantly rushing into the condenser M, has a perpetual tendency to heat that vessel, as well as the water of the cistern ss, in which it stands: the whole of the steam, if this were unchecked, would not be condensed, or the condensation would not be sufficiently rapid, because the injection-water itself flows out of this cistern. A part of the water is therefore allowed to flow from this cistern by a waste pipe, and an equal quantity of cold is constantly supplied by the pump P.

In Newcomen's, or the atmospherical engine the cylinder was open at the top, and therefore, during the descent of the piston, the air exerted a great power in cooling it; but in the modern engines, where steam is the active power both in raising and depressing the piston, the top of the cylinder is closed with an iron lid, and not an atom of steam can escape, except at the proper time, into the condenser. In order that the connecting rod *e*, may work freely, and yet possess this desirable property of being steam-tight, it passes through what is called a *stuffing* or *packing box*. This stuffing consists of some material which the steam will rather adapt to its office than injure; leather, which is used for the stuffing or collars of machines never to be subjected to heat, will not answer here; hempen yarn is the material usually employed. The rod of the piston *l*, passes through a stuffing box of the same kind as that of the piston E; and the pistons themselves are surrounded with stuffing.

The cylinder D is surrounded by a case, to keep it from being cooled by contact with the external air. The extremity, or any given point removed from the centre of the great beam, can describe only the arc of a circle; but it is necessary that the piston rod *e* should rise and fall vertically. Newcomen effected this object, by making the end of the beam into the arc of a circle, the radius of which was equal to the distance from the centre of the beam: a chain went over this arc, and was fastened on the higher end of it; this simple contrivance effectually answered his purpose, because in his engine the effective stroke was only downwards; but here, in a double-power engine, where the stroke is both upwards and downwards, a chain would yield in rising, and be altogether unsuitable. An apparatus is therefore used, called the parallel joint, which is easily understood by inspection. By this means the rod *e*, not only rises and falls perpendicularly, but is perfectly rigid, and communicates all its motion to the great beam in each direction of its motion. The connecting rod *g* does not require the same contrivance, because it does not rise and fall perpendicularly; its lower end, with the outer end of the crank, describing a circle: it has therefore only a simple joint, admitting of this deviation.

In order to communicate a rotary motion to the fly-wheel, instead of the crank may be used a contrivance giving twice the rapidity to the fly. For this purpose, on the outside of the axis of the fly, where the crank is shown in the plate, a small toothed wheel is fixed, and can only be moved with the fly; at the extremity of the rod *g*, and on that side of it which is next the fly-wheel, another toothed wheel is fixed, in such a manner that it cannot turn round on its axis, but must rise and fall with the rod to which it is attached. These two wheels work in each other, and that attached to the connecting rod cannot leave its fellow, because their centres are connected by a strap or bar of iron. When, therefore, the connecting rod rises, the wheel upon it moves round the circumference of the wheel upon the axis of the fly. By this means the fly makes an entire revolution for every stroke of the piston, and some mechanics are apt to think that they are great gainers by such an arrangement: the contrivance is certainly elegant; but with respect to utility, the fact is, that a crank is preferable; for it is more simple, cheaper, and less likely to be out of order, while, if the fly be large enough to receive, with less velocity, all the momentum that can be communicated to it, the effect will certainly not be inferior.

10. *Locomotive steam-engines*, or those which will propel both themselves when placed upon wheels, and any suitable carriages attached to them, were invented by Mr. *Trevithick*, in Cornwall, about the year 1804. They are now in constant use in the northern coal-districts; their peculiar construction will be very evident from the following description, by Mr. *Tredgold*, of the engines employed on the Hetton rail-way.

The wheels of the coal-waggons drawn by the engines are 2 feet 11 inches in diameter, with 10 spokes, and weigh $2\frac{3}{4}$ cwt. Their axles are 3 inches in diameter, and revolve in fixed bushes.

The weight of each engine is about 8 tons. It consists of a boiler 4 feet in diameter, with an internal fire-place. The smoke ascends from the fire by a chimney about 12 feet high: the lower 8 feet of the chimney is formed of sheet iron of 6lbs. to the square foot, and the rest of iron $2\frac{1}{2}$lbs. to the oot. There are two cylinders, which work alternately. The diameter of the pistons is 9 inches, and the length of the stroke 2 feet; the pistons make about 45 double strokes per minute. The steam is admitted to the cylinders by slide valves worked by eccentric wheels on the axis of the engine carriage. The pressure of the steam in the boiler is from 40 to 50lbs. on the square inch.

The wheels of the engine carriage are 3 feet 2 inches diameter, with 12 spokes in each, and each weighs 3¾ cwt. The axles are 3½ inches diameter, and are connected by an endless chain working into a wheel on each axle, so that both the axles of the carriage may be turned at the same rate. The boiler is supported on the carriage by four floating pistons, which answer the purpose of springs in equalizing the pressure on the wheels, and softening the jerks of the carriage. A floating piston is packed as the steam piston of a steam-engine, and has a short piston rod of 1¼ inches diameter, which rests upon the brass bush, in which the axle of the wheel turns. The water in the boiler presses on the upper side of the piston; and whatever elevation or depression the wheel follows, the pressure upon it is nearly the same. This ingenious substitute for a spring, as well as the other peculiarities of this engine, were invented by Messrs. Losh and Stephenson, of Newcastle-upon-Tyne, and made the subject of a patent in 1816.

Fig. 2. plate II. is a sketch of the steam-carriage, &c. A is the boiler, and B B the steam cylinders; the fire-place is within the boiler, and F is the entrance to it; C is the chimney; D D the floating pistons which support the carriage on the axles, and answer as springs in making it press equally on the rails. As the moving force is not equal at the same time on the wheels of both axles, it is necessary to connect the axles by a pitch chain G, working into toothed wheels on the axles. The water for supplying the boiler, and the coals at *b* for the fire, are carried by a small carriage, called the *tender*; I is the water barrel, and *a* is a hose pipe which conveys the water to the force-pump H, which is worked by the engine; W W are coal waggons, each of which carries 53 cwt. of coals. From 13 to 17 of these waggons are drawn in a train by one steam carriage; they are connected by the short chains *c c*. The connecting rods which communicate the power from the pistons to the wheels of the steam carriage are attached to the wheels, so that one piston is at half the length of its stroke, when the other is at the commencement of its stroke.

Fig. 3. is a vertical transverse section of the steam carriage; the comparison of which with fig. 2, will render the whole more intelligible.

11. We shall terminate our *description* of steam-engines, by exhibiting one invented by Mr. John Nancarrow, an American, and which is intended to give motion to water-wheels, in places where there is no fall, and but a very small stream or spring.

A, fig. 4. plate II, the receiver, which may be made either of wood or iron.

B, B, B, B, B, wooden or cast iron pipes, for conveying the water to the receiver, and thence to the penstock.

C, the penstock or cistern.

D, the water wheel.

E, the boiler, which may be either iron or copper.

F, the hot well for supplying the boiler with water.

G, G, two cisterns, under the level of the water, in which the small bores B, B, and the condenser are contained.

H, H, H, the surface of the water with which the steam-engine and the water-wheel are supplied.

a, *a*, the steam-pipe, through which the steam is conveyed from the boiler to the receiver.

b, the feeding-pipe, for supplying the boiler with hot water.

c, *c*, *c*, *c*, *c*, the condensing apparatus.

d, *d*, the pipe which conveys the hot water from the con denser to the hot well.

e, *e*, *e*, valves for admitting and excluding the water.

f, *f*, the injection-pipe; and *g*, the injection cock.

h, the condenser.

It must be remarked that the receiver, penstock, and all the pipes, must be previously filled before any water can be delivered on the wheel; and when the steam in the boiler has required a sufficient strength, the valve at *i* is opened, and the steam immediately rushes from the boiler at E into the receiver A; the water descends through the tubes A and B, and ascends through the valve *k*, and the other pipe or tube B, into the pentstock C. This part of the operation being performed, and the valve *i* shut, that at *e* is suddenly opened, through which the steam rushes down the condensing-pipe *c*, and in its passage meets with a jet of cold water from the injection-cock *g*, by which it is condensed. A vacuum being made by this means in the receiver, the water is driven up to fill it a second time through the valves *e*, *e*, by the pressure of the external air, when the steam-valve at *i* is again opened, and the operation repeated for any length of time the machine is required to work.

There are many advantages which a a steam-engine on this construction possesses beyond any thing of the kind hitherto invented; a few of which the inventor thus enumerates.

(1.) It is subject to little or no friction.

(2.) It may be erected at a small expense when compared with any other sort of steam-engine.

(3.) It has every advantage which may be attributed to Boulton and Watt's engines, by condensing out of the receiver, either in the pentstock or at the level of the water.

(4.) Another very great advantage is, that the water in the upper part of the pipe adjoining the receiver acquires a heat, by its being in frequent contact with the steam, very nearly equal to that of boiling water; hence the receiver is always kept uniformly hot, as in the case of *Boulton* and *Watt's* engines.

(5.) A very small stream of water is sufficient to supply this engine, (even where there there is no fall,) for all the water raised by it is returned into the reservoir H, H, H.

From the foregoing reasons it would seem that no kind of steam-engine is better adapted to give rotary motion to machinery of every kind than this. Its form is simple, and the materials of which it is composed are cheap; the power is more than equal to any other machine of the kind, because there is no deduction to be made for friction, except on account of turning the cocks, which is but trifling.

Its great utility is therefore evident in supplying water for every kind of work performed by a water-wheel, such as grist-mills, saw-mills, blast-furnaces, forges, &c.*

12. The theory of steam-power in reference to the mechanical energy of engines, is, as yet, in a very imperfect state. The best formulæ which we have hitherto seen, are exhibited by Mr. Tredgold, in his judicious and valuable work on *Rail-Roads*. As they are found to furnish results which agree very nearly with those of experiment, we shall insert them here.

If f be the measure of the force of steam in inches of the mercurial column, and t the corresponding temperature measured on Fahrenheit's thermometer; f' the resistance from the friction of the steam piston, and the condensed vapour in the cylinder, or the atmospheric pressure in high-pressure engines, and n the bulk or capacity of the steam cylinder, when the bulk of the steam admitted at the pressure f is unity. Then the power of the steam generated from a cubic foot of water is

$$4873\,(459 + t) \times \left(1 - \frac{nf'}{f}\right) + \text{hyp. log. } n\Big).$$

When the steam does not act by expansion $n = 1$.

When the expanding force of the steam is employed, the above equation has a maximum, which will obtain when hyp.

* For descriptive information on the subject of steam-engines, the reader may consult the historical treatises of *Partington* and *Stuart*.

log. $n - \frac{nf'}{f}$ is a minimum, which is evidently the case when $n = \frac{f}{f'}$. In that case, inserting, $\frac{f}{f'}$, for n, we have

$4873\,(459 + t).\ (\text{hyp.log}.\ \frac{f}{f'}$ = the maximum power of a cubic foot of water converted into steam.

When $f = f'$, then hyp. log. $\frac{f}{f'} = 0$, and the power is nothing.

And, when $1 - \frac{f'}{f}$ is greater then hyp. log. $\frac{f}{f'}$ it is disadvantageous to work by expansion.

13. To calculate the quantity of fuel, let c be the quantity which converts a cubic foot of water into steam that will bear the pressure of the atmosphere; let s be the specific heat of the steam, a the specific heat of the air and smoke which escape up the chimney, and w the weight of fuel that will heat one cubic foot of water one degree: then

$$c + [(t - 212^\circ) \times (a + s)\, w] =$$

the least quantity of fuel that will produce steam of the force f and temperature t.

Mr. Tredgold, by assuming $c = 8{\cdot}4$ lbs. of Newcastle coals, $w = {\cdot}0075$ lbs. $s = {\cdot}847$, and $a = {\cdot}753$, reduces the preceding to $8{\cdot}4 + {\cdot}012\,(t - 212^\circ)$ = the lbs. of coal to produce steam of the temperature t.

14. For a high-pressure engine, taking 30 inches for the measure of atmospheric pressure, $\frac{1}{7}$ of the pressure of the steam for the friction of the steam piston, and $\frac{1}{10} f$ for the plus pressure in the boiler, the whole loss becomes $\frac{1}{4} f$. But, one side of the piston of a high-pressure engine is acted upon by the same pressure as that of the external atmosphere: hence $f' = \frac{1}{4} f + 30$ = the resistance to the moving force f.

Consequently, when a high-pressure engine is worked expansively, we have

$$4873\,(459 + t) \times \left(\text{hyp. log.}\ \frac{f}{\frac{1}{4} f + 30}\right) =$$

the mechanical power of a cubic foot of water converted into steam.

Hence, there is no advantage in making a high-pressure steam-engine work expansively, when the force of the steam is less than 60 inches of the mercurial column; because the above hyp. log. is then less than $1 - \frac{\frac{1}{4} f + 30}{f}$.

When an engine does not employ the expansive power of steam, we have

$4873\,(459 + t) \times \left(1 - \frac{\frac{1}{4}f + 30}{f}\right) =$ the mechanical power of a cubic foot of water converted into steam.

15. Mr. Tredgold illustrates these formulæ by the following

Example.—Let the force of the steam be 120 inches of mercury; the corresponding temperature is 292·8°. Then

$$4873\,(459 + 292{\cdot}8) \times \left(1 - \frac{(\frac{1}{4} \times 120) + 30}{120}\right) =$$

1,830,000lbs. raised one foot high.

The quantity of coal is 8·4 + ·012 (292·8 − 212) = 9·37lbs. of coal.

Now if the horse power be 16,000,000lbs. raised one foot in a day of 8 hours; then

1,830,000, : 9·37lbs. :: 16,000,000 : 82lbs.

Therefore, working with steam of 44½lbs. on the square inch on the piston, above the pressure of the atmosphere, 82lbs. of Newcastle coal ought to do the day's work of a horse.

But if the engine works expansively with the same force of steam, then

$$4873\,(459 + 292{\cdot}8) \times \text{hyp. log. } 2) = 2{,}540{,}000\text{lbs.}$$

raised one foot high by 9·37lbs. of coal; and consequently 59lbs. of coal ought to do the day's work of a horse.

16. With regard to the maximum of useful effect in steam-engines, it will be found, according to Mr. Tredgold, by taking $v = 120\,\sqrt{l}$, for the working velocity of an engine in feet per minute, l being the length of the stroke in feet.

If an engine has a 2 feet stroke, then $v = 170$ feet per minute, and the number of strokes per minute 42½.

By increasing the stroke to 3·4 feet we get a velocity of 220 feet per minute, with 32 strokes per minute.

If any variation be made from the maximum power the decrease of effect is the same as in horse power; but, as Mr. T. remarks, we have this advantage in an engine, it can be made for any velocity, by attending to the relative proportions of its parts; those of a horse we cannot alter.

17. A horse when he treads a mill-path at the rate of 2½ miles an hour, will, on an average, raise about 150 lbs. by a cord hanging over a pulley; which is equivalent to 33, 000 lbs. one foot high in an hour. Boulton and Watt estimate this at 32,000. Tredgold, still lower, at 27,500. Taking the first measure, however, as a basis of comparison; putting d

for the diameter of the piston in inches, p for the pressure of the steam upon each square inch) diminished usually by about $\frac{1}{5}$ for friction and inertia), l for the length of the stroke of the piston in feet, n for the number of strokes in a minute: then, the power of the engine in "Horse-powers," (H P), is

$$\left.\begin{array}{l}(\text{H P}) = \cdot0000238\, d^2\, n\, p\, l, \text{ if it be a single stroke} \\ \text{H P} = \cdot000476\, d^2\, n\, p\, l, \text{ if it be a double stroke}\end{array}\right\} \text{engine.}$$

Example. Suppose $d = 20$ inches, $l = 3$ feet, $n = 36$, $p = 50$, and the engine one of double stroke. Then

$\cdot0000476 \times 20^2 \times 36 \times 50 \times 3 = 102\cdot816$, or nearly 103 horse-powers, the measure of the energy of the engine.

Mr. Boulton states that 1 bushel of Newcastle coals, containing 84 pounds, will raise 30 million pounds 1 foot high; that it will grind and dress 11 bushels of wheat; that it will slit and draw into nails 5 cwt. of iron; that it will drive 1000 cotton spindles, with all the preparation machinery, with the proper velocity; and that these effects are equivalent to the work of 10 horses.

18. The rule usually given to adjust the weight of the *fly-wheel*, is this:

Multiply the number of *Horse-powers* in the machine by 2000; divide the product by the square of the velocity in feet, per second, of the fly's circumference; the quotient will give its weight in hundredweights.

$$\text{Or, } 2000\,(\text{H P}) \div \left(\frac{\pi\, d\, n}{60}\right)^2 = w^t \text{ of fly.}$$

Thus, suppose the fly-wheel of a 20 horse-power engine to be 18 feet diameter, and to revolve 22 times in a minute; what should be its weight?

