College Entrance Examination Board

EXAMINATION QUESTIONS

IN

MATHEMATICS

FOURTH SERIES

1916-1920

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON
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PREFACE

While the annual volume of examination questions published by the College Entrance Examination Board has met the needs of many candi- dates for examination and their teachers, the Board is constantly in receip of communications asking for the questions set in certain subjects in suc cessive years. In order to meet this demand the Board has prepared pam phlets containing the questions in certain subjects from 1916 to 192 inclusive. These pamphlets are as follows:

1. Examination questions in Latin and Greek, 1916–1920.
2. Examination questions in English and other modern languages 1916–1920.
3. Examination questions in mathematics, 1916–1920.
4. Examination questions in history, 1916–1920.
5. Examination questions in the natural sciences and in drawing 1916–1920.

Quite apart from meeting the needs of candidates for examination an their teachers, these publications ought to have a beneficial influence upo teaching, for the reason that they illustrate in concrete form principles agree upon by many leading teachers of the subjects represented.

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1916

MATHEMATICS *A*—ELEMENTARY ALGEBRA COMPLETE

Monday 9:00 a.m. Two hours

1. *a*) Factor

$$6x^2-5x-4,$$
$$30m+4p^2-25-9m^2,$$
$$x(x+2)(2x-1)-(x+2).$$

b) Find the value of

$$\frac{x-a}{x}\sqrt{x^2-4},$$

when

$$x=a+\frac{1}{a}.$$

2. *a*) What are the values of

$$(a+b)^0,\ (8)^{-\frac{2}{3}},\ \frac{1}{2^{-1}+3^{-1}}.$$

b) Simplify

$$\frac{\sqrt{5-2\sqrt{6}}+\sqrt{5+\sqrt{24}}}{3-\sqrt{5}}.$$

3. Solve

$$\frac{x+1}{2x^2+3x-2}+\frac{2-x}{1-x-2x^2}+\frac{x-2}{x^2+3x+2}=0.$$

4. Plot the equations $xy=4$ and $2x-3y=5$ on the same axes, and from the figure estimate the solutions of the equations.

5. Solve

$$\begin{cases} x^2-xy=2x+5, \\ x^2-xy=3y+9. \end{cases}$$

6. The difference between two numbers is 3, and the square of their geometric mean exceeds 75 per cent of their arithmetic mean by $\frac{1}{4}$. Find the numbers.

7. Two bodies, one starting from A and the other from B at the same time, move toward each other and meet in $17\frac{1}{2}$ sec. If it takes one 12 sec. longer than the other to move between A and B, and the rate of the faster is 7 ft. per sec., how long does it take each to traverse the distance?

MATHEMATICS *AI*—ALGEBRA TO QUADRATICS

Monday 9:00 a.m. Two hours

1. *a*) Factor

$$6x^2-5x-4,$$
$$30m+4p^2-25-9m^2,$$
$$x(x+2)(2x-1)-(x+2).$$

b) Find the value of

$$\frac{x-a}{x}\sqrt{x^2-4}, \quad \text{when } x=a+\frac{1}{a}.$$

2. *a*) Simplify

$$\frac{x-\dfrac{x^2}{x+y}}{x-y}+\frac{x^2+xy+y^2}{x^2-y^2}.$$

b) Simplify

$$2+\frac{1}{x}-\left[2x-3\left(\frac{1}{x+1}-\frac{1}{1-x}\right)-\left(\frac{1}{x-1}-\frac{1}{x}-2\right)\right]$$

and test the correctness of your answer by placing $x=2$ in the answer and in the original expression.

3. *a*) Solve

$$\frac{5x-.4}{.3}+\frac{1.5x-.05}{2}=\frac{4.15-8x}{1.2}.$$

b) Solve

$$\begin{cases}\dfrac{2a}{x}+\dfrac{5a}{y}=\dfrac{3}{2},\\[2mm] \dfrac{5a}{x}-\dfrac{2a}{y}=\dfrac{17}{20}.\end{cases}$$

4. *a*) What are the values of

$$(a+b)^0,\ (8)^{-\frac{2}{3}},\ \frac{1}{2^{-1}+3^{-1}}.$$

b) Simplify

$$\frac{\sqrt{5-2\sqrt{6}}+\sqrt{5+\sqrt{24}}}{3-\sqrt{5}}.$$

5. Two boys have together 252 marbles; one arranges his in groups of 6 and the other in groups of 9. They have between them 34 groups. How many marbles has each?

6. A man derives an income of \$165 a year from some money invested at 3% and some at $3\frac{1}{2}$%. If the amounts of the respective investments were interchanged, he would receive \$160. How much has he in each investment?

MATHEMATICS *A2*—QUADRATICS AND BEYOND

Monday 9:00 a.m. **Two hours**

1. *a*) Solve

$$\frac{x+1}{2x^2+3x-2}+\frac{2-x}{1-x-2x^2}+\frac{x-2}{x^2+3x+2}=0\,.$$

b) Find the middle term of

$$\left(2x^3-\frac{y}{4x}\right)^8.$$

2. Solve

$$\sqrt{x+1}-\sqrt{3x+7}+6=0.$$

3. Solve

$$\begin{cases} x^2-xy=2x+5\,, \\ x^2-xy=3y+9\,. \end{cases}$$

4. *a*) Plot the equations $xy=4$ and $2x-3y=5$ on the same axes, and from the figure estimate the solutions of the equations.

   b) A body falls 16 ft. in the first second, three times as far in the next second, five times as far in the third second, and so on. How far would it fall in a minute?

5. The difference between two numbers is 3, and the square of their geometric mean exceeds 75 per cent of their arithmetic mean by $\frac{1}{4}$. Find the numbers.

6. Two bodies, one starting from A and the other from B at the same time, move toward each other and meet in $17\frac{1}{2}$ sec. If it takes one 12 sec. longer than the other to move between A and B and the rate of the faster is 7 ft. per second, how long does it take each to traverse the distance?

MATHEMATICS *A*—ELEMENTARY ALGEBRA COMPLETE

Monday 9:30 a.m. Three hours

1. *a*) Factor

$$6x^2-x-77,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}+2,$$

$$a^3+b^3+a+b.$$

b) Simplify

$$3x^2-[7x-2-(2x-1)(3-x)].$$

2. *a*) Simplify

$$\left[\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right]\div\left[\frac{a+x}{a-x}-\frac{a-x}{a+x}\right].$$

b) Solve, and verify your answer,

$$\frac{4(x+3)}{9}=\frac{8x+37}{18}-\frac{7x-29}{5x-12}.$$

3. *a*) Find, and express in its simplest form, the sixth term of $\left(\frac{x}{2}-\frac{4}{x^2}\right)^8$.

b) Divide $2a-a^{\frac{1}{2}}b^{-\frac{1}{2}}-3b^{-1}$ by $2a^{\frac{1}{2}}-3b^{-\frac{1}{2}}$, and verify the result by placing $a=9$ and $b=4$ in dividend, divisor, and quotient.

4. *a*) Solve $.5x^2+1.5x+.88=0$.

b) Find the values of k for which $x^2-6kx+12=0$ has equal roots.

5. Solve, and verify the positive answer,

$$\sqrt{3x+1}+\frac{35}{\sqrt{3x+1}}=3\sqrt{x}.$$

6. If A gives B \$6, B will have $\frac{2}{3}$ as much as A has left; but if B gives A \$5, B will have $\frac{3}{5}$ as much as A then has. How much had each?

7. The sum of three terms in arithmetical progression beginning with $\frac{3}{2}$ is equal to the sum of three terms of a geometrical progression beginning with $\frac{3}{2}$, and the common difference is equal to the ratio. What are the two series?

8. A man bought a certain number of oranges for \$2. If he had paid 5 cents more per dozen, he would have received two dozen less for the same money. How much did he pay for a dozen?

MATHEMATICS *A1*—ALGEBRA TO QUADRATICS

Monday 9:30 a.m. Two hours

1. *a*) Factor

$$6x^2-x-77,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}+2,$$

$$a^3+b^3+a+b.$$

b) Simplify

$$3x^2-[7x-2-(2x-1)(3-x)].$$

2. *a*) Simplify

$$\left[\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right]\div\left[\frac{a+x}{a-x}-\frac{a-x}{a+x}\right].$$

b) Solve, and verify your answer,

$$\frac{4(x+3)}{9}=\frac{8x+37}{18}-\frac{7x-29}{5x-12}.$$

3. Solve the following simultaneous equations, and verify your solution.

$$\begin{cases} ax+by=n, \\ 2ax+3by=n. \end{cases}$$

4. *a*) Divide $2a-a^{\frac{1}{2}}b^{-\frac{1}{2}}-3b^{-1}$ by $2a^{\frac{1}{2}}-3b^{-\frac{1}{2}}$, and check the result by placing $a=9$ and $b=4$ in dividend, divisor, and quotient.

b) Express $\sqrt{6}-\sqrt{\frac{3}{2}}+\sqrt{54}$ in a form involving a single radical and then find the value of the expression to two decimal places.

5. A can do half as much work as B and B half as much as C, and together they can complete a piece of work in 24 days. How long would it take each to do it alone?

6. If A gives B \$6, B will have $\frac{2}{3}$ as much as A has left; but if B gives A \$5, B will have $\frac{3}{5}$ as much as A then has. How much had each?

MATHEMATICS *A2*—QUADRATICS AND BEYOND

Monday 9:30 a.m. Two hours

1. *a*) Solve

$$.5x^2+1.5x+.88=0.$$

b) Solve

$$abx^2-a^2x=b^2x-ab.$$

2. *a*) Solve

$$\sqrt[5]{x^2}+8\sqrt[5]{x}=9$$

b) Find, and express in its simplest form, the sixth term of

$$\left(\frac{x}{2}-\frac{4}{x^2}\right)^8.$$

3. Solve, and verify the positive answer

$$\sqrt{3x+1}+\frac{35}{\sqrt{3x+1}}=3\sqrt{x}.$$

4. Solve

$$\begin{cases} x^2+y^2+2x-4y+1=0, \\ 3x+5y-1=0. \end{cases}$$

5. The sum of three terms in arithmetical progression beginning with $\frac{3}{2}$ is equal to the sum of three terms in geometrical progression beginning with $\frac{3}{2}$, and the common difference is equal to the ratio. What are the two series?

6. A man bought a certain number of oranges for \$2. If he had paid 5 cents more per dozen, he would have received two dozen less for the same money. How much did he pay for a dozen?

MATHEMATICS A—ELEMENTARY ALGEBRA COMPLETE

Monday, June 17 — 9:30 a.m. Three hours

1. *a*) Factor

$$3x^2+5x-12,$$
$$(a^2-ab)^2-(ab-b^2)^2,$$
$$x^2+y^2+abx-(2x+ab)y.$$

b) Simplify

$$\frac{\frac{1}{x}-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}\div\frac{\frac{1}{x}+\frac{y}{x^2}}{1-\frac{y^2}{x^2}}.$$

2. Solve the following simultaneous equations and verify the solution:

$$x+y+z=-2,$$
$$3x+2z=\ \ 0,$$
$$5y-3z=\ \ 4.$$

3. *a*) Simplify the expression $\sqrt{28}+\frac{1}{8-3\sqrt{7}}$ and calculate its value correct to the nearest hundredth.

b) Find, and express in its simplest form, the fourth term of $(\sqrt[4]{m}-\sqrt[3]{5})^{11}$.

4. *a*) Solve $x+\frac{1}{x-1}=\frac{9}{2}$.

b) Solve

$$(x^2-1)^2-11(x^2-1)+24=0.$$

5. *a*) Solve the simultaneous equations:

$$y=x^2+2x-5,$$
$$3x=y+3.$$

b) Plot the graphs of these two equations on the same set of axes and show how the figure checks the solutions found in *a*).

6. Solve $\sqrt{2x+1}+\sqrt{x-3}=\sqrt{5x+4}$. Show whether your results satisfy the given equation.

(SEE NEXT PAGE)

7. A dealer bought a certain number of grapefruit for $1.04. After throwing away 4 bad ones, he sold the others for 6 cents apiece more than he paid for them and made a profit of 22 cents. How many did he buy? (No credit will be given for merely guessing the answer.)

8. *a*) The sum of the first and third terms of a geometrical progression is 15; the sum of the second and fourth terms is 30. Find the first term and the common ratio.

b) In the equation $ax^2+bx+c=0$ prove that the sum of the roots is $-\frac{b}{a}$ and that the product of the roots is $\frac{c}{a}$.

MATHEMATICS *AI*—ALGEBRA TO QUADRATICS

Monday, June 17 9:30 a.m. Two hours

1. *a*) Factor

$$3x^2+5x-12,$$
$$(a^2-ab)^2-(ab-b^2)^2,$$
$$x^2+y^2+abx-(2x+ab)y.$$

b) Simplify $\frac{1}{(x+1)^2}+\frac{1}{(x-1)^2}-\frac{2}{x^2-1}$ and test the result by putting $x=2$.

2. *a*) Simplify

$$\frac{\frac{1}{x}-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}\div\frac{\frac{1}{x}+\frac{y}{x^2}}{1-\frac{y^2}{x^2}}.$$

b) Find the value of

$$x^2+\frac{(ab+b^2-1)x}{a+b}+\frac{a}{a+b}$$

when $x=\frac{1}{a+b}$.

3. Solve the following simultaneous equations and verify the solution:

$$x+y+z=-2,$$
$$3x+2z=0,$$
$$5y-3z=4.$$

4. *a*) Simplify the expression $\sqrt{28}+\frac{1}{8-3\sqrt{7}}$ and calculate its value correct to the nearest hundredth.

b) Multiply

$$x^{\frac{3}{4}}-2x^{\frac{1}{4}}+4x^{-\frac{1}{4}} \text{ by } x^{\frac{1}{4}}+2x^{-\frac{1}{4}}$$

5. Two automobiles starting 180 miles apart, and running toward each other with different velocities, meet at the end of four hours. If one had gone twice as fast and the other only three-fourths as fast, they would have met after three hours. What was the speed of each?

6. On an inclined railway the fare is 75 cents for the upward and 25 cents for the downward trip. One day a certain number of passengers rode up; some of them rode down, while the rest walked. The receipts of the railway were $155. Next day three times as many passengers rode up as on the first day. All of these either walked or rode down during the day, but there were 120 more who walked than on the previous day. The receipts for the second day were $445. How many passengers rode up and how many walked down the first day?

MATHEMATICS *A2*—QUADRATICS AND BEYOND

Monday, June 17 9:30 a.m. Two hours

1. *a*) Solve $x+\frac{1}{x-1}=\frac{9}{2}$; find the value of x correct to the nearest hundredth.

   b) Find the values of t in

$$(3t+a)^2=4(2t^2-a^2).$$

2. *a*) Solve the simultaneous equations:

$$y=x^2+2x-5,$$
$$3x=y+3.$$

   b) Plot the graphs of these two equations on the same set of axes and show how the figure checks the solutions found in *a*).

3. Solve $\sqrt{2x+1}+\sqrt{x-3}=\sqrt{5x+4}$. Show whether your results satisfy the given equation.

4. *a*) Find, and express in its simplest form, the fourth term of

$$(\sqrt[4]{m}-\sqrt[3]{5})^{11}.$$

   b) Solve $(x^2-1)^2-11(x^2-1)+24=0$.

5. A dealer bought a certain number of grapefruit for $1.04. After throwing away 4 bad ones, he sold the others at 6 cents apiece more than he had paid for them and made a profit of 22 cents. How many did he buy? (No credit will be given for merely guessing the answer.)

6. *a*) If the first term of an arithmetical progression is a and the common difference d, what is the sixteenth term? What is the sum of the first sixteen terms? Evaluate these formulas for $a=11$ and $d=-2$.

   b) The sum of the first and third terms of a geometrical progression is 15 and the sum of the second and fourth terms is 30. Find the first term and the common ratio.

1919

MATHEMATICS *A*—ELEMENTARY ALGEBRA COMPLETE

Monday, June 16 9:30 a.m. Three hours

[For separate question papers in Mathematics *A1* and *A2* see pages 3 and 4]

1. *a*) Factor

$$3mx^2-8mx-3m,$$
$$x^4+64x,$$
$$1-2ax-(c-a^2)x^2+acx^3.$$

b) Simplify

$$\frac{\frac{1}{a}-\frac{b+a}{b^2+a^2}}{\frac{1}{b}-\frac{b+a}{b^2+a^2}},$$

and test the result by setting $a=1$ and $b=2$ in the given expression and in your result.