Here $\frac{18 \times 3\cdot1416 \times 22}{60} = 20\frac{1}{2}$ feet, nearly, velocity of circumference per second.

Whence $\frac{20 \times 2000}{(20\frac{1}{2})^2} = 90\cdot4$ cwt., weight of the fly-wheel required.

*** See, farther, pa. 261, preceding.

CHAP. XIV.

COMPARATIVE TABLES AND REMARKS ON STEAM-ENGINES, RAIL-ROADS, CANALS, AND TURNPIKE-ROADS.*

TABLE I.—*Maximum power of the steam of a cubic foot of water, in high pressure steam-engines.*

Temperature of steam.	Total force of steam in inches of mercury.	Force of steam in lbs. per sq. inch above the atmos.	Maximum mechanical power of the steam of a cubic ft. of water, in lbs. raised one foot high.		Proportion of the stroke to cut off the steam to obtain the maximum by expansion.	Pounds of Newcastle coal to convert 1 cubic foot of water into steam.
			When working at full pressure.	Acting expansively.		
	inches.	lbs.	lbs.	lbs.		lbs.
220°	35	2·5	negative.	negative.		8·5
234½	45	7·4	287,000			8·67
251	60	14·8	864,000	985,000	¾	8·87
275	90	29·7	1,495,000	1,927,000	7/12	9·16
292·8	120	44·5	1,830,000	2,540,000	½	9·37
307·7	150	59·3	2,054,000	2,988,000	9/20	9·55
320·2	180	74·2	2,202,000	3,326,000	5/12	9·7
343·6	240	104	2,444,000	3,832,000	⅜	9·98

This table shows the whole power of the steam of a cubic foot of water, when generated at different temperatures, in pounds raised one foot high, acting in a high pressure engine. The 6th column shows what proportion of the stroke the piston should work at full pressure, to obtain the maximum power by working expansively. But to obtain the greatest useful effect, the full pressure will require to be continued longer; and to an extent which depends on the quantity of friction of the additional machinery necessary to produce the useful effect. An engine ought to be made so that the communication between the cylinder and boiler could be cut off at any part of the stroke, from the one given in the table to full pressure, at the pleasure of the attendant, or according to the stress on the engine; instead of the usual method of straightening the steam passage by that bungling contrivance which has been very properly termed a *throttle-valve.*

* These valuable tables are extracted from *Tredgold on Rail-Roads.*

TABLE II.—*Maximum power of the steam of a cubic foot of water, in a condensing steam-engine.*

Temperature of steam.	Total force of steam in inches of mercury.	Force of steam in lbs. per sq. inch above the atmos.	Maximum mechanical power of the steam of a cubic ft. of water, in lbs. raised one foot high. When working at full pressure.	Maximum mechanical power of the steam of a cubic ft. of water, in lbs. raised one foot high. Acting expansively.	Proportion of the stroke to cut off the steam to obtain the maximum expansively.	Pounds of Newcastle coal to convert 1 cubic foot of water into steam.
	inches.	lbs.	lbs.	lbs.		lbs.
220°	35	2·5	2,134,000	3,350,000	4/11	8·5
234·5	45	7·4	2,230,000	3,636,000	[illegible]	8·67
251	60	14·8	2,366,000	3,961,000	7/33	8·87
275	90	29·7		4,379,000	10/51	9·16
292·8	120	44·5		4,590,000	2/9	9·37
307·7	150	59·3		4,819,000	10/56	9·55
320·2	180	74·2		4,932,000	3/17	9·7
343·6	240	104		5,162,000	4/15	9·98

TABLE III.—*Quantity of Coals equivalent to the horse power of 33,000lbs. raised one foot per minute in high pressure steam-engines, when the greatest possible effect is obtained.*

Temperature of steam.	Total force in inches of mercury.	Force in lbs. per square inch above the atmosphere.	Quantity of coal equivalent to a horse power. When working at full pressure.	Quantity of coal equivalent to a horse power. When acting expansively.	Pounds raised one foot high equivalent to the immediate power of the steam produced by 84lbs. of coal. When working at full pressure.	Pounds raised one foot high equivalent to the immediate power of the steam produced by 84lbs. of coal. When working expansively.
°	inches.	lbs.	lbs.	lbs.	lbs.	lbs.
234·5	45	7·4	480		2,780,000	
251	60	14·8	163	143	8,200,000	9,300,000
275	90	29·7	98	77	13,700,000	17,700,000
292·8	120	44·5	82	59	16,600,000	22,700,000
307·7	150	59·3	74	51	18,000,000	26,200,000
320·2	180	74·2	70	48	19,200,000	28,700,000
343·6	240	104	65	41½	20,500,000	32,200,000

TABLE IV.—*Quantity of Coals equivalent to the horse power of 33,000lbs. raised one foot per minute in condensing steam-engines, when the greatest possible effect is obtained.*

Temperature of steam.	Total force in inches of mercury.	Force in lbs. per square inch above the atmosphere.	Quantity of coal equivalent to a horse power. When working at full pressure.	Quantity of coal equivalent to a horse power. When acting expansively.	Pounds raised one foot high equivalent to the immediate power of the steam produced by 84lbs. of coal. When working at full pressure.	Pounds raised one foot high equivalent to the immediate power of the steam produced by 84lbs. of coal. When working expansively.
	inches.	lbs.	lbs.	lbs.	lbs.	lbs.
220°	35	2·5	63½	40½	21,000,000	33,100,000
234·5	45	7·4	62	38¼	21,400,000	35,200,000
251	60	14·8	60	35¾	22,400,000	37,500,000
275	90	29·7		33½		40,000,000
292·8	120	44·5		32½		41,000,000
307·7	150	59·3		32		42,400,000
320·2	180	74·2		31½		42,700,000
343·6	240	104·0		31		43,500,000

Remarks on Tables III *and* IV.—The columns showing the pounds an engine ought to raise one foot high, by the heat of one bushel of coals, were added chiefly for the purpose of comparison with actual practice. Now, it is stated, that after the most impartial examination for several years in succession, it was found that Woolf's engine at Wheal Abraham Mine, raised 44 millions of pounds of water, one foot high, with a bushel of coals. And, "the burning of one bushel of good Newcastle or Swansea coals, in Mr. Watt's reciprocating engines, working more or less expansively, was found, by the accounts kept at the Cornish mines, to raise from 24 to 32 millions of pounds of water one foot high; the greater or less effect depending upon the state of the engine, its size, and rate of working, and the quality of the coal."

We shall further add the results of half a year's reports taken, without selection, from Messrs. Lean's Monthly Reports on the work performed by the steam engines in Cornwall, with each bushel of coals. The numbers show the pounds of water raised one foot high with each bushel, from January to June 1818.

	January.	February.	March.	April.	May.	June.
	lbs. raised one foot.	lbs. raised one foot.	lbs. raised one foot.	lbs. raised one foot.	lbs. raised one foot.	lbs. raised one foot.
22 to 25 Common Engines average	22,188,000	22,424,000	21,898,000	22,982,000	23,608,000	23,826,000
Wheal Vor (Woolf's Engine)	30,834,000	26,158,000	29,511,000	26,064,000	29,032,000	30,336,000
Wheal Abraham (ditto)	41,847,000	35,364,000	30,445,000	32,723,000	31,520,000	34,352,000
Ditto (ditto)	27,942,000	28,009,000	26,978,000	23,626,000	29,702,000	34,846,000
Wheal Unity (ditto)	31,900,000	32,306,000				
Dalcouth Engine	42,622,000	41,354,000	40,429,000	41,888,000	38,233,000	38,143,000
Wheal Abraham Engine	32,239,000	36,180,000	35,715,000	33,934,000	33,714,000	34,291,000
United Mines Engine	36,296,000	31,830,000	31,427,000	38,564,000	33,967,000	30,105,000
Treskirby Engine	38,733,000	39,375,000	41,867,000	41,823,000	40,615,000	42,098,000
Wheal Chance Engine	28,496,000	32,319,000	33,594,000	33,932,000		35,797,000

These numbers are less than the immediate power of the engines, by the friction and loss of effect in working the pumps; hence in comparing them with Mr. Tredgold's table, it will be evident that he made his calculations from such data as can be realised in practice. It is known from experience, that a cubic foot of water can be converted into steam equal in force to the atmosphere, with 7 lbs. of Newcastle coal; but we also know the attention necessary to produce that effect, and therefore have assumed that $8\frac{1}{10}$ lbs. will be required for that purpose.

TABLE V.—*Showing the effects of a force of traction of 100 lbs. at different velocities, on canals, rail-roads, and turnpike-roads.**

Velocity of Motion.		Load moved by a power of 100 lbs.					
		On a Canal.		On a level Rail-way.		On a level Turnpike Road.	
Miles per hour.	Feet per second.	Total mass moved.	Useful effect.	Total mass moved.	Useful effect.	Total mass moved.	Useful effect.
		lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
2½	3·66	55,500	39,400	14,400	10,800	1,800	1,350
3	4·40	38,542	27,361	14,400	10,800	1,800	1,350
3½	5·13	28,316	20,100	14,400	10,800	1,800	1,350
4	5·86	21,680	15,390	14,400	10,800	1,800	1,350
5	7·33	13,875	9,850	14,400	10,800	1,800	1,350
6	8·80	9,635	6,840	14,400	10,800	1,800	1,350
7	10·26	7,080	5,026	14,400	10,800	1,800	1,350
8	11·73	5,420	3,848	14,400	10,800	1,800	1,350
9	13·20	4,282	3,040	14,400	10,800	1,800	1,350
10	14·66	3,468	2,462	14,400	10,800	1,800	1,350
13·5	19·9	1,900	1,350	14,400	10,800	1,800	1,350

This table is intended to exhibit the work that may be performed by the same mechanical power, at different velocities, on canals, rail-roads, and turnpike-roads. Ascending and descending by locks or canals, may be considered equivalent to the ascent and descent of inclinations on rail-roads and turnpike-roads. The load carried, added to the weight of the vessel or carriage which contains it, forms the total mass moved; and the useful effect is the load. To find the effect on canals at different velocities, the effect of the given power at one velocity being known, it will be as $3^2 : 2{\cdot}5^2 :: 55,500 : 38,542$. The mass moved being very nearly inversely as the square of the velocity.

This table shows, that when the velocity is 5 miles per hour, it requires less power to obtain the same effect on a rail-way than on a canal; and the lower range of figures is added to show the velocity at which the effect on a canal is only equal to that on a turnpike-road. By comparing the power and tonnage of steam vessels, it will be found that the rate of decrease of power by increase of velocity, is not very

* The force of traction on a canal varies as the square of the velocity; but the mechanical power necessary to move the boat is usually reckoned to increase as the cube of the velocity. On a rail-road or turnpike, the force of traction is constant; but the mechanical power necessary to move the carriage, increases as the velocity.

distant from the the truth; but we know that in a narrow canal the resistance increases in a more rapid ratio than as the square of the velocity.*

TABLE VI.—*Showing the maximum quantity of labour a Horse of average strength is capable of performing, at different velocities, on canals, rail-ways, and turnpike-roads.*

Velocity in miles per hour.	Duration of the day's work at the preceding velocity.	Force of traction in lbs.	Useful effect of one horse working one day, in tons drawn one mile.		
			On a canal.	On a level rail-way.	On a good level turn-pike road.
miles.	hours.	lbs.	tons.	tons.	tons.
$2\frac{1}{2}$	$11\frac{1}{2}$	$83\frac{1}{3}$	520	115	14
3	8	$83\frac{1}{3}$	243	92	12
$3\frac{1}{2}$	$5\frac{9}{16}$	$83\frac{1}{3}$	153	82	10
4	$4\frac{1}{2}$	$83\frac{1}{3}$	102	72	9
5	$2\frac{9}{10}$	$83\frac{1}{3}$	52	57	7·2
6	2	$83\frac{1}{3}$	30	48	6·0
7	$1\frac{1}{6}$	$83\frac{1}{3}$	19	41	5·1
8	$1\frac{1}{8}$	$83\frac{1}{3}$	12·8	36	4·5
9	$\frac{9}{10}$	$83\frac{1}{3}$	9·0	32	4·0
10	$\frac{3}{4}$	$83\frac{1}{3}$	6·6	28·8	3·6

Where horse power is employed for the higher velocities, the animals ought to be allowed to acquire the speed as gradually as possible at the first starting. This simple expedient will save the proprietors of horses much more than they are aware of; and it deserves their attention to consider the best

* According to the interesting researches of Du Buat, the resistance to the motion of boats even in canals may be regarded as proportional to the square of the velocity, or R as V^2 nearly, provided R be made to depend upon the transverse sections of the vessel and the canal in which it moves, and a constant quantity K determinable by experiment. If C be the vertical section of the canal, b the vertical section of the immersed portion of the boat or barge; then $R = K \div \left(\frac{C}{b} + 2\right)$. The medium of Du Buat's experiments gives $K = 8·46$, or $R = 8·46 \div \left(\frac{C}{b} + 2\right)$; but those experiments were not so numerous and varied as might be wished. See *Principes d'Hydraulique*, tom. ii. pp. 340, 342, &c.

In cases connected with these, or kindred inquiries, where velocities in miles per hour, are to be reduced into velocities in feet per second, or the contrary; the rules at pa. 100 of this work will be found useful.

mode of feeding and training horses for performing the work with the least injury to their animal powers.

To compare the preceding table with practice at the higher velocities, it will be necessary to have the total mass moved, which is one-third more than the useful effect in this table. Now, the actual rate at which some of the quick coaches travel, is 10 miles an hour; the stages average about 9 miles; and a coach with its load of luggage and passengers amounts to about 3 tons; therefore the average day's work of 4 coach horses is 27 tons drawn one mile, or $6\frac{3}{4}$ tons drawn one mile by one horse. The table gives 3·6 tons, added $\frac{1}{3}$ of 3·6 = 4·8 tons drawn one mile for the extreme quantity of labour for a horse at that speed, upon a good level road; from which should be deducted the loss of effect in ascending hills, heavy roads, &c. which will make the actual labour performed by a coach-horse average about double the maximum given by the table. The consequences are well known.

According to Mr. Bevan's observations, the horses on the Grand Junction Canal draw 617 tons one mile, at the velocity of 2·45 miles per hour.

According to Mr. Tredgold, if v be the maximum velocity of a horse, and v any other velocity, the immediate power of a horse is $250\, v \left(1 - \frac{v}{\mathrm{v}}\right)$; and, when the weight of the vessel or carriage is to the weight of the load, as $n : 1$, we have $\frac{250\, v \left(1 - \frac{v}{\mathrm{v}}\right)}{1 + n}$ = the effective power; and d being the hours the horse works in one day, the day's work will be $\frac{250\, d v \left(1 - \frac{v}{\mathrm{v}}\right)}{1 + n}$ in lbs. raised 1 mile, and $250 \left(1 - \frac{v}{\mathrm{v}}\right)$ = the force of traction in lbs. But if the force were immediately applied, the value of v would be $\frac{14{\cdot}7}{\sqrt{d}}$; and to find the value when the waggons alone are moved, we have $1 : \frac{1}{\sqrt{1 +}} :: \frac{14{\cdot}7}{\sqrt{d}} : \frac{14{\cdot}7}{\sqrt{d(1+n)}} = \mathrm{v}$; whence the day's work is $\frac{250\, d v}{1 + n} \left(1 - \frac{v \sqrt{d(1 + n)}}{14{\cdot}7}\right)$; which is a maximum when

$\frac{96}{v^2(1+n)} = d$. Consequently, when the velocity is given, we have $\frac{96}{v^2(1+n)}$ equal the duration of the day's work in hours; and $\frac{8000}{v(1+n)^2}$ = the effective day's work; and 250 $\left(1 - \frac{9 \cdot 8}{14 \cdot 7}\right) = 83\frac{1}{3}$ lbs. But we may assume n to be always so near $\frac{1}{3}$, as not to affect the result; and then, $\frac{72}{v^2} = d$, and $\frac{4500}{v}$ = the day's work in lbs. or very nearly $\frac{2}{v}$ tons raised one mile. This being conbined with the numbers of the preceding table, gives the effect of a horse on canals, rail-roads, and turnpike-roads.

*** For farther information on the subject of rail-roads, their nature, construction, and comparative utility, we beg to refer the reader to Mr. Tredgold's work, as sober and judicious in its comparisons, and sound and correct in its scientific deductions. Some interesting information, also, and a comparative table of the resistances on several rail-roads now in use, may be seen in Mr. H. R. Palmer's "Description of a Rail-way on a new Principle," published in 1824.

CHAP. XV.

ACTIVE AND PASSIVE STRENGTH.

SECTION I. *Active Strength, or Animal Energy, as of Men, Horses, &c.*

1. The force obtained through the medium of animal agency, evidently varies, not only in different species of animals, but, also, in different individuals. And this variation depends, first, on the particular constitution of the individual, and upon the complication of causes, which may influence it; secondly, upon the particular dexterity acquired by habit. It is plain, that such a variation cannot be subjected to any law, and that there is no expedient to which we can have recourse, but that of seeking mean results.

Secondly, the force varies according to the nature of the labour. Different muscles are brought into action in different gestures and positions of an animal which labours; the weight itself of the animal machine is an aid in some kinds of labour, and a disadvantage in others: whence it is not surprising that the force exerted is different, in different kinds of work. Thus the force exerted by a man is different, in carrying a weight, in drawing or pushing it horizontally, and in drawing or pushing it vertically.

Thirdly, the force varies according to the duration of the labour. The force, for example, which man can exert in an effort of a few instants, is different from that which he can maintain equably in a course of action continued, or interrupted only by short intervals, for a whole day of labour, without inducing excessive fatigue. The former of these, may be called *Absolute Force*, the latter *Permanent Force*. It is of use to become acquainted with them both, as it is often advantageous to avail ourselves sometimes of the one, sometimes of the other.

Lastly, the force varies according to the different degrees of velocity, with which the animal, in the act of labouring, moves either its whole body, or that part of it which operates. The force of the animal is the greatest, when it stands still; and becomes weaker as it moves forward, in proportion to its speed; the animal acquiring, at last, such a degree of velocity as renders it incapable of exerting any force.

2. Let φ be a weight equivalent to the force, which a man can exert, standing still; and let v be the velocity with which,

if he proceeds, he is no longer capable of exerting any force: also, let F be a weight equivalent to the force, which he exerts, when he proceeds, equably, with a velocity v.

Then F will be a function of v, such that, 1st, it decreases whilst v increases; 2ndly, when $v = 0$, then $F = \phi$; 3rdly, when $v = V$, $F = 0$.

3. Upon the nature of this function, we have the three following suppositions

1. $F = \phi . \left(1 - \frac{v}{V}\right)$. (Bouguer, *Man. des. Vais.*)

2. $F = \phi \left(1 - \frac{v^2}{V^2}\right)$. (Euler, *Nov. Comm. Pet.* tom. III).

3. $F = \phi \left(1 - \frac{v}{V}\right)^2$ (Ib. tom. VIII; and *Act. of Rowers*).

4. *Coroll.* 1. The effect of the permanent force being measured by the product F v, the expression for the effect will be one of the three following, accordingly as one or other of the suppositions is adopted.

1. $FV . \left(1 - \frac{F}{\phi}\right)$, or $\phi v \left(1 - \frac{v}{V}\right)$.

2. $FV . \sqrt{\left(1 - \frac{F}{\phi}\right)}$, or $\phi v \left(1 - \frac{v^2}{V^2}\right)$.

3. $FV . \left(1 - \sqrt{\frac{F}{\phi}}\right)$, or $\phi v \left(1 - \frac{v}{V}\right)^2$.

5. *Coroll.* 2. To know the weight, with which a man should be loaded, or the velocity, with which he ought to move, in order to produce the greatest effect, we must make $d . Fv = 0$.

Whence we shall have

1. $F = \frac{1}{2}\phi$; and $v = \frac{1}{2}V$.

2. $F = \frac{2}{3}\phi$; and $v = \frac{1}{\sqrt{3}} . V = 0{\cdot}5773\ V$.

3. $F = \frac{4}{9}\phi$; and $v = \frac{1}{3}V$.

6. *Coroll.* 3. And the value of the greatest effect will be, according to the 1st hypothesis. . . . $= \frac{1}{4}\phi V$:

. . . . 2nd . . . . $= \frac{2}{3\sqrt{3}}\phi V = 0{\cdot}3836\,\phi V$:

. . . . 3rd . . . . $= \frac{4}{27}\phi V$.

But which of the three suppositions ought we to prefer? And are we certain that any of them approximates to the true law of nature?

M. Schulze made a series of experiments with a view to the determination of this point,* and with regard to men decided in favour of the last of Euler's formulæ:

$$\text{viz. } F = \varphi\left(1 - \frac{v}{V}\right)^2.$$

As the experiments of this philosopher are very little known in England, I shall here present his brief account of them.

7. To make the experiments on human strength, he took at random, 20 men of different sizes and constitutions, whom he measured and weighed. The result is exhibited in the following table.

Order	Height			Weight	Order	Height			Weight
1	5′	3″	4‴	122	11	5′	9″	7‴	132
2	5	2	8	134	12	5	1	4	157
3	5	7	2	165	13	5	3	2	175
4	5	5	0	131	14	5	4	1	117
5	5	11	2	177	15	5	10	8	192
6	6	0	4	158	16	5	0	3	133
7	5	8	3	180	17	4	11	2	147
8	5	2	1	117	18	5	3	9	124
9	5	4	8	140	19	5	6	0	163
10	5	0	4	126	20	5	10	1	181

Here the heights are expressed in feet (marked ′), inches (″), twelfths (‴), the feet being those of Rhinland, each 12·35 English inches. The weights are in pounds, which are to our avoirdupois lbs. as 30 to 29.

To find the strength that each of these men might exert to raise a weight vertically, Mr. Schulze made the following experiments:

He took various weights increasing by 10 lbs. from 150 lbs. up to 260 lbs.; all these weights were of lead having circular and equal bases. To use them with success in the proposed experiments he had at the same time a kind of bench made, in the middle of which was a hole of the same size as the base of the weights: this hole was shut by a circular cover when pressed against the bench; at other times it was kept at about the distance of a foot and a half above the bench

* Mem. Acad. Scienc. Berlin, for 1783.

by means of a spring and some iron bars. To prevent the weight with which this cover was loaded during the experiment from forcing down the cover, lower than the level of the surface of the bench, he had several grooves made in the four iron bars, which sustained the cover, and which at the same time served to hold up the cover at any height where it might arrive by the pressure of the springs as soon as the pressure of the weight ceased.

After having laid the 150 lbs. weight on the cover, and the other weights in succession increasing by 10 lbs. up to 250 lbs. he made the following experiments with the men whose size and weight are given above, by making them lift up the weights as vertically as possible all at once, and by observing the height to which they were able to lift them. The annexed table gives the heights observed for the different weights marked at its head.

	150		160		170		180		190		200		210		220		230		240		250
	″	‴	″	‴	″	‴	″	‴	″	‴	″	‴	″	‴							
1	7	9	6	4	4	11	4	4	3	8	2	8	1	1							
2	7	10	6	6	5	7	4	7	3	11	2	5	0	5	″	‴	″	‴			
3	7	9	7	3	6	5	5	9	4	11	4	0	3	0	1	7	0	3	″	‴	
4	8	3	7	6	7	2	5	10	5	3	4	7	4	0	3	8	3	1	1	4	
5	12	4	11	1	9	7	8	5	7	10	7	1	5	10	4	7	3	2	1	3	
6	14	5	14	0	13	5	12	8	11	5	10	1	8	6	6	6	4	1	0	1	
7	12	11	11	3	10	5	9	3	8	1	6	9	5	3	3	8	1	11	0	2	
8	11	9	10	2	9	4	8	11	8	1	6	11	5	10	5	1	3	2	1	0	
9	9	5	8	3	7	1	5	6	4	1	2	9	1	3							
10	8	1	6	5	4	7	3	9	2	5	1	7	0	4							

This table proves that the size of the men employed to raise the weights vertically has considerable influence on the height to which they brought the same weight. We find also that the height diminishes in a much more considerable ratio than the weight increases: and we may therefore conclude, that it is advantageous to employ large men when it becomes necessary to draw vertically from below upwards: and on the contrary, it is more advantageous to employ men of a considerable *weight*, when it is required to lift up loads by means of a pulley about which a cord passes, that the workmen draw n a vertical direction, from above downwards. To find the aosolute strength of these men in a horizontal direction, Mr. Schulze proceeded thus:

Having fixed over an open pit a brass pulley extremely well made of 15 inches diameter, whose axis, made of well polished steel to diminish the friction, was $\frac{3}{4}$ inch in diameter; he passed over this pulley a silk cord, worked with care, to give it both the necessary strength and flexibility. One of the ends of this cord carried a hook to hang a weight to it which hung vertically in the pit, whilst the other end was held by one of the 20 men, who in the first order of the following experiments made it pass above his shoulders; instead of which, in the second, he simply held it by his hands.

Mr. S. had taken the precaution to construct this in such a manner that the pulley might be raised or lowered at pleasure, in order to keep the end of the cord held by the man always in a horizontal direction, according as the man was tall or short, and exerted his strength in any given direction.

He had made the necessary arrangements so as to be able to load successively the basin of a balance which was attached to the hook at the end of the cord which descended into the pit, whilst the man who held the other end of this cord employed all his strength without advancing or receding a single inch.

The following table gives the weights placed in the basin when the workmen were obliged to give up, having no longer sufficient strength to sustain the pressure occasioned by the weight. To proceed with certainty, Mr. S. increased the weight each time by five pounds, beginning from 60, and took the precaution to make this augmentation in equal intervals of time; having always precisely a space of 10 seconds between them. The result of these observations repeated several days in succession, is contained in the following tables.

I. When the cord passed over the shoulders of the workmen:

Order.	lbs.	Order.	lbs.	Order.	lbs.	Order.	lbs.
1	95	6	100	11	95	16	95
2	105	7	115	12	100	17	100
3	110	8	105	13	110	18	90
4	100	9	95	14	90	19	100
5	105	10	90	15	110	20	100

II. When the cord was simply held before the man:

Order.	lbs.	Order.	lbs.	Order.	lbs.	Order.	lbs.
1	90	6	100	11	90	16	90
2	105	7	110	12	90	17	90
3	105	8	100	13	100	18	85
4	90	9	90	14	85	19	100
5	95	10	85	15	105	20	100

These two tables show that men have less power in drawing a cord before them than when they make it pass over their shoulders; they show, also, that the largest men have not always the greatest strength to hold, or to draw in a horizontal direction, by means of a cord. To obtain the absolute velocity of these 20 men, Mr. S. proceeded as follows:

Having measured very exactly a distance of 12000 Rhinland feet in a plain nearly level, he caused these 20 men to march with a fair pace but without running, and so as to continue during the period of four or five hours; the following is the time employed in describing this space, with the velocity resulting for each of them.

Order	Time.	Veloc.	Order	Time.	Veloc.	Order	Time.	Veloc.
1	40·18	4·94	8	40· 9	4·99	15	36·17	5·51
2	41·12	4·85	9	40·20	4·96	16	41·28	4·82
3	39· 8	5·55	10	40·51	4·90	17	42·25	4·71
4	39·40	5·04	11	36·17	5·51	18	40·19	4·98
5	34·19	5·88	12	38·11	5·24	19	39·57	5·01
6	35·11	5·68	13	38· 5	5·25	20	37·51	5·29
7	38· 7	5·25	14	37· 1	5·40			

It is necessary to mention with regard to these experiments, that Mr. Schulze took care to place at certain distances persons in whom he could place confidence, in order to observe whether these men marched uniformly, and sufficiently quick, without running.

Having thus obtained, not only the absolute force, but the absolute velocity also, of several men, he took the following method to determine their relative force.

He made use of a machine composed of two large cylinders of very hard marble, which turned round a vertical cylinder of wood, and moved by a horse, which described in his march a circle of 10 Rhinland feet. This machine appeared the most proper to make the subsequent experiments, which serve to determine the relative strength that the men had employed to move this machine, and which is used hereafter to determine which of Euler's two formula ought to be preferred.

To obtain this relative force, he took here the same pulley which served in the preceding experiments, by applying a cord to the vertical cylinder of wood, and attaching to the other end of this cord, which entered into an open pit, a sufficient weight to give successively to the machine different velocities.

Having applied in this manner a weight of 215 lbs., the machine acquired a motion, which after being reduced to an uniform velocity taking into account the acceleration of the weight, of the friction, and of the stiffness of the cord, gave 2·41 feet velocity; and having applied in the same manner a weight of 220 lbs. the resulting uniform motion gave a velocity of 2·47 feet. These two limits are mentioned because they serve as a comparison with what immediately follows: Mr. S. began these experiments with a weight of 100 lbs. and increased it by 5 every time from that number up to 400 lbs.

He made this machine move by the seven first of his workmen, placing them in such a way that their direction remained almost always perpendicular to the arm on which was attached the cord which passed over their shoulders in an almost horizontal direction.

Thus situated, they made 281 turns with this machine in two hours, which gave for their relative velocity $v = 2{\cdot}45$ feet per second. We have also the absolute force, or ϕ, from these 7 men, by the above table $= 730$ lbs.: and their absolute velocity or $\text{v} = 5{\cdot}30$ feet.

Therefore, by substituting these values in the first formula, we find the relative force $\text{F} = 205$ lbs. which agrees very well with what we have just found above.

If instead of this first formula the second be taken, it gives $\text{F} = 153$ lbs. which is far too little.

By this it is evident, that the last of Euler's two formulæ is to be preferred in all respects. Mr. Schulze made a great number of combinations, and almost always found the same effect.

Dividing the 205 lbs. which we have just found by 7 the number of workmen, we get 29 lbs. for the relative force with 2·45 feet relative velocity for each man, which is rather more than the values commonly adopted in the computation of machinery. A number of other observations on different machines, have given the same result; that is to say, we must value the mean human strength at 29 or 30 lbs. with a velocity of $2\frac{1}{10}$ feet per second.

To obtain the ratio of the strength of a horse to that of a man, Mr. Schulze proceeded in a similar manner; but his results, in reference to that enquiry, are neither so correct nor so interesting.

8. In the first volume of my *Mechanics*, I stated the average force of a man at rest to be 70 lbs., and his utmost walking velocity when unloaded to be about 6 feet per second; and thence inferred that a man would produce the greatest momentum when drawing $31\frac{1}{9}$ lbs. along a horizontal plane with a velocity of 2 feet per second. But that is not the most advantageous way of applying human strength.

9. Dr. Desaguliers asserts, that a man can raise of water or any other weight about 550 lbs., or one hogshead (weight of the vessel included), 10 feet high in a minute; this statement, though he says it will hold good for 6 hours, appears from his own facts to be too high; and is certainly such as could not be continued one day after another. Mr. Smeaton considers this work as the effort of haste or distress; and reports that 6 good English labourers will be required to raise 21141 solid feet of sea water to the height of four feet in four hours: in this case the men will raise a very little more than 6 cubic feet of fresh water each to the height of 10 feet in a minute. Now, the hogshead containing about $8\frac{1}{2}$ cubic feet, Smeaton's allowance of work proves less than that of Desaguliers in the ratio of 6 to $8\frac{1}{2}$ or 3 to $4\frac{1}{4}$. And as his good English labourers who can work at this rate are estimated by him to be equal to a double set of common men picked up at random, it seems proper to state that, with the probabilities of voluntary interruption, and other incidents, a man's work for several successive days ought not to be valued at more than half a hogshead raised 10 feet high in a minute. Smeaton likewise states that two ordinary horses will do the work in three hours and twenty minutes, which amounts to little more than two hogsheads and a half raised 10 feet high in a minute. So that, if these statements be accurate, one horse will do the work of five men.

Mr. Emerson affirms, that a man of ordinary strength turning a roller by the handle can act for a whole day against a resistance equal to 30 lbs. weight; and if he works 10 hours a day he will raise a weight of 30 lbs. through $3\frac{1}{2}$ feet in a second of time; or, if the weight be greater, he will raise it to a proportionally less height. If two men work at a windlass or roller, they can more easily draw up 70 lbs. than one man can 30 lbs.; provided the elbow of one of the handles be at right angles to that of the other. Men used to bear loads, such as porters, will carry from 150 lbs. to 200 or 250 lbs. according to their strength. A man cannot well draw more than 70 lbs. or 80 lbs. horizontally: and he cannot thrust with a greater force acting horizontally at the height of his shoulders than 27 or 30 lbs. But one of the most advantageous ways in which a man can exert his force is to sit and pull towards him nearly horizontally, as in the action of rowing.

M. Coulomb communicated to the French National Institute the results of various experiments on the quantity of action which men can afford by their daily work, according to the different manners in which they employ their strength. In the first place he examined the quantity of action which men can produce when, during a day, they mount a set of steps or stairs, either with or without a burthen. He found that the quantity of action of a man who mounts without a burthen, having only his own body to raise, is double that of a man loaded with a weight of 68 kilogrammes, or 150 lbs. averdupois, both continuing at work for a day. Hence it appears how much, with equal fatigue and time, the total or absolute effort may obtain different value by varying the combinations of effort and velocity.

But the word *effect* here denotes the total quantity of labour employed to raise, not only the burthen, but the man himself; and as Coulomb observes, what is of the greatest importance to consider is the *useful effect*, that is to say, the total effect deducting the value which represents the transference of the weight of the man's body. This total effect is the greatest possible when the man ascends without a burthen; but the *useful* effect is then nothing: it is also nothing if the man be so much loaded as to be scarcely capable of moving: and consequently there exists between these two limits a value of the load such that the useful effect is a maximum. M. Coulomb supposes that the loss of quantity of action is proportional to the load (an hypothesis which experience confirms), whence he obtains an equation which, treated according to the rules of maxima and minima, gives 53 kilogrammes (117 lbs. averd.) for the weight with which the man ought to be loaded, in order to produce during one day, by ascending stairs, the greatest useful effect: the quantity of action which results from this determination has for its value 56 kilogrammes ($123\frac{1}{2}$ lbs. averd.) raised through one kilometer, or nearly 1094 yards. But this method of working is attended with a loss of three-fourths of the total action of men, and consequently costs four times as much as work in which, after having mounted a set of steps without any burthen, the man should suffer himself to fall by any means, so as to raise a weight nearly equal to that of his own body.