2. *a*) Find the value of each of the following expressions:

$$\frac{(x+y)^0}{2^{-3}}, \quad (4)^{\frac{3}{2}}, \quad \frac{1^{3x}}{2^{-1}-3^{-1}}.$$

b) Given $x^4(x^2y-xyz)^{-2}$. Re-write the expression, without changing its value, so that x appears only within the parenthesis and y only outside the parenthesis.

3. *a*) Solve $3x^2+2x=11$. Give the values of x correct to the nearest hundredth.

b) Find the relation between p and q, for which one root of $x^2+px+q=0$ is three times the other.

4. *a*) Find the value of

$$\sqrt{13+\sqrt{2}+\frac{7}{3+\sqrt{2}}}$$

b) Solve $x^2-2\sqrt{x^2+16}=-1$

Substitute your results in the equation and explain the failure of some of them to satisfy the equation.

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

5. *a*) Solve

$$\frac{1}{x}+\frac{1}{y}=5,$$
$$x-y=0.3.$$

b) Write, and simplify, that term of the expansion of $\left(\frac{x}{\sqrt{y}}-y^{\frac{3}{2}}\right)^8$ which has the greatest coefficient.

6. *a*) Solve

$$y=3x-12,$$
$$x^2-y^2=16.$$

b) Plot the curves represented by the two equations in 6 *a*) and check the solution of 6 *a*) from the graph.

7. *a*) Write down, and prove, the formula for the sum of n terms of the series a, $a+d$, $a+2d$, etc.

b) A farmer sowed $\frac{1}{8}$ bushel of wheat and used the whole produce for seed the following year, the produce of this second year for seed the third year, and the produce of the third year for seed the fourth year. The fourth harvest provided 12,800 bushels. Assuming the rate of increase to have been the same for each of the four years, how many bushels did the second harvest yield?

8. An automobile makes a trip of 150 miles at a constant speed. Returning, it travels $2\frac{1}{2}$ miles an hour faster and returns in 40 minutes' less time than it took to go out. Required, the speed of the car on the outward trip.

1919

MATHEMATICS *AI*—ALGEBRA TO QUADRATICS

Monday, June 16 9:30 a.m. Two hours

[For question paper in Mathematics *A* see page 1.]

1. *a*) Factor

$$3mx^2-8mx-3m,$$
$$x^4+64x,$$
$$1-2ax-(c-a^2)x^2+acx^3.$$

b) Find the H.C.F. and L.C.M. of $9(1-x)^3(2+x)$, $3(x+1)(x-1)(x-2)$, and $6(x+1)^2(x-1)^2(2x+3)$. Leave the results in factored form.

2. *a*) Solve

$$\frac{x}{x-2}-\frac{4}{x+3}=1$$

and verify the solution.

b) Simplify

$$\frac{\frac{1}{a}-\frac{b+a}{b^2+a^2}}{\frac{1}{b}-\frac{b+a}{b^2+a^2}}.$$

and test the result by putting $a=1$ and $b=2$ in the given expression and in your result.

3. A boatman rowing down a river makes 23 miles in 3 hours, and returns at the rate of $3\frac{1}{2}$ miles an hour. How fast does the river flow?

4. *a*) Given $x^4(x^2y-xyz)^{-2}$. Re-write the expression, without changing its value so that x appears only within the parenthesis and y only outside the parenthesis.

b) Solve

$$\sqrt{x}+\sqrt{9+x}=\frac{45}{\sqrt{9+x}}.$$

5. Solve the following equations for x and y:

$$\frac{x}{y}=\frac{a}{b},\quad \frac{x+1}{y+1}=\frac{c}{d}.$$

6. At two stations, A and B, six miles apart on a line of railway the prices of coal are \$10 per ton and \$12 per ton respectively. The rates of cartage of coal are \$1 per ton per mile from A and \$1.50 per ton per mile from B. Find the distance from A of a place on the railroad from A to B at which a consumer lives, if the cost of a ton is the same whether delivered from A or B.

MATHEMATICS *A2*—QUADRATICS AND BEYOND

Monday, June 16 9:30 a.m. Two hours

1. *a*) Solve $3x^2+2x=11$. Give the values of x correct to the nearest hundredth.

   b) Find the relation between p and q for which one root of $x^2+px+q=0$ is three times the other.

2. *a*) Find the value of

$$\sqrt{13+\sqrt{2}+\frac{7}{3+\sqrt{2}}}.$$

   b) Solve

$$x^2-\sqrt{x^2+16}=-1.$$

   Substitute your results in the equation and explain the failure of some of them to satisfy the equation.

3. *a*) Solve

$$\frac{1}{x}+\frac{1}{y}=5,$$
$$x-y=0.3.$$

   b) Write, and simplify, that term of the expansion of $\left(\frac{x}{\sqrt{y}}-y^{\frac{3}{2}}\right)^8$ which has the greatest coefficient.

4. *a*) Solve

$$y=3x-12,$$
$$x^2-y^2=16.$$

   b) Plot the curves represented by the two equations in 4 *a*) and check the solutions of 4 *a*) from the graph.

5. *a*) Write down, and prove, the formula for the sum of n terms of the series a, $a+d$, $a+2d$, etc.

   b) A farmer sowed $\frac{1}{8}$ bushel of wheat and used the whole produce for seed the following year, the produce of this second year for seed the third year, and the produce of the third year for seed the fourth year. The fourth harvest provided 12,800 bushels. Assuming the rate of increase to have been the same for each of the four years, how many bushels did the second harvest yield?

6. An automobile makes a trip of 150 miles at a constant speed. Returning, it travels $2\frac{1}{2}$ miles an hour faster and returns in 40 minutes' less time than it took to go out. Required, the speed of the car on the outward trip.

1920

MATHEMATICS *A*—ELEMENTARY ALGEBRA COMPLETE

Monday, June 21 9:30 a.m. Three hours

[For separate question papers in Mathematics *A1* and *A2* see pages 3 and 4.]

1. Factor

$$35+11x-6x^2,$$
$$x^2-a^2+y^2-b^2-2xy+2ab.$$

2. Express as a fraction in its lowest terms

$$\frac{1}{ax}\left(\frac{a}{x}-\frac{x}{a}\right)\div\frac{a^3+x^3}{a^3x^3}.$$

3. Solve the equation

$$3x^2-4x-5=0,$$

computing the roots to the nearest hundredth.

4. Solve the equation

$$\frac{x-1}{x-2}-2-\frac{x}{1-x}=0.$$

Test your answer.

5. The sides of a rectangle are in the ratio 5:2. If 2 inches are added to each side, the ratio is 4:3. Find the sides.

6. Solve the simultaneous equations:

$$y=x^2-4x+3,$$
$$4x+4y-7=0.$$

Pair the results clearly.

7. Plot the graphs of the equations of question 6, and mark the values of x and y at the points of intersection.

8. Simplify

$$(x^{\frac{1}{2}}-y^{-\frac{3}{2}})^2+2\left(\sqrt{\frac{x}{y}}\right)^3.$$

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

9. A man wishing to travel from a town A to a town B, 65 miles away, finds it convenient to go by automobile from A to another town C and thence by trolley to B. The triangle ABC is assumed to have a right angle at C. If the automobile averages 20 miles an hour and the trolley 10 miles an hour, and the total time of the trip, exclusive of the time spent at C, is $5\frac{1}{2}$ hours, find the distances AC and CB.

(NOTE.—Make use of the fact that in the triangle ABC the square of the side opposite the right angle is equal to the sum of the squares of the other two sides.)

10. A car slipping backward down hill moves 2 inches in the first second, 6 inches in the second second, 10 inches in the third second and so on. How far will it move in 20 seconds?

1920

MATHEMATICS *AI*—ALGEBRA TO QUADRATICS

Monday, June 21 9:30 a.m. Two hours

[For question paper in Mathematics *A* see page 1.]

1. Factor

$$35+11x-6x^2,$$
$$x^2-a^2+y^2-b^2-2xy+2ab.$$

2. Express as a fraction in its lowest terms

$$\frac{1}{ax}\left(\frac{a}{x}-\frac{x}{a}\right)\div\frac{a^3+x^3}{a^3x^3}.$$

3. Solve the equation

$$\frac{x-1}{x-2}-2-\frac{x}{1-x}=0.$$

Test your answer.

4. Simplify

$$(x^{\frac{3}{2}}-y^{-\frac{3}{2}})^2+2\left(\sqrt{\frac{x}{y}}\right)^3.$$

5. The sides of a rectangle are in the ratio 5:2. If 2 inches are added to each side the ratio is 4:3. Find the sides.

6. A photographer has two mixtures of a certain chemical and water, the one containing 50 per cent of the chemical, the other 10 per cent. He wishes to obtain 8 oz. of a mixture which shall contain 25 per cent of the chemical. How much shall he take of each of the mixtures he has?

MATHEMATICS *A2*—QUADRATICS AND BEYOND

Monday, June 21 9:30 a.m. Two hours

[For question paper in Mathematics *A* see page 1.]

1. Solve the equation

$$3x^2-4x-5=0,$$

computing the roots to the nearest hundredth.

2. Solve the simultaneous equations:

$$y=x^2-4x+3,$$
$$4x+4y-7=0.$$

Pair the results clearly.

3. Plot the graphs of the equations of question 2, and mark the values of x and y at the points of intersection.

4. Find the term of the expansion of

$$\left(\sqrt{x}-\sqrt{\frac{y}{x}}\right)^{10}$$

which does not contain x.

5. A man wishing to travel from a town A to a town B, 65 miles away, finds it convenient to go by automobile from A to another town C and thence by trolley to B. The triangle ABC is assumed to have a right angle at C. If the automobile averages 20 miles an hour and the trolley 10 miles an hour, and the total time of the trip, exclusive of the time spent at C, is $5\frac{1}{2}$ hours, find the distances AC and CB.

(NOTE.—Make use of the fact that in the triangle ABC the square of the side opposite the right angle is equal to the sum of the squares of the other two sides.)

6. A car slipping backward down hill moves 2 inches in the first second, 6 inches in the second second, 10 inches in the third second, and so on. How far will it move in 20 seconds?

MATHEMATICS *B*

ADVANCED ALGEBRA

MATHEMATICS *B*—ADVANCED ALGEBRA

Tuesday 2:00 p.m. Two hours

1. *a*) Solve

$$3x^4+20x^3+46x^2+41x+10=0.$$

b) Write the equation of which the roots are the reciprocals of those of the equation

$$5x^5+4x^4+3x^3-x+6=0.$$

2. *a*) Solve the equation

$$x^4-4x^3+10x^2-12x+8=0,$$

knowing that one root is $1+i$.

b) From a study of the signs of the equation

$$x^5+3x^3-2x^2-x+11=0,$$

what can be said of its roots?

3. Locate the roots of

$$x^3-3x^2-x+5=0,$$

and find the largest one to two decimal places.

4. *a*) Reduce to the form $a+bi$

$$\frac{4+\sqrt{-3}}{2-\sqrt{-3}},\ (2-3\sqrt{-4})^2,\ \sqrt{5-12i}.$$

b) Prove that the sum of the roots of an equation is minus the quotient of the second coefficient by the first.

5. Expand

$$\begin{vmatrix} 1+a & 2 & 3 & 4 \\ 1 & 2+a & 3 & 4 \\ 1 & 2 & 3+a & 4 \\ 1 & 2 & 3 & 4+a \end{vmatrix}$$

6. From the letters of the word "triangles" how many arrangements of 5 letters may be made? Of these how many consist of 2 vowels and 3 consonants? Of these latter, how many contain "s"?

MATHEMATICS *B*—ADVANCED ALGEBRA

Tuesday 2 p.m. Two hours

1. *a*) Solve the equation

$$4x^3-12x^2+11x-3=0.$$

b) Construct the graph of $y=x^3+x-1$, and state what information it gives in regard to the roots of the equation $x^3+x-1=0$.

2. *a*) Find the value of the minor of 12 in the determinant

$$\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}$$

b) Write the value of z in the equations

$$\begin{aligned} 2x-3y+4z&=0, \\ 5x+7y-3z&=2, \\ x-2z-3&=2y, \end{aligned}$$

leaving the answer in the form of the quotient of two determinants.

3. *a*) Find all the integral roots of the equation

$$x^4+x^3-5x^2+x-6=0.$$

b) If a is a root of an equation formed by equating a polynomial to zero, prove that $x-a$ is a factor of the polynomial.

4. *a*) How many arrangements of the first six letters of the alphabet can be made in which the two vowels shall precede the consonants?

b) A committee of five is to be chosen from a group of seven Englishmen and eight Americans. In how many ways can the committee be chosen if it is to contain just two Englishmen?

5. *a*) Locate the roots of the equation

$$x^3+x^2-3x-1=0,$$

and find the value of a positive root to two places of decimals.

b) Knowing that one of the roots of the equation

$$x^4+4x^3+6x^2+4x+5=0$$

is $\sqrt{-1}$, solve the equation completely. Represent all the complex roots on one diagram, and find their sum by graphic methods.

MATHEMATICS *B*—ADVANCED ALGEBRA

Tuesday, June 18 2 p.m. Two hours

1. *a*) Put the following expressions into the form $a+bi$, add them, and express the result graphically:

$$(2+3i)^2, \qquad \frac{3+\sqrt{-4}}{1-\sqrt{-4}}, \qquad \sqrt{8-6i}.$$

b) Find all the values of x which satisfy

$$2x^4-9x^3+16x^2-14x+4=0.$$

2. *a*) From the signs of $x^5+2x^2-x-11=0$ what may be inferred about the roots?
   b) If a polynomial in x is divided by $x-a$, prove that the remainder is the result obtained by substituting a for x in the dividend.

3. One root of $x^3-4x^2+kx-6=0$ is $1+\sqrt{-2}$. Find the other roots and the value of k.

4. *a*) Graph

$$y=x^3+2x^2-5x-9.$$

b) Find to two decimal places the positive root of

$$x^3+2x^2-5x-9=0.$$

5. From the letters of the word "importance" how many permutations of five letters each may be formed? How many of these will include the letter p? How many permutations of five letters could be formed if p and m must not appear in the same permutation?

6. *a*) Write a determinant in which an element is zero and which is equal to

$$\begin{vmatrix} 3 & -2 & -3 & 1 \\ -4 & 1 & 2 & -3 \\ 2 & 3 & -4 & 1 \\ 3 & 1 & -1 & 4 \end{vmatrix}$$

b) Evaluate the above determinant.

MATHEMATICS *B*—ADVANCED ALGEBRA

Tuesday, June 17 **2 p.m. Two hours**

1. Find, correct to the nearest hundredth, the real root of

$$x^3+x^2-2x-10=0.$$

2. *a*) Solve the equation

$$x^4-4x^2+8x+35=0$$

completely, having given that one root is $2-\sqrt{-3}$

b) From a study of the signs of

$$x^3+ax^2+a^2=0$$

say what you can about the roots of the equation under the two hypotheses (1) that a is positive and (2) that a is negative.

3. *a*) Solve completely the equation

$$(x-1)(x-2)(x+3)=12.$$

b) If a is a root of

$$x^3+mx^2+mx+1=0$$

prove that $\frac{1}{a}$ is also a root.

4. *a*) Find in the form $a+bi$ the value of $\frac{2z+1}{z^2-3z+2}$ when $z=3+2i$.

b) Find all the roots of $x^3=8$ and represent each root graphically.

5. *a*) Express the value of x in the solution of the equations

$$x-2y+1=0$$
$$3y-z-2=0$$
$$z+5x+4=0$$

as the quotient of two determinants, and evaluate.

b) Prove that if two rows of a determinant are equal, the value of the determinant is zero.

6. *a*) With 12 consonants and 5 vowels, how many arrangements of four letters with two vowels in the middle and a consonant at each end can be made if no letter is repeated in each arrangement? How many if any letter may be repeated?

b) If the number of permutations of n things taken 4 at a time is equal to twice the number of combinations of $n+1$ things taken 5 at a time, what is the value of n?

MATHEMATICS *B*—ADVANCED ALGEBRA

Tuesday, June 22 — 2 p.m. Two hours

1. Reduce the following expressions to the form $a+bi$, find their sum, and represent the addition graphically:

$$\frac{1-2i}{2+i}, \qquad (2-i)^2.$$

2. How many odd numbers of 5 figures each, without repetitions, can be formed from the figures, 1, 2, 3, 4, 5, 6, 7?

3. By means of determinants find the value of z which satisfies the following simultaneous equations:

$$5x+3y=65,$$
$$2y-\ z=11,$$
$$3x+4z=57.$$

4. One root of the equation

$$4x^4-20x^3+25x^2+70x-39=0$$

is $3-2i$. Factor the polynomial.