From an examination of the work of men walking on a horizontal path, with or without a load, M. Coulomb concludes that the greatest quantity of action takes place when the men walk being loaded: and is to that of men walking under a load of 58 kilogrammes (128 lbs. averd.) nearly as 7 to 4. The weight which a man ought to carry in order to produce the greatest *useful effect*, namely, that effect in which the quantity of action relative to the carrying his own weight is deducted from the total effect, is 50·4 kilogrammes, or 111·18 lbs. averdupois.

There is a particular case which always obtains with respect to burthens carried in towns, viz. that in which the men, after having carried their load, return unloaded for a new burthen. The weight they should carry in this case, to produce the greatest effect, is 61·25 kilogrammes ($135\frac{1}{8}$ lbs. averd.) The quantity of useful action in this case compared with that of a man who walks freely and without a load, is nearly as 1 to 5, or, in other words, he employs to pure loss $\frac{4}{5}$ of his power. By causing a man to mount a set of steps freely and without burthen, his quantity of action is at least double of what he affords in any other method of employing his strength.

When men labour in cultivating the ground, the whole quantity afforded by one during a day amounts to 100 kilogrammes elevated to one kilometer, that is 220·6 lbs. raised 1094 yards. M. Coulomb comparing this work with that of men employed to carry burthens up an ascent of steps, or at the pile-engine, finds a loss of about $\frac{1}{20}$ part only of the quantity of action, which may be neglected in researches of this kind.

In estimating mean results we should not determine from experiments of short duration, nor should we make any deductions from the exertions of men of more than ordinary strength. The mean results have likewise a relation to climate. "I have caused," says M. Coulomb, "extensive works to be executed by the troops at Martinico, where the thermometer (of Reaumur) is seldom lower than 20° (77° of Fahrenheit). I have executed works of the same kind by the troops in France: and I can affirm that under the fourteenth degree of latitude, where men are almost always covered with perspiration, they are not capable of performing half the work they could perform in our climate."

10. Entirely according with these are the experiments of Regnier, by means of a Dynamometer, the results of which not only establish the superiority of civilized men over savages, but that of the Englishman over the Frenchman. The following is reduced from one of Regnier's tables of mean results.

STRENGTH.	With the hands.		With the reins.	
	lbs.	oz.	lbs.	oz.
Savages, of Van Dieman's Land..	30	6	0	0
——— New Holland.........	51	8	14	8
——— Timor	58	7	16	2
Civilized men: French	69	2	22	1
——— English	71	4	23	8

11. A porter in London is accustomed to carry a burthen of 200 lbs. at the rate of three miles an hour: and a couple of chairmen continue at the pace of four miles an hour, under a load of 300 lbs. Yet these exertions, Professor Leslie remarks are greatly inferior to the subsultory labour performed by porters in Turkey, the Levant, and generally on the shores of the Mediterranean. At Constantinople, an Albanian porter will carry 800 or 900 lbs. on his back, stooping forward, and assisting his steps by a short staff. Such loads, however, are carried for very short intervals. At Marseilles it is affirmed that four porters carry the immense load of nearly two tons, by means of soft hods passing over their heads, and resting on their shoulders, with the ends of poles from which the goods are suspended.

12. With regard to the magnitude of the comparative efforts of man in different employments, the late Mr. Robertson Buchanan ascertained, that in working a pump, in turning a winch, in ringing a bell, and rowing a boat, the Dynamic results are as the numbers 100, 167, 227, and 248.

According to the interesting experiments described in M. Hachette's *Traité des Machines*, the dynamic unit being *the weight of a cubic metre of water raised to the height of one metre*, [that is, 2208 lbs. avoird. or 4 hogsheads raised to the height of 3·281 feet, or 1·3124 hogsheads to the height of 10 feet], we have the following measures, at a medium, of the daily actions of men.

	Dyn. Unit.
1. A man marching $7\frac{1}{2}$ hours on a slope of 7 degrees, with a load of from 15 to 18 lbs.	225
2. Marching in a mountainous country without load	140
3. Carrier of wood up a ladder, his weight 128, his load 117 lbs.	109
4. Carrier of peat up steps, his own weight comprized, 112 to	120
5. Man working at the cord of a pulley to raise the ram of a pile engine: three examples	75 40 48
6. A man drawing water from a well by means of a cord	$7\frac{1}{4}$
7. Man working at a capstan	116
8. Man working at a capstan to raise water, mean of 24	110

The unit of transport being the weight of a cubic metre of water, carried a metre (2208 lbs. 3·281 feet) upon a horizontal road, we have for the daily action.

	Dyn. Unit.
1. A man travelling without load on a flat road, his weight 154 lbs. his journey $31\frac{1}{4}$ miles	3500
2. A soldier, carring from 44 to 55 lbs. travelling $12\frac{1}{2}$ miles, 1600 to	1900
3. Ditto a forced march of 25 miles	2800
4. A French coal-porter, weight of the man not included, 792 to	880
5. Porter with wheel-barrow, weight of the man not included	1015
6. Porters with a sledge	627
7. A man drawing a boat on a canal; 110310 lbs. conveyed $6\frac{7}{8}$ miles	550000

14. Mr. *B. Bevan*, an able engineer, has made experiments on the application of human energy to the use of augers, gimblets, screw-drivers, &c. He has presented to the public the following list, as a specimen; premising that many ordinary operations are performed in a short space of time, and may therefore be done by greater exertion than if a longer time was necessary. Thus a person, for a short time, is able to use a tool or instrument called

	lbs.
A drawing-knife, with a force of	100
An auger, with two hands	100
A screw-driver, one hand	84

	lbs.
A common bench vice handle	72
A chisel and awl, vertical pressure	72
A windlass, handle revolving	60
Pincers and pliers, compression	60
A hand-plane, horizontally	50
A hand or thumb-vice	45
A hand-saw	36
A stock-bit, revolving	16
Small screw-drivers, or twisting by the thumb and fore-finger only	14

15. M. Morisot, informs us that the time employed by a French stone masons' sawyer, to make a section of a square toise (40·89 sq. feet Eng.) in different stones, is as below: viz.

	hours.
Calcareous stone, equal grain, spec. grav. 2200	45
——————— hard, spec. grav. 2300	62
Liais, ditto hard, fine grain, spec. grav. 2400	67
Pyrenean alabaster, the softest of the marbles	56
Normandy granite	504
Granite from Vosges	700
Red and green porphyry	1177

The workmen ordinarily made 50 oscillations in a minute; each stroke about 15¾ inches.

16. Hassenfratz assigns 13 kilogrammes as the mean effort of such a man; but M. Navier, in his new edition of *Belidor, Architecture Hydraulique*, regards this estimate as too high. If this were correct, the daily quantity of action of the sawyer would be equivolent to 376 kilogrammes elevated to a kilometre (818 lbs. raised ⅝ of a mile), a quantity more than triple that of a man working at a winch. M. Navier gives, as a more correct measure of this labour for 12 hours, 188 kilogrammes raised a kilometre; half the former measure. But all this is probably very vague.

17. Among quadrupeds, those which are employed to produce a mechanical effect are, the dog, the ass, the mule, the ox, the camel, and the horse. Of these the *horse* is the only one, so far as we are aware, whose animal energy has been subjected to cautious experiments; and even with regard to this noble animal, opinions as to actual results are very much afloat. The dynamic effort of the horse is, however, probably, about 6 times that of a strong and active labourer. Desaguliers states the proportion as 5 to 1, coinciding with the deductions of Smeaton. The French authors usually

regard 7 men as equivalent to one horse. As a fair mean between these, I assumed in vol. I. of my *Mechanics* the proportion of 6 to 1, and stated the strength of a horse as equivalent to 420 lbs. at a dead pull. But the proportion must not be regarded as constant; but, obviously, varies much according to the breed and training of the animal, as well as according to the nature of the work about which he is employed. Thus the worst way, as De la Hire observed, of applying the strength of a horse is to make him carry a weight up a steep hill; while the organization of a man fits him very well for that kind of labour: hence *three* men climbing up such a hill with a weight of 100 lbs. each, will proceed faster than a horse with a load of 300 lbs.

18. In the memoirs of the French Academy for 1703, are inserted the comparative observations of M. Amontons, on the velocity of men and of horses; in which he states the velocity of a horse loaded with a man and walking to be rather more than $5\frac{1}{4}$ feet per second, or $3\frac{1}{2}$ miles per hour, and when going a moderate trot with the same weight to be about $8\frac{1}{2}$ feet per second, or about 6 miles per hour. These velocities, however, are somewhat less than what might have been taken for the mean velocities.

19. But the best way of applying the strength of horses is to make them draw weights in carriages, &c. To this kind of labour, therefore, the enquiries of experimentalists should be directed. A horse put into harness and making an effort to draw, bends himself forward, inclines his legs, and brings his breast nearer to the earth; and this so much the more as the effort is the more considerable. So that when a horse is employed in drawing, his effort will depend, in some measure, both upon his own weight and that which he carries on his back.

Indeed it is highly useful to load the back of a drawing horse to a certain extent; though this, on a slight consideration, might be thought to augment unnecessarily the fatigue of the animal: but it must be recollected that the mass with which the horse is charged vertically is added in part to the effort which he makes in the direction of traction, and thus dispenses with the necessity of his inclining so much forward as he must otherwise do: and may, therefore, under this point of view, relieve the draught more than to compensate for the additional fatigue occasioned by the vertical pressure. Carmen, and waggoners in general, are well aware of this, and are commonly very careful to dispose of the load in such a manner that the shafts shall throw a due proportion of the weight on the back of the shaft horse. This is most efficaciously accomplished at Yarmouth, in Norfolk, where a number of narrow streets connecting the market-place with the quay, have led to the invention and use of the low, strong, narrow carts, thence denominated *Yarmouth carts*, drawn by one horse; and on which the loads are frequently shifted, especially when the vehicles pass over the bridge, in order to give the animals better foot-hold, and consequently a greater dynamic effort.

20. The best disposition of the traces during the time a horse is drawing is to be perpendicular to the position of the collar upon his breast and shoulders: when the horse stands at ease, this position of the traces is rather inclined upwards from the direction of the road; but when he leans forward to draw the load, the traces should then become nearly parallel to the plane over which the carriage is to be drawn; or, if he be employed in drawing a sledge, or any thing without wheels, the inclination of the traces to the road, should (from the table at pa. 241) be about $18\frac{1}{2}°$, when the friction is one-third of the pressure. If the relation of the friction to the pressure be different from this, the same table will exhibit the angle which the traces must make with the road.

21. When a horse is made to move in a circular path, as is often practised in mills and other machines moved by horses, it will be necessary to give the circles which the animal has to walk round the greatest diameter that will comport with the local and other conditions to which the motion must be subjected. It is obvious, indeed, that, since a rectilinear motion is the most easy for the horse, the less the line in which he moves is curved, with the greater facility he will walk over it, and the less he need recline from a vertical position: and besides this, with equal velocity the centrifugal force will be less in the greatest circle, which will proportionally diminish the friction of the cylindrical part of the trunnions, and the labour of moving the machine. And, further, the greater the diameter of the horse-walk, the nearer the chord of the circle in which the horse draws is to coincidence with the tangent, which is the most advantageous position of the line of traction. On these accounts it is that, although a horse *may* draw in a circular walk of 18 feet diameter, yet in general it is advisable that the diameter of such a walk should not be less than 25 or 30 feet; and in many instances 40 feet would be preferable to either.

22. It has been stated by Desaguliers and some others, that a horse employed daily in drawing nearly horizontally can move, during eight hours in the day, about 200 lbs. at the rate of $2\frac{1}{2}$ miles per hour, or $3\frac{2}{3}$ feet per second. If the weight be augmented to about 240 or 250 lbs., the horse cannot work

more than six hours a day, and that with a less velocity. And, in both cases, if he carry some weight, he will draw better than if he carried none. M. Sauveur estimates the mean effort of a horse at 175 French, or 189 averd. pounds, with a velocity of rather more than three feet per second. But all these are probably too high to be continued for eight hours, day after day. In another place Desaguliers states the mean work of a horse as equivalent to the raising a hogshead full of water (or 550 lbs.) 50 feet high in a minute. But Mr. Smeaton, to whose authority much is due, asserts, from a number of experiments, that the greatest effect is the raising 550 lbs. forty feet high in a minute. And, from some experiments made by the Society for the Encouragement of Arts, under the direction of Mr. Samuel Moore, it was concluded, that a horse moving at the rate of three miles an hour can exert a force of 80 lbs. Unluckily, we are not sufficiently acquainted with the nature of the experiments and observations from which these deductions were made, to institute an accurate comparison of their results. Neither of them ought to express what a horse can draw upon a carriage; because in that case friction only is to be overcome (after the load is once put into motion); so that a middling horse, well applied to a cart, will often draw much more than 1000 lbs. The proper estimate would be that which measures the weight that a horse would draw up out of a well; the animal acting by a horizontal line of traction turned into the vertical direction by a simple pulley, or roller, whose friction should be reduced as much as possible.

23. Mr. Tredgold, in his recent publication on Rail-roads, has directed his attention to the subject of "*horse power.*" We have already quoted (pa. 867) his expression for the power of a horse, $250\,v\left(1-\frac{v}{V}\right)$; and $\frac{250\,d\,v\left(1-\frac{v}{V}\right)}{1+n}$ for the day's work in lbs. raised one mile; d being the hours which the horse works in a day, and the weight of the carriage to that of the load as $n:1$. He also gives $\frac{14{\cdot}7}{\sqrt{d}}$, for the greatest speed in miles per hour, when the horse is unloaded. These expressions, must, at present, be regarded as tentative; and with all our respect for that truly ingenious and indefatigable author, we, as yet, suspend our judgment in reference to their perfect accuracy. But Mr. Tredgold is a man of experience

as well as of competent science; so that whatever he advances on a subject of mixed research deserves attention. We select, therefore, the following tablet of the comparison of the duration of a horse's daily labour and maximum velocity, unloaded.

Duration of labour: *Hours.*	Max. velocity unloaded in miles per hour.
1	14·7
2	10·4
3	8·5
4	7·3
5	6·6
6	6·0
7	5·5
8	5·2
9	4·9
10	4·6

Taking the hours of labour at 6 per diem, the utmost that Mr. Tredgold would recommend, the maximum of useful effect he assigns at 125 lbs. moving at the rate of three miles per hour, and regarding the expense of carriage, in that case, as unity; then,—

Miles per hour.	Proportional expense.	Moving force.
2	$1\frac{1}{8}$ or 1·125	166 lbs.
3	1	125 —
$3\frac{1}{2}$	$1\frac{1}{35}$ or 1·0285	104 —
4	$1\frac{1}{8}$ or 1·125	83 —
$4\frac{1}{2}$	$1\frac{1}{3}$ or 1·333	$62\frac{1}{2}$ —
5	$1\frac{4}{5}$ or 1·8	$41\frac{2}{3}$ —
$5\frac{1}{8}$	2	$36\frac{1}{2}$ —

That is, the expense of conveying goods at 3 miles per hour, being 1; the expense at $4\frac{1}{2}$ miles per hour, will be $1\frac{1}{3}$; and so on, the expense being doubled when the speed is $5\frac{1}{8}$ miles per hour.

24. Thus, according to Mr. Tredgold, we have for the day of 6 hours 2250 lbs. raised one mile. And Mr. Bevan, who has made many experiments on the force of traction to move canal boats on the Grand Junction Canal, found the force of traction 80 lbs., and the space travelled in a day 26 miles; hence, it is only equivalent to 26 × 80 = 2080 lbs. raised one mile for the day's work; the rate of travelling being 2·45 miles per hour; and the result a little less than Mr. Tredgold's, the difference probably arising from the deviation of the angle of the catenary from the horizon. See page 178 of this volume.

Section II. *Passive Strength.*

1. When a weight is supported by a bar resting on two fulcrums, the pressure on each is inversely as its distance from the weight.

2. The strain on a given point of a bar, placed horizontally, and supported at both ends, from a weight placed on it, is proportional to the rectangle of the segments into which the point divides the bar.

3. Hence that strain is greatest in the middle of the bar or beam; or, in other words, if the bar be prismatic, it is most likely to break in the middle, or it is weakest there.

4. The strain produced by the weight of an equable bar, at any point of its length, is equal to the strain produced by half the weight of one segment acting at the end of a lever equal to the other segment.