5. Find by Horner's method a positive root of the following equation, correct to the nearest hundredth:

$$x^4-3x^3+2x-3=0.$$

6. State and prove the theorem for transforming

$$a_0x^n+a_1x^{n-1}+\ .\ .\ .\ .\ +a_{n-1}x+a_n=0$$

into an equation whose roots shall be m times the roots of the given equation.

MATHEMATICS *CD, C, D*

*MATHEMATICS *CD*—PLANE AND SOLID GEOMETRY

MATHEMATICS *C*—PLANE GEOMETRY

MATHEMATICS *D*—SOLID GEOMETRY

* Not given in 1916

M

MATHEMATICS *C*—PLANE GEOMETRY

Monday 2:00 p.m. Two hours

The candidate is requested to state on the cover of the answer-book what textbook of Geometry was used in preparation.

Group A

(Answer all the questions of this group.)

1. *a*) Prove that the diagonals of a rhombus bisect each other at right angles.
   b) What is the sum of the angles of a polygon of n sides? How many diagonals in all may be drawn in such a figure? Prove your answers correct.

2. Prove that two triangles are similar if the three sides of the one are proportional respectively to the three sides of the other.
   Make a separate statement of the hypothesis of this theorem. Point out all places in the course of the proof where use is made of the hypothesis.

3. State and prove a theorem concerning the area of a parallelogram.

4. An equilateral triangle, a square, and a regular hexagon, are inscribed in a circle whose radius is 2 inches. Find the areas of the four figures mentioned. (Roots need not be extracted.)

Group B

(Answer two questions from this group.)

5. Construct an equilateral triangle with given altitude, and prove the construction correct.

6. $ABCD$ is a rhombus with A, C as opposite vertices. O is a point within the rhombus such that $OB=OD$. Prove that A, O, and C are on the same straight line, and that $OA \cdot OC=AB^2-OB^2$.

7. A right triangle has the vertex and the sides of its right angle fixed in position and the length of its hypotenuse constant. Find the locus of the middle point of the hypotenuse. Prove that your answer is correct.

MATHEMATICS *D*—SOLID GEOMETRY

Monday 2:00 p.m. Two hours

The candidate is requested to state on the cover of the answer-book what textbook of Geometry was used in preparation.

1. Prove that a perpendicular can be drawn to a given plane from an external point.

2. Prove that an oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to the lateral edge of the oblique prism.

3. *a*) Prove that a spherical angle is measured by the arc of a great circle drawn from its vertex as a pole and included between its sides or sides produced.

   b) State, without proof, the conditions under which two spherical triangles on the same sphere are congruent.

4. Prove that if a plane and a line are parallel, a plane perpendicular to the line is also perpendicular to the plane.

5. A regular hexagon and its circumscribed circle are revolved about a diameter of the circle which is also a diagonal of the hexagon. Find the ratio of the areas and the ratio of the volumes of the solids generated.

6. An oil tank, in the shape of a right circular cylinder, 30 ft. long and 8 ft. in diameter, lies on its side and is filled to a depth of 2 ft. Find the number of gallons of oil it contains. The number of gallons in one cubic foot is 7.48. Give the result to the nearest gallon.

MATHEMATICS *CD*—PLANE AND SOLID GEOMETRY

Monday 2 p.m. Three hours

This paper will be rated as a whole; separate credits will not be given on this paper for plane geometry and solid geometry.

Group A

(*Answer four questions from this group.*)

1. Prove: Two triangles are congruent if the three sides of one are equal, respectively, to the three sides of the other.

2. *a*) Prove: If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

b) A and B are two points on a railway curve which is an arc of a circle. If the length of the chord AB is 200 ft. and the shortest distance from the midpoint of the curve AB to the chord is 4 ft., find the radius of the arc.

3. Prove: The line segment joining the midpoints of the legs of a trapezoid is parallel to the bases of the trapezoid and is equal to half their sum.

Point out all places in the course of the proof where "what is given" is used. For convenience number the steps of the proof.

4. *a*) A circular pond has a circumference of 88 yd. What is the diameter of the pond? Find to the nearest dollar the cost at $2.00 per sq. yd. of constructing a cement walk 3 ft. wide around the pond. $\left(\text{Use } \pi = \frac{22}{7}.\right)$

b) Given a line segment a, explain how to construct a segment of length $a\sqrt{3}$, and prove that your method is correct.

5. A chord of given length, 8 in., moves with its extremities always on the circumference of a circle whose radius is 5 in. The chord is trisected by the points P and Q. What is the locus of P? (Give no proof.) Find the total length of this locus.

(SEE NEXT PAGE)

Group B

(Answer four questions from this group.)

6. Prove: The sum of the face angles of any convex polyhedral angle is less than four right angles.

7. *a*) Can a straight line be perpendicular to each of two intersecting planes? Why?

   b) Can a straight line be perpendicular to each of two intersecting straight lines? Why?

   c) If a straight ruler be held parallel to a blackboard, will its shadow on the blackboard be parallel to the ruler? Why?

   d) Can a plane be passed through a right circular cone in such a way as to make the section formed a triangle? Explain.

8. Two similar right triangles ABC and $A'B'C'$, the right angles being at C and C', are rotated about AB and $A'B'$ respectively as axes. Prove that the total areas of the solids thus formed are to each other as the squares of any two homologous sides of the triangles.

9. *a*) Between what two limits does the sum of the angles of a spherical triangle lie? Prove your statement.

   b) A right circular cylinder of radius r contains a certain quantity of water. If a volume of water equal to the volume of a sphere of radius r be added to the water in the cylinder, the height of the water rises to a height $2r$ above the bottom of the cylinder. How deep was the water originally in the cylinder?

10. On one wall of a rectangular room is a point A and on the opposite wall is a point B. State, without proof, what points on the floor are equidistant from A and B.

MATHEMATICS *C*—PLANE GEOMETRY

Monday 2 p.m. Two hours

1. Prove: Two triangles are congruent if the three sides of one are equal, respectively, to the three sides of the other.

2. *a*) Prove: If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

   b) *A* and *B* are two points on a railway curve which is an arc of a circle. If the length of the chord *AB* is 200 ft. and the shortest distance from the midpoint of the curve *AB* to the chord is 4 ft., find the radius of the arc.

3. Prove: The line segment joining the midpoints of the legs of a trapezoid is parallel to the bases of the trapezoid and is equal to half their sum.

   Point out all places in the course of the proof where "what is given" is used. For convenience number the steps of the proof.

4. Explain how to construct a mean proportional between two given line segments. Prove that your method is correct.

5. *a*) A circular pond has a circumference of 88 yd. What is the diameter of the pond? Find to the nearest dollar the cost at $2.00 per sq. yd. of constructing a cement walk 3 ft. wide around the pond. $\left(\text{Use } \pi = \frac{22}{7}.\right)$

   b) In laying out a tennis court it is necessary to construct accurately the right angle formed by a base line and a side line. A base line being given and assuming that you have at your disposal a 50-ft. tape measure and a number of stakes, describe a practical method for determining the direction of a side line.

6. A chord of given length, 8 in., moves with its extremities always on the circumference of a circle whose radius is 5 in. The chord is trisected by the points *P* and *Q*. What is the locus of *P*? (Give no proof.) Find the total length of this locus.

MATHEMATICS *D*—SOLID GEOMETRY

Monday 2 p.m. Two hours

1. Prove: The sum of the face angles of any convex polyhedral angle is less than four right angles.

2. Prove: If two perpendicular planes intersect in a line x, a line through any point of x, perpendicular to one of the planes, will lie in the other plane.

3. *a*) Can a straight line be perpendicular to each of two intersecting planes? Why?

   b) Can a straight line be perpendicular to each of two intersecting straight lines? Why?

   c) If a straight ruler be held parallel to a blackboard, will its shadow on the blackboard be parallel to the ruler? Why?

   d) Can a plane be passed through a right circular cone in such a way as to make the section formed a triangle? Explain.

4. Two similar right triangles ABC and $A'B'C'$, the right angles being at C and C', are rotated about AB and $A'B'$, respectively, as axes. Prove that the total areas of the solids thus formed are to each other as the squares of any two homologous sides of the triangles.

5. *a*) Between what two limits does the sum of the angles of a spherical triangle lie? Prove your statement.

   b) A right circular cylinder of radius r contains a certain quantity of water. If a volume of water equal to the volume of a sphere of radius r be added to the water in the cylinder, the height of the water rises to a height $2r$ above the bottom of the cylinder. How deep was the water originally in the cylinder?

6. On one wall of a rectangular room is a point A and on the opposite wall is a point B. State without proof, what points on the floor are equidistant from A and B.

MATHEMATICS *CD*—PLANE AND SOLID GEOMETRY

Monday, June 17 2 p.m. Three hours

This paper will be rated as a whole; separate credits will not be given on this paper for plane geometry and solid geometry.

Group A

(*Answer five questions from this group.*)

1. Prove: Two right triangles are congruent if a leg and the hypotenuse of one are equal respectively to a leg and the hypotenuse of the other.
2. If two of the altitudes of a triangle are equal, prove that the triangle is isosceles.
3. Prove: An angle between two chords which intersect within the circle is measured by half the sum of the arcs intercepted by the angle and by its vertical angle.
4. To a circle with diameter AB, a tangent BD and a secant DA are drawn. The secant cuts the circle at E. Prove that the triangles AEB and ADB are similar.
5. Construct a square having three times the area of a given square. Show all construction lines.
6. Through a point, A, within a circle whose center is B, chords are drawn. Prove that the locus of the middle points of these chords is the circle on AB as diameter.

Group B

(*Answer five questions from this group.*)

7. Prove: If two straight lines are cut by three parallel planes the corresponding segments are proportional.
8. Prove: The sum of two face angles of a trihedral is greater than the third face angle.
9. State a theorem for spherical triangles that may be derived from the theorem in (8). Explain briefly the method by which theorems about spherical polygons may be derived from theorems about polyhedral angles.
10. Prove: A triangular prism may be divided into three equivalent triangular pyramids.
11. A cone of revolution 3 inches high with a base whose area is 81π square inches is cut by a plane parallel to the base and 2 inches from it. Find the volume of the frustum.
12. A rectangle whose sides are a and b is rotated first about one side and then about the other. Prove that the total area of the solids thus generated is half the area of a circle whose radius is the perimeter of the rectangle.

MATHEMATICS *C*—PLANE GEOMETRY

Monday, June 17 2 p.m. Two hours

1. **Prove:** Two right triangles are congruent if a leg and the hypotenuse of one are equal respectively to a leg and the hypotenuse of the other.
2. If two of the altitudes of a triangle are equal, prove that the triangle is isosceles.
3. Prove: An angle between two chords which intersect within the circle is measured by half the sum of the arcs intercepted by the angle and by its vertical angle.
4. Prove: The area of a regular polygon is equal to one-half the product of its perimeter and its apothem.
5. To a circle with diameter AB, a tangent BD and a secant DA are drawn. The secant cuts the circle at E. Prove that the triangles AEB and ADB are similar.
6. Construct a square equivalent to the sum of two given squares. Show all construction lines.
7. Through a point A, within a circle whose center is B, chords are drawn. Prove that the locus of the middle points of these chords is the circle on AB as diameter.
8. *a*) If the sum of the squares of two sides of a triangle is equal to the square of the third side, prove that the triangle is a right triangle.

   b) Prove that a triangle whose sides are 9 ft., 10 ft., and 17 ft. can be placed adjacent to a right triangle whose sides are 6 ft., 8 ft., and 10 ft., so that the whole figure forms a right triangle.

MATHEMATICS *D*—SOLID GEOMETRY

Monday, June 17 2 p.m. Two hours

1. Prove: If two straight lines are cut by three parallel planes, the corresponding segments are proportional.

2. Prove: All straight lines perpendicular to a given straight line at a given point in that straight line lie in a plane perpendicular to the given line.

3. Prove: The sum of two face angles of a trihedral is greater than the third face angle.

4. State a theorem for spherical triangles that may be derived from the theorem in (3). Explain briefly the method by which theorems about spherical polygons may be derived from theorems about polyhedral angles.

5. Prove: A triangular prism may be divided into three equivalent triangular pyramids.

6. A cone of revolution 3 inches high with a base whose area is 81π square inches is cut by a plane parallel to the base and 2 inches from it. Find the volume of the frustum.

7. A and B are points 8 inches apart. What is the locus of a point which is 5 inches from both A and B? Give reasons for your answer.

8. A rectangle whose sides are a and b is rotated first about one side and then about the other. Prove that the total area of the solids thus generated is half the area of the circle whose radius is the perimeter of the rectangle.

1919

MATHEMATICS *CD*—PLANE AND SOLID GEOMETRY

Monday, June 16 2 p.m. Three hours

[For separate question papers in Mathematics *C* and Mathematics *D* see pages 3 and 4.

This paper will be rated as a whole; separate credits will not be given on this paper for plane geometry and solid geometry.

Group A

(*Answer five questions from this group.*)

1. Prove: In any circle the perpendicular to a radius at its outer extremity is tangent to the circle.

2. Prove: If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

3. A plot of ground, consisting of two parallel straight sides and two semicircular ends, can be so inscribed in a rectangle that the straight sides of the plot lie in the longer sides of the rectangle. If the distance around the plot is half a mile and the longer sides of the rectangle are 1,000 ft., find the distance between the parallel sides of the plot and also the length of those sides. (Use $\pi = \frac{22}{7}$, and 1 mile = 5,280 ft.)

4. Prove: If from a point outside a circle a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.

5. The sum of the areas of two similar triangles is 255 sq. in., and the ratio of their sides is 1:4. Find the area of each triangle.

6. In the triangle ABC the side BA is extended through A to D. The bisector of the angle B meets the bisector of the angle CAD at E. Prove that the angle BEA is one-half the angle C.

Group B

(*Answer five questions from this group.*)

7. Prove: If two straight lines are perpendicular to the same plane, they are parallel.

8. Prove: Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle. What is the locus of all points equidistant from two intersecting planes? Give reason.

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

9. Show how to draw through a fixed point a plane which shall be parallel to a given line and perpendicular to a given plane. Give proof of the method. Under what conditions will there be more than one solution? No solution?

10. Prove: If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.

11. The area of an equilateral spherical triangle is $\frac{1}{10}$ of the entire surface of the sphere. How many degrees are there in each of the angles of the triangle?

12. A right circular cone whose slant height is 5 times the radius of its base has the same total area as a sphere. Find the ratio of the volume of the sphere to that of the cone.

MATHEMATICS *C*—PLANE GEOMETRY

Monday, June 16 2 p.m. Two hours

[The question paper Mathematics *CD*, Plane and Solid Geometry, is printed on page 1.]

1. Prove: The perimeters of two regular polygons of the same number of sides are to each other as the radii or the apothems of the polygons.

2. Prove: In any circle the perpendicular to a radius at its outer extremity is tangent to the circle.

3. Construct a circle tangent to a given circle at a given point and also passing through a given point not on the circle.

4. Prove: If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

5. A plot of ground, consisting of two parallel straight sides and two semicircular ends, can be so inscribed in a rectangle that the straight sides of the plot lie in the longer sides of the rectangle. If the distance around the plot is half a mile and the longer sides of the rectangle are 1,000 ft., find the distance between the parallel sides of the plot and the length of those sides. (Use $\pi = \frac{22}{7}$, and 1 mile = 5,280 ft.)

6. Prove: If from a point outside a circle a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.

7. The sum of the areas of two similar triangles is 255 sq. in., and the ratio of their sides is 1:4. Find the area of each triangle.

8. In the triangle ABC the side BA is extended through A to D. The bisector of the angle B meets the bisector of the angle CAD at E. Prove that the angle BEA is one-half the angle C.

MATHEMATICS *D*—SOLID GEOMETRY

Monday, June 16 2 p.m. Two hours

1. Prove: If two straight lines are perpendicular to the same plane, they are parallel.

2. Prove: Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle. What is the locus of all points which are equidistant from two intersecting planes? Give reason.

3. Show how to draw through a fixed point a plane which shall be parallel to a given line and perpendicular to a given plane. Give proof of the method. Under what conditions will there be more than one solution? No solution?

4. Prove: The sections of a prism made by two parallel planes which cut all the lateral edges of the prism are congruent polygons.

5. Find the area and the volume of a regular square pyramid each of whose edges is 6 inches.

6. Prove: If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.

7. The area of an equilateral spherical triangle is $\frac{1}{10}$ of the entire surface of the sphere. How many degrees are there in each of the angles of the triangle?

8. A right circular cone whose slant height is 5 times the radius of its base has the same total area as a sphere. Find the ratio of the volume of the sphere to that of the cone.

1920

MATHEMATICS *CD*—PLANE AND SOLID GEOMETRY

Monday, June 21 2 p.m. Three hours

[For separate question papers in Mathematics *C* and Mathematics *D* see pages 2 and 3.]

This paper will be rated as a whole; separate credits will not be given on this paper for plane geometry and solid geometry.

1. Prove: The diagonals of a parallelogram bisect each other.

2. Prove: The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

3. A right triangle, ABC, has a fixed hypotenuse, AB, 2 inches long. The vertex of the right angle, C, is allowed to take all possible positions. Construct accurately, and describe fully, the locus of the mid-point, P, of the leg, AC. A proof is not required.

4. A loop of cord is passed over a wheel with a radius of 3 inches, and, when pulled taut, reaches a point 6 inches from the center of the wheel. Find the total length of the cord.

   Compute the answer correct to the nearest one one-hundredth of an inch.

5. Prove: If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

6. Prove: The angle formed by two arcs of great circles is measured by the arc of the great circle described from the vertex of the angle as a pole and included between its sides.

7. A cylindrical bar, 6 feet long and 2 inches in diameter, is melted and cast into spherical bullets of $\frac{1}{2}$-inch diameter. How many bullets can be made from the bar?

8. Explain how, by carpenters' squares, a screen can be set up so that it will be vertical, and perpendicular to a given horizontal line. What theorems of solid geometry are used?

   NOTE.—A carpenter's square is a tool consisting of two arms forming a right angle.

MATHEMATICS *C*—PLANE GEOMETRY

Monday, June 21 2 p.m. Two hours

[The question paper Mathematics *CD*, Plane and Solid Geometry, is printed on page 1.]

1. Prove: The diagonals of a parallelogram bisect each other.

2. Prove: The area of a trapezoid is equal to the product of its altitude and half the sum of its parallel sides.

3. Prove: The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

4. A right triangle, ABC, has a fixed hypotenuse, AB, 2 inches long. The vertex of the right angle, C, is allowed to take all possible positions. Construct accurately, and describe fully, the locus of the midpoint, P, of the leg AC. A proof is not required.

5. A loop of cord is passed over a wheel with a radius of 3 inches, and, when pulled taut, reaches a point 6 inches from the center of the wheel. Find the total length of the cord.

   Compute the answer correct to the nearest one one-hundredth of an inch.

6. On one side of a given angle with vertex O, two points P and P' are given, and on the other side two points Q and Q' are given, so that

$$OP \times OP' = OQ \times OQ'.$$

Prove that the triangles OPQ and $OP'Q'$ are similar.

MATHEMATICS *D*—SOLID GEOMETRY

Monday, June 21 2 p.m. Two hours

[The question paper Mathematics *CD*, Plane and Solid Geometry, is printed on page 1.]

1. Prove: If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

2. Prove: The angle formed by two arcs of great circles is measured by the arc of the great circle described from the vertex of the angle as a pole and included between its sides.

3. A cylindrical bar 6 feet long and 2 inches in diameter is melted and cast into spherical bullets of $\frac{1}{2}$-inch diameter. How many bullets can be made from the bar?

4. Prove: Through four points in space, not lying in one and the same plane, one spherical surface, and only one, can be passed.

5. Explain how, by use of carpenters' squares, a screen can be set up so that it will be vertical, and perpendicular to a given horizontal line. What theorems of solid geometry are used?

   Note.—A carpenter's square is a tool consisting of two arms forming a right angle.

6. Give the sides of the spherical triangle whose angles are 90°, 90°, and 30°.

   Give the angles of the polar triangle of this given triangle.

   What portion of the total surface of the sphere does the original triangle enclose?

MATHEMATICS *E*

TRIGONOMETRY (PLANE AND SPHERICAL)

MATHEMATICS E—TRIGONOMETRY (PLANE AND SPHERICAL)

Tuesday 2:00 p.m. Two hours

Group A

(Answer any four questions in this group.)

1. *a*) If $\cos x = -\frac{4}{5}$ and $\tan x$ is positive, find the values of $\sin x$ and $\cot x$. Use no tables.

   b) Describe a method involving the use of a protractor and a scale for finding an approximate value for $\tan 40°$.

2. *a*) State the Law of Sines, the Law of Cosines, and the Law of Tangents.

   b) Given the sides a, c, and the included angle B of an oblique triangle, write formulas for finding the angles A and C and the side b.

3. *a*) Prove the identity $\tan \frac{x}{2} + \cot \frac{x}{2} = 2 \operatorname{cosec} x$.

   b) Find all positive values of x less than four right angles for which $\tan 2x + \cot 2x = 2$. Express the results both in degrees and radians.

4. Prove directly from a figure that in a right spherical triangle $\tan A = \frac{\tan a}{\sin b}$.

5. Solve the isosceles spherical triangle for which

$$b = 54° \, 33', \qquad c = 54° \, 33', \qquad a = 73°.$$

Group B

(Answer both questions in this group.)

6. If the distance from New York City to Boston is 200 miles; from Boston to Albany is 168 miles; and from Albany to New York City is 143 miles, what is the size of the angle at Boston, and what is the area of the plane triangle of which the three cities are the vertices?

7. Solve the right spherical triangle for which

$$a = 61°, \qquad B = 123° \, 40', \qquad C = 90°.$$

MATHEMATICS *E*—TRIGONOMETRY (PLANE AND SPHERICAL)

Tuesday 2 p.m. Two hours

Group A

(*Answer three questions from this group.*)

1. *a*) An angle y starts with the value 0° and increases to 360°. Describe how $2-\sin y$ varies.

   b) Certain of the following statements are untrue. Which are these? Give your reasons.

$$\sin x=.5;\quad \cos z=-2.3;\quad \operatorname{cosec}\theta=\frac{1}{3};\quad \cot y=-27.3;$$

$$\arctan 1=\frac{\pi}{2},\ \text{or}\ \tan^{-1}1=\frac{\pi}{2}.$$

2. *a*) Find all the positive values of x which are not greater than 180° and which satisfy the equation $\cos 2x+\sin x=1$. Express the results in degrees and in radians.

   b) In a circle of radius 10 inches what is the circular (or radian) measure of an angle subtended at the center of the circle by an arc whose length is 25 inches?

3. *a*) By aid of a figure show how to reduce $\sin(270°-x)$ to a function of the angle x. Assume x to be positive and less than 90°.

   b) The angles of a triangle are A, B, and C, and A is obtuse. By aid of a figure show that $\sin A : \sin B=a : b$.

4. *a*) Describe, and illustrate by means of a figure, the construction of the least positive angle x if $x=\arctan\left(-\frac{5}{12}\right)$, or if $x=\tan^{-1}\left(-\frac{5}{12}\right)$. Find the sine and the cosine of x.

   b) Prove that $\dfrac{1+\sin 2x-\cos 2x}{(\cos x+\sin x)\cos x}=2\tan x$.

Group B

(*Answer all questions from this group.*)

5. *a*) How would you solve an isosceles spherical triangle for which $b=c=60°$, and $a=40°$? Do not perform the computations.

   b) How would you solve a spherical triangle for which one side is 90° and one angle of which is given?

6. Two points A and C are on opposite sides of a lake. A third point B is chosen so that A and C are visible and accessible from B. If $BC=5$ miles, $AB=4$ miles, and the angle ABC is 101° 32′, what is the distance from A to C?

7. In a right spherical triangle $B=33°\ 50'$, $a=108°$, and $C=90°$. Find b and c.

8. Using logarithms compute the value of

   a) $\sqrt[4]{\dfrac{269.3}{8046}}$ and of *b*) $5-\sqrt[5]{21}$.

MATHEMATICS *E*—TRIGONOMETRY (PLANE AND SPHERICAL)

Tuesday, June 18 — 2 p.m. Two hours

1. *a*) Evaluate (without the use of tables)

$$\frac{\sin 30°+\tan 225°+\sin 270°}{\sec(-60°)-\cot 90°+\cos \frac{2}{3}\pi}.$$

b) Show how the value of $1+2\cos\theta$ changes as θ changes from 90° to 180°.

2. If A is a second quadrant angle such that $\sin A=\frac{5}{13}$, construct A (marking A in the figure). Find (without using tables) the values of $\cos A$ and $\tan 2A$. Also if B is an acute angle such that $B=\arctan\frac{3}{4}$ (or $\tan^{-1}\frac{3}{4}$), find the value of $\sin(A-B)$.

3. Prove

$$\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta}=\cot\theta.$$

4. Find all values of θ between 0° and 360° which satisfy $2\csc\theta-3\sec^2\theta=0$.

5. In the triangle ABC, $a=23.68$, $b=34.57$, and $c=28.93$; find A.

6. In a spherical triangle ABC, $C=90°$, $a=53°\,40'$, and $b=73°\,12'$; find A, B, and c and check the solution.

7. In which of the following cases is it possible to have a spherical triangle ABC having parts of the type indicated?

1)	2)	3)
$C=90°$	$C=90°$	$C=90°$
c acute	a acute	A obtuse
a acute	A obtuse	B obtuse

In each possible case state whether the remaining parts are acute or obtuse.

MATHEMATICS *E*—TRIGONOMETRY (PLANE AND SPHERICAL)

Tuesday, June 17 2 p.m. Two hours

1. Calculate by means of logarithms

$$\sqrt[3]{\frac{2.047\times\cos 64^\circ}{3.652\times\sin^2 72^\circ}}.$$

2. An angle lies in the third quadrant and its sine is numerically equal to three times its cosine. Without using tables or extracting roots, write the six trigonometric functions of this angle. Then compute the sine to four decimal places, and use the tables to find the angle.

3. Prove that

$$\sec 2\theta=\frac{1+\tan^2\theta}{2-\sec^2\theta}.$$

4. Without using tables find all positive angles x less than 180° which satisfy the equation

$$\sin 3x=\tfrac{1}{2}\sqrt{3}.$$

5. In the triangle ABC, $a=203.6$, $b=154.2$, $B=35^\circ\ 15'$ (35.25°). Find all possible values for A.

6. What formulas would you use to find A in the spherical triangle ABC, if $C=90^\circ$ and a and c are known? If B and b are known? If b and c are known?

7. A vessel is at the equator in longitude 33° 54′ (33.9°) W. How far must she sail, in nautical miles, to reach a point in latitude 30° 45′ (30.75°) S., longitude 10° 30′ (10.5°) W.? (A nautical mile is the length of 1′ of arc on the earth's surface.)

MATHEMATICS E—TRIGONOMETRY (PLANE AND SPHERICAL)

Tuesday, June 22 2 p.m. Two hours

1. Two stations, A and B, are four miles apart, and A is directly south of B. An aeroplane is observed from A at an angle of elevation of 33° 47′ (or 33.78°) and due north. At the same instant the plane is observed from B at an angle of elevation of 48° 16′ (or 48.27°) and due south. How far is the plane from A?

2. Using logarithms, compute the value of the quantity

$$\sqrt[7]{.08147 \cos 33° 26'} \quad \text{or} \quad \sqrt[7]{.08147 \cos 33.43°}$$

3. *a*) Find the value of

$$\sin x + \cos x$$

when $x = 90°, 120°, 135°, 180°$.

b) By the aid of a figure show how to reduce $\cos (180° + A)$ to a trigonometric function of the angle A. Assume A to be an angle of the first quadrant.

4. Prove the following identity:

$$\tan x + \cot x = \frac{2}{\sin 2x}.$$

5. Prove that in a right spherical triangle whose right angle is C

$$\cos A = \cos a \sin B.$$

6. Solve the right spherical triangle in which $A = 43°$, $c = 102°$, $C = 90°$.

MATHEMATICS *F*

PLANE TRIGONOMETRY

MATHEMATICS *F*—PLANE TRIGONOMETRY

Tuesday 2:00 p.m. Two hours

Group A

(*Answer any four questions of this group.*)

1. *a*) If $\cos x = -\frac{4}{5}$ and $\tan x$ is positive, find the values of $\sin x$ and $\cot x$. (Use no tables.)

   b) Describe a method involving the use of a protractor and scale for finding an approximate value for $\tan 40°$.

2. *a*) Find the value of $2 \sin 0° + \tan 225° \sin 90° - \cos 180° \cos(-420°) + \sec\frac{\pi}{6} \tan\frac{\pi}{3} \sin\frac{3\pi}{2}$.

   b) Find a value of $\tan^{-1} 1 + \sin^{-1} \frac{1}{2}$, or of arc tan 1+arc sin $\frac{1}{2}$.

3. Draw a figure showing how the magnitude and sign of each of the functions, sine, cosine, and tangent, of an angle in the second quadrant can be represented by a line. Explain why the cosine is represented by the line which you designate.

4. *a*) State the Law of Sines, the Law of Cosines, and the Law of Tangents.

   b) Given the sides a and c and the included angle B of an oblique triangle, write formulas for finding the angles A and C and the side b.

5. *a*) Prove the identity $\tan\frac{x}{2} + \cot\frac{x}{2} = 2 \operatorname{cosec} x$.

   b) Find all positive values of x less than four right angles for which

$$\tan 2x + \cot 2x = 2.$$

Group B

(*Answer all the questions of this group.*)

6. A tower is situated on a level plain. At a certain point on the plain the angle of elevation of the top of the tower is 20° 18′. At a point 500 ft. farther away from the tower and in a straight line with the foot of the tower and the first point the angle of elevation is 16° 48′. How high is the tower?

7. If the distance from New York City to Boston is 200 miles; from Boston to Albany is 168 miles; and from Albany to New York is 143 miles, what is the size of the angle at Boston of the plane triangle of which the three cities are the vertices? What is the area of the triangle?

8. Using logarithms, compute the value of

$$\left[\frac{256.4\sqrt[3]{.07834}}{\sqrt{90.04}\,(8.634)^2}\right]^2.$$

MATHEMATICS *F*—PLANE TRIGONOMETRY

Tuesday 2 p.m. Two hours

Group A

(*Answer four questions from this group.*)

1. *a*) An angle y starts with the value $0°$ and increases to $360°$. Describe how the value of $2-\sin y$ varies.

   b) Certain of the following statements are untrue. Which are these? Give your reasons.

$$\sin x = .5;\quad \cos z = -2.3;\quad \operatorname{cosec}\theta = \frac{1}{3};\quad \cot y = -27.3;$$

$$\arctan 1 = \frac{\pi}{2},\quad \text{or } \tan^{-1} 1 = \frac{\pi}{2}.$$

2. *a*) Find all positive values of x which are not greater than $180°$ and which satisfy the equation $\cos 2x + \sin x = 1$. Express the results in degrees and in radians.

   b) In a circle of radius 10 inches what is the circular (i.e., radian) measure of an angle subtended at the center of the circle by an arc whose length is 25 inches?

3. *a*) By aid of a figure show how to reduce $\sin(270° - x)$ to a function of the angle x. Assume x to be positive and less than $90°$.

   b) The angles of a triangle are A, B, and C, and A is obtuse. By aid of a figure show that $\sin A : \sin B = a : b$.

4. Prove that $\dfrac{1+\sin 2x - \cos 2x}{(\cos x + \sin x)\cos x} = 2\tan x$.

5. *a*) Describe, and illustrate by means of a figure, the construction of the least positive angle x if $x = \arctan\left(-\frac{5}{12}\right)$, or if $x = \tan^{-1}\left(-\frac{5}{12}\right)$. Find the sine and the cosine of x.

   b) Without using any tables find the value of $\cos 225° - \tan(-1050°) + \sin 15°$.

Group B

(*Answer all the questions in this group.*)

6. From the foot of a pole 32.5 ft. high the angle of elevation of the top of a tower is $63° 43'$, and from the top of the pole the angle of depression of the foot of the tower is $49° 12'$. If the pole and the tower are on the same horizontal plane, find the height of the tower and the distance from the foot of the tower to the foot of the pole.

7. Two points A and C are on opposite sides of a lake. A third point B is chosen so that A and C are visible and accessible from B. If $BC = 5$ miles, $AB = 4$ miles, and the angle ABC is $101° 32'$, what is the distance from A to C?

8. Using logarithms compute the value of

   a) $\sqrt[4]{\dfrac{269.3}{8046}}$ and of *b*) $5 - \sqrt[5]{21}$.