5. Def. A substance perfectly elastic is initially extended and compressed in equal degrees by equal forces, and proportionally by proportional forces.

6. Def. The *modulus of the elasticity* of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a certain degree of compression, as the length of the substance is to the diminution of its length.

The modulus of elasticity is the measure of the elastic force of any substance.

A practical notion of the *modulus of elasticity* may be readily obtained. Let ε be the quantity a bar of wood, iron, or other substance, an inch square and a foot in length would be extended or diminished by the force f; and let l be any other length of a bar of equal base and like substance; then

$1 : l :: \varepsilon : \Delta$, or $l\varepsilon = \Delta$, the extension or diminution in the length l.

The modulus of elasticity is found by this analogy: as the diminution of the length of any substance, is to its length, so is the force that produced that diminution to the modulus of elasticity. Or, denoting the weight of the modulus in lbs. for a base of an inch square by m; it is

$$\varepsilon : f :: 1 : m = \frac{f}{\varepsilon}$$

And, if w be the weight of a bar of the substance one inch square and 1 foot in length; then, if M be the height of the modulus of elasticity in feet, we have

$$\frac{f}{w\varepsilon} = \text{M}.$$

7. When a force is applied to an elastic column of a rectangular prismatic form in a direction parallel to the axis, the parts nearest to the line of direction of the force exert a resistance in an opposite direction; those particles, which are at a distance beyond the axis, equal to a third proportional to the depth, and twelve times the distance of the line of direction of the force, remain in their natural state; and the parts beyond them act in the direction of the force.

8. The weight of the modulus of the elasticity of a column being m, a weight bending it in any manner f, the distance of the line of its application from any point of the axis D, and the depth of the column, d, the radius of curvature will be $\frac{d^2 m}{12 \, D f}$.

9. The distance of the point of greatest curvature of a prismatic beam, from the line of direction of the force, is twice the versed sine of that arc of the circle of greatest curvature, of which the extremity is parallel to that of the beam.

When the force is longitudinal, and the curvature inconsiderable, the form coincides with the harmonic curve, the curvature being proportional to the distance from the axis: and the distance of the point of indifference from the axis becomes the secant of an arc proportional to the distance from the middle of the column.

10. If a beam is naturally of the form which a prismatic beam would acquire, if it were slightly bent by a longitudinal force, calling its depth, d, its length, l, the circumference of a circle of which the diameter is unity, c, the weight of the modulus of elasticity, m, the natural deviation from the rectilinear form, Δ, and a force applied at the extremities of the axis, f, the total deviation from the rectilinear form will be

$$\Delta' = \frac{d^2 c^2 \Delta m}{d^2 c^2 m - 12 l^2 f}$$

Scholium. It appears from this formula, that when the other quantities remain unaltered, Δ' varies in proportion to Δ, and if $\Delta = 0$, the beam cannot be retained in a state of inflection, while the denominator of the fraction remains a finite quantity: but when $d^2 c^2 m = 12 l^2 f$, Δ' becomes infinite, whatever may be the magnitude of Δ, and the force will overpower the beam, or will at least cause it to bend so much as to derange the operation of the forces concerned. In this case $f = \left(\frac{dc}{l}\right)^2 \cdot \frac{m}{12}, .8225 \frac{d^2}{l^2} m$, which is the force capable of holding the beam in equilibrium in any inconsiderable degree of curvature. Hence the modulus being known for any substance, we may determine at once the weight which a given bar nearly straight is capable of supporting. For instance, in fir wood, supposing its height 10,000,000 feet, a bar an inch square and ten feet long may begin to bend with the weight of a bar of the same thickness, equal in length to $.8225 \times \frac{1}{120 \times 120} \times 10,000,000$ feet, or 571 feet; that is, with a weight of about 120 pounds; neglecting the effect of the weight of the bar itself. In the same manner the strength of a bar of any other substance may be determined, either from direct experiments on its flexure, or from the sounds that it produces. If $f = \frac{m}{n}, \frac{l^2}{d^2} = .8225\ n$, and $\frac{l}{d} = \sqrt{(.8225\,n)} = .907 \sqrt{n}$; whence, if we know the force required to crush a bar or column, we may calculate what must be the proportion of its length to its depth, in order that it may begin to bend rather than be crushed.

11. When a longitudinal force is applied to the extremities of a straight prismatic beam, at the distance D from the axis, the deflection of the middle of the beam will be D . (sec. arc $\left(\sqrt{\left(\frac{3f}{m}\right)}.\frac{l}{d}\right) - 1$).

12. The form of an elastic bar, fixed at one end, and bearing a weight at the extremity, becomes ultimately a cubic parabola, and the depression is $\frac{2}{3}$ of the versed sine of an equal arc, in the smallest circle of curvature.

13. The weight of the modulus of the elasticity of a bar is to a weight acting at its extremity only, as four times the cube of the length to the product of the square of the depth and the depression.

14. If an equable bar be fixed horizontally at one end, and bent by its own weight, the depression at the extremity will be half the versed sine of an equal arc in the circle of curvature at the fixed point.

15. The height of the modulus of the elasticity of a bar, fixed at one end, and depressed by its own weight, is half as much more as the fourth power of the length divided by the product of the square of the depth and the depression.

16. The depression of the middle of a bar supported at both ends, produced by its own weight, is five-sixths of the versed sine of half the equal arc in the circle of least curvature.

17. The height of the modulus of the elasticity of a bar,

supported at both ends, is $\frac{5}{32}$ of the fourth power of the length, divided by the product of the depression and the square of the depth.

From an experiment made by Mr. Leslie on a bar in these circumstances, the height of the modulus of the elasticity of deal appears to be about 9,328,000 feet. Chladni's observations on the sounds of fir wood, afford very nearly the same result.

18. The weight under which a vertical bar not fixed at the end, may begin to bend, is to any weight laid on the middle of the same bar, when supported at the extremities in a horizontal position, nearly in the ratio of $\frac{514}{10000}$ of the length to the depression.

19. Def. The stiffness of bodies is measured by their resistance at an equal linear deviation from their natural position.

20. The stiffness of a beam is directly as its breadth, and as the cube of its depth, and inversely as the cube of its length.

21. The direct cohesive or repulsive strength of a body is in the joint ratio of its primitive elasticity, of its toughness, and the magnitude of its section.

Though most natural substances appear in their intimate constitution to be perfectly elastic, yet it often happens that their toughness with respect to extension and compression differs very materially. In general, bodies are said to have less toughness in resisting extension than compression.

22. The transverse strength of a beam is directly as the breadth and as the square of the depth, and inversely as the length.

Scholium. If one of the surfaces of a beam were incompressible, and the cohesive force of all its strata collected in the other, its strength would be six times as great as in the natural state; for the radius of curvature would be $\frac{d^2 m}{\text{D} f}$, which could not be less than twice as great as in the natural state, because the strata would be twice as much extended, with the same curvature, as when the neutral point is in the axis; and f would then be six times as great.

23. Def. The resilience of a beam may be considered as proportional to the height from which a given body must fall to break it.

24. The resilience of prismatic beams is simply as their bulk.

25. The stiffest beam that can be cut out of a given cylinder is that of which the depth is to the breadth as the square root of 3 to 1, and the strongest as the square root of 2 to 1;

but the most resilient will be that which has its depth and breadth equal.

26. Supposing a tube of evanescent thickness to be expanded into a similar tube of greater diameter, but of equal length, the quantity of matter remaining the same, the strength will be increased in the ratio of the diameter, and the stiffness in the ratio of the square of the diameter, but the resilience will remain unaltered.

27. The stiffness of a cylinder is to that of its circumscribing prism as three times the bulk of the cylinder to four times that of the prism.

28. If a column, subjected to a longitudinal force, be cut out of a plank or slab of equable depth, in order that the extension and compression of the surfaces may be initially every where equal, its outline must be a circular arc.

29. If a column be cut out of a plank of equable breadth, and the outline limiting its depth be composed of two triangles, joined at their bases, the tension of the surfaces produced by a longitudinal force, will be every where equal, when the radius of curvature at the middle becomes equal to half the length of the column; and in this case the curve will be a cycloid.

When the curvature at the middle differs from that of the cycloid, the figure of the column becomes of more difficult investigation. It may however be delineated mechanically, making both the depth of the column and its radius of curvature proportional always to $\surd a$. If the breadth of the column vary in the same proportion as the depth, they must both be every where as the cube root of a, the ordinate. (*Young's Nat. Phil. vol,* ii.)

30. The modulus of elasticity has not yet been ascertained in reference to so many subjects as could be wished. Professor Leslie exhibits several, however, as below. That of white marble is 2,150,000 feet, or a weight of 2,520,000 pounds avoirdupois on the square inch; while that of Portland stone is only 1,570,000 feet, corresponding on the square inch to the weight of 1,530,000 lb.

White marble and Portland stone are found to have, for every square inch of section, a cohesive power of 1811 lb. and 857 lb.; wherefore, suspended columns of these stones, of the altitude of 1542 and 945 feet, or only the 1394th and 1789th part of their respective measure of elasticity, would be torn asunder by their own weight.

31. Of the principal kinds of timber employed in building and carpentry, the annexed table will exhibit their respective Modulus of Elasticity, and the portion of it which limits their cohesion, or which lengthwise would tear them asunder.

Teak,	6,040,000 feet.	168th.
Oak,	4,150,000 feet.	144th.
Sycamore,	3,860,000 feet.	108th.
Beech,	4,180,000 feet.	107th.
Ash,	4,617,000 feet.	109th.
Elm,	5,680,000 feet.	146th.
Memel Fir,	8,292,000 feet.	205th.
Christiana Deal,	8,118,000 feet.	146th.
Larch,	5,096,000 feet.	121th.

The Professor gives also this tabular view of their absolute cohesion, or the load which would rend a prism of an inch square; and the altitude of the prism which would be severed by the action of its own weight.

Teak,	12,915 lb.	36,049 feet.
Oak,	11,880 lb.	32,900 feet.
Sycamore,	9,630 lb.	35,800 feet.
Beech,	12,225 lb.	38,940 feet.
Ash,	14,130 lb.	42,080 feet.
Elm,	9,720 lb.	39,050 feet.
Memel Fir,	9,540 lb.	40,500 feet.
Christiana Deal,	12,346 lb.	55,500 feet.
Larch,	12,240 lb.	42,160 feet.

32. The modulus of the elasticity of hempen fibres has not been ascertained, but may probably be reckoned about 5,000,000 feet. Their cohesion is, for every square inch of transverse section, 6,400 lb. The best mode of estimating the strength of a rope of hemp, is to multiply by 200 the square of its number of inches in girth, and the product will express in pounds the practical strain it may be safely loaded with; for cables multiply by 120, instead of 200. The ultimate strain is probably double this; as will appear from the account following of Du Hamel's experiments. If yarns of 180 yards long be worked up into a rope of only 120 yards, it will lose one-fourth of its strength, the exterior fibres alone resisting the greatest part of the strain. The register cordage of the late Captain Huddart exerts nearly the whole force of the strands, since they suffer a contraction of only the eighth part in the process of combining.

33. For the *utmost* strength that a rope will bear before it breaks, a good estimate will be found by taking *one-fifth of the square of the girth of the rope*, to express the tons it will carry. This is about double the rule for practice which we have just given (art. 32); and is, even, for an ulterior measure, too great for tarred cordage, which is always weaker than white.

The following experiments were made by Mons. Du Hamel, at Rochfort, on cordage of 3-inch French circumference, made of the best Riga hemp, August 8th, 1741.

White.		Tarred.
Broke with strain of 4500 pounds	......	3400 pounds.
4000	....	3300
4800	....	3258
August 25th, 1743.		
4600	....	3500
5000	....	3400
5000	....	3400
September 23d, 1746.		
3880	....	3000
4000	....	2700
4200	....	2800

A parcel of white and tarred cordage was taken out of a quantity which had been made February 12, 1746.

It was laid up in the magazine, and comparisons were made from time to time, as—

	White bore.	Tarred.	Difference.
1746 April 14th	2645	2312	333
1747 May 18th	2762	2155	607
1747 October 21st	2710	2050	660
1748 June 19th	2575	1752	823
1748 October 2d	2425	1837	588
1749 Sept. 25th	2917	1865	1052

M. Du Hamel says, that it is decided by experience, that white cordage, in continued service, is one-third more durable than tarred; secondly, it retains its force much longer while kept in store; thirdly, it resists the ordinary injuries of the weather one-fourth longer. These observations deserve the attention of practical men.

34. The metals differ more widely from each other in their elastic force and cohesive strength, than the several species of wood or vegetable fibres. Thus, the cohesion of fine steel is about 135,000 lb. for the square inch, while that of cast lead amounts only to about the hundred and thirtieth part, or 1800 lb.

According to the accurate experiments of Mr. George Rennie in 1817, the cohesive power of a rod an inch square of different metals, in pounds avoirdupois, with the corresponding length in feet, is as follows:

Cast Steel,	134,256 lb.	39,455 feet.
Swedish Malleable Iron,	72,064 lb.	19,740 feet.
English ditto,	55,872 lb.	16,938 feet.
Cast Iron	19,096 lb.	6,110 feet.

Cast Copper	19,072 lb.	5,003 feet.
Yellow Brass, ...	17,958 lb.	5,180 feet.
Cast Tin, ...	4,736 lb.	1,496 feet.
Cast Lead, ...	1,824 lb.	348 feet.

It thus appears, as Professor Leslie remarks, that a vertical rod of lead 348 feet long, would be rent asunder by its own weight. The best steel has nearly twice the strength of English soft iron, and this again is about three times stronger than cast iron. Copper and brass have almost the same cohesion as cast iron. This tenacity is sometimes considerably augmented by hammering or wire-drawing, that of copper being thus nearly doubled, and that of lead, according to Eytelwein, more than quadrupled. The consolidation is produced chiefly at the surface, and hence a slight notch with a file will materially weaken a hard metallic rod. English malleable iron has 7,550,000 feet for its modulus of elasticity, or the weight of 24,920,000 lb. on the square inch, while cast iron has 5,895,000 feet, and 18,421,000 lb. Of other metals, the modulus of elasticity is probably smaller, but has not yet been well ascertained.

35. The *Longitudinal Compression* which any column suffers, is at first equal to the dilatation occasioned by an equal and opposite strain, being in both cases proportional to the modulus of elasticity. But while the incumbent weight is increased the power of resistance likewise augments, as long as the column withstands flexure. After it begins to bend, a lateral disruption soon takes place. A slender vertical prism is hence capable of supporting less pressure than the tension which it can bear. Thus, a cubic inch of English oak was crushed only by the load of 3860 lb. but a bar of an inch square and 5 inches high gave way under the weight of 2572 lb. It would evidently have been still feebler if it had been longer. On the other hand, if the breadth of a column be considerable in proportion to its height, it will sustain a greater pressure than its cohesive power. Thus, though the cohesion of a rod of cast iron of the quarter of an inch square is only 300 lb. a cube of that dimension will require 1440 lb. to crush it.

In general, while the resisting mass preserves its erect form, the several sections are compressed and extended by additional weight, and their repellent particles are not only brought nearer, but multiplied. This repulsion is likewise increased by the lateral action arising from the confined ring of detrusion. The primary resistance becomes hence greatly augmented in the progress of loading the pillar.

36. The most precise experiments on this subject seem to

be those of Mr. Rennie. The weights required to crush cubes of the quarter of an inch of certain metals, are as follow:

Iron cast vertically,	11,140 lb.
Iron cast horizontally,	10,110 lb.
Cast Copper	7,318 lb.
Cast Tin,	966 lb.
Cast Lead,	483 lb.

Cubes of an inch are crushed by the weights annexed:

Elm,	1,284 lb.
White Deal,	1,928 lb.
English Oak,	3,860 lb.
Craigleith freestone,	8,688 lb.

Cubes of an inch and a half, and consequently presenting a section of $2\frac{1}{4}$ times greater than the former, might be expected to resist compression in that ratio. They are crushed, however, with loads considerably less.

Red brick,	1,817 lb.
Yellow baked brick,	2,254 lb.
Fire brick,	3,864 lb.
Craigleith stone, direction of the strata	15,560 lb.
Ditto, across the strata	12,346 lb.
White Statuary Marble,	13,632 lb.
White-veined Italian Marble,	21,783 lb.
Purbeck Limestone,	20,610 lb.
Cornish Granite,	14,302 lb.
Peterhead Granite,	18,636 lb.
Aberdeen Blue Granite,	24,536 lb.

These facts show the comparative firmness of different materials, but it is to be regretted, that such results are not of much practical value, since they are confined to a very narrow scale, and applicable only to cubical blocks. While the breadth remains the same, the resistance appears to depend on some unascertained ratio of the altitude of the column.

Nay, as Professor Leslie observes, the absolute height itself has probably a material influence on the effect. Thus, from some experiments made in France, it appears, that prisms of seasoned oak, two inches square, and two, four or six feet high, would be crushed by the vertical pressures of 17,500 lb. 10,500 lb. and 7,000 lb.; but, if four inches square, and of the same altitudes, they would give way under loads of only 80,000 lb. 70,000 lb. and 50,000 lb. In the first set of trials, the mean cohesive power amounts to 130,000 lb. and in the second to 520,000 lb. The vertical support is therefore greatly inferior to these limits. When the length of the pillar exceeds 36 times its breadth, the resistance to longitudinal compression appears to be diminished 18 times.—(*Leslie's Elements*, and *Duhamel in Mem. Paris. Acad.*)

37. Mr. B. Bevan has favoured the author with a tabular view of his results with regard to the *modulus of cohesion*, or the length in feet of any prismatic substance required to break its cohesion, or tear it asunder.

Bevan's Results.

	feet.
Tanned cow's skin	10,250
——— calf skin	5,050
——— horse skin	7,000
——— cordovan	3,720
——— sheep skin	5,600
Untanned horse skin	8,900
Old harness of 30 years	5,000
Hempen twine	75,000
Catgut, some years old	23,000
Garden matting	27,000
Writing-paper, foolscap	8,000
Brown wrapping-paper, thin	6,700
Bent grass, (holcus)	79,000
Whalebone	14,000
Bricks, (Fenny Stratford)	970
——— (Leighton)	144
Ice	300
Leicestershire slate	7,300

The following, also, exhibits Mr. Bevan's results as to the *modulus of elasticity*.

	feet.
Steel	9,300,000
Bar iron	9,000,000
Ditto	8,450,000
Yellow pine	9,150,000
Ditto	11,840,000
Finland deal	6,000,000
Mahogany	7,500,000
Rose wood	3,600,000
Oak, dry	5,100,000
Fir bottom, 25 years old	7,400,000
Petersburg deal	6,000,000
Lance wood	5,100,000
Willow	6,200,000
Oak	4,350,000

	feet.
Satin wood	2,290,000
Lincolnshire bog oak	1,710,000
Lignum Vitæ	1,850,000
Teak wood	4,780,000
Yew	2,220,000
Whalebone	1,000,000
Cane	1,400,000
Glass tube	4,440,000
Ice	6,000,000
Limestone	
——— Dinton, Buck ...	2,400,000
——— Ketton	1,600,000
——— Jetternoe	625,000
Ryegate	621,000
Yorkshire paving	1,320,000
Cork	3,300
Slate, Leicestershire	7,800,000

Many other results are collected in Mr. Tredgold's *Essay on the Strength of Cast Iron*, a work which may be consulted with advantage on this and kindred topics.

Indeed, we have not aimed at more in this section than a brief summary of the leading principles involved in the consideration of passive strength, and a corresponding exhibition of the best ascertained facts. The subject is one of great and growing interest to all concerned in the erection of extensive structures. Such may consult, in addition to the volume just mentioned, *Tredgold's Carpentry*, *Barlow on the Strength of Timber*, *Gregory's Mechanics*, Vol. I, the Lectures of *Dr. T. Young* and *Professor Leslie*, already quoted, and *Girard, Traité Analytique de la Résistance des Solids.*

SUPPLEMENTARY TABLES.

TABLE I.—*Useful Factors, extended to several places of Decimals, in which* p *represents the Circumference of a Circle, whose Diameter is* 1.

Then

p =	3·1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679821480865132723066470938446 +
$2p$	6·283185307179586476925286766559
$4p$	12·566370614359172953850573533118
$\frac{1}{2}p$	1·570796326794896619231321691639
$\frac{1}{4}p$	0·785398163397448309615660845819
$\frac{4}{3}p$	4·188790204786390984616857844372
$\frac{1}{6}p$	0·523598775598298873077107230546
$\frac{1}{8}p$	0·392699081698724154807830422909
$\frac{1}{12}p$	0·261799387799149436538553615273
$\frac{1}{360}p$	0·008726646259971647884618453842
$\frac{1}{p}$	0·3183098861837906715377675267450
$\frac{2}{p}$	0·636619772367581343075535053490
$\frac{4}{p}$	1·273239544735162686151070106980
$\frac{1}{4p}$	0·0795774715459476678844418816862
$\sqrt{2}$	1·4142135623730950488016887242097 nearly
$\sqrt{\frac{1}{2}}$	0·707106781186547524400844362104
$p\sqrt{2}$	4·4428829381583662470158809900605
$p\sqrt{\frac{1}{2}}$	2·2214414690791831235079404950302
$\frac{1}{p}\sqrt{\frac{1}{2}}$	0·2250790790392765173887997751
$\frac{1}{p}\sqrt{2}$	0·4501581580785530347775995955
$\sqrt{p}$	1·772453850905516027298167483341
$\frac{1}{2}\sqrt{p}$	0·886226925452758013649083741670
$2\sqrt{p}$	3·544907701811032054596334966682
$\sqrt{\frac{p}{2}}$	1·253314137315500251207882642402
$\sqrt{\frac{2}{p}}$	0·797884560802865355879892119868
$\sqrt{\frac{1}{p}}$	0·564189583547756286948079451560

$2\sqrt{\frac{1}{p}}$ = 1·128379167095512573896158903120

$\frac{1}{2}\sqrt{\frac{1}{p}}$ .... 0·282094791773878143474039725780

p^2 .... 9·869604401089358618834490999876

$\frac{1}{p^2}$ .... 0·101321183642337771443879463209

$\frac{1}{2p^2}$ .... 0·050660591821168885721939731604

$\frac{1}{6p^2}$ .... 0·016886863940389628573979910534

$2\sqrt{\frac{1}{p}}$ .... 1·128379167095512573896158903120

$\frac{1}{6}\sqrt{\frac{1}{p}}$ .... 0·094031597257959381158015241926

$\frac{1}{8}\sqrt{\frac{1}{p}}$ .... 0·070523697943469535868509931445

$\frac{1}{6p}$ .... 0·053051647697298445256294587790

$\frac{360}{p}$ .. 114·591559026164641753596309628200

$\frac{2}{3}p$ .... 2·094395102393195492308428922186

$\frac{p}{24}$ .... 0·130899693899574718269276807636

$\frac{6}{p}$ .... 1·909859317102744029226605160470

$\sqrt[3]{\frac{6}{p}}$ .... 1·2407009819 &c.

$6p^2$ .... 59·217626406536151713006945999256

$\sqrt{6p^2}$ .... 3·89777707 &c.

$36p$ .. 113·097335529232556584655161798062

$\sqrt[3]{36p}$ .... 4·83597586204 &c.

$\frac{1}{8}\sqrt{p}$ .... 0·221556731363189503412270935418

$\frac{1}{2p}$ .... 0·159154943091895335768883763372

$\sqrt[3]{\frac{p}{6}}$ .... 0·805995977007 &c.

$\sqrt{231}$ .. 15·1986841535706636

$\sqrt{282}$ .. 16·7928556237466 + &c.

$\sqrt{10152}$ .. 100·75713374

$\sqrt{\frac{282}{\cdot 785398 \text{ \&c.}}}$ = 18·948708 &c.

$\sqrt{277\cdot 274}$ = 16·65154647474

TABLE II.—*A Table of Circles, from which knowing the diameters, the areas, circumferences, and sides of equal squares, are found.*

Given	Required		
Diam.	Area.	Circumference.	side of eq. sq.
1·00	0·7853981	3·14159265	0·88622692
·25	1·22718463	3·92699081	1·10778365
·5	1·76714586	4·71238898	1·32934038
·75	2·40528187	5·49778714	1·55089711
2·	3·14159265	6·28318530	1·77245384
·25	3·97607820	7·06858347	1·99401058
·5	4·90873852	7·85398163	2·21556731
·75	5·93957361	8.63937979	2·43712404
3·	7·06858347	9·42477796	2·65868077
·25	8·29576810	10·21017612	2·88023750
·5	9·62112750	10·99557428	3·10179423
·75	11·04466167	11·78097245	3·32335096
4·	12·56637061	12·56637061	3·54490769
·25	14·18625432	13·35176877	3·76646442
·5	15·90431280	14·13716694	3·98802116
·75	17·72054606	14·92256510	4·20957789
5·	19·63495408	15·70796326	4·43113462
·25	21·64753687	16·49336143	4·65269135
·5	23·75829444	17·27875959	4·87424808
·75	25·96722677	18·06415775	5·09580482
6·	28·27433388	18·8495559	5·31736155
·25	30·67961575	19·63495408	5·53891828
·5	33·18307240	20·42035224	5·76047501
·75	35·78470382	21·20575041	5·98203174
7·	38·48456000	21·99114857	6·20358847
·25	41·28249096	22·77654673	6·42514520
·5	44·17864669	23·56194490	6·64670193
·75	47·17297718	24·34734306	6.86825866
8·	50·26548245	25·13274122	7·08981539
·25	53·45616249	25·91813939	7·31137213
·5	56·74501730	26·70353755	7·53292886
·75	60·13204688	27·48893571	7·75448559
9·	63·61725123	28·27433388	7·97604232
·25	67·20063035	29·05973204	8·19759905
·5	70·88218424	29·84513020	8·41915578
·75	74·66191290	30·63052837	8·64071251
10·	78·53981633	31·41592653	8·86226925
·25	82·51589454	32·20132469	9·08382598
·5	86·59014751	32·98672286	9·30538270
·75	90·76257525	33·77212102	9·52693944
11·	95·03317777	34·55751918	9·74849617
·25	99·40195505	35·34291735	9·97005290
·5	103·86890711	36·12831551	10·19160964
·75	108·43403393	36·91371367	10·41316637
12·	113·09733553	37·69911184	10·63472310
·25	117·85881189	38·48451000	10·85627983

Given	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
12·5	122·71846303	39·26990816	11·07783656
·75	127·67628893	40·05530633	11·29939329
13·	132·73228961	40·84070449	11·52095002
·25	137·88646506	41·62610265	11·74250675
·5	143·13881527	42·41150082	11 96406348
·75	148·48934026	43·19689898	12·18562021
14·	153·93804002	43·98229714	12·40717695
·25	159·48491455	44·76769531	12·62873368
·5	165·12996385	45·55309347	12·85029041
·75	170·87318792	46·33849163	13·07184714
15·	176·71458676	47·12388980	13·29340388
·25	182·65416028	47·90928796	13·51496061
·5	188·69190875	48·69468613	13·73651734
·75	194·82783190	49·48008429	13·95807407
16·	201·06192982	50·26548245	14·17963080
·25	207·39420252	51·05088062	14·40118753
·5	213·82464998	51·83627878	14·62274426
·75	220·35327221	52·62167694	14·84430099
17·	226·98006922	53·40707511	15·06585772
·25	233·70504099	54·19247327	15·28741446
·5	240·52818753	54·97787143	15·50897119
·75	247·44950885	55·76326960	15·73052792
18·	264·46900493	56·54866776	15·95208465
·25	266·58667578	57·33406592	16·17364138
·5	268·80252140	58·11946409	16·39519811
·75	276·11654180	58·90486225	16·61675484
19·	283·52873696	59·69026041	16·83831157
·25	291·03910692	60·47565858	17·05986830
·5	298·64765163	61·26105674	17·28142503
·75	306·35437111	62·04645490	17·50298177
20·	314·15926535	62·83185307	17·72453850
·25	322·06233437	63·61725123	17·94609524
·5	330·06357816	64·40264939	18·16765197
·75	338·16299672	65·18804756	18·38920870
21·	346·36059005	65·97344572	18·61076543
·25	354·65635814	66·75884388	18·83232216
·5	363·05030101	67·54424205	19·05387889
·75	371·54241865	68·32964021	19·27543562
	380·13271106	69·11503837	19·49699235
·25	388·82117826	69·90043654	19·71854908
·5	397·60782021	70·68583470	19·94010581
·75	406·49263694	71·47123286	20·16166255
23·	415·47562843	72·25663103	20·38321928
·25	424·55679467	73·04202919	20·60477601
·5	433·73613573	73·82742735	20·82633274
·75	443·01365154	74·61282552	21·04788945
4·	452·38934207	75·39822368	21·26944618
2 ·25	461·86320745	76·18362184	21·49100291
·5	471·43524757	76·96902001	21.71255964
·75	481·10546239	77·75441817	21·93411637
· ·	490·87385212	78·53981634	22·15567313
25·25	500·74041655	79·32521450	22·37722986
·5	510·70515575	80·11061266	22·59878659
·75	520·76806971	80·89601083	22·82034332
26·	530·92915845	81·68140899	23·04190006
·25	541·18842196	82·46680715	23·26345679

Given	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
26·5	551·54586024	83·25220532	23·48501352
·75	562·00147328	84·03760348	23·70657025
27·	572·55526110	84·82300164	23·92812698
·25	583·20722369	85·60839981	24·14968371
·5	593·95736105	86·39379797	24·37124044
·75	604·80567318	87·17919613	24·59279717
28·	615·75216017	87·96459430	24·81435390
·25	626·79682177	88·74999246	25·03591063
·5	637·93965822	89·53539062	25·25746737
·75	649·18066943	90·32078879	25·47902410
29·	660·51985541	91·10618695	25·70058083
·25	671·95721616	91·89158511	25·92213756
·5	683·49275169	92·67698328	26·14369429
·75	695·12646198	93·46238144	26·36525102
30·	706·85834704	94·24777960	26·58680776
·25	718·68840688	95·03317777	26·80836449
·5	730·61664148	95·81857593	27·02992122
·75	742·64305085	96·60397409	27·25147794
31·	754·76763502	97·38937226	27·47303468
·25	766·99039394	98·17477042	27·69459141
·5	779·31132762	98·96016858	27·91614814
·75	791·73043607	99·74556675	28·13770488
32·	804·24771930	100·53096491	28·35926161
·25	816·86317729	101·31636307	28·58081834
·5	829·57681005	102·10176124	28·80237507
·75	842·38861759	102·88715940	29·02393180
33·	855·29859989	103·67255756	29·24548853
·25	868·30675696	104·45795573	29·46704526
·5	881·41308881	105·24335389	29·68860199
·75	894·61759542	106·02875205	29·91015872
34·	907·92027688	106·81415022	30·13171545
·25	921·32113305	107·59954838	30·35327219
·5	934·82016398	108·38494654	30·57482892
·75	948·41736968	109·17034471	30·79638565
35·	962·11275016	109·95574287	31·01794239
·25	975·90630540	110·74114103	31·23949912
·5	989·79803541	111·52653920	31·46105585
·75	1003·78794019	112·31193736	31·68261258
36·	1017·87601975	113·09733552	31·90416931
·25	1032·06227407	113·88273369	32·12572604
·5	1046·34670316	114·66813185	32·34728277
·75	1060·72930703	115·45353001	32·56883950
37·	1075·21008569	116·23892818	32·79039623
·25	1089·78903909	117·02432634	33·01195296
·5	1104·46616727	117·80972450	33·23350970
·75	1119·24147022	118·59512267	33·45506643
38·	1134·11494794	119·38052083	33·67662316
·25	1149·08660043	120·16591899	33·89817989
·5	1164·15642768	120·95131716	34·11973662
·75	1179·32442971	121·73671532	34·34129335
39·	1194·59060651	122·52211348	34·56285008
·25	1209·95495809	123·30751165	34·78440681
·5	1225·41748449	124·09290981	35·00596354
·75	1240·97818552	124·87830797	35·22752027
40·	1256·63704143	125·66370614	35·44907701
·25	1272·39411208	126·44910430	35·67063374

Given	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
40·5	1288·24933751	127·23450246	35·89219048
·75	1304·20273770	128·01990063	36·11374721
41·	1320·25431266	128·80529879	36·33530394
·25	1336·40406240	129·59069695	36·55686067
·5	1352·65198690	130·37609512	36·77841740
·75	1368·99808617	131·16149328	36·99997413
42·	1385·44236022	131·94689144	37·22153086
·25	1401·98480903	132·73228961	37·44308759
·5	1418·62543261	133·51768777	37·66464432
·75	1435·36423096	134·30308593	37·88620105
43·	1452·20120412	135·08848410	38·10775779
·25	1469·13635202	135·87388226	38·32931452
·5	1486·16967468	136·65928042	38·55087125
·75	1503·30117212	137·44467859	38·77242798
44·	1520·53084433	138·23007675	38·99398471
·25	1537·85869131	139·01547491	39·21554144
·5	1556·28471306	139·80087308	39·43709817
·75	1572·80890957	140·58627124	39·65865490
45·	1590·43128086	141·37166940	39·88021164
·25	1608·15182692	142·15706757	40·10176837
·5	1625·97054775	142·94246573	40·32332510
·75	1643·88744335	143·72786390	40·54488183
46·	1661·90251374	144·51326206	40·76643856
·25	1680·01575889	145·29866022	40·98799530
·5	1698·22717880	146·08405839	41·20955203
·75	1716·53677348	146·86945655	41·43110876
47·	1734·94454294	147·65485471	41·65266549
·25	1753·45048716	148·44025288	41·87422222
·5	1772·05460615	149·22565104	42·09577891
·75	1790·75689992	150·01104920	42·31733568
48·	1809·55736845	150·79644797	42·53889241
·25	1828·45601175	151·58184553	42·76044914
·5	1847·45282982	152·36724369	42·98200587
·75	1866·54782267	153·15264186	43·20356261
49·	1885·74099031	153·93804002	43·42511934
·25	1905·83233270	154·72343818	43·64667607
·5	1924·42184986	155·50883635	43·86823280
·75	1943·90954179	156·29423451	44·08978953
50·	1963·49540848	157·07963268	44·31134627
·25	1983·17944995	157·96503084	44·53290300
·5	2002·96166619	158·65042900	44·75445973
·75	2022·84205720	159·43582717	44·97601646
51·	2042·82062298	160·22122533	45·19757319
·25	2062·89736352	161·00662349	45·41912992
·5	2083·07227884	161·79202166	45·64068665
·75	2103·34536893	162·57741982	45·86224338
52·	2123·71663382	163·36281798	46·08380012
·25	2144·18607346	164·14821615	46·30535685
·5	2164·75368786	164·93361431	46·52691358
·75	2185·41947703	165·71901247	46·74847031
53·	2206·18344098	166·50441064	46·97002704
·25	2227·04557969	167·28980880	47·19158377
·5	2248·00589318	168·07520696	47·41314050
·75	2269·06438143	168·86060513	47·63469723
54·	2290·22104445	169·64600329	47·85625386
·25	2311·47588225	170·43140145	48·07781069