MATHEMATICS *F*—PLANE TRIGONOMETRY

Tuesday, June 18 2 p.m. Two hours

1. *a*) Evaluate (without the use of tables)

$$\frac{\sin 30°+\tan 225°+\sin 270°}{\sec(-60°)-\cot 90°+\cos \frac{2}{3}\pi}.$$

b) Show how the value of $1+2\cos\theta$ changes as θ changes from 90° to 180°.

2. If A is a second quadrant angle such that $\sin A=\frac{5}{13}$, construct A (marking A in the figure). Find (without using tables) the values of $\cos A$ and $\tan 2A$. Also if B is an acute angle such that $B=\arctan\frac{3}{4}$ (or $\tan^{-1}\frac{3}{4}$), find the value of $\sin(A-B)$.

3. *a*) What is the value of $\sin(180°+\theta)$ in terms of θ? Prove your answer by means of a figure in which θ is acute.

b) Prove

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}.$$

4. Prove

$$\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta}=\cot\theta.$$

5. Find all values of θ between 0° and 360° which satisfy $2\csc\theta-3\sec^2\theta=0$.

6. In the triangle ABC, $a=23.68$, $b=34.57$, and $c=28.93$; find A.

7. At a point 286 ft. from the foot of a tower the angle of elevation of the top of the tower is 38° 18′ and the angle of elevation of the top of a flagstaff which stands on the tower is 48° 20′. Find the height of the tower and the length of the flagstaff.

MATHEMATICS *F*—PLANE TRIGONOMETRY

Tuesday, June 17 — 2 p.m. Two hours

1. Calculate by means of logarithms

$$\sqrt[3]{\frac{2.047\times\cos 64^\circ}{3.652\times\sin^2 72^\circ}}.$$

2. An angle lies in the third quadrant, and its sine is numerically equal to three times its cosine. Without using tables or extracting roots, write the six trigonometric functions of this angle. Then compute the sine to four decimal places and use the tables to find the angle.

3. Prove that

$$\sec 2\theta=\frac{1+\tan^2\theta}{2-\sec^2\theta}.$$

4. Without using tables, find all positive angles x less than 180° which satisfy the equation

$$\sin 3x=\tfrac{1}{2}\sqrt{3}\,.$$

5. Find in degree measure all values of x which satisfy the equation

$$3\sin^2 x+5\cos x=5.$$

6. In the triangle ABC, $a=203.6$, $b=154.2$, $B=35^\circ\ 15'$ (35.25°). Find all possible values for A.

7. A river flows between two vertical cliffs. One of the cliffs is known to be 250 feet high. An observer standing on the other finds that the top of the 250-foot cliff has an angle of elevation of 20° and the bottom an angle of depression of 10°. What is the width of the river?

MATHEMATICS *F*—PLANE TRIGONOMETRY

Tuesday, June 22 2 p.m. Two hours

1. Two stations, A and B, are four miles apart, and A is directly south of B. An aeroplane is observed from A at an angle of elevation of 33° 47′ (or 33.78°) and due north. At the same instant the plane is observed from B at an angle of elevation of 48° 16′ (or 48.27°) and due south. How far is the plane from A?

2. Using logarithms, compute the value of the quantity

$$\sqrt[7]{.08147 \cos 33° 26'} \quad \text{or} \quad \sqrt[7]{.08147 \cos 33.43°}$$

3. *a*) Find the value of

$$\sin x + \cos x,$$

when $x=90°$, $120°$, $135°$, $180°$.

b) By the aid of a figure show how to reduce $\cos(180°+A)$ to a trigonometric function of the angle A. Assume A to be an angle of the first quadrant.

4. Find all positive values of x between 0° and 360° which satisfy the following equation:

$$3 \cos x - 4 \sin x = 3.$$

Use tables when necessary.

5. Prove the following identity:

$$\tan x + \cot x = \frac{2}{\sin 2x}.$$

6. The top of a mountain near the seashore is 3 miles above sea-level. From its top the angle of depression of the sea-horizon is 2° 15′ (or 2.25°). Determine the radius of the earth.

COMPREHENSIVE MATHEMATICS

ELEMENTARY AND ADVANCED

Comprehensive Examination

ELEMENTARY MATHEMATICS

Monday, June 19 9:00 a.m.–12:00 m.

For candidates who have not studied advanced mathematics.

1. *a*) Factor each of the following:

$8x^2+2x-15; \quad x^2+2xy+y^2-p(x+y).$

b) Simplify $\left(\dfrac{1}{4x^0+4^{\frac{3}{2}}}\right)^{-2}$.

2. Solve $\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}}=\dfrac{x+2}{2}$.

3. Solve the simultaneous equations

$$\frac{5}{3}(x-2y)=2x-y-1,$$
$$x^2+2y^2-3x+y+1=0.$$

Group the answers in corresponding pairs and check one pair.

4. The larger of two numbers when divided by the smaller gives a quotient 3 and a remainder 8. The square of the smaller exceeds twice the larger by 11. Find the numbers.

5. A steamer makes a trip of 3,240 miles. The time of the trip would have been a day and a half less if the speed had been 3 miles per hour more. Find the number of days required for the trip.

6. In a given right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse. Prove that this perpendicular divides the given right triangle into two similar triangles, and that the perpendicular is the mean proportional between the segments of the hypotenuse.

7. Two chords of a circle are drawn forming an angle whose vertex is inside the circle. Prove that the angle formed by the chords is measured by half the sum of the arc intercepted by the sides of the angle and the arc intercepted by the sides of its vertical angle.

8. Let AB and AC be chords of a circle, and D and E the middle points of the arcs AB and AC, respectively, D and C being on opposite sides of AB, and E and B on opposite sides of AC. If the chord DE intersects AB and AC in F and G, respectively, prove that $AF=AG$. (Use the theorem of Question 7.)

9. Let AB be the chord of an arc of 60° in a circle of radius a. Describe a circle on AB as a diameter and find the area common to the two circles.

Comprehensive Examination

ELEMENTARY MATHEMATICS

Wednesday, September 20 9:00 a.m.–12:00 m.

For candidates who have not studied advanced mathematics.

1. Solve the simultaneous equations

$$x-4y+3z=5, \qquad \tfrac{1}{2}x+3y+\tfrac{3}{4}z=-\tfrac{1}{2}, \qquad 2y-3z=-3.$$

Check your answers.

2. Simplify $\left(a^{\frac{9}{2}}\sqrt{a^{-3}}\right)^{\frac{1}{3}} \div \sqrt{a^{-\frac{7}{3}}\,a^{\frac{1}{3}}}$.

3. Find the least common multiple of the denominators in the following equation, and solve the equation

$$\frac{x+7}{2x^2-7x+3}+\frac{x}{x^2-2x-3}-\frac{x+3}{1-x-2x^2}=0.$$

4. The sum of the first 4 terms of an arithmetical progression is 24 and the sum of the first 9 terms is 99. Find the first term and the 20th term.

5. Separate 24 into two parts such that one of them increased by 50 per cent of itself is to the square of the other as 3:4.

6. Prove that the area of a triangle equals half the product of its base and altitude.
If the perimeter of a rhombus is 20 in. and one of the diagonals is 6 in., what is the area?

7. Prove that a circle may be circumscribed about any regular polygon.

8. Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals inscribed respectively in two concentric circles in such a way that the points A', B', C', and D' are on the radii OA, OB, OC, and OD, respectively. Prove that the quadrilaterals are similar.

9. Given a right triangle ABC with C the vertex of the right angle and BC the shortest side. Let CD be the perpendicular from C on AB, and E a point on AC such that $\angle CBE = \angle BCD$. Show that CD bisects BE.

Comprehensive Examination

ADVANCED MATHEMATICS

Monday, June 19 9:00 a.m.–12:00 m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

PART I. ELEMENTARY

1. *a*) Factor each of the following:
   $8x^2+2x-15$; $x^2+2xy+y^2-p(x+y)$.

   b) Simplify $\left(\dfrac{1}{4x^0+4^{\frac{3}{2}}}\right)^{-2}$.

2. Solve the simultaneous equations

$$\frac{5}{3}(x-2y)=2x-y-1,$$
$$x^2+2y^2-3x+y+1=0.$$

   Group the answers in corresponding pairs and check one pair.

3. Two chords of a circle are drawn forming an angle whose vertex is inside the circle. Prove that the angle formed by the chords is measured by half the sum of the arc intercepted by the sides of the angle and the arc intercepted by the sides of its vertical angle.

   Let AB and AC be chords of a circle, and D and E the middle points of the arcs AB and AC, respectively, D and C being on opposite sides of AB, and E and B on opposite sides of AC. If the chord DE intersects AB and AC in F and G, respectively, prove that $AF=AG$. (Use the theorem given in the first part of this question.)

4. On the sides of a square $ABCD$, the points A', B', C', and D' are chosen so that A' is two-fifths of the way from A to B, B' is two-fifths of the way from B to C, C' is two-fifths of the way from C to D, and D' two-fifths of the way from D to A. Prove that $A'B'C'D'$ is a square, and find its area, if the length of AB is a.

PART II. ADVANCED

SOLID GEOMETRY

5. If two straight lines are cut by three parallel planes, the corresponding segments intercepted on the lines are proportional.

6. A spherical angle is measured by an arc of a great circle having its vertex as a pole and included between its sides, produced if necessary.

(SEE NEXT PAGE)

7. If two parallel planes are cut by a third plane, the alternate-interior dihedral angles are equal.

8. Two spheres of lead, of radii a and $2a$ respectively, are melted and recast into a cylinder of revolution whose altitude is $3a$. Prove that the total surface is unchanged in amount.

LOGARITHMS AND TRIGONOMETRY

9. Given that x is an angle of the third quadrant and that $\sin x = -\frac{3}{5}$. Construct the angle. Find $\tan x$, $\sin 2x$, $\tan \frac{1}{2}x$, and $\cos(\pi - x)$.

10. Prove the formulas necessary for solving an oblique triangle when a side and two angles are given. Find BC in $\triangle ABC$, if $AB = 10.82$, angle $A = 58°.25$ (58° 15′), and angle $B = 65°$, using logarithms.

11. Prove the identity $\cos(60° + x) + \cos(60° - x) = \cos x$.

12. Find the values of

$$(a)\ \log_3 9 + \log_2 \frac{1}{16} + \log_{10} \sqrt{10};\quad (b)\ \sqrt{0 \cdot 85}\ \sin 91°.$$

ADVANCED ALGEBRA

13. How many numbers between 100 and 1,000 can be formed from the digits 1, 2, 3, ... 7, no digit being repeated in any one number?
How many between 100 and 10,000?

14. Locate the roots of $x^3 - 2x^2 - 7x - 1 = 0$, and find the positive root to two decimal places.

15. Solve $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$, making use of the fact that $1 + i$ is a root.

16. In the simultaneous equations

$$\begin{aligned} 2x - 3y - z + 3w &= 1, \\ 3x + y + 4z - 2w &= 2, \\ x + 2y - 3z + w &= 3, \\ 4x - 3y + 2z + w &= 4, \end{aligned}$$

write the value of y as the quotient of two determinants. Evaluate the determinant which forms the denominator.

Comprehensive Examination

ADVANCED MATHEMATICS

Wednesday, September 20 — 9:00 a.m.–12:00 m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. Find the least common multiple of the denominators in the following equation, and solve the equation

$$\frac{x+7}{2x^2-7x+3}+\frac{x}{x^2-2x-3}-\frac{x+3}{1-x-2x^2}=0.$$

2. Separate 24 into two parts such that one of them increased by 50 per cent of itself is to the square of the other as 3:4.

3. Prove that the area of a triangle equals half the product of its base and altitude.
   If the perimeter of a rhombus is 20 in. and one of the diagonals is 6 in., what is the area?

4. Given a right triangle ABC with C the vertex of the right angle and BC the shortest side. Let CD be the perpendicular from C on AB, and E a point on AC such that $\angle CBE = \angle BCD$. Show that CD bisects BE.

Part II. Advanced

SOLID GEOMETRY

5. The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.

6. In two polar spherical triangles an angle of one is measured by the supplement of that side of the other of which it is the pole.

7. Given a quadrilateral whose sides do not all lie in the same plane. Prove that the lines joining the middle points of the sides of the quadrilateral form a parallelogram.

8. Through a straight line parallel to a given plane two planes are passed so as to intersect the given plane. Prove that the two lines of intersection thus formed are parallel.

(SEE NEXT PAGE)

9. Given cosec $x=\frac{13}{5}$, and x an angle of the second quadrant, find sin $(90°-x)$, cos $2x$, tan $\frac{1}{2}x$.

10. Write the formula for $\sin(x+y)$, and prove it for the case that x, y, and $x+y$ are each less than a right angle. From this formula find sin $2x$.

11. Prove the identity $(\tan x+\sec x)^2=\frac{1+\sin x}{1-\sin x}$.

12. (*a*) State and prove the rules for simplifying the logarithms of a product and of a power.
    (*b*) Find one angle of the triangle whose sides are 10, 12, and 15.

ADVANCED ALGEBRA

13. Solve $x^4-x^3-6x^2+14x-12=0$.

14. Apply Descartes' Rule of Signs to the equation $3x^4+4x^3-36x^2-5=0$. Draw the graph of $10y=3x^4+4x^3-36x^2-5$, and show how the results obtained in the first part of this question are in agreement with the graph.

15. Evaluate
$$\begin{vmatrix} 2 & 3 & 5 & -4 \\ 1 & -2 & -3 & 2 \\ 3 & 1 & 1 & -3 \\ -5 & 2 & 4 & -1 \end{vmatrix}.$$

16. Given 20 points in a plane, no three of which lie in the same straight line. How many triangles are there of which the given points are the vertices? If, on the other hand, half of the given points are on a straight line, how many triangles can be formed?

Comprehensive Examination

ELEMENTARY MATHEMATICS

Monday, June 18 9:30 a.m.–12:30 p.m.

For candidates who have not studied advanced mathematics.

1. *a*) Factor each of the following:

$12x^2+x-6$; x^2-x-y^2+y.

b) Find the numerical value of $ax+a^2$,

$$\text{if } x=a+\frac{1}{a}, \text{ and } a=\frac{1}{\sqrt{2}-1}.$$

2. Solve

$$\frac{1}{1+\frac{1}{x}}-\frac{3}{1-\frac{1}{x}}=2,$$

calculating the values of x correct to two places of decimals.

3. The sum of a number of terms in an arithmetical progression is 126. The last term is 10, and the common difference is $\frac{2}{5}$. Find the number of terms and the first term.

4. The diagonal of a rectangle is equal to the width increased by two-thirds of the length. The area is 60 sq. ft. Find the dimensions of the rectangle.

5. The sum of the squares of two numbers when divided by the larger number gives a quotient equal to $\frac{25}{12}$ of the smaller number. The difference of the numbers is 2. Find the numbers.

6. If two sides of a triangle are equal, prove that the opposite angles are equal. State and prove the converse theorem.

7. If two triangles are mutually equiangular, prove that the triangles are similar.

8. In an acute triangle ABC let the altitudes BD and CE intersect in the point O. Prove that $BO:OC=OE:OD$.

9. A regular hexagon $ABCDEF$ is inscribed in a given circle, and the straight lines BD and EA are drawn. Prove that the quadrilateral $BDEA$ is a rectangle.

Assuming the truth of the proposition just stated, and denoting a side of the hexagon by a, find the area of the rectangle in terms of a.

Comprehensive Examination

ELEMENTARY MATHEMATICS

Wednesday, September 19 9 a.m.–12 m.

For New Plan candidates who have not studied advanced mathematics.
New Plan candidates will omit all questions marked with an asterisk (*).
Old Plan candidates will answer all questions set in any subject taken by them.

ELEMENTARY ALGEBRA

To Quadratics

1. *a*) Find the Highest Common Factor and Least Common Multiple of $2-8x^2$, $8x^3-1$, and $6x^4+5x^3-4x^2$.

   b) Simplify $5a^0\sqrt{12}+8^{-\frac{2}{3}}\ (2^2)^3+\dfrac{4-3\sqrt{3}}{2+\sqrt{3}}$.

2. Solve
$$x-y+5z=0,$$
$$5x+3z+2=0,$$
$$3x+2y-z+1=0.$$
Test the answers.

3.* Solve
$$\frac{b+x}{b+a}+\frac{b-x}{b-a}=\frac{b^2-ax}{a^2-b^2}.$$
Test the answer.

4.* A sum of money is placed at compound interest (interest compounded annually). The first year $2\frac{1}{2}$ per cent is paid, the second year $3\frac{1}{2}$ per cent. At the end of two years the money amounts to $254.61. Find the sum of money placed at interest.