Given	Required		
Diam.	Area.	Circumference.	side of eq. sq.
54·5	2332·82889481	171·21679962	48·29936743
·75	2354·28008215	172·00219778	48·52092416
55·	2375·82944427	172·78759594	48·74248089
·25	2397·47698115	173·57299411	48·96403763
·5	2419·22269280	174·35839227	49·18559436
·75	2441·06657922	175·14379043	49·40715109
56·	2463·00864068	175·92918860	49·62870782
·25	2485·04887637	176·71458676	49·85026455
·5	2507·18728710	177·49998492	50·07182128
·75	2520·42387260	178·28538309	50·29337801
57·	2551·75863286	179·07078125	50·51493474
·25	2574·19156790	179·85617941	50·73649147
·5	2596·72267781	180·64157758	50·95804820
·75	2619·35196239	181·42697574	51·17960494
58·	2642·07942166	182·21237390	51·40116167
·25	2664·90505579	182·99777207	51·62271840
·5	2687·82886464	183·78317023	51·84427513
·75	2710·85084834	184·56856839	52·06583186
59·	2733·97100678	185·35396656	52·28738859
·25	2757·18933998	186·13936472	52·50894532
·5	2780·50584795	186·92476288	52·73050205
·75	2803·92053070	187·71016105	52·95205878
60·	2827·43338821	188·49555921	53·17364552
·25	2851·04442049	189·28095737	53·39517225
·5	2874·75362754	190·06635554	53·61672898
·75	2898·56100937	190·85175370	53·83828572
61·	2922·46656692	191·63715186	54·05984245
·25	2946·47029734	192·42255003	54·28139918
·5	2970·57220350	193·20794819	54·50295591
·75	2994·77228444	193·99334635	54·72451264
62·	3019·07054008	194·77874452	54·94606937
·25	3043·46697053	195·56414268	55·16762610
·5	3067·96157576	196·34954084	55·38918283
·75	3092·55435572	197·13493901	55·61073956
63·	3117·24531051	197·92033717	55·83229629
·25	3142·03444002	198·70573533	56·05385303
·5	3166·92174434	199·49113350	56·27540976
·75	3191·90722341	200·27653166	56·49696649
64·	3216·99087720	201·06192982	56·71852322
·25	3242·17270581	201·84732799	56·94007995
·5	3267·45270920	202·63272615	57·16163668
·75	3292·83088742	203·41812431	57·38319341
65·	3318·30724030	204·20352248	57·60475015
·25	3343·88176802	204·98892064	57·82630688
·5	3369·55447036	205·77431880	58·04786361
·75	3395·32534774	206·55971697	58·26942034
66·	3421·19439974	207·34511513	58·49097707
·25	3447·16162652	208·13051329	58·71253380
·5	3473·22702806	208·91591146	58·93409054
·75	3499·39060438	209·70130962	59·15564727
67·	3525·65235549	210·48670778	59·37720400
·25	3552·01228135	211·27210595	59·59876073
·5	3578·47038197	212·05750411	59·82031746
·75	3605·02665737	212·84290227	60·04187419
68·	3631·68110754	213·62830044	60·26343092
·25	3658·43373248	214·41369860	60·48498765

Given	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
68·5	3685·28453219	215·19909676	60·70654438
·75	3712·23350667	215·98449493	60·92810111
69·	3739·28065592	216·76989309	61·14965785
·25	3766·42597994	217·55529125	61·37121458
·5	3793·66947873	218·34068942	61·59277131
·75	3821·01115229	219·12608758	61·81432804
70·	3848·45100064	219·91148574	62·03588478
·25	3875·98902375	220·69688391	62·25744151
·5	3903·62522162	221·48228207	62·47899824
·75	3931·35959426	222·26768023	62·70055497
71·	3959·19214168	223·05307840	62·92211170
·25	3987·12286386	223·83847656	63·14366843
·5	4015·15176082	224·62387472	63·36522516
·75	4043·27883254	225·40927289	63·58678189
72·	4071·50407903	226·19467105	63·80833862
·25	4099·82750030	226·98006921	64·02989536
·5	4128·24909633	227·76546738	64·25145209
·75	4156·76886714	228·55086554	64·47300882
73·	4185·38681274	229·33626370	64·69456555
·25	4214·10293309	230·12166187	64·91612228
·5	4242·91722821	230·90706003	65·13767901
·75	4271·82969810	231·69245819	65·35923574
74·	4300·84034275	232·47785636	65·58079247
·25	4329·94916219	233·26325452	65·80234920
·5	4359·15615638	234·04865268	66·02390593
·75	4388·46132535	234·83405085	66·24546267
75·	4417·86466909	235·61944901	66·46701940
·25	4447·36618760	236·40484717	66·68857613
·5	4476·96588088	237·19024534	66·91043287
·75	4506·66374893	237·97564350	67·13168960
76·	4536·45979178	238·76104166	67·35324633
·25	4566·35400937	239·54643983	67·57480306
·5	4596·34640174	240·33183799	67·79635979
·75	4626·43696897	241·11723615	68·01791652
77·	4656·62571078	241·90263432	68·23947325
·25	4686·91262745	242·68803248	68·46102998
·5	4717·29771890	243·47343064	68·68258671
·75	4747·78098511	244·25882881	68·90414344
78·	4778·36242610	245·04422697	69·12570018
·25	4809·04204185	245·82962513	69·34725691
·5	4839·81983238	246·61502330	69·56881364
·75	4870·79579767	247·40042146	69·79037037
79·	4901·66993776	248·18581962	70·01192710
·25	4932·74225260	248·97121779	70·23348383
·5	4963·91274221	249·75661595	70·45504056
·75	4995·18140659	250·34201411	70·67659729
80·	6026·54824574	251·32741228	70·89815403
·25	5058·01325066	252·11281044	71·11971076
·5	5089·57644835	252·89820860	71·34126749
·75	5121·23781181	253·68360677	71·56282422
81·	5152·99735004	254·46900493	71·78438096
·25	5184·85506304	255·25440309	72·00593769
·5	5216·81095081	256·03980126	72·22749442
·75	5248·86501335	256·82519942	72·44905115
82·	5281·01725068	257·61059758	72·67060788
·25	5313·26766276	258·39599575	72·89216461

Given	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
82·5	5345·61624962	259·18139391	73·11372134
·75	5378·06301124	259·96679207	73·33527807
83·	5410·60794761	260·75219024	73·55683480
·25	5443·25105880	261·53758840	73·77839153
·5	5475·99234474	262·32298656	73·99994827
·75	5508·83180544	263·10838473	74·22150500
84·	5541·76944092	263·89378269	74·44306173
·25	5574·80525116	264·67918105	74·66161846
·5	5607·93923618	265·46457922	74·88617519
·75	5641·17139596	266·24997738	75·10773192
85·	5674·50173054	267·03537554	75·32928866
·25	5707·93023987	267·82077371	75·55084539
·5	5741·45692397	268·60617187	75·77240212
·75	5775·08178284	269·39157003	75·99395885
86·	5808·80481648	270·17696820	76·21551558
·25	5842·62602489	270·96236636	76·43707232
·5	5876·54540807	271·74776452	76·65862904
·75	5910·56296602	272·53316269	76·88018578
87·	5944·67869874	273·31856085	77·10174251
·25	5978·89260623	274·10395901	77·32329924
·5	6013·20468849	274·88935718	77·54485597
·75	6047·61494552	275·67475534	77·76641270
88·	6082·12337734	276·46015350	77·98796943
·25	6116·72998392	277·24555167	78·20952616
·5	6151·43476526	278·03094983	78·43103289
·75	6186·23772138	278·81634799	78·65263962
89·	6221·13885226	279·60174616	78·87419635
·25	6256·13815792	280·38714432	79·09575309
·5	6291·23563834	281·17254248	79·31730982
·75	6326·43129354	281·95794065	79·53886655
90·	6361·72512350	282·74333881	79·76042329
·25	6397·11712824	283·52873697	79·98198002
·5	6432·60730774	284·31413514	80·20353673
·75	6468·19566202	285·09953330	80·42509348
91·	6503·88219109	285·88493146	80·64669021
·25	6539·66689491	286·6703296	80·86820694
·5	6575·54977350	287·45572779	81·08976367
·75	6611·53082686	288·24112595	81·31132040
92·	6647·61005499	289·02652412	81·53287713
·25	6683·78745789	289·81192228	81·75443366
·5	6720·06303556	290·59732044	81·97599060
·75	6756·43678800	291·38271861	82·19754733
93·	6792·90871521	292·16811677	82·41910406
·25	6829·47881719	292·95351493	82·64066079
·5	6866·14709394	293·73891310	82·86221752
·75	6902·91354546	294·52431126	83·08377425
94·	6939·77817177	295·30970912	83·30533096
·25	6976·74097284	296·09510759	83·52688771
·5	7013·80194367	296·88050579	83·74844444
·75	7050·96109928	297·66590391	83·97000117
95·	7088·21842465	298·45130208	84·19155791
·25	7125·57992480	299·23670024	84·41311464
·5	7163·02759971	300·02209840	84·63467136
·75	7200·57944940	300·80749657	84·8562284
96·	7238·22947380	301·59289473	85·07778484
·25	7275·97767308	302·37829269	85·29934157

Given.	Required		
Diam.	Area.	Circumference.	Side of eq. sq.
96·5	7313·82404707	303·16369106	85·52089830
·75	7351·76[illegible]	303·949[illegible]	85·74245503
97·	7389·81131940	304·73448738	85·96401176
·25	7427·95221771	305·51988555	86·18556849
·5	7466·19129079	306·30528371	86·40718522
·75	7504·52853864	307·09068187	86·62868195
98·	7542·96396126	307·87608004	86·85023869
·25	7581·49755865	308·66147820	87·07179542
·5	7620·19983081	309·44081635	87·29335215
·75	7658·85927774	310·23227453	87·51490888
99·	7697·68789944	311·01767269	97·73646568
·25	7736·61369291	311·80307085	87·95802234
·5	7775·63816715	312·58846902	88·17957907
·75	7814·76081816	313·37386718	88·40113580
100·	7853·98168397	314·15926535	88·62269254
·25	7893·30062952	314·94466352	88·84424927
·5	7932·71779985	315·73006168	89·06580600
·75	7972·23814494	316·51545984	89·28736273

The preceding tables were computed with great care by the author's esteemed friend, the late H. Goodwin, Esq. of Blackheath, a gentleman whose indefatigable perseverance and remarkable accuracy in reference to numerical computations cannot be too highly characterized. They are inserted here to supersede the necessity of consulting some erroneous tables of the areas, &c. of circles recently put into circulation.

Supposing the unit to be an inch, for example, the last table exhibits the area, circumference, and side of equal square, corresponding to diameters varying by a quarter of an inch, from 1 inch to 100.

But the same table may also be made to serve for many intermediate diameters, as well as for diameters beyond its apparent reach, by simply recollecting that *the circumferences and sides of equal squares are as the diameters*, while *the areas are as the squares of the diameters*.

Thus, divide any

circumf. or side of equal squ.	by	2, 3, 4, 5, 6, 10 &c.	the quotient will give the	circumf. or side of equal square to	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$	the original diameter.

Or, multiply by 2, 3, 4, 5, &c. the product will give the circumference or side of equal square to 2, 3, 4, 5, &c. times the assumed diameter.

For areas, take $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, &c. for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. the assumed diameter, and, contrarily 4 times, 9 times, 16 times, 25 times, 100 times, &c. an area, for the one which agrees to twice, 3 times, 4 times, 5 times, 10 times, &c. the diameter to which the assumed area corresponds.

Example. Find the area of a circle whose diameter is $8\frac{1}{8}$ or 8·125.

Area to diameter 16·25 is	207·39420252
Divide this by 2^2 =	4
The quotient is the area required	51·84855063

Ex. 2. Find the area of a circle whose diameter is 8·1.

Area to diameter 81, is	5152·99735604
Divide this by 10^2 =	100
The quotient is the area required	51·5299735604

Ex. 3. The exterior and the interior diameters of a circular ring, are 8·75 and 8·1. Required its area.

Area to diameter 8·75, from table	60·13204688
——————— 8·1, from above	51·52997356
Their diff. is the area of the ring . .	8·60205332

Areas of circular rings thus found, being multiplied into their thickness, will give their capacity, as in the rim of a fly-wheel, &c.; whence, knowing the weight of a cubic inch, or foot, the weight of the whole becomes known: but the method already given at pa. 293, will usually be found preferable, unless great accuracy be required; in which case recourse may be had to this table.

Table III. *Relations of the Arc, Abscissa, Ordinate, and Subnormal, in the Catenary; useful in the construction of Catenarian equilibrated arches for bridges or powder magazines.*

Arc AP	Abscissa AG	Ordinate GP	Subnormal GK	Arc AP	Abscissa AG	Ordinate GP	Subnormal GK
1	1·00	0·02	25·00	40	31·22	22·17	19·51
2	2·00	0·08	24·97	41	31·74	23·02	19·36
3	2·99	0·18	24·94	42	32·26	23·88	19·20
4	3·99	0·32	24·91	43	32·77	24·74	19·05
5	4·97	0·50	24·86	44	33·27	25·61	18·90
6	5·95	0 71	24·79	45	33·76	26·48	18·75
7	6·92	0·96	24·71	46	34·24	27·36	18·61
8	7·87	1·25	24·60	47	34·71	28·24	18·46
9	8·82	1·57	24·50	48	35·18	29·12	18·32
10	9·75	1·93	24·37	49	35·64	30·01	18·18
11	10·67	2·31	24·25	50	36·09	30·90	18·05
12	11·58	2·73	24·12	51	36·53	31·80	17·91
13	12·47	3·18	23·98	52	36·97	32·70	17·77
14	13·36	3·65	23·86	53	37·40	33·60	17·64
15	14·23	4·16	23·72	54	37·82	34·51	17·51
16	15·07	4·68	23·55	55	38·24	35·42	17·38
17	15·90	5·23	23·38	56	38·65	36·33	17·25
18	16·72	5·81	23·22	57	39·06	37·24	17·13
19	17·52	6·40	23·05	58	39·46	38·16	17·01
20	18·31	7·01	22·89	59	39·85	39·08	16·89
21	19·08	7·65	22·72	60	40·24	40·00	16 77
22	19·84	8·31	22·55	61	40·62	40·93	16 65
23	20·59	8·98	22·38	62	40·99	41·85	16·53
24	21·32	9·66	22·21	63	41·36	42·78	16·42
25	22·03	10·36	22·03	64	41·73	43·71	16·30
26	22·73	11·08	21·86	65	42·09	44·65	16·19
27	23·42	11·80	21·68	66	42·45	45·58	16·08
28	24·09	12·54	21·51	67	42·80	46·51	15·97
29	24 76	13·29	21·34	68	43·15	47·45	15·86
30	25·40	14·05	21·17	69	43·49	48·39	15·76
31	26·04	14·83	21·00	70	43·83	49·33	15·65
32	26·66	15·61	20·83	71	44·17	50·28	15·55
33	27·27	16·41	20·66	72	44·50	51·22	15·45
34	27·87	17·21	20·49	73	44·82	52·17	15·35
35	28·45	18·02	20·33	74	45·14	53·11	15·25
36	29·03	18·83	20·16	75	45·46	54·06	15·16
37	29·59	19·66	20·00	76	45·77	55·01	15·06
38	30·14	20·49	19·83	77	46·08	55·96	14·96
39	30·68	21·33	19·67	78	46·39	56·91	14·87

Table III. *Relations of the Catenary, continued.*

Arc AP	Abscissa AG	Ordinate GP	Subnormal GK	Arc AP	Abscissa AG	Ordinate GP	Subnormal GK
79	46·69	57·86	14·77	115	55·77	92·68	12·13
80	46·99	58·81	14·68	116	55·98	93·66	12·07
81	47·29	59·77	14·59	117	56·19	94·64	12·01
82	47·58	60·72	14·50	118	56·39	95·62	11·95
83	47·87	61·68	14·42	119	56·60	96·59	11·90
84	48·16	62·64	14·33	120	56·80	97·57	11·84
85	48·44	63·60	14·25	121	57·00	98·55	11·78
86	48·73	64·56	14·16	122	57·21	99·53	11·73
87	49·00	65·52	14·08	123	57·41	100·51	11·67
88	49·28	66·48	14·00	124	57·61	101·49	11·62
89	49·55	67·44	13·92	125	57·81	102·47	11·57
90	49·82	68·41	13·84	126	58·00	103·45	11·51
91	50·09	69·37	13·76	127	58·20	104·42	11·46
92	50·35	70·34	13·68	128	58·39	105·42	11·41
93	50·61	71·30	13·60	129	58·58	106·40	11·36
94	50·87	72·27	13·53	130	58·77	107·38	11·30
95	51·13	73·24	13·45	131	58·96	108·35	11·25
96	51·38	74·20	13·38	132	59·15	109·35	11·20
97	51·63	75·17	13·31	133	59·34	110·33	11·15
98	51·88	76·14	13·23	134	59·52	111·31	11·10
99	52·13	77·11	13·16	135	59·71	112·28	11·06
100	52·37	78·08	13·09	136	59·89	113·28	11·01
101	52·61	79·05	13·02	137	60·07	114·26	10·96
102	52·85	80·02	12·95	138	60·25	115·25	10·91
103	53·09	81·00	12·89	139	60·43	116·23	10·87
104	53·32	81·97	12·82	140	60·60	117·22	10·82
105	53·55	82·94	12·75	141	60·77	118·20	10·77
106	53·78	83·91	12·69	142	60·95	119·18	10·73
107	54·01	84·89	12·62	143	61·12	120·15	10·68
108	54·24	85·86	12·56	144	61·29	121·15	10·64
109	54·47	86·83	12·50	145	61·46	122·14	10·59
110	54·69	87·81	12·43	146	61·63	123·12	10·55
111	54·91	88·78	12·37	147	61·79	124·11	10·50
112	55·13	89·76	12·31	148	61·96	125·10	10·46
113	55·35	90·73	12·25	149	62·12	126·09	10·42
114	55·56	91·71	12·19	150	62·29	127·08	10·38

At pa. 177 of this volume there is given a table of relations of catenarian curves, so arranged as to be useful in the erection of suspension bridges, and in determining the weight and tension of the several parts. The table now presented in this supplement agrees with that in principle, but presents a totally different aspect, in order that it may be subservient to the construction of catenarian arches of equilibration, whether for bridges or for powder magazines.

It is very well known with regard to arches of equilibration, when the substance of the structure presses vertically downwards by the force of gravity, that for a parabolic arch assumed as the intrados, the principles of equiponderance require a similar and equal parabolic curve, situated throughout at the same vertical distance from their inner curve, as the extrados: and in that case, the curves may be easily constructed and the joints of the voussoirs found, by the methods explained in p. 163 and 165. But it is equally true, that upon the same general principles of equipollence a catenarian curve may be assumed for the arch, and the intrados and extrados be two similar and equidistant curves, provided that equidistance be measured upon the radius of curvature at each point, and be but small compared with the span of the arch. Constructions founded upon the knowledge of this fact, have been long rejected, but have of late been re-introduced; on which account Table III is given, that the time of practical men may not be wasted in *needless* calculation.

Suppose that the marginal figure in the lower part of pa. 162, represented a catenary, in which, as indicated in the table, A P represented any portion of the curve, A G, and G P, the corresponding abscissa and ordinate, and G K, the subnormal, p representing the parameter, or constant horizontal tension at the vertex of the curve. Then, in addition to the equations exhibited in pa. 173, &c. it is known that $AP = \sqrt{(AG)^2 + 2p\,AG}$.

$$\text{and } \frac{1}{p} = \frac{2\,AG}{(GP)^2} - \frac{2\,(AG)^3}{3\,(GP)^4} + \frac{26\,(AG)^5}{45\,(GP)^6} - \frac{662\,(AG)^7}{945\,(GP)^8} + \text{\&c.}$$

Now, the table presents values of A G, G P, G K, corresponding to several values of A P, the arch, on the supposition that P, the parameter, or tension at the vertex, is 25.

Suppose, as an illustration of the use of the table, it were required to construct a vault whose semibase should be 8 (yards, for example) and the height 10; we shall find its model, by searching that part of the table where the abscissa, A G, and the ordinate, G P, are in the ratio of 10 to 8. This is, where the arc A P is 80; for there the abscissa and ordinate are 58·81 and 46·99, which are in the ratio of 10·01 to 8, sufficiently near for practical purposes. Hence then, the arch may be regarded as divided into 80 equal parts, and the table will present the computed proportions for 80 voussoirs, on each side of the vertex, or rather for 79 voussoirs and half the key-stone. Or they may be reduced to 40, or to 20, or to 10, on each side the crown of the structure. The actual values to the dimensions 10 and 8, will be found by multiplying each number in the table by $\frac{8}{46\cdot99}$ or its equivalent 0·1702; or, to have the dimensions in feet, let the multiplier be 3 × 0·1702 or 0·5106. Thus, we should have the values of A G, G P, A P, for every voussoir from the vertex of the curve downwards, while the corresponding values of A G + G K would give the points K from which the line K P must be drawn to give in each case the direction of the joint. In the example assumed the value of the parameter, p, would be 25 × 0·1702 or 4·255; and this would be the measure of the horizontal thrust.

The curve thus sketched will be posited in the middle of the arch, half way between the intrados and the extrados. Trace above and below this, at the distance $\frac{1}{2}$ t, (half the thickness at the crown) measured upon the respective positions of the joints, two curves parallel to the mean catenary; so shall you obtain the proposed arch of equilibration.

In cases where the proposed ratio of the height and semi-span of the arch, cannot be found with sufficient accuracy in the tables, it may be approximated to by the usual methods employed in reference to proportional parts. As, suppose the semi-span and the altitude were to be equal; this condition lies between 60 and 61 in the values of the arch; and, by the method just alluded to, we shall find that when the arc is equal to 60·44, the height and half base are each 40·40, the horizontal thrust being 25.

If an equilibrated circular arch were to be erected, of the same span and height, and the same thickness at the crown, the horizontal thrust would be 31·6. Whence, by the way, it appears that an equilibrated catenarian arch in these proportions, would produce a less horizontal thrust than an equilibrated circular arch: contrary to the opinion of Bossut, *Equilibre des voûtes*.

THE END.

OTHER WORKS BY THE AUTHOR OF THIS VOLUME.

1. LESSONS, ASTRONOMICAL AND PHILOSOPHICAL, for the Amusement and Instruction of British Youth, 12mo. 5th edition. Price 5s. boards.

2. AN ELEMENTARY TREATISE ON NATURAL PHILOSOPHY; translated from the French of M. R. J. Hauy, with Notes, &c. in 2 vols. 8vo. with twenty-four plates. Price 1*l*. 4*s*. boards.

3. Dr. GREGORY'S TREATISE ON ASTRONOMY; in which the Elements of the Science are deduced in a natural Order from the Appearances of the Heavens to an Observer on the Earth; demonstrated on Mathematical Principles, and explained by an application to the various Phenomena, 8vo. with nine Plates. Price 15*s*. boards.

4. Dr. GREGORY'S TREATISE ON MECHANICS, THEORETICAL, PRACTICAL, and DESCRIPTIVE; in two large octavo volumes, besides a volume containing sixty-three quarto Plates. The first volume is devoted to the theory of Statics, Dynamics, Hydrostatics, Hydrodynamics, and Pneumatics: the second volume contains Remarks on the Nature, Construction, and Simplification of Machinery, on Friction, and the Rigidity of Cords, First Movers, &c. with Tables, and Descriptions of nearly two hundred curious and useful Machines, 4th edition. Price 2*l*. 2*s*. boards.

5. In conjunction with Dr. Hutton, Volume Third, of A COURSE OF MATHEMATICS, composed for the Use of the Royal Military Academy. Price 12s. bound.

6. In conjunction with John Mason Good, M. D. F. R. S. and Mr. Newton Bosworth, PANTOLOGIA; a General Dictionary of Arts, Sciences, and Words, completed in 12 vols. royal 8vo. Of this Work, Dr. Gregory was the general Editor.

7. LETTERS TO A FRIEND ON THE EVIDENCES, DOCTRINES, AND DUTIES OF THE CHRISTIAN RELIGION, in 2 vols. crown 8vo. Fourth Edition, 14*s*. boards.

This work is intended to guard young persons of liberal education against infidelity and heterodoxy, and to evince the peculiar importance and reasonableness of Christianity, as manifested in its distinguishing doctrines, and in their powerful tendency to influence the conduct.

8. THE ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY, 12mo. Price 5*s*. bound.

9. A new and improved edition of Dr. HUTTON'S COURSE OF MATHEMATICS, Vols. I. and II. The 3d vol. is also in the press, with additions and improvements.

C. Baldwin, Printer,
New Bridge Street, London.

Plate 1.

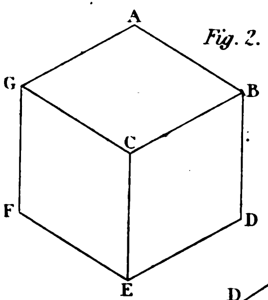


Fig. 2.

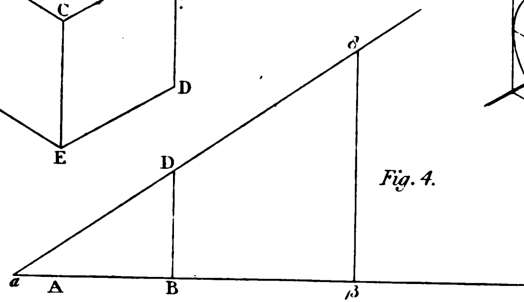


Fig. 4.

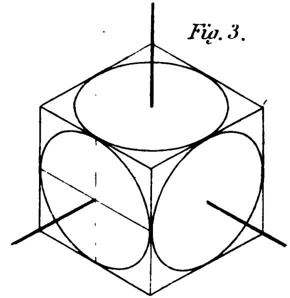


Fig. 3.

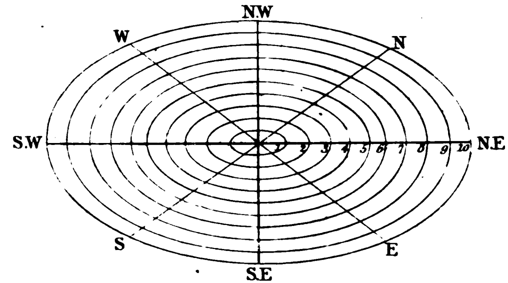


Fig. 7.

Fig. 1.

Fig. 6.

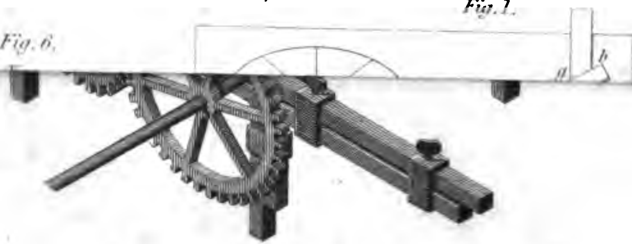
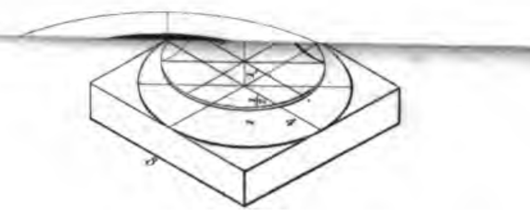


Fig. 5.



ES.

Plate II.

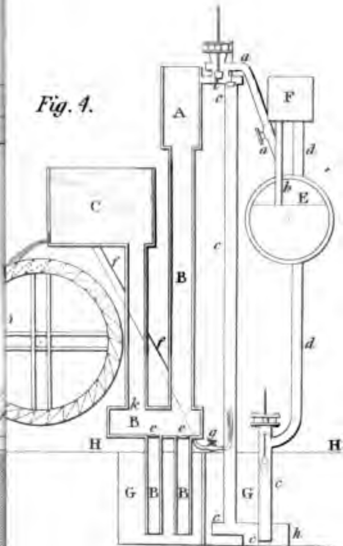


Fig. 4.

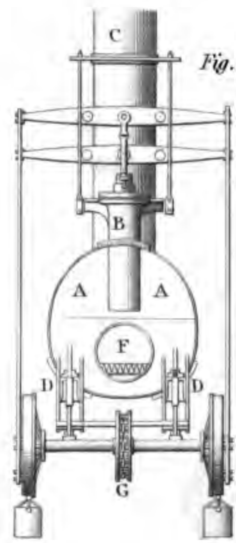
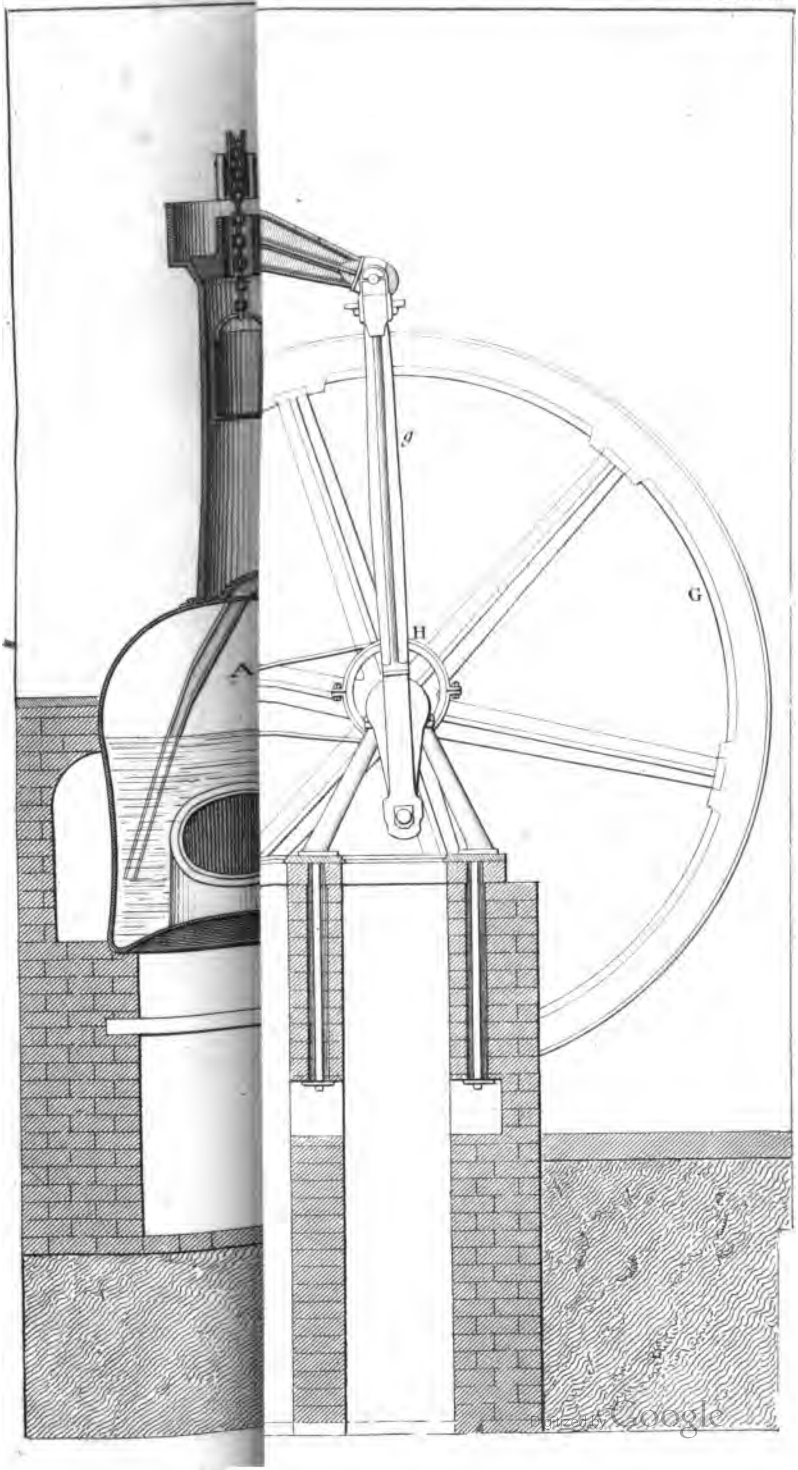


Fig. 3.


A
g
H
G