Quadratics and Beyond

5. *a*) Solve $2x^2-3x+\frac{2}{3}=0$, and calculate each root to two decimal places.

   b) Find the fourth term in the expansion of $\left(\frac{1}{2}x^2-\frac{1}{\sqrt[3]{x^2}}\right)^{12}$, and simplify it.

6. If the numerator of a certain fraction is diminished by 9 and the denominator diminished by 2, the result is 1 less than the reciprocal of the fraction. If the numerator is diminished by 4 and the denominator increased by 1, the result is $\frac{1}{2}$. Find the fraction.

7.* Solve
$$\frac{\sqrt{x}}{2-\sqrt{x}}+\frac{2-\sqrt{x}}{\sqrt{x}}+4=0.$$

(SEE NEXT PAGE)

8. A reservoir has a supply pipe A and an exhaust pipe B. The pipe A takes four minutes less time to fill the reservoir than the pipe B takes to empty it. When both are open the reservoir is filled in 24 minutes. Find the time needed for A to fill the reservoir when B is closed.

PLANE GEOMETRY

9. If the opposite sides of a quadrilateral are equal, prove that the figure is a parallelogram.

10.* *a*) Show how to draw a tangent to a given circle from an exterior point, and prove your construction.

* *b*) If a secant and a tangent are drawn to a circle from an exterior point, prove that the tangent is the mean proportional between the whole secant and the exterior segment.

11. Explain what is meant by saying that two polygons are similar. If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, prove that the polygons are similar.

12. On a certain circle the arc AB is 60°. At A a tangent is drawn, and the radius through B is prolonged both ways, namely, to meet the circle at C and to meet the tangent at T. Find the angles of the triangle ACT. If the radius of the circle is 5, find the length of AT.

13. *a*) What is the locus of a point which moves so that it is always one foot distant either from one or from the other of two fixed points one foot apart? Give a reason for your answer.

b) If two right triangles are inscribed in a circle, with the same diameter as hypotenuse, and if one of the triangles is isosceles, while the other is not, prove that the isosceles triangle has the greater area.

Comprehensive Examination

ELEMENTARY AND ADVANCED MATHEMATICS

Monday, June 18 9:30 a.m.–12:30 p.m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. Solve
$$\frac{1}{1+\frac{1}{x}}-\frac{3}{1-\frac{1}{x}}=2,$$
calculating the values of x correct to two places of decimals.

2. The diagonal of a rectangle is equal to the width increased by two-thirds of the length. The area is 60 sq. ft. Find the dimensions of the rectangle.

3. If two triangles are mutually equiangular, prove that the triangles are similar.

4. A regular hexagon $ABCDEF$ is inscribed in a given circle, and the straight lines BD and EA are drawn. Prove that the quadrilateral $BDEA$ is a rectangle.

   Assuming the truth of the proposition just stated, and denoting a side of the hexagon by a, find the area of the rectangle in terms of a.

Part II. Advanced

SOLID GEOMETRY

5. If two intersecting planes are each perpendicular to a third plane, prove that the line in which the first two planes intersect is perpendicular to the third plane.

6. The plane passed through two diagonally opposite edges of any parallelepiped divides the parallelepiped into two equivalent triangular prisms.

7. Prove that all tangents drawn to a given sphere from a point outside the sphere are of the same length.

8. An isosceles triangle, with sides 5, 5, and 8 inches respectively, is revolved about its shortest altitude as an axis. What is the total area of the solid so generated?

   What is the total area of the solid that would have been generated if the triangle had been revolved about its longest side?

(SEE NEXT PAGE)

9. Construct an angle x in the second quadrant for which $\cos x = -\frac{3}{5}$. Calculate $\cot x$, $\sin 2x$, $\tan(270° + x)$, $\cos \frac{1}{2}x$, $\csc(-x)$.

10. If a, b, and c are the sides of an oblique triangle, and A, B, and C, respectively, the corresponding opposite angles, show that $a^2 = b^2 + c^2 - 2bc \cos A$. Using this formula, find one angle of the triangle whose sides are 8, 6, and 12.

11. *a*) Prove the identity $1 + \tan x \tan 2x = \sec 2x$.
    b) Determine an angle x in the second quadrant for which
$$\tan(45° - x) + \tan(45° + x) = 4.$$

12. *a*) State and prove the rules for simplifying the logarithms of a quotient and of a root.
    b) Find by logarithms one of the unknown sides and the area of the $\triangle ABC$, if $AB = 36.23$, angle $A = 28°.15$ $(28° \, 9')$, angle $C = 62°.35$ $(62° \, 21')$.

ADVANCED ALGEBRA

13. Solve $2x^5 + 11x^4 + 23x^3 + 25x^2 + 16x + 4 = 0$.

14. Apply Descartes' Rule of Signs to $x^3 - 4x^2 - 4x + 12 = 0$. Locate the roots between successive integers, and find the negative root to two decimal places.

15. Evaluate
$$\begin{vmatrix} 3 & -2 & -4 & 2 \\ 1 & 3 & -2 & 2 \\ -4 & 1 & 3 & -1 \\ 2 & 2 & 1 & -3 \end{vmatrix}$$

16. Of all the permutations of the letters of "favorites" how many begin with *f* and end with *s*? How many begin with two vowels?

Comprehensive Examination

ADVANCED MATHEMATICS

Thursday, September 20 2 p.m.–5 p.m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Old Plan candidates will answer all questions in any subject which they are offering.

New Plan candidates will omit all questions marked with an asterisk (*). Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. If $x=\frac{2-\sqrt{6}}{2}$, find the numerical value of $\frac{2x^2-4x+1}{2x-1}$.

2. Solve the simultaneous equations
$2x+3y-3=0,\quad 2x^2-xy-3y^2+x+2y-3=0.$
Arrange the answers in corresponding pairs, and check.

3. If two sides of a triangle are unequal, prove that the angles opposite are unequal, and that the greater angle is opposite the greater side.
State and prove the converse theorem.

4. If ABC is a right triangle with the right angle at B, and if D is a point on AC such that AB is the mean proportional between AC and AD, prove that ADB is a right angle.

Part II. Advanced

SOLID GEOMETRY

5. In any pyramid a plane section parallel to the base is similar to the base.

6. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

7. Q is the center of the circumscribed circle of the triangle ABC, and QX is perpendicular to the plane ABC. Prove that the lines XA, XB, and XC make equal angles with the plane ABC.

8. The right triangle ABC has $BC=20$ ft., $AB=15$ ft., $AC=25$ ft. There is an unlimited number of points that are equally distant from B and C and are at the same time 26 feet distant from A. Tell what you can of the shape and position of the locus of such points.

9.* The altitude of a frustum of a regular square pyramid is 4 in., and the sides of the bases are respectively 12 in. and 6 in. Calculate the total surface and the volume. What is the volume of the pyramid itself?

(SEE NEXT PAGE)

10. Given that $\tan x = \frac{4}{3}$, and x an angle in the third quadrant. Find $\cos x$, $\sin(180°+x)$, $\cot 2x$, $\csc 2x$, $\sec(-x)$.

11. *a*) Prove $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$.

b) Making use of the preceding equation, and assuming the Law of Sines, derive the formula,

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)},$$

where a and b are the sides of the triangle ABC opposite the angles A and B respectively.

12. *a*) Prove the identity $\cos(30°+y) - \cos(30°-y) = \cos(90°+y)$.

b) Determine an angle A in the first quadrant for which

$$\sin^2 A - 2 \cos A + \frac{1}{4} = 0.$$

13. *a*) What is the value of $\log_2 \frac{1}{2} - \log_{10}\sqrt{0.1}$?

b) Find by logarithms one of the unknown angles of $\triangle ABC$ when angle $A = 78°.55$ $(78°\ 33')$, $AB = 26.18$, and $AC = 54.83$.

14.**a*) Calculate by logarithms $\left[\frac{2.186\sqrt{0.965}}{38.23}\right]^3$.

**b*) Write the formula for $\cos(x+y)$, and prove it for the case when angles x, y, and $x+y$ are each less than a right angle. Using this formula find $\cos 2x$, $\sin \frac{1}{2}x$, and $\tan \frac{1}{2}x$ in terms of $\cos x$.

ADVANCED ALGEBRA

15. How many numbers between 100 and 1,000 may be formed from 0, 1, 2, 3, 4, 5, 6, no digit being repeated in any one number? How many of these are even?

16. In an equation of even degree, in which the sign of the constant term is different from that of the highest power, prove that there must be an odd number of positive roots and an odd number of negative roots.

17. The equation $x^3+2x^2-13x+11=0$ has two positive roots. Prove this, and find the larger one to two decimal places.
Graph the equation $10y = x^3+2x^2-13x+11$.

18. In the equations

$$\begin{aligned} 2x - y + 3z &= 1, \\ x + 3y - 4z &= 0, \\ 3x + 2y + z &= 2, \end{aligned}$$

write the ratio of x to y as the quotient of two determinants. Evaluate the determinants.

19.* Transform the equation

$$2x^4 - x^3 - 29x^2 + 34x + 24 = 0,$$

so that the coefficient of x^4 is unity, and the other coefficients integral. Solve this new equation, and from the solutions write down the roots of the original equation.

Comprehensive Examination

ELEMENTARY MATHEMATICS

Monday, June 17 9:30 a.m.–12:30 p.m.

For candidates who have not studied advanced mathematics.

1. Reduce $\frac{10x^2+13xy-3y^2}{4(2x+y)^2-(x-3y)^2}$ to its lowest terms. Verify the result by putting $x=2$ and $y=1$.

2. *a*) Solve $2\sqrt{x+1}-\sqrt{2x+3}-\sqrt{x-2}=0$. Check the integral answer.

   b) Simplify the following equation and solve it for x:

$$\frac{7}{x^{-\frac{3}{2}}}+4^0-\frac{1}{27^{-\frac{2}{3}}}=0.$$

3. In an arithmetical progression, in which the common difference is -2, the sum of a certain number of terms is 240, the first of these terms being 33. Find the last term and the number of terms.

4. A man can row twice as fast downstream as he can row upstream. He can row down 12 miles and back in $4\frac{1}{2}$ hours. Find his rate in still water, and also the rate of the stream.

5. *a*) If $\frac{x}{q+r}=\frac{y}{r+p}=\frac{z}{p-q}$, prove that $x-y+z=0$.

   b) In the equation $x^2-(d+2)x+d^2=0$, find the values of d which will make the sum of the roots equal to three times the product of the roots.

6. Prove that in any parallelogram the diagonals bisect each other.
   Through the point of intersection of the diagonals of a parallelogram draw a line terminating in opposite sides. Prove that this line is bisected by the diagonals.

7. Prove that a circle can be circumscribed about any triangle.
   Is the above theorem true when the word "triangle" is replaced by "quadrilateral"? State a theorem, or draw a figure, to justify your answer.

8. *AB* is the base of an isosceles triangle *ABC*. The side *AC* is extended through *C* to a point *D*, and the line *DB* is drawn. A line drawn through *C* parallel to *AB* meets *DB* at *E*.

   a) Prove that *CE* bisects the angle *DCB*.

   b) State a general theorem about the bisector of an angle of a triangle and the segments into which it divides the opposite side.

   c) Using part (*a*) of this question, prove this theorem for the bisector *CE* in the triangle *DCB* of the figure which you have drawn.

9. Each side of a regular hexagon is trisected, and a circle is passed through all the points of division. If a side of the hexagon is 6 inches, find the area of the circle.

Comprehensive Examination

ELEMENTARY MATHEMATICS

Wednesday, September 18 **9 a.m.–12 m.**

For New Plan candidates who have not studied advanced mathematics.
New Plan candidates will omit all questions marked with an asterisk (*).
Old Plan candidates will answer all questions set in any subject taken by them.

ELEMENTARY ALGEBRA

To Quadratics

1. Solve

$$\begin{aligned} 2x-3y+2&=0,\\ y-4z\quad &=1,\\ x+3z+1&=0. \end{aligned}$$

Check your answers.

2.* *a*) Factor a^2-b^2+2b-1, and $(x-2y)x^3-(2y-x)y^3$.

* *b*) Simplify $$\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}-\frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}-\frac{4x^0}{x-1}.$$

Verify the result by using $x=4$.

3. Simplify each of the two following expressions and multiply the results:

$$\tfrac{5}{6}\sqrt{216}-3\sqrt{\tfrac{1}{6}}+\sqrt[6]{125}\,;\quad \sqrt{\tfrac{4}{5}}+3(5^{-\frac{1}{2}})-\frac{2}{\sqrt{6}}.$$

Calculate the value of the answer correct to the nearest hundredth.

4.* A and B working together can do a piece of work in 8 days. If A works 3 days and B 6 days, half the work will be done. How long would it take each to do the work alone? Prove your answers.

Quadratics and Beyond

5. Solve $$\frac{15-4x-x^2}{1-x^2}=\frac{2x-1}{x+1}-\frac{3x-4}{x-1}.$$

Verify one of the answers.

6.* Find three numbers in geometric progression whose sum is 52 and such that the ratio of the third to the sum of the first and second is $\frac{9}{4}$.

7. A farmer bought a certain number of lambs for $90. Had the price per lamb been one dollar more the number of lambs received for the same money would have been three less. Find the original price of one lamb.

(SEE NEXT PAGE)

8. *a*) Plot the graphs of $x-2y+1=0$ and $x^2+x-4y-4=0$, using the same axes for both. From the figure estimate the values of x and y which satisfy the two equations simultaneously.

b) Solve the equations and compare your answers with the estimated values found in (*a*).

PLANE GEOMETRY

9. Prove that two right triangles are equal if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.

10.* *a*) If one side of an angle inscribed in a circle is a diameter, prove that the angle is measured by half its intercepted arc.

* *b*) The points A, B, C, D, and E are five consecutive vertices of a regular inscribed octagon. If the diagonal AE is drawn, what is angle BAE in degrees? Give reasons for your answer.

11. *a*) Prove that two triangles are similar if the three sides of one are proportional respectively to the three sides of the other.

b) The sides of one triangle are 10, 12, and 15 inches long; in a triangle similar to it, the shortest side is 16 inches long. Find the lengths of the other two sides.

12. *a*) If a diagonal of a quadrilateral bisects the angles through which it is drawn, prove that the other two angles of the quadrilateral are equal to each other.

b) If both diagonals of a quadrilateral bisect the angles through which they are drawn, prove that all four sides of the quadrilateral are equal.

13. In a regular hexagon, one side of which is 10 in., two diagonals of different lengths are drawn from the same vertex. What is the length of the shorter diagonal? What fraction of the area of the hexagon do the two diagonals include? State your reasons.

Comprehensive Examination

ELEMENTARY AND ADVANCED MATHEMATICS

Monday, June 17 **9:30 a.m.–12:30 p.m.**

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. *a*) Solve $2\sqrt{x+1}-\sqrt{2x+3}-\sqrt{x-2}=0$. Check the integral answer.

b) Simplify the following equation and solve it for x:

$$\frac{1}{x^{-\frac{3}{2}}}+4^0-\frac{1}{27^{-\frac{2}{3}}}=0.$$

2. A man can row twice as fast downstream as he can row upstream. He can row down 12 miles and back in $4\frac{1}{2}$ hours. Find his rate in still water, and also the rate of the stream.

3. Prove that a circle can be circumscribed about any triangle.
Is the above theorem true when the word "triangle" is replaced by "quadrilateral"? State a theorem, or draw a figure, to justify your answer.

4. AB is the base of an isosceles triangle ABC. The side AC is extended through C to a point D, and the line DB is drawn. A line drawn through C parallel to AB meets DB at E.

a) Prove that CE bisects the angle DCB.

b) State a general theorem about the bisector of an angle of a triangle and the segments into which it divides the opposite side.

c) Using part (*a*) of this question, prove this theorem for the bisector CE in the triangle DCB of the figure which you have drawn.

Part II. Advanced

SOLID GEOMETRY

5. If two straight lines are perpendicular to the same plane they are parallel; state the converse of this theorem and give for either theorem a proof which does not depend upon the other.

6. Prove that a sphere can be inscribed in any given tetrahedron.

7. On a sphere 280 sq. ft. in area an equilateral spherical triangle is 7 sq. ft. in area. How many degrees are there in each angle of that triangle? Explain your solution.

(SEE NEXT PAGE)

8. A triangle ABC has $AC=4\sqrt{3}$ ft., $BC=4\sqrt{2}$ ft., and angle $C=90°$. At the middle point X of AB a perpendicular is erected to the plane ABC, and a point Y is marked on it 4 ft. from X. What is the distance AY? Tell what you can of the shape and position of the locus of points in the plane ABC that are precisely 6 ft. distant from Y.

LOGARITHMS AND TRIGONOMETRY

9. *a*) Prove the relations $\sin^2 x+\cos^2 x=1$, $\sec^2 x=1+\tan^2 x$, for an acute angle x.
   b) Find an expression for each of the following, $\sin x$, $\cos x$, $\sin 2x$, and $\tan (270°-x)$, in terms of $\tan x$.

10. *a*) For what positive values of x less than 360° does the equation

$$\tan (45°+x)=1+\sin 2x$$

hold? Prove your answer.
   b) Show without tables that $\sin 32°+\sin 28°=\cos 2°$.

11. *a*) If a and b are two sides of a triangle, and C the included angle, show that the area is $\frac{1}{2}\, ab \sin C$.
   b) Calculate the area if $a=25.64$, $b=38.82$, $C=31°\ 30'$ (31.5°).

12. *a*) Find the value of $\log_2 8+\log_{10} \sqrt[3]{0.1}$.
   b) Find one of the unknown angles in the triangle of 11 (*b*).

ADVANCED ALGEBRA

13. Locate the roots of $x^3-2x^2-5x+7=0$ between consecutive integers, and find the larger positive root to the nearest hundredth.

14. Write a determinant which has zero for one element and which is equal to

$$\begin{vmatrix} 2 & -3 & 1 & 5 \\ 3 & 2 & -4 & -1 \\ -1 & 4 & -1 & 2 \\ 2 & -1 & 3 & 2 \end{vmatrix}.$$

Evaluate the determinant.

15. *a*) If the roots of $2x^3+4x^2-x-11=0$ are a, b, c, find the values of $a+b+c$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.
   b) One root of $x^3+x^2+kx-3=0$ is -3. Find the value of k and the other roots.

16. How many different permutations, each consisting of two vowels and three consonants, could be formed from the vowels *a*, *e*, *i* and eight different consonants? Of these permutations how many would begin with *a*?

Comprehensive Examination

ADVANCED MATHEMATICS

Thursday, September 19 2 p.m.–5 p.m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Old Plan candidates will answer all questions in any subject which they are offering.

New Plan candidates will omit all questions marked with an asterisk (*). Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

PART I. ELEMENTARY

1. Solve $$\sqrt{\frac{2x+1}{4x+3}}+\sqrt{\frac{4-2x}{4x+3}}=1.$$
   Check the answers obtained.

2. Solve the simultaneous equations
$$3x+4y-3=0,\ x^2+4xy+3x-2y-3=0.$$
   Write the answers so that the proper value of y is associated with each value of x, and check.

3. Through a point A on a circle a chord AB and the tangent at A are drawn. From a point D on this tangent a line is drawn parallel to AB, meeting the circle at P and again at Q. Prove that the triangles ADP and ABQ are similar.

4. Show how to construct a triangle equivalent to a given pentagon, giving reasons for your construction.

PART II. ADVANCED

SOLID GEOMETRY

5. Prove that a plane through a point on the surface of a sphere perpendicular to the radius at that point is tangent to the sphere.

6. If two intersecting planes are each perpendicular to a third plane, prove that their intersection is also perpendicular to that plane.

7.*If two spheres are tangent externally, prove that their point of contact is in line with their centers.

8. Two straight lines AB and PQ are not in the same plane, and X is a point not in either of the lines. The plane ABX cuts PQ in Y and the plane PQX cuts AB in Z. Prove that X, Y, and Z are in line.

9. Two sections made by parallel planes cutting all the edges of a regular hexagonal pyramid are respectively $216\sqrt{3}$ sq. ft. and $486\sqrt{3}$ sq. ft. in area. They are 8 ft. apart. How far is the vertex of the pyramid from the larger section?

(SEE NEXT PAGE)

10. Solve $4 \tan x - 3 \cot x + 1 = 0$ for $\tan x$, and write down a general expression for all values of x corresponding to the negative result. If x is an acute angle satisfying this equation, find $\sin x$, $\cos 2x$, and $\sec (180° - x)$.

11. *a*) Prove the Law of Sines for an obtuse triangle.

    b) Calculate AB in the $\triangle ABC$ if $BC = 28.39$, angle $A = 34.5°$ (34° 30′), and angle $C = 94.5°$ (94° 30′).

12. Prove the identity $\sin (x+y) \sin (x-y) = \sin^2 x - \sin^2 y$.

13. *a*) State and prove the rule for finding the logarithm of a product, and also that for finding the logarithm of a power.

    b) Find a value of x which satisfies the equation $2 \log x \doteq \sqrt{8.345}$.

14.*Prove $\cos A - \cos B = -2 \sin \frac{1}{2} (A+B) \sin \frac{1}{2} (A-B)$. Check this formula for $A = 30°$, $B = 90°$.

ADVANCED ALGEBRA

15. *a*) From a study of the signs of $x^6 - 2x^3 + x^2 - 11 = 0$, what can be inferred as to the signs and reality of the roots?

    b) Tell what you mean by an *imaginary number*, a *rational number*, a *root* of an *equation*, a *determinant*, and give an example in each case.

16.**a*) In the formula $V = \pi h^2 \left(r - \frac{h}{3}\right)$, if $V = 198$, $\pi = \frac{22}{7}$, and $r = 8$, find a value of h.

    **b*) Solve $2x^4 - 3x^3 + 4x^2 + 8x - 8 = 0$, using the fact that $1 + \sqrt{-3}$ is a root.

17. Graph $10y = 2x^3 - 7x^2 + x + 2$.
    Prove that $2x^3 - 7x^2 + x + 2 = 0$ has a root between 3 and 4, and find this root to the nearest hundredth.

18. From 12 men how many committees of 4 can be formed? How many of these would include one particular man A? How many would include A but exclude B?

19. By the aid of determinants express the solution of the following equations for x, y, z, and calculate the value of x.

$$\frac{5}{x} - \frac{1}{y} = 4\tfrac{1}{2},$$

$$\frac{3}{y} + \frac{5}{z} = -1,$$

$$\frac{4}{x} - \frac{2}{z} = 0.$$

1919

Comprehensive Examination

ELEMENTARY MATHEMATICS

Monday, June 16 9:30 a.m.–12:30 p.m.

For candidates who have not studied advanced mathematics.

1. *a*) Solve $\frac{8x+23}{20}-\frac{5x+2}{3x+4}=\frac{2x+3}{5}-1$,

   and check the answer.

   b) Solve and check $\sqrt{5-x}+\sqrt{6-x-\sqrt{x}}=1$.

2. Solve the simultaneous equations:

$$x^2-xy+y^2=28, \quad 2x+y=14.$$

   Arrange the solutions for x and y in corresponding pairs, and check.

3. Simplify the following equation, solve it for x, and reduce the answers to their simplest form:

$$4x^{\frac{4}{3}}-\frac{13}{x^{-\frac{2}{3}}}=12x^0.$$

   Check one answer.

4. A line 2 in. long is divided into two parts so that the longer segment is the mean proportional between the whole line and the shorter segment. Find the lengths of the segments correct to the nearest hundredth of an inch.

5. Four numbers are in geometric progression. The fourth exceeds the first by 189, and the fourth exceeds the third by 144. Find the numbers.

6. State and prove the theorem by which we measure an angle between two chords that intersect within a circle.

7. *a*) In a circle a triangle ABC is inscribed. A straight line is drawn through A, cutting BC at E and the circle at D. Prove that the triangles CDE and ABE are similar.

   b) If $AE=6$ in., $ED=2$ in., and $BE=3$ in., what is the ratio of the areas of the two triangles?

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

8. *a*) Prove that the sum of the squares of the two perpendicular sides of a right triangle is equal to the square of the hypotenuse.

b) The three sides of each of three triangles are given as follows: 17, 8, 15; 5, 14, 13; 1, 2, $\sqrt{3}$. Is there a right triangle among them? Give a reason for your answer.

c) Explain how you might construct a line whose length in inches is represented by $\sqrt{5}$.

9. Find the area of a crescent-shaped region bounded by a semicircumference of a circle whose radius is 20 ft. and another arc whose center lies on this circle.

1919

ELEMENTARY MATHEMATICS

Wednesday, September 17 — 9 a.m.–12 m.

For New Plan candidates who have not studied advanced mathematics.
New Plan candidates will omit all questions marked with an asterisk (*).
Old Plan candidates will answer all questions set in any subject taken by them.

ELEMENTARY ALGEBRA

To Quadratics

1. Find the highest common factor and the least common multiple of $9x^2+9xy-10y^2$, $27x^3-8y^3$, and $(5x-4y)^2-4(x-y)^2$.

*2. Solve the simultaneous equations

$$x+y+z=27, \ \frac{y+z}{4}=\frac{z+x}{3}=\frac{x+y}{2}.$$

Check the answers.

3. *a*) Simplify the following expression, and calculate its value to the nearest hundredth:

$$\left[\tfrac{3}{8}\sqrt{48}-\tfrac{1}{5}\sqrt{33\tfrac{1}{3}}+\sqrt[4]{4}\right]\div\left[\tfrac{1}{6}\sqrt{75}-2^{\frac{1}{2}}\right].$$

b) Simplify $$\frac{a^{-1}b^{-1}}{a^{-2}-b^{-2}}\times\frac{a^0}{(a+b)^{-1}}.$$

*4. A grocer wishes to mix sugar worth 9 cents a pound with another kind worth 12 cents a pound so as to make 60 pounds worth 11 cents per pound. What quantity of each kind of sugar must be used?

Quadratics and Beyond

5. Solve $\sqrt{8x+1}-\sqrt{2x+3}=\sqrt{x+1}$, and check.

6. *a*) Solve $27x^{\frac{3}{2}}+26x^{\frac{3}{4}}=1$, and reduce the answers to their simplest form.

b) For what values of m are the roots of the equation $x^2+25=9x+mx$ equal? Prove your answers.

7. Two stations, A and B, on a railroad are 306 mi. apart. Trains start simultaneously from each station toward the other, each train running at a uniform speed. The train from A reaches B in $9\frac{9}{16}$ hours. The train from B arrives at A four hours after passing the train from A. Find the point between the stations at which the trains pass, and the speed of each train.

*8. The fifth term of an arithmetic progression is 13, and the thirteenth term is 37. Find the twenty-first term, and the sum of the first 21 terms.

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

PLANE GEOMETRY

9. *a*) If the diagonals of a quadrilateral bisect each other, prove that the quadrilateral is a parallelogram.

b) Two circles have a common center. A diameter is drawn in one with extremities A and B. In the other, a diameter, with end-points X and Y, is drawn, making an angle with AB. Show that the line joining A and X is parallel to the line joining B and Y.

***10.** *a*) From a point P outside a circle two straight lines are drawn, one cutting the circle at X and at Y, the other cutting it at A and at B, so that $PA=20$ in. and $PB=45$ in. We know that the product of PX and PY is 900 square inches. State and prove the theorem that gives us this fact.

b) A chord 5 in. long is prolonged to a point P, from which a tangent 6 in. long is drawn to the circle. How far is the chord prolonged?

11. *a*) State three theorems which may be used to prove two oblique triangles similar. Prove one of these theorems.

b) In the triangle ABC, $BC=9$ in., $CA=15$ in. On the side BC a point Y is marked, so that $CY=5$ in.; and on CA the point X is marked, so that $CX=3$ in. Draw XY, and give reasons for concluding that the triangles ABC and XYC are similar. What is the ratio of their areas?

12. *a*) In the quadrilateral $ABCD$, $AB=AD$ and $CB=CD$. If $BD=x$, and $AC=y$, what is the area of $ABCD$? Give reasons for your answer.

b) The length of the arc subtended by the side of a regular inscribed polygon of 20 sides is 44 inches. What is the radius of the circle? (Use $\pi=\frac{22}{7}$).

1919

Comprehensive Examination

ELEMENTARY AND ADVANCED MATHEMATICS

Monday, June 16 9:30 a.m.–12:30 p.m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. Solve the simultaneous equations:

$$x^2-xy+y^2=28, \quad 2x+y=14.$$

Arrange the solutions for x and y in corresponding pairs, and check.

2. Simplify the following equation, solve it for x, and reduce the answers to their simplest form:

$$4x^{\frac{4}{3}}-\frac{13}{x^{-\frac{2}{3}}}=12x^0.$$

Check one answer.

3. *a*) Prove that the sum of the squares of the two perpendicular sides of a right triangle is equal to the square of the hypotenuse.

b) The three sides of each of three triangles are given as follows: 17, 8, 15; 5, 14, 13; 1, 2, $\sqrt{3}$. Is there a right triangle among them? Give a reason for your answer.

c) Explain how you might construct a line whose length in inches is represented by $\sqrt{5}$.

4. Find the area of a crescent-shaped region bounded by a semicircumference of a circle whose radius is 20 ft. and another arc whose center lies on this circle.

Part II. Advanced

SOLID GEOMETRY

5. Prove that through a given straight line oblique to a plane, one and only one plane can be passed perpendicular to the given plane.

Is the conclusion still true if the given line is parallel to the given plane? Or perpendicular to it? State reasons in each case.

(THIS EXAMINATION IS CONTINUED ON PAGE 4)

6. Answer each of the following questions, and if your answer is in the negative give a reason by means of an illustration or otherwise:

a) If two planes are perpendicular to the same line, are they necessarily parallel?

b) If two lines are perpendicular to the same line, are they necessarily parallel?

c) If two lines are perpendicular to each other, is every plane through either necessarily perpendicular to the other?

d) If two lines are parallel to each other, is a plane containing one and not the other necessarily parallel to the other?

e) If a line is parallel to a plane, is every plane through the line necessarily parallel to the plane?

7. *a*) The angles of a spherical triangle are 98°, 72°, and 52°, respectively. Find its area in square inches, if the radius of the sphere is 30 inches. (Use $\pi = \frac{22}{7}$.)

b) What facts about the triangle polar to the one given in *a*) are established by the given data?

c) Find the altitude of the zone whose area is equal to that of the triangle given in *a*).

8. A fixed plane is 15 ft. from the center of a sphere whose radius is 6 ft. Describe and locate accurately the locus of a point which moves so that it is always at a distance of 3 ft. from the plane and 7 ft. from the surface of the sphere. (The distance of an exterior point from the surface of a sphere equals the difference between the distance of the point from the center of the sphere and the radius of the sphere.)

LOGARITHMS AND TRIGONOMETRY

9. *a*) In the triangle ABC the sides are of length a, b, and c respectively, and the angle A opposite a is an obtuse angle. Prove the law of cosines $a^2 = b^2 + c^2 - 2bc \cos A$.

b) If $a=10$, $b=6$, and $c=5$, find $\cos A$ by the formula preceding, and find A in degrees by the Tables.

10. *a*) Derive formulas expressing each function of an angle in terms of its cosine.

b) Show that $\cos 2A \cot(A+45°) = 1 - \sin 2A$ is an identity.

(THIS EXAMINATION IS CONTINUED ON PAGE 5)

11. Find the numerical value of each of the following without using the Tables:

a) $\sin 120°+\cos(-300°)$.

b) $\sin 270° \cos 180°+\csc 90° \cot 270°$.

c) $2 \tan(\cos^{-1} \frac{3}{5})$.

d) $2 \log_2 16+3 \log_{10} \sqrt{0.01}$.

12. Two buoys, A and B, are 2,685 ft. apart. A ship, C, is half a mile from the buoy A, and the angle *BCA* is 62.14° (62° 8.4′). Find the distance from the ship to the other buoy.

ADVANCED ALGEBRA

13. Graph the equation

$$y=x^3-5x^2+2x+6.$$

Point out the relation between the graph and the roots of $x^3-5x^2+2x+6=0$. Calculate one root to two decimal places.

14. Without solving $x^6-5x^4+3x-7=0$ state any four facts with regard to its roots.

15. *a*) Given the polynomial $2x^3-11x^2+62x+34$. Calculate its value for $x=3+4i$. What will be its value for $x=3-4i$?

b) What general fact concerning the roots of an equation is illustrated by the results of part *a*) of this question?

16. *a*) Prove the following theorem on determinants: If each of the elements of any row (or column) is increased by the same multiple of the corresponding element of another row (or column), the value of the determinant is unaltered.

b) How many different numbers of six figures each can be formed from the digits 1, 2, 3, 4, 5, 6, each number being such that the digits 1 and 2 are never separated?

ADVANCED MATHEMATICS

Thursday, September 18 2 p.m.–5 p.m.

For candidates who have studied any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Old Plan candidates will answer all questions in any subject which they are offering.

New Plan candidates will omit all questions marked with an asterisk (*). Correct answers to eight questions constitute a full paper. Candidates are advised to devote at least half of the examination period to the advanced part beginning with question 5.

Part I. Elementary

1. Simplify the following, and calculate its value to the nearest hundredth:

$$\frac{3}{7}\sqrt{16\tfrac{1}{3}}+15(3)^{-\frac{1}{2}}+\frac{33}{6-\sqrt{3}}.$$

2. Solve the simultaneous equations:

$$x^2+y^2=xy+37, \quad x-3y=-2.$$

Group the answers for x and y in corresponding pairs, and check for one pair.

3. Show how to construct a figure similar to a given irregular convex quadrilateral, but inclosing an area only one-ninth as great.

4. Two circles intersect in the points A and B. From a point M on the line AB, produced, two straight lines are drawn, one line intersecting one circle at X and Y, the other line cutting the other circle at P and Q. If A, X, and P are equally distant from M, prove that B, Y, and Q are also.

Part II. Advanced

SOLID GEOMETRY

5. Prove that the acute angle which a straight line makes with its own projection on a plane is the least angle which it makes with any line of the plane.

6. Prove that the lateral area of *any* pyramid (not necessarily regular) circumscribed about a right circular cone is equal to half the product of the perimeter of the base of the pyramid by the slant height of the cone.

7. A right circular cone has an altitude of 12 ft., and the radius of its base is 16 ft. How far from the vertex must a plane parallel to the base be passed if it is to cut the cone into two parts of equal volume? (Extraction of roots not required.)

   What is the radius of the sphere whose area is equal to the lateral area of the given cone?

*8. Prove that on a spherical surface any point equidistant from the extremities of a given arc of a great circle lies on the great circle which bisects the given arc at right angles.

   State the converse theorem.

9. *a*) Describe accurately the locus of a point which is twice as far from a fixed plane as from a fixed line perpendicular to the plane.

   b) What is the locus of the center of a sphere tangent to two given intersecting planes?

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

LOGARITHMS AND TRIGONOMETRY

10. *a*) Write expressions for each of the following in terms of functions of x: $\sin(90°+x)$, $\sec(180°-x)$, $\sin 2x$, $\tan \frac{1}{2}x$, $\cos(45°+x)$, $\cot(x-90°)$.

   b) Derive numerical values for the sine, cosine, and tangent of 60° and 45°, and using these values find sin 105°, cos 15°, tan $22\frac{1}{2}°$.

11. *a*) Find all angles less than 360° for which $5 \sin x - 3 \csc x = 2$, using the Tables where necessary.

   b) Compute the value of

$$\frac{0.08962\sqrt{0.35 \cos 38°}}{26 \sin 98°}.$$

12. Two stations, A and B, on one side of a river are 268.5 ft. apart. To find the distances from A and B of a station C on the opposite side of the river, the angles *ABC* and *BAC* are measured and found to be 37.4° (37° 24′), and 52.6° (52° 36′), respectively. Calculate *AC* and *BC*.

*13. *a*) Derive an expression for each function of an angle in terms of its tangent.

   b) If $\tan x = \frac{3}{4}$, and x is an angle between 180° and 270°, construct x, and write down the values of the other functions.

14. The circular measure of an angle is 2.85. If a circle of radius 2.16 in. is drawn about the vertex of the angle as a center, what is the length of the arc intercepted by the sides of the angle? How many degrees in the angle? Find the sine and tangent of the angle, using Tables.

ADVANCED ALGEBRA

15. Reduce each of the following to the form $a+ib$:

$$\frac{2-4i}{1-i}; \quad (1-i)^3.$$

Represent graphically the resulting numbers, and also their sum and their difference.

16. By the use of determinants find whether the value of y or of z satisfying the following simultaneous equations is numerically the greater:

$$2x+3y-5z=4, \quad 3x-2y+3z=5, \quad 4x+y-4z=8.$$

17. *a*) Three roots of $2x^4-6x^3+kx^2+mx+n=0$ are -1, 2, and 4. What is the fourth root?

   b) If the roots of $x^4-4x^3+6x-8=0$ are a, b, c, and d, write the two equations of which the roots are respectively $2a$, $2b$, $2c$, $2d$, and $-a$, $-b$, $-c$, $-d$.

18. Find the real roots of the following equation, either exactly or to the nearest tenth:

$$x^4-7x^3+14x^2+2x-20=0.$$

*19. Given 11 points, no three of them lying on a line. How many triangles are there with these points as vertices?

1920

Comprehensive Examination

ELEMENTARY MATHEMATICS

Monday, June 21 9:30 a.m.–12:30 p.m.

For candidates whose school records submitted for admission include Elementary Algebra Complete and Plane Geometry only.

1. *a*) Simplify $\left(x-y-\frac{4y^2}{x-y}\right)\left(x^2-\frac{x^2y+y^3}{x+y}\right)\div\left(x-y+\frac{3xy}{x-y}\right)$.

   b) Write and simplify the first three terms in the expansion of $(x^{\frac{2}{3}}-2x^{\frac{1}{3}})^{10}$.

2. Solve $\sqrt{2x+1}+\sqrt{x-3}-\sqrt{3x+4}=0$. Test one of your results.

3. Solve $2x-3y-4=0$, $2x^2-3y^2-x-2y+1=0$. Write the answers so that corresponding values are paired. Test one pair.

4. There are three numbers in arithmetic progression, the common difference being 3. If the first be decreased by 1, the second increased by 3, and the third be doubled, the resulting numbers are in geometric progression. Find the numbers.

5. A can do a piece of work in two hours less than it would take B; together they would take $2\frac{1}{12}$ hours less than A alone. How long would it take each working alone to do the work?

6. If two sides of a quadrilateral are equal and parallel, prove that the other two sides are equal and parallel and the figure is a parallelogram.

7. *a*) Two chords intersect within a circle. State and prove the theorem concerning the relation between the segments of these chords.

   b) Find the length of the longest chord and of the shortest chord that can be drawn through a point 6 inches from the center of a circle whose radius is 10 inches.

8. If AB is the hypotenuse of a right triangle, and the leg BC is bisected at K, prove that $\overline{AB}^2-\overline{AK}^2=3\overline{CK}^2$.

9. A circle is described on one side of an equilateral triangle as a diameter. If one side of the triangle is 6 inches, find the area of that part of the triangle that lies without the circle.

ELEMENTARY MATHEMATICS

Wednesday, September 22 9 a.m.–12 m.

For New Plan candidates whose school records submitted for admission include Elementary Algebra Complete and Plane Geometry only.

New Plan candidates will omit all questions marked with an asterisk (*).

Old Plan candidates will answer all questions set in any subject taken by them.

ELEMENTARY ALGEBRA

To Quadratics

1. *a*) Factor $12x^3-26x^2-16x$; x^3-8; $x^2+2x-4y^2-4y$.

   b) Simplify

$$\frac{x(16-x)}{x^2-4}+\frac{2x+3}{2-x}-\frac{2-3x}{x+2}.$$

*2. Solve for x

$$\frac{x}{x+b-a}+\frac{b}{x+b-c}=1.$$

3. Simplify the following and find its value to the nearest hundredth:

$$9^{\frac{3}{2}}+\sqrt[4]{16}-\frac{5}{2^{-2}-3^{-2}}+\sqrt{66\tfrac{2}{3}}+32(8^{-\frac{4}{3}})-3^0(4^{\frac{1}{2}}).$$

*4. A, B, C, and D have $176 between them. A and D together have as much as C; C and D together have $16 less than B; B and C together have four times as much as A. How much has each?

Quadratics and Beyond

5. *a*) Solve $\sqrt{5-2x}-\sqrt{x+3}-2=0$. Test one of your results.

   b) Solve for x $\quad 9x^{-\frac{4}{3}}+4-37x^{-\frac{2}{3}}=0.$

6. Plot the graphs of $x+2y=5$ and $y=x^2+1$ on the same axes. Estimate the values of x and y which satisfy the equations simultaneously.
   Solve the equations and compare the answers with those found from the graph.

*7. Find the 100th term of the progression $1\frac{1}{2}$, 3, $4\frac{1}{2}$, etc. How many terms must be taken to make the sum $1912\frac{1}{2}$?

8. A, B, and C start to walk from the same point. A starts at 10 o'clock at a rate of a mile and a half per hour. B follows after him at 11 o'clock; C after both at 12:30, walking 2 miles per hour faster than B. C and B overtake A at the same time. Find B's rate of walking and the time during which A walked.

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

PLANE GEOMETRY

9. Prove that an angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.

10. Prove that two triangles are similar if their homologous sides are in proportion.

*11. *a*) Prove that the area of a parallelogram is equal to the product of its base by its altitude.

b) Construct a parallelogram whose area is 12 square inches, and having its base equal to 4 inches, and an angle equal to 45°.

12. Two circles intersect at P and Q. The lines PA and PB are diameters. Prove that QA and QB form a straight line.

13. If squares are constructed on the sides of a regular hexagon and external to the hexagon, prove that the lines joining their exterior vertices form a regular polygon of twelve sides. Find the area of this regular polygon if the side of the hexagon is 3 inches.

1920

Comprehensive Examination

ELEMENTARY AND ADVANCED MATHEMATICS

Monday, June 21 9:30 a.m.–12:30 p.m.

For candidates whose school records submitted for admisson include any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

Correct answers to nine questions constitute a full paper.

Part I. Elementary

1. Solve $2x-3y-4=0$, $2x^2-3y^2-x-2y+1=0$. Write the answers so that corresponding values are paired. Test one pair.

2. There are three numbers in arithmetic progression, the common difference being 3. If the first be decreased by 1, the second increased by 3, and the third be doubled, the resulting numbers are in geometric progression. Find the numbers.

3. A can do a piece of work in two hours less than it would take B; together they would take $2\frac{1}{12}$ hours less than A alone. How long would it take each working alone to do the work?

4. *a*) Two chords intersect within a circle. State and prove the theorem concerning the relation between the segments of these chords.

   b) Find the length of the longest chord and of the shortest chord that can be drawn through a point 6 inches from the center of a circle whose radius is 10 inches.

5. A circle is described on one side of an equilateral triangle as a diameter. If one side of the triangle is 6 inches, find the area of that part of the triangle that lies without the circle.

Part II. Advanced

SOLID GEOMETRY

6. Prove that a line and a plane which are perpendicular to the same line are parallel.

7. Prove that a triangular prism may be divided into three triangular pyramids having equal volumes.

8. A right circular cone is inscribed in a sphere, its base being a great circle of the sphere, and its vertex lying in the surface of the sphere. Find the ratio of the lateral surface of the cone to the surface of the sphere, and also the ratio of their volumes.

9. Describe a method of passing a line through a given point which shall cut two given lines not lying in the same plane.

(THIS EXAMINATION IS CONTINUED ON PAGE 4)

LOGARITHMS AND TRIGONOMETRY

10. *a*) Define the logarithm of a positive number to a given base. State and prove the rule concerning the logarithm of the product of two numbers.

b) Find the value of $2 \log_4 2 + 5 \log \sin 90° + \log \tan 225°$.

11. *a*) Prove the formula $\sec^2 x = 1 + \tan^2 x$, using a figure, when x is an angle of the third quadrant.

b) Solve $2 \sec^2 x + \tan x - 3 = 0$ for $\tan x$, and determine all values of x between 0° and 540° which satisfy the equation, using tables when necessary.

12. Given the triangle ABC with $C = 58° \; 37'$ (58.62°), $AC = 611.4$, and $BC = 483.4$. Find the remaining parts.

13. Prove that the relation $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$ holds for all values of x.

ADVANCED ALGEBRA

14. *a*) By a study of the signs of $x^6 - 4x^3 + x^2 - 11 = 0$ what may be inferred about the roots?

b) Locate the roots of $x^3 + 3x^2 - 4x - 8 = 0$ between consecutive integers, and find the positive one to the nearest hundredth.

15. In how many ways may four men be chosen from 5 Americans and 6 Frenchmen so as to include (*a*) just one American? (*b*) at least one American? (*c*) one particular American, Mr. A?

16. Evaluate

$$\begin{vmatrix} 3 & -2 & -2 & 1 \\ 2 & 3 & -3 & 4 \\ 0 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \end{vmatrix}.$$

17. Solve $x^4 + 6x^3 + 9x^2 + 8x - 6 = 0$, knowing that the sum of two of the roots is -4 and that the product of the other two is 3.

1920

ADVANCED MATHEMATICS

Thursday, September 23 2 p.m.–5 p.m.

For New Plan candidates whose school records submitted for admission include any branch of advanced mathematics, namely, solid geometry, trigonometry, or advanced algebra.

New Plan candidates will omit all questions marked with an asterisk (*). Correct answers to nine questions constitute a full paper.

Old Plan candidates will answer all questions in any subject which they are offering.

Part I. Elementary

1. Solve

$$\frac{x}{x+1}-\frac{2x+1}{1-x}+\frac{3x-2}{x^2-1}+3=0.$$

2. Solve $3x+2y-2=0$, $3x^2-2y^2-5x-y+4=0$. Arrange the answers in corresponding pairs and test one pair.

3. A certain number contains two digits. If the number is multiplied by the digit in unit's place the result is sixteen times the sum of the digits. Twice the digit in ten's place exceeds five times the digit in unit's place by 4. Find the number. (No credit will be given for merely guessing the answer.)

4. State and prove the theorem concerning the area of a regular polygon.

5. Two circles are tangent externally at the point A. A line is drawn through A meeting the circles in P and Q. Show that the ratio $AP:AQ$ is the same for all lines through A.

Part II. Advanced

SOLID GEOMETRY

6. Prove that all the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.

7. *a*) Prove that a section of a pyramid made by a plane parallel to the base is similar to the base.

   b) If the altitude of the pyramid is 12 inches, how far from the vertex must the plane be passed in order that the area of the section formed may be half that of the base?

*8. Prove that every point in a plane which bisects a diedral angle is equally distant from the faces of the angle.

(THIS EXAMINATION IS CONTINUED ON PAGE 2)

9. On a sphere 9 inches in diameter the sides of a spherical triangle are arcs of 90°, 90°, and 40°. What are the angles? Find the area of the triangle. (Use $\pi=\frac{22}{7}$.)

   How would you pass a circle through the three vertices of this triangle? Does the plane of this circle pass through the center of the sphere? Give a reason for your answer.

10. A point P is at a distance of 4 inches from a given plane. A second plane forms a diedral angle of 45° with the given plane. Describe a method of drawing a line 6 inches long from P to the first plane so that this line shall be parallel to the second plane.

LOGARITHMS AND TRIGONOMETRY

11. *a*) If $\sin A=0.8$, and A is an angle of the second quadrant, find the numerical values of $\cos A$, $\tan 2A$, $\csc (90°+A)$, $\cot (180°+A)$.

    b) Find a value for $\sin^{-1} 0.8+\cos^{-1} 0.8$.

*12. Prove that the area of any triangle equals the product of two sides by half the sine of the included angle. What is the area of the triangle ABC if $AB=2$ inches, $BC=3$ inches, and angle $B=150°$?

13. Show that $\frac{\cos 2x}{1-\sin 2x}=\frac{1+\tan x}{1-\tan x}$ is true for all values of x.

14. In a parallelogram the sides are respectively 12.18 in. and 25.22 in. One diagonal is 28.58 in. Find the angles.

15. *a*) What is the circular measure of an angle of 85°?

    b) Find all values of x between 0° and 450° for which $\sin (120°+x)+\sin (x+60°)=1.5$.

ADVANCED ALGEBRA

16. *a*) Draw the graph of $y=x^3-4x^2-4x+12$.

    b) Locate the roots of $x^3-4x^2-4x+12=0$ between consecutive integers and find the larger positive one to the nearest hundredth.

17. Write the value of y in the following equations as the quotient of two determinants. Evaluate the denominator.

$$2x-3y+z+4w=2;$$
$$x+4y-3z=1;$$
$$3x+2z-w=-3;$$
$$2y+z-3w=4.$$

18. One root of $x^4-4x^3+5x^2+2x+52=0$ is $3-2i$. Find the others.

*19. Find all the roots of $2x^5+5x^4-19x^3-44x^2+2x+12=0$.

20. From the digits 0, 1, 2, 3, 4, 5, 6, how many numbers between 100 and 10,000 may be formed, no digit being repeated in any one number? How many of these are odd